

Advanced Measurements for Quantum Photonics and Quantum Technologies

David S. Phillips
St. Hugh's College, Oxford



Submitted for the degree of Doctor of Philosophy
Hilary term 2020

Supervised by
Prof. Ian A. Walmsley

Clarendon Laboratory
University of Oxford
United Kingdom

Advanced Measurements for Quantum Photonics and Quantum Technologies

David S. Phillips
St. Hugh's College, Oxford

Submitted for the degree of Doctor of Philosophy
Hilary Term 2020

Abstract

Measurements on states of light underpins the entire fields of classical and quantum optics. Without measurement we are unable to gain any information about the light we are studying or about the processes it has undergone. The property of the density matrix of light prior to measurement that we want to measure also relies heavily on the measurement technique we employ. We can use a single-photon detector to measure the particle nature of light—the photon—the discrete quanta of the electromagnetic field. We can conversely measure the wave nature of the light field by making phase-sensitive measurements. In this thesis, we shall explore some advanced implementations and applications of these measurement schemes.

First, we shall focus on a specific type of single-photon detector: the Superconducting-Nanowire Single-Photon Detector. We will first consider the integration of these detectors on optical waveguides, and discuss their optical characterisation. Then we will consider a practical application of these detectors by investigating a method to perform broadband spectroscopy with a single detector.

We shall then progress to phase-sensitive measurements, and consider the intermediate regime between measuring the wave-like and particle-like nature of light. First we will perform a fundamental investigation of the transition between both measurement regimes. We will see how varying the phase reference itself allows us to tune between both regimes. Following this we will discuss an application of this intermediate regime, by considering an experimental setup to perform state tomography with a weak phase reference.

The final consideration of this thesis will be a theoretical benchmarking scheme for a practical application of quantum optics. The manipulation of certain quantum states of light followed by a photon-number measurement can have practical applications in a variety of fields, and therefore validating the results is of extreme importance. On this final topic, we will discuss an experimentally friendly technique to benchmark the output of such measurements by considering smaller subsystems, and the practical requirements of this technique.

Acknowledgements

On the first morning of my D.Phil, I was sorting admin at the secretary's office when I heard heavy footsteps approaching the door. A tall, bearded man burst in and exclaimed: "I'm looking for David Phillips!". Jelmer, you took me under your wing from the very start and always had my back. Same goes for William, I learned so much from you and for that I will always be grateful. As time progressed Jasleen, Tom W and Jake joined our corner of the group, it was a pleasure working with you all! I especially enjoyed the conference trips, namely being a tourist in Rome with Jasleen and in Paris with Jake, William, and Jelmer.

I would like to thank my supervisor Ian for all of his guidance. Although we didn't have much interaction at the very start of my D.Phil, as time progressed you became an important source of knowledge and direction. Additionally, I would like to thank Prof Robert Taylor for looking out for me for the latter part of my D.Phil. Also, I would like to thank everyone who overlapped with me in the Ultrafast group; I know if I try to name individuals I will forget someone, as the group got so large at one point we had more Toms than females... Thank you all for the lunchtime conversation, lab help, and pub and punting trips! I will now go through each chapter of my thesis to mop up everyone else I would like to thank.

Chapter 4: first a thanks to our collaborators at Southampton for making the waveguides, Paolo and Devin, and to Varun at NIST for making the SNSPDs. Raj, thank you for your help with installation, co-ordination, and generally overseeing me for this project

(and everything else!). A special thanks to Jamie for fibre splicing and computational device control help, and also to Santi for your help with data acquisition.

Chapter 5: Jelmer this was your experimental idea, thank you for gifting it to me! Andreas, I imagine I irritated you beyond belief with my incompetence at computational data acquisition, but thank you so much for everything you did for me on this.

Chapter 6: thank you so much to Tom W and Andreas for dragging the laser, kicking and screaming, to the end of the experiment. Guillaume, it was great working with you on the sources and learning about them in the process, plus thanks for all of your work with the model.

Chapter 7: this was the experimental project I conducted at NIST. Experiment wise, thank you to Mike for helping with the setup, to Adriana and Matt for detector support, and to Sae Woo and Rich for your help. Theory wise thanks to Arik, Scott and Manny, our meetings were always very productive and it was lovely working with you all. But most of all a huge thanks to Thomas, without you my visit to NIST would not have been possible. I had such a wonderful time and learned so much from you. You oversaw and looked after me the whole time, and I can't thank you enough for this.

Chapter 8: Jelmer, again this project was your idea, thanks! Mattia, thank you so much for your contribution to this project, and also for hosting me during my collaboration trip to Paris for this project. Jan, I don't know where to start with thanking you. I approached you to supervise me on this project, and you took me on without any hesitation at all. You supported me through what was my first theory project, which can't have been easy given I'm an experimentalist. Thank you for everything you did for me during this project and afterwards as well!

On the non-academic side, I would like to thank everyone I know through bell ringing in Oxford (OUSCR and OSCR) for the fun times, and for the support I had from the people Wesley Memorial Methodist Church. Finally, thank you to my family for their continual support.

Journal Publications and Conference Presentations

Journal Publications

1. D. S. Phillips, M. Walschaers, J. J. Renema, I. A. Walmsley, N. Treps, and J. Sperling, “**Benchmarking of Gaussian boson sampling using two-point correlators**”, *Physical Review A*, 99:023836, 2019.
2. J. Sperling, D. S. Phillips, J. F. F. Bulmer, G. S. Thekkadath, A. Eckstein, T. A. W. Wolterink, J. Lugani, S. W. Nam, A. E. Lita, T. Gerrits, W. Vogel, G. S. Agarwal, C. Silberhorn, and I. A. Walmsley, “**Detector-Agnostic Phase-Space Distributions**”, *Physical Review Letters*, 124:013605, 2020.
3. G. S. Thekkadath, D. S. Phillips, J. F. F. Bulmer, W. R. Clements, A. Eckstein, B. A. Bell, J. Lugani, T. A. W. Wolterink, A. E. Lita, S. W. Nam, T. Gerrits, C. G. Wade, and I. A. Walmsley, “**Tuning between photon-number and quadrature measurements with weak-field homodyne detection**”, *Physical Review A*, 101:031801(R), 2020.
4. P. L. Mennea, D. H. Smith, D. S. Phillips, V. B. Verma, A. E. Lita, T. Gerrits, J. C. Gates, R. P. Mirin, R. H. S. Bannerman, P. C. Gow, J. J. Renema, R. J. A. Francis-Jones, S. Sempere-Llagostera, R. B. Patel, I. A. Walmsley, S. W. Nam, and P. G. R. Smith, “**Multiplexed superconducting nanowire single photon detectors integrated on silica waveguides**”, *manuscript in preparation*.

Conference Presentations

1. D. S. Phillips, M. Walschaers, J. J. Renema, I. A. Walmsley, N. Treps, and J. Sperling, “**Certification of Gaussian Boson Sampling using Two-Point Correlators**”, *Quantum Controlled Ultrafast Multimode Entanglement and Measurement Conference*, July 2018.
2. D. S. Phillips, M. Walschaers, J. J. Renema, I. A. Walmsley, N. Treps, and J. Sperling, “**Certification of Gaussian Boson Sampling using Two-Point Correl-**

-
- ators**", *2nd International Conference on Integrated Quantum Photonics*, October 2018.
3. D. S. Phillips, G. S. Thekkadath, J. F. F. Bulmer, W. R. Clements, B. A. Bell, A. Eckstein, J. Lugani, T. A. W. Wolterink, A. Lita, S. W. Nam, T. Gerrits, C. G. Wade, I. A. Walmsley, "**Photon-Number-Resolved Homodyne Detection**", *Central European Workshop on Quantum Optics 2019*, June 2019.

Contents

Acknowledgements	v
Journal Publications and Conference Presentations	vii
List of Figures	xiii
1 Introduction	1
1.1 Thesis Outline	2
1.2 Declaration and Disclaimer	4
2 Quantum Theory of Light and Linear Optics	5
2.1 What is Light?	5
2.1.1 Classical Description of Light	6
2.1.2 Thermal Radiation	7
2.1.3 Laser Radiation	8
2.2 Quantum Description of Light	9
2.2.1 Field Quantisation	10
2.2.2 Fock States	12
2.2.3 Coherent States	13
2.2.4 Thermal States	14
2.2.5 Squeezed Vacuum	15
2.3 Linear Optics	18
2.3.1 Passive Transformations	19
2.3.2 Hong-Ou-Mandel Interference	21
2.3.3 Boson Sampling	22
2.4 Single-Photon Detection	24
2.4.1 Photon-Number Projection	25
2.4.2 Superconducting-Nanowire Single-Photon Detectors	27
2.4.3 Transition Edge Sensors	31
3 Gaussian State Formalism	37
3.1 Phase Space	37
3.1.1 Quadrature Operators	38

3.1.2	Hamiltonian Revisited	39
3.2	Quasiprobability Distributions	39
3.2.1	Characteristic Functions	40
3.2.2	The Wigner Function	41
3.2.3	Other Quasiprobability Distributions	44
3.3	The Gaussian State Formalism	46
3.3.1	Displacement Vectors and Covariance Matrices	46
3.3.2	States and Operators	48
3.3.3	Factorisation and Applications	55
3.4	Measurement	58
3.4.1	Balanced Homodyne Detection	59
3.4.2	Photon-Number Projection	61
3.5	Gaussian Boson Sampling	62
3.5.1	Scattershot Boson Sampling	62
3.5.2	The Hafnian	62
3.5.3	Applications	64
4	Characterisation of SNSPDs on a Buried Waveguide in Silica Glass	67
4.1	Introduction	68
4.2	Fabrication and Layout	69
4.2.1	Fabrication	69
4.2.2	Overall Structure	71
4.3	Experimental Setup	73
4.3.1	Dilution Refrigeration	73
4.3.2	Installation and Operation	76
4.4	Characterisation Results	77
4.4.1	Voltage Detection Thresholds	78
4.4.2	Calibration Curves and Dark Counts	80
4.4.3	Dead Time	83
4.4.4	Jitter	85
4.4.5	Efficiency	88
4.5	Conclusion	91
5	Broadband Spectroscopy with a Single SNSPD	93
5.1	Background	94
5.1.1	Current Spectroscopy Techniques	94
5.1.2	Spectroscopy using SNSPDs	96
5.2	Spectral Decomposition of Broadband Optical Signals	96
5.2.1	Mathematical Formulation	97
5.2.2	Spectrometer Wavelength Resolution	98
5.2.3	Motivation	100
5.3	Experimental Realisation	101
5.3.1	Sorption Refrigeration	101
5.3.2	Optical Setup	102
5.4	Inversion Results	107
5.4.1	1.3 μm , 1.25 μm and 1.35 μm Signals	107

5.4.2	Broadband Signal	108
5.4.3	Discussion	109
5.4.4	Subsequent Work	110
5.5	Conclusion and Outlook	111
6	Weak Field Homodyne Detection with a Gaussian State	113
6.1	Background	114
6.1.1	Introduction	114
6.1.2	Theoretical Background	115
6.1.3	Experimental Background	116
6.2	Gaussian State Picture	117
6.2.1	The Signal	117
6.2.2	The Local Oscillator	119
6.2.3	The Full System	121
6.2.4	Classical Approximation	123
6.3	Experimental Methods	123
6.3.1	The Full Setup	124
6.3.2	Characterisation of the Single-Photon Sources	125
6.3.3	Additional Measurement Parameters	129
6.4	Results and Discussion	130
6.4.1	$k=6$ Signal	130
6.4.2	Quantifying the Transition	132
6.5	Conclusion and Outlook	133
7	State Tomography of a Multimode State	135
7.1	Introduction	136
7.1.1	Background	136
7.1.2	Theoretical Outline	137
7.1.3	Experimental Implementation	139
7.2	The Experiment	139
7.2.1	Optical Setup	140
7.2.2	Signal and Local Oscillator Visibility	142
7.2.3	Phase Stability	144
7.2.4	Adiabatic Demagnetisation Refrigeration	146
7.2.5	Data Acquisition	147
7.3	Results	149
7.3.1	Phase Tracking and Calibration	149
7.3.2	Tomographic Results	152
7.4	Conclusion and Outlook	153
8	Benchmarking Gaussian Boson Sampling via Two-Point Correlators	155
8.1	Introduction	156
8.1.1	The Verification Problem	157
8.1.2	Correlation Functions	158
8.1.3	Benchmarking Boson Sampling	158
8.1.4	Extension to Gaussian Boson Sampling	160

8.2	Algebraic Results	161
8.2.1	The Two-Point Correlator for Gaussian States	161
8.2.2	Benchmarking Assumptions	162
8.2.3	Randomisation Over Unitaries	165
8.3	Simulation Results	166
8.3.1	Simulation Methods	166
8.3.2	Squeezed and Thermal State Comparison	167
8.4	Experimental Considerations	172
8.4.1	Network Loss	173
8.4.2	Additive Noise	175
8.4.3	Discrimination via Statistical Significance	176
8.4.4	Detectors with Finite Photon-Number Resolution	178
8.5	Conclusion and Outlook	181
9	Conclusion	185
9.1	Summary	186
9.1.1	Superconducting-Nanowire Single-Photon Detectors	186
9.1.2	Weak Field Homodyne Detection	187
9.1.3	Gaussian Boson Sampling	189
9.2	Outlook	190
	Bibliography	193

List of Figures

2.1	Diagram of an electromagnetic wave	6
2.2	Planck's Law at different temperatures	8
2.3	Simple laser cavity illustration	9
2.4	Phase shifter schematic	19
2.5	Beamsplitter schematic	19
2.6	Boson sampler schematic	22
2.7	Click detector visualisation	25
2.8	SNSPD structure	27
2.9	SNSPD vortex and detection voltage pulse	29
2.10	SNSPD circuitry	29
2.11	TES images	31
2.12	TES circuitry	33
2.13	TES readout	35
3.1	Examples of Wigner functions	42
3.2	Balanced homodyne detector diagram	59
3.3	Quadrature projection plot	60
3.4	Gaussian Boson samples schematic	63
4.1	On-chip SNSPDs waveguides	70
4.2	On-chip SNSPDs layout and electronics	72
4.3	Dilution refrigeration diagram	74
4.4	Dilution refrigerator	75
4.5	On-chip SNSPDs setup installed in dilution refrigerator	77
4.6	Example detection voltage trace from an on-chip SNSPD	78
4.7	SNSPD voltage detection threshold comparison	79
4.8	On-chip SNSPDs calibration curves	82
4.9	Dark count rates for on-chip SNSPDs	83
4.10	On-chip SNSPDs dead time	84
4.11	Jitter measurement for on-chip SNSPDs	87
4.12	SNSPDs on waveguide for efficiency measurement	89
5.1	Grating spectroscopy setup	94

5.2	Time-of-flight spectroscopy setup	95
5.3	Water transmission spectra	100
5.4	Sorption refrigerator images	101
5.5	SNSPD spectroscopy optical setup	103
5.6	SNSPD spectroscopy calibration curves	105
5.7	Power spectra for inversion	106
5.8	Spectral inversions of bandpass filters	107
5.9	Spectral inversions of a broadband signal	109
6.1	Signal preparation diagram	117
6.2	Signal theory properties plots	118
6.3	Local oscillator preparation diagram	119
6.4	WFHD full schematic	122
6.5	WFHD full experimental setup	124
6.6	Joint spectral intensities of the SPDC sources	126
6.7	Signal and local oscillator HOM dip	128
6.8	$k = 6$ transition plots	131
6.9	Sum of residuals plots	132
7.1	Multimode tomography schematic	138
7.2	Optical setup schematic	141
7.3	Image of the optical setup	142
7.4	Signal and LO visibility	143
7.5	Phase stability consideration	145
7.6	Schematic of an ADR	147
7.7	Local oscillator phase calibration	150
7.8	State tomography fidelity	152
8.1	Squeezed and thermal state histogram comparison	168
8.2	Moments comparison for small and large systems	169
8.3	Constant dilution histogram comparison	171
8.4	Constant dilution NM comparison	172
8.5	Loss heatmaps of NM, CV and Sk	174
8.6	Noise heatmaps of NM, CV and Sk	176
8.7	Discrimination via statistical significance	177
8.8	Effect of detector finite photon-number resolution	180

Chapter 1

Introduction

Light is a very powerful tool. For centuries we have utilised the different sections of the electromagnetic spectrum to perform tasks from the sterilisation of medical equipment with gamma rays, to using radio waves for communication and entertainment, and everything that comes in between. Of the many applications of light, one aspect that they share is the requirement for detection. For instance, when x-rays are used to diagnose a broken bone, an x-ray detector is required to view the contrast image. This is additionally true for the applications of quantum states of light, and a myriad of different methods and devices are actively being used and researched for these purposes.

More recently, quantum states of light have promised to be extremely useful for a variety of applications. These include quantum computing^[1–4] and simulation^[5,6] for fast calculations, quantum metrology^[7,8] and sensing^[9] to beat the shot noise limit, and quantum cryptography^[10] for e.g. secure data transfer. Again, the above applications require some form of measurement on the quantum states of light, requiring a range of detection schemes for the relevant protocols. As all of the above examples are currently areas of intense research, it motivates the study and development of the different types of detection schemes in quantum optics.

One of the major areas in detecting quantum states of light is single-photon detectors (SPDs), as current technologies enable us to detect single quanta of the light field^[11,12].

Since the first gamma and x-ray photomultiplier tubes in the late 19th century^[13], SPD technologies have come on a long way. The mid-20th century saw the development of semiconductor SPDs, exploiting the avalanche effect in some semiconductors to detect single photons in the UV, visible and IR regions at room temperature^[14]. Such detectors are still used today for experiments requiring single-photon detection^[12].

More recently, there have been large developments in the field of superconducting SPDs. Although their cryogenic requirements make them more difficult to operate, photon-absorbing superconductors enable the detection of lower energy photons with higher efficiency and lower noise compared to the most advanced room temperature SPDs^[11,12]. One of the most commonly used superconducting SPD is the ‘Superconducting Nanowire Single-Photon Detector’ (SNSPD)^[15]. SNSPDs will superconduct at liquid helium temperatures (≈ 4.2 K) making the cryogenic requirements fairly straightforward, but they do not give photon-number resolution for photons shorter than the detector recovery time. Some quantum optical applications require number resolution (see ref.^[16] for example), and for this we can use the more advanced ‘Transition Edge Sensor’ (TES)^[17,18]. TESs are more complex to operate compared to SNSPDs and require < 100 mK temperatures, but provide the number-resolution that SNSPDs cannot. Both are very attractive as SPDs as they can detect at telecommunications wavelengths (≈ 1550 nm) at $> 95\%$ efficiencies^[19,20].

Whereas SPDs can determine the photon-number distribution of a quantum state of light, they cannot always determine all possible information about a optical state. To do this, we require a phase-sensitive type of measurement, the most common being ‘Balanced Homodyne Detection’ (BHD)^[21]. This technique, along with others, reveals phase information of an optical state and can enable full state characterisation.

1.1 Thesis Outline

In this thesis, we shall explore advanced types of the measurements listed above. We shall investigate both technological advancements and improvements in methodology to both

characterise quantum states of light and for wider applications.

This thesis is organised as follows. Chapters 2 and 3 are background introductory chapters. In chapter 2 we will discover the basic properties of light in the classical picture, and then move to the quantum picture through quantising the electromagnetic field. We will then discuss some common and useful quantum states of light, transformations on optical fields, and discuss single photon detection in detail. The 3rd chapter gives the background to quantum optics in phase space and introduces the concept of a Gaussian state. We shall discover some useful phase space distributions, and then there will be a very detailed introduction to the ‘Gaussian State Formalism’ which will have much relevance to the latter half of this thesis. We shall also discuss the concept of a Gaussian measurement, and why photon-number measurements are considered non-Gaussian.

Chapters 4 and 5 concern advancements in SNSPDs. In chapter 4 we shall discuss the characterisation of SNSPDs deposited on silica waveguides, a scheme with potential to unify the manipulation and detection of optical states. Then in chapter 5 we will see a novel method to exploit the intrinsic properties of SNSPDs to perform broadband spectroscopy on weak light signals.

Chapters 6 and 7 are about variants of the BHD detection scheme. The 6th chapter investigates BHD with a weak phase references, putting us in the intermediate regime of a Gaussian and non-Gaussian measurement. Then in chapter 7, we shall discuss the experimental implementation of a potential application of weak phase reference BHD to characterise optical states in the presence of certain limitations.

Then, with a slight change of tone, chapter 8 is a purely theoretical investigation of the photon-number detection of Gaussian states of light after manipulation. Specifically, we shall benchmark the photon-number distributions obtained from interfering different types of Gaussian states, and discuss how this is affected by experimental imperfections.

Finally, in chapter 9 I shall conclude the thesis and set its content in the context of current advancements and challenges.

1.2 Declaration and Disclaimer

The work in this thesis has come from a large number of projects which have been highly collaborative in nature. I played a significant role in each, however at the beginning of each chapter I outline exactly who contributed to each aspect of the relevant project. As a result, when presenting the material in this thesis I tend to use the pronoun ‘we’ to reflect this collaborative nature, despite the fact that I mostly pay closer attention to my specific contributions. In addition, for each figure I either state the source or name the creator when I am not the person responsible. Please note that I will also sometimes use the pronoun ‘we’ to refer to the reader and myself (e.g. “we shall discuss...”), separate to the ‘we’ as project contributors (e.g. “we measured...”), though this should be obvious from context.

Now a note on notation and nomenclature. A very large number of quantities are discussed and used in this thesis, and in order to align with accepted conventions there will occasionally be some symbols which are used twice for different quantities (and is the reason why I am not including a list of symbols). In each case the quantity a symbol represents should be obvious from the super/subscript and/or variable dependence. This avoids, say, using Ξ to denote optical power, when we could just deal with the fact that $P(p(\varphi))$ refers to a phase-space quadrature distribution (Chaps. 3 and 6) and P^{SNSPD} refers to optical power measured by an SNSPD (Chap. 4) (‘ P ’ is in fact the most overused symbol...). Also, some concepts or quantities have different names within the literature, and sometimes I use these names interchangeably. Before this happens I clearly state the different names for the same thing, then I have the freedom to select the most appropriate for each situation.

Chapter 2

Quantum Theory of Light and Linear Optics

In this chapter we will start by considering the fundamental question “What is light?”, leading us to the classical description of light and light sources. We will then investigate the steps needed to transition to a quantum description, consider some useful quantum states of light and how to produce them. We will finish by considering how quantum states of light interact with each other and are detected.

2.1 What is Light?

When people without a physical science background ask me “So what area of physics are you researching?”, my usual response is “Light!” and then I just hope they don’t pry any further. But it does beg the important question of what light actually is and what physical models do we have to describe it. In this section we shall explore this question from a classical perspective, and consider this in terms of classical sources of light.

2.1.1 Classical Description of Light

Throughout history there have been many different descriptions of light, though most notable are the particle (or ‘corpuscule’) theory and the wave theory. The double slit experiment attributed to Dr Thomas Young in 1801 demonstrated interference between two coherent light sources^[22], seemingly proving that light was a propagating wave.

In the 1860s James Clerk Maxwell described light as an electromagnetic phenomenon, suggesting that light is propagating electric and magnetic waves at right-angles to each other.

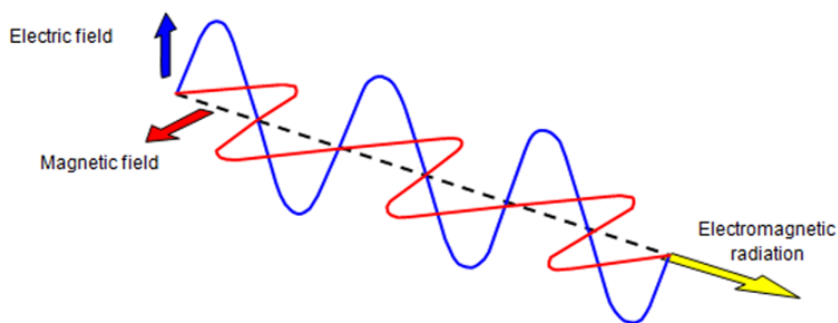


Figure 2.1: Diagram of an electromagnetic (or ‘light’) wave. Both the electric and magnetic waves are transverse and orthogonal to each other. Source: http://www.schoolphysics.co.uk/age16-19/Wave%20properties/Polarisation/text/Polarisation_/images/2.png

Maxwell’s equations that govern an electric field $\mathbf{E}(\mathbf{r}, t)$ and a magnetic field $\mathbf{B}(\mathbf{r}, t)$ and determine their propagation are given by

$$\nabla \cdot \mathbf{E}(\mathbf{r}, t) = \frac{\rho(\mathbf{r}, t)}{\epsilon_0} \quad (2.1)$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0 \quad (2.2)$$

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t} \quad (2.3)$$

$$\nabla \times \mathbf{B}(\mathbf{r}, t) = \mu_0 \left(\mathbf{J}(\mathbf{r}, t) + \epsilon_0 \frac{\partial \mathbf{E}(\mathbf{r}, t)}{\partial t} \right), \quad (2.4)$$

where $\rho(\mathbf{r}, t)$ is electric charge density, ϵ_0 and μ_0 are physical constants (permittivity and

permeability of free space respectively), and $\mathbf{J}(\mathbf{r}, \mathbf{t})$ is electric current density^[23]. The solutions to these equations with appropriate boundary conditions lead to the propagating wave seen in Fig. 2.1. In the next parts of this section I shall briefly discuss the classical radiation mechanisms that are relevant to this thesis.

2.1.2 Thermal Radiation

Thermal radiation is the light emitted by all matter at temperatures above absolute zero. It arises due to the thermal motion of the atoms that make up the material^[24]. It is also related to black-body radiation which is essentially the idealised scenario of thermal radiation, in that a black body is a perfect absorber of all incident electromagnetic radiation.

In 1900 Max Planck formulated the relationship between the spectral radiance (light intensity emitted per unit solid angle) of a black body $B(\lambda, T)$ at a given wavelength due to its temperature^[25]. The relation is given by

$$B(\lambda, T) = \frac{2hc^2}{\lambda^5} \frac{1}{\exp\left(\frac{hc}{\lambda k_B T}\right) - 1}, \quad (2.5)$$

where λ is wavelength, T is temperature, h is Planck's constant, c is the speed of light in vacuum ($c = 1/\sqrt{\epsilon_0\mu_0}$) and k_B is the Boltzmann constant.

The visualisation of Planck's law can be seen in Fig. 2.2. Examples of objects that are approximately black bodies are the sun and an incandescent light bulb (incandescent referring to thermal radiation in the visible range). The sun is very close to the 5000 K curve, though temperatures of traditional incandescent light bulbs are lower than the lowest shown in Fig. 2.2 (3000 K) and the peak of their emission is in the infrared region^[24].

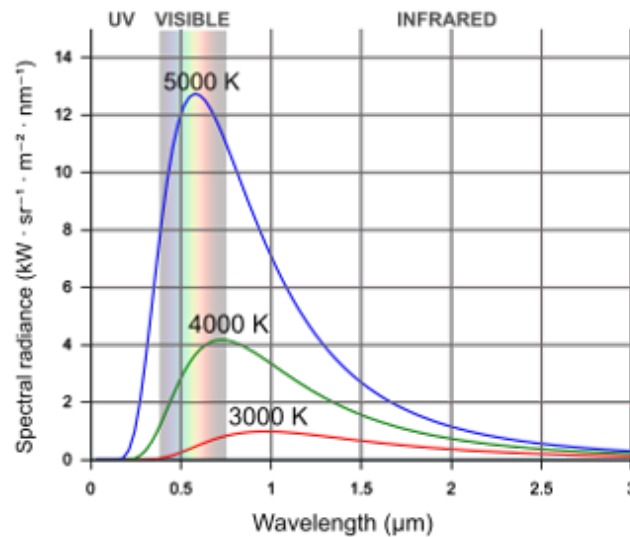


Figure 2.2: Visualisation of Planck's law. This is the spectral radiance plotted against wavelength for various different temperatures, with the visible spectrum shown for reference. Source: <http://www.tikalon.com/blog/2017/Planck.law.png>

2.1.3 Laser Radiation

The term 'laser' is thrown around a lot, and I'm sure most will have at least heard of one. But beyond describing it as 'a very powerful light beam', which strictly speaking is rarely true, I doubt that the actual process is well-known to most. 'Laser' is actually an acronym for 'light amplification by stimulated emission of radiation', and in this section I will give a very brief outline of what this means.

In contrast to thermal radiation which is disordered and broadband, radiation from a laser is much more ordered and at a much smaller wavelength range. In very simple terms, imagine supplying energy to a material which has some defined electronic energy structure such that the majority of the electrons are in the same excited state. Suppose one of those electrons were to transition to a lower energy level, and in the process emits a photon. Through a process called stimulated emission this photon can cause an excited electron to undergo the same transition, emitting another photon which is at the same wavelength, has the same polarisation, and in the same direction of travel as the original photon. Now imagine these photons travelling backwards and forwards through this material and the

process continuing, and very quickly there is an intense lightbeam which is essentially the output of a laser^[26].

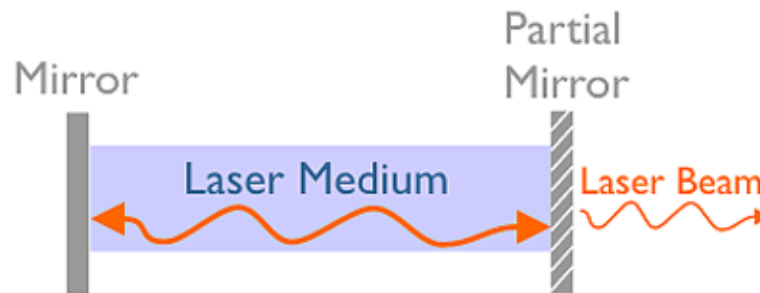


Figure 2.3: Heavily simplified schematic of a laser cavity. Source: <https://www.orc.soton.ac.uk/sites/www.orc.soton.ac.uk/files/laser.gif>

A very simplified schematic of a laser can be seen in Fig. 2.3, but it illustrates the basic physics. The lightbeam traverses the cavity many times in order for the stimulated emission process to continue. One of the cavity mirrors is partially transmissive, allowing some of the beam to be emitted. Lasers can emit beams continuously or in pulses depending on their setup.

2.2 Quantum Description of Light

Now when those who have some physical sciences background ask me what my D.Phil topic is (I tend to draw the line at those who are likely to know what a photon is), my answer is ‘Quantum Optics!’. But for those from the previous category of no physical sciences background who do actually try to pry further, I end up explaining what a photon is and continuing until I can see the look of “your answer is now satisfactory” on their faces. But the comparison between the two answers I give alludes to some fundamental aspects of light, namely how to go from the classical wave picture to the quantum (or ‘particle’/‘corpuscule’) description of light. In fact, the idea that light was constituted of particles, or ‘corpuscule’, first came about from Sir Isaac Newton in 1704^[27]. Though as a result of the double slit experiment nearly 100 years later (discussed in Sec. 2.1.1) the

scientific community changed to view light as comprised of waves.

In order to derive the relation in Eq. (2.5) Planck had to assume that the energy of the light was released in ‘packets’, but even before this the suggestion of a discrete quanta of light (i.e. a photon) came from Einstein in 1905 to describe the photoelectric effect^[28]. In this section I will begin by overviewing how Maxwell’s equations (Eqs. (2.1) to (2.4)) are used to quantise the electromagnetic field, giving rise to the ‘field quanta’ known as photons. I will also introduce some common quantum states of light which will be discussed throughout this thesis, and describe how they are produced.

2.2.1 Field Quantisation

It is common to describe fields in terms of a potential instead of a field strength. A well-known example would be the electric potential $V(\mathbf{r}, t)$, which is related to $\mathbf{E}(\mathbf{r}, t)$ in Eqs. (2.1) to (2.4) by

$$\mathbf{E}(\mathbf{r}, t) = -\nabla V(\mathbf{r}, t). \quad (2.6)$$

To illustrate this as a practical example $V(\mathbf{r}, t)$ is related to voltage in electric circuits^[23].

An analogous quantity to $V(\mathbf{r}, t)$ in electromagnetism is the vector potential $\mathbf{A}(\mathbf{r}, t)$, which is defined as:

$$\mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t). \quad (2.7)$$

In the presence of $\mathbf{A}(\mathbf{r}, t)$, Eq. (2.6) is adjusted to:

$$\mathbf{E}(\mathbf{r}, t) = -\nabla V(\mathbf{r}, t) - \frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t}, \quad (2.8)$$

rendering Eq. (2.6) applicable to electrostatics only^[29]. Then by considering the case where we are in vacuum ($\mathbf{J}(\mathbf{r}, t) = 0$) and using the Coulomb gauge ($\nabla \cdot \mathbf{A}(\mathbf{r}, t) = 0$), we can rewrite Eq. (2.4) as:

$$\nabla^2 \mathbf{A}(\mathbf{r}, t) = \frac{1}{c^2} \frac{\partial^2 \mathbf{A}(\mathbf{r}, t)}{\partial t^2}. \quad (2.9)$$

Noting that Eq. (2.9) is a wave equation, by following the procedure in Sec. 2.1 of Ref. [30] we may write $\mathbf{A}(\mathbf{r}, t)$ as a sum of normal modes:

$$\mathbf{A}(\mathbf{r}, t) = \sum_k \sqrt{\frac{\hbar}{2\omega_k \epsilon_0}} (a_k \mathbf{u}_k(\mathbf{r}) e^{-i\omega_k t} + a_k^* \mathbf{u}_k^*(\mathbf{r}) e^{i\omega_k t}), \quad (2.10)$$

where the prefactor multiplied by $a_k \mathbf{u}_k(\mathbf{r})$ are the Fourier amplitudes at angular frequency ω_k (same for complex conjugates), and the prefactor (where $\hbar = h/2\pi$) is such that a_k (and a_k^*) are dimensionless amplitudes, and $\mathbf{u}_k(\mathbf{r})$ (and $\mathbf{u}_k^*(\mathbf{r})$) is the spatial distribution of mode k . From this an expression for $\mathbf{E}(\mathbf{r}, t)$ can be derived using Eq. (2.8). Note that the above assumes discrete angular frequencies ω_k , for a continuum an integral replaces the sum in Eq. (2.10).

It is at this point we canonically quantise, replacing $\mathbf{A}(\mathbf{r}, t)$ with $\hat{\mathbf{A}}(\mathbf{r}, t)$, a_k with $\hat{a}_k$ and a_k^* with $\hat{a}_k^\dagger$ (where $\dagger$ denotes the Hermitian conjugate of an operator). $\hat{a}_k$ and $\hat{a}_k^\dagger$ are known as ladder operators on mode k , and obey the commutation relations:

$$[\hat{a}_j, \hat{a}_k] = [\hat{a}_j^\dagger, \hat{a}_k^\dagger] = 0 \quad \text{and} \quad [\hat{a}_j, \hat{a}_k^\dagger] = \delta_{jk}. \quad (2.11)$$

Their action on optical states will be seen in Eqs. (2.15) to (2.18) in Sec. 2.2.2. This is standard procedure, but full rigour is given in Refs. [29–31]. This now allows us to write the now-quantised electric field $\hat{\mathbf{E}}(\mathbf{r}, t)$ as:

$$\begin{aligned} \hat{\mathbf{E}}(\mathbf{r}, t) &= \hat{\mathbf{E}}^{(+)}(\mathbf{r}, t) + \hat{\mathbf{E}}^{(-)}(\mathbf{r}, t) \\ &= i \sum_k \sqrt{\frac{\hbar \omega_k}{2\epsilon_0}} \left(\hat{a}_k \mathbf{u}_k(\mathbf{r}) e^{-i\omega_k t} - \hat{a}_k^\dagger \mathbf{u}_k^*(\mathbf{r}) e^{i\omega_k t} \right), \end{aligned} \quad (2.12)$$

where I have split $\hat{\mathbf{E}}(\mathbf{r}, t)$ into the complex parts $\hat{\mathbf{E}}^{(+)}(\mathbf{r}, t)$ and $\hat{\mathbf{E}}^{(-)}(\mathbf{r}, t)$ which oscillate as $e^{-i\omega_k t}$ and $e^{i\omega_k t}$ respectively, a consequence of the Fourier analysis of a complex function. We can also write a similar for the now-quantised magnetic field $\hat{\mathbf{B}}(\mathbf{r}, t)$.

To end the quantisation let's consider the Hamiltonian for the electromagnetic field.

The classical Hamiltonian H is given by:

$$H = \frac{\epsilon_0}{2} \int \left[|\mathbf{E}(\mathbf{r}, t)|^2 + c^2 |\mathbf{B}(\mathbf{r}, t)|^2 \right] d^3\mathbf{r}, \quad (2.13)$$

which is the volume integral over the electromagnetic field energy density^[31]. For the quantised Hamiltonian $\hat{H}$ we need only insert Eq. (2.12) for $\hat{\mathbf{E}}(\mathbf{r}, t)$ and the equivalent for $\hat{\mathbf{B}}(\mathbf{r}, t)$. We obtain:

$$\begin{aligned} \hat{H} &= \sum_k \frac{\hbar\omega_k}{2} \left(\hat{a}_k \hat{a}_k^\dagger + \hat{a}_k^\dagger \hat{a}_k \right) \\ &= \sum_k \hbar\omega_k \left(\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2} \right) \end{aligned} \quad (2.14)$$

where the commutation relations in Eq. (2.11) have been used, and the integral has disappeared because $\mathbf{u}_k(\mathbf{r})$ and $\mathbf{u}_k^*(\mathbf{r})$ are orthonormal. Eq. (2.14) is the energy operator on all of the modes in question, and in the subsequent sections we will see which quantum states this gives rise to.

2.2.2 Fock States

In Dirac (or second quantisation) notation we can write a pure quantum state as $|\psi\rangle$. Restricting ourselves to a single mode (i.e. dropping the subscript k), we write a Fock state or a photon-number state as $|n\rangle$ where n is the number of photons (i.e. quanta) in the single mode. At this point I will define precisely what I mean by the word ‘mode’ as it can be used rather broadly. If two photons are in the same mode then there is nothing to distinguish between them, i.e. they have the same wavelength and same spectral and temporal profile, are at the same point in space, and occupy the same volume.

The ladder operators whose commutation relations were defined in Eq. (2.11) act on

Fock states as follows:

$$\hat{a} |n\rangle = \sqrt{n} |n-1\rangle, \quad (2.15)$$

$$\hat{a} |0\rangle = 0, \quad (2.16)$$

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle, \quad (2.17)$$

$$\hat{a}^\dagger \hat{a} |n\rangle = \hat{n} |n\rangle = n |n\rangle. \quad (2.18)$$

Intuitively, from Eqs. (2.15) to (2.18) $\hat{a}$ is known as the ‘lowering’ or ‘annihilation’ operator, $\hat{a}^\dagger$ as the ‘raising’ or ‘creation’ operator, and $\hat{a}^\dagger \hat{a} = \hat{n}$ as the number operator of which $|n\rangle$ is an eigenstate. Using Eq. (2.17), we can write $|n\rangle$ as:

$$|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}} |0\rangle, \quad (2.19)$$

where $|0\rangle$ is the vacuum state.

As individual photons are added to the mode the energy increases by $\hbar\omega$ each time, therefore $\hbar\omega$ is known as the photon energy. $\hat{H}$ will be discussed in the context of the quantum harmonic oscillator in Sec. 3.1.2. In addition, as $\hat{a}$ and $\hat{a}^\dagger$ are essentially the building blocks of quantising radiation, how these operators transform with other state operators is a very important consideration and will arise multiple times throughout this thesis.

2.2.3 Coherent States

A coherent state $|\alpha\rangle$ is defined as the eigenstate of the annihilation operator, i.e.

$$\hat{a} |\alpha\rangle = \alpha |\alpha\rangle, \quad (2.20)$$

for some complex amplitude α . This state is sometimes referred to as ‘quasi-classical’^[31] or ‘most classical’^[32] as its expectation value with the electric field has the same form as

a classical wave (i.e. the solutions to Eqs. (2.1) to (2.4) in vacuum). This can be seen by considering Eq. (2.12) for a single mode. In practise a coherent state is what describes the output of a laser^[21] described in Sec. 2.1.3.

From the relation in Eq. (2.20) it is possible to write $|\alpha\rangle$ in the photon-number basis:

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \quad (2.21)$$

Two noteworthy characteristics of coherent states are that their average photon number is $\langle\alpha|\hat{n}|\alpha\rangle = |\alpha|^2$, and that their photon-number distribution $P(n|\alpha)$ (i.e. probability of measuring the photon number n) is:

$$P(n|\alpha) = |\langle n|\alpha\rangle|^2 = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!}, \quad (2.22)$$

which is a Poissonian distribution^[32].

Coherent states can also be described by the displacement operator $\hat{D}(\alpha)$ on vacuum, where:

$$\hat{D}(\alpha) = \exp\left(\alpha\hat{a}^\dagger - \alpha^*\hat{a}\right), \quad (2.23)$$

and $\hat{D}(\alpha)$ transforms $\hat{a}$ and $\hat{a}^\dagger$ as:

$$\begin{aligned} \hat{D}^\dagger(\alpha) \hat{a} \hat{D}(\alpha) &= \hat{a} + \alpha \\ \hat{D}^\dagger(\alpha) \hat{a}^\dagger \hat{D}(\alpha) &= \hat{a}^\dagger + \alpha^*. \end{aligned} \quad (2.24)$$

In the next chapter it will become apparent why the word ‘displacement’ is appropriate in this context.

2.2.4 Thermal States

So far we have only discussed pure quantum optical states using Dirac notation, $|\psi\rangle$ being an example of a pure state. A thermal state is the quantum description of a single mode of black body radiation (see Sec. 2.1.2) at wavelength λ (or angular frequency ω). Given

that Eq. (2.5) was derived using statistical mechanics, a thermal state is therefore not a pure state but a mixed state, and is described using the density matrix $\hat{\rho}_{\text{Th}}$. Any pure state can be described by a density matrix, e.g. $\hat{\rho} = |\psi\rangle\langle\psi|$, but a state is mixed if there exists no $|\psi\rangle$ such that the density matrix can be written in that way. A state $\hat{\rho}$ is pure if $\text{Tr}(\hat{\rho}^2) = 1$, but if that quantity is less than 1 then $\hat{\rho}$ is mixed. Additionally, we can write a mixed state $\hat{\rho}$ as a sum of pure states $\hat{\rho}_j = |\psi_j\rangle\langle\psi_j|$ weighted by probabilities p_j .

A single mode thermal state at angular frequency ω is given by:

$$\hat{\rho}_{\text{Th}} = \left(1 - e^{-\frac{\hbar\omega}{k_B T}}\right) \sum_{n=0}^{\infty} e^{-\frac{n\hbar\omega}{k_B T}} |n\rangle\langle n| = \frac{1}{1 + \bar{n}} \sum_{n=0}^{\infty} \left(\frac{\bar{n}}{1 + \bar{n}}\right)^n |n\rangle\langle n|, \quad (2.25)$$

where the average photon number $\bar{n}$ is:

$$\bar{n} = \text{Tr}(\hat{\rho}_{\text{Th}} \hat{n}) = \frac{1}{\exp(\hbar\omega/k_B T) - 1}, \quad (2.26)$$

which also shows the form of an expectation value for a mixed state. The full derivation of $\hat{\rho}_{\text{Th}}$ can be found in Sec. 2.5 of Ref. [32], and is essentially a statistical mechanics analysis using $\hat{H}$ in (2.14) for a single mode. It is also possible to write down the density matrix for a black body radiation field in terms of products of the single mode thermal state [33,34].

2.2.5 Squeezed Vacuum

We have considered coherent states and thermal states in terms of photon numbers, but they were described as quasi-classical and classical respectively. We will now consider physical examples of states in terms of photon numbers which have quantum features.

Non-linear optics is the study of light inside media with non-linear electric susceptibilities, and can be considered in both a classical and quantum picture. The phenomenon of parametric down-conversion (PDC) is when a photon with angular frequency ω_p (the ‘pump’) in a non-linear medium is converted into two daughter photons with angular frequencies ω_s (the ‘signal’) and ω_i (the ‘idler’) due to a second order interaction, where

$\omega_p = \omega_s + \omega_i$ due to energy conservation. A full theoretical model of this for both the classical and quantum case can be found in Sec. 11.2 of Ref.^[12], but I shall proceed by giving the Hamiltonian.

By following the procedure in Sec. 7.2 of Ref.^[32], in the interaction picture the Hamiltonian for PDC $\hat{H}_{\text{PDC}}$ is:

$$\hat{H}_{\text{PDC}} \propto i\hbar\chi^{(2)} \left(\hat{a}_s \hat{a}_i \hat{a}_p^\dagger - \hat{a}_s^\dagger \hat{a}_i^\dagger \hat{a}_p \right), \quad (2.27)$$

where $\chi^{(2)}$ is the second-order electric susceptibility of the non-linear medium, and the subscripts p , s and i denote the pump, signal and idler modes respectively. From here we consider the case where the pump photon comes from a coherent state and assume that its magnitude is large so it can be approximated as classical. In this case the operators $\hat{a}_p^\dagger$ and $\hat{a}_p$ can be approximated as complex numbers. Using this we can use the Schrödinger equation to write the unitary evolution operator in the interaction picture $\hat{U}_I$ as:

$$\hat{U}_I = \exp \left(-\frac{i\hat{H}_{\text{PDC}}t}{\hbar} \right) \approx \exp \left(\Gamma^* t \hat{a}_s \hat{a}_i - \Gamma t \hat{a}_s^\dagger \hat{a}_i^\dagger \right), \quad (2.28)$$

where Γ (and Γ^*) are an accumulation of constants taking care of the proportionality in Eq. (2.27), and t is the time duration of the non-linear interaction (i.e. the time it takes the laser pulse to traverse the non-linear medium).

Single-Mode Squeezed Vacuum

The expressions that are about to come are well-known in quantum optics, but I have mainly used Ref.^[35] to within a sign. Let's first consider the case where both the signal and idler are in the same mode, so we'll drop the subscripts. In this case we call $\hat{U}_I$ the squeezing operator $\hat{S}(\zeta)$:

$$\hat{S}(\zeta) = \exp \left(\frac{1}{2} \left(\zeta^* \hat{a}^2 - \zeta \hat{a}^{\dagger 2} \right) \right), \quad (2.29)$$

where ζ is a complex parameter that depends on various quantities such as the power and phase of the pump laser, the material non-linearity, the length of the material (related to the interaction time of the pump with the material), etc. It is usually experimentally determined, and written as $\zeta = re^{i\phi}$ where r is a positive real number and ϕ is the phase. Note that ϕ will be used as phase in other contexts throughout this thesis. The reason for the term ‘squeezing’ will become apparent in the next chapter.

When $\hat{S}(\zeta)$ acts on vacuum, the result is:

$$\hat{S}(\zeta)|0\rangle = \frac{1}{\sqrt{\cosh(r)}} \sum_{n=0}^{\infty} (-\tanh(r))^n \frac{\sqrt{(2n)!}}{2^n (n!)} e^{in\phi} |2n\rangle, \quad (2.30)$$

which is known as a ‘single-mode squeezed vacuum’ (SMSV), and because of the $\hat{a}^{\dagger 2}$ term only has even photon number contributions. The probability of measuring $2n$ photons is given by:

$$P(2n|r) = \frac{(2n)!}{2^{2n} (n!)^2} \frac{\tanh^{2n}(r)}{\cosh(r)}, \quad (2.31)$$

and the average photon number in an SMSV is $\langle \hat{n} \rangle = \sinh^2(r)$. One final important quantity to consider is how $\hat{a}$ and $\hat{a}^\dagger$ transform with $\hat{S}(\zeta)$:

$$\begin{aligned} \hat{S}^\dagger(\zeta) \hat{a} \hat{S}(\zeta) &= \cosh(r) \hat{a} - \sinh(r) e^{i\phi} \hat{a}^\dagger, \\ \hat{S}^\dagger(\zeta) \hat{a}^\dagger \hat{S}(\zeta) &= \cosh(r) \hat{a}^\dagger - \sinh(r) e^{-i\phi} \hat{a}. \end{aligned} \quad (2.32)$$

Two-Mode Squeezed Vacuum

Now let’s consider when the signal and idler are in different modes. This could be because they are at a different wavelength, have different polarisations, are in different spatial modes, etc. We now call $\hat{U}_I$ the two-mode squeezing operator $\hat{S}_2(\zeta)$:

$$\hat{S}_2(\zeta) = \exp \left(r \left(e^{-i\phi} \hat{a}_s \hat{a}_i - e^{i\phi} \hat{a}_s^\dagger \hat{a}_i^\dagger \right) \right), \quad (2.33)$$

where ζ has been written in terms of r and ϕ . This acts on vacuum as:

$$\hat{S}_2(\zeta) |0\rangle_s |0\rangle_i = \frac{1}{\cosh(r)} \sum_{n=0}^{\infty} (-\tanh(r))^n e^{in\phi} |n\rangle_s |n\rangle_i, \quad (2.34)$$

which is known as ‘two-mode squeezed vacuum’ (TMSV). We can see there is perfect photon-number correlation, i.e. if we measure k photons in the signal mode we collapse the idler mode into $|k\rangle_i$. $\hat{a}_s$, $\hat{a}_s^\dagger$, $\hat{a}_i$ and $\hat{a}_i^\dagger$ transform with $\hat{S}_2(\zeta)$ as:

$$\begin{aligned} \hat{S}_2^\dagger(\zeta) \hat{a}_s \hat{S}_2(\zeta) &= \cosh(r) \hat{a}_s - \sinh(r) e^{i\phi} \hat{a}_i^\dagger, \\ \hat{S}_2^\dagger(\zeta) \hat{a}_s^\dagger \hat{S}_2(\zeta) &= \cosh(r) \hat{a}_s^\dagger - \sinh(r) e^{-i\phi} \hat{a}_i, \\ \hat{S}_2^\dagger(\zeta) \hat{a}_i \hat{S}_2(\zeta) &= \cosh(r) \hat{a}_i - \sinh(r) e^{i\phi} \hat{a}_s^\dagger, \\ \hat{S}_2^\dagger(\zeta) \hat{a}_i^\dagger \hat{S}_2(\zeta) &= \cosh(r) \hat{a}_i^\dagger - \sinh(r) e^{-i\phi} \hat{a}_s. \end{aligned} \quad (2.35)$$

Note that when you trace over either signal or idler mode in Eq. (2.34) (in practise this corresponds to ignoring or ‘throwing away’ one of the modes), you end up with a thermal state given in Eq. (2.25) with $\bar{n} = \sinh^2(r)$.

As a concluding remark, the above is a simplification as it treats the pump, signal and idler as monochromatic. In practise the the signal and idler will have a certain wavelength spectrum, and usually the pump will too, but the above treatment does generalise (see Refs.^[12,35] for more details).

2.3 Linear Optics

In the previous section we discussed how various states of light are generated and described. In this section we shall consider some passive transformations on single and multiple optical modes.

2.3.1 Passive Transformations

We will consider two important passive transformations - phase shifters and beamsplitters. We will also consider loss, which while not being passive can be modeled with a beamsplitter, and is a very important experimental consideration.

Phase Shifters

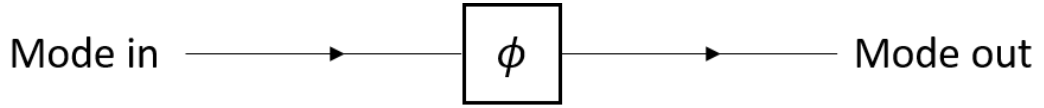


Figure 2.4: Simple schematic of a phase shift of ϕ on a single optical mode.

The phase shift operator acts on each mode individually (see Fig. 2.4). It is given by $e^{i\phi\hat{n}}$ and transforms the annihilation operator as:

$$\hat{a} \mapsto e^{-i\phi\hat{n}}\hat{a}e^{i\phi\hat{n}} = e^{i\phi}\hat{a}. \quad (2.36)$$

In practical terms this would involve changing the density of the material the light is traversing. By considering Eq. (2.12), we can think of phase shifts as changing which point of the wave we are at.

Beamsplitters

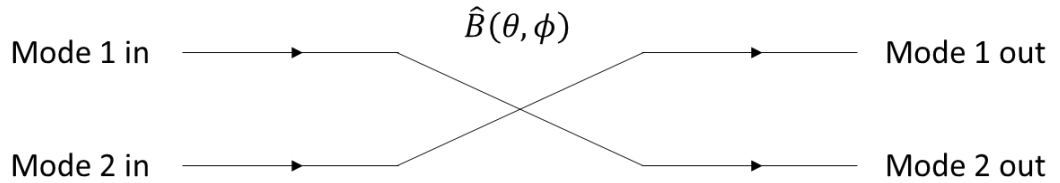


Figure 2.5: Simple schematic of a beamsplitter on modes 1 and 2 with arbitrary splitting ratio and phase shift.

Consider bringing two modes light together at a device which partially reflects and partially

transmits (see Fig. 2.5). We call this device a beamsplitter, and examples include a partially silvered mirror or two waveguides close enough together that the field from one can evanescently couple into the other (and vice versa). To make things more general we now call the modes ‘1’ and ‘2’. In the quantum description the beamsplitter operator $\hat{B}(\theta, \phi)$ which describes the modal transformation is given by:

$$\hat{B}(\theta, \phi) = \exp \left(\frac{\theta}{2} \left(e^{-i\phi} \hat{a}_1^\dagger \hat{a}_2 - e^{i\phi} \hat{a}_1 \hat{a}_2^\dagger \right) \right), \quad (2.37)$$

where θ quantifies the mixing of both modes and ϕ is the relative phase shift imparted. It is such that the input annihilation operators $\hat{a}_1$ and $\hat{a}_2$ transform to the output annihilation operators $\hat{b}_1$ and $\hat{b}_2$ as:

$$\begin{pmatrix} \hat{b}_1 \\ \hat{b}_2 \end{pmatrix} = \begin{pmatrix} \hat{B}^\dagger(\theta, \phi) \hat{a}_1 \hat{B}(\theta, \phi) \\ \hat{B}^\dagger(\theta, \phi) \hat{a}_2 \hat{B}(\theta, \phi) \end{pmatrix} = \begin{pmatrix} \cos(\frac{\theta}{2}) & \sin(\frac{\theta}{2}) e^{-i\phi} \\ -\sin(\frac{\theta}{2}) e^{i\phi} & \cos(\frac{\theta}{2}) \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}, \quad (2.38)$$

where the matrix is a unitary transformation on the optical modes (i.e. total photon number is conserved). The ‘50:50’ or ‘balanced’ beamsplitter case is when $\theta = \frac{\pi}{2}$. The relative phase shift imparted depends on the construction of the beamsplitter^[36], for example a symmetric beamplitter is when $\phi = -\frac{\pi}{2}$ and is typically the case for directional couplers in fibre beamsplitters or on photonic chips^[37].

It is interesting to note that we can go from a TMSV in Eq. (2.34) to a product state of two SMSV in Eq. (2.30) (provided both initial modes are identical apart from location in space) and vice versa with the beamsplitter operator in Eq. (2.37) with $\theta = \frac{\pi}{2}$. We are essentially swapping the quantum resource of entanglement (perfect photon number correlation in Eq. (2.34)) for sub-vacuum variance which we will discuss in Sec. 3.2.2.

Loss

Optical loss can arise in a variety of ways, such as material absorption, reflection or inefficient coupling. Loss on a single mode can be modelled by mixing that mode with

vacuum on a beamsplitter with transmission $T = \cos^2\left(\frac{\theta}{2}\right)$ and reflection $R = \sin^2\left(\frac{\theta}{2}\right)$, where T is the system efficiency (and conversely R is the loss fraction). The reflected mode is then traced over (i.e. ignored), and the transmitted mode now describes the state after loss^[21]. If the initial state was pure it is now obviously mixed.

This treatment in the Fock basis can become rather cumbersome. If we start with n photons, after loss we have a mixture of n (no photons lost), $n - 1$ (1 photon lost), $n - 2$ (2 photons lost), and so on, with the weightings depending on T . This can be formulated by a Bernoulli transformation, we will not be discussing it here as loss will be considered in the Gaussian state formalism in this thesis (see Sec. 3.3.2), but the details can be found in Refs.^[21,38,39].

2.3.2 Hong-Ou-Mandel Interference

In their seminal paper^[40], Hong, Ou and Mandel demonstrated a quantum interference phenomenon with photons which comes about due to their bosonic nature which arises when single photons are inserted into a beamsplitter. Imagine the scenario from Fig. 2.5 where the inputs to modes 1 and 2 are single photons which are otherwise identical. We can describe the input state $|\psi\rangle_{\text{in}}$ as:

$$|\psi\rangle_{\text{in}} = |1\rangle_1 |1\rangle_2 = \hat{a}_1^\dagger \hat{a}_2^\dagger |0\rangle_1 |0\rangle_2. \quad (2.39)$$

The output state $|\psi\rangle_{\text{out}} = \hat{B}(\theta, \phi) |\psi\rangle_{\text{in}}$ is then given by:

$$|\psi\rangle_{\text{out}} = \frac{\sin(\theta)}{\sqrt{2}} e^{-i\phi} |2\rangle_1 |0\rangle_2 - \frac{\sin(\theta)}{\sqrt{2}} e^{i\phi} |0\rangle_1 |2\rangle_2 + \cos(\theta) |1\rangle_1 |1\rangle_2. \quad (2.40)$$

In the case of a balanced beamsplitter ($\theta = \frac{\pi}{2}$), the $|1\rangle_1 |1\rangle_2$ term in Eq. (2.40) vanishes no matter the value of ϕ . This is rather significant as this is a quantum interference effect which completely suppresses the case where one photon emerges from each output mode. When the input photons have some degree of distinguishability other than their location

in space (e.g. in polarisation, wavelength, etc.) the suppression of the $|1\rangle_1 |1\rangle_2$ term is reduced depending on their distinguishability. As a result performing this experiment and measuring the ‘coincidences’ (i.e. the relative instances of detecting an output photon in modes 1 and 2 at the same time) is a useful tool for characterising photon distinguishability for more complex experiments.

2.3.3 Boson Sampling

Previously we considered the case of single photon inputs to two modes of a beamsplitter. Now let’s consider the case of Fock states inserted into an array of beamsplitters and phase shifters on an arbitrary number of modes. On M modes, the input is now $|\psi\rangle_{\text{in}} =$

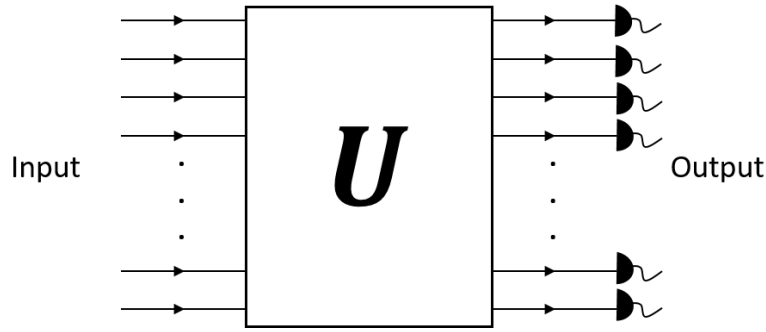


Figure 2.6: Abstract diagram of a Boson Sampler. The Boson Sampler imparts a unitary transformation on the input modes, and the output modes are measured in the photon number basis.

$|n_1\rangle_1 |n_2\rangle_2 \dots |n_M\rangle_M$ where $|n_j\rangle_j$ means n_j photons in mode j (which could be zero), with $\sum_{j=1}^M n_j = N$. Eqs. (2.36) and (2.38) are unitary transformations, therefore in the absence of loss an array of beamsplitters and phase shifters can impart a unitary transformation $\mathbf{U}$ on M modes. The general expression for this unitary transformation is:

$$\hat{\mathbf{b}} = \mathbf{U}\hat{\mathbf{a}} \quad \text{or} \quad \hat{b}_j = \sum_{k=1}^M U_{jk} \hat{a}_k, \quad (2.41)$$

i.e. this is the relation between the annihilation operators of the input modes $\{\hat{a}_k\}$ and output modes $\{\hat{b}_j\}$. It is possible to construct the array of beamsplitters and phase shifters in such a way to impart any arbitrary unitary^[41], and the two main schemes for

doing this are the Reck^[42] and the Clements^[43] decompositions.

As a brief aside, Eq. (2.41) is unitary on the optical modes as $\hat{a}_k$ and $\hat{a}_k^\dagger$ transform separately, i.e. the transformation of any $\hat{a}_j$ does not contain any $\hat{a}_k^\dagger$ terms. This signifies a passive process, one in which the total photon number is conserved, unlike squeezing or loss. Active transformations where $\hat{a}_j$ and $\hat{a}_k^\dagger$ mix or there is an additional constant are known as Bogoliubov transformations^[44], Eqs. (2.24), (2.32) and (2.35) are examples of this.

The calculation process in Eq. (2.40) extends to M modes, but it becomes rather cumbersome to calculate this way. For a given input configuration $|\psi\rangle_{\text{in}} = |\mathbf{X}\rangle$, we can calculate the probability of some output configuration $|\mathbf{Y}\rangle$, $P(\mathbf{Y}|\mathbf{X})$. $\mathbf{X}$ and $\mathbf{Y}$ are vectors of numbers that denote the Fock occupation numbers of the input and output configurations respectively. $P(\mathbf{Y}|\mathbf{X})$ is given by:

$$P(\mathbf{Y}|\mathbf{X}) = \frac{|\text{Per}(\mathbf{U}^{(\mathbf{Y},\mathbf{X})})|^2}{\prod_{j=1}^M (X_j!) \prod_{k=1}^M (Y_k!)}, \quad (2.42)$$

where $\mathbf{U}^{(\mathbf{Y},\mathbf{X})}$ is an $N \times N$ submatrix of $\mathbf{U}$ that maps $\mathbf{X}$ to $\mathbf{Y}$ ^[45–47], and $\text{Per}(\cdots)$ is a matrix permanent^[48]. You can think of a matrix permanent as like a matrix determinant but with all ‘+’ signs rather than alternating ‘+’ and ‘−’.

