CORRIGENDUM: "COMPENSATED COMPACTNESS METHODS IN THE STUDY OF COMPRESSIBLE FLUID FLOW"

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Abstract. We correct an error in Chapter 3 of the thesis "Compensated compactness methods in the study of compressible fluid flow", which was made at six distinct points in the chapter. The main result therein (Theorem 3.2) is correct as stated, with the only exception that the requirement $\alpha > 1/2$ must be imposed in point (3) of Definition 3.1 of the chapter. We also make note of some minor typos found in the rest of the thesis.

In this document, we correct an error in Chapter 3 of the thesis "Compensated compactness methods in the study of compressible fluid flow" [4], which was made at six distinct points in the chapter, and which had a minor repercussion on the main theorem; Theorem 3.2. The only essential lemmas that were affected were Lemmas 3.52, 3.60, and 3.66 of [4]. In the following sections, we correct this mistake.

We begin by stating the minor change to our main theorem in Section 1. Then we explain the error in detail, in Section 2, which concerns a Grönwall-type estimate. Section 3 shows how to correct this mistake, and how to obtain an alternative estimate. Sections 4-6 then show how to recover the conclusions of Lemmas 3.52, 3.60, and 3.66 of Chapter 3, respectively. Finally, Section 7 contains a small list of minor typos found in the rest of the thesis.

In Sections 3-6, M denotes a positive constant, independent of ρ, which may change from line to line. The proofs are written for $\alpha < 1$, as the case $\alpha \geq 1$ is easier and follows directly from the same strategy.

Acknowledgements. The author wishes to thank Matthew Schrecker for pointing out this error.

1. Change to main result

Our main result (Theorem 3.2 in [4]) is correct as stated in the original thesis chapter, with the only exception that the requirement $\alpha > 1/2$ must be imposed in point (3) of Definition 3.1 of [4].

The minor lemmas affected are: Lemma 3.23, Remark 3.24, equation (3.53) in Lemma 3.27, Lemmas 3.28 and 3.30, Lemmas 3.41-3.45, Corollaries 3.46 and 3.47, equations (3.116) and (3.117) in Lemma 3.54, Corollary 3.55, Lemma 3.59, equation (3.132) in Lemma 3.62, Remark 3.63, Corollary 3.64, equation (3.149) in Lemma 3.65, and equation (3.159) in Lemma 3.66 of [4]. These are not correct as originally stated, and Sections 4-6 herein rectify them.

As already mentioned, the only essential lemmas that build on these minor lemmas are: Lemmas 3.52, 3.60, and 3.66 of [4]. The statements of Lemmas 3.52 and 3.60 are correct as written Chapter 3, but Lemma 3.66 has one minor mistake in equation (3.159); which is rectified in Section 6 of this document.

2. Explanation of the error

It is wrongly claimed, after equation (3.41) in Chapter 3, that one can apply the Grönwall Lemma to get a decay bound when one has an estimate of the form:

$$f(\rho_0) \leq g(\rho_0) + \frac{1}{\rho_0 - \rho_*}\int_{\rho_*}^{\rho_0} f(\rho)\,\mathrm{d}\rho \qquad \forall \rho_0 > \rho_*, \tag{1}$$

where it is assumed that $f, g \geq 0$ are locally bounded, $f(\rho_*) = g(\rho_*) = 0$, and g itself satisfies a decay bound of the form $|g(\rho_0)| \leq M\rho_0^{-q}$ for some $M, q > 0$. More precisely, it is claimed that an application of Grönwall's Lemma yields $|f(\rho_0)| \leq \tilde{M}\rho_0^{-q}$ with the same exponent q, but with a possibly different constant $\tilde{M}$. This is false, as shown by choosing, for instance, $\rho_* = 0$ along with

$$f(\rho_0) = \operatorname{erf}(\rho_0), \qquad g(\rho_0) = \rho_0^{-1}\bigg(\frac{2}{\sqrt{\pi}}\int_0^{\rho_0} y e^{-y^2}\,\mathrm{d}y\bigg).$$

This example yields equality in (1) with $f(0) = g(0) = 0$, along with $q = 1$, but the error function tends to the constant function 1 in the limit as $\rho_0 \to \infty$ and thus does not decay; it is merely bounded.

This mistake was then repeated in the proof of Lemmas 3.41, 3.44, 3.59, 3.62 and 3.65 of the chapter. As already mentioned, the only results essential to our approach that this affects are Lemmas 3.52, 3.60, and 3.66 of Chapter 3. In the subsequent sections, we show how to recover the conclusions of these lemmas using an alternative estimate, which is the topic of the next section.

3. Alternative estimate to the error

We prove the following lemma, which will be at the core of the estimates of Sections 4 and 6.

Lemma 3.1. *Assume that $f \geq 0$ is measurable, locally bounded, and satisfies, for some $M, q > 0$, the estimate*

$$f(\rho_0) \leq M\rho_0^{-q} + \frac{1}{\rho_0 - \rho_*}\int_{\rho_*}^{\rho_0} d(\rho)f(\rho)\,\mathrm{d}\rho \qquad \forall \rho_0 > \rho_*, \tag{2}$$

where $d(\rho)$ is the coefficient defined in Definition 3.8 of [4, Chapter 3]. *Then, for some possibly larger $\tilde{M} > 0$ independent of ρ_0, there holds*

$$|f(\rho_0)| \leq \tilde{M} \qquad \forall \rho_0 \geq \rho_*. \tag{3}$$

Proof. The main observation is that $d(\rho) = 1 + o(\rho)$. Observe firstly that, due to the local boundedness of the integrand and the mean-value inequality, there exists a positive constant M, independent of ρ_0, such that

$$\frac{1}{\rho_0 - \rho_*}\int_{\rho_*}^{\rho_0} d(\rho)f(\rho)\,\mathrm{d}\rho \leq M \qquad \text{for } \rho_0 \in [\rho_*, 2\rho_*], \tag{4}$$

and it follows immediately from the above and the hypothesis of the lemma that $|f(\rho_0)| \leq M$ on the interval $\rho_0 \in [\rho_*, 2\rho_*]$. Suppose now that $\rho_0 \geq 2\rho_*$. Then, by dividing the integral over two regions, we have

