Compensated compactness methods in the study of compressible fluid flow

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To my family and friends.

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Statement of Originality

I declare that the contents of this thesis are, to the best of my knowledge, original and my own work, except where otherwise indicated, cited, or commonly known, nor has any part of this thesis been submitted for a degree at another university. The work in Chapter 2 is the product of a collaboration with M.R.I. Schrecker, which has been published in the SIAM Journal on Mathematical Analysis (*cf.* [80]).

Examiners

Prof. Endre Süli	Prof. Denis Serre
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Abstract

This thesis comprises an introduction and three subsequent chapters; each focusing on a particular problem, and containing novel results that complement the existing theory. Throughout, we are concerned with the existence of solutions of the governing equations of inviscid compressible fluid mechanics; the Euler equations. As these equations form an archetypal system of conservation laws, we study them using approaches from the theory of hyperbolic differential equations. Specifically, we implement methods of compensated compactness developed by Tartar and Murat.

To begin with, in Chapter 2, we prove the existence of a finite relative energy entropy solution of the one-dimensional compressible Euler equations under the assumption of an approximately isothermal pressure law. In particular, we obtain this solution as the vanishing viscosity limit of solutions of the one-dimensional compressible Navier–Stokes equations. This procedure can in fact be carried out for a more general class of pressure laws, which we call asymptotically isothermal. This is the subject of Chapter 3, where we show the existence of a finite relative energy entropy solution of the Euler equations in this new setting.

The focus of Chapter 4 is a related two-dimensional stationary problem. Therein, we consider the existence of bounded entropy solutions of the steady compressible potential flow equations in two dimensions; the Morawetz problem for transonic flow. We provide partial results for a γ-law gas of index $\gamma \in [3, \infty)$. This involves a detailed analysis of the singular ordinary differential equations that arise when considering the Lax entropy pairs of the system in the presence of a vacuum, which had yet to be performed.

Contents

Chapter 1

Introduction

This research memoir is concerned with the mathematical theory of the mechanics of compressible fluids. In the absence of viscosity, the motion of such fluids is modelled by the Euler equations (*cf.* [32]). As is well-known, these equations can be written as a hyperbolic system of conservation laws. As such, their analysis is particularly amenable to methods stemming from the field of hyperbolic partial differential equations. We begin this introduction with a succinct overview of the theory of hyperbolic conservation laws in Section 1.1. Having introduced these elementary notions we then, in Section 1.2, reframe the Euler equations as a hyperbolic system, and introduce the necessary terminology to state the main results of Chapters 2 and 3 (*cf.* Section 1.3).

In a similar vein, the equations of steady two-dimensional potential flow can be recast as a hyperbolic system. This is the content of Section 1.4, which introduces the necessary vocabulary to state the main results of Chapter 4 (*cf.* Section 1.5).

An explanation of the notation used in later chapters is given in Section 1.6.

1.1 Hyperbolic systems of conservation laws

Many physical phenomena are modelled mathematically using balance laws. In the monodimensional setting, a general system of conservation laws may be written as

$$\begin{cases} U_t + F(U)_x = 0 & \text{for } (t,x) \in \mathbb{R}^2_+, \\ U(0,x) = U_0(x) & \text{for } x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $U : \mathbb{R}^2_+ \to \mathbb{R}^n$ is the *vector field of conserved quantities* and $F : \mathbb{R}^n \to \mathbb{R}^n$ is called the *flux*. Throughout, we adopt the convention $\mathbb{R}^2_+ = (0,\infty) \times \mathbb{R}$. Note that a more thorough treatment of the material contained in this section may be found in the encyclopaedic accounts of Dafermos (*cf.* [26]) and Serre (*cf.* [82]).

The following two definitions are key, and were introduced by Lax (*cf.* [54, 55, 56]).

Definition 1.1. We say that system (1.1) is *strictly hyperbolic* if for all $U \in \mathbb{R}^n$, the matrix $\nabla F(U)$ has n real and distinct eigenvalues $\{\lambda_j(U)\}_{j=1}^n$. For each $j \in \{1, \dots, n\}$, we denote by r_j a corresponding non-zero right eigenvector of $\nabla F(U)$; similarly l_j denotes a corresponding non-zero left eigenvector.

The notion of strict hyperbolicity is related to the system having n distinct travelling wave solutions, where we recall that n is the dimension of the vector field U of conserved quantities.

Definition 1.2. The pair (λ_k, r_k) is called *genuinely nonlinear* if, for all $U \in \mathbb{R}^n$,

$$\nabla \lambda_k(U) \cdot r_k(U) \neq 0.$$

On the other hand, we say that (λ_k, r_k) is *linearly degenerate* if, for all $U \in \mathbb{R}^n$,

$$\nabla \lambda_k(U) \cdot r_k(U) = 0.$$

Typically, one is able to track the solution U of (1.1) for small times using the *method of characteristics*. This entails finding curves in the (t, x)-plane along which particular quantities related to U remain constant; the *characteristic curves*. The constant quantities in question are called the *Riemann invariants* of system (1.1), and a precise definition is given underneath (*cf.* [76]).

Definition 1.3. We call $w^j : \mathbb{R}^2 \to \mathbb{R}$ a j-th Riemann invariant for $j \in \{1, 2\}$ if, for all $U \in \mathbb{R}^2$,

$$\nabla w^j(U) \cdot r_j(U) = 0. \tag{1.2}$$

Remark 1.4. When treating viscous approximations of (1.1), it is possible to show that the Riemann invariants satisfy parabolic equations, from which one can often deduce *invariant regions* via an application of the maximum principle (*cf.* [22]).

If the initial data U_0 and the flux are sufficiently regular, then the method of characteristics yields the existence of an equally regular solution of (1.1) up to some positive maximal time of existence, T_*. This solution is called *classical*, on the interval $[0, T_*)$, as it satisfies (1.1) as an equality between continuous functions on this interval. If $T_* < \infty$, the solution of (1.1) becomes discontinuous beyond this time; a phenomenon commonly called *breakdown of classical solutions*. For this reason, when investigating global existence of solutions, one must rely on a weaker notion of solution, which does not require so much differentiability. For instance, one may ask that, for all test functions $\phi \in \mathcal{D}(\overline{\mathbb{R}^2_+}; \mathbb{R}^n)$, there holds

$$\int_0^\infty \int_{\mathbb{R}} \big(U(t,x) \cdot \phi_t(t,x) + F(U(t,x)) \cdot \phi_x(t,x) \big)\, dx\, dt + \int_{\mathbb{R}} U_0(x) \cdot \phi(0,x)\, dx = 0, \tag{1.3}$$

in which case we say that U is a *weak solution* of (1.1). Naturally, integration by parts shows that any global classical solution is also a weak solution in the sense of (1.3). As such, the available class of functions that are allowed to be solutions of (1.1) has been enlarged, which makes existence more easily attainable. This comes at a price, which is that uniqueness is now more difficult to show. Therefore, in order to make the problem well-posed in the sense of Hadamard (*cf.* [44]), one must impose an extra admissibility criterion; ideally, one that selects the most physically appropriate weak solution. With this in mind, we develop the notion of an entropy solution, first introduced by Kružkov in [52]. We begin by defining entropies and entropy-fluxes, and then introduce the entropy solution.

Definition 1.5. An *entropy pair* is a pair of C^2 functions $(\eta, q) : \mathbb{R}^n \to \mathbb{R}^2$ such that, for all $U \in \mathbb{R}^n$,

$$\nabla q(U) = \nabla\eta(U)\nabla F(U). \tag{1.4}$$

The function η is called an *entropy*, while q is a corresponding *entropy-flux*. When η is convex, we call the couple (η, q) a *convex entropy pair*.

Remark 1.6. Note that (1.4) and the fact that the mixed partial derivatives commute ($q_{U_1U_2} = q_{U_2U_1}$) give rise to a wave equation on η, called the *entropy equation*.

Definition 1.7. We call U an *entropy solution* of (1.1) if it satisfies (1.3) and

$$\eta(U)_t + q(U)_x \le 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^2_+) \tag{1.5}$$

holds for all convex entropy pairs, i.e.,

$$\int_0^\infty \int_{\mathbb{R}} \big(\eta(U)\phi_t + q(U)\phi_x\big)\, dx\, dt \ge 0 \quad \text{for all } \phi \in \mathcal{D}(\mathbb{R}^2_+) \text{ such that } \phi \ge 0.$$

Our objective in later chapters is to obtain entropy solutions of the Euler equations. A common tactic for finding such solutions is to study the vanishing viscosity limit of a related parabolic equation; a *viscous approximate* system. For instance, one may try to identify the solution of (1.1) as the limit of solutions U^ε of

$$\begin{cases} U^\varepsilon_t + F(U^\varepsilon)_x = \varepsilon U^\varepsilon_{xx} & \text{for } (t,x) \in \mathbb{R}^2_+, \\ U^\varepsilon(0,x) = U^\varepsilon_0(x) & \text{for } x \in \mathbb{R}, \end{cases} \tag{1.6}$$

where U_0^ε converges to U_0 in some sense to be made precise. The existence of the *viscous approximates* U^ε is typically ensured by the standard parabolic theory, along with uniform estimates on U^ε in some appropriate function space.

Having established the existence of the viscous approximates and the uniform bounds, the usual compactness theorems yield the existence of a subsequence converging in some weak sense to a limit function U^*. The main difficulty then lies in showing that U^* is a solution of the original problem (1.1), since the weak convergence alone does not allow us to pass to the limit in the term $F(U^{\varepsilon'})_x$, generally. To complete this final step, one must employ the theory of compensated compactness (*cf.* [34]), originally developed by Tartar and Murat (*cf.* [70, 86, 87, 88]), which relies on the use of Young measures to characterise the limit U^* (*cf.* [3, 90, 91]). A crucial component of this method is to generate sufficiently many entropies of system (1.1), and to estimate them appropriately. We will develop these ideas in detail in the subsequent chapters.

1.2 The Euler equations of compressible fluid flow

1.2.1 The Euler equations as a hyperbolic system

The **Euler equations** encapsulate the dynamics of an inviscid compressible fluid; such as air. When posed in one spatial dimension, they take the form

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p(\rho))_x = 0, \end{cases} \tag{1.7}$$

where the coordinates $(t, x) \in \mathbb{R}^2_+$ correspond to time and space, respectively. Here, the variables ρ and u designate the density and velocity of the fluid, and the above equations give a quantified meaning to the physical principles of conservation of mass and momentum. One supplements the above system with appropriate initial data (ρ_0, u_0). The fluid is said to be *barotropic* when the pressure p is assumed to depend solely on the density, and a precise description of the interaction between p and ρ is essential to solve these equations. A common choice is to have $p(\rho) \propto \rho^\gamma$ for some *adiabatic exponent* γ. When $\gamma > 1$ the fluid is called *polytropic*, while when $\gamma = 1$ it is called *isothermal*; in accordance with the ideal gas equation (*cf.* [23]).

Definition 1.8. For $\gamma \in (1, 3)$, we define

$$\theta := \frac{\gamma - 1}{2}, \qquad \lambda := \frac{3 - \gamma}{2(\gamma - 1)}. \tag{1.8}$$

By rewriting (1.7) in terms of the momentum $m = \rho u$, one can study the Euler equations as a first-order hyperbolic system (*cf.* [33, Section 11.3]). Indeed, we have

$$U_t + F(U)_x = 0 \qquad \text{for } (t, x) \in \mathbb{R}^2_+, \tag{1.9}$$

where $U = (\rho, m)^t$ and $F(U) = (m, \frac{m^2}{\rho} + p(\rho))^t$. By requiring that, for $\rho > 0$,

$$p'(\rho) > 0, \qquad \rho p''(\rho) + 2p'(\rho) > 0, \tag{1.10}$$

the system (1.9) is strictly hyperbolic and genuinely nonlinear, in the sense of Definitions 1.1 and 1.2, respectively. While there exist particular global existence results for classical solutions of (1.9) (*cf.* [42, 49, 81]), it is more convenient to keep to entropy solutions when considering global existence, not least because of the aforementioned breakdown of classical solutions (*cf.* [2, 83]). To this end, appealing to Definition 1.5, a pair of functions $(\eta, q) : \mathbb{R}^2_+ \to \mathbb{R}^2$ forms an entropy pair for system (1.9) provided

$$\nabla q(\rho, m) = \nabla \eta(\rho, m) \nabla \begin{pmatrix} m \\ \frac{m^2}{\rho} + p(\rho) \end{pmatrix}.$$

We will recurrently use the mechanical energy and its flux, (η^*, q^*), defined below.

Definition 1.9. We define the *mechanical energy* and its *flux* by

$$\eta^*(\rho, m) := \frac{1}{2}\frac{m^2}{\rho} + \rho e(\rho), \qquad q^*(\rho, m) := \frac{1}{2}\frac{m^3}{\rho^2} + m e(\rho) + \rho m e'(\rho), \tag{1.11}$$

where the *internal energy* is defined to be $e(\rho) := \int_0^\rho \frac{p(y)}{y^2}\, dy$.

In [30], DiPerna proved the existence of bounded entropy solutions of (1.9) in the case $\gamma = (N+2)/N$ for $N > 3$ an odd integer, assuming bounded initial data $\rho_0, u_0 \in L^\infty(\mathbb{R})$. DiPerna's underlying strategy, which is common to his other work [29], was to study the vanishing viscosity limit of an artificial viscous approximate system. He then proved that solutions of this artificial system converge to an entropy solution of the Euler equations. In order to do this, he acquired suitable estimates on the entropy pairs of the system to employ the compensated compactness framework of Tartar and Murat (*cf.* [86]). For an overview of this work, see [9]. Chen then extended the result to the case $\gamma \in (1, 5/3]$ (*cf.* [11, 12]). The next development was due to Lions, Perthame, and Tadmor (*cf.* [64]), who reframed the problem as a kinetic equation. In doing so, they proved the existence of bounded entropy solutions provided that $\gamma \in [3, \infty)$. The gap $\gamma \in (5/3, 3)$ was then closed by Lions, Perthame, and Souganidis (*cf.* [63]), using again a kinetic formulation of the problem. As an alternative to the vanishing viscosity method, one can employ Lax–Friedrichs or Godunov schemes to obtain similar results (*cf.* [8, 28, 27]).

We also note that, in the context of functions of bounded variation, the existence of global weak solutions of related problems has been investigated (see, for example, the work of Liu [65]). In this case, one proceeds by means of a Glimm scheme (*cf.* [40]). However, such an approach is often impractical, as it requires smallness assumptions on the total variation norm of the initial data.

1.2.2 Entropies of the Euler system

As mentioned previously, a key step in DiPerna's approach was to obtain appropriate estimates on the entropy pairs in order to use the theory of compensated compactness. One way of obtaining such estimates is to study the linear wave equation that the entropies must satisfy (*cf.* Remark 1.6), namely,

$$\eta_{\rho\rho} - \frac{p'(\rho)}{\rho^2}\eta_{uu} = 0 \qquad \text{for } (\rho, u) \in \mathbb{R}^2_+. \tag{1.12}$$

In fact, in view of (1.12), any *weak entropy* (an entropy vanishing at $\rho = 0$) may be generated by the convolution of a test function ψ with the fundamental solution $\chi(\rho, u, s)$ of the entropy equation vanishing at the vacuum, i.e.,

$$\eta^\psi(\rho, \rho u) = \int_{\mathbb{R}} \psi(s)\chi(\rho, u, s)\, ds, \tag{1.13}$$

provided such a χ exists (*cf.* [14, Corollary 2.1]). Given the discussion regarding the need for sufficiently many entropies at the end of Section 1.1, this motivates the study of this fundamental solution, defined precisely below (*cf.* [75, Chapters 2 and 7]).

Definition 1.10. We define the *entropy kernel* χ to be the solution of

$$\begin{cases} \chi_{\rho\rho} - k'(\rho)^2\chi_{uu} = 0 & \text{for } (\rho, u) \in \mathbb{R}^2_+, \\ \chi(0, u, s) = 0, \\ \chi_\rho(0, u, s) = \delta_{u=s}, \end{cases} \tag{1.14}$$

in the sense of distributions, where we define $k(\rho) := \int_0^\rho \frac{\sqrt{p'(y)}}{y}\, dy$.

Since $\chi(\rho, u, s) = \chi(\rho, u-s, 0)$, we write $\chi = \chi(\rho, u-s)$ in slight abuse of notation. Correspondingly, one can generate an associated entropy-flux,

$$q^\psi(\rho, \rho u) = \int_{\mathbb{R}} \psi(s)\sigma(\rho, u, s)\, ds, \tag{1.15}$$

where $\sigma(\rho, u, s)$ is an entropy-flux associated to $\chi(\rho, u, s)$, defined below.

Definition 1.11. We define the *entropy-flux kernel* σ to be the solution of

$$\begin{cases} (\sigma - u\chi)_{\rho\rho} - k'(\rho)^2(\sigma - u\chi)_{uu} = \dfrac{p''(\rho)}{\rho}\chi_u & \text{for } (\rho, u) \in \mathbb{R}^2_+, \\ (\sigma - u\chi)(0, u, s) = 0, \\ (\sigma - u\chi)_\rho(0, u, s) = 0, \end{cases} \tag{1.16}$$

in the sense of distributions, and we define $h(\rho, u, s) := \sigma(\rho, u, s) - u\chi(\rho, u, s)$. Again, $h(\rho, u, s) = h(\rho, u - s, 0)$.

An essential feature of equations (1.14) and (1.16) is that they contain coefficients that are singular at the vacuum. In view of this, establishing the global existence of such kernels is a nontrivial task. Nevertheless, in [14], Chen–LeFloch were able to overcome these difficulties for a very general class of pressure laws, dubbed *general pressure laws*, which are approximately polytropic around the vacuum. Therein, they prove the global existence of these kernels, by computing detailed asymptotics for the comportment of the kernels in the vicinity of the vacuum. This is summarised in the next theorem, which we shall make crucial use of in both Chapters 2 and 3.

Theorem 1.12 (Theorems 2.1, 2.2, and 2.3 of [14]). *Assume that the pressure* $p \in C^1([0,\infty)) \cap C^4((0,\infty))$ *satisfies the assumptions of strict hyperbolicity and genuine nonlinearity* (1.10)*, and that there exists constants* $\gamma \in (1,3)$ *and* $\kappa > 0$ *such that*

$$p(\rho) = \kappa\rho^\gamma(1 + P(\rho)) \qquad \text{for } \rho \in [0, r), \tag{1.17}$$

for some fixed $r > 0$*, and there exists a positive constant* C_r *such that*

$$|P^{(j)}(\rho)| \le C_r \rho^{2\theta - j} \qquad \text{for } \rho \in [0, r), \text{ and } j \in \{0, \dots, 4\}. \tag{1.18}$$

Then (1.14) *and* (1.16) *admit global unique Hölder continuous solutions* $\chi(\rho, u, s) = \chi(\rho, u - s)$ *and* $h(\rho, u, s) = h(\rho, u - s)$*. Additionally,*

$$\operatorname{supp}\chi(\rho,\cdot), \operatorname{supp} h(\rho,\cdot) = [-k(\rho), k(\rho)] \qquad \text{for } \rho \ge 0, \tag{1.19}$$

and χ *is positive on* $(-k(\rho), k(\rho))$*. Further, the entropy kernel admits the expansion*

$$\chi(\rho, u) = a_\sharp(\rho) G_\lambda(\rho, u) + a_\flat(\rho) G_{\lambda+1}(\rho, u) + g_1(\rho, u) \qquad \text{for } \rho \ge 0, \tag{1.20}$$

where $G_\lambda(\rho, u) := [k(\rho)^2 - u^2]_+^\lambda$*. The coefficient* $a_\sharp(\rho) > 0$ *for* $\rho > 0$*, and* $a_\sharp, a_\flat \in C^3((0,\infty))$*. On the other hand, the entropy-flux kernel admits the expansion*

$$h(\rho, u) = -u\big(b_\sharp(\rho) G_\lambda(\rho, u) + b_\flat(\rho) G_{\lambda+1}(\rho, u)\big) + g_2(\rho, u) \qquad \text{for } \rho \ge 0, \tag{1.21}$$

where the coefficient $b_\sharp(\rho) > 0$ *for* $\rho > 0$*, and* $b_\sharp, b_\flat \in C^3((0,\infty))$*. Additionally,*

$$a_\sharp(\rho) + b_\sharp(\rho) + \frac{k(\rho)^2}{\rho}\big(|a_\flat(\rho)| + |b_\flat(\rho)|\big) \le C_r \qquad \text{for } \rho \in (0, r). \tag{1.22}$$

The remainders g_1, g_2*, along with* $\partial_u^{\lambda+1} g_1, \partial_u^{\lambda+1} g_2$*, are Hölder continuous. Also,*

$$|g_1(\rho, u)| + |g_2(\rho, u)| \le C_r G_{\lambda+1+\alpha_0}(\rho, u) \quad \text{for some } \alpha_0 \in (0,1), \text{ for } \rho \in [0, r). \tag{1.23}$$

An explanation of the fractional derivative $\partial_u^{\lambda+1}$ is given in Definition 2.71. Finally, we note the following result, which also plays a crucial role in Chapters 2 and 3.

Lemma 1.13 (Proposition 2.4 of [14])**.** *Assume that the pressure satisfies the assumptions of Theorem 1.12, and define*

$$D(\rho) := a_\sharp(\rho)b_\sharp(\rho) - k(\rho)^2\big(a_\sharp(\rho)b_\flat(\rho) - a_\flat(\rho)b_\sharp(\rho)\big). \tag{1.24}$$

Then, $D(\rho) > 0$ for every $\rho \geq 0$.

1.2.3 Finite relative energy solutions of the Euler equations

As an alternative to DiPerna's original idea, instead of looking at an artificial viscous system approximating the Euler equations, one can opt to study the vanishing viscosity limit of the **compressible Navier–Stokes equations**, i.e., the limit as $\varepsilon \to 0$ of solutions $(\rho^\varepsilon, u^\varepsilon)$ of

$$\begin{cases} \rho^\varepsilon_t + (\rho^\varepsilon u^\varepsilon)_x = 0, \\ (\rho^\varepsilon u^\varepsilon)_t + (\rho^\varepsilon (u^\varepsilon)^2 + p(\rho^\varepsilon))_x = \varepsilon u^\varepsilon_{xx}. \end{cases} \tag{1.25}$$

Provided we impose appropriate initial data and a reasonable pressure law, the global existence of unique smooth solutions of (1.25) is guaranteed by [45] (*cf.* [80, Theorem 3.1]). In higher dimensions, the existence of global weak solutions of the compressible Navier–Stokes equations was studied by P.-L. Lions [61, 62], and Feireisl et al. [36, 37] (*cf.* [72]). The global well-posedness of the one-dimensional full compressible system (with the temperature equation) was studied by Kanel [50] and Kazhikhov–Shelukhin [51]; and the three-dimensional case by Tani [85] and Matsumura–Nishida [67].

Demanding that solutions of the Euler equations are obtained as inviscid limits of solutions of (1.25) is a rather popular selection criterion, due to its obvious physical motivation (*cf.* [84]). However, obtaining uniformly bounded approximate solutions from bounded initial data is not known to be possible. Indeed, the compressible Navier–Stokes equations do not straightforwardly admit invariant regions in the sense of Chueh, Conley, and Smoller (*cf.* [22]). This motivates the framework of finite relative energy entropy solutions, which we develop herein.

To begin with, we fix end-states $(\rho_\pm, u_\pm)$ such that $\lim_{x\to\pm\infty}(\rho, u) = (\rho_\pm, u_\pm)$. We choose smooth, monotone functions $(\bar\rho(x), \bar u(x))$ such that, for some $L_0 > 1$,

$$(\bar\rho(x), \bar u(x)) = \begin{cases} (\rho_-, u_-), & x \leq -L_0, \\ (\rho_+, u_+), & x \geq L_0. \end{cases}$$

These reference functions are fixed from this point onward. With this mind, and by letting $\bar m(x) := \bar\rho(x)\bar u(x)$, we give the following definition.

Definition 1.14. We define the *relative mechanical energy* with respect to $(\bar{\rho}, \bar{m})$ by the formula

$$\overline{\eta^*}(\rho, m) := \eta^*(\rho, m) - \eta^*(\bar{\rho}, \bar{m}) - \nabla\eta^*(\bar{\rho}, \bar{m}) \cdot (\rho - \bar{\rho}, m - \bar{m}). \tag{1.26}$$

Correspondingly, we define the *relative internal energy* to be

$$e^*(\rho, \bar{\rho}) := \rho e(\rho) - \bar{\rho} e(\bar{\rho}) - (\bar{\rho} e'(\bar{\rho}) + e(\bar{\rho}))(\rho - \bar{\rho}), \tag{1.27}$$

and the *total relative mechanical energy*, relative to the end-states $(\rho_\pm, u_\pm)$, to be

$$E[\rho, u](t) := \int_{\mathbb{R}} \overline{\eta^*}(\rho, \rho u)(t, x)\, dx. \tag{1.28}$$

We then say that a pair (ρ, m) with $m = \rho u$ is of *finite relative energy* if $E[\rho, u] < \infty$.

Remark 1.15. Note that $e^*(\rho, \bar{\rho}) \geq 0$ in view of the convexity of $\rho e(\rho)$, which is due to $p'(\rho) > 0$. Also, a routine calculation shows that

$$\overline{\eta^*}(\rho, m) = \frac{1}{2}\rho|u - \bar{u}|^2 + e^*(\rho, \bar{\rho}) \geq 0.$$

We now define the notion of solution considered in [16, 58, 80], which is the one that will be used throughout Chapters 2 and 3.

