Moments of the Riemannn Zeta Function and Log-Correlated Random Variables

Michael J. Curran
St Hilda's College
University of Oxford

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Recent developments in random matrix theory and the theory of log-correlated random variables have led to several interesting conjectures in analytic number theory. This conjectures primarily concern the asymptotic behavior of several quantities related to the moments of the zeta function. In this dissertation, we prove estimates of the conjectured order of magnitude for several of these quantities. First we will give upper bounds for the joint moments of the $2k^{\text{th}}$ power of the zeta function with the $2h^{\text{th}}$ power of its logarithmic derivative when $k \in [1, 2]$ and $h \in [0, 1]$. Then in joint work with André Heycock, we extend these upper bounds to all $0 \leq h \leq k \leq 2$ unconditionally and for all $0 \leq h \leq k$ assuming the Riemann hypothesis. We also prove unconditional lower bounds for all $0 \leq h \leq k$, demonstrating that our upper bounds are of the correct order of magnitude.

We then move on to study the problem of moments of moments of zeta and the closely related problem of estimating shifted moments of zeta on the half line. Moments of moments of zeta were introduced by Fyodorov and Keating [42] in order to understand the value distribution of zeta in short intervals on the critical line. By calculating moments of moments of random matrices, they formed a conjecture for the asymptotic behavior of the moments of moments of zeta and the distribution of the maximum of zeta in random short intervals. We prove upper bounds of the conjectured order for the $(2, \beta)$-moment of moments of zeta when $0 \leq \beta \leq 1$ and lower bounds of the conjectured size for all $\beta \geq 0$. In particular, we demonstrate that there is a phase transition that occurs at $\beta = \frac{1}{\sqrt{2}}$.

The key quantity we need to estimate to understand the (k, β)-moments of moments are averages of k shifted products of zeta raised to the power β. Unconditionally, we can only estimate these shifted moments when $k = 1$ and $\beta \leq 2$ or when $k = 2$ and $0 \leq \beta \leq 1$. Assuming the Riemann hypothesis, however, we give sharp upper and lower bounds for more general shifted moments of zeta. This improves upon previous work of Chandee [21] and Ng, Shen, and Wong [77]. Our bounds are especially interesting when the the shifts are unbounded and the random matrix theory model is no longer appropriate.

Statement of Originality

I confirm that the work in this thesis has not been submitted for any other academic award. The work presented is my own unless otherwise stated.

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Notation

$f \ll g$ or $f = O(g)$	$\lvert f(x)\rvert \leq Cg(x)$ for some $C > 0$
$f \ll_\beta g$ or $f = O_\beta(g)$	$\lvert f(x)\rvert \leq C_\beta g(x)$ for some $C_\beta > 0$
$f \gg g$	$g \ll f$
$f \gg_\beta g$	$g \ll_\beta f$
$f \asymp g$	$f \ll g$ and $g \ll f$
$f \asymp_\beta g$	$f \ll_\beta g$ and $g \ll_\beta f$
$f \sim g$	$\lim_{x\to\infty} f(x)/g(x) = 1$
$a \mid b$	a divides b
$p^a \\| b$	p^a divides b and p^{a+1} does not divide b for a prime p
$\Omega(n)$	Number of divisors with multiplicity $\sum_{p^a \\| n} a$
$\int_{(c)} f(s)ds$	Integral of $f(s)$ over the contour Re $s = c$
$\\|f\\|_p$	L^p norm of f for $p > 0$
$\log_j x$	j-fold iterated logarithm

Contents

Chapter 1

Introduction

1.1 The zeta function

One of the most important functions in all of mathematics is arguably the Riemann zeta function, defined by

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}$$

for complex s such that Re $s > 1$. Its importance is a consequence of the following consequence of the unique factorization of integers into primes observed by Euler:

$$\sum_{n\geq 1} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \text{ when Re } s > 1,$$

where here and throughout this thesis p will denote prime numbers. The left hand side is defined solely in terms of the equally spaced positive integers, while the right hand side– the so called Euler product– is defined using only the set of prime numbers, a set which is much more difficult to understand. One of the central goals of the subject of analytic number theory is to use analytic information about $\zeta(s)$ to deduce interesting information about the set of prime numbers.

One of the biggest successes of analytic number theory to date is the prime number theorem:

$$\pi(x) := \#\{p \leq x\} \sim \int_2^x \frac{dt}{\log t}.$$

The first proof of this theorem came from an idea due to Riemann. Riemann found that there is a natural way to extend $\zeta(s)$ to a function on the entire complex plane that satisfies the functional equation

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

Then by using Fourier analysis, Riemann gave a formula for $\pi(x)$ in terms of all the zeros of $\zeta(s)$. Therefore if one could determine enough information about the zeros of $\zeta(s)$, one would obtain good estimates for the prime counting function $\pi(x)$. It is straightforward to show that $\zeta(s)$ has no zeros in the half plane $\mathrm{Re}\ s > 1$, so by the functional equation the only zeros in the half plane $\mathrm{Re}\ s < 0$ lie at the negative even integers. All the remaining zeros therefore lie in the so-called critical strip $0 \leq \mathrm{Re}\ s \leq 1$. Riemann showed that the number of zeros in the critical strip with imaginary part between 0 and T is approximately

$$\frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi},$$

and he conjectured that all of the zeros in the critical strip in fact lie on the critical line $\mathrm{Re}\ s = \frac{1}{2}$. This conjecture is the well known Riemann hypothesis. To this day, however, no one has been able to come close to proving it. One interesting consequence of the Riemann hypothesis is the following much more precise estimate on the prime counting function

$$\pi(x) = \int_2^x \frac{dt}{\log t} + O(\sqrt{x}\log x).$$

Without the assumption of the Riemann hypothesis, the strongest estimate for $\pi(x)$ to date is due to Vinogradov and Korobov [86]:

$$\pi(x) = \int_2^x \frac{dt}{\log t} + O(xe^{-c(\log x)^{3/5}/(\log\log x)^{1/5}})$$

for some $c > 0$.

The two most important features of any complex function are its zeros and its growth rate. While the Riemann hypothesis concerns the zeros of $\zeta(s)$, there is another conjecture concerning the growth rate of $\zeta(s)$ on the critical line: the Lindelöf hypothesis. This states that

$$\zeta(\tfrac{1}{2} + it) \ll_\varepsilon t^\varepsilon$$

for any $\varepsilon > 0$. Using interpolation theory and the functional equation, one can use the Lindelöf hypothesis to determine the growth rate of $\zeta(s)$ throughout the entire complex plane. It is well known that the Riemann hypothesis implies the Lindelöf hypothesis [86]. The Lindelöf hypothesis is a very difficult question nonetheless, and the strongest progress made to date is the bound $\zeta(\frac{1}{2}+it) \ll_\varepsilon t^{13/84+\varepsilon}$ due to Bourgain [20].

To get an idea of why one might believe that the Lindelöf hypothesis is true, we will make a comparison between $\zeta(\frac{1}{2} + it)$ and a random model. Even though the Euler product for zeta is only convergent when Re $s > 1$, we can still consider trunctations of this product in the critical strip. Take t uniformly at random from 0 to T and consider

$$\prod_{p\le T}\left(1-\frac{1}{p^{1/2+it}}\right)^{-1}.$$

We can think of this random Euler product as a model for $\zeta(\frac{1}{2}+it)$. Taking the logarithm of the absolute value, Taylor expanding, and discarding terms of order p^{-2} or smaller, we obtain the sum

$$\sum_{p\le T}\text{Re }\frac{1}{p^{1/2+it}}.$$

Now for each p, the quantity Re $p^{-1/2-it}$ is a random variable on the circle of radius $p^{-1/2}$ in the complex plane. Its mean is approximately zero and its variance is approximately given by

$$\frac{1}{T}\int_0^T\frac{\cos^2(t\log p)}{p}\,dt\sim\frac{1}{2p}.$$

The central limit theorem therefore leads us to predict that the value distribution of $\log|\zeta(\frac{1}{2}+it)|$ for $t\in[0,T]$ is approximately Gaussian with mean 0 and variance

$$\sum_{p\le T}\frac{1}{2p}.$$

By Merten's estimate, this sum is asymptotically equal to $\frac{1}{2}\log\log T$. It turns out this heuristic can be made rigorous, and it is a theorem of Selberg [84] that as $T\to\infty$

$$\frac{1}{T}\text{ meas}\left\{t\in[T,2T]:\frac{\log|\zeta(\frac{1}{2}+it)|}{\sqrt{\frac{1}{2}\log\log T}}\ge V\right\}\sim\frac{1}{\sqrt{2\pi}}\int_V^\infty e^{-x^2/2}dx.$$

The reader may consult Tsang's thesis [87] for a detailed proof of this result. Therefore typically the size of $\log|\zeta(\frac{1}{2}+it)|$ is of the order $\sqrt{\log\log T}$, which is consistent with the prediction of the Lindelöf hypothesis.

One of the earliest approaches to make progress on the Lindelöf hypothesis was the introduction of the moments of zeta by Hardy and Littlewood [48]. For $\beta\ge0$, define the $2\beta^{\text{th}}$ moment of zeta by

$$M_\beta(T)=\frac{1}{T}\int_T^{2T}|\zeta(\tfrac{1}{2}+it)|^{2\beta}dt.$$

The Lindelöf hypothesis is equivalent to the estimate

$$M_\beta(T)\ll_{\varepsilon,\beta}T^\varepsilon$$

for every $\beta \geq 0$ and $\varepsilon > 0$. Notice that $M_\beta(T)$ may also be thought of as the moment generating function of $\log|\zeta(\frac{1}{2}+it)|$. Because the moment generating function $M(t)$ of a Gaussian random variable of mean zero and variance σ^2 is equal to $e^{\sigma^2 t^2/2}$, in view of Selberg's central limit theorem it should not be so surprising that it has long been conjectured that

$$M_\beta(T) \sim C_\beta (\log T)^{\beta^2}$$

for certain constants C_β. This heuristic reasoning cannot, however, lead to a proof of the Lindelöf hypothesis: Selberg's central only describes the typical values of $\zeta(\frac{1}{2}+it)$, while the Lindelöf hypothesis concerns the extreme values of zeta.

Asymptotics for $M_\beta(T)$ are known for $\beta = 1$ due to Hardy and Littlewood [48]

$$M_1(T) \sim \log T$$

and for $\beta = 2$ due to Ingham [58]

$$M_2(T) \sim \frac{1}{2\pi^2}(\log T)^4.$$

Lower bounds of the expected order are known for all $\beta \geq 0$ due to work of Heap and Soundararajan [53] and Radziwiłł and Soundararajan [82]

$$M_\beta(T) \gg_\beta (\log T)^{\beta^2}.$$

Sharp upper bounds are known for all $0 \leq \beta \leq 2$ due to Heap, Radziwiłł, and Soundararajan [52]

$$M_\beta(T) \ll_\beta (\log T)^{\beta^2},$$

and Harper [50] showed this bound holds for all $\beta \geq 0$ assuming the Riemann hypothesis. For the remaining moments, the best estimates available come from either Bourgain's subconvexity estimate [20] or Heath-Brown's bound for the twelfth moment of zeta (recall that we are rescaling the moments of zeta by a factor of $1/T$) [54]

$$M_6(T) \ll T(\log T)^{17}.$$

For a long time, the values of the constants C_β were very mysterious and there were only a few guesses for how they depend on β; this is where random matrix theory enters the picture.

1.2 Random matrix theory

While random matrix theory goes back to Wishart's work in multivariate statistics, the study of random matrices has largely been driven by applications to physics, particularly by statistical models for heavy nuclei. In this setting, the energy levels of a system are determined by the time independent Schrödinger equation

$$H\psi_i = E_i\psi_i,$$

where the Hamiltonian H is a Hermitian operator on an infinite dimensional Hilbert space containing the stationary states ψ_i. In practice, the operator H is unknown and far too complicated to obtain solutions to the corresponding Schrödinger equation. It is natural to approximate the infinite dimensional operator H by a large dimensional matrix, however this doesn't address the problem that the corresponding matrix is typically unknown and too complicated to analyze.

Wigner proposed that instead of analyzing the eigenvalues of any fixed Hamiltonian matrix H, one should look at the average behavior of the eigenvalues of families of random matrices. We must impose the constraint that the random matrices are Hermitian, but otherwise we can take the upper triangular entries to be independent copies of any distribution. Through a Monte Carlo analysis, Porter and Rosenzweig [80] showed that eigenvalues of random Hermitian matrices shared many properties with energy levels of some simple systems. For example, both the energy levels and the eigenvalues of random matrices seem repel their neighbors: very rarely did they observe gaps between energy levels or eigenvalues that were significantly smaller than the average gap. This is in stark contrast to the spacing of points that are generated uniformly at random from a given interval or from Poisson point processes.

The numerical analysis also showed that the eigenvalue distributions of random matrices did not seem to depend on the underlying distribution of the matrix entries. This observation helped lead to what is now known as *Wigner's semicircle law* [1]: Let M be an N by N Hermitian matrix whose off-diagonal entries M_{jk} with $j < k$ are independent and identically distributed (i.i.d.) random variables with mean zero and variance 1, and whose diagonal entries M_{jj} are i.i.d. with mean zero and finite variance. Then if $\lambda_1, \dots, \lambda_N$ are the eigenvalues of the normalized matrix $\frac{1}{\sqrt{N}}M$, define the empirical spectral distribution by

$$\mu_N = \frac{1}{N}\sum_{j\le N}\delta_{\lambda_j}.$$

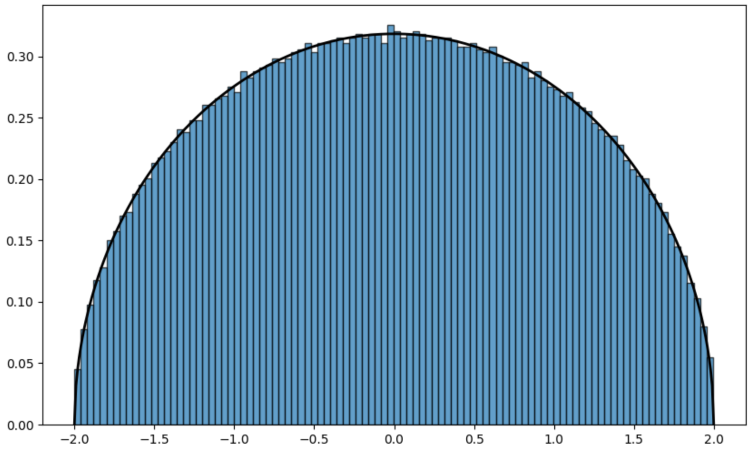

Figure 1.1: A histogram of the empirical spectral density of a 10,000 by 10,000 random matrix generated from the Gaussian unitary ensemble alongside the semicircle density.

In other words μ_N attaches a point mass of weight $1/N$ to each eigenvalue. Wigner's semicircle law states that the probability measures μ_N converge weakly in probability to the semicircle distribution on $\mathbb{R}$, which has density

$$\frac{1}{2\pi}\sqrt{4-x^2}$$

for $|x| \leq 2$. The same limiting distribution is observed regardless of whether the underlying random matrices are real symmetric, complex Hermitian, or even symplectic. The local statistics, however, do differ depending on the symmetry class of the random matrix ensemble. This includes statistics such as the spacing between consecutive eigenvalues and the largest eigenvalue. We will now discuss the three canonical ensembles of each symmetry type: the Gaussian ensembles.

1.2.1 The Gaussian ensembles and n-point correlations

The Gaussian random matrix ensembles are determined by two properties. We first require that for each matrix M in our ensemble that the entries M_{jk} must be independent random variables for $j \leq k$. The second property is that the distribution of M must be invariant under certain changes of basis. There are three types of symmetry that give rise to the three Gaussian ensembles. If the law of M is invariant under the transformation

$$M \mapsto O^{-1}MO$$

for all N by N orthogonal matrices O, then the resulting ensemble is called the Gaussian orthogonal ensemble (GOE). If instead we require that the law of M is invariant under the transformation

$$M \mapsto U^{-1}MU$$

for all N by N unitary matrices U, we will obtain the Gaussian unitary ensemble (GUE). The final ensemble arises when we require invariance under

$$M \mapsto S^{-1}MS$$

for all N by N unitary symplectic matrices S. Not surprisingly, it is called the Gaussian symplectic ensemble (GSE). While the local statistics of each of the three Gaussian ensembles have different limiting distributions, most of the general principles by which one analyzes these statistics are the same. Therefore going forward we will focus our discussion on the GUE, which is the most important case for applications to the moments of the zeta function.

First we shall discuss the distribution of the entries of a GUE matrix M. The independence assumption on the entries ensures that the density $P(M)dM$ factors as the product of the densities $P(M_{jk})dM_{jk}$ of each entry for $j \leq k$. Then if $M' = U^{-1}MU$, the unitary invariance property implies that

$$P(M')dM' = P(M)dM.$$

By considering the unitary matrices in a small neighborhood of the identity matrix, one can then differentiate this expression at the identity matrix to derive a partial differential equation for the densities $P(M_{jk})$. In this manner, one may show (see Theorem 2.6.3 of [71]), that $P(M)$ must be of the form

$$P(H) = \exp\left(-a\mathrm{Tr}M^2 + b\mathrm{Tr}M + c\right)$$

for certain constants $a > 0$, $b, c \in \mathbb{R}$. In fact the same result holds for the GOE and the GSE. After shifting and rescaling the entries M_{jk}, we may assume the density takes the form

$$P(H) = \frac{1}{Z_{\mathrm{GUE}}} \exp\left(-\mathrm{Tr}M^2\right)$$

for an appropriate normalization constant Z_{GUE}.

Now we will consider the eigenvalue density for the GUE. Instead of parameterizing M in terms of its entries, we can parameterize it in terms of its eigenvalues and a unitary matrix, and then find the marginal distribution of the eigenvalues alone. Indeed if the eigenvalues of M are distinct then we may diagonalize

$$M = U^{-1}DU$$

where D is a diagonal matrix containing the eigenvalues of M and U is a unitary matrix. Assuming M has distinct eigenvalues is of no issue, for if we think of M as an element of $\mathbb{R}^N \times \mathbb{C}^{N(N-1)/2}$, then the matrices with repeated eigenvalues are contained in a finite union of hypersurfaces of codimension at least 1. Therefore M has distinct eigenvalues with probability 1. We would like to change variables from the entries M_{jk} to pairs of a unitary matrix U and a diagonal matrix D. Even though the mapping

$$(U, D) \mapsto U^{-1}DU$$

is many to one, this will not cause issues when changing variables as we can readjust the normalization constant after the dust settles. Because of the identity

$$\mathrm{Tr} M^2 = \sum_{j \le N} \lambda_j^2,$$

in order to find the eigenvalue density, we must compute the Jacobian $|J(\lambda_1, \ldots, \lambda_N, U)|$ of the change of variables and then integrate the expression

$$\frac{1}{Z_{\mathrm{GUE}}} \exp\left(-\sum_{j \le N} \lambda_j^2\right) |J(\lambda_1, \ldots, \lambda_N, U)|$$

over all unitary matrices U. For this choice of parameters, it turns out that the Jacobian factors as [71]

$$\prod_{1 \le j < k \le N} (\lambda_j - \lambda_k)^2 \times f(U)$$

for some function f. Therefore the eigenvalue density for the GUE is equal to

$$\rho(\lambda_1, \ldots, \lambda_N) := \frac{1}{Z_N} \exp\left(-\sum_{j \le N} \lambda_j^2\right) \prod_{1 \le j < k \le N} (\lambda_j - \lambda_k)^2$$

for an appropriate normalization constant Z_N. This formula is an example of the more general Weyl integration formula, which may also be used to find the eigenvalue density of many other random matrix ensembles [2]. Notice in the eigenvalue density that the factors $(\lambda_j - \lambda_k)^2$ vanish as two eigenvalues approach one another, which makes rigorous the observation of Porter and Rosenberg [80] that the eigenvalues appear to repel each other. Having an exact formula for the eigenvalue density for all N is extremely valuable, and we will now discuss some of its consequences.

Many interesting statistics about the eigenvalues of a GUE matrix M can be written in terms of the k-point correlation function

$$R_k(\lambda_1, \ldots, \lambda_k) = \frac{N!}{(N-k)!} \int_{\mathbb{R}^{N-k}} \rho(\lambda_1, \ldots, \lambda_N)\ d\lambda_{k+1} \cdots d\lambda_N.$$

The 1-point correlation function $R_1(x)$ is called the level density, which describes the probability density of finding an eigenvalue in $[x, x + dx]$. Calculating the 1 level density allows one to describe the bulk distribution of the eigenvalues for finite N, not just in the limit of large dimension. This will be helpful later when using random matrices to model the zeta function. There is a similar interpretation for all the higher point correlation functions.

What is particularly remarkable about the Gaussian ensembles is that they are determinantal point processes [71]. This means that is there some function $K : \mathbb{R}^2 \to \mathbb{R}$ called a kernel such that

$$R_k(\lambda_1, \ldots, \lambda_k) = \det\left(K(\lambda_j, \lambda_k)\right)_{j,k \leq N}.$$

For the GUE, this kernel can be described in terms of the Hermite polynomials

$$H_N(x) := (-1)^N e^{-x^2/2} \frac{d^N}{dx^N} e^{x^2/2}.$$

If we set

$$\phi_N(x) = \frac{1}{\sqrt{N!\sqrt{2\pi}}} H_N(x) e^{-x^2/4},$$

then the kernel for the GUE is given by

$$K(x, y) = \sqrt{N} \frac{\phi_N(x)\phi_{N-1}(y) - \phi_{N-1}(x)\phi_N(y)}{x - y}$$

for $x \neq y$ and

$$K(x, x) = \sqrt{N}(\phi_N'(x)\phi_{N-1}(x) - \phi_{N-1}'(x)\phi_N(x)).$$

Therefore for each N, the normalized level density of GUE eigenvalues is given by

$$\frac{1}{\sqrt{N}} K\left(\sqrt{N}x, \sqrt{N}x\right).$$

Figure 1.2 demonstrates this for $N = 5$. Using this formula directly, one can use asymptotics for the Hermite polynomials to show that that $R_1(\sqrt{N}x)/\sqrt{N}$ tends to the semicircle distribution as $N \to \infty$ to give a direct proof of Wigner's semicircle law for the Gaussian ensembles.

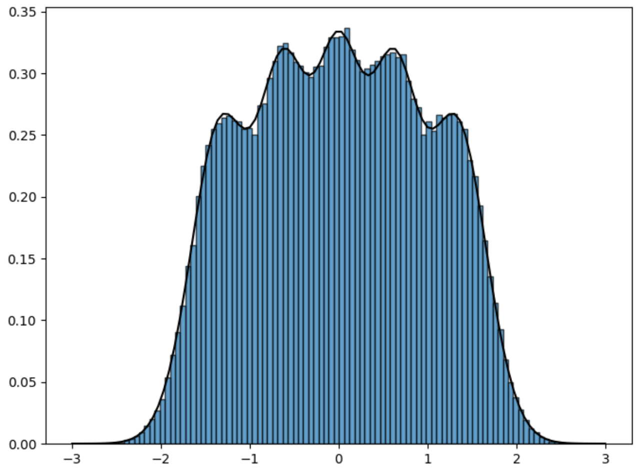

Figure 1.2: Histogram of the density of 50,000 GUE matrices of dimension 5 compared to the level density for 5 by 5 GUE matrices

One can also use the correlation functions to analyze local statistics of the GUE eigenvalues. By the semicircle law the average spacing between eigenvalues is of size $1/\sqrt{N}$. To study the eigenvalue distribution at this scale, the relevant object of interest is now

$$\frac{1}{\sqrt{N}} K\left(x/\sqrt{N}, x/\sqrt{N}\right).$$

As $N \to \infty$, this kernel approaches the sine kernel [71, eq. 6.3.13]

$$\frac{\sin(\pi(x-y))}{\pi(x-y)}.$$

Therefore the two point correlation, also known as the pair correlation of the eigenvalues, is equal to

$$1 - \frac{\sin^2(\pi(x-y))}{\pi^2(x-y)^2}.$$

We will encounter this function again when we look at both the circular unitary ensemble and surprisingly enough the zeros of the zeta function.

1.2.2 The circular ensembles

The downside of the Gaussian ensembles is that the set of N by N matrices is not compact, so it is impossible to assign each matrix an equal weight. There is another family of ensembles, however, that does not face this issue: the so-called circular ensembles. Because the unitary group is compact, we can assign to this group a uniform probability measure commonly known as the Haar measure. This is also true for the orthogonal group and the group of unitary symplectic matrices. These give rise to the circular unitary ensemble (CUE), the circular orthogonal ensemble (COE), and the circular symplectic ensemble (CSE) respectively.

While the circular ensembles have the advantage of compactness, it is not immediately clear how to actually generate matrices from any of the circular ensembles. To generate a random CUE matrix, one first generates an N by N matrix with standard complex Gaussian entries, otherwise known as a Ginibre matrix. Then one can apply the Gram-Schmidt process to obtain a unitary matrix. The resulting matrix will be a CUE matrix [70]. The entries of the circular ensembles do not have a very clean description like the Gaussian ensembles. Nonetheless, all N^2 entries are identically distributed, and as $N \to \infty$ the entries of a CUE matrix are approximately Gaussian with variance $1/N$ [70, Corollary 2.6].

While the entries of CUE matrices do not have a very clean exact formula, there is still a clean expression for the density of CUE eigenvalues derived by the Weyl integration formula [70, Theorem 3.1]. All eigenvalues of a unitary matrix lie on the unit circle, so we may parameterize them by the eigenangles $e^{i\theta_k}$, where $0 \leq \theta_k < 2\pi$ for $k = 1, \ldots, N$. The density of CUE eigenangles is then given by

$$\rho(\theta_1, \ldots, \theta_k) = \frac{1}{n!(2\pi)^n} \prod_{1 \leq j < k \leq N} |e^{i\theta_j} - e^{i\theta_k}|^2.$$

As with the GUE, the presence of the $|e^{i\theta_j} - e^{i\theta_k}|^2$ factors implies that the eigenvalues will repel one another.

Like the Gaussian ensembles, the eigenangles of CUE matrices are also a determinantal point process. The kernel for the CUE is equal to

$$\sum_{j=0}^{N-1} e^{ij(x-y)}$$

for $x, y \in [0, 2\pi)$ [70, Proposition 3.7]. Now the level density is simpler than the one we saw for the GUE: any given CUE eigenvalue is uniformly distributed over the unit circle. Things become much more interesting when we look at the two point correlation and the eigenvalue spacings. As $N \to \infty$, the normalized kernel satisfies the following limit

$$\frac{1}{N} \sum_{j=0}^{N-1} e^{ij(x-y)/N} \to \frac{\sin(\pi(x-y))}{\pi(x-y)}.$$

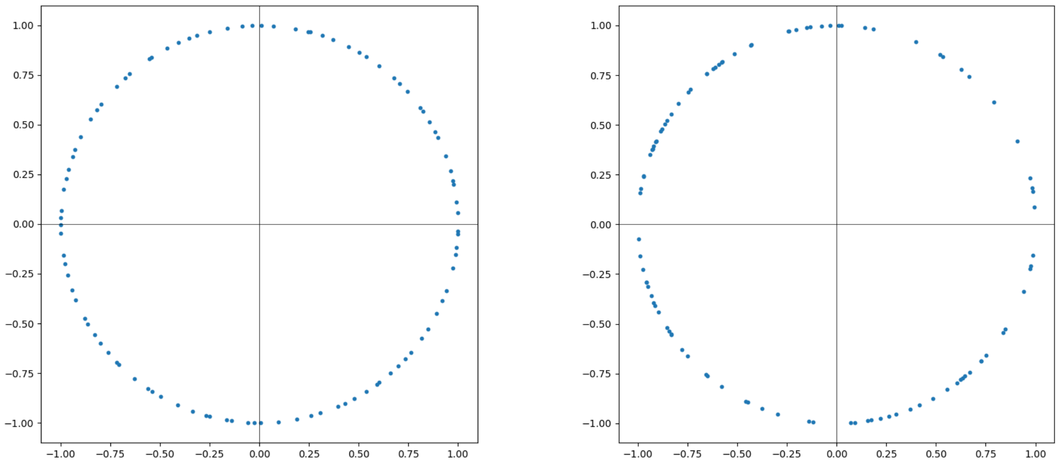

Figure 1.3: On the left are eigenvalues of a 100 by 100 CUE matrix. On the right are 100 points on the circle with angles generated uniformly at random from 0 to 2π. There are noticeably fewer clusters in the eigenvalues than in the points generated uniformly at random.

This is the same as what we found for the GUE. Therefore the pair correlation, and therefore the spacing distribution, of the CUE eigenvalues is the same as that of the GUE eigenvalues in the large N limit. In this regime, only the underlying symmetry group seems to matter, and a similar phenomenon occurs whether one is comparing the GOE to the COE or the GSE to the CSE. Using the theory of Fredholm determinants, one can show that as $N \to \infty$ the normalized spacings between CUE eigenangles converges to a limiting distribution known as the Gaudin distribution [71, eq. 6.4.35]. Figure 1.4 compares the normalized spacings of eigenangles to the normalized spacings of points generated uniformly on the unit interval. Note that for the CUE the density vanishes as the gap size tends to zero, unlike the density of the uniform random points.

1.3 Zeta and random matrix theory

Now that we have discussed the Gaussian and circular ensembles, we will connect them back to the Riemann zeta function. A link between random matrices and zeta was first suspected after a paper of Montgomery studying the distribution of gaps between zeros of zeta assuming the Riemann hypothesis [73]. Montgomery showed that if the

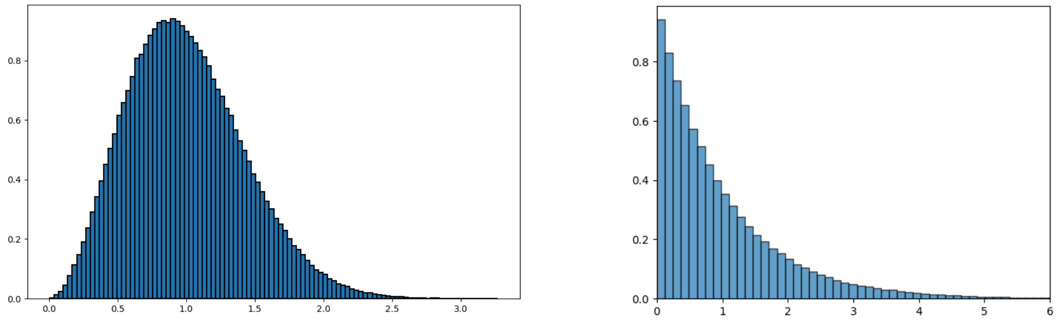

Figure 1.4: On the left is a density histogram of the normalized gaps $(\theta_{j+1} - \theta_j)/2\pi$ between the eigenangles of 10,000 CUE matrices of dimension 50. On the right is a density histogram of the gaps between 50,000 points generated uniformly at random on the unit interval.

Fourier transform of f is supported in $(-1, 1)$ then as $T \to \infty$ the normalized pair correlation of the zeros of zeta satisfies the asymptotic

$$\frac{2\pi}{T \log T} \sum_{0 \le \gamma_1, \gamma_2 \le T} f\left((\gamma_1 - \gamma_2) \frac{\log T}{2\pi} \right) \sim \int_{\mathbb{R}} f(x) \left(1 - \frac{\sin^2(\pi x)}{\pi^2 x^2} \right) dx.$$

Here the sum is taken over all pairs of zeros $\frac{1}{2} + i\gamma_j$ of zeta. Montgomery also gave a heuristic argument using the Hardy–Littlewood prime k-tuples conjecture to show that this asymptotic also likely holds without the restriction on the Fourier transform of f. Dyson pointed out to Montgomery that this coincides with the pair correlation of the GUE in the large N limit, and the study of connections between random matrices and zeros of the zeta function was born.

We are interested in what this connection can be used to say about the moments of zeta. Recall the conjecture

$$M_\beta(T) \sim C_\beta (\log T)^{\beta^2}.$$

It was first conjectured that C_β can be factored into $a_\beta g_\beta$, where the factor

$$a_\beta := \prod_p \left(\left(1 - \frac{1}{p} \right)^{\beta^2} \sum_{m \ge 0} \left(\frac{\Gamma(\beta + m)^2}{m! \Gamma(\beta)} \right)^2 \frac{1}{p^m} \right)$$

comes from number theoretic considerations. The factor g_β was much more mysterious for a long time. By the work of Hardy and Littlewood [48] it was known that $g_1 = 1$, and by the work of Ingham [58] it was known that $g_2 = \frac{1}{12}$. Beyond these results, the only other conjecture for values of g_β were due to Conrey and Ghosh [27] and Conrey and Gonek [29], who argued respectively that

$$g_3 = \frac{42}{9!} \quad \text{and} \qquad g_4 = \frac{24024}{16!}.$$

One explanation for the factorization $C_\beta = a_\beta g_\beta$ is the hybrid Euler-Hadamard product formula for zeta of Gonek, Hughes, and Keating [45], which states that assuming the Riemann hypothesis

$$\zeta(\tfrac{1}{2}+it) \approx \prod_{p\le X}\left(1-\frac{1}{p^{1/2+it}}\right)^{-1} \prod_{\substack{\gamma_n \\ |\gamma_n - t|<1/\log X}} (i(t-\gamma_n)e^{\gamma}\log X).$$

where $X \ge 2$ and the product is taken of zeros $\frac{1}{2}+i\gamma_n$ of zeta and γ is the Euler-Mascheroni constant. The product over primes explains the factor a_β in the moment conjecture, so it is natural to expect that the zeros of zeta could explain the factor g_β. With the benefit of hindsight, the hybrid Euler-Hadamard product and Montgomery's pair correlation conjecture gives us reason to expect that random matrix theory could be used to explain the values of the constants g_β. After all the pair correlation of the zeros of zeta appears to be the same as the pair correlation of the eigenvalues GUE and the CUE, the moments of characteristic polynomials of random matrices could serve as a good model for the moments of zeta. Keating and Snaith [61] accomplished this even before the work of Gonek, Hughes, and Keating [45].