The reason why this is significant is that the permanent is a computationally difficult task, i.e. it takes a classical computer exponential time to calculate it^[49]. Scott Aaronson and Alex Arkhipov made the connection between the output of a linear optical network and a matrix permanent in their 2013 paper^[45], concluding that it is not possible for a classical computer to efficiently simulate this setup. The argument is that generating samples from the distribution $P(\mathbf{Y}|\mathbf{X})$ is a difficult task for a classical computer, but an experimental setup would do this naturally. Many experiments have attempted multiple variations on this (see for example^[50–54] for a by no means exhaustive selection), however the criteria for the quality of the photons, noise and loss tolerance in order for the setup to not be efficiently simulable by a classical computer is very stringent^[55–58].

Single photon quality is arguably the area in which the most improvement is needed, as generating single photons on demand is currently very difficult^[50–52,59]. Most experimental schemes have relied on the TMSV state in Eq. (2.34), where due to the perfect photon-number correlations one mode can be sent to a single-photon detector (known as heralding) and the other into the interferometer. The results can then be post-processed based on the herald detector outcome. This is known as post-selection, but does not scale favourably in processing time. There are additional challenges of this realisation which we will discuss in Sec. 6.3.2.

2.4 Single-Photon Detection

So far we have mainly discussed generating states of light. In Sec. 2.3.3 we discussed measuring the output distribution of a linear optical network, but at no point did I say how this could be achieved, I simply assumed we could detect Fock states of arbitrary size. In this section we will discuss how measurements in the Fock or photon-number basis are achieved, and discuss the different types of detectors with emphasis on those most relevant to this thesis.

For monochromatic or quasi-monochromatic single-mode light, a photon-number measurement is equivalent to an energy measurement, using for instance a powermeter. However, even the most sensitive conventional powermeters aren't usually sensitive enough to detect single photons. The most frequently used wavelength in this thesis will be 1550 nm which is known as the telecom wavelength due to low loss in optical fibres. A single photon at this wavelength has an energy of 1.28×10^{-19} J which is too small to be detected by a conventional powermeter.

A common type of room temperature single photon detector is the 'Avalanche Photodiode' (APD)^[11,60]. These are made of voltage-biased semiconductors which give a detection event when an absorbed photon excites an electron-hole pair. This creates a current which is amplified by a bias current through an avalanche multiplication process, and it is this

amplified current that gives the detection event^[61]. For 1550 nm light APDs are usually constructed from the semiconductor Indium Gallium Arsenide (InGaAs). InGaAs APDs work at room temperature which makes them easier to operate, but they suffer from high dark counts (i.e. false detection events) of $\approx 1 \text{ ks}^{-1}$, and low efficiency of $\approx 20\%$ ^[62]. APDs for lower wavelengths (i.e. higher energy photons) are made from different materials, and tend to have lower dark counts and higher efficiencies^[11,60].

In order to get high efficiencies and low dark counts at 1550 nm it is necessary to use superconducting detectors at cryogenic temperatures. In this section I shall give a practical overview of the two main ones that are used in this thesis: the ‘Superconducting Nanowire Single-Photon Detector’ (SNSPD), and the ‘Transition Edge Sensor’ (TES). But first, I shall introduce photon-number measurements (i.e. Fock basis projection) in a quantum optical context.

2.4.1 Photon-Number Projection

In quantum mechanics, the act of measurement is described by ‘Positive Operator Valued Measures’ (POVMs). For a state $\hat{\rho}$ the probability of outcome m , $P(m|\hat{\rho})$, is given by:

$$P(m|\hat{\rho}) = \text{Tr}(\hat{\rho}\hat{\pi}_m), \quad (2.43)$$

where $\hat{\pi}_m$ is the POVM for outcome m (for whatever m is!)^[12,63,64] (see Sec. 2.2.6 of Ref.^[63] for a more thorough discussion). The main purpose of this section is to discuss the difference between ‘click’ detectors and ‘number-resolved’ detectors, and their associated POVMs.

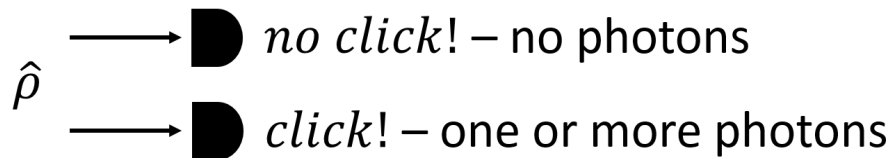


Figure 2.7: Visualisation of a perfect ‘click’ detector on the state $\hat{\rho}$.

A click detector visualisation can be seen in Fig. 2.7. When an optical state $\hat{\rho}$ is impinged on it, it gives a ‘click’ if one or more photons are detected, and ‘no click’ if no photons are detected. The click is usually in the form of an electrical pulse.

For an idealised lossless click detector in the absence of dark counts, in the Fock basis the POVMs are given by:

$$\{ \hat{\pi}_{no\ click}, \hat{\pi}_{click} \} = \left\{ |0\rangle\langle 0|, \sum_{n=1}^{\infty} |n\rangle\langle n| = \mathbf{I} - |0\rangle\langle 0| \right\}, \quad (2.44)$$

where $\mathbf{I}$ is the identity operator. For the full treatment including imperfections see Ref. [12].

An idealised photon-number resolving detector would give a discrete different outcome when detecting different photon numbers from 0 to ∞ photons. Such an idealised detector would have POVMs^[65]:

$$\{ \hat{\pi}_n = |n\rangle\langle n| : n = 0, 1, \dots, \infty \}. \quad (2.45)$$

but sadly it does not exist. In reality, number-resolved detectors truncate at some upper value. If this value is, say, $|10\rangle\langle 10|$, then the POVMs would be^[65]:

$$\{ \hat{\pi}_0, \hat{\pi}_1, \dots, \hat{\pi}_{10}, \hat{\pi}_{\geq 11} \} = \left\{ |0\rangle\langle 0|, |1\rangle\langle 1|, \dots, |10\rangle\langle 10|, \mathbf{I} - \sum_{n=0}^{10} |n\rangle\langle n| \right\}. \quad (2.46)$$

A full treatment of photon-number resolved POVMs in the presence of imperfections can also be seen in Ref. [12].

There is also an intermediate regime called ‘pseudo-number resolution’ where click detectors are multiplexed either spatially^[66] or temporally^[67] to gain at least some photon number information. The spatial case for example involves inserting the state into one mode of a 50:50 beamsplitter, then inserting both outputs into separate 50:50 beamsplitters, and so on (imagine it growing like the roots of a tree) with a click detector at each of the eventual outputs. This approach scales rather badly as the number of click detectors required increases with the degree of number resolution desired, though this is dependent

on error tolerance. For temporal multiplexing, the analogous poor scaling comes in time as it necessitates slower experimental operation rates. The POVMs and limitations of this scheme are discussed in detail in Ref. [65], but one of the main advantages is that generally click detectors are much easier to operate compared to intrinsically number-resolving detectors as we shall see.

2.4.2 Superconducting-Nanowire Single-Photon Detectors

Superconducting-nanowire single photon detectors (SNSPDs) are a type of cryogenic single-photon detector which have been reported to have efficiencies as high as 98% at 1550 nm^[20]. For typical PDC photons they have no (or very little) number resolution and are therefore click detectors. They usually operate at temperatures around a few Kelvin, and sometimes it is sufficient to simply immerse them in liquid Helium at 4.2 K^[68].

Structure

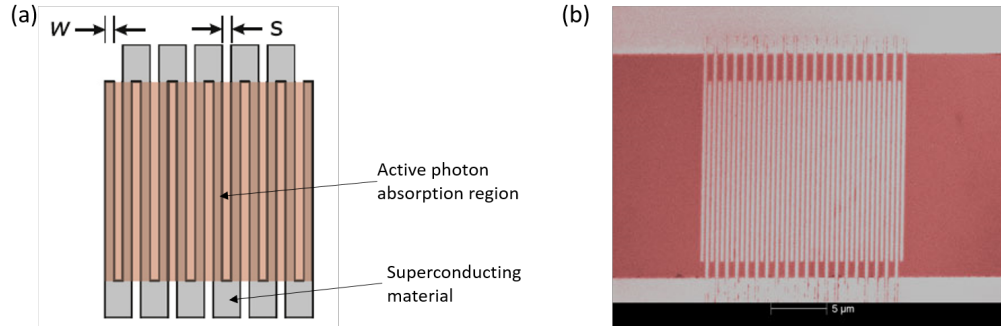


Figure 2.8: (a) Schematic of the layout of the superconducting material in an SNSPD, adapted from Ref. [69]. The electrical connections are made with the two loose ends. w is the width of the SNSPD and s is the distance between each wire. (b) Scanning electron micrograph (SEM) of a Tungsten Silicide (WSi) SNSPD. Source: National Institute of Standards and Technology (NIST).

The structure of an SNSPD can be seen in Fig. 2.8 (a), and an SEM image of one in Fig. 2.8 (b). The first SNSPD nanowires had more of a ‘bow-tie’ structure^[15], but several layouts have been trialed to maximise photon absorption^[69]. Currently SNSPDs are constructed from an ultra-thin (typically ≈ 5 nm thick) superconducting material which is

typically ≈ 80 nm wide (i.e. the ‘nanowire’) arranged in a meander structure deposited on a semiconductor layer^[69,70].

A variety of materials can be used for the superconducting material, there are two broad categories which are amorphous superconductors and poly-crystalline materials^[11]. The most commonly used material from the former group is Tungsten Silicide (WSi)^[71], and from the latter group is Niobium Nitride (NbN)^[15]. The meander structure is placed inside an optical cavity in order to maximise photon absorption efficiency, where the cavity is specifically manufactured for the desired photon wavelength^[72]. Optical coupling is usually done via an optical fibre, where the light is focused onto the meander structure itself (i.e. ‘into the page’ with respect to Fig. 2.8). It is necessary to coil the fibre inside the cryostat to reduce dark counts due to blackbody radiation^[73].

Detection Mechanism

The basic detection mechanism of an SNSPD is that a photon is absorbed at a point along the nanowire which causes a small portion to go from the superconducting state to the normal state^[74]. For operation, a bias current I_b is applied to an SNSPD, therefore a voltage pulse results due to the normal state section of the nanowire. The exact mechanism between photon absorption and voltage pulse was not known exactly for many years, however there is a growing understanding. Experimental verification work in Ref.^[74] tested some models put forward in Ref.^[75] for the different groups of SNSPDs.

For amorphous superconductors (e.g. for WSi, the most relevant for this thesis) the mechanism is as follows: a photon is absorbed in a section of the nanowire which causes Cooper pairs to break. This causes quasiparticles (i.e. the broken Cooper pairs) to form and diffuse away from the photon absorption spot, causing what is known as a ‘hotspot’. This causes a vortex to form, which is a single magnetic flux quantum. The vortex is subject to the Lorentz force due to I_b and moves across the nanowire, where dissipative motion cause the vortex to dissipate energy into the nanowire^[74,75]. The above is a slight simplification, for more sophisticated and recent proposals see Refs.^[76–79].

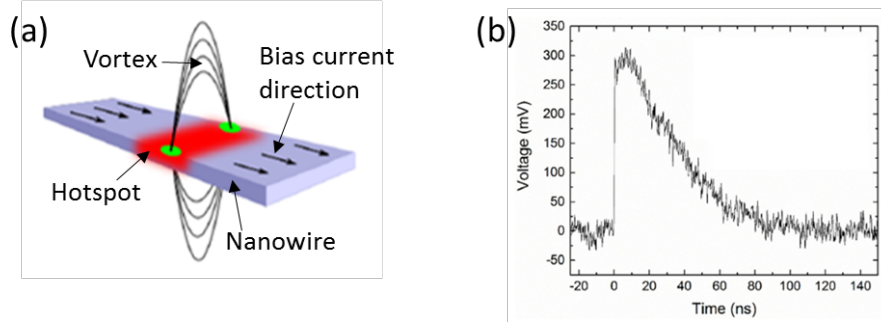


Figure 2.9: (a) Diagram of a hotspot and vortex forming across a section of a nanowire, adapted from Ref. [74]. (b) Image of a typical detection event signal on an oscilloscope, taken from Ref. [80].

A diagram of a hotspot and vortex forming on a nanowire can be seen in Fig. 2.9 (a). If the combination of an incident photon and I_b causes a vortex to form, the nanowire undergoes a transition to the normal state, which gives rise to a voltage pulse V_{det} due to I_b and the kinetic inductance of the wire^[81]. The structure of V_{det} can be seen in Fig. 2.9 (b), where the height of the pulse is proportional to I_b ^[81].

Electronic Operation

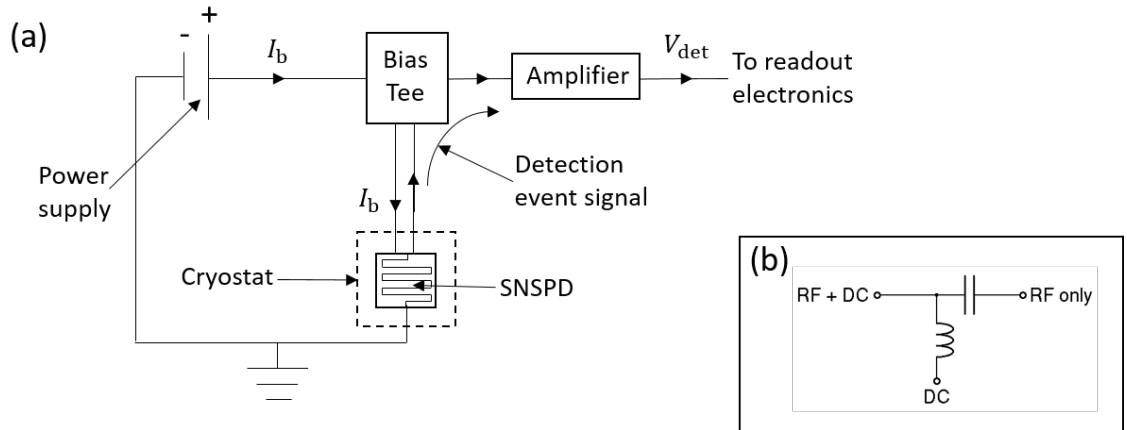


Figure 2.10: (a) Schematic of the electronic circuit for SNSPD operation, including biasing and readout. Fibre-coupling for light is absent in this schematic. (b) Diagram of a bias tee for separating DC and RF (i.e. AC) electrical signals. Source: https://upload.wikimedia.org/wikipedia/commons/6/6c/Bias_t.png

The standard circuit to operate an SNSPD can be seen in Fig. 2.10 (a). Two electrical connections are made, usually one at each end of the nanowire (see Fig. 2.8), and one of

them has to lead to ground. It is necessary to supply I_b to an SNSPD and to have a V_{det} readout connection, therefore a device called a ‘bias tee’ is required^[68], which can be seen in Fig. 2.10 (b). The electrical cable leading to an SNSPD is able to take currents flowing in opposite directions as one of them is direct (DC) and the other is radio frequency (RF). i.e. an alternating current. The function of the bias tee is to separate these two signals. I_b is supplied from the DC region of Fig. 2.10 (b) through the inductor, it is unable to take the capacitor route at the junction due to high impedance, therefore it travels to the SNSPD which is located at the RF + DC region. V_{det} from a SNSPD detection event travels from the RF + DC region, is unable to take the inductor route also due to high impedance, so travels the capacitor route to the RF region where the readout electronics would be located. An oscilloscope should be used to view V_{det} (usually only for setup or diagnostic purposes), or to count detection events a pulse counter should be used. The above electrical circuitry works well at comparatively lower count rates ($\lesssim 10$ MHz), but at higher count rates there are some non-linear effects^[82]. A more advanced circuit at cryogenic temperatures which decouples the SNSPD itself from the room temperature electronics is presented in Ref.^[82].

Although SNSPDs are traditionally click detectors, some parallel operation schemes have been devised to obtain some number resolution in a similar fashion to multiplexing, see for example Ref.^[83]. Also it has been shown in Refs.^[84,85] that there is some photon-number information in the rise time of V_{det} , though the resolution beyond 3 photons is poor and the electrical readout for this is considerably more complex compared to simple pulse counting. Additionally, typical PDC photons have a temporal spread that is much shorter than the SNSPD recovery time (see Secs. 4.4.3 and 6.3), but one could imagine that for photonic states with much longer temporal profiles some photon-number information could be obtained by temporally binning the detector outcomes.

SNSPDs have several characteristics which are important to consider for their operation. These characteristics and how to measure them will be described in detail in Chap. 4

with the supporting raw data.

2.4.3 Transition Edge Sensors

A transition edge sensor (TES) is also a type of superconducting single-photon detector requiring cryogenic temperatures. Though unlike an SNSPD, due to the way they are constructed and operated TESs are intrinsically photon-number resolving. Although detecting photons with TESs pre-dates SNSPDs^[86], their capabilities as photon number-resolving detectors weren't fully investigated until a few years later^[87]. As such they are a very important measurement device in the field of quantum optics^[12], but also in fields such as x-ray spectroscopy, particle and astrophysics physics^[17,18,88].

Structure

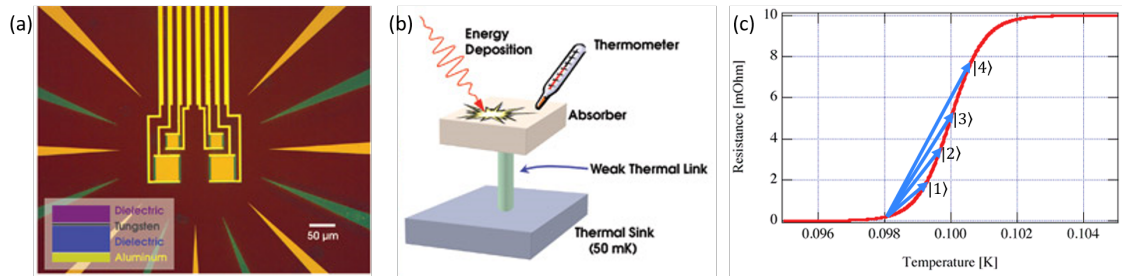


Figure 2.11: (a) Image of four tungsten TES films. Insert: Schematic of the TES deposition layers. (b) Schematic of the thermal process of photon absorption by a TES. Source for (a) and (b): https://www.photonics.com/images/Web/Articles/2008/11/1/Counting_Fig4a.jpg. (c) Plot of the change in resistance of a TES due to different photon number absorption. The axis values are typical for a tungsten TES. Source: Adapted from <https://www.the-athena-x-ray-observatory.eu/images/Nuggets/AN21.gif>.

A TES is different to an SNSPD in that it is effectively a bolometer. In contrast to an SNSPD with its thin nanowire structure, a TES is made from a continuous thin film of material typically measuring $20\text{ }\mu\text{m} \times 20\text{ }\mu\text{m} \times 20\text{ nm}$. An image of these films can be seen in Fig. 2.11 (a), tungsten is the typically used material^[18] though titanium has also been used for detection at optical and NIR wavelengths^[89,90]. Similarly to SNSPDs, TESs are constructed within an optical cavity (see insert of Fig. 2.11 (a)) designed to maximise

absorption at a given wavelength^[91], and through this efficiencies of $> 95\%$ have been achieved^[19,90]. They are also fibre-coupled in the same fashion as SNSPDs^[92].

Basic Principle of Operation

The basic principle of operation is as follows. The film is cooled until it is superconducting, this is typically below 100 mK (see Fig. 2.11 (b)). A TES usually has to be cooled in a dilution refrigerator (DR) or an adiabatic demagnetisation refrigerator (ADR) to achieve the required low temperatures. It is then biased with a current to place it at the bottom of its transition edge, i.e. the point at which the material starts to transition to its normal state (see Fig. 2.11 (c)). At this point very small changes in temperature of the material give rise to large changes in resistance. The film material is picked such that the change in resistance when absorbing single photons of the desired wavelength is measurable. As seen in Fig. 2.11 (c), absorbing different photon-number states gives rise to different changes in resistance, and measuring this resistance gives the photon-number information. This is the reason for the thermometer in Fig. 2.11 (b), as the photon-number information essentially comes from measuring the TES's temperature.

Electronic Readout: Superconducting Quantum Interference Devices

The basic physics behind TES operation sounds rather simple, however the process of measuring a photon number from measuring a resistance is more involved. This is unsurprising given that at, say, 1550 nm we're trying to energy-resolve 1.28×10^{-19} J (0.8 eV). A simplified circuit for TES operation can be seen in Fig. 2.12, and from this we can understand how number-resolution is achieved.

The TES itself is in series with an inductor coil (with inductance L_{TES}) and in parallel with a shunt resistor (with resistance R_{Shunt}). R_{Shunt} is determined such that it is comparable to the TES resistance on its transition edge, therefore around 20 m Ω ^[86]. The TES is held at its transition edge through Joule heating by biasing it with a current I_{TES} . The setup with R_{Shunt} exploits the electrothermal feedback effect. If the temperature

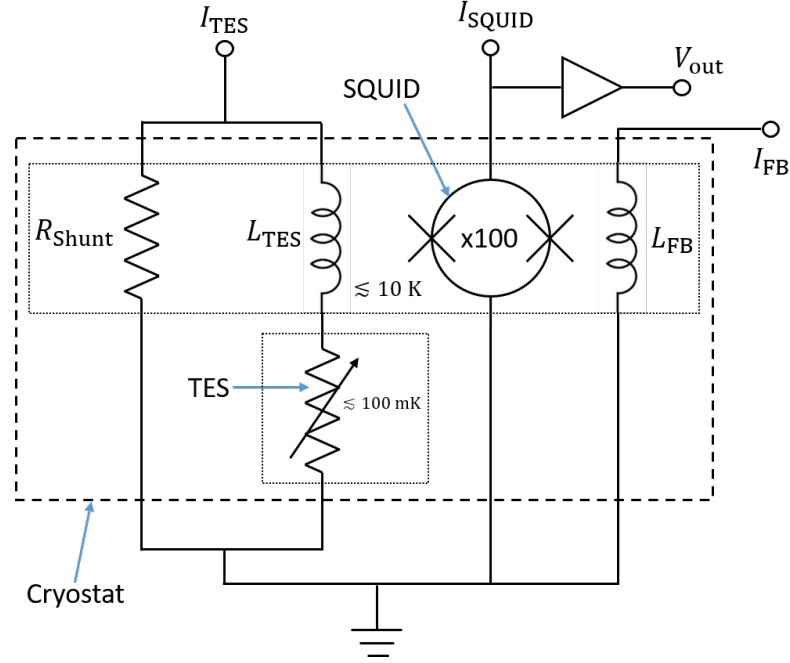


Figure 2.12: Circuit diagram for TES operation. All non-standard electronic components are labelled.

(and subsequently the resistance) of the TES drops, more current flows to the TES, thus maintaining its temperature. A full analysis of this effect in the context of TES operation can be found in Refs. [17,18,93,94].

Although we are using the TES's resistance to infer a photon number, we don't actually read out from the TES itself. When a photon is absorbed the TES heats up leading to a resistance increases, which directs less current through the TES and more through the shunt resistor. When the TES cools back down due to being (weakly) thermally-coupled to the cryostat, the original current flow is restored. The inductor in series with the TES converts this change in current into a magnetic flux, which is how readout occurs.

This magnetic flux is picked up by a superconducting quantum interference device (SQUID), a highly-sensitive magnetic flux to voltage converter^[95]. By looking at the SQUID in Fig. 2.12, we can see it consists of a loop with two Josephson junctions (denoted by $\times$) in parallel. For more information on the operational principles of SQUIDs see Refs. [95–97]. For the experiments in this thesis a SQUID array of 100 individual SQUIDs

was used (hence the $\times 100$ in the SQUID in Fig. 2.12) for greater voltage signal amplification^[98]. For optimal operation the SQUIDS are biased with a current (I_{SQUID}), and are held in a constant magnetic field by a DC current I_{FB} running through a feedback inductor coil (with inductance L_{FB}). The SQUIDS, TES inductor and feedback inductor are also cooled such that they are superconducting, but only require temperatures lower than $\lesssim 10\text{K}$. In our experiments, for each TES those three components were on the same $\sim 5\text{mm} \times 5\text{mm}$ chip, and those chips were affixed and electrically connected to a printed circuit board incorporating the shunt resistors. This is represented by the ' $\lesssim 10\text{K}$ ' box in Fig. 2.12

Obtaining a Photon-Number Distribution

The voltage output from the SQUID array is further amplified and filtered at room temperature. The upper right section of fig 2.12 is a simplification of the actual circuitry, but it illustrates the basic process. The amplifier shown in the figure is in practise usually two operational amplifiers (op-amps) followed by a commercial bandpass filter and amplifier (e.g. a Stanford Research Systems SR560)^[94]. This outputs the voltage used for data processing (V_{out}), examples of the traces viewed on an oscilloscope can be seen in Fig. 2.13 (a).

The final stage is to go from V_{out} (Fig. 2.13 (a)) to a photon-number distribution (Fig. 2.13 (b)). We can see that the V_{out} traces in Fig. 2.13 (a) are grouped together representing the different photon-number outcomes. For extremely low noise systems measuring the pulse heights can be sufficient, however more sophisticated techniques exist. The one used for the TES data in this thesis is matched filter analysis^[99]. This involves first measuring some V_{out} traces and calculating an average trace. Following this the overlap integral between the average trace and each individual trace is taken, giving an arbitrary number known as the 'matched filter value' (MFV). The MFVs are then binned into a histogram, an example of this can be seen in Fig. 2.13 (b). We can see from the histogram that the MFVs are grouped based on photon number, and from the troughs

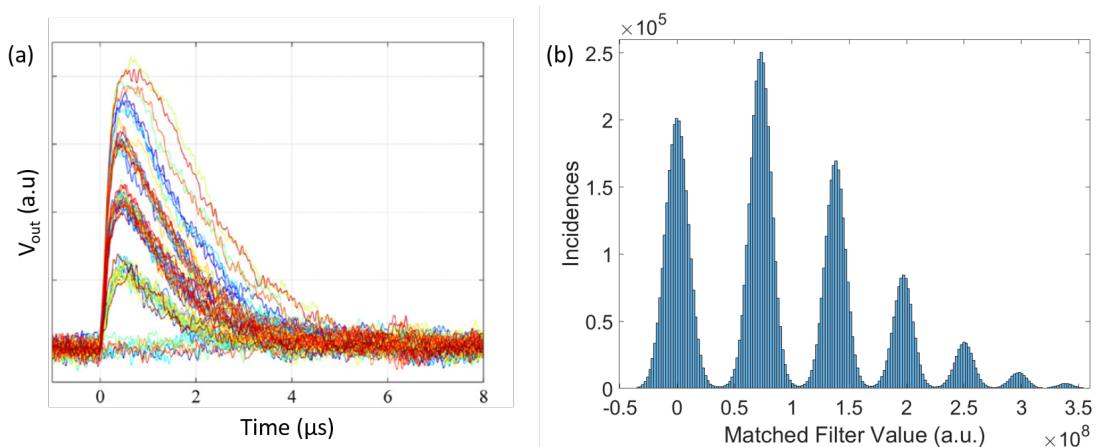


Figure 2.13: (a) Example output voltage (V_{out}) traces from TES detection events. Adapted from Fig. 6.9 of Ref. [12]. (b) Example of a histogram of TES matched filter outputs, taken from a small data sample from Chap. 7.

we can determine the MFV boundaries to estimate the photon number. From these boundaries a photon number is assigned to each MFV. More details on this procedure can be found in Ref. [99], and there are also other techniques to assign photon numbers to V_{out} traces such as principle component analysis [100].

The main draw to using TESs is obviously that they can measure photon numbers rather than just clicks, and also that they can have very high efficiencies [19,90]. However there are some disadvantages of using them. We can see in Fig. 2.13 (b) that the trace duration is $\approx 5 \mu\text{s}$ which limits rates to a maximum of $\approx 200 \text{ kHz}$, which is very slow considering that SNSPDs can be operated as fast as $\approx 1 \text{ GHz}$ [11]. Another disadvantage is that they are much more difficult to operate. To work properly TESs need to be operated below 100 mK , but SNSPDs only require a $\approx 2 \text{ K}$. Not only this, but by comparing Figs. 2.10 and 2.12 we can see that the readout process is far more involved for TESs. As a final note TESs have an upper limit for number resolution. Limits are typically between 11 and 20 photons from histogram binning [101–103], but it is possible to extract more information with more advanced data processing techniques [100,104].

Chapter 3

Gaussian State Formalism

In the previous chapter we started by discussing what exactly is light in the classical sense, then we quantised the electromagnetic field and discussed some optical states in terms of the electromagnetic field quantum - the photon. We saw that some optical states are quite messy when written in terms of photon numbers: coherent states (Eq. (2.21)), thermal states (Eq. (2.25)) and both squeezed vacuums (Eqs. (2.30) and (2.34)) are all infinite superpositions of photon numbers.

We will discover in this chapter what a Gaussian state of light is, how to describe it in the better-suited Gaussian state formalism, and discuss how to implement the linear optical transformations in Sec. 2.3. We will also consider how photon-number projection (see Sec. 2.4) is described in this formalism, and extend the Boson Sampling discussed in Sec. 2.3.3 to Gaussian states. But first we shall discuss the building blocks of this formalism: phase space and quadrature operators.

3.1 Phase Space

In classical mechanics we are used to writing quantities in terms of two canonical variables, usually position ‘ q ’ and momentum ‘ p ’, where we can add subscripts when more than one mode is present. These variables are used extensively in classical Hamiltonian mechanics,

and also in the study of phase portraits of dynamical systems^[105]. It is natural to ask how this extends to quantum mechanics given that we have already come across the quantised electromagnetic Hamiltonian.

3.1.1 Quadrature Operators

In quantum mechanical phase space, we define the quadrature operators $\hat{q}_k$ and $\hat{p}_k$ on mode k as:

$$\hat{q}_k = \sqrt{\frac{\hbar}{2}} (\hat{a}_k^\dagger + \hat{a}_k) \quad \text{and} \quad \hat{p}_k = i\sqrt{\frac{\hbar}{2}} (\hat{a}_k^\dagger - \hat{a}_k), \quad (3.1)$$

or conversely:

$$\hat{a}_k = \frac{\hat{q}_k + i\hat{p}_k}{\sqrt{2\hbar}} \quad \text{and} \quad \hat{a}_k^\dagger = \frac{\hat{q}_k - i\hat{p}_k}{\sqrt{2\hbar}}. \quad (3.2)$$

We can think of $\hat{q}_k$ and $\hat{p}_k$ as two canonical conjugate operators for mode k . The definition in Ref.^[21] is that they are the ‘real’ and ‘imaginary’ parts of $\hat{a}_k$, and that they represent the ‘in-phase’ and ‘out-of-phase’ components of $\hat{\mathbf{E}}(\mathbf{r}, t)$ in Eq. (2.12).

We can also write down the commutation relations similar to Eq. (2.11):

$$[\hat{q}_j, \hat{q}_k] = [\hat{p}_j, \hat{p}_k] = 0 \quad \text{and} \quad [\hat{q}_j, \hat{p}_k] = i\hbar\delta_{jk}, \quad (3.3)$$

which is obtained from using Eqs. (2.11) and (3.1). It is customary to set $\hbar$ to either 1/2, 1, or 2 in Eqs. (3.1) to (3.3). $\hbar$ is important when measuring absolute energy or power, but irrelevant when considering photon numbers at an arbitrary wavelength. With exception of Sec. 3.1.2 we will be setting $\hbar = 1$ throughout this thesis, the main benefit of this is that it’s easy to remember how to switch between Eq. (3.1) and Eq. (3.2) - it’s just $\sqrt{2}$ in the denominator for both. There will be a brief note on the advantages of the two other conventions later on in this chapter.

3.1.2 Hamiltonian Revisited

For completeness, $\hat{H}$ from Eq. (2.14) in terms of $\hat{q}_k$ and $\hat{p}_k$ becomes^[31]:

$$\hat{H} = \sum_k \frac{\omega_k}{2} (\hat{q}_k^2 + \hat{p}_k^2). \quad (3.4)$$

This has a striking similarity to the Hamiltonian of a quantum harmonic oscillator, with $\hat{q}_k$ and $\hat{p}_k$ being the analogies to the ‘position’ and ‘momentum’ operators on mode k . It is for this reason that photons are the excitations the quantum field described by the Hamiltonian in Eq. (3.4). This is discussed in much greater detail in Refs. ^[29,31,106].

3.2 Quasiprobability Distributions

In classical mechanics it is common to analyse the dynamics of a system in terms of its phase space probability distribution, where each possible position and momentum of an object can be represented as a point in phase space^[105]. Similar treatment is possible for systems such as gasses in equilibrium, e.g. through the Boltzmann partition function^[107].

Given that we have just defined phase space in quantum optics it is natural to think about phase space distributions for quantum optical systems. This was first proposed by Eugene Wigner in 1932^[108]. In quantum mechanics it is not possible to represent a system by a single point in phase space due Heisenberg’s uncertainty relation, therefore such a phase space distribution would not satisfy the axioms of a conventional probability distribution^[109]. Such distributions are therefore known as ‘quasiprobability distributions’, and in this section we will consider the different types and their relevance to measurement and state characterisation.

As a note for clarity, sometimes these functions have the quadrature variables q and p , and sometimes the coherent state variables α and α^* , as phase space representation can be seen as describing states in the coherent state basis (see Sec. 3.2.3). They are related

in the same way as Eqs. (3.1) and (3.2):

$$\alpha = \frac{q + ip}{\sqrt{2}} \quad \text{and} \quad \alpha^* = \frac{q - ip}{\sqrt{2}}, \quad (3.5)$$

using the convention $\hbar = 1$. The coming expressions are always true for α and α^* , but in terms of q and p they depend on your definition of $\hbar$. Throughout this thesis the definition in Eq. (3.5) will be used.

3.2.1 Characteristic Functions

It is well-known that a characteristic function is the Fourier transform of a probability distribution. In quantum mechanics we can also have a characteristic function which fully describes a given state. At this point we introduce the s parameter which is used to parameterise quantum mechanical characteristic functions, such that the s -parameterised characteristic function $\chi(\alpha; s)$ of a single-mode state $\hat{\rho}$ is defined as:

$$\begin{aligned} \chi(\alpha; s) &= e^{\frac{s}{2}|\alpha|^2} \text{Tr} \left(\hat{\rho} \hat{D}(\alpha) \right) = e^{\frac{s}{2}|\alpha|^2} \text{Tr} \left(\hat{\rho} e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}} \right) \\ &= e^{\frac{s-1}{2}|\alpha|^2} \text{Tr} \left(\hat{\rho} e^{\alpha \hat{a}^\dagger} e^{-\alpha^* \hat{a}} \right) = e^{\frac{s+1}{2}|\alpha|^2} \text{Tr} \left(\hat{\rho} e^{-\alpha^* \hat{a}} e^{\alpha \hat{a}^\dagger} \right), \end{aligned} \quad (3.6)$$

where $\hat{D}(\alpha)$ is the displacement operator from Eq. (2.23) such that the variables α and α^* are being used to parameterise phase space. In practise, the s -parameter can vary smoothly between -1 and 1 depending on the desired operator ordering^[109,110]. As a brief note on the above, operator ordering is an important consideration in quantum mechanics. For instance, normal ordering is when all terms involving $\hat{a}^\dagger$ are moved to the left of an expression, and all terms involving $\hat{a}$ are moved to the right. This is important for, e.g., Wick's theorem in quantum field theory^[111], and is discussed further in Sec. 3.3.1. Conversely, antinormal ordering is when terms involving $\hat{a}$ are moved to the left and $\hat{a}^\dagger$ to the right.

We will discuss in the rest of this section how $\chi(\alpha; s)$ relates to the various quasiprob-

ability distributions depending on the selected value of s , but before moving on I will state without deriving how we can use it to evaluate some expectation values. For $s = 1$ we find:

$$\left\langle \left(\hat{a}^\dagger \right)^m \left(\hat{a} \right)^n \right\rangle = \frac{\partial^{m+n}}{(\partial \alpha^*)^m (\partial \alpha)^n} \chi(\alpha; 1) \Big|_{\alpha=0}, \quad (3.7)$$

and for $s = -1$ we find:

$$\left\langle \left(\hat{a} \right)^m \left(\hat{a}^\dagger \right)^n \right\rangle = \frac{\partial^{m+n}}{(\partial \alpha)^m (\partial \alpha^*)^n} \chi(\alpha; -1) \Big|_{\alpha=0}, \quad (3.8)$$

which is sometimes a more straight-forward way of calculating these quantities compared to calculating them directly (e.g. evaluating $\text{Tr}(\hat{\rho}(\hat{a}^\dagger)^m(\hat{a})^n)$ etc.) when we know $\chi(\alpha; s)$ for the state $\hat{\rho}$ ^[109]. We will see why the expressions in Eqs. (3.7) and (3.8) are important for state characterisation in Sec. 3.3.1, and we will briefly discuss how they can be used in Chap. 8.

Although probability distribution characteristic functions are common, the appearance of the s -parameter feels almost arbitrary. While this will be briefly discussed further in the next sections, the above is intended only as a very short introduction to present the formalism rather than to provide in-depth mathematical explanations. For this I refer the interested reader to Sec. 4.2.3 of Ref.^[109] and Ref.^[112].

3.2.2 The Wigner Function

The Wigner function $W(q, p)$ was the original quasiprobability distribution for quantum states, and can be thought of as the analogy to a classical phase space distribution^[108,113]. Given that it is not a proper probability distribution it can have negativities, and such negativities can be interpreted as a quantum signature of a state. Wigner functions are also normalised, smooth, and are bounded by $[-1/\pi, 1/\pi]$ ^[21].

$W(q, p)$ is the Fourier transform of $\chi(\alpha; s = 0)$, and for completeness I will state

without deriving that the Wigner function for a state $\hat{\rho}$ is given by^[21]:

$$W_{\hat{\rho}}(q, p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ipy} \left\langle q - \frac{y}{2} \left| \hat{\rho} \right| q + \frac{y}{2} \right\rangle dy. \quad (3.9)$$

This formula looks rather abstract, so let's consider some examples. The formulae for all examples have been taken from Ref.^[21].

Fock States

The Wigner function for a Fock state $|n\rangle$ is given by:

$$W_n(q, p) = \frac{(-1)^n}{\pi} \exp(-q^2 - p^2) L_n(2(q^2 + p^2)), \quad (3.10)$$

where $L_n(\dots)$ is a Laguerre polynomial of order n . The Wigner function for $|1\rangle$, $W_1(q, p)$,

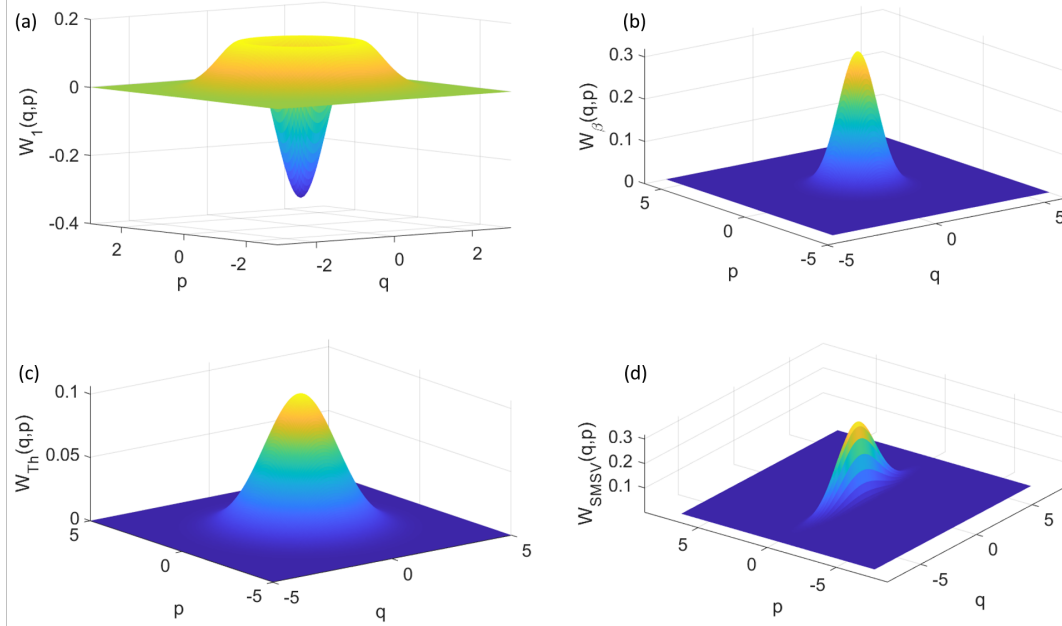


Figure 3.1: Wigner functions for some common states. (a) For the $n = 1$ Fock state $|1\rangle$. (b) A coherent state with $\beta = e^{i\frac{\pi}{4}}$ ($\langle \hat{n} \rangle = 1$). (c) Thermal state with $\bar{n} = 1$. (d) Single-mode squeezed vacuum with $\zeta = \ln(1 + \sqrt{2})$ ($\langle \hat{n} \rangle = 1$).

is shown in Fig. 3.1 (a), where we can see that it has negative parts. From Eq. (3.10) we can see that for vacuum $|0\rangle$ it's just a 2D Gaussian distribution, as $L_0(\dots) = 1$ for any

argument.

Coherent States

Let us denote the coherent state with a complex amplitude β to avoid confusion with the α and α^* variables defined in Eq. (3.5). Therefore for $|\beta\rangle$ the Wigner function is:

$$W_\beta(q, p) = \frac{1}{\pi} \exp\left(- (q - q_0)^2 - (p - p_0)^2\right), \quad (3.11)$$

where $q_0 = \sqrt{2}\text{Re}(\beta)$ and $p_0 = \sqrt{2}\text{Im}(\beta)$. With reference to Eq. (3.10) with $n = 0$, we can see that $|\beta\rangle$ is just a vacuum state that has been displaced from the origin in phase space by absolute distance $\sqrt{2}|\beta|$. Now we can see why we refer to $\hat{D}(\beta)$ in Eq. (2.23) as the ‘displacement operator’. We can see an example for $\beta = e^{i\frac{\pi}{4}}$ in Fig. 3.1 (b), though the displacement from the origin isn’t easy to see so this can serve as the $|0\rangle$ example as well.

It is here we see the logic behind the $\hbar = 1/2$ convention, as in this case the absolute displacement in q, p -space is $|\beta|$ which feels fitting. When writing the Wigner function in terms of the α and α^* variables the absolute displacement is always $|\beta|$.

Thermal States

Thermal states are the only mixed states we shall consider. The Wigner function of $\hat{\rho}_{\text{Th}}$ is:

$$W_{\text{Th}}(q, p) = \frac{1}{\pi(2\bar{n} + 1)} \exp\left(-\frac{q^2 + p^2}{2\bar{n} + 1}\right). \quad (3.12)$$

It is rotationally symmetric in phase space due to only being dependent on $q^2 + p^2$, and is therefore phase insensitive. It can be seen in Fig. 3.1 (c) for $\bar{n} = 1$, it’s a Gaussian broader than that of vacuum, and gets broader as $\bar{n}$ increases.

Single-Mode Squeezed Vacuum

The final state we shall consider is single-mode squeezed vacuum (SMSV). For $\phi = 0$ (i.e. $\zeta = r$ in Eqs. (2.29) and (2.30)) the Wigner function is:

$$W_{\text{SMSV}}(q, p) = \frac{1}{\pi} \exp(-e^{2r} q^2 - e^{-2r} p^2). \quad (3.13)$$

The expression for a general ϕ is not as elegant and if anything gives less insight, but just corresponds to a rotation in phase space. With reference to Fig. 3.1 (d), we can see that the variance (or spread) along the p -axis has been ‘squeezed’ inwards, and along the q -axis has been ‘antisqueezed’ outwards with respect to the Wigner function of vacuum. This is the reason for the term ‘squeezed’. The product of the quadrature variances still satisfies the uncertainty principle (we have $\Delta\hat{q}\Delta\hat{p} = 1/2$), yet the squeezed quadrature has a variance less than that of vacuum ($\Delta\hat{p} = e^{-r}/\sqrt{2} < 1/\sqrt{2}$ in this case), which is considered a non-classical feature^[109].

3.2.3 Other Quasiprobability Distributions

We have discussed the quasiprobability of the Fourier transform of $\chi(\alpha; s = 0)$, but let’s briefly consider the distributions arising for other values of s .

The P Function

The distribution for $s = 1$ is known as the P function, which was independently discussed by both Sudarshan^[114] and Glauber^[115]. This is the distribution function that allows the state $\hat{\rho}$ to be written in the coherent basis, i.e.:

$$\hat{\rho} = \int P(\alpha) |\alpha\rangle \langle \alpha| d^2\alpha, \quad (3.14)$$

where $d^2\alpha = d(\text{Re}(\alpha)) d(\text{Im}(\alpha)) = d\alpha d\alpha^*/2$ ^[109]. $P(\alpha)$ cannot usually be interpreted as a classical probability distribution, for some states it can be negative and/or singular

which are considered non-classical features^[109]. There are some regularisation procedures to determine non-classicality of P functions under realistic experimental conditions^[116].

The Q Function

The Husimi Q function is the case for $s = -1$ and is defined as:

$$Q(\alpha) = \frac{1}{\pi} \langle \alpha | \hat{\rho} | \alpha \rangle, \quad (3.15)$$

it is bounded by $[0, 1/\pi]$, and is always regular^[117]. They are a useful tool for experimentally determining detector POVMs (see Sec. 2.4.1) as coherent states are readily available^[64,65,118].

Miscellaneous

W , P and Q are related to one another by convolutions. W and Q can be obtained by convolving P with a Gaussian, and Q from convolving W with a Gaussian. P can therefore be found by deconvolving W or Q with a Gaussian^[110].

When you take the Fourier transform of $\chi(\alpha; s)$ from Eq. (3.6) but leave s as a parameter rather than setting it to -1 , 0 or 1 , you arrive at something called the s -parameterised P function: $P(\alpha; s)$ ^[109,119]. From this it is possible to calculate the overlaps between two states $\hat{\rho}_1$ and $\hat{\rho}_2$, for the single-mode case:

$$\text{Tr}(\hat{\rho}_1 \hat{\rho}_2) = \pi \int P_{\hat{\rho}_1}(\alpha; s) P_{\hat{\rho}_2}(\alpha; -s) d^2\alpha, \quad (3.16)$$

where one of $\hat{\rho}_1$ or $\hat{\rho}_2$ could be a POVM quasiprobability function^[109,112]. When $s = 0$ Eq. (3.16) reduces to the integral over two Wigner functions.

Ref.^[120] states that (in the general multimode case) if there is a value of s such that $P_{\hat{\rho}_1}(\alpha; s)$ and $P_{\hat{\rho}_2}(\alpha; -s)$ are both always positive, then it is efficient to simulate samples from the probability distribution corresponding to $\text{Tr}(\hat{\rho}_1 \hat{\rho}_2)$ on a classical computer. This

is directly related to the Boson Sampling problem discussed in Sec. 2.3.3, and to the Gaussian case that we will meet in Sec. 3.5.

The study and use of quasiprobability distributions in quantum optics is a huge field of which I have barely scratched the surface in this section. I have focused mainly on single-mode cases for simplicity, and concentrated only on the aspects directly applicable to future chapters. For further details and multimode generalisations I will refer the interested reader to Sec. 4 of Ref. [109].

3.3 The Gaussian State Formalism

This chapter is titled ‘Gaussian State Formalism’, yet the word ‘Gaussian’ has hardly appeared! If we look back to the Wigner functions in Sec. 3.2.2, we see that the ones for coherent states, thermal states and SMSV are all 2-D Gaussians in q and p . We therefore define a Gaussian state as a state with a Gaussian Wigner function, or conversely a state which can be fully described by a mean and a variance [106]. In this section we shall discuss a very useful formalism to describe Gaussian states, and discover how linear optical transformations (Sec. 2.3) and measurements (Sec. 2.4) are described in this formalism. From now on we shall be using multimode notation, and most of the expressions have been adapted from Refs. [2,3].

3.3.1 Displacement Vectors and Covariance Matrices

In this section we shall see how for Gaussian states we can replace the inconvenient Fock basis representation with its sums to infinity (e.g. Eqs. (2.21), (2.25), (2.30) and (2.34)) with a more compact formalism. We introduce the displacement vector $\mathbf{d}$ and covariance matrix $\mathbf{Q}$, which respectively quantify the mean and variance of a state. For M modes let us define the quadrature operator vectors as $\hat{\mathbf{q}} = (\hat{q}_1, \hat{q}_2, \dots, \hat{q}_M)^T$ and

$\hat{\mathbf{p}} = (\hat{p}_1, \hat{p}_2, \dots, \hat{p}_M)^T$. Let us also define the general quadrature operator $\hat{\mathbf{d}}$ as:

$$\hat{\mathbf{d}} = (\hat{\mathbf{q}}, \hat{\mathbf{p}})^T. \quad (3.17)$$

$\mathbf{d}$ and $\mathbf{Q}$ for a state $\hat{\rho}$ are then defined as:

$$\mathbf{d} = \langle \hat{\mathbf{d}} \rangle, \quad (3.18)$$

and

$$Q_{jk} = \frac{1}{2} \langle \hat{d}_j \hat{d}_k + \hat{d}_k \hat{d}_j \rangle - \langle \hat{d}_j \rangle \langle \hat{d}_k \rangle, \quad (3.19)$$

where $\langle \dots \rangle = \text{Tr}(\hat{\rho} \dots)^{[106]}$. There exists different definitions, for instance $\hat{\mathbf{d}} = (\hat{\mathbf{p}}, \hat{\mathbf{q}})^T$ or $\hat{\mathbf{d}} = (\hat{q}_1, \hat{p}_1, \hat{q}_2, \hat{p}_2, \dots, \hat{q}_M, \hat{p}_M)^T$ (i.e. unitary transformations of Eq. (3.17)). We notice that for an M -mode state $\mathbf{d}$ is a $2M$ -dimensional vector and $\mathbf{Q}$ is a $2M \times 2M$ matrix. Also note that expressions for $\mathbf{d}$ and $\mathbf{Q}$ for a given state depend on what you set $\hbar$ to (see Eqs. (3.1) and (3.2)) and your definition of $\hat{\mathbf{d}}$. In this thesis we will always use $\hbar = 1$ and the definition of $\mathbf{d}$ in Eq. (3.17).

With quadrature variables $\mathbf{q} = (q_1, q_2, \dots, q_M)^T$, $\mathbf{p} = (p_1, p_2, \dots, p_M)^T$, and $\boldsymbol{\delta} = (\mathbf{q}, \mathbf{p})^T$, the Wigner function of a Gaussian state is given by:

$$W(\mathbf{q}, \mathbf{p}) = \frac{1}{(2\pi)^M \sqrt{\det(\mathbf{Q})}} \exp\left(-\frac{1}{2}(\boldsymbol{\delta} - \mathbf{d})^T \mathbf{Q}^{-1}(\boldsymbol{\delta} - \mathbf{d})\right), \quad (3.20)$$

from which Eqs. (3.11) to (3.13) can be derived^[21,106]. One final note is that we can find the purity of a state $\hat{\rho}$ (see Sec. 2.2.4) from $\mathbf{Q}$:

$$\text{Tr}(\hat{\rho}^2) = \frac{1}{2^M \sqrt{\det(\mathbf{Q})}}, \quad (3.21)$$

therefore this notation is valid for both pure and mixed Gaussian states^[121]. The above is one of the reasons that I find the Gaussian state formalism so elegant, but also because it is neater in the respect that we can do away with the sums to infinity in Eqs. (2.21),

(2.25), (2.30) and (2.34) which we shall see. Of course the major drawback is that it is only valid for states which are Gaussian.

3.3.2 States and Operators

In this section we will first consider how $\mathbf{d}$ and $\mathbf{Q}$ transform with a general operator. We will then discuss the three components of a Gaussian state: displacement, adding (thermal) noise, and squeezing, and how they appear in this formalism. We will also consider transformations in addition to the above that retain Gaussianity, namely unitary transformation of optical modes and loss. The main aim is to describe the relevant states in Sec. 2.2 in the Gaussian state formalism.

General Operator Transformations

For any number of modes, when the initial state $\hat{\rho}$ is acted on by operator $\hat{O}$, the transformed state $\hat{\rho}'$ is:

$$\hat{\rho}' = \hat{O}\hat{\rho}\hat{O}^\dagger. \quad (3.22)$$

The transformed displacement vector, $\mathbf{d}'$, is then given by:

$$\mathbf{d}' = \text{Tr} \left(\hat{O}\hat{\rho}\hat{O}^\dagger \hat{\mathbf{d}} \right) = \text{Tr} \left(\hat{\rho}\hat{O}^\dagger \hat{\mathbf{d}} \hat{O} \right), \quad (3.23)$$

using the cyclical properties of the matrix trace. $\mathbf{Q}$ transforms to $\mathbf{Q}'$ in a similar fashion. For the most general transformation that is relevant for our consideration, the quantity $\hat{O}^\dagger \hat{\mathbf{d}} \hat{O}$ in Eq. (3.23) can be written as:

$$\hat{O}^\dagger \hat{\mathbf{d}} \hat{O} = \mathbf{O} \hat{\mathbf{d}} + \mathbf{D}, \quad (3.24)$$

where $\mathbf{O}$ is a $2M \times 2M$ symplectic^[3] real matrix, and $\mathbf{D}$ is a real $2M$ -component vector. This therefore shows us that $\mathbf{d}$ and $\mathbf{Q}$ transform as:

$$\mathbf{d}' = \mathbf{O}\mathbf{d} + \mathbf{D} \quad \text{and} \quad \mathbf{Q}' = \mathbf{O}\mathbf{Q}\mathbf{O}^T. \quad (3.25)$$

$\mathbf{O}$ and $\mathbf{D}$ depend on the operator $\hat{O}$, therefore different symbols for both will be used depending on what $\hat{O}$ is. If $\mathbf{O}$ is a unitary matrix and $\mathbf{D} = \mathbf{0}$, then the transformation is unitary on the optical modes. But if $\mathbf{O}$ is not unitary or $\mathbf{D} \neq \mathbf{0}$ then we have a Bogoliubov transformation^[44].

Vacuum

In the photon number basis the vacuum is quite tidily represented by $|0\rangle$. In the Gaussian state formalism, $\mathbf{d}_{vac}$ and $\mathbf{Q}_{vac}$ are given by:

$$\mathbf{d}_{vac} = \mathbf{0} \quad \text{and} \quad \mathbf{Q}_{vac} = \frac{1}{2}\mathbf{I}, \quad (3.26)$$

i.e. zero displacement and $\mathbf{Q}_{vac}$ is the identity divided by 2. In Sec. 3.1.1 we discussed the different conventions for $\hbar$ and why my preference is setting $\hbar = 1$, but here we see the benefit of the $\hbar = 2$ convention: $\mathbf{Q}_{vac} = \mathbf{I}$ in this case which is rather elegant.

Displacement Operator and Coherent States

Consider an M -mode displacement operator $\hat{D}(\boldsymbol{\alpha})$ with $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_M)^T$ (i.e. mode k has amplitude α_k). From this we find:

$$\mathbf{O}_{\boldsymbol{\alpha}} = \mathbf{I} \quad \text{and} \quad \mathbf{D}_{\boldsymbol{\alpha}} = \sqrt{2} \begin{pmatrix} \text{Re}(\boldsymbol{\alpha}) \\ \text{Im}(\boldsymbol{\alpha}) \end{pmatrix}, \quad (3.27)$$

which means that when displacing, $\mathbf{D}_{\boldsymbol{\alpha}}$ is added to $\mathbf{d}$, and $\mathbf{Q}$ remains unchanged. The resulting multimode coherent state from Sec. 2.2.3 therefore has $\mathbf{d}_{\boldsymbol{\alpha}} = \mathbf{D}_{\boldsymbol{\alpha}}$ and $\mathbf{Q}_{\boldsymbol{\alpha}} = \mathbf{Q}_{vac}$. In

a similar sense to Sec. 3.2.2, the representation of the coherent state in the GSF shows the logic behind the $\hbar = 1/2$ convention. In this case we find $\mathbf{D}_\alpha = \begin{pmatrix} \text{Re}(\alpha), & \text{Im}(\alpha) \end{pmatrix}^T$, which is quite easy to remember.

Thermal Noise and Thermal States

Adding thermal (i.e. Gaussian) noise does not affect $\mathbf{d}$, but transforms $\mathbf{Q}$ as:

$$\mathbf{Q}' = \mathbf{Q} + \mathbf{Q}_{noise}, \quad (3.28)$$

where:

$$\mathbf{Q}_{noise} = \text{diag} \begin{pmatrix} \bar{n}_1, & \bar{n}_2, & \dots, & \bar{n}_M \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (3.29)$$

i.e. both quadratures are affected in the same way, and $\bar{n}_k$ is the average photon number due to noise in mode k ^[122,123]. This means that if we add thermal noise in M modes to vacuum, we create the thermal state in Eq. (2.25) from Sec. 2.2.4, which has:

$$\mathbf{Q}_{Th} = \frac{1}{2} \text{diag} \begin{pmatrix} 1 + 2\bar{n}_1, & 1 + 2\bar{n}_2, & \dots, & 1 + 2\bar{n}_M \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (3.30)$$

Single-Mode Squeezing

I shall write this down for the single-mode case, then generalise to squeezing one mode of an M -mode system. With reference to Eqs. (2.29) and (2.32) in Sec. 2.2.5, we can write:

$$\hat{S}^\dagger(\zeta) \begin{pmatrix} \hat{q} \\ \hat{p} \end{pmatrix} \hat{S}(\zeta) = \mathbf{S} \begin{pmatrix} \hat{q} \\ \hat{p} \end{pmatrix}, \quad (3.31)$$

with:

$$\mathbf{S} = \begin{pmatrix} \cosh(r) - \sinh(r) \cos(\phi) & -\sinh(r) \sin(\phi) \\ -\sinh(r) \sin(\phi) & \cosh(r) + \sinh(r) \cos(\phi) \end{pmatrix}. \quad (3.32)$$

If $\phi = 0$, Eq. (3.32) reduces to $\mathbf{S} = \text{diag} \begin{pmatrix} e^{-r} & e^r \end{pmatrix}$. When squeezing only mode k of an M -mode system, $\mathbf{S}$ becomes:

$$\begin{aligned} (\mathbf{S})_{kk} &= \cosh(r) - \sinh(r) \cos(\phi) \\ (\mathbf{S})_{k(k+M)} &= (\mathbf{S})_{(k+M)k} = -\sinh(r) \sin(\phi) \\ (\mathbf{S})_{(k+M)(k+M)} &= \cosh(r) + \sinh(r) \cos(\phi) \\ (\mathbf{S})_{\mu\nu} &= \delta_{\mu\nu} \text{ for } \mu, \nu \neq k \text{ or } k+M, \end{aligned} \quad (3.33)$$

where k runs from 1 to M and μ, ν run from 1 to $2M$. The final result for the single-mode squeezed vacuum from Eq. (2.30) in Sec. 2.2.5 is:

$$\mathbf{Q}_{\text{SMSV}} = \frac{1}{2} \begin{pmatrix} \cosh(2r) - \sinh(2r) \cos(\phi) & -\sinh(2r) \sin(\phi) \\ -\sinh(2r) \sin(\phi) & \cosh(2r) + \sinh(2r) \cos(\phi) \end{pmatrix}, \quad (3.34)$$

which we notice is just Eq. (3.32) with $r \mapsto 2r$ and a factor of $1/2$ in front.

Two-Mode Squeezing

In a similar fashion to the single-mode case, by using the operator $\hat{S}_2(\zeta)$ in Eq. (2.33), we can write:

$$\hat{S}_2^\dagger(\zeta) \begin{pmatrix} \hat{q}_s \\ \hat{q}_i \\ \hat{p}_s \\ \hat{p}_i \end{pmatrix} \hat{S}_2(\zeta) = \mathbf{S}_2 \begin{pmatrix} \hat{q}_s \\ \hat{q}_i \\ \hat{p}_s \\ \hat{p}_i \end{pmatrix}, \quad (3.35)$$

and with reference to Eq. (2.35) we find:

$$\mathbf{S}_2 = \begin{pmatrix} \cosh(r) & -\sinh(r) \cos(\phi) & 0 & -\sinh(r) \sin(\phi) \\ -\sinh(r) \cos(\phi) & \cosh(r) & -\sinh(r) \sin(\phi) & 0 \\ 0 & -\sinh(r) \sin(\phi) & \cosh(r) & \sinh(r) \cos(\phi) \\ -\sinh(r) \sin(\phi) & 0 & \sinh(r) \cos(\phi) & \cosh(r) \end{pmatrix}. \quad (3.36)$$

Eq. (3.36) generalises to squeezing two out of M modes in the same way as the single mode case in Eq. (3.33), I will not write it down as it will not give any further insight. Finally the covariance matrix for the two-mode squeezed vacuum in Eq. (2.34) is simply:

$$\mathbf{Q}_{\text{TMSV}} = \frac{1}{2} \mathbf{S}_2 \mathbf{S}_2^T, \quad (3.37)$$

and I will also not write the full form as it is just as cumbersome as Eq. (3.36), but similar to Eq. (3.34) $\mathbf{Q}_{\text{TMSV}}$ is just $\mathbf{S}_2$ with $r \mapsto 2r$ and a factor of $1/2$ in front.