Definition 1.16. Given initial data $(\rho_0, u_0) \in L^1_{loc}(\mathbb{R}^2_+)$ with finite relative energy, $E[\rho_0, u_0] \leq E_0 < \infty$, we say that a pair of functions $(\rho, u) \in L^1_{loc}(\mathbb{R}^2_+)$ with $\rho \geq 0$ is a *finite relative energy entropy solution* of the Euler equations (1.7) if:

1. There exists a constant $M(E_0, t)$, monotonically increasing and continuous with respect to t, such that
$$E[\rho, u](t) \leq M(E_0, t) \qquad \text{for almost every } t \geq 0;$$
2. For any $\phi \in C^\infty_c(\overline{\mathbb{R}^2_+})$,
$$\begin{aligned} &\int_{\mathbb{R}^2_+} (\rho\phi_t + \rho u \phi_x)\, dx\, dt + \int_{\mathbb{R}} \rho_0(x)\phi(0, x)\, dx = 0, \\ &\int_{\mathbb{R}^2_+} (\rho u \phi_t + (\rho u^2 + p(\rho))\phi_x)\, dx\, dt + \int_{\mathbb{R}} \rho_0(x) u_0(x) \phi(0, x)\, dx = 0; \end{aligned} \tag{1.29}$$
3. There exists a bounded Radon measure $\mu(t, x, s)$ on $\mathbb{R}^2_+ \times \mathbb{R}$ such that
$$\mu(U \times \mathbb{R}) \leq 0 \qquad \text{for any open subset } U \subset \mathbb{R}^2_+,$$
and the corresponding entropy kernel and its flux satisfy
$$\partial_t \chi(\rho(t, x), u(t, x), s) + \partial_x \sigma(\rho(t, x), u(t, x), s) = \partial_s^2 \mu(t, x, s), \tag{1.30}$$
in the sense of distributions on $\mathbb{R}^2_+ \times \mathbb{R}$.

Our aim in Chapters 2 and 3 is to obtain such a finite relative energy entropy solution (ρ, u) as the limit of classical solutions $(\rho^\varepsilon, u^\varepsilon)$ of (1.25). In view of the adaptation of the fundamental theorem of Young measures (*cf.* [3]) due to Alberti–Müller in [1] (*cf.* [58, Proposition 2.3]), there exists a family of probability measures $\{\nu_{(t,x)}\}_{(t,x)\in\mathbb{R}^2_+}$ which characterises the limit of $\{(\rho^\varepsilon, u^\varepsilon)\}_{\varepsilon>0}$. We say that the sequence $\{(\rho^\varepsilon, u^\varepsilon)\}_{\varepsilon>0}$ *generates the Young measure* $\nu_{(t,x)}$. If one is able to show that, for almost all $(t,x) \in \mathbb{R}^2_+$, the measure $\nu_{(t,x)} \in \text{Prob}(\mathbb{R}^2_+)$ is only supported at a point, then one can deduce that $\nu_{(t,x)}$ is a Dirac mass. This procedure is called the *Young measure reduction*. In this case, the convergence $(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon) \to (\rho, \rho u)$ occurs in measure (and, up to a subsequence, almost everywhere), and one can pass to the limit in (1.25). Thus, an admissible solution is obtained as an inviscid limit of viscous solutions.

1.3 The focus of Chapters 2 and 3: Vanishing viscosity limit of the Navier–Stokes equations

We mentioned in Section 1.2.3 that an appropriate class of solutions for the compressible Navier–Stokes equations (1.25) is that of finite relative energy entropy solutions, originally studied by LeFloch–Westdickenberg in [58]. In this work, they proved the existence of finite relative energy entropy solutions of the Euler equations when $\gamma \in (1, 5/3]$. Their strategy was then adapted by Chen–Perepelitsa (*cf.* [16]), who showed the existence of finite relative energy solutions for $\gamma \in (1, 3)$, via a vanishing viscosity limit of the compressible Navier–Stokes equations. Chen–Perepelitsa also considered the inviscid limit of an artificial viscous approximate system to the Euler equations under the assumption of spherical symmetry (*cf.* [17]). Further results assuming additional symmetries can be found in [10, 18, 25, 47, 59], while other inviscid limits are studied in [39, 43, 46]. Thus far, similar approaches have not been fruitful for the isothermal system, with the exception of [48], which motivates the study of toy models that approximate the behaviour of an isothermal gas.

One such toy model is to assume that the fluid is isothermal for large densities (i.e. $p(\rho) \propto \rho$ for ρ large enough), while being approximately polytropic near the vacuum, in the sense of (1.17). This constitutive assumption on $p(\rho)$ is what we refer to as an *approximately isothermal pressure law* (*cf.* Definition 2.1) in Chapter 2. Note that the study of gases with isothermal comportment for large densities is of interest in its own right, as discussed by Nobel physicist K.S. Thorne in [89]. These are believed to model more accurately the behaviour of real gases than pure polytropic laws, especially in the context of star formation and other astrophysical phenomena.

With this in mind, we present a rough statement of the main result of Chapter 2; obtained in collaboration with M.R.I. Schrecker in [80].

Theorem 1.17 (Main Theorem of Chapter 2)**.** *Suppose that the initial data* $(\rho_0, u_0) \in L^1_{loc}(\mathbb{R}^2_+)$ *with* $\rho_0 \geq 0$ *and end-states* $(\rho_\pm, u_\pm)$ *is of finite relative energy, and suppose that the pressure function* $p(\rho)$ *behaves according to an approximately isothermal pressure law. Then there exists a sequence of regularised initial data* $(\rho_0^\varepsilon, u_0^\varepsilon)$ *such that the sequence of unique, smooth solutions* $(\rho^\varepsilon, u^\varepsilon)$ *of the Navier–Stokes equations* (1.25) *with this initial data converges as* $\varepsilon \to 0$*,* $(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon) \to (\rho, \rho u)$*, to a finite relative energy entropy solution of the Euler equations* (1.7) *with initial data* $(\rho_0, \rho_0 u_0)$*.*

Alternatively, one could ask that the gas be approximately polytropic near the vacuum, while being asymptotically isothermal in the limit of infinitely large densities (i.e. $p(\rho)/\rho = O(1)$ as $\rho \to \infty$). This latter modelling assumption is what we refer to as an *asymptotically isothermal pressure law* (*cf.* Definition 3.1), and is the focus of Chapter 3. Below, we provide a rough statement of the main result contained therein.

Theorem 1.18 (Main Theorem of Chapter 3)**.** *Suppose that the initial data* $(\rho_0, u_0) \in L^1_{loc}(\mathbb{R}^2_+)$ *with* $\rho_0 \geq 0$ *and end-states* $(\rho_\pm, u_\pm)$ *is of finite relative energy, and suppose that the pressure function* $p(\rho)$ *behaves according to an asymptotically isothermal pressure law. Then there exists a sequence of regularised initial data* $(\rho_0^\varepsilon, u_0^\varepsilon)$ *such that the sequence of unique, smooth solutions* $(\rho^\varepsilon, u^\varepsilon)$ *of the Navier–Stokes equations* (1.25) *with this initial data converges as* $\varepsilon \to 0$*,* $(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon) \to (\rho, \rho u)$*, to a finite relative energy entropy solution of the Euler equations* (1.7) *with initial data* $(\rho_0, \rho_0 u_0)$*.*

1.4 The equations of steady potential flow and the Morawetz problem

The dynamics of a steady barotropic fluid in the plane are encapsulated in the steady two-dimensional compressible Euler equations,

$$\begin{cases} \operatorname{div} \rho \mathbf{u} = 0, \\ \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) = -\nabla p, \end{cases} \tag{1.31}$$

where ρ is the density of the fluid, $\mathbf{u} = (u, v)$ is its velocity, and $p = p(\rho)$ is the pressure. In what follows, we will assume the fluid to be *irrotational* (i.e. curl-free). Including the irrotationality criterion, the system of equations (1.31) becomes

$$\begin{cases} (\rho u)_x + (\rho v)_y = 0, \\ u_y - v_x = 0, \\ (\rho \mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p. \end{cases} \tag{1.32}$$

Provided the fluid is polytropic, i.e., there holds the constitutive relation

$$p(\rho) = \frac{\rho^\gamma}{\gamma} \qquad \text{for } \gamma \in (1, \infty), \tag{1.33}$$

then the final equation of (1.32) can be integrated to give the relation

$$\frac{\rho^{\gamma-1}}{\gamma - 1} = B - \frac{1}{2}q^2, \tag{1.34}$$

where $q = |\mathbf{u}|$, and the real number B is called the *Bernoulli constant.*

For the density to remain non-negative, we must enforce $B \geq 0$. After rescaling by the positive quantity $\gamma - 1$, we thereby obtain *Bernoulli's equation*,

$$\rho(q) = \left(1 - \frac{\gamma - 1}{2}q^2\right)^{\frac{1}{\gamma-1}}. \tag{1.35}$$

Remark 1.19. The above shows that, when the flow is fast enough, we will reach the vacuum. The process by which a vacuum is formed is called *cavitation.* For this reason, we define the *cavitation speed* to be the speed at which this is achieved, namely

$$q_{cav} := \sqrt{\frac{2}{\gamma - 1}} = \theta^{-1/2}. \tag{1.36}$$

The *local speed of sound* is defined to be

$$c(\rho) := \sqrt{p'(\rho)}, \tag{1.37}$$

and the *Mach number* is defined to be the ratio of the flow speed to the local sound speed, i.e.,

$$M := q/c. \tag{1.38}$$

Additionally, we define the *critical speed* as the speed at which the local sound speed is achieved

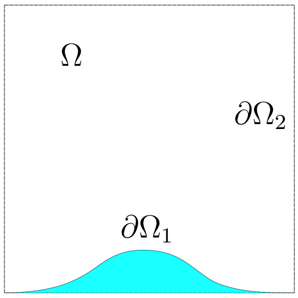
$$q_{cr} := \sqrt{\frac{2}{\gamma + 1}}, \tag{1.39}$$

and the density to which this speed corresponds as $\rho_{cr} := \rho(q_{cr})$; the *critical density.*

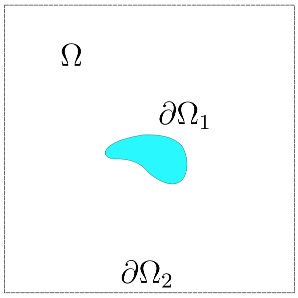
Our objective is then to find a solution $\mathbf{u} = (u, v)$ of the system

$$\begin{cases} u_y - v_x = 0, \\ (\rho(q)u)_x + (\rho(q)v)_y = 0, \end{cases} \tag{1.40}$$

in a domain $\Omega \subset \mathbb{R}^2$, where $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$, as drawn at the top of the next page, and $\rho(q)$ as prescribed by the Bernoulli law (1.35).



Domain (a)



Domain (b)

This is what we refer to as the *Morawetz problem* for transonic flow, in view of the seminal works of C.S. Morawetz on this matter (*cf.* [68, 69]). We say that the flow is *supersonic* at a point if $q > c$ at this point; if $q < c$ at this point, then the flow is *subsonic*. If some of the flow is supersonic in some regions of the domain and subsonic in others, then we say that the flow is *transonic*. The system (1.40) is hyperbolic in supersonic regions and elliptic in subsonic regions. In view of this, we say that it is a system of *mixed type*. For an overview of mixed elliptic-hyperbolic problems, see [74].

Mixed type problems are particularly prevalent in fluid mechanics and geometry. In fact, when the flow is purely subsonic, the equations of steady two-dimensional potential flow can be studied with the same complex analytic methods as minimal surfaces. Indeed, this is illustrated by the works of Bers [4, 5], written in the 1950s. Since then, the two fields have mostly evolved independently. That said, in [19], Chen, Slemrod, and Wang employed methods of compensated compactness to investigate the theory of isometric immersions, and exploited a correspondence between the equations of steady planar potential flow and the Gauß–Codazzi equations. In turn, the authors were able to establish a compensated compactness framework to study the global existence of solutions to the isometric embedding problem. The ideas from this work were then developed further in [21], where a weak L^p continuity result was established.

Returning to the potential flow equations, by introducing the angle variable $t := \arctan(v/u)$, the system (1.40) can be recast (in polar form) into

$$A \begin{pmatrix} q \\ t \end{pmatrix}_x + B \begin{pmatrix} q \\ t \end{pmatrix}_y = 0, \tag{1.41}$$

where

$$A = \begin{pmatrix} -\sin t & -q\cos t \\ \frac{c^2-q^2}{c^2 q}\cos t & -\sin t \end{pmatrix}, \qquad B = \begin{pmatrix} \cos t & -q\sin t \\ \frac{c^2-q^2}{c^2 q}\sin t & \cos t \end{pmatrix}, \tag{1.42}$$

which is strictly hyperbolic, in the sense of Definition 1.1, provided $q > c$. In [20], Chen, Slemrod, and Wang give a framework for solving the previous system when $\gamma \in [1,3)$, and proceed via a vanishing viscosity method. Specifically, by adding an artificial viscosity to the right-hand side of (1.41), they transform this equation into an elliptic system. Using fixed-point methods, and the elliptic estimates of Lieberman in [60], they show the existence of viscous approximate solutions $\mathbf{u}^\varepsilon = (u^\varepsilon, v^\varepsilon)$, and prove that these are bounded uniformly away from cavitation. Their method of proof involves a careful analysis of the Riemann invariants, $W_\pm$, of system (1.41), which shows that (provided one makes a judicious choice of boundary data) the viscous solutions are constrained to lie in "apple-shaped" regions of the phase space, as shown in [20, Figures 1-5]. These regions are in fact the areas trapped inside intersecting level set curves of W_+ and W_-.

The viscous approximate solutions are carefully constructed so that they can, in turn, be employed within the compensated compactness framework established by Morawetz in [68, 69]. At the core of Morawetz's strategy lies an intricate hodograph transformation. Indeed, in [69, Section 3], Morawetz introduces the new variable μ, defined to be the unique solution of the problem

$$\begin{cases} \mu'(\rho) = \dfrac{c^2}{q^2}, \\ \mu(\rho_{cr}) = 0. \end{cases} \tag{1.43}$$

Using this new variable, entropy pairs (Q_1, Q_2) of system (1.40) may be constructed using the *Loewner–Morawetz relations*, as in the following result.

Lemma 1.20 (Lemma 7.1 in [20]). *Let H be a solution of the linear equation*

$$H_{\mu\mu} - \left(\frac{M^2-1}{\rho^2}\right) H_{tt} = 0. \tag{1.44}$$

Provided such a solution exists, define

$$Q_1 := \rho q H_\mu \cos t - q H_t \sin t, \qquad Q_2 := \rho q H_\mu \sin t + q H_t \cos t. \tag{1.45}$$

Then (Q_1, Q_2) forms an entropy pair of system (1.40) in the (u, v)-plane.

Since the sequence of viscous approximate solutions $\mathbf{u}^\varepsilon$ constructed by Chen et al. [20] is uniformly bounded away from the vacuum, the coefficient in (1.44) is regular. Generating entropies then becomes a relatively straightforward task, especially when one considers separable solutions of (1.44) (*cf.* [20, Section 9]). With this in hand, Chen, Slemrod, and Wang were able to proceed with the compensated compactness framework of Morawetz, and proved the partial result [20, Theorem 9.1].

1.5 The focus of Chapter 4: Lax entropies of the Morawetz problem for $\gamma \geq 3$

As already mentioned in Section 1.4, one possible way of generating entropies for the potential flow system (1.40) is to look for separable solutions of (1.44), i.e., $H_n(\mu,t) = f_n(\mu)e^{\pm int}$ or $g_n(\mu)e^{\pm nt}$ (*cf.* [66]). In this case the functions f_n and g_n must solve the ordinary differential equations

$$\ddot{f}_n(\mu) + n^2\left(\frac{M^2-1}{\rho^2}\right)f_n(\mu) = 0, \qquad \ddot{g}_n(\mu) - n^2\left(\frac{M^2-1}{\rho^2}\right)g_n(\mu) = 0. \tag{1.46}$$

These separable entropies are called the *Lax entropies*. The subject matter of Chapter 4 is to study the Lax entropies generated by $(g_n)_{n\in\mathbb{N}}$ for the potential flow system when $\gamma \geq 3$, which will employ techniques discussed in Chapters 2 and 3. Note that, in some cases, one does not need to construct so many entropies. Indeed, in [13], Chen, Dafermos, Slemrod, and Wang were able to complete the Young measure reduction for planar sonic-subsonic flows using only four entropy pairs. However, since they only consider sonic-subsonic flows, the compensated compactness framework that they develop is only suitable for situations that avoid cavitation.

As previously stated, in the case $\gamma \in [1,3)$, Chen et al. [20] were able to select suitable viscosities and boundary data such that cavitation was avoided, since an analysis of the Riemann invariants showed that the viscous approximate solutions remained inside particular invariant regions of the phase space. As such, there was no need to understand the behaviour of solutions of (1.44) near the vacuum. However, one finds that when $\gamma \in [3,\infty)$, the most natural choices of viscosities and boundary values lead to very different invariant regions. In fact, the approximate solutions $\mathbf{u}^\varepsilon$ are now constrained to stay outside of the apple-shaped regions, instead of inside. In view of this, suitable choices of boundary data lead to $M^2 > 1$ throughout the domain Ω. However, unlike in [20], cavitation is now attainable, and it is therefore of primordial importance to develop methods capable of tracking the behaviour of the solutions in the vicinity of $\rho = 0$. Indeed, the problem of generating solutions of the ordinary differential equations (1.46) is now considerably more delicate, since the coefficient $(M^2-1)/\rho^2$ becomes singular at the vacuum. One must therefore perform a thorough asymptotic analysis of the term $(M^2-1)/\rho^2$ near the vacuum, and appeal to techniques developed for general pressure laws (*cf.* [14]). This analysis is contained in Chapter 4, the main result of which is the following.

Theorem 1.21 (Main Theorem of Chapter 4). *Assume that, for each $\varepsilon > 0$, there exists a regular solution $\mathbf{u}^\varepsilon = (q^\varepsilon \cos t^\varepsilon, q^\varepsilon \sin t^\varepsilon)$ of the viscous problem*

$$\begin{cases} v_x^\varepsilon - u_y^\varepsilon = \varepsilon \operatorname{div}(\sigma_1(\rho^\varepsilon)\nabla t^\varepsilon), \\ (\rho^\varepsilon u^\varepsilon)_x + (\rho^\varepsilon v^\varepsilon)_y = \varepsilon \operatorname{div}(\sigma_2(\rho^\varepsilon)\nabla \rho^\varepsilon), \end{cases} \tag{1.47}$$

in Ω depicted by Domain (a) such that $q^\varepsilon \leq q_{cav}$, where $\rho^\varepsilon = \rho(q^\varepsilon)$ is prescribed by the Bernoulli law (1.35) *with $\gamma \geq 3$, with $\sigma_1(\rho) = 1$, $\sigma_2(\rho) = \left(1 - \frac{c^2}{q^2}\right)$ for $q > q_{cr}$, and the boundary conditions:*

$$\begin{cases} \nabla t^\varepsilon \cdot \mathbf{n} = 0 & \text{on } \partial\Omega_1, \\ \varepsilon\sigma_2(\rho^\varepsilon)\nabla\rho^\varepsilon \cdot \mathbf{n} = |\rho^\varepsilon(u^\varepsilon, v^\varepsilon) \cdot \mathbf{n}| & \text{on } \partial\Omega_1, \\ (u^\varepsilon, v^\varepsilon) - (u_\infty, v_\infty) = 0 & \text{on } \partial\Omega_2 \text{ with } q_{cr} < q_\infty < q_{cav}. \end{cases} \tag{1.48}$$

Then, for each $n \in \mathbb{N}$, there exists a solution g_n of the equation

$$\ddot{g}_n(\mu) - n^2 \left(\frac{M^2 - 1}{\rho^2}\right) g_n(\mu) = 0,$$

so that, if (Q_1^n, Q_2^n) is the entropy pair generated by $H_n(\mu, t) = g_n(\mu)e^{\pm nt}$ via (1.45), *then the entropy dissipation measures*

$$\partial_x Q_1^n(\mathbf{u}^\varepsilon) + \partial_y Q_2^n(\mathbf{u}^\varepsilon)$$

are confined to a compact set of $H^{-1}(\Omega)$.

Theorem 1.21 is a first step towards proving the existence of a bounded entropy solution of the Morawetz problem subject to a polytropic pressure law with adiabatic exponent $\gamma \geq 3$.

1.6 Notation

Throughout this thesis, M is a positive constant independent of ε. In Chapters 2 and 3, the parameter $\gamma \in (1, 3)$ (and associated θ, λ as per Definition 1.8) will always correspond to the law imposed in the vicinity of the vacuum (*cf.* Theorem 1.12). In Chapters 2 and 3, and in Appendices A and B, the functions I_0 and I_1 will always refer to the zeroth and first modified Bessel functions of the first kind. Throughout Section 2.4 and Appendix B, we have the shorthand $R = \log \rho$ in the region $\rho \geq 1$. Throughout Section 2.4, we write $X(s) = \chi(1, s)$ and $Y(s) = \chi_\rho(1, s)$ for all $s \in \mathbb{R}$, and $c_{total} = \int_{\mathbb{R}} X(t)\, dt = \int_{\mathbb{R}} Y(t)\, dt$ (this equality is justified in Lemma 2.10). Throughout Chapter 4, C is a positive constant independent of ε, that may change from one line to the next.

Chapter 2

Vanishing viscosity of the Navier–Stokes equations for an approximately isothermal gas

The results contained herein can be found in [80], a paper written by M.R.I. Schrecker and the author. The main purpose of this chapter is to provide complete proofs of Theorem 3.5 and Lemma 3.8 of [80], since these were omitted for the sake of succinctness. These proofs are contained in Section 2.4 of this document. We also give a new approach to obtain equation (2.7) of [80]; an essential ingredient in the proof of the kinetic formulation of the main result.

2.1 Introduction

In [80], the authors addressed the issue of recovering finite relative energy entropy solutions of the one-dimensional, isentropic Euler equations from a vanishing viscosity limit of solutions of the one-dimensional, isentropic compressible Navier–Stokes equations under a particular pressure law assumption. In detail, we produce these solutions by considering the limit as $\varepsilon \to 0$ of classical solutions $(\rho^\varepsilon, u^\varepsilon)$ of (1.25), i.e.,

$$\begin{cases} \rho^\varepsilon_t + (\rho^\varepsilon u^\varepsilon)_x = 0, \\ (\rho^\varepsilon u^\varepsilon)_t + (\rho^\varepsilon (u^\varepsilon)^2 + p(\rho^\varepsilon))_x = \varepsilon u^\varepsilon_{xx}, \end{cases}$$

where $(t, x) \in \mathbb{R}^2_+$, while ρ is the density of the fluid and u its velocity. Recall that the existence of classical solutions of (1.25) is ensured by the result of Hoff in [45] (*cf.* [80, Theorem 3.1]). The quantity p is the pressure, which depends solely on the density through assumptions (1.6)-(1.8) of [80], which are contained in the next definition.

Definition 2.1. We say that a fluid behaves according to an *approximately isothermal pressure law* if it satisfies the following constitutive assumptions.

1. The pressure, $p \in C^1([0,\infty)) \cap C^4((0,\infty))$, is such that $p(\rho) > 0$ for all $\rho > 0$, and satisfies the assumptions of strict hyperbolicity and genuine nonlinearity, i.e.,
$$p'(\rho) > 0, \qquad \rho p''(\rho) + 2p'(\rho) > 0 \qquad \text{for } \rho > 0. \tag{2.1}$$
2. There exist constants $\gamma \in (1,3)$ and $\kappa > 0$, and a function $P \in C^4((0,\infty))$ such that
$$p(\rho) = \kappa \rho^\gamma (1 + P(\rho)) \qquad \text{for } \rho \in [0,r), \tag{2.2}$$
for some fixed $r > 0$, and there exists a positive constant C_r such that
$$|P^{(j)}(\rho)| \leq C_r \rho^{2\theta - j} \qquad \text{for } \rho \in [0,r), \text{ and } j \in \{0,\dots,4\}. \tag{2.3}$$
3. For $\rho \geq r$, we have $p(\rho) = c_* \rho$, for some constant $c_* > 0$.

Via the rescaling $(\rho, u) \mapsto \left(\frac{\rho}{r}, \frac{u}{\sqrt{c_*}}\right)$, it suffices to consider the case $c_* = r = 1$.

Since our work was confined to the finite energy setting (as opposed to the L^∞ framework), it was necessary to develop a thorough understanding of the behaviour of the entropy pairs in the large density regime, i.e., $\rho \in [1,\infty)$. The result of this analysis is encapsulated in the first half of Theorem 3.5 of [80]. As previously discussed, the details of the proof were omitted due to the length and technicality of the arguments, and the objective of Section 2.4 of this chapter is to go through these in detail. Therein, we also give a complete proof of Lemma 3.8 of [80]. Then, we use a modified version of the finite energy method developed by LeFloch–Westdickenberg in [58] to deduce the almost everywhere convergence of the viscous approximate solutions of (1.25) to a finite relative energy entropy solution of (1.7).

The main result of this chapter is the following, which is a precise version of Theorem 1.17 stated in Section 1.3, and was proved in [80].

Theorem 2.2. *Suppose that the initial data $(\rho_0, u_0) \in L^1_{loc}(\mathbb{R}^2_+)$ with $\rho_0 \geq 0$ and end-states $(\rho_\pm, u_\pm)$ is of finite relative energy,*
$$E[\rho_0, u_0] = \int_{\mathbb{R}} \overline{\eta^*}(\rho_0, \rho_0 u_0)\, dx \leq E_0 < \infty,$$
and suppose that the pressure function $p(\rho)$ satisfies the criteria for an approximately isothermal gas, in the precise sense of Definition 2.1. Then there exists a sequence of regularised initial data $(\rho_0^\varepsilon, u_0^\varepsilon)$ such that the sequence of unique, smooth solutions $(\rho^\varepsilon, u^\varepsilon)$ of (1.25) with this initial data converges as $\varepsilon \to 0$, $(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon) \to (\rho, \rho u)$, to a finite relative energy entropy solution of the Euler equations (1.7) with initial data $(\rho_0, \rho_0 u_0)$, in the precise sense of Definition 1.16. The convergence is almost everywhere and $L^p_{loc}(\mathbb{R}^2_+) \times L^q_{loc}(\mathbb{R}^2_+)$ for $p \in [1,2)$ and $q \in [1, 3/2)$.