Given a CUE matrix U, denote by

$$Z(U,\theta) = \det(I - Ue^{-i\theta})$$

its characteristic polynomial. Using Selberg's integral formula, Keating and Snaith [61] showed that

$$M_N(s) := \int_{U(N)} |Z(U,\theta)|^s dU = \prod_{j\le N} \frac{\Gamma(j)\Gamma(j+s)}{\Gamma(j+s/2)^2}$$

where $U(N)$ is the group of N by N unitary matrices, dU is the Haar measure, $\mathrm{Re}\, s > -1$, and $0 \le \theta < 2\pi$. When $s = 2\beta$ for a positive integer β, the right hand side is a polynomial in N of degree β^2. In the large N limit, this implies the asymptotic

$$M_N(2\beta) \sim \frac{G(1+\beta)^2}{G(1+2\beta)} N^{\beta^2},$$

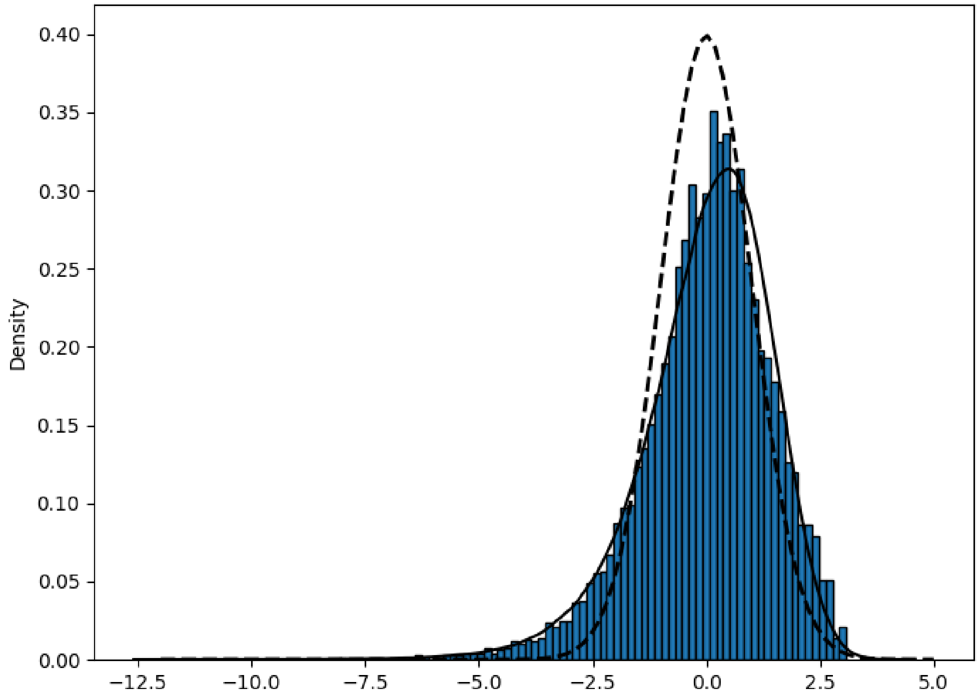

Figure 1.5: A histogram of the value distribution of $\log|\zeta(\frac{1}{2}+it)|$ for t taken randomly at uniform between 10^6 and 10^6+100. The dotted line is the density of a standard Gaussian, while the solid line is the density of $\log|Z(U,0)|$ where U is an 11 by 11 CUE matrix. The latter density gives a noticeably better fit.

where $G(s)$ is the Barnes G function. What is particularly remarkable is that the ratio of Barnes G functions appearing in the large N limit is equal to $\frac{42}{9!}$ when $\beta = 3$ and $\frac{24024}{16!}$ when $\beta = 4$. This has led to the now widely believed conjecture that

$$M_\beta(T) \sim a_\beta \frac{G(1+\beta)^2}{G(1+2\beta)} (\log T)^{\beta^2}.$$

Since this conjecture was made Conrey, Farmer, Keating, Rubenstein, and Snaith [26] gave a heuristic derivation of the asymptotics for $M_\beta(T)$ when β is a positive integer. In fact, their derivation showed that likely $M_\beta(T)$ is asymptotically given by a polynomial of degree β^2 in $\log T$. This is in accord with the finite N expression for the moments of characteristic polynomials for CUE matrices, so one might also try to model the value distribution of the zeta function at height T on the critical line by a CUE matrix of finite size N determined by T.

Since the average spacing between CUE eigenangles is $N/2\pi$, and the average spacing between zeros of zeta up to height T on the critical line is $\frac{1}{2\pi}\log(T/2\pi)$, it is natural to model the zeta function at height T by the characteristic polynomial of a CUE matrix of dimension

$$N = \log \frac{T}{2\pi}.$$

This is particularly interesting, because Fourier inversion and the exact formula of Keating and Snaith [61] for $M_N(s)$ implies that the random variable $\log|Z(U,0)|$ for a N by N CUE matrix U is given by

$$\rho_N(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-iyx} M_N(iy)dy.$$

Figure 1.5 compares the value distribution of $\log|\zeta(\frac{1}{2}+it)|$ for t near 10^6 with both $\rho_{11}(x)$, predicted by random matrix theory, and the Gaussian density, predicted by Selberg's central limit theorem. Notably, the CUE density appears to give a much better fit. While this might seem puzzling at first glance, it is not in conflict with Selberg's central limit theorem for two reasons. The first reason is that the convergence of $\log|\zeta(\frac{1}{2}+it)|/\sqrt{\frac{1}{2}\log\log T}$ to a standard Gaussian is slow. In Tsang's thesis [87, Theorem 6.1], it is shown that the Kolmogorov distance between $\log|\zeta(\frac{1}{2}+it)|/\sqrt{\frac{1}{2}\log\log T}$ and a standard normal is $\ll (\log\log\log T)^2/\sqrt{\log\log T}$. The true rate of convergence is probably of the order $1/(\log\log T)^{3/2}$. The second reason is that Keating and Snaith [61] showed the random variables $\log|Z(U,0)|/\sqrt{\frac{1}{2}\log N}$ converge to a standard Gaussian at the rate of $1/(\log N)^{3/2}$, so the observed behavior is still consistent with Selberg's central limit theorem.

1.4 Overview

Despite the evidence discussed so far, the connection between random matrices and zeta remains mysterious and largely conjectural. We note however, that there is an explicit connection between zeta functions of curves over finite fields and random matrices, see [60]. In the case of zeta and more general number fields however, random matrix theory remains but a source of interesting conjectures. We will be making progress on a few of these conjectures. In the remainder of this dissertation, we will discuss the work contained in the authors papers [31, 32, 33, 34] and a joint paper of the author and Heycock [35].

The first topic we will discuss are the joint moments of zeta as well as some natural generalizations to higher derivatives. The primary goal of Chapters 2 and 3 is to prove the estimate

$$\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2k-2h} \left|\zeta'(\tfrac{1}{2}+it)\right|^{2h} dt \asymp_{h,k} T(\log T)^{k^2+2h},$$

The lower bounds hold for all $0 \le h \le k$, while the upper bounds hold when $0 \le h \le k \le 2$. Assuming the Riemann hypothesis, we extend the upper bounds to all $0 \le h \le k$.

The second topic we will discuss are the moments of moments of zeta given by

$$\mathrm{MoM}_T(k,\beta) := \frac{1}{T}\int_T^{2T} \Bigg(\int\limits_{|h|\le 1} |\zeta(\tfrac{1}{2}+it+ih)|^{2\beta}\, dh \Bigg)^k dt$$

for $\beta \ge 0$ and positive integers k. Fyodorov and Keating [42] introduced these quantities to study the size of the typical maximum of zeta in short intervals on the critical line, and made conjectures for their asymptotic behavior using random matrix theory. An interesting feature of these asymptotics is that when $k > 1$ they undergo a phase transition at $\beta = 1/\sqrt{k}$: for $\beta < 1/\sqrt{k}$ the moments of moments are dominated by the average values of zeta, while for $\beta > 1/\sqrt{k}$ the moments of moments are dominated by a few large values. The case of $k = 1$ is essentially equivalent to evaluating the $2\beta^{\text{th}}$ moment of zeta. In Chapter 4 we analyze the behavior of $\mathrm{MoM}_T(2,\beta)$. We will give lower bounds of the conjectured size for all $\beta \ge 0$ and upper bounds of the correct order for $0 \le \beta \le 1$. A consequence of these bounds is that $\mathrm{MoM}_T(2,\beta)$ indeed undergoes a phase transition at $\beta = \frac{1}{\sqrt{2}}$.

To estimate $\mathrm{MoM}_T(2,\beta)$, the key quantity we need to control is

$$\int_T^{2T} |\zeta(\tfrac{1}{2}+it+i\alpha_1)\zeta(\tfrac{1}{2}+it+i\alpha_2)|^{2\beta}\, dt$$

for $\beta \le 1$. Unconditionally, this appears to be the limit of what current techniques can estimate. Assuming the Riemann hypothesis, however, we can give estimates for more general shifted moments

$$M_{\boldsymbol{\alpha},\boldsymbol{\beta}}(T) = \int_T^{2T} \prod_{k=1}^{m} |\zeta(\tfrac{1}{2}+i(t+\alpha_k))|^{2\beta_k} dt.$$

for $\beta_j \ge 0$ and $\alpha_k \in \mathbb{R}$. Estimates for these shifted moments can be used to give bounds for $\mathrm{MoM}_T(m,\beta)$ for all positive integers m. In the final two chapters we will turn our attention to these shifted moments. When each $\alpha_j = O(\log T)$ and $|a_j - \alpha_k| = O(1)$, Chandee [21] used random matrix theory to predict

$$M_{\boldsymbol{\alpha},\boldsymbol{\beta}}(T) \asymp_{\boldsymbol{\beta}} T(\log T)^{\beta_1^2+\cdots+\beta_m^2} \prod_{1\le j<k\le m} \min(\log T, 1/|\alpha_j-\alpha_k|)^{2\beta_j\beta_k}.$$

For larger shifts α_j however, the random matrix model is no longer correct, and we need some arithmetic input to obtain bounds of the correct order. We shall prove that assuming the Riemann hypothesis

$$M_{\boldsymbol{\alpha},\boldsymbol{\beta}}(T) \asymp_{\boldsymbol{\beta}} T(\log T)^{\beta_1^2+\cdots+\beta_m^2} \prod_{1\le j<k\le m} |\zeta(1+i(\alpha_j-\alpha_k)+1/\log T)|^{2\beta_j\beta_k}$$

for $|\alpha_k| \le T/2$. This agrees with Chandee's conjecture when the shifts are bounded, but the arithmetic factor on the right hand side shows that the values of zeta have long range correlations due to the primes.

Chapter 2

Joint Moments I

2.1 Introduction

In the past two decades, conjectural connections between the zeros of the Riemann zeta function $\zeta(s)$ and eigenvalues of random unitary matrices have led to many interesting developments in understanding the moments of the zeta function. In the recent random matrix theory literature, there has been a fair bit of interest in understanding the joint moments of the characteristic polynomial of a random unitary matrix with its derivative. In this chapter, the primary objects are the joint moments of $\zeta(s)$, given by

$$\mathcal{I}_T(k,h) = \int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2k-2h} \left|\zeta'(\tfrac{1}{2}+it)\right|^{2h} dt,$$

as well as the joint moments of the Hardy Z function

$$\mathcal{J}_T(k,h) = \int_T^{2T} |Z(t)|^{2k-2h} |Z'(t)|^{2h} dt,$$

where

$$Z(t) = e^{i\theta(t)}\zeta(\tfrac{1}{2}+it), \quad \theta(t) = \arg\left(\Gamma\left(\tfrac{1}{4}+\tfrac{it}{2}\right)\right) - \frac{\log\pi}{2}t.$$

Note in particular that $|Z(t)| = |\zeta(\frac{1}{2}+it)|$, and that $Z(t)$ is real valued for $t \in \mathbb{R}$. The work of Keating and Snaith [61, 62], Hughes [56], and Hall [55] has led to the conjecture that whenever $k > -\frac{1}{2}$ and $-\frac{1}{2} < h \leq k + \frac{1}{2}$

$$\mathcal{I}_T(k,h) \sim \mathfrak{C}_\zeta(k,h)T(\log T)^{k^2+2h}, \quad \mathcal{J}_T(k,h) \sim \mathfrak{C}_Z(k,h)T(\log T)^{k^2+2h} \tag{2.1}$$

for certain constants $\mathfrak{C}_\zeta(k,h), \mathfrak{C}_Z(k,h)$ as $T \to \infty$. There are conjectured values for the constants $\mathfrak{C}_Z(k,h)$ for general real h,k, but values for $\mathfrak{C}_\zeta(k,h)$ are only conjectured for integral h,k. In both cases, the constants split as a product of an arithmetic factor and a random matrix factor. The arithmetic factor is a well understood product over primes. The random matrix factor has many different expressions including combinatorial sums [36, 37, 56], a multiple contour integral in the case $h = k$ [30], and a determinant of Bessel functions [10, 30]. For h, k not necessarily equal, the random matrix factor can be solved for finite N and is related to the solution of a Painlevé V type differential equation [15]. Furthermore, the limit as $N \to \infty$ is related to the solution of a certain Painlevé III equation [8, 10, 15, 40].

Previously, the asymptotics (2.1) for $\mathcal{I}_T(h, k)$ and $\mathcal{J}_T(h, k)$ were known for $h, k \in \{0, 1, 2\}$ with $h \leq k$ due to Ingham [58] and Conrey [24]. Assuming the Riemann hypothesis, Conrey and Ghosh [28] established the conjectured asymptotic (2.1) for $\mathcal{J}_T(1, \frac{1}{2})$ and come close to establishing the corresponding asymptotic for $\mathcal{I}_T(1, \frac{1}{2})$. Upper bounds of the conjectured order of magnitude were only known for very few values of h and k. Upper bounds for half-integer valued $h, k \leq 2$ are available due to work of Conrey [24] and Conrey and Ghosh [28] . In the case $h = 0$, quite a bit more is known. Unconditionally, upper bounds of the correct order of magnitude are known for all real $0 \leq k \leq 2$ due to work of Heap, Radziwiłł, and Soundararajan [52]. Upper bounds of the correct order for all real $k \geq 0$ are also known conditionally on the Riemann hypothesis due to work of Harper [50], which builds on the work of Soundararajan [85]. The aim of this chapter is to establish upper bounds for $\mathcal{I}_T(k, h)$ and $\mathcal{J}_T(k, h)$ of the expected order in a larger range of h and k.

Theorem 2.1.1. *Let $1 \leq k \leq 2$ and $0 \leq h \leq 1$. Then for large T*

$$\mathcal{I}_T(k, h) \ll T(\log T)^{k^2+2h},$$

and the same bound holds for $\mathcal{J}_T(k, h)$.

The proof we give is based on the work of Heap, Radziwiłł and Soundararajan [52] which in turn is based on a method introduced by Radziwiłł and Soundararajan in [83]. The general principle in these works is that if one can compute the $2k^{\text{th}}$ moment of a given L-function twisted by an arbitrary Dirichlet polynomial, then one can find upper bounds of the right order for all of its lower order moments. In particular, Heap, Radziwiłł, and Soundararajan [52] used this approach to prove Theorem 2.1.1 in the case $h = 0$. We combine the ideas of the paper [52] with twisted joint moment calculations to deduce Theorem 2.1.1 in the case of $h = 1$ and then deduce the result from Hölder's inequality– the bounds we obtain are of the conjectured order of magnitude since the exponent of $\log T$ in (2.1) is linear in h. We are forced to take $k \in [1, 2]$ because $2k - 2h$ is only non-negative when $k \geq 1$ at the boundary case $h = 1$.

2.2 Outline of the Proof

We will deduce Theorem 2.1.1 from the following.

Proposition 2.2.1. *Let T be large and $1 \leq k \leq 2$. Then*

$$\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2k-2}|\zeta'(\tfrac{1}{2}+it)|^2 dt \ll T(\log T)^{k^2+2},$$

and the same bound holds when $\zeta(\frac{1}{2}+it)$ is replaced by $Z(t)$.

Proof of Theorem 2.1.1. Recall Theorem 1 of [52] gives for $0 \leq k \leq 2$

$$\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2k} dt \ll T(\log T)^{k^2}.$$

Therefore by Hölder's inequality with $p = \frac{1}{h}$ and $q = \frac{1}{1-h}$, this estimate and Theorem 2.1.1 give

$$\begin{aligned}\mathcal{I}_T(k,h) \leq \Big(\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2k-2}|\zeta'(\tfrac{1}{2}+it)|^2 dt\Big)^h \Big(\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2k} dt\Big)^{1-h} \\ \ll T(\log T)^{k^2+2h}.\end{aligned}$$

The case of the joint moments of $Z(t)$ is similar since $|Z(t)| = |\zeta(\frac{1}{2}+it)|$. ☐

Remark. The bound for $\mathcal{J}_T(k,1)$ in Proposition 2.2.1 can be deduced from the corresponding bound for $\mathcal{I}_T(k,1)$. The following argument is due to an anonymous referee: The product rule implies

$$Z'(t) = i\theta'(t)e^{i\theta(t)}\zeta(\tfrac{1}{2}+it) + ie^{i\theta(t)}\zeta'(\tfrac{1}{2}+it).$$

Combining this with the inequality $|a+b|^2 \leq 2|a|^2 + 2|b|^2$ implies

$$|Z'(t)|^2 \ll |\theta'(t)|^2|\zeta(\tfrac{1}{2}+it)|^2 + |\zeta'(\tfrac{1}{2}+it)|^2.$$

An application of Stirling's formula gives $\theta'(t) \sim \frac{1}{2}\log t$ and thus

$$|Z'(t)|^2 \ll (\log T)^2|\zeta(\tfrac{1}{2}+it)|^2 + |\zeta'(\tfrac{1}{2}+it)|^2$$

for $t \in [T, 2T]$. Then Proposition 2.2.1 for $Z(t)$ readily follows from Theorem 1 of [52] and the conclusion of Proposition 2.2.1 for $\zeta(\frac{1}{2}+it)$.

To prove Proposition 2.2.1, we will approximate the logarithm of $\zeta(s)$ by a truncated sum over primes $\sum_{p\le X} p^{-s}$. Following the works [50, 81, 85], we will break up this sum into increments that have progressively smaller variance. This in turn allows us to work with a Dirichlet polynomial of length T^θ for some small but fixed $\theta > 0$, which is long enough to give a good enough approximation of $\zeta(s)$.

We follow the notation introduced in [52]. Take ℓ to be the largest integer so that $\log_\ell T \ge 10^5$. Now define a sequence T_j for $1 \le j \le \ell$ by $T_1 = e^2$ and

$$T_j = \exp\left(\frac{\log T}{(\log_j T)^2}\right)$$

for $2 \le j \le \ell$, and for $2 \le j \le \ell$ and $s \in \mathbb{C}$ set

$$\mathcal{P}_j(s) = \sum_{T_{j-1}\le p<T_j} \frac{1}{p^s}, \quad \text{and} \quad P_j = \sum_{T_{j-1}\le p<T_j} \frac{1}{p}.$$

The hope is then that on average $\log\zeta(s)$ will be controlled by the sum of the increments $\mathcal{P}_j(s)$, where P_j is the variance of the j^{th} increment on the half line. In essence, we are approximating $\zeta(s)$ with a truncated Euler product (though we are omitting the terms p^{-ks} with $k \ge 2$ when truncating the Dirichlet series for $\log\zeta(s)$). A modern theme in analytic number theory is that for $\sigma \ge 1/2$ the zeta function $\zeta(\sigma+it)$ ought to behave like a truncated Euler product– for example, see the work of Gonek [44] assuming the Riemann hypothesis. Thus we expect that $\exp(\alpha\sum_{2\le j\le\ell}\mathcal{P}_j(s))$ should be a good approximation to $\zeta(s)^\alpha$ so long as s is not near a zero of ζ. Because the zeros of zeta do not contribute to the moments of zeta, one might expect that this approximation will be sufficient for estimating moments of zeta.

To facilitate computations, it will be convenient to replace $\exp(\alpha\mathcal{P}_j(s))$ with a Dirichlet polynomial. We will accomplish this by expanding the exponential as a Taylor series. By Mertens' second estimate

$$\sum_{p\le x}\frac{1}{p} = \log\log x + O(1),$$

it follows that

$$P_j \sim 2\log_j T - 2\log_{j+1} T.$$

We then define for $2 \le j \le \ell$ the truncated Taylor expansion

$$\mathcal{N}_j(s;\alpha) = \sum_{\substack{p|n \Rightarrow T_{j-1}\le p<T_j \\ \Omega(n)\le 500P_j}} \frac{\alpha^{\Omega(n)} g(n)}{n^s}$$

where g is the multiplicative function given by $g(p^m) = 1/m!$ on prime powers. For most $t \in [T, 2T]$ the size of $\mathcal{P}_j(\frac{1}{2}+it)$ will be smaller than $50P_j$, say, so $\mathcal{N}_j(\frac{1}{2}+it;\alpha)$ will be a good proxy for $\exp(\alpha\mathcal{P}_j(\frac{1}{2}+it))$. Therefore we expect $\prod_{2\le j\le \ell}\mathcal{N}_j(\frac{1}{2}+it;\alpha)$ to behave similarly to $\prod_{2\le j<\ell} e^{\alpha\mathcal{P}_j(1/2+it)}$ which, as mentioned earlier, should be a good approximation for $\zeta(s)^\alpha$. Now each $\mathcal{N}_j$ is a Dirichlet polynomial of length at most $T_j^{500P_j}$ so $\prod_{2\le j\le\ell}\mathcal{N}_j(\frac{1}{2}+it;\alpha)$ is a Dirichlet polynomial of length at most T^c, where

$$c = \sum_{2\le j\le \ell}\frac{500P_j}{(\log_j T)^2} \le \sum_{j\le\ell}\frac{1000}{\log_j T}.$$

The ℓ^{th} term in this series is at most $1/100$ by choice of ℓ. So by summing in reverse and bounding the $(\ell-j)^{\text{th}}$ term by $e^{e^j}/100$, we see that $\prod_{2\le j\le\ell}\mathcal{N}_j(\frac{1}{2}+it;\alpha)$ has length at most $T^{1/20}$, which is amenable to analysis.

We will deduce Proposition 2.2.1 in two steps. First we bound the integrand by a product of integral powers of ζ and ζ' with short Dirichlet polynomials.

Proposition 2.2.2. *For $1\le k\le 2$ and $s=\frac{1}{2}+it$ with $t\in\mathbb{R}$*

$$\begin{aligned}
|\zeta(s)|^{2k-2}|\zeta'(s)|^2 \ll \\
&|\zeta(s)|^2|\zeta'(s)|^2\prod_{2\le j\le\ell}|\mathcal{N}_j(s;k-2)|^2 + |\zeta'(s)|^2\prod_{2\le j\le\ell}|\mathcal{N}_j(s;k-1)|^2 \\
&+\sum_{2\le v\le\ell}\Bigg(|\zeta(s)|^2|\zeta'(s)|^2\prod_{2\le j<v}|\mathcal{N}_j(s;k-2)|^2 \\
&\qquad + |\zeta'(s)|^2\prod_{2\le j<v}|\mathcal{N}_j(s;k-1)|^2\Bigg)\left|\frac{\mathcal{P}_v(s)}{50P_v}\right|^{2\lceil 50P_v\rceil}.
\end{aligned}$$

The same bound holds when $\zeta(s)$ is replaced by $Z(t)$.

The proof of Proposition 2.2.2 is almost identical to the proof of Proposition 1 in [52], so it is omitted. The only difference is that one uses the conjugate exponents $p=\frac{1}{k-1}$ and $q=\frac{1}{2-k}$, and then one multiplies the resulting inequality by $|\zeta'(s)|^2$ or $|Z'(t)|^2$. This reduces the proof of Proposition 2.2.1 to the calculation of two types of twisted moments.

Proposition 2.2.3. *For $1\le k\le 2$*

$$\int_T^{2T}|\zeta'(\tfrac{1}{2}+it)|^2\prod_{2\le j\le\ell}|\mathcal{N}_j(\tfrac{1}{2}+it;k-1)|^2dt \ll T(\log T)^{k^2+2} \qquad (2.2)$$

and for $2 \le v \le \ell$ *and* $0 \le r \le \lceil 50P_v \rceil$

$$\int_T^{2T} |\zeta'(\tfrac{1}{2}+it)|^2 \prod_{2\le j<v} |\mathcal{N}_j(\tfrac{1}{2}+it;k-1)|^2 |\mathcal{P}_v(\tfrac{1}{2}+it)|^{2r} dt \qquad (2.3)$$
$$\ll T(\log T)^3 (\log T_{v-1})^{k^2-1} \left(2^r r! P_v^r \exp(P_v)\right),$$

and the same bounds hold when $\zeta(\frac{1}{2}+it)$ *is replaced by* $Z(t)$*.*

Proposition 2.2.4. *For* $1 \le k \le 2$

$$\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^2 |\zeta'(\tfrac{1}{2}+it)|^2 \prod_{2\le j\le \ell} |\mathcal{N}_j(\tfrac{1}{2}+it;k-2)|^2 dt \ll T(\log T)^{k^2+2} \qquad (2.4)$$

and for $2 \le v \le \ell$ *and* $0 \le r \le \lceil 50P_v \rceil$

$$\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^2 |\zeta'(\tfrac{1}{2}+it)|^2 \prod_{2\le j<v} |\mathcal{N}_j(\tfrac{1}{2}+it;k-2)|^2 |\mathcal{P}_v(\tfrac{1}{2}+it)|^{2r} dt \qquad (2.5)$$
$$\ll T(\log T)^6 (\log T_{v-1})^{k^2-4} \left(18^r r! P_v^r \exp(P_v)\right),$$

and the same bounds hold when $\zeta(\frac{1}{2}+it)$ *is replaced by* $Z(t)$*.*

We will derive estimates for general twisted joint moments of ζ in the following section, and then use these estimates to prove Propositions 2.2.3 and 2.2.4 in the final section. Before we undertake this, let us see how these estimates imply Proposition 2.2.1.

Proof of Proposition 2.2.1. Our estimates give

$$\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2k-2} |\zeta'(\tfrac{1}{2}+it)|^2 dt \ll$$
$$T(\log T)^{k^2+2} + \sum_{2\le v\le \ell} T(\log T_{v-1})^{k^2+2} \Bigg(\left(\frac{\log T}{\log T_{v-1}}\right)^3 \frac{2^{\lceil 50P_v\rceil} \lceil 50P_v \rceil! P_v^{\lceil 50P_v\rceil} \exp(P_v)}{(50P_v)^{2\lceil 50P_v\rceil}}$$
$$+ \left(\frac{\log T}{\log T_{v-1}}\right)^6 \frac{18^{\lceil 50P_v\rceil} \lceil 50P_v \rceil! P_v^{\lceil 50P_v\rceil} \exp(P_v)}{(50P_v)^{2\lceil 50P_v\rceil}} \Bigg) \ll T(\log T)^{k^2+2},$$

where the final bound follows by the same reasoning used in [52]. The conclusion for the Z function is the same.

□

2.3 Twisted Moment Formulae

We will derive the necessary twisted joint moment formulae from formulae for twisted moments of $\zeta(s)$ with small shifts off of the critical line. Fortunately there are many known formulae for computing twisted moments of ζ due to connections with the proportion of zeros of ζ lying on the critical line [23, 69]. Then following work of Young [88], we can differentiate these formulae with respect to the shifts to obtain the desired twisted joint moments. The formula in [88] is valid for Dirichlet polynomials of length $T^{1/2-\varepsilon}$, and we note that work of Bettin, Chandee, and Radziwiłł [18] provides asymptotics for the twisted second moment without shifts for any Dirichlet polynomial of length at most $T^{17/33-\varepsilon}$. The twisted fourth moment formula we use was first proven by Hughes and Young [57] for Dirichlet polynomials of length at most $T^{1/11-\varepsilon}$, which was later increased to $T^{1/4-\varepsilon}$ by Bettin, Bui, Li, and Radziwiłł [17].

Following these works, we will bound the desired twisted moments by introducing a smooth cutoff. Going forward, we fix a smooth nonnegative $\phi : \mathbb{R} \to \mathbb{R}$ such that supp $\phi \subset [3/4, 9/4]$ and $\phi(t) = 1$ for all $t \in [1, 2]$.

Lemma 2.3.1. *Given a Dirichlet polynomial $A(s) = \sum_{h \le T^\theta} \frac{a_h}{h^s}$ with $\theta < 1/2$ and $a_h \ll_\varepsilon h^\varepsilon$, if*

$$F(z_1, z_2) = \sum_{h,k \le T^\theta} \frac{a_h \overline{a_k}}{[h,k]} \frac{(h,k)^{z_1+z_2}}{h^{z_1} k^{z_2}},$$

then

$$\widetilde{I}^{(1)}(T) := \int_{\mathbb{R}} |\zeta'(\tfrac{1}{2}+it)|^2 |A(\tfrac{1}{2}+it)|^2 \phi(t/T) dt \ll T(\log T)^3 \max_{|z_j| = 3^j/\log T} |F(z_1, z_2)|,$$

and the same bound holds when $\zeta(\frac{1}{2}+it)$ is replaced by $Z(t)$.

Proof. Let $\alpha, \beta \in \mathbb{C}$ have modulus less than $1/\log T$. Then by using Lemma 5 in [88] and reasoning in a similar way to the proof of Lemma 6 in [88] we may write

$$\begin{aligned} I_T(\alpha,\beta) &:= \int_{\mathbb{R}} \zeta(\tfrac{1}{2}+\alpha+it)\zeta(\tfrac{1}{2}+\beta-it)|A(\tfrac{1}{2}+it)|^2\phi(t/T)dt \\ = \sum_{h,k \le T^\theta} \frac{a_h \overline{a_k}}{[h,k]} \int_{\mathbb{R}} \Bigg(&\frac{(h,k)^{\alpha+\beta}}{h^\alpha k^\beta}\zeta(1+\alpha+\beta) \\ &+ \left(\frac{t}{2\pi}\right)^{-\alpha-\beta} \frac{(h,k)^{-\alpha-\beta}}{h^{-\beta}k^{-\alpha}}\zeta(1-\alpha-\beta)\Bigg)\phi(t/T)dt + O(T^{1-\delta}) \end{aligned}$$

for some $\delta > 0$. The main term is holomorphic in α, β sufficiently small since the principal parts of $\zeta(1+\alpha+\beta)$ and $\zeta(1-\alpha-\beta)$ cancel. We may express the main term as a multiple contour integral around α and $-\beta$: by Lemma 2.5.1 of [26]

$$I_T(\alpha,\beta) = -\frac{1}{(2\pi i)^2} \oint_{|z_2|=9/\log T} \oint_{|z_1|=3/\log T} F(z_1,-z_2)\frac{\zeta(1+z_1-z_2)(z_1-z_2)^2}{(z_1-\alpha)(z_1+\beta)(z_2-\alpha)(z_2+\beta)} \times \left(\int_{\mathbb{R}} \left(\frac{t}{2\pi}\right)^{\frac{z_1-z_2-\beta-\alpha}{2}} \phi(t/T)dt\right) dz_1 dz_2 + O(T^{1-\delta}).$$

The shifts α and $-\beta$ are enclosed in these contours because we have assumed $|\alpha|, |\beta| < 1/\log T$. Now since $I_T(\alpha,\beta)$ is holomorphic with respect to small α and β, as in [88] the derivatives of $I_T(\alpha,\beta)$ with respect to α and β can be obtained via Cauchy's theorem as contour integrals along circles of radii $\asymp 1/\log T$. Since the error term holds uniformly on these contours, we conclude

$$\begin{aligned}\widetilde{I}_T(\alpha,\beta) &:= \int_{\mathbb{R}} \zeta'(\tfrac12+\alpha+it)\zeta'(\tfrac12+\beta+it)|A(\tfrac12+it)|^2\phi(t/T)dt \\ &= \frac{d}{d\alpha}\frac{d}{d\beta}\Bigg[-\frac{1}{(2\pi i)^2} \oint_{|z_2|=9/\log T} \oint_{|z_1|=3/\log T} F(z_1,-z_2)\frac{\zeta(1+z_1-z_2)(z_1-z_2)^2}{(z_1-\alpha)(z_1+\beta)(z_2-\alpha)(z_2+\beta)} \\ &\qquad \times \left(\int_{\mathbb{R}} \left(\frac{t}{2\pi}\right)^{\frac{z_1-z_2-\beta-\alpha}{2}} \phi(t/T)dt\right) dz_1 dz_2\Bigg] + O(T^{1-\delta}).\end{aligned}$$

To compute $\widetilde{I}^{(1)}(T)$, we evaluate these derivatives and then set $\alpha=\beta=0$, obtaining

$$\begin{aligned}\widetilde{I}^{(1)}(T) &= \frac{1}{(2\pi i)^2} \oint_{|z_2|=9/\log T} \oint_{|z_1|=3/\log T} F(z_1,-z_2)\zeta(1+z_1-z_2)(z_1-z_2)^2 \\ &\int_{\mathbb{R}} \left[\left(z_1+z_2+\frac{z_1z_2}{2}\log\frac{t}{2\pi}\right) \times \left(z_1+z_2-\frac{z_1z_2}{2}\log\frac{t}{2\pi}\right)\left(\frac{t}{2\pi}\right)^{\frac{z_1-z_2}{2}} \phi(t/T)dt\right] \\ &\qquad \times \frac{dz_1}{z_1^4}\frac{dz_2}{z_2^4} + O(T^{1-\delta}).\end{aligned}$$

Finally, since $|z_j| = 3^j/\log T$ and $\operatorname{supp}\phi \subset [3/4, 9/4]$, notice that

$$\zeta(1+z_1-z_2) \ll \log T, \quad (z_1-z_2)^2 \ll (\log T)^{-2},$$

and

$$\int_{\mathbb{R}} \left(z_1+z_2+\frac{z_1z_2}{2}\log\frac{t}{2\pi}\right)\left(z_1+z_2-\frac{z_1z_2}{2}\log\frac{t}{2\pi}\right)\left(\frac{t}{2\pi}\right)^{\frac{z_1-z_2}{2}} \phi(t/T)dt \ll T(\log T)^{-2},$$

so the claim now follows.

The case for twisted moments of Z is similar. This time following the argument of [88] gives the more symmetric formula (up to terms of order $O(T^{1-\delta})$)

$$\int_{\mathbb{R}} Z(\alpha+t)Z(\beta-t)|A(\tfrac{1}{2}+it)|^2\phi(t/T)dt$$
$$= \sum_{h,k\le T^\theta} \frac{a_h\overline{a_k}}{[h,k]}\int_{\mathbb{R}}\Bigg(\left(\frac{t}{2\pi}\right)^{\frac{\alpha+\beta}{2}}\frac{(h,k)^{\alpha+\beta}}{h^\alpha k^\beta}\zeta(1+\alpha+\beta)$$
$$+\left(\frac{t}{2\pi}\right)^{\frac{-\alpha-\beta}{2}}\frac{(h,k)^{-\alpha-\beta}}{h^{-\beta}k^{-\alpha}}\zeta(1-\alpha-\beta)\Bigg)\phi(t/T)dt.$$

Now applying Lemma 2.5.1 of [26] gives up to a power savings the simpler formula

$$-\frac{1}{(2\pi i)^2}\oint_{|z_2|=9/\log T}\oint_{|z_1|=3/\log T} F(z_1,-z_2)\frac{\zeta(1+z_1-z_2)(z_1-z_2)^2}{(z_1-\alpha)(z_1+\beta)(z_2-\alpha)(z_2+\beta)}$$
$$\times\left(\int_{\mathbb{R}}\left(\frac{t}{2\pi}\right)^{\frac{z_1-z_2}{2}}\phi(t/T)dt\right)dz_1dz_2.$$

Then differentiating with respect to α and β and setting the shifts to zero we obtain

$$\frac{1}{(2\pi i)^2}\oint_{|z_2|=9/\log T}\oint_{|z_1|=3/\log T} F(z_1,-z_2)\zeta(1+z_1-z_2)(z_1^2-z_2^2)^2$$
$$\left(\int_{\mathbb{R}}\left(\frac{t}{2\pi}\right)^{\frac{z_1-z_2}{2}}\phi(t/T)dt\right)\frac{dz_1}{z_1^4}\frac{dz_2}{z_2^4},$$

which satisfies the same bound. □

Lemma 2.3.2. *Given a Dirichlet polynomial $A(s)=\sum_{h\le T^\theta}\frac{a_h}{h^s}$ with $\theta<1/4$ and $a_h\ll_\varepsilon h^\varepsilon$, if*

$$G(z_1,z_2,z_3,z_4)=\sum_{h,k\le T^\theta}\frac{a_h\overline{a_k}}{[h,k]}B_{z_1,z_2,z_3,z_4}\left(\frac{h}{(h,k)}\right)B_{z_3,z_4,z_1,z_2}\left(\frac{k}{(h,k)}\right),$$

where

$$B_{z_1,z_2,z_3,z_4}(n)=\prod_{p^m\|n}\left(\sum_{j\ge0}\frac{\sigma_{z_1,z_2}(p^{j+m})\sigma_{z_3,z_4}(p^j)}{p^j}\right)\left(\sum_{j\ge0}\frac{\sigma_{z_1,z_2}(p^j)\sigma_{z_3,z_4}(p^j)}{p^j}\right)^{-1}$$

and $\sigma_{z_1,z_2}(n)=\sum_{ab=n}a^{-z_1}b^{-z_2}$, then

$$\widetilde{I}^{(2)}(T):=\int_{\mathbb{R}}|\zeta(\tfrac{1}{2}+it)|^2|\zeta'(\tfrac{1}{2}+it)|^2|A(\tfrac{1}{2}+it)|^2\phi(t/T)dt$$
$$\ll T(\log T)^6\max_{|z_j|=3^j/\log T}|G(z_1,z_2,z_3,z_4)|.$$

The same bound holds when $\zeta(\frac{1}{2}+it)$ is replaced by $Z(t)$.