$$\begin{aligned}\frac{1}{\rho_0 - \rho_*}\int_{\rho_*}^{\rho_0} d(\rho)f(\rho)\,\mathrm{d}\rho &= \frac{1}{\rho_0 - \rho_*}\int_{\rho_*}^{2\rho_*} d(\rho)f(\rho)\,\mathrm{d}\rho + \frac{1}{\rho_0 - \rho_*}\int_{2\rho_*}^{\rho_0} d(\rho)f(\rho)\,\mathrm{d}\rho \\ &\leq M\rho_0^{-1} + \frac{1}{\rho_0 - \rho_*}\int_{2\rho_*}^{\rho_0} d(\rho)f(\rho)\,\mathrm{d}\rho,\end{aligned}$$

whenever $\rho_0 \geq 2\rho_*$ (as the singularity at $\rho_0 = \rho_*$ is not encountered), where again M is independent of ρ_0. It follows that, over the interval $[2\rho_*, \infty)$, the estimate (2) may be rewritten as

$$f(\rho_0) \leq M\rho_0^{-\tilde{q}} + \frac{1}{\rho_0 - \rho_*}\int_{2\rho_*}^{\rho_0} d(\rho)f(\rho)\,\mathrm{d}\rho \qquad \text{for } \rho_0 \in [2\rho_*, \infty),$$

for a larger constant M independent of ρ_0, and where $\tilde{q} = \min\{q, 1\} > 0$. Using the variant of the Grönwall Lemma from [3, Theorem 1, Section 4, Chapter XII] and the previous inequality, we obtain

$$f(\rho_0) \leq M\rho_0^{-\tilde{q}} + \frac{1}{\rho_0 - \rho_*}\int_{2\rho_*}^{\rho_0} M\rho^{-\tilde{q}}d(\rho)\exp\left(\int_{\rho}^{\rho_0}\frac{d(r)}{r - \rho_*}\,\mathrm{d}r\right)\mathrm{d}\rho \qquad \text{for } \rho_0 \in [2\rho_*, \infty), \tag{5}$$

and the integral inside the exponential is bounded as

$$\left|\int_{\rho}^{\rho_0}\frac{d(r)}{r - \rho_*}\,\mathrm{d}r\right| \leq \int_{\rho}^{\rho_0}\frac{1}{r - \rho_*}\,\mathrm{d}r + \int_{\rho}^{\rho_0}\frac{|d_*(r) - 1|}{r - \rho_*}\,\mathrm{d}r + \int_{\rho}^{\rho_0}\frac{|d_*(r) - d(r)|}{r - \rho_*}\,\mathrm{d}r, \tag{6}$$

and we emphasise that $\rho \geq 2\rho_*$ in the above. Using the decay bound provided by Lemma 3.12 of Chapter 3, it follows that there exists a positive constant M, independent of r, such that

$$\frac{|d_*(r) - d(r)|}{r - \rho_*} \leq Mr^{-\alpha-1} \qquad \forall r \geq 2\rho_*.$$

Similarly, again from Lemma 3.12 Chapter 3, $d_*(r) - 1 = \rho_*/r$, and so

$$\frac{|d_*(r) - 1|}{r - \rho_*} = \rho_*\frac{1}{r(r - \rho_*)} \leq Mr^{-2} \qquad \forall r \geq 2\rho_*,$$

again for some positive constant $M = M(\rho_*)$. Hence, it follows from (6) that

$$\exp\left(\int_{\rho}^{\rho_0}\frac{d(r)}{r - \rho_*}\,\mathrm{d}r\right) \leq M\left(\frac{\rho_0 - \rho_*}{\rho - \rho_*}\right) \qquad \forall \rho \geq 2\rho_*.$$

Returning to (5) and using $|d(\rho)| \leq 3$ from Lemma 3.12 of Chapter 3, we obtain

$$f(\rho_0) \leq M\left(1 + \int_{2\rho_*}^{\rho_0}\frac{\rho^{-\tilde{q}}}{\rho - \rho_*}\,\mathrm{d}\rho\right) \qquad \text{for } \rho_0 \in [2\rho_*, \infty),$$

and hence, using the finiteness of $\int_{2\rho_*}^{\infty}\rho^{-1-\tilde{q}}\,\mathrm{d}\rho$ as $\tilde{q} > 0$, we obtain $|f(\rho_0)| \leq M$ in the interval $\rho_0 \in [2\rho_*, \infty)$. Combining this estimate with the one for the region $\rho_0 \in [\rho_*, 2\rho_*]$ yields the result. ∎

Remark 3.2. If $q=0$ in Lemma 3.1, then, by following the same proof to the letter, one obtains the weaker estimate

$$|f(\rho_0)| \le M \log \rho_0 \qquad \forall \rho_0 \ge \rho_*.$$

4. Recovering Lemma 3.52 of Chapter 3

Using Lemma 3.1 after equation (3.41) in Chapter 3, we get $\|k'(\rho_0)\chi^{error}(\rho_0,\cdot)\|_{L^\infty(\mathbb{R})} \le M$, i.e.,

$$\|\chi^{error}(\rho_0,\cdot)\|_{L^\infty(\mathbb{R})} \le M\rho_0 \qquad \forall \rho \in [\rho_*,\infty),$$

and we do not obtain equation (3.42) in Chapter 3, nor do we obtain the conclusion of Lemma 3.23 in Chapter 3. Instead, we have:

Lemma 4.1 (Correction of Lemma 3.23 and Remark 3.24 of [4])**.** *There exists a positive constant M, independent of ρ, such that*

$$\|\chi^{error}(\rho,\cdot)\|_{L^\infty(\mathbb{R})} \le M\rho \qquad \forall \rho \in [\rho_*,\infty). \tag{7}$$

Nonetheless, following Lemma 3.27 of Chapter 3, we observe that

$$\hat{\eta}^{error}_{uuu}(\rho,u) = 2\chi^{error}(\rho,u) \qquad \text{for } (\rho,u) \in [\rho_*,\infty)\times\mathbb{R},$$

as per equation (3.52) of the chapter. It follows from (7), using also [4, Lemma 3.17], that we have:

Lemma 4.2 (Correction of (3.53) in Lemma 3.27 of [4])**.** *There exists a positive constant M, independent of ρ, such that*

$$\|\hat{\eta}^{error}_{uuu}(\rho,\cdot)\|_{L^\infty(\mathbb{R})} + \|\hat{\eta}^{iso}_{uuu}(\rho,\cdot)\sqrt{\log\rho}\|_{L^\infty(\mathbb{R})} \le M\rho. \tag{8}$$