The remainder of this chapter is devoted to proving Theorem 2.2, and we proceed as follows. In Section 2.2, we give explicit characterisations of the entropies of the Euler equations with an approximately isothermal pressure law. In Section 2.3, we state the required pointwise estimates for a special entropy pair, and for entropies generated by compactly supported test functions. These pointwise estimates on the entropy pairs then enable us to obtain uniform estimates on the solutions of the Navier–Stokes equations, in Section 2.5. The precise details of the pointwise bounds of Section 2.3 are contained in Section 2.4. In Section 2.6, we show that the sequence of viscous approximate solutions converges to a limit, in some weak sense to be made precise. This limit can be characterized by a Young measure, and we show that this measure satisfies the Tartar–Murat commutation relation, in Section 2.7. In Section 2.8, we use this commutation relation to show that the Young measure is only supported at a point; the Young measure reduction. This implies that the measure is in fact a Dirac mass, which guarantees that the convergence of the approximate solutions to the limit happens in measure. This mode of convergence is then strong enough to pass to the limit in the nonlinear equations, thereby proving Theorem 2.2, which concludes this chapter.

2.2 The entropy kernels

The Young measure reduction of Section 2.8 makes use of the Tartar–Murat relation, and requires that we generate suitably many entropies. To this end, we recall that all weak entropies may be generated using the entropy kernel χ, as discussed in Subsection 1.2.2 and [14, Corollary 2.1]. Observe that a gas satisfying the assumptions of an approximately isothermal gas, as per Definition 2.1, satisfies the assumptions of Theorem 1.12. Hence we deduce the global existence of χ, which solves (1.12) for all $(\rho, u) \in \mathbb{R}^2_+$, with the assumptions on the pressure prescribed by Definition 2.1.

In order to obtain finer control on the entropies for large densities, we need a more accurate characterisation of the entropy kernel. One way to proceed is by exploiting the fact that the kernel solves

$$\begin{cases} \chi_{\rho\rho}(\rho,u) - \dfrac{1}{\rho^2}\chi_{uu}(\rho,u) = 0 \qquad \text{for } (\rho,u) \in [1,\infty)\times\mathbb{R}, \\ \chi|_{\rho=1} = \chi(1,\cdot), \\ \chi_\rho|_{\rho=1} = \chi_\rho(1,\cdot), \end{cases} \tag{2.4}$$

due to the semigroup property of linear wave equations. In view of this, we have

$$\chi(\rho,u) = \int_{\mathbb{R}} \big(\chi_\rho(1,s)\chi^\sharp(\rho,u-s) + \chi(1,s)\chi^\flat(\rho,u-s)\big)\, ds,$$

where the fundamental solutions $\chi^\sharp$ and $\chi^\flat$ solve

$$\begin{cases} \chi^\sharp_{\rho\rho} - \dfrac{1}{\rho^2}\chi^\sharp_{uu} = 0, \\ \chi^\sharp|_{\rho=1} = 0, \\ \chi^\sharp_\rho|_{\rho=1} = \delta_{u=0}, \end{cases} \qquad \begin{cases} \chi^\flat_{\rho\rho} - \dfrac{1}{\rho^2}\chi^\flat_{uu} = 0, \\ \chi^\flat|_{\rho=1} = \delta_{u=0}, \\ \chi^\flat_\rho|_{\rho=1} = 0, \end{cases}$$

in the sense of distributions. Next we recall [80, Theorem 2.2], the proof of which we omit, which provides explicit formulas for the kernels $\chi^\sharp$ and $\chi^\flat$.

Theorem 2.3 (Theorem 2.2 of [80])**.** *For $\rho > 0$ and $u, s \in \mathbb{R}$, the function*

$$\chi^\sharp(\rho, u-s) = \frac{\sqrt{\rho}}{2} I_0\left(\frac{\sqrt{(\log\rho)^2 - (u-s)^2}}{2}\right)\mathbb{1}_{|u-s|<|\log\rho|}, \tag{2.5}$$

and the measure

$$\begin{aligned} \chi^\flat(\rho, u-s) =& \frac{\sqrt{\rho}}{2}(\delta_{u-s=\log\rho} + \delta_{u-s=-\log\rho}) \\ &- \frac{\sqrt{\rho}}{4} I_0\left(\frac{\sqrt{(\log\rho)^2 - (u-s)^2}}{2}\right)\mathbb{1}_{|u-s|<|\log\rho|} \\ &+ \frac{\sqrt{\rho}}{4}\frac{\log\rho}{\sqrt{(\log\rho)^2 - (u-s)^2}} I_1\left(\frac{\sqrt{(\log\rho)^2 - (u-s)^2}}{2}\right)\mathbb{1}_{|u-s|<|\log\rho|}, \end{aligned} \tag{2.6}$$

solve the problems

$$\begin{cases} \chi^\sharp_{\rho\rho} - \dfrac{1}{\rho^2}\chi^\sharp_{uu} = 0, \\ \chi^\sharp|_{\rho=1} = 0, \\ \chi^\sharp_\rho|_{\rho=1} = \delta_{u=s}, \end{cases} \qquad \begin{cases} \chi^\flat_{\rho\rho} - \dfrac{1}{\rho^2}\chi^\flat_{uu} = 0, \\ \chi^\flat|_{\rho=1} = \delta_{u=s}, \\ \chi^\flat_\rho|_{\rho=1} = 0, \end{cases} \tag{2.7}$$

respectively, in the sense of distributions for $(\rho, u) \in \mathbb{R}^2_+$. Moreover,

$$\lim_{\rho\to 0}\chi^\sharp(\rho, \cdot) = \lim_{\rho\to 0}\chi^\flat(\rho, \cdot) = 0.$$

Note that I_ν is the ν^{th}-modified Bessel function of the first kind (*cf.* Appendix A). Similarly, we have explicit characterisations of the entropy-flux kernels, as per [80, Theorem 2.4], which is rewritten below without proof.

Theorem 2.4 (Theorem 2.4 of [80])**.** *The entropy-flux kernels $\sigma^\sharp$ and $\sigma^\flat$ corresponding to $\chi^\sharp$ and $\chi^\flat$ respectively can be written as*

$$\begin{aligned} \sigma^\sharp(\rho, u, s) &= u\chi^\sharp(\rho, u-s) + h^\sharp(\rho, u-s), \\ \sigma^\flat(\rho, u, s) &= u\chi^\flat(\rho, u-s) + h^\flat(\rho, u-s), \end{aligned}$$

where

$$h^\sharp(\rho, u-s) = \frac{1}{2}\operatorname{sgn}(u-s) + \frac{\partial}{\partial u}\int_0^\rho \chi^\sharp(r, u-s)\,\frac{dr}{r},$$
$$h^\flat(\rho, u-s) = \chi^\sharp_u(\rho, u-s) - h^\sharp(\rho, u-s).$$

In summary, for $\rho \geq 1$, the entropy and entropy-flux kernels for the approximately isothermal pressure law may be written as

$$\begin{aligned}\chi(\rho,u) &= \int_{\mathbb{R}} \big(\chi_\rho(1,s)\chi^\sharp(\rho,u-s) + \chi(1,s)\chi^\flat(\rho,u-s)\big)\,ds,\\ \sigma(\rho,u,0) &= \int_{\mathbb{R}} \big(\chi_\rho(1,s)\sigma^\sharp(\rho,u,s) + \chi(1,s)\sigma^\flat(\rho,u,s)\big)\,ds.\end{aligned} \tag{2.8}$$

2.3 Pointwise estimates

In order to employ tools from the theory of compensated compactness in later sections, we need to obtain suitable uniform estimates on the viscous approximate solutions. One way of obtaining such estimates is by constructing appropriate entropy pairs through judicious choices of test functions ψ (*cf.* (1.13)), and understanding their behaviour for large densities. Proceeding in this manner requires the explicit representation for the entropy kernel provided by (2.8) and Theorem 2.3. We first consider a special entropy pair; the subject of the next lemma, for which we recall that γ (and its corresponding θ) was fixed in Definition 2.1.

Lemma 2.5 (Lemma 3.5 of [80]). *Let $\hat\psi(s) := \frac{1}{2}s|s|$. The associated entropy pair $(\hat\eta, \hat q)$, via* (1.13) *and* (1.15)*, satisfies the following for $\rho \geq \rho_*$, with $\rho_* > 1$ fixed,*

$$\begin{aligned} &|\hat\eta(\rho,m)| \leq M\eta^*(\rho,m), && \hat q(\rho,\rho u) \geq M^{-1}\rho|u|^3 - M\left(\rho|u|^2 + \rho + \rho(\log\rho)^4\right),\\ &|\hat\eta_m(\rho,\rho u)| \leq M(|u| + \sqrt{\log\rho}), && |\rho\hat\eta_{mm}(\rho,m)| \leq M,\end{aligned} \tag{2.9}$$

and, with $\hat\eta_{mu}(\rho,\rho u) = \partial_u\hat\eta_m(\rho,\rho u)$ and $\hat\eta_{m\rho}(\rho,\rho u) = \partial_\rho\hat\eta_m(\rho,\rho u)$,

$$|\hat\eta_{mu}(\rho,\rho u)| \leq M\frac{1}{\sqrt{\log\rho}}, \qquad |\hat\eta_{m\rho}(\rho,\rho u)| \leq M\rho^{-1}. \tag{2.10}$$

Moreover, on the complement region $\rho \leq \rho_*$, we have

$$\begin{aligned} &|\hat\eta(\rho,m)| \leq M\eta^*(\rho,m), && \hat q(\rho,\rho u) \geq M^{-1}\left(\rho|u|^3 + \rho^{\gamma+\theta}\right) - M\left(\rho|u|^2 + \rho^\gamma\right),\\ &|\hat\eta_m(\rho,\rho u)| \leq M(|u| + \rho^\theta), && |\rho\hat\eta_{mm}(\rho,m)| \leq M\rho^{\theta-1},\\ &|\hat\eta_{mu}(\rho,\rho u)| \leq M, && |\hat\eta_{m\rho}(\rho,\rho u)| \leq M\rho^{\theta-1},\end{aligned} \tag{2.11}$$

where in the final line we consider $\hat\eta_m(\rho,\rho u)$ as a function of (ρ,u), as in (2.10)*. Finally,*

$$\rho|\hat\eta_m(\rho,0) - \hat\eta_m(\bar\rho,0)|^2 \leq Me^*(\rho,\bar\rho) \qquad \text{for } \rho,\bar\rho \geq 0. \tag{2.12}$$

In addition, we will require precise pointwise bounds on entropies generated by compactly supported test functions. These are contained in the following result.

Lemma 2.6 (Lemma 3.8 of [80]). *Let $\psi \in C_c^2(\mathbb{R})$ be a compactly supported test function such that the support of ψ is contained in an interval $[z_*, w_*]$. Then the corresponding entropy pair (η^ψ, q^ψ), via (1.13) and (1.15), has support*

$$\operatorname{supp}\eta^\psi, \operatorname{supp} q^\psi \subset \{(\rho, u) : w(\rho, u) \geq z_*, z(\rho, u) \leq w_*\},$$

where $w(\rho, u) = u+k(\rho)$ and $z(\rho, u) = u-k(\rho)$ are the Riemann invariants. Moreover,

$$|\eta^\psi(\rho, m)| \leq M_\psi \rho \min\{1, \frac{1}{\sqrt{\log(\rho+1)}}\} \quad \textit{and} \quad |q^\psi(\rho, m)| \leq M_\psi \rho, \tag{2.13}$$

and

$$|\eta^\psi_m(\rho, m)| + |\rho\eta^\psi_{mm}(\rho, m)| \leq M_\psi \min\{1, \frac{1}{\sqrt{\log(\rho+1)}}\}. \tag{2.14}$$

Considering η^ψ_m as a function of ρ and u,

$$|\eta^\psi_{mu}(\rho, \rho u)| \leq M_\psi \min\{1, \frac{1}{\sqrt{\log(\rho+1)}}\}, \quad |\rho\eta^\psi_{m\rho}(\rho, \rho u)| \leq M_\psi \min\{\rho^\theta, \frac{1}{\log(\rho+1)}\}, \tag{2.15}$$

where, for example, $\eta^\psi_{mu}(\rho, \rho u) = \partial_u \eta^\psi_m(\rho, \rho u)$. In particular, $|\eta^\psi_{m\rho}(\rho, \rho u)| \leq M_\psi \frac{\sqrt{p'(\rho)}}{\rho}$.

2.4 Details of the pointwise estimates

The purpose of this section is to give detailed proofs of the pointwise estimates stated in the previous section, which were not contained in [80]. Prior to that, we make a basic observation concerning the parity of $\chi(\rho, \cdot)$ and its derivatives, and prove two results concerning the mass of these quantities, which will be crucial later.

Lemma 2.7. *For every $\rho > 0$, $\chi(\rho, \cdot)$ and $\chi_\rho(\rho, \cdot)$ are even, while $\chi_u(\rho, \cdot)$ is odd.*

Proof. Observe that the function

$$\tilde{\chi}(\rho, u) := \chi(\rho, u) - \chi(\rho, -u), \tag{2.16}$$

solves the linear wave equation $\tilde{\chi}_{\rho\rho} - k'(\rho)^2 \tilde{\chi}_{uu} = 0$ with zero initial data. As such, one possible solution is $\tilde{\chi}(\rho, u) \equiv 0$. In view of the uniqueness ensured by [14, Theorem 2.1], which applies since the assumptions of this theorem are satisfied for an approximately isothermal gas, this is the only solution. The result for χ_ρ and χ_u follows by differentiating (2.16) with respect to ρ and u, respectively, in the sense of distributions. □

Lemma 2.8. *For every $\rho > 0$, the function $\chi(\rho, \cdot)$ and its derivatives, $\chi_u(\rho, \cdot)$ and $\chi_\rho(\rho, \cdot)$, are compactly supported and belong to $L^1(\mathbb{R})$.*

Proof. The compact support follows directly from Theorem 1.12. Since $\chi(\rho, \cdot)$ is Hölder continuous and compactly supported, it is integrable. Making use of the expansion at the vacuum provided by (1.20), we get, for $\rho > 0$,

$$\|\chi_\rho(\rho, \cdot)\|_{L^1(\mathbb{R})} \leq 4\lambda a_\sharp(\rho) k'(\rho) k(\rho)^{2\lambda} \int_0^1 (2-y)^{\lambda-1} y^{\lambda-1} \, dy + C_1(\rho, \lambda),$$
$$\|\chi_u(\rho, \cdot)\|_{L^1(\mathbb{R})} \leq 4\lambda a_\sharp(\rho) k(\rho)^{2\lambda-1} \int_0^1 (2-y)^{\lambda-1} y^{\lambda-1} \, dy + C_2(\rho, \lambda),$$

where $C_j(\rho, \lambda) > 0$ ($j = 1, 2$) correspond to integrals of Hölder continuous compactly supported functions, and are therefore finite. The result follows. □

Remark 2.9. We use the **notation** $X(s) = \chi(1, s)$ and $Y(s) = \chi_\rho(1, s)$. Lemma 2.8 showed that $\|X\|_{L^\infty(\mathbb{R})}, \|X\|_{L^1(\mathbb{R})}, \|Y\|_{L^1(\mathbb{R})} < \infty$, and **all calculations in this section rely solely on the finiteness of these three quantities**; note that $Y \notin L^\infty(\mathbb{R})$ for $\gamma > 5/3$. Herein and in Appendix B, we use the **shorthand** $R = \log \rho$.

Lemma 2.10. *Define*

$$F(\rho, u) := \int_{-\infty}^{\infty} \big(\rho \chi_\rho(\rho, u-\tau) - \chi(\rho, u-\tau)\big) \, d\tau \qquad \text{for } (\rho, u) \in \mathbb{R}^2_+. \tag{2.17}$$

Then, $F(\rho, u) = 0$ for every $(\rho, u) \in \mathbb{R}^2_+$. It follows that

$$\int_{\mathbb{R}} X(t) \, dt = \int_{\mathbb{R}} Y(t) \, dt =: c_{total}. \tag{2.18}$$

Proof. In view of Lemma 2.8, the integral is well-defined. Now, taking a distributional derivative in ρ, we find that, for any $\phi \in \mathcal{D}(\mathbb{R}^2_+)$,

$$\begin{aligned}\langle F, \partial_\rho \phi \rangle_{\mathcal{D}'(\mathbb{R}^2_+) \times \mathcal{D}(\mathbb{R}^2_+)} &= \int_{\mathbb{R}^2_+} \partial_\rho \phi(\rho, u) \left(\int_{\mathbb{R}} (\rho \chi_\rho(\rho, u-\tau) - \chi(\rho, u-\tau)) \, d\tau \right) d\rho \, du \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}_+} \partial_\rho \phi(\rho, u) (\rho \chi_\rho(\rho, u-\tau) - \chi(\rho, u-\tau)) \, d\rho \right) d\tau \, du,\end{aligned}$$

by appealing to the Tonelli–Fubini theorem. Integrating by parts in ρ gives

$$\begin{aligned}&\int_{\mathbb{R}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}_+} \partial_\rho \phi(\rho, u) \big(\rho \chi_\rho(\rho, u-\tau) - \chi(\rho, u-\tau)\big) \, d\rho \right) du \, d\tau \\ &\qquad = - \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}_+} \chi(\rho, u-\tau) \partial^2_{\rho\rho}(\rho \phi(\rho, u)) \, d\rho \right) du \, d\tau, \\ &\qquad = - \int_{\mathbb{R}} \big\langle \chi(\cdot, \cdot - \tau), \partial^2_{\rho\rho}(\rho \phi(\cdot, \cdot)) \big\rangle_{\mathcal{D}'(\mathbb{R}^2_+) \times \mathcal{D}(\mathbb{R}^2_+)} \, d\tau.\end{aligned}$$

Passing both derivatives in ρ onto χ and using the entropy equation (1.14), this final quantity is equal to $-\int_{\mathbb{R}} \langle \chi_{uu}(\cdot, \cdot - \tau), \rho k'(\rho)^2 \phi(\cdot,\cdot)\rangle_{\mathcal{D}'(\mathbb{R}^2_+)\times\mathcal{D}(\mathbb{R}^2_+)}\, d\tau$. Hence,

$$\langle \partial_\rho F, \phi\rangle_{\mathcal{D}'\times\mathcal{D}} = -\int_{\mathbb{R}_+} \rho k'(\rho)^2 \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \partial_u \phi(\rho,u)\chi_u(\rho, u-\tau)\, du\, d\tau \right) d\rho,$$

where the integral is well-defined since $\chi_u(\rho,\cdot)$ is integrable, in view of Lemma 2.8 (*cf.* [14, Theorem 2.2]). Thus,

$$\langle \partial_\rho F, \phi\rangle_{\mathcal{D}'\times\mathcal{D}} = \int_{\mathbb{R}_+} \rho k'(\rho)^2 \left(\int_{\mathbb{R}} \partial_u \phi(\rho,u) \left(\int_{-\infty}^{\infty} \frac{d}{d\tau}\left(\chi(\rho, u-\tau)\right) d\tau \right) du \right) d\rho = 0,$$

since $\chi(\rho,\cdot)$ is compactly supported, which shows that $\partial_\rho F(\rho,u) = 0$ in the sense of distributions. We thereby have $F(\rho,u) = F(u)$ in $\mathcal{D}'(\mathbb{R}^2_+)$. Additionally,

$$\begin{aligned}\langle \partial_u F, \phi\rangle_{\mathcal{D}'\times\mathcal{D}} &= -\int_{\mathbb{R}}\int_{\mathbb{R}} \left(\int_{\mathbb{R}_+} \partial_u\phi(\rho,u)\left(\rho\chi_\rho(\rho,u-\tau) - \chi(\rho,u-\tau)\right) d\rho \right) d\tau\, du \\ &= -\int_{\mathbb{R}^2_+} \left(\partial_\rho(\rho\phi(\rho,u)) + \phi(\rho,u)\right) \left(\int_{-\infty}^{\infty} \chi_u(\rho,u-\tau)\, d\tau \right) du\, d\rho = 0,\end{aligned}$$

since, by Lemma 2.7, $\chi_u(\rho,\cdot)$ is odd. Hence, we deduce that F is constant. Since it is manifestly null at the vacuum (as χ is a weak entropy and $\rho\delta_{u=0}$ vanishes as a distribution when $\rho = 0$), we deduce that $F(\rho,u) = 0$ for all $(\rho,u) \in \mathbb{R}^2_+$. □

Corollary 2.11. *Define*

$$F(\rho,u,s) := \int_{-\infty}^{s} \big(\rho\chi_\rho(\rho,u-\tau) - \chi(\rho,u-\tau)\big)\, d\tau. \tag{2.19}$$

Then for every $(\rho,u) \in \mathbb{R}^2_+$ the function $F(\rho,u,\cdot) : \mathbb{R} \to \mathbb{R}$ is compactly supported and $\int_{\mathbb{R}} F(\rho,u,s)\, ds = 0$.

Proof. Fix $(\rho,u) \in \mathbb{R}^2_+$. Theorem 1.12 shows that $\operatorname{supp}\chi(\rho, u-\cdot) \subset [u-k(\rho), u+k(\rho)]$. Hence, $F(\rho,u,s) = 0$ for all $s \le u - k(\rho)$, and $F(\rho,u,s) = F(\rho,u,\infty)$ for all $s \ge u + k(\rho)$. From Lemma 2.10, we know that $F(\rho,u,\infty) = F(\rho,u) = 0$. Thus, we deduce that $F(\rho,u,\cdot)$ is compactly supported for each fixed $(\rho,u) \in \mathbb{R}^2_+$.

Additionally, observe that

$$F(\rho,u,s) = \int_{u-s}^{\infty} \big(\rho\chi_\rho(\rho,z) - \chi(\rho,z)\big)\, dz.$$

Then, using the evenness of $\chi(\rho,\cdot)$ and $\chi_\rho(\rho,\cdot)$ derived in Lemma 2.7, we obtain

$$\begin{aligned}\int_{\mathbb{R}} F(\rho,u,s)\, ds &= \int_{\mathbb{R}} \left(\int_{w}^{\infty} \big(\rho\chi_\rho(\rho,z) - \chi(\rho,z)\big)\, dz \right) dw \\ &= \int_{\mathbb{R}} \left(\int_{-\infty}^{w} \big(\rho\chi_\rho(\rho,z) - \chi(\rho,z)\big)\, dz \right) dw.\end{aligned}$$

Adding the above and using Lemma 2.10 yields $\int_{\mathbb{R}} F(\rho,u,s)\, ds = 0$, as required. □

2.4.1 Computing entropies and their derivatives

In view of (2.8) and Theorem 2.3, we have the following lemma.

Lemma 2.12. *In the isothermal regime $\{\rho \geq 1\}$, the entropy kernel can be written as*

$$\chi(\rho, u) = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3, \tag{2.20}$$

where

$$\begin{aligned} \mathcal{L}_1 &= \int_{\mathbb{R}} \frac{\sqrt{\rho}}{2} I_0\left(\frac{\sqrt{(\log\rho)^2 - (u-t)^2}}{2}\right) \mathbb{1}_{|u-t|<\log\rho} Y(t)\, dt, \\ \mathcal{L}_2 &= \frac{\sqrt{\rho}}{2}\big(X(u-\log\rho) + X(u+\log\rho)\big), \end{aligned} \tag{2.21}$$

and

$$\begin{aligned} \mathcal{L}_3 = \int_{\mathbb{R}} \frac{\sqrt{\rho}}{4}\Bigg[&\frac{\log\rho}{\sqrt{\log\rho^2 - (u-t)^2}} I_1\left(\frac{\sqrt{(\log\rho)^2 - (u-t)^2}}{2}\right) \\ &- I_0\left(\frac{\sqrt{(\log\rho)^2 - (u-t)^2}}{2}\right)\Bigg] \mathbb{1}_{|u-t|<\log\rho} X(t)\, dt. \end{aligned} \tag{2.22}$$

Since X and Y are supported in $[-k(1), k(1)]$, by Theorem 1.12, it follows from (2.21)-(2.22) that $\operatorname{supp}\chi(\rho,\cdot) \subset \{u \in \mathbb{R} : |u| \leq k(1) + \log\rho\}$ for $\rho \geq 1$.

Remark 2.13. Since χ is given as a function of (ρ, u), we will also generate entropies as functions of (ρ, u), even though these are technically functions of (ρ, m). **We adopt this convention throughout Section 2.4 only**, in order to simplify computations.

Lemma 2.14. *In the isothermal regime $\{\rho \geq 1\}$, the weak entropies are generated by χ through test functions ψ via $\eta^\psi(\rho, u) = \mathcal{J}_1^\psi + \mathcal{J}_2^\psi + \mathcal{J}_3^\psi$, where*

$$\begin{aligned} \mathcal{J}_1^\psi &= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\sqrt{\rho}}{2} I_0\left(\frac{\sqrt{(\log\rho)^2 - (u-s-t)^2}}{2}\right) \mathbb{1}_{|u-s-t|<\log\rho} Y(t)\psi(s)\, dt\, ds, \\ \mathcal{J}_2^\psi &= \int_{\mathbb{R}} \frac{\sqrt{\rho}}{2}\big(X(u-s-\log\rho) + X(u-s+\log\rho)\big)\psi(s)\, ds, \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} \mathcal{J}_3^\psi = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\sqrt{\rho}}{4}\Bigg[&\frac{\log\rho}{\sqrt{(\log\rho)^2 - (u-s-t)^2}} I_1\left(\frac{\sqrt{(\log\rho)^2 - (u-s-t)^2}}{2}\right) \\ &- I_0\left(\frac{\sqrt{(\log\rho)^2 - (u-s-t)^2}}{2}\right)\Bigg] \mathbb{1}_{|u-s-t|<\log\rho} X(t)\psi(s)\, dt\, ds. \end{aligned} \tag{2.24}$$

Proof. The assertion is immediate from Lemma 2.12 and (1.13). □

Choosing $\hat{\psi}(s) = \frac{1}{2}s|s|$, we obtain

$$\hat{\eta}(\rho, u) = \hat{\mathcal{J}}_1(\rho, u) + \hat{\mathcal{J}}_2(\rho, u) + \hat{\mathcal{J}}_3(\rho, u) \qquad \text{for } (\rho, u) \in [1, \infty) \times \mathbb{R}, \tag{2.25}$$

where $\hat{\mathcal{J}}_i = \mathcal{J}_i^{\hat{\psi}}$ for $i = 1, 2, 3$, where $\hat{\eta}$ is the special entropy of Lemma 2.5. We use this notation throughout the estimates of this chapter.