Proof. This is similar to the proof of Lemma 2.3.1. Using the twisted 4th moment formula with shifts in [17] and Lemma 2.5.1 of [26], we can write up to a power savings in T

$$\begin{aligned}
&\int_{\mathbb{R}} \zeta(\tfrac{1}{2}+\alpha+it)\zeta(\tfrac{1}{2}+it)\zeta(\tfrac{1}{2}+\beta-it)\zeta(\tfrac{1}{2}-it)|A(\tfrac{1}{2}+it)|^2\phi(t/T)dt \\
&= \frac{1}{4(2\pi i)^4} \oint\limits_{|z_j|=3^j/\log T} A(z_1,z_2,-z_3,-z_4)G(z_1,z_2,-z_3,-z_4)\Delta(z_1,z_2,z_3,z_4)^2 \\
&\quad \times \left(\int_{\mathbb{R}} \left(\frac{t}{2\pi}\right)^{\frac{z_1+z_2-z_3-z_4-\alpha-\beta}{2}} \phi(t/T)dt\right) \prod_{m=1}^{4} \frac{dz_m}{z_m^2(z_m-\alpha)(z_m+\beta)},
\end{aligned}$$

where $\Delta(z_1,z_2,z_3,z_4)=\prod_{1\le j<k\le 4}(z_k-z_j)$ is the Vandermonde determinant and

$$A(z_1,z_2,z_3,z_4)=\frac{\zeta(1+z_1+z_3)\zeta(1+z_1+z_4)\zeta(1+z_2+z_3)\zeta(1+z_2+z_4)}{\zeta(2+z_1+z_2+z_3+z_4)}.$$

Now differentiating with respect to α and β and then setting $\alpha=\beta=0$ gives up to a power savings in T

$$\begin{aligned}
\widetilde{I}^{(2)}(T) &= \frac{1}{4(2\pi i)^4} \oint\limits_{|z_j|=3^j/\log T} A(z_1,z_2,-z_3,-z_4)G(z_1,z_2,-z_3,-z_4)\Delta(z_1,z_2,z_3,z_4)^2 \\
&\times \Bigg[\int_{\mathbb{R}} \left(z_1^2z_2^2z_3^2z_4^2\left(\log\frac{t}{2\pi}\right)^2-(z_1z_2z_3+z_1z_2z_4+z_1z_3z_4+z_2z_3z_4)^2\right) \\
&\qquad \times \left(\frac{t}{2\pi}\right)^{\frac{z_1+z_2-z_3-z_4}{2}} \phi(t/T)dt\Bigg] \prod_{m=1}^{4}\frac{dz_m}{z_m^6}.
\end{aligned}$$

Now to deduce the claim, notice that

$$A(z_1,z_2,-z_3,-z_4) \ll (\log T)^4, \quad \Delta(z_1,z_2,z_3,z_4)^2 \ll (\log T)^{-12},$$

$$\begin{aligned}
&\int_{\mathbb{R}} \left(z_1^2z_2^2z_3^2z_4^2\left(\log\frac{t}{2\pi}\right)^2-(z_1z_2z_3+z_1z_2z_4+z_1z_3z_4+z_2z_3z_4)^2\right) \\
&\qquad \times \left(\frac{t}{2\pi}\right)^{\frac{z_1+z_2-z_3-z_4}{2}} \phi(t/T)dt \ll T(\log T)^{-6}
\end{aligned}$$

for $|z_j|=3^j/\log T$ and $t\in[3T/4,9T/4]$. As in the previous proof, the analysis for the Z function is simpler, and the same bound holds. $\square$

2.4 Proof of Propositions 2.2.3 and 2.2.4

The proofs of Propositions 2.2.3 and 2.2.4 are straightforward modifications of the proof of Proposition 3 in [52]. In fact we will see that Proposition 2.2.4 is an immediate consequence of Lemma 2.3.2 and a bound for $G(z_1, z_2, z_3, z_4)$ proven in [52]. This will then conclude the proof of Theorem 2.1.1.

Proof of Proposition 2.2.3. We will apply Lemma 2.3.1 to the Dirichlet polynomials

$$\prod_{2\le j\le \ell} \mathcal{N}_j(s; k-1)$$

and

$$\left(\prod_{2\le j<v} \mathcal{N}_j(\tfrac{1}{2}+it; k-1)\right) \mathcal{P}_v(\tfrac{1}{2}+it)^r.$$

By multiplicativity, it suffices to bound the sums

$$\sum_{\substack{p|mn\Rightarrow T_{j-1}\le p<T_j\\ \Omega(m),\Omega(n)\le 500P_j}} \frac{(k-1)^{\Omega(m)+\Omega(n)} g(n)g(m)}{[n,m]} \cdot \frac{(m,n)^{z_1+z_2}}{m^{z_1}n^{z_2}} \tag{2.6}$$

arising from the $\mathcal{N}_j(s; k-1)$ terms and

$$\sum_{\substack{p|mn\Rightarrow T_{v-1}\le p<T_v\\ \Omega(m)=\Omega(n)=r}} \frac{r!^2 g(n)g(m)}{[n,m]} \cdot \frac{(m,n)^{z_1+z_2}}{m^{z_1}n^{z_2}} \tag{2.7}$$

coming from the $\mathcal{P}_v(\frac{1}{2}+it)^r$ term. In both cases, we will use the estimate

$$\frac{(m,n)^{z_1+z_2}}{m^{z_1}n^{z_2}} \ll 1,$$

which holds under the assumptions $|z_j| \le 9/\log T$ and $m, n \le T^{1/20}$.

First we handle (2.6). We drop the condition $\Omega(m), \Omega(n) \le 500P_j$ using Rankin's trick: Since $|k-1| \le 1$ and $\exp(\Omega(m)+\Omega(n)-500P_j) \ge 1$ when either $\Omega(m)$ or $\Omega(n)$ is larger than $500P_j$, the terms with either $\Omega(m)$ or $\Omega(n)$ exceeding $500P_j$ contribute an error of size at most

$$\begin{aligned} e^{-500P_j} &\sum_{p|mn\Rightarrow T_{j-1}\le p<T_j} \frac{((k-1)e)^{\Omega(n)+\Omega(m)}}{[n,m]} \\ &\ll e^{-500P_j} \prod_{T_{j-1}\le p<T_j} \left(1+\frac{e+e+e^2}{p}+O\left(\frac{1}{p^2}\right)\right) \\ &\ll e^{-100P_j}. \end{aligned}$$

Now write

$$\sum_{p|mn \Rightarrow T_{j-1} \le p < T_j} \frac{(k-1)^{\Omega(n)+\Omega(m)} g(n)g(m)}{[n,m]}$$
$$= \prod_{T_{j-1} \le p < T_j} \left(1 + \frac{2(k-1)+(k-1)^2}{p} + O\left(\frac{1}{p^2}\right)\right)$$
$$= \prod_{T_{j-1} \le p < T_j} \left(1 + \frac{k^2-1}{p} + O\left(\frac{1}{p^2}\right)\right).$$

Therefore by Lemma 2.3.1 we conclude that the integral in (2.2) is

$$\ll T(\log T)^3 \prod_{2 \le j \le \ell} \left(\prod_{T_{j-1} \le p < T_j} \left(1 + \frac{k^2-1}{p} + O\left(\frac{1}{p^2}\right)\right) + O(e^{-100P_j}) \right)$$
$$\ll T(\log T)^{k^2+2}.$$

Now we handle the sums (2.7). Write

$$\sum_{\substack{p|mn \Rightarrow T_{v-1} \le p < T_v \\ \Omega(m)=\Omega(n)=r}} \frac{r!^2 g(n)g(m)}{[n,m]} \le r!^2 \sum_{j=0}^{r} \sum_{\substack{p|d \Rightarrow T_{v-1} \le p < T_v \\ \Omega(d)=j}} \frac{1}{d} \Bigg(\sum_{\substack{p|n \Rightarrow T_{v-1} \le p < T_v \\ \Omega(n)=r-j}} \frac{g(nd)}{n} \Bigg)^2.$$

By the inequalities $\binom{r}{j} \le 2^r$ and $g(nd) \le g(n)g(d)$, we may further bound this by

$$r!^2 \sum_{j=0}^{r} \left(\frac{1}{j!} P_v^j\right) \left(\frac{1}{(r-j)!} P_v^{r-j}\right)^2 = r! P_v^r \sum_{j=0}^{r} \binom{r}{j} \frac{P_v^{r-j}}{(r-j)!} \le 2^r r! P_v^r \exp(P_v)$$

The claim now readily follows by Lemma 2.3.1. □

Proof of Proposition 2.2.4. This is a direct consequence of Lemma 2.3.2 and the proof of Proposition 3 of [52], where it is shown in the first case that

$$\max_{|z_j|=3^j/\log T} |G(z_1,z_2,z_3,z_4)| \ll T(\log T)^{k^2-4},$$

and in the second case that

$$\max_{|z_j|=3^j/\log T} |G(z_1,z_2,z_3,z_4)| \ll (\log T_{v-1})^{k^2-4} \left(18^r r! P_v^r \exp(P_v)\right).$$

□

Chapter 3

Joint moments II

3.1 Introduction

This chapter contains joint work with André Heycock. We will provide upper and lower bounds for joint moments of the Riemann zeta function in a larger range than that obtained in the previous chapter. Our primary aim is to prove the following result.

Theorem 3.1.1. *If* $0 \leq h \leq k \leq 2$ *then*

$$\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2k-2h} |\zeta'(\tfrac{1}{2}+it)|^{2h} \; dt \ll T(\log T)^{k^2+2h}.$$

Remark. The method here can be generalized in a straightforward manner to provide sharp bounds for the joint moments of the $(2k-2h)^{\text{th}}$ power of ζ with the $2h^{\text{th}}$ power of $\zeta^{(n)}$ for all natural numbers n, but we focus on the case of $n = 1$ for simplicity.

We will also extend the bound to all $0 \leq h \leq k$ assuming the Riemann hypothesis. Our proof easily generalizes to give upper bounds for joint moments of zeta with its higher order derivatives.

Theorem 3.1.2. *Suppose that* $k \geq 1/2$*,* $h_j \geq 0$ *for* $1 \leq j \leq m$*, and* $h_1 + \cdots + h_m \leq k$*. Then*

$$\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2k-2\sum_{j=1}^m h_j} \prod_{j=1}^m |\zeta^{(j)}(\tfrac{1}{2}+it)|^{2h_j} \; dt \ll_{k,h_j} T(\log T)^{k^2+2\sum_{j=1}^m jh_j},$$

where we assume the Riemann hypothesis when $k > 2$*.*

The upper bound in Theorem 3.1.2 is the same order of magnitude one would predict using random matrix theory. This result also likely holds without the restriction $k \geq 1/2$, but our proof fails because $x \mapsto x^{2k}$ is not convex for $k < 1/2$. The case of integral joint moments of a product of two derivatives of characteristic random unitary matrices has been extensively studied by Keating and Wei [63, 64].

Combining Theorems 3.1.1 and 3.1.2, we have upper bounds of the expected order for all $0 \le h \le k$ assuming the Riemann hypothesis. We also obtain lower bounds for all $k \ge 0$ and $0 \le h \le k + \frac{1}{2}$, so the upper bounds are of the correct order.

Theorem 3.1.3. *If $k \ge 0$ and $0 \le h \le k + \frac{1}{2}$ then*

$$\int_T^{2T} |\zeta(\tfrac{1}{2} + it)|^{2k-2h} |\zeta'(\tfrac{1}{2} + it)|^{2h} \, dt \gg_{k,h} T(\log T)^{k^2+2h}.$$

We will begin by showing that Theorem 3.1.3 is a short consequence of the lower bounds for the $2k^{\text{th}}$ moments of zeta for $k \ge 0$ obtained by Heap and Soundararajan [53]. The argument we give is essentially the same argument used in Conrey's [24] proof of the explicit lower bound

$$\int_T^{2T} |\zeta(\tfrac{1}{2} + it)|^3 |\zeta'(\tfrac{1}{2} + it)| \, dt \ge (1 + o(1)) \frac{173}{672\pi^2} T(\log T)^5.$$

Since obtaining good implicit constants is not our concern, a slightly simpler version of the argument in [24] will suffice.

We will then move onto proving the upper bounds. The proof of Theorem 3.1.1 splits into two cases. Using Hölder's inequality, it suffices to prove Theorem 3.1.1 first in the case where $1 \le h \le k \le 2$, and then in the case where $0 \le h \le k \le 1$. When $1 \le h \le k \le 2$, we will represent the derivative of zeta by a contour integral and can use Hölder's or Jensen's inequality to essentially bound the joint moments by $(\log T)^{2h}$ times the $2k^{\text{th}}$ moment of zeta. The argument here is similar to one in work of Milinovich [72] on moments of derivatives of zeta, and the proof fails for small h because $x \mapsto x^{2h}$ is not convex when $h < 1/2$. We will use this same technique to prove Theorem 3.1.2.

To prove Theorem 3.1.1 when $0 \le h \le k \le 1$, we will use the work of Heap, Radziwiłł, and Soundararajan [52] to bound the $(2k - 2h)^{\text{th}}$ power of zeta by the square of zeta times the exponential of a Dirichlet polynomial plus the exponential of another Dirichlet polynomial. A key difference between our method and the one in [52] is that we will first decompose the interval $[T, 2T]$, and then apply a separate interpolation inequality on each piece of the decomposition. This trick allows us to circumvent the issue of bounding non-integer moments of Dirichlet polynomials. Finally we will bound the exponentials of Dirichlet polynomials using the method in [52] to reduce our theorem to a variety of twisted moment calculations, all of which are essentially computed in either [52] or in the previous section.

3.2 Lower Bounds: Proof of Theorem 3.1.3

We begin by writing

$$\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2k-2h}|\zeta'(\tfrac{1}{2}+it)|^{2h}\,dt$$

$$= \int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2k}\left(\left|\mathrm{Re}\,\frac{\zeta'}{\zeta}(\tfrac{1}{2}+it)\right|^2+\left|\mathrm{Im}\,\frac{\zeta'}{\zeta}(\tfrac{1}{2}+it)\right|^2\right)^h\,dt.$$

It is a standard consequence of Stirling's approximation and the functional equation that

$$\mathrm{Re}\,\frac{\zeta'}{\zeta}(\tfrac{1}{2}+it) = -\frac{1}{2}\log\frac{t}{2\pi}+O(|t|^{-1}).$$

Therefore for large T

$$\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2k-2h}|\zeta'(\tfrac{1}{2}+it)|^{2h}\,dt \gg (\log T)^{2h}\int_T^{2T}|\zeta(\tfrac{1}{2}+it)|^{2k}\,dt.$$

Theorem 3.1.3 now follows after an application of the main theorem in [53]. □

3.3 Unconditional Upper Bounds

3.3.1 Proof of Theorem 3.1.1: $h \geq 1$ case

The starting point of the proof is the formula

$$\zeta'(s) = \frac{1}{2\pi i}\int_{|z|=1/\log T}\frac{\zeta(s+z)}{z^2}dz.$$

A couple of applications of Hölder's inequality give

$$\begin{aligned}
&\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2k-2h}|\zeta'(\tfrac{1}{2}+it)|^{2h}\,dt\\
&\quad\ll \int_T^{2T}|\zeta(\tfrac{1}{2}+it)|^{2k-2h}\left(\int_{|z|=1/\log T}\frac{|\zeta(\frac{1}{2}+it+z)|}{|z|^2}\,d|z|\right)^{2h}\,dt\\
&\quad\ll (\log T)^{2h}\max_{|z|=1/\log T}\int_T^{2T}|\zeta(\tfrac{1}{2}+it)|^{2k-2h}|\zeta(\tfrac{1}{2}+it+z)|^{2h}\,dt\\
&\quad\ll (\log T)^{2h}\max_{|z|=1/\log T}\left(\int_T^{2T}|\zeta(\tfrac{1}{2}+it)|^{2k}\,dt\right)^{1-h/k}\left(\int_T^{2T}|\zeta(\tfrac{1}{2}+it+z)|^{2k}\,dt\right)^{h/k}.
\end{aligned}$$

The first integral may be bounded by $\ll T(\log T)^{k^2}$ by Theorem 1 of [52]. The second integral may also be bounded by $\ll T(\log T)^{k^2}$ uniformly in $|z| = 1/\log T$. To see this, we may reduce to the case that $\mathrm{Re}\ z \geq 0$ by using the functional equation (see Lemma

22 of [4]). The claim now follows from combining Theorem 1 of [52] and Theorem 7.1 of [86] with Theorem 2 of [43]. While Theorem 2 of Gabriel's paper [43] only holds for functions analytic in a strip, we may still use his result by approximating the indicator function of the rectangle $1/2 \leq \sigma \leq 3/2, T \leq t \leq 2T$ by a suitable analytic function, see Lemma 3.5 of [6]. Alternatively, one may show this follows by a straightforward modification of the argument in [52]. This concludes the proof when $h \geq 1$, and in fact when $h \geq 1/2$. □

3.3.2 Proof of Theorem 3.1.1: $h \leq k \leq 1$ case

We will first set up our notation. Set $T_0 = 1$,

$$T_j := \exp\left(\frac{\log T}{(\log_{j+1} T)^2}\right)$$

for $j \geq 1$, and

$$J := \max\left\{j : \log_j T \geq 10^5\right\}.$$

For $1 \leq j \leq J$ set

$$\mathcal{P}_j(s) := \sum_{T_{j-1} < p \leq T_j} \frac{1}{p^s}, \quad P_j := \sum_{T_{j-1} < p \leq T_j} \frac{1}{p}.$$

Next define the Dirichlet polynomials

$$\mathcal{N}_j(s;\beta) = \sum_{\substack{p|n \Rightarrow p \in (T_{j-1},T_j] \\ \Omega(n) \leq 10K_j}} \frac{\beta^{\Omega(n)} g(n)}{n^s},$$

where $|\beta| \leq 1$,

$$K_j := 50P_j \ \text{ and } g(n) = \prod_{p^r \| n} \tfrac{1}{r!}.$$

Note $\mathcal{N}_j(s;\beta)$ is a Dirichlet polynomial of length $\leq T_j^{500P_j}$, so $\prod_{j \leq J} \mathcal{N}_j(s;\beta)$ has length at most $T_1^{500P_1} T_2^{500P_2} \cdots T_J^{500P_J} \leq T^{1/20}$.

We will decompose the interval $[T, 2T]$ depending on the sizes of the $\mathcal{P}_j(s)$. Define the good set

$$\mathcal{G} := \left\{t \in [T, 2T] : |\mathcal{P}_j(\tfrac{1}{2} + it)| \leq K_j \text{ for all } j \leq J\right\},$$

and the bad sets

$$\mathcal{B}_r := \{t \in [T, 2T] : |\mathcal{P}_j(\tfrac{1}{2} + it)| \leq K_j \text{ for all } 1 \leq j < r \text{ but } |\mathcal{P}_r(\tfrac{1}{2} + it)| > K_r\}.$$

A similar decomposition occurs in the work of Harper [50] and also occurs implicitly in the work of Radziwiłł and Soundararajan [83] and related papers [34, 52, 53]. In our argument, it will be important to keep careful track of which values t are good or bad because we will use a separate interpolation inequality on each set of our partition of $[T, 2T]$. This will allow us to avoid having to calculate non-integer moments of Dirichlet polynomials. While we could compute non-integer moments in the same manner as Heap and Soundararajan do in [53], circumventing these calculations will simplify our proof. Throughout we will use the following lemmata, which are essentially contained in Proposition 1 of [52].

Lemma 3.3.1. *For* $0 \leq \beta \leq 1$ *and* $r \leq J+1$,

$$|\zeta(s)|^{2\beta} \ll |\zeta(s)|^2 \exp\left((2\beta-2)\sum_{j<r} \text{Re } \mathcal{P}_j(s)\right) + \exp\left(2\beta \sum_{j<r} \text{Re } \mathcal{P}_j(s)\right).$$

Proof. This is an immediate consequence of Young's inequality $xy \leq x^p/p + y^q/q$ for $x, y \geq 0$ and conjugate exponents $p, q > 1$, noting that the result is trivial when β is 0 or 1. □

Lemma 3.3.2. *If* $|\beta| \leq 1$, $j \leq J$ *and* $|\mathcal{P}_j(s)| \leq K_j$, *then*

$$\exp\left(2\beta \text{Re } \mathcal{P}_j(s)\right) = (1 + O(e^{-P_j}))^{-1} |\mathcal{N}_j(s;\beta)|^2.$$

Proof. First note that if $|z| \leq X/10$, then $|e^z - \sum_{j=0}^{X} \frac{z^j}{j!}| \leq e^{-X}$. Therefore we find that

$$|\exp(\beta \mathcal{P}_j(s)) - \mathcal{N}_j(s;\beta)| \leq e^{-10K_j}$$

since $|\beta \mathcal{P}_j(s)| \leq K_j$. Therefore

$$\exp\left(2\beta \text{Re } \mathcal{P}_j(s)\right) = |\exp(\beta \mathcal{P}_j(s))|^2 = |\mathcal{N}_j(s;\beta) + \theta e^{-10K_j}|^2$$

where θ denotes a quantity depending on s with $|\theta| \leq 1$. Finally by assumption $|\exp(\beta \mathcal{P}_j(s))| \geq e^{-K_j}$ and therefore also $|\mathcal{N}_j(s;\beta)| \geq e^{-2K_j}$, so we may turn the additive error into a multiplicative error. □

The proof of Theorem 3.1.1 for $0 \leq h \leq k \leq 1$ will follow from the bounds

$$\int_{\mathcal{G}} |\zeta(\tfrac{1}{2}+it)|^{2k-2h} |\zeta'(\tfrac{1}{2}+it)|^{2h} \, dt \ll T(\log T)^{k^2+2h}$$

and

$$\int_{\mathcal{B}_r} |\zeta(\tfrac{1}{2}+it)|^{2k-2h} |\zeta'(\tfrac{1}{2}+it)|^{2h} \, dt \ll T(\log T)^{k^2+2h} e^{-10P_r}$$

after summing over $1 \le r \le J$. To bound joint moments over the good set, we will apply Lemma 3.3.1 with $r = J + 1$:

$$\begin{aligned}
&\int_{\mathcal{G}} |\zeta(\tfrac{1}{2}+it)|^{2k-2h}|\zeta'(\tfrac{1}{2}+it)|^{2h}\,dt \\
&\quad\ll \int_{\mathcal{G}} |\zeta(\tfrac{1}{2}+it)|^{2}|\zeta'(\tfrac{1}{2}+it)|^{2h}\exp\left(2(k-h-1)\sum_{j\le J}\mathrm{Re}\,\mathcal{P}_j(\tfrac{1}{2}+it)\right)\,dt \\
&\qquad + \int_{\mathcal{G}} |\zeta'(\tfrac{1}{2}+it)|^{2h}\exp\left(2(k-h)\sum_{j\le J}\mathrm{Re}\,\mathcal{P}_j(\tfrac{1}{2}+it)\right)\,dt.
\end{aligned}$$

Now applying Hölder's inequality, then Lemma 3.3.2, and finally extending the range of integration to all of $[T, 2T]$ for both summands we find

$$\begin{aligned}
&\int_{\mathcal{G}} |\zeta(\tfrac{1}{2}+it)|^{2k-2h}|\zeta'(\tfrac{1}{2}+it)|^{2h}\,dt \\
&\quad\ll \left(\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2}|\zeta'(\tfrac{1}{2}+it)|^{2}\prod_{j\le J}|\mathcal{N}_j(\tfrac{1}{2}+it;k-2)|^2\,dt\right)^{h} && (3.1)\\
&\qquad\times\left(\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2}\prod_{j\le J}|\mathcal{N}_j(\tfrac{1}{2}+it;k-1)|^2\,dt\right)^{1-h} && (3.2)\\
&\quad+\left(\int_T^{2T} |\zeta'(\tfrac{1}{2}+it)|^{2}\prod_{j\le J}|\mathcal{N}_j(\tfrac{1}{2}+it;k-1)|^2\,dt\right)^{h} && (3.3)\\
&\qquad\times\left(\int_T^{2T} \prod_{j\le J}|\mathcal{N}_j(\tfrac{1}{2}+it;k)|^2\,dt\right)^{1-h}. && (3.4)
\end{aligned}$$

Here we have used that $\prod_{j\le J}(1+O(e^{-P_j})) = O(1)$. The order in which we apply Hölder's inequality and Lemma 3.3.2 is crucial here, for when the dust settles all of the Dirichlet polynomials are simply raised to the power two.

The moments over the sets $\mathcal{B}_r$ are handled in a completely analogous manner. The only differences are that we apply Lemma 3.3.1 with the sum truncated at r instead of $J+1$ and that we multiply the integrand by $|\mathcal{P}_r(s)/50P_r|^{2\lceil 50P_r\rceil}$ before extending the range of integration to all of $[T, 2T]$. The corresponding inequality obtained is

$$\int_{\mathcal{B}_r} |\zeta(\tfrac{1}{2}+it)|^{2k-2h}|\zeta'(\tfrac{1}{2}+it)|^{2h}\, dt$$

$$\ll \Bigg(\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^2|\zeta'(\tfrac{1}{2}+it)|^2 \prod_{j<r}|\mathcal{N}_j(\tfrac{1}{2}+it;k-2)|^2 \left|\frac{\mathcal{P}_r(s)}{50P_r}\right|^{2\lceil 50P_r\rceil} dt\Bigg)^h \tag{3.5}$$

$$\times \Bigg(\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^2 \prod_{j<r}|\mathcal{N}_j(\tfrac{1}{2}+it;k-1)|^2 \left|\frac{\mathcal{P}_r(s)}{50P_r}\right|^{2\lceil 50P_r\rceil} dt\Bigg)^{1-h} \tag{3.6}$$

$$+\Bigg(\int_T^{2T} |\zeta'(\tfrac{1}{2}+it)|^2 \prod_{j<r}|\mathcal{N}_j(\tfrac{1}{2}+it;k-1)|^2 \left|\frac{\mathcal{P}_r(s)}{50P_r}\right|^{2\lceil 50P_r\rceil} dt\Bigg)^h \tag{3.7}$$

$$\times \Bigg(\int_T^{2T} \prod_{j<r}|\mathcal{N}_j(\tfrac{1}{2}+it;k)|^2 \left|\frac{\mathcal{P}_r(s)}{50P_r}\right|^{2\lceil 50P_r\rceil} dt\Bigg)^{1-h}. \tag{3.8}$$

Thus the problem is reduced to a handful of twisted moment or mean value calculations. Since all the Dirichlet polynomials appearing have length at most $T^{1/4-\varepsilon}$ we may bound (3.1) and (3.5) with Proposition 2.2.4 in Chapter 2, and we may bound (3.3) and (3.7) with Proposition 2.2.3 in Chapter 2 (while these propositions are only stated for $1 \le k \le 2$, they are still valid in the range $0 \le k \le 2$). We may bound (3.4) and (3.8) with Proposition 2 of [52], and we may bound (3.2) with Proposition 2 of [53]. The only moment left to evaluate is (3.6), which can be quickly computed using the formulae in the preceding chapter. Before we proceed with the calculation, given a Dirichlet polynomial $A(s) = \sum_{h\le T^\theta} a_h h^{-s}$ with $\theta < 1/2$, define

$$F(z_1, z_2) = \sum_{h,k\le T^\theta} \frac{a_h\overline{a_k}}{[h,k]}\frac{(h,k)^{z_1+z_2}}{h^{z_1}k^{z_2}}.$$

Proposition 3.3.3. *For $1 \le r \le J$,*

$$\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^2 \prod_{j<r}|\mathcal{N}_j(\tfrac{1}{2}+it;k-1)|^2 \left|\frac{\mathcal{P}_r(s)}{50P_r}\right|^{2\lceil 50P_r\rceil} dt \ll e^{-10P_r}T(\log T)^{k^2}. \tag{3.9}$$

Proof. If one takes $\phi(t)$ to be a smooth majorant of $1_{[1,2]}$ supported on $[1/2, 3/2]$, the proof of lemma 2.3.1 in Chapter 2 shows that for a Dirichlet polynomial $A(s)$ of length $\le T^{1/2-\varepsilon}$ satisfying $a_h \ll_\varepsilon h^\varepsilon$

$$\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^2|A(\tfrac{1}{2}+it)|^2\, dt \ll T\log T \max_{|z_1|,|z_2|\asymp 1/\log T} |F(z_1,z_2)|.$$

We wish to apply this to study the Dirichlet polynomial

$$\left(\prod_{j<r}\mathcal{N}_j(\tfrac{1}{2}+it;k-1)\right)\mathcal{P}_r(s)^{\lceil 50P_r\rceil}.$$

Now we may simply note that in the proof of Proposition 2.2.3 in Chapter 2, it is shown for this choice of Dirichlet polynomial that $|F(z_1, z_2)|$ is of order at most

$$\prod_{j<r}\left(\prod_{T_{j-1}<p\le T_j}\left(1+\frac{k^2-1}{p}+O\left(\frac{1}{p^2}\right)\right)+O(e^{-100P_j})\right)\times 2^{\lceil 50P_r\rceil}\lceil 50P_r\rceil! P_r^{\lceil 50P_r\rceil}\exp(P_r)$$

uniformly in $|z_j| \asymp 1/\log T$. The claim now follows from Mertens' estimates and Stirling's approximation. □

We now have all the necessary estimates to prove Theorem 3.1.1.

Proof of Theorem 3.1.1. Combining Proposition 3.3.3 with the aforementioned bounds in [52, 53] and Chapter 2, we see that the integral over the $\mathcal{G}$ satisfies

$$\int_{\mathcal{G}} |\zeta(\tfrac12+it)|^{2k-2h}|\zeta'(\tfrac12+it)|^{2h}\,dt \ll \left(T(\log T)^{k^2+2}\right)^h \left(T(\log T)^{k^2}\right)^{1-h} \ll T(\log T)^{k^2+2h}.$$

Similarly for each $1 \le r \le J$, an application of Stirling's approximation gives

$$\int_{\mathcal{B}_r} |\zeta(\tfrac12+it)|^{2k-2h}|\zeta'(\tfrac12+it)|^{2h}\,dt$$
$$\ll \left(T(\log T)^{k^2+2}e^{-10P_r}\right)^h \left(T(\log T)^{k^2}e^{-10P_r}\right)^{1-h} \ll T(\log T)^{k^2+2h}e^{-10P_r}.$$

The claim now follows since $\mathcal{G}$ and $\mathcal{B}_r$ partition $[T, 2T]$ while $\sum_{r\le J} e^{-10P_r} = O(1)$. □

3.4 Conditional upper bounds

We will now prove Theorem 3.1.2. This is essentially the same as the proof of Theorem 3.1.1 when $h \ge 1/2$. For simplicity we will write $h = h_1 + \cdots + h_m$. First, apply Hölder's inequality to separate out each derivative:

$$\int_T^{2T} |\zeta(\tfrac12+it)|^{2k-2h}\prod_{j=1}^m |\zeta^{(j)}(\tfrac12+it)|^{2h_j}\,dt$$
$$\ll \left(\int_T^{2T} |\zeta(\tfrac12+it)|^{2k}\,dt\right)^{1-h/k} \times \prod_{j=1}^m \left(\int_T^{2T} |\zeta^{(j)}(\tfrac12+it)|^{2k}\,dt\right)^{h_j/k}. \quad (3.10)$$

The result then follows once we show that, for any fixed $j \in \mathbb{Z}_{>0}$

$$\int_T^{2T} |\zeta^{(j)}(\tfrac12+it)|^{2k}\,dt \ll T(\log T)^{k^2+2jk}. \quad (3.11)$$

For $t \in [T, 2T]$ we have that

$$\zeta^{(j)}(\tfrac{1}{2}+it) = \frac{j!}{2\pi i} \oint_{|z|=1/\log T} \frac{\zeta(\frac{1}{2}+it+z)}{z^{j+1}}\, dz.$$

Applying this to the moment in (3.11) and arguing as in Section 3.3.1 yields

$$\begin{aligned}\int_T^{2T} |\zeta^{(j)}(\tfrac{1}{2}+it)|^{2k}\, dt &\ll (\log T)^{2jk} \int_T^{2T} \left| \oint_{|z|=1/\log T} |\zeta(\tfrac{1}{2}+it+z)|\, \frac{d|z|}{|z|} \right|^{2k} dt \\ &\ll (\log T)^{2jk} \max_{|z|=1/\log T} \int_T^{2T} |\zeta(\tfrac{1}{2}+it+z)|^{2k}\, dt,\end{aligned}$$

with the penultimate step by Hölder's inequality. Finally, the shifted moments is $\ll T(\log T)^{k^2}$ uniformly in z. This again uses the corresponding argument in Section 3.3.1, appealing to Harper's bound [50] for the range $k > 2$ instead of Theorem 1 in [52]. Inserting these bounds into the right hand side of (3.10) proves Theorem 3.1.2. □

Chapter 4

Moments of moments and correlations of zeta

4.1 Introduction

While relatively little is known unconditionally about the moments of zeta for $\beta > 2$, in recent literature there has been interest in moments and the value distribution of zeta in short intervals on the critical line. Instead of studying zeta along an interval $[T, 2T]$, one instead chooses a point $t \in [T, 2T]$ uniformly at random and then tries to understand the typical behavior of the moments or maximum of zeta in the shorter interval $[t-1, t+1]$, say. The question of the maximum of zeta in short intervals is the subject of the Fyodorov-Hiary-Keating conjecture [41, 42], which predicts that if t is chosen uniformly at random from $[T, 2T]$ then as $T \to \infty$

$$\mathbb{P}\left(\max_{|h|\leq 1} |\zeta(\tfrac{1}{2} + it + ih)| > e^y \frac{\log T}{(\log\log T)^{3/4}}\right) \to 1 - F(y), \tag{4.1}$$

where $F(y)$ is a cumulative distribution function with tail decay $1 - F(y) \asymp ye^{-2y}$ as $y \to \infty$. The fraction $\frac{3}{4}$ and the tail decay are manifestations of correlations of nearby shifts of zeta. If the shifts were independent, probabilistic heuristics would predict a $\frac{1}{4}$ in place of $\frac{3}{4}$ and a tail decay of e^{-2y} instead of ye^{-2y}. An analogous conjecture is known to hold for the circular-β ensemble in the random matrix setting– see the work of Paquette and Zeitouni [78, 79]. The current state of the art on this problem is the work of Arguin, Bourgade, and Radziwiłł [4, 5] which, among other results, shows that

$$\mathbb{P}\left(\max_{|h|\leq 1} |\zeta(\tfrac{1}{2} + it + ih)| > e^y \frac{\log T}{(\log\log T)^{3/4}}\right) \asymp ye^{-2y - y^2/\log\log T}$$

for $y \ll \log\log T / \log\log\log T$. For fixed y, this gives a tail bound of the conjectured order as $T \to \infty$.

Moments of zeta in short intervals were also recently studied by Fyodorov and Keating [42] followed by Arguin, Ouimet, and Radziwiłł [6]. The problem now is to understand the behavior of the short moments or the partition function

$$\int_{|h|\leq(\log T)^\theta} |\zeta(\tfrac{1}{2}+it+ih)|^{2\beta} dt \tag{4.2}$$

where $\theta > -1$ and t is chosen uniformly at random from $[T, 2T]$. The typical behavior of these short moments was investigated by Fyodorov and Keating [42] when $\theta = 0$ using random matrix theory. The typical behavior of these moments for all $\theta > -1$ was determined by Arguin, Ouimet, and Radziwiłł [6]. One particularly interesting conclusion of the analysis is that there is a so-called freezing transition as one varies the parameter β. More precisely, for fixed $\theta > -1$ there is a critical value $\beta_c(\theta)$ such that when $\beta < \beta_c(\theta)$, the moments (4.2) are governed by the typical values of $\zeta(\frac{1}{2}+it)$, yet when $\beta > \beta_c(\theta)$ the moments (4.2) are dominated by just a few large values of $\zeta(\frac{1}{2}+it)$.