Phase Rotation

The phase rotation in Sec. 2.3.1 can now be described by the matrix $\mathbf{R}(\phi)$ by using Eq. (2.36) and its conjugate. On a single mode it is given by:

$$\mathbf{R}(\phi) = \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix}. \quad (3.38)$$

By considering the Wigner functions in Fig. 3.1, we can see that performing $\mathbf{R}(\phi)$ corresponds to a rotation of angle ϕ in phase space. Note that ϕ in $\hat{S}(\zeta)$ from Eq. (2.29) is half whatever the corresponding phase rotation is, as rotating the SMSV in Fig. 3.1 (d) by 180° gets you back where you started. Mathematically we can think of this as applying Eq. (2.36) to $\hat{a}$ and $\hat{a}^\dagger$ in Eq. (2.29), as the squaring accounts for the doubling of the rotation angle.

General Unitary Transformations on Optical Modes

We can generalise the unitary transformation on the annihilation operators in Eq. (2.41) from Sec. 2.3.3 to a transformation on $\mathbf{d}$ and $\mathbf{Q}$. For a unitary $\mathbf{U}$, the transformation operator $\mathbf{O}$ from Eqs. (3.24) and (3.25) is given by:

$$\mathbf{O} = \begin{pmatrix} \text{Re}(\mathbf{U}) & -\text{Im}(\mathbf{U}) \\ \text{Im}(\mathbf{U}) & \text{Re}(\mathbf{U}) \end{pmatrix}. \quad (3.39)$$

$\mathbf{R}(\phi)$ in Eq. (3.38) is constructed this way from Eq. (2.36), and we can also construct a beamsplitter operator in the same way by using Eq. (2.38).

Loss

We briefly discussed in Sec. 2.3.1 that dealing with loss in the Fock basis can become a notational nightmare. Here, we shall see that in the Gaussian state formalism it is much neater to do. Consider an M -mode system with total efficiency η_k in mode k . We construct the efficiency matrix $\mathbf{T}$ as:

$$\mathbf{T} = \text{diag} \left(\sqrt{\eta_1}, \sqrt{\eta_2}, \dots, \sqrt{\eta_M} \right) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (3.40)$$

and in the presence of loss $\mathbf{d}$ and $\mathbf{Q}$ are affected as:

$$\mathbf{d} \mapsto \mathbf{T}\mathbf{d} \quad \text{and} \quad \mathbf{Q} \mapsto \mathbf{T}\mathbf{Q}\mathbf{T} + \frac{1}{2}(\mathbf{I} - \mathbf{T}^2). \quad (3.41)$$

We can visualise this in phase space (with reference to Fig. 3.1) as any displacement being pushed back toward the origin (i.e. no displacement), and $\mathbf{Q}$ getting to look more like vacuum (i.e. $\mathbf{I}/2$).

Tracing Over Subsystems

In Sec. 2.2.5 we discussed that if we trace over one mode of a TMSV then we are left with a single-mode thermal state. While for that particular example the tracing-over procedure is straight forward, in general it can get complicated in the Fock basis as it involves a sum to infinity per unwanted mode.

In the Gaussian state formalism however it is remarkably easy. You simply ignore all entries in $\mathbf{d}$ and $\mathbf{Q}$ that contain indices of the modes you don't want. For instance, if you have an M -mode system and you want to trace over N modes ($M > N$), by throwing away all entries with the indices of the unwanted modes you will naturally end up with a $2(M - N)$ component $\mathbf{d}$ and a $2(M - N) \times 2(M - N)$ $\mathbf{Q}$.

Non-Gaussian Processes

There are some processes which transform initially Gaussian states into non-Gaussian ones. These processes can be active (e.g. photon addition or subtraction, see Refs. [124,125] for example), but the main one I will focus on here is the passive process of phase diffusion. Both displacement and squeezing are operations with a well-defined phase, therefore it is unsurprising that phase-diffusion leads to loss of Gaussianity.

Often in experiments it is the relative phase between two states that is important, and this can drift (i.e. diffuse) when two light sources are not phase-locked to each other. Also, a relative or absolute phase can drift due to path length variation on the order of a wavelength, which can occur due to temperature fluctuations for example. When phase diffusion is desired it can be implemented by e.g. vibrating a piezo on the relevant optical path [126,127].

When the phase of a state $\hat{\rho}(\phi)$ completely diffuses over the full $[0, 2\pi]$ range, the new phase-averaged state $\hat{\rho}_{\bar{\phi}}$ is given by [128]:

$$\hat{\rho}_{\bar{\phi}} = \frac{1}{2\pi} \int_0^{2\pi} \hat{\rho}(\phi) \, d\phi. \quad (3.42)$$

When a phase is not fully diffused it is possible to include a phase distribution function in Eq. (3.42)^[127]. This has been experimentally investigated for both SMSV^[126,127], and coherent states^[128], and for both the Wigner function loses the form in Eq. (3.20).

3.3.3 Factorisation and Applications

A very useful characteristic of variables (or operators) that follow a Gaussian distribution is that they factorise. This means that we can write the expectation value of the product of the variables in terms of sums of the products of the expectation value of pairs. The above is a bit of a mouthful, so let's illustrate with an example.

For an M -mode system, consider components of the general quadrature operator $\hat{\mathbf{d}}$ from Eq. (3.17). The k th component of this $2M$ vector is $\hat{d}_k$. The 4th order expectation value factorises as:

$$\begin{aligned} \langle \Delta \hat{d}_j \Delta \hat{d}_k \Delta \hat{d}_l \Delta \hat{d}_m \rangle &= \langle \Delta \hat{d}_j \Delta \hat{d}_k \rangle \langle \Delta \hat{d}_l \Delta \hat{d}_m \rangle \\ &+ \langle \Delta \hat{d}_j \Delta \hat{d}_l \rangle \langle \Delta \hat{d}_k \Delta \hat{d}_m \rangle \\ &+ \langle \Delta \hat{d}_j \Delta \hat{d}_m \rangle \langle \Delta \hat{d}_k \Delta \hat{d}_l \rangle, \end{aligned} \quad (3.43)$$

where $\Delta \hat{d}_k = \hat{d}_k - \langle \hat{d}_k \rangle$. For Gaussian variables this is known as Isserlis' theorem^[129,130]. The generalisation to an N th order moment can be seen in Ref.^[131], but it is worth noting that the value is zero for odd N . A similar theorem exists for ladder operators which is Wick's theorem mentioned in Sec. 3.2.1^[111], though this does not allow for complete factorisation and is used more for normal ordering.

From Eq. (3.43) we can see that we can write higher order moments in terms of components of $\mathbf{d}$ and $\mathbf{Q}$, which is a very useful property of Gaussian states. We will utilise this in Chap. 8, but for now let us consider an important quantity in quantum optics: the second-order correlation function.

The Second-Order Correlation Function

In quantum optics this is known as $g^{(2)}(\tau)$. It is derived from the measurements of a single mode of light from the same source at two different times, but here we shall consider it a diagnostic tool for optical states. A comprehensive introduction to optics correlation functions can be found in Secs. 3.6 and 3.7 of Ref. [30], and examples for higher orders and two modes can be found in Ref. [132].

For a single mode, $g^{(2)}(0)$ is given by:

$$g^{(2)}(0) = \frac{\langle : \hat{n}^2 : \rangle}{\langle \hat{n} \rangle^2} = \frac{\langle \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} \rangle}{\langle \hat{a}^\dagger \hat{a} \rangle^2} = \frac{\langle \hat{n}^2 \rangle - \langle \hat{n} \rangle}{\langle \hat{n} \rangle^2}, \quad (3.44)$$

where $: \dots :$ is the symbol for operator normal ordering. Some common values can be

Table 3.1: $g^{(2)}(0)$ values for some common states

State	$g^{(2)}(0)$ Value
Fock State $ n \geq 1\rangle$	$1 - \frac{1}{n}$
Coherent State $ \alpha\rangle$	1
Thermal State $\hat{\rho}_{\text{Th}}$	2
SMSV $\hat{S}(\zeta) 0\rangle$	$3 + \text{cosech}^2(r)$

found in table 3.1, and can be used as a reference when characterising a single-mode state (e.g. by measuring in the photon-number basis). In addition to this, $g^{(2)}(0)$ can determine the statistical properties of the photon-number distribution of a state. If $g^{(2)}(0) = 1$ the distribution is Poissonian (e.g. the coherent state in Eq. (2.22)), and if $g^{(2)}(0) < 1$ or $g^{(2)}(0) > 1$ then it's respectively sub- or super-Poissonian. Sub-Poissonian statistics can sometimes indicate non-classicality [30], but can also serve as an indicator of whether more than one mode is present [132]. If we can perfectly measure a state's photon-number distribution then we can trivially calculate its $g^{(2)}(0)$, but otherwise Ref. [133] describes some methods to estimate $g^{(2)}(0)$ of a given state in the presence of experimental limitations.

I will now go through the procedure of writing $g^{(2)}(0)$ in terms of $\mathbf{d}$ and $\mathbf{Q}$ in full detail. I will not leave this for an appendix as I will be quoting results in Chap. 8 without

derivation which use the same method. As a starting point, we first trace over all modes except the relevant one, meaning $\mathbf{Q}$ is a 2×2 matrix and $\mathbf{d}$ is a 2-component vector.

We do not have to invoke Isserlis' theorem to find the denominator of Eq. (3.44). Using Eqs. (2.18), (3.2) and (3.3) we find:

$$\langle \hat{n} \rangle = \frac{\langle \hat{q}^2 \rangle + \langle \hat{p}^2 \rangle - 1}{2}, \quad (3.45)$$

and we can simply read off the components in Eqs. (3.18) and (3.19) to find:

$$\langle \hat{n} \rangle = \frac{Q_{11} + d_1^2 + Q_{22} + d_2^2 - 1}{2}. \quad (3.46)$$

We do however require factorisation to write the numerator of Eq. (3.44) in terms of $\mathbf{d}$ and $\mathbf{Q}$. Given we already have $\langle \hat{n} \rangle$, let's calculate $\langle \hat{n}^2 \rangle$. We find:

$$\langle \hat{n}^2 \rangle = \frac{\langle \hat{q}^4 \rangle + \langle \hat{p}^4 \rangle + \langle \hat{q}^2 \hat{p}^2 \rangle + \langle \hat{p}^2 \hat{q}^2 \rangle - 2 \langle \hat{q}^2 \rangle - 2 \langle \hat{p}^2 \rangle + 1}{4}. \quad (3.47)$$

We now need to factorise the 4th order moments in Eq. (3.47). To do this we write $\hat{d}_k = \Delta \hat{d}_k + \langle \hat{d}_k \rangle$ and substitute into the relevant quantities, invoke Eq. (3.43), then substitute back with $\Delta \hat{d}_k = \hat{d}_k - \langle \hat{d}_k \rangle$. The factorised quantities are:

$$\langle \hat{q}^4 \rangle = 3 \langle \hat{q}^2 \rangle^2 - 2 \langle \hat{q} \rangle^4 \quad (3.48)$$

$$\langle \hat{p}^4 \rangle = 3 \langle \hat{p}^2 \rangle^2 - 2 \langle \hat{p} \rangle^4 \quad (3.49)$$

$$\langle \hat{q}^2 \hat{p}^2 \rangle = \langle \hat{q}^2 \rangle \langle \hat{p}^2 \rangle + 2 \langle \hat{q} \hat{p} \rangle^2 - 2 \langle \hat{q} \rangle^2 \langle \hat{p} \rangle^2 \quad (3.50)$$

$$\langle \hat{p}^2 \hat{q}^2 \rangle = \langle \hat{q}^2 \rangle \langle \hat{p}^2 \rangle + 2 \langle \hat{p} \hat{q} \rangle^2 - 2 \langle \hat{q} \rangle^2 \langle \hat{p} \rangle^2, \quad (3.51)$$

then finally by inserting Eqs. (3.48) to (3.51) into Eq. (3.47), using the commutation

relation in Eq. (3.3), and Eqs. (3.18) and (3.19), we arrive at:

$$\begin{aligned} \langle \hat{n}^2 \rangle = & \frac{3}{4} \left((Q_{11} + d_1^2)^2 + (Q_{22} + d_2^2)^2 \right) - \frac{1}{2} (d_1^2 + d_2^2)^2 \\ & + \frac{1}{2} (Q_{11} + d_1^2) (Q_{22} + d_2^2) + (Q_{12} + d_1 d_2)^2 \\ & - \frac{1}{2} (Q_{11} + d_1^2 + Q_{22} + d_2^2). \end{aligned} \quad (3.52)$$

We can now find $g^{(2)}(0)$ in terms of $\mathbf{d}$ and $\mathbf{Q}$ using Eqs. (3.44), (3.46) and (3.52). If we have an M -mode system and we do not trace over all but the relevant mode, if we require the $g^{(2)}(0)$ of the j th mode then we simply replace the indices in Eqs. (3.46) and (3.52) as $1 \mapsto j$ and $2 \mapsto j + M$. We find similar expressions in Refs. ^[134,135], though derived from slightly different (but equivalent) methods.

As a brief but relevant aside, in Eq. (3.44) we can see that the normally-ordered quantity has been written in terms of powers of $\hat{n}$. The formula to go the other way (i.e. to write $\hat{n}^m$ in terms of a sum of $(\hat{a}^\dagger)^p (\hat{a})^p$ for $p = 1$ to m) can be found in e.g. Ref. ^[136], but to undo normal ordering the formula is:

$$: \hat{n}^m : = \left(\hat{a}^\dagger \right)^m (\hat{a})^m = \sum_{p=1}^m (-1)^{m-p} S_1(m, p) \hat{n}^p, \quad (3.53)$$

where $S_1(m, p)$ are Stirling numbers of the first kind ^[137]. I include this formula as $\langle \hat{n}^p \rangle$ are much easier to calculate from an experimentally-determined photon-number distribution.

3.4 Measurement

So far we have considered states and evolution in terms of the Gaussian state formalism, but now will discuss measurement. We will discuss ‘Balanced Homodyne Detection’ which is a Gaussian measurement, and we will also discuss how photon-number projection (see Sec. 2.4.1), a non-Gaussian measurement, is formulated for Gaussian states.

3.4.1 Balanced Homodyne Detection

Balanced Homodyne Detection (BHD) is a commonly-used optical state tomography method^[21,138]. It is a Gaussian measurement, but can be used to characterise any state whether Gaussian or non-Gaussian. I will now briefly go through the derivation of the BHD measurement, following the procedure in Sec. 4.2.2 of Ref. ^[21].

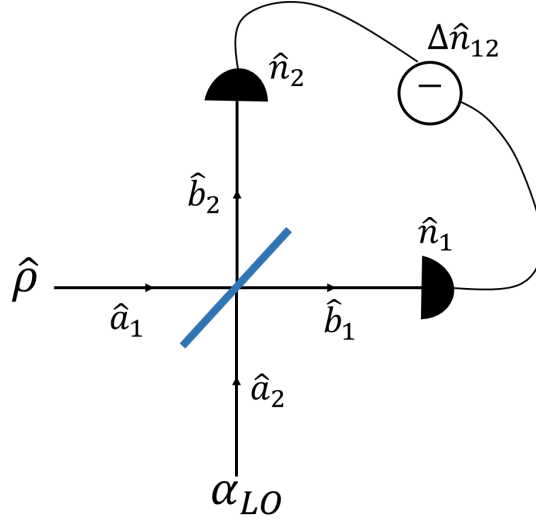


Figure 3.2: Schematic of a Balanced Homodyne Detector. A state $\hat{\rho}$ is impinged on a 50:50 beamsplitter with a coherent state of amplitude α_{LO} . Both output modes are measured in the photon-number basis, and the difference is taken (hence the minus sign).

Consider the setup in Fig. 3.2. The state we wish to analyse, $\hat{\rho}$, is initially in mode 1. It is impinged on a balanced beamsplitter with a coherent state (initially in mode 2) of amplitude α_{LO} , which we will refer to as the ‘local oscillator’ (LO). We measure the photon-number difference between the output modes:

$$\Delta \hat{n}_{12} = \hat{n}_1 - \hat{n}_2 = \hat{b}_1^\dagger \hat{b}_1 - \hat{b}_2^\dagger \hat{b}_2. \quad (3.54)$$

Using Eq. (2.38) with $\theta = \frac{\pi}{2}$, we can write Eq. (3.54) in terms of the operators to before the beamsplitter:

$$\Delta \hat{n}_{12} = e^{-i\phi} \hat{a}_1^\dagger \hat{a}_2 + e^{i\phi} \hat{a}_1 \hat{a}_2^\dagger. \quad (3.55)$$

ϕ is the relative phase shift imparted by the beamsplitter and will remain constant, for now let's assume we have a symmetric beamsplitter and set $\phi = -\frac{\pi}{2}$ [36,37].

The next step is to consider the case when the LO is strong, i.e. $|\alpha_{\text{LO}}|^2 \gg \text{Tr}(\hat{\rho}\hat{n})$. This is usually a requirement, as traditionally photodiodes are used as detectors which are insensitive at the single-photon level due to electronic noise [139]. In this case we make the classical approximation (essentially undoing the quantisation in Eqs. (2.10) to (2.12) from Sec. 2.2.1) of $\hat{a}_2 \mapsto \alpha_{\text{LO}} = |\alpha_{\text{LO}}| e^{i\varphi}$, where φ is the phase of the LO. When we insert this approximation into Eq. (3.55), and swap $\hat{a}_1$ and $\hat{a}_1^\dagger$ for $\hat{q}_1$ and $\hat{p}_1$, we arrive at:

$$\Delta\hat{n}_{12} = \sqrt{2} |\alpha_{\text{LO}}| (-\sin(\varphi) \hat{q}_1 + \cos(\varphi) \hat{p}_1) = \sqrt{2} |\alpha_{\text{LO}}| \hat{p}_1(\varphi), \quad (3.56)$$

i.e. we are measuring $\hat{p}_1$ that has been phase rotated by Eq. (3.38). Note that the expression in Eq. (3.56) depends on the beamsplitter phase ϕ , and for some values we end up with $\hat{q}_1(\varphi)$.

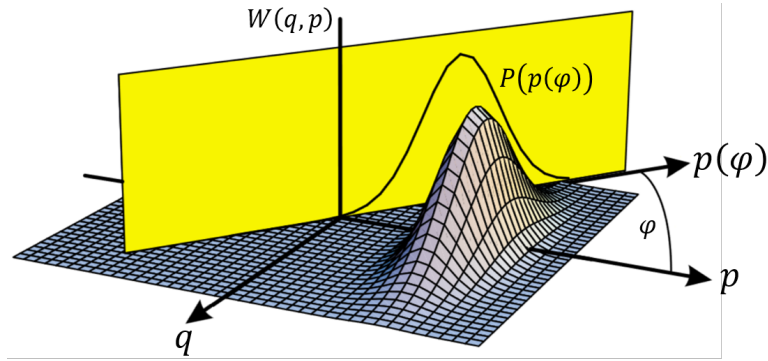


Figure 3.3: Projection of $W(q, p)$ onto the axis defined by φ , the phase of the LO. This figure has been adapted from Ref. [138], as it is far nicer than anything I could make myself.

When we perform multiple measurements of BHD we build up the distribution of $\Delta\hat{n}_{12}$. Although $\Delta\hat{n}_{12}$ is a discrete quantity, for large LO magnitudes and noisy detectors it can be considered continuous. By using Eq. (3.56) we can find the distribution of $\hat{p}_1(\varphi)$ which

we shall call $P(p(\varphi))$. $P(p(\varphi))$ is given by:

$$P(p(\varphi)) = \int_{-\infty}^{\infty} W(\cos(\varphi)q + \sin(\varphi)p, -\sin(\varphi)q + \cos(\varphi)p) dq, \quad (3.57)$$

which is the projection of the Wigner function onto the axis defined by φ . This can be visualised in Fig. 3.3. Note that we are free to define the LO complex amplitude for which $\varphi = 0$ as it is only the relative phase between the LO and signal we are able to measure.

It is possible to reconstruct $W(q, p)$ from $P(p(\varphi))$, and the method used depends on whether or not $\hat{\rho}$ is rotationally symmetric in phase space. If $\hat{\rho}$ is not rotationally symmetric then multiple LO phases are required to obtain a set of $P(p(\varphi))$ for many φ . Details on some of the reconstruction algorithms can be found in Refs. [21, 140–143].

3.4.2 Photon-Number Projection

Unlike BHD which is a Gaussian measurement, photon-number projection is a non-Gaussian measurement regardless of the state. This is related to the fact that $W_n(q, p)$ for a Fock state $|n\rangle$ in eq (3.10) is not a Gaussian for $n \geq 1$.

It turns out to be rather tricky to project a multimode Gaussian state onto a photon number distribution. The procedure for a single-mode Gaussian state was first presented in Ref. [144], and then generalised to multiple modes in Ref. [121]. Ref. [121] explains that the photon-number distribution of a Gaussian state is related to multidimensional Hermite polynomials, where the arguments to these polynomials are related to $\mathbf{d}$ and $\mathbf{Q}$. Hermite polynomials are related to the Laguerre polynomials [145], which is unsurprising given the form of $W_n(q, p)$ in Eq. (3.10). There is further information in Ref. [146] on how to calculate multidimensional Hermite polynomials, either from generating functions, or iteratively. The main result for photon-number projection is Eq. (9) of Ref. [121], and in Ref. [147] the same method is used to also find the off-diagonal entries of the density matrix. Despite their huge importance, the exact functional forms themselves are neither particularly insightful nor inspiring, so I will not include them here. An alternative,

yet equivalent, method for calculating the photon-number distribution for a (multimode) Gaussian state is presented in the next section.

3.5 Gaussian Boson Sampling

In this final section of the Gaussian State Formalism chapter we will discuss an important topic which will feature heavily in this thesis: Gaussian Boson Sampling (GBS). This is, unsurprisingly, the Boson Sampling case from Sec. 2.3.3 but with Gaussian state inputs.

3.5.1 Scattershot Boson Sampling

There is an article from 2014 titled ‘Boson Sampling from a Gaussian State’^[148], but this article discusses the scenario of using a Gaussian state input (namely TMSV from Eq. (2.34)) to perform Fock state Boson sampling, rather than what would later be considered GBS. The article presents the scheme of Boson sampling with M lots of TMSV, with one mode of each TMSV entering an M -mode interferometer and the other mode being sent to a single photon detector. The single photon detectors would reveal information of the state entering the interferometer due to the photon-number entanglement of TMSV. The article argues that it is still computationally difficult to simulate output samples, even with the input state being non-deterministic. This scheme became known as ‘Scattershot Boson Sampling’, and has been implemented experimentally (see for example Refs.^[149,150]).

3.5.2 The Hafnian

We will now discuss what is currently considered to be GBS: Gaussian state inputs evolved by a unitary as per Eqs. (2.41) and (3.39), with measurement in the photon-number basis. This is arguably a better experimental framework as it is possible to produce Gaussian states deterministically (see Sec. 2.2.5), rather than relying on getting single photons via post-selection.

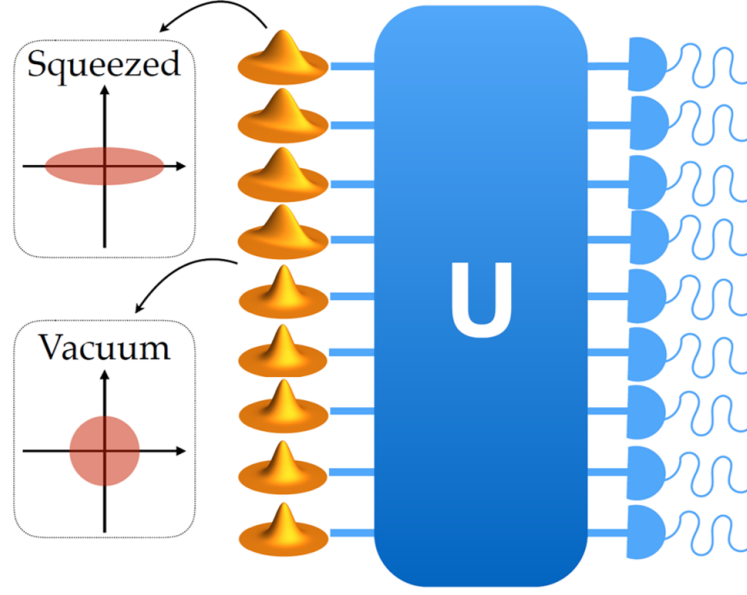


Figure 3.4: An illustration of Gaussian Boson Sampling. This has four SMSV inputs into a 9-mode interferometer, with photon-number projection at the output. This figure has been adapted from Ref.^[151].

The essence of GBS can be visualised in Fig. 3.4. Although evolving Gaussian states followed by photon-number projection was not a new idea^[16], its formulation in terms of a Boson sampler was first properly put forward in Refs.^[152,153]. Eqs. (3)-(6) in Ref.^[152] present the photon-number projection formulae for an M -mode Gaussian state with an arbitrary $\mathbf{Q}$ but $\mathbf{d} = \mathbf{0}$, and Eq. (22) of Ref.^[153] gives the formula for arbitrary $\mathbf{d}$. It is a different (but equivalent) approach to that in Ref.^[121], and is equally as difficult to compute.

Both Refs.^[152,153] introduce a matrix operation known as a ‘Hafnian’, which is performed on a matrix related to $\mathbf{Q}$ to obtain a photon-number distribution. This operation is not a new concept^[154], but Ref.^[152] is the first time it was used in the context of quantum optics. Ref.^[152] also relates the matrix Hafnian to the matrix permanent (Eq. (7)), and thus concludes that GBS is also computationally difficult to simulate in the same way as Boson sampling with Fock states. Note that the displacement terms are not included within the Hafnian (Eq. (22) of Ref.^[153]). This is intuitive: for an input of a product of

coherent states, the output would also be a product of coherent states but with amplitudes jumbled by the unitary evolution (consider unitary evolution on Eq. (3.27)). The output photon-number distribution is thus easy to compute as per Eq. (2.21).

There has subsequently been more in-depth work on the complexity of classical GBS simulation, confirming that GBS is at least as difficult to simulate on a classical computer as regular Boson sampling^[155,156]. Also, there have been studies on the simulation complexity in the presence of imperfections, suggesting that certain imperfections (e.g. loss) may be easier to compensate for compared to regular Boson sampling^[157]. Finally, the case of GBS using click detectors (i.e. without number-resolution) has also been studied in Ref.^[158], which they say is an attractive setup due to the experimental difficulty of photon-number resolution (see Sec. 2.4.3). The result was that if the probability of measuring photon numbers higher than 1 is small enough then it is still a difficult calculation for a classical computer.

3.5.3 Applications

Other than providing an advantage over classical computers^[45], to date there have been no reported practical applications for either Fock state or scattershot Boson sampling. Gaussian Boson Sampling however is a different story. Let us briefly review the applications of a GBS system as a quantum simulator^[5,6].

The first reported application was on computing molecular vibronic spectra. Refs.^[16,159] detail how to compute the Franck-Condon profile for a given molecule based on photon-number basis measurements of squeezed displaced states that have undergone unitary evolution. A proof-of-principle experiment has also been performed on this^[160]. On a similar topic, it has also been shown that a GBS system can be used to simulate molecular docking problems, i.e. modeling how different molecules bind together^[161].

Another class of problems GBS can simulate are related to graph theory. According to Refs.^[162,163] GBS can be used to find dense subgraphs, a computationally difficult task in graph theory. The molecular docking task actually comes under this category, another

example is the computation of point processes^[164].

One aspect that the above applications have in common is the requirement to vary the unitary evolution the input states undergo. For example, with reference to molecular vibronic spectra simulation application, being able to change the unitary evolution increases the number of types of molecules for which the vibronic spectra can be simulated. Experimentally this involves being able to tune the splitting ratios and relative phase shifters that make up the interferometer. This was briefly discussed in Sec. 2.3.3, and is further discussed in Chap. 8.

All of the above is to provide motivation for the study of GBS and for the improvement of the platforms to perform it, not to detail how GBS solves each of the problems. The references provided give this insight. However, other applications of GBS and GBS platform diagnostics will feature heavily in this thesis.

Chapter 4

Characterisation of SNSPDs on a Buried Waveguide in Silica Glass

In Sec. 2.3.3 we discussed how a unitary transformation on optical modes could be realised through an array of beamsplitters and phase shifters, yet we did not discuss any form of practical implementation. One very robust and stable approach to do this is through waveguides on photonic chips, i.e. fabricated structures which confine and guide light^[23,165]. Some examples of using waveguides for quantum optical processes can be seen in Refs.^[166,167], but this is by no means exhaustive. In addition, we did not discuss the types of detector required at the output other than stating that it should be a ‘photon-number measurement’. Some experiments have used multiplexed SNSPDs or APDs (see Sec. 2.4)^[50–52,59], but in principle TESs could also be used.

In this chapter, I will present an experimental implementation of combining both components by means of depositing SNSPDs on waveguides. This is not a new idea^[168], but we will demonstrate the first instance of Tungsten Silicide (WSi) SNSPDs on silica glass waveguides. For this project the waveguides were fabricated in the Optoelectronics Research Centre at the University of Southampton, and the SNSPDs were fabricated directly onto the waveguides at the National Institute of Standards and Technology (NIST) in Boulder,

Colorado. The part we at Oxford played was to perform the cryogenic characterisation of the SNSPDs. As a result this chapter will focus more on these measurements and how we performed them, but I will first motivate this project further and briefly discuss fabrication.

4.1 Introduction

For telecom wavelengths (≈ 1550 nm) the most efficient single-photon detectors to date are SNSPDs and TESs^[19,20], i.e. ones which require cryogenic temperatures to operate. This means that regardless of the procedure the light has to be sent into some form of cryostat for detection, which is usually by means of fibre coupling. This can cause additional loss, whether doing this from free space (see Sec. 6.3.3), or from waveguides on a photonic chip. Inefficient fibre-coupling in waveguides is mainly due to scattering loss caused by optical mode mis-match, but this is dependent on the waveguide material^[169].

Ref.^[170] proposed to overcome this by means of placing detectors on top of waveguides, where light is detected through evanescent absorption. This results in a setup free of fibre or end coupling from chip to detector, though necessitates placing the whole system inside a cryostat. Additionally, detectors on waveguides have the further application of performing partial measurements on an optical field. Their size and geometry can be fabricated to control optical absorption, making it possible to perform weak measurements such as photon subtraction without disturbing the rest of the optical mode (i.e. a non-destructive measurement)^[170]. It is also possible to obtain a degree of number resolution from SNSPDs by placing several along a single waveguide (equivalent to spatial multiplexing)^[171].

There have already been many implementations of superconducting detectors on waveguides. TESs on silica waveguides have been demonstrated with efficiencies as high as $88 \pm 3\%$ (with an integrated cavity)^[172,173], and a large number of SNSPD-on-waveguide schemes have been realised. Ref.^[168] is a review article on the very subject of SNSPDs on waveguides, and Table 3 from this article summarises the implementations to date (at the

time of writing this thesis).

One waveguide material missing from this table is silica. Silica has very low loss at 1550 nm (a propagation loss of 1.1 dB cm^{-1} was measured for the waveguides in this chapter), and the fabrication method used gave very low surface roughness which was ideal for positioning SNSPDs. Sometimes it is possible to generate light on waveguides (see Refs.^[174,175] for example), but for the setup we used it was necessary to couple externally-generated light in. In this instance, silica is an ideal material as the waveguides can be fabricated to properly mode-match to telecom optical fibre^[176], therefore in theory coupling into the waveguides should be efficient. We also benefited from the well-established WSi SNSPD fabrication method^[177], meaning we could compare our data to standard fibre-coupled SNSPDs of the same material.

4.2 Fabrication and Layout

We will now discuss the steps involved for the fabrication of the whole SNSPDs-on-chip setup. I will only give a brief overview of the fabrication methods as I played no part in this. I will go into more detail on the structure of the overall setup as this is important to consider for characterisation.

4.2.1 Fabrication

The waveguides were fabricated at the University of Southampton through direct UV laser writing in doped silica^[178]. A $15 \mu\text{m}$ layer of Boron-doped silica was deposited on a 1 mm silicon wafer substrate to act as undercladding, and on top of this a $4.2 \mu\text{m}$ layer of Germanium-doped silica was deposited to form the waveguides. We'll refer to this as the chip. A CW UV laser was then focused into the Germanium-doped layer while the chip was spatially translated. This created a region of slightly higher refractive index which was the waveguide. The refractive index contrast in this case was 1.2%, and a core layer thickness of $4.2 \mu\text{m}$ was chosen such that there was some modal overlap with the SNSPDs

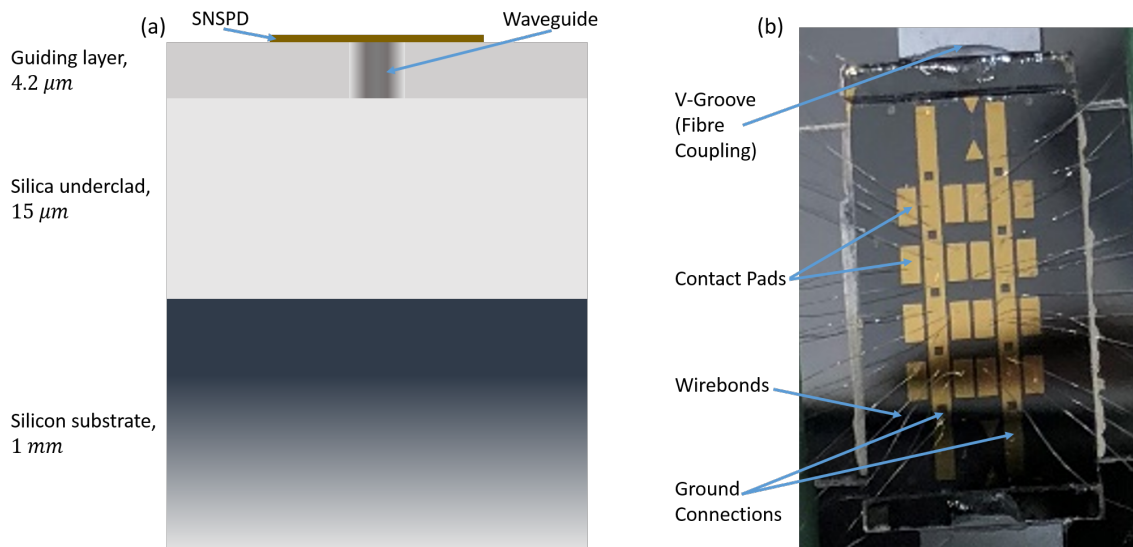


Figure 4.1: (a) Schematic of the layers making up the chip, with one waveguide and SNSPD shown as an example. The meander structure of the SNSPD is not visible from the angle the chip is viewed at. (b) Picture of the chip viewed top down, the waveguides run from top to bottom. We cannot see the SNSPDs or waveguides, but we can see the SNSPD electrical connectors and the v-grooves coupling the waveguides to optical fibre.

which were deposited directly on top of this layer. On silica waveguides there is usually an overladding layer of Boron-doped silica of similar thickness to the undercladding, but this was absent to enable SNSPD fabrication directly above the waveguides. The layers can be visualised in Fig. 4.1 (a), and the specifics of the fabrication method are given in Ref. [179].

Five 20 mm-long waveguides were written into the chip. Five reflective Bragg gratings were also inscribed into each waveguide for SNSPD absorption measurements^[180], one at either end and one in between each SNSPD. Each was fabricated to reflect light at a different wavelength (all different to 1550 nm such that it wouldn't cause reflections when calibrating the SNSPDs at that wavelength) such that the waveguide loss between each SNSPD could be estimated.

Four WSi SNSPDs were deposited onto four of the waveguides at NIST, with one free waveguide for comparison. The fabrication method is described in Ref. [177], and the standard $16\mu\text{m}\times 16\mu\text{m}$ design was used for this first proof-of-principle device rather than

optimising the structure for maximum optical absorption. Following this, electrical contact pads were deposited on the chip surface and connected to the SNSPDs to enable external electrical connection. There are 16 ‘positive’ pads for each SNSPD and two common grounds. Fibre-coupling was achieved with commercial v-grooves with polarisation-maintaining (PM) 1550 nm optical fibres. One was glued at each end of the chip to allow optical access in both directions, and for throughput measurements. We see a close-up picture of the chip in Fig. 4.1 (b), we cannot see the waveguides or SNSPDs but we can see the chip surface, electrical contact pads and v-groove fibre-coupling.

4.2.2 Overall Structure

A printed circuit board (PCB) was produced by our collaborators at the University of Southampton to enable electrical connections to the SNSPDs, the design can be seen in Fig. 4.2 (a). The chip and PCB were electrically connected via wirebonds (black lines on figure) between their respective contact pads. Wirebonds are usually fragile, but these ones were more so as they were comparatively long.

I have simplified this into the schematic in Fig. 4.2 (b) to show the placement of each SNSPD along the waveguides. The ‘X...’ labels correspond to the SMA plug labels in (a), also the ‘F...’ fibre labels in (a) and (b) match up. First we note that the furthest left waveguide was not fibre coupled (there were only four fibres in the v-groove), though we characterised the SNSPDs on that waveguide through scatter coupling by sending more powerful light into the adjacent waveguide. Second, unfortunately the fibre furthest right on the lower v-groove broke off (what would have been F13), meaning we could only access that waveguide in one direction. Finally we notice that not all SNSPDs maintained their electrical connection. X2 and X15 registered a short, presumably due to their supply and ground wirebonds touching as they were long.

Fig. 4.2 (c) shows a picture of the whole chip-on-PCB setup. The chip was thermally-pasted with cryogenic heatsinking varnish to the PCB and bolted with two brass rods. There was a brass rectangle fastened to the back of the PCB to improve thermal conduct-

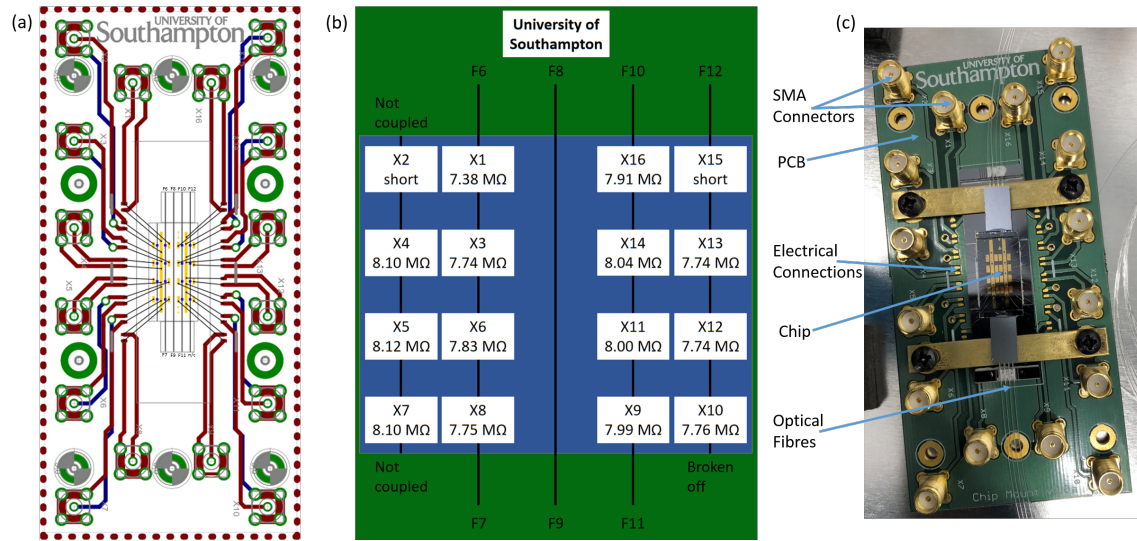


Figure 4.2: (a) Schematic of the printed circuit board with the chip attached, created by our collaborators at the University of Southampton. This maps the SNSPDs to their electrical port. The labels on the fibres correspond to the labels in the actual setup, they are irrelevant other than for reference with (b). (b) Simplified schematic of the SNSPD distribution on the chip, including their room temperature resistances. The ‘X...’ labels map the SNSPDs to the SMA sockets depicted in (a) and (c), and the fibre labels ‘F...’ match with (a). The ‘University of Southampton’ label gives the orientation with respect to (a) and (c). (c) Annotated picture of the setup.

ivity when cooling the setup to cryogenic temperatures.

4.3 Experimental Setup

Here we will discuss the practical laboratory setup for characterising the SNSPDs, including the cryogenic system, installation, and room temperature electronics.

4.3.1 Dilution Refrigeration

We cooled the whole setup down to ≈ 250 mK in a dilution refrigerator (DR). Cryogenics and cryostats have been a large part of my D.Phil, therefore I will go into detail on how a DR works and is operated. Ref.^[181] has been the primary source for this section.

We used a commercial Oxford Instruments Triton Dry Dilution Refrigerator. The coldest part of these DRs are capable of reaching temperatures lower than 10 mK, though ours was at ≈ 100 mK for the characterisation. There are two main cooling mechanisms in a DR: pulse tube cooling and dilution cooling. Pulse tube cooling is firstly used to achieve temperatures around 1 – 2 K, then dilution cooling gets it down to the mK level. We will discuss both mechanisms, then briefly review the physical structure of a DR to implement this.

Pulse Tube Cooling

In a DR, pulse tube cooling is usually achieved with a ‘compressor’ and a ‘cold head’. The compressor is a commercial device at room temperature (in our case from Cryomech) which pumps gaseous Helium-4 (^4He). The gaseous ^4He is pumped to the cold head, where it is cooled by adiabatic expansion. The first stage (PT1) reaches around 60 K, and at the second stage (PT2) it liquefies and reaches 4.2 K (see Fig. 4.4 for both). The liquid ^4He is then pumped on and further cools PT2 to 1 – 2 K by evaporative cooling. For more detailed information see Ref.^[182].

As an alternative to this, some DRs are described as ‘wet’, meaning they instead rely on an external supply of liquid ^4He which needs frequent topping up. This is beneficial when cooling devices which are extremely sensitive to vibrations, as very small vibrations can

occur with pulse tube cooling. However liquid ^4He is expensive and it is labour intensive to refill it every few days.

Dilution Cooling

The dilution section of a DR is the part that enables it to reach the mK range. Unlike the pulse tube section which only has ^4He , the dilution section has a mixture of Helium-3 (^3He) and ^4He . This mixture is cooled by the pulse tube cooler and liquefies into a container known as the ‘Mixing Chamber’ (MC) on the coldest part of the DR, the MC plate.

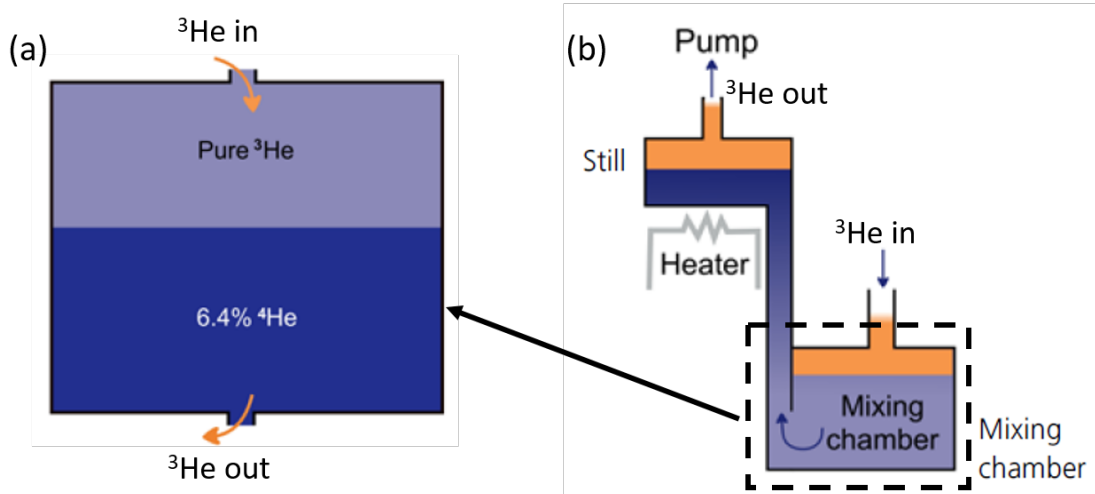


Figure 4.3: Diagram of the Mixing Chamber and Still, adapted from Ref.^[181]. (a) Phase (of matter) diagram for the mixture of ^3He and ^4He . The upper section is pure ^3He , the lower section is 6.4 % ^3He dissolved in ^4He . (b) The connection between the Still and the MC. Orange is ‘pure’ ^3He (in practise $\approx 90\%$) and blue is ^4He with a small fraction of ^3He .

When the mixture liquefies in the MC it undergoes a phase transition with ^3He on top and ^4He at the bottom (see Fig. 4.3 (a)). For clarity, in this instance I mean ‘phase of matter’, it has absolutely nothing to do with ‘phase space’ introduced in Chap. 3. A fraction of 6.4% of ^3He dissolves into the ^4He layer, which is because ^3He is a Fermion and will only dissolve in ^4He until it reaches the same chemical potential as the pure ^3He state. This layer is known as the ‘dilute phase’.

The cooling power comes from removing ^3He atoms from the dilute phase (the ‘out’ arrow at the bottom of Fig. 4.3 (a)). When this happens ^3He atoms from the pure ^3He phase come to take the place of the removed ones which is an endothermic process, thus cooling the MC plate.

This leaves us with how ^3He atoms are removed from the dilute phase. The MC is connected to the ‘Still’ (on the Still plate), which is heated such that the mixture evaporates and circulates (Fig. 4.3 (b)). ^3He evaporates first as it has a higher vapour pressure, and is pumped away from the Still. This creates an osmotic pressure which gives the required gradient to remove ^3He atoms from the dilute phase.

I have only given a very brief overview of this process, and as a result have glossed over some of the finer but important details. For a thorough description of the process please see Ref.^[181].

Structure and Operation of a Dilution Refrigerator

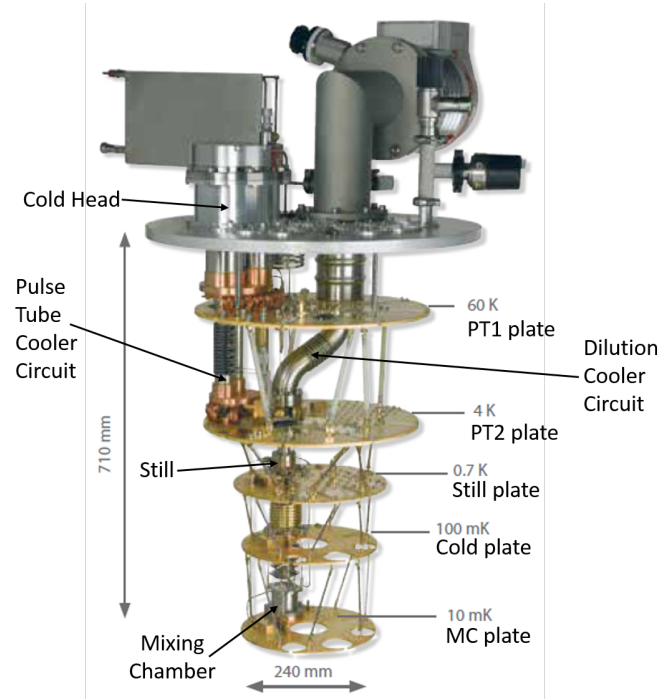


Figure 4.4: Image of a typical DR with plate names and approximate temperatures and dimensions, adapted from Ref.^[181]. Description in main text.

We see the typical layout of a DR in Fig. 4.4, this one being from Oxford Instruments. The first two plates, PT1 and PT2 (PT standing for pulse tube), are the plates that are cooled by pulse tube cooling. We can see the lines for the ^4He in the figure. Beneath this is the Still plate on which sits the Still chamber. Then beneath this there is an intermediate plate called the Cold plate, and at the bottom we see the MC plate with the MC itself. There is a further circuit called the ‘Pre-Cool Circuit’ which is not really visible in this figure, but it is a gas line separate to the dilution system in which a small amount of the mixture is circulated during the pulse tube cooling. Its purpose is to act as a heat exchange between the cold head and the remainder of the plates.

For operation there are usually up to three radiation shields attached to the PT1, PT2 and Still plates. These are non-vacuum tight metal shields to reduce RF noise inside the DR, which is usually necessary for the devices inside. Over the whole setup we attach an outer vacuum chamber (OVC), which is evacuated to $< 10^{-4}$ mbar before cooling commences to ensure ice crystals don’t form on the plates. Separate to this is an external device called a ‘Cold Trap’ which is a sealed piece of charcoal submerged in liquid nitrogen at 77 K. When the mixture is pumped away from the Still it is forced through the charcoal which acts as a decontaminator, absorbing impurities that may have entered the mixture. Once at base temperature a DR can be continuously operated for long periods of time.

4.3.2 Installation and Operation

The whole setup mounted in the DR can be seen in Fig. 4.5. We screwed the PCB onto the Cold plate of the DR which reached a base temperature of ≈ 250 mK, and used GE varnish to improve thermal contact as the SNSPDs were thermally-coupled through the silica and the PCB. As silica is a poor heat conductor we suspect the SNSPDs were at a higher temperature than the Cold plate, though all of the (connected) SNSPDs demonstrated superconductivity. Commercial SMA-terminated coaxial cables connected the PCB to the build-in semi-rigid coaxial cables of the DR. The fibres from the v-grooves were spliced to the fibres of the DR fibre feedthrough. The feedthrough fibres were SMF28 (i.e. 1550 nm

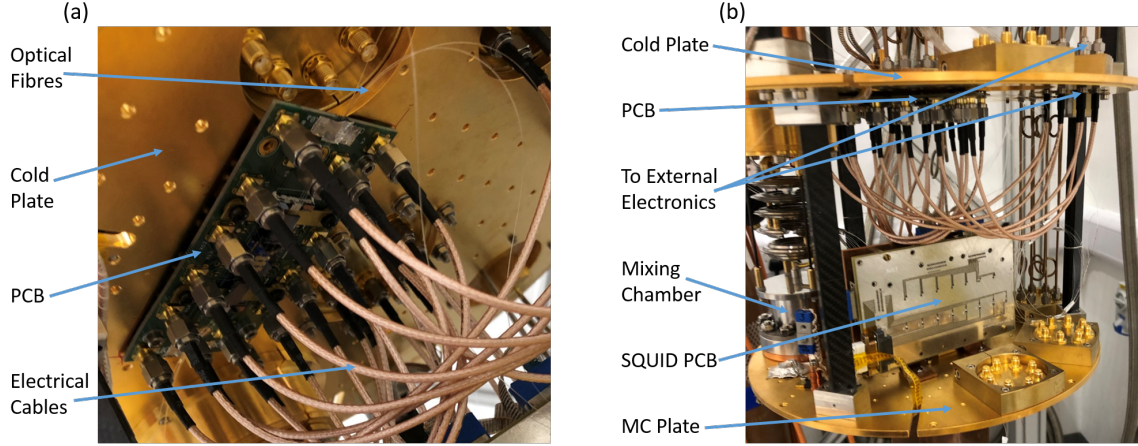


Figure 4.5: Setup pictures. (a) The setup fastened to the Cold plate of the DR before cooling. We can see the optical fibres coming to and from the chip, as well as the electrical connections for biasing and readout. (b) The setup in relation to the DR structure. We see the built-in DR coaxial cables, as well as the Mixing Chamber, MC Plate and SQUID PCB (see Sec. 2.4.3).

but not PM), therefore these splices will have introduced some loss.

The SNSPDs were biased by SIM928 isolated voltage sources mounted in a SIM900 mainframe, both from Stanford Research Systems. At the output of each voltage source was a $100\text{ k}\Omega$ resistor to convert it into a current source, along with a Mini-Circuits lowpass filter (DC-1.9 MHz) to attenuate unwanted RF. This was fed into a Mini-Circuits bias-tee (ZFBT-4R2GW+) to send the bias current to the SNSPDs as per Figs. 2.10 (a) and (b). The RF output signals from the SNSPDs were amplified by two Mini-Circuits low noise amplifiers (ZFL-1000LN+ then ZFL-500LN+) after exiting the AC part of the bias tee. These amplified electrical signals were then sent for processing, the device used depending on the quantity being measured.

4.4 Characterisation Results

Here I shall present the experimental results from the SNSPD characterisation at cryogenic temperatures. The quantities we shall consider are voltage detection thresholds, calibration curves, dead time, jitter, and efficiency. We discussed some of these in Sec. 2.4.2, but in each instance I shall introduce the property and describe how we measured it. Before

starting I will note that unfortunately there was an issue coupling light to the waveguides. With reference to Figs. 4.1 (b) and 4.2 (c) we suspect that the v-grooves shifted during cooldown, thermal contraction was the most probable cause. This resulted in a drastically reduced transmission onto the chip and off the chip, with ~ 25 dB loss per v-groove. The only positive from this is that we could send a measurable optical power (with a commercial powermeter) to the SNSPDs without saturating them.

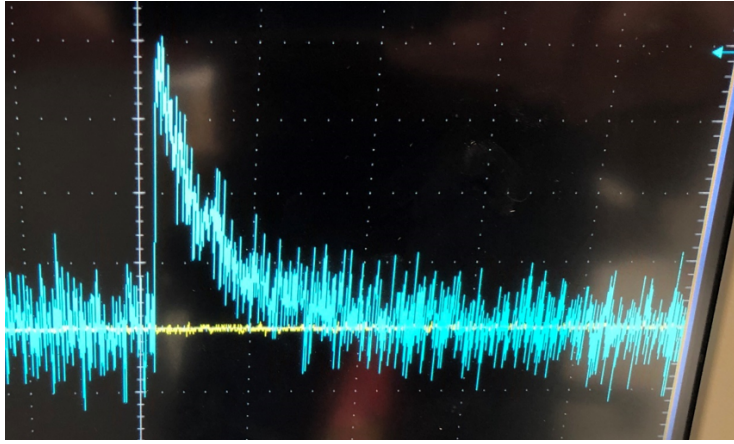


Figure 4.6: Detection voltage trace V_{det} from one of the on-chip SNSPDs viewed on a Tektronix Oscilloscope (70804C). This is to serve as a shape example only, but the pulse height is ≈ 40 mV.

The very first check we performed was to confirm that the detection voltage traces V_{det} looked as they should for an SNSPD (see Fig. 2.9 (b)). An example is shown in Fig. 4.6, and all of the connected SNSPDs followed suit.

4.4.1 Voltage Detection Thresholds

Consider the SNSPD V_{det} in Fig. 4.6. When counting detection pulses for an experiment we need to set a voltage detection threshold (VDT) that is above the noise floor but below the pulse peak. We also need to be conscious of the bias current I_b as the pulse height is proportional to it. Now consider an SNSPD with a constant I_b illuminated with a constant optical power, and think of how the count rate will vary when changing the VDT. When starting with the VDT near zero there will be a significant contribution from noise making the count rate artificially high. Then as the VDT increases the count rate will reduce until

the VDT is above the noise floor. As it continues to increase the count rate will remain constant while it is still below the V_{det} peak, we will call this the ‘operating plateau’. There is a small variation in the V_{det} height from the SNSPD from shot-to-shot, therefore as the VDT increases to the pulse peak height and beyond it will not step sharply to zero, but will smoothly decrease to zero. This variation of count rate with VDT was the first characterisation we performed to set an appropriate VDT for other measurements.

To perform this measurement we illuminated the waveguides with a tunable CW laser (Santec TSL-550) at 1550 nm, which we attenuated using Thorlabs fibre attenuators as to not saturate the SNSPDs. We used a fibre paddle polarisation controller before the attenuators to optimise polarisation, though this was not strictly necessary for this measurement as efficiency was irrelevant. We biased the SNSPDs appropriately by checking the V_{det} traces on an oscilloscope. The amplified traces were sent to a time-to-digital converter (a quTAU from qutools) to measure the count rate. We varied the VDT of the quTAU and recorded the count rate at each setting. This was done for each SNSPD to find the appropriate operational VDTs.

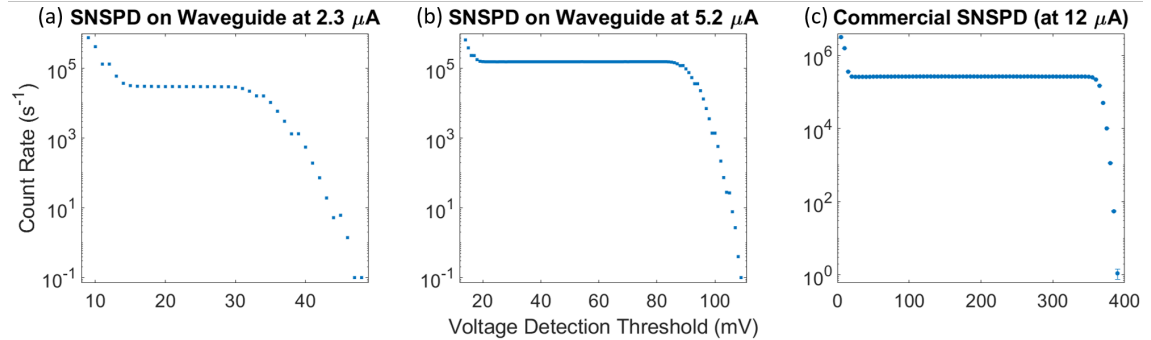


Figure 4.7: Examples of count rate vs voltage detection threshold plots for SNSPDs. (a) Shortest measured operating plateau from an on-chip SNSPD. (b) Longest measured operating plateau for an on-chip SNSPD. (c) Commercial PhotonSpot SNSPD of the same material (Tungsten Silicide) for comparison.

Some examples of count rate vs VDT plots can be seen in Fig. 4.7, where we see the behaviour previously described including the operating plateau. In a way the axes of this plot are arbitrary as they are partly dependent on factors other than the SNSPDs: the y-

axis has some dependence on optical power, and the x-axis on the amount of amplification used. We see that the length of the operating plateau increases with I_b which is intuitive as the pulse height is proportional to I_b . Figs. 4.7 (a) and (b) show the measured shortest and longest extremes of operating plateaus respectively. All SNSPDs were biased at their approximate optimal I_b , and we suspect that the temperature on the chip was non-uniform which resulted in the large variation in optimal I_b for the different SNSPDs. This will be discussed further in Sec. 4.4.2. Fig. 4.7 (c) shows the same plot but for a commercial (PhotonSpot) fibre-coupled SNSPD made of the same material (but different design) for comparison.

Fig. 4.7 (a) shows the ‘worst case scenario’, but there is still a clear operating plateau which is encouraging. Therefore for other measurements on this particular SNSPD we would set the VDT to ≈ 25 mV. If we disregard efficiency, the operating plateau sets a lower bound on I_b to measure a meaningful count rate with an SNSPD. There comes a point when I_b is reduced so much that there is no operating plateau, meaning where to set the VDT it is ambiguous. This value of I_b can be estimated by measuring operating plateau length vs I_b and extrapolating to zero. We did not measure this as it was not required to measure any of the other quantities.

4.4.2 Calibration Curves and Dark Counts

Next we shall discuss what is arguably the most defining feature of an SNSPD: the calibration curve. When an SNSPD is illuminated at a constant optical power and a constant VDT is set, the calibration curve is the plot of count rate vs I_b . A calibration curve can be normalised to the total SNSPD efficiency, and in this case can be thought of as an efficiency vs I_b plot.

With reference to Sec. 2.4.2, at a low I_b the probability for an incoming photon to cause a detection event is lower. This is partly because a superconductor’s critical temperature T_C reduces when a current is applied^[183,184]. As I_b increases the probability of a detection event given a photon increases, therefore the efficiency (and hence count rate) will increase

until it reaches a maximum. Any further increase in I_b will not increase the efficiency, but once it reaches the critical current I_C (i.e. the current at which T_C is equal to the SNSPD temperature) the SNSPD will go to the normal state and not register any further detection events.

In the long wavelength limit the calibration curve has the form of an error function, and in the short wavelength limit a step function with a ‘turn on’ I_b (see Fig. 2 of Ref.^[185]). In addition, when normalising for absorption efficiency the ‘turn on’ I_b is smaller for shorter wavelengths due to their larger energy^[186]. A similar relationship exists for SNSPD temperature as higher temperatures results in lower values of I_C . This means that at fixed photon wavelengths the I_b required for ‘turn on’ is lower at higher temperatures as less photon energy is required to reach T_C ^[186]. However at higher temperatures, when turning up I_b it is possible to reach I_C before attaining maximum efficiency^[71].

Through preliminary testing by varying I_b and viewing the V_{det} traces on an oscilloscope we determined a suitable I_b range for each SNSPD to measure the calibration curves. The optical setup for these measurements was identical as for the VDT measurement. We first measure the calibration curves at 1550 nm, but to test the variation with wavelength we varied the tunable laser. We fixed the VDT on the quTAU for each SNSPD based on the results of the previous section, then we varied I_b and recorded the count rate for each SNSPD on a given waveguide.