2.4.2 The term $u\hat{\eta}(\rho, u)$

In order to estimate the entropy-flux $\hat{q}$, it is crucial to have an exact expansion for $u\hat{\eta}(\rho, u)$. This subsection provides this in full detail.

Explicit computation shows that the term $\hat{\eta}$ can be decomposed into three terms; one with no u powers, one that is linear in u, and one that is of quadratic order for large u. Hence,

$$u\hat{\eta}(\rho, u) = \Sigma_1(\rho, u) + \Sigma_2(\rho, u) + \Sigma_3(\rho, u) \qquad \text{for } (\rho, u) \in [1, \infty) \times \mathbb{R}, \tag{2.26}$$

where Σ_1 is linear in u, Σ_2 quadratic in u, and $\Sigma_3 = O(u^3)$ as $|u| \to \infty$. The rest of this subsection is concerned with computing and estimating Σ_i, for $i = 1, 2, 3$.

Lemma 2.15. *Explicit computation shows that the term of order u in the expansion of $u\hat{\eta}(\rho, u)$, which we denote by Σ_1, is given by*

$$\Sigma_1 = \mathcal{I}_1 + \mathcal{II}_1 + \mathcal{III}_1 + \mathcal{IV}_1, \tag{2.27}$$

where

$$\begin{aligned}
\mathcal{I}_1 :=& \frac{\sqrt{\rho}R^3 u}{2} \int_{-\infty}^{u-R} Y(t) \left(\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right) \cos\theta \sin^2\theta \, d\theta \right) dt \\
&+ \frac{\sqrt{\rho}Ru}{2} \int_{-\infty}^{u-R} t^2 Y(t) \left(\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right) \cos\theta \, d\theta \right) dt \\
&- \sqrt{\rho}R^2 u \int_{u-R}^{u} tY(t) \left(\int_{\arcsin\left(\frac{u-t}{R}\right)}^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right) \cos\theta \sin\theta \, d\theta \right) dt \\
&+ \frac{\sqrt{\rho}Ru}{2} \int_{u-R}^{u} t^2 Y(t) \left(\int_0^{\arcsin\left(\frac{u-t}{R}\right)} I_0\left(\frac{R}{2}\cos\theta\right) \cos\theta \, d\theta \right) dt \\
&+ \frac{\sqrt{\rho}R^3 u}{2} \int_{u-R}^{u} Y(t) \left(\int_0^{\arcsin\left(\frac{u-t}{R}\right)} I_0\left(\frac{R}{2}\cos\theta\right) \cos\theta \sin^2\theta \, d\theta \right) dt \\
&- \sqrt{\rho}R^2 u \int_{u}^{u+R} tY(t) \left(\int_{\arcsin\left(\frac{t-u}{R}\right)}^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right) \cos\theta \sin\theta \, d\theta \right) dt
\end{aligned}$$

$$
\begin{aligned}
&- \frac{\sqrt{\rho} R^3 u}{2} \int_u^{u+R} Y(t) \left(\int_0^{\arcsin\left(\frac{t-u}{R}\right)} I_0\left(\frac{R}{2}\cos\theta\right) \cos\theta \sin^2\theta \, d\theta \right) dt \\
&- \frac{\sqrt{\rho} R u}{2} \int_u^{u+R} t^2 Y(t) \left(\int_0^{\arcsin\left(\frac{t-u}{R}\right)} I_0\left(\frac{R}{2}\cos\theta\right) \cos\theta \, d\theta \right) dt \\
&- \frac{\sqrt{\rho} R^3 u}{2} \int_{u+R}^{\infty} Y(t) \left(\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right) \cos\theta \sin^2\theta \, d\theta \right) dt \\
&- \frac{\sqrt{\rho} R u}{2} \int_{u+R}^{\infty} t^2 Y(t) \left(\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right) \cos\theta \, d\theta \right) dt,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{II}_1 := &- \frac{\sqrt{\rho} u}{4} \int_{u-R}^{\infty} t^2 X(t) \, dt - \frac{\sqrt{\rho} R u}{2} \int_{u-R}^{\infty} X(t) t \, dt - \frac{\sqrt{\rho} R^2 u}{4} \int_{u-R}^{\infty} X(t) \, dt \\
&- \frac{\sqrt{\rho} u}{4} \int_{u+R}^{\infty} t^2 X(t) \, dt + \frac{\sqrt{\rho} R u}{2} \int_{u+R}^{\infty} t X(t) \, dt - \frac{\sqrt{\rho} R^2 u}{4} \int_{u+R}^{\infty} X(t) \, dt \\
&+ \frac{\sqrt{\rho} u}{4} \int_{-\infty}^{u-R} t^2 X(t) \, dt + \frac{\sqrt{\rho} R u}{2} \int_{-\infty}^{u-R} t X(t) \, dt + \frac{\sqrt{\rho} R^2 u}{4} \int_{-\infty}^{u-R} X(t) \, dt \\
&+ \frac{\sqrt{\rho} u}{4} \int_{-\infty}^{u+R} t^2 X(t) \, dt - \frac{\sqrt{\rho} R u}{2} \int_{-\infty}^{u+R} t X(t) \, dt + \frac{\sqrt{\rho} R^2 u}{4} \int_{-\infty}^{u+R} X(t) \, dt,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{III}_1 := &\frac{\sqrt{\rho} R^3 u}{4} \int_{-\infty}^{u-R} X(t) \left(\int_0^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right) \sin^2\theta \, d\theta \right) dt \\
&+ \frac{\sqrt{\rho} R u}{4} \int_{-\infty}^{u-R} t^2 X(t) \left(\int_0^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right) \, d\theta \right) dt \\
&- \frac{\sqrt{\rho} R^2 u}{2} \int_{u-R}^{u} t X(t) \left(\int_{\arcsin\left(\frac{u-t}{R}\right)}^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right) \sin\theta \, d\theta \right) dt \\
&+ \frac{\sqrt{\rho} R^3 u}{4} \int_{u-R}^{u} X(t) \left(\int_0^{\arcsin\left(\frac{u-t}{R}\right)} I_1\left(\frac{R}{2}\cos\theta\right) \sin^2\theta \, d\theta \right) dt \\
&+ \frac{\sqrt{\rho} R u}{4} \int_{u-R}^{u} t^2 X(t) \left(\int_0^{\arcsin\left(\frac{u-t}{R}\right)} I_1\left(\frac{R}{2}\cos\theta\right) \, d\theta \right) dt \\
&- \frac{\sqrt{\rho} R^2 u}{2} \int_{u}^{u+R} t X(t) \left(\int_{\arcsin\left(\frac{u-t}{R}\right)}^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right) \sin\theta \, d\theta \right) dt \\
&- \frac{\sqrt{\rho} R^3 u}{4} \int_{u}^{u+R} X(t) \left(\int_0^{\arcsin\left(\frac{t-u}{R}\right)} I_1\left(\frac{R}{2}\cos\theta\right) \sin^2\theta \, d\theta \right) dt
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{\sqrt{\rho} R u}{4} \int_u^{u+R} t^2 X(t) \left(\int_0^{\arcsin\left(\frac{t-u}{R}\right)} I_1\left(\frac{R}{2}\cos\theta\right) d\theta \right) dt \\
&- \frac{\sqrt{\rho} R^3 u}{4} \int_{u+R}^{\infty} X(t) \left(\int_0^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right) \sin^2\theta \, d\theta \right) dt \\
&- \frac{\sqrt{\rho} R u}{4} \int_{u+R}^{\infty} t^2 X(t) \left(\int_0^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right) d\theta \right) dt,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{IV}_1 := &- \frac{\sqrt{\rho} R^3 u}{4} \int_{-\infty}^{u-R} X(t) \left(\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right) \cos\theta \sin^2\theta \, d\theta \right) dt \\
&- \frac{\sqrt{\rho} R u}{4} \int_{-\infty}^{u-R} t^2 X(t) \left(\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right) \cos\theta \, d\theta \right) dt \\
&+ \frac{\sqrt{\rho} R^2 u}{2} \int_{u-R}^{u} t X(t) \left(\int_{\arcsin\left(\frac{u-t}{R}\right)}^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right) \cos\theta \sin\theta \, d\theta \right) dt \\
&- \frac{\sqrt{\rho} R u}{4} \int_{u-R}^{u} t^2 X(t) \left(\int_0^{\arcsin\left(\frac{u-t}{R}\right)} I_0\left(\frac{R}{2}\cos\theta\right) \cos\theta \, d\theta \right) dt \\
&- \frac{\sqrt{\rho} R^3 u}{4} \int_{u-R}^{u} X(t) \left(\int_0^{\arcsin\left(\frac{u-t}{R}\right)} I_0\left(\frac{R}{2}\cos\theta\right) \cos\theta \sin^2\theta \, d\theta \right) dt \\
&+ \frac{\sqrt{\rho} R^2 u}{2} \int_{u}^{u+R} t X(t) \left(\int_{\arcsin\left(\frac{t-u}{R}\right)}^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right) \cos\theta \sin\theta \, d\theta \right) dt \\
&+ \frac{\sqrt{\rho} R^3 u}{4} \int_{u}^{u+R} X(t) \left(\int_0^{\arcsin\left(\frac{t-u}{R}\right)} I_0\left(\frac{R}{2}\cos\theta\right) \cos\theta \sin^2\theta \, d\theta \right) dt \\
&+ \frac{\sqrt{\rho} R u}{4} \int_{u}^{u+R} t^2 X(t) \left(\int_0^{\arcsin\left(\frac{t-u}{R}\right)} I_0\left(\frac{R}{2}\cos\theta\right) \cos\theta \, d\theta \right) dt \\
&+ \frac{\sqrt{\rho} R^3 u}{4} \int_{u+R}^{\infty} X(t) \left(\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right) \cos\theta \sin^2\theta \, d\theta \right) dt \\
&+ \frac{\sqrt{\rho} R u}{4} \int_{u+R}^{\infty} t^2 X(t) \left(\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right) \cos\theta \, d\theta \right) dt.
\end{aligned}
$$

Note that $\mathcal{I}_1$ is the contribution from $\hat{\mathcal{J}}_1$; $\mathcal{II}_1$ from $\hat{\mathcal{J}}_2$; all other terms are from $\hat{\mathcal{J}}_3$.

Lemma 2.16. *The term Σ_1 admits the following lower bound,*

$$
\Sigma_1(\rho, u) \geq -C\left(\rho|u|^2 + \rho + \rho(\log\rho)^2\right) \qquad \text{for } (\rho, u) \in [1, \infty) \times \mathbb{R}. \tag{2.28}
$$

Proof. The first term is bounded above as follows,

$$
\begin{aligned}
|\mathcal{I}_1| \leq & 2\sqrt{\rho}R^3|u|\left(\int_{\mathbb{R}}|Y(t)|\,dt\right)\left(\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\sin^2\theta\,d\theta\right) \\
& + 2\sqrt{\rho}R^2|u|\left(\int_{\mathbb{R}}t|Y(t)|\,dt\right)\left(\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\sin\theta\,d\theta\right) \\
& + 2\sqrt{\rho}R|u|\left(\int_{\mathbb{R}}t^2|Y(t)|\,dt\right)\left(\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\,d\theta\right),
\end{aligned}
$$

and, using Lemma A.1, the right-hand side is itself bounded above by

$$
\begin{aligned}
C\sqrt{\rho}|u|\bigg(R\cosh\left(\frac{R}{2}\right) + \sinh\left(\frac{R}{2}\right) + RI_1\left(\frac{R}{2}\right)\bigg) & \\
\leq C\left(\rho\log\rho|u| + \rho|u| + \rho\sqrt{\log\rho}|u|\right) & \\
\leq C\left(\rho|u|^2 + \rho + \rho(\log\rho)^2\right), &
\end{aligned}
$$

where we used the Cauchy–Schwarz inequality in the final line, along with the compact support and integrability of Y, and C is independent of (ρ, u). On the other hand, $\mathcal{II}_1$ is bounded above by

$$
\begin{aligned}
\sqrt{\rho}|u|\left(\int_{\mathbb{R}}t^2X(t)\,dt\right) + 2\sqrt{\rho}R|u|\left(\int_{\mathbb{R}}tX(t)\,dt\right) + \sqrt{\rho}R^2|u|\left(\int_{\mathbb{R}}X(t)\,dt\right) & \\
\leq C\left(\rho|u|^2 + 1 + (\log\rho)^2 + (\log\rho)^4\right) \leq C\left(\rho|u|^2 + \rho\right), &
\end{aligned}
$$

where we made use of Lemma A.8 for the final inequality. For $\mathcal{III}_1$, we have

$$
\begin{aligned}
|\mathcal{III}_1| \leq & \sqrt{\rho}R^3|u|\left(\int_{\mathbb{R}}X(t)\,dt\right)\left(\int_0^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right)\sin^2\theta\,d\theta\right) \\
& + \sqrt{\rho}R^2|u|\left(\int_{\mathbb{R}}tX(t)\,dt\right)\left(\int_0^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right)\sin\theta\,d\theta\right) \\
& + \sqrt{\rho}R|u|\left(\int_{\mathbb{R}}t^2X(t)\,dt\right)\left(\int_0^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right)\,d\theta\right),
\end{aligned}
$$

and, using Lemmas A.2 and A.3, the right-hand side is itself bounded above by

$$
\begin{aligned}
C\sqrt{\rho}|u|\bigg(R\sinh\left(\frac{R}{2}\right) + R^2 + RI_0\left(\frac{R}{2}\right) + \cosh\left(\frac{R}{2}\right) + 1\bigg) & \\
\leq C\left(\rho|u|^2 + \rho + \rho(\log\rho)^2\right), &
\end{aligned}
$$

as claimed. The final term $\mathcal{IV}_1$ is dealt with in an identical fashion to the first term $\mathcal{I}_1$ (with all integrals involving Y replaced by the corresponding ones with X). □

Lemma 2.17. *Explicit computation shows that the term of order u^2 in the expansion of $u\hat{\eta}(\rho,u)$, which we denote by Σ_2, is given by*

$$\Sigma_2 = \mathcal{I}_2 + \mathcal{II}_2 + \mathcal{III}_2 + \mathcal{IV}_2, \tag{2.29}$$

where

$$\begin{aligned}
\mathcal{I}_2 :=& -\sqrt{\rho}Ru^2 \int_{-\infty}^{u-R} tY(t) \left(\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\,d\theta \right) dt \\
&+ \sqrt{\rho}R^2u^2 \int_{u-R}^{u} Y(t) \left(\int_{\arcsin\left(\frac{u-t}{R}\right)}^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\sin\theta \right) dt \\
&- \sqrt{\rho}Ru^2 \int_{u-R}^{u} tY(t) \left(\int_0^{\arcsin\left(\frac{u-t}{R}\right)} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\,d\theta \right) dt \\
&+ \sqrt{\rho}R^2u^2 \int_{u}^{u+R} Y(t) \left(\int_{\arcsin\left(\frac{t-u}{R}\right)}^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\sin\theta \right) dt \\
&+ \sqrt{\rho}Ru^2 \int_{u}^{u+R} tY(t) \left(\int_0^{\arcsin\left(\frac{t-u}{R}\right)} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta \right) dt \\
&+ \sqrt{\rho}Ru^2 \int_{u+R}^{\infty} tY(t) \left(\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta \right) dt,
\end{aligned}$$

$$\begin{aligned}
\mathcal{II}_2 :=& \frac{\sqrt{\rho}u^2}{2}\int_{u-R}^{\infty} tX(t)\,dt + \frac{\sqrt{\rho}Ru^2}{2}\int_{u-R}^{\infty} X(t)\,dt + \frac{\sqrt{\rho}u^2}{2}\int_{u+R}^{\infty} tX(t)\,dt \\
&- \frac{\sqrt{\rho}Ru^2}{2}\int_{u+R}^{\infty} X(t)\,dt - \frac{\sqrt{\rho}u^2}{2}\int_{-\infty}^{u-R} tX(t)\,dt - \frac{\sqrt{\rho}Ru^2}{2}\int_{-\infty}^{u-R} tX(t)\,dt \\
&- \frac{\sqrt{\rho}u^2}{2}\int_{-\infty}^{u+R} tX(t)\,dt + \frac{\sqrt{\rho}Ru^2}{2}\int_{-\infty}^{u+R} X(t)\,dt,
\end{aligned}$$

$$\begin{aligned}
\mathcal{III}_2 :=& -\frac{\sqrt{\rho}Ru^2}{2}\int_{-\infty}^{u-R} tX(t) \left(\int_0^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right) d\theta \right) dt \\
&+ \frac{\sqrt{\rho}R^2u^2}{2}\int_{u-R}^{u} X(t) \left(\int_{\arcsin\left(\frac{u-t}{R}\right)}^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right)\sin\theta \right) dt \\
&- \frac{\sqrt{\rho}Ru^2}{2}\int_{u-R}^{u} tX(t) \left(\int_0^{\arcsin\left(\frac{u-t}{R}\right)} I_1\left(\frac{R}{2}\cos\theta\right) d\theta \right) dt \\
&+ \frac{\sqrt{\rho}R^2u^2}{2}\int_{u}^{u+R} X(t) \left(\int_{\arcsin\left(\frac{u-t}{R}\right)}^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right)\sin\theta \right) dt
\end{aligned}$$

$$+\frac{\sqrt{\rho}Ru^2}{2}\int_u^{u+R} tX(t)\left(\int_0^{\arcsin\left(\frac{t-u}{R}\right)} I_1\left(\frac{R}{2}\cos\theta\right)\,d\theta\right)\,dt$$

$$+\frac{\sqrt{\rho}Ru^2}{2}\int_{u+R}^{\infty} tX(t)\left(\int_0^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right)\,d\theta\right)\,dt,$$

$$\begin{aligned}
\mathcal{IV}_2 :=& \frac{\sqrt{\rho}Ru^2}{2}\int_{-\infty}^{u-R} tX(t)\left(\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\,d\theta\right)\,dt \\
&- \frac{\sqrt{\rho}R^2u^2}{2}\int_{u-R}^{u} X(t)\left(\int_{\arcsin\left(\frac{u-t}{R}\right)}^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\sin\theta\right)\,dt \\
&+ \frac{\sqrt{\rho}Ru^2}{2}\int_{u-R}^{u} tX(t)\left(\int_0^{\arcsin\left(\frac{u-t}{R}\right)} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\,d\theta\right)\,dt \\
&- \frac{\sqrt{\rho}R^2u^2}{2}\int_{u}^{u+R} X(t)\left(\int_{\arcsin\left(\frac{t-u}{R}\right)}^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\sin\theta\right)\,dt \\
&- \frac{\sqrt{\rho}Ru^2}{2}\int_{u}^{u+R} tX(t)\left(\int_0^{\arcsin\left(\frac{t-u}{R}\right)} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\right)\,dt \\
&- \frac{\sqrt{\rho}Ru^2}{2}\int_{u+R}^{\infty} tX(t)\left(\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\right)\,dt.
\end{aligned}$$

Note that the first term $\mathcal{I}_2$ is a contribution from $\hat{\mathcal{J}}_1$; the second term $\mathcal{II}_2$ from $\hat{\mathcal{J}}_2$; and all remaining terms from $\hat{\mathcal{J}}_3$.

Lemma 2.18. *Provided $Y \geq 0$, the term Σ_2 admits the following lower bound,*

$$\Sigma_2(\rho, u) \geq -C\rho|u|^2 \qquad \text{for } (\rho, u) \in [1, \infty) \times \mathbb{R}. \tag{2.30}$$

Proof. Begin by observing that the largest terms due to $\hat{\mathcal{J}}_1$, which can be grouped as

$$\sqrt{\rho}R^2u^2\int_{u-R}^{u+R} Y(t)\left(\int_{\arcsin\left(\frac{|u-t|}{R}\right)}^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\sin\theta\right)\,dt \geq 0, \tag{2.31}$$

are non-negative, and can thus be dropped from the lower bound. The largest terms due to $\hat{\mathcal{J}}_3$ can be grouped as

$$\frac{\sqrt{\rho}R^2u^2}{2}\int_{u-R}^{u+R} X(t)\cdot$$
$$\left(\int_{\arcsin\left(\frac{|u-t|}{R}\right)}^{\pi/2}\left[I_1\left(\frac{R}{2}\cos\theta\right)\sin\theta - I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\sin\theta\right]\,d\theta\right)\,dt,$$

which can be evaluated explicitly as

$$\sqrt{\rho}u^2 \int_{u-R}^{u+R} X(t) \cdot$$
$$\left[RI_0\left(\frac{\sqrt{R^2-(u-t)^2}}{2}\right) - R - \sqrt{R^2-(u-t)^2} I_1\left(\frac{\sqrt{R^2-(u-t)^2}}{2}\right)\right] dt.$$

The previous line can be split into

$$\sqrt{\rho}Ru^2 \int_{u-R}^{u+R} X(t) \cdot$$
$$\left[I_0\left(\frac{\sqrt{R^2-(u-t)^2}}{2}\right) - \sqrt{1-\left(\frac{|u-t|}{R}\right)^2} I_1\left(\frac{\sqrt{R^2-(u-t)^2}}{2}\right)\right] dt \geq 0,$$

where the inequality follows from $I_0(x) \geq I_1(x)$ for all $x \geq 0$, and

$$-\sqrt{\rho}Ru^2 \int_{u-R}^{u+R} X(t)\, dt \geq -C\sqrt{\rho}Ru^2 \geq -C\rho u^2,$$

where we made use of Lemma A.8 for the final inequality. The terms due to $\hat{\mathcal{J}}_2$ are manifestly bounded above by $C\rho u^2$, and all remaining terms are bounded above by

$$C\sqrt{\rho}Ru^2 \int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\, d\theta + C\sqrt{\rho}Ru^2 \int_0^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right) d\theta \leq C\rho u^2,$$

where we used the results of Lemmas A.1 and A.2, along with the integrability and compact support of X and Y. □

Remark 2.19. The term in (2.31) is the only place where we appeal to the non-negativity of Y. However, we can still make progress if we do not have Y non-negative. Indeed, should we have $|u| > R + k(1)$, then the whole term in (2.31) vanishes. On the other hand, if $|u| \leq R + k(1)$, then the term in (2.31) becomes bounded above by

$$\sqrt{\rho}R^2\left(R+k(1)\right)^2\left(\int_{\mathbb{R}}|Y(t)|\, dt\right)\frac{\pi}{2}\frac{C\sqrt{\rho}}{\sqrt{R}} \leq C\rho R^{3/2}\left(1+R^2\right) \leq C\left(1+\rho(\log\rho)^{7/2}\right)$$
$$\leq C\left(\rho+\rho(\log\rho)^4\right).$$

where we used Lemma A.6. In this case, our result still holds, in the following non-optimal form,

$$\Sigma_2(\rho,u) \geq -C\left(\rho|u|^2+\rho+\rho(\log\rho)^4\right).$$

Lemma 2.20. *Following the discussion in the previous remark, we have that, irrespective of the sign of Y,*

$$\Sigma_2(\rho,u) \geq -C\left(\rho|u|^2+\rho+\rho(\log\rho)^4\right) \qquad \textit{for } (\rho,u)\in[1,\infty)\times\mathbb{R}. \tag{2.32}$$

In the rest of this chapter we use Lemma 2.20, as was done in [80], where we need not assume that Y is non-negative.

Lemma 2.21. *Explicit computation shows that the term of order u^3 in the expansion of $u\hat{\eta}(\rho, u)$, which we denote by Σ_3, is given by*

$$\Sigma_3 = \mathcal{I}_3 + \mathcal{II}_3 + \mathcal{III}_3 + \mathcal{IV}_3, \tag{2.33}$$

where

$$\begin{aligned}\mathcal{I}_3 :=& \frac{\sqrt{\rho}Ru^3}{2}\int_{-\infty}^{u-R} Y(t)\left(\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\right)\,dt \\ &+ \frac{\sqrt{\rho}Ru^3}{2}\int_{u-R}^{u} Y(t)\left(\int_0^{\arcsin\left(\frac{u-t}{R}\right)} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\right)\,dt \\ &- \frac{\sqrt{\rho}Ru^3}{2}\int_{u}^{u+R} Y(t)\left(\int_0^{\arcsin\left(\frac{t-u}{R}\right)} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\right)\,dt \\ &- \frac{\sqrt{\rho}Ru^3}{2}\int_{u+R}^{\infty} Y(t)\left(\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\right)\,dt,\end{aligned}$$

$$\begin{aligned}\mathcal{II}_3 :=& \frac{\sqrt{\rho}u^3}{4}\int_{-\infty}^{u-R} X(t)\,dt + \frac{\sqrt{\rho}u^3}{4}\int_{-\infty}^{u+R} X(t)\,dt - \frac{\sqrt{\rho}u^3}{4}\int_{u-R}^{\infty} X(t)\,dt \\ &- \frac{\sqrt{\rho}u^3}{4}\int_{u+R}^{\infty} X(t)\,dt,\end{aligned}$$

$$\begin{aligned}\mathcal{III}_3 :=& \frac{\sqrt{\rho}Ru^3}{4}\int_{-\infty}^{u-R} X(t)\left(\int_0^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right)\,d\theta\right)\,dt \\ &+ \frac{\sqrt{\rho}Ru^3}{4}\int_{u-R}^{u} X(t)\left(\int_0^{\arcsin\left(\frac{u-t}{R}\right)} I_1\left(\frac{R}{2}\cos\theta\right)\,d\theta\right)\,dt \\ &- \frac{\sqrt{\rho}Ru^3}{4}\int_{u}^{u+R} X(t)\left(\int_0^{\arcsin\left(\frac{t-u}{R}\right)} I_1\left(\frac{R}{2}\cos\theta\right)\,d\theta\right)\,dt \\ &- \frac{\sqrt{\rho}Ru^3}{4}\int_{u+R}^{\infty} X(t)\left(\int_0^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right)\,d\theta\right)\,dt,\end{aligned}$$

$$\begin{aligned}
\mathcal{IV}_3 :=& -\frac{\sqrt{\rho}Ru^3}{4}\int_{-\infty}^{u-R} X(t)\left(\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\right)\,dt \\
&-\frac{\sqrt{\rho}Ru^3}{4}\int_{u-R}^{u} X(t)\left(\int_0^{\arcsin\left(\frac{u-t}{R}\right)} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\,d\theta\right)\,dt \\
&+\frac{\sqrt{\rho}Ru^3}{4}\int_{u}^{u+R} X(t)\left(\int_0^{\arcsin\left(\frac{t-u}{R}\right)} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\right)\,dt \\
&+\frac{\sqrt{\rho}Ru^3}{4}\int_{u+R}^{\infty} X(t)\left(\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\right)\,dt.
\end{aligned}$$

Note that the first term $\mathcal{I}_3$ is a contribution from $\hat{\mathcal{J}}_1$; the second term $\mathcal{II}_3$ is from $\hat{\mathcal{J}}_2$; and all remaining terms are from $\hat{\mathcal{J}}_3$.