Returning to the representation formula (3.50) in Lemma 3.26 of Chapter 3, we set ψ to be $\frac{1}{2}s|s|$ so as to obtain the representation formula for $\hat{\eta}^{error}$, i.e.,

$$\begin{aligned} &2(\rho_0-\rho_*)k'(\rho_0)\hat{\eta}^{error}(\rho_0,u_0) = \\ &\int_{\rho_*}^{\rho_0} d(\rho)k'(\rho)\big(\hat{\eta}^{error}(\rho,u_0+k(\rho_0)-k(\rho)) + \hat{\eta}^{error}(\rho,u_0-k(\rho_0)+k(\rho))\big)\,\mathrm{d}\rho \\ &+\int_{\rho_*}^{\rho_0}(\rho-\rho_*)\big(k'(\rho)^2-\frac{\kappa_2}{\rho^2}\big)\big(\hat{\eta}^{iso}_u(\rho,u_0+k(\rho_0)-k(\rho)) - \hat{\eta}^{iso}_u(\rho,u_0-k(\rho_0)+k(\rho))\big)\,\mathrm{d}\rho. \end{aligned} \tag{9}$$

Differentiating the above twice with respect to u_0, we obtain an identical formula for $\hat{\eta}^{error}_{uu}$ involving $\hat{\eta}^{iso}_{uuu}$ on the right-hand side. Then, using the estimate (8) on $\|\hat{\eta}^{iso}_{uuu}(\rho,\cdot)\|_{L^\infty(\mathbb{R})}$ and Corollary 3.10 of Chapter 3, we obtain

$$\|k'(\rho_0)\hat{\eta}^{error}_{uu}(\rho_0,\cdot)\|_{L^\infty(\mathbb{R})} \le \frac{1}{\rho_0-\rho_*}\int_{\rho_*}^{\rho_0} d(\rho)\|k'(\rho)\hat{\eta}^{error}_{uu}(\rho,\cdot)\|_{L^\infty(\mathbb{R})}\,\mathrm{d}\rho + \frac{M}{\rho_0-\rho_*}\int_{\rho_*}^{\rho_0}\rho^{-\alpha}\,\mathrm{d}\rho, \tag{10}$$

whence, using the fact that $\frac{1}{\rho_0-\rho_*}\int_{\rho_*}^{\rho_0}\rho^{-\alpha}\,\mathrm{d}\rho \le M\rho_0^{-\alpha}$, a direct application of Lemma 3.1 yields

$$\|k'(\rho_0)\hat{\eta}^{error}_{uu}(\rho_0,\cdot)\|_{L^\infty(\mathbb{R})} \le M, \text{ i.e.,} \quad \|\hat{\eta}^{error}_{uu}(\rho_0,\cdot)\|_{L^\infty(\mathbb{R})} \le M\rho_0 \qquad \forall \rho_0 \ge \rho_*.$$

In turn, using the chain rule with $m=\rho u$, we immediately obtain

$$\|\rho_0\hat{\eta}^{erro}_{mm}(\rho_0,\cdot)\|_{L^\infty(\mathbb{R})} + \|\hat{\eta}^{error}_{mu}(\rho_0,\cdot)\|_{L^\infty(\mathbb{R})} \le M \qquad \forall \rho_0 \ge \rho_*. \tag{11}$$

We may now repeat this argument identically to estimate $\hat{\eta}^{error}_u$, with the representation formula given by differentiating (9) only once with respect to u_0. In turn, now using the estimate $\|\hat{\eta}^{iso}_{uu}(\rho,\cdot)\|_{L^\infty(\mathbb{R})} \le M\rho$ (*cf.* Lemma 2.5 in Chapter 2) to bound the right-hand side, we obtain

$$\|\hat{\eta}^{error}_u(\rho_0,\cdot)\|_{L^\infty(\mathbb{R})} \le M\rho_0, \text{ i.e.,} \quad \|\hat{\eta}^{error}_m(\rho_0,\cdot)\|_{L^\infty(\mathbb{R})} \le M \qquad \forall \rho_0 \ge \rho_*. \tag{12}$$

The same argument does not work immediately for $\hat{\eta}^{error}$. Returning to the representation formula (9), we rewrite, using the fundamental theorem of calculus, the final term on the right-hand side as being equal to

$$\int_{\rho_*}^{\rho_0}(\rho-\rho_*)\big(k'(\rho)^2-\frac{\kappa_2}{\rho^2}\big)\Big(\int_{u_0-(k(\rho_0)-k(\rho))}^{u_0+k(\rho_0)-k(\rho)}\hat{\eta}^{iso}_{uu}(\rho,y)\,\mathrm{d}y\Big)\,\mathrm{d}\rho.$$

In turn, using again the estimate $\|\hat{\eta}^{iso}_{uu}(\rho,\cdot)\|_{L^\infty(\mathbb{R})} \le M\rho$ (*cf.* Lemma 2.5 in Chapter 2), we get from (9):

$$\|k'(\rho_0)\hat{\eta}^{error}(\rho_0,\cdot)\|_{L^\infty(\mathbb{R})} \le \frac{1}{\rho_0-\rho_*}\int_{\rho_*}^{\rho_0} d(\rho)\|k'(\rho)\hat{\eta}^{error}(\rho,\cdot)\|_{L^\infty(\mathbb{R})}\,\mathrm{d}\rho + \frac{Mk(\rho_0)}{\rho_0-\rho_*}\int_{\rho_*}^{\rho_0}\rho^{-\alpha}\,\mathrm{d}\rho.$$

Applying Lemma 3.1, we obtain

$$\|\hat{\eta}^{error}(\rho_0,\cdot)\|_{L^\infty(\mathbb{R})} \le M\rho_0 \qquad \forall \rho_0 \ge \rho_*, \tag{13}$$

from which it follows, since $\eta^*(\rho,u)=\frac{1}{2}\rho u^2+\rho e(\rho)$ with $\frac{e(\rho)}{\log\rho}\to$ const. >0 as $\rho\to\infty$, that

$$\|\hat{\eta}^{error}(\rho_0,\cdot)\|_{L^\infty(\mathbb{R})}\le M\eta^*(\rho_0,u_0)\qquad\forall\rho_0\ge\rho_*. \tag{14}$$