We see example calibration curves in Fig. 4.8 (a) for four SNSPDs on the same waveguide. The bias currents are of the same order as for the fibre-coupled version of the same material and design which is reassuring^[71,177]. They are unnormalised so the curves are as we expect: the first SNSPD the light passes has the highest count rate, and so on down to the fourth SNSPD with the lowest count rate. We see the 1st, 3rd and 4th SNSPD reach maximum efficiency, but the 2nd goes to the normal state before this which suggests it is at a higher temperature. In addition to this the 4th SNSPD looks to be at a lower temperature compared to the rest as it has a higher ‘turn on’ I_b despite seeing the same

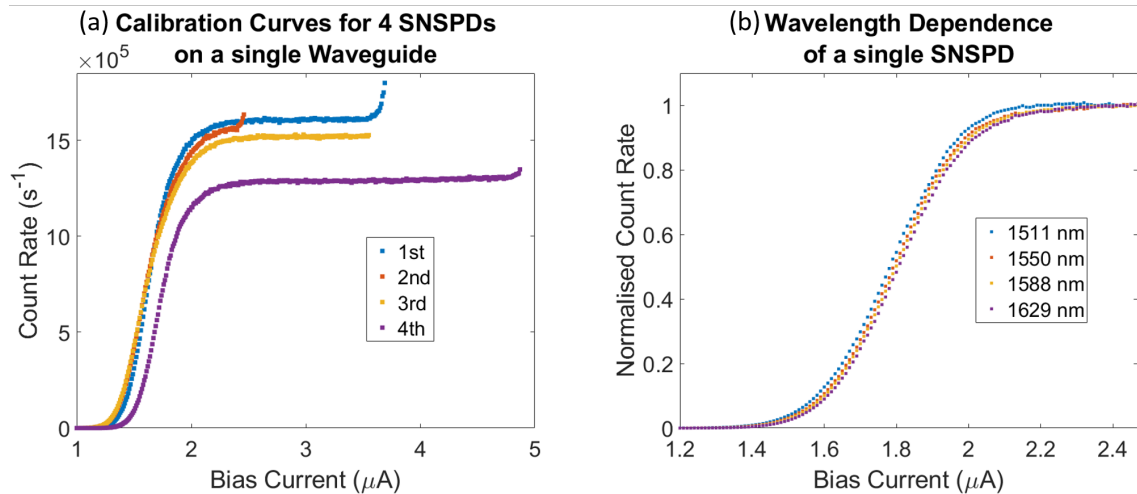


Figure 4.8: Example of calibration curves. (a) Curves from four SNSPDs on the same waveguide. The legend indicates the order in which the light passes the SNSPDs on the waveguide. (b) Calibration curves for different wavelengths incident on the same SNSPD. Each has been normalised to 1 for comparison.

wavelength as the other three. This all supports the earlier claim that the chip has a non-uniform temperature.

In Fig. 4.8 (b) we see the wavelength dependence of a single SNSPD. Even though we used the full range of the laser, unfortunately the wavelengths are quite close together so the difference between the curves isn't very pronounced. We do, however, see a lower 'turn on' I_b for the higher wavelength and vice versa.

The next thing to consider is varying I_b with the SNSPDs in total darkness to measure the dark count rate. As we discussed in Sec. 2.4.2, the dark counts are higher for higher values of I_b , and we can already see the effect of dark counts in Fig. 4.8 (a) (it is most pronounced for the 1st SNSPD). To measure this we varied I_b and measured the count rate of the SNSPDs with metal caps on the optical fibre breakout box and the room in total darkness.

Figs. 4.9 (a) and (b) show the dark count rate vs I_b for the 2nd and 4th SNSPD along the waveguide respectively (see Fig. 4.8 (a)). These two in particular have been shown as the demonstrate both extremes of I_C on the same waveguide. The plots have the same form as the dark count rate plots in Figs. 1 (b) and 3 (b) of Ref.^[71], which is reassuring

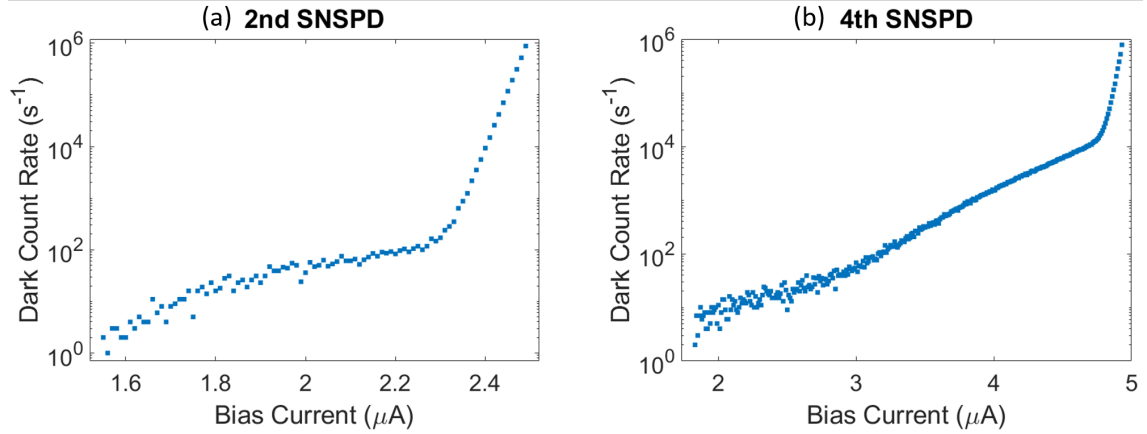


Figure 4.9: Dark count rate vs bias current for on-chip SNSPDs. (a) and (b) are the 2nd and 4th SNSPD along waveguide respectively, with reference to Fig. 4.8 (a).

as in both cases the SNSPDs were fabricated at NIST and have the same design and material. Due to the suspected temperature difference we notice that the dark count rate is significantly higher in Fig. 4.9 (a) for the same I_b compared to (b).

From the calibration curves and dark count rates we gain three main insights into optimal operation of SNSPDs. The first is that it is necessary to bias at a high enough I_b such that maximum efficiency is achieved. The second is that to minimise dark counts we must bias at as low an I_b as possible. Both of those combined indicate that we need to select I_b such that we are as far as possible to the left on the maximum efficiency plateaus in Fig. 4.8. This means that, for example, for the 4th SNSPD we would select $I_b \approx 2.5\mu\text{A}$. The third is that the colder the SNSPDs are the better, as the dark count rate is lower at lower temperatures.

4.4.3 Dead Time

Single-photon detector dead time was discussed in Sec. 2.4 when considering the duration of SNSPD and TES detection voltage traces, as the trace duration (i.e. the time it takes to return to the superconducting state) limits the detection rate. In this section we will quantify the dead time for on-chip SNSPDs by measuring the inter-arrival time of photons. Note that in the literature dead time is sometimes referred to as ‘reset time’ or ‘recovery

time’. To do this we again used the same optical setup as Secs. 4.4.1 and 4.4.2 with CW light at 1550 nm, and we biased each detector at its optimal I_b . This time we recorded the arrival time of the V_{det} pulses with the quTAU as opposed to measuring just the number of counts in a given time interval. We then calculated the pulse inter-arrival time from the difference between the successive arrival times. For CW light this should follow an exponential distribution, but due to the dead time it will rise from zero and peak before falling exponentially^[187].

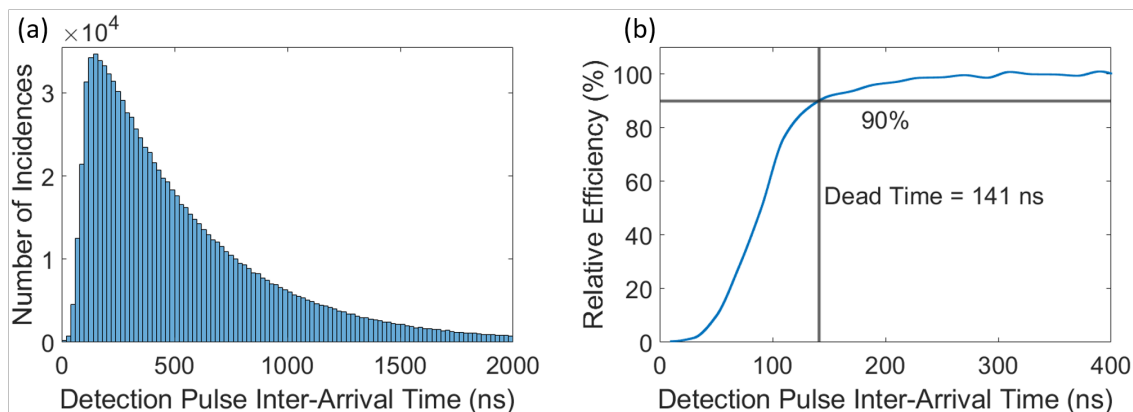


Figure 4.10: Dead time determination plots for the 2nd SNSPD from figs 4.8 and 4.9. (a) Histogram of V_{det} pulse inter-arrival time. (b) Data from (a) normalised by an exponential fit of the tail of (a). We define the dead time as the time it takes to reach 90% relative efficiency which is shown on the plot.

We bin the inter-arrival times in a histogram to see this, and an example can be seen in Fig. 4.10 (a). This example is the ‘2nd SNSPD’ from figs 4.8 and 4.9. We then fitted exponentials to the tails of the histograms, and normalised each histogram with its exponential fit (Fig. 4.10 (b)). This is a similar procedure to Ref.^[188] for calculating the dead time. This normalised histogram can be thought of as a relative efficiency vs pulse inter-arrival time, as the detection probability is decreased when an SNSPD absorbs a photon before it has completely returned to the superconducting state. We define the dead time after a detection event the SNSPD takes to reach 90% of its total efficiency.

We suspect that the chip (and hence SNSPDs) had poor thermal contact which would have resulted in longer dead times than expected for an SNSPD of the same design and

material due to taking longer to cool back down after a detection event. For the example in Fig. 4.10 the dead time was $141 \pm 2\text{ns}$, and this was one of the SNSPDs whose calibration curve indicated that it was at a higher temperature. The longest dead time we measured was $144 \pm 3\text{ns}$, and the shortest was $65.3 \pm 0.1\text{ns}$. With better thermal contact the dead time for WSi SNSPDs is typically $\approx 30\text{ns}$ ^[189]. There was also a strong correlation between a shorter dead time and a higher I_C , further supporting the uneven temperature on the chip.

In terms of SNSPD characterisation the dead time tells us which repetition rate we can run at and still achieve maximum efficiency, which is important to consider when designing an experiment. Due to the poor heat-sinking and low thermal conductivity of silica we lose approximately a factor of 4 with the waveguide configuration. For more information on the specifics of dead time in relation to SNSPD structure see Ref.^[81].

4.4.4 Jitter

When doing an experiment involving the detection of single photons, it is common to use a pulsed laser at a constant repetition rate for their generation (see Secs. 2.2.5 and 6.3.2). In this instance we can expect the V_{det} pulses to arrive at a fixed time delay with respect to the laser pulses. There is actually a small variation in this time delay which is known as ‘jitter’. This is important as it sets the arrival time resolution of an SNSPD^[11]. In an experiment multiple factors correspond to the total jitter, such as laser pulse duration, laser electrical trigger jitter, electronic amplification jitter, and so on.

There is, however, jitter intrinsic to the SNSPDs themselves. This is partly due to the detector geometry, which we can see in Fig. 2.8. Light is usually focused onto the centre of the meander structure, but some photons will be absorbed at the extreme ends. This results in a spread of arrival times for the V_{det} pulses as the arrival time will depend on how far the pulse has to travel along the meander^[186,190]. In addition, for a fixed VDT the higher the V_{det} pulse the lower the jitter. By considering Fig. 4.6, the jitter is the horizontal (i.e. temporal) variation in this pulse. If the pulse is larger then the rising

edge is steeper, leading to a smaller time duration to reach the VDT. The above is a simplification, for an in-depth theoretical analysis of SNSPD jitter see Ref.^[191].

Currently the lowest reported SNSPD jitter is for a Niobium Nitride (NbN) device, and at 1550 nm it was 4.3 ± 0.2 ps^[192]. This was achieved because NbN SNSPDs operate at higher values of I_b , but the disadvantage is they have lower efficiencies compared to WSi^[177]. Another common material is Molybdenum Silicide (MoSi) which is intermediate, with higher jitter than NbN (10.6 ps^[193]), but lower efficiency ($87.1 \pm 0.5\%$ ^[80]) than WSi. Typical jitter values for WSi are 150 ps^[71], but we recall from Sec. 2.4.2 they can have efficiencies as high as 98% ^[20]. The values quoted above are the best-case scenarios meaning that they were obtained with I_b close to I_C , therefore the jitter at a more optimal I_b will be higher.

To measure the jitter we followed the procedure in Ref.^[194]. We illuminated the waveguides with pulses from a Menlo Systems C-Fiber laser, which gave <90 fs pulses at 1560 nm with full-width at half maximum (FWHM) 40 nm at a repetition rate of 100 MHz. We attenuated the pulses such that only one in every ≈ 100 pulses resulted in a detection event. The laser system also gave an RF trigger synchronised with the optical pulses which we used as a time reference. The RF trigger and the voltage detection pulses were collected by an arrival time correlator (HydraHarp 400 by PicoQuant), which measured the relative time delay between the laser RF pulse and the SNSPD detection pulse. Ref.^[194] explains that the relative time delays follow a Gaussian distribution, and the total system jitter is defined as the FWHM of this distribution. We therefore fitted Gaussians to histograms of the relative time delays to compute this.

Ref.^[194] also explains that as the components that make up the total system jitter add in quadrature, in order to make an accurate measurement it is necessary to ensure that the SNSPD jitter is dominant. We estimated the jitter of the laser RF trigger by comparing it with the signals from the laser pulse on a photodiode, and found it to be <1 ps. The laser pulse itself had a short enough duration to be negligible, and the arrival

time correlator had an intrinsic jitter of <12 ps. Given that there is a relation between pulse amplification and jitter, we decided to include this as part of the SNSPD jitter as the pulse will always require amplification before processing. We used the same readout electronics as described in Sec. 4.3.2 for this measurement. As we expected the SNSPD jitter to be ≥ 150 ps, we neglected all other contributions and took the Gaussian FWHM as the actual value. We did this for a representative selection of SNSPDs and a range of I_b values.

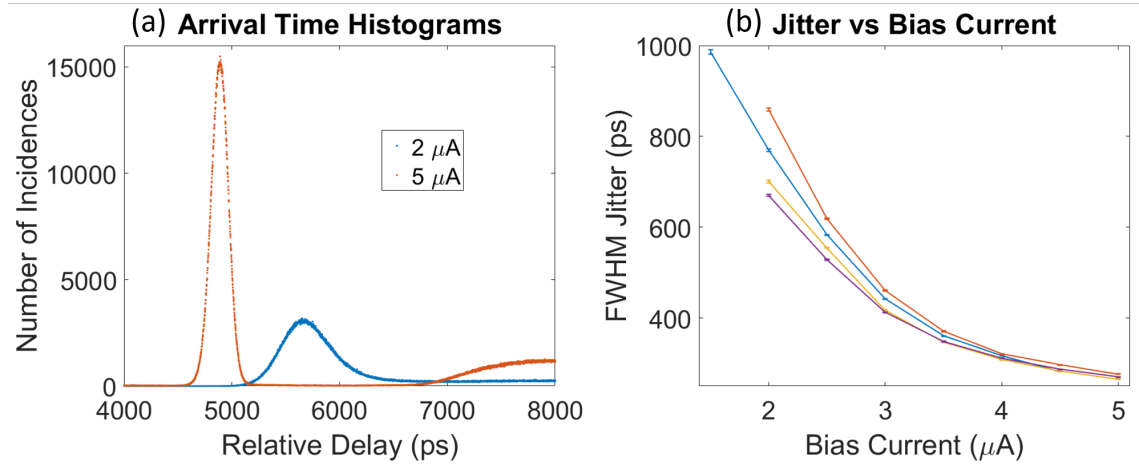


Figure 4.11: (a) Histograms of relative arrival time delays between the laser RF triggers and V_{det} pulses for the same SNSPD at $2 \mu\text{A}$ and $5 \mu\text{A}$. The bin size is 1 ps. The FWHM of the Gaussian fits to the peaks is what we take to be the jitter. (b) Measured jitter vs bias current for a selection of SNSPDs across the chip. The error bars are from the uncertainties in the Gaussian fits.

First we show example histograms in Fig. 4.11 (a). These are from the same SNSPD at two different values of I_b . There is some noise present but the peaks due to detection events are clear. We see that the peak for the lower I_b value is broader compared to the higher one which is expected. A higher bias current results in a faster-travelling detection pulse, hence the shift to the left for the peak of $5 \mu\text{A}$ compared to $2 \mu\text{A}$. When I_b is very low (and hence the count rate is lower) there comes a point where the Gaussian peak is not discernable from the background. In Fig. 4.11 (b) we see jitter vs I_b for a selection of SNSPDs. For a fair comparison the four examples shown have similar values of I_C . The reason for this jitter vs I_b relationship is still not fully understood, but possible causes are

discussed in the caption of Fig. 10 from Ref.^[195].

The lowest FWHM jitter we measured was 265.1 ± 0.6 ps which was at $5\mu\text{A}$. The optimal I_b for that SNSPD was $\approx 3\mu\text{A}$, and at that value the jitter was 418 ± 1 ps. This is somewhat larger, though of the order of magnitude, as the best case WSi value quoted in Ref.^[71] of 150 ps. There is one additional component in our setup which could account for this which is the PCB. In a standard SNSPD setup the meander is wirebonded to contact pads which are directly connected to an electrical socket (e.g. SMP or SSMCX). But in our setup the SNSPDs were wirebonded to contact pads on a PCB which tracked to SMA sockets (see Fig. 4.2), and the resistance and capacitance of this additional path could have broadened the electrical pulse.

4.4.5 Efficiency

The final measurement we will consider is efficiency. There are two separate SNSPD efficiencies which are absorption efficiency η_{abs} and quantum efficiency η_Q . η_{abs} is the efficiency of photon absorption by the SNSPD, and regular fibre-coupled SNSPDs can be optimised by placing the SNSPD in a cavity (see Sec. 2.4.2). η_Q is the efficiency of a detection event resulting from photon absorption. In Sec. 4.2.1 and previously in this section it was mentioned that the SNSPD design was not optimised for absorption, and that the v-grooves shifted during cooldown, therefore we were not expecting particularly good efficiency results when attempting to couple light onto the chip. The measurement we will consider is the total efficiency for one waveguide $\eta_{\text{total}} = \eta_{\text{abs}}\eta_Q$, which is the probability a photon is registered as a detection event in any of the SNSPDs given it made it onto the waveguide. Note that absorption by the waveguide itself has been neglected. Given the setup it was impossible to determine that quantity, but measurements on similar silica waveguides have shown that the absorption is expected to be low enough such that it can be neglected^[196].

Measuring η_{total} for a particular waveguide involved measuring optical powers and count rates, and some of the quantities can be seen in Fig. 4.12. They can be summarised

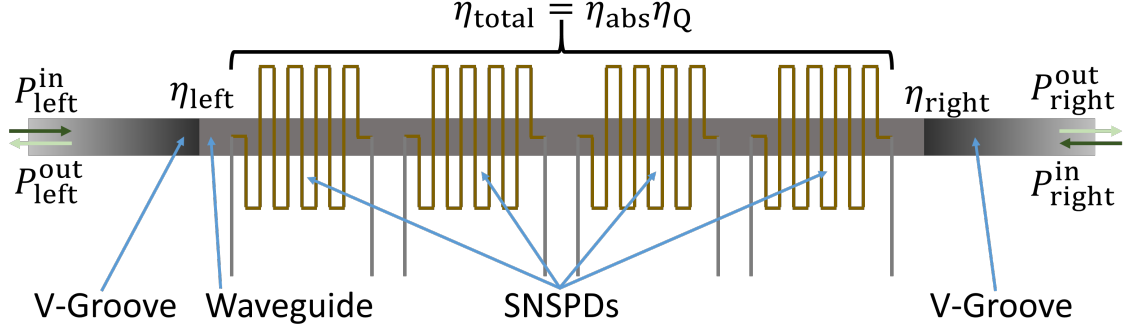


Figure 4.12: Schematic of four SNSPDs on a waveguide showing optical powers and all efficiencies we will consider, description in main text. The SNSPDs with respect to the waveguide diameter are to scale, but their spacing and the waveguide length are not to scale (also the actual number of meanders is ≈ 64).

by the three equation:

$$P_{\text{right}}^{\text{out}} = \eta_{\text{left}} \eta_{\text{right}} (1 - \eta_{\text{abs}}) P_{\text{left}}^{\text{in}} \rightarrow \eta_{\text{left}} \eta_{\text{right}} (1 - \eta_{\text{abs}}) = K_{\text{total}}, \quad (4.1)$$

$$P_{\text{left}}^{\text{SNSPD}} = \eta_{\text{left}} \eta_{\text{abs}} \eta_Q P_{\text{left}}^{\text{in}} \rightarrow \eta_{\text{left}} \eta_{\text{abs}} \eta_Q = K_{\text{left}}, \quad (4.2)$$

$$P_{\text{right}}^{\text{SNSPD}} = \eta_{\text{right}} \eta_{\text{abs}} \eta_Q P_{\text{right}}^{\text{in}} \rightarrow \eta_{\text{right}} \eta_{\text{abs}} \eta_Q = K_{\text{right}}. \quad (4.3)$$

When we send light in from the left with power $P_{\text{left}}^{\text{in}}$, the output power on the right $P_{\text{right}}^{\text{out}}$ is given by Eq. (4.1). This takes into account fibre coupling efficiency onto the chip from the left and off the chip on the right (η_{left} and η_{right} respectively), and the fraction of the light that is not absorbed by any of the four SNSPDs ($1 - \eta_{\text{abs}}$). The total optical power from all four SNSPDs combined P^{SNSPD} is given by $P^{\text{SNSPD}} = hcR_{\text{count}}/\lambda$, where R_{count} is the sum of count rates from the four SNSPDs at $\lambda = 1550$ nm. When sending light in from the left we expect Eq. (4.2), and from the right Eq. (4.3). η_Q is the combined SNSPD quantum efficiency, i.e. the probability that a photon will cause a detection event given it has been absorbed by an SNSPD. The quantities K_{total} , K_{left} and K_{right} are constant ratios that can be obtained by dividing the powers. We have three equations with four unknowns, so we also require an independent measurement of η_{abs} using the Bragg gratings.

Unfortunately there was only one candidate waveguide to measure the maximum pos-

sible η_{total} with this setup. With reference to Fig. 4.2 (b), there were only two waveguides with SNSPDs with bi-directional optical access (F6 & F7 and F10 & F11). Of these two waveguides on only one of them did all four SNSPDs have their optimal I_b below I_C , which was the waveguide with SNSPDs X1, X3, X6 and X8. The calibration curves in Fig. 4.8 (a) are from the other waveguide with bi-directional access, and the 2nd SNSPD in this (which is X14) goes to the normal state before reaching its optimal I_b , meaning its η_Q does not saturate. We used the CW Santec laser at 1550 nm as a light source and optimised polarisation with fibre paddles. The only advantage of the poor v-groove couplings was that we could send in optical powers that we could measure with a commercial powermeter, rather than having to use a more advanced technique to measure small powers^[197]. We used a well-calibrated Thorlabs InGaAs photodiode (S154C) for this purpose. The count rates were measured with each SNSPD at its optimal I_b (i.e. at the start of the plateaus in Fig. 4.8 (a)) using the quTAU.

Our collaborators at the University of Southampton attempted to measure η_{abs} at room temperature through the Bragg gratings, but unfortunately it was not possible to obtain an accurate value. The measurement did show that it was very small ($\eta_{\text{abs}} < 1\%$), which agrees with simulations of SNSPD absorption performed by our collaborators at NIST for this configuration. Based on this information, and by combining Eqs. (4.1) to (4.3), we can estimate η_{total} from:

$$\frac{1 - \eta_{\text{abs}}}{\eta_{\text{total}}^2} = \frac{K_{\text{total}}}{K_{\text{left}} K_{\text{right}}} \rightarrow \eta_{\text{total}} \approx \sqrt{\frac{K_{\text{left}} K_{\text{right}}}{K_{\text{total}}}}, \quad (4.4)$$

where we have made the approximation $1 - \eta_{\text{abs}} \approx 1$. We estimated $\eta_{\text{total}} = (0.98 \pm 0.03)\%$, the uncertainty was obtained by propagating the power and count rate measurement uncertainties using Eq. (4.4). This results in a mean η_{total} per SNSPD of $(0.245 \pm 0.006)\%$. The same material and design of SNSPD has been measured to have a near unity η_Q ^[177], therefore η_{abs} is usually the limiting factor.

4.5 Conclusion

In this chapter, we have discussed the characterisation results of WSi SNSPDs deposited on silica waveguides. The materials were a promising choice, as WSi SNSPDs have very high η_Q ^[177], and silica waveguides couple well to single-mode optical fibres^[176]. Despite this, we were limited by the poor thermal conductivity of silica, which resulted in a uneven chip temperature and slower SNSPD re-thermalisation time after detection events. The higher temperatures resulted in lower values of I_C , which compromised the ability to reach the optimal I_b and increased dark count rates. Also, the slower re-thermalisation time resulted in a longer dead time, decreasing the maximum operating rate. In addition, as the SNSPD design was not optimised for absorption its η_{abs} was very low.

In a wider sense we also investigated the practicalities of operating SNSPDs in general. We learned that a lower I_b helps to reduce dark counts, but that a higher I_b gives a larger VDT range, a higher η_Q (up to saturation) and lowers jitter. The most common approach is to use what I've been calling the 'optimal I_b ' which is biasing at the lowest current value which saturates η_Q , i.e. as far to the left as possible on the plateau region of the calibration curve. We also found the colder the temperature (with good thermal contact) the better, as colder temperatures reduce dark count rates and dead times, and enable operation at higher values of I_b with less compromise. We mainly considered WSi SNSPDs, but we also discussed the benefits and drawbacks of other materials, for example such as reduced jitter but lower η_Q for NbN.

Despite the shortcomings of the on-chip SNSPD setup the SNSPDs still worked as SNSPDs and registered detection events when we sent light into the waveguides. Also, although the measured quantities were worse compared to the standard fibre-coupled devices, where relevant they were of the same order of magnitude. In light of this there are some areas for further advancement. First, we believe the type of adhesive used when gluing the fibre v-grooves to the chip contributed to the poor coupling after cooling, therefore a first step would be to investigate alternative adhesives. Should a high-efficiency

measurement be desired over a weak non-demolition measurement, one way to improve this would be to optimise the SNSPD design to improve modal overlap. Another addition would be to introduce a Boron-doped silica overclad on top of the waveguide layer after depositing the SNSPDs, which would work to reduce waveguide propagation loss. Using the Bragg grating technology it would also be possible to design a cavity around the SNSPDs in the waveguide itself optimised for the desired wavelength to increase η_{abs} , similar to Ref.^[176] for TESs. As a final note on the subject of scaling up, a way to increase the total efficiency and provide a degree of number resolution would be to add many more SNSPDs per waveguide. Even with the possibility of a re-designed SNSPD geometry there would be space for hundreds on one waveguide, therefore in this instance the electronic readout would be the limiting factor.

Chapter 5

Broadband Spectroscopy with a Single SNSPD

In Sec. 2.4.2 we discussed the concept and operation of a Superconducting-Nanowire Single Photon Detector (SNSPD), and in the previous chapter we investigated their properties and characteristics in depth. In this chapter we will consider the potential application of using SNSPDs to measure optical power spectra through exploiting their intrinsic wavelength dependence (see Fig. 4.8 (b)). We will build on the work of Reiger *et al.* in Ref.^[68] to obtain a broadband power spectrum from measuring count rate vs SNSPD bias current of weak optical signals.

The work in this chapter is based on the theory developed by Jelmer Renema in Ref.^[198], who worked very closely with me on this project as a mentor. We both contributed greatly to the experimental setup and data analysis, and the data acquisition software was developed by Andreas Eckstein.

We will first discuss some common spectroscopy methods for both weak and strong optical signals, and then consider the methods in Ref.^[68] for spectroscopy using SNSPDs alone. I will then introduce the more advanced theoretical framework of Ref.^[198] used in this chapter. Following this we will consider the experimental setup used to implement

the framework, and end by discussing the results and limitations of this scheme.

5.1 Background

As a background to the work in this chapter, we will first discuss the different types of optical spectroscopy, and then discuss the idea of spectroscopy using the SNSPD wavelength dependence alone in terms of the previous work of Ref. [68].

5.1.1 Current Spectroscopy Techniques

There are currently two broad family of spectrometers in the visible to near infrared (NIR) optical range, which are grating spectrometers and time-of-flight spectrometers.

Grating Spectroscopy

Grating spectroscopy is when (broadband) light is spatially split into its spectral components, essentially mapping frequency to space. This could be with a diffraction grating or a prism. We see an example setup to perform this in Fig. 5.1. The light we wish to spectrally analyse is split into its spectral components, which propagate to a detector or multiple detectors. The type of detector used will depend on the optical power and wavelength

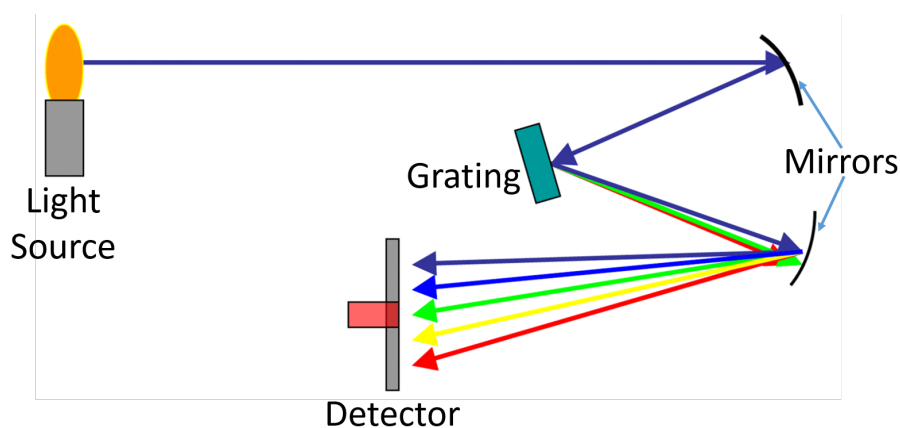


Figure 5.1: Example setup schematic for performing grating spectroscopy, description in main text. Source: adapted from https://upload.wikimedia.org/wikipedia/commons/thumb/f/f6/Spectrometer_schematic.gif/1920px-Spectrometer_schematic.gif.

range. Its spectrum can then be obtained from the detection outcomes and knowledge of how different wavelengths are separated with the setup.

It is possible to obtain very high wavelength resolution using this technique, for instance the optical spectrum analyser we used for our experiment can resolve ≈ 10 pm. In addition it will work for continuous wave (CW) or pulsed light signals. However a disadvantage is that multiple detectors or moving parts are required as the light is spatially split. This can be done with a stationary multiple pixel camera, or with one detector that spatially scans (imagine the red detector in Fig. 5.1 scanning up and down).

Time-of-Flight Spectroscopy

Time-of-flight spectroscopy works by mapping frequency to time. With reference to Fig. 5.2, a light pulse is stretched in time by a temporally dispersive optical fibre, and sent to a detector with a very high temporal resolution. The different spectral components will arrive at the detector at different times, therefore time bins can be assigned and the optical power in each time bin is measured by the detector. The spectrum is obtained from these measured powers and knowledge of the fibre dispersion^[199,200].

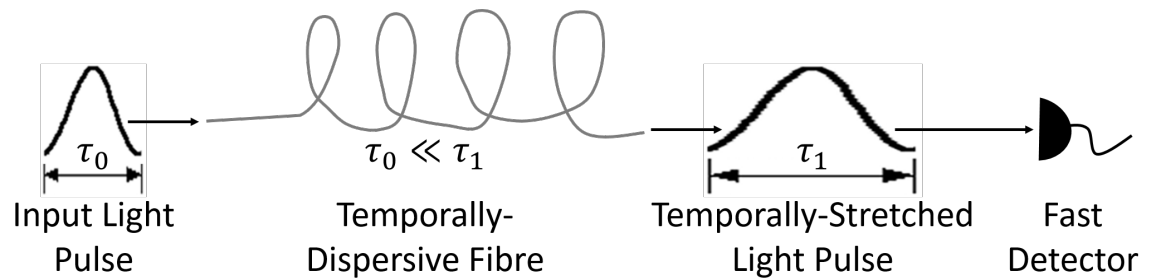


Figure 5.2: Example setup schematic for time-of-flight spectroscopy, description in main text.

Only a single detector with no moving parts is required to perform time-of-flight spectroscopy, however it is only possible to obtain spectra for pulsed light sources. Also, the spectral resolution from this technique tends to be worse compared to that from grating spectroscopy. The resolution depends on how much the light pulse is dispersed and on the temporal resolution of the detector and measurement apparatus, though it is typically

≈ 0.1 nm.

5.1.2 Spectroscopy using SNSPDs

Consider the plot of SNSPD calibration curve wavelength dependence in Fig. 4.8 (b). The wavelength range depicted there is very small, but we can discern a difference in the curves. Now consider plotting this for a much larger wavelength range at larger wavelength intervals (see for example Fig. 2 from Ref.^[185]). If we measure the calibration curve for some broadband optical signal, it is plausible that we could use the individual wavelength calibration curves to decompose the signal into spectral components. This procedure only requires a single SNSPD, and as it relies on the intrinsic wavelength dependence alone the optical signal could be continuous or pulsed. The requirements on the signal are that it has a low enough power such that the SNSPD non-linear effects are negligible^[82], and a spectrum lies within the calibrated wavelength range.

This idea was first presented by Reiger *et al.* in Ref.^[68]. They measured calibration curves for the wavelength range 700 nm-1000 nm at intervals of 50 nm, then measured the calibration curves again at the same wavelengths but arbitrary intensities compared to the initial ones. They then attempted to pick out the correct wavelengths by calculating normalised sums of residuals with the original calibration curves and finding the minimum (Fig. 9 of Ref.^[68]). They report that this method was mostly successful. As a further step they measure calibration curves for 530 nm and 900 nm, and then measure the same for a signal of the two wavelengths combined. They qualitatively show that the combined signal calibration curve is the sum of the two individual calibration curves, but this is an initial proof-of-principle with little mathematical insight.

5.2 Spectral Decomposition of Broadband Optical Signals

In this section we will mathematically formalise a procedure to decompose a broadband optical signal into its spectral components, and provide motivation for doing so based on

potential practical applications.

5.2.1 Mathematical Formulation

The majority of this section had been adapted from Ref.^[198]. For a low-power broadband photon source (i.e. incident photon flux <10 MHz), the measured count rate $R(I_b)$ of an SNSPD at a given bias current I_b is:

$$R(I_b) = \int S(E) \eta(E, I_b) dE, \quad (5.1)$$

where E is photon energy, $S(E)$ is photon power spectrum (power in a given wavelength interval) and $\eta(E, I_b)$ is the total efficiency. Now consider discretising Eq. (5.1) (i.e. $\int \rightarrow \sum$ and $d \rightarrow \Delta$) for k specific values of photon energy E_i . We obtain:

$$R(I_b) = \sum_{i=1}^k M(I_b, E_i) S(E_i), \quad (5.2)$$

where $M(I_b, E_i) = \eta(E_i, I_b) \Delta E_i$ is the measured count rate at I_b and E_i per unit power. Eq. (5.2) is for only one value of I_b , so we can turn it into a vector equation for j values of I_b :

$$\mathbf{R}(I_b) = \mathbf{M}(I_b, \lambda) \mathbf{S}(\lambda), \quad (5.3)$$

where photon energy has been interchanged for photon wavelength λ . In this equation $\mathbf{R}(I_b)$ is a j -component column vector, $\mathbf{M}(I_b, \lambda)$ is a $j \times k$ matrix and $\mathbf{S}(\lambda)$ a k -component column vector of optical power in a given wavelength interval.

When performing spectroscopy the quantity we wish to obtain is $\mathbf{S}(\lambda)$, and the principle of this technique is to obtain it from Eq. 5.3 given we can measure $\mathbf{R}(I_b)$ and $\mathbf{M}(I_b, \lambda)$. We first obtain $\mathbf{M}(I_b, \lambda)$ by measuring calibration curves at j different values of I_b for k different values of λ . For each different λ we measure optical power with which we normalise each respective calibration curve. Therefore for the optical signal we wish to spectrally decompose we measure $\mathbf{R}(I_b)$, i.e. its calibration curve at the same j values

of I_b , and then invert Eq. (5.3) to obtain $\mathbf{S}(\lambda)$.

If we select $j = k$ then obtaining $\mathbf{S}(\lambda)$ from Eq. (5.3) is simply a matrix inversion. It is obvious that if $j < k$ then no form of inversion is possible as we have an under-constrained problem, but for $j > k$ we have an over-constrained problem and can use a pseudoinversion such as least-squares minimisation^[201]. We write:

$$\mathbf{S}(\lambda) = \mathbf{M}^+(I_b, \lambda) \mathbf{R}(I_b), \quad (5.4)$$

where $\mathbf{M}^+(I_b, \lambda)$ is the pseudoinverse of $\mathbf{M}(I_b, \lambda)$ and is now a $k \times j$ matrix. The matrix $\mathbf{M}^+(I_b, \lambda)$ is never actually calculated when using least-squares minimisation, the process simply takes $\mathbf{M}(I_b, \lambda)$ and $\mathbf{R}(I_b)$ as arguments and outputs $\mathbf{S}(\lambda)$.

An issue with regular matrix inversion is that if a matrix is very close to being singular then errors are greatly amplified when performing an inversion^[198,202]. By considering Eq. (5.3) for $j = k$, even the slightest of error in the components of $\mathbf{R}(I_b)$ lead to greatly amplified errors in the components of $\mathbf{S}(\lambda)$. We see worked examples of this in Refs.^[198,202], but the situation is equally true when performing pseudoinversions^[203]. We are able to quantify how ill-conditioned an inversion is with the matrix condition number, in this case of $\mathbf{M}(I_b, \lambda)$. This quantity is introduced and explained in Sec. 1.1.1 of Ref.^[202], but in short the larger the condition number the more sensitive the inversion is to small deviations, and the identity matrix has a condition number of 1.

5.2.2 Spectrometer Wavelength Resolution

One final consideration is the wavelength spacing $\Delta\lambda$, which sets the resolution of SNSPD spectroscopy. If all of the calibration curves were step functions with a well-defined λ -dependent ‘turn-on’ I_b then we could make $\Delta\lambda$ arbitrarily small. However, there is some overlap between calibration curves at different values of λ , which is larger for smaller $\Delta\lambda$. This sets a maximum resolution.

By considering a calibration curve (e.g. Fig. 4.8 (b)), the region where the count rate

starts to increase from zero is approximately exponential, and can be modelled by the equation^[204]:

$$R(I_b) \propto \exp\left(\frac{I_b - I_{th}}{I^*}\right), \quad (5.5)$$

for $I_b < I_{th}$. I_{th} is the threshold current which is defined as the value of I_b required to reach a certain detection probability that we can arbitrarily define, and I^* is a characteristic current scale which is determined experimentally^[74,205]. In other words, with reference to a calibration curve, I_{th} is the ‘turn-on’ I_b , and I^* characterises how sharp this turn-on is. From Eq. (5.5), a calibration curve will approximate a step function with turn-on I_{th} when I^* is very small.

We then linearise the photon energy dependence of I_{th} to $I_{th} = I_0 - \gamma E$, where E denotes the incident photon energy^[74,205]. The parameters I_0 and γ can be experimentally determined, but I_0 depends on SNSPD temperature^[74], and γ depends on the nanowire width^[204]. As the components of $R(I_b)$ at different wavelengths make up the entries of $\mathbf{M}(I_b, \lambda)$ for $I_b < I_{th}$, our goal is to estimate the minimum energy spacing ΔE (corresponding to a wavelength spacing $\Delta \lambda$) without significantly compromising the invertability of $\mathbf{M}(I_b, \lambda)$. To make a qualitative argument, the larger the energy spacing the greater the difference between the columns of $\mathbf{M}(I_b, \lambda)$. This gets us close to having a series of step functions as calibration curves (the ideal case), and is similar to the case of having a very small I^* . However, the above results in a lower wavelength resolution (i.e. a large $\Delta \lambda$), so it is necessary to find a trade-off between both.

The numerical simulations of Ref.^[198] compare $R(I_b)$ at E and $E + \Delta E$. They suggest that the best compromise is to make the numerator and denominator of the argument of Eq. (5.5) approximately equal, leading to an energy resolution of:

$$\Delta E \approx \frac{I^*}{\gamma}. \quad (5.6)$$

ΔE can be converted into $\Delta \lambda$, which can then be calculated based on SNSPD parameters.

Using typical values for WSi SNSPDs and a base wavelength of 1300 nm, we find $\Delta\lambda \approx 50 \text{ nm}$ -100 nm.

5.2.3 Motivation

The wavelength spacing required for spectroscopy with SNSPDs is quite large, meaning the resolution is quite low. However we are able to scan a very large wavelength range, as SNSPDs are sensitive from optical to NIR wavelengths^[70]. For these reasons, the main motivation for this technique is broadband spectroscopy, i.e. when only broad features of an optical spectrum over a large wavelength range are of interest. An application of

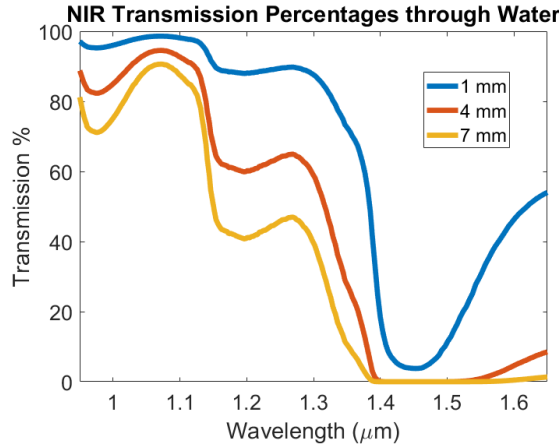


Figure 5.3: NIR transmission spectra through 1 mm, 4 mm and 7 mm of water.

this could be measuring the water content of an object (e.g. a crop) from its reflected light. Fig. 5.3 shows that optical transmission (and hence absorption) through water have broad features, making it a promising exploration avenue for this technique. This of course extends to other materials with similar properties.

An important consideration for this technique is measuring optical power when calculating $\mathbf{M}(I_b, \lambda)$, as the matrix entries have the dimension count rate per unit power. The optical power for count rates that do not saturate an SNSPD will be of the order of $10^{-14} - 10^{-13} \text{ W}$, and recalling our discussion in Sec. 2.4 this is very difficult to measure with conventional methods. A common way to measure low optical powers is by using a

well-calibrated attenuator, however this method is very prone to systematic errors. Despite this, imperfections such as SNSPD detection loss do not affect this technique, as both $\mathbf{R}(I_b)$ and $\mathbf{M}(I_b, \lambda)$ will be affected in the same way.

5.3 Experimental Realisation

In this section I will present the experimental setup we used to perform broadband spectroscopy with an SNSPD. We will first discuss the cryogenic system, then the optical setup.

5.3.1 Sorption Refrigeration

We used a PhotonSpot Helium Sorption Refrigerator (HSR) to cool the SNSPDs to ≈ 0.78 K. As with Sec. 4.3.1, cryogenics have been a large part of my D.Phil so I will outline the cooling mechanisms of a HSR.

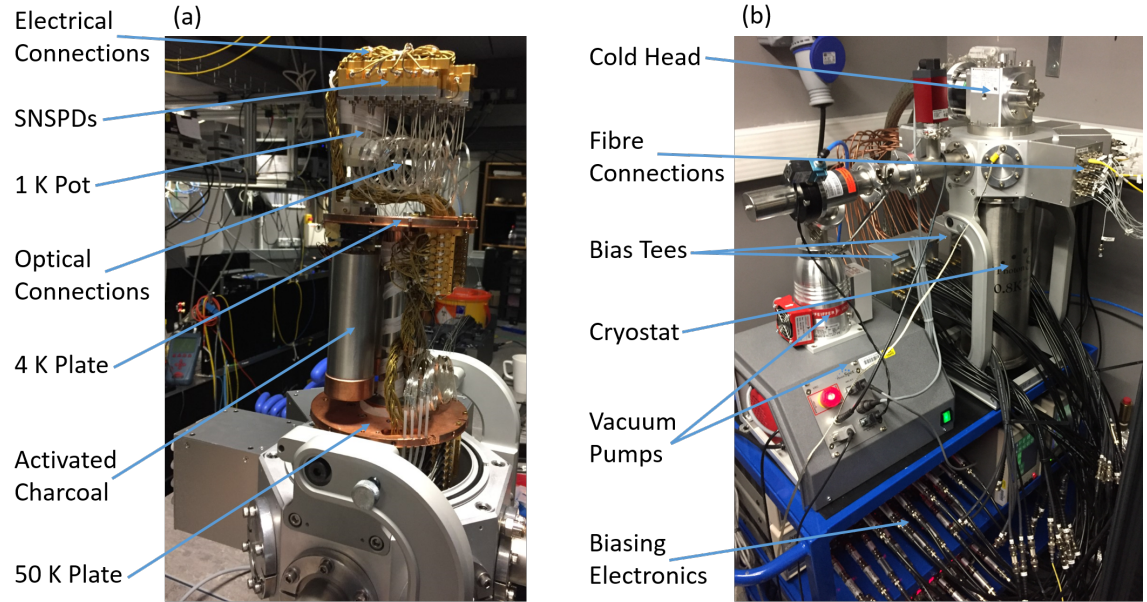


Figure 5.4: Pictures of our Helium Sorption Refrigerator (HSR). (a) The inside of our HSR with the radiation and vacuum shields removed. We can see the internal structure as well as the SNSPDs and their connections. (b) The whole closed-up setup with relevant components attached. The cryostat is upsidedown with respect to (a).

Our HSR has three temperature sections which are at approximately 50 K, 4 K and

1 K. We see these in Fig. 5.4 (a), where the SNSPDs are attached to the 1 K section (in reality at ≈ 0.78 K). The 50 K and 4 K plates are cooled by pulse tube cooling in the same fashion as for a dilution refrigerator which has already been explained in Sec. 4.3.1. To cool from 4 K to ≈ 0.78 K the HSR uses a process called Helium-4 (^4He) adsorption cooling^[206]. The parts labelled ‘1 K pot’ and ‘Activated charcoal’ in Fig. 5.4 (a) form a sealed system containing ^4He . The ^4He is condensed by the 4 K plate and the liquid collects into the 1 K pot, bearing in mind that during operation everything in Fig. 5.4 (a) is upside down. At this point the 1 K pot is at a temperature of ≈ 2.8 K. When enough liquid ^4He has condensed a thermal link cools the part with activated charcoal inside. At low temperatures charcoal adsorbs Helium which provides a low pressure, which results in the evaporation of ^4He from the 1 K pot^[207]. This process removes heat from the system and cools the SNSPDs to ≈ 0.78 K, and will continue until the reserve of liquid ^4He has been depleted. At this point the charcoal is heated which causes it to release the adsorbed ^4He ^[207], and the process repeats as the ^4He is again condensed into the 1 K pot.

The main disadvantage of this process is that, unlike a DR, a HSR is not able to continually maintain its base temperature and must be operated in cycles. For our system it is possible to maintain base temperature for 18 h when operating on a 24 h cycle. However, using activated charcoal as opposed to manually pumping avoids having moving parts (e.g. a mechanical pump) to perform evaporative cooling. As a final note, lower base temperatures (≈ 0.3 K^[206]) can be achieved using Helium-3 (^3He) as it has a higher vapour pressure compared to ^4He at the same temperature, and thus provides greater cooling power^[208]. ^3He is significantly more expensive than ^4He , therefore for SNSPD operation ^4He is perfectly adequate.

5.3.2 Optical Setup

We required light at several different wavelengths for the calibration curves, for which we used a broadband light source (a 100 W incandescent light bulb) along with optical filters. This gave us approximately a black body optical spectrum (see Sec. 2.1.2), and therefore

the multimode thermal state described in Sec. 2.2.4^[33,34], i.e. a Gaussian state.

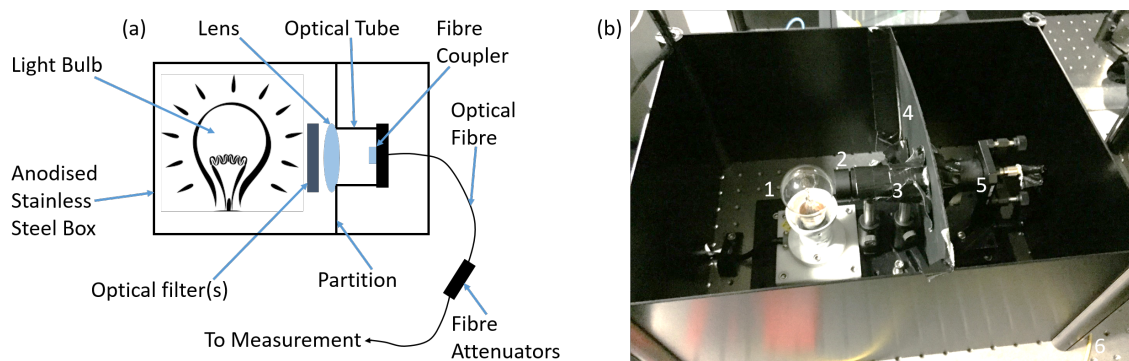


Figure 5.5: (a) Schematic of the optical setup to perform SNSPD spectroscopy, with descriptions in the main text. The light bulb image was adapted from: https://cdn.clipart.email/9d439b996f6d8df82b27dc0785661858.light-bulb-line-drawing-at-getdrawingscom-free-for-personal-use-_628-800.jpeg. (b) Picture of the optical setup. 1. 100 W incandescent light bulb. 2. Lens and filter(s). 3. Optical tube. 4. Partition. 5. Fibre coupling package. 6. SMF28 fibre.

We see a schematic and image of the experimental setup used in Figs. 5.5 (a) and (b) respectively. We focused light from the light bulb using a 3.5 mm focal length lens into the fibre coupling package of an SMF28 optical fibre. The distances and alignment were optimised by coupling the end of the fibre to a powermeter. As the containing box was partly reflective, a partition and optical tube inside the box were used to ensure that only light focused through the lens would enter the fibre. We screwed whichever combinations of filters that were being used onto the lens holder in order to minimise unfiltered light from entering the fibre. We estimated that approximately 1 W of the light bulb power was optical, and this was radiated over a whole 4π sr. With the fibre coupling efficiencies optimised, the measured power of the broadband light at the end of the optical fibre was of the order of 100 nW with no filters or attenuators.

We decided to measure calibration curves for the central wavelengths of 1.0 μm -1.6 μm , initially in intervals of $\Delta\lambda = 100$ nm, i.e. $k = 7$. To first bring the light source into this wavelength range we used a silicon filter as a high pass filter at ≈ 1.1 μm , and an Edmund Optics low pass filter at 1.6 μm . Then for the calibration curves themselves we used Thorlabs bandpass filters centred at the relevant wavelengths, each with full-width at

half-maximum ≈ 12 nm.

To measure the components of $\mathbf{M}(I_b, \lambda)$ we measured optical powers and calibration curves. After positioning the relevant filter for each wavelength, we first measured the spectral density (power per unit wavelength) with an Anritsu Optical Spectrum Analyser (OSA). We then corrected this with the wavelength-dependent transmission fractions for the Thorlabs fibre attenuators used to bring the power down to a level suitable for an SNSPD. After this we integrated the spectral density to obtain the total power at each wavelength. We did this with the light bulb switched on (signal) and switched off (background) so that we could do a background subtraction as there was some intrinsic noise on the OSA.

Following this we measured the calibration curves. We used PhotonSpot Tungsten Silicide (WSi) SNSPDs for the implementation. In Sec. 2.4.2 we learned that SNSPDs are usually fabricated within a wavelength-specific optical cavity, therefore the SNSPD sensitivity for the range 950 nm-1650 nm was an important consideration. The peak sensitivity was 90% at 1550 nm, and the minimum absorption within the defined range was above 28%. It does not matter that the sensitivity is different at different wavelengths as it will affect both $\mathbf{M}(I_b, \lambda)$ and $\mathbf{R}(I_b)$ in the same way. We then measured the calibration curves at each wavelength. PhotonSpot WSi SNSPDs have a larger nanowire width compared to the NIST fabricated ones in Chap. 4 which means their efficiency plateaus at higher values of I_b (≈ 12 μ A at 1550 nm). Due to this we measured at $I_b = 5.5$ μ A-11.0 μ A at intervals of 0.5 μ A, and at 13.0 μ A so that saturation is achieved at all wavelengths, giving $j = 13$. For each calibration curve we measured at each value of I_b for 10 s, and repeated the process 10 times and averaged to account for light bulb power fluctuations on the order of minutes (which were very small). We measured the count rates with a QTau time-to-digital converter (from QTools), the same way as in Chap. 4.

For visualisation purposes the normalised calibration curves can be seen in Fig. 5.6. They can be thought of as the columns of $\mathbf{M}(I_b, \lambda)$ normalised by their last components

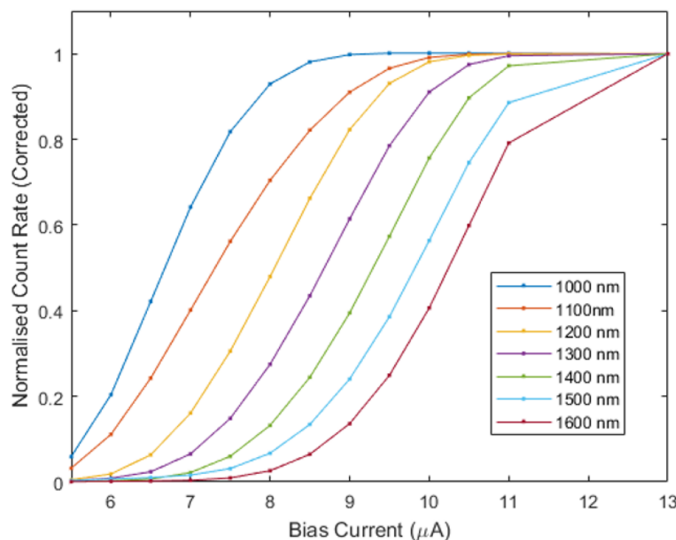


Figure 5.6: The calibration curves used to construct $\mathbf{M}(I_b, \lambda)$, for $\lambda = 1.0\text{--}1.6\mu\text{m}$ at 100 nm intervals. They have been normalised for the purpose of plotting, and the curve for $\lambda = 1.1\mu\text{m}$ has been corrected by interpolation due to a systematic error.

plotted against I_b . We notice is that the curves are much more discernable from each other compared to Fig. 4.8 (b), which is reassuring. However, the condition number (introduced in Sec. 5.2.1) for $\mathbf{M}(I_b, \lambda)$ is of the order of 10^{16} . This large value illustrates the difficulty of this inversion.

Signals for Inversion

In order to perform the inversion of Eq. (5.3) we require some form of optical signal to generate $\mathbf{R}(I_b)$. First we decided to test our procedure in a similar fashion to Ref. [68] by selecting a wavelength that made up one of the calibration curves. We selected $1.3\mu\text{m}$ and used the same bandpass filter as for the $1.3\mu\text{m}$ calibration curve, but we recorded this data separately. As a further step we decided to select wavelengths in between two calibration curves. We picked $1.25\mu\text{m}$ and $1.35\mu\text{m}$, and used Thorlabs bandpass filters with a FWHM of $\approx 12\text{ nm}$ (similar to those used for the calibration curves). Then to test the inversion with a more intricate signal we used broadband light. For this we used light from the light bulb with only the silicon long pass and Edmund Optics short pass filters

such that the signal had broader features.

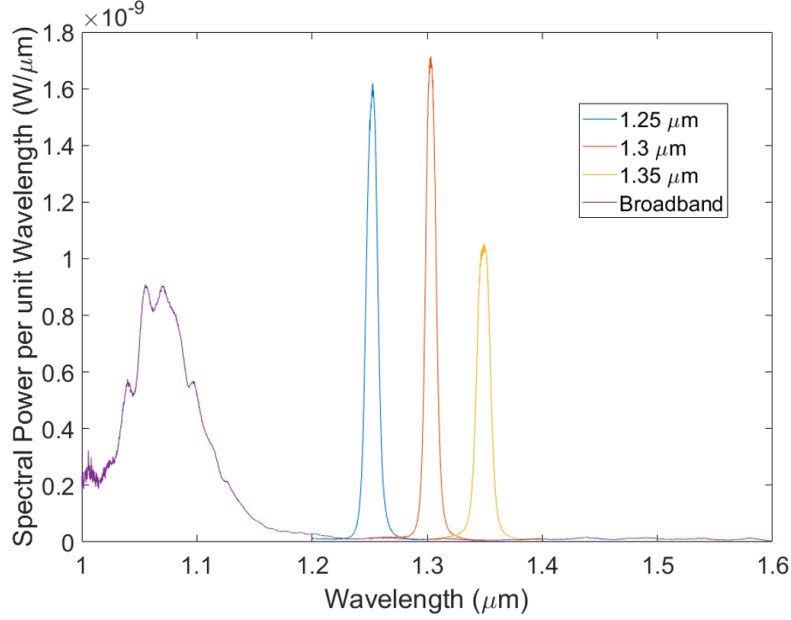


Figure 5.7: Power spectral density of the optical signals used to test the inversion.

We see the spectral density of all four optical signals used to test the inversion in Fig. 5.7. All were first measured by the OSA, and the numerical data was then corrected for fibre attenuation. More attenuators were required for the broadband signal compared to the others to achieve an SNSPD-suitable count rate, and the fibre attenuators used had greater attenuation above $\lambda = 1.2\mu\text{m}$ which is why the signal is much lower above those wavelengths. There is also noise present on the far-left of the broadband signal which was intrinsic to the OSA (i.e. it was there with the OSA in darkness), which was constant and therefore was subtracted from each spectrum measurement. The spectral densities used for the calibration curves were very similar to that of $1.3\mu\text{m}$ in Fig. 5.7, just centred at different wavelengths. $\mathbf{R}(I_b)$ for the signals were then measured in exactly the same fashion as the calibration curves, except we repeated this measurement 50 times in attempt to reduce the effect of random uncertainty on the inversion.

5.4 Inversion Results

After calculating $\mathbf{M}(I_b, \lambda)$ and measuring $\mathbf{R}(I_b)$ for each signal, we inverted Eq. (5.3) using a Matlab least-squares minimisation function with the constraint that the components of $\mathbf{S}(\lambda)$ had to be non-negative. We did this for each measurement run (i.e. 50 times) and calculated a mean and uncertainty in the components of $\mathbf{S}(\lambda)$ for each signal. We also integrated the spectral densities in Fig. 5.7 between the same wavelength boundaries to compare with the power values from the inversion. For clarity, as the calibration curve wavelengths are centred at $1.0\mu\text{m}$, $1.1\mu\text{m}$, ..., and $1.6\mu\text{m}$, the spectral bins have wavelength ranges $0.95\text{--}1.05\mu\text{m}$, $1.05\text{--}1.15\mu\text{m}$, ..., and $1.55\text{--}1.65\mu\text{m}$. Therefore the first component of $\mathbf{S}(\lambda)$ is optical power in the wavelength range $0.95\text{--}1.05\mu\text{m}$, and so on.

5.4.1 $1.3\mu\text{m}$, $1.25\mu\text{m}$ and $1.35\mu\text{m}$ Signals

The results for the bandpass filter spectral inversions can be seen in Fig. 5.8. First we

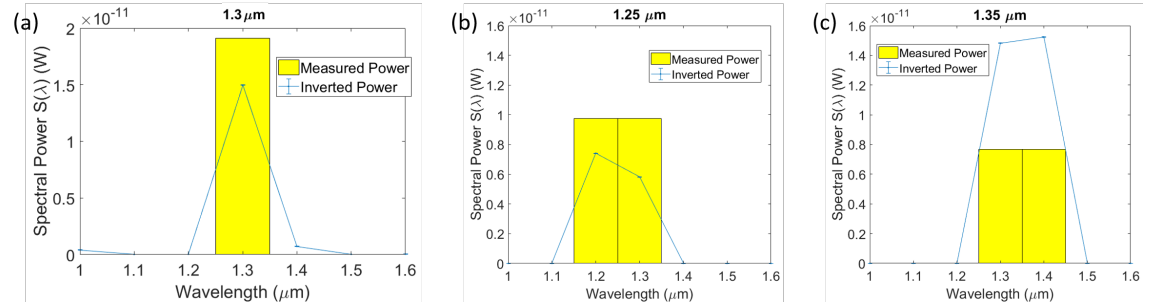


Figure 5.8: Comparison of spectral power $\mathbf{S}(\lambda)$ from optical spectrum analyser (OSA) measurements (yellow bars) and Eq. (5.3) inversions (blue lines) for $\mathbf{R}(I_b)$ as measured with bandpass filters of (a) $1.3\mu\text{m}$, (b) $1.25\mu\text{m}$ and (c) $1.35\mu\text{m}$.

see the result from the $1.3\mu\text{m}$ bandpass filter in (a). The orange curve in Fig. 5.7 was integrated, and the power obtained was assigned to $1.25\text{--}1.35\mu\text{m}$ bin (single yellow bar in Fig. 5.8 (a)). We see that the inversion agrees qualitatively as the inverted power at $1.3\mu\text{m}$ is by far the most prominent. The power measured from the OSA after correcting for attenuation and background was 19.1pW , but the sum of the inversion powers (i.e. sum of $\mathbf{S}(\lambda)$ components) was $16.2 \pm 0.2\text{pW}$, and the $1.3\mu\text{m}$ peak was $15.1 \pm 0.1\text{pW}$.

The inversion results from the 1.25 μm bandpass filter can be seen in Fig. 5.8 (b). After integrating the blue curve in Fig. 5.7 we assigned the powers to the 1.15-1.25 μm and 1.25-1.35 μm bins. Again we see qualitative agreement as only the 1.2 μm and 1.3 μm components of $\mathbf{S}(\lambda)$ are non-zero, but we also see an asymmetry and a quantitative discrepancy similar to Fig. 5.8 (a). The spectral density integration gave 19.5 pW over the whole range, whereas the sum of $\mathbf{S}(\lambda)$ components from the inversion falls short at 13.22 ± 0.02 pW. There was a slight asymmetry in the bandpass filter, though this was towards the 1.25-1.35 μm bin and was negligible in terms of optical power.