Here we will not be interested in the typical values of the moments (4.2), but instead the extreme values. More precisely, we will investigate the $k = 2$ case of the so-called *moments of moments* of zeta:

$$\mathrm{MoM}_T(k,\beta) := \frac{1}{T}\int_T^{2T} \Bigg(\int\limits_{|h|\leq(\log T)^\theta} |\zeta(\tfrac{1}{2}+it+ih)|^{2\beta}\, dh \Bigg)^k dt$$

where $\beta \geq 0$ and $\theta > -1$. In other terms, we are interested in the variance of the moments (4.2). The problem of moments of moments was first studied by Fyodorov, Hiary, and Keating in the random matrix setting [41, 42], who conjectured asymptotics for

$$\mathrm{MoM}_{U(N)}(k,\beta) := \int_{U(N)} \left(\frac{1}{2\pi}\int_0^{2\pi} |P_N(A,\theta)|^{2\beta} d\theta \right)^k dA$$

as $N \to \infty$, where dA is the Haar measure on the unitary group $U(N)$ of size N and $P_N(A,\theta) := \det(I - Ae^{-i\theta})$ is the characteristic polynomial of A. There has been a lot of recent progress on these conjectures, and it has now been proven that (see [7, 12, 22, 39]) for $\beta \geq 0$ and $k \in \mathbb{N}$ that

$$\mathrm{MoM}_{U(N)}(k,\beta) \sim \begin{cases} A(k,\beta)N^{k\beta^2} & k < 1/\beta^2 \\ B(k,\beta)N^{k^2\beta^2-k+1} & k > 1/\beta^2 \end{cases}$$

for certain constants $A(k,\beta), B(k,\beta)$ as $N \to \infty$. For $k \geq 2$, the transition at the critical exponent $\beta = 1/\sqrt{k}$ was analyzed by Keating and Wong [65], who showed

$$\mathrm{MoM}_{U(N)}(1/\beta^2,\beta) \sim D_\beta N \log N$$

for certain explicit constants D_β that also appear in the theory of Gaussian multiplicative chaos.

In the case $\theta = 0$, the random matrix philosophy of Keating and Snaith [61] leads us to the prediction that if $k \geq 2$ is an integer and $\beta \geq 0$ then

$$\mathrm{MoM}_T(k,\beta) \sim \begin{cases} A'(k,\beta)(\log T)^{k\beta^2} & k < 1/\beta^2 \\ B'(k,\beta)(\log T)(\log\log T) & k = 1/\beta^2 \\ C'(k,\beta)(\log T)^{k^2\beta^2-k+1} & k > 1/\beta^2 \end{cases}$$

for certain constants $A'(k,\beta), B'(k,\beta), C'(k,\beta)$ as $T \to \infty$. These conjectures were also shown to hold for $\beta, k \in \mathbb{N}$ by Bailey and Keating [13] assuming the standard conjectures on shifted moments of zeta due to Conrey, Farmer, Keating, Rubinstein, and Snaith [26]. These asymptotics do not however include the critical or subcritical regimes where $\beta \leq 1/\sqrt{k}$.

The goal of this chapter is to provide upper and lower bounds of the expected order of magnitude in the case $k = 2$ and non-integer β. In particular we will show that there is indeed a phase transition at $\beta = \frac{1}{\sqrt{2}}$, although this transition occurs in the sub leading order terms when $\theta > 0$. The starting point of the method is to rewrite the integral as

$$\mathrm{MoM}_T(2,\beta) = \frac{1}{T} \iint\limits_{|h_1|,|h_2| \leq (\log T)^\theta} \int_T^{2T} |\zeta(\tfrac{1}{2}+it+ih_1)\zeta(\tfrac{1}{2}+it+ih_2)|^{2\beta}\, dt\, dh_1\, dh_2. \quad (4.3)$$

Therefore we can understand $\mathrm{MoM}_T(2,\beta)$ by understanding the correlation structure of zeta in short intervals and then integrating over all of the possible shifts h_1, h_2. Morally, we will find that if $|h_1 - h_2| \ll 1/\log T$ then

$$\frac{1}{T}\int_T^{2T} |\zeta(\tfrac{1}{2}+it+ih_1)\zeta(\tfrac{1}{2}+it+ih_2)|^{2\beta}\, dt \approx (\log T)^{4\beta^2},$$

while when $|h_1 - h_2| \gg 1/\log T$ that

$$\frac{1}{T}\int_T^{2T} |\zeta(\tfrac{1}{2}+it+ih_1)\zeta(\tfrac{1}{2}+it+ih_2)|^{2\beta}\, dt \approx (\log T)^{2\beta^2}|\zeta(1+i(h_1-h_2))|^{2\beta^2}.$$

So $\zeta(\frac{1}{2}+it+ih_1)$ and $\zeta(\frac{1}{2}+it+ih_2)$ decorrelate as $|h_1 - h_2|$ grows. Indeed one may see that the factor of $|\zeta(1+i(h_1-h_2))|^{2\beta^2}$ is bounded on average by computing the moments of zeta on the one line. A similar correlation structure for $\log|\zeta(\frac{1}{2}+it)|$ was proven by Bourgade [19], and this correlation structure is what gives rise to the phase transition at $\beta = 1/\sqrt{2}$. However as we look at larger shifts, we also see that a large value of $\zeta(1+ih)$ on the one line will cause $\zeta(\frac{1}{2}+it)$ and $\zeta(\frac{1}{2}+it+ih)$ to have an unusually large average correlation.

For $\beta = 1$, an asymptotic formula for $\mathrm{MoM}_T(2,1)$ as $T \to \infty$ is obtained work of Kovaleva, which also studies the fourth moment of zeta with shifts as large as $T^{3/2-\varepsilon}$ [67]. However, this asymptotic does not address the particularly interesting critical case $\beta = 1/\sqrt{2}$. The author has also studied more general shifted moments introduced by Chandee [21]

$$M_{\boldsymbol{\alpha},\boldsymbol{\beta}}(T) = \frac{1}{T}\int_T^{2T} \prod_{k=1}^{m} |\zeta(\tfrac{1}{2}+i(t+\alpha_k))|^{2\beta_k} dt$$

where $\boldsymbol{\alpha} = \boldsymbol{\alpha}(T) = (\alpha_1, \ldots, \alpha_m)$ and $\boldsymbol{\beta} = (\beta_1 \ldots, \beta_m)$ satisfy $|\alpha_k| \leq T/2$ and $\beta_k \geq 0$. In the final two chapters of this thesis, the author will show that assuming the Riemann hypothesis

$$M_{\boldsymbol{\alpha},\boldsymbol{\beta}}(T) \asymp_{\boldsymbol{\beta}} (\log T)^{\beta_1^2+\cdots+\beta_m^2} \prod_{1\leq j<k\leq m} |\zeta(1+1/\log T + i(\alpha_j - \alpha_k))|^{2\beta_j\beta_k}.$$

Using this result, one may obtain sharp conditional bounds for $\mathrm{MoM}_T(k,\beta)$ whenever $\beta \geq 0$ and k is an integer.

To unconditionally obtain upper bounds for $\mathrm{MoM}_T(2,\beta)$ when β is not an integer, we will use a principle pioneered in works of Heap, Radziwiłł, and Soundararajan [52] and Radziwiłł and Soundararajan [83]; to obtain lower bounds we will use a principle in the works of Heap and Soundararajan [53] and Radziwiłł and Soundararajan [82]. Overall, these works demonstrate that if one can asymptotically evaluate the twisted $2k^{\text{th}}$ moment of a family of L-functions for some $k > 0$, then one can obtain upper bounds of the correct order for the $2\beta^{\text{th}}$ moment in that family when $\beta \leq k$ as well as sharp lower bounds for all $\beta \geq 0$. This method has been used to give conjecturally sharp upper and lower bounds for moments of several families of L-functions; see [34, 52, 53, 82, 83] for example.

The first task will be to establish the corresponding twisted fourth moment formula with unbounded shifts, which will follow from some minor modifications to the work of Hughes and Young [57]. We may then follow the principle of [52] and [83] to establish sharp upper bounds for $\mathrm{MoM}_T(2,\beta)$ for all $\beta \leq 1$, and lower bounds for all $\beta \geq 0$.

Theorem 4.1.1. *Let $-1 < \theta \leq 0$. If $0 \leq \beta < 1/\sqrt{2}$ then*

$$MoM_T(2,\beta) \ll \frac{1}{1-2\beta^2}(\log T)^{2\beta^2(1-\theta)+2\theta}.$$

For $1/\sqrt{2} < \beta \leq 1$

$$MoM_T(2,\beta) \ll \frac{1}{2\beta^2-1}(\log T)^{4\beta^2+\theta-1},$$

and for $\beta = 1/\sqrt{2}$

$$MoM_T(2,\tfrac{1}{\sqrt{2}}) \ll (1+\theta)(\log T)^{1+\theta}(\log\log T).$$

Instead if $\theta > 0$*, then if* $0 \leq \beta \leq \min(\sqrt{(1+\theta)/2}, 1)$

$$MoM_T(2,\beta) \ll (\log T)^{2\beta^2+2\theta} + \frac{1}{1-2\beta^2}(\log T)^{2\beta^2+\theta},$$

where the sub leading term is replaced by $(1+\theta)(\log T)^{1+\theta}(\log\log T)$ *if* $\beta = 1/\sqrt{2}$*. In the case* $\min((\sqrt{1+\theta})/2, 1) \leq \beta \leq 1$ *we have*

$$MoM_T(2,\beta) \ll \frac{1}{2\beta^2-1}(\log T)^{4\beta^2-1+\theta}.$$

All implied constants are absolute assuming T *is taken sufficiently large in terms of* θ*.*

Remark. For $\theta > 0$ and $0 \leq \beta \leq \min(\sqrt{(1+\theta)/2}, 1)$, the sub leading order term can be removed provided that T is sufficiently large in terms of β or if one allows the implicit constant to depend on β. This technicality is necessary because the phase transition occurs in the sub leading order terms of the second moment of moments.

Theorem 4.1.2. *Let* $-1 < \theta \leq 0$*. If* $0 \leq \beta < 1/\sqrt{2}$ *then*

$$MoM_T(2,\beta) \gg_\beta (\log T)^{2\beta^2(1-\theta)+2\theta}.$$

For $\beta > 1/\sqrt{2}$

$$MoM_T(2,\beta) \gg_\beta (\log T)^{4\beta^2+\theta-1},$$

and for $\beta = 1/\sqrt{2}$

$$MoM_T(2,\tfrac{1}{\sqrt{2}}) \gg (1+\theta)(\log T)^{1+\theta}(\log\log T).$$

Instead if $\theta > 0$*, then if* $0 \leq \beta \leq \sqrt{(1+\theta)/2}$

$$MoM_T(2,\beta) \gg_\beta (\log T)^{2\beta^2+2\theta}.$$

When $\beta \geq \sqrt{(1+\theta)/2}$*, we have*

$$MoM_T(2,\beta) \gg_\beta (\log T)^{4\beta^2-1+\theta},$$

where all implied constants depend only on β *provided* T *is large enough in terms of* θ*.*

Theorems 4.1.1 and 4.1.2 are consequences of the following result, which may be of independent interest.

Theorem 4.1.3. *If* $0 \le \beta \le 1$ *and* $|h_1|, |h_2| \le T^{1/2-\varepsilon}$ *then*

$$\frac{1}{T}\int_T^{2T} |\zeta(\tfrac{1}{2}+it+ih_1)\zeta(\tfrac{1}{2}+it+ih_2)|^{2\beta}\, dt \ll_\varepsilon (\log T)^{2\beta^2}|\zeta(1+i(h_1-h_2)+1/\log T)|^{2\beta^2}.$$

For all $\beta \ge 0$ *and* $|h_1|, |h_2| \le T/2$

$$\frac{1}{T}\int_T^{2T} |\zeta(\tfrac{1}{2}+it+ih_1)\zeta(\tfrac{1}{2}+it+ih_2)|^{2\beta}\, dt \gg_{\beta,\varepsilon} (\log T)^{2\beta^2}|\zeta(1+i(h_1-h_2)+1/\log T)|^{2\beta^2}.$$

We will first establish the necessary twisted moment estimate in section 4.2. Subsequently we will prove Theorem 4.1.1 in section 4.3 and then Theorem 4.1.2 in section 4.4. We now introduce some notation we will use throughout this chapter. Given some positive integer ℓ and parameters $T_0 < T_1 < \ldots < T_\ell$, recall that $1 \le j \le \ell$

$$\mathcal{P}_j(s) := \sum_{T_{j-1}<p\le T_j} p^{-s}, \qquad P_j := \sum_{T_{j-1}<p\le T_j} p^{-1}.$$

If T_ℓ is chosen to be some small power of T, then the sum of all the $\mathcal{P}_j(s)$ will be a good proxy for $\log|\zeta(s)|$ on average [52, 53], and each P_j can be thought of as the variance of each $\mathcal{P}_j(s)$ on the half line. Then given parameters $K_j \ge 0$ for $1 \le j \le \ell$ let

$$\mathcal{N}_j(s;\beta) := \sum_{m=0}^{K_j} \frac{1}{m!}(\beta\mathcal{P}_j(s))^m = \sum_{\substack{p|n \Rightarrow p\in(T_{j-1},T_j] \\ \Omega(n)\le K_j}} \frac{\beta^{\Omega(n)}g(n)}{n^s}$$

where $g(n)$ is the multiplicative function such that $g(p^a) = 1/a!$. Notice that $\mathcal{N}_j(s;\beta)$ is simply a truncation of the series expansion for $\exp(\beta\mathcal{P}_j(s))$. Since $\mathcal{P}_j(s)$ is a reasonable proxy for $\log|\zeta(s)|$, if we choose parameters K_j to be large multiples of the variances P_j then one might expect

$$\mathcal{N}(s;\beta) := \prod_{j\le\ell} \mathcal{N}_j(s;\beta)$$

to behave like $\zeta(s)^\beta$ on average on the half line. The main difficulty here is that if the $\mathcal{P}_j(s)$ are unusually large for some bad set of s, then the series approximation for $\exp(\beta\mathcal{P}_j(s))$ will be weaker. Fortunately, such events are rare, and we can discard the contribution of such bad s using the incremental structure of $\mathcal{N}(s;\beta)$.

4.2 Twisted fourth moment with shifts

We will now introduce the relevant notation necessary to apply the main result of Hughes and Young [57], and then mention the necessary modifications to obtain an asympotic for larger shifts. Let

$$\lambda(s) = \pi^{s-1/2}\frac{\Gamma((1-s)/2)}{\Gamma(s/2)},$$

write $\sigma_{z_1,z_2}(n) = \sum_{ab=n} a^{-z_1}b^{-z_2}$ for the generalized divisor sum functions, and for the sake of concreteness let

$$G(s) = e^{s^2}\left(1-\left(\frac{2s}{\alpha_1+\alpha_3}\right)^2\right)\left(1-\left(\frac{2s}{\alpha_1+\alpha_4}\right)^2\right)$$
$$\times\left(1-\left(\frac{2s}{\alpha_2+\alpha_3}\right)^2\right)\left(1-\left(\frac{2s}{\alpha_2+\alpha_4}\right)^2\right).$$

Given $\boldsymbol{\alpha} = (\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ denote $\pi\boldsymbol{\alpha} = (\alpha_3,\alpha_4,\alpha_1,\alpha_2)$. Write

$$g_{\boldsymbol{\alpha}}(s,t) = \frac{\Gamma\left(\frac{\frac{1}{2}+\alpha_1+s+it}{2}\right)\Gamma\left(\frac{\frac{1}{2}+\alpha_2+s+it}{2}\right)\Gamma\left(\frac{\frac{1}{2}+\alpha_3+s-it}{2}\right)\Gamma\left(\frac{\frac{1}{2}+\alpha_4+s-it}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}+\alpha_1+it}{2}\right)\Gamma\left(\frac{\frac{1}{2}+\alpha_2+it}{2}\right)\Gamma\left(\frac{\frac{1}{2}+\alpha_3-it}{2}\right)\Gamma\left(\frac{\frac{1}{2}+\alpha_4-it}{2}\right)},$$

$$X_{\boldsymbol{\alpha},t} = \lambda(\tfrac{1}{2}+\alpha_1+it)\lambda(\tfrac{1}{2}+\alpha_2+it)\lambda(\tfrac{1}{2}+\alpha_3-it)\lambda(\tfrac{1}{2}+\alpha_4-it),$$

and

$$V_{\boldsymbol{\alpha}}(x,t) = \frac{1}{2\pi i}\int_{(1)}\frac{G(s)}{s}g_{\boldsymbol{\alpha}}(s,t)x^{-s}ds.$$

By Stirling's approximation (refer to the formula for $\kappa_{x,y}(t)$ preceding (2.4) in [67]), for $|x|,|y| < T/2$ and $t \in [T,2T]$

$$\lambda(\tfrac{1}{2}+it+ix)\lambda(\tfrac{1}{2}-it-iy) = \left(\frac{t+x}{2\pi e}\right)^{i(y-x)}\left(1+O\left(\frac{|y-x|^2}{T}\right)\right).$$

Therefore when $|\alpha_j| \le T^{1/2-\varepsilon}$ for all j

$$X_{\boldsymbol{\alpha},t} = \left(\frac{t}{2\pi e}\right)^{-\alpha_1-\alpha_2-\alpha_3-\alpha_4}\left(1+O(T^{-\varepsilon})\right).$$

It is here in the proof that we are forced to take shifts smaller than $T^{1/2-\varepsilon}$. An additional application of Stirlings formula shows that for $t \in [T,2T]$ and $|\alpha_j| \le T/2$

for all j

$$g_{\boldsymbol{\alpha}}(s,t) = \left(\frac{t+\mathrm{Im}\ \alpha_1}{2}\right)^{s/2}\left(\frac{t+\mathrm{Im}\ \alpha_2}{2}\right)^{s/2}$$
$$\times\left(\frac{t-\mathrm{Im}\ \alpha_3}{2}\right)^{s/2}\left(\frac{t-\mathrm{Im}\ \alpha_4}{2}\right)^{s/2}\left(1+O(|s|^2/T)\right).$$

Therefore $V_{\boldsymbol{\alpha}}$ satisfies the decay estimate

$$t^j\frac{\partial^j}{\partial t^j}V_{\boldsymbol{\alpha}}(x,t) \ll_{A,j} (1+|x|/t^2)^{-A}. \tag{4.4}$$

We will make use of the following version of the approximate functional equation from theorem 5.3 of [59]:

Lemma 4.2.1. *Let $t\in[T/4,4T]$, $\mathrm{Re}\ \alpha_j \ll 1/\log T$, and $|\mathrm{Im}\ \alpha_j|\le T/2$ for all j. Then for all $A>0$*

$$\zeta(\tfrac12+\alpha_1+it)\zeta(\tfrac12+\alpha_2+it)\zeta(\tfrac12+\alpha_3-it)\zeta(\tfrac12+\alpha_4-it)$$
$$=\sum_{m,n}\frac{\sigma_{\alpha_1,\alpha_2}(m)\sigma_{\alpha_3,\alpha_4}(n)}{\sqrt{mn}}\left(\frac{m}{n}\right)^{-it}V_{\boldsymbol{\alpha}}(\pi^2 mn,t)$$
$$+X_{\boldsymbol{\alpha},t}\sum_{m,n}\frac{\sigma_{-\alpha_3,-\alpha_4}(m)\sigma_{-\alpha_1,-\alpha_2}(n)}{\sqrt{mn}}\left(\frac{m}{n}\right)^{-it}V_{-\pi\boldsymbol{\alpha}}(\pi^2 mn,t)+O_A((1+t)^{-A}).$$

Finally, we set

$$B_{z_1,z_2,z_3,z_4}(n)=\prod_{p^m\|n}\left(\sum_{j\ge0}\frac{\sigma_{z_1,z_2}(p^j)\sigma_{z_3,z_4}(p^{j+m})}{p^j}\right)\left(\sum_{j\ge0}\frac{\sigma_{z_1,z_2}(p^j)\sigma_{z_3,z_4}(p^j)}{p^j}\right)^{-1},$$

$$A(z_1,z_2,z_3,z_4)=\frac{\zeta(1+z_1+z_3)\zeta(1+z_1+z_4)\zeta(1+z_2+z_3)\zeta(1+z_2+z_4)}{\zeta(2+z_1+z_2+z_3+z_4)},$$

and let

$$\Delta(z_1,z_2,z_3,z_4)=\prod_{1\le j<k\le4}(z_k-z_j)$$

denote the Vandermonde determinant.

Theorem 4.2.2. *Let w be a smooth non-negative function supported on $[1/2,4]$, and*

$$A_{\boldsymbol{\alpha}}(s)=\sum_{n\le T^\eta}\frac{a_{\boldsymbol{\alpha}}(n)}{n^s}$$

be a Dirichlet polynomial with $\eta < 1/11$ and $a_{\boldsymbol{\alpha}}(n) \ll_\varepsilon n^\varepsilon$. Then uniformly for $\text{Re } \alpha_j \ll 1/\log T$ *and* $|\text{Im } \alpha_j| \leq T^{1/2-\varepsilon}$

$$\int_{\mathbb{R}} \zeta(\tfrac{1}{2}+\alpha_1+it)\zeta(\tfrac{1}{2}+\alpha_2+it)\zeta(\tfrac{1}{2}+\alpha_3-it)\zeta(\tfrac{1}{2}+\alpha_4-it)|A_{\boldsymbol{\alpha}}(\tfrac{1}{2}+it)|^2 w(t/T)\, dt$$
$$= \frac{1}{4(2\pi i)^4} \oint_{|z_j-\alpha_j| \asymp 1/\log T} A(\boldsymbol{z})\Delta(z_1,z_2,-z_3,-z_4)^2 F_{\boldsymbol{\alpha}}(\boldsymbol{z})$$
$$\times \int_{\mathbb{R}} \left(\frac{t}{2\pi}\right)^{\frac{z_1+z_2-z_3-z_4-\alpha_1-\alpha_2-\alpha_3-\alpha_4}{2}} w(t/T)\, dt \prod_{j\leq 4}\prod_{k\leq 4} \frac{dz_j}{(z_j-\alpha_k)} + O_\varepsilon(T^{1-\delta})$$

for some $\delta > 0$, where $\boldsymbol{z} = (z_1,z_2,z_3,z_4)$, $-\pi\boldsymbol{z} = (-z_3,-z_4,-z_1,-z_2)$ and

$$F_{\boldsymbol{\alpha}}(\boldsymbol{z}) = \sum_{n,m\leq T^\eta} \frac{a_{\boldsymbol{\alpha}}(n)\overline{a_{\boldsymbol{\alpha}}(m)}}{[n,m]} B_{\boldsymbol{z}}\left(\frac{n}{(n,m)}\right) B_{-\pi\boldsymbol{z}}\left(\frac{m}{(n,m)}\right).$$

The proof of the preceding Theorem is essentially the same as the proof of the main result of [57]. We now sketch the differences in the argument. During the set up of the calculation in section 2 of [57], one first applies Lemma 4.2.1 to approximate the product of shifted zeta functions by a Dirichlet polynomial, and then integrates the resulting sum term by term (see (37) of [57]). There are only a few minor differences. The first is that now in place of (40) and (41) of [57] we have the aforementioned weaker approximations for $X_{\boldsymbol{\alpha},t}$ and $g_{\boldsymbol{\alpha}}$ section that hold uniformly for $\text{Re } \alpha_j \ll 1/\log T$ and $|\text{Im } \alpha_j| \leq T^{1/2-\varepsilon}$. These weaker approximations, however, are still sufficient to carry out the proof without any further complications. The only remaining differences in the calculation occur in the application of the delta method in section 5 and 6 of [57]. Since the work of Duke, Friedlander, and Iwaniec [38] uses the Weil bound for Kloosterman sums to control the error terms that arise, it only matters that the real parts of the shifts are small. Therefore the rest of the calculation in [57] is valid uniformly in the range of shifts considered here. The final difference is that in place of (94) of [57], we must use the approximation

$$\frac{\Gamma(\frac{1}{2}-\alpha-s+it)}{\Gamma(\frac{1}{2}+\gamma+s+it)} = t^{-\alpha-\gamma-2s} \exp\left(\frac{\pi i}{2}\text{sgn}(t)(-\alpha-\gamma-2s)\right)$$
$$\times \left(1+\left(\frac{1+|s+\alpha||s+\gamma|}{T}\right)\right)$$

valid when $t \in [T,2T]$ and $|\alpha|,|\gamma| \leq T/2$. Therefore theorem 1.1 of [57] is valid when the shifts satisfy $|\alpha_j| \leq T^{1/2-\varepsilon}$ for some $\varepsilon > 0$. We can now rewrite the main term of theorem 1.1 of [57] as the desired contour integral by applying lemma 2.5.1 of [26] as done in [34] or [52].

4.3 Upper Bounds

To prove Theorem 4.1.1, we must estimate how zeta is correlated on average along intervals of size $(\log T)^\theta$. When the difference $|h_1 - h_2|$ is of size $\ll 1/\log T$, one could simply appeal to a $4\beta^{\text{th}}$ moment bound for zeta and the Cauchy-Schwarz inequality to obtain a bound of the right order of magnitude. To handle larger shifts, stronger techniques are needed however since $\zeta(\frac{1}{2}+it+ih_1)$ and $\zeta(\frac{1}{2}+it+ih_2)$ decouple. We will adapt the method of Heap, Radziwiłł, and Soundararajan [52] and use twisted moments to efficiently estimate the correlation. Throughout this section we will select the following parameters: take ℓ to be the largest integer so that $\log_\ell T \geq 10^5$. Let $T_0 = e^2$ and for $1 \leq j \leq \ell$ take

$$T_j = \exp\left(\frac{\log T}{(\log_{j+1} T)^2}\right)$$

and $K_j = 500P_j$ for $1 \leq j \leq \ell$.

The first step will be to bound the integrand in (4.3) by products of integral powers of zeta and Dirichlet polynomials following [52].

Proposition 4.3.1. *For $0 \leq \beta \leq 1$ and $s, w \in \mathbb{C}$*

$$|\zeta(s)|^{2\beta}|\zeta(w)|^{2\beta} \ll |\zeta(s)|^2|\zeta(w)|^2 \prod_{1\leq j\leq \ell} |\mathcal{N}_j(s;\beta-1)|^2|\mathcal{N}_j(w;\beta-1)|^2 +$$

$$\prod_{1\leq j\leq\ell} |\mathcal{N}_j(s;\beta)|^2|\mathcal{N}_j(w;\beta)|^2 + \sum_{1\leq v\leq\ell}\Bigg(|\zeta(s)|^2|\zeta(w)|^2 \prod_{1\leq j<v} |\mathcal{N}_j(s;\beta-1)|^2|\mathcal{N}_j(w;\beta-1)|^2$$

$$+\prod_{1\leq j<v}|\mathcal{N}_j(s;\beta)|^2|\mathcal{N}_j(w;\beta)|^2\Bigg)\left(\left|\frac{\mathcal{P}_v(s)}{50P_v}\right|^{2\lceil 50P_v\rceil} + \left|\frac{\mathcal{P}_v(w)}{50P_v}\right|^{2\lceil 50P_v\rceil}\right).$$

Proof. The claim is immediate if β is 0 or 1, so assume $0 < \beta < 1$. Take $1 \leq v \leq \ell$ to be the smallest index such that either $|\mathcal{P}_v(s)| > 50P_v$ or $|\mathcal{P}_v(w)| > 50P_v$, and set $v = \ell + 1$ if no such index exists. Applying Young's inequality $ab \leq a^p/p + b^q/q$ for conjugate exponents $p = 1/\beta$, $q = 1/(1-\beta)$ gives

$$|\zeta(s)|^{2\beta}|\zeta(w)|^{2\beta}\prod_{1\leq j<v} e^{2(1-\beta)(\text{Re }\mathcal{P}_j(s)+\text{Re }\mathcal{P}_j(w))}$$

$$\ll |\zeta(s)|^2|\zeta(w)|^2 + \prod_{1\leq j<v} e^{2(\text{Re }\mathcal{P}_j(s)+\text{Re }\mathcal{P}_j(w))},$$

hence

$$|\zeta(s)|^{2\beta}|\zeta(w)|^{2\beta}$$

$$\ll |\zeta(s)|^2|\zeta(w)|^2\prod_{1\leq j<v} e^{2(\beta-1)(\text{Re }\mathcal{P}_j(s)+\text{Re }\mathcal{P}_j(w))} + \prod_{1\leq j<v} e^{2\beta(\text{Re }\mathcal{P}_j(s)+\text{Re }\mathcal{P}_j(w))}.$$

The next step is to expand the exponentials into Dirichlet polynomials. Since $|\mathcal{P}_j(s)| \leq 50P_j$ and $|\mathcal{P}_j(w)| \leq 50P_j$ for $j < v$, lemma 1 of [52] furnishes the bound

$$|\zeta(s)|^{2\beta}|\zeta(w)|^{2\beta} \ll |\zeta(s)|^2|\zeta(w)|^2 \prod_{1\leq j<v} |\mathcal{N}_j(s;\beta-1)|^2|\mathcal{N}_j(w;\beta-1)|^2(1-e^{-P_j})^{-2} + \prod_{1\leq j<v} |\mathcal{N}_j(s;\beta)|^2|\mathcal{N}_j(w;\beta)|^2(1-e^{-P_j})^{-2}.$$

To conclude, note that when $1 \leq v \leq \ell$ we may multiply the right-hand side by

$$\left|\frac{\mathcal{P}_v(s)}{50P_v}\right|^{2\lceil 50P_v\rceil} + \left|\frac{\mathcal{P}_v(w)}{50P_v}\right|^{2\lceil 50P_v\rceil} \geq 1.$$

Since $\prod_{1\leq j\leq\ell}(1-e^{-P_j})^{-2} \leq 4$, the claim follows by summing over $1 \leq v \leq \ell+1$.

□

Therefore proving Theorem 4.1.1 is now reduced to a handful of moment calculations. To simplify the notation, we will write

$$\mathcal{N}_{h_1,h_2,j}(s;\beta) := \mathcal{N}_j(s+ih_1;\beta)\mathcal{N}_j(s+ih_2;\beta).$$

Proposition 4.3.2. *If $\beta \leq 1$ and $|h_1|, |h_2| \leq T^{1/2-\varepsilon}$, then for large T*

$$\frac{1}{T}\int_T^{2T} |\zeta(\tfrac{1}{2}+it+ih_1)\zeta(\tfrac{1}{2}+it+ih_2)|^2 \prod_{1\leq j\leq\ell} |\mathcal{N}_{h_1,h_2,j}(\tfrac{1}{2}+it;\beta-1)|^2 dt \ll_\varepsilon (\log T)^{2\beta^2}|\zeta(1+1/\log T+i(h_1-h_2))|^{2\beta^2} \tag{4.5}$$

and for $1 \leq v \leq \ell$, $k \in \{1,2\}$ and $0 \leq r \leq \lceil 50P_v\rceil$

$$\frac{1}{T}\int_T^{2T} |\zeta(\tfrac{1}{2}+it+ih_1)\zeta(\tfrac{1}{2}+it+ih_2)|^2 \times \prod_{1\leq j<v} |\mathcal{N}_{h_1,h_2,j}(\tfrac{1}{2}+it;\beta-1)|^2|\mathcal{P}_v(\tfrac{1}{2}+i(t+h_k))|^{2r} dt$$
$$\ll_\varepsilon (\log T)^2\left(\frac{(\log T_{v-1})^2}{\log T}\right)^{2\beta^2-2} |\zeta(1+1/\log T+i(h_1-h_2))|^{2\beta^2}\left(18^r r! P_v^r \exp(P_v)\right).$$

Proposition 4.3.3. *If $\beta \leq 1$ and $|h_1|, |h_2| \leq T$, then for large T*

$$\frac{1}{T}\int_T^{2T} \prod_{1\leq j\leq\ell} |\mathcal{N}_{h_1,h_2,j}(\tfrac{1}{2}+it;\beta)|^2 dt \ll (\log T)^{2\beta^2}|\zeta(1+1/\log T+i(h_1-h_2))|^{2\beta^2} \tag{4.6}$$

and for $1 \leq v \leq \ell$, $k \in \{1,2\}$ and $0 \leq r \leq \lceil 50P_v\rceil$

$$\frac{1}{T}\int_T^{2T} \prod_{1\leq j<v} |\mathcal{N}_{h_1,h_2,j}(\tfrac{1}{2}+it;\beta)|^2|\mathcal{P}_v(\tfrac{1}{2}+i(t+h_k))|^{2r} dt \ll (\log T)^{2\beta^2}|\zeta(1+1/\log T+i(h_1-h_2))|^{2\beta^2}\left(r! P_v^r\right).$$

Before embarking on the proofs of these propositions, we first show how to deduce the upper bound in Theorem 4.1.3 and then Theorem 4.1.1.

Proof of Theorem 4.1.3: upper bound case. Propositions 4.3.1, 4.3.2, and 4.3.3 imply

$$\frac{1}{T}\int_T^{2T} |\zeta(\tfrac{1}{2}+it+ih_1)\zeta(\tfrac{1}{2}+it+ih_2)|^{2\beta}\, dt$$
$$\ll_\varepsilon (\log T)^{2\beta^2}|\zeta(1+1/\log T+i(h_1-h_2))|^{2\beta^2}$$
$$\times\left[1+\sum_{1\le v\le \ell}\left((\log_v T)^{4(2-2\beta^2)}\frac{18^{\lceil 50P_v\rceil}\lceil 50P_v\rceil!P_v^{\lceil 50P_v\rceil}\exp(P_v)}{(50P_v)^{2\lceil 50P_v\rceil}}+\frac{\lceil 50P_v\rceil!P_v^{\lceil 50P_v\rceil}}{(50P_v)^{2\lceil 50P_v\rceil}}\right)\right].$$

A quick calculation shows the sum over v is $O(1)$, so the result follows.

□

Proof of Theorem 4.1.1. Split the range of integration in (4.3) depending on whether $|h_1-h_2|<1/\log T$ or $|h_1-h_2|\ge 1/\log T$. By the Laurent expansion for zeta near 1, the contribution of the region where $|h_1-h_2|<1/\log T$ to the integral (4.3) is $\ll (\log T)^{4\beta^2+\theta-1}$, which is admissible. To handle the integral when $|h_1-h_2|\ge 1/\log T$, we treat the cases $\theta\le 0$ and $\theta>0$ separately. When $\theta\le 0$, applying the upper bound in Theorem 4.1.3 the Laurent expansion for zeta gives the bound

$$\mathrm{MoM}_T(2,\beta)\ll(\log T)^{2\beta^2+\theta}\int_{1/\log T}^{2(\log T)^\theta}\frac{dh}{h^{2\beta^2}}.$$

Therefore when $\theta\le 0$ Theorem 4.1.1 immediately follows. When $\theta>0$, standard estimates for the moments of zeta to the right of the one line [86, Theorem 7.9] now show that

$$\mathrm{MoM}_T(2,\beta)\ll(\log T)^{2\beta^2+\theta}\int_{1/\log T}^{1}\frac{dh}{h^{2\beta^2}}+(\log T)^{2\beta^2+2\theta}.$$

The implicit constant is absolute because $\beta\le 1$.