Finally, for the bound concerning the mixed partial derivative $\hat{\eta}^{error}_{m\rho}$, we follow the proof of Lemma 3.45 in Chapter 3, and obtain equation (3.98) in Chapter 3, i.e.,

$$\begin{aligned}
&2(\rho_0-\rho_*)\rho_0 k'(\rho_0)\hat{\eta}^{error}_{m\rho}(\rho_0,u_0)=\\
&\quad-\partial_{\rho_0}\big(2(\rho_0-\rho_*)\rho_0k'(\rho_0)\big)\hat{\eta}^{error}_{m}(\rho_0,u_0)\\
&\quad+k'(\rho_0)\int_{\rho_*}^{\rho_0}d(\rho)\rho k'(\rho)\big(\hat{\eta}^{error}_{mu}(\rho,u_0+k(\rho_0)-k(\rho))-\hat{\eta}^{error}_{mu}(\rho,u_0-k(\rho_0)+k(\rho))\big)\,\mathrm{d}\rho\\
&\quad+2d(\rho_0)\rho_0k'(\rho_0)\hat{\eta}^{error}_{m}(\rho_0,u_0)\\
&\quad+k'(\rho_0)\int_{\rho_*}^{\rho_0}(\rho-\rho_*)\rho\big(k'(\rho)^2-\frac{\kappa_2}{\rho^2}\big)\big(\hat{\eta}^{iso}_{muu}(\rho,u_0+k(\rho_0)-k(\rho))+\hat{\eta}^{iso}_{muu}(\rho,u_0-k(\rho_0)+k(\rho))\big)\,\mathrm{d}\rho.
\end{aligned}$$

Hence, proceeding as per the original proof of Lemma 3.45 of Chapter 3, using the bounds provided by (8) and (11)-(12) to estimate the right-hand side of the previous equality, we get

$$(\rho_0-\rho_*)\rho_0k'(\rho_0)\|\hat{\eta}^{error}_{m\rho}(\rho_0,\cdot)\|_{L^\infty(\mathbb{R})}\le M\rho_0^{-2}+M+M\rho_0^{-\alpha},$$

and therefore

$$\|\hat{\eta}^{error}_{m\rho}(\rho_0,\cdot)\|_{L^\infty(\mathbb{R})}\le M\rho_0^{-1}\qquad\forall\rho_0\ge\rho_*. \tag{15}$$

Collating the results of (11)-(15), we obtain the following.

Lemma 4.3 (Correction of Lemmas 3.28, 3.30, 3.44, 3.45 and Corollary 3.46 of [4]). *There exists a positive constant M, independent of ρ, such that*

$$\|\hat{\eta}^{error}_{m}(\rho,\cdot)\|_{L^\infty(\mathbb{R})}+\|\hat{\eta}^{error}_{mu}(\rho,\cdot)\|_{L^\infty(\mathbb{R})}+\|\rho\hat{\eta}^{error}_{mm}(\rho,\cdot)\|_{L^\infty(\mathbb{R})}+\|\rho\hat{\eta}^{error}_{m\rho}(\rho,\cdot)\|_{L^\infty(\mathbb{R})}\le M\qquad\forall\rho\ge\rho_*,$$

and

$$\|\hat{\eta}^{error}(\rho,\cdot)\|_{L^\infty(\mathbb{R})}\le M\eta^*(\rho,u)\qquad\forall\rho\ge\rho_*.$$

Remark 4.4. Note that the final estimate in Lemma 4.3 proves the validity of Lemma 3.29 in Chapter 3, although the proof given in the chapter uses the original statement of Lemma 3.23 in Chapter 3, which had a mistake in it.

Next, recall from Lemma 3.39 of Chapter 3 the representation formula for $\hat{H}^{error}$, i.e.,

$$\begin{aligned}
&2(\rho_0-\rho_*)k'(\rho_0)\hat{H}^{error}(\rho_0,u_0)=\\
&\qquad\int_{\rho_*}^{\rho_0}d(\rho)k'(\rho)\big(\hat{H}^{error}(\rho,u_0+k(\rho_0)-k(\rho))+\hat{H}^{error}(\rho,u_0-k(\rho_0)+k(\rho))\big)\,\mathrm{d}\rho\\
&\qquad+\int_{\rho_*}^{\rho_0}(\rho-\rho_*)\big(k'(\rho)^2-\frac{\kappa_2}{\rho^2}\big)\big(\hat{H}^{iso}_{u}(\rho,u_0+k(\rho_0)-k(\rho))-\hat{H}^{iso}_{u}(\rho,u_0-k(\rho_0)+k(\rho))\big)\,\mathrm{d}\rho\\
&\qquad+\int_{\rho_*}^{\rho_0}\frac{p''(\rho)}{\rho}(\rho-\rho_*)\big(\hat{\eta}(\rho,u_0+k(\rho_0)-k(\rho))-\hat{\eta}(\rho,u_0-k(\rho_0)+k(\rho))\big)\,\mathrm{d}\rho,
\end{aligned}$$

where we emphasise that in the final term on the right-hand side, $\hat{\eta}=\hat{\eta}^{iso}+\hat{\eta}^{error}$. Using the bounds provided by Corollary 3.10 and Lemma 3.35 of Chapter 3 for the bound on $\|\hat{H}^{iso}_{uuu}(\rho,\cdot)\|_{L^\infty(\mathbb{R})}$, Lemma 2.5 of Chapter 2 for the bound on $\|\hat{\eta}^{iso}_{uu}(\rho,\cdot)\|_{L^\infty(\mathbb{R})}$ and Lemma 4.3 for the bound on $\|\hat{\eta}^{error}_{uu}(\rho,\cdot)\|_{L^\infty(\mathbb{R})}$, we estimate the right-hand side by proceeding exactly as per the proof of the estimates (11)-(12). We obtain:

$$\|\hat{H}^{error}_{uu}(\rho,\cdot)\|_{L^\infty(\mathbb{R})}\le M\rho\qquad\forall\rho\ge\rho_*. \tag{16}$$

We cannot repeat the same argument to bound $\hat{H}^{error}$. Instead, using the same strategy as in the proof of Lemma 3.43 of Chapter 3, by integrating the wave equation for $\hat{H}^{error}_{\rho\rho}$ twice with respect to ρ and using the zero initial data posed at ρ_*, we obtain the bound

$$|\hat{H}^{error}(\rho,u)|\le M\big(\rho|u|+\rho+\rho\log\rho\big)\qquad\forall\rho\ge\rho_*. \tag{17}$$

To summarise, we have:

Lemma 4.5 (Correction of Lemmas 3.41-3.43 of [4])**.** *There exists a positive constant M, independent of ρ, such that*

$$\|\hat{H}^{error}_{uu}(\rho,\cdot)\|_{L^\infty(\mathbb{R})} \leq M\rho \qquad \forall \rho \geq \rho_*,$$

and

$$|\hat{H}^{error}(\rho,u)| \leq M\big(\rho|u| + \rho + \rho\log\rho\big) \qquad \forall(\rho,u)\in[\rho_*,\infty)\times\mathbb{R}.$$

By combining with the estimate (13) to bound $u\hat{\eta}^{error}$, we immediately deduce:

Corollary 4.6 (Correction of Corollary 3.47 of [4])**.** *There exists a positive constant M, independent of ρ, such that*

$$|\hat{q}^{error}(\rho,u)| \leq M\big(\rho|u|^2 + \rho + \rho\log\rho\big) \qquad \forall(\rho,u)\in[\rho_*,\infty)\times\mathbb{R}.$$

Corollary 3.48 of Chapter 3 then follows from the above lemma, exactly as in Chapter 3. Hence, collating the results from Lemmas 4.3 and 4.5, we recover Lemma 4.1 of Chapter 3, with the exception of the decay bound on $\hat{\eta}_{mu}$. The boundedness of $\hat{\eta}_{mu}$ by a constant established in (12) is sufficient for all other parts of the chapter; as it did already in Chapter 2, and in the earlier work [2]. Nevertheless, the approach of the next section will show that we can also recover this decay bound (*cf.* Remark 5.2).