Lemma 2.22. *Assume $\rho \geq 1$. Provided that $|u| \leq 2k(1)$, we have that $\Sigma_3 \geq -C\rho|u|^2$ for some positive constant C. If, on the other hand, $|u| \geq 2k(1)$, then*

$$\begin{cases} \Sigma_3 \geq C\rho|u|^3, & \textit{provided } |u| \geq R + k(1) \textit{ and } |u| \geq 2k(1), \\ \Sigma_3 \geq -C\left(\rho|u|^2 + \rho + \rho(\log\rho)^4\right), & \textit{provided } |u| < R + k(1) \textit{ and } |u| \geq 2k(1). \end{cases} \tag{2.34}$$

Proof. For the first result, observe that, by bounding bluntly,

$$\begin{aligned}
|\Sigma_3(\rho,u)| \leq& C\sqrt{\rho}R|u|^3\left(\int_{\mathbb{R}}|Y(t)|\,dt + \int_{\mathbb{R}} X(t)\,dt\right)\left(\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\,d\theta\right) \\
&+ C\sqrt{\rho}R|u|^3\left(\int_{\mathbb{R}} X(t)\,dt\right)\left(\int_0^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right)\,d\theta\right) + C\sqrt{\rho}|u|^3.
\end{aligned}$$

Using Lemmas A.1 and A.2, the above simplifies to $C\sqrt{\rho}|u|^3\left(\sqrt{\rho}+1\right) \leq C\rho|u|^3$. So, with the assumption $|u| \leq 2k(1)$, we get that $|\Sigma_3| \leq C\rho|u|^2$, thereby yielding

$$\Sigma_3(\rho,u) \geq -C\rho|u|^2, \quad \text{provided } |u| \leq 2k(1).$$

Assume for the remainder of the proof that $u \geq 2k(1)$. Then, Σ_3 is equal to

$$\begin{aligned}
&\frac{\sqrt{\rho}Ru^3}{2}\int_{-k(1)}^{u-R} Y(t)\left(\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\right)\,dt \\
&+\frac{\sqrt{\rho}Ru^3}{2}\int_{u-R}^{k(1)} Y(t)\left(\int_0^{\arcsin\left(\frac{u-t}{R}\right)} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\right)\,dt \\
&+\frac{\sqrt{\rho}u^3}{4}\int_{-k(1)}^{u-R} X(t)\,dt + \frac{\sqrt{\rho}u^3}{4}\int_{-k(1)}^{k(1)} X(t)\,dt - \frac{\sqrt{\rho}u^3}{4}\int_{u-R}^{k(1)} X(t)\,dt
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\sqrt{\rho}Ru^3}{4}\int_{-k(1)}^{u-R} X(t)\left(\int_0^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right)\,d\theta\right)\,dt\\
&+\frac{\sqrt{\rho}Ru^3}{4}\int_{u-R}^{k(1)} X(t)\left(\int_0^{\arcsin\left(\frac{u-t}{R}\right)} I_1\left(\frac{R}{2}\cos\theta\right)\,d\theta\right)\,dt\\
&-\frac{\sqrt{\rho}Ru^3}{4}\int_{-k(1)}^{u-R} X(t)\left(\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\,d\theta\right)\,dt\\
&-\frac{\sqrt{\rho}Ru^3}{4}\int_{u-R}^{k(1)} X(t)\left(\int_0^{\arcsin\left(\frac{u-t}{R}\right)} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\,d\theta\right)\,dt.
\end{aligned}
$$

Suppose firstly that $u-R\geq k(1)$, then Σ_3 reduces to

$$
\begin{aligned}
&\frac{\sqrt{\rho}Ru^3}{2}\int_{-k(1)}^{k(1)} Y(t)\left(\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\,d\theta\right)\,dt+\frac{\sqrt{\rho}u^3}{2}\int_{-k(1)}^{k(1)} X(t)\,dt\\
&+\frac{\sqrt{\rho}Ru^3}{4}\int_{-k(1)}^{k(1)} X(t)\left(\int_0^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right)\,d\theta\right)\,dt\\
&-\frac{\sqrt{\rho}Ru^3}{4}\int_{-k(1)}^{k(1)} X(t)\left(\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\,d\theta\right)\,dt,
\end{aligned}
$$

which, due to (2.18), is equal to

$$
\frac{c_{total}\sqrt{\rho}u^3}{2}+\frac{c_{total}\sqrt{\rho}Ru^3}{4}\left(\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\,d\theta+\int_0^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right)\,d\theta\right).
$$

Evaluating the latter integrals exactly using Lemmas A.1 and A.2, we deduce that

$$
\Sigma_3(\rho,u)=\frac{c_{total}}{2}\rho u^3,\quad \text{provided } u-R\geq k(1) \text{ and } u\geq 2k(1).
$$

If, on the other hand, we suppose that $u-R<k(1)$ (we note that this can only be true provided $R>k(1)$), then $u\leq R+k(1)$. Thus, we bound Σ_3 by

$$
\begin{aligned}
|\Sigma_3|\leq &C\sqrt{\rho}R|u|\left(R+k(1)\right)^2\left(\int_{\mathbb{R}}|Y(t)|\,dt+c_{total}\right)\left(\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\,d\theta\right)\\
&+C\sqrt{\rho}R|u|\left(R+k(1)\right)^2\left(\int_{\mathbb{R}}X(t)\,dt\right)\left(\int_0^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right)\,d\theta\right)\\
&+C\sqrt{\rho}|u|(R+k(1))^2,
\end{aligned}
$$

which, using Lemmas A.1 and A.2, is bounded by $C\rho\left(1+(\log\rho)^2\right)|u|$. Using the Cauchy–Young inequality then yields

$$
\Sigma_3(\rho,u)\geq -C\left(\rho|u|^2+\rho+\rho(\log\rho)^4\right),\quad \text{provided } u-R<k(1) \text{ and } u\geq 2k(1).
$$

Following the same strategy of proof with $u\leq -2k(1)$ gives the results for $u<0$. □

Lemma 2.23. *There exists a positive constant C such that*

$$\Sigma_3(\rho, u) \geq C^{-1}\rho|u|^3 - C\left(\rho|u|^2 + \rho + \rho(\log\rho)^4\right) \qquad \text{for } (\rho, u) \in [1, \infty) \times \mathbb{R}. \quad (2.35)$$

Consequently,

$$u\hat{\eta}(\rho, u) \geq C^{-1}\rho|u|^3 - C\big(\rho|u|^2 + \rho + \rho(\log\rho)^4\big) \qquad \text{for } (\rho, u) \in [1, \infty) \times \mathbb{R}. \quad (2.36)$$

Proof. We have split the analysis of Σ_3 over three distinct regions $F_1 := A$, $F_2 := A^c \cap B$, and $F_3 := A^c \cap B^c$, where

$$A := \{u \in \mathbb{R} : |u| < 2k(1)\}, \qquad B := \{u \in \mathbb{R} : |u| \geq \log\rho + k(1)\}.$$

Note that $F_1 \cup F_2 \cup F_3 = A \cup (A^c \cap (B \cup B^c)) = A \cup A^c = \mathbb{R}$. Additionally, $F_i \cap F_j = \emptyset$ whenever $i \neq j$, so these sets are disjoint. Then, from Lemma 2.22, we have

$$\begin{aligned}\Sigma_3 =& \Sigma_3 \mathbb{1}_{F_1} + \Sigma_3 \mathbb{1}_{F_2} + \Sigma_3 \mathbb{1}_{F_3} \\ \geq& - C_1\rho|u|^2\mathbb{1}_{F_1} + C_2\rho|u|^3\mathbb{1}_{F_2} - C_3\left(\rho|u|^2 + \rho + \rho(\log\rho)^4\right)\mathbb{1}_{F_3},\end{aligned}$$

and the above can be bounded from below by

$$-C\big(\rho|u|^2 + \rho + \rho(\log\rho)^4\big) + C^{-1}\rho|u|^3\mathbb{1}_{F_2}.$$

However, observe that

$$\rho|u|^3\mathbb{1}_{F_2} = \rho|u|^3\left(1 - \mathbb{1}_{F_2^c}\right),$$

and $F_2^c = A \cup B^c$, so $\mathbb{1}_{F_2^c} \leq \mathbb{1}_A + \mathbb{1}_{B^c}$. Therefore,

$$\rho|u|^3\mathbb{1}_{F_2^c} \leq \rho|u|^3\mathbb{1}_A + \rho|u|^3\mathbb{1}_{B^c} \leq C\rho\big(1 + (\log\rho)^3\big).$$

As a result,

$$\rho|u|^3\mathbb{1}_{F_2} \geq \rho|u|^3 - C\big(\rho + \rho(\log\rho)^3\big),$$

from which the conclusion of the lemma is immediate. □

2.4.3 Bounds on the entropy and its derivatives

In what follows, we estimate the special entropy $\hat{\eta}$ and its derivatives.

Lemma 2.24. *In the isothermal region $\{\rho \geq 1\}$, the entropy satisfies*

$$|\hat{\eta}(\rho, u)| \leq C\left(\rho|u|^2 + \rho + \rho\log\rho\right). \quad (2.37)$$

Proof. Following the decomposition (2.25) and Lemmas B.1-B.3, we observe the following

$$
\begin{aligned}
|\hat{\mathcal{J}}_1(\rho,u)| \leq & 2\sqrt{\rho}R\int_{\mathbb{R}} |Y(t)||u-t|^2 \left(\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\,d\theta\right) dt \\
& + 2\sqrt{\rho}R^3 \int_{\mathbb{R}} |Y(t)| \left(\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\sin^2\theta\,d\theta\right) dt \\
& + 2\sqrt{\rho}R^2 \int_{\mathbb{R}} |Y(t)||u-t| \left(\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\sin\theta\,d\theta\right) dt.
\end{aligned}
$$

Evaluating each of the inner integrals using Lemmas A.1 and A.2, we get

$$
\begin{aligned}
|\hat{\mathcal{J}}_1(\rho,u)| \leq 2(\rho-1)\int_{\mathbb{R}} |Y(t)||u-t|^2\,dt + 4\big((\rho+1)R - 2(\rho-1)\big)\int_{\mathbb{R}} |Y(t)|\,dt \\
+ 4\sqrt{\rho}RI_1\left(\frac{R}{2}\right)\int_{\mathbb{R}} |Y(t)||u-t|\,dt,
\end{aligned}
$$

which shows that

$$
|\hat{\mathcal{J}}_1(\rho,u)| \leq C\left(\rho|u|^2 + \rho + \rho\log\rho\right) + 4\sqrt{\rho}RI_1\left(\frac{R}{2}\right)\int_{\mathbb{R}} |Y(t)||u-t|\,dt.
$$

Now using Lemma A.6, we get that $\sqrt{\rho}RI_1(R/2) \leq C\rho\sqrt{\log\rho}$. By the Cauchy–Schwarz inequality, we have

$$
|\hat{\mathcal{J}}_1(\rho,u)| \leq C\left(\rho|u|^2 + \rho + \rho\log\rho\right). \tag{2.38}
$$

Remark 2.25. Throughout the above manipulations, we relied on the finiteness of $\|Y\|_{L^1(\mathbb{R})}$ and on the compact support of Y, along with the relations $\sinh(\frac{R}{2}) = \frac{\rho-1}{2\sqrt{\rho}}$ and $\cosh(\frac{R}{2}) = \frac{\rho+1}{2\sqrt{\rho}}$.

The term $\hat{\mathcal{J}}_3$ is manifestly bounded by the two quantities

$$
\begin{aligned}
A := & \sqrt{\rho}R\int_{\mathbb{R}} X(t)|u-t|^2 \left(\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\,d\theta\right) dt \\
& + \sqrt{\rho}R^3 \int_{\mathbb{R}} X(t) \left(\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\sin^2\theta\,d\theta\right) dt \\
& + \sqrt{\rho}R^2 \int_{\mathbb{R}} X(t)|u-t| \left(\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\sin\theta\,d\theta\right) dt,
\end{aligned}
$$

and

$$
\begin{aligned}
B :=&\sqrt{\rho}R\int_{\mathbb{R}} X(t)|u-t|^2\left(\int_0^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right)\,d\theta\right)\,dt\\
&+\sqrt{\rho}R^3\int_{\mathbb{R}} X(t)\left(\int_0^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right)\sin^2\theta\,d\theta\right)\,dt\\
&+\sqrt{\rho}R^2\int_{\mathbb{R}} X(t)|u-t|\left(\int_0^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right)\sin\theta\,d\theta\right)\,dt.
\end{aligned}
$$

It is immediate from our bound on $|\hat{\mathcal{J}}_1|$ that $|A|\leq C\left(\rho|u|^2+\rho+\rho\log\rho\right)$. Evaluating the inner integrals of B using Lemmas A.2 and A.3, we find

$$
\begin{aligned}
B =&2\sqrt{\rho}\left(\cosh\left(\frac{R}{2}\right)-1\right)\int_{\mathbb{R}} X(t)|u-t|^2\,dt\\
&+2\sqrt{\rho}R\left(2\sinh\left(\frac{R}{2}\right)-R\right)\int_{\mathbb{R}} X(t)\,dt\\
&+2\sqrt{\rho}R\left(I_0\left(\frac{R}{2}\right)-1\right)\int_{\mathbb{R}} X(t)|u-t|\,dt.
\end{aligned}
$$

Thus, using the relations $\sinh(\frac{R}{2})=\frac{\rho-1}{2\sqrt{\rho}}$ and $\cosh(\frac{R}{2})=\frac{\rho+1}{2\sqrt{\rho}}$, we obtain

$$
\begin{aligned}
|B|\leq (\sqrt{\rho}-1)^2\int_{\mathbb{R}} X(t)|u-t|^2\,dt+2\log\rho\,(\rho-1-\sqrt{\rho}\log\rho)\int_{\mathbb{R}} X(t)\,dt&\\
+2\sqrt{\rho}\log\rho I_0\left(\frac{\log\rho}{2}\right)\int_{\mathbb{R}} X(t)|u-t|\,dt.&
\end{aligned}
$$

It follows that $|B|\leq C\left(\rho|u|^2+\rho+\rho\log\rho\right)$. Hence,

$$
|\hat{\mathcal{J}}_3(\rho,u)|\leq C\left(\rho|u|^2+\rho+\rho\log\rho\right),
$$

as claimed. Manifestly, we also have

$$
|\hat{\mathcal{J}}_2(\rho,u)|\leq C\sqrt{\rho}\left(1+(\log\rho)^2+|u|^2\right)\leq C\left(\rho|u|^2+\rho+\rho\log\rho\right),
$$

since $\rho>1$. Hence, the lemma is proved. □

We now bound the derivatives of $\hat{\eta}$. In particular, we have the following.

Lemma 2.26. *The partial derivatives with respect to m of the entropy satisfy the following bounds in the isothermal region $\{\rho\geq 1\}$,*

$$
|\hat{\eta}_m(\rho,u)|\leq C\left(|u|+1+\sqrt{\log\rho}\right),\qquad |\hat{\eta}_{mm}(\rho,u)|\leq C\rho^{-1}. \tag{2.39}
$$

Proof. Note firstly that, by the chain rule, we have

$$\eta_u = \left.\frac{\partial \eta}{\partial u}\right|_\rho = \left.\frac{\partial \eta}{\partial \rho}\right|_m \left.\frac{\partial \rho}{\partial u}\right|_\rho + \left.\frac{\partial \eta}{\partial m}\right|_\rho \left.\frac{\partial m}{\partial u}\right|_\rho = \rho \left.\frac{\partial \eta}{\partial m}\right|_\rho = \rho \eta_m,$$

where, for instance, $\frac{\partial \eta}{\partial u}|_\rho$ is the partial derivative with respect to u while keeping ρ constant. Similarly, $\eta_{uu} = \rho^2 \eta_{mm}$. Hence, it suffices to show that

$$|\hat{\eta}_u(\rho, u)| \leq C\left(\rho|u| + \rho + \rho\sqrt{\log \rho}\right), \qquad |\hat{\eta}_{uu}(\rho, u)| \leq C\rho, \tag{2.40}$$

Following the decomposition for $\hat{\eta}$ in (2.25), we split $\hat{\eta}_u$ into three pieces $\hat{\eta}_u(\rho, u) = \partial_u \hat{\mathcal{J}}_1(\rho, u) + \partial_u \hat{\mathcal{J}}_2(\rho, u) + \partial_u \hat{\mathcal{J}}_3(\rho, u)$. Explicit calculation (*cf.* Appendix B.1) yields

$$\begin{aligned}\partial_u \hat{\mathcal{J}}_1 =& \frac{\sqrt{\rho}}{4}\int_{-\infty}^{u-R} Y(t)\partial_u L_1^+(\rho, u; t)\,dt + \frac{\sqrt{\rho}}{4}\int_{u-R}^{u} Y(t)\partial_u K_1^+(\rho, u; t)\,dt \\ &+ \frac{\sqrt{\rho}}{4}\int_{u}^{u+R} Y(t)\partial_u K_1^-(\rho, u; t)\,dt + \frac{\sqrt{\rho}}{4}\int_{u+R}^{\infty} Y(t)\partial_u L_1^-(\rho, u; t)\,dt,\end{aligned}$$

where $K_1^\pm, L_1^\pm$ are defined in Appendix B.1. From this, we find, using Lemma A.1,

$$\begin{aligned}|\partial_u \hat{\mathcal{J}}_1| \leq& C\sqrt{\rho}R\int_{\mathbb{R}} |Y(t)||u-t|\left(\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\,d\theta\right)dt \\ &+ C\sqrt{\rho}R I_1\left(\frac{R}{2}\right)\int_{\mathbb{R}} |Y(t)|\,dt \\ \leq& C(\rho-1)(1+|u|) + C\sqrt{\rho}R I_1\left(\frac{R}{2}\right) \leq C\left(\rho|u| + \rho + \rho\sqrt{\log\rho}\right).\end{aligned}$$

Observe that, since we control $\hat{\mathcal{J}}_1$, the term $\hat{\mathcal{J}}_3$ will be controlled provided we can bound the term $\int_{\mathbb{R}} \mathcal{K}_2(\rho, u; t)Y(t)\,dt =: \hat{J}$ appropriately (refer to Appendix B.3 for the definition of $\mathcal{K}_2$). To this end, we compute

$$\begin{aligned}\partial_u \hat{J} =& \frac{\sqrt{\rho}}{4}\int_{-\infty}^{u-R} X(t)\partial_u L_2^+(\rho, u; t)\,dt + \frac{\sqrt{\rho}}{4}\int_{u-R}^{u} X(t)\partial_u K_2^+(\rho, u; t)\,dt \\ &+ \frac{\sqrt{\rho}}{4}\int_{u}^{u+R} X(t)\partial_u K_2^-(\rho, u; t)\,dt + \frac{\sqrt{\rho}}{4}\int_{u+R}^{\infty} X(t)\partial_u L_2^-(\rho, u; t)\,dt,\end{aligned}$$

where the terms $K_2^\pm, L_2^\pm$ are defined in Appendix B.3. Thus, using Lemmas B.24 and B.26, in conjunction with Lemmas A.2 and A.3,

$$\begin{aligned}|\partial_u \hat{J}| \leq& C\sqrt{\rho}R\int_{\mathbb{R}} X(t)|u-t|\left(\int_0^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right)d\theta\right)dt \\ &+ C\sqrt{\rho}R\left(I_0\left(\frac{R}{2}\right) - 1\right)\int_{\mathbb{R}} X(t)\,dt \\ \leq& C(\rho-1)(1+|u|) + C\sqrt{\rho}R\left(I_0\left(\frac{R}{2}\right) - 1\right) \leq C\left(\rho|u| + \rho + \rho\sqrt{\log\rho}\right).\end{aligned}$$

In addition we manifestly have, using Lemma B.16,

$$|\partial_u \hat{\mathcal{J}}_2| \leq C\sqrt{\rho}\,(1+\log\rho+|u|)\,,$$

which, using Lemma A.8, confirms that we do indeed obtain the bound (2.40).

In addition, Appendix B.1 shows that

$$\begin{aligned}\partial_{uu}\hat{\mathcal{J}}_1 =& \frac{\sqrt{\rho}}{4}\int_{-\infty}^{u-R} Y(t)\partial_{uu}L_1^+(\rho,u;t)\,dt + \frac{\sqrt{\rho}}{4}\int_{u-R}^{u} Y(t)\partial_{uu}K_1^+(\rho,u;t)\,dt \\ &+ \frac{\sqrt{\rho}}{4}\int_{u}^{u+R} Y(t)\partial_{uu}K_1^-(\rho,u;t)\,dt + \frac{\sqrt{\rho}}{4}\int_{u+R}^{\infty} Y(t)\partial_{uu}L_1^-(\rho,u;t)\,dt.\end{aligned}$$

Thus, an application of Lemmas A.1, B.9, and B.10 yields, using $\|Y\|_{L^1(\mathbb{R})}$ finite,

$$|\partial_{uu}\hat{\mathcal{J}}_1| \leq C\sqrt{\rho}R\int_{\mathbb{R}}|Y(t)|\left(\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\,d\theta\right)\,dt = C(\rho-1).$$

We now consider the term $\partial_{uu}\hat{\mathcal{J}}_2$. By explicit computation (*cf.* Lemma B.17), we obtain $\partial_{uu}\hat{\mathcal{J}}_2 = \partial_{uu}\hat{\mathcal{J}}_2^- + \partial_{uu}\hat{\mathcal{J}}_2^+$, where

$$\begin{aligned}\partial_{uu}\hat{\mathcal{J}}_2^- &= -\frac{\sqrt{\rho}}{2}\int_u^{\infty}[X(t-R)+X(t+R)]\,dt, \\ \partial_{uu}\hat{\mathcal{J}}_2^+ &= \frac{\sqrt{\rho}}{2}\int_{-\infty}^{u}[X(t-R)+X(t+R)]\,dt.\end{aligned}$$

Thus,

$$\begin{aligned}\partial_{uu}\hat{\mathcal{J}}_2 =& -\frac{\sqrt{\rho}}{2}\int_{u-R}^{\infty}X(t)\,dt - \frac{\sqrt{\rho}}{2}\int_{u+R}^{\infty}X(t)\,dt \\ &+ \frac{\sqrt{\rho}}{2}\int_{-\infty}^{u-R}X(t)\,dt + \frac{\sqrt{\rho}}{2}\int_{-\infty}^{u+R}X(t)\,dt,\end{aligned}$$

thereby implying that $|\partial_{uu}\hat{\mathcal{J}}_2| \leq C\sqrt{\rho}$.

Now turn to $\partial_{uu}\hat{\mathcal{J}}_3$, which is controlled by $\partial_{uu}\hat{\mathcal{J}}_1$ and by $\partial_{uu}\hat{J}$, where

$$\begin{aligned}\partial_{uu}\hat{J} =& \frac{\sqrt{\rho}}{8}\int_{-\infty}^{u-R} X(t)\partial_{uu}L_2^+(\rho,u;t)\,dt + \frac{\sqrt{\rho}}{8}\int_{u-R}^{u} X(t)\partial_{uu}K_2^+(\rho,u;t)\,dt \\ &+ \frac{\sqrt{\rho}}{8}\int_{u}^{u+R} X(t)\partial_{uu}K_2^-(\rho,u;t)\,dt + \frac{\sqrt{\rho}}{8}\int_{u+R}^{\infty} X(t)\partial_{uu}L_2^-(\rho,u;t)\,dt,\end{aligned}$$

as shown in Appendix B.3. Thus, using Lemmas B.28 and B.29,

$$|\partial_{uu}\hat{J}| \leq C\sqrt{\rho}R\int_{\mathbb{R}}X(t)\left(\int_0^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right)\,d\theta\right)\,dt = C\sqrt{\rho}\left(\cosh\left(\frac{R}{2}\right)-1\right),$$

where we used Lemma A.2 in the final equality. In view of $\cosh(\frac{R}{2}) = \frac{\rho+1}{2\sqrt{\rho}}$, the right-hand side is bounded by a constant multiple of ρ. This concludes the proof. □

It is important for the uniform estimates of Section 2.5 to view $\hat{\eta}_m$ as a function of (ρ, u), in order to integrate by parts. This leads to the following definition.