□

Proof of Proposition 4.3.2

We will apply Theorem 4.2.2 with $\boldsymbol{\alpha}=(ih_1,ih_2,-ih_1,-ih_2)$ and $w(t)$ a smooth majorant of $1_{t\in[1,2]}$ to the Dirichlet polynomials

$$\prod_{1\le j\le \ell}\mathcal{N}_{h_1,h_2,j}(s;\beta-1) \tag{4.7}$$

and

$$\prod_{1\le j\le v}\mathcal{N}_{h_1,h_2,j}(s;\beta-1)\mathcal{P}_v(\tfrac{1}{2}+i(t+h_k))^r \tag{4.8}$$

for $k \in \{1,2\}$. By our choice of parameters both of these Dirichlet polynomials have length at most $T^{1/20}$, which is admissible. We may choose circular contours for the z_j so that

$$A(\boldsymbol{z}) \ll (\log T)^2 |\zeta(1 + 1/\log T + i(h_1 - h_2))|^2,$$
$$\Delta(z_1, z_2, -z_3, -z_4)^2 \ll (\log T)^{-4}(|h_1 - h_2| + 1/\log T)^8.$$

Because each Re $z_j \ll 1/\log T$ it follows that

$$\int_{\mathbb{R}} \left(\frac{t}{2\pi}\right)^{\frac{z_1+z_2-z_3-z_4-\alpha_1-\alpha_2-\alpha_3-\alpha_4}{2}} w(t/T)\, dt \ll T.$$

Now the Dirichlet polynomials $\mathcal{N}_{h_1,h_2,j}$ and $\mathcal{P}_v$ are supported on integers composed of primes in $(T_{j-1}, T_j]$ and $(T_{v-1}, T_v]$, respectively, which are disjoint intervals. Therefore by multiplicativity we are left with the task of bounding the product of the sums

$$F_{h_1,h_2,j}(\boldsymbol{z}) = \sum_{n,m \le T^\eta} \frac{a_{h_1,h_2,j}(n)\overline{a_{h_1,h_2,j}(m)}}{[n,m]} B_{\boldsymbol{z}}\left(\frac{n}{(n,m)}\right) B_{\pi\boldsymbol{z}}\left(\frac{m}{(n,m)}\right)$$

and

$$F_{v,r,h_k}(\boldsymbol{z}) = \sum_{n,m \le T^\eta} \frac{b_{v,r,h_k}(n)\overline{b_{v,r,h_k}(m)}}{[n,m]} B_{\boldsymbol{z}}\left(\frac{n}{(n,m)}\right) B_{\pi\boldsymbol{z}}\left(\frac{m}{(n,m)}\right),$$

uniformly for $|z_j - \alpha_j| \ll 1/\log T$, where $a_{h_1,h_2,j}(n)$ are the coefficients of the Dirichlet polynomials $\mathcal{N}_{h_1,h_2,j}(s)$, and $b_{v,r,h_k}(n)$ are the coefficients of $\mathcal{P}_v(s+ih_k)^{2r}$. Multiplying all of these bounds for the terms in the integrand of Theorem 4.2.2 will then give upper bounds of

$$\ll T(\log T)^2 |\zeta(1 + 1/\log T + i(h_1 - h_2))|^2 \sup_{|z_j - \alpha_j| \ll 1/\log T} \prod_{1 \le j \le \ell} F_{h_1,h_2,j}(\boldsymbol{z})$$

and

$$\ll T(\log T)^2 |\zeta(1 + 1/\log T + i(h_1 - h_2))|^2 \sup_{|z_j - \alpha_j| \ll 1/\log T} F_{v,r,h_k}(\boldsymbol{z}) \prod_{1 \le j < v} F_{h_1,h_2,j}(\boldsymbol{z})$$

for the shifted fourth moment of zeta twisted by (4.7) and (4.8) respectively.

First we estimate $F_{h_1,h_2,j}(\boldsymbol{z})$. Note we can write

$$\mathcal{N}_{h_1,h_2,j}(\tfrac{1}{2} + it; \beta - 1) = \sum_{p | n \Rightarrow p \in (T_{j-1}, T_j]} \frac{1}{n^{1/2+it}} \sum_{\substack{cd = n \\ \Omega(c), \Omega(d) \le K_j}} \frac{(\beta - 1)^{\Omega(c)+\Omega(d)} g(c) g(d)}{c^{ih_1} d^{ih_2}},$$

so $a_{h_1,h_2,j}(p) = (\beta - 1)(p^{-ih_1} + p^{-ih_2}) = (\beta - 1)\sigma_{ih_1,ih_2}(p)$ for $p \in (T_{j-1}, T_j]$. Since $\beta \le 1$ it also follows that

$$|a_{h_1,h_2,j}(p^m)| \le \sum_{cd = p^m} g(c) g(d) = \sum_{c+d=m} \frac{1}{c!d!} = \frac{2^m}{m!}. \tag{4.9}$$

We will require the following estimates for $B_{\boldsymbol{z}}(n)$.

Lemma 4.3.4. *For $m \geq 1$, $B_{\boldsymbol{\alpha}}(p^m)$ is equal to*

$$\frac{\sigma_{\alpha_3,\alpha_4}(p^m) - \sigma_{\alpha_3,\alpha_4}(p^{m-1})p^{-1-\alpha_3-\alpha_4}(p^{-\alpha_1}+p^{-\alpha_2}) + \sigma_{\alpha_3,\alpha_4}(p^{m-2})p^{-2-\alpha_1-\alpha_2-2\alpha_3-2\alpha_4}}{1-p^{-2-\alpha_1-\alpha_2-\alpha_3-\alpha_4}},$$

where by convention we set $\sigma_{\alpha_3,\alpha_4}(p^{-1}) = 0$. Furthermore, for integers n composed of primes at most $T^{10^{-8}}$ and $\mathrm{Re}\ \alpha_j \ll 1/\log T$ *for each j*

$$|B_{\boldsymbol{\alpha}}(n)| \ll d_3(n),$$

where d_3 is the ternary divisor function.

Proof. The first formula follows from taking $s = 0$ and $h = p^m$ in lemma 6.9 of [57] and using the formula

$$\sigma_{z_1,z_2}(p^m) = \frac{p^{-z_3(m+1)} - p^{-z_4(m+1)}}{p^{-z_3} - p^{-z_4}}.$$

To prove the second bound, first note by assumption on the size of the shifts and the size of the primes p

$$|\sigma_{\alpha_3,\alpha_4}(p^m)| \leq \sum_{ab=p^m} a^{-\mathrm{Re}\ \alpha_3} b^{-\mathrm{Re}\ \alpha_4} \leq (1+m)\left(1+O\left(\frac{m\log p}{\log T}\right)\right).$$

Whence

$$|B_{\boldsymbol{\alpha}}(p^m)| \leq d(p^m)\left(1+O\left(\frac{m\log p}{\log T}+\frac{1}{p}\right)\right),$$

where d is the divisor function. It now follows by assumption on n that

$$|B_{\boldsymbol{\alpha}}(n)| \ll d(n)\prod_{p^m\|n}\left(1+O\left(\frac{m\log p}{\log T}\right)\right)\left(1+O\left(\frac{1}{p}\right)\right) \ll d_3(n)$$

where we have used

$$\prod_{p^m\|n}\left(1+O\left(\frac{m\log p}{\log T}\right)\right) \ll \exp(\log n/\log T)$$

and that

$$\prod_{p|n}\left(1+O\left(\frac{1}{p}\right)\right) \ll (3/2)^{\omega(n)}.$$

□

The following lemma is almost identical to the proof of Lemma 24 of [4]. The proof is a straightforward application of Cauchy's theorem. The only necessary modification is to use Lemma 4.3.4 to bound $B_{\boldsymbol{z}}(p)$ because here z need not have small imaginary part.

Lemma 4.3.5. *Let $m \geq 1$ be an integer and $\boldsymbol{z} = (z_1, z_2, z_3, z_4)$, $\boldsymbol{w} = (w_1, w_2, w_3, w_4)$ vectors such that $|w_j - z_j| \ll 1/\log T$ for all j. Then*

$$|B_{\boldsymbol{z}}(p) - B_{\boldsymbol{w}}(p)| \ll \frac{\log p}{\log T}.$$

We will now bound $F_{h_1,h_2,j}(\boldsymbol{z})$ by a product over primes. To accomplish this, define the multiplicative coefficients

$$a'_{h_1,h_2,j}(n) = \sum_{cd=n} \frac{(\beta-1)^{\Omega(c)+\Omega(d)} g(c)g(d)}{c^{ih_1} d^{ih_2}}, \;\; a''_{h_1,h_2,j}(n) = \sum_{cd=n} (\beta-1)^{\Omega(c)+\Omega(d)} g(c)g(d).$$

Note that $|a_{h_1,h_2,j}(n)|, |a'_{h_1,h_2,j}(n)| \leq a''_{h_1,h_2,j}(n)$. We use Rankin's trick to show the error incurred when replacing $a_{h_1,h_2,j}(n)$ with $a'_{h_1,h_2,j}(n)$ is negligible. First note that if $n = cd$ with either $\Omega(c)$ or $\Omega(d)$ larger than K_j, then $\Omega(n) \geq K_j$. Furthermore $\exp(\Omega(m) + \Omega(n) - K_j) \geq 1$ when either $\Omega(m)$ or $\Omega(n)$ exceeds K_j. Now using Lemma 4.3.4, that $d_3(n) \ll 3^{\Omega(n)}$, and noticing that $a'_{h_1,h_2,j}(n)$ also satisfies the bound (4.9), we see that replacing $a_{h_1,h_2,j}(n)$ with $a'_{h_1,h_2,j}(n)$ in the expression for $F_{h_1,h_2,j}(\boldsymbol{z})$ contributes an error of at most

$$\begin{aligned}
&\ll e^{-K_j} \sum_{p|m,n \Rightarrow p \in (T_{j-1},T_j]} \frac{|a''_{h_1,h_2,j}(m)||a''_{h_1,h_2,j}(n)| e^{\Omega(n)+\Omega(m)}}{[n,m]} d_3(n) d_3(m) \\
&\ll e^{-K_j} \prod_{T_{j-1}<p\leq T_j} \sum_{c,d\geq 0} \frac{(6e)^{c+d}}{c!d!p^{\max(c,d)}} \\
&\ll e^{-K_j} \prod_{T_{j-1}<p\leq T_j} \left(1 + \frac{12e+36e^2}{p} + O\left(\frac{1}{p^2}\right)\right) \ll e^{-100P_j}.
\end{aligned}$$

Therefore, upon replacing $a_{h_1,h_2,j}(n)$ with $a'_{h_1,h_2,j}(n)$ in the definition of $F_{h_1,h_2,j}(\boldsymbol{z})$ and using multiplicativity to write the resulting sum as a product, we find $F_{h_1,h_2,j}(\boldsymbol{z})$ equals

$$\begin{aligned}
\prod_{T_{j-1}<p\leq T_j} \Big(1 + \frac{1}{p}\big(&a'_{h_1,h_2,j}(p) B_{ih_1,ih_2,-ih_1,-ih_2}(p) + a'_{-h_1,-h_2,j}(p) B_{-ih_1,-ih_2,ih_1,ih_2}(p) \\
&+ |a'_{h_1,h_2,j}(p)|^2\big) + O\left(\frac{\log p}{p\log T} + \frac{1}{p^2}\right)\Big) + O(e^{-100P_j})
\end{aligned}$$

uniformly for $|z_j - \alpha_j| \ll 1/\log T$. Note here we have used (4.9), Lemma 4.3.4, and Lemma 4.3.5. Using Lemma 4.3.4 and that $a_{h_1,h_2,j}(p) = a'_{h_1,h_2,j}(p) = (\beta-1)\sigma_{ih_1,ih_2}(p)$, routine algebra shows that the numerator of the second term in parentheses equals

$$\begin{aligned}
&\frac{(\beta-1)(\beta-1+p+\beta p)}{1+p}(2 + 2\cos((h_1-h_2)\log p)) \\
&\qquad = (2\beta^2 - 2)(1 + \cos((h_1-h_2)\log p)) + O\left(\frac{1}{p}\right).
\end{aligned}$$

Therefore, we must control the products

$$\prod_{T_{j-1}<p\le T_j}\left(1+\frac{(2\beta^2-2)(1+\cos((h_1-h_2)\log p))}{p}+O\left(\frac{\log p}{p\log T}+\frac{1}{p^2}\right)\right)$$

To this end, we will need the following special case of lemma 3.2 of [66].

Lemma 4.3.6. *Given $h\in\mathbb{R}$ and $X\ge 2$*

$$\sum_{p\le X}\frac{\cos(h\log p)}{p}=\log|\zeta(1+1/\log X+ih)|+O(1).$$

Remark. While similar results have appeared in previous literature, this formulation is due to Granville and Soundararajan [46].

Proof. Because the Euler product of $\zeta(s)$ is convergent for Re $s>1$ we may write

$$\begin{aligned}\log|\zeta(1+1/\log X+ih)| &= \mathrm{Re}\left(\sum_p\frac{1}{p^{1+1/\log X+ih}}+\sum_p\sum_{m\ge2}\frac{1}{mp^{m(1+1/\log X+ih)}}\right)\\ &= \sum_p\frac{\cos(h\log p)}{p^{1+1/\log X}}+O(1).\end{aligned}$$

The primes larger than X contribute at most

$$\sum_{p>X}\frac{1}{p^{1+1/\log X}}\ll 1.$$

For $p\le X$ the mean value theorem gives $p^{1/\log X}=1+O(\log p/\log X)$, so the claim follows from Mertens' first estimate $\sum_{p\le X}\frac{\log p}{p}=\log X+O(1)$. □

Therefore uniformly for $|z_j-\alpha_j|\ll 1/\log T$, the product $\prod_{1\le j<v}F_{h_1,h_2,j}(\boldsymbol{z})$ is of order at most

$$\begin{aligned}&\prod_{1\le j<v}\left(\prod_{T_{j-1}<p\le T_j}\left(1+\frac{(2\beta^2-2)(1+\cos((h_1-h_2)\log p))}{p}+O\left(\frac{\log p}{p\log T}+\frac{1}{p^2}\right)\right)\right.\\ &\qquad\qquad\left.+O(e^{-100P_j})\right)\\ &\ll(\log T_{v-1})^{2\beta^2-2}\left(\frac{\log T_{v-1}}{\log T_\ell}\right)^{2\beta^2-2}|\zeta(1+1/\log T_\ell+i(h_1-h_2))|^{2\beta^2-2}\\ &\ll\left(\frac{(\log T_{v-1})^2}{\log T_\ell}\right)^{2\beta^2-2}|\zeta(1+1/\log T_\ell+i(h_1-h_2))|^{2\beta^2-2}.\end{aligned}$$

Since $\log T_\ell \asymp \log T$, this proves the first bound in Proposition 4.3.2. All that remains is to estimate the $F_{v,r,h_k}(\boldsymbol{z})$. Recall $|d_3(n)| \ll 3^{\Omega(n)}$, and we can trivially estimate $|n^{-ih_k}| \le 1$, so

$$|F_{v,r,h_k}(\boldsymbol{z})| \le 9^r \sum_{\substack{p|m,n \Rightarrow p\in(T_{v-1},T_v] \\ \Omega(n)=\Omega(m)=r}} \frac{r!^2 g(n)g(m)}{[n,m]}.$$

In the proof of proposition 3 of [52], it was shown that the right-hand side is at most $18^r r! P_v^r \exp(P_v)$, which concludes the proof of the second bound of Proposition 4.3.2.

Proof of Proposition 4.3.3

To handle the first bound in Proposition 4.3.3, we will use the mean value theorem for Dirichlet polynomials [59, Theorem 9.1] in place of Theorem 4.2.2. We will abuse notation and also denote the coefficients of $N_{h_1,h_2,j}(s;\beta)$ by $a_{h_1,h_2,j}(n)$ now with β in place of $\beta - 1$. Therefore

$$\frac{1}{T}\int_T^{2T} \prod_{1\le j\le \ell} |\mathcal{N}_{h_1,h_2,j}(\tfrac{1}{2}+it;\beta)|^2 dt \ll \prod_{1\le j\le \ell} \sum_{\substack{p|n\Rightarrow p\in(T_{j-1},T_j] \\ \Omega(n)\le 2K_j}} \frac{|a_{h_1,h_2,j}(n)|^2}{n}.$$

The proof is similar to the proof of Proposition 4.3.2, but simpler because there is no need to use Rankin's trick. Indeed all of the terms we add in when replacing $a_{h_1,h_2,j}(n)$ with $a'_{h_1,h_2,j}(n)$ are positive since $\beta \ge 0$. The argument in the proof of Proposition 4.3.2 now gives the bound

$$\ll \prod_{1\le j\le \ell} \prod_{T_{j-1}<p\le T_j} \left(1 + \frac{2\beta^2(1+\cos((h_1-h_2)\log p))}{p} + O\left(\frac{\log p}{p\log T} + \frac{1}{p^2}\right)\right)$$
$$\ll (\log T)^{2\beta^2} |\zeta(1+1/\log T + i(h_1-h_2))|^{2\beta^2}.$$

To prove the second bound of Proposition 4.3.3, note that the mean value theorem for Dirichlet polynomials and the same reasoning as the proof of Proposition 4.3.2 now gives a bound of

$$\ll \prod_{1\le j< v} \prod_{T_{j-1}<p\le T_j} \left(1 + \frac{2\beta^2(1+\cos((h_1-h_2)\log p))}{p} + O\left(\frac{\log p}{p\log T} + \frac{1}{p^2}\right)\right)$$
$$\times \sum_{\substack{p|n\Rightarrow p\in(T_{v-1},T_v] \\ \Omega(n)=r}} \frac{(r!g(n))^2}{n}$$
$$\ll (\log T)^{2\beta^2} |\zeta(1+1/\log T + i(h_1-h_2))|^{2\beta^2} (r! P_v^r).$$

Here we have used that the sum over n above is bounded by $r! P_v^r$, which is a porism of proposition 2 of [52]. This concludes the proofs of Propositions 4.3.2 and 4.3.3, so Theorem 4.1.3 hence the upper bound of Theorem 4.1.1 follows. □

4.4 Lower Bounds

Theorem 4.1.2 follows by integrating the lower bound given by Theorem 4.1.3 over the range $0 \le |h_1 - h_2| \le 2(\log T)^\theta$ using the Laurent expansion of ζ and standard moment estimates for ζ to the right of the one line. We will first prove Theorem 4.1.3 in the case $\beta \le 1$.

4.4.1 Proof of Theorem 4.1.3 for $\beta \le 1$

Throughout this subsection, we will let $T_0 = e^2$, let ℓ be the largest integer such that $\log_\ell T \ge 10^5$, and for $1 \le j \le \ell$ set

$$T_j = \exp\left(\frac{\beta \log T}{(\log_j T)^2}\right)$$

and $K_j = 500P_j$. With this choice of parameters, $\mathcal{N}(s;\alpha)$ is a Dirichlet polynomial of length at most $T^{\beta/18}$. Inspired by the method of Heap and Soundararajan [53], our key inequality is the following consequence of Hölder's inequality:

$$\begin{aligned}
&\Big| \int_{\mathbb{R}} \zeta(\tfrac{1}{2}+it+ih_1)\zeta(\tfrac{1}{2}+it+ih_2)\mathcal{N}(\tfrac{1}{2}+it+ih_1;\beta-1)\mathcal{N}(\tfrac{1}{2}+it+ih_2;\beta-1) \\
&\qquad \times \mathcal{N}(\tfrac{1}{2}-it-ih_1;\beta)\mathcal{N}(\tfrac{1}{2}-it-ih_2;\beta)w(t/T)\,dt\Big| \\
&\le \Big(\int_{\mathbb{R}} |\zeta(\tfrac{1}{2}+it+ih_1)\zeta(\tfrac{1}{2}+it+ih_2)|^{2\beta} w(t/T)\,dt\Big)^{\frac{1}{2}} \\
&\times\Big(\int_{\mathbb{R}} |\zeta(\tfrac{1}{2}+it+ih_1)\zeta(\tfrac{1}{2}+it+ih_2)|^2 \\
&\qquad \times |\mathcal{N}(\tfrac{1}{2}+it+ih_1;\beta-1)\mathcal{N}(\tfrac{1}{2}+it+ih_2;\beta-1)|^2 w(t/T)\,dt\Big)^{\frac{1-\beta}{2}} \\
&\times\Big(\int_{\mathbb{R}} |\mathcal{N}(\tfrac{1}{2}+it+ih_1;\beta-1)\mathcal{N}(\tfrac{1}{2}+it+ih_2;\beta-1)|^2 \\
&\qquad \times |\mathcal{N}(\tfrac{1}{2}+it+ih_1;\beta)\mathcal{N}(\tfrac{1}{2}+it+ih_2;\beta)|^{2/\beta} w(t/T)\,dt\Big)^{\frac{\beta}{2}},
\end{aligned} \tag{4.10}$$

where $0 \le w(t) \le 1$ is a smooth function supported on the interval $[1.1, 1.9]$ with $w(t) = 1$ for $t \in [1.2, 1.8]$, say. Note that Proposition 4.3.2 gives the upper bound

$$\begin{aligned}
&\int_{\mathbb{R}} |\zeta(\tfrac{1}{2}+it+ih_1)\zeta(\tfrac{1}{2}+it+ih_2)|^2 \\
&\qquad \times |\mathcal{N}(\tfrac{1}{2}+it+ih_1;\beta-1)\mathcal{N}(\tfrac{1}{2}+it+ih_2;\beta-1)|^2 w(t/T)\,dt \\
&\qquad\qquad \ll_\varepsilon T(\log T)^{2\beta^2}|\zeta(1+1/\log T + i(h_1-h_2))|^{2\beta^2}
\end{aligned}$$

for $|h_1 - h_2| \le T^{1/2-\varepsilon}$, so we are left with two moment computations.

Before carrying out these computations, we will introduce some simplifying notation. Denote

$$A(\tfrac{1}{2}+it) := \mathcal{N}(\tfrac{1}{2}+it+ih_1;\beta-1)\mathcal{N}(\tfrac{1}{2}+it+ih_2;\beta-1) = \sum_{n\le N} \frac{a(n)}{n^{1/2+it}},$$

$$B(\tfrac{1}{2}+it) := \mathcal{N}(\tfrac{1}{2}+it+ih_1;\beta)\mathcal{N}(\tfrac{1}{2}+it+ih_2;\beta) = \sum_{n\le N} \frac{b(n)}{n^{1/2+it}}$$

where $N \le T^{\beta/9}$. For $1 \le j \le \ell$, set

$$A_j(\tfrac{1}{2}+it) := \mathcal{N}_j(\tfrac{1}{2}+it+ih_1;\beta-1)\mathcal{N}_j(\tfrac{1}{2}+it+ih_2;\beta-1),$$

$$B_j(\tfrac{1}{2}+it) := \mathcal{N}_j(\tfrac{1}{2}+it+ih_1;\beta)\mathcal{N}_j(\tfrac{1}{2}+it+ih_2;\beta).$$

These are Dirichlet polynomials supported on integers n composed of primes $p \in (T_{j-1}, T_j]$ with coefficients

$$a_j(n) := \sum_{\substack{cd=n\\ \Omega(c),\Omega(d)\le K_j}} \frac{(\beta-1)^{\Omega(c)+\Omega(d)} g(c)g(d)}{c^{ih_1}d^{ih_2}},$$

$$b_j(n) := \sum_{\substack{cd=n\\ \Omega(c),\Omega(d)\le K_j}} \frac{\beta^{\Omega(c)+\Omega(d)} g(c)g(d)}{c^{ih_1}d^{ih_2}}.$$

and for $a = 1, 2$ define

$$M_a(t) = |\mathcal{N}(\tfrac{1}{2}+it+ih_a;\beta-1)||\mathcal{N}(\tfrac{1}{2}+it+ih_a;\beta)|^{1/\beta}.$$

The lower bound of Theorem 4.1.3 will follow from the following two propositions.

Proposition 4.4.1. *Uniformly in* $|h_1|, |h_2| \le T/2$

$$\left|\frac{1}{T}\int_{\mathbb{R}} \zeta(\tfrac{1}{2}+it+ih_1)\zeta(\tfrac{1}{2}+it+ih_2)A(\tfrac{1}{2}+it)\overline{B(\tfrac{1}{2}+it)}w(t/T)\ dt\ \right| \gg_\beta (\log T)^{2\beta^2}|\zeta(1+1/\log T + i(h_1-h_2))|^{2\beta^2}.$$

Proposition 4.4.2. *Uniformly in* $|h_1|, |h_2| \le T$

$$\frac{1}{T}\int_{\mathbb{R}} M_1M_2(t)^2 w(t/T)dt \ll_\beta (\log T)^{2\beta^2}|\zeta(1+1/\log T + i(h_1-h_2))|^{2\beta^2}.$$

Proof of Proposition 4.4.1

Write

$$\mathcal{I}(\beta, h_1, h_2) = \int_{\mathbb{R}} \zeta(\tfrac{1}{2} + it + ih_1)\zeta(\tfrac{1}{2} + it + ih_2)A(\tfrac{1}{2} + it)\overline{B(\tfrac{1}{2} + it)}w(t/T)\ dt.$$

We will evaluate this integral with an appropriate approximate functional equation. Set

$$\widetilde{g}_{\alpha_1,\alpha_2}(s,t) = \pi^{-s}\frac{\Gamma\left(\frac{\frac{1}{2}+i\alpha_1+s+it}{2}\right)\Gamma\left(\frac{\frac{1}{2}+i\alpha_2+s+it}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}+i\alpha_1+it}{2}\right)\Gamma\left(\frac{\frac{1}{2}+i\alpha_2+it}{2}\right)},$$

$$\widetilde{X}_{\alpha_1,\alpha_2,t} = \lambda(\tfrac{1}{2} + i\alpha_1 + it)\lambda(\tfrac{1}{2} + i\alpha_2 + it),$$

and

$$\widetilde{V}_{\alpha_1,\alpha_2}(x,t) = \frac{1}{2\pi i}\int_{(1)} \frac{e^{s^2}}{s}\widetilde{g}_{\alpha_1,\alpha_2}(s,t)x^{-s}\ ds,$$

which satisfies the decay estimate

$$t^j\frac{\partial^j}{\partial t^j}\widetilde{V}_{\alpha_1,\alpha_2}(x,t) \ll_{A,j} (1 + |x|/t)^{-A}.$$

Lemma 4.4.3 ([59] Theorem 5.3)**.** *Let $t \in [T, 2T]$, $\mathrm{Re}\ \alpha_j \ll 1/\log T$ and $|\mathrm{Im}\ \alpha_j| \le T/2$ for $j = 1, 2$. Then for all $A > 0$*

$$\zeta(\tfrac{1}{2} + \alpha_1 + it)\zeta(\tfrac{1}{2} + \alpha_2 + it) = \sum_{m,n} \frac{1}{m^{1/2+\alpha_1}n^{1/2+\alpha_2}}\,(mn)^{-it}\,\widetilde{V}_{\alpha_1,\alpha_2}(mn,t)$$
$$+\widetilde{X}_{\alpha_1,\alpha_2,t}\sum_{m,n}\frac{1}{m^{1/2-\alpha_2}n^{1/2-\alpha_1}}\,(mn)^{it}\,\widetilde{V}_{-\alpha_2,-\alpha_1}(mn,t) + O_A((1+t)^{-A}).$$

Therefore, up to a negligible error, $\mathcal{I}(\beta, h_1, h_2)$ is given by

$$\sum_{h,k,m,n}\frac{a(h)\overline{b(k)}}{m^{ih_1}n^{ih_2}\sqrt{hkmn}}\int_{\mathbb{R}}\left(\frac{hmn}{k}\right)^{-it}\widetilde{V}_{h_1,h_2}(mn,t)w(t/T)\ dt \tag{4.11}$$
$$+\sum_{h,k,m,n}\frac{a(h)\overline{b(k)}}{m^{-ih_2}n^{-ih_1}\sqrt{hkmn}}\int_{\mathbb{R}}\widetilde{X}_{h_1,h_2,t}\left(\frac{h}{kmn}\right)^{-it}\widetilde{V}_{-h_2,-h_1}(mn,t)w(t/T)\ dt.$$

Using the decay of $\widetilde{V}_{\alpha_1,\alpha_2}$, we can discard the contribution of terms with $hmn \neq k$ and $h \neq kmn$ in the first and second lines of (4.11) respectively. Next we can parameterize the diagonal sums in the first and second lines by $\sum_{h|k,mn=k/h}$ and

$\sum_{k|h,mn=h/k}$ respectively. Then after using the definition of $\widetilde{V}_{\alpha,\beta}$, shifting the contour, and recognizing the sum over m and n as a familiar Dirichlet convolution we find

$$\begin{aligned}\mathcal{I}(\beta, h_1, h_2) &= \sum_{h|k} \frac{a(h)\overline{b(k)}}{k}\sigma_{ih_1,ih_2}(k/h) \int_{\mathbb{R}} w(t/T)\ dt \\ &\quad + \sum_{k|h} \frac{a(h)\overline{b(k)}}{h}\sigma_{-ih_2,-ih_1}(h/k) \int_{\mathbb{R}} \widetilde{X}_{h_1,h_2,t}\ w(t/T)\ dt + O(T^{1-\delta}).\end{aligned} \tag{4.12}$$

for some $\delta > 0$.

Due to the oscillating term $\widetilde{X}_{h_1,h_2,t}$ the second line in (4.12) will not contribute to the leading order, and we will control this term first. Because $w(t/T)$ is supported on $[T, 2T]$ and that $|h_1|, |h_2| \leq T/2$, Stirling's approximation gives $\widetilde{X}_{h_1,h_2,t} = e^{-if(t)}(1 + O(1/T))$ where

$$f(t) = (h_1 + t)\log\left(\frac{t+h_1}{2\pi}\right) + (h_2 + t)\log\left(\frac{t+h_2}{2\pi}\right) - 2t - h_1 - h_2 - \frac{\pi}{2}.$$

For $t \in [T, 2T]$ note that $f'(t) \gg \log T$ and that $f''(t) \ll 1/T$. Therefore, an integration by parts à la van der Corput gives the bound

$$\int_{\mathbb{R}} \widetilde{X}_{h_1,h_2,t}\ w(t/T)\ dt \ll 1/\log T.$$

Therefore because $A(s)$ and $B(s)$ are Dirichlet polynomials of length $\leq T^{\beta/18}$ and $a(n), b(n) \ll_\varepsilon n^\varepsilon$, we may absorb the second line of (4.12) into the error term $O(T^{1-\delta})$.

We can factor the main term as

$$T\|w\|_1 \sum_{h|k} \frac{a(h)\overline{b(k)}}{k}\sigma_{ih_1,ih_2}(k/h) = T\|w\|_1 \prod_{j\leq \ell}\sum_{h|k} \frac{a_j(h)\overline{b_j(k)}}{k}\sigma_{ih_1,ih_2}(k/h).$$

It therefore remains to analyze the sums

$$S_j := \sum_{h|k} \frac{a_j(h)\overline{b_j(k)}}{k}\sigma_{ih_1,ih_2}(k/h) = \sum_{\substack{h,k \\ p|h,k \Rightarrow p\in(T_{j-1},T_j]}} \frac{\sigma_{ih_1,ih_2}(a)a_j(h)\overline{b_j(hk)}}{hk}$$

for $j \leq \ell$. We will accomplish this by replacing $a_j(n)$ and $b_j(n)$ with the multiplicative coefficients

$$\begin{aligned} a'(n) &:= \sum_{cd=n} \frac{(\beta-1)^{\Omega(c)+\Omega(d)}g(c)g(d)}{c^{ih_1}d^{ih_2}}, \\ b'(n) &:= \sum_{cd=n} \frac{\beta^{\Omega(c)+\Omega(d)}g(c)g(d)}{c^{ih_1}d^{ih_2}}, \end{aligned}$$

and show that the difference between S_j and

$$S_j' = \sum_{\substack{h,k \\ p|h,k \Rightarrow p\in(T_{j-1},T_j]}} \frac{\sigma_{ih_1,ih_2}(a)a'(h)\overline{b'(hk)}}{hk}$$

is small using Rankin's trick. To see this, denote

$$a''(n) := \sum_{cd=n} (\beta-1)^{\Omega(c)+\Omega(d)} g(c)g(d),$$

$$b''(n) := \sum_{cd=n} \beta^{\Omega(c)+\Omega(d)} g(c)g(d).$$

Notice that if $a_j(h)\overline{b_j(hk)} \neq a'(h)\overline{b'(hk)}$ where $p|h,k \Rightarrow p \in (T_{j-1}, T_j]$, then it must follow that $\exp(2\Omega(h) + \Omega(k) - K_j) \geq 1$. Therefore $|S_j - S_j'|$ is bounded by

$$\begin{aligned} e^{-K_j} &\prod_{p\in(T_{j-1},T_j]} \sum_{r,s\geq 0} \frac{|\sigma_{ih_1,ih_2}(p^r)a''(p^s)b''(p^{r+s})|}{p^{r+s}} e^{2r+s} \\ &\ll e^{-K_j} \prod_{p\in(T_{j-1},T_j]} \left(1 + \frac{2e+2e^2}{p} + O\left(\frac{1}{p^2}\right)\right) \ll e^{-100P_j}, \end{aligned}$$

whence

$$S_j = \prod_{p\in(T_{j-1},T_j]} \sum_{r,s\geq 0} \frac{\sigma_{ih_1,ih_2}(p^r)a'(p^s)\overline{b'(p^{r+s})}}{p^{r+s}} + O(e^{-100P_j}).$$

Now since $a'(p) = (\beta-1)\sigma_{ih_1,ih_2}(p)$ and $b'(p) = \beta\sigma_{ih_1,ih_2}(p)$ it follows that

$$S_j = \prod_{p\in(T_{j-1},T_j]} \left(1 + \frac{2\beta^2(1+\cos((h_1-h_2)\log p))}{p} + O\left(\frac{1}{p^2}\right)\right) + O(e^{-100P_j}),$$

so up to a power savings in T the integral $\mathcal{I}(\beta, h_1, h_2)$ is equal to

$$T\prod_{j\leq\ell}\left(\prod_{p\in(T_{j-1},T_j]} \left(1 + \frac{2\beta^2(1+\cos((h_1-h_2)\log p))}{p} + O\left(\frac{1}{p^2}\right)\right) + O(e^{-100P_j})\right) \|w\|_1.$$

Using Lemma 4.3.6 and that $\|w\|_1 > 0.6$ we conclude

$$|\mathcal{I}(\beta, h_1, h_2)| \gg_\beta T(\log T)^{2\beta^2} |\zeta(1 + 1/\log T_\ell + i(h_1-h_2))|^{2\beta^2},$$

so Proposition 4.4.1 follows. □

Proof of Proposition 4.4.2

Since we only need an upper bound, we will bound the weight $w(t/T)$ by the characteristic function of $[T, 2T]$. Therefore, we must simply evaluate the mean value of the Dirichlet polynomial $M_1M_2(t)$ on $[T, 2T]$. We will decompose each

$$M_a(t) = \prod_{j\le\ell} M_{a,j}(t)$$

where

$$M_{a,j}(t) := |\mathcal{N}_j(\tfrac{1}{2} + it + ih_a; \beta - 1)||\mathcal{N}_j(\tfrac{1}{2} + it + ih_a; \beta)|^{1/\beta}.$$

Following [53] we will now bound $|M_{1,j}M_{2,j}|^2$ by squares of Dirichlet polynomials.

Lemma 4.4.4. *For $j \le \ell$*

$$\begin{aligned}|M_{1,j}(t)M_{2,j}(t)|^2 \le |\mathcal{N}_j(\tfrac{1}{2} + it + ih_1; \beta)\mathcal{N}_j(\tfrac{1}{2} + it + ih_2; \beta)|^2(1 + O(e^{-K_j/10}))\\ +O(2^{2/\beta}Q_{1,j}(t)|\mathcal{N}_j(\tfrac{1}{2} + it + ih_2; \beta)|^2(1 + O(e^{-K_j/10}))\\ +2^{2/\beta}Q_{2,j}(t)|\mathcal{N}_j(\tfrac{1}{2}+it + ih_2; \beta)|^2(1 + O(e^{-K_j/10}))\\ +2^{4/\beta}Q_{1,j}(t)Q_{2,j}(t)),\end{aligned}$$

where for $a = 1, 2$

$$Q_{a,j}(t) := \left(\frac{12|\mathcal{P}_j(\frac{1}{2} + it + ih_a)|}{K_j}\right)^{2K_j} \sum_{r=0}^{K_j/\beta} \left(\frac{2e|\mathcal{P}_j(\frac{1}{2} + it + ih_a)|}{r + 1}\right)^{2r}$$

Proof. The proof is nearly identical to the proof of lemma 1 of [53]. If $|\mathcal{P}_j(\frac{1}{2} + it + ih_a)| \le K_j/10$, then we may write

$$\begin{aligned}|M_{a,j}(t)|^2 &= \exp\left(2\beta\mathrm{Re}\ \mathcal{P}_j(\tfrac{1}{2} + it + ih_a)+\right)(1 + O(e^{-K_j/10}))^2\\ &= |\mathcal{N}_j(\tfrac{1}{2} + it + ih_a; \beta)|^2(1 + O(e^{-K_j/10})).\end{aligned}$$

In the case $|\mathcal{P}_j(\frac{1}{2} + it + ih_a)| > K_j/10$ for some a then the proof of lemma 1 of [53] provides the bounds

$$|\mathcal{N}_j(\tfrac{1}{2} + it + ih_a; \beta - 1)| \le \left(\frac{12|\mathcal{P}_j(\frac{1}{2} + it + ih_a)|}{K_j}\right)^{2K_j}$$

and

$$|\mathcal{N}_j(\tfrac{1}{2} + it + ih_a; \beta)|^{1/\beta} \ll 2^{2/\beta} \sum_{r=0}^{K_j/\beta} \left(\frac{2e|\mathcal{P}_j(\frac{1}{2} + it + ih_a)|}{r + 1}\right)^{2r}.$$

The claim now follows by summing over all four cases where $|\mathcal{P}_j(\frac{1}{2} + it + ih_1)|$ and $|\mathcal{P}_j(\frac{1}{2} + it + ih_2)|$ are either smaller or larger than $K_j/10$. □

Lemma 4.4.5. *For $a = 1, 2$*

$$\frac{1}{T}\int_T^{2T} Q_{a,j}(t)dt \ll e^{-K_j}, \quad \frac{1}{T}\int_T^{2T} Q_{a,j}(t)^2 dt \ll e^{-K_j}.$$

Proof. To handle the first estimate, note for $0 \leq r \leq K_j/\beta$

$$\mathcal{P}_j(\tfrac{1}{2}+it+ih_a)^{K_j+r} = \sum_{\substack{\Omega(n)=K_j+r \\ p|n \Rightarrow p \in (T_{j-1},T_j]}} \frac{(K_j+r)!g(n)}{n^{1/2+it+ih_a}}.$$

This is a Dirichlet polynomial of length at most $T_j^{K_j(1+1/\beta)}$ whose coefficients have magnitude $(K_j+r)!g(n)/n^{1/2}$, so the claim follows by the same argument used in lemma 2 of [53].