The final bandpass filter inversion result for 1.35 μm can be seen Fig. 5.8 (c). The OSA-measured power was obtained in the same fashion as for the 1.25 μm bandpass filter. There is again qualitative agreement between the measured and inverted power as only the 1.3 μm and 1.4 μm components of $\mathbf{S}(\lambda)$ are non-zero. In addition there is less asymmetry between the 1.25-1.35 μm and 1.35-1.45 μm bins. However, this time the measured power falls short of the inverted power by approximately half. We calculated the total power to be 15.3 pW, but the sum of $\mathbf{S}(\lambda)$ components gave 30.05 ± 0.02 pW.

5.4.2 Broadband Signal

The final and more advanced optical signal we attempted to invert was the broadband signal, the purple curve in Fig. 5.7. We see the OSA-measured power for each wavelength range bin (yellow bars) and the inverted power (blue line) in Fig. 5.9. Qualitatively this has the worst agreement between both; the inversion correctly identifies that the 1.05-1.15 μm bin has the peak power, but other than this the agreement is poor. Unfortunately there were some issues with the 1.1 μm calibration curve which required correction (Fig. 5.6), which may account for the under- and over-compensation in the bins on and around that point. Let us now discuss all of the results together, the progress made and the limitations of this method.

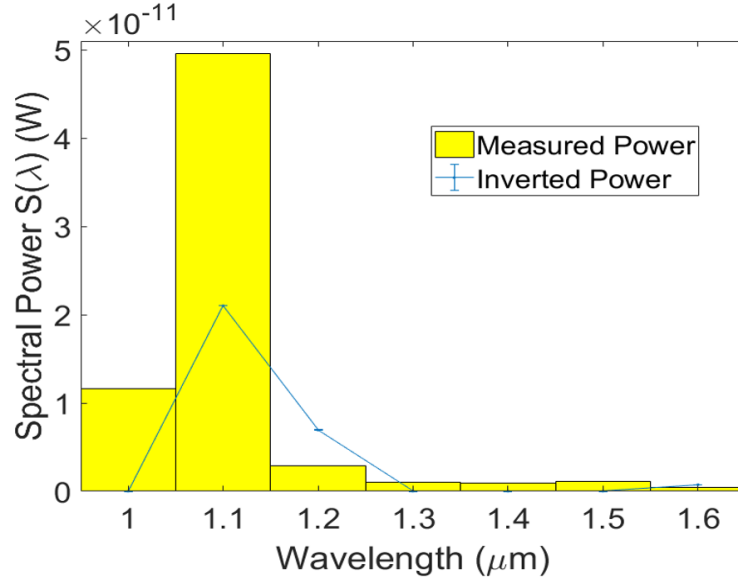


Figure 5.9: Comparison of OSA-measured and inverted optical spectrum of the broadband signal from Fig. 5.7 (purple curve).

5.4.3 Discussion

All four inversions have a common theme which is that they are able to identify the wavelength bins containing the largest quantity of optical power. It is able to do this regardless of whether the corresponding optical power wavelength is centred on a calibration curve wavelength or not. In this respect, the presented technique has made progress compared to the sum-of-residuals method in Ref. [68] which was only able to identify wavelengths at which calibration curves were measured. In addition, the error bars for all of the inversions are relatively small ($< 5\%$), indicating that the random fluctuations when measuring $\mathbf{R}(I_b)$ were very low.

Despite this, the spectroscopy method presented here was not able to correctly measure the wavelength-range optical power within error bars for any of the signals. Although the inverted powers were of the same order of magnitude as the measured ones, it still indicates a systematic error within the measurement. The most likely cause was the OSA optical power measurement, which we can understand in terms of $\mathbf{M}(I_b, \lambda)$. We recall that we construct the columns of $\mathbf{M}(I_b, \lambda)$ by measuring count rate vs I_b at a fixed wavelength

then normalising by the optical power used to measure the count rate. We do this to account for SNSPD wavelength-dependent efficiency and it corresponds only to a power-normalisation of the calibration curves. But measuring the powers inaccurately leads to systematic errors in the columns of $\mathbf{M}(I_b, \lambda)$. We recall from Sec. 5.3.2 that the condition number of $\mathbf{M}(I_b, \lambda)$ was very large, which results in the large systematic errors in the powers assigned by the inversion. However, the above only changes each calibration curve by a constant factor and does not affect the I_b and wavelength dependence, which is what determines the wavelength assignment of the inversion.

We briefly discussed in Sec. 4.4.5 the difficulties of performing absolute calibration of single-photon detectors due to the necessity of measuring very small powers. For this spectroscopy technique we did this by wavelength-calibrating fibre attenuators such that we could send measurable powers to the OSA, then correct for attenuation. This process involved repeatedly plugging and unplugging fibre FC connectors which even with cleaning we found is not a repeatable process. This introduced a systematic error in both the columns of $\mathbf{M}(I_b, \lambda)$ and in the measurement of $\mathbf{R}(I_b)$ for each signal. We can see this clearly by comparing Figs. 5.8 (b) and (c), as the inversion in (b) under-assigns the power and over-assigns for (c) despite using the same $\mathbf{M}(I_b, \lambda)$. This issue will equally affect the calculation of $\mathbf{M}(I_b, \lambda)$ and the power measurement of the signal for comparison.

In order to improve this, a more robust power measurement method could be used, such as that of Ref.^[197]. As an initial step simply daisy-chaining beamsplitters could be an option, provided the setup could be calibrated at high power and operated at a lower power.

5.4.4 Subsequent Work

We attempted more iterations of this experiment with an altered setup and a smaller $\Delta\lambda$, but unfortunately the results did not yield any significant improvement on what was presented here. In an attempt to reduce the background without the need for the partition seen in Fig. 5.5, we first coupled light from the light bulb into fibre and then filtered in

a separate fibre to free space to fibre setup. This particular method was very sensitive to beam path changes induced by filters being inserted which introduced similar systematic errors for the power measurements as the fibre attenuation did. We also measured calibration curves with $\Delta\lambda = 50$ nm, and used Edmund Optics bandpass filters with a top-hat profiles covering the whole 50 nm range, such that the calibration curves would encompass the features of the whole wavelength bin. Unfortunately the inversion results for the smaller $\Delta\lambda$ were worse, and when we used the new filters with $\Delta\lambda = 100$ nm there was no noticeable improvement to the results presented here.

5.5 Conclusion and Outlook

In this chapter we explored the practical application of using a single SNSPD to perform broadband spectroscopy on weak optical signals. We showed that the wavelength dependence of SNSPD calibration curves can be used to decompose a broadband optical signal into spectral components of $\Delta\lambda = 100$ nm from count rate vs bias current measurements. We first considered the previous work on this topic in Ref.^[68], then developed a more advanced mathematical formalism to perform the inversion. The inversion procedure was able to qualitatively identify the main components of the optical signals we tested, but could not accurately reconstruct the spectra due to systematic errors in power calibration. Despite this, the presented inversion method surpassed that of Ref.^[68] by identifying spectral components from signals whose central wavelengths were not those of the calibration curves.

The main advantages of this technique is that it works for optical signals (both pulsed and continuous) far weaker than current spectrometers can measure, and that it can determine a broadband spectrum with a single SNSPD rather than using multiple pixels. In addition, this method has the potential to work remotely with the correct light collecting apparatus. This could include scanning an area on an aeroplane (provided it is able to host a cryostat), or on a satellite in outer space in temperatures low enough that SNSPDs

will superconduct without a cryostat.

This method however does have some drawbacks which limits its applications. First, it is only suitable for identifying broad features of an optical spectrum within the range that SNSPDs are sensitive. Also, the fact that it only works for weak signals could be viewed as a disadvantage. A more powerful signal could be attenuated to a level suitable for SNSPDs to determine its spectrum, but there would be little point in doing this unless the initial signal is too weak to be analysed using a pre-existing method with better resolution. In addition, we showed that computing the power-normalised calibration curves $\mathbf{M}(I_b, \lambda)$ from measurements is very difficult to do accurately due to the effect of error amplification from the inversion. One would require a method to measure optical powers of order 10 pW with far reduced systematic uncertainty. Despite all of this, this technique shows improvement on the previous work which suggests further advancement could be made with more accurate calibration and potentially a more sophisticated inversion procedure.

Weak Field Homodyne Detection with a Gaussian State

In Sec. 3.4.1 we discussed how balanced homodyne detection is performed, and its importance for state characterisation in phase space. In this chapter we will consider ‘Weak Field Homodyne Detection’ (WFHD), when the amplitude of the local oscillator (LO) is no longer large enough to be considered classical. In this case, the approximation in going from Eq. (3.55) to Eq. (3.56) is no longer valid, and full quantum treatment is required. We will ultimately show that by increasing the strength of the local oscillator that we can continuously tune between a non-Gaussian measurement (photon-number projection) and a Gaussian measurement (balanced homodyne detection). We will demonstrate this for a signal of a Fock state heralded from TMSV (Eq. (2.34)) with TESs as our detectors to resolve photon numbers.

Given that we will be considering both the signal and LO as quantum entities, we will analyse the setup as a simple Gaussian Boson Sampling (GBS) circuit with identity on the herald TMSV mode and a beamsplitter mixing the signal and LO. We will discover that there are certain mathematical advantages to formalising the system in this way, but it will also outline the drawback of the post-selection method mentioned in Secs. 2.3.3 and 3.5.1

when heralding from a TMSV. Nevertheless, we will show that by post-selecting in a certain way we can use the phase-sensitive Gaussian state formalism (GSF) to model an experiment which is phase-insensitive due to the signal in question. This ultimately gives us the best of both worlds, as phase control in experiments can be extremely challenging (see Chap. 7).

The work in this chapter is based on the publication ‘Tuning between photon-number and quadrature measurements with weak-field homodyne detection’^[103]. It was a highly-collaborative project, therefore in order to tell a coherent story this chapter will contain contributions from the other authors. William Clements is responsible for the initial idea and initiating the project. In terms of experimental contributions, Guillaume Thekkadath and I worked on the optical setup, and Jacob Bulmer, William Clements and I set up, maintained and operated the cryogenic detector system. Guillaume Thekkadath, Jacob Bulmer and I performed the actual data acquisition. Jacob Bulmer processed the raw detector data into photon numbers. Guillaume Thekkadath did the Fock-basis theory and processed the results based on this (ultimately what ended up in the paper), but I will present my own theoretical framework using the GSF. I will also be using some figures from Ref.^[103], I will clearly state who is responsible for each one when it was not me.

6.1 Background

In this section we will do a brief literature review of the work to-date on WFHD, both theoretical and experimental.

6.1.1 Introduction

BHD and photon-number projection are fundamentally different measurements. Measuring the quadrature distribution $P(p(\varphi))$ for an optical state as per Eq. (3.57) for only one value of φ reveals no photon-number information, unless the state is rotationally invariant in phase space. Conversely, measuring the state’s photon-number distribution reveals

no phase information. Note that measuring $P(p(\varphi))$ for multiple φ then reconstructing $W(q, p)$ will reveal a full photon-number distribution, but is far more labour-intensive than measuring the photon-number distribution directly.

For a state $\hat{\rho}$, the case where $\text{Tr}(\hat{\rho}\hat{n}) \ll |\alpha_{\text{LO}}|^2$ is not true is an interesting one. It is not a proper Gaussian quadrature measurement as the approximation that leads to Eq. (3.56) does not hold, but it is also not a completely phase-insensitive measurement as $|\alpha_{\text{LO}}\rangle$ provides some relative phase reference. We are in some form of intermediate hybrid regime, and in this chapter we will show that we can continuously tune between both extremes.

There is further motivation to studying this regime other than showing that we can tune between a non-Gaussian and a Gaussian measurement with a single setup. There is interest in the field of hybrid quantum information, where information can be encoded in both the discrete and continuous variable (CV) degrees of freedom of a state^[209]. Given that WFHD operates in the intermediate region of a discrete and CV measurement it has the potential for applications in hybrid quantum information. Some applications we will mention in Sec. 6.1.3, and some more specific to our setup in Sec. 6.5.

6.1.2 Theoretical Background

The variation of the BHD signal as a function of the power of the LO was first studied theoretically for coherent state signals in Ref.^[210], and for single photon signals with loss in Ref.^[211]. Both publications contain expressions for $P(\Delta n)$, i.e. the probability distribution for the photon-number difference measurement which is the discrete version of $P(p(\varphi))$ up to a constant factor (see Eq. (3.56)). Fig. 2 in Ref.^[210] and Figs. 2 and 3 in Ref.^[211] show how $P(\Delta n)$ converges to $P(p(\varphi))$ as $|\alpha_{\text{LO}}|$ increases. Ref.^[211] is more relevant for this chapter as it considers a single-photon signal, though we will be generalising to higher Fock states. That publication also takes non-unity detector efficiency into account which will be an important consideration in our experiment.

We see a practical formulation of the BHD observables we measure in Ref.^[212], though

this publication only considers the classical approximation case. Ref. ^[213] builds on this work by formulating the POVMs (see Sec. 2.4.1) without making the classical approximation. They also address the issue of the required size of $|\alpha_{\text{LO}}|^2$ compared to $\text{Tr}(\hat{\rho}\hat{n})$ for the classical approximation to be valid. They note in Sec. 4.4 that this would be difficult to determine for an arbitrary state.

More recently, though not directly in the context of BHD, Ref. ^[214] investigates the ‘size’ of the measurement device required to measure the quantum features of a state. They show that in general a classical measuring device needs to be ‘quadratically larger’, but for this they are referring to a single-shot measurement rather than building up a distribution such as $P(p(\varphi))$ or $P(\Delta n)$. Though the specific example they give concerns the physical size of a measurement device as opposed to the size or dimension of any probe state, it is still an interesting consideration which we will revisit when discussing our experimental results.

6.1.3 Experimental Background

One of the first reported experimental implementations of WFHD was in Ref. ^[215]. They derived the POVM elements for the WFHD measurement on a signal state $\hat{\rho}$, and reconstruct a coherent state signal from the measurement outcomes. In this instance, number resolution was achieved by multiplexing click detectors (see Sec. 2.4.1). In Ref. ^[216] the authors report full characterisation of the WFHD setup via detector tomography, in which they experimentally reconstruct the setup POVM Wigner functions.

Ref. ^[217] demonstrates the first instance of using the WFHD setup as a hybrid detector. The authors successfully measure optical coherence (i.e. phase relationship, a CV property) when varying the LO phase by observing modulation in the photon-number detection outcomes. As for another application, in Ref. ^[218] the authors report the ability to measure the relative phase between the signal and LO with a coherent state signal. In this scheme hybrid photodetectors were used; a type of APD (see Sec. 2.4) with partial number-resolving abilities. The final application we will consider is density matrix

reconstruction in Ref. ^[219]. The authors present using WFHD also with hybrid photodetectors to reconstruct a quantum state using BHD methods, which they demonstrate with a coherent state signal.

In contrast to the above demonstrations, in this chapter we will demonstrate the use of fully photon-number resolving detectors in this application for the first time. This will allow a proper assessment of the WFHD in respect to theory, and to determine the role of imperfections on the measurements.

6.2 Gaussian State Picture

Let us now consider the whole setup of our experiment in the Gaussian state picture.

6.2.1 The Signal

At the beginning of the chapter, we discussed that for our signal we considered heralded Fock states from TMSV. Under perfect experimental conditions the signal would be a pure photon-number state, but with loss this is no longer the case. Consider the TMSV equation in Eq. (2.34). If we measure the state $|k\rangle_1$ on the herald TES, due to loss our signal is a weighted mixture of Fock states of order k and above. Fig. 6.1 illustrates the

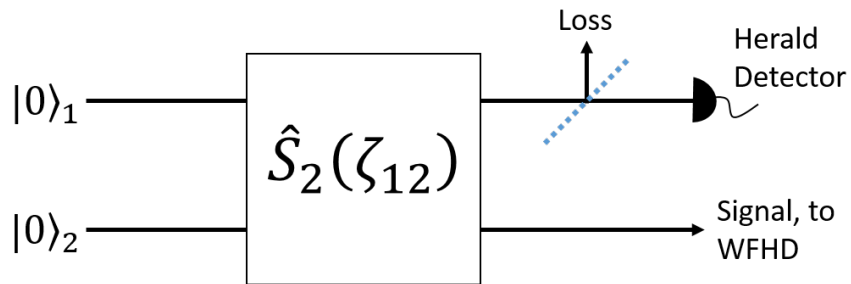


Figure 6.1: Schematic of obtaining a heralded Fock state. Mode 1 is the herald and mode 2 is the signal. The loss on the herald mode is modeled as a beamsplitter, and for our experiment the detector was a number-resolving TES.

signal preparation. In the GSF, after two-mode squeezing we have $\mathbf{Q}_{TMSV}$ (Eq. (3.37))

and $\mathbf{d} = \mathbf{0}$ with $\zeta_{12} = r_{12}e^{i\phi_{12}}$ as the squeezing parameter on modes 1 and 2. We then apply the loss transformation from Eqs. (3.40) and (3.41). For the signal preparation, we will only consider loss in the herald mode for now and we will consider the signal loss as part of the WFHD beamsplitter. We then project the transformed covariance matrix onto the photon number basis as per Secs. 3.4.2 and 3.5.2 to obtain the joint photon-number distribution $P(n_1, n_2)$. If we then post-select on detecting k photons in the herald TES, our signal photon-number distribution $P(n_2|k)$ is:

$$P(n_2|k) = \frac{P(k, n_2)}{\sum_{n_2=0}^{\infty} P(k, n_2)}. \quad (6.1)$$

In practise we only calculate $P(n_1, n_2)$ up to some maximum value of n_1 and n_2 , in which case the sum in Eq. (6.1) is finite. To visualise this state let's consider the specific example of $k = 6$, the resulting signal we will call $\hat{\rho}^{(6)}$. As $\hat{\rho}^{(6)}$ contains photon-number

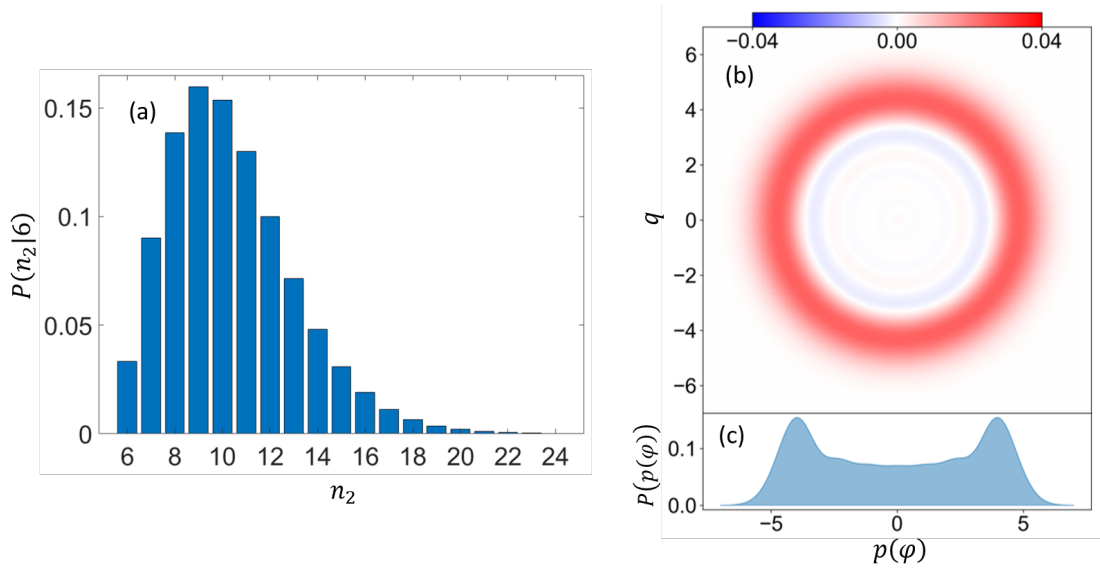


Figure 6.2: Properties of the signal $\hat{\rho}^{(6)}$ for experimental parameters $|\zeta_{12}| = r_{12} = 1.09 \pm 0.03$ (squeezing parameter) and $\eta_1 = 0.395 \pm 0.002$ (herald mode efficiency). (a) Photon-number distribution $P(n_2|k)$. (b) Heatmap of the Wigner function for $\hat{\rho}^{(6)}$. (c) Quadrature projection (see Eq. (3.57)) $P(p(\varphi))$. This figure is taken from Ref. [103]; (a) was made by me, and (b) and (c) were originally made by Guillaume Thekkadath (with slight modifications by me for this chapter).

contributions from 6 to ∞ , $\text{Tr}(\hat{\rho}^{(6)}\hat{n}_2)$ is in fact 10.4. We can see this photon-number

distribution in Fig. 6.2 (a). Figs. 6.2 (b) and (c) depict the Wigner function $W(q, p)$ and $P(p(\varphi))$ respectively. We see that $W(q, p)$ is phase-insensitive (i.e. radially-symmetric) which is unsurprising given $\hat{\rho}^{(6)}$ is a mixture of phase-insensitive Fock states (see Sec. 3.2.2). This means Fig. 6.2 (c) would look the same whatever the value of φ .

We note that the treatment above is for the purposes of visualising the signal state only; for the actual formalism we will not project onto the Fock basis until after the WFHD step.

6.2.2 The Local Oscillator

When mixing the signal and LO on a (balanced) beamsplitter, in order for Eq. (3.55) to be valid they must both be well mode-matched to one another. This means they must have the same polarisation, be in the same spatial mode, and be well-matched temporally and spectrally. A common method to achieve spectral overlap is to use the same laser for both the signal and LO. When homodyning a coherent state the laser beam needs only to be split then recombined at the beamsplitter. Otherwise, when homodyning a state from an SPDC process (see Sec. 2.2.5) such as a squeezed state, a Fock state heralded from TMSV, etc., it is common to use the same laser to pump the SPDC process as for the LO. For this case it is sometimes necessary to double the laser frequency using a non-linear crystal to pump the SPDC process^[138].

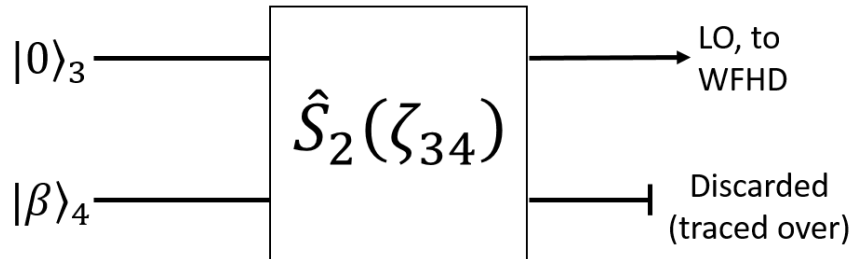


Figure 6.3: Schematic of LO preparation using seeded SPDC.

For our experiment we took a different approach. Consider the scenario in Fig. 6.3. Initially there is vacuum in mode 3, and a coherent state with amplitude β in mode 4. The two-mode squeezing operator acts on both modes, the state in mode 3 is used as the LO, and mode 4 is discarded.

In the Gaussian state formalism, the initial $\mathbf{Q}$ and $\mathbf{d}$ are:

$$\mathbf{Q}_{34} = \frac{1}{2}\mathbf{I} \quad \text{and} \quad \mathbf{d}_{34} = \sqrt{2} \begin{pmatrix} 0, & \text{Re}(\beta), & 0, & \text{Im}(\beta) \end{pmatrix}^T. \quad (6.2)$$

After two-mode squeezing, $\mathbf{Q}$ is the same as Eq. (3.37), and $\mathbf{d}_{34} \rightarrow \mathbf{d}'_{34}$:

$$\mathbf{d}'_{34} = \sqrt{2} \begin{pmatrix} -\sinh(r_{34}) [\cos(\phi_{34}) \text{Re}(\beta) + \sin(\phi_{34}) \text{Im}(\beta)] \\ \cosh(r_{34}) \text{Re}(\beta) \\ \sinh(r_{34}) [\cos(\phi_{34}) \text{Im}(\beta) - \sin(\phi_{34}) \text{Re}(\beta)] \\ \cosh(r_{34}) \text{Im}(\beta) \end{pmatrix}, \quad (6.3)$$

where $\zeta_{34} = r_{34}e^{i\phi_{34}}$ is the squeezing parameter for modes 3 and 4. After tracing over mode 4, we are left with $\mathbf{d}_3$ and $\mathbf{Q}_3$:

$$\mathbf{Q}_3 = \frac{\cosh(2r_{34})}{2}\mathbf{I} \quad \text{and} \quad \mathbf{d}_3 = \sqrt{2} \sinh(r_{34}) \begin{pmatrix} -\cos(\phi_{34}) \text{Re}(\beta) - \sin(\phi_{34}) \text{Im}(\beta) \\ \cos(\phi_{34}) \text{Im}(\beta) - \sin(\phi_{34}) \text{Re}(\beta) \end{pmatrix}. \quad (6.4)$$

Eq. (6.4) is a displaced thermal state, but when $r_{34} \ll 1$ and $|\beta| \gg 1$ it is approximately a coherent state. Another way to look at this is to consider the two-mode squeezing operator from Eq. (2.33). For a strong coherent state in mode i we make the classical approximation $\hat{a}_i \rightarrow \beta$, then the two-mode squeezing operator now has the form of a displacement operator (Eq. (2.23)) on only mode s with amplitude $-\zeta\beta^*$.

This is in fact quite a well-known difference frequency generation process^[220], and we refer to $|\beta\rangle_4$ as the seed. An advantage of producing the LO this way, other than equipment

availability, is that we can fully control the seed amplitude completely independently of the pump amplitude (i.e. ζ_{12} and β have no relation). Another advantage is that we achieve mode-matching through producing the signal and LO through similar SPDC processes. A disadvantage is that there is no phase relationship between the signal TMSV and Eq. (6.4), but given we homodyne a phase-insensitive state (see Figs. 6.2 (b) and (c)) the LO phase becomes irrelevant.

A couple of final notes on the LO: despite Eq. (6.4), we will still refer to the LO magnitude as $|\alpha_{\text{LO}}|$. For the case of $r_{34} \ll 1$, we make the approximation $\cosh(2r_{34}) \approx 1$ such that in the absence of loss or attenuation we have:

$$|\alpha_{\text{LO}}|^2 \approx \sinh^2(r_{34}) (1 + |\beta|^2). \quad (6.5)$$

Also, given that we do not require a controllable LO phase, we make no attempt to set or track it during the experiment. As a result it will drift over time and phase diffusion will occur (see Sec. 3.3.2), introducing some non-Gaussianity into the system. For a full description of fully phase-diffused coherent states see Ref. [128], where the form of their density matrix elements and Wigner functions can be seen.

6.2.3 The Full System

Now that we have discussed the theoretical models for the signal and LO, let's put them both together and consider the system as a whole. We see the full system in Fig. 6.4. To ensure our LO was approximately a coherent state, it was necessary to have $|\zeta_{34}| \ll 1$ and $|\beta| \gg 1$. This required us to attenuate our LO after the SPDC process to ensure that it was of the same order as the signal. This procedure means that we apply Eqs. (3.40) and (3.41) (Gaussian state efficiency equation) to Eq. (6.4) to model the attenuation. This preserves the coherent state approximation, whereas controlling the LO magnitude through controlling the size of $|\beta|$ might compromise this.

We model the system by first putting together $\mathbf{d}$ and $\mathbf{Q}$ from Eq. (3.37) for the TMSV

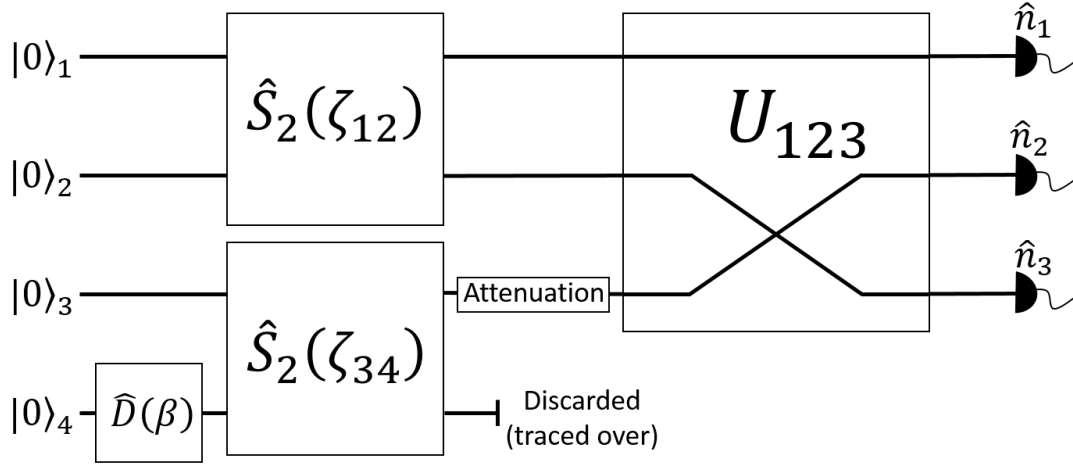


Figure 6.4: Schematic of the whole system for the WFHD experiment.

and Eq. (6.4) for the LO to get the 3-mode system. Loss is not depicted in Fig. 6.4, but can be included as per Eqs. (3.40) and (3.41). In practise, we measure the squeezing parameters, $|\alpha_{\text{LO}}|^2$, and efficiency experimentally; the methods will be given in Sec. 6.3. Following this we apply a unitary transformation, U_{123} , to the 3-mode system as per Eq. (3.39). This is given by:

$$U_{123} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & i \\ 0 & i & 1 \end{pmatrix}, \quad (6.6)$$

which corresponds to ‘do nothing’ on the herald mode (mode 1) and a symmetric balanced beamsplitter on modes 2 and 3 (we used a fibre 50:50 beamsplitter for the actual experiment^[36,37]).

The final step is photon-number projection. We project onto the photon-number basis as per Sec. 3.4.2 to obtain the joint probability distribution $P(n_1, n_2, n_3)$. Firstly we select the herald photon-number k by setting $n_1 = k$, and obtain the distribution $P(n_2, n_3|k)$ in a similar way to Eq. (6.1). We then obtain the difference probability $P(\Delta n|k)$ by setting

$\Delta n = n_2 - n_3$ and performing the sum:

$$P(\Delta n|k) = \sum_{n_2=\max(0,\Delta n)}^{\infty} P(n_2, n_2 - \Delta n|k). \quad (6.7)$$

This can equivalently be done by setting $n_2 = n_3 + \Delta n$ and summing over n_3 . As a concluding remark, $\mathbf{d}$ and $\mathbf{Q}$ can be written down after the full evolution, but the calculation of $P(n_1, n_2, n_3)$ up until $P(\Delta n|k)$ was a computational task.

6.2.4 Classical Approximation

Though not technically using the GSF, it is now fitting to discuss the classical expression in going from Eq. (3.55) to Eq. (3.56). Using the LO amplitude α_{LO} , the classical expression $P_{\text{classical}}(\Delta n|k, \alpha_{\text{LO}})$ is given by:

$$P_{\text{classical}}(\Delta n|k, \alpha_{\text{LO}}) = \frac{1}{\sqrt{2\pi}|\alpha_{\text{LO}}|} \frac{1}{2^k k!} \left| H_k \left(\frac{\Delta n}{\sqrt{2}\alpha_{\text{LO}}} \right) e^{-\left(\frac{\Delta n}{2\alpha_{\text{LO}}}\right)^2} \right|^2, \quad (6.8)$$

where $H_k(\dots)$ denotes a Hermite polynomial of order k . This is taken from Ref.^[211], where we can also see the expression including (imbalanced) loss. More information on this can also be found in the supplementary material of Ref.^[103], where we discuss the inclusion of herald mode loss and imperfect overlap. We will use this as a comparison for the full quantum description.

6.3 Experimental Methods

In this section we will discuss the experimental implementation of Fig. 6.4. We will start by considering the setup in full, then proceed to analyse the individual elements. Additionally we will consider some imperfections and discuss how we included them in the theoretical model.

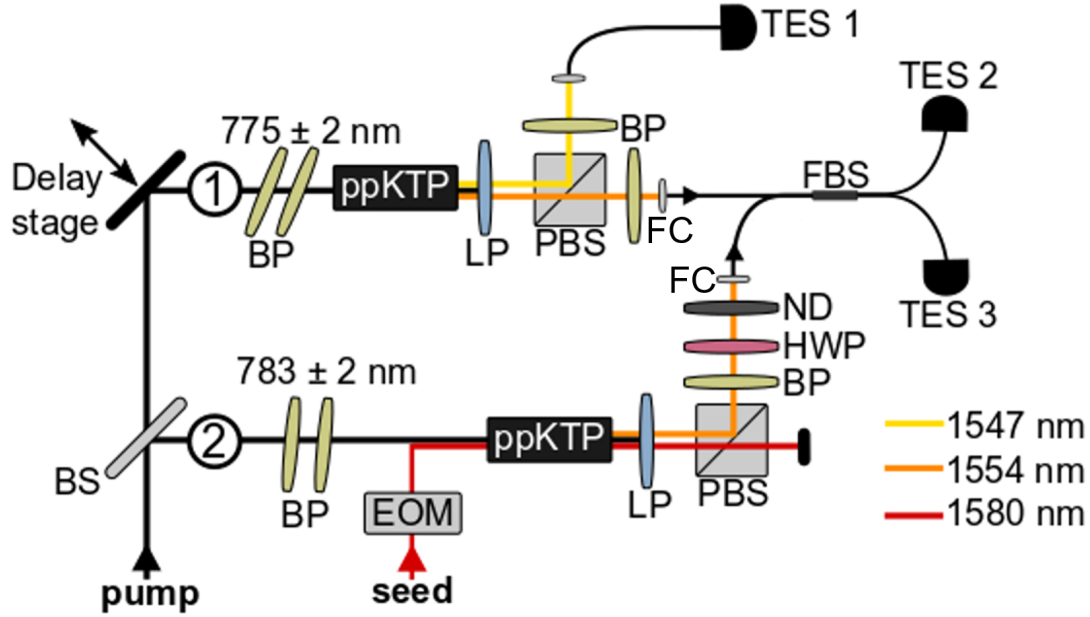


Figure 6.5: Overview of the full experimental setup. Path 1: Signal preparation. Path 2: Local oscillator preparation. BS: beamsplitter, BP: bandpass filter, EOM: electro-optic modulator, ppKTP: periodically-poled potassium titanyl phosphate (waveguide), LP: longpass filter, PBS: polarising beam splitter, FC: fibre coupler, HWP: half-wave plate, ND: neutral-density filter, FBS: fiber beamsplitter, TES: transition edge sensor. This figure is taken from ref^[103], and was originally made by Guillaume Thekkadath (with some modifications by me for this chapter).

6.3.1 The Full Setup

The full experimental setup can be seen in Fig. 6.5. Both SPDC sources were pumped by the same titanium sapphire laser system (a Mira pumped by a Verdi, both from Coherent) which was split by a free-space beamsplitter to couple to both waveguides. Before filtering, the pulse duration was ≈ 50 fs and it was operated at a repetition rate of 100 kHz to avoid saturation of the TESs (see Sec. 2.4.3).

In path 1, we prepared the signal. There was a delay stage in the optical path of the pump pulse to ensure temporal overlap of the signal and LO at the fibre beamsplitter (FBS). The pulse was filtered using two angle-tuned bandpass filters to 775 ± 2 nm then focused into the first periodically-poled potassium titanyl phosphate (ppKTP) waveguide sources. This produced the signal at 1554 nm and idler at 1547 nm, following which a longpass filter attenuated the pump. The signal and idler were split by a polarising

beamsplitter (PBS) and both were filtered by bandpass filters to ensure a pure spectrum. Following this, both were coupled into single-mode polarisation-maintaining fibres, where the signal went to the FBS and the idler to the herald TES (mode 1).

We prepared the LO in path 2. The pump was filtered to 783 ± 2 nm and coupled into a second ppKTP waveguide source in a different crystal. The seed coherent state $(|\beta\rangle)_4$ from Sec. 6.2.2) was derived from a tunable continuous wave (CW) laser (a Tunics from Yenista) at 1580 nm at ≈ 1 mW. We carved 2 ns pulses from the CW beam using an electro-optic modulator, and overlapped this pulse with the pump pulse in the waveguide. After the stimulated SPDC process, the pump was attenuated by a longpass filter and both modes were separated by a PBS, and it was at this point the seed mode was discarded. The LO was then further filtered and its polarisation was rotated to match the signal. Following this, it was attenuated by a neutral density filter down to the magnitude of the signal, and coupled into single-mode optical fibre to meet the signal at the FBS. The above generated an LO at 1554 nm to match that of the signal.

A Thorlabs 1550 nm polarisation-maintaining 50:50 FBS was used for the WFHD balanced beamsplitter. At 1554 nm we measured the splitting ratio to be approximately 49.3:50.7. The FBS outputs were then impinged on two TESs. The herald and FBS output TESs were all operated in a dilution refrigerator at ≈ 80 mK, see Sec. 2.4.3 for operation and data processing of TESs, and Sec. 4.3.1 for the basics of dilution refrigeration.

6.3.2 Characterisation of the Single-Photon Sources

The ppKTP crystals used to generate the signal and LO were phase matched such that light was generated via type-II SPDC. This means that the signal and idler photons were generated in orthogonal polarisations (see Eq. (2.34))^[221]. For more information on the types of SPDC and sources see Chap. 11 of Ref.^[12]. Let us now discuss some properties of the sources which are important for consideration.

Seeded Joint Spectral Intensities

We already discussed in Sec. 2.2.5 that only two modes in Eq. (2.34) is an idealisation. In reality we need to take into account the pump laser bandwidth, photon energy conservation, and phase matching (photon momentum conservation). To herald a pure Fock state from a TMSV, we require there to be no spectral correlations between the signal spectrum and the idler spectrum. We introduce the quantity called the ‘joint spectral amplitude’ (JSA) $f(\omega_s, \omega_i)$ for signal and idler angular frequencies ω_s and ω_i respectively. When there are no spectral correlations between the signal and idler we say that the JSA is ‘factorable’, i.e. we can write $f(\omega_s, \omega_i) = f_1(\omega_s) f_2(\omega_i)$ ^[12,222].

To check this, we measure the ‘seeded joint spectral intensity’ (seeded JSI). The JSI is the modulus squared of the JSA. This procedure involves stimulating the SPDC process with a monochromatic laser in one mode, and measuring the spectrum of the light produced in the other mode (similarly to Eq. (6.3)) using e.g. an optical spectrum analyser. By varying the seed wavelength at a fixed pump wavelength and measuring the spectrum of the stimulated emission, we build up the JSI of a source slice by slice. From this we can determine if the source is factorable.

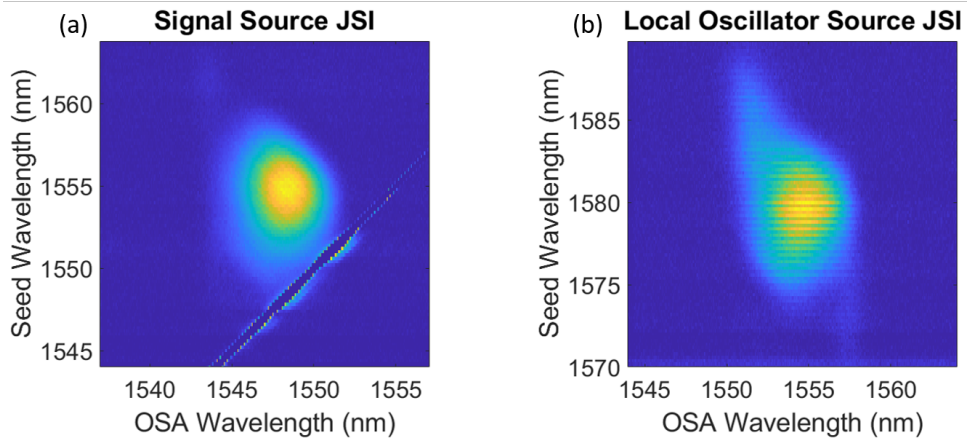


Figure 6.6: Joint spectral intensities for both SPDC sources, intensities are in arbitrary units. (a) Waveguide used to generate the signal. As the signal and idler wavelengths were very close, the seed wavelength was visible on this plot. The diagonal ‘tear’ is where the seed has been removed. (b) Waveguide used to generate the local oscillator. The signal and idler wavelengths here are far enough apart that the seed is not visible on the plot.

The JSI plots for both ppKTP waveguide sources can be seen in Fig. 6.6. A qualitative way to see if a source is factorable is if the JSI looks uncorrelated, i.e. if the plot appears symmetric. We can see this is approximately the case for both sources. A more quantitative way is to perform a single-valued decomposition on the JSI data and calculate the quantity known as the Schmidt number^[223,224]. The full method behind this is given in Chap. 11 of Ref.^[12] and in Ref.^[222], but for simplicity I will quote that a Schmidt number of 1 implies spectral purity. We calculated the Schmidt numbers to be 1.196 and 1.192 for the signal and local oscillator sources respectively, showing that they are both approximately spectrally pure. Chap. 11 of Ref.^[12] discusses optical filtering to improve purity, which we did to obtain the JSIs.

Indistinguishability

We discussed the requirements for good modal overlap between the signal and LO in Sec. 6.2.2, but now we will investigate this using data. To measure the modal overlap (i.e. signal and LO indistinguishability) we follow the procedure in Refs.^[225,226], which is an example of the Hong-Ou-Mandel interference we discussed in Sec. 2.3.2.

We mixed one mode of the signal TMSV and LO on a balanced beamsplitter, and measured the other TMSV mode (the herald) and both beamsplitter output modes on three SNSPDs (see Sec. 2.4.2). We then post-select on a single photon in the herald mode and consider the measurements at the beamsplitter outputs. If the signal and LO are perfectly overlapped then the number of incidences of measuring exactly one photon in each output of the beamsplitter is zero (Eq. (3) of Ref.^[225]). We varied the temporal overlap of the signal and LO by using a delay stage in the signal optical path to get the comparison between the best case overlap and no overlap. As we used SNSPDs for this measurement we could not number-resolve, but for weak pump powers and a very weak coherent state the higher photon number contributions are negligible.

We plot in fig 6.7 the number of times all three SNSPDs click in 30 s against the stage position in the signal path (see Fig. 6.5). We then fitted Eq. (3) from Ref.^[226]

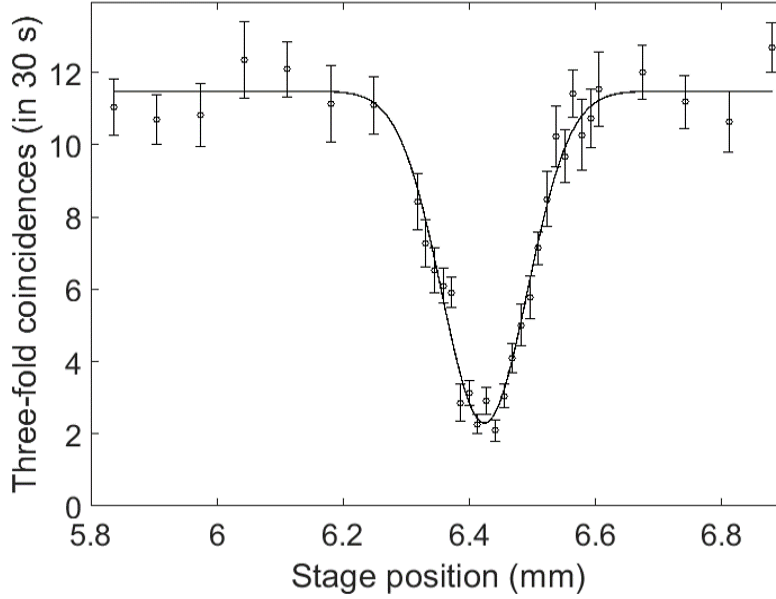


Figure 6.7: Plot of the number of three fold coincidences in 30 s vs stage position in the signal optical path. Guillaume Thekkadath performed the fitting for this plot.

(up to a constant factor), and used Eq. (4) (also from Ref. ^[226]) to estimate the overlap which we will call $\mathcal{M}$. $\mathcal{M}$ is unity for perfect overlap, but we measured it to be 0.800 ± 0.060 . In practise, unity overlap is very difficult to achieve, as experiments are affected by imperfections such as background counts, modal impurity, etc. The minimum of the dip in Fig. 6.7 gives the stage position for optimal temporal overlap at the FBS.

We model imperfect overlap in WFHD using the Gram-Schmidt orthonormalisation procedure in Sec. 2.2 of Ref. ^[227]. We decompose the LO into a component with perfect overlap with the signal which now has amplitude $\sqrt{\mathcal{M}}\alpha_{\text{LO}}$, and a component orthogonal to the signal with amplitude $\sqrt{1 - \mathcal{M}}\alpha_{\text{LO}}$. We then evolve the system twice as per Fig. 6.4: with the signal and LO amplitude $\sqrt{\mathcal{M}}\alpha_{\text{LO}}$, and and no signal (i.e. vacuum) and LO amplitude $\sqrt{1 - \mathcal{M}}\alpha_{\text{LO}}$. Convolving the photon-number distribution for both cases gives the full photon-number distribution, and the same procedure can also be used for the classical model in Eq. (6.8).

Local Oscillator $g^{(2)}(0)$

We saw in Sec. 3.3.3 that $g^{(2)}(0) = 1$ for a coherent state. Therefore to check that the LO was approximately a coherent state we measured its photon-number distribution on a single TES, and calculated its $g^{(2)}(0)$ through Eq. (3.44). We measured it to be 1.005 ± 0.002 , indicating that the approximation is appropriate. Note that the effect of loss on a coherent state is reduced amplitude, therefore loss does not affect this measurement.

6.3.3 Additional Measurement Parameters

We have already discussed how we measured $\mathcal{M}$ and $g^{(2)}(0)$, but there are some additional parameters still required to simulate the results which are the efficiencies, the signal squeezing parameter $|\zeta_{12}|$, and $|\alpha_{\text{LO}}|^2$.

As we used a TMSV to generate the signal, we exploited the perfect photon-number correlations (see Eq. (2.34)) to measure the transmission efficiency of each mode. This is known as a Klyshko measurement^[228]. We first blocked the LO and turned the pump laser power down such that $|\zeta_{12}| \ll 1$, meaning that single photon pairs were the dominant emission from the source. With reference to Figs. 6.4 and 6.5, to measure the mode 1 efficiency η_1 we calculated the fraction of detection events at TES 1 given TES 2 or 3 had registered a detection event. We measured $\eta_1 = 0.395 \pm 0.002$. Then we measured the efficiencies of modes 2 and 3 (η_2 and η_3 respectively) from the fraction of detection events at TES 2 or 3 given than TES 1 had registered a detection event, which gave the average efficiency for both modes due to the 50:50 FBS. We could also measure relative efficiency between modes 2 and 3 (η_2/η_3) by comparing the fraction of detection events between TES 2 and 3 given a detection event at TES 1. From those two measurements we obtained $\eta_2 = 0.274 \pm 0.001$ and $\eta_3 = 0.352 \pm 0.002$. Note that η_1 , η_2 and η_3 are total efficiencies from the source to the TESs which will include loss from the waveguide, coupling to fibre, fibre transmission and at the detectors. The dominant loss was from coupling to fibre as fibre transmission has low loss and TESs have efficiencies $> 90\%$ ^[19].

We calculated the squeezing parameter for the signal source ($|\zeta_{12}| = r_{12}$) with the laser pump at the same strength as used for the actual data collection. We measured the average photon number at TES 1, and used the fact that by ignoring mode 2 we're measuring a thermal state with $\langle \hat{n}_1 \rangle = \eta_1 \sinh^2(r_{12})$ (see Sec. 2.2.5). We obtained $r_{12} = 1.09 \pm 0.03$.

We measured $|\alpha_{\text{LO}}|^2$ by simply measuring the average photon numbers at TESs 2 and 3 with the signal blocked, then adjusting for loss. We find $|\alpha_{\text{LO}}|^2 = \langle \hat{n}_2 \rangle / \eta_2 + \langle \hat{n}_3 \rangle / \eta_3$. This was the parameter we varied during the experiment, which we did through varying the power of the CW seed laser as this gave us the most stable control.

6.4 Results and Discussion

After photon-number-binning the experimental data from the TESs (see Sec. 2.4.3), we obtained $P(n_1, n_2, n_3)$ for each value of $|\alpha_{\text{LO}}|^2$. From this we obtained $P(n_2, n_3 | k)$ (i.e. post-selecting on $n_1 = k$) from Eq. (6.1), and then $P(\Delta n | k)$ from Eq. (6.7). We then compared these experimental results to the quantum model (Sec. 6.2.3) and the classical model (Sec. 6.2.4) including the aforementioned experimental imperfections, and it is these results we will now discuss.

6.4.1 k=6 Signal

We first consider the case of post-selecting on detecting 6 photons in TES 1 to serve as an example, before considering other values of k . This corresponds to the signal $\hat{\rho}^{(6)}$ we saw in Fig. 6.2, which we recall has an average photon number of 10.4.

We see $P(\Delta n | 6)$ plotted against Δn from the data, quantum, and classical LO model for three different values of $|\alpha_{\text{LO}}|^2$ in Fig. 6.8. There is a slight asymmetry about $\Delta n = 0$ in the plots due to a small imbalance in the efficiencies in modes 2 and 3. Firstly we see good agreement between the data and the full quantum model for all $|\alpha_{\text{LO}}|^2$ values, which is expected as this model will always describe the measurement. Also we can qualitatively see that the data (and hence the quantum model) has better agreement with the classical

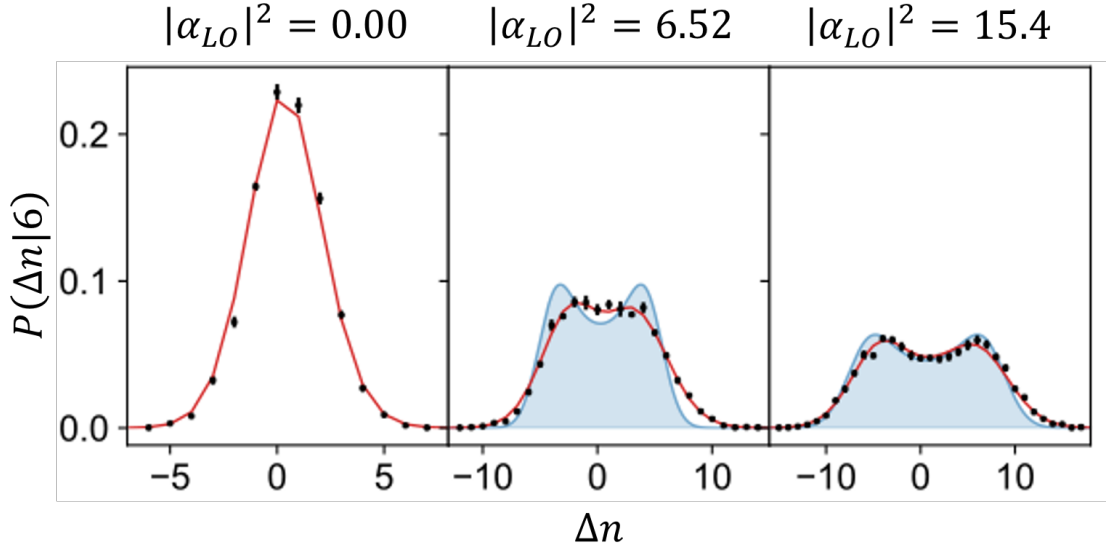


Figure 6.8: Comparison of $P(\Delta n|6)$ from the experimental data (black dots), full quantum model (red curves) and classical LO model (blue area) for three different values of $|\alpha_{LO}|^2$. The error bars for the experimental data are based on the standard error for 10 trials. The expression for $P_{\text{classical}}(\Delta n|k, \alpha_{LO})$ is singular when $|\alpha_{LO}|^2 = 0$ even when we include the experimental imperfections not shown in Eq. (6.8). This plot is taken from Ref. ^[103], and was originally produced by Guillaume Thekkadath (with light modifications by me for this chapter) from data we collectively took.

LO model for $|\alpha_{LO}|^2 = 15.4$ compared to $|\alpha_{LO}|^2 = 6.52$ which is to be expected, but the fact that they are very similar for $|\alpha_{LO}|^2 = 15.4$ is surprising. Even though the average photon number of the LO is larger than that of the signal, they are of the same order of magnitude. Simulations indicate that the close resemblance at lower $|\alpha_{LO}|^2$ values than expected is due to loss in the system. We can see the similarities between $P(\Delta n|6)$ for $|\alpha_{LO}|^2 = 15.4$ and the quadrature distribution of $\hat{\rho}^{(6)}$ in Fig. 6.2 (c) up to a constant factor. $P(p(\varphi))$ for a signal of $|6\rangle\langle 6|$ has distinct fringes, which in Fig. 6.2 (c) have been smoothed out by loss. For a system with higher efficiency simulations suggest that we would require a larger value of $|\alpha_{LO}|^2$ to see convergence between the quantum and classical LO models.

6.4.2 Quantifying the Transition

We have considered the transition from quantum to classical for $\hat{\rho}^{(6)}$ in a qualitative way, so now let's consider other values of k and quantify the transition. To do this we introduce the quantity $\mathcal{S}_{\text{classical}}$ which is the sum of residuals between the data and classical model:

$$\mathcal{S}_{\text{classical}} = \frac{1}{\Lambda} \sum_{\Delta n} |P_{\text{exp}}(\Delta n|k, \alpha_{\text{LO}}) - P_{\text{classical}}(\Delta n|k, \alpha_{\text{LO}})|^2. \quad (6.9)$$

In this expression Λ is the number of data points (i.e. number of values of Δn used), $P_{\text{exp}}(\Delta n|k, \alpha_{\text{LO}})$ is now explicitly the photon-number difference distribution from experimental data, and $P_{\text{classical}}(\Delta n|k, \alpha_{\text{LO}})$ is from Eq. (6.8) including experimental imperfections.

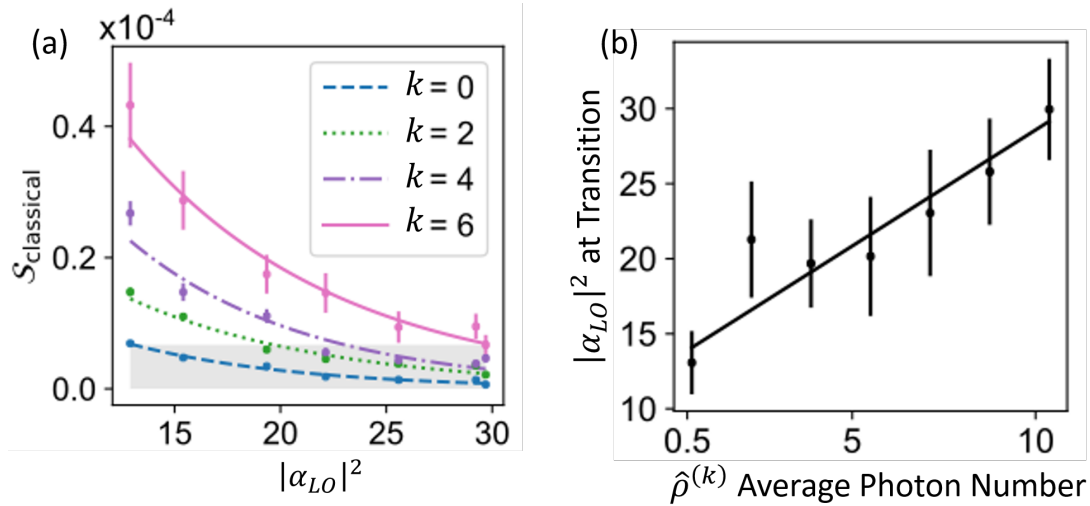


Figure 6.9: (a) Values of $\mathcal{S}_{\text{classical}}$ plotted against $|\alpha_{\text{LO}}|^2$ for different values of k . The grey area is an average of $\mathcal{S}_{\text{quantum}}$, i.e. Eq. (6.9) but for the quantum model, which is the threshold we define as the transition. The error bars are from the standard error of 10 trials, and the fits are exponential which heuristically models $\mathcal{S}_{\text{classical}}$. (b) Values of $|\alpha_{\text{LO}}|^2$ where the curves from (a) cross the threshold plotted against the average photon number of the corresponding $\hat{\rho}^{(k)}$, with a linear fit. The error bars were obtained from the uncertainties in the exponential fit parameters from (a). Both plots are taken from Ref. [103], and were originally produced by Guillaume Thekkadath (with modifications by me for this chapter) from data we collectively took.

Fig. 6.9 (a) shows $\mathcal{S}_{\text{classical}}$ plotted against $|\alpha_{\text{LO}}|^2$ for different values of k . It shows that for larger values of k (i.e. states with a larger average photon number), a larger

value of $|\alpha_{\text{LO}}|^2$ is required to converge to the classical approximation. This is expected from Ref. [214]. Now consider Eq. (6.9) but for the quantum model: $\mathcal{S}_{\text{quantum}}$. In a perfect experiment, or a situation in which we could accurately quantify every single imperfection, we would expect $\mathcal{S}_{\text{quantum}}$ to be zero for all cases. Practically this is not the case, therefore we took an average value of $\mathcal{S}_{\text{quantum}}$ from all values of k and $|\alpha_{\text{LO}}|^2$ (the value is 6.7×10^{-6}) and used it to define the point of transition to a quadrature measurement. The transition point is subjective, but we felt that the point which $\mathcal{S}_{\text{classical}}$ gets smaller than $\mathcal{S}_{\text{quantum}}$ was fitting.

Fig. 6.9 (b) shows the value of $|\alpha_{\text{LO}}|^2$ at which the fits from Fig. 6.9 (a) cross the transition point, plotted against the average photon number for the corresponding $\hat{\rho}^{(k)}$. This we define as the value of $|\alpha_{\text{LO}}|^2$ at which we are performing a quadrature measurement on $\hat{\rho}^{(k)}$. The two most notable features are that we are far from the regime of $|\alpha_{\text{LO}}|^2 \gg \text{Tr}(\hat{\rho}\hat{n})$ [213], and we observe a linear relation rather than quadratic [214]. Similarly to the point in Sec. 6.4.1, we believe that the relaxation on the size requirements of $|\alpha_{\text{LO}}|^2$ is due to loss in mode 1 (the herald mode) which smooths the signal quadrature distribution making the features more subtle. This argument is supported by Ref. [210] which explains that lower values of $|\alpha_{\text{LO}}|^2$ are required for quadrature projection for signals with a smooth quadrature distribution (Ref. [210] shows this for a coherent state signal).

6.5 Conclusion and Outlook

In this chapter we showed that we can tune between a non-Gaussian photon-number projection measurement (i.e. with $|\alpha_{\text{LO}}|^2 = 0$) to a Gaussian quadrature distribution measurement with a single setup. We demonstrated this for signals with an average photon number greater than 10. This shows that the current dynamic range and noise levels of TES detectors is sufficient to probe both the discrete and continuous nature of the signals we considered.

We also showed that it is possible to use the Gaussian state formalism with photon-

number projection to describe both regimes. When post-selecting on the herald mode we saw that we could use this phase-sensitive formalism to model a phase-insensitive measurement. Furthermore we showed that the phase of the LO is irrelevant when probing signals that are radially-symmetric in phase space, thus relaxing the difficult experimental requirement of phase-control.

An application of this exact setup is to herald interesting states in mode 1 (the original herald mode) by post-selecting on photon numbers in modes 2 and 3. This builds on the work of Ref.^[229], is discussed in the publication this chapter is based on (Ref.^[103]), and is investigated in more detail in Ref.^[230].

As a further application, in addition to the work presented in this chapter we used a slightly-modified version of the same setup to measure generalised phase-space distributions, as presented in Ref.^[231]. The scheme consists of replacing the 50:50 beamsplitter in Fig. 6.5 with a 90:10 beamsplitter, and spatially multiplexing one of the outputs with a 50:50 beamsplitter. Ref.^[231] explains how, with the modified setup, one can reconstruct a generalised phase-space distribution of an arbitrary signal without being constrained to use a particular type of detector. The theoretical framework for this scheme is beyond the scope of this thesis.

We will now continue to investigate another application of this type of setup in more detail: weak field homodyne detection for tomography of multimode states.

State Tomography of a Multimode State

Quantum state tomography is the procedure of determining an unknown quantum state from some form of measurement^[232,233]. For quantum states of light, one of the most well-known procedure is the reconstruction of the Wigner function (see Sec. 3.2.2) using the balanced-homodyne detection (BHD) method discussed in Sec. 3.4.1^[138]. In this chapter I will present an experimental realisation of a new tomographic method for optical states which is a similar yet more general version of BHD. It still involves probing a signal state with a local oscillator (LO) and measuring in the photon-number basis, yet has some distinct differences. This new method is designed to characterise a multimode signal, and the LO is of comparable strength to the signal (similar to the weak field homodyne detection (WFHD) in the previous chapter). In addition, the balanced beamsplitter can be replaced with an arbitrary (yet known) unitary transformation, and the output photon-number outcomes are no longer subtracted.

The key word in the above paragraph is ‘experimental realisation’, as the theory was not developed by me and I played no part in the tomographical analysis. Instead, the main subject of this chapter will be how we worked with theorists to construct an experiment to test their new tomography method with a known state. This project was part of my secondment to the National Institute of Standards and Technology (NIST) in Boulder, Colorado, and was also a highly collaborative undertaking. The theoretical framework

for the tomography method was developed by Arik Avagyan, Scott Glancy and Emanuel Knill, and the experiment was constructed and performed by Thomas Gerrits, Michael Mazurek, and me.