Definition 2.27. We define the *fake mixed derivatives* $\hat{\eta}_{m\rho}$ and $\hat{\eta}_{mu}$ to be those of Lemma 2.5, i.e.,

$$\hat{\eta}_{m\rho} := \frac{\partial}{\partial\rho}\bigg|_u \frac{\partial}{\partial m}\bigg|_\rho \hat{\eta}, \qquad \hat{\eta}_{mu} := \frac{\partial}{\partial u}\bigg|_\rho \frac{\partial}{\partial m}\bigg|_\rho \hat{\eta}. \tag{2.41}$$

Lemma 2.28. *The fake mixed derivative* $\hat{\eta}_{m\rho}$ *admits the representation*

$$\begin{aligned}
\hat{\eta}_{m\rho} =& \frac{1}{\rho^2}\int_{-\infty}^{u-R} tX(t)\,dt - \frac{1}{\rho^2}\int_{-\infty}^{u-R} tY(t)\,dt - \frac{1}{\rho^2}\int_{u+R}^{\infty} tX(t)\,dt + \frac{1}{\rho^2}\int_{u+R}^{\infty} tY(t)\,dt \\
&- \frac{R}{2\rho^{3/2}}\int_{u-R}^{u+R} X(t)\,dt + \frac{1}{\rho^{3/2}}\int_{u-R}^{u+R} X(t)\,dt \\
&+ \int_{u-R}^{u} Y(t)\left(\frac{\partial_{\rho u}K_1^+(\rho,u;t)}{4\sqrt{\rho}} - \frac{\partial_u K_1^+(\rho,u;t)}{8\rho^{3/2}}\right)dt \\
&+ \int_{u}^{u+R} Y(t)\left(\frac{\partial_{\rho u}K_1^-(\rho,u;t)}{4\sqrt{\rho}} - \frac{\partial_u K_1^-(\rho,u;t)}{8\rho^{3/2}}\right)dt \\
&+ \int_{u-R}^{u} X(t)\left(\frac{\partial_{\rho u}K_2^+(\rho,u;t)}{8\sqrt{\rho}} - \frac{\partial_u K_2^+(\rho,u;t)}{16\rho^{3/2}}\right)dt \\
&+ \int_{u}^{u+R} X(t)\left(\frac{\partial_{\rho u}K_2^-(\rho,u;t)}{8\sqrt{\rho}} - \frac{\partial_u K_2^-(\rho,u;t)}{16\rho^{3/2}}\right)dt \\
&- \int_{u-R}^{u} X(t)\left(\frac{\partial_{\rho u}K_1^+(\rho,u;t)}{8\sqrt{\rho}} - \frac{\partial_u K_1^+(\rho,u;t)}{16\rho^{3/2}}\right)dt \\
&- \int_{u}^{u+R} X(t)\left(\frac{\partial_{\rho u}K_1^-(\rho,u;t)}{8\sqrt{\rho}} - \frac{\partial_u K_1^-(\rho,u;t)}{16\rho^{3/2}}\right)dt,
\end{aligned} \tag{2.42}$$

where the functions $K_1^\pm$ *and* $K_2^\pm$ *are defined in Appendix B.1 and B.3, respectively.*

Proof. Refer to the explicit calculations given in Appendix B.4. □

The next result is indispensable for the uniform estimates of Section 2.5; in particular for the main energy estimate Lemma 2.47, and for the higher integrability of the velocity (*cf.* Lemma 2.51). The proof of this result relies on intricate bounds on the zeroth and first modified Bessel functions of the first kind (*cf.* Appendix A).

Lemma 2.29. *There exists a positive constant* C *such that*

$$\|\hat{\eta}_{m\rho}(\rho,\cdot)\|_{L^\infty(\mathbb{R})} \leq C\rho^{-1} \qquad \text{for } \rho \geq 1. \tag{2.43}$$

Proof. To begin with, we make use of the compact support of X and Y and their integrability to deduce that Lines 1 and 2 of (2.42) are bounded by $CR\rho^{-3/2}$. Next, using Lemma B.36, Lines 3 and 4 of (2.42) are grouped as follows,

$$\begin{aligned}
&\frac{1}{\rho^{3/2}}\int_{u-R}^{u+R}|t-u|Y(t)\Bigg\{\left(1-\frac{R}{2}\right)\int_0^{\arcsin\left(\frac{|t-u|}{R}\right)} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\,d\theta \\
&\qquad\qquad + \frac{R}{2}\int_0^{\arcsin\left(\frac{|t-u|}{R}\right)} I_1\left(\frac{R}{2}\cos\theta\right)\cos^2\theta\,d\theta\Bigg\}\,dt \\
&-\frac{1}{R\rho^{3/2}}\int_{u-R}^{u+R}(t-u)^2Y(t)I_0\left(\frac{\sqrt{R^2-(t-u)^2}}{2}\right)dt \\
&+\frac{1}{\rho^{3/2}}\int_{u-R}^{u+R}Y(t)\Bigg[RI_0\bigg(\frac{\sqrt{R^2-(t-u)^2}}{2}\bigg) \\
&\qquad\qquad - \sqrt{R^2-(t-u)^2}\,I_1\left(\frac{\sqrt{R^2-(t-u)^2}}{2}\right)\Bigg]\,dt.
\end{aligned} \tag{2.44}$$

The inner integral of the first term in (2.44) can be rewritten as

$$\begin{aligned}
&\int_0^{\arcsin\left(\frac{|t-u|}{R}\right)} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\,d\theta \\
&\qquad -\frac{R}{2}\int_0^{\arcsin\left(\frac{|t-u|}{R}\right)}\left[I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta - I_1\left(\frac{R}{2}\cos\theta\right)\cos^2\theta\right]\,d\theta.
\end{aligned}$$

Each integrand is non-negative, as $I_0(x)\geq I_1(x)$ for $x\geq 0$, and thus we may bound the above (in absolute value) by

$$\begin{aligned}
&\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\,d\theta \\
&\qquad +\frac{R}{2}\int_0^{\pi/2}\left[I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta - I_1\left(\frac{R}{2}\cos\theta\right)\cos^2\theta\right]\,d\theta \\
&\qquad = \frac{4\sinh\left(\frac{R}{2}\right)}{R} - e^{-R/2} \\
&\qquad \leq \left(\frac{\sqrt{\rho}}{R}+\frac{1}{\sqrt{\rho}}\right) \leq C\frac{\sqrt{\rho}}{R},
\end{aligned}$$

where we used Lemmas A.1 and A.2 to evaluate the integrals exactly. Hence, the whole of the first term in (2.44) is bounded (in absolute value) by

$$\frac{C}{\rho R}\int_{u-R}^{u+R}|t-u||Y(t)|\,dt \leq \frac{C}{\rho}\int_{u-R}^{u+R}|Y(t)| \leq C\rho^{-1},$$

where we used the facts that Y is compactly supported and integrable.

The second term in (2.44) is bounded by

$$\frac{\sup_{v\in[0,R]} vI_0\left(\frac{\sqrt{R^2-v^2}}{2}\right)}{R\rho^{3/2}}\int_{u-R}^{u+R}|t-u||Y(t)|\,dt \le \rho^{-1}\int_{u-R}^{u+R}|Y(t)|\,dt$$
$$\le C\rho^{-1},$$

where we again made use of the facts that Y is compactly supported and integrable, and appealed to Lemma A.9 to bound the supremum.

The final term in (2.44) is bounded above (in absolute value) by

$$\frac{1}{\rho^{3/2}}\sup_{v\in[0,R]}\left|RI_0\left(\frac{\sqrt{R^2-v^2}}{2}\right)-\sqrt{R^2-v^2}I_1\left(\frac{\sqrt{R^2-v^2}}{2}\right)\right|\int_{u-R}^{u+R}|Y(t)|\,dt$$
$$\le C\rho^{-1}\|Y\|_{L^1(\mathbb{R})}.$$

where we once again made use of the compact support of Y and $\|Y\|_{L^1(\mathbb{R})}<\infty$, along with Lemma A.9 to bound the supremum.

Again using Lemma B.36, Lines 7 and 8 of (2.42) can be written as

$$\begin{aligned}
&-\frac{1}{2\rho^{3/2}}\int_{u-R}^{u+R}|t-u|X(t)\Bigg\{\left(1-\frac{R}{2}\right)\int_0^{\arcsin\left(\frac{|t-u|}{R}\right)}I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\,d\theta\\
&\qquad\qquad+\frac{R}{2}\int_0^{\arcsin\left(\frac{|t-u|}{R}\right)}I_1\left(\frac{R}{2}\cos\theta\right)\cos^2\theta\,d\theta\Bigg\}\\
&+\frac{1}{2R\rho^{3/2}}\int_{u-R}^{u+R}(t-u)^2X(t)I_0\left(\frac{\sqrt{R^2-(t-u)^2}}{2}\right)dt\\
&-\frac{1}{2\rho^{3/2}}\int_{u-R}^{u+R}X(t)\Bigg[RI_0\left(\frac{\sqrt{R^2-(t-u)^2}}{2}\right)\\
&\qquad\qquad-\sqrt{R^2-(t-u)^2}I_1\left(\frac{\sqrt{R^2-(t-u)^2}}{2}\right)\Bigg]dt.
\end{aligned}\tag{2.45}$$

We rewrite the first line of (2.45) as

$$\begin{aligned}
&-\frac{1}{2\rho^{3/2}}\int_{u-R}^{u+R}|t-u|X(t)\left(\int_0^{\arcsin\left(\frac{|t-u|}{R}\right)}I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\,d\theta\right)dt\\
&+\frac{R}{4\rho^{2/3}}\int_{u-R}^{u+R}|t-u|X(t)\left(\int_0^{\arcsin\left(\frac{|t-u|}{R}\right)}I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\,d\theta\right)dt\\
&-\frac{R}{4\rho^{2/3}}\int_{u-R}^{u+R}|t-u|X(t)\left(\int_0^{\arcsin\left(\frac{|t-u|}{R}\right)}I_1\left(\frac{R}{2}\cos\theta\right)\cos^2\theta\,d\theta\right)dt.
\end{aligned}\tag{2.46}$$

The first term in (2.46) is bluntly bounded by

$$\frac{R}{2\rho^{3/2}}\left(\int_0^{\pi/2}I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\,d\theta\right)\int_{u-R}^{u+R}X(t)\,dt\le C\rho^{-1},$$

where we made use of Lemma A.1 to evaluate the integral exactly. The other two terms we keep, with the aim of adding them to Lines 5 and 6 of (2.42). The last two lines of (2.45) are handled in the same fashion as the corresponding terms in Lines 3 and 4.

Using Lemma B.38, Lines 5 and 6 of (2.42) are grouped as written underneath,

$$\begin{aligned}
&\frac{1}{\rho^{3/2}}\int_{u-R}^{u+R} X(t) I_0\left(\frac{\sqrt{R^2-(t-u)^2}}{2}\right) dt - \frac{1}{\rho^{3/2}}\int_{u-R}^{u+R} X(t)\, dt \\
&-\frac{1}{2\rho^{3/2}}\int_{u-R}^{u+R} X(t)\Bigg[R I_0\left(\frac{\sqrt{R^2-(t-u)^2}}{2}\right) \\
&\qquad\qquad - \sqrt{R^2-(t-u)^2}\, I_1\left(\frac{\sqrt{R^2-(t-u)^2}}{2}\right)\Bigg] dt \\
&+\frac{R}{2\rho^{3/2}}\int_{u-R}^{u+R} X(t)\, dt \\
&+\frac{R}{4\rho^{3/2}}\int_{u-R}^{u+R} |t-u| X(t)\left(\int_0^{\arcsin\left(\frac{|t-u|}{R}\right)} \left[I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta - I_1\left(\frac{R}{2}\cos\theta\right)\right] d\theta\right) dt.
\end{aligned} \tag{2.47}$$

The second term on the first line of (2.47) is obviously bounded by $C\rho^{-3/2}$, and, using the fact that $\|X\|_{L^\infty(\mathbb{R})}$ is finite, the first term on the first line of (2.47) is bounded by

$$\frac{2R\sup_{t\in\mathbb{R}} X(t)}{\rho^{3/2}}\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\, d\theta = \frac{4\sinh\left(\frac{R}{2}\right)\sup_{t\in\mathbb{R}} X(t)}{\rho^{3/2}} \leq C\rho^{-1},$$

where the equality follows from Lemma A.1. Also, since $I_0(x) \geq I_1(x)$ for $x \geq 0$, the integrand of the term on the second line of (2.47) is positive, and is therefore bounded by

$$\begin{aligned}
\frac{R^2\sup_{t\in\mathbb{R}} X(t)}{\rho^{3/2}}&\int_0^{\pi/2}\left[I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta - I_1\left(\frac{R}{2}\cos\theta\right)\cos^2\theta\right] d\theta \\
&= \frac{2R^2\sup_{t\in\mathbb{R}} X(t)}{\rho^{3/2}}\left(\frac{2\sinh\left(\frac{R}{2}\right)}{R^2} - \frac{e^{-R/2}}{R}\right) \\
&\leq \frac{C}{\rho} + \frac{CR}{\rho^2} \leq C\rho^{-1},
\end{aligned}$$

where we used Lemmas A.1 and A.2 to evaluate the integrals exactly.

Meanwhile, the term on the third line of (2.47) is manifestly bounded above by $CR\rho^{-3/2} \leq C\rho^{-1}$.

On the other hand, the final term of (2.47) is

$$\frac{R}{4\rho^{3/2}}\int_{u-R}^{u+R}|t-u|X(t)\left(\int_0^{\arcsin\left(\frac{|t-u|}{R}\right)} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\,d\theta\right)dt$$
$$-\frac{R}{4\rho^{3/2}}\int_{u-R}^{u+R}|t-u|X(t)\left(\int_0^{\arcsin\left(\frac{|t-u|}{R}\right)} I_1\left(\frac{R}{2}\cos\theta\right)d\theta\right)dt,$$

which, when adding the remaining two terms from (2.45), becomes

$$\frac{R}{4\rho^{3/2}}\int_{u-R}^{u+R}|t-u|X(t)\Bigg(\int_0^{\arcsin\left(\frac{|t-u|}{R}\right)}\Bigg[2I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta - I_1\left(\frac{R}{2}\cos\theta\right)\cos^2\theta - I_1\left(\frac{R}{2}\cos\theta\right)\Bigg]d\theta\Bigg)dt,$$

which, using the identity $\sin^2\theta+\cos^2\theta=1$, can be expanded as

$$\frac{R}{2\rho^{3/2}}\int_{u-R}^{u+R}|t-u|X(t)\Bigg(\int_0^{\arcsin\left(\frac{|t-u|}{R}\right)}\Bigg[I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta - I_1\left(\frac{R}{2}\cos\theta\right)\cos^2\theta\Bigg]d\theta\Bigg)dt \tag{2.48}$$
$$-\frac{R}{4\rho^{3/2}}\int_{u-R}^{u+R}|t-u|X(t)\left(\int_0^{\arcsin\left(\frac{|t-u|}{R}\right)} I_1\left(\frac{R}{2}\cos\theta\right)\sin^2\theta\,d\theta\right)dt.$$

Note that the integrand of the inner integral of the first term in (2.48) is positive, as $I_0(x)\geq I_1(x)$ for $x\geq 0$. Thus, we may bound the first term in (2.48) by

$$\begin{aligned}\frac{R^2}{2\rho^{3/2}}\left(\int_{u-R}^{u+R}X(t)\,dt\right)&\left(\int_0^{\pi/2}\left[I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta - I_1\left(\frac{R}{2}\cos\theta\right)\cos^2\theta\right]d\theta\right)\\ &=\frac{C\|X\|_{L^1(\mathbb{R})}R^2}{\rho^{3/2}}\left[\frac{4\sinh\left(\frac{R}{2}\right)}{R^2}-\frac{2e^{-R/2}}{R}\right]\\ &\leq C\left(\frac{1}{\rho}-\frac{R}{\rho^2}\right)\leq C\rho^{-1},\end{aligned}$$

where we made use of Lemmas A.1 and A.2, and the finiteness of $\|X\|_{L^1(\mathbb{R})}$. As the integrand of the second term in (2.48) is also non-negative, this term is bounded by

$$\begin{aligned}\frac{R^2}{4\rho^{3/2}}\left(\int_{u-R}^{u+R}X(t)\,dt\right)\left(\int_0^{\pi/2}I_1\left(\frac{R}{2}\cos\theta\right)\sin^2\theta\,d\theta\right)&\\ =\frac{CR^2}{\rho^{3/2}}\left(\frac{4}{R^2}\sinh\left(\frac{R}{2}\right)-\frac{2}{R}\right)&\\ \leq C\left(\frac{1}{\rho}+\frac{R}{\rho^{3/2}}\right)\leq C\rho^{-1},&\end{aligned}$$

where we made use of Lemma A.2. This concludes the proof. □

2.4.4 The lower bound on the entropy-flux

Lemma 2.30. *The entropy-flux has the following decomposition,*

$$\begin{aligned}\hat{q}(\rho,u) = u\hat{\eta}(\rho,u) &+ \frac{1}{2}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} h^\sharp(\rho,u-s-t)s|s|\,ds\right)Y(t)\,dt \\ &+ \frac{1}{2}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} h^\flat(\rho,u-s-t)s|s|\,ds\right)X(t)\,dt.\end{aligned} \tag{2.49}$$

Proof. For the entropy-flux we have

$$\sigma(\rho,u,0) = \int_{\mathbb{R}} \sigma^\sharp(\rho,u,t)Y(t)\,dt + \int_{\mathbb{R}} \sigma^\flat(\rho,u,t)X(t)\,dt,$$

and, in view of the decomposition $\sigma(\rho,u,s) = u\chi(\rho,u,s) + h(\rho,u,s)$, we have

$$\begin{aligned}\sigma(\rho,u,s) =& \sigma(\rho,u-s,0) + s\chi(\rho,u-s) \\ =& \int_{\mathbb{R}} \sigma^\sharp(\rho,u-s,t)Y(t)\,dt + \int_{\mathbb{R}} \sigma^\flat(\rho,u-s,t)X(t)\,dt \\ &+ s\int_{\mathbb{R}} \chi^\sharp(\rho,u-s,t)Y(t)\,dt + s\int_{\mathbb{R}} \chi^\flat(\rho,u-s,t)X(t)\,dt.\end{aligned}$$

It therefore follows from (1.15) that

$$\begin{aligned}q^\psi(\rho,u) = \int_{\mathbb{R}} \sigma(\rho,u,s)\psi(s)\,ds = &\int_{\mathbb{R}}\int_{\mathbb{R}} \sigma^\sharp(\rho,u-s,t)Y(t)\psi(s)\,dt\,ds \\ &+ \int_{\mathbb{R}}\int_{\mathbb{R}} \sigma^\flat(\rho,u-s,t)X(t)\psi(s)\,dt\,ds \\ &+ \int_{\mathbb{R}}\int_{\mathbb{R}} s\chi^\sharp(\rho,u-s,t)Y(t)\psi(s)\,dt\,ds \\ &+ \int_{\mathbb{R}}\int_{\mathbb{R}} s\chi^\flat(\rho,u-s,t)X(t)\psi(s)\,dt\,ds.\end{aligned}$$

By applying the Tonelli–Fubini theorem and substituting for $\hat{\psi}(w) = \frac{1}{2}w|w|$, we obtain

$$\begin{aligned}\hat{q}(\rho,u) = &\int_{\mathbb{R}}\left(\int_{\mathbb{R}} \frac{1}{2}\sigma^\sharp(\rho,u-s,t)s|s|\,ds\right)Y(t)\,dt \\ &+ \int_{\mathbb{R}}\left(\int_{\mathbb{R}} \frac{1}{2}\sigma^\flat(\rho,u-s,t)s|s|\,ds\right)X(t)\,dt \\ &+ \int_{\mathbb{R}}\left(\int_{\mathbb{R}} \frac{1}{2}\chi^\sharp(\rho,u-s,t)Y(t)s^2|s|\,ds\right)Y(t)\,dt \\ &+ \int_{\mathbb{R}}\left(\int_{\mathbb{R}} \frac{1}{2}\chi^\flat(\rho,u-s,t)s^2|s|\,ds\right)X(t)\,dt.\end{aligned}$$

Hence, recalling the decompositions in terms of $h^\sharp$ and $h^\flat$ (*cf.* Theorem 2.4), we get

$$\begin{aligned}\hat{q}(\rho,u) = &\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\frac{1}{2}\left((u-s)\chi^\sharp(\rho,u-s-t)+h^\sharp(R,u-s-t)\right)s|s|\,ds\right)Y(t)\,dt\\ &+\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\frac{1}{2}\left((u-s)\chi^\flat(\rho,u-s-t)+h^\flat(R,u-s-t)\right)s|s|\,ds\right)X(t)\,dt\\ &+\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\frac{1}{2}\chi^\sharp(\rho,u-s-t)Y(t)s^2|s|\,ds\right)Y(t)\,dt\\ &+\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\frac{1}{2}\chi^\flat(\rho,u-s-t)s^2|s|\,ds\right)X(t)\,dt,\end{aligned}$$

from which the result follows straightforwardly. □

The content of the following lemma is to show that $h^\sharp$ and $h^\flat$ are indeed entropies in the isothermal region $\{\rho \geq 1\}$. The proof is by direct calculation.

Lemma 2.31. *In the region* $\{\rho > 1\}$, *we have*

$$\begin{cases} h^\sharp_{\rho\rho} - \dfrac{1}{\rho^2}h^\sharp_{uu} = 0 \quad \text{in } (1,\infty)\times\mathbb{R},\\ h^\sharp(1,u,s) = \dfrac{1}{2}\operatorname{sgn}(u-s),\\ h^\sharp_\rho(1,u,s) = 0,\end{cases} \qquad \begin{cases} h^\flat_{\rho\rho} - \dfrac{1}{\rho^2}h^\flat_{uu} = 0 \quad \text{in } (1,\infty)\times\mathbb{R},\\ h^\flat(1,u,s) = -\dfrac{1}{2}\operatorname{sgn}(u-s),\\ h^\flat_\rho(1,u,s) = \delta'_{u=s},\end{cases} \tag{2.50}$$

since $p''(\rho) = 0$ *in this region, where*

$$h^\sharp(\rho,u,s) = \sigma^\sharp(\rho,u,s) - u\chi^\sharp(\rho,u,s), \qquad h^\flat(\rho,u,s) = \sigma^\flat(\rho,u,s) - u\chi^\flat(\rho,u,s). \tag{2.51}$$

Lemma 2.32. *In the isothermal region* $\{\rho \geq 1\}$, *the convolutions of* $h^\sharp$ *and* $h^\flat$ *admit the bounds*

$$\begin{aligned}\left|\frac{1}{2}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}h^\sharp(\rho,u-s-t)s|s|\,ds\right)Y(t)\,dt\right| &\leq C\left(\rho|u|^2+\rho+\rho\log\rho\right),\\ \left|\frac{1}{2}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}h^\flat(\rho,u-s-t)s|s|\,ds\right)X(t)\,dt\right| &\leq C\left(\rho|u|^2+\rho+\rho\log\rho\right).\end{aligned} \tag{2.52}$$

Proof. Since $h^\sharp$ and $h^\flat$ obey the entropy equation in the region $\rho > 1$, we can generate these functions using the entropy kernel and their initial data at $\rho = 1$. Specifically,

$$\begin{aligned} h^\sharp(\rho,u) &= h^\sharp(1,u,\cdot)*\chi^\flat(\rho,u,\cdot) + h^\sharp_\rho(1,u,\cdot)*\chi^\sharp(\rho,u,\cdot),\\ h^\flat(\rho,u) &= h^\flat(1,u,\cdot)*\chi^\flat(\rho,u,\cdot) + h^\flat_\rho(1,u,\cdot)*\chi^\sharp(\rho,u,\cdot),\end{aligned}$$

i.e.,

$$\begin{aligned} h^\sharp(\rho,u) &= h^\sharp(1,u,\cdot)*\chi^\flat(\rho,u,\cdot),\\ h^\flat(\rho,u) &= h^\flat(1,u,\cdot)*\chi^\flat(\rho,u,\cdot) + \chi^\sharp_u(\rho,u),\end{aligned}$$

where, by the convolution symbol $*$, we mean

$$h^\sharp(1,u,\cdot)*\chi^\flat(\rho,u,\cdot)=\int_{\mathbb{R}}h^\sharp(1,s,0)\chi^\flat(\rho,u,s)\,ds=\frac{1}{2}\int_{\mathbb{R}}\operatorname{sgn}(s)\chi^\flat(\rho,u,s)\,ds.$$

Thus,

$$h^\sharp(\rho,u)=\frac{1}{2}\int_{\mathbb{R}}\operatorname{sgn}(s)\chi^\flat(\rho,u,s)\,ds,\quad h^\flat(\rho,u)=\chi_u^\sharp(\rho,u)-h^\sharp(\rho,u)\quad \text{for } \rho>1.$$

Hence,

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}}\Big(\int_{\mathbb{R}}h^\sharp(\rho,u-s-t)s|s|\,ds\Big)Y(t)\,dt=\\ \frac{1}{4}\int_{\mathbb{R}}\bigg(\int_{\mathbb{R}}\bigg(\int_{\mathbb{R}}\operatorname{sgn}(r)\chi^\flat(\rho,u-s-t,r)\,dr\bigg)\,s|s|\,ds\bigg)\,Y(t)\,dt.\end{aligned}\tag{2.53}$$

It is not obvious that the above integral is nicely controlled. However, since χ and σ obey (1.14) and (1.16), respectively, they are each compactly supported in u for fixed ρ by Theorem 1.12. Hence, $\operatorname{supp} h(\rho,\cdot)\subset[-k(\rho),k(\rho)]$, and the non-zero contributions due to $h^\sharp$ and $h^\flat$ need only be considered on this set.

In particular, in (2.53), $h^\sharp(1,s,0)$ need only be considered for $|s|<k(1)$. Hence,

$$\begin{aligned}&\left|\frac{1}{2}\int_{\mathbb{R}}\Big(\int_{\mathbb{R}}h^\sharp(\rho,u-s-t)s|s|\,ds\Big)Y(t)\,dt\right|\\ &=\frac{1}{4}\left|\int_{\mathbb{R}}\operatorname{sgn}(r)\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\chi^\flat(\rho,u-r-s,t)Y(t)\,dt\right)s|s|\,ds\right)\,dr\right|\\ &\le\frac{1}{2}\int_{-k(1)}^{k(1)}\left|\frac{1}{2}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\chi^\flat(\rho,u-r-s,t)Y(t)\,dt\right)s|s|\,ds\right|\,dr.\end{aligned}\tag{2.54}$$

We note that the quantity inside the absolute value is precisely equal to

$$\begin{aligned}&\int_{\mathbb{R}}\frac{\sqrt{\rho}}{4}\big(Y(u-r-s-\log\rho)+Y(u-r-s+\log\rho)\big)s|s|\,ds\\ &\quad+\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\sqrt{\rho}}{8}\Bigg[\frac{\log\rho}{\sqrt{(\log\rho)^2-(u-r-t)^2}}I_1\Bigg(\frac{\sqrt{(\log\rho)^2-(u-r-t)^2}}{2}\Bigg)\\ &\quad-I_0\left(\frac{\sqrt{R^2-(u-r-t)^2}}{2}\right)\Bigg]\mathbb{1}_{|u-r-t|<\log\rho}Y(t)s|s|\,dt\,ds,\end{aligned}$$

which is controlled in the same way as $|\hat{\mathcal{J}}_2(\rho,u-r)|+|\hat{\mathcal{J}}_3(\rho,u-r)|$ (*cf.* Lemma 2.24). Indeed, the only difference between this and the term above is the presence of Y instead of X in the convolutions.