5. Recovering Lemma 3.60 of Chapter 3

By direct computation using the formula (3.48) of Chapter 3 and the estimate on the perturbation kernel provided by equation (7) in Lemma 4.1, we have

$$\|\eta^{error,\psi}(\rho,\cdot)\|_{L^\infty(\mathbb{R})}+\|\eta^{error,\psi}_u(\rho,\cdot)\|_{L^\infty(\mathbb{R})}+\|\eta^{error,\psi}_{uu}(\rho,\cdot)\|_{L^\infty(\mathbb{R})}+\|\eta^{error,\psi}_{uuu}(\rho,\cdot)\|_{L^\infty(\mathbb{R})} \leq M\rho \quad \forall\rho\geq\rho_*,$$

for any $\psi\in C^3_c(\mathbb{R})$. Likewise, considering again (3.48) of Chapter 3 and the bound provided by Lemma 3.17 of Chapter 3, we obtain

$$\|\eta^{iso,\psi}(\rho,\cdot)\|_{L^\infty(\mathbb{R})}+\|\eta^{iso,\psi}_u(\rho,\cdot)\|_{L^\infty(\mathbb{R})}+\|\eta^{iso,\psi}_{uu}(\rho,\cdot)\|_{L^\infty(\mathbb{R})}+\|\eta^{iso,\psi}_{uuu}(\rho,\cdot)\|_{L^\infty(\mathbb{R})} \leq M\rho \qquad \forall\rho\geq\rho_*.$$

In turn, we observe that, by summing the contributions from $\eta^{error,\psi}$ and $\eta^{iso,\psi}$, we already have

$$\|\eta^{\psi}(\rho,\cdot)\|_{L^\infty(\mathbb{R})}+\|\eta^{\psi}_u(\rho,\cdot)\|_{L^\infty(\mathbb{R})}+\|\eta^{\psi}_{uu}(\rho,\cdot)\|_{L^\infty(\mathbb{R})}+\|\eta^{\psi}_{uuu}(\rho,\cdot)\|_{L^\infty(\mathbb{R})} \leq M\rho \qquad \forall\rho\geq\rho_*, \tag{18}$$

for any $\psi\in C^3_c(\mathbb{R})$. With the help of a new representation formula obtained by an application of the Duhamel principle, we refine these latter estimates and recover Lemma 3.60 of Chapter 3.

By convolving the wave equation (3.11) of Chapter 3 with $\psi\in C^3_c(\mathbb{R})$ and rewriting the coefficient on the left-hand side, we obtain

$$\begin{cases} \eta^{error,\psi}_{\rho\rho}(\rho,u) - \dfrac{\kappa_2}{\rho^2}\eta^{error,\psi}_{uu}(\rho,u) = \big(k'(\rho)^2 - \dfrac{\kappa_2}{\rho^2}\big)\eta^{\psi}_{uu}(\rho,u) & \text{for } (\rho,u)\in[\rho_*,\infty)\times\mathbb{R},\\ \eta^{error,\psi}(\rho_*,u)=0 & \text{for } u\in\mathbb{R},\\ \eta^{error,\psi}_{\rho}(\rho_*,u)=0 & \text{for } u\in\mathbb{R}, \end{cases} \tag{19}$$

where we emphasise that the term on the right-hand side is $\eta^{\psi}_{uu}=\eta^{iso,\psi}_{uu}+\eta^{error,\psi}_{uu}$. Using the kernel $\chi^\sharp$ from Definition 3.3 of Chapter 3 and the Duhamel principle, we obtain from (19) the (implicit) representation formula

$$\eta^{error,\psi}(\rho,u) = \int_{\rho_*}^{\rho}\int_{\mathbb{R}} r\chi^\sharp\big(\frac{\rho}{r},\frac{u-s}{\sqrt{\kappa_2}}\big)\big(k'(r)^2-\frac{\kappa_2}{r^2}\big)\eta^{\psi}_{uu}(r,s)\,\mathrm{d}s\,\mathrm{d}r \qquad \text{for } (\rho,u)\in[\rho_*,\infty)\times\mathbb{R}.$$

Using the explicit representation of $\chi^\sharp$ (which was presented in Theorem 2.3 of Chapter 2) and a trigonometric change of variables, we obtain

$$\eta^{error,\psi}(\rho,u) = \sqrt{\rho}\frac{\sqrt{\kappa_2}}{2}\int_{\rho_*}^{\rho}\sqrt{r}\big(k'(r)^2-\frac{\kappa_2}{r^2}\big)\log(\frac{\rho}{r})\Big(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\eta^{\psi}_{uu}(r,u-\sqrt{\kappa_2}\log(\frac{\rho}{r})\sin\theta)I_0\big(\frac{\log(\frac{\rho}{r})}{2}\cos\theta\big)\cos\theta\,\mathrm{d}\theta\Big)\,\mathrm{d}r,$$

for $(\rho,u)\in[\rho_*,\infty)\times\mathbb{R}$, where I_0 is the zeroth modified Bessel function. Integrating by parts once in the innermost integral, we obtain

$$\begin{aligned}\eta^{error,\psi}(\rho,u) =& \frac{\sqrt{\rho}}{2}\int_{\rho_*}^{\rho}\sqrt{r}\big(k'(r)^2-\frac{\kappa_2}{r^2}\big)\big(\eta^{\psi}_u(r,u+\sqrt{\kappa_2}\log(\frac{\rho}{r}))-\eta^{\psi}_u(r,u-\sqrt{\kappa_2}\log(\frac{\rho}{r}))\big)\,\mathrm{d}r\\ &+\frac{\sqrt{\rho}}{2}\int_{\rho_*}^{\rho}\sqrt{r}\big(k'(r)^2-\frac{\kappa_2}{r^2}\big)\Big(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\eta^{\psi}_u(r,u-\sqrt{\kappa_2}\log(\frac{\rho}{r})\sin\theta)\partial_\theta\big(I_0\big(\frac{\log(\frac{\rho}{r})}{2}\cos\theta\big)\big)\,\mathrm{d}\theta\Big)\,\mathrm{d}r.\end{aligned} \tag{20}$$