To handle the second estimate an application of the Cauchy-Schwarz inequality yields

$$Q_{a,j}(t)^2 \leq \left(\frac{12|P_j(\frac{1}{2}+it+ih_a)|}{K_j}\right)^{4K_j} \cdot \frac{K_j}{\beta}\sum_{r=0}^{K_j/\beta}\left(\frac{2e|P_j(\frac{1}{2}+it+ih_a)|}{r+1}\right)^{4r}.$$

Next note that

$$\mathcal{P}_j(\tfrac{1}{2}+it+ih_a)^{2K_j+2r} = \sum_{\substack{\Omega(n)=2K_j+2r \\ p|n \Rightarrow p \in (T_{j-1},T_j]}} \frac{(2K_j+2r)!g(n)}{n^{1/2+it+ih_a}},$$

which is also a short Dirichlet polynomial. Now the mean value theorem for Dirichlet polynomials gives

$$\frac{1}{T}\int_T^{2T} |\mathcal{P}_j(\tfrac{1}{2}+it+ih_a)|^{4K_j+4r}dt \ll \sum_{\substack{\Omega(n)=2K_j+2r \\ p|n \Rightarrow p \in (T_{j-1},T_j]}} \frac{(2K_j+2r)!^2 g(n)^2}{n} \ll (2K_j+2r)!P_j^{2K_j+2r},$$

where we have used that $g(n) \leq 1$ and the definition of P_j. Whence

$$\frac{1}{T}\int_T^{2T} Q_{a,j}(t)^2 dt \ll \frac{K_j}{\beta}\left(\frac{12}{K_j}\right)^{4K_j}\sum_{r=0}^{K_j/\beta}\left(\frac{2e}{r+1}\right)^{4r}(2K_j+2r)!P_j^{2K_j+2r}.$$

The summand is maximized near r satisfying $r^2 = 4P_j(2K_j+2r)$. Since $K_j = 500P_j$ any such r necessarily lies in $[2\sqrt{P_jK_j}, 2.1\sqrt{P_jK_j}]$, and we conclude in the same manner as lemma 2 of [53].

□

To conclude, we will use the following splitting lemma, which appears in equation (16) of [53]

Lemma 4.4.6. *Suppose for $1 \le j \le \ell$ we have j disjoint intervals I_j and Dirichlet polynomials $A_j(s) = \sum_n a_j(n) n^{-s}$ such that $a_j(n)$ vanishes unless n is composed of primes in I_j. Then if $\prod_{j\le\ell} A_j(s)$ is a Dirichlet polynomial of length $\le N$, then*

$$\frac{1}{T}\int_T^{2T} \prod_{j\le \ell} |A_j(\tfrac12+it)|^2 dt = (1+O(N/T)) \prod_{j\le\ell}\left(\frac{1}{T}\int_T^{2T} |A_j(\tfrac12+it)|^2 dt\right)$$

In [53] equation (16), there is an additional factor of $\log N$ in the error term which arises because [53] uses a simpler treatment of the mean value theorem for Dirichlet polynomials. This however can be easily removed by using a version of the mean value theorem for Dirichlet polynomials with a slightly stronger error term, i.e. theorem 9.1 of [59]. In fact a version of this splitting lemma has also appeared in [4, lemma 14].

To conclude the proof, note the mean value theorem for Dirichlet polynomials gives

$$\begin{aligned}
&\frac{1}{T}\int_T^{2T} |\mathcal{N}_j(\tfrac12+it+ih_1;\beta)\mathcal{N}_j(\tfrac12+it+ih_2;\beta)|^2 dt \\
&\quad = (1+O(T^{-8/9})) \sum_{\substack{p|n \Rightarrow p\in(T_{j-1},T_j] \\ \Omega(n)\le K_j}} \frac{|b_j(n)|^2}{n} \\
&\quad \le (1+O(T^{-8/9})) \prod_{p\in(T_{j-1},T_j]} \left(1+\frac{2\beta^2(1+\cos((h_1-h_2)\log p))}{p} + O\left(\frac{1}{p^2}\right)\right).
\end{aligned} \tag{4.13}$$

Similar reasoning gives that for $a = 1, 2$

$$\begin{aligned}
\frac{1}{T}\int_T^{2T} |\mathcal{N}_j(\tfrac12+it+ih_a;\beta)|^2 dt &\le (1+O(T^{-8/9})) \prod_{p\in(T_{j-1},T_j]} \left(1+\frac{\beta^2}{p}+O\left(\frac{1}{p^2}\right)\right) \\
&\le (1+O(T^{-8/9}))\left(\frac{\log T_j}{\log T_{j-1}}\right)^{\beta^2}.
\end{aligned} \tag{4.14}$$

Combining these calculations with lemma 4.4.4, 4.4.5, and 4.4.6 and bounding integrals of products using the Cauchy-Schwarz inequality, we find

$$\begin{aligned}
&\frac{1}{T}\int_{\mathbb{R}} M_1 M_2(t)^2 w(t/T) dt \\
&\ll_\beta \prod_{1\le j\le\ell}\left(\prod_{T_{j-1}<p\le T_j}\left(1+\frac{2\beta^2(1+\cos((h_1-h_2)\log p))}{p}+O\left(\frac{1}{p^2}\right)\right)+O(e^{-K_j/10})\right) \\
&\ll_\beta (\log T)^{2\beta^2} |\zeta(1+1/\log T_\ell + i(h_1-h_2))|^{2\beta^2}.
\end{aligned}$$

This completes the proof of Proposition 4.4.2.

4.4.2 Proof of Theorem 4.1.3 for $\beta \geq 1$

The case of $\beta \geq 1$ is a bit simpler than the case $\beta \leq 1$. Now let $T_0 = \beta^4 e^2$, let ℓ be the largest integer such that $\log_\ell T \geq 10^5$, for $1 \leq j \leq \ell$ set

$$T_j = \exp\left(\frac{\log T}{\beta^2 (\log_j T)^2}\right),$$

and take $K_j = 250\beta^2 P_j$. With this choice of parameters, $\mathcal{N}(s;\alpha)$ is a Dirichlet polynomial of length at most $T^{1/18}$. Now because $\beta \geq 1$, Hölder's inequality gives

$$\Big| \int_{\mathbb{R}} \zeta(\tfrac{1}{2}+it+ih_1)\zeta(\tfrac{1}{2}+it+ih_2)\mathcal{N}(\tfrac{1}{2}+it+ih_1;\beta-1)\mathcal{N}(\tfrac{1}{2}+it+ih_2;\beta-1) \tag{4.15}$$

$$\times \mathcal{N}(\tfrac{1}{2}-it-ih_1;\beta)\mathcal{N}(\tfrac{1}{2}-it-ih_2;\beta)w(t/T)\ dt\Big|$$

$$\leq \Big(\int_{\mathbb{R}} |\zeta(\tfrac{1}{2}+it+ih_1)\zeta(\tfrac{1}{2}+it+ih_2)|^{2\beta} w(t/T)\ dt\Big)^{\frac{1}{2\beta}}$$

$$\times\Big(\int_{\mathbb{R}} |\mathcal{N}(\tfrac{1}{2}+it+ih_1;\beta-1)\mathcal{N}(\tfrac{1}{2}+it+ih_2;\beta-1)$$

$$\times\mathcal{N}(\tfrac{1}{2}+it+ih_1;\beta)\mathcal{N}(\tfrac{1}{2}+it+ih_2;\beta)|^{\frac{2\beta}{2\beta-1}} w(t/T)\ dt\Big)^{1-\frac{1}{2\beta}}.$$

Therefore, if we denote

$$N_{a,j}(t) := |\mathcal{N}_j(\tfrac{1}{2}+it+ih_1;\beta-1)\mathcal{N}_j(\tfrac{1}{2}+it+ih_1;\beta)|^{\frac{2\beta}{2\beta-1}}$$

all we must show is

Proposition 4.4.7. *For $\beta \geq 1$ and $|h_1|, |h_2| \leq T$*

$$\frac{1}{T}\int_{\mathbb{R}} \prod_{j\leq \ell} N_{1,j}(t)N_{2,j}(t)w(t/T)\ dt$$

$$\ll_\beta (\log T)^{2\beta^2}|\zeta(1+1/\log T + i(h_1-h_2))|^{2\beta^2}.$$

This is a lot like Proposition 4.4.2, but the proof is even simpler. The analog of Lemma 4.4.4 is

Lemma 4.4.8. *For $j \leq \ell$*

$$N_1, j(t)N_{2,j}(t) \leq |\mathcal{N}_j(\tfrac{1}{2}+it+ih_1;\beta)\mathcal{N}_j(\tfrac{1}{2}+it+ih_2;\beta)|^2(1+O(e^{-K_j/10}))$$

$$+O(R_{1,j}(t)|\mathcal{N}_j(\tfrac{1}{2}+it+ih_2;\beta)|^2(1+O(e^{-K_j/10}))$$

$$+R_{2,j}(t)|\mathcal{N}_j(\tfrac{1}{2}+it+ih_2;\beta)|^2(1+O(e^{-K_j/10}))$$

$$+R_{1,j}(t)R_{2,j}(t)),$$

where for $a = 1, 2$

$$R_{a,j}(t) := \left(\frac{12\beta |\mathcal{P}_j(\frac{1}{2} + it + ih_a)|}{K_j} \right)^{2K_j}.$$

Proof. Following the proof of Lemma 4.4.4, if $|\mathcal{P}_j(\frac{1}{2} + it + ih_a)| \leq K_j/10$, then we may write

$$N_{a,j}(t) = |\mathcal{N}_j(\tfrac{1}{2} + it + ih_a; \beta)|^2 (1 + O(e^{-K_j/10})).$$

In the case $|\mathcal{P}_j(\frac{1}{2} + it + ih_a)| > K_j/10$ for some a then simply note for $\alpha = \beta - 1$ or $\alpha = \beta$

$$|\mathcal{N}_j(\tfrac{1}{2} + it + ih_a; \alpha)|^{\frac{2\beta}{2\beta-1}} \leq \left(\sum_{r=0}^{K_j} \frac{\beta^r |\mathcal{P}_j(\frac{1}{2} + it + ih_a)|^r}{r!} \right)^{\frac{2\beta}{2\beta-1}}$$
$$\leq \left(\sum_{r=0}^{K_j} \frac{\beta^r |\mathcal{P}_j(\frac{1}{2} + it + ih_a)|^r}{r!} \right)^2 \leq \left(\frac{12\beta |\mathcal{P}_j(\frac{1}{2} + it + ih_a)|}{K_j} \right)^{2K_j},$$

where we have used that $\beta \geq 1$ and that $|\mathcal{P}_j(\frac{1}{2} + it + ih_a)|$ is large. The claim now follows. □

The analog of Lemma 4.4.4 is

Lemma 4.4.9. *For* $a = 1, 2$

$$\frac{1}{T} \int_T^{2T} R_{a,j}(t) dt \ll e^{-K_j}, \quad \frac{1}{T} \int_T^{2T} R_{a,j}(t)^2 dt \ll e^{-K_j}.$$

The proof is a simpler version of Lemma 4.4.4, and is omitted. Now we may conclude the proof of Proposition 4.4.7 when $\beta \geq 1$ by combining Lemma 4.4.8, Lemma 4.4.9 and Lemma 4.4.6 with the mean value theorem for Dirichlet polynomials as we did in the $\beta \leq 1$ case. This concludes the proof of 4.4.7, hence the proof of Theorems 4.1.2 and 4.1.3.

Chapter 5

Conditional upper bounds for shifted moments of zeta

Shifted moments of the Riemann zeta function

$$M_{\boldsymbol{\alpha},\boldsymbol{\beta}}(T) = \int_T^{2T} \prod_{k=1}^{m} |\zeta(\tfrac{1}{2} + i(t+\alpha_k))|^{2\beta_k} dt, \tag{5.1}$$

where $\boldsymbol{\alpha} = \boldsymbol{\alpha}(T) = (\alpha_1, \dots, \alpha_m)$ and $\boldsymbol{\beta} = (\beta_1 \dots, \beta_m)$ satisfy $|\alpha_k| \le T/2$ and $\beta_k \ge 0$, were first studied in general by Chandee [21]. Chandee gave lower bounds assuming the β_k are integers, $\alpha_k = O(\log\log T)$, and $|\alpha_j - \alpha_k| = O(1)$. Chandee also gave upper bounds assuming the Riemann hypothesis when $|\alpha_j - \alpha_k| = O(1)$ and $\alpha_k = O(\log T)$ which are sharp up to a $(\log T)^{\varepsilon}$ loss. Subsequently Ng, Shen, and Wong [77] removed the $(\log T)^{\varepsilon}$ loss in the special case where $\boldsymbol{\beta} = (\beta, \beta)$ by using the work of Harper [50] on the moments of the zeta, and they also gave bounds in the larger regime $|\alpha_1 + \alpha_2| \le T^{0.6}$. More precisely, in this range they proved

$$M_{(\alpha_1,\alpha_2),(\beta,\beta)}(T) \ll T(\log T)^{2\beta^2} F(\alpha_1, \alpha_2, T)^{2\beta^2}$$

where

$$F(\alpha_1, \alpha_2, T) = \begin{cases} \min\left(|\alpha_1 - \alpha_2|^{-1}, \log T\right) & |\alpha_1 - \alpha_2| \le 1/100 \\ \log(2 + |\alpha_1 - \alpha_2|) & |\alpha_1 - \alpha_2| > 1/100 \end{cases}.$$

Some special cases of the shifted moments $M_{\boldsymbol{\alpha},\boldsymbol{\beta}}(T)$ and related objects have been studied unconditionally. For example the integral

$$\int_T^{2T} \zeta(\tfrac{1}{2} + i(t+\alpha_1))\zeta(\tfrac{1}{2} - i(t+\alpha_2)) dt$$

akin to $M_{\boldsymbol{\alpha},\boldsymbol{\beta}}(T)$ with $\boldsymbol{\beta} = (\frac{1}{2}, \frac{1}{2})$ is fairly well understood. Here the current state of the art comes from Atkinson's formula for the mean square of zeta [9] and Bettin's work on the second moment of zeta with shifts of size $T^{2-\varepsilon}$ [16]. The current state of the art for $M_{\boldsymbol{\alpha},\boldsymbol{\beta}}(T)$ with $\boldsymbol{\beta} = (1,1)$ is due to Motohashi's explicit formula for the fourth moment of zeta [74, 75] and Kovaleva's work on the fourth moment of zeta with shifts of size up to $T^{3/2-\varepsilon}$ [67]. Finally in the case where $\boldsymbol{\beta} = (\beta,\beta)$, sharp upper bounds for $\beta \le 1$ and lower bounds for all $\beta \ge 0$ with shifts of size up to $T^{1/2-\varepsilon}$ were obtained in Chapter 4. The goal of this chapter is, assuming the Riemann hypothesis, to extend the work of Ng, Shen, and Wong [77] to arbitrary $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ and to give stronger bounds in the regime where the differences $|\alpha_j - \alpha_k|$ are unbounded.

Theorem 5.0.1. *Assume the Riemann hypothesis. If $\beta_k \ge 0$ and $|\alpha_k| \le T/2$ for $k = 1, \ldots, m$, then*

$$M_{\boldsymbol{\alpha},\boldsymbol{\beta}}(T) \ll_{\boldsymbol{\beta}} T(\log T)^{\beta_1^2+\cdots+\beta_m^2} \prod_{1\le j<k\le m} |\zeta(1+i(\alpha_j-\alpha_k)+1/\log T)|^{2\beta_j\beta_k}.$$

Remark. Throughout this chapter we will assume T is sufficiently large in terms of $\boldsymbol{\beta}$.

Our bound is the same order of magnitude predicted by the famous recipe of Conrey, Farmer, Keating, Rubinstein, and Snaith [26]. We obtain lower bounds of the same order in the following chapter, so the bound is sharp. At heart, Theorem 5.0.1 is a statement about how $\zeta(\frac{1}{2}+it)$ and $\zeta(\frac{1}{2}+i(t+\alpha))$ are correlated for $t \in [T,2T]$ and $|\alpha| \le T/2$. More precisely, it predicts that $\zeta(\frac{1}{2}+it)$ and $\zeta(\frac{1}{2}+i(t+\alpha))$ are perfectly correlated on average when $|\alpha| \le 1/\log T$, and decorrelate like $|\zeta(1+i\alpha)|$ for $|\alpha| > 1/\log T$. When $\alpha \le 1$, the Laurent expansion for zeta shows that we obtain the same correlations predicted from random matrix theory. For larger α, the correlation is of order 1 on average, which can be seen by calculating the moments of zeta to the right of the 1-line. There are, however, long range correlations coming from the primes; more precisely, from the extreme values of zeta on the one line. This is not so surprising, for the Keating Snaith philosophy only predicts that random matrix theory is a good model for $\zeta(\frac{1}{2}+it)$ in short intervals.

The starting point of the proof is to use the method of Soundararajan [85] and Harper [50] to bound $\log|\zeta(\frac{1}{2}+it)|$ by a short Dirichlet polynomial. Instead of following the argument of Harper, however, we treat the exponential of this short Dirichlet polynomial in a manner similar to the approach taken in the work of Heap, Radziwiłł, and Soundararajan [52]. Using this method, the integrals that arise can be evaluated by simply using the mean value theorem for Dirichlet polynomials. These mean values are much easier to evaluate uniformly in the shifts α_k than the integrals of products of shifted cosines that appear when using Harper's method [50] as Ng, Shen, and Wong do in [77]. This difference allows us to obtain upper bounds for general shifts $\boldsymbol{\alpha}$ and exponents $\boldsymbol{\beta}$. The final ingredient is a more precise estimate of the following sum

$$\sum_{p\le X}\frac{\cos(\delta\log p)}{p}$$

coming from the theory of pretentious multiplicative functions, see Lemma 4.3.6. This idea appeared in the author's previous work on studying the second moment of moments of zeta in short intervals [32]. This more precise estimate is what allows us to improve the bound of Ng, Shen, and Wong [77] in the regime where $|\alpha_j-\alpha_k|$ is unbounded.

5.1 Preliminary tools and notation

We will start by using the following lemma, which is due to Soundararajan [85] and Harper [50].

Lemma 5.1.1. *Assume the Riemann hypothesis, let $t\in[T,2T]$, and $|\alpha|\le T/2$. Then for $2\le X\le T^2$*

$$\log|\zeta(\tfrac12+i(t+\alpha))|\le \mathrm{Re}\sum_{p\le X}\frac{1}{p^{1/2+1/\log X+i(t+\alpha)}}\frac{\log X/p}{\log X}$$
$$+\sum_{p\le\min(\sqrt{X},\log T)}\frac{1}{2p^{1+2i(t+\alpha)}}+\frac{\log T}{\log X}+O(1).$$

Throughout it will be useful to break the set of primes into certain intervals. Set

$$\beta_*:=\sum_{k\le m}\max(1,\beta_k).$$

We choose a sequence of parameters $T_j=T^{c_j}$, where

$$c_0=0 \text{ and } c_j=\frac{e^j}{(\log_2 T)^2}$$

for $j>0$. We will choose L to be the largest integer such that $T_L\le T^{e^{-1000\beta_*}}$. Let

$$\mathcal{P}_{1,X}(s)=\sum_{p\le T_1}\frac{1}{p^{s+1/\log X}}\frac{\log X/p}{\log X}+\sum_{p\le\log T}\frac{1}{2p^{2s}},$$

and given any $2\le j\le L$ define

$$\mathcal{P}_{j,X}(s)=\sum_{p\in(T_{j-1},T_j]}\frac{1}{p^{s+1/\log X}}\frac{\log X/p}{\log X}.$$

If $\mathcal{P}_{j,X}(s)$ is not too large, then we will be able to efficiently approximate $\exp(\beta\mathcal{P}_{j,X}(s))$ with its Taylor series. Indeed, if we choose cutoff parameters $K_j = c_j^{-3/4}$ for $j \geq 1$ and set

$$\mathcal{N}_{j,X}(s;\beta) := \sum_{m \leq 100\beta_*^2 K_j} \frac{\beta^m \mathcal{P}_{j,X}(s)^m}{m!}$$

then we have the following analog of lemma 1 of [52]:

Lemma 5.1.2. *If* $\beta \leq \beta_*$ *and* $|\mathcal{P}_{j,X}(s)| \leq K_j$ *for some* $1 \leq j \leq L$*, then*

$$\exp(\beta\mathcal{P}_{j,X}(s)) = (1 + O(e^{-50\beta_*^2 K_j}))^{-1}\mathcal{N}_{j,X}(s;\beta).$$

Proof. Since $|\mathcal{P}_{j,X}(s)| \leq 2K_j$, Taylor expansion gives

$$\mathcal{N}_{j,X}(s;\beta) = \exp(\beta\mathcal{P}_{j,X}(s)) + O(e^{-100\beta_*^2 K_j}).$$

By assumption $\exp(-2K_j\beta_*) \leq |\exp(\beta\mathcal{P}_{j,X}(s))| \leq \exp(2K_j\beta_*)$, so the claim follows. □

We will first bound the shifted moment of zeta when all of the shifts $t + \alpha_k$ lie in the "good" set

$$\mathcal{G} := \left\{t \in [T/2, 5T/2] : |P_{j,T_L}(\tfrac{1}{2} + it)| \leq K_j \text{ for all } 1 \leq j \leq L\right\}. \tag{5.2}$$

In this case we may use Lemma 5.1.1 with $X = T_L$ in tandem with Lemma 5.1.2 to reduce the problem to computing the mean value of certain Dirichlet polynomial. We will accomplish this with the following mean value theorem of Montgomery and Vaughan (see for example theorem 9.1 of [59]).

Lemma 5.1.3. *Given any complex numbers* a_n

$$\int_T^{2T} \left|\sum_{n \leq N} \frac{a_n}{n^{it}}\right|^2 dt = (T + O(N)) \sum_{n \leq N} |a_n|^2.$$

The following variant due to Soundararajan [85, lemma 3] will also be useful for handling moments of Dirichlet polynomials supported on primes.

Lemma 5.1.4. *Let* r *be a natural number and suppose* $N^r \leq T/\log T$*. Then given any complex numbers* a_p

$$\int_T^{2T} \left|\sum_{p \leq N} \frac{a_p}{p^{it}}\right|^{2r} dt \ll Tr! \left(\sum_{p \leq N} |a_p|^2\right)^r.$$

During the main mean value calculation, we will need to bound a certain product over primes. This product will using Lemma 4.3.6.

To handle the shifted moment of zeta when some of the shifts $t + \alpha_k$ lie in the "bad" set $[T/2, 5T/2] \setminus \mathcal{G}$, we take advantage of the incremental structure present. For each $1 \leq j \leq L$, define

$$\mathcal{B}_j := \{t \in [T/2, 5T/2] : |\mathcal{P}_{r,T_s}(\tfrac{1}{2} + it)| \leq K_j \text{ for all } 1 \leq r < j \text{ and } r \leq s \leq L$$
$$\text{but } |\mathcal{P}_{j,T_s}(\tfrac{1}{2} + it)| > K_j \text{ for some } j \leq s \leq L\}.$$

Notice that

$$[T/2, 5T/2] \setminus \mathcal{G} = \bigsqcup_{j \leq L} \mathcal{B}_j.$$

On the bad sets $\mathcal{B}_j$ the series expansion $\mathcal{N}_{j,T_s}$ of $\exp(P_{j,T_s})$ is a poor approximation, so we are forced to estimate $\log \zeta$ using only the primes up to T_{j-1}. While the resulting Dirichlet polynomial is too short to obtain sharp bounds, we can overcome this loss by multiplying by a suitably large even power of $|P_{j,T_s}|/K_j$, which is larger than 1 on $\mathcal{B}_j$. If we then extend the range of integration to all of $[T, 2T]$, we can still win as the event $|\mathcal{P}_{j,T_s}(\frac{1}{2} + it)| > K_j$ is quite rare. For example, we will make use of the following bound.

Lemma 5.1.5. *If $T_1^r \leq T/\log T$ then*

$$\int_{T/2}^{5T/2} |\mathcal{P}_{1,X}(\tfrac{1}{2} + it)|^{2r} dt \ll 2^{2r} r! T (\log_2 T)^r.$$

Therefore

$$\textit{meas}(\mathcal{B}_1) \ll T \log_2 T e^{-K_1^2/4\log_2 T} \ll_A T(\log T)^{-A}.$$

Proof. Write $\mathcal{P}_{1,X} = \mathcal{P}_{1,X}^{(1)} + \mathcal{P}_{1,X}^{(2)}$, where $\mathcal{P}_{1,X}^{(1)}$ is the sum of primes up to T_1 and $\mathcal{P}_{1,X}^{(2)}$ is the sum of squares of primes up to $\log T$. Then

$$\int_{T/2}^{5T/2} |\mathcal{P}_{1,X}(\tfrac{1}{2} + it)|^{2r} dt \leq 2^{2r} \int_{T/2}^{5T/2} |\mathcal{P}_{1,X}^{(1)}(\tfrac{1}{2} + it)|^{2r} dt + 2^{2r} \int_{T/2}^{5T/2} |\mathcal{P}_{1,X}^{(2)}(\tfrac{1}{2} + it)|^{2r} dt.$$

By Lemma 5.1.4, this is at most

$$\ll 2^{2r} r! T (\log_2 T_1 + O(1))^r + 2^{2r} r! T (\zeta(2)/4)^r \ll 2^{2r} r! T (\log_2 T)^r.$$

To deduce the second bound we note that

$$\text{meas}(\mathcal{B}_1) \ll \max_{s \leq L} \frac{1}{K_1^{2r}} \int_{T/2}^{5T/2} |\mathcal{P}_{1,T_s}(\tfrac{1}{2} + it)|^{2r} dt.$$

We may now conclude by taking $r = \lceil K_1^2/4\log_2 T \rceil$ and using Stirling's approximation. □

The proof of Theorem 5.0.1 is based on the following partition of $[T, 2T]$: Given a subset A of $[m] := \{1, \dots, m\}$ define

$$\mathcal{G}_A := \{t \in [T, 2T] : t + \alpha_k \in \mathcal{G} \text{ if and only if } k \in A\}.$$

Then we can decompose $[T, 2T]$ into the disjoint union

$$[T, 2T] = \bigsqcup_{A \subseteq [m]} \mathcal{G}_A. \tag{5.3}$$

In section 5.2, we will handle the integral over the set $\mathcal{G}_{[m]}$ where all of the shifts $t+\alpha_k$ are good. In section 5.3, we will handle the cases where some of the shifts $t+\alpha_k$ are bad. We will have to further partition the sets $\mathcal{G}_A$ with $A \subsetneq [m]$ according to which of the sets $\mathcal{B}_j$ the bad shifts $t+\alpha_k$ lie in.

5.2 Moments over good shifts

By Lemma 5.1.1 with $X = T_L$ we find

$$\int_{\mathcal{G}_{[m]}} \prod_{k=1}^{m} |\zeta(\tfrac{1}{2} + i(t+\alpha_k))|^{2\beta_k} dt \ll_{\boldsymbol{\beta}} \int_{\mathcal{G}_{[m]}} \prod_{k=1}^{m} \exp\left(2\beta_k \mathrm{Re} \sum_{j=1}^{L} \mathcal{P}_{j,T_L}(\tfrac{1}{2} + i(t+\alpha_k))\right) dt$$

By definition of $\mathcal{G}_{[m]}$ the hypotheses of Lemma 5.1.2 are satisfied for all $j \le L$, so the integral over $\mathcal{G}_{[m]}$ can be bounded by

$$\begin{aligned} &\ll_{\boldsymbol{\beta}} \int_{\mathcal{G}_{[m]}} \prod_{k=1}^{m} \prod_{j=1}^{L} (1 + e^{-50\beta_*^2 K_j})^{-1} |\mathcal{N}_{j,T_L}(\tfrac{1}{2} + i(t+\alpha_k); \beta_k)|^2 dt \\ &\ll_{\boldsymbol{\beta}} \int_{T}^{2T} \prod_{k=1}^{m} \prod_{j=1}^{L} |\mathcal{N}_{j,T_L}(\tfrac{1}{2} + i(t+\alpha_k); \beta_k)|^2 dt. \end{aligned} \tag{5.4}$$

We are now in a setting where me may use the mean value theorem for Dirichlet polynomials. First note that $\prod_{k=1}^{m} \mathcal{N}_{j,T_L}(s + i\alpha_k; \beta_k)$ has length at most $T_j^{200m\beta_*^2 K_j}$ for $j = 1$ and $T_j^{100m\beta_*^2 K_j}$ for $2 \le j \le L$. Therefore the integrand $\prod_{j \le L} \prod_{k=1}^{m} \mathcal{N}_{j,T_L}(s + i\alpha_k; \beta_k)$ has length at most $T_1^{200m\beta_*^2 K_1} T_2^{100m\beta_*^2 K_2} \cdots T_L^{100m\beta_*^2 K_L} \le T^{1/2}$ as $T_L \le T^{e^{-1000\beta_*}}$, so it is a short Dirichlet polynomial. By Lemma 4.4.6, we are left with the task of computing

$$\int_{T}^{2T} \prod_{k=1}^{m} |\mathcal{N}_{j,T_L}(\tfrac{1}{2} + i(t+\alpha_k); \beta_k)|^2 dt$$

for each $1 \leq j \leq L$. To do this, we must analyze the coefficients of the $\mathcal{N}_{j,X}$. Denote $a_X(p) := \log(X/p)p^{-1/\log X}/\log X$ and define multiplicative functions g_X and h_X satisfying

$$g_X(p^r;\beta) := \frac{\beta^r a_X(p)^r}{r!},$$

and

$$h_X(p^r;\beta) := g_X(p^r;\beta) + 1_{p\leq \log T}\sum_{t=1}^{r/2}\frac{\beta^{r-t}a_X(p)^{r-t}}{2^t t!(r-2t)!}.$$

Next define $c_1(n)$ to be 1 if n can be written as $n = n_1 \cdots n_r$ where $r \leq 100\beta_*^2 K_1$ and each n_i is either a prime $\leq T_1$ or a prime square $\leq \log T$. Finally for $2 \leq j \leq L$ set $c_j(n)$ to be 1 if n is the product of at most $100\beta_*^2 K_j$ not necessarily distinct primes in $(T_{j-1}, T_j]$.

Proposition 5.2.1. *For $2 \leq j \leq L$*

$$\mathcal{N}_{j,X}(s;\beta) = \sum_{p|n \Rightarrow p\in(T_{j-1},T_j]} \frac{g_X(n;\beta)c_j(n)}{n^s}.$$

If

$$\mathcal{N}_{1,X}(s;\beta) = \sum_{p|n \Rightarrow p\in(T_{j-1},T_j]} \frac{f_X(n;\beta)}{n^s}$$

then $f_X(n;\beta) \leq h_X(n;\beta)c_1(n)$ and $f_X(p;\beta) = g_X(p;\beta)$.

Proof. When $j \geq 1$ write $p_1, \ldots, p_a$ for the primes in $(T_{j-1}, T_j]$. First assume $j \geq 2$. By applying the multinomial theorem to the definition of $\mathcal{N}_{j,X}(s;\beta)$ we find it equals

$$\sum_{m\leq 100\beta_*^2 K_j} \frac{\beta^m}{m!}\left(\sum_{p\in(T_{j-1},T_j]}\frac{a_X(p)}{p^s}\right)^m = \sum_{m\leq 100\beta_*^2 K_j}\frac{\beta^m}{m!}\sum_{\substack{u_1+\ldots+u_a=m\\ u_r\geq 0}}\binom{m}{u_1,\ldots,u_a}\prod_{r=1}^{a}\frac{a_X(p)^{u_r}}{p_r^{u_r s}}.$$

Therefore if $n = p_1^{u_1}\cdots p_a^{u_a}$ with $u_1 + \cdots + u_a = m$, the coefficient of n^{-s} in $\mathcal{N}_{j,X}(s)$ is

$$c_j(n)\frac{\beta^m}{m!}\binom{m}{u_1,\ldots,u_a}\prod_{r=1}^{a} a_X(p)^{u_r} = g_X(n;\beta)c_j(n).$$

Next we will handle the case of $j = 1$. Now we will also denote the primes up to $\log T$ by $p_1, \ldots p_b$ with $b < a$. The multinomial theorem tells us that $\mathcal{N}_{1,X}$ equals

$$\sum_{m\leq 100\beta_*^2 K_2}\frac{\beta^m}{m!}\left(\sum_{p\leq T_1}\frac{a_X(p)}{p^s} + \sum_{p\leq \log T}\frac{1}{2p^{2s}}\right)^m$$

$$= \sum_{m\leq 100\beta_*^2 K_1}\frac{\beta^m}{m!}\sum_{\substack{u_1+\ldots+u_a+v_1+\ldots+v_b=m\\ u_r,v_r\geq 0}}\binom{m}{u_1,\ldots,u_a,v_1,\ldots,v_b}\prod_{r=1}^{a}\frac{a_X(p)^{u_r}}{p_r^{u_r s}}\prod_{r=1}^{b}\frac{1}{2^{v_r}p_r^{2v_r s}}.$$

The claim now follows by considering the possible ways to write $n = p_1^{u_1} \cdots p_a^{u_a}$ as a product of the p_r with $r \le a$ or of p_r^2 with $r \le b$.

□

We may write

$$\prod_{k=1}^{m} \mathcal{N}_{j,X}(s + i\alpha_k; \beta_k) = \sum_{n\ge 1} \frac{b_{j,X,\boldsymbol{\alpha},\boldsymbol{\beta}}(n)}{n^s},$$

where $b_{1,X,\boldsymbol{\alpha},\boldsymbol{\beta}}(n)$ is the m-fold Dirichlet convolution of $f_X(n;\beta_k)n^{-i\alpha_k}$ and $b_{j,X,\boldsymbol{\alpha},\boldsymbol{\beta}}(n)$ is the m-fold convolution of $g_X(n;\beta_k)c_j(n)n^{-i\alpha_k}$ for $2 \le j \le L$. For technical reasons, we will need to use two other sets of coefficients. First define $b'_{j,X,\boldsymbol{\alpha},\boldsymbol{\beta}}(n)$ to be the m-fold convolution of $h_X(n;\beta_k)n^{-i\alpha_k}1_{p|n \Rightarrow p\in(T_0,T_1]}$ when $j = 1$ and the m-fold convolution of $g_X(n;\beta_k)n^{-i\alpha_k}1_{p|n\Rightarrow p\in(T_{j-1},T_j]}$ when $2 \le j \le L$. Finally, let $b''_{1,X,\boldsymbol{\alpha},\boldsymbol{\beta}}(n)$ be the m-fold convolution of $h_X(n;\beta_k)1_{p|n\Rightarrow p\in(T_0,T_1]}$ when $j = 1$ and the m-fold convolution of $g_X(n;\beta_k)1_{p|n\Rightarrow p\in(T_{j-1},T_j]}$ when $2 \le j \le L$. Unlike $b_{j,X,\boldsymbol{\alpha},\boldsymbol{\beta}}$, the coefficients $b'_{j,X,\boldsymbol{\alpha},\boldsymbol{\beta}}$ and $b''_{j,X,\boldsymbol{\alpha},\boldsymbol{\beta}}$ are multiplicative, and they satisfy the bound $|b_{j,X,\boldsymbol{\alpha},\boldsymbol{\beta}}(n)|, |b'_{j,X,\boldsymbol{\alpha},\boldsymbol{\beta}}(n)| \le b''_{j,X,\boldsymbol{\alpha},\boldsymbol{\beta}}(n)$. We will require the following information about these coefficients.