Now, since

$$\begin{aligned}|\hat{\mathcal{J}}_2(\rho,u-r)|+|\hat{\mathcal{J}}_3(\rho,u-r)| &\le C\left(\rho|u-r|^2+\rho+\rho\log\rho\right)\\ &\le C\left(\rho|u|^2+\rho+\rho\log\rho\right)+C\rho|r|^2,\end{aligned}$$

we see that

$$\begin{aligned}&\left|\frac{1}{2}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}h^\sharp(\rho,u-s-t)s|s|\,ds\right)Y(t)\,dt\right|\\ &\qquad\le C\left(\rho|u|^2+\rho+\rho\log\rho\right)+C\rho\int_{-k(1)}^{k(1)}|r|^2\,dr\\ &\qquad\le C\left(\rho|u|^2+\rho+\rho\log\rho\right),\end{aligned}$$

as anticipated. Similarly, for the term

$$\frac{1}{2}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}h^\flat(R,u-s-t)s|s|\,ds\right)X(t)\,dt,$$

the term involving $\chi_u^\sharp$ is

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\chi_u^\sharp(\rho,u-s-t)s|s|\,ds\right)X(t)\,dt\\ &\qquad=\frac{1}{2}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\chi_u^\sharp(\rho,u-s-t)X(t)\,dt\right)s|s|\,ds\\ &\qquad=\frac{\partial}{\partial u}\frac{1}{2}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\chi^\sharp(\rho,u-s-t)X(t)\,dt\right)s|s|\,ds.\end{aligned}$$

We notice that the term on the right-hand side is precisely

$$\frac{\partial}{\partial u}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\sqrt{\rho}}{4}I_0\left(\frac{\sqrt{(\log\rho)^2-(u-s-t)^2}}{2}\right)\mathbb{1}_{|u-s-t|<\log\rho}X(t)s|s|\,dt\,ds,$$

which is controlled in the same way as $\partial_u\hat{\mathcal{J}}_1$ (*cf.* Lemma 2.26). Indeed, the only difference between the term in question and $\partial_u\hat{\mathcal{J}}_1$ is the presence of X instead of Y in the convolution. Since $|\partial_u\hat{\mathcal{J}}_1(\rho,u)|\le C\left(\rho|u|+\rho+\rho\sqrt{\log\rho}\right)$, we get

$$\begin{aligned}\left|\frac{1}{2}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\chi_u^\sharp(\rho,u-s-t)s|s|\,ds\right)X(t)\,dt\right| &\le C\left(\rho|u|+\rho+\rho\sqrt{\log\rho}\right)\\ &\le C\left(\rho|u|^2+\rho+\rho\log\rho\right),\end{aligned}$$

where we applied the Cauchy–Schwarz inequality to obtain the final line. As a result, we deduce

$$\left|\frac{1}{2}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}h^\flat(\rho,u-s-t)s|s|\,ds\right)X(t)\,dt\right|\le C\left(\rho|u|^2+\rho+\rho\log\rho\right),$$

which concludes the proof of the lemma. □

Lemma 2.33. *In light of Lemmas 2.30 and 2.32, we have*

$$\hat{q}(\rho, u) \geq u\hat{\eta}(\rho, u) - C\left(\rho|u|^2 + \rho + \rho\log\rho\right) \qquad \text{for } (\rho, u) \in [1, \infty) \times \mathbb{R}. \tag{2.55}$$

From Lemma 2.23, the special entropy-flux admits the lower bound

$$\hat{q}(\rho, u) \geq C^{-1}\rho|u|^3 - C\left(\rho + \rho(\log\rho)^4\right) \qquad \text{for } (\rho, u) \in [1, \infty) \times \mathbb{R}. \tag{2.56}$$

Remark 2.34. Note that Lemmas 2.24, 2.26, 2.29, and 2.33 have completely shown the validity of the first half of Lemma 2.5. Meanwhile, the second half of Lemma 2.5 follows easily by combining the results in [64] for the leading order terms of the expansions (1.20)-(1.21), along with the bounds on the remainders.

2.4.5 Two useful results for Taylor expansions

Lemma 2.35. *For any* $\rho \geq 0$*, we have* $\hat{\eta}(\rho, 0) = 0$.

Proof. Observe that, since $\chi(\rho, \cdot)$ is even due to Lemma 2.7, we have $\hat{\eta}(\rho, 0) = \frac{1}{2}\int_{\mathbb{R}} \chi(\rho, s)s|s|\,ds = 0$, as the integrand is odd. □

Lemma 2.36. *By direct calculation, we have that, for* $\rho > 1$,

$$\begin{aligned} \hat{q}(\rho, 0) =& \frac{1}{2}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} h^{\sharp}(\rho, -s-t)s|s|\,ds\right)Y(t)\,dt \\ &+ \frac{1}{2}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} h^{\flat}(\rho, -s-t)s|s|\,ds\right)X(t)\,dt. \end{aligned} \tag{2.57}$$

By Lemma 2.32, we get $|\hat{q}(\rho, 0)| \leq C\left(\rho + \rho\log\rho\right)$.

2.4.6 Results regarding the relative internal energy

We focus on the relative internal energy with respect to end-states $(\rho^{\pm}, u^{\pm})$ through $(\bar{\rho}, \bar{u})$, i.e., $e^*(\rho, \bar{\rho})$, defined by the formula (1.27) (*cf.* Definition 1.14).

Lemma 2.37. *Assuming that* p *satisfies the assumptions of Definition 2.1, the following results regarding* e *and* e^* *hold:*

1. $e(\rho) \geq 0$ *and* $e'(\rho) \geq 0$*, and* $e(\rho) = 0$ *iff* $\rho = 0$*;*
2. $e^*(\rho, \bar{\rho}) \geq 0$*, and* $e^*(\bar{\rho}, \bar{\rho}) = e^*_{\rho}(\bar{\rho}, \bar{\rho}) = 0$*;*
3. $e^*_{\bar{\rho}}(\rho, \bar{\rho}) = -(\rho - \bar{\rho})\frac{p'(\bar{\rho})}{\bar{\rho}}$, $e^*_{\rho\bar{\rho}}(\rho, \bar{\rho}) = -\frac{p'(\bar{\rho})}{\bar{\rho}}$*, and* $e^*_{\rho\rho}(\rho, \bar{\rho}) = \frac{p'(\rho)}{\rho}$*;*
4. $\operatorname{sgn} e^*_{\rho}(\rho, \bar{\rho}) = \operatorname{sgn}(\rho - \bar{\rho})$*, and hence* $e(\rho, \bar{\rho}) = e_{\rho}(\rho, \bar{\rho}) = 0$ *iff* $\rho = \bar{\rho}$.

In particular, due to the last point, we have that if $\rho' \leq \rho < \bar{\rho} \leq \bar{\rho}'$, then

$$e^*(\rho, \bar{\rho}) \leq e^*(\rho', \bar{\rho}'). \tag{2.58}$$

Proof. The first three points follow directly from Definition 1.14 and the assumptions of strict hyperbolicity and genuine nonlinearity. For the fourth point, note that the fundamental theorem of calculus gives, by letting $g(y) := ye(y)$,

$$e^*_\rho(\rho, \bar{\rho}) = g'(\rho) - g'(\bar{\rho}) = \int_{\bar{\rho}}^{\rho} g''(y)\, dy,$$

provided $\rho \geq \bar{\rho}$. Note also that $g''(y) = p'(y)/y$, which is locally integrable in $[0, \infty)$, since $g''(y) = O(y^{\gamma-1})$ as $y \to 0$, and $\gamma > 1$. Further, g'' is strictly positive by the assumption of strict hyperbolicity, from which the result for e^*_ρ follows, since the integral of a strictly positive integrand is itself strictly positive. Additionally, suppose that $e^*(\rho, \bar{\rho}) = 0$ but that $\rho \neq \bar{\rho}$. Assume without loss of generality that $\bar{\rho} < \rho$, then

$$0 = e(\rho, \bar{\rho}) = \int_{\bar{\rho}}^{\rho} e_\rho(y, \bar{\rho})\, dy,$$

but the right-hand side is strictly positive since it is the integral of a strictly positive integrand. This is a contradiction.

Finally, suppose that $\rho' \leq \rho < \bar{\rho} \leq \bar{\rho}'$. Then $e^*(\rho, \bar{\rho}) = \int_\rho^{\bar{\rho}} e^*_{\bar{\rho}}(\rho, y)\, dy$ and, using the third point of the Lemma,

$$\begin{aligned} e^*(\rho, \bar{\rho}) = \int_\rho^{\bar{\rho}} (y - \rho)\frac{p'(y)}{y}\, dy &\leq \int_\rho^{\bar{\rho}} (y - \rho')\frac{p'(y)}{y}\, dy \\ &\leq \int_{\rho'}^{\bar{\rho}'} (y - \rho')\frac{p'(y)}{y}\, dy = \int_{\rho'}^{\bar{\rho}'} e^*_{\bar{\rho}}(\rho', y)\, dy = e^*(\rho', \bar{\rho}'), \end{aligned}$$

where we made use of the non-negativity of the integrand. □

The following result plays a crucial role in our later energy estimates.

Lemma 2.38. *There exists a positive constant C such that*

$$\rho + \rho \log \rho \leq C\big(1 + e^*(\rho, \bar{\rho})\big) \qquad \textit{for } \rho \geq 1. \tag{2.59}$$

Proof. Recall from Lemma 2.37 that

$$e^*_{\rho\rho}(\rho, \bar{\rho}) = \frac{p'(\rho)}{\rho} = \frac{1}{\rho} \qquad \text{for } \rho \geq 1.$$

Integrating in ρ twice yields $e^*(\rho, \bar{\rho}) - (\rho - 1)e^*_\rho(1, \bar{\rho}) - e^*(1, \bar{\rho}) = \rho \log \rho - (\rho - 1)$.

Recall that the reference function $\bar\rho$ was chosen to be smooth and constant outside of a compact set. Given that the functions $e(\rho)$ and $\rho e'(\rho)$ are continuous on $[0,\infty)$, there exists a positive constant $M = M(\bar\rho,\gamma)$ such that $|e(\bar\rho)| + |\bar\rho e'(\bar\rho)| \leq M$. Thus, there is a positive $M = M(\bar\rho,\gamma)$ such that $|e^*_\rho(1,\bar\rho)| + e^*(1,\bar\rho) \leq M$. Hence,

$$\rho\log\rho \leq M\big(e^*(\rho,\bar\rho) + \rho + 1\big). \tag{2.60}$$

Additionally, since $\frac{\rho}{1+\rho\log\rho} \to 0$ as $\rho\to\infty$, there exists a positive $\tilde R = \tilde R(M) \geq 1$ such that

$$0 < \frac{\rho}{1+\rho\log\rho} \leq \frac{1}{2M} \qquad \text{for all } \rho \geq \tilde R.$$

Hence, $\rho \leq (\tilde R + \frac{1}{2M}) + \frac{1}{2M}\rho\log\rho$ for all $\rho\geq 1$. In view of this and (2.60), there exists a positive constant $\tilde M = \tilde M(M,\gamma,\bar\rho)$ such that

$$\rho\log\rho \leq \tilde M\big(e^*(\rho,\bar\rho)+1\big) \qquad \text{for all } \rho \geq 1.$$

Combining the above inequality with $\rho \leq (\tilde R + \frac{1}{2M}) + \frac{1}{2M}\rho\log\rho$ yields the result. □

We straightforwardly deduce the following result, which features in Lemma 2.47.

Corollary 2.39. *There exists a positive constant C such that*

$$0 \leq \rho + p(\rho) \leq C\big(1 + e^*(\rho,\bar\rho)\big) \qquad \textit{for } \rho \geq 0. \tag{2.61}$$

Finally, we conclude the proof of Lemma 2.5 by showing the validity of (2.12).

Proof of (2.12). Arguing as we did in Lemma 2.38, there exists $\tilde R$ such that $e^*(\rho,\bar\rho) \geq \frac{1}{2}\rho\log\rho$ for all $\rho \geq \tilde R$. Hence, in view of Lemma 2.26, the inequality is verified for all $\rho \geq \tilde R$. The inequality is also verified for $\rho \leq 1$, in view of the expansion at the vacuum provided by Theorem 1.12, as stated in [16, Section 3.1]. It remains is to consider the interval $\rho \in [1,\tilde R]$. Let $\rho\in[1,\tilde R]$ and, for fixed $\bar\rho$, and $M>0$ to be chosen,

$$g(\rho) := \rho|\hat\eta_m(\rho,0) - \hat\eta_m(\bar\rho,0)|^2, \qquad f(\rho) := Me_*(\rho,\bar\rho) - g(\rho).$$

There are now three cases to consider, depending on where $\bar\rho$ lies. Suppose firstly that $\bar\rho\in[0,1)$. Then, since $e^*(\rho,\bar\rho)=0$ if and only if $\rho=\bar\rho$, we know that, in this case, $e^*(\rho,\bar\rho)$ is strictly positive on $[1,\tilde R]$. Thus, by continuity, there exists a small constant $\delta>0$ such that $e^*(\rho,\bar\rho)\geq\delta$ for all $\rho\in[1,\tilde R]$. Since g is continuous on a compact set, it achieves its bounds, thus there exists $\tilde M > 0$ such that $|g(\rho)| \leq \tilde M$ for every $\rho\in[1,\tilde R]$. Thus, letting $M = 2\tilde M/\delta$, we see that $f(\rho)\geq\tilde M$ for all $\rho\in[1,\tilde R]$. The same argument works when $\bar\rho\in(\tilde R,\infty)$. It remains to consider the case $\bar\rho\in[1,\tilde R]$.

Firstly, observe that, by differentiating the entropy equation (1.12) in u, we get

$$\hat{\eta}_{u\rho\rho} - \frac{1}{\rho^2}\hat{\eta}_{uuu} = 0, \tag{2.62}$$

where $\hat{\eta}_{uuu}$ is well-defined and continuous since $\hat{\eta}_{uuu}(\rho,u) = 2\chi(\rho,u)$, by direct computation (*cf.* Lemma 3.27). From (2.62), it follows that the derivative $\hat{\eta}_{u\rho\rho}$ is well-defined and continuous for $\rho \geq 1$. In view of this, we deduce that $g \in C^2([1,\infty))$. Hence, we have

$$f''(\rho) = Me^*_{\rho\rho}(\rho,\bar{\rho}) - g''(\rho).$$

Since $e^*_{\rho\rho}(\rho,\bar{\rho}) = \frac{p'(\rho)}{\rho}$ is strictly positive and continuous on $[1,\infty)$, we know that it achieves its infimum on the compact set $[1,\tilde{R}]$, and that this infimum is positive. Similarly, g'' achieves its bounds on this compact set. Thus, by choosing

$$M = 2 \sup_{\rho\in[1,\tilde{R}]} |g''(\rho)| \left(\inf_{\rho\in[1,\tilde{R}]} e^*_{\rho\rho}(\rho,\bar{\rho}) \right)^{-1},$$

we get $f''(\rho) > 0$ for $\rho \in [1,\tilde{R}]$. Thus, for $\rho \in [1,\bar{\rho}]$, using $f(\bar{\rho}) = f'(\bar{\rho}) = 0$, the fundamental theorem of calculus yields

$$0 - f(\rho) = \int_\rho^{\bar{\rho}} f'(y)\,dy = -\int_\rho^{\bar{\rho}}\int_y^{\bar{\rho}} f''(z)\,dz \leq 0.$$

Similarly, if $\rho \in [\bar{\rho},\tilde{R}]$,

$$f(\rho) - 0 = \int_{\bar{\rho}}^{\rho} f'(y)\,dy = \int_{\bar{\rho}}^{\rho}\int_{\bar{\rho}}^{y} f''(z)\,dz \geq 0.$$

In both cases, we get $f(\rho) \geq 0$ for our choice of M, as required. □

2.4.7 Entropies generated by C_c^2 test functions

This section is concerned with the case of entropies generated by compactly supported test functions. In what follows, we give a complete proof of Theorem 2.6.

Lemma 2.40. *For a compactly supported test function $\psi \in C_c^2(\mathbb{R})$ with $\operatorname{supp}\psi \subset [a,b]$, we have that, in the region $\{\rho \geq 1\}$,*

$$\operatorname{supp}\eta^\psi(\rho,\cdot) \subset \big\{u \in \mathbb{R}^2_+ : \log\rho + u \geq a - k(1),\, u - \log\rho \leq b + k(1)\big\}. \tag{2.63}$$

Furthermore, there exists a positive constant $C_\psi > 0$ such that, for any $\rho \geq 1$ and $u \in \mathbb{R}$,

$$|\eta^\psi(\rho,u)| \leq \frac{C_\psi \rho}{1+\sqrt{\log\rho}}. \tag{2.64}$$

In particular, recalling the decomposition in Lemma 2.14, we have that

$$|\mathcal{J}_1^\psi(\rho,u)| \leq \frac{C_\psi\rho}{1+\sqrt{\log\rho}}, \quad |\mathcal{J}_2^\psi(\rho,u)| \leq C_\psi\sqrt{\rho}, \;\; \textit{and}\; |\mathcal{J}_3^\psi(\rho,u)| \leq \frac{C_\psi\rho}{1+\log\rho}. \tag{2.65}$$

Proof. Writing out the integral limits of the first term explicitly, we have that

$$\mathcal{J}_1^\psi = \frac{\sqrt{\rho}}{2}\int_{-k(1)}^{k(1)}\left(\int_{\max\{-R,a+t-u\}}^{\min\{R,b+t-u\}} I_0\left(\frac{\sqrt{R^2-z^2}}{2}\right)\psi(z+u-t)\,dz\right)Y(t)\,dt. \quad (2.66)$$

It is immediate from the above formula that, if $b+t-u \le -R$, or if $a+t-u \ge R$, this first term vanishes. This means that, in order for $\mathcal{J}_1^\psi$ to be non-zero, we require $a-R-k(1) \le u \le b+R+k(1)$, as in (2.63). Note that, by first bounding the integrand using Lemma A.6 and then relaxing the integral endpoints, we obtain

$$|\mathcal{J}_1^\psi(\rho,u)| \le \frac{\sqrt{\rho}\|\psi\|_{L^\infty(\mathbb{R})}}{2}\int_{-k(1)}^{k(1)}\left(\int_{a+t-u}^{b+t-u} I_0\left(\frac{R}{2}\right)dz\right)|Y(t)|\,dt \le \frac{C_\psi \rho}{1+\sqrt{\log\rho}}, \quad (2.67)$$

as required, where we used $\|Y\|_{L^1(\mathbb{R})} < \infty$. For the second term,

$$\begin{aligned}|\mathcal{J}_2^\psi(\rho,u)| &= \left|\int_a^b \frac{\sqrt{\rho}}{2}\big(X(u-s-\log\rho)+X(u-s+\log\rho)\big)\psi(s)\,ds\right| \\ &\le \sqrt{\rho}\|X\|_{L^\infty(\mathbb{R})}\int_a^b |\psi(s)|\,ds.\end{aligned}$$

Hence, $|\mathcal{J}_2^\psi(\rho,u)| \le C_\psi\sqrt{\rho}$. Note again that, since the support of X is contained within the interval $[-k(1),k(1)]$, we have that $\mathcal{J}_2^\psi$ vanishes if both $u-s-\log\rho \notin [-k(1),k(1)]$ and $u-s+\log\rho \notin [-k(1),k(1)]$, which again verifies (2.63).

For the third term, we note again that the support of $\mathcal{J}_3^\psi$ is contained in the interval prescribed by (2.63), and it is controlled as shown underneath.

$$\begin{aligned}|\mathcal{J}_3^\psi(\rho,u)| \le \frac{\sqrt{\rho}}{4}\int_{\mathbb{R}}\int_{\mathbb{R}}\Bigg|&\frac{R}{\sqrt{R^2-(u-s-t)^2}}I_1\left(\frac{\sqrt{R^2-(u-s-t)^2}}{2}\right) \\ &-I_0\left(\frac{\sqrt{R^2-(u-s-t)^2}}{2}\right)\Bigg|\mathbb{1}_{|u-s-t|<R}X(t)\psi(s)\,dt\,ds,\end{aligned}$$

where the right-hand side of the above is bounded by

$$\begin{aligned}\frac{\sqrt{\rho}\|\psi\|_{L^\infty(\mathbb{R})}}{4}\int_{-k(1)}^{k(1)}\Bigg(\int_{\max\{-R,a+t-u\}}^{\min\{R,b+t-u\}}\Bigg|&I_0\left(\frac{\sqrt{R^2-z^2}}{2}\right) \\ &-\frac{R}{\sqrt{R^2-z^2}}I_1\left(\frac{\sqrt{R^2-z^2}}{2}\right)\Bigg|\,dz\Bigg)X(t)\,dt.\end{aligned}$$

The innermost integral of the previous expression can be bounded above by

$$2R\int_0^{\pi/2}\left|I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta - I_1\left(\frac{R}{2}\cos\theta\right)\right|\,d\theta,$$

by allowing the limits of the integral to be $\pm R$. This itself is bounded by

$$2R\int_0^{\pi/2}\left|I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta - I_1\left(\frac{R}{2}\cos\theta\right)\cos^2\theta\right|d\theta \\ + 2R\int_0^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right)\sin^2\theta\,d\theta,$$

using the identity $\sin^2\theta + \cos^2\theta = 1$. Since the integrand of the first term is non-negative, as $I_0(x) \geq I_1(x)$ for $x \geq 0$, we can drop the absolute value sign, and, using Lemmas A.1 and A.2, evaluate the whole of the above line to be equal to

$$\frac{16}{R}\sinh\left(\frac{R}{2}\right) - \frac{4}{\sqrt{\rho}} - 4 \leq \frac{\sqrt{\rho}}{1+\log\rho}.$$

This proves that $|\mathcal{J}_3^{\psi}(\rho,u)| \leq \frac{C_\psi \rho}{1+\log\rho}$. This concludes the proof of the lemma. □

Remark 2.41. This immediately yields that $|q^{\psi}(\rho,u)| \leq C_\psi \rho\sqrt{\log\rho}$. However, this latter estimate does not yield enough decay on the entropy-flux for later use. We need the next lemma.

Lemma 2.42. *For a compactly supported test function $\psi \in C_c^2(\mathbb{R})$ with* $\operatorname{supp}\psi \subset [a,b]$*, there exists a positive constant $C_\psi > 0$ such that*

$$|u\mathcal{J}_1^{\psi}(\rho,u)| \leq C_\psi \rho \qquad \textit{for } (\rho,u) \in [1,\infty)\times\mathbb{R}. \tag{2.68}$$

Proof. By extending the compact set which contains the support of ψ if necessary, assume without loss of generality that $b > k(1)$ and that $a < -k(1)$. Recall that

$$\mathcal{J}_1^{\psi}(\rho,u) = \frac{\sqrt{\rho}}{2}\int_{-k(1)}^{k(1)}\left(\int_{\max\{-R+u-t,a\}}^{\min\{R+u-t,b\}} I_0\left(\frac{\sqrt{R^2-(u-s-t)^2}}{2}\right)\psi(s)\,ds\right)Y(t)\,dt,$$

so that

$$\begin{aligned} &u\mathcal{J}_1^{\psi}(\rho,u) = \\ &\frac{\sqrt{\rho}}{2}\int_{-k(1)}^{k(1)}\left(\int_{\max\{-R+u-t,a\}}^{\min\{R+u-t,b\}}(u-s-t)I_0\left(\frac{\sqrt{R^2-(u-s-t)^2}}{2}\right)\psi(s)\,ds\right)Y(t)\,dt \\ &+\frac{\sqrt{\rho}}{2}\int_{-k(1)}^{k(1)}\left(\int_{\max\{-R+u-t,a\}}^{\min\{R+u-t,b\}}(s+t)I_0\left(\frac{\sqrt{R^2-(u-s-t)^2}}{2}\right)\psi(s)\,ds\right)Y(t)\,dt. \end{aligned} \tag{2.69}$$

Using the estimate provided by Lemma A.9 to bound $\sup_{v\in[0,R]} vI_0\left(\frac{\sqrt{R^2-v^2}}{2}\right)$ in the integrand of the first term in (2.69), and the integral limits for the second, we get

$$\begin{aligned}|u\mathcal{J}_1^\psi(\rho,u)| \le& C\rho \int_{-k(1)}^{k(1)} \left(\int_a^b |\psi(s)|\,ds\right)|Y(t)|\,dt \\ &+ \frac{\sqrt{\rho}I_0\left(\frac{R}{2}\right)}{2}\int_{-k(1)}^{k(1)}\left(\int_{\max\{-R+u-t,a\}}^{\min\{R+u-t,b\}} |s+t||\psi(s)|\,ds\right)|Y(t)|\,dt \\ \le& C_\psi\rho + \frac{C\rho}{1+\sqrt{\log\rho}}\int_{-k(1)}^{k(1)}\left(\int_a^b |s+t||\psi(s)|\,ds\right)|Y(t)|\,dt \\ \le& C_\psi\rho + \frac{C_\psi\rho}{1+\sqrt{\log\rho}},\end{aligned}$$

where we used Lemma A.6 to get the second inequality. The result follows easily. □

Corollary 2.43. *For a compactly supported test function $\psi \in C_c^2(\mathbb{R})$ with $\operatorname{supp}\psi \subset [a,b]$, there exists a positive constant $C_\psi > 0$ such that*

$$|u\hat{\eta}(\rho,u)| \le C_\psi\rho \qquad \text{for } (\rho,u)\in[1,\infty)\times\mathbb{R}. \tag{2.70}$$

Proof. Since $\mathcal{J}_2^\psi$ and $\mathcal{J}_3^\psi$ are supported on $|u| \le \big(|a|+|b|+k(1)\big)+\log\rho$, we see from (2.65) that $|u\mathcal{J}_2^\psi| + |u\mathcal{J}_3^\psi| \le C_\psi\rho$. Now combine with Lemma 2.42. □

Next, we estimate a corresponding entropy-flux, using the bound on $u\eta^\psi$ provided by Corollary 2.43.