Using (18) and Corollary 3.10 of Chapter 3, we estimate the right-hand side of the previous formula,

$$|\eta^{error,\psi}(\rho,u)| \leq M\sqrt{\rho}\int_{\rho_*}^{\rho} r^{-\frac{1}{2}-\alpha}\,\mathrm{d}r + M\sqrt{\rho}\int_{\rho_*}^{\rho} r^{-\frac{1}{2}-\alpha}\Big(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \big|\partial_\theta\big(I_0(\frac{\log(\frac{\rho}{r})}{2}\cos\theta)\big|\,\mathrm{d}\theta\Big)\,\mathrm{d}r. \tag{21}$$

We now observe that

$$\partial_\theta\big(I_0(\frac{\log(\frac{\rho}{r})}{2}\cos\theta\big) = -\frac{\log(\frac{\rho}{r})}{2}\sin\theta I_1\big(\frac{\log(\frac{\rho}{r})}{2}\cos\theta\big),$$

and the latter is an odd function with fixed sign on the interval $[0,\frac{\pi}{2}]$. Thus,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \big|\partial_\theta\big(I_0(\frac{\log(\frac{\rho}{r})}{2}\cos\theta\big)\big|\,\mathrm{d}\theta = 2\int_0^{\frac{\pi}{2}} \frac{\log(\frac{\rho}{r})}{2}\sin\theta I_1\big(\frac{\log(\frac{\rho}{r})}{2}\cos\theta\big)\,\mathrm{d}\theta = 2\big(I_0(\frac{\log(\frac{\rho}{r})}{2}) - 1\big).$$

Since the logarithm and I_0 are increasing functions, it follows that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \big|\partial_\theta\big(I_0(\frac{\log(\frac{\rho}{r})}{2}\cos\theta\big)\big|\,\mathrm{d}\theta \leq 2\big(I_0(\frac{\log(\frac{\rho}{\rho_*})}{2}) - 1\big) \leq \frac{M\rho}{\sqrt{\log(\rho/\rho_*+1)}},$$

where the final inequality follows from the asymptotic properties of the modified Bessel functions (*cf.* Lemma A.6 in Appendix A). Using this estimate in (21), we obtain

$$|\eta^{error,\psi}(\rho,u)| \leq \frac{M\rho}{\sqrt{\log(\rho/\rho_*+1)}} \qquad \text{for } (\rho,u)\in[\rho_*,\infty)\times\mathbb{R}, \tag{22}$$

where we made use of $\alpha > \frac{1}{2}$. We note that, by differentiating the formula (20) with respect to u, we obtain analogous formulas for $\eta_u^{error,\psi}$ and $\eta_{uu}^{error,\psi}$. Then, using the bounds provided by (18) to estimate as we did with (21), we also recover

$$|\eta_u^{error,\psi}(\rho,u)| + |\eta_{uu}^{error,\psi}(\rho,u)| \leq \frac{M\rho}{\sqrt{\log(\rho/\rho_*+1)}} \qquad \text{for } (\rho,u)\in[\rho_*,\infty)\times\mathbb{R}. \tag{23}$$

We have therefore proved:

Lemma 5.1 (Correction of (3.116) and (3.117) in Lemma 3.54 of [4])**.** *There exists a positive constant M, independent of ρ, such that*

$$|\eta^{error,\psi}(\rho,u)| + |\eta_u^{error,\psi}(\rho,u)| + |\eta_{uu}^{error,\psi}(\rho,u)| \leq \frac{M\rho}{\sqrt{\log(\rho/\rho_*+1)}} \qquad \textit{for } (\rho,u)\in[\rho_*,\infty)\times\mathbb{R}.$$

Remark 5.2. Considering the special entropy generated by $\psi(s) = \frac{1}{2}s|s|$, we note that $\hat{\eta}^{error}$ satisfies the same equation as (19), with the exception that the term on the right-hand side is $\hat{\eta}_{uu}$ instead of η_{uu}^{ψ}. Also, using the compact supports of χ^{iso} and χ^{error} and the formula (3.48) of Chapter 3, we get

$$\|\hat{\eta}(r,\cdot)\|_{L^\infty(\mathbb{R})} + \|\hat{\eta}_u(r,\cdot)\|_{L^\infty(\mathbb{R})} + \|\hat{\eta}_{uu}(r,\cdot)\|_{L^\infty(\mathbb{R})} + \|\hat{\eta}_{uuu}(r,\cdot)\|_{L^\infty(\mathbb{R})} \leq Mr(\log r)^3 \qquad \forall r\geq\rho_*.$$

Using the above bounds to estimate the analogous representation formula to (20) for $\hat{\eta}$, we obtain

$$|\hat{\eta}^{error}(\rho,u)| + |\hat{\eta}_u^{error}(\rho,u)| + |\hat{\eta}_{uu}^{error}(\rho,u)| \leq \frac{M\rho}{\sqrt{\log(\rho/\rho_*+1)}} \qquad \text{for } (\rho,u)\in[\rho_*,\infty)\times\mathbb{R}.$$

Thus, we have recovered Lemma 3.52 of Chapter 3 in full.

Using the wave equation for the perturbation of the entropy-flux kernel $H^{error,\psi}$, i.e., equation (3.122) in Chapter 3, we obtain an analogous representation formula for $H^{error,\psi}$, namely

$$\begin{aligned} &H^{error,\psi}(\rho,u) = \\ &\sqrt{\rho}\frac{\sqrt{\kappa_2}}{2}\int_{\rho_*}^{\rho}\sqrt{r}\big(k'(r)^2 - \frac{\kappa_2}{r^2}\big)\log(\frac{\rho}{r})\Big(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} H_{uu}^{iso,\psi}(r,u-\sqrt{\kappa_2}\log(\frac{\rho}{r})\sin\theta)I_0\big(\frac{\log(\frac{\rho}{r})}{2}\cos\theta\big)\cos\theta\,\mathrm{d}\theta\Big)\,\mathrm{d}r \\ &\quad + \sqrt{\rho}\frac{\sqrt{\kappa_2}}{2}\int_{\rho_*}^{\rho}\sqrt{r}\frac{p''(r)}{r}\log(\frac{\rho}{r})\Big(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \eta_u^{\psi}(r,u-\sqrt{\kappa_2}\log(\frac{\rho}{r})\sin\theta)I_0\big(\frac{\log(\frac{\rho}{r})}{2}\cos\theta\big)\cos\theta\,\mathrm{d}\theta\Big)\,\mathrm{d}r. \end{aligned}$$