This chapter is organised as follows. After briefly reviewing the topic, I shall describe a suitable experimental setup to implement the tomography method. Following this, I shall give a very detailed account of the construction of this setup and data acquisition, including the optical and cryogenic components, and characterisation of the relevant parts. Finally, I shall use the raw data to present a processing method to calibrate the LO phase, a crucial part of the tomography method. I shall then briefly allude to the tomography outcomes before concluding.

7.1 Introduction

In this section, I shall first give some background to the tomography concept. Some of the work summarised in Sec. 6.1 is also relevant, though I shall not state it again here. Following this, I will give a very brief heuristic overview of the tomography setup with the view to translate it into a practical experimental setup.

7.1.1 Background

In BHD, only the mode of the signal which is in common with the LO will be homodyned, hence the signal is usually constrained to be single mode^[234]. Note that mode in this context is referring to anything that can distinguish between the signal and LO, e.g. polarisation, spatial or temporal profile, etc. From the analysis in Sec. 3.4.1, it is straightforward to see why this is the case. Any mode of the signal that does not interfere with the LO will be split equally by the balanced beamsplitter, and thus have equal optical power at both detectors when averaging over multiple shots. This component of the signal will then cancel out during the subtraction. The same will occur for a multimode LO, but it will have no affect on the homodyning provided it has a mode in common with the

signal. More advanced schemes exist for characterising multimode signals using BHD as a basis, see Refs. [235–237] for example.

Now consider the case when we either have a beamsplitter with unequal splitting, or consider something other than the photon-number difference at the outputs (i.e. only consider the output from one detector or the full joint probability distribution). In this instance we have ‘unbalanced homodyne detection’ where the above reasoning will not apply, and the modality of both the signal and LO becomes important. Refs. [238,239] discuss the theoretical framework of state tomography (specifically the determination of quasiprobability distributions) using unbalanced homodyne detection, and shows that a weak LO is desirable in this instance as fluctuations will not cancel during subtraction. In addition, Ref. [240] studies the characterisation of multimode states with unbalanced weak field homodyne detection. The authors show that full state characterisation is possible when the overlap between the signal and LO is varied in a controlled way.

7.1.2 Theoretical Outline

The experiment presented in this chapter is an implementation of the tomography protocol in Ref. [241]. Unfortunately, at the time of writing this thesis Ref. [241] is unpublished, so here I will attempt to explain for which situation the protocol is intended, the advantages over other current methods, and then the experimental idea for a proof-of-principle implementation.

Consider the setup in Fig. 7.1. A signal $\hat{\rho}_S$ is impinged on one input port of an interferometer implementing a known unitary $\mathbf{U}$ on the annihilation operators of the input modes (see Sec. 2.3). $\hat{\rho}_S$ could occupy multiple modes, including spectral, spatio-temporal, polarisation, etc. In the other input port, a LO $|\alpha_{LO}\rangle$ is impinged. The assumptions of this method about the LO are that its amplitude (both phase and magnitude) are controllable, it is single-mode, and that this mode is one of the ones occupied by $\hat{\rho}_S$. After unitary evolution, at each of the two output ports is a photon-number resolving detector. With these the joint photon-number distribution of the outputs are measured, where it is

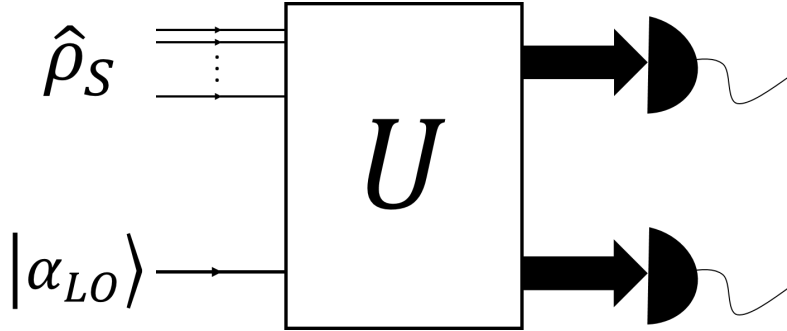


Figure 7.1: Visualisation of the multimode tomography setup. An interferometer that implements a unitary transformation U on the annihilation operators of the input modes has two distinct input ports and two distinct output ports. A (potentially multimode) state $\hat{\rho}_S$ occupies one input, and a known, single-mode local oscillator $|\alpha_{LO}\rangle$ occupies the other. The two distinct output ports each have a photon-number resolving detector.

assumed that the detectors have the same efficiency at each mode.

With the described setup and assumptions, and without changing the modality of the LO, the tomographic method is able to reconstruct the reduced density matrix of the signal $\hat{\rho}_r$. $\hat{\rho}_r$ is the original $\hat{\rho}_S$ written as a component in the same mode as $|\alpha_{LO}\rangle$ (the in-mode component), and a component not in the same mode (the out-of-mode component), i.e. the out-of-mode components have been compressed into one if the original signal has more than two modes. This does not give full information about $\hat{\rho}_S$, but given the LO mode is not varied it gives more information than BHD would. It also gives us correlation information between the in-mode and out-of-mode components of $\hat{\rho}_r$, but not between the different modes of the out-of-mode component (should there be more than one). The only additional requirement compared to BHD is the ability to vary the magnitude of the LO as well as the phase. There is a relation between the total number of photons in the signal and the number of LO amplitudes required to reconstruct $\hat{\rho}_r$, and this number increases when only using photon number information from one detector.

To summarise what this tomography method is capable of, when a state is multimode and the LO is single mode, more information can be gained about the state compared to BHD provided the signal and LO have a mode in common and the LO amplitude can be varied. It will also work for any two-input two-output interferometer as opposed to

only for a balanced beamsplitter, provided the unitary transformation it imparts is known. The only additional requirements are that the LO is weak (i.e. of comparable size to the signal) and that both its magnitude and phase can be varied. This is in contrast to BHD where the LO magnitude remains constant, only the relative phase between the signal and the LO needs to be varied. In addition, provided more LO amplitudes are used, it is possible to use the photon-number distribution from only one of the detectors to infer the information about the input state.

7.1.3 Experimental Implementation

The above description of the tomographic method is more of a ‘what it is able to do’ rather than ‘how it does it’, and the latter is indeed beyond the scope of this chapter. Therefore, let us discuss a possible experimental setup that could serve as a proof-of-principle to test the tomography method.

A very simple implementation would be to use a coherent state signal such that we could derive the signal and LO from the same laser pulse. This would give us good control over the overlap as the signal and LO as initially they would be well mode-matched. There are several ways to obtain partial overlap, and we decided to use the polarisation degree of freedom. This way there would be only two modes to worry about, and we would keep the LO polarisation fixed and then vary the signal polarisation. We decided to use a balanced beamsplitter for simplicity, and we would spatio-temporally overlap the signal and LO on this. We would then vary the intensity and phase of the LO, and measure both beamsplitter outputs with photon-number resolving detectors (TESs in this case) for each LO amplitude used.

7.2 The Experiment

In this section we will discuss in detail the physical implementation of the plan in Sec. 7.1.3. We will first review the optical setup in terms of obtaining a coherent state signal and

LO with variable overlap. Following this, we will discuss the method used to calibrate the best-case overlap between the signal and LO. The next step will be to investigate phase stability, a requirement for this tomography method. We will then briefly discuss the cryogenic setup, and finish with the full data acquisition protocol.

7.2.1 Optical Setup

In order to implement the above idea, we built what was essentially a large Mach-Zehnder interferometer with the ability to vary the polarisation of one of the paths. We derived the coherent state signal and LO from the same laser, such that in principle we could obtain good mode matching between both. A schematic and image of the setup can be seen in Figs. 7.2 and 7.3 respectively. We first coupled 1550 nm 100 ps pulsed light from a PicoQuant Sepia II laser diode into polarisation-maintaining (PM) single-mode (SM) 1550 nm optical fibre (not shown in either figure). The laser was externally triggered by a function generator at 100 kHz due to TES requirements (see Sec. 2.4.3), and the coupled light was of the order of 300 μ W. This was first transmitted through a Tektronix variable attenuator to bring the power down to the level of a few photons per pulse. Although the laser light was (vertically) polarised, the fibre coupling and attenuation had a measurable de-polarising effect, so due to this we placed a Thorlabs fibre vertical polariser immediately after the attenuator. Following this, the pulses were impinged on a Thorlabs 90 : 10 fibre beam splitter (FBS), the 10% arm for the signal and the 90% arm for the LO, and de-coupled into free space using Thorlabs Nanomax stages and lenses.

In path 1 we prepared the signal. The pulses first passed through an arrangement of four silver mirrors to help with alignment stability and path-length matching. Following this they passed through a polarising-beam splitter (PBS), which in this instance was used to polarise the light. The PBS was orientated such that the vertical polarisation was transmitted, and the horizontal component was reflected downwards with respect to Fig. 7.2. The PBS was aligned to the vertical axis of the collection fibre in path 1 in the absence of the half-wave plate (HWP). The final optic was a motorised HWP such that

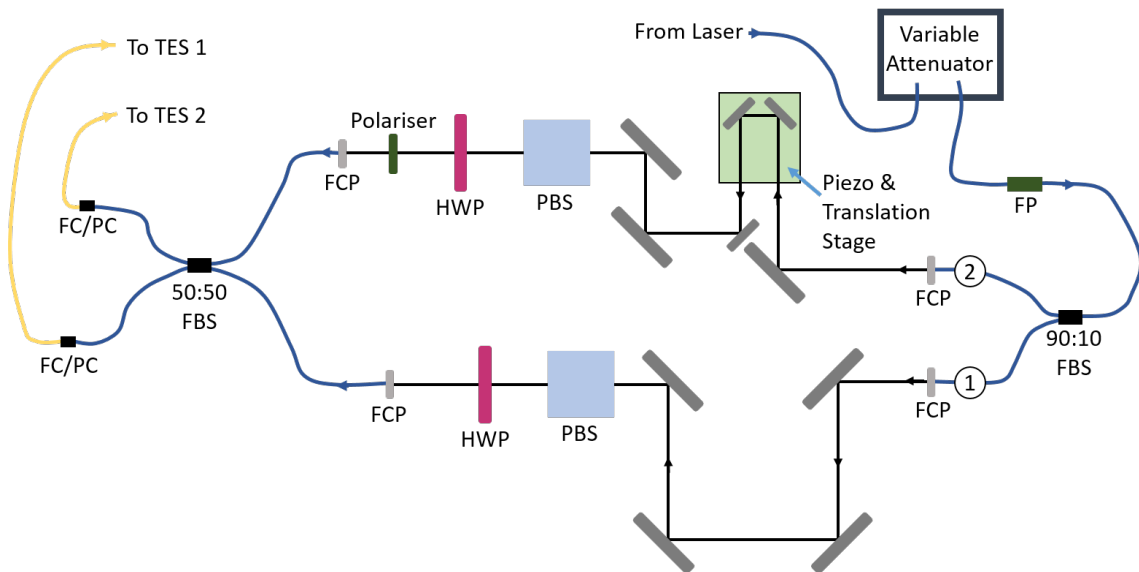


Figure 7.2: Schematic of the optical setup for state (path 1) and LO (path 2) generation and multimode tomography. The blue lines correspond to polarisation-maintaining (PM) fibre, the black lines are the free-space optical paths, and the yellow lines are non-PM fibre. FP: fibre polariser, FBS: fibre beamsplitter, FCP: fibre coupling package (on Nanomax stage), PBS: polarising beamsplitter, HWP: (motorised) half-wave plate, FC/PC: fibre mating coupler, TES: transition edge sensor. The unlabelled grey rectangles are silver mirrors, and the PBSs are orientated to reflect the undesired polarisation downwards.

we could vary the polarisation overlap between the signal and LO.

In path 2 we prepared the LO. We first passed the pulses through an ‘L’ mirror arrangement on top of a piezo on top of a translation stage (green box in Fig. 7.2). The translation stage enabled us to make more coarse adjustments to overlap the signal and LO on the 50:50 FBS, and the piezo allowed us control over LO phase with respect to the signal. For the tomographical reconstruction we were required to use different LO strengths for each signal, therefore to vary this we had a PBS-HWP-polariser arrangement to give us variable attenuation. The PBS served the same function as in path 1, and both the PBS and polariser were aligned to the vertical axis of the collection fibre in path 2. The motorised HWP in between served to change the polarisation of the LO, thus varying the amount that made it through the polariser, and therefore the strength.

The next component was the interferometer itself. Both signal and LO were coupled back into PM SM fibre using the same arrangement as for coupling out of fibre, and were

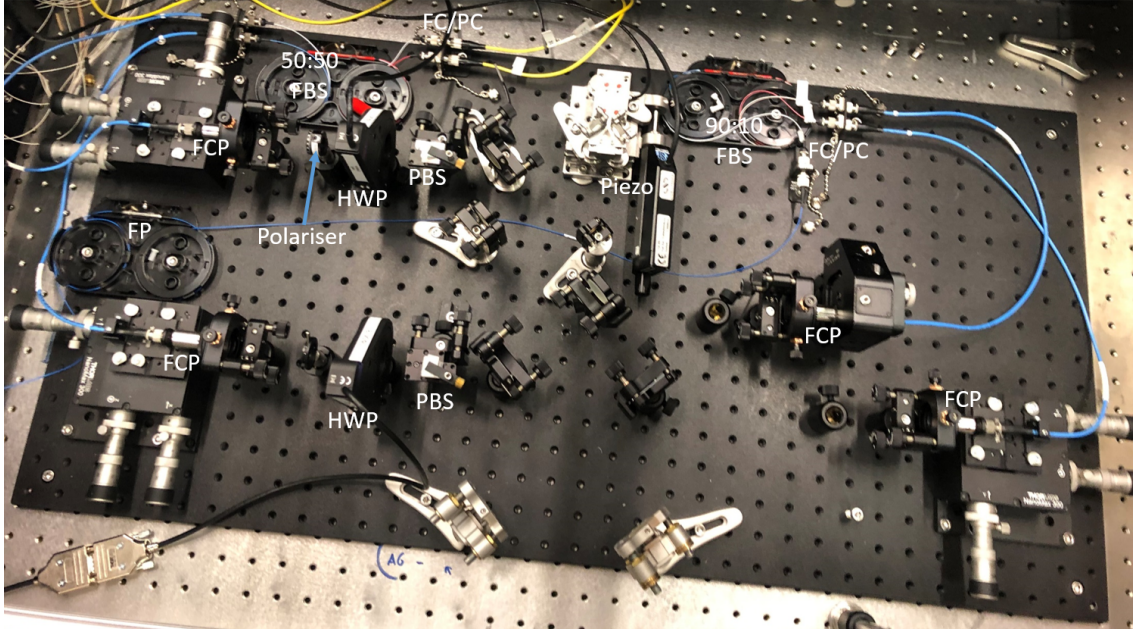


Figure 7.3: Picture of the setup in Fig. 7.2 with various components labelled. The free-space optical path follows the schematic, but some fibre components are in a different location due to space on the breadboard. The variable attenuator is not visible in this image.

then overlapped on a Thorlabs PM 50:50 FBS. The outputs of the FBS were then each sent to a different TES. The fibres leading to the TESs were not PM, therefore rather than splicing PM to non-PM, we connected non-PM fibres to the PM output fibres of the FBS using commercial mating couplers. This induced some loss which we measured.

7.2.2 Signal and Local Oscillator Visibility

We used two separate measurements to estimate the maximum interference visibility between the signal and LO. The first was to measure which signal HWP angle gave the best overlap with the LO. To do this, with reference to Fig. 7.2, we placed an additional polariser between the HWP and the FCP which we aligned to the horizontal axis of the signal collection PM fibre. In this arrangement, the HWP angle that gave the minimum transmitted power corresponded to the angle which aligned the signal polarisation to the vertical axis of the collection PM fibre. To perform this measurement we bypassed the variable attenuator, and measured the power with a Thorlabs InGaAs photodiode (S154C)

after coupling into fibre but before the 50:50 FBS. We measured power vs HWP angle, and the results are plotted in Fig. 7.4 (a). When fitting a sinusoid, we found the HWP angle for a vertically-polarised signal to be $54.78 \pm 0.01^\circ$.

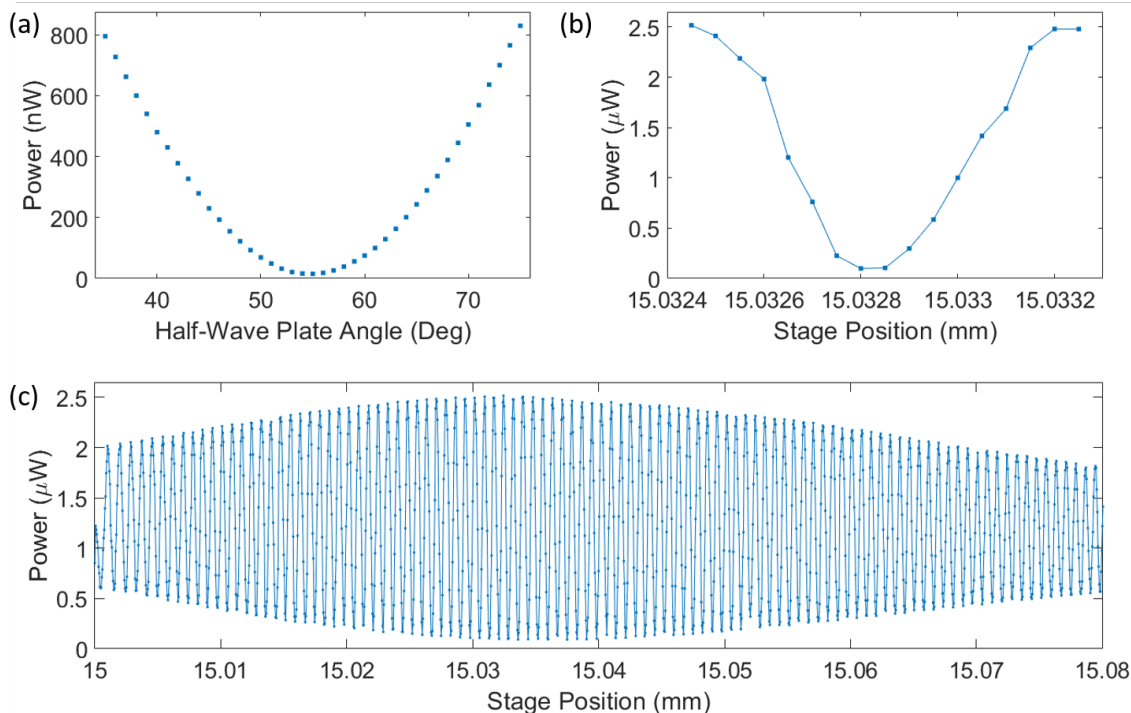


Figure 7.4: Data for measuring the best overlap between the signal and LO. (a) HWP angle vs measured power for the signal (path 1). The minimum corresponds to the HWP angle that transmits a signal with vertical polarisation with respect to the collection PM fibre. (b) Power at one output of the 50:50 FBS vs stage position, both signal and LO have the same polarisation and are at very similar powers. The stage position is around the point of optimal overlap of the signal and LO at the FBS, and we are scanning $\approx 2\pi$ of phase. (c) Zoomed out version of (b). The peak (and trough) of this envelope sets the stage position.

Following this, we removed the additional polariser in path 1, set the signal HWP to 54.78° , and adjusted the LO HWP in path 2 such that the power of the signal and LO were the same (we measured to within 0.1%). Then we measured the power at one of the outputs of the 50:50 FBS while we swept the translation stage. From this we would expect to see interference fringes between the signal and LO, which would both indicate the optimal stage position for the best overlap and quantify the overlap (i.e. the fringe visibility).

We see a plot of the power from one of the 50:50 FBS output ports vs stage position

in Figs. 7.4 (b) and (c). (c) shows a larger range of stage positions, and (b) is zoomed in around the optimum stage position value to show the phase variation on the order of half a wavelength (i.e. $\approx \pi$ of phase variation). We fit polynomials to the envelope of (c), from which we estimate the optimum stage position to be 15.033 ± 0.003 mm, and the overlap (from (max-min)/max) to be $95.6 \pm 0.2\%$. It is crucial to note that the FBS splitting ratio was not uniform with polarisation. For this measurement where both signal and LO were in the vertical polarisation the splitting was (very close to) 50:50, however this was not the general case. The polarisation-dependent splitting ratio was calibrated separately, and as the tomography method only required a known unitary (i.e. rather than specifically a 50:50 one like in balanced homodyne detection) it does not affect the outcome of the state tomography reconstruction.

7.2.3 Phase Stability

For the relative phase between the signal and LO to stay stable, we require the path lengths in path 1 and 2 to stay stable with respect to each other on the order of much less than a wavelength (i.e. $\ll 1550$ nm). From considering Fig. 7.3, we can see that both paths are of the order of 1 m, so keeping them stable to within much less than a wavelength would be extremely difficult. Things such as small breezes and small changes in temperature can change the optical path length in free space and in fibre respectively enough to completely change the relative phase. Although the laboratory was temperature controlled and we kept the setup in an enclosure, phase stability was still an important consideration.

To check the phase stability we performed a similar measurement to finding the optimal stage position in Sec. 7.2.2. With the stage set at its optimal position of 15.033 mm, we swept the piezo position and recorded the power at one of the 50:50 FBS output ports. When the piezo moves a certain distance, the path length changes by twice that distance due to the ‘L-shape’ setup (see Fig. 7.2). Therefore to measure the power for a full phase range of 0 to 2π , we were required to move the piezo by half a wavelength (i.e. 775 nm). Each piezo sweep lasted approximately 25 seconds, and we recorded the power at 42

evenly-spaced piezo positions, We did this 1,000 times, the whole measurement lasted approximately 30 minutes. For each sweep we expect to see a sinusoid interference fringe, similar to Figs. 7.4. We expect the period to stay the same, however the change in the offset will give insight to the phase drift over time.

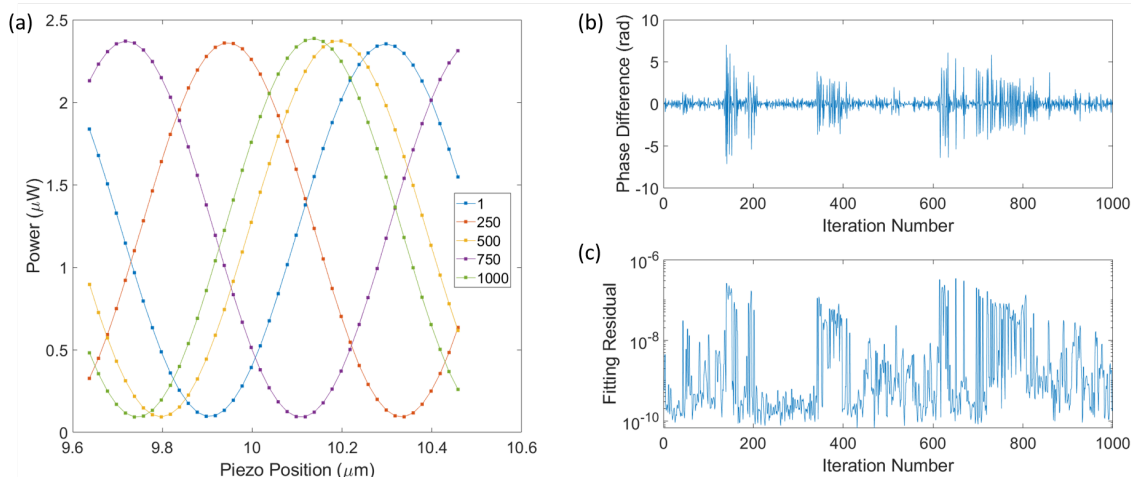


Figure 7.5: (a) Piezo position vs measured power at one of the outputs of the 50:50 FBS with signal and LO input. 5 example iterations of piezo sweep are plotted, the iteration number is shown in the legend. (b) The difference between the phase offset from successive iterations, as obtained from sinusoidal fitting of (a). (c) The fitting residual from the piezo sweep vs power sinusoidal fit for each iteration.

We see example plots of piezo position vs measured power at one of the outputs of the 50:50 FBS with signal and LO input in Fig. 7.5 (a). The iterations selected were approximately 8 minutes apart, and although we can see significant phase drift, it is not unexpected at this time scale. We performed a sinusoidal fit for each iteration, and as the period is constant we can use the phase offset extracted from the fit to quantify the drift. The difference between the phase offset for successive iterations is plotted in Fig. 7.5 (b). From this we see that there are periods of relative stability, but there are also some periods with large fluctuations. Note that absolute phase differences of larger than 2π have been adjusted for. We see the sum of residuals from the fitting procedure for each iteration in Fig. 7.5 (c), where we immediately notice a correlation between a higher residual (i.e. a worse fit) and a larger phase difference. When examining some individual iterations with comparatively higher residuals we notice that the sinusoid appears either stretched

or compressed, showing that the phase is drifting within that iteration. From this, we see that despite our attempts to stabilise laboratory conditions that the phase drift is sometimes over timescales shorter than ≈ 25 seconds. A possible cause for this was slight temperature variations causing the optical fibres and the FC connections to contract and expand. Due to this we tracked the phase during data acquisition, and we will discuss the method in Sec. 7.2.5.

7.2.4 Adiabatic Demagnetisation Refrigeration

The TESs used for this experiment were cooled to 100 mK in an Adiabatic Demagnetisation Refrigeration (ADR), a different type of cryostat compared to the Dilution Refrigerator (DR) from Chap. 4 and the Helium Sorption Refrigerator (HSR) from Chap. 5. In this section we will briefly discuss the cooling mechanism of an ADR, Ref.^[242] has been the primary source for this.

Unlike DRs and HSRs which rely entirely on the circulation of gasses/liquids for cooling, ARDs obtain cooling from the magnetocaloric effect^[243,244]. When a magnetic field is applied to a paramagnetic material, the electron spins of the material line up with the magnetic field, and are in an ordered state of higher entropy. When the magnetic field is removed, the orientation of the electron spins become disordered to achieve lower entropy, an endothermic process which removes (thermal) energy from the system.

In Fig. 7.6 we see a very basic schematic of the structures that provide the cooling inside an ADR. The location of the TESs are shown, but optical and electrical connections are absent. First, the entire system is cooled to ≈ 3 K by a pulse tube cooling system, the same as described in Sec. 4.3.1. The ‘Heat Bath’ part of Fig. 7.6 is the connection to this system. The ‘Salt Pill’ contains the paramagnetic material, usually some form of paramagnetic salt (examples can be seen in table 9.1 of Ref.^[242]). When the system is cooled to ≈ 3 K, the magnets seen in the figure are superconducting, and a current is applied which generates a magnetic field of ≈ 1.5 T. This is known as the ‘recycle time’ when the electron spins are aligning, and is typically ≈ 2 hours. After this time, the

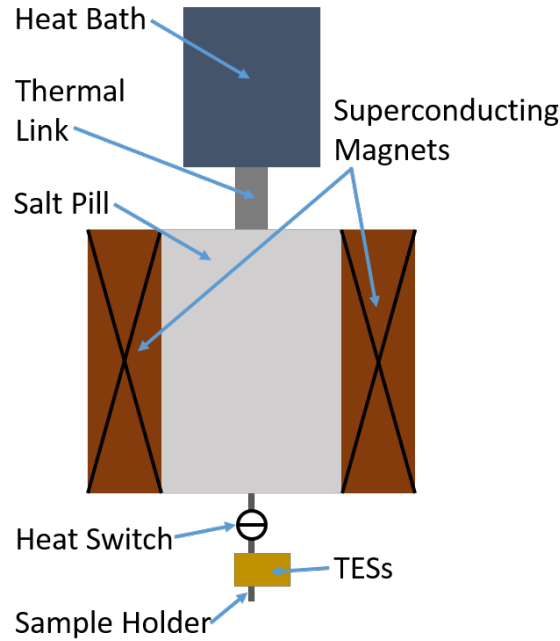


Figure 7.6: A very basic schematic of the internal structures of an ADR. Description in main text.

‘Heat Switch’ is first opened, then the current supply is removed. The heat switch creates a thermal connection between the ‘Salt Pill’ and the ‘Sample Holder’, and disconnects the ‘Salt Pill’ from the ‘Heat Bath’. The ‘Sample Holder’ is the part that reaches < 100 mK, and is where the TESs were affixed for this experiment. When the current supply is removed the orientation of the electron spins randomise, resulting in thermal energy being drawn from the Sample Holder (and whatever is attached to it), cooling it down to a base temperature of ≈ 50 mK. The duration the Sample Holder remains at base temperature is known as the ‘hold time’, and for the system we used was approximately 8 hours. This time can be increased by maintaining a small current through the magnets, resulting in a slightly higher base temperature, and for this experiment we regulated at 100 mK.

7.2.5 Data Acquisition

We collected the data in a similar fashion to Chap. 6, based on the overview in Sec. 2.4.3. The TES traces were recorded by an AlazarTech waveform digitizer (ATS9440), which

was operated at a 10 MHz acquisition rate provided by an external clock. Pulses from the signal generator that provided the external clock were fed into an Analog Devices electrical pulse divider, which we set to 100:1 to provide the repetition rate of 100 kHz required for TES operation. These pulses triggered both the waveform digitizer and the laser.

We collected data at five different signal HWP settings to provide a range of overlaps. These overlaps (estimated from fringe visibility) approximately corresponded to 95.6%, 84.5%, 50.0%, 25.0% and 10.0%. With reference to Fig. 7.4 (a), these corresponded to HWP angles of 54.78° , 64.80° , 76.74° , 84.59° and 90.66° . For each HWP angle we attempted to keep the signal average photon number ($|\alpha_{\text{sig}}|^2$) at a constant value of ≈ 0.7 , which we controlled using the variable attenuator and calibrated using TESs.

For each signal, we collected tomography data for three different LO magnitudes. We controlled the average photon number ($|\alpha_{\text{LO}}|^2$) using the HWP in path 2 of Fig. 7.2. We aimed for average photon numbers of approximately 1.2, 1.7 and 2.3, but this was also calibrated each time using TESs.

We then had the task of setting several LO phases for each magnitude. Given the phase stability issues discussed in Sec. 7.2.3, an active approach for phase control was required. We supplied the piezo with saw-tooth electrical pulses at 1 Hz, the amplitude was selected such that the motion cycled through a little more than the full phase range of 0 to 2π . The piezo controller outputted a voltage reading proportional to its position which we also recorded with the waveform digitizer, meaning when we collected data we recorded three voltage values per shot: ‘TES1’, ‘TES2’ and ‘piezo’. We then divided the data into pairs of piezo ramps. From the first ramp we calibrated the phase from the average photon number (equivalent to measuring power), then we used this calibration to set the phase for the second ramp, meaning we only required phase stability on the order of 2 seconds. For each LO magnitude we recorded data in this fashion for approximately an hour to ensure that we had enough samples at a decent range of LO phases. We will discuss this procedure in more detail in Sec. 7.3.1 of the results.

7.3 Results

In this section we shall discuss the photon-number distribution results. We shall first see how the raw photon-number results can be used to track then retrospectively set relative phases, and some preliminary tomography results will be hinted at.

7.3.1 Phase Tracking and Calibration

The result of this section is to provide insight into the procedure only, for the actual tomographic analysis a more rigorous procedure was used based on the detector POVMs and the polarisation-dependent unitary. For simplicity, let us consider a signal $|\alpha_{\text{sig}}\rangle$ and LO $|\alpha_{\text{LO}}\rangle$ which overlap perfectly at the beamsplitter. As only the relative phase between the signal and LO is important, we will define the signal as having zero phase ($\alpha_{\text{sig}} = |\alpha_{\text{sig}}|$), and the LO as having phase φ ($\alpha_{\text{LO}} = |\alpha_{\text{LO}}| e^{i\varphi}$) which we (partially) control with the piezo.

Now consider the beamsplitter transformation in Eq. (2.38). As we used a fibre beamsplitter the transformation will be symmetric, i.e. the relative phase shift imparted by the beamsplitter, ϕ , is $\phi = -\pi/2$ ^[36,37]. When we transform the amplitudes, the average photon number at the outputs are:

$$\bar{n}_1 = \cos^2\left(\frac{\theta}{2}\right) |\alpha_{\text{sig}}|^2 + \sin^2\left(\frac{\theta}{2}\right) |\alpha_{\text{LO}}|^2 - \sin(\theta) |\alpha_{\text{sig}}| |\alpha_{\text{LO}}| \sin(\varphi), \quad (7.1)$$

$$\bar{n}_2 = \sin^2\left(\frac{\theta}{2}\right) |\alpha_{\text{sig}}|^2 + \cos^2\left(\frac{\theta}{2}\right) |\alpha_{\text{LO}}|^2 + \sin(\theta) |\alpha_{\text{sig}}| |\alpha_{\text{LO}}| \sin(\varphi), \quad (7.2)$$

where $\bar{n}_1$ and $\bar{n}_2$ are the average photon numbers in the paths leading to TES 1 and TES 2 respectively (see Fig. 7.2), and θ quantifies the splitting ratio.

We can now calibrate and set the LO phase φ from the data mentioned in Sec. 7.2.5 and Eqs. (7.1) and (7.2). From the first piezo ramp we calibrate the phase-piezo voltage relation. After processing the data into photon number outcomes, we have a three-column data array of: ‘TES 1 photon-number’, ‘TES 2 photon-number’ and ‘average piezo voltage’.

There were 100 data points for each TES trace used to calculate the matched filter value and subsequently photon number (see Fig. 2.13 (a) from Sec. 2.4.3), and hence the piezo voltage for each photon-number outcome is an average of the corresponding 100 values. As we were recording photon numbers at a rate of 100 kHz and ramping the piezo at 1 Hz, there were approximately 10^5 data points for each phase calibration. In order to use Eqs. (7.1) and (7.2), we averaged over 200 photon-number outcomes (and piezo voltages) to calculate average photon numbers.

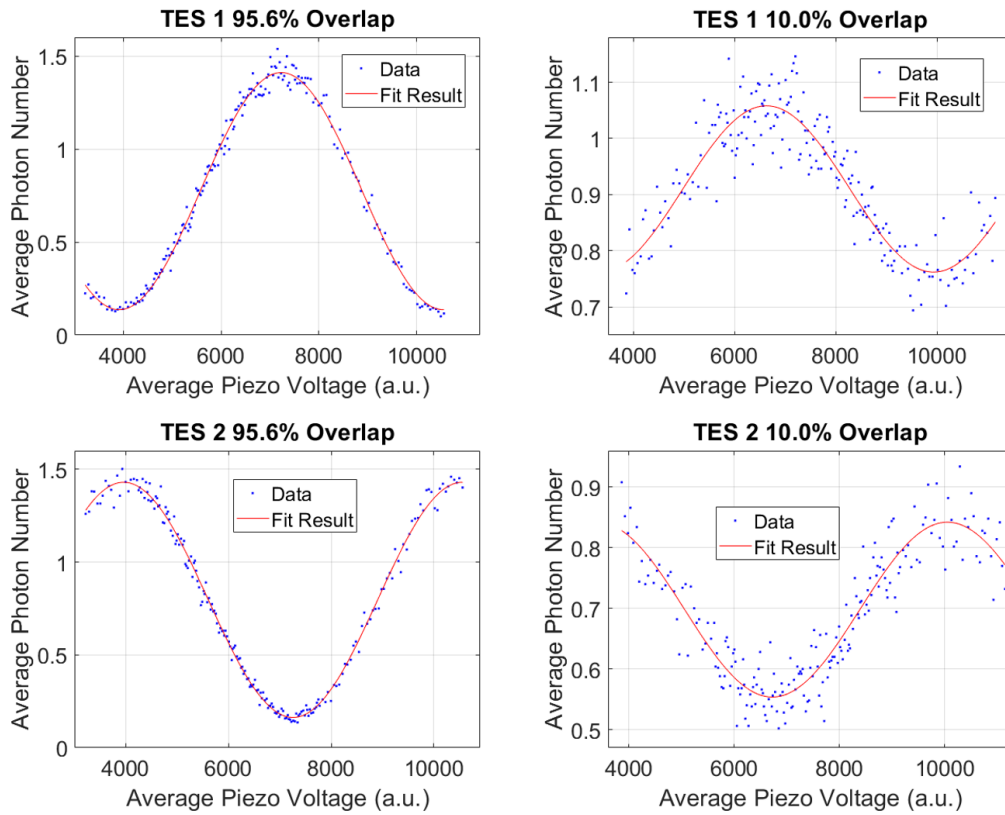


Figure 7.7: Average piezo voltage vs. average photon numbers with fitting results for LO phase calibration. Left column: results for TES 1 and 2 with $|\alpha_{\text{sig}}|^2 \approx 0.70$, $|\alpha_{\text{LO}}|^2 \approx 1.2$, and 95.6% overlap between the signal and LO as determined by visibility. Right column: $|\alpha_{\text{sig}}|^2 \approx 0.72$, $|\alpha_{\text{LO}}|^2 \approx 1.2$, and 10.0% overlap between the signal and LO.

Following this, we fitted Eqs. (7.1) and (7.2) to the results. We see the data and fitting plotted for each TES for two signal and LO overlap examples (95.6% and 10.0%)

in Fig. 7.7. The actual equation we fitted was:

$$\bar{n}_{1/2} = A \mp B \sin(aV_{piezo} + b), \quad (7.3)$$

where A , B , a and b are all fitting parameters, and V_{piezo} is the piezo voltage in arbitrary units. Imperfect overlap is manifested as worse sinusoid visibility, i.e. a larger A and smaller B , which is demonstrated when comparing the left and right columns of Fig. 7.7. Also, the polarisation-dependent splitting ratio is evident from these plots. For 95.6% overlap, the signal and LO are both in the vertical polarisation and the splitting ratio is very close to 50:50. As a result, the midway average photon number (i.e. halfway between the peak and the trough) is approximately the same for both. For 10.0% overlap however, the signal is mostly in the horizontal polarisation, and we characterised the splitting ratio to be approximately 80:20. The result is that midway average photon number is higher for TES 1 compared to TES 2, and is manifested by different values for B from the fitting results. As long as there is measurable overlap, the above does not affect the phase calibration as we only require a and b . Only when the overlap is so small that a sinusoidal fit is no longer possible it becomes problematic, but we see from the right column of Fig. 7.7 that successful fits were obtained at 10.0% overlap.

After calibrating the phase for one piezo ramp we applied this calibration to the next ramp through $\varphi = aV_{piezo} + b$. We continue this process for the rest of the data set, resulting in half of the data being used for phase calibration, and the other half for the state reconstruction algorithm. One advantage of this method is that poor fits can be rejected. We recall from Sec. 7.2.3 (specifically Fig. 7.5) that occasionally the phase will drift during one ramp. In this event, the error on the fit will be worse and we can define a cut-off error such that all fits which are poorer are discounted, though this does result in discounting some of the data.

7.3.2 Tomographic Results

At the time of writing, state reconstruction from the data using the tomographic method is still a work in progress. Some preliminary results from looking at the specific case of 50.0% polarisation overlap between $|\alpha_{\text{sig}}\rangle$ and $|\alpha_{\text{LO}}\rangle$ reconstruct the signal with 98.6% fidelity to what it was measured to be. From the signal photon-number distribution for this particular overlap, we estimate the signal amplitude to be $|\alpha_{\text{sig}}|^2 \approx 0.73$. For the purpose

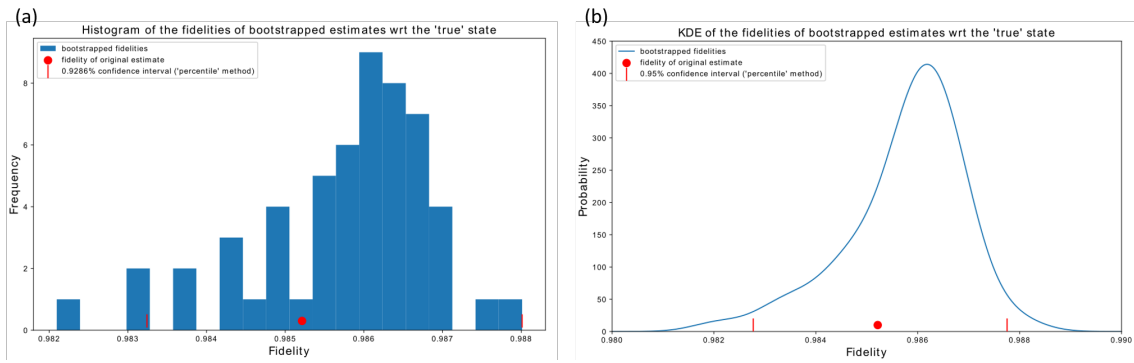


Figure 7.8: Fidelity estimation using bootstrapping technique with 95% confidence intervals shown. (a) Histogram of fidelities from 55 bootstraps. (b) Interpolated fidelities using Kernel Density Estimation. Both plots were produced by Arik Avagyan.

of including some form of graphical result, Figs. 7.8 (a) and (b) give some insight into calculating the fidelity of the ‘true’ state and the state obtained from the tomographical method. Fig. 7.8 (a) shows a histogram of fidelities obtained from 55 bootstraps of the samples, and Fig. 7.8 (b) shows this histogram interpolated by a Kernel Density Estimation (KDE) method. We can see this histogram is peaked at a fidelity of 98.6% as expected. The reason for bootstrapping this way is to estimate a confidence interval in the fidelity.

While initially this result is very promising, the modal overlap between $|\alpha_{\text{sig}}\rangle$ and $|\alpha_{\text{LO}}\rangle$ inferred from the HWP measurement and from the tomographical reconstruction are different. We would expect to infer an overlap close to 50%, but in reality it was calculated to be 25%. This indicates that there may be additional modes present or that the temporal overlap at the beamsplitter drifted during data collection, however this is in contradiction to the visibility measured from Fig. 7.4 (c). The reason for this is still an

open question. Despite this, as the tomographic method only requires non-zero overlap between the state and the LO, it was still possible to determine the magnitude of $|\alpha_{\text{sig}}\rangle$ tomographically with excellent agreement to measuring it directly.

7.4 Conclusion and Outlook

In this chapter we discussed an experimental setup to perform a proof-of-principle demonstration of a novel state tomography procedure. The procedure is very similar to the weak field homodyne detection in Chap. 6, however it is no longer necessary for the signal to occupy the same mode as the LO; only partial overlap is required. In order to provide this initial experimental demonstration, we required a signal and weak variable LO with non-zero modal overlap, and a homodyne setup with photon-number resolving detectors. As this was proof-of-principle it was desirable to have the ability to control and vary the overlap between the signal and LO. This chapter concerned the experimental challenges to realise this demonstration.

We decided to use a coherent state as a signal, and as such we derived the signal and LO from the same laser. This gave the highest degree of overlap tunability as both signal and LO were initially well mode-matched. We varied the overlap using the polarisation degree of freedom, and tuned the polarisation of the signal using a half-waveplate while keeping that of the LO constant. An additional requirement of the tomographic reconstruction technique was the ability to vary the magnitude and phase of the LO in a controlled way, which we did using attenuation and a piezo controller respectively. This outlined the major experimental challenge of phase stability, which required the relative path lengths between the signal and LO to be stable to within much less than the order of a wavelength. In practise this is generally extremely difficult, but given both the signal and LO were coherent states with some overlap we were able to track the LO phase from the photon-number statistics using the interference at the beamsplitter.

Although the data analysis is still a work in progress, we discussed a preliminary result

from 50.0% polarisation overlap between the signal and LO. The signal magnitude obtained from the tomographical state reconstruction procedure was in excellent agreement to direct measurement with a fidelity of 98.6%. Despite the issues with overlap discrepancy this result is very promising, and at the time of writing the effort is continuing.

Once the full proof-of-principle demonstration is complete, an interesting extension would be to test this tomographic method with more complex signal states. This could include on both modes of a two-mode squeezed vacuum, or single-mode states with more of a quantum nature compared to coherent states. In most of these instances, good phase control would be crucial given tracking would become a more difficult task when the signal is not a coherent state. Additionally, phase tracking requires prior knowledge of the signal state, therefore having a fully-controllable LO phase relaxes this requirement and makes the technique more robust.

Benchmarking Gaussian Boson Sampling via Two-Point Correlators

In their seminal work, Hanbury Brown and Twiss analysed two-point correlators to improve the apparent angular size estimation of distant stars^[245]. On the quantum level, two-point correlations render it possible to experimentally uncover non-classical properties of light, e.g., photon antibunching^[246]. Currently, general quantum correlations form the foundation of quantum information and communication science^[1,63,247]. For example, continuous variable (CV) entanglement offers a robust resource for quantum protocols when optical modes propagate through the turbulent atmosphere^[248,249]. While quantum correlations enable us to perform certain tasks (e.g. quantum teleportation^[250,251]), the problem of whether or not there is a true advantage of quantum protocols over classical information processing is still debated as quantum correlations can be significantly diminished in the presence of imperfections and require error correction (see Ref.^[252] for a recent discussion). For these reasons, the application-oriented study of realistic quantum correlations is a timely problem of fundamental importance for the development of quantum technologies.

In this chapter, we will consider the problem of experimentally verifying the output of

the Gaussian Boson Sampler from Sec. 3.5 using two-point correlators. Instead of coming up with a robust verification protocol, we will instead present an experimentally friendly benchmarking procedure to discriminate the output of a Gaussian Boson Sampler from its classical analogue (i.e. with thermal states) in the presence of experimental imperfections. We will follow the work of Walschaers *et al.* from Ref.^[253] by considering the two-point correlator C_{jk} between the output modes of a Gaussian Boson Sampler.

The work for this chapter is based on the publication ‘Benchmarking of Gaussian boson sampling using two-point correlators’^[151]. The original project idea was conceived by Jelmer Renema and is based on applying the analysis of Ref.^[253] for Boson Sampling to Gaussian Boson Sampling (GBS). The mathematical formulation was supervised by Mattia Walschaers and Jan Sperling, and the numerical studies were carried out by me. For this chapter I have adapted and expanded the formulae from Ref.^[151] to be consistent with the definitions in Chap. 3, and to allow for the consideration of other scenarios. Section V onwards describes (for the most part) my contribution to Ref.^[151], though I inherited part of the Gaussian State Formalism code I used for simulations from William Clements. Due to this, in this chapter we will focus on Secs. V and VI of the publication, but first we will discuss the verification problem and Ref.^[253], and then develop the mathematical tools required for the analysis.

8.1 Introduction

Following the instructive introduction in Ref.^[151], first we shall discuss verification of Boson Sampling in general, then correlation functions and their moments, and finally bring both elements together by discussing Ref.^[253], i.e. the precursor to this work. Note that the terms ‘verification’, ‘validation’ and ‘certification’ are used interchangeably.

8.1.1 The Verification Problem

Signatures of many-particle interference serve an important purpose as a tool for the benchmarking of Boson Sampling. The debate on how to validate a Boson Sampler (see Fig. 2.6) started with the concern that it would be impossible to distinguish data from a Boson Sampling setup from data that were drawn from a uniform distribution. Due to this, the first certification protocols aimed at making the distinction between these scenarios^[254–256]. Even though this led to the development of several certification protocols^[257,258], the main focus in research on validation of Boson Sampling has shifted to signatures of many-Boson interference^[253,259–264]. Further to this, other similar sampling scenarios that are easier to simulate on a classical computer have been studied, such as sampling with distinguishable or partially distinguishable Bosons^[55].

In general, we can single out two approaches to benchmarking many-particle interference. In the first case, we could construct highly symmetric unitary circuits (e.g. the Fourier interferometer^[260]) that manifest totally destructive interference, which are unique benchmarks of many-Boson interference. In the second case, we could instead use Haar-random circuits that are common in Boson Sampling, and employ statistical analysis on the data (e.g. by studying two-point correlators^[253]) to find statistical signatures of many-particle interference. By ‘Haar-random circuits’ we are referring to photonic circuits that implement unitary matrices which are random with respect to the Haar measure^[265,266].

Recent developments^[267,268] in the understanding of total destructive interference may provide a potential pathway for constructing a benchmark for GBS. Nevertheless, the statistical signatures^[253,264,269] are found by analyzing intensity correlations between output detectors, which can be calculated for arbitrary initial states. Therefore, this approach is a viable candidate for a benchmark of GBS.

8.1.2 Correlation Functions

In statistical physics, a two-point correlator quantifies the correlation between two measured quantities. In general, correlators are second-order cumulants over multiple random variables, and higher orders can be generalized by the Ursell function^[270]. These higher order correlators have a long history in quantum statistical mechanics as they characterize many-body states^[271–274], and are commonly referred to as truncated correlation functions. For two classical random variables X and Y , the two-point correlator $\mathbb{C}(X, Y)$ is commonly defined as:

$$\mathbb{C}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y), \quad (8.1)$$

where $\mathbb{E}(\dots)$ denotes the expectation value.

8.1.3 Benchmarking Boson Sampling

The most obvious (and potentially inefficient) way to certify the output of an experimental Boson Sampler is to simulate the output as per Eq. (2.42) (including imperfections), and compare to the experimental results. As discussed in Sec. 2.3.3, this is a computationally difficult task, and having to simulate the experimental results defeats the purpose of the experiment. Therefore, it is attractive to have a measure to benchmark the results against alternative hypotheses.

In Refs.^[253,264,269], the two-point correlator is used to benchmark the output of a Boson Sampler by considering Eq. (8.1) in terms of particle numbers. The two-point correlator of particle numbers between output modes j and k of an interferometer, C_{jk} , is given by:

$$C_{jk} = \langle \hat{n}_j \hat{n}_k \rangle - \langle \hat{n}_j \rangle \langle \hat{n}_k \rangle, \quad (8.2)$$

which we can see has a very similar form to Eq. (8.1).

Ref.^[253] benchmarks Boson Sampling against sampling with Fermions, distinguishable particles and ‘Simulated Bosons’. They explain that Simulated Bosons are a mean-field

approximation of Bosonic particle input with randomised phases, which can mimic Bosonic bunching but not many-body interference effects^[259,260,275–277]. The authors first build up a data set of C_{jk} values for each particle type by simulating N particles in M modes of an interferometer for many different unitaries, where each particle type obeys its respective statistics (i.e. Bose–Einstein statistics for Bosons, Fermi–Dirac statistics for Fermions, etc.). There are two approaches for this. One is to fix the outputs considered (e.g. $j = 1$ and $k = 2$ for example) and simulate C_{12} for many different Haar-random unitaries. The second approach is to choose only one Haar-random unitary $\mathbf{U}$, and build a data set of C_{jk} using all of the different output combinations. The authors state that both approaches are equivalent in the limit of a large number of modes, as swapping output modes is equivalent to a unitary transformation, and a Haar-random unitary multiplied by a unitary gives another Haar-random unitary. For the first case, the number of C_{jk} values obtained depends of the number of Haar-random unitaries considered, but for the second we are restricted to $M(M - 1)/2$ values. In Ref.^[253] the explicit formulae for C_{jk} for each particle type is given in terms of the components of $\mathbf{U}$ and N , thus avoiding calculating the computationally-difficult permanent for the Bosonic case.

With the data sets of C_{jk} for each particle type, the authors calculate the first three normalised moments. The first moment is the normalised mean **NM**:

$$\text{NM} = \frac{\mathbb{E}_U(C) M^2}{N}, \quad (8.3)$$

the second is the coefficient of variation **CV**:

$$\text{CV} = \frac{\sqrt{\mathbb{E}_U(C^2) - \mathbb{E}_U(C)^2}}{\mathbb{E}_U(C)}, \quad (8.4)$$

and the third is the skewness **Sk**:

$$\text{Sk} = \frac{\mathbb{E}_U(C^3) - 3\mathbb{E}_U(C)\mathbb{E}_U(C^2) + 2\mathbb{E}_U(C)^3}{\sqrt{(\mathbb{E}_U(C^2) - \mathbb{E}_U(C)^2)^3}}. \quad (8.5)$$

These three quantities correspond to the normalised first three moments of the distribution of C_{jk} for a fixed system (N particles in M modes) averaged over many Haar-random unitaries, labeled as $\mathbb{E}_U(\cdots)$. The indices of C_{jk} have been dropped in Eqs. (8.3) to (8.5) as all of the permutations are taken into account during the averaging process, meaning C denotes an arbitrary element C_{jk} .

The benchmark itself comes in Fig. 4 of Ref. [253], which shows 500 values of CV plotted against its corresponding Sk for each particle type. We see that the data points form clusters for each particle type, thus distinguishing them. The authors show this for $N = 6$ in $M = 120$, though provide a discussion on the implications for more experimentally-feasible sizes. The authors also calculate the explicit forms of $\mathbb{E}_U$, $\mathbb{E}_U(C^2)$ and $\mathbb{E}_U(C^3)$ for each particle type in terms of N and M by averaging over the whole set of Haar-random unitaries using random matrix theory (RMT) techniques (see Refs. [278–284]). This allows them to calculate the exact values of NM, CV and Sk for each system size using Eqs. (8.3) to (8.5) for comparison. Ref. [269] then extends this analysis to partial overlap between Bosons, i.e. investigating the intermediate regime between the Bosonic and distinguishable particle cases, and Ref. [264] applies these techniques to an experiment.

8.1.4 Extension to Gaussian Boson Sampling

In this chapter, we discuss the above analysis applied to GBS. Unlike the scenario in Ref. [253], for Gaussian state inputs the particle number is not fixed, and the input states are now phase sensitive. Therefore, in contrast to Ref. [253], we will formulate the systems in question using the Gaussian State Formalism introduced in Chap. 3. Instead of concocting CV analogies of Fermions and Simulated Bosons, we will take a more practical route by benchmarking true GBS with single-mode squeezed vacuum (SMSV) inputs against more likely laboratory mishaps.

We discussed in Sec. 2.3.1 that it is possible to interchange between a two-mode squeezed vacuum (TMSV) and a product of two SMSVs using a 50:50 beamsplitter, provided both modes of the TMSV are otherwise identical. As TMSV is easily avail-

able from spontaneous parametric down conversion (SPDC) processes^[221], it is common to produce SMSV in this way. Also, we discussed in Sec. 2.2.5 that discarding (i.e. tracing over) one mode of a TMSV results in a thermal state (see Secs. 2.2.4 and 3.3.2), i.e. a classical state, in the other mode. For this reason we will benchmark GSB with SMSV against classical thermal states. We will do this initially for the perfect experiment case, then extend to include imperfections such as loss, noise, and detectors with finite photon-number resolution.

8.2 Algebraic Results

We shall first derive the explicit form of C_{jk} for Gaussian states evolved under an arbitrary unitary $\mathbf{U}$ as per Eq. (2.41), and then randomise over the set of Haar-random unitaries to obtain explicit expressions for $\mathbb{E}_U(C)$, $\mathbb{E}_U(C^2)$ and $\mathbb{E}_U(C^3)$.

8.2.1 The Two-Point Correlator for Gaussian States

Let us first note that Eq. (8.2) is written in terms of the ladder operators of the output state, and we wish for an expression in terms of the input state. Rather than following the procedure in Ref.^[151], we will first relate Eq. (8.2) to the quadrature operators defined in Sec. 3.1.1, and then to the covariance matrix $\mathbf{Q}$ and displacement vector $\mathbf{d}$ elements (also defined in Sec. 3.3.1). This way is more intuitive and less mathematically cumbersome.

After substituting the quadrature operators from Eq. (3.2) (with $\hbar = 1$) into Eq. (8.2), and invoking the factorisation discussed in Sec. 3.3.3, we obtain:

$$\begin{aligned} C_{jk} = \frac{1}{2} & (\langle \hat{q}_j \hat{q}_k \rangle^2 - \langle \hat{q}_j \rangle^2 \langle \hat{q}_k \rangle^2 + \langle \hat{q}_j \hat{p}_k \rangle^2 - \langle \hat{q}_j \rangle^2 \langle \hat{p}_k \rangle^2 \\ & + \langle \hat{p}_j \hat{q}_k \rangle^2 - \langle \hat{p}_j \rangle^2 \langle \hat{q}_k \rangle^2 + \langle \hat{p}_j \hat{p}_k \rangle^2 - \langle \hat{p}_j \rangle^2 \langle \hat{p}_k \rangle^2). \end{aligned} \quad (8.6)$$

We then write this in terms of the components of $\mathbf{Q}$ and $\mathbf{d}$:

$$\begin{aligned}
C_{jk} = & \frac{1}{2} [Q_{jk} (Q_{jk} + 2d_j d_k) - \frac{1}{2} \delta_{jk} \\
& + Q_{j(k+M)} (Q_{j(k+M)} + 2d_j d_{(k+M)}) \\
& + Q_{(j+M)k} (Q_{(j+M)k} + 2d_{(j+M)} d_k) \\
& + Q_{(j+M)(k+M)} (Q_{(j+M)(k+M)} + 2d_{(j+M)} d_{(k+M)})],
\end{aligned} \tag{8.7}$$

where we have assumed an M -mode system. Eqs. (8.6) and (8.7) are still in terms of the output state, but it is now easier to see how we can obtain C_{jk} from the input state and $\mathbf{U}$. We simply start with an initial Gaussian state with a $\mathbf{Q}$ and $\mathbf{d}$, evolve the state under $\mathbf{U}$ using Eqs. (3.25) and (3.39), and then calculate C_{jk} from Eq. (8.7) (with $\mathbf{Q} \mapsto \mathbf{Q}'$ and $\mathbf{d} \mapsto \mathbf{d}'$).

8.2.2 Benchmarking Assumptions

Eq. (8.7) gives a good insight on how to calculate C_{jk} for an arbitrary Gaussian state, but lacks the physical intuition of how the quantity differs for the states under consideration. In other words, for this benchmarking procedure to work for GBS we require a measurable difference in C_{jk} between SMSV and thermal state inputs. To consider this, we will first make a few assumptions.

First, we will assume that the initial $\mathbf{Q}$ is diagonal. $\mathbf{Q}$ can be diagonalised by a unitary transformation (compare Eq. (3.25) to the eigen-decomposition equation), so because we average C_{jk} over a large number of different unitary evolutions, starting with a diagonal $\mathbf{Q}$ will not affect this result. Physically, for SMSV this corresponds to a squeezing angle of $\phi = 0$ or π (see Eq. (3.34)), and $\mathbf{Q}$ is always diagonal for thermal states (see Eq. (3.30)). Under this assumption, we have:

$$\mathbf{Q} = \text{diag} \left(v_{q,1}, \dots, v_{q,M}, v_{p,1}, \dots, v_{p,M} \right), \tag{8.8}$$

where $v_{q,l} = \langle \hat{q}_l^2 \rangle - \langle \hat{q}_l \rangle^2$, and similarly for $v_{p,m}$. In this form the covariance matrix corresponds to a physical state if the uncertainty relation $v_{q,j}v_{p,j} \geq 1/4$ holds true. For $v_{q,j} = v_{p,j} = 1/2$, we have a coherent state or vacuum (the latter for zero displacement). In the case that the variances are identical but larger than one, the input is a (displaced) thermal state. A (displaced) squeezed state is described when one of the variances is below the vacuum fluctuation, i.e. $v_{q,j} < 1/2$ or $v_{p,j} < 1/2$. For completeness, we could also have a classical state ($v_q, v_p \geq 1/2$) which exhibits an unequal noise distribution in the two quadratures ($v_p \neq v_q$). Such a state could simulate a squeezed vacuum state by having an asymmetric Wigner function (see Fig. 3.1 (d)) that still has classical (i.e. not squeezed) variances. Ref.^[164] coins the term ‘squashed states’ for these states.

Second, we will assume zero displacement (i.e. $\mathbf{d} = \mathbf{0}$). We discussed in Sec. 3.5.2 that with an input of a product of purely coherent states (could be with different amplitudes), the output after unitary evolution will also be a product of coherent states with amplitudes altered by the unitary evolution. Due to this, we see from Eq. (8.2) that C_{jk} is always zero for coherent state inputs.

This now enables us to write C_{jk} in terms of $v_{q,l}$ and $v_{p,m}$ for the initial states and $\mathbf{U}$. The quadrature operators evolve as Eq. (3.24), and when this evolution is unitary Eq. (3.39) gives the relation to $\mathbf{U}$. We can use this to write C_{jk} in Eq. (8.6) for the stated assumptions as:

$$C_{jk} = \frac{1}{4} \sum_{w,w'=1}^M [(v_{q,w}+v_{p,w})(v_{q,w'}+v_{p,w'})U_{jw}U_{kw'}U_{jw'}^*U_{kw}^* + (v_{q,w}-v_{p,w})(v_{q,w'}-v_{p,w'})U_{jw}U_{kw}U_{jw'}^*U_{kw'}^*] - \frac{1}{4}\delta_{jk}. \quad (8.9)$$

This provides us a direct comparison to many-Boson interference using Fock states as inputs^[253,285]. For this the correlators C_{jk} are associated solely with two-particle interference processes, arising from terms proportional to $U_{jw}U_{kw'}U_{jw'}^*U_{kw}^*$ (see Eq. (A.4) in Ref.^[253]). These terms also appear in Eq. (8.9), and can be considered signatures of similar interference processes appearing in GBS. However, there are also considerable differences

between the Fock state correlators and the Gaussian scenario. For instance, the terms proportional to $U_{jw}U_{kw}U_{jw}^*U_{kw}^*$ (for $w' = w$) are added in the Gaussian case, whereas they are subtracted in the Fock state case.

Even more profound is the appearance of a completely new class of terms proportional to $U_{jw}U_{kw}U_{jw'}^*U_{kw'}^*$, which are absent in scenarios with fixed particle numbers. As their contribution is weighted with the difference of the variances, they reflect the non-rotational invariance of the initial Gaussian states when compared to Fock states (and mixtures thereof) in phase space. The appearance of this new class of terms may indicate the presence of a new type of interference phenomenon that can manifest itself in GBS. Although this is beyond the scope of this chapter, it is an indication that Gaussian interference experiments may show new physics beyond the many-particle interference processes for Boson sampling with Fock states.