Lemma 5.2.2. *For $1 \le j \le L$ and $p \in (T_{j-1}, T_j]$*

$$b_{j,X,\boldsymbol{\alpha},\boldsymbol{\beta}}(p) = a_X(p)\sum_{k=1}^{m} \beta_k p^{-i\alpha_k},$$

and $b''_{j,X,\boldsymbol{\alpha},\boldsymbol{\beta}}(p) \le \beta_*$. If $r \ge 2$

$$b''_{j,X,\boldsymbol{\alpha},\boldsymbol{\beta}}(p^r) \le \frac{\beta_*^r m^r}{r!}$$

holds whenever $2 \le j \le L$ or $p > \log T$, and otherwise

$$b''_{1,X,\boldsymbol{\alpha},\boldsymbol{\beta}}(p^r) \le m\beta_*^r r^{2m} e^{-r\log(r/m)/2m+2r}.$$

Proof. The first two assertions are immediate from the definition of the Dirichlet convolution. To prove the upper bound when $r \ge 2$ and $j \ne 1$, first note that

$$b_{j,X,\boldsymbol{\alpha},\boldsymbol{\beta}}(p^r) \le \sum_{r_1+\cdots+r_m=r} \prod_{l=1}^{m} \frac{\beta_l^{r_l}}{r_l!} \le \frac{\beta_*^r}{r!} \sum_{r_1+\cdots+r_m=r} \binom{r}{r_1,\ldots,r_m} = \frac{\beta_*^r m^r}{r!}.$$

To handle the $j = 1$ case, we can bound $h_X(p^r;\beta)$ by

$$\sum_{t=0}^{r/2} \frac{\beta^{r-t}}{2^t t!(r-2t)!} \le \beta_*^r \sum_{t=0}^{r/2} \frac{1}{2^t t!(r-2t)!}.$$

In fact when $p > \log T$ we have the stronger bound $\beta_r/r!$. To bound the sum on the right hand side, note by Stirling's formula the maximum summand occurs near the solution to $(r-2t)^2 = 2t$. One more application of Stirling's formula shows that the maximum is $\le e^{-r\log r/2+2r}$, so this sum is bounded by $re^{-r\log r/2+2r}$. It now follows that

$$|b_{1,X,\boldsymbol{\alpha},\boldsymbol{\beta}}(p^r)| \le \beta_*^r r^m \sum_{r_1+\cdots+r_m=r} \prod_{l=1}^m e^{-r_l\log r_l/2+2r_l} \le \beta_*^r r^m e^{-r\log(r/m)/2m+2r}\binom{m+r-1}{r},$$

where we have used the fact that at least one r_l must exceed r/m. To conclude, notice that $\binom{m+r-1}{r}$ is a polynomial of degree $m-1$ in r with coefficients all bounded by 1, so it is at most mr^{m-1}.

□

We can now compute

Proposition 5.2.3. *For $1 \le j \le L$*

$$\int_T^{2T} \prod_{k=1}^m |\mathcal{N}_{j,X}(\tfrac{1}{2}+i(t+\alpha_k);\beta_k)|^2 dt$$
$$\le (T+O(T^{1/2})) \prod_{p\in(T_{j-1},T_j]} \left(1+\frac{|b_{j,X,\boldsymbol{\alpha},\boldsymbol{\beta}}(p)|^2}{p}+O_{\boldsymbol{\beta}}\left(\frac{1}{p^2}\right)\right)+O_{\boldsymbol{\beta}}(e^{-50\beta_*^2K_j}).$$

Proof. By Lemma 5.1.3 the mean value of interest equals

$$(T+O(T^{1/2}))\sum_n \frac{|b_{j,X,\boldsymbol{\alpha},\boldsymbol{\beta}}(n)|^2}{n}.$$

We will now show that we may replace b with b' at a negligible cost. If $b_{j,X,\boldsymbol{\alpha},\boldsymbol{\beta}}(n) \ne b'_{j,X,\boldsymbol{\alpha},\boldsymbol{\beta}}(n)$ then it follows that $\Omega(n) \ge 100\beta_*^2K_j$. Therefore when we replace b with b', we incur an error of at most

$$e^{-100\beta_*^2K_j} \sum_{p|n\Rightarrow p\in(T_{j-1},T_j]} \frac{b''_{j,X,\boldsymbol{\alpha},\boldsymbol{\beta}}(n)^2 e^{\Omega(n)}}{n}.$$

Since the coefficients b'' are multiplicative, this is

$$\ll e^{-100\beta_*^2K_j} \prod_{p\in(T_{j-1},T_j]} \left(1+\frac{\beta_*^2 e}{p}+O\left(\frac{\beta_*^4 m^4}{p^2}\right)\right) \ll_{\boldsymbol{\beta}} e^{-50\beta_*^2K_j},$$

where we have used Lemma 5.2.2 to bound the sum over prime powers. Therefore the mean value of interest is

$$\le (T+O(T^{1/2})) \sum_{p|n\Rightarrow p\in(T_{j-1},T_j]} \sum_{r\ge0} \frac{|b'_{j,X,\boldsymbol{\alpha},\boldsymbol{\beta}}(p^r)|^2}{p^r}+O_{\boldsymbol{\beta}}(e^{-50\beta_*^2K_j}).$$

The claim now follows by Lemma 5.2.2 and multiplicativity.

□

We may finally deduce

Proposition 5.2.4. *Assuming the Riemann hypothesis,*

$$\int_{\mathcal{G}_{[m]}} \prod_{k\le m} |\zeta(\tfrac{1}{2}+i(t+\alpha_k))|^{2\beta_k} dt \\ \ll_{\boldsymbol{\beta}} T(\log T)^{\beta_1^2+\cdots+\beta_m^2} \prod_{1\le j<k\le m} |\zeta(1+i(\alpha_j-\alpha_k)+1/\log T)|^{2\beta_j\beta_k}.$$

Proof. We have shown that the shifted moment on the good set is bounded by

$$\ll T\prod_{j\le L}\left(\prod_{p\in(T_{j-1},T_j]}\left(1+\frac{|b_{j,X,\boldsymbol{\alpha},\boldsymbol{\beta}}(p)|^2}{p}+O_{\boldsymbol{\beta}}\left(\frac{1}{p^2}\right)\right)+O_{\boldsymbol{\beta}}(e^{-50\beta_*^2K_j})\right).$$

To conclude, we first note that

$$|b_{j,X,\boldsymbol{\alpha},\boldsymbol{\beta}}(p)|^2 \le \sum_{j,k\le m}\frac{\beta_j\beta_k}{p^{i(\alpha_j-\alpha_k)}} = \sum_{j\le m}\beta_j^2 + 2\sum_{1\le j<k\le m}\beta_j\beta_k\cos((\alpha_j-\alpha_k)\log p)$$

and then use 4.3.6 to bound the resulting products over primes. □

5.3 Moments over bad shifts

We now consider the integral over $\mathcal{G}_A$ where A is a proper subset of $[m]$. Without loss of generality we will write $A=[m]\setminus[a]$. For each $t\in\mathcal{G}_A$, there is a function $F_t:[a]\to[L]$ such that $t+\alpha_j\in\mathcal{B}_{F_t(j)}$. We will further partition $\mathcal{G}_A$ into the sets

$$\mathcal{B}_{A,n}=\{t\in\mathcal{G}_A:\min_{j\in[a]}F_t(j)=n\}.$$

First we handle the case of $n=1$.

Proposition 5.3.1. *Assuming the Riemann hypothesis*

$$\int_{\mathcal{B}_{A,1}}\prod_{k=1}^m|\zeta(\tfrac{1}{2}+i(t+\alpha_k))|^{2\beta_k}dt \ll_{A,\boldsymbol{\beta}} T(\log T)^{-A}.$$

Proof. Because $\mathcal{B}_{A,1}$ is contained in the union of the $a\le m$ translates $\mathcal{B}_1-\alpha_j$ for $j\le a$, by the Cauchy-Schwarz inequality

$$\int_{\mathcal{B}_{A,1}}\prod_{k=1}^m|\zeta(\tfrac{1}{2}+i(t+\alpha_k))|^{2\beta_k}dt \le (m\cdot\text{meas }\mathcal{B}_1)^{1/2}\left(\int_T^{2T}\prod_{k=1}^m|\zeta(\tfrac{1}{2}+i(t+\alpha_k))|^{4\beta_k}dt\right)^{1/2}.$$

By Lemma 5.1.5 the first factor is $\ll_A T(\log T)^{-A}$. To bound the second factor, Hölder's inequality implies that

$$\int_T^{2T} \prod_{k=1}^m |\zeta(\tfrac{1}{2}+i(t+\alpha_k))|^{4\beta_k} dt \le \prod_{k=1}^m \left(\int_{T/2}^{5T/2} |\zeta(\tfrac{1}{2}+it)|^{4m\beta_k} dt \right)^{1/m}.$$

The right hand side is $\ll_{\boldsymbol{\beta}} T(\log T)^{O(m\beta_*^2)}$, which follows from the following result of Harper [50]:

$$\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2\beta} dt \ll_\beta T(\log T)^{\beta^2}.$$

□

Now for fixed $n > 1$ we may use Lemma 5.1.1 with $X = T_{n-1}$ to find

$$\begin{aligned}
&\int_{\mathcal{B}_{A,n}} \prod_{k=1}^m |\zeta(\tfrac{1}{2}+i(t+\alpha_k))|^{2\beta_k} dt \\
&\quad \ll \int_{\mathcal{B}_{A,n}} \prod_{k=1}^m \exp\left(2\beta_k \mathrm{Re}\left(\sum_{j<n} \mathcal{P}_{j,T_{n-1}}(\tfrac{1}{2}+i(t+\alpha_k)) + 2\beta_k/c_{n-1} \right) dt \right. \\
&\quad \ll e^{2\beta_*/c_{n-1}} \int_{\mathcal{B}_{A,n}} \prod_{k=1}^m \prod_{j<n} \exp\left(2\beta_k \mathrm{Re}\ \mathcal{P}_{j,T_{n-1}}(\tfrac{1}{2}+i(t+\alpha_k))\right) dt \\
&\quad \ll e^{2\beta_*/c_{n-1}} \max_{\substack{\ell\in[a] \\ s\in[L]}} \int_T^{2T} \prod_{k=1}^m \prod_{j<n} |\mathcal{N}_{j,T_{n-1}}(\tfrac{1}{2}+i(t+\alpha_k);\beta_k)|^2 \\
&\qquad\qquad \times |\mathcal{P}_{n,T_s}(\tfrac{1}{2}+i(t+\alpha_\ell))/K_n|^{2\lceil 1/10c_n \rceil} dt
\end{aligned}$$

Unlike the previous section, we have now also used the definition of the bad set $\mathcal{B}_{A,n}$. By Lemma 4.4.6, all that remains is to control the moments of $\mathcal{P}_{n,T_s}$ on the half line.

Proposition 5.3.2. *Uniformly for $\ell \in [m]$ and $s \in [L]$*

$$\int_{T/2}^{5T/2} |\mathcal{P}_{n,T_s}(\tfrac{1}{2}+i(t+\alpha_\ell))/K_n|^{2\lceil 1/10c_n \rceil} dt \ll Te^{-\log(1/c_n)/20c_n}.$$

Proof. Trivially bounding $p^{-i\alpha_\ell}$ and $a_{X_s}(p)$ by 1, Lemma 5.1.4 gives a bound of

$$TK_n^{-2r} r! \left(\sum_{p\in(T_{n-1},T_n]} \frac{1}{p} \right)^r$$

where $r = \lceil 1/10c_n \rceil$. The sum in parentheses is asymptotic to $\log(c_n/c_{n-1}) = e$, so is at most $2e$ for large T, say. The conclusion follows by recalling $K_n = c_n^{-3/4}$ and applying Stirling's approximation. □

We now have all the necessary tools to bound the shifted moment (5.1) over the bad sets.

Proposition 5.3.3. *Assuming the Riemann hypothesis*

$$\int_{[T,2T]\setminus\mathcal{G}_{[m]}}\prod_{k=1}^{m}|\zeta(\tfrac{1}{2}+i(t+\alpha_k))|^{2\beta_k}dt$$
$$\ll_{A,\boldsymbol{\beta}} T(\log T)^{\beta_1^2+\cdots+\beta_m^2}\prod_{1\le j<k\le m}|\zeta(1+i(\alpha_j-\alpha_k)+1/\log T)|^{2\beta_j\beta_k}.$$

Proof. Applying Lemma 4.4.6 along with Propositions 5.2.3, 5.3.1, and 5.3.2, we may bound the relevant integral by

$$\ll_{A,\boldsymbol{\beta}} T\sum_{2\le n\le L}\exp\left(\frac{2\beta_*}{c_{n-1}}-\frac{\log(1/c_n)}{20c_n}\right)\prod_{p\le T_{n-1}}\left(1+\frac{1}{p}\sum_{1\le j,k\le m}\frac{\beta_j\beta_k}{p^{i(\alpha_j-\alpha_k)}}+O_{\boldsymbol{\beta}}\left(\frac{1}{p^2}\right)\right)$$
$$+T(\log T)^{-A}$$
$$\ll_{A,\boldsymbol{\beta}} T\sum_{n\le L}\exp\left(e^{-n}(\log_2 T)^2(2\beta_*e+\tfrac{1}{20}n-\tfrac{1}{10}\log_3 T)\right)$$
$$\times\prod_{p\le T_{n-1}}\left(1+\frac{1}{p}\sum_{1\le j,k\le m}\frac{\beta_j\beta_k}{p^{i(\alpha_j-\alpha_k)}}\right)+T(\log T)^{-A},$$

where we have applied a union bound over all bad subsets A of $[m]$. Because the shifts satisfy $|\alpha_j-\alpha_k|\le T$, the $T(\log T)^{-A}$ term is negligible by the estimate

$$|\zeta(1+1/\log T+it)|\gg\zeta(2+2/\log T)/\zeta(1+1/\log T)\gg 1/\log T$$

for $|t|\le 2T$. To simplify remaining term, note that because $T_L\le T^{e^{-1000\beta_*}}$ it follows that $L\le 2\log_3 T-1000\beta_*$. Therefore the latter term is

$$\ll_{A,\boldsymbol{\beta}} T\sum_{n\le L}\exp\left(-4\beta_*e^{-n}(\log_2 T)^2\right)\prod_{p\le T_{n-1}}\left(1+\frac{1}{p}\sum_{1\le j,k\le m}\frac{\beta_j\beta_k}{p^{i(\alpha_j-\alpha_k)}}\right)$$
$$\ll_{A,\boldsymbol{\beta}} T(\log T)^{\beta_1^2+\cdots+\beta_m^2}\prod_{1\le j<k\le m}|\zeta(1+i(\alpha_j-\alpha_k)+1/\log T)|^{2\beta_j\beta_k}$$
$$\times\sum_{n\le L}\exp\left(-4\beta_*e^{-n}(\log_2 T)^2\right)\prod_{p\in(T_{n-1},T_L]}\left(1+\frac{1}{p}\sum_{1\le j,k\le m}\frac{\beta_j\beta_k}{p^{i(\alpha_j-\alpha_k)}}\right)^{-1}$$
$$\ll_{A,\boldsymbol{\beta}} T(\log T)^{\beta_1^2+\cdots+\beta_m^2}\prod_{1\le j<k\le m}|\zeta(1+i(\alpha_j-\alpha_k)+1/\log T)|^{2\beta_j\beta_k}$$
$$\times\sum_{n\le L}\exp\left(-4\beta_*e^{-n}(\log_2 T)^2+\beta_*^2(L-n)\right).$$

Note we used Mertens' estimate when passing to final line. By summing in reverse, one readily sees the sum over n is convergent, and the claim now follows.

$\square$

In view of (5.3), this completes the proof of Theorem 5.0.1.

Chapter 6

Conditional lower bounds for shifted moments of zeta

6.1 Introduction

This chapter is concerned with obtaining the corresponing lower bounds the shifted moments

$$M_{\boldsymbol{\alpha},\boldsymbol{\beta}}(T) = \int_T^{2T} \prod_{k=1}^{m} |\zeta(\tfrac{1}{2} + i(t+\alpha_k))|^{2\beta_k} dt,$$

considered in the previous chapter. Our goal will be the following result, which shows that the upper bounds obtained in Chapter 5 are of the correct order or magnitude.

Theorem 6.1.1. *Assume the Riemann hypothesis. If $\beta_k \geq 0$ and $|\alpha_k| \leq T/2$ for $k = 1, \dots, m$, then*

$$M_{\boldsymbol{\alpha},\boldsymbol{\beta}}(T) \gg_{\boldsymbol{\beta}} T(\log T)^{\beta_1^2+\cdots+\beta_m^2} \prod_{1\leq j<k\leq m} |\zeta(1 + i(\alpha_j - \alpha_k) + 1/\log T)|^{2\beta_j\beta_k}$$

for T sufficiently large in terms of $\boldsymbol{\beta}$.

The method of proof is similar to a principle pioneered in the works of Heap and Soundararajan [53] and Radziwiłł and Soundararajan [82]. These works demonstrate that if one can asymptotically evaluate the twisted $2k^{\text{th}}$ moment of a family of L-functions for some integer k, then one can obtain sharp lower bounds for the $2\beta^{\text{th}}$ moments for all $\beta \geq 0$. We will need a more elaborate argument in our setting since there are more parameters for the shifted moments. To explain the method, we will first introduce some notation. Set

$$\beta_* := \sum_{k\leq m} \max(1, \beta_k).$$

We will choose a sequence of parameters $T_j = T^{c_j}$, where

$$c_0 = 0 \text{ and } c_j = \frac{e^j}{(\log_2 T)^2}$$

for $j > 0$. Let L be the largest integer such that $T_L \le T^\delta$ where $0 < \delta < e^{-1000\beta_*}$ is some small constant depending on $\boldsymbol{\beta}$ to be chosen later. For any X let

$$\mathcal{P}_{1,X}(s) = \sum_{p \le T_1} \frac{1}{p^{s+1/\log X}} \frac{\log X/p}{\log X} + \sum_{p \le \log T} \frac{1}{2p^{2s}},$$

and given any $2 \le j \le L$ define

$$\mathcal{P}_{j,X}(s) = \sum_{p \in (T_{j-1}, T_j]} \frac{1}{p^{s+1/\log X}} \frac{\log X/p}{\log X}.$$

The only difference between this notation and that used in the previous chapter is that L will smaller since the value of δ is more important in this chapter.

Remark. In our proof, we could have also used the simpler Dirichlet polynomial

$$\sum_{p \in (T_{j-1}, T_j]} \frac{1}{p^s}$$

in place of $\mathcal{P}_{j,X}(s)$ for all $1 \le j \le L$. However the $\mathcal{P}_{j,X}$ will appear when using the Riemann hypothesis, and using the $\mathcal{P}_{j,X}$ from the onset will reduce the total number of mean value calculations.

If $\mathcal{P}_{j,X}(s)$ is not too large, then (see lemma 5.1.2) we will be able to efficiently approximate $\exp(\beta \mathcal{P}_{j,X}(s))$ with the following Dirichlet polynomial

$$\mathcal{N}_{j,X}(s;\beta) := \sum_{m \le 100\beta_*^2 K_j} \frac{\beta^m \mathcal{P}_{j,X}(s)^m}{m!},$$

where $K_j = c_j^{-3/4}$ for $j \ge 1$. The set

$$\mathcal{G} := \left\{ t \in [T/2, 5T/2] : |P_{j,T_L}(\tfrac{1}{2} + it)| \le K_j \text{ for all } 1 \le j \le L \right\}.$$

consists of the $t \in [T/2, 5T/2]$ for which $\mathcal{N}_{j,T_L}(s;\beta)$ is a good approximation to $\exp(\beta \mathcal{P}_{j,T_L}(s))$. We will be computing moments over the set

$$\mathcal{G}_m := \{ t \in [T, 2T] : t + \alpha_k \in \mathcal{G} \text{ for all } 1 \le k \le m \}.$$

Fix a smooth function w such that $1_{[5/4,7/4]}(t) \le w(t) \le 1_{[1,2]}$, and suppose for a moment that we could compute the integral

$$\mathcal{I}_0 = \int_{\mathcal{G}_m} \prod_{k\le m} \zeta(\tfrac{1}{2} + i(t+\alpha_k))$$
$$\times \prod_{j\le L} \exp\big((\beta_k - 1)\mathcal{P}_{j,T_L}(\tfrac{1}{2} + i(t+\alpha_k)) + \beta_k \mathcal{P}_{j,T_L}(\tfrac{1}{2} - i(t+\alpha_k))\big)\, w(t/T)\ dt.$$

Then Hölder's inequality implies

$$|\mathcal{I}_0| \le M_{\boldsymbol{\alpha},\boldsymbol{\beta}}(T)^{1/p} \times |\mathcal{J}|^{1/q} \times \prod_{k\le m} |\mathcal{I}_k|^{1/r_k},$$

where

$$\frac{1}{p} = \frac{1}{4\beta_*}, \quad \frac{1}{r_k} = \frac{1}{2m} - \frac{\beta_k}{4m\beta_*}, \quad \frac{1}{q} = 1 - \frac{1}{p} - \sum_{k\le m} \frac{1}{r_k}$$

are conjugate exponents,

$$\mathcal{I}_k = \int_{\mathcal{G}_m} |\zeta(\tfrac{1}{2} + i(t+\alpha_k))|^{2m} \prod_{j\le L} \exp\big(2(\beta_k - m)\mathrm{Re}\ \mathcal{P}_{j,T_L}(\tfrac{1}{2} + i(t+\alpha_k))\big)$$
$$\times \prod_{\substack{\ell\le m\\ \ell\ne k}} \prod_{j\le L} \exp\big(2\beta_\ell \mathrm{Re}\ \mathcal{P}_{j,T_L}(\tfrac{1}{2} + i(t+\alpha_\ell))\big)\, w(t/T)dt,$$

and

$$\mathcal{J} = \int_{\mathcal{G}_m} \prod_{k\le m} \prod_{j\le L} \exp\big(2\beta_k \mathrm{Re}\ \mathcal{P}_{j,T_L}(\tfrac{1}{2} + i(t+\alpha_k))\big)\, w(t/T)dt.$$

Therefore, Theorem 6.1.1 will follow from the following three propositions.

Proposition 6.1.2. *For large T*

$$\mathcal{J} \ll_{\boldsymbol{\beta}} T(\log T)^{\beta_1^2+\cdots+\beta_m^2} \prod_{1\le j<k\le m} |\zeta(1 + i(\alpha_j - \alpha_k) + 1/\log T)|^{2\beta_j\beta_k}.$$

Proposition 6.1.3. *Assuming the Riemann hypothesis, for large T*

$$|\mathcal{I}_0| \gg_{\boldsymbol{\beta}} T(\log T)^{\beta_1^2+\cdots+\beta_m^2} \prod_{1\le j<k\le m} |\zeta(1 + i(\alpha_j - \alpha_k) + 1/\log T)|^{2\beta_j\beta_k}.$$

Proposition 6.1.4. *Assuming the Riemann hypothesis, for large T and $1 \le k \le m$*

$$\mathcal{I}_k \ll_{\boldsymbol{\beta}} T(\log T)^{\beta_1^2+\cdots+\beta_m^2} \prod_{1\le j<k\le m} |\zeta(1 + i(\alpha_j - \alpha_k) + 1/\log T)|^{2\beta_j\beta_k}.$$

To prove Propositions 6.1.2 and 6.1.4, we will use the definition of $\mathcal{G}_m$ to show that

$$\mathcal{J} \ll \int_T^{2T} \prod_{k\le m} |\mathcal{N}_{T_L}(\tfrac12 + i(t+\alpha_k);\beta_k)|^2 dt$$

and

$$\mathcal{I}_k \ll \int_{\mathcal{G}_m} |\zeta(\tfrac12+i(t+\alpha_k))|^{2m} |\mathcal{N}_{T_L}(\tfrac12+i(t+\alpha_k);\beta_k-m)|^2 \prod_{\substack{\ell\le m\\ \ell\ne k}} |\mathcal{N}_{T_L}(\tfrac12+i(t+\alpha_\ell);\beta_\ell)|^2 dt,$$

where

$$\mathcal{N}_X(s;\beta) := \prod_{j\le L} \mathcal{N}_{j,X}(s;\beta).$$

Therefore $\mathcal{J}$ can be controlled by computing the mean value of a Dirichlet polynomial, and the $\mathcal{I}_k$ can be bounded by computing twisted $2m^{\text{th}}$ moments of the zeta function. Both of these quantities can be controlled as $\mathcal{N}_{T_L}(s;\beta)$ is a short Dirichlet polynomial. For the latter task, we will need to invoke the Riemann hypothesis when $m > 2$.

The proof of Proposition 6.1.3 is a bit more difficult. The first step is to show that $\mathcal{I}_0$ is approximately equal to

$$\int_{\mathcal{G}_m} \prod_{k\le m} \zeta(\tfrac12 + i(t+\alpha_k))\mathcal{N}_{T_L}(\tfrac12 + i(t+\alpha_k);\tfrac12(\beta_k-1))^2 \\ \times \overline{\mathcal{N}_{T_L}(\tfrac12 + i(t+\alpha_k);\tfrac12\beta_k)}^2 w(t/T)\, dt$$

using the definition of $\mathcal{G}_m$ and the Riemann hypothesis. To compute this quantity, we will write

$$\int_{\mathcal{G}_m} = \int_T^{2T} - \int_{[T,2T]\setminus\mathcal{G}_m}.$$

The integral over the entire interval $[T,2T]$ is a shifted pure m^{th} moment of zeta twisted by a Dirichlet polynomial, which we can calculate unconditionally because $\mathcal{N}_{T_L}(s;\beta)$ is a short Dirichlet polynomial. To bound the integral over $[T,2T]\setminus\mathcal{G}_m$, we will first decompose this set according to which subsum $\mathcal{P}_{j,X}(\frac12 + i(t+\alpha_\ell);\beta)$ is unusually large for some ℓ, and then bound the contribution of such t by

$$\frac{1}{K_j^{2r}} \int_T^{2T} \prod_{k\le m} |\zeta(\tfrac12 + i(t+\alpha_k))| \\ \times|\mathcal{N}_{T_L}(\tfrac12 + i(t+\alpha_k);\tfrac12(\beta_k-1))\mathcal{N}_{T_L}\tfrac12 + i(t+\alpha_k);\tfrac12\beta_k)|^2 \\ \times |\mathcal{P}_{j,X}(\tfrac12 + i(t+\alpha_\ell))|^{2r}\, dt$$

for some large integer r. To bound this quantity we are again forced to use the Riemann hypothesis, at least when $m > 2$. We can estimate this integral using the same method in the previous chapter.

We now have all the necessary tools to begin our proof of Theorem 6.1.1. To help simplify the notation going forward, we will omit subscripts depending on $\boldsymbol{\beta}$ from all big-O or Vinogradov asymptotic notation. The reader should keep in mind that all of the implicit constants in this notation can, and usually will, depend on $\boldsymbol{\beta}$. We will also implicitly assume that T is sufficiently large in terms of $\boldsymbol{\beta}$. We will first prove Proposition 6.1.2, followed by Proposition 6.1.3, and we will conclude the proof by establishing Proposition 6.1.4.

6.2 Proof of Proposition 6.1.2

We will begin with the evaluation of

$$\mathcal{J} = \int_{\mathcal{G}_m} \prod_{k \le m} \prod_{j \le L} \exp\left(2\beta_k \mathrm{Re}\, \mathcal{P}_{j,T_L}(\tfrac{1}{2} + i(t + \alpha_k))\right) w(t/T) dt,$$

which is the simplest computation. To compute this mean value, we apply lemma 5.1.2 and then extend the integral over the entire set $[T, 2T]$ to find

$$\mathcal{J} \ll \int_T^{2T} \prod_{k \le m} |\mathcal{N}_{T_L}(\tfrac{1}{2} + i(t + \alpha_k); \beta_k)|^2 dt.$$

Here we have used that $\prod_{j \le L}(1 + O(e^{-50\beta_*^2 K_j}))^2 = O(1)$. Note $\mathcal{N}_{j,X}(s;\beta)$ is a Dirichlet polynomial of length $\le T_j^{100m\beta_*^2 K_j}$ when $j > 1$ and of length $\le T_1^{200m\beta_*^2 K_1}$ when $j = 1$, so

$$\prod_{k \le m} \mathcal{N}_X(s + i\alpha_k; \beta_k)$$

has length at most $T_1^{200m\beta_*^2 K_1} T_2^{100m\beta_*^2 K_2} \cdots T_L^{100m\beta_*^2 K_L} \le T^{1/10}$ by choice of T_j, K_j and L. Therefore using lemma 4.4.6 in tandem with Proposition 5.2.3 we find

$$\begin{aligned}\mathcal{J} &\ll T \prod_{j \le L} \left(\prod_{p \in (T_{j-1}, T_j]} \left(1 + \frac{|b_{j,T_L,\boldsymbol{\alpha},\boldsymbol{\beta}}(p)|^2}{p} + O\left(\frac{1}{p^2}\right)\right) + O(e^{-50\beta_*^2 K_j}) \right) \\ &\ll T(\log T)^{\beta_1^2 + \cdots + \beta_m^2} \prod_{1 \le j < k \le m} |\zeta(1 + i(\alpha_j - \alpha_k) + 1/\log T)|^{2\beta_j \beta_k},\end{aligned}$$

where the latter bound follows from an application of lemma 4.3.6. This concludes the proof of Proposition 6.1.2. □

6.3 Proof of Proposition 6.1.3

Recall

$$\mathcal{I}_0 = \int_{\mathcal{G}_m} \prod_{k\le m} \zeta(\tfrac{1}{2}+i(t+\alpha_k))$$
$$\times \prod_{j\le L} \exp\big((\beta_k-1)\mathcal{P}_{j,T_L}(\tfrac{1}{2}+i(t+\alpha_k))+\beta_k\mathcal{P}_{j,T_L}(\tfrac{1}{2}-i(t+\alpha_k))\big)\, w(t/T)\ dt.$$

To compute this, we will first need to replace the exponential by a Dirichlet polynomial. To accomplish this, we will write $\mathcal{I}_0$ as a telescoping sum and control the size of the intermediate increments. For $0 \le J \le L$ denote

$$\mathcal{I}_0^{(J)} = \int_{\mathcal{G}_m} \prod_{k\le m} \zeta(\tfrac{1}{2}+i(t+\alpha_k))$$
$$\times \prod_{j\le J} \mathcal{N}_{j,T_L}(\tfrac{1}{2}+i(t+\alpha_k);\tfrac{1}{2}(\beta_k-1))^2\overline{\mathcal{N}_{j,T_L}(\tfrac{1}{2}+i(t+\alpha_k);\tfrac{1}{2}\beta_k)}^2$$
$$\times \prod_{J<j\le L} \exp\big((\beta_k-1)\mathcal{P}_{j,T_L}(\tfrac{1}{2}+i(t+\alpha_k))+\beta_k\mathcal{P}_{j,T_L}(\tfrac{1}{2}-i(t+\alpha_k))\big)\, w(t/T)\ dt.$$

Since $\mathcal{I}_0 = \mathcal{I}_0^{(0)}$, we may decompose

$$\mathcal{I}_0 = \mathcal{I}_0^{(L)} - \sum_{J\le L}\left(\mathcal{I}_0^{(J)} - \mathcal{I}_0^{(J-1)}\right).$$

To prove Proposition 6.1.3 we will give a lower bound for $|I_0^{(L)}|$ and then show that

$$\sum_{J\le L} |\mathcal{I}_0^{(J)} - \mathcal{I}_0^{(J-1)}| \le \frac{|I_0^{(L)}|}{2}.$$

To control these differences, we will use the following consequence of lemma 5.1.2 and the definition of $\mathcal{G}_m$.

Lemma 6.3.1. *For* $0 \le J \le L$

$$|\mathcal{I}_0^{(J)} - \mathcal{I}_0^{(J-1)}|$$
$$\ll e^{-50\beta_*^2 K_J}\int_{\mathcal{G}_m} \prod_{k\le m} |\zeta(\tfrac{1}{2}+i(t+\alpha_k))|\cdot|\mathcal{N}_{T_L}(\tfrac{1}{2}+i(t+\alpha_k);\tfrac{1}{2}(\beta_k-1))\mathcal{N}_{T_L}\tfrac{1}{2}+i(t+\alpha_k);\tfrac{1}{2}\beta_k)|^2 dt.$$

To prove Proposition 6.1.3, we will now just need the following two estimates.

Proposition 6.3.2.

$$|\mathcal{I}_0^{(L)}| \gg T\prod_{p\le T_L}\left(1+\sum_{1\le j,k\le m}\frac{\beta_j\beta_k}{p^{1+i(\alpha_j-\alpha_k)}}\right).$$

Proposition 6.3.3. *Assuming the Riemann hypothesis*

$$\int_{\mathcal{G}_m} \prod_{k\le m} |\zeta(\tfrac12+i(t+\alpha_k))|\cdot|\mathcal{N}_{T_L}(\tfrac12+i(t+\alpha_k);\tfrac12(\beta_k-1))\mathcal{N}_{T_L}\tfrac12+i(t+\alpha_k);\tfrac12\beta_k)|^2 dt$$

$$\ll T\prod_{p\le T_L}\left(1+\sum_{1\le j,k\le m}\frac{\beta_j\beta_k}{p^{i(\alpha_j-\alpha_k)}}\right).$$

Before proving these two statements, let us briefly see how they imply Proposition 6.1.3.

Proof of Proposition 6.1.3. Combining Propositions 6.3.3 and 6.3.2 with lemma 6.3.1, we see that

$$\sum_{J\le L}|\mathcal{I}_0^{(J)}-\mathcal{I}_0^{(J-1)}|\ll|I_0^{(L)}|\sum_{J\le L}e^{-50\beta_*^2K_J}.$$

By definition of K_J, the sum on the right hand side is equal to

$$\sum_{J\le L}\exp\left(-50\beta_*^2\frac{(\log_2 T)^{3/2}}{e^{3J/4}}\right).$$

Because $T_L\le T^\delta$, it also follows that $L\le 2\log_3 T+\log\delta$. Therefore by summing in reverse starting at $J=L$, we see that this sum is bounded by

$$\sum_{j\ge1}\exp\left(-50\beta_*^2e^{-3\log\delta/4}e^{3j/4}\right)\le\sum_{j\ge1}\exp\left(-50\beta_*^2e^{-3\log\delta/4}j\right)\ll\exp\left(-50\beta_*^2e^{-3\log\delta/4}\right).$$

Therefore if we choose $\delta>0$ sufficiently small in terms of $\boldsymbol{\beta}$ we may ensure that

$$\sum_{J\le L}|\mathcal{I}_0^{(J)}-\mathcal{I}_0^{(J-1)}|\le\frac{|I_0^{(L)}|}{2}.$$

Therefore it follows that

$$|I_0|\gg T\prod_{p\le T_L}\left(1+\sum_{1\le j,k\le m}\frac{\beta_j\beta_k}{p^{1+i(\alpha_j-\alpha_k)}}\right).$$

□

6.3.1 Proof of Proposition 6.3.2

To estimate $\mathcal{I}_0^{(L)}$, we will first write $\mathcal{I}_0^{(L)}=\mathcal{J}_1-\mathcal{J}_2$, where

$$\mathcal{J}_1=\int_T^{2T}\prod_{k\le m}\zeta(\tfrac12+i(t+\alpha_k))$$
$$\times\mathcal{N}_{T_L}(\tfrac12+i(t+\alpha_k);\tfrac12(\beta_k-1))^2\overline{\mathcal{N}_{T_L}(\tfrac12+i(t+\alpha_k);\tfrac12\beta_k)}^2w(t/T)\,dt$$

and

$$\mathcal{J}_2 = \int_{[T,2T]\setminus\mathcal{G}_m} \prod_{k\le m} \zeta(\tfrac12 + i(t+\alpha_k))$$
$$\times \mathcal{N}_{T_L}(\tfrac12 + i(t+\alpha_k); \tfrac12(\beta_k - 1))^2 \overline{\mathcal{N}_{T_L}(\tfrac12 + i(t+\alpha_k); \tfrac12\beta_k)}^2 w(t/T)\ dt.$$

We will evaluate $\mathcal{J}_1$ first. To do this, we will make use of the following approximate functional equation.