Then, using Lemma 3.34 of Chapter 3 along with the formula (3.67) in Chapter 3, we see that

$$|H_{uu}^{iso,\psi}(r,u)| \leq \int_{\mathbb{R}} |h^{iso}(r,s)||\psi''(u-s)|\,\mathrm{d}s \leq M\frac{r}{\sqrt{\log(r/\rho_*+1)}} \qquad \forall r\geq\rho_*,$$

for any $\psi\in C_c^3(\mathbb{R})$. Using also the estimates of (18), we get

$$|H^{error,\psi}(\rho,u)| \leq M\sqrt{\rho}\int_{\rho_*}^{\rho} r^{-\frac{1}{2}-\alpha}\sqrt{\log r}\log(\frac{\rho}{r})\Big(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \big|I_0\big(\frac{\log(\frac{\rho}{r})}{2}\cos\theta\big)\cos\theta\big|\,\mathrm{d}\theta\Big)\,\mathrm{d}r. \tag{24}$$

Due to the non-negativity and the parity of the integrand, using also Lemma A.1 in Appendix A,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \big|I_0\big(\frac{\log(\frac{\rho}{r})}{2}\cos\theta\big)\cos\theta\big|\,\mathrm{d}\theta = 2\int_0^{\frac{\pi}{2}} I_0\big(\frac{\log(\frac{\rho}{r})}{2}\cos\theta\big)\cos\theta\,\mathrm{d}\theta = \frac{4}{\log(\frac{\rho}{r})}\sinh\big(\frac{\log(\frac{\rho}{r})}{2}\big).$$

It therefore follows from the above and (24), using also $\alpha > \frac{1}{2}$, that we have:

Lemma 5.3 (Correction of Lemma 3.59 of [4])**.** *There exists a positive constant M, independent of ρ, such that*

$$|H^{error,\psi}(\rho,u)| \le M\rho \qquad \textit{for } (\rho,u)\in[\rho_*,\infty)\times\mathbb{R}. \tag{25}$$

We consider the term $u\eta^{error,\psi}$. From the representation formula (20), we have

$$\begin{aligned} u\eta^{error,\psi}(\rho,u) = &\frac{\sqrt{\rho}}{2}\int_{\rho_*}^{\rho}\sqrt{r}\big(k'(r)^2-\frac{\kappa_2}{r^2}\big)\big(u\eta_u^{\psi}(r,u+\sqrt{\kappa_2}\log(\frac{\rho}{r})) - u\eta_u^{\psi}(r,u-\sqrt{\kappa_2}\log(\frac{\rho}{r}))\big)\,\mathrm{d}r \\ &+\frac{\sqrt{\rho}}{2}\int_{\rho_*}^{\rho}\sqrt{r}\big(k'(r)^2-\frac{\kappa_2}{r^2}\big)\Big(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u\eta_u^{\psi}(r,u-\sqrt{\kappa_2}\log(\frac{\rho}{r})\sin\theta)\partial_\theta\big(I_0\big(\frac{\log(\frac{\rho}{r})}{2}\cos\theta\big)\big)\,\mathrm{d}\theta\Big)\,\mathrm{d}r. \end{aligned}$$

In view of the bounds of Lemma 5.1 and the compact support of ψ and the entropy kernel (*cf.* equation (3.33) in Chapter 3), we have that

$$|u\eta_u^{\psi}(r,u\pm\sqrt{\kappa_2}\log(\frac{\rho}{r}))| \le Mr\frac{(1+\log r+\log(\frac{\rho}{r}))}{\sqrt{\log(r/\rho_*+1)}} \qquad \forall r\ge\rho_*,$$

where M also depends on the endpoints of the support of ψ, and we used Corollary 3.11 of Chapter 3 for the bound $k(r)\le M\log r$. Similarly,

$$|u\eta_u^{\psi}(r,u-\sqrt{\kappa_2}\log(\frac{\rho}{r})\sin\theta)| \le Mr\frac{(1+\log r+\log(\frac{\rho}{r})|\sin\theta|)}{\sqrt{\log(r/\rho_*+1)}} \qquad \forall r\ge\rho_*.$$

Note that $1+\log r\le M\log r$ for some $M=M(\rho_*)$, for all $r\ge\rho_*$. Returning to the representation formula and expanding the θ-derivative term,

$$\begin{aligned} |u\eta^{error,\psi}(\rho,u)| \le& M\sqrt{\rho}\log\rho\int_{\rho_*}^{\rho} r^{-\frac{1}{2}-\alpha}\log r\,\mathrm{d}r \\ &+M\sqrt{\rho}\int_{\rho_*}^{\rho} r^{-\frac{1}{2}-\alpha}\Big(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\log r+\log(\frac{\rho}{r})|\sin\theta|)\log(\frac{\rho}{r})|\sin\theta I_1\big(\frac{\log(\frac{\rho}{r})}{2}\cos\theta\big)|\,\mathrm{d}\theta\Big)\,\mathrm{d}r \\ \le& M\sqrt{\rho}\log\rho + M\sqrt{\rho}\int_{\rho_*}^{\rho} r^{-\frac{1}{2}-\alpha}\log r\Big(\int_0^{\frac{\pi}{2}}\log(\frac{\rho}{r})\sin\theta I_1\big(\frac{\log(\frac{\rho}{r})}{2}\cos\theta\big)\,\mathrm{d}\theta\Big)\,\mathrm{d}r \\ &+M\sqrt{\rho}\int_{\rho_*}^{\rho} r^{-\frac{1}{2}-\alpha}(\log(\frac{\rho}{r}))^2\Big(\int_0^{\frac{\pi}{2}} I_1\big(\frac{\log(\frac{\rho}{r})}{2}\cos\theta\big)\sin^2\theta\,\mathrm{d}\theta\Big)\,\mathrm{d}r \\ \le& M\sqrt{\rho}\log\rho + M\rho, \end{aligned}$$

where we used $\alpha>\frac{1}{2}$, the non-negativity of the modified Bessel functions and of sine over the interval $[0,\frac{\pi}{2}]$ along with the parity of these factions, and employed Lemma A.2 of Appendix A to evaluate the Bessel integrals in order to obtain the final line. By combining with Lemma 5.3, we have therefore shown:

Lemma 5.4 (Correction of Corollary 3.55 of [4])**.** *There exists a positive constant M, independent of ρ, such that*

$$|u\eta^{error,\psi}(\rho,u)|\le M\rho \quad \textit{for } (\rho,u)\in[\rho_*,\infty)\times\mathbb{R}, \; \textit{thus} \quad |q^{error,\psi}(\rho,u)|\le M\rho \quad \textit{for } (\rho,u)\in[\rho_*,\infty)\times\mathbb{R}.$$

Finally, the term $\eta_{mp}^{error,\psi}$ is bounded as shown in Lemma 3.56 of Chapter 3, using only the estimates of (18). Combining this with Lemmas 5.1-5.4, we have fully recovered Lemma 3.60 of Chapter 3.