Given the above statements, we now wish to transform Eq. (8.9) into something which gives further physical insight. This involves making two further assumptions. First, we assume that each occupied mode has identical input. This is in line with Ref. [253], as they only consider single-particle input (i.e. only one quantum excitation) to each occupied mode. Second, for N occupied modes, we assume that modes 1 to N are occupied and modes $N + 1$ to M have vacuum input (see Fig. 3.4). This will not affect the averaging of C_{jk} over many unitaries for the same reason as assuming a diagonal $\mathbf{Q}$. Both of these assumptions mean we can write:

$$v_{s,j} = \begin{cases} v_s & \text{for } j = 1, \dots, N, \\ \frac{1}{2} & \text{for } j = N + 1, \dots, M, \end{cases} \quad (8.10)$$

for $s = p, q$.

We will now factor out $v_{q,l}$ and $v_{p,m}$ from Eq. (8.9), and write them in terms of insightful quantities. We choose the average photon number in each occupied mode $\langle \hat{n} \rangle$ (see Eq. (3.45) for this quantity in terms of $v_{s,j}$), and a new operator we introduce called

the eccentricity $\hat{\epsilon}$. On mode j this is given by:

$$\hat{\epsilon}_j = \frac{\hat{q}_j^2 - \hat{p}_j^2}{2}, \quad (8.11)$$

which is the difference between both quadratures of mode j , and quantifies the phase-dependence of the input states. With all of this, we can finally write:

$$C_{jk} = \langle \hat{n} \rangle^2 \sum_{w,w'=1}^N U_{jw} U_{kw'} U_{jw'}^* U_{kw}^* + \langle \hat{\epsilon} \rangle^2 \sum_{w,w'=1}^N U_{jw} U_{kw} U_{jw'}^* U_{kw'}^*, \quad (8.12)$$

where the sum now only runs to N due to the unoccupied modes. From this we notice that the second term will vanish for phase-invariant Gaussian states, which is key to benchmarking SMSV from thermal state inputs.

8.2.3 Randomisation Over Unitaries

We now obtain expressions for $\mathbb{E}_U(C)$, $\mathbb{E}_U(C^2)$ and $\mathbb{E}_U(C^3)$ for the formula for C_{jk} in Eq. (8.12). For $\mathbb{E}_U(C)$ for example, we first write:

$$\mathbb{E}_U(C) = \langle \hat{n} \rangle^2 \sum_{w,w'=1}^N \mathbb{E}_U(U_{jw} U_{kw'} U_{jw'}^* U_{kw}^*) + \langle \hat{\epsilon} \rangle^2 \sum_{w,w'=1}^N \mathbb{E}_U(U_{jw} U_{kw} U_{jw'}^* U_{kw'}^*), \quad (8.13)$$

then use RMT^[278–284] to compute the $\mathbb{E}_U(\dots)$ terms. An example formula for this is seen in Eq. (30) of Ref.^[151], and as we demonstrated in our work^[151], we find:

$$\mathbb{E}_U(C) = \frac{N(M-N)}{(M-1)M(M+1)} \langle \hat{n} \rangle^2 + \frac{N}{M(M+1)} \langle \hat{\epsilon} \rangle^2, \quad (8.14)$$

$$\begin{aligned} \mathbb{E}_U(C^2) = & \frac{2N(N+1)(M-N+1)(M-N)}{(M-1)M^2(M+1)(M+2)(M+3)} \langle \hat{n} \rangle^4 \\ & + \frac{2N(M-N)(MN+3M-N+1)}{(M-1)M^2(M+1)(M+2)(M+3)} \langle \hat{n} \rangle^2 \langle \hat{\epsilon} \rangle^2 \\ & + \frac{2N(M^2N+M^2+NM-3M+2N-2)}{(M-1)M^2(M+1)(M+2)(M+3)} \langle \hat{\epsilon} \rangle^4, \end{aligned} \quad (8.15)$$

and:

$$\begin{aligned}
\mathbb{E}_U(C^3) = & \frac{6(N+1)N(N+2)(M-N+2)(M-N+1)(M-N)}{(M-1)M^2(M+1)^2(M+2)(M+3)(M+4)(M+5)} \langle \hat{n} \rangle^6 \\
& + \frac{6N(N+2)(M-N)(M-N+1)(MN+5M-N+7)}{(M-1)M^2(M+1)^2(M+2)(M+3)(M+4)(M+5)} \langle \hat{n} \rangle^4 \langle \hat{e} \rangle^2 \\
& + \frac{6N(N+2)(M-N)(M^2N+MN+5M^2+5M+4N-4)}{(M-1)M^2(M+1)^2(M+2)(M+3)(M+4)(M+5)} \langle \hat{n} \rangle^2 \langle \hat{e} \rangle^4 \\
& + \frac{6N(N+2)(M^2N+5MN+M^2-7M+12N-12)}{(M-1)M^2(M+1)(M+2)(M+3)(M+4)(M+5)} \langle \hat{e} \rangle^6.
\end{aligned} \tag{8.16}$$

These expressions can then be inserted into Eqs. (8.3) to (8.5) to calculate NM, CV and Sk.

8.3 Simulation Results

Now that we have developed the above formalism, we simulate an experimental setup with reconfigurable linear optics and photon-number-resolved detectors and investigate the impact of the properties of the input states on the benchmarking scheme. Initially, we study perfect squeezed and thermal states as two paradigmatic examples of relevance for experimental implementations. In addition, for all simulations and without loss of generality, we set $j = 1$ and $k = 2$, meaning that we are working with C_{12} the whole time.

8.3.1 Simulation Methods

We use two different methods to simulate the values of C_{12} for different Haar-random unitaries. The first one is closest to what would be done in a laboratory. We first obtain $\mathbf{Q}$ for the state under consideration in line with the assumptions made in Sec. 8.2.2 (see Sec. 3.3.2 for examples). Following this, we perform unitary evolution as per Eqs. (3.25) and (3.39). As only modes 1 and 2 are required, all other modes are traced over using the procedure described in Sec. 3.3.2. This results in a 4×4 covariance matrix. We then project this onto the photon-number basis using the procedure described in Sec. 3.4.2^[121,144,146], from which we obtain the joint photon-number probability distribution for modes 1 and

2, $P(n_1, n_2)$. From this, C_{12} is estimated through:

$$C_{12} = \sum_{n_1, n_2=0}^{n_{\max}} n_1 n_2 P(n_1, n_2) - \left(\sum_{n_1=0}^{n_{\max}} n_1 P(n_1) \right) \left(\sum_{n_2=0}^{n_{\max}} n_2 P(n_2) \right), \quad (8.17)$$

where the marginal distributions are given by $P(n_1) = \sum_{n_2=0}^{n_{\max}} P(n_1, n_2)$ and $P(n_2) = \sum_{n_1=0}^{n_{\max}} P(n_1, n_2)$. This is a straightforward, yet a highly computationally inefficient approach as it corresponds to simulating GBS, a problem considered to be computationally difficult^[152,153]. To determine the true value of C_{12} this way, we would require $n_{\max} \rightarrow \infty$ as Gaussian states do not generally have a fixed photon number, however this is not feasible neither experimentally nor through simulation. We investigate the effect of the value of $n_{\max}$ on C_{12} in Sec. 8.4.4.

The second method of simulating C_{12} is to use our results from Sec. 8.2 directly. This avoids the intermediate calculation of photon-number distributions altogether, resulting in much faster simulation times. From the set of randomly generated C_{12} values, NM, CV and Sk are obtained from Eqs. (8.3) to (8.5). These values can be compared to the exact values obtained via Eqs. (8.14) to (8.16) for the same systems to get an idea of how many Haar-random unitary evolutions one requires to determine good estimates for NM, CV and Sk in simulations and future experiments. Initially, both methods to simulate the statistics were used in parallel for comparison. Otherwise, other than for Sec. 8.4.4, it was mainly the second method that was used as it was (unsurprisingly) significantly faster.

8.3.2 Squeezed and Thermal State Comparison

Pure squeezed states form a class of Gaussian states that cannot be modeled with classical light. They have been produced in experiments for decades and thus serve as a good starting point to develop an intuition for our benchmarking scheme. Furthermore, SMSV inputs are the archetypal scenario for GBS^[152,153]. On the other hand, thermal light behaves in a highly classical way, rendering it an ideal example to contrast against SMSV. It is worth emphasizing that GBS with thermal input states can be simulated in an efficient

way.

With reference to Eq. (8.12), we recall from Sec. 2.2.5 that $\langle \hat{n}_j \rangle = \sinh^2(r_j)$ for SMSV. We also note that $\langle \hat{\epsilon}_j \rangle = \sinh(2r_j)/2$ for SMSV, and $\langle \hat{\epsilon}_j \rangle = 0$ for thermal states.

Small and Large Systems

First, we separately consider squeezed and thermal states as inputs for Boson Sampling to gather insights into their characteristic features. This is done for a small system of $M = 8$ and $N = 2$, typical for what is currently achievable in a laboratory^[286], as well as in a large system of $M = 120$ and $N = 10$ to compare with the results from Ref.^[253]. These numbers were selected due to the technical requirement in Boson Sampling of having many more modes available than non-vacuum input states, i.e. $M \gtrsim N^\varsigma$ for $\varsigma = 2$. As it was shown that Boson Sampling cannot be hard for $\varsigma < 2$ ^[45], we specifically focus on this borderline case.

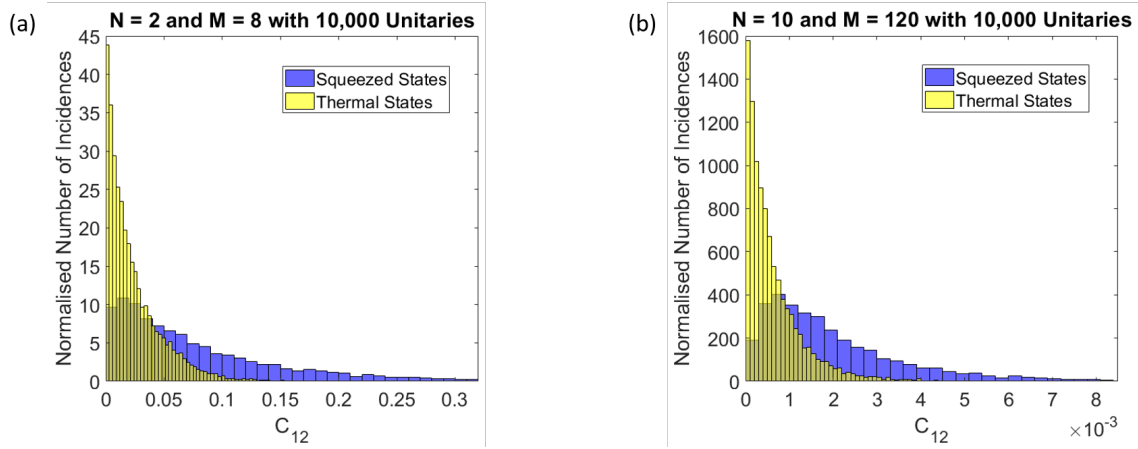


Figure 8.1: Comparison of histograms of C_{12} for squeezed and thermal states. Each occupied mode is identical with $\langle \hat{n} \rangle = 1$, resulting in $r = \ln(1 + \sqrt{2})$ and $\langle \hat{\epsilon} \rangle = \sqrt{2}$ for each SMSV. For both plots, a sample of 10,000 different Haar-random unitaries was generated. (a) The histogram for two occupied modes out of eight available modes, $N = 2$ and $M = 8$, respectively. (b) $N = 10$ and $M = 120$.

For M identical input modes, we use Eq. (8.12) to obtain typical histograms of C_{12} for squeezed and thermal states for small and large systems. This is shown in Figs. 8.1 (a) and (b) respectively. In both cases, we have the same mean photon number per occupied

mode for the input states, $\langle \hat{n} \rangle = 1$. For both system sizes, there is a distinctively broader distribution for SMSV compared to thermal states with the same number of photons. We can understand this in terms of the phase-dependence of the former, resulting in $\langle \hat{\epsilon} \rangle = \sqrt{2}$ and hence giving an additional contribution to C_{12} (see Eq. (8.12)).

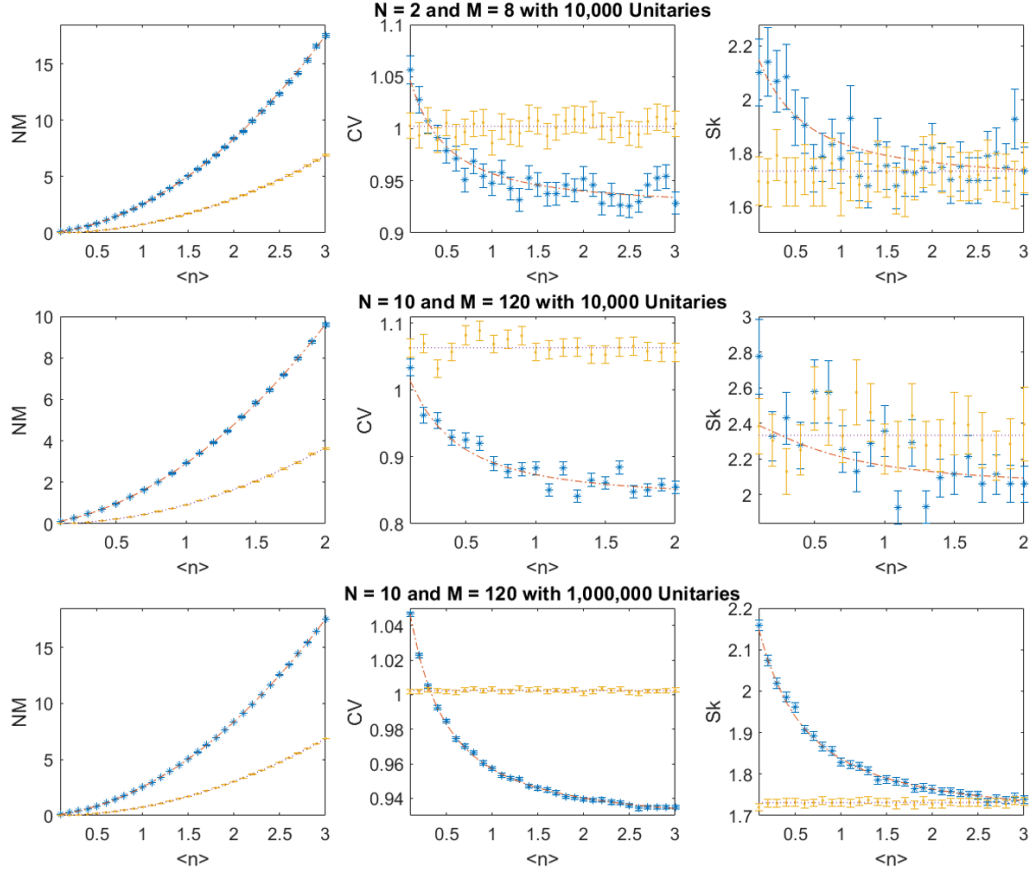


Figure 8.2: MN, CV, and Sk (columns from left to right) for squeezed and thermal states. The analytical expressions (orange dot-dashed lines for squeezed states and purple dotted lines for thermal states) and the values obtained from the simulated data with 1σ error bars (blue stars for squeezed states and yellow dots for thermal states) are plotted. Top row: results for a small system, $N = 2$ and $M = 8$, and 10,000 Haar-random unitaries are generated to sample C_{12} . Middle row: large system, $N = 10$ and $M = 120$, and 10,000 Haar-random unitaries. Bottom row: small system, $N = 2$ and $M = 8$, however, for an increased sample size of 1,000,000 values for C_{12} compared to the first row.

In Fig. 8.2 we see a comparison of the variation of NM, CV and Sk, as defined in Eqs. (8.3) to (8.5), for both types of state and for large and small systems. We do this by averaging over the simulation values of C_{12} , C_{12}^2 and C_{12}^3 , and compare the results

to the values of NM , CV and Sk that are obtained via the relations in Eqs. (8.14) to (8.16).

These results allow us to probe and distinguish the features of squeezed and thermal states. For thermal states, CV and Sk are constant with varying average photon number. This can be understood from Eq. (8.12) for $\langle \hat{\epsilon} \rangle = 0$, as the $\langle \hat{n} \rangle$ terms cancel out in the final expression for CV and Sk . However for squeezed states, we do observe an effect of altering $\langle \hat{n} \rangle$, which can be used as a method to distinguish both types of states.

Moreover, through all of Fig. 8.2, we gauge the number of iterations that are required to let the statistics of the simulated data converge to the analytical predictions as marked by the standard error. Due to the increased standard error, it is obvious that the uncertainties are larger in Fig. 8.2 (top) compared to the corresponding plot in Fig. 8.2 (bottom), describing 10,000 iterations versus 1,000,000, respectively. Furthermore, by comparing Fig. 8.2 (top) and Fig. 8.2 (middle), we observe that system size does not affect the relative uncertainties for the corresponding moment for the same number of iterations. This demonstrates that we need to be conscious of the number of iterations performed depending on which moment we wish to consider. Though with a tunable photonic circuit, it would be experimentally possible to generate sufficiently many random unitaries to reach the necessary statistical error on the measured data. However, in a situation with a fixed $\mathbf{U}$ where we are limited to $M(M-1)/2$ iterations, this may become problematic.

From the different results in Fig. 8.2, it seems appropriate to use NM to distinguish between squeezed and thermal states at large $\langle \hat{n} \rangle$ and Sk to tell them apart at small $\langle \hat{n} \rangle$. However, due to the relation $\sigma_{\text{NM}} < \sigma_{\text{CV}} < \sigma_{\text{Sk}}$ (where σ_x is the standard error in $x \in \{\text{NM}, \text{CV}, \text{Sk}\}$), it becomes apparent that it is most efficient to use NM to discriminate between squeezed and thermal input states. Also there will always be implementation-dependent sources of error in addition to the statistical uncertainties. Therefore, further error analysis directly on experimental data would be required^[264]. The problem of statistical discrimination is discussed in more detail in Sec. 8.4.3.

Constant Dilution

So far we have studied the impact of the type of state and the sample and system sizes on the implementation of Boson Sampling protocols with Gaussian states. We now investigate the influence of the distribution of a fixed amount of energy (i.e. total average photon number) into a varying number of occupied modes, which we refer to as constant dilution. The motivation to study such a problem comes from Ref. ^[153], where the impact of multi-photon events in the same mode is considered. Specifically, we address the question of whether our benchmarking method works better for increased squeezing with a few occupied inputs, or less squeezing distributed over a larger number of modes.

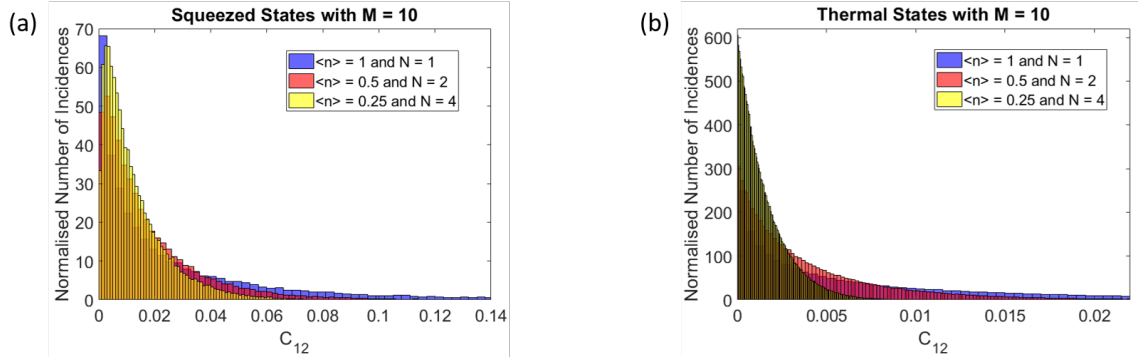


Figure 8.3: Histograms of the two-point correlators C_{12} for $\langle \hat{n}_\Sigma \rangle = 1$, with $N = 1, 2$, and 4. (a) Squeezed states. (b) Thermal states.

For this reason, we consider the mean total photon number $\langle \hat{n}_\Sigma \rangle$, where:

$$\hat{n}_\Sigma = \sum_{j=1}^M \hat{n}_j. \quad (8.18)$$

This mean value reads $\langle \hat{n}_\Sigma \rangle = N \langle \hat{n} \rangle$ for our scenario of N occupied input modes with identical states. For investigating the impact of the number of occupied modes, we keep the total energy constant while altering N , yielding $\langle \hat{n} \rangle = \langle \hat{n}_\Sigma \rangle / N$ for each non-vacuum input. Typical histograms for dilution can be seen in Fig. 8.3. For a fair comparison, we additionally fix the number of modes M , regardless of the choice of occupied modes N . We make the particular choice of $M = 10$ modes to satisfy the minimal constraint

$$M \gtrsim N^2.$$

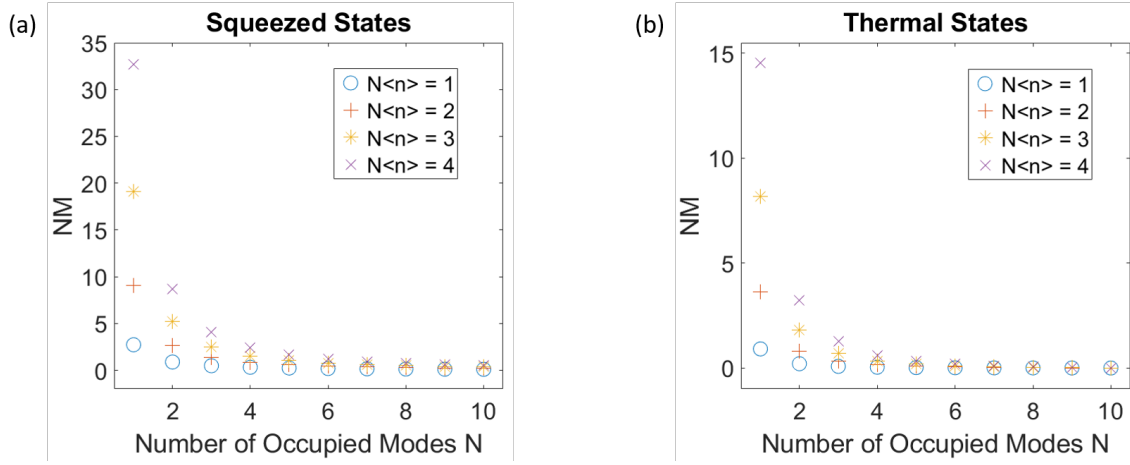


Figure 8.4: Variation of NM with N for $\langle \hat{n}_\Sigma \rangle \in \{1, 2, 3, 4\}$. (a) Squeezed states. (b) Thermal states.

In Fig. 8.4, we use eqs. (8.3) and (8.14) to plot the variation of NM for several values of $\langle \hat{n}_\Sigma \rangle$. We observe that the correlations are most pronounced for fewer occupied inputs with a higher mean photon number. This can be understood from the following considerations. For full dilution and spreading over all modes ($N = M$), the terms proportional to $\langle \hat{n}_\Sigma \rangle$ vanish in Eq. (8.13) according to the RMT calculations. Since these terms are always positive, their vanishing reduces the value of C_{12} . In addition, $\langle \hat{\epsilon} \rangle$ also takes its smallest value in the case of full dilution such that this scenario must lead to the lowest C_{12} . If we treat unoccupied modes as asymmetries in the system, then larger asymmetries (i.e. small N or large $\langle \hat{n}_\Sigma \rangle$) lead to larger two-point correlators. Due to this, our method works best when remaining within the Boson Sampling limit (i.e. $M \gtrsim N^2$) in order to obtain stronger correlations.

8.4 Experimental Considerations

So far we have only considered the perfect theoretical scenario. We have assumed that all states are created perfectly, all components are lossless, there is no noise, all unitaries are ideal, and each detector is 100% efficient with complete photon-number resolution.

In reality this is not the case. In this section, we will explore how the results from the previous sections are affected by such impurities, and the tolerances required to obtain statistically significant results.

We can quantify the effect of impurities by considering how they degrade the states, which we can measure through a state's purity. We recall from Sec. 3.3.1 that the purity of a (multimode) state $\hat{\rho}$ is given by Eq. (3.21), and we can determine how various impurities affect this quantity. In general, the influence of a given imperfection onto $\mathbf{Q}$ determines the impact on the correlators.

In the most general case, imperfections can affect each individual mode in a different manner. This can be considered by using our general results from Sec. 8.2. However, in practice one can assume that all prepared states are subjected to almost the same amount of impurities when passing similar optical elements and being measured with similar detectors. Due to this, for the sake of getting a fundamental idea of what the influence of different imperfections is, we can approximate imperfections by modeling them to affect each mode equally.

8.4.1 Network Loss

In Sec. 3.3.2, we formulated how loss affects a Gaussian state through Eqs. (3.40) and (3.41), and we can use this in the above formalism to consider its effect on C_{12} . If we assume identical efficiency η for each mode ($\eta = 1$ for no loss and $\eta = 0$ for no transmission), then Eq. (3.41) simplifies to:

$$\mathbf{d} \mapsto \sqrt{\eta} \mathbf{d} \quad \text{and} \quad \mathbf{Q} \mapsto \eta \mathbf{Q} + \frac{1}{2} (1 - \eta) \mathbf{I}. \quad (8.19)$$

This affects $\langle \hat{n} \rangle$ and $\langle \hat{\epsilon} \rangle$ as $\langle \hat{n} \rangle \mapsto \eta \langle \hat{n} \rangle$ and $\langle \hat{\epsilon} \rangle \mapsto \eta \langle \hat{\epsilon} \rangle$ respectively, and hence C_{12} as:

$$C_{jk} \mapsto \eta^2 C_{jk}, \quad (8.20)$$

which we can now use to study the affect on loss on our benchmarking procedure.

Let us first consider how loss degrades the purity of SMSV. For this single-mode state, we evaluate the purity through Eq. (3.21) using Eqs. (3.34) and (8.19), and obtain:

$$\text{Tr}(\hat{\rho}^2) = [4\eta(1-\eta)\sinh^2(r) + 1]^{-1/2}. \quad (8.21)$$

In addition, we can also characterise the purity through the uncertainty relation, which is minimally satisfied (i.e. $v_q v_p = 1/4$) for pure Gaussian states. In the diagonal form we have assumed for $\mathbf{Q}$, it is easy to see that the uncertainty relation is simply the inverse squared of Eq. (8.21). The impact of identical loss in each mode on NM, CV and Sk for

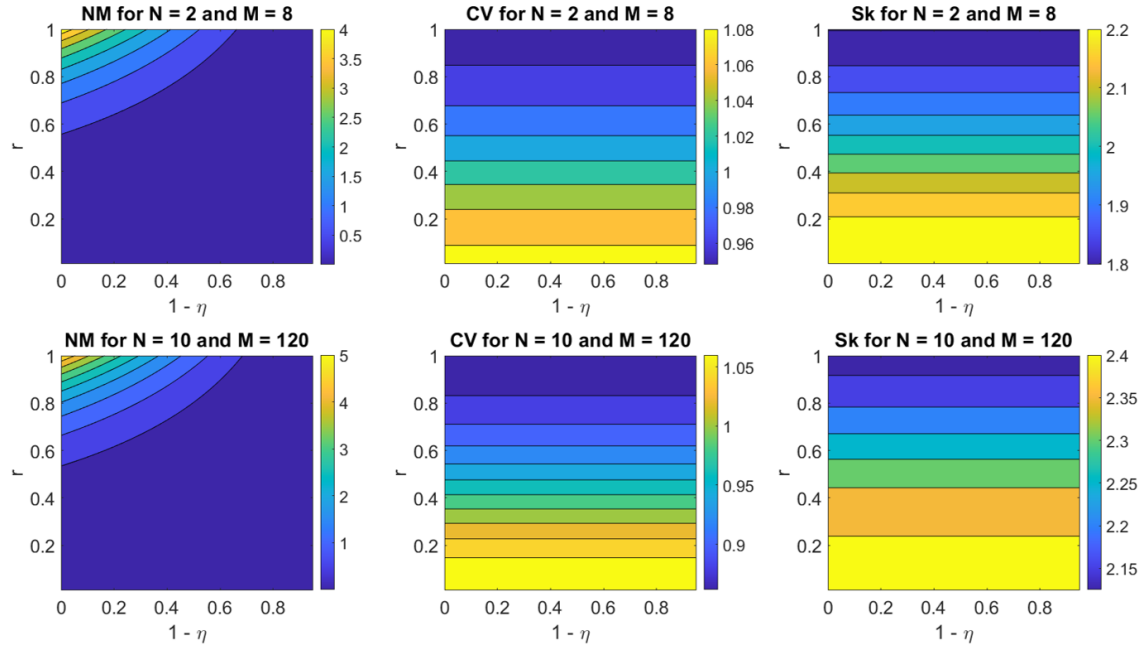


Figure 8.5: Heat maps of NM, CV, and Sk (columns from left to right) for SMSV inputs with varying efficiency η and squeezing parameter r for small ($N = 2$ and $M = 8$, top row) and large ($N = 10$ and $M = 120$, bottom row) optical networks. Note that $1 - \eta$ is plotted on the horizontal axis for an increasing loss fraction.

SMSV input states is depicted in Fig. 8.5, as calculated through Eq. (8.20). We see that CV and Sk are unaffected by loss, which is intuitive as η will cancel out when $\eta^2 C_{jk}$ is inserted into Eqs. (8.4) and (8.5). From this, we might think that CV and Sk are good measures to benchmark between sampling with SMSV and thermal state inputs, but we

will see in Sec. 8.4.3 that even with loss we get the most information with the least effort out of NM.

8.4.2 Additive Noise

When working at telecom wavelengths, the thermal noise due to the environment (i.e. room temperature) is negligible. However in an experimental scenario, noise on the input states could occur due to e.g. non-ideally filtered pump laser of the SPDC process used to create SMSV. We consider the affect of thermal noise on our benchmarking process using Eqs. (3.28) and (3.29), and similarly to loss, for simplicity we assume the same noise in each mode. For thermal noise with an average photon number of ν in each mode, $\mathbf{Q}$ is affected as:

$$\mathbf{Q} \mapsto \mathbf{Q} + \frac{\nu}{2}\mathbf{I}, \quad (8.22)$$

and $\mathbf{d}$ is unchanged. This results in $\langle \hat{n} \rangle \mapsto \langle \hat{n} \rangle + \nu/2$, and $\langle \hat{\epsilon} \rangle$ is unchanged.

In the same fashion as for loss, we consider how noise affects the input state purity for the occupied modes. For SMSV input states, we arrive at:

$$\text{Tr}(\hat{\rho}^2) = [1 + \nu^2 + 2\nu \cosh(2r)]^{-1/2}. \quad (8.23)$$

From this, we see that we only have a pure state when $\nu = 0$, and when $\nu > 0$ any squeezing will exacerbate the purity. In addition, we recall from Sec. 3.2.2 that the sub-vacuum variance of SMSV is a non-classical feature, for which we can use the quadrature variances to investigate. When the squeezing is along the p quadrature, we have $v_p = e^{-2r}/2 \leq 1/2$ for a pure SMSV state (with $r \geq 0$). In the presence of thermal noise, the condition to maintain sub-vacuum variance (i.e. $v_p < 1/2$) is given by:

$$r > -\frac{1}{2}\ln(1 - \nu), \quad (8.24)$$

which defines the boundary separating classical from squeezed states.

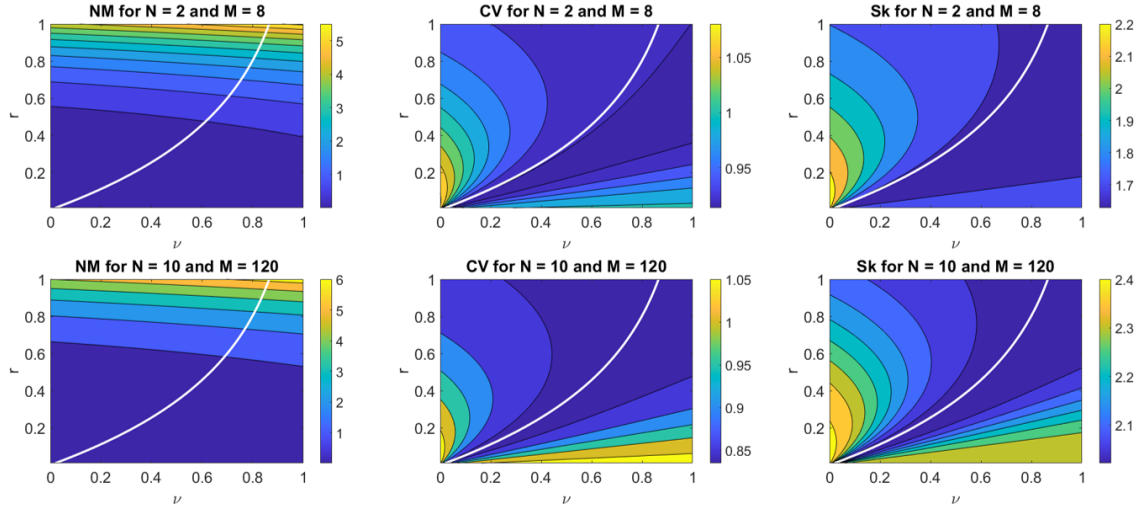


Figure 8.6: Heat maps of NM, CV, and Sk (columns left to right) for noisy squeezed states obtained by varying squeezing parameter r and noise parameter ν , for a small ($N = 2$ and $M = 8$, top row) and large ($N = 10$ and $M = 120$, bottom row) systems. The (white) lines for the sub-vacuum variance boundary in Eq. (8.24) are additionally depicted, to the left of those lines we have a sub-vacuum variance.

Fig. 8.6 shows the dependence of NM, CV and Sk on the noise and squeezing parameters. Compared to the loss heat maps in Fig. 8.5, the functional landscape is more complex. Specifically, in this instance CV (center row) and Sk (right row) are affected by noise. Comparing the lower right-hand corner of the CV and Sk plots for both small (top row) and large (bottom row) systems, it appears that the noise is suppressed by higher moments as the change with ν is more shallow in the Sk plots compared to the CV plots.

8.4.3 Discrimination via Statistical Significance

In Sec. 8.3.2, we alluded to the effect of error when calculating NM, CV and Sk, and in Sec. 8.4.1 we hinted that NM would be the most efficient measure to benchmark between squeezed and thermal states despite the invariance of CV and Sk with loss. In this section, we shall discover why this is the case through considering the effect of finite sample size when computing the moments of C_{jk} . In an experimental context, the ideal situation involves benchmarking using the fewest number of Haar-random unitaries (i.e. getting away with the fewest data points).

To determine this, we approximate NM, CV and Sk by generating a set of C_{12} from several Haar-random unitaries and using the relations in Eqs. (8.3) to (8.5) and (8.13). The variation in values of C_{12} enables us to assign statistical uncertainties to those quantities, using the typical propagation of errors. As a measure for benchmarking, we consider $\Delta\text{NM} = \text{NM}_S - \text{NM}_T$ (and equivalent for CV and Sk), which is the difference between the normalised mean for squeezed and thermal states. The metric we impose is the minimum number of iterations required for ΔNM to be non-zero with a 3σ statistical significance. This statistical bound is enough to tell the considered families of states apart.

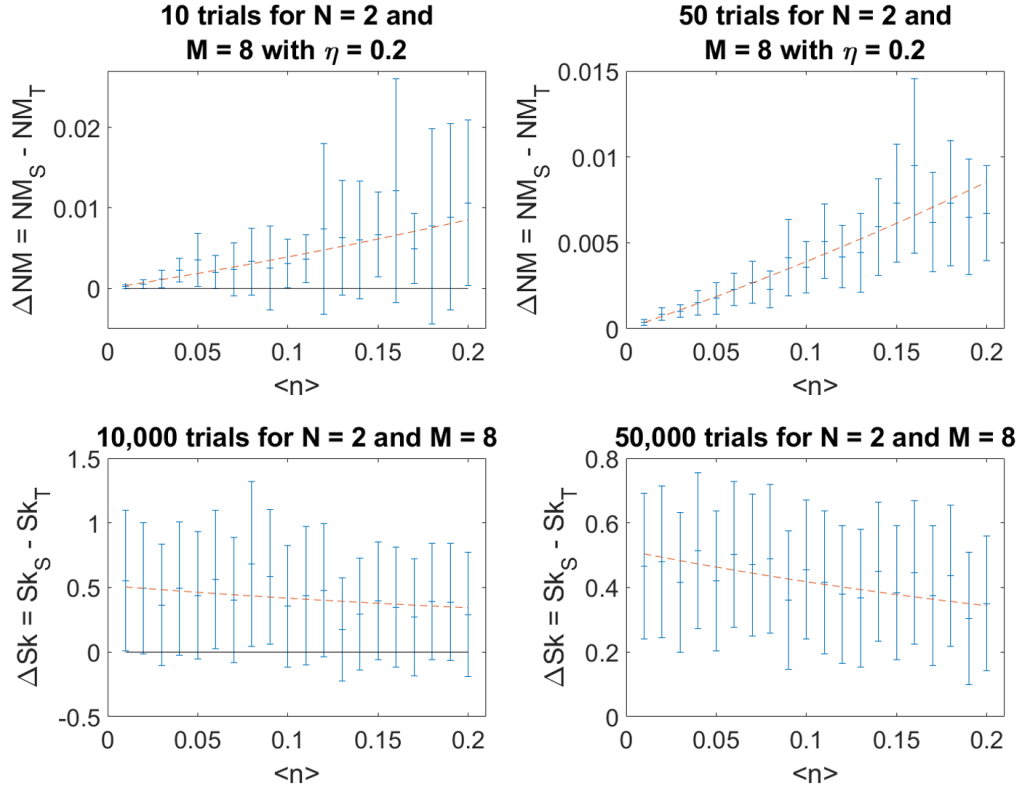


Figure 8.7: The top row shows ΔNM with 3σ error bars plotted against average photon number $\langle \hat{n} \rangle$ for small systems ($N = 2$ in $M = 8$) with $\eta = 0.2$ (i.e. 80% loss). On the left, the system evolved under 10 different Haar-random unitaries (i.e. 10 trials), and 50 trials for the plot on the right. The bottom row depicts ΔSk with 3σ error bars plotted against average photon number $\langle \hat{n} \rangle$ for small systems ($N = 2$ in $M = 8$). The left and right plots have 10,000 and 50,000 trials respectively. The blue error bars are from the simulated data, and orange dotted lines are from the analytical expressions. Black zero lines have been added to the plots in the left column to show that the error bars go through zero.

We see plots of ΔNM and ΔSk for a range of number of trials in Fig. 8.7. From our

previous analysis, we can see that ΔNM scales with $\langle\hat{n}\rangle$, so only low values of $\langle\hat{n}\rangle$ are considered. The top two plots contain parameters typical for current photonic architectures^[286]. We can see that 10 trials are not enough, but 50 trials discriminate between squeezed and thermal states with a 3σ significance. In addition to this, further simulations showed that the degree of system loss does not affect the number of trials required for discrimination. This result is encouraging as 50 trials seems to be a feasible number to undertake in a laboratory. This does however show that for $M = 8$ it is not sufficient to use one fixed Haar-random unitary and interchange between the outputs, as this only gives 28 values of C_{jk} . To overcome this, more values of C_{jk} could be generated by varying the input modes.

In Sec. 8.3.2, we initially proposed that Sk might be a good measure to discriminate between squeezed and thermal states at low values of $\langle\hat{n}\rangle$ through considering Fig. 8.2. Therefore, to test this we consider the same for ΔSk with varying $\langle\hat{n}\rangle$. The results can be seen at the bottom of Fig. 8.7 for small systems ($N = 2$ in $M = 8$), we did not consider losses as they do not effect Sk . We see that 10,000 trials are insufficient for discrimination, but 50,000 allow this with a statistical significance of 3σ . The meaning of this result is that on the order of 10^3 more trials are required when using Sk compared to NM to benchmark between GBS with squeezed and thermal states, regardless of the value of $\langle\hat{n}\rangle$. Therefore, NM is clearly the most experimentally friendly metric even in the presence of system loss.

8.4.4 Detectors with Finite Photon-Number Resolution

In Sec. 8.3.1, we discussed that when calculating C_{jk} from a photon number distribution (Eq. (8.17)) we are required to truncate the sum at some finite photon number $n_{\max}$. This is because Gaussian states do not have a fixed number of photons, and in principle $n_{\max}$ could be infinite. Despite this, the contributions for high photon numbers become arbitrarily small for both squeezed and thermal states, therefore we can consider a highest sensible $n_{\max}$ and truncate our statistics without affecting the result. In this section, we will consider how C_{jk} is affected by truncating the statistics at different values of $n_{\max}$.

This is an important consideration when experimentally measuring C_{jk} , as such a truncation resembles a necessary restriction when operating photon-number detectors. For example, when detecting using the transition edge sensor (TES) presented in Sec. 2.4.3, we are limited to around 15 photons per TES. It is possible to multiplex detectors with a finite maximal photon-number resolution^[101,287], this will increase the maximally measurable photon number but increases experimental overheads.

We consider the specific example of C_{12} to test the effect of n_{max} on the two-point correlator. For a given system evolved under a given Haar-random unitary, we first calculate $P(n_1, n_2)$ up to a level much higher than a TES is capable of (here, for up to 40 photons per mode). This sample enables us to approximate C_{12} using Eq. (8.17) by truncating at successively higher values of n_{max} ($n_1, n_2 \leq n_{max}$). These values are then compared to the analytical value for the same system and Haar-random unitary (using either Eq. (8.7) or Eq. (8.12), for example), and the relative distance $[C_{12}^{(\text{analytical})} - C_{12}^{(\text{estimated})}]/C_{12}^{(\text{analytical})}$ between the exact and estimated result can be calculated.

In the top row of Fig. 8.8, we see two plots of normalised Eq. (8.17) vs n_{max} . Both are for a small system ($N = 2$ and $M = 8$), with $\langle \hat{n} \rangle = 1$ and a randomly-selected Haar-random unitary. 100% efficiency is shown on the left, and 20% on the right. The y-axis of both plots have been normalised against the true value of C_{12} with the respective parameters.

The degree of photon-number resolution required to measure C_{12} is subjective as it depends on the error we are willing to tolerate. To serve as an example, we set the error tolerance to a maximum relative distance of 10^{-3} , i.e.:

$$-\log_{10} \left(\frac{C_{12}^{(\text{analytical})} - C_{12}^{(\text{estimated})}}{C_{12}^{(\text{analytical})}} \right) > 3. \quad (8.25)$$

The bottom row of Fig. 8.8 depicts the number of times Eq. (8.25) is satisfied for a given n_{max} for 500 Haar-random unitaries, for the same systems as above. The spread in values arises due to the variation in scattering of different unitaries to the output ports under

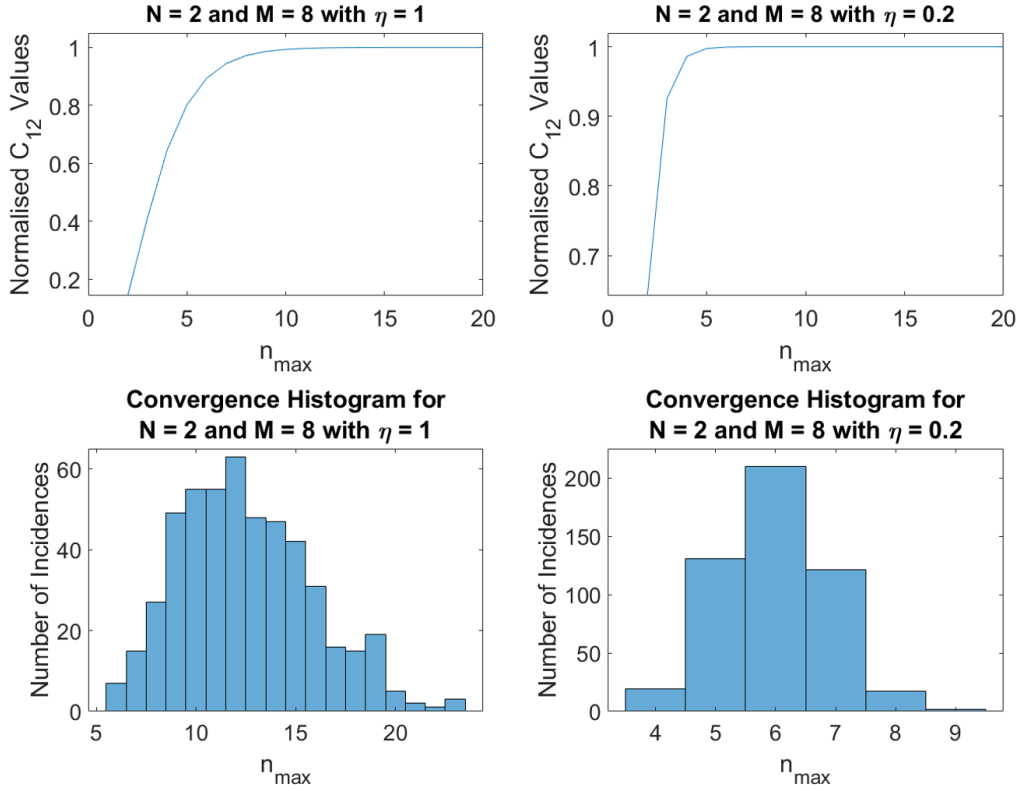


Figure 8.8: Top row: convergence plots of C_{12} for an example Haar-random unitary (normalised against $C_{12}^{(\text{analytical})}$) plotted against maximum photon number resolution $n_{\max}$. Both are for a small system ($N = 2$ in $M = 8$) with SMSV inputs of $\langle \hat{n} \rangle = 1$. Left: system with full transmission $\eta = 1$. Right: system with $\eta = 0.2$, typical of current architectures^[286]. Bottom row: histograms for the number of incidences of $n_{\max}$ that satisfy convergence condition in Eq. (8.25) for 500 different Haar-random unitaries. Left: $\eta = 1$. Right: $\eta = 0.2$.

consideration, for fixed input and output ports and system size. In addition, a lossy system will have a lower $\langle \hat{n} \rangle$ on average compared to a lossless system, so a lower resolution would be required for the same convergence. This is clearly demonstrated in the right column of Fig. 8.8.

Fig. 8.8 presents only a small system for SMSV input states, however other system sizes and states were investigated. We observe that for $N = 10$ occupied modes in an $M = 120$ -mode system, we have a more dilute photon number distribution at the output compared to $N = 2$ and $M = 8$. Therefore, this larger system size also requires a lower resolution compared to the small system.

Even with only 500 different Haar-random unitaries in the bottom row of Fig. 8.8, it

shows there is a mean resolution for the convergence condition in Eq. (8.25). Interestingly, we noted that SMSV inputs require a slightly higher resolution compared to a thermal state with the same $\langle \hat{n} \rangle$. This is because higher photon-number correlations scale differently for these classes of states, even though the mean photon number is the same. This is demonstrated in Eq. (8.12) as $\langle \hat{\epsilon} \rangle$ is zero for thermal states, but non-zero (and related to $\langle \hat{n} \rangle$) for SMSV. Moreover, typical experimental parameters for current architecture would be two SMSV inputs with $\langle \hat{n} \rangle \approx 1$ each with 80% loss per mode, hence why we chose those parameters for the right column of Fig. 8.8. Therefore, TES resolution ($n_{\max} \approx 15$) is sufficient to measure C_{12} to within our chosen error bound with typical experimental parameters. For improved transmission, the squeezing could be even further reduced to lower $\langle \hat{n} \rangle$, which would in turn reduce the requirement on $n_{\max}$.

8.5 Conclusion and Outlook

In this chapter, we formulated a method to benchmark between squeezed and thermal states as input to a Gaussian Boson Sampler. This was motivated by the fact that sampling from thermal state inputs can be efficiently simulated on a classical computer, which is not the case for squeezed states. We followed the same procedure as Ref. [253] for phase-insensitive Fock input states, by studying the photon-number two-point correlator between output modes of a photonic circuit. We derived analytical expressions for these two-point correlators and their average moments, and considered the effects of experimental imperfections on them. In particular, we found that these correlators are affected by the eccentricity (i.e. phase space asymmetry) of the initial states, an effect not present in the standard Boson Sampling setup. Following this we derived expressions for the moments of the two-point correlators by averaging over the whole unitary group, a process known as random matrix theory. It was these moments, and their simulated estimations, that set the path for us to distinguish between squeezed and thermal state inputs.

We first considered the effects of system size on the moments of the two-point correl-

ators for squeezed and thermal states. This gave us some indication of which moments are most suitable for benchmarking in under different conditions. Following this, we compared the results of few occupied modes with highly squeezed input states are compared to many occupied modes with weakly squeezed input states for a constant expectation value of the total particle number. We observed a clear difference in the correlations, notably that our benchmarking procedure favours fewer occupied modes.

We next considered the effects of some common experimental imperfections on the correlators and their moments. We first considered loss, an impurity which can be eliminated through post-selection in standard Boson Sampling (though decreasing sampling efficiency), but not for Gaussian Boson Sampling. We found, under the assumption of uniform loss, that the two-point correlator scales with the square of the efficiency. This results in a reduction of the first moment, but leaves the second and third unchanged, which made us initially suspect that the higher moments would prove more robust for benchmarking purposes. The second imperfection we studied was thermal noise. In contrast to loss, we found that the higher moments are affected by noise, making it an important experimental consideration when benchmarking using this method. In particular, we show that the second and third moments can be used as probes for transitions from non-classical to classical light, which occurs when the classical noise drives the quadrature fluctuations beyond those of vacuum.

The next (and arguably most important) property we investigated was the effect of finite sample size on calculating the moments of the two-point correlator. This is a crucial consideration as it underpins the experimental overheads of the benchmarking procedure, but also gives the best insight into the measure that lets us discriminate between squeezed and thermal state inputs most efficiently. We discovered that the first moment (the normalised mean) allows us to benchmark with fewest number of trials. For parameters typical of current architecture, we found that approximately 50 iterations (i.e. 50 different unitary evolutions) is sufficient to discriminate between sampling with squeezed and thermal

states, compared to the $\approx 50,000$ iterations required when using the third moment (the skewness).

The final experimental limitation we considered was finite detector photon-number resolution. When using state-of-the-art number-resolving transition edge sensors, we found that their resolution of ≈ 15 photons is sufficient to calculate the two-point correlator to within a relative error of 10^{-3} for current typical systems.

Although our parameter space for exploration already is very large, with many variables and imperfections that we can quantify, there are still many further avenues to investigate when considering experimental impurities. One is the problem of partial or full distinguishability as analogy to Ref. [269], which gives rise to additional degrees of freedom. The analysis in this chapter is only applicable when the input states are in the same mode, i.e. they have the same polarisation, spectral distribution, and are in the same spatial and temporal mode. There exists framework to consider this (see Ref. [227] for example), though it is very cumbersome to implement in the Gaussian state formalism. We came across a basic analysis of this in Sec. 6.3.2 of Chap. 6.

An additional experimental challenge is phase stability, which was first alluded to in Sec. 3.3.2. The phase drift of radially-asymmetric Gaussian states induces non-Gaussianity, meaning that the derivations in this chapter would no longer be valid. We have already come across phase-diffusion (of specifically coherent states) in this thesis: in Chap. 6 where phase-control was not necessary for the measurement, and Chap. 7 where active tracking of the phase was required. In contrast to distinguishability however, it would be comparatively straightforward to derive the two-point correlators using the characteristic functions of these phase-averaged states (see Eqs. (3.7) and (3.8)). It would be particularly interesting to benchmark between squeezed and phase-averaged squeezed states (see Refs. [126,127]), as phase stability affects all phase-sensitive optical measurements. In this instance it might even be necessary to utilise different types of correlators [288], depending on how alike the input states are, but for now I leave the effect of phase diffusion of

squeezed states when benchmarking as an open question.

Conclusion

All experiments in quantum and classical optics require some form of detection. Whether it be a photon-number measurement, a quadrature measurement, or something more advanced in between, without measurement an experiment is rendered pointless^[21]. The same applies to practical applications of quantum optics. Regardless of the protocol or the quantity being simulated, a measurement of some form or another on the system is essential to obtain the necessary results^[1,5,6].

In this thesis, we have explored some advanced types of measurement in quantum optics, with a particular focus on the outcomes with Gaussian states of light. First came superconducting nanowire single-photon detectors (SNSPDs), where we considered a more advanced implementation and then an application exploiting their intrinsic properties. We then explored the regime of weak field homodyne detection (WFHD), first by investigating tunability, and then by considering an experimental implementation of a novel tomographic method. Finally, we delved into the regime of Gaussian Boson Sampling (GBS). We discussed an experimentally friendly benchmarking protocol, with some focus on the number of measurements and the photon-number resolution required to benchmark against classical analogues. In this final section, I shall first summarise the key points of each chapter, and then finish with an outlook into future directions.

9.1 Summary

There are five chapters of new material in this thesis which can be grouped into three themes: SNSPDs, WFHD, and GBS. In this section, I shall be summarising the advancements made in this thesis relating to each of three fields.

9.1.1 Superconducting-Nanowire Single-Photon Detectors

Chapters 4 and 5 concerned some sophisticated implementations of SNSPDs. In chapter 4, we saw the optical characterisation results of Tungsted Silicide (WSi) SNSPDs on silica glass waveguides, and chapter 5 explored the possibility of using a single SNSPD to perform broadband spectroscopy with a weak signal.

When characterising the on-chip SNSPDs, we considered five main quantities: voltage detection threshold, calibration curves (including dark counts), dead time, jitter, and efficiency. Initially the setup looked promising, as silica glass waveguides are well mode matched to optical fibres^[176], and WSi SNSPDs are robust and have a high quantum efficiency^[177]. However, due to the poor thermal conductivity of silica, unfortunately the chip gave an indication of being at a higher temperature compared to the cryostat whilst having an uneven temperature profile. This resulted in longer dead times and higher dark count rates compared to standard operation. In addition, measurements indicated that the fibre-coupling shifted during cooldown resulting in very poor optical coupling. Although this did not affect the majority of the optical calibrations, it dramatically reduced the total system efficiency. Despite this, once light was in the waveguide, we calculated an average SNSPD efficiency of approximately $0.245 \pm 0.006\%$. This was expected as the shape of the SNSPDs were not optimised for optical absorption. Despite all of the system's shortcomings, we found the SNSPDs behaved as SNSPDs should, and the relevant characteristics were of the same order of magnitude as typical fibre-coupled WSi SNSPDs. This showed promise of system improvement, which could be achieved with better thermal contact and an optimised SNSPD design.

After investigating some important properties of SNSPDs, we then explored using the intrinsic SNSPD wavelength dependence to perform spectroscopy on weak optical systems. The procedure involved first measuring normalised calibration curves (bias current vs. count rate normalised by incident optical power) at a range of wavelengths, and then attempting to infer the spectrum of a broadband optical signal from its calibration curve through using a least-squares minimisation with the normalised basis calibration curves. While this procedure was mostly successful in identifying the correct spectral components of the inversion signals, it fell short when determining spectral powers. This we believe is due to the extreme difficulty of measuring the optical powers required to normalise the calibration curves. These were of the order of 10 pW, and the requirement to use optical attenuation for these measurements would have introduced some systematic errors. On the other hand, this spectroscopy method surpassed the ability of the previous work using a similar procedure^[68] as it was able to identify spectral components which did not correspond to the normalised calibration curve wavelengths. Therefore, with the inclusion of more advanced optical power measurements, I believe that spectroscopy using SNSPDs has the potential for advancement.

9.1.2 Weak Field Homodyne Detection

Chapters 6 and 7 were concerned with the composite detection scheme of WFHD, a measurement similar to balanced homodyne detection (BHD) to measure quadrature distributions except with a weak phase reference local oscillator (LO). In this WFHD regime, we first examined the effect of tuning the LO strength in chapter 6, and then considered the experimental implementation of a tomographical application in chapter 7.

BHD requires the assumption of a strong LO in order to approximate the photon-number difference measurement as a quadrature projection. In chapter 6, we examined the situation where the LO was of comparable strengths to the signal being probed, and its effect on the photon-number difference measurement as measured with number-resolving transition edge sensors (TESs). We considered a photon-number state heralded from a

two-mode squeezed vacuum (TMSV) as our signal, and examined the setup using the Gaussian state formalism (GSF) to model the circuit due to the ease of dealing with imperfections such as loss. We then varied the LO strength and calculated the photon-number difference distribution from TES measurements, and compared the results with the classical LO formalism. For our particular signal, we showed that the LO average photon number required for good agreement with the quadrature distribution grew linearly with signal average photon number. We also showed that the LO strength at the transition is larger than that of the signal, but still of the same order of magnitude. In addition to this, the system in question also demonstrated the versatility of the GSF. We were not required to set or track the LO phase as heralded photon-number states are symmetric in phase space, and also heralded photon-number states are non-Gaussian. Through using TMSV and a LO at an arbitrary phase we were able to use the GSF with photon-number projection to model the system, which was possible as we lost the phase information when we post-selected on photon number in the herald mode of the TMSV.

Then in chapter 7, we discussed the experimental implementation of an application of WFHD. This novel application was previously only a theoretical idea, and is a technique to perform state tomography using WFHD with only partial overlap between the signal and LO. This chapter mainly concerned the experimental implementation of this technique and the challenges it brought, but some preliminary tomography results were discussed. To perform a proof-of-concept demonstration we used a coherent state signal derived from the same laser as the LO, and varied the overlap in the polarisation degree of freedom. We interfered the signal and LO on a 50:50 fibre beamsplitter and measured both outputs with polarisation-insensitive TESs. Instead of calculating the photon-number difference distribution like BHD and WFHD, this technique is subtly different in that it requires the joint photon-number distribution of the output spatial modes. A challenge of this implementation was the phase control of the LO. Despite attempting to control this with a piezo the laboratory conditions were too unstable, therefore we instead tracked the rel-

ative phase between the signal and LO. From the photon-number statistics we were able to infer the relative phase, and retrospectively assign it to the LO. Although topographical reconstruction is, at the time of writing, an ongoing effort, initial results suggest a similarity with the measured state with a fidelity of 98.6%, which is a promising result.

9.1.3 Gaussian Boson Sampling

Finally, chapter 8 was on the theme of Gaussian Boson Sampling (GBS), a promising avenue for using optical quantum systems to simulate certain computationally difficult problems^[5], with many practical applications^[6].

We looked specifically at benchmarking the output of a Gaussian Boson Sampler against similar situations which are computationally easy to simulate, i.e. sampling with thermal states. We used the two-point correlator between the photon-number distributions of two output ports as our benchmarking measure, a quantity reliant on photon-number measurements. Using the GSF, we derived analytical expressions for the two-point correlator in terms of the systems considered, giving insight into how the values differ when comparing the quantum and classical cases. Then using random matrix theory, the first three moments of the two-point correlator were taken to study the affect of averaging over a large number of unitary evolutions. It was these moments that gave us the benchmark comparison.

After examining the ideal cases, we incorporated experimental impurities such as loss, noise and finite detector resolution. We found that, even in the presence of loss, that the first moment of the two-point correlator allows us to benchmark against the classical case with the fewest number of iterations. In addition, the requirement on detector resolution to accurately measure two-point correlators was considered. With typical current photonic sources and circuitry architecture^[286], we found that the number resolution provided by TES detectors is sufficient to measure the two-point correlator to within a relative distance of 10^{-3} .

Although the benchmarking scheme we presented does not certify GBS against all

possible alternative hypotheses, it offers an insightful check against common experimental mishaps. Also, it can be performed with no additional experimental overheads, as all is required is the photon-number distribution data from a GBS experiment.

9.2 Outlook

Of the above topics, for the majority of them there is a clear aspect: the implementation or the importance of photon-number resolution. The on-chip SNSPD setup provides a scalable way to place many detectors on one waveguide, allowing for pseudo-number resolution with fewer temporal and spatial overheads. Demonstrating the tunability of WHFD was only possible with the use of TES detectors for number resolution, which was also crucial for the tomographic application. Additionally, the benchmarking procedure for GBS requires a degree of number resolution for the calculation of the two-point correlators as there is a trade-off between degree of number resolution and accuracy.

Through all of this, we see that the TES detector plays an important role due to its intrinsic photon-number resolving abilities. Despite its shortcomings of slow operating rates, milliKelvin temperature requirements, and complex electronic readout, the ability to resolve up to 15 – 20 photons with a single detector is invaluable. Therefore in light of this, an important area for future research into detection in quantum optics is the continual improvement of TES detectors. By making them faster and more user-friendly, they would become more accessible for the many applications requiring photon-number resolution.

A field that greatly benefits from photon-number resolving detectors is experimental Gaussian Boson Sampling^[152,153]. Due to its numerous applications, it is currently an area of intense research^[289,290]. Simulation schemes involving GBS have applications from quantum chemistry to graph theory calculations, making it a promising platform for near-term quantum devices^[6], as well as the longer term goal of a full quantum computer^[2,3]. In addition to improved detectors, this field (along with other optical quantum

computing schemes^[1]) also benefits from improvements in photonic sources (as in chap. 6) and photonic circuits (as in chap. 4). It is only with the development of these three components together that the whole field can be properly advanced.

We live in an exciting age of growing quantum awareness. When exactly quantum optical devices will become mainstream is still an open question, but the current interest and growing public awareness enables the continual progress towards this goal.