Lemma 6.3.4. *Let*

$$V(x,t) = \frac{1}{2\pi i}\int_{(1)} \frac{e^{s^2}}{s}\left(\frac{t^{3m}}{x}\right)^s ds$$

and $\tau_{\boldsymbol{\alpha}}(n) = \sum_{n_1\cdots n_m = n} n_1^{-i\alpha_1}\cdots n_m^{-i\alpha_m}$. *For* $\alpha_j \le T/2$ *and* $t\in[T,2T]$

$$\prod_{k\le m}\zeta(\tfrac12 + it + i\alpha_k) = \sum_n \frac{\tau_{\boldsymbol{\alpha}}(n)}{n^{1/2+it}}V(n,t) + O(1/T).$$

Proof. We will evaluate the integral

$$I = \frac{1}{2\pi i}\int_{(1)}\frac{e^{s^2}}{s}t^{3ms}\prod_{k\le m}\zeta(\tfrac12 + it + i\alpha_k + s)ds$$

in two ways. Expanding $\prod_{k\le m}\zeta(\frac12+it+i\alpha_k+s)$ into its Dirichlet series and simplifying shows

$$I = \sum_n \frac{\tau_{\boldsymbol{\alpha}}(n)}{n^{1/2+it}}V(n,t).$$

Alternatively, by shifting the contour to Re $s=-1$, we pass over poles at $s=0$ and $s=-\frac12 - it - i\alpha_k$. Only the residue at $s=0$ contributes because of the rapid decay of e^{s^2} and because $|t+\alpha_k|\ge T/2$. Therefore

$$I = \prod_{k\le m}\zeta(\tfrac12 + it + i\alpha_k) + \frac{1}{2\pi i}\int_{(-1)}\frac{e^{s^2}}{s}t^{3ms}\prod_{k\le m}\zeta(\tfrac12 + it + i\alpha_k + s)ds + O_A(T^{-A}).$$

To conclude, by the standard estimate $\zeta(-\frac12 + it) \ll 1 + |t|$, we may bound this final integral by

$$\ll \int_{\mathbb{R}} \frac{e^{-y^2}}{y+1}T^{-3m}\prod_{j\le s}(1+|t+\alpha_j+y|)dy.$$

The integral over the region $|y|\le T$ can be bounded by

$$\ll \int_0^T \frac{e^{-y^2}}{y+1}T^{-3m}\times T^m\ dy \ll T^{-1},$$

and the integral over the region $|y| \geq T$ is

$$\ll \int_T^\infty \frac{e^{-y^2}}{y} T^{-2m} \times y^m \, dy \ll T^{-1}.$$

□

Next we will need to understand the coefficients of the Dirichlet polynomials

$$\prod_{k \leq m} \mathcal{N}_{j,T_L}(\tfrac{1}{2} + i(t + \alpha_k); \tfrac{1}{2}(\beta_k - 1))^2 = \sum_n \frac{q_j(n)}{n^{1/2+it}},$$

$$\prod_{k \leq m} \mathcal{N}_{T_L}(\tfrac{1}{2} + i(t + \alpha_k); \tfrac{1}{2}(\beta_k - 1))^2 = \sum_n \frac{q(n)}{n^{1/2+it}},$$

and

$$\prod_{k \leq m} \mathcal{N}_{j,T_L}(\tfrac{1}{2} + i(t + \alpha_k); \tfrac{1}{2}\beta_k)^2 = \sum_n \frac{r_j(n)}{n^{1/2+it}},$$

$$\prod_{k \leq m} \mathcal{N}_{T_L}(\tfrac{1}{2} + i(t + \alpha_k); \tfrac{1}{2}\beta_k)^2 = \sum_n \frac{r(n)}{n^{1/2+it}}.$$

The reader may want to review the definition of the coefficients $b_{j,T_L,\boldsymbol{\alpha},\boldsymbol{\beta}}(n)$ found in section 5.2. Notice that each $q_j(n)$ is the twofold Dirichlet convolution of $b_{j,T_L,\boldsymbol{\alpha},\frac{1}{2}(\boldsymbol{\beta}-1)}(n)$ with itself and each $r_j(n)$ is the twofold Dirichlet convolution of $b_{j,T_L,\boldsymbol{\alpha},\frac{1}{2}\boldsymbol{\beta}}(n)$ with itself, where $\frac{1}{2}(\boldsymbol{\beta} - 1)$ and $\frac{1}{2}\boldsymbol{\beta}$ denote the vectors with j^{th} element $\frac{1}{2}(\beta_j - 1)$ and $\frac{1}{2}\beta_j$ respectively. As before, we will define two more sets of coefficients: Let $q'_j(n)$ be the twofold Dirichlet convolution of $b'_{j,T_L,\boldsymbol{\alpha},\frac{1}{2}(\boldsymbol{\beta}-1)}(n)$ with itself and let $r'_j(n)$ be the twofold Dirichlet convolution of $b'_{j,T_L,\boldsymbol{\alpha},\frac{1}{2}\boldsymbol{\beta}}(n)$ with itself. Finally define $q''_j(n)$ and $r''_j(n)$ in an analogous fashion. Again the key point is that q'_j and r'_j are multiplicative approximations of q_j and r_j and that q''_j and r''_j are non-negative multiplicative coefficients satisfying $|q_j|, |q'_j| \leq q''$ and $|r_j|, |r'_j| \leq r''$. Before proceeding further, we will need the following estimate.

Lemma 6.3.5. *For $p \in (T_{j-1}, T_j]$*

$$q_j(p^l), r_j(p^l) \ll m^2 \beta_*^l l^{4m} e^{-l \log(l/2m)/4m + 2l}.$$

Proof. The bound for q_j follows by combining the formula

$$q_j(p^l) = \sum_{x+y=l} b_{j,T_L,\boldsymbol{\alpha},\frac{1}{2}(\boldsymbol{\beta}-1)}(p^x) b_{j,T_L,\boldsymbol{\alpha},\frac{1}{2}(\boldsymbol{\beta}-1)}(p^y)$$

with the bound for $b_{j,T_L,\boldsymbol{\alpha},\frac{1}{2}(\boldsymbol{\beta}-1)}(p^x)$ given in lemma 5.2.2 and noting that either $x \geq l/2$ or $y \geq l/2$. The argument for r_j is the same. □

We can now compute

$$\mathcal{J}_1 = \frac{1}{2\pi i}\sum_n \sum_{h,k<T^{1/2}} \frac{\tau_{\boldsymbol{\alpha}}(n)q(h)\overline{r(k)}}{n^{1/2}h^{1/2}k^{1/2}}\int_{(1)}\frac{e^{s^2}}{s}\int_T^{2T}\left(\frac{nh}{k}\right)^{-it}\frac{t^{3ms}}{n^s}w(t/T)\ dt\ ds$$
$$+O\left(\frac{1}{T}\int_T^{2T}\prod_{k\le m}|\mathcal{N}_{T_L}(\tfrac12+i(t+\alpha_k);\tfrac12(\beta_k-1))\mathcal{N}_{T_L}(\tfrac12+i(t+\alpha_k);\tfrac12\beta_k)|^2dt\right).$$

Because

$$\prod_{k\le m}\mathcal{N}_{T_L}(\tfrac12+i(t+\alpha_k);\tfrac12(\beta_k-1))\mathcal{N}_{T_L}(\tfrac12+i(t+\alpha_k);\tfrac12\beta_k)$$

is a short Dirichlet polynomial whose coefficients $c(n)$ satisfy $c(n)\ll_\varepsilon n^\varepsilon$, it follows that the error term is $\ll_\varepsilon T^\varepsilon$ by lemma 5.1.3. Returning now to the main term, the next step is to discard the terms where $nh\neq k$. An exercise in contour integration shows that

$$t^j\frac{\partial^j}{\partial t^j}V(x,t)\ll_{A,j}\left(1+|x/t^{3m}|\right)^{-A}.$$

Therefore, because w is smooth, repeated integration by parts implies that

$$\int_T^{2T}w(t/T)\left(\frac{nh}{k}\right)^{-it}V(n,t)\ dt\ll_{j,A}\frac{(1+n/T^{3m})^{-A}}{|\log(nh/k)|^jT^j}.$$

Because h and k are at most $T^{1/2}$, it follows that $\log(nh/k)\gg T^{-1/2}$ if $nh\neq k$. Therefore the contribution to the sum of terms with $hn\neq k$ is $O_A(T^{-A})$ because $h,k<T^{1/2}$.

Now by shifting the contour to Re $s=-1/4$ and simplifying the diagonal terms, we find that

$$\mathcal{J}_1 = T\|w\|_1\sum_{\substack{h,k\le T^{1/2}\\ h|k}}\frac{\tau_{\boldsymbol{\alpha}}(k/h)q(h)\overline{r(k)}}{k}+O(T^{1-\varepsilon}).$$

By multiplicativity, we may factor the inner sum as

$$\sum_{\substack{h,k\le T^{1/2}\\ h|k}}\frac{\tau_{\boldsymbol{\alpha}}(k/h)q(h)\overline{r(k)}}{k}=\prod_{j\le L}\sum_{\substack{p|h,k\Rightarrow p\in(T_{j-1},T_j]\\ h|k}}\frac{\tau_{\boldsymbol{\alpha}}(k/h)q_j(h)\overline{r_j(k)}}{k}.$$

Next notice if $q_j(h)\neq q_j'(h)$, then it must be that $\Omega(h)\ge 100\beta_*^2K_j$. So replacing q_j and r_j with q_j' and r_j' respectively incurs an error of order

$$e^{-100\beta_*^2K_j}\sum_{\substack{p|h,k\Rightarrow p\in(T_{j-1},T_j]\\ h|k}}\frac{e^{\Omega(h)}|\tau_{\boldsymbol{\alpha}}(k/h)|q_j''(h)r_j''(k)}{k}$$
$$\ll e^{-100\beta_*^2K_j}\prod_{p\in(T_{j-1},T_j]}\left(1+\frac{2e\beta_*^2}{p}+O\left(\frac{1}{p^2}\right)\right)\ll e^{-50\beta_*^2K_j}.$$

for each j. Here we have used the divisor bound $|\tau_{\boldsymbol{\alpha}}(p^y)| \ll_\varepsilon p^{\varepsilon y}$ and lemma 6.3.5 to bound the contribution of the terms of order smaller than $1/p^2$. Therefore, suppressing terms of order $T^{1-\varepsilon}$, we see that

$$\mathcal{J}_1 = T\|w\|_1 \prod_{j \le L} \left(\prod_{p \in (T_{j-1}, T_j]} \left(\sum_{0 \le x \le y} \frac{\tau_{\boldsymbol{\alpha}}(p^{y-x}) q_j(p^x) \overline{r_j(p^y)}}{p^y} \right) + O(e^{-50\beta_*^2 K_j}) \right)$$

$$= T\|w\|_1 \prod_{j \le L} \left(\prod_{p \in (T_{j-1}, T_j]} \left(1 + \frac{q_j(p)\overline{r_j(p)} + \tau_{\boldsymbol{\alpha}}(p)\overline{r_j(p)}}{p} + O\left(\frac{1}{p^2}\right) \right) + O(e^{-50\beta_*^2 K_j}) \right).$$

A quick computation shows that

$$q_j(p) + \tau_\alpha(p) = a_{T_L}(p) \sum_{k \le m} \beta_k p^{-i\alpha_k} = r_j(p).$$

Therefore, recalling $a_{T_L}(p) = p^{-1/\log T_L}(1 - \log p / \log T_L)$, we deduce that $\mathcal{J}_1$ is equal to

$$T\|w\|_1 \prod_{j \le L} \left(\prod_{p \in (T_{j-1}, T_j]} \left(1 + \frac{1}{p} \left| \sum_{k \le m} \beta_k p^{-i\alpha_k} \right|^2 + O\left(\frac{\log p}{p \log T_L} + \frac{1}{p^2} \right) \right) + O(e^{-50\beta_*^2 K_j}) \right),$$

and we may readily conclude

Proposition 6.3.6.

$$\mathcal{J}_1 \gg T \prod_{p \le T_L} \left(1 + \sum_{1 \le j,k \le m} \frac{\beta_j \beta_k}{p^{1+i(\alpha_j - \alpha_k)}} \right).$$

We will now show, for δ sufficiently small, that $|\mathcal{J}_2| \le |\mathcal{J}_1|/2$. To do this we will recall some of the notation from the previous chapter. First recall that

$$\mathcal{G} := \left\{ t \in [T/2, 5T/2] : |P_{j,T_L}(\tfrac{1}{2} + it)| \le K_j \text{ for all } 1 \le j \le L \right\}.$$

Next given a subset A of $[m] := \{1, \ldots, m\}$

$$\mathcal{G}_A := \{ t \in [T, 2T] : t + \alpha_k \in \mathcal{G} \text{ if and only if } k \in A \},$$

and for each $1 \le j \le L$ let

$$\mathcal{B}_j := \{ t \in [T/2, 5T/2] : |\mathcal{P}_{r,T_s}(\tfrac{1}{2} + it)| \le K_j \text{ for all } 1 \le r < j \text{ and } r \le s \le L \\ \text{but } |\mathcal{P}_{j,T_s}(\tfrac{1}{2} + it)| > K_j \text{ for some } j \le s \le L\}.$$

To evaluate $\mathscr{J}_2$, we must evaluate an integral over all of the $\mathcal{G}_A$ with A ranging over all proper subsets of $[m]$. Without loss of generality we will write $A = [m] \setminus [a]$. For each $t \in \mathcal{G}_A$, there is a function $F_t : [a] \to [L]$ such that $t + \alpha_j \in \mathcal{B}_{f(j)}$. We further partition $\mathcal{G}_A$ into the sets

$$\mathcal{B}_{A,n} = \{t \in \mathcal{G}_A : \min_{j\in[a]} F_t(j) = n\}.$$

With this new notation, we may write

$$[T, 2T] \setminus \mathcal{G}_m = \bigsqcup_{n\leq L} \bigsqcup_{A\subsetneq[m]} \mathcal{B}_{A,n}.$$

When $n > 1$, the calculation is analogous to the one in section 5.3. Using Lemma 5.1.1 with $X = T_{n-1}$ and the definition of $\mathcal{B}_{A,n}$ we find

$$\begin{aligned}
&\int_{\mathcal{B}_{A,n}} \prod_{k\leq m} |\zeta(\tfrac{1}{2} + i(t + \alpha_k))| \\
&\qquad\qquad \times |\mathcal{N}_{T_L}(\tfrac{1}{2} + i(t + \alpha_k); \tfrac{1}{2}(\beta_k - 1))\mathcal{N}_{T_L}(\tfrac{1}{2} + i(t + \alpha_k); \tfrac{1}{2}\beta_k)|^2 w(t/T)\ dt \\
&\quad \ll \int_{\mathcal{B}_{A,n}} \prod_{k=1}^{m} \exp\left(\mathrm{Re}\left(\sum_{j<n} \mathcal{P}_{j,T_{n-1}}(\tfrac{1}{2} + i(t + \alpha_k)) + 1/c_{n-1}\right)\right. \\
&\qquad\qquad\qquad \times |\mathcal{N}_{T_L}(\tfrac{1}{2} + i(t + \alpha_k); \tfrac{1}{2}(\beta_k - 1))\mathcal{N}_{T_L}(\tfrac{1}{2} + i(t + \alpha_k); \tfrac{1}{2}\beta_k)|^2\ dt \\
&\quad \ll e^{1/c_{n-1}} \int_{\mathcal{B}_{A,n}} \prod_{k=1}^{m}\prod_{j<n} \exp\left(\mathrm{Re}\ \mathcal{P}_{j,T_{n-1}}(\tfrac{1}{2} + i(t + \alpha_k))\right) \\
&\qquad\qquad\qquad \times |\mathcal{N}_{T_L}(\tfrac{1}{2} + i(t + \alpha_k); \tfrac{1}{2}(\beta_k - 1))\mathcal{N}_{T_L}(\tfrac{1}{2} + i(t + \alpha_k); \tfrac{1}{2}\beta_k)|^2\ dt \\
&\quad \ll e^{1/c_{n-1}} \max_{\substack{\ell\in[a]\\ s\in[L]}} \int_T^{2T} \prod_{k=1}^{m}\prod_{j<n} |\mathcal{N}_{j,T_{n-1}}(\tfrac{1}{2} + i(t + \alpha_k); \tfrac{1}{2})|^2 \\
&\qquad\qquad\qquad \times |\mathcal{P}_{n,T_s}(\tfrac{1}{2} + i(t + \alpha_\ell))/K_n|^{2\lceil 1/20c_n\rceil} \\
&\qquad\qquad\qquad \times |\mathcal{N}_{T_L}(\tfrac{1}{2} + i(t + \alpha_k); \tfrac{1}{2}(\beta_k - 1))\mathcal{N}_{T_L}(\tfrac{1}{2} + i(t + \alpha_k); \tfrac{1}{2}\beta_k)|^2\ dt.
\end{aligned}$$

So by lemma 4.4.6 we need the following mean value calculations.

Proposition 6.3.7. *For $j < n$*

$$\begin{aligned}
&\int_T^{2T} \prod_{k=1}^{m} |\mathcal{N}_{j,T_{n-1}}(\tfrac{1}{2} + i(t + \alpha_k); \tfrac{1}{2})\mathcal{N}_{j,T_L}(\tfrac{1}{2} + i(t + \alpha_k); \tfrac{1}{2}(\beta_k - 1))|^2 \\
&\qquad\qquad \times |\mathcal{N}_{j,T_L}(\tfrac{1}{2} + i(t + \alpha_k); \tfrac{1}{2}\beta_k)|^2 dt \\
&\quad \leq T \prod_{p\in(T_{j-1},T_j]} \left(1 + \frac{1}{p} \sum_{1\leq j,k\leq m} \frac{\beta_j\beta_k}{p^{i(\alpha_j-\alpha_k)}} + O\left(\frac{\log p}{p\log T_{n-1}} + \frac{1}{p^2}\right)\right) + O(e^{-50\beta_*^2 K_j}).
\end{aligned}$$

Proof. All of the following calculations are similar to earlier computations, so we will just sketch their proofs. The coefficients of the Dirichlet polynomial

$$\prod_{k=1}^{m} \mathcal{N}_{j,T_{n-1}}(\tfrac{1}{2}+i(t+\alpha_k);\tfrac{1}{2})\mathcal{N}_{j,T_L}(\tfrac{1}{2}+i(t+\alpha_k);\tfrac{1}{2}(\beta_k-1))\mathcal{N}_{j,T_L}(\tfrac{1}{2}+i(t+\alpha_k);\tfrac{1}{2}\beta_k)$$

are given by the triple convolution

$$a(n) := b_{j,T_{n-1},\boldsymbol{\alpha},\mathbf{1/2}} * b_{j,T_L,\boldsymbol{\alpha},\frac{1}{2}(\boldsymbol{\beta}-1)} * b_{j,T_L,\boldsymbol{\alpha},\frac{1}{2}\boldsymbol{\beta}}(n),$$

where $\mathbf{1/2}$ is a vector of m copies of 1/2. Using Rankin's trick, one can replace these with the multiplicative coefficients

$$a'(n) := b'_{j,T_{n-1},\boldsymbol{\alpha},\mathbf{1/2}} * b'_{j,T_L,\boldsymbol{\alpha},\frac{1}{2}(\boldsymbol{\beta}-1)} * b'_{j,T_L,\boldsymbol{\alpha},\frac{1}{2}\boldsymbol{\beta}}(n)$$

at a cost of $O(e^{-50\beta_*^2 K_j})$. Using lemma 5.2.2, one may then show that

$$|a'(p)|^2 = \sum_{1\le j,k\le m} \frac{\beta_j\beta_k}{p^{i(\alpha_j-\alpha_k)}} + O\left(\frac{\log p}{\log T_{n-1}}\right)$$

and

$$\sum_{r\ge 2} \frac{|a(p^r)|^2}{p^r} \ll \frac{1}{p^2}.$$

The claim now follows by multiplicativity and lemma 5.1.3. □

Proposition 6.3.8. *For* $1 < n \le L$

$$\int_T^{2T} \prod_{k=1}^{m} |\mathcal{N}_{n,T_L}(\tfrac{1}{2}+i(t+\alpha_k);\tfrac{1}{2}(\beta_k-1))\mathcal{N}_{n,T_L}(\tfrac{1}{2}+i(t+\alpha_k);\tfrac{1}{2}\beta_k)|^2$$
$$\times |\mathcal{P}_{n,T_s}(\tfrac{1}{2}+i(t+\alpha_\ell))/K_n|^{2\lceil 1/20c_n\rceil}\, dt \ll Te^{-\log(1/c_n)/40c_n}.$$

Proof. By Cauchy Schwarz, the relevant mean value is at most

$$\left(\int_T^{2T} |\mathcal{P}_{n,T_s}(\tfrac{1}{2}+i(t+\alpha_\ell))/K_n|^{2\lceil 1/20c_n\rceil}\, dt\right)^{1/2}$$
$$\times \left(\int_T^{2T} \prod_{k=1}^{m} |\mathcal{N}_{T_L}(\tfrac{1}{2}+i(t+\alpha_k);\tfrac{1}{2}(\beta_k-1))\mathcal{N}_{T_L}(\tfrac{1}{2}+i(t+\alpha_k);\tfrac{1}{2}\beta_k)|^2\, dt\right)^{1/2}.$$

By Proposition 5.3.2, the first integral is $\ll Te^{-\log(1/c_n)/20c_n}$. Using the same reasoning in the previous proof, one may show that the second integral is

$$\ll \prod_{p\in(T_{n-1},T_n]} \left(1+O\left(\frac{1}{p}\right)\right) \ll 1$$

because $\log T_n/\log T_{n-1} = e$ for $n>1$. The claim now follows.

□

Proposition 6.3.9. *For $j > n$*

$$\int_T^{2T} \prod_{k=1}^{m} |\mathcal{N}_{j,T_L}(\tfrac{1}{2} + i(t+\alpha_k); \tfrac{1}{2}(\beta_k - 1))\mathcal{N}_{j,T_L}(\tfrac{1}{2} + i(t+\alpha_k); \tfrac{1}{2}\beta_k)|^2 dt$$
$$\le T \prod_{p\in(T_{j-1},T_j]} \left(1 + O\left(\frac{1}{p}\right)\right) + O(e^{-50\beta_*^2 K_j}).$$

Proof. The proof is a simpler version of the proof of Proposition 6.3.7. One now has a double convolution instead of a triple convolution, and uses a cruder bound for the Dirichlet polynomial coefficients. The details are omitted. □

The previous three propositions and Mertens' estimate now imply that

$$\int_{\mathcal{B}_{A,n}} \prod_{k\le m} |\zeta(\tfrac{1}{2} + i(t+\alpha_k))|$$
$$\times |\mathcal{N}_{T_L}(\tfrac{1}{2} + i(t+\alpha_k); \tfrac{1}{2}(\beta_k - 1))\mathcal{N}_{T_L}(\tfrac{1}{2} + i(t+\alpha_k); \tfrac{1}{2}\beta_k)|^2 w(t/T)\ dt$$
$$\ll T \prod_{j<n} \left(\prod_{p\in(T_{j-1},T_j]} \left(1 + \sum_{1\le j,k\le m} \frac{\beta_j\beta_k}{p^{1+i(\alpha_j-\alpha_k)}} + O\left(\frac{\log p}{p\log T_{n-1}} + \frac{1}{p^2}\right)\right) + O(e^{-50\beta_*^2 K_j})\right)$$
$$\times e^{1/c_{n-1} - \log(1/c_n)/40c_n} \prod_{n<j\le L} \left(\prod_{p\in(T_{j-1},T_j]} \left(1 + O\left(\frac{1}{p}\right)\right) + O(e^{-50\beta_*^2 K_j})\right)$$
$$\ll T \exp\left(1/c_{n-1} - \log(1/c_n)/40c_n + O\left((\log_2 T)^2 e^{-n}\right)\right) \prod_{p\le T_{n-1}} \left(1 + \frac{1}{p}\sum_{1\le j,k\le m} \frac{\beta_j\beta_k}{p^{i(\alpha_j-\alpha_k)}}\right)$$
$$\ll T \exp\left(1/c_{n-1} - \log(1/c_n)/40c_n + O\left((\log_2 T)^2 e^{-n} + L - n\right)\right)$$
$$\times \prod_{p\le T_L} \left(1 + \frac{1}{p}\sum_{1\le j,k\le m} \frac{\beta_j\beta_k}{p^{i(\alpha_j-\alpha_k)}}\right).$$

When $n = 1$, we will instead use the estimate

Proposition 6.3.10. *Assuming the Riemann hypothesis, for any $A \subsetneq [m]$*

$$\int_{\mathcal{B}_{A,1}} \prod_{k\le m} \zeta(\tfrac{1}{2} + i(t+\alpha_k))\mathcal{N}_{T_L}(\tfrac{1}{2} + i(t+\alpha_k); \tfrac{1}{2}(\beta_k - 1))^2$$
$$\times \overline{\mathcal{N}_{T_L}(\tfrac{1}{2} + i(t+\alpha_k); \tfrac{1}{2}\beta_k)}^2 w(t/T)\ dt \ll_A T(\log T)^{-A}.$$

Proof. By Hölder's inequality, the integral over $\mathcal{B}_{A,1}$ is at most

$$(\text{meas } \mathcal{B}_{A,1})^{1/3} \left(\int_T^{2T} \prod_{k\le m} |\zeta(\tfrac{1}{2} + i(t+\alpha_k))|^3\ dt\right)^{1/3}$$

$$\times \left(\int_T^{2T} |\mathcal{N}_{T_L}(\tfrac{1}{2} + i(t+\alpha_k); \tfrac{1}{2}(\beta_k - 1))\mathcal{N}_{T_L}(\tfrac{1}{2} + i(t+\alpha_k); \tfrac{1}{2}\beta_k)|^6 \, dt \right)^{1/3}$$

By the same reasoning used in the proof of Proposition 6.3.7, one may show the final integral is $\ll T(\log T)^{O(1)}$. By Theorem 5.0.1, the first integral is also $\ll T(\log T)^{O(1)}$. The claim now follows by using 5.1.5 to bound meas $\mathcal{B}_{A,1}$. □

We now have all the necessary tools to show

Proposition 6.3.11. *Assuming the Riemann hypothesis, if δ is sufficiently small in terms of $\boldsymbol{\beta}$*

$$|\mathcal{J}_2| \leq |\mathcal{J}_1|/2.$$

Proof. The preceding calculations in tandem with Proposition 6.3.6 imply that for $n > 1$

$$\begin{aligned}
&\int_{\mathcal{B}_{A,n}} \prod_{k \leq m} |\zeta(\tfrac{1}{2} + i(t+\alpha_k))| \\
&\qquad \times |\mathcal{N}_{T_L}(\tfrac{1}{2} + i(t+\alpha_k); \tfrac{1}{2}(\beta_k - 1))\mathcal{N}_{T_L}(\tfrac{1}{2} + i(t+\alpha_k); \tfrac{1}{2}\beta_k)|^2 w(t/T) \, dt \\
&\qquad \ll \exp\left(1/c_{n-1} - \log(1/c_n)/40c_n + O\left((\log_2 T)^2 e^{-n} + L - n\right)\right) |\mathcal{J}_1|.
\end{aligned}$$

Therefore by summing over all $A \subsetneq [m]$ and applying Proposition 6.3.10 we see that

$$\begin{aligned}
|\mathcal{J}_2| \ll |\mathcal{J}_1| \sum_{2 \leq n \leq L} &\exp\left(1/c_{n-1} - \log(1/c_n)/40c_n + O\left((\log_2 T)^2 e^{-n} + L - n\right)\right) \\
&+ O_A\left(T(\log T)^{-A}\right). \qquad (6.1)
\end{aligned}$$

The $T(\log T)^{-A}$ term is negligible. To bound the sum over n, we use the definition of the c_n to find

$$\begin{aligned}
&\sum_{2 \leq n \leq L} \exp\left(1/c_{n-1} - \log(1/c_n)/40c_n + O\left((\log_2 T)^2 e^{-n} + L - n\right)\right) \\
&= \sum_{2 \leq n \leq L} \exp\left(e^{-n}(\log_2 T)^2 (O(1) + \tfrac{1}{40} n - \tfrac{1}{20}\log_3 T)\right) + O(L-n)).
\end{aligned}$$

Because $T_L \leq T^\delta$ it follows that $L \leq 2\log_3 T + \log \delta$ so the sum is at most

$$\sum_{2 \leq n \leq L} \exp\left(e^{-n}(\log_2 T)^2 (O(1) + \tfrac{1}{40}\log \delta)\right) + O(L-n)).$$

By summing in reverse, we may bound this sum by

$$\begin{aligned}
\sum_{j \geq 0} \exp\left(e^j (O(1) + \tfrac{1}{40}\log \delta) + O(j)\right) &\leq \sum_{j \geq 1} \exp\left((O(1) + \tfrac{1}{40}\log \delta) j\right) \\
&\leq \exp\left(O(1) + \tfrac{1}{40}\log \delta\right).
\end{aligned}$$

Therefore by taking δ sufficiently small in terms of $\boldsymbol{\beta}$, we can ensure that this sum times the implicit constant in (6.1) is at most 1/3, so the claim follows. □

Combining this with Proposition 6.3.6 completes the proof of Proposition. 6.3.2 $\square$

6.3.2 Proof of Proposition 6.3.3

Now only Proposition 6.3.3 is needed to complete the proof of Proposition 6.1.3. To accomplish this, we will now use Lemma 5.1.1 with $X = T_L$ and the definition of $\mathcal{G}_m$ to deduce

$$\begin{aligned}
&\int_{\mathcal{G}_m} \prod_{k\leq m} |\zeta(\tfrac{1}{2}+i(t+\alpha_k))| \\
&\qquad \times |\mathcal{N}_{T_L}(\tfrac{1}{2}+i(t+\alpha_k);\tfrac{1}{2}(\beta_k-1))\mathcal{N}_{T_L}(\tfrac{1}{2}+i(t+\alpha_k);\tfrac{1}{2}\beta_k)|^2 w(t/T)\ dt \\
&\ll \int_{\mathcal{G}_m} \prod_{k=1}^{m}\prod_{j\leq L} \exp\left(\text{Re } \mathcal{P}_{j,T_L}(\tfrac{1}{2}+i(t+\alpha_k))\right) \\
&\qquad\qquad \times |\mathcal{N}_{T_L}(\tfrac{1}{2}+i(t+\alpha_k);\tfrac{1}{2}(\beta_k-1))\mathcal{N}_{T_L}(\tfrac{1}{2}+i(t+\alpha_k);\tfrac{1}{2}\beta_k)|^2\ dt \\
&\ll \int_{T}^{2T} \prod_{k=1}^{m} |\mathcal{N}_{T_L}(\tfrac{1}{2}+i(t+\alpha_k);\tfrac{1}{2})|^2 \\
&\qquad\qquad \times |\mathcal{N}_{T_L}(\tfrac{1}{2}+i(t+\alpha_k);\tfrac{1}{2}(\beta_k-1))\mathcal{N}_{T_L}(\tfrac{1}{2}+i(t+\alpha_k);\tfrac{1}{2}\beta_k)|^2\ dt.
\end{aligned}$$

So all that remains to is compute the mean square of the Dirichlet polynomial

$$\prod_{k=1}^{m} \mathcal{N}_{T_L}(\tfrac{1}{2}+i(t+\alpha_k);\tfrac{1}{2})\mathcal{N}_{T_L}(\tfrac{1}{2}+i(t+\alpha_k);\tfrac{1}{2}(\beta_k-1))\mathcal{N}_{T_L}(\tfrac{1}{2}+i(t+\alpha_k);\tfrac{1}{2}\beta_k).$$

Proposition 6.3.7 with $n = L+1$ and lemma 4.4.6 imply that this mean value is

$$\ll T \prod_{p\leq T_L}\left(1+\frac{1}{p}\sum_{1\leq j,k\leq m}\frac{\beta_j\beta_k}{p^{i(\alpha_j-\alpha_k)}}\right).$$

This now concludes the proof of Proposition 6.3.3, and therefore also the proof of Proposition 6.1.3.

$\square$

6.4 Proof of Proposition 6.1.4

All that remains is to bound the quantities

$$\begin{aligned}
\mathcal{I}_k = \int_{\mathcal{G}_m} |\zeta(\tfrac{1}{2}+i(t+\alpha_k))|^{2m} &\prod_{j\leq L} \exp\left(2(\beta_k-m)\text{Re } \mathcal{P}_{j,T_L}(\tfrac{1}{2}+i(t+\alpha_k))\right) \\
&\times \prod_{\substack{\ell\leq m\\ \ell\neq k}}\prod_{j\leq L} \exp\left(2\beta_\ell \text{Re } \mathcal{P}_{j,T_L}(\tfrac{1}{2}+i(t+\alpha_\ell))\right) w(t/T) dt.
\end{aligned}$$

By applying lemma 5.1.1 with $X = T_L$ and lemma 5.1.2 it follows that

$$\mathcal{I}_k \ll \int_T^{2T} |\mathcal{N}_{T_L}(\tfrac{1}{2} + i(t+\alpha_k); m)\mathcal{N}_{T_L}(\tfrac{1}{2} + i(t+\alpha_k); \beta_k - m)|^2 \times \prod_{\substack{\ell \le m \\ \ell \ne k}} |\mathcal{N}_{T_L}(\tfrac{1}{2} + i(t+\alpha_\ell); \beta_\ell)|^2 \, dt.$$

Before proceeding it may be helpful to review the definitions made preceding Proposition 5.2.1. When $j > 1$, the coefficients $a_j(n)$ of the Dirichlet polynomial

$$\mathcal{N}_{j,T_L}(\tfrac{1}{2} + i(t+\alpha_k); m)\mathcal{N}_{j,T_L}(\tfrac{1}{2} + i(t+\alpha_k); \beta_k - m) \prod_{\substack{\ell \le m \\ \ell \ne k}} \mathcal{N}_{j,T_L}(\tfrac{1}{2} + i(t+\alpha_\ell); \beta_\ell)$$

are given by the $m+1$ fold Dirichlet convolution of $g_{T_L}(n; m)n^{-i\alpha_k}c_j(n)$ and $g_{T_L}(n; \beta_k - m)n^{-i\alpha_k}c_j(n)$ with $g_{T_L}(n; \beta_\ell)n^{-i\alpha_\ell}c_j(n)$ for all $\ell \le m$ not equal to k. When $j = 1$ a similar formula holds with f_{T_L} in place of g_{T_L}. As before, one may replace $c_j(n)$ with $1_{p|n \Rightarrow p \in (T_{j-1}, T_j]}$ at a cost of $O(e^{-50\beta_*^2 K_j})$ to obtain multiplicative coefficients $a'_j(n)$ which

$$|a'_j(p)|^2 = \sum_{1 \le j,k \le m} \frac{\beta_j \beta_k}{p^{i(\alpha_j - \alpha_k)}} + O\left(\frac{\log p}{\log T_L}\right)$$

and

$$\sum_{r \ge 2} \frac{|a('p^r)|^2}{p^r} \ll \frac{1}{p}$$

for $T_{j-1} < p \le T_j$. As before, this allows us to conclude that

$$\mathcal{I}_k \ll T \prod_{p \le T_L} \left(1 + \sum_{1 \le j,k \le m} \frac{\beta_j \beta_k}{p^{i(\alpha_j - \alpha_k)}} \frac{1}{p}\right).$$

Proposition 6.1.4, and therefore Theorem 6.1.1, readily follows.

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