6. Recovering Lemma 3.66 of Chapter 3

It remains to control the growth of the fractional derivatives of the remainder terms g_1 and g_2 in the asymptotic expansions for the entropy/entropy-flux kernels (*cf.* Lemmas 3.62 and 3.65 of Chapter 3) for large densities. Following the proof of Lemma 3.62 of Chapter 3, we arrive at the representation formula

$$\begin{aligned} k'(\rho_0)g_1(\rho_0,u_0) =& \frac{1}{2\rho_0}\int_0^{\rho_0} d(\rho)k'(\rho)\big(g_1(\rho,u_0+k(\rho_0)-k(\rho))+g_1(\rho,u_0-k(\rho_0)+k(\rho))\big)\,\mathrm{d}\rho \\ &+\frac{1}{2\rho_0}\int_0^{\rho_0}\rho A(\rho)k(\rho)^{-1}\Big(\int_{u_0-k(\rho_0)+k(\rho)}^{u_0+k(\rho_0)-k(\rho)} f_{\lambda+1}\big(\frac{y}{k(\rho)}\big)\,\mathrm{d}y\Big)\,\mathrm{d}\rho, \end{aligned} \tag{26}$$

where $f_\lambda(s) = [1-y^2]_+^\lambda$ for $\lambda > 0$. In the above, A was defined in (3.140) of Chapter 3 and, due to inequalities (3.135) and (3.138) of Chapter 3, satisfies the estimate

$$A(\rho) \leq M\rho^{-\frac{3}{2}}k(\rho)^{\lambda+3} \qquad \forall \rho \geq \rho_*. \tag{27}$$

Then, by applying the fractional derivative of order $\lambda+1$ in the variable u_0 to (26) and then applying absolute values, we obtain equation (3.144) in Chapter 3.

Remark 6.1. In the proof of Lemma 3.62 of Chapter 3, in the lines that follow equation (3.141), it is claimed that $\partial_s^\lambda f_{\lambda+1}$ is a compactly supported continuous function. This is not correct. However, it is a uniformly bounded continuous function. To see this, observe that, using the Fourier transform relation in [1, Proof of Proposition 2.4],

$$|\widehat{\partial_\lambda f_{\lambda+1}}(\xi)| = |c(\lambda+1)|\xi|^{-\frac{3}{2}}J_{\lambda+\frac{3}{2}}(|\xi|)| \leq \frac{\tilde{c}(\lambda+1)}{1+\xi^2},$$

for constants $c, \tilde{c}$ depending only on $\lambda+1$, where we used the asymptotic relations for the Bessel functions to obtain the final inequality. The latter is integrable, from which the Fourier Inversion Theorem yields the required boundedness. The same mistake is present in the proof of Lemma 3.65, and is rectified in the same way.

Then, using the estimate (27), equation (3.144) in Chapter 3 becomes

$$\|k'(\rho_0)\partial_u^{\lambda+1}g_1(\rho_0,\cdot)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\rho_0}\int_0^{\rho_0} d(\rho)\|k'(\rho)\partial_u^{\lambda+1}g_1(\rho,\cdot)\|_{L^\infty(\mathbb{R})}\,\mathrm{d}\rho + Mk(\rho_0)^2\rho_0^{-\frac{1}{2}}.$$

A direct application of Lemma 3.1 then yields

$$\|k'(\rho)\partial_u^{\lambda+1}g_1(\rho,\cdot)\|_{L^\infty(\mathbb{R})} \leq M, \quad \text{i.e.,} \quad \|\partial_u^{\lambda+1}g_1(\rho,\cdot)\|_{L^\infty(\mathbb{R})} \leq M\rho \qquad \forall \rho \geq \rho_*.$$

The same estimate for g_2 is proved in the very same way using its own evolution equation (see the proof of Lemma 3.65 in Chapter 3). Therefore, we have:

Lemma 6.2 (Correction of (3.132) of Lemma 3.62, Corollary 3.64, and (3.149) of Lemma 6.2 of [4])**.** *There exists a positive constant M, independent of ρ, such that*

$$|\partial_u^{\lambda+1}g_1(\rho,u)| + |\partial_u^{\lambda+1}g_2(\rho,u)| \leq M\rho \qquad \forall \rho > 0.$$

In turn, we deduce, in accordance with equation (2.112) in Lemma 2.73 of Chapter 2:

Lemma 6.3 (Correction of (3.159) of Lemma 3.66 of [4])**.** *There exists a positive constant M, independent of ρ, such that*

$$\|r_\chi(\rho,\cdot)\|_{L^\infty(\mathbb{R})} + \|r_\sigma(\rho,\cdot)\|_{L^\infty(\mathbb{R})} \leq M\rho \qquad \forall \rho > 0.$$

The proof of the reduction of the Young measure, and also that of the main theorem, follows as they did in Section 3.6 of Chapter 3. We emphasise that the choice $C_c^3(\mathbb{R})$ for the test functions of Section 5 (as opposed to $C_c^2(\mathbb{R})$ in the original chapter) has no consequences on the Young measure reduction.

7. Minor typos

(1) Section 1.6: the first sentence should be "Throughout the estimates of this thesis, M is a positive constant independent of ε, except in Chapter 4 where it denotes the Mach number".
(2) The quantity estimated in Corollary 2.43 is $|u\eta^\psi(\rho,u)| \leq C_\psi\rho$.
(3) In equation (2.46), the prefactors in the last two terms should be $\pm\frac{R}{4\rho^{3/2}}$.
(4) In Lemmas 3.52 and 3.60, the denominators should be written in terms of $\log(\rho/\rho_*+1)$.
(5) Equation (3.111): the numerator of the first term should have an x-derivative.
(6) In Lemma B.19, the fake mixed derivative in question is $\partial_{m\rho}\hat{\mathcal{J}}_2$.

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