Bibliography

- [1] E. Knill, R. Laflamme, and G. J. Milburn. A scheme for efficient quantum computation with linear optics. *Nature*, 409(6816):46–52, 2001.
- [2] S. L. Braunstein and P. van Loock. Quantum information with continuous variables. *Rev. Mod. Phys.*, 77:513–577, 2005.
- [3] C. Weedbrook, S. Pirandola, R. García-Patrón, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd. Gaussian quantum information. *Rev. Mod. Phys.*, 84:621–669, 2012.
- [4] S. Barz, E. Kashefi, A. Broadbent, J. F. Fitzsimons, A. Zeilinger, and P. Walther. Demonstration of blind quantum computing. *Science*, 335(6066):303–308, 2012.
- [5] A. Aspuru-Guzik and P. Walther. Photonic quantum simulators. *Nat. Phys.*, 8:285 EP –, 2012. Review Article.
- [6] T. R. Bromley, J. M. Arrazola, S. Jahangiri, J. Izaac, N. Quesada, A. D. Gran, M. Schuld, J. Swinerton, Z. Zabaneh, and N. Killoran. Applications of near-term photonic quantum computers: software and algorithms. *Quantum Sci. Technol.*, 2020.
- [7] M. Szczykulska, T. Baumgratz, and A. Datta. Multi-parameter quantum metrology. *Adv. Phys. X*, 1(4):621–639, 2016.
- [8] S. Slussarenko, M. M. Weston, H. M. Chrzanowski, L. K. Shalm, V. B. Verma, S. W. Nam, and G. J. Pryde. Unconditional violation of the shot-noise limit in photonic quantum metrology. *Nat. Photonics*, 11(11):700–703, 2017.
- [9] The LIGO Scientific Collaboration. Enhanced sensitivity of the ligo gravitational wave detector by using squeezed states of light. *Nat. Photonics*, 7(8):613–619, 2013.
- [10] C. Gobby, Z. L. Yuan, and A. J. Shields. Quantum key distribution over 122 km of standard telecom fiber. *Appl. Phys. Lett.*, 84(19):3762–3764, May 2004.
- [11] R. H. Hadfield. Single-photon detectors for optical quantum information applications. *Nat. Photonics*, 3(12):696–705, 2009.
- [12] A. Migdall, S. V. Polyakov, J. Fan, and J. C. Bienfang. *Single-Photon Generation and Detection: Physics and Applications*. Elsevier Science, Burlington, 2013.
- [13] H. Hertz. Ueber einen einfluss des ultraviolettten lichtes auf die electriche entladung. *Ann. Phys. Chem.*, 267(8):983–1000, 1887.

- [14] A. Goetzberger, B. McDonald, R. H. Haitz, and R. M. Scarlett. Avalanche effects in silicon p—n junctions. II. structurally perfect junctions. *J. Appl. Phys.*, 34(6):1591–1600, 1963.
- [15] G. N. Gol’tsman, O. Okunev, G. Chulkova, A. Lipatov, A. Semenov, K. Smirnov, B. Voronov, A. Dzardanov, C. Williams, and R. Sobolewski. Picosecond superconducting single-photon optical detector. *Appl. Phys. Lett.*, 79(6):705–707, 2001.
- [16] J. Huh, G. G. Guerreschi, B. Peropadre, J. R. McClean, and A. Aspuru-Guzik. Boson sampling for molecular vibronic spectra. *Nat. Photon.*, 9:615 EP –, 2015. Article.
- [17] K. D. Irwin. *Phonon-mediated particle detection using superconducting tungsten transition-edge sensors*. PhD thesis, Stanford University, 1995.
- [18] K. D. Irwin. An application of electrothermal feedback for high resolution cryogenic particle detection. *Appl. Phys. Lett.*, 66(15):1998–2000, 1995.
- [19] A. E. Lita, A. J. Miller, and S. W. Nam. Counting near-infrared single-photons with 95% efficiency. *Opt. Express*, 16(5):3032, 2008.
- [20] D. V. Reddy, R. R. Nerem, S. W. Nam, R. P. Mirin, and V. B. Verma. Superconducting nanowire single-photon detectors with 98% system detection efficiency at 1550 nm. *Optica*, 7(12):1649, 2020.
- [21] Leonhardt. *Measuring Quantum State of Light (Cambridge Studies in Modern Optics)*. Cambridge University Press, 2008.
- [22] E. Hecht. *Optics, 5th ed.* Pearson, 2017.
- [23] D. J. Griffiths. *Introduction to Electrodynamics, 4th ed.* Pearson, 2014.
- [24] S. J. Blundell and K. M. Blundell. *Concepts in Thermal Physics*. Oxford University Press, 2009.
- [25] M. Planck. *The Theory of Heat Radiation*. Dover Publications, Inc., 1914.
- [26] O. Svelto. *Principles of Lasers, 5th ed.* Springer, 2010.
- [27] I. Newton. *Opticks: or, A treatise of the reflections, refractions, inflexions and colours of light : also two treatises of the species and magnitude of curvilinear figures*. Printed for Sam. Smith, and Benj. Walford, 1704.
- [28] A. Einstein. Über einen die erzeugung und verwandlung des liches betreffenden heuristischen gesichtspunkt. *Ann. Phys.*, 322(6):132–148, 1905.
- [29] R. Loudon. *The Quantum Theory of Light, 3rd ed.* Oxford University Press, 2000.
- [30] D.F. Walls and G. J. Milburn. *Quantum Optics*. Springer, 2008.
- [31] G. Grynberg, A. Aspect, and C. Fabre. *Introduction to Quantum Optics: From the Semi-classical Approach to Quantized Light*. Cambridge University Press, 2010.
- [32] C. C. Gerry and P. L. Knight. *Introductory Quantum Optics*. Cambridge University Press, 2004.
- [33] L. Mandel. Statistical properties of light and the sudarshan representation. *Phys. Lett.*, 7(2):117–119, 1963.

- [34] C. L. Mehta and E. Wolf. Coherence properties of blackbody radiation. ii. correlation tensors of the quantized field. *Phys. Rev.*, 134:A1149–A1153, 1964.
- [35] G. S. Agarwal. *Quantum Optics*. Cambridge University Press, 2009.
- [36] M. W. Hamilton. Phase shifts in multilayer dielectric beam splitters. *Am. J. Phys.*, 68(2):186–191, 2000.
- [37] X. Fang, R. O. Claus, and G. Indebetouw. Interferometric model for phase analysis in fiber couplers. *Appl. Opt.*, 35(22):4510, 1996.
- [38] C. T. Lee. External photodetection of cavity radiation. *Phys. Rev. A*, 48(3):2285–2291, 1993.
- [39] T. Kiss, U. Herzog, and U. Leonhardt. Compensation of losses in photodetection and in quantum-state measurements. *Phys. Rev. A*, 52(3):2433–2435, 1995.
- [40] C. K. Hong, Z. Y. Ou, and L. Mandel. Measurement of subpicosecond time intervals between two photons by interference. *Phys. Rev. Lett.*, 59(18):2044–2046, 1987.
- [41] J. Carolan *et al.* Universal linear optics. *Science*, 349(6249):711–716, 2015.
- [42] M. Reck, A. Zeilinger, H. J. Bernstein, and P. Bertani. Experimental realization of any discrete unitary operator. *Phys. Rev. Lett.*, 73(1):58–61, 1994.
- [43] W. R. Clements, P. C. Humphreys, B. J. Metcalf, W. S. Kolthammer, and I. A. Walsmley. Optimal design for universal multiport interferometers. *Optica*, 3(12):1460, 2016.
- [44] S. L. Braunstein. Squeezing as an irreducible resource. *Phys. Rev. A*, 71(5), 2005.
- [45] S. Aaronson and A. Arkhipov. The computational complexity of linear optics. *Theor. Comput.*, 9(4):143–252, 2013.
- [46] S. Scheel. Permanents in linear optical networks. *arXiv:0406127 [quant-ph]*, 2004.
- [47] B. T. Gard, K. R. Motes, J. P. Olson, P. P. Rohde, and J. P. Dowling. An introduction to boson-sampling. In *From Atomic to Mesoscale*, pages 167–192. WORLD SCIENTIFIC, 2015.
- [48] M. Marcus and H. Minc. Permanents. *Am. Math. Mon.*, 72(6):577, 1965.
- [49] L. G. Valiant. The complexity of computing the permanent. *Theor. Comput. Sci.*, 8(2):189 – 201, 1979.
- [50] M. A. Broome, A. Fedrizzi, S. Rahimi-Keshari, J. Dove, S. Aaronson, T. C. Ralph, and A. G. White. Photonic boson sampling in a tunable circuit. *Science*, 339(6121):794–798, 2013.
- [51] J. B. Spring *et al.* Boson sampling on a photonic chip. *Science*, 339(6121):798–801, 2013.
- [52] M. Tillmann, B. Dakić, R. Heilmann, S. Nolte, A. Szameit, and P. Walther. Experimental boson sampling. *Nat. Photon.*, 7:540 EP –, May 2013.
- [53] H. Wang *et al.* High-efficiency multiphoton boson sampling. *Nat. Photon.*, 11:361 EP –, 2017. Article.
- [54] H. Wang, J. Qin, X. Ding, M.-C. Chen, S. Chen, X. You, Y.-M. He, X. Jiang, Z. Wang, L. You, J. J. Renema, S. Höfling, C.-Y. Lu, and J.-W. Pan. Boson sampling with 20 input photons and a 60-mode interferometer in a 10^{14} -dimensional hilbert space. *Phys. Rev. Lett.*, 123:250503, 2019.

- [55] J. J. Renema, A. Menssen, W. R. Clements, G. Triginer, W. S. Kolthammer, and I. A. Walmsley. Efficient classical algorithm for boson sampling with partially distinguishable photons. *Phys. Rev. Lett.*, 120(22), 2018.
- [56] R. García-Patrón, J. J. Renema, and V. Shchesnovich. Simulating boson sampling in lossy architectures. *Quantum*, 3:169, 2019.
- [57] V. S. Shchesnovich. Noise in boson sampling and the threshold of efficient classical simulatability. *Phys. Rev. A*, 100(1), 2019.
- [58] Alexandra E Moylett, Raúl García-Patrón, Jelmer J Renema, and Peter S Turner. Classically simulating near-term partially-distinguishable and lossy boson sampling. *Quantum Sci. Technol.*, 5(1):015001, 2019.
- [59] A. Crespi, R. Osellame, R. Ramponi, D. J. Brod, E. F. Galv ao, N. Spagnolo, C. Vitelli, E. Maiorino, P. Mataloni, and F. Sciarrino. Integrated multimode interferometers with arbitrary designs for photonic boson sampling. *Nat. Photon.*, 7:545 EP –, 2013.
- [60] M. D. Eisaman, J. Fan, A. Migdall, and S. V. Polyakov. Invited review article: Single-photon sources and detectors. *Rev. Sci. Instrum.*, 82(7):071101, 2011.
- [61] S. M. Sze and K. K. Ng. *Physics of Semiconductor Devices*. John Wiley & Sons, Inc., 2006.
- [62] M. A. Itzler, X. Jiang, M. Entwistle, K. Slomkowski, A. Tosi, F. Acerbi, F. Zappa, and S. Cova. Advances in InGaAsP-based avalanche diode single photon detectors. *J. Mod. Opt.*, 58(3-4):174–200, 2011.
- [63] Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information: 10th Anniversary Edition*. Cambridge University Press, 2010.
- [64] J. S. Lundeen, A. Feito, H. Coldenstrodt-Ronge, K. L. Pregnell, C. Silberhorn, T. C. Ralph, J. Eisert, M. B. Plenio, , and I. A. Walmsley. Tomography of quantum detectors. *Nat. Phys.*, 5(1):27–30, 2008.
- [65] A. Feito, J. S. Lundeen, H. Coldenstrodt-Ronge, J. Eisert, M. B. Plenio, and I. A. Walmsley. Measuring measurement: theory and practice. *New J. Phys.*, 11(9):093038, 2009.
- [66] H. Paul, P. Törmä, T. Kiss, and I. Jex. Photon chopping: New way to measure the quantum state of light. *Phys. Rev. Lett.*, 76:2464–2467, 1996.
- [67] D. Achilles, C. Silberhorn, C. Sliwa, K. Banaszek, I. A. Walmsley, M. J. Fitch, B. C. Jacobs, T. B. Pittman, , and J. D. Franson. Photon-number-resolving detection using time-multiplexing. *J. Mod. Opt.*, 51(9-10):1499–1515, 2004.
- [68] E. Reiger *et al.* Spectroscopy with nanostructured superconducting single photon detectors. *IEEE J. Sel. Top. Quantum Electron.*, 13(4):934–943, 2007.
- [69] A. Engel, J. J. Renema, K. Il'in, and A. Semenov. Detection mechanism of superconducting nanowire single-photon detectors. *Supercond. Sci. Technol.*, 28(11):114003, 2015.
- [70] C. M. Natarajan, M. G. Tanner, and R. H. Hadfield. Superconducting nanowire single-photon detectors: physics and applications. *Supercond. Sci. Technol.*, 25(6):063001, 2012.
- [71] F. Marsili, V. B. Verma, J. A. Stern, S. Harrington, A. E. Lita, T. Gerrits, I. Vayshenker, B. Baek, M. D. Shaw, R. P. Mirin, and S. W. Nam. Detecting single infrared photons with 93% system efficiency. *Nat. Photon.*, 7(3):210–214, 2013.

- [72] K. M. Rosfjord, J. K. W. Yang, E. A. Dauler, A. J. Kerman, V. Anant, B. M. Voronov, G. N. Gol'tsman, and K. K. Berggren. Nanowire single-photon detector with an integrated optical cavity and anti-reflection coating. *Opt. Express*, 14(2):527, 2006.
- [73] K. Smirnov, Y. Vachtomin, A. Divochiy, A. Antipov, and G. Goltsman. Dependence of dark count rates in superconducting single photon detectors on the filtering effect of standard single mode optical fibers. *Appl. Phys. Express*, 8(2):022501, 2015.
- [74] J. J. Renema, R. Gaudio, Q. Wang, Z. Zhou, A. Gaggero, F. Mattioli, R. Leoni, D. Sahin, M. J. A. de Dood, A. Fiore, and M. P. van Exter. Experimental test of theories of the detection mechanism in a nanowire superconducting single photon detector. *Phys. Rev. Lett.*, 112(11), 2014.
- [75] A. Engel and A. Schilling. Numerical analysis of detection-mechanism models of superconducting nanowire single-photon detector. *J. Appl. Phys.*, 114(21):214501, 2013.
- [76] D. Yu. Vodolazov. Current dependence of the red boundary of superconducting single-photon detectors in the modified hot-spot model. *Phys. Rev. B*, 90:054515, 2014.
- [77] D. Yu. Vodolazov, Yu. P. Korneeva, A. V. Semenov, A. A. Korneev, and G. N. Goltsman. Vortex-assisted mechanism of photon counting in a superconducting nanowire single-photon detector revealed by external magnetic field. *Phys. Rev. B*, 92:104503, 2015.
- [78] F. Marsili, M. J. Stevens, A. Kozorezov, V. B. Verma, C. Lambert, J. A. Stern, R. D. Horansky, S. Dyer, S. Duff, D. P. Pappas, A. E. Lita, M. D. Shaw, R. P. Mirin, and S. W. Nam. Hotspot relaxation dynamics in a current-carrying superconductor. *Phys. Rev. B*, 93:094518, 2016.
- [79] D. Yu. Vodolazov. Single-photon detection by a dirty current-carrying superconducting strip based on the kinetic-equation approach. *Phys. Rev. Applied*, 7:034014, 2017.
- [80] V. B. Verma, B. Korzh, F. Bussi eres, R. D. Horansky, S. D. Dyer, A. E. Lita, I. Vayshenker, F. Marsili, M. D. Shaw, H. Zbinden, R. P. Mirin, and S. W. Nam. High-efficiency superconducting nanowire single-photon detectors fabricated from MoSi thin-films. *Opt. Express*, 23(26):33792, 2015.
- [81] A. J. Kerman, E. A. Dauler, W. E. Keicher, J. K. W. Yang, K. K. Berggren, G. Gol'tsman, and B. Voronov. Kinetic-inductance-limited reset time of superconducting nanowire photon counters. *Appl. Phys. Lett.*, 88(11):111116, 2006.
- [82] A. J. Kerman, D. Rosenberg, R. J. Molnar, and E. A. Dauler. Readout of superconducting nanowire single-photon detectors at high count rates. *J. Appl. Phys.*, 113(14):144511, 2013.
- [83] A. Divochiy, F. Marsili, D. Bitauld, A. Gaggero, R. Leoni, F. Mattioli, A. Korneev, V. Seleznev, N. Kaurova, O. Minaeva, G. Gol'tsman, K. G. Lagoudakis, M. Benkhaoul, F. L vy, and A. Fiore. Superconducting nanowire photon-number-resolving detector at telecommunication wavelengths. *Nat. Photonics*, 2(5):302–306, 2008.
- [84] C. Cahall, K. L. Nicolich, N. T. Islam, G. P. Lafyatis, A. J. Miller, D. J. Gauthier, and J. Kim. Multi-photon detection using a conventional superconducting nanowire single-photon detector. *Optica*, 4(12):1534, 2017.
- [85] K. L. Nicolich, C. Cahall, N. T. Islam, G. P. Lafyatis, J. Kim, A. J. Miller, and D. J. Gauthier. Universal model for the turn-on dynamics of superconducting nanowire single-photon detectors. *Phys. Rev. Appl.*, 12(3), 2019.

- [86] B. Cabrera, R. M. Clarke, P. Colling, A. J. Miller, S. W. Nam, and R. W. Romani. Detection of single infrared, optical, and ultraviolet photons using superconducting transition edge sensors. *Appl. Phys. Lett.*, 73(6):735–737, 1998.
- [87] A. J. Miller, S. W. Nam, J. M. Martinis, and A. V. Sergienko. Demonstration of a low-noise near-infrared photon counter with multiphoton discrimination. *Applied Physics Letters*, 83(4):791–793, 2003.
- [88] R. W. Romani, A. J. Miller, B. Cabrera, S. W. Nam, and J. M. Martinis. Phase-resolved crab studies with a cryogenic transition-edge sensor spectrophotometer. *Astrophys. J.*, 563(1):221–228, 2001.
- [89] D. Fukuda, G. Fujii, A. Yoshizawa, H. Tsuchida, R. M. T. Damayanthi, H. Takahashi, S. Inoue, and M. Ohkubo. High speed photon number resolving detector with titanium transition edge sensor. *J. Low Temp. Phys.*, 151(1-2):100–105, 2008.
- [90] D. Fukuda, G. Fujii, T. Numata, K. Amemiya, A. Yoshizawa, H. Tsuchida, H. Fujino, H. Ishii, T. Itatani, S. Inoue, and T. Zama. Titanium-based transition-edge photon number resolving detector with 98% detection efficiency with index-matched small-gap fiber coupling. *Opt. Express*, 19(2):870, 2011.
- [91] D. Rosenberg, A. E. Lita, A. J. Miller, S. W. Nam, and R. E. Schwall. Performance of photon-number resolving transition-edge sensors with integrated 1550 nm resonant cavities. *IEEE Trans. Appl. Supercond.*, 15(2):575–578, 2005.
- [92] A. J. Miller, A. E. Lita, B. Calkins, I. Vayshenker, S. M. Gruber, and S. W. Nam. Compact cryogenic self-aligning fiber-to-detector coupling with losses below one percent. *Opt. Express*, 19(10):9102, 2011.
- [93] K. D. Irwin and G. C. Hilton. *Cryogenic Particle Detection*, chapter Transition-Edge Sensors, pages 0–67. Springer, 2005.
- [94] Silica on-silicon waveguide circuits and superconducting detectors for integrated quantum information processing. B. J. Metcalf. PhD thesis, University of Oxford, 2014.
- [95] J. Clarke and A. I. Braginski, editors. *The SQUID Handbook*. Wiley, 2004.
- [96] D. Drung, C. Assmann, J. Beyer, A. Kirste, M. Peters, F. Ruede, and T. Schurig. Highly sensitive and easy-to-use SQUID sensors. *IEEE Trans. Appl. Supercond.*, 17(2):699–704, 2007.
- [97] R. W. Simon, M. J. Burns, M. S. Colclough, G. Zaharchuk, and R. Cantor. *Mr. SQUID User's Guide*. STAR Cryoelectronics, Santa Fe, NM, version 6.6 edition, 2012.
- [98] R. P. Welty and J. M. Martinis. A series array of DC SQUIDs. *IEEE Trans. Magn.*, 27(2):2924–2926, 1991.
- [99] E. Figueroa-Feliciano, B. Cabrera, A. J. Miller, S. F. Powell, T. Saab, and A. B. C. Walker. Optimal filter analysis of energy-dependent pulse shapes and its application to TES detectors. *Nucl. Instrum. Methods Phys. Res. A*, 444(1-2):453–456, 2000.
- [100] P. C. Humphreys, B. J. Metcalf, T. Gerrits, T. Hiemstra, A. E. Lita, J. Nunn, S. W. Nam, A. Datta, W. S. Kolthammer, and I. A. Walmsley. Tomography of photon-number resolving continuous-output detectors. *New J. Phys.*, 17(10):103044, 2015.

- [101] J. Sperling, W. R. Clements, A. Eckstein, M. Moore, J. J. Renema, W. S. Kolthammer, S. W. Nam, A. E. Lita, T. Gerrits, W. Vogel, G. S. Agarwal, and I. A. Walmsley. Detector-independent verification of quantum light. *Phys. Rev. Lett.*, 118(16), 2017.
- [102] J. Sperling, A. Eckstein, W. R. Clements, M. Moore, J. J. Renema, W. S. Kolthammer, S. W. Nam, A. E. Lita, T. Gerrits, I. A. Walmsley, G. S. Agarwal, and W. Vogel. Identification of nonclassical properties of light with multiplexing layouts. *Phys. Rev. A*, 96(1), 2017.
- [103] G. S. Thekkadath, D. S. Phillips, J. F. F. Bulmer, W. R. Clements, A. Eckstein, B. A. Bell, J. Lugani, T. A. W. Wolterink, A. E. Lita, S. W. Nam, T. Gerrits, C. G. Wade, and I. A. Walmsley. Tuning between photon-number and quadrature measurements with weak-field homodyne detection. *Phys. Rev. A*, 101:031801(R), 2020.
- [104] Z. H. Levine, B. L. Glebov, A. L. Migdall, T. Gerrits, B. Calkins, A. E. Lita, and S. W. Nam. Photon-number uncertainty in a superconducting transition edge sensor beyond resolved-photon-number determination. *J. Opt. Soc. Am. B*, 31(10):B20, 2014.
- [105] T. W. B. Kibble and F. H. Berkshire. *Classical Mechanics*, 5th ed. Icp, 2004.
- [106] A. Serafini. *Quantum Continuous Variables: A Primer of Theoretical Methods*. CRC Press, 2017.
- [107] N. M. Laurendeau. *Statistical Thermodynamics: Fundamentals and Applications*. Cambridge University Press, 2005.
- [108] E. Wigner. On the quantum correction for thermodynamic equilibrium. *Phys. Rev.*, 40(5):749–759, 1932.
- [109] W. Vogel and D.-G. Welsch. *Quantum Optics*, 3rd ed. Wiley-VCH Verlag GmbH & Co. KGaA, 2006.
- [110] Y. Lai and H. A. Haus. Characteristic functions and quantum measurements of optical observables. *Quantum Opt.*, 1(2):99–115, 1989.
- [111] G. C. Wick. The evaluation of the collision matrix. *Phys. Rev.*, 80:268–272, 1950.
- [112] K. E. Cahill and R. J. Glauber. Density operators and quasiprobability distributions. *Phys. Rev.*, 177(5):1882–1902, 1969.
- [113] H. J. Groenewold. On the principles of elementary quantum mechanics. *Physica*, 12(7):405–460, 1946.
- [114] E. C. G. Sudarshan. Equivalence of semiclassical and quantum mechanical descriptions of statistical light beams. *Phys. Rev. Lett.*, 10:277–279, 1963.
- [115] Roy J. Glauber. Coherent and incoherent states of the radiation field. *Phys. Rev.*, 131(6):2766–2788, 1963.
- [116] T. Kiesel and W. Vogel. Nonclassicality filters and quasiprobabilities. *Phys. Rev. A*, 82(3), 2010.
- [117] K. Husimi. Some formal properties of the density matrix. *Proc. Phys. Math. Soc. Jpn.*, 22(4):264–314, 1940.
- [118] L. Zhang, A. Datta, H. B. Coldenstrodt-Ronge, X.-M. Jin, J. Eisert, M. B. Plenio, and I. A. Walmsley. Recursive quantum detector tomography. *New J. Phys.*, 14(11):115005, November 2012.

-
- [119] G. S. Agarwal and E. Wolf. Quantum dynamics in phase space. *Phys. Rev. Lett.*, 21(3):180–183, 1968.
 - [120] S. Rahimi-Keshari, T. C. Ralph, and C. M. Caves. Sufficient conditions for efficient classical simulation of quantum optics. *Phys. Rev. X*, 6(2), 2016.
 - [121] V. V. Dodonov, O. V. Man’ko, and V. I. Man’ko. Multidimensional hermite polynomials and photon distribution for polymode mixed light. *Phys. Rev. A*, 50(1):813–817, 1994.
 - [122] J. Eisert and M. M. Wolf. Gaussian quantum channels. In *Quantum Information with Continuous Variables of Atoms and Light*, pages 23–42. PUBLISHED BY IMPERIAL COLLEGE PRESS AND DISTRIBUTED BY WORLD SCIENTIFIC PUBLISHING CO., 2007.
 - [123] M. Walschaers, C. Fabre, V. Parigi, and N. Treps. Statistical signatures of multimode single-photon-added and -subtracted states of light. *Phys. Rev. A*, 96:053835, 2017.
 - [124] V. A. Averchenko, V. Thiel, and N. Treps. Nonlinear photon subtraction from a multimode quantum field. *Phys. Rev. A*, 89:063808, 2014.
 - [125] D. Braun, P. Jian, O. Pinel, and N. Treps. Precision measurements with photon-subtracted or photon-added gaussian states. *Phys. Rev. A*, 90:013821, 2014.
 - [126] B. Hage, A. Franzen, J. DiGuglielmo, P. Marek, J. Fiurášek, and R. Schnabel. On the distillation and purification of phase-diffused squeezed states. *New J. Phys.*, 9(7):227–227, 2007.
 - [127] T. Kiesel, W. Vogel, B. Hage, J. DiGuglielmo, A. Samblowski, and R. Schnabel. Experimental test of nonclassicality criteria for phase-diffused squeezed states. *Phys. Rev. A*, 79:022122, 2009.
 - [128] A. Allevi, M. Bondani, P. Marian, T. A. Marian, and S. Olivares. Characterization of phase-averaged coherent states. *J. Opt. Soc. Am. B*, 30(10):2621–2627, 2013.
 - [129] L. Isserlis. On certain probable errors and correlation coefficients of multiple frequency distributions with skew regression. *Biometrika*, 11(3):185–190, 1916.
 - [130] L. Isserlis. On a formula for the product-moment coefficient of any order of a normal frequency distribution in any number of variables. *Biometrika*, 12(1-2):134–139, 1918.
 - [131] J. V. Michalowicz, J. M. Nichols, F. Bucholtz, and C. C. Olson. An issleris’ theorem for mixed gaussian variables: Application to the auto-bispectral density. *J. Stat. Phys.*, 136(1):89–102, 2009.
 - [132] A. Christ, K. Laiho, A. Eckstein, K. N. Cassemiro, and C. Silberhorn. Probing multimode squeezing with correlation functions. *New J. Phys.*, 13(3):033027, 2011.
 - [133] M. Beck. Comparing measurements of $g^{(2)}(0)$ performed with different coincidence detection techniques. *J. Opt. Soc. Am. B*, 24(12):2972, 2007.
 - [134] M. Alexanian. Temporal second-order coherence function for displaced-squeezed thermal states. *J. Mod. Opt.*, 63(10):961–967, 2016.
 - [135] S. Olivares, S. Cialdi, and M. G. A. Paris. Homodyning the $g^{(2)}(0)$ of gaussian states. *Opt. Commun.*, 426:547 – 552, 2018.
 - [136] P. Blasiak, A. Horzela, K. A. Penson, A. I. Solomon, and G. H. E. Duchamp. Combinatorics and boson normal ordering: A gentle introduction. *Am. J. Phys.*, 75(7):639–646, 2007.

- [137] K. Goldberg, M. Newman, and E. Haynsworth. *Handbook of Mathematical Functions: with Formulas, Graphs, and Mathematical Tables*. Dover Publications, 1965.
- [138] A. I. Lvovsky and M. G. Raymer. Continuous-variable optical quantum-state tomography. *Rev. Mod. Phys.*, 81:299–332, 2009.
- [139] H. Hansen, T. Aichele, C. Hettich, P. Lodahl, A. I. Lvovsky, J. Mlynek, and S. Schiller. Ultrasensitive pulsed, balanced homodyne detector: application to time-domain quantum measurements. *Opt. Lett.*, 26(21):1714–1716, 2001.
- [140] K. Vogel and H. Risken. Determination of quasiprobability distributions in terms of probability distributions for the rotated quadrature phase. *Phys. Rev. A*, 40:2847–2849, 1989.
- [141] D. T. Smithey, M. Beck, M. G. Raymer, and A. Faridani. Measurement of the wigner distribution and the density matrix of a light mode using optical homodyne tomography: Application to squeezed states and the vacuum. *Phys. Rev. Lett.*, 70:1244–1247, 1993.
- [142] M. G. Raymer, J. Cooper, H. J. Carmichael, M. Beck, and D. T. Smithey. Ultrafast measurement of optical-field statistics by dc-balanced homodyne detection. *J. Opt. Soc. Am. B*, 12(10):1801, 1995.
- [143] A. I. Lvovsky. Iterative maximum-likelihood reconstruction in quantum homodyne tomography. *J. Opt. B: Quantum Semiclass. Opt.*, 6(6):S556–S559, 2004.
- [144] V. V. Dodonov, O. V. Man’ko, and V. I. Man’ko. Photon distribution for one-mode mixed light with a generic gaussian wigner function. *Phys. Rev. A*, 49:2993–3001, 1994.
- [145] G. Szegő. *Orthogonal Polynomials*. American Mathematical Society, 1939.
- [146] P. Kok and S. L. Braunstein. Multi-dimensional hermite polynomials in quantum optics. *J. Phys. A: Math. Gen.*, 34(31):6185–6195, 2001.
- [147] Olga Man’ko and V. I. Man’ko. Photon-number tomography of multimode states and positivity of the density matrix. *Journal of Russian Laser Research*, 24(5):497–506, 2003.
- [148] A. P. Lund, A. Laing, S. Rahimi-Keshari, T. Rudolph, J. L. O’Brien, and T. C. Ralph. Boson sampling from a gaussian state. *Phys. Rev. Lett.*, 113:100502, 2014.
- [149] M. Bentivegna, N. Spagnolo, C. Vitelli, F. Flamini, N. Viggianiello, L. Latmiral, P. Mataloni, D. J. Brod, E. F. Galvão, A. Crespi, R. Ramponi, R. Osellame, and F. Sciarrino. Experimental scattershot boson sampling. *Sci. Adv.*, 1(3), 2015.
- [150] L. Latmiral, N. Spagnolo, and F. Sciarrino. Towards quantum supremacy with lossy scattershot boson sampling. *New J. Phys.*, 18(11):113008, 2016.
- [151] D. S. Phillips, M. Walschaers, J. J. Renema, I. A. Walmsley, N. Treps, and J. Sperling. Benchmarking of gaussian boson sampling using two-point correlators. *Phys. Rev. A*, 99:023836, 2019.
- [152] C. S. Hamilton, R. Kruse, L. Sansoni, S. Barkhofen, C. Silberhorn, and I. Jex. Gaussian boson sampling. *Phys. Rev. Lett.*, 119:170501, 2017.
- [153] R. Kruse, C. S. Hamilton, L. Sansoni, S. Barkhofen, C. Silberhorn, and I. Jex. Detailed study of gaussian boson sampling. *Phys. Rev. A*, 100(3), 2019.

- [154] E. R. Caianiello. On quantum field theory — i: explicit solution of dyson’s equation in electro-dynamics without use of feynman graphs. *Il Nuovo Cimento (1943-1954)*, 10(12):1634–1652, 1953.
- [155] N. Quesada and J. M. Arrazola. Exact simulation of gaussian boson sampling in polynomial space and exponential time. *Phys. Rev. Research*, 2:023005, 2020.
- [156] B. Wu, B. Cheng, J. Zhang, M.-H. Yung, and X. Sun. Speedup in classical simulation of gaussian boson sampling. *Sci. Bull.*, 2020.
- [157] H. Qi, D. J. Brod, N. Quesada, and R. García-Patrón. Regimes of classical simulability for noisy gaussian boson sampling. *Phys. Rev. Lett.*, 124:100502, 2020.
- [158] N. Quesada, J. M. Arrazola, and N. Killoran. Gaussian boson sampling using threshold detectors. *Phys. Rev. A*, 98(6), 2018.
- [159] J. Huh and M.-H. Yung. Vibronic boson sampling: Generalized gaussian boson sampling for molecular vibronic spectra at finite temperature. *Sci. Rep.*, 7(1):7462, 2017.
- [160] W. R. Clements, J. J. Renema, A. Eckstein, A. A. Valido, A. E. Lita, T. Gerrits, S. W. Nam, W. S. Kolthammer, J. Huh, , and I. A. Walmsley. Approximating vibronic spectroscopy with imperfect quantum optics. *J. Phys. B: At. Mol. Opt. Phys.*, 51(24):245503, 2018.
- [161] L. Banchi, M. Fingerhuth, T. Babej, C. Ing, and J. M. Arrazola. Molecular docking with gaussian boson sampling. *Sci. Adv.*, 6(23):eaax1950, 2020.
- [162] J. M. Arrazola and T. R. Bromley. Using gaussian boson sampling to find dense subgraphs. *Phys. Rev. Lett.*, 121:030503, 2018.
- [163] J. M. Arrazola, T. R. Bromley, and P. Rebentrost. Quantum approximate optimization with gaussian boson sampling. *Phys. Rev. A*, 98:012322, 2018.
- [164] S. Jahangiri, J. M. Arrazola, N. Quesada, and N. Killoran. Point processes with gaussian boson sampling. *Phys. Rev. E*, 101:022134, 2020.
- [165] E. A. J. Marcatili. Dielectric rectangular waveguide and directional coupler for integrated optics. *Bell Labs Tech. J.*, 48(7):2071–2102, 1969.
- [166] B. J. Metcalf, J. B. Spring, P. C. Humphreys, N. Thomas-Peter, M. Barbieri, W. S. Kolthammer, X.-M. Jin, N. K. Langford, D. Kundys, J. C. Gates, B. J. Smith, P. G. R. Smith, and I. A. Walmsley. Quantum teleportation on a photonic chip. *Nat. Photon.*, 8(10):770–774, 2014.
- [167] N. C. Harris, G. R. Steinbrecher, M. Prabhu, Y. Lahini, J. Mower, D. Bunandar, C. Chen, F. N. C. Wong, T. Baehr-Jones, M. Hochberg, S. Lloyd, and D. Englund. Quantum transport simulations in a programmable nanophotonic processor. *Nat. Photonics*, 11(7):447–452, 2017.
- [168] S. Ferrari, C. Schuck, and W. Pernice. Waveguide-integrated superconducting nanowire single-photon detectors. *Nanophotonics*, 7(11):1725–1758, 2018.
- [169] V. R. Almeida, R. R. Panepucci, and M. Lipson. Nanotaper for compact mode conversion. *Opt. Lett.*, 28(15):1302, 2003.
- [170] X. Hu, C. W. Holzwarth, D. Masciarelli, E. A. Dauler, and K. K. Berggren. Efficiently coupling light to superconducting nanowire single-photon detectors. *IEEE Trans. Appl. Supercond.*, 19(3):336–340, 2009.

- [171] D. Sahin, A. Gaggero, Z. Zhou, S. Jahanmirinejad, F. Mattioli, R. Leoni, J. Beetz, M. Lerner, M. Kamp, S. Höfling, and A. Fiore. Waveguide photon-number-resolving detectors for quantum photonic integrated circuits. *Appl. Phys. Lett.*, 103(11):111116, 2013.
- [172] T. Gerrits *et al.* On-chip, photon-number-resolving, telecommunication-band detectors for scalable photonic information processing. *Phys. Rev. A*, 84:060301, 2011.
- [173] B. Calkins, P. L. Mennea, A. E. Lita, B. J. Metcalf, W. S. Kolthammer, A. Lamas-Linares, J. B. Spring, P. C. Humphreys, R. P. Mirin, J. C. Gates, P. G. R. Smith, I. A. Walmsley, T. Gerrits, and S. W. Nam. High quantum-efficiency photon-number-resolving detector for photonic on-chip information processing. *Opt. Express*, 21(19):22657, 2013.
- [174] G. Reithmaier, M. Kaniber, F. Flassig, S. Lichtmannecker, K. Müller, A. Andrejew, J. Vučković, R. Gross, and J. J. Finley. On-chip generation, routing, and detection of resonance fluorescence. *Nano Lett.*, 15(8):5208–5213, 2015.
- [175] J. W. Silverstone, D. Bonneau, J. L. O’Brien, and M. G. Thompson. Silicon quantum photonics. *IEEE J. Sel. Top. Quantum Electron.*, 22(6):390–402, 2016.
- [176] C. Sima, J. C. Gates, H. L. Rogers, P. L. Mennea, C. Holmes, M. N. Zervas, and P. G. R. Smith. Ultra-wide detuning planar bragg grating fabrication technique based on direct UV grating writing with electro-optic phase modulation. *Opt. Express*, 21(13):15747, 2013.
- [177] B. Baek, A. E. Lita, V. Verma, and S. W. Nam. Superconducting a-WxSi1-x nanowire single-photon detector with saturated internal quantum efficiency from visible to 1850 nm. *Appl. Phys. Lett.*, 98(25):251105, 2011.
- [178] M. Svalgaard, A. Bjarklev, C. V. Poulsen, and O. Poulsen. Direct UV writing of buried singlemode channel waveguides in ge-doped silica films. *Electron. Lett.*, 30(17):1401–1403, 1994.
- [179] B. J. Smith, D. Kundys, N. Thomas-Peter, P. G. R. Smith, and I. A. Walmsley. Phase-controlled integrated photonic quantum circuits. *Opt. Express*, 17(16):13516, 2009.
- [180] H. L. Rogers, C. Holmes, J. C. Gates, and P. G. R. Smith. Analysis of dispersion characteristics of planar waveguides via multi-order interrogation of integrated bragg gratings. *IEEE Photon. J.*, 4(2):310–316, 2012.
- [181] G. Batey and G. Teleberg. Principles of dilution refrigeration: A brief technology guide. Technical report, Oxford Instruments NanoScience, 2015.
- [182] R. Radebaugh. Development of the pulse tube refrigerator as an efficient and reliable cryocooler. In *Proceedings of the Institute of Refrigeration (London)*, 2000.
- [183] J. W. Ekin. *Experimental Techniques: Cryostat Design, Material Properties and Superconductor Critical-Current Testing*. Oxford University Press, 2006.
- [184] J. K. W. Yang, A. J. Kerman, E. A. Dauler, V. Anant, K. M. Rosfjord, and K. K. Berggren. Modeling the electrical and thermal response of superconducting nanowire single-photon detectors. *IEEE Transactions on Applied Superconductivity*, 17(2):581–585, 2007.
- [185] M. Caloz, B. Korzh, N. Timoney, M. Weiss, S. Gariglio, R. J. Warburton, C. Schönenberger, J. J. Renema, H. Zbinden, and F. Bussi eres. Optically probing the detection mechanism in a molybdenum silicide superconducting nanowire single-photon detector. *Appl. Phys. Lett.*, 110(8):083106, 2017.

-
- [186] A. G. Kozorezov, C. Lambert, F. Marsili, M. J. Stevens, V. B. Verma, J. P. Allmaras, M. D. Shaw, R. P. Mirin, and S. W. Nam. Fano fluctuations in superconducting-nanowire single-photon detectors. *Phys. Rev. B*, 96:054507, 2017.
 - [187] C. Meeks and P. B. Siegel. Dead time correction via the time series. *Am. J. Phys.*, 76(6):589–590, 2008.
 - [188] E. E. Wollman, V. B. Verma, A. D. Beyer, R. M. Briggs, B. Korzh, J. P. Allmaras, F. Marsili, A. E. Lita, R. P. Mirin, S. W. Nam, and M. D. Shaw. UV superconducting nanowire single-photon detectors with high efficiency, low noise, and 4 k operating temperature. *Opt. Express*, 25(22):26792, 2017.
 - [189] R. Valivarthi, I. Lucio-Martinez, A. Rubenok, P. Chan, F. Marsili, V. B. Verma, M. D. Shaw, J. A. Stern, J. A. Slater, D. Oblak, S. W. Nam, and W. Tittel. Efficient bell state analyzer for time-bin qubits with fast-recovery WSi superconducting single photon detectors. *Opt. Express*, 22(20):24497, 2014.
 - [190] N. Calandri, Q.-Y. Zhao, D. Zhu, A. Dane, and K. K. Berggren. Superconducting nanowire detector jitter limited by detector geometry. *Appl. Phys. Lett.*, 109(15):152601, 2016.
 - [191] J. P. Allmaras, A. G. Kozorezov, B. A. Korzh, K. K. Berggren, and M. D. Shaw. Intrinsic timing jitter and latency in superconducting nanowire single-photon detectors. *Phys. Rev. Appl.*, 11:034062, 2019.
 - [192] B. Korzh *et al.* Demonstration of sub-3 ps temporal resolution with a superconducting nanowire single-photon detector. *Nat. Photonics*, 2020.
 - [193] M. Caloz, B. Korzh, E. Ramirez, C. Schönenberger, R. J. Warburton, H. Zbinden, M. D. Shaw, and F. Bussi eres. Intrinsically-limited timing jitter in molybdenum silicide superconducting nanowire single-photon detectors. *J. Appl. Phys.*, 126(16):164501, 2019.
 - [194] L. You, X. Yang, Y. He, W. Zhang, D. Liu, W. Zhang, L. Zhang, L. Zhang, X. Liu, S. Chen, Z. Wang, and X. Xie. Jitter analysis of a superconducting nanowire single photon detector. *AIP Adv.*, 3(7):072135, 2013.
 - [195] E. A. Dauler, M. E. Grein, A. J. Kerman, F. Marsili, S. Miki, S. W. Nam, M. D. Shaw, H. Terai, V. B. Verma, and T. Yamashita. Review of superconducting nanowire single-photon detector system design options and demonstrated performance. *Opt. Eng.*, 53(8):081907, 2014.
 - [196] D. Zauner, M. Svalgaard, and M. Kristensen. Directly UV written silica-on-silicon planar waveguides with low insertion loss. *Opt. Soc. America*, 1998.
 - [197] T. Lunghi, B. Korzh, B. Sanguinetti, and H. Zbinden. Absolute calibration of fiber-coupled single-photon detector. *Opt. Express*, 22(15):18078, 2014.
 - [198] J. J. Renema and A. Fiore. Notes on using ssps as a spectrometers - extended edition. Unpublished.
 - [199] W. B. Whitten and H. H. Ross. Fiber optic waveguides for time-of-flight optical spectrometry. *Anal. Chem.*, 51(3):417–419, 1979.
 - [200] W. B. Whitten. Time-of-flight optical spectrometry with fiber optic waveguides. *Anal. Chem.*, 54(7):1026–1028, 1982.
 - [201] A. Ben-Israel and T. N. E. Greville. *Generalized Inverses*. Springer-Verlag, 2003.

- [202] M. Benning. Inverse problems, 2015. Lecture notes: <http://www.damtp.cam.ac.uk/user/mb941/Inverse>
- [203] R. Hagen, S. Roch, and B. Silbermann. *C* - Algebras and Numerical Analysis*. CRC Press, 2000.
- [204] J. J. Renema, Q. Wang, R. Gaudio, I. Komen, K. op 't Hoog, D. Sahin, A. Schilling, M. P. van Exter, A. Fiore, A. Engel, and M. J. A. de Dood. Position-dependent local detection efficiency in a nanowire superconducting single-photon detector. *Nano Lett.*, 15(7):4541–4545, 2015.
- [205] J. J. Renema, G. Frucci, Z. Zhou, F. Mattioli, A. Gaggero, R. Leoni, M. J. A. de Dood, A. Fiore, and M. P. van Exter. Universal response curve for nanowire superconducting single-photon detectors. *Phys. Rev. B*, 87(17), 2013.
- [206] J. P. Torre and G. Chanin. Miniature liquid-3He refrigerator. *Rev. Sci. Instrum.*, 56(2):318–320, 1985.
- [207] L. Duband, L. Clerc, E. Ercolani, L. Guillemet, and R. Vallcorba. Herschel flight models sorption coolers. *Cryogenics*, 48(3-4):95–105, 2008.
- [208] F. Pobell. *Matter and Methods at Low Temperatures*. Springer Berlin Heidelberg, 2007.
- [209] U. L. Andersen, J. S. Neergaard-Nielsen, P. van Loock, and A. Furusawa. Hybrid discrete- and continuous-variable quantum information. *Nat. Phys.*, 11(9):713–719, 2015.
- [210] S. L. Braunstein. Homodyne statistics. *Phys. Rev. A*, 42:474–481, 1990.
- [211] W. Vogel and J. Grabow. Statistics of difference events in homodyne detection. *Phys. Rev. A*, 47:4227–4235, 1993.
- [212] K. Banaszek and K. Wódkiewicz. Operational theory of homodyne detection. *Phys. Rev. A*, 55:3117–3123, 1997.
- [213] T. Tyc and B. C. Sanders. Operational formulation of homodyne detection. *J. Phys. A: Math. Gen.*, 37(29):7341–7357, 2004.
- [214] M. Skotiniotis, W. Dür, and P. Sekatski. Macroscopic superpositions require tremendous measurement devices. *Quantum*, 1:34, 2017.
- [215] G. Puentes, J. S. Lundeen, M. P. A. Branderhorst, H. B. Coldenstrodt-Ronge, B. J. Smith, and I. A. Walmsley. Bridging particle and wave sensitivity in a configurable detector of positive operator-valued measures. *Phys. Rev. Lett.*, 102:080404, 2009.
- [216] L. Zhang, H. B. Coldenstrodt-Ronge, A. Datta, G. Puentes, J. S. Lundeen, X.-M. Jin, B. J. Smith, M. B. Plenio, and I. A. Walmsley. Mapping coherence in measurement via full quantum tomography of a hybrid optical detector. *Nat. Photon.*, 6(6):364–368, 2012.
- [217] G. Donati, T. J. Bartley, X.-M. Jin, M.-D. Vidrighin, A. Datta, M. Barbieri, and I. A. Walmsley. Observing optical coherence across fock layers with weak-field homodyne detectors. *Nat. Commun.*, 5(1), 2014.
- [218] A. Allevi, M. Bina, S. Olivares, and M. Bondani. Homodyne-like detection scheme based on photon-number-resolving detectors. *Int. J. Quantum Inf.*, 15(08):1740016, 2017.
- [219] S. Olivares, A. Allevi, G. Caiazzo, M. G. A. Paris, and M. Bondani. Homodyne tomography with homodyne-like detection. *arXiv:1809.00818v1 [quant-ph]*, 2018.

- [220] M. Liscidini and J. E. Sipe. Stimulated emission tomography. *Phys. Rev. Lett.*, 111:193602, 2013.
- [221] A. Eckstein, A. Christ, P. J. Mosley, and C. Silberhorn. Highly efficient single-pass source of pulsed single-mode twin beams of light. *Phys. Rev. Lett.*, 106:013603, 2011.
- [222] A. M. Brańczyk, T. C. Ralph, W. Helwig, and C. Silberhorn. Optimized generation of heralded fock states using parametric down-conversion. *New J. Phys.*, 12(6):063001, 2010.
- [223] C. K. Law, I. A. Walmsley, and J. H. Eberly. Continuous frequency entanglement: Effective finite hilbert space and entropy control. *Phys. Rev. Lett.*, 84:5304–5307, 2000.
- [224] J. H. Eberly. Schmidt analysis of pure-state entanglement. *Laser Phys.*, 16(6):921–926, 2006.
- [225] J. G. Rarity, P. R. Tapster, and R. Loudon. Non-classical interference between independent sources. *J. Opt. B: Quantum Semiclass. Opt.*, 7(7):S171–S175, 2005.
- [226] R.-B. Jin, J. Zhang, R. Shimizu, N. Matsuda, Y. Mitsumori, H. Kosaka, and K. Edamatsu. High-visibility nonclassical interference between intrinsically pure heralded single photons and photons from a weak coherent field. *Phys. Rev. A*, 83:031805, 2011.
- [227] M. C. Tichy. Interference of identical particles from entanglement to boson-sampling. *J. Phys. B: At. Mol. Opt. Phys.*, 47(10):103001, 2014.
- [228] D. N. Klyshko. Use of two-photon light for absolute calibration of photoelectric detectors. *Quantum Electron.*, 10(9):1112–1117, 1980.
- [229] T. J. Bartley, G. Donati, J. B. Spring, X.-M. Jin, M. Barbieri, A. Datta, B. J. Smith, and I. A. Walmsley. Multiphoton state engineering by heralded interference between single photons and coherent states. *Phys. Rev. A*, 86:043820, 2012.
- [230] G. S. Thekkadath, B. A. Bell, I. A. Walmsley, and A. I. Lvovsky. Engineering schrödinger cat states with a photonic even-parity detector. *Quantum*, 4:239, 2020.
- [231] J. Sperling, D. S. Phillips, J. F. F. Bulmer, G. S. Thekkadath, A. Eckstein, T. A. W. Wolterink, J. Lugani, S. W. Nam, A. E. Lita, T. Gerrits, W. Vogel, G. S. Agarwal, C. Silberhorn, and I. A. Walmsley. Detector-agnostic phase-space distributions. *Phys. Rev. Lett.*, 124:013605, 2020.
- [232] M. G. A. Paris and J. Řeháček, editors. *Quantum State Estimation*. Springer Berlin Heidelberg, 2004.
- [233] J. B. Altepeter, E. R. Jeffrey, and P. G. Kwiat. Photonic state tomography. In *Advances In Atomic, Molecular, and Optical Physics*, pages 105–159. Elsevier, 2005.
- [234] D. T. Smithey, M. Beck, J. Cooper, M. G. Raymer, and A. Faridani. Complete experimental characterization of the quantum state of a light mode via the wigner function and the density matrix: application to quantum phase distributions of vacuum and squeezed-vacuum states. *Phys. Scr.*, T48:35–44, 1993.
- [235] V. Delaubert, N. Treps, M. Lassen, C. C. Harb, C. Fabre, P. K. Lam, and H.-A. Bachor. tem_{10} homodyne detection as an optimal small-displacement and tilt-measurement scheme. *Phys. Rev. A*, 74:053823, 2006.
- [236] F. A. S. Barbosa, A. S. Coelho, K. N. Cassemiro, P. Nussenzeig, C. Fabre, M. Martinelli, and A. S. Villar. Beyond spectral homodyne detection: Complete quantum measurement of spectral modes of light. *Phys. Rev. Lett.*, 111:200402, 2013.

- [237] F. A. S. Barbosa, A. S. Coelho, K. N. Cassemiro, P. Nussenzveig, C. Fabre, A. S. Villar, and M. Martinelli. Quantum state reconstruction of spectral field modes: Homodyne and resonator detection schemes. *Phys. Rev. A*, 88:052113, 2013.
- [238] W. Vogel. Homodyne correlation measurements with weak local oscillators. *Phys. Rev. A*, 51:4160–4171, 1995.
- [239] S. Wallentowitz and W. Vogel. Unbalanced homodyning for quantum state measurements. *Phys. Rev. A*, 53:4528–4533, 1996.
- [240] J. Tiedau, V. S. Shchesnovich, D. Mogilevtsev, V. Ansari, G. Harder, T. J. Bartley, N. Korolkova, and C. Silberhorn. Quantum state and mode profile tomography by the overlap. *New J. Phys.*, 20(3):033003, 2018.
- [241] A. Avagyan, E. Knill, S. Glancy, and H. Vasconcelos. State tomography with photon counting after a beam splitter. Unpublished.
- [242] G. K. White and P. J. Meeson. *Experimental Techniques In Low-Temperature Physics*, 3rd ed. Oxford University Press, U.S.A., 2002.
- [243] E. Warburg. Magnetische untersuchungen. *Ann. Phys.*, 249(5):141–164, 1881.
- [244] P. Weiss and A. Piccard. Le phénomène magnétocalorique. *Journal de Physique Théorique et Appliquée*, 7(1):103–109, 1917.
- [245] R. Hanbury Brown and R. Q. Twiss. A test of a new type of stellar interferometer on sirius. *Nature*, 178(4541):1046–1048, 1956.
- [246] H. J. Kimble, M. Dagenais, and L. Mandel. Photon antibunching in resonance fluorescence. *Phys. Rev. Lett.*, 39:691–695, 1977.
- [247] S. Lloyd and S. L. Braunstein. Quantum computation over continuous variables. *Phys. Rev. Lett.*, 82:1784–1787, 1999.
- [248] M. Bohmann, A. A. Semenov, J. Sperling, and W. Vogel. Gaussian entanglement in the turbulent atmosphere. *Phys. Rev. A*, 94:010302, 2016.
- [249] F. Steinlechner, S. Ecker, M. Fink, B. Liu, J. Bavaresco, M. Huber, T. Scheidl, and R. Ursin. Distribution of high-dimensional entanglement via an intra-city free-space link. *Nat. Commun.*, 8:15971 EP –, 2017. Article.
- [250] BC. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters. Teleporting an unknown quantum state via dual classical and einstein-podolsky-rosen channels. *Phys. Rev. Lett.*, 70:1895–1899, 1993.
- [251] J.-G. Ren *et al.* Ground-to-satellite quantum teleportation. *Nature*, 549:70 EP –, 2017.
- [252] A. Bleicher. Quantum algorithms struggle against old foe: Clever computers, 2018.
- [253] M. Walschaers, J. Kuipers, J.-D. Urbina, K. Mayer, M. C. Tichy, K. Richter, and A. Buchleitner. Statistical benchmark for bosonsampling. *New J. Phys.*, 18(3):032001, 2016.
- [254] C. Gogolin, M. Kliesch, L. Aolita, and J. Eisert. Boson-sampling in the light of sample complexity. *arXiv:1306.3995v3 [quant-ph]*, 2013.
- [255] S. Aaronson and A. Arkhipov. Bosonsampling is far from uniform. *Quantum Info. Comput.*, 14(15-16):1383–1423, 2014.

- [256] N. Spagnolo, C. Vitelli, M. Bentivegna, D. J. Brod, A. Crespi, F. Flamini, S. Giacomini, G. Milani, R. Ramponi, P. Mataloni, R. Osellame, E. F. Galvão, and F. Sciarrino. Experimental validation of photonic boson sampling. *Nat. Photonics*, 8(8):615–620, 2014.
- [257] L. Aolita, C. Gogolin, M. Kliesch, and J. Eisert. Reliable quantum certification of photonic state preparations. *Nat. Commun.*, 6(1), 2015.
- [258] M. Bentivegna, N. Spagnolo, C. Vitelli, D. J. Brod, A. Crespi, F. Flamini, R. Ramponi, P. Mataloni, R. Osellame, E. F. Galvão, and F. Sciarrino. Bayesian approach to boson sampling validation. *Int. J. Quantum Inform.*, 12(07n08):1560028, 2014.
- [259] J. Carolan, J. D. A. Meinecke, P. J. Shadbolt, N. J. Russell, N. Ismail, K. Wörhoff, T. Rudolph, M. G. Thompson, J. L. O’Brien, J. C. F. Matthews, and A. Laing. On the experimental verification of quantum complexity in linear optics. *Nat. Photonics*, 8(8):621–626, 2014.
- [260] M. C. Tichy, K. Mayer, A. Buchleitner, and K. Mølmer. Stringent and efficient assessment of boson-sampling devices. *Phys. Rev. Lett.*, 113:020502, 2014.
- [261] V. S. Shchesnovich. Universality of generalized bunching and efficient assessment of boson sampling. *Phys. Rev. Lett.*, 116(12), 2016.
- [262] L. Rigovacca, C. Di Franco, B. J. Metcalf, I. A. Walmsley, and M. S. Kim. Nonclassicality criteria in multiport interferometry. *Phys. Rev. Lett.*, 117(21), 2016.
- [263] N. Viggianiello, F. Flamini, L. Innocenti, D. Cozzolino, M. Bentivegna, N. Spagnolo, A. Crespi, D. J. Brod, E. F. Galvão, R. Osellame, and F. Sciarrino. Experimental generalized quantum suppression law in sylvester interferometers. *New J. Phys.*, 20(3):033017, 2018.
- [264] T. Giordani, F. Flamini, M. Pompili, N. Viggianiello, N. Spagnolo, A. Crespi, R. Osellame, N. Wiebe, M. Walschaers, A. Buchleitner, and F. Sciarrino. Experimental statistical signature of many-body quantum interference. *Nat. Photonics*, 12(3):173–178, 2018.
- [265] A. Haar. Der massbegriff in der theorie der kontinuierlichen gruppen. *Ann. Math.*, 34(1):147, 1933.
- [266] K. Życzkowski and M. Kus. Random unitary matrices. *J. Phys. A*, 27(12):4235–4245, 1994.
- [267] C. Dittel, G. Dufour, M. Walschaers, G. Weihs, A. Buchleitner, and R. Keil. Totally destructive many-particle interference. *Phys. Rev. Lett.*, 120(24), 2018.
- [268] C. Dittel, G. Dufour, M. Walschaers, G. Weihs, A. Buchleitner, and R. Keil. Totally destructive interference for permutation-symmetric many-particle states. *Phys. Rev. A*, 97(6), 2018.
- [269] M. Walschaers, J. Kuipers, and A. Buchleitner. From many-particle interference to correlation spectroscopy. *Phys. Rev. A*, 94(2), 2016.
- [270] S. B. Shlosman. Signs of the ising model ursell functions. *Commun. Math. Phys.*, 102(4):679–686, 1986.
- [271] D. W. Robinson. The ground state of the bose gas. *Commun. Math. Phys.*, 1(2):159–174, 1965.
- [272] O. Bratteli, A. Kishimoto, and D. W. Robinson. Stability properties and the KMS condition. *Commun. Math. Phys.*, 61(3):209–238, 1978.

- [273] D. Goderis, A. Verbeure, and P. Vets. Non-commutative central limits. *Probab. Th. Rel. Fields*, 82(4):527–544, 1989.
- [274] B. Nachtergaele, Y. Ogata, and R. Sims. Propagation of correlations in quantum lattice systems. *J. Stat. Phys.*, 124(1):1–13, 2006.
- [275] M. C. Tichy, M. Tiersch, F. Mintert, and A. Buchleitner. Many-particle interference beyond many-boson and many-fermion statistics. *New J. Phys.*, 14(9):093015, 2012.
- [276] Y.-S. Ra, M. C. Tichy, H.-T. Lim, O. Kwon, F. Mintert, A. Buchleitner, and Y.-H. Kim. Nonmonotonic quantum-to-classical transition in multiparticle interference. *Proc. Natl. Acad. Sci. U.S.A.*, 110(4):1227–1231, 2013.
- [277] M. Chuchem, K. Smith-Mannschott, M. Hiller, T. Kottos, A. Vardi, and D. Cohen. Quantum dynamics in the bosonic josephson junction. *Phys. Rev. A*, 82:053617, 2010.
- [278] 3rd ed. M. L. Mehta. *Random Matrices*. Academic Press, 2004.
- [279] S. Samuel. $U(n)$ integrals, $1/n$, and the de wit–t hooft anomalies. *J. Math. Phys.*, 21(12):2695–2703, 1980.
- [280] P. A. Mello. Averages on the unitary group and applications to the problem of disordered conductors. *J. Phys. A: Math. Gen.*, 23(18):4061–4080, 1990.
- [281] P. W. Brouwer and C. W. J. Beenakker. Diagrammatic method of integration over the unitary group, with applications to quantum transport in mesoscopic systems. *J. Math. Phys.*, 37(10):4904–4934, 1996.
- [282] G. Berkolaiko and J. Kuipers. Combinatorial theory of the semiclassical evaluation of transport moments. i. equivalence with the random matrix approach. *J. Math. Phys.*, 54(11):112103, 2013.
- [283] M. Creutz. On invariant integration over $SU(n)$. *J. Math. Phys.*, 19(10):2043, 1978.
- [284] D. Weingarten. Asymptotic behavior of group integrals in the limit of infinite rank. *J. Math. Phys.*, 19(5):999–1001, 1978.
- [285] K. Mayer, M. C. Tichy, F. Mintert, T. Konrad, and A. Buchleitner. Counting statistics of many-particle quantum walks. *Phys. Rev. A*, 83(6), 2011.
- [286] C. Taballione, T. A. W. Wolterink, J. Lugani, A. Eckstein, B. A. Bell, R. Grootjans, I. Vischer, D. Geskus, C. G. H. Roeloffzen, J. J. Renema, I. A. Walmsley, P. W. H. Pinkse, and K.-J. Boller. 8×8 reconfigurable quantum photonic processor based on silicon nitride waveguides. *Opt. Express*, 27(19):26842, 2019.
- [287] G. Harder, T. J. Bartley, A. E. Lita, S. W. Nam, T. Gerrits, and C. Silberhorn. Single-mode parametric-down-conversion states with 50 photons as a source for mesoscopic quantum optics. *Phys. Rev. Lett.*, 116(14), 2016.
- [288] A. Miranowicz, M. Bartkowiak, X. Wang, Y.-X. Liu, and F. Nori. Testing nonclassicality in multimode fields: A unified derivation of classical inequalities. *Phys. Rev. A*, 82:013824, 2010.
- [289] H.-S. Zhong *et al.* Experimental gaussian boson sampling. *Sci. Bull.*, 64(8):511–515, 2019.
- [290] H.-S. Zhong *et al.* Quantum computational advantage using photons. *Science*, 370(6523):1460–1463, 2020.