Lemma 2.44. *For a compactly supported test function $\psi \in C_c^2(\mathbb{R})$ with $\operatorname{supp}\psi \subset [a,b]$, we have that, in the region $\rho > 1$,*

$$\operatorname{supp} q^\psi \subset \big\{(\rho,u)\in\mathbb{R}_+^2 : \log\rho + u \ge a - k(1), u - \log\rho \le b + k(1)\big\}. \tag{2.71}$$

Furthermore, there exists a positive constant $C_\psi > 0$ such that

$$\|q^\psi(\rho,\cdot)\|_{L^\infty(\mathbb{R})} \le C_\psi\rho \qquad \text{for } \rho \ge 1. \tag{2.72}$$

Proof. Recall (1.15). Following a procedure analogous to the one in Lemma 2.30, we get

$$\begin{aligned}q^\psi(\rho,u) = u\eta^\psi(\rho,u) &+ \int_{\mathbb{R}}\left(\int_{\mathbb{R}} h^\sharp(R,u-s-t)\psi(s)\,ds\right)Y(t)\,dt \\ &+ \int_{\mathbb{R}}\left(\int_{\mathbb{R}} h^\flat(R,u-s-t)\psi(s)\,ds\right)X(t)\,dt.\end{aligned}$$

An argument identical to the one in Lemma 2.32 then yields that the latter two integrals on the right-hand side are controlled by $C|\eta^\psi(\rho,u)|$. Hence, $|q^\psi(\rho,u)| \le C|\eta^\psi(\rho,u)|\,(1+|u|)$, which shows that $\operatorname{supp} q^\psi \subset \operatorname{supp}\eta^\psi$, as claimed. An application of Corollary 2.43 yields the result. □

Lemma 2.45. *For $\psi \in C_c^2(\mathbb{R})$ a compactly supported test function, there exists a positive constant $C_\psi > 0$ such that*

$$\|\eta_m^\psi(\rho,\cdot)\|_{L^\infty(\mathbb{R})} + \|\rho\eta_{mm}^\psi(\rho,\cdot)\|_{L^\infty(\mathbb{R})} \le \frac{C_\psi}{1+\sqrt{\log\rho}} \qquad \text{for } \rho \ge 1. \tag{2.73}$$

Proof. For the first bound, as $\eta_m = \rho^{-1}\eta_u$, it suffices to show that $|\eta_u| \le C_\psi\rho$. To this end, we differentiate the terms $\mathcal{J}_1^\psi, \mathcal{J}_2^\psi, \mathcal{J}_3^\psi$ in u. We begin with $\partial_u\mathcal{J}_1^\psi$, for which, by first bounding the inner integrand and then relaxing the inner integral endpoints,

$$\begin{aligned}
|\partial_u\mathcal{J}_1^\psi| &= \frac{\sqrt{\rho}}{2}\left|\int_{-k(1)}^{k(1)}\left(\int_{\max\{-R,a+t-u\}}^{\min\{R,b+t-u\}} I_0\left(\frac{\sqrt{R^2-z^2}}{2}\right)\psi'(z+u-t)\,dz\right)Y(t)\,dt\right| \\
&\le C\sqrt{\rho}\|Y\|_{L^1(\mathbb{R})}\|\psi'\|_{L^\infty(\mathbb{R})}\int_{a-k(1)-u}^{b+k(1)-u} I_0\left(\frac{R}{2}\right)\,dz \\
&\le C_\psi\sqrt{\rho}I_0\left(\frac{R}{2}\right) \le \frac{C_\psi\rho}{1+\sqrt{\log\rho}},
\end{aligned}$$

where we used Lemma A.6. Similarly,

$$|\partial_u\mathcal{J}_2^\psi| = \left|\frac{\sqrt{\rho}}{2}\int_{\mathbb{R}} X(s)\psi'(u+s-R)\,ds + \frac{\sqrt{\rho}}{2}\int_{\mathbb{R}} X(s)\psi'(u+s+R)\,ds\right| \le C_\psi\sqrt{\rho}.$$

Finally, we see that

$$\begin{aligned}
&|\partial_u\mathcal{J}_3^\psi| \le \\
&\quad \frac{\sqrt{\rho}}{4}\int_{\mathbb{R}}\left(\int_{\max\{-R,a+t-u\}}^{\min\{R,b+t-u\}} I_0\left(\frac{\sqrt{R^2-z^2}}{2}\right)|\psi'(z+u-t)|\,dz\right)X(t)\,dt \\
&\quad + \frac{\sqrt{\rho}}{4}\int_{\mathbb{R}}\left(\int_{\max\{-R,a+t-u\}}^{\min\{R,b+t-u\}} \frac{R}{\sqrt{R^2-z^2}}I_1\left(\frac{\sqrt{R^2-z^2}}{2}\right)|\psi'(z+u-t)|\,dz\right)X(t)\,dt.
\end{aligned}$$

As such, by first bounding the integrands and then relaxing the integral endpoints, we obtain

$$\begin{aligned}
|\partial_u\mathcal{J}_3^\psi| \le& \frac{c_{total}\sqrt{\rho}\|\psi'\|_{L^\infty(\mathbb{R})}}{2}\left(\int_{a-k(1)-u}^{b+k(1)-u} I_0\left(\frac{R}{2}\right)\,dz + \int_{a-k(1)-u}^{b+k(1)-u} I_1\left(\frac{R}{2}\right)\,dz\right) \\
\le& \frac{C_\psi\rho}{1+\sqrt{\log\rho}},
\end{aligned}$$

in view of the fact that $x^{-1}I_1(x/2)$ is monotonically increasing, which concludes the proof of the first bound. The same estimates hold for the second bound, with $\|\psi'\|_{L^\infty(\mathbb{R})}$ replaced by $\|\psi''\|_{L^\infty(\mathbb{R})}$. This concludes the proof of the lemma. □

Lemma 2.46. *The fake mixed derivatives admit the bounds*

$$\|\eta_{mu}^\psi(\rho,\cdot)\|_{L^\infty(\mathbb{R})} \le \frac{C_\psi}{1+\sqrt{\log\rho}}, \quad \|\eta_{m\rho}^\psi(\rho,\cdot)\|_{L^\infty(\mathbb{R})} \le \frac{C_\psi}{\rho\,(1+\log\rho)} \quad \text{for } \rho \ge 1. \tag{2.74}$$

Proof. Recall that the fake mixed derivatives are given by

$$\eta^{\psi}_{m\rho} = \sum_{i=1}^{3} \left(\rho^{-1}\partial_{\rho u}\mathcal{J}^{\psi}_{i} - \rho^{-2}\partial_{u}\mathcal{J}^{\psi}_{i} \right), \qquad \eta^{\psi}_{mu} = \sum_{i=1}^{3} \left(\rho^{-1}\partial_{uu}\mathcal{J}^{\psi}_{i} \right) = \rho\eta^{\psi}_{mm}.$$

It is immediate from the second expression and the bound on $\rho\eta^{\psi}_{mm}$ in Lemma 2.45 that $|\eta^{\psi}_{mu}| \le \frac{C_\psi}{1+\sqrt{\log\rho}}$. For the first term, we have

$$\begin{aligned}
\rho^{-1}\partial_{\rho u}\mathcal{J}^{\psi}_{1} - \rho^{-2}\partial_{u}\mathcal{J}^{\psi}_{1} = &\\
&\frac{1}{2\rho^{3/2}} \int_{-k(1)}^{k(1)} \left(\psi'(R+u-t) + \psi'(-R+u-t)\right) Y(t)\, dt \\
&+ \frac{1}{4\rho^{3/2}} \int_{-k(1)}^{k(1)} \left(\int_{-R}^{R} \frac{R}{\sqrt{R^2-z^2}} I_1\left(\frac{\sqrt{R^2-z^2}}{2}\right) \psi'(z+u-t) \right) Y(t)\, dt \\
&- \frac{1}{4\rho^{3/2}} \int_{-k(1)}^{k(1)} \left(\int_{-R}^{R} I_0\left(\frac{\sqrt{R^2-z^2}}{2}\right) \psi'(z+u-t)\, dz \right) Y(t)\, dt.
\end{aligned} \tag{2.75}$$

The first term in (2.75) is manifestly bounded, in absolute value, by $C_\psi\rho^{-3/2}$. The last two terms in (2.75) can be grouped as follows

$$\begin{aligned}
-\frac{1}{4\rho^{3/2}} \int_{-k(1)}^{k(1)} \Bigg(\int_{-R}^{R} \Bigg[I_0\left(\frac{\sqrt{R^2-z^2}}{2}\right) \\
- \frac{R}{\sqrt{R^2-z^2}} I_1\left(\frac{\sqrt{R^2-z^2}}{2}\right) \Bigg] \psi'(z+u-t)\, dz \Bigg) Y(t)\, dt,
\end{aligned}$$

which is bounded above (in absolute value) by

$$\frac{\left(\int_{\mathbb{R}} |Y(t)|\, dt\right) \|\psi'\|_{L^\infty(\mathbb{R})}}{2\rho^{3/2}} \int_0^R \left| I_0\left(\frac{\sqrt{R^2-z^2}}{2}\right) - \frac{R}{\sqrt{R^2-z^2}} I_1\left(\frac{\sqrt{R^2-z^2}}{2}\right) \right| dz,$$

which can be rewritten as

$$\frac{C_\psi R}{\rho^{3/2}} \int_0^{\pi/2} \left| I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta - I_1\left(\frac{R}{2}\cos\theta\right) \right| d\theta.$$

Using the identity $\sin^2\theta + \cos^2\theta = 1$, the above term is dominated by

$$\begin{aligned}
&\frac{C_\psi R}{\rho^{3/2}} \int_0^{\pi/2} \left[I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta - I_1\left(\frac{R}{2}\cos\theta\right)\cos^2\theta \right] d\theta \\
&+ \frac{C_\psi R}{2\rho^{3/2}} \int_0^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right)\sin^2\theta\, d\theta.
\end{aligned}$$

Evaluating the previous integrals using Lemmas A.1-A.3, we obtain $|\partial_{m\rho}\mathcal{J}^{\psi}_{1}| \le \frac{C_\psi}{\rho\log\rho}$. Alternatively, bluntly bounding this term by

$$\frac{C_\psi R}{\rho^{3/2}} \left(\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\, d\theta + \int_0^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right) d\theta \right),$$

we get the bound $|\partial_{m\rho}\mathcal{J}_1^\psi| \le C_\psi \rho^{-1}$. Interpolating between this and the previous bound gives the required estimate on $\partial_{m\rho}\mathcal{J}_1^\psi$. For the second term, $\mathcal{J}_2^\psi$, we have

$$\begin{aligned}
&\rho^{-1}\partial_{\rho u}\mathcal{J}_2^\psi - \rho^{-2}\partial_u \mathcal{J}_2^\psi \\
&\qquad = -\frac{1}{4\rho^{3/2}}\int_{\mathbb{R}} X(s)\psi'(u+s-R)\,ds - \frac{1}{4\rho^{3/2}}\int_{\mathbb{R}} X(s)\psi'(u+s+R)\,ds \\
&\qquad\quad - \frac{1}{2\rho^{3/2}}\int_{\mathbb{R}} X(s)\psi''(u+s-R)\,ds + \frac{1}{2\rho^{3/2}}\int_{\mathbb{R}} X(s)\psi''(u+s+R)\,ds,
\end{aligned}$$

from which it is clear that $|\partial_{m\rho}\mathcal{J}_2^\psi(\rho,\rho u)| \le C_\psi \rho^{-3/2}$. It remains to bound $\partial_{m\rho}\mathcal{J}_3^\psi$. We have, as before,

$$\begin{aligned}
\partial_u \mathcal{J}_3^\psi = -\frac{\sqrt{\rho}}{4}\int_{-k(1)}^{k(1)} \Bigg(\int_{-R}^{R}\Bigg[& I_0\left(\frac{\sqrt{R^2-z^2}}{2}\right) \\
& - \frac{R}{\sqrt{R^2-z^2}} I_1\left(\frac{\sqrt{R^2-z^2}}{2}\right)\Bigg]\psi'(z+u-t)\,dz\Bigg)X(t)\,dt,
\end{aligned}$$

from which it follows that

$$\begin{aligned}
&\rho^{-1}\partial_{\rho u}\mathcal{J}_3^\psi - \rho^{-2}\partial_u \mathcal{J}_3^\psi = \\
&\quad \frac{1}{8\rho^{3/2}}\int_{-k(1)}^{k(1)}\Bigg(\int_{-R}^{R}\Bigg[I_0\left(\frac{\sqrt{R^2-z^2}}{2}\right) - \frac{2(R-1)}{\sqrt{R^2-z^2}} I_1\left(\frac{\sqrt{R^2-z^2}}{2}\right) \\
&\qquad + \frac{R^2}{R^2-z^2} I_2\left(\frac{\sqrt{R^2-z^2}}{2}\right)\Bigg]\psi'(z+u-t)\,dz\Bigg)X(t)\,dt \\
&\quad -\frac{1}{4\rho^{3/2}}\int_{-k(1)}^{k(1)}\left(1-\frac{R}{4}\right)\left(\psi'(R+u-t)+\psi'(-R+u-t)\right)X(t)\,dt.
\end{aligned} \tag{2.76}$$

The second term in (2.76) is manifestly bounded by $\frac{C_\psi(1+\log\rho)}{\rho^{3/2}}$. The first term in (2.76) is bounded by

$$\begin{aligned}
\frac{c_{total}\|\psi'\|_{L^\infty(\mathbb{R})}}{8\rho^{3/2}}\int_{-R}^{R}\Bigg| & I_0\left(\frac{\sqrt{R^2-z^2}}{2}\right) - \frac{2(R-1)}{\sqrt{R^2-z^2}} I_1\left(\frac{\sqrt{R^2-z^2}}{2}\right) \\
& + \frac{R^2}{R^2-z^2} I_2\left(\frac{\sqrt{R^2-z^2}}{2}\right)\Bigg|\,dz,
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
\frac{c_{total}\|\psi'\|_{L^\infty(\mathbb{R})}}{4\rho^{3/2}}\int_0^{\pi/2}\Bigg| & R I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta - 2R I_1\left(\frac{R}{2}\cos\theta\right) + 2 I_1\left(\frac{R}{2}\cos\theta\right) \\
& + \frac{R}{\cos\theta} I_2\left(\frac{R}{2}\cos\theta\right)\Bigg|\,d\theta,
\end{aligned} \tag{2.77}$$

which can be split up into

$$
\begin{aligned}
\frac{C_\psi}{\rho^{3/2}}\bigg\{ R\int_0^{\pi/2} & \left| I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta - I_1\left(\frac{R}{2}\cos\theta\right)\cos^2\theta\right| d\theta \\
&+ R\int_0^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right)\sin^2\theta\, d\theta \\
&+ R\int_0^{\pi/2} \left| I_1\left(\frac{R}{2}\cos\theta\right) - I_2\left(\frac{R}{2}\cos\theta\right)\cos\theta\right| d\theta \\
&+ R\int_0^{\pi/2} \frac{\sin^2\theta}{\cos\theta} I_2\left(\frac{R}{2}\cos\theta\right) d\theta \\
&+ 2\int_0^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right) d\theta\bigg\}.
\end{aligned}
$$

In the above, each integrand is non-negative. We can thus remove the absolute values and, using Lemmas A.1-A.5 to evaluate the integrals, bound (2.77) by $\frac{C_\psi}{\rho\log\rho}$. □

Altogether, Lemmas 2.40-2.46 completely prove Lemma 2.6, as desired.

2.5 Uniform estimates on solutions of the Navier–Stokes equations

2.5.1 Standard energy estimates

Below, $m^\varepsilon = \rho^\varepsilon u^\varepsilon$, and recall that the smooth solutions $(\rho^\varepsilon, u^\varepsilon)$ of the Navier–Stokes equations (1.25) are provided by the main theorem in [45] (*cf.* [80, Theorem 3.1]), with $(\rho_0^\varepsilon, u_0^\varepsilon)$ smooth initial data uniformly bounded away from the vacuum.

Lemma 2.47 (Lemma 3.2 of [80])**.** *Let $E[\rho_0^\varepsilon, u_0^\varepsilon] \leq E_0 < \infty$, for some constant $E_0 > 0$ independent of ε, and suppose that $(\rho^\varepsilon, u^\varepsilon)$ is the smooth solution of the Cauchy problem* (1.25) *with initial data $(\rho_0^\varepsilon, u_0^\varepsilon)$. Then, for any $T > 0$, there exists a constant $M > 0$, independent of ε, but depending on $E_0, T, \bar\rho, \bar u$, such that*

$$\sup_{t\in[0,T]} E[\rho^\varepsilon, u^\varepsilon](t) + \int_0^T\int_{\mathbb{R}} \varepsilon|u_x^\varepsilon|^2\,dx\,dt \leq M. \tag{2.78}$$

Proof. For brevity, we omit the superscript ε in this proof. Direct calculation yields

$$\frac{d}{dt}E[\rho,u](t) = \frac{d}{dt}\int_{\mathbb{R}} \eta^*(\rho,m)\,dx - \int_{\mathbb{R}} \nabla\eta^*(\bar\rho,\bar m)\cdot(\rho_t, m_t)\,dx,$$

as the reference functions $(\bar\rho,\bar m)$ are independent of t. Meanwhile, multiplying the first equation in (1.25) by $\eta^*_\rho(\rho,m)$ and the second by $\eta^*_m(\rho,m)$ and adding, we get

$$\eta^*(\rho,m)_t + q^*(\rho,m)_x = \varepsilon\eta^*_m(\rho,m)u_{xx}.$$

Recall that $\eta^*_m(\rho, m) = u$, from which an integration by parts yields

$$\frac{d}{dt}E[\rho,u](t) = q^*(\rho_-, m_-) - q^*(\rho_+, m_+) - \varepsilon\int_{\mathbb{R}} |u_x|^2\,dx - \int_{\mathbb{R}} \nabla\eta^*(\bar\rho,\bar m)\cdot(\rho_t, m_t)\,dx.$$

Using both equations in (1.25), we control the last term on the right-hand side,

$$\begin{aligned}
\left|\int_{\mathbb{R}} \nabla\eta^*(\bar\rho,\bar m)\cdot(\rho_t,m_t)\,dx\right| &= \left|\int_{\mathbb{R}} \nabla\eta^*(\bar\rho,\bar m)\cdot(m_x, (\rho u^2+p(\rho))_x - \varepsilon u_{xx})\,dx\right| \\
&\leq \int_{\mathbb{R}} |(\nabla\eta^*(\bar\rho,\bar m))_x\cdot(m, \rho u^2 + p(\rho) - \varepsilon u_x)|\,dx \\
&\quad + |\nabla\eta^*(\bar\rho,\bar m)\cdot(m,\rho u^2+p(\rho)-\varepsilon u_x)|_{x=-\infty}^{x=\infty}| \\
&\leq \frac{\varepsilon}{2}\int_{\mathbb{R}}|u_x|^2\,dx + M\int_{\mathbb{R}}\rho|u-\bar u|^2\,dx + M\left(1+\int_{-L_0}^{L_0}(\rho+p(\rho))\,dx\right),
\end{aligned}$$

since the reference functions are constant outside the interval $[-L_0, L_0]$ and $u_x(t,\cdot)\in L^2(\mathbb{R})$. Finally, applying Corollary 2.39 yields

$$\frac{d}{dt}E[\rho,u](t) + \frac{\varepsilon}{2}\int_{\mathbb{R}}|u_x|^2\,dx \leq C\big(1+E[\rho,u](t)\big),$$

and the result follows at once from an application of the Grönwall lemma. □

The second uniform estimate is the following, and is based on a slight modification of the standard argument of [16, Lemma 3.2], which relied on the previous ideas of Kanel' in [49]. As such, we do not include a proof, and simply state the result.

Lemma 2.48 (Lemma 3.3 of [80])**.** *Let $(\rho_0^\varepsilon, u_0^\varepsilon)$ be initial data such that*

$$\varepsilon^2\int_{\mathbb{R}}\frac{|\rho^\varepsilon_{0,x}(x)|^2}{\rho_0^\varepsilon(x)^3}\,dx \leq E_1 < \infty, \tag{2.79}$$

for a constant $E_1 > 0$ independent of ε. Then, for any $T > 0$, there exists a constant $M > 0$, depending on $E_0, E_1, \bar\rho, \bar u, T$, but independent of $\varepsilon > 0$, such that

$$\varepsilon^2\int_{\mathbb{R}}\frac{|\rho_x^\varepsilon(T,x)|^2}{\rho^\varepsilon(T,x)^3}\,dx + \varepsilon\int_0^T\int_{\mathbb{R}}\frac{p'(\rho^\varepsilon)}{(\rho^\varepsilon)^2}|\rho_x^\varepsilon|^2\,dx\,dt \leq M. \tag{2.80}$$

The third estimate is concerned with the higher integrability of the density. Once again, we omit the proof, as it is standard (*cf.* [16, Lemma 3.3]).

Lemma 2.49 (Lemma 3.4 of [80])**.** *Let $E_0[\rho_0^\varepsilon, u_0^\varepsilon] \leq E_0 < \infty$ with E_0 independent of ε and let $K \subset \mathbb{R}$ be compact. Then, for any $T > 0$, there exists a constant $M = M(E_0, K, \bar\rho, \bar u, T) > 0$, independent of $\varepsilon > 0$, such that*

$$\int_0^T\int_K \rho^\varepsilon p(\rho^\varepsilon)\,dx\,dt \leq M. \tag{2.81}$$

2.5.2 Higher integrability of the velocity

The final estimate of this section is concerned with the higher integrability of the velocity. The method of proof follows that of [16, Lemma 3.4] to the letter, but it now requires the results contained in Lemma 2.5. In order to perform the necessary manipulations, we need to derive a Taylor expansion for the entropy pairs near the end-states. This is encapsulated in the next lemma.

Lemma 2.50. *Define*

$$\begin{aligned}\check{\eta}(\rho, m) &:= \hat{\eta}(\rho, m - \rho u_-),\\ \check{q}(\rho, m) &:= \hat{q}(\rho, m - \rho u_-) + u_- \hat{\eta}(\rho, m - \rho u_-).\end{aligned} \tag{2.82}$$

Then, $\check{\eta}$ has the following Taylor expansion, about $m = \rho u_-$,

$$\check{\eta}(\rho, m) = \hat{\eta}_m(\rho, 0)\rho(u - u_-) + \check{r}(\rho, m), \tag{2.83}$$

where $\check{r}(\rho, \rho u) \leq M\rho|u - u_-|^2$ for some positive constant M independent of ε.

Proof. From Lemma 2.35, we know that $\hat{\eta}(\rho, 0) = 0$. As this is true for every $\rho > 0$, we also have $\hat{\eta}_\rho(\rho, 0) = 0$. Now for $\rho > 0$ fixed, we have, in the vicinity of $m = 0$,

$$\hat{\eta}(\rho, m) = m\hat{\eta}_m(\rho, 0) + r(\rho, m),$$

where, using the integral form of Taylor's theorem,

$$|r(\rho, m)| = \left| \int_0^m \hat{\eta}_{mm}(\rho, w)(m - w)\, dw \right| \leq M\frac{m^2}{\rho} = M\rho|u|^2,$$

where the inequality follows from Lemma 2.5. Translating by ρu_- and defining $\check{r}(\rho, m) := r(\rho, m - \rho u_-)$ readily yields the desired result. □

Lemma 2.51 (Lemma 3.6 of [80])**.** *Let the initial data $(\rho_0^\varepsilon, u_0^\varepsilon)$ satisfy that, in addition to the conditions required for Lemmas 2.47-2.49,*

$$\int_{\mathbb{R}} \rho_0^\varepsilon(x)|u_0^\varepsilon(x) - \bar{u}(x)|\, dx \leq M_0 < \infty, \tag{2.84}$$

where $M_0 > 0$ is a constant independent of ε. Then, for any compact set $K \subset \mathbb{R}$ and $T > 0$, there exists an $M > 0$ depending on K but not on ε, such that

$$\int_0^T \int_K \rho^\varepsilon |u^\varepsilon|^3\, dx\, dt \leq M. \tag{2.85}$$

Proof. In this proof we omit the ε superscript. Testing the governing equations for the viscous solutions (1.25) against $\check{\eta}_\rho$ and $\check{\eta}_m$, respectively, and adding, one obtains

$$\int_0^T \int_{-\infty}^x \big(\check{\eta}(\rho,m)_t + \check{q}(\rho,m)_x - \varepsilon u_{xx}\check{\eta}_m(\rho,m)\big)\,dy\,dt = 0.$$

Integrating by parts yields

$$\begin{aligned}\int_{-\infty}^x \big(\check{\eta}(\rho,m)(T,y) - \check{\eta}(\rho_0,m_0)(y)\big)\,dy - \varepsilon\int_0^T u_x\check{\eta}_m\,dt + \varepsilon\int_0^T\int_{-\infty}^x \check{\eta}_{mu}|u_x|^2\,dy\,dt \\ +\varepsilon\int_0^T\int_{-\infty}^x \rho_x u_x\check{\eta}_{m\rho}\,dy\,dt + \int_0^T \check{q}(\rho,m)\,dt - T\check{q}(\rho_-,m_-) = 0,\end{aligned}$$

where we emphasise again that $\check{\eta}_{m\rho}(\rho,m) = \partial_\rho\check{\eta}_m(\rho,\rho u)$ and $\check{\eta}_{mu}(\rho,m) = \partial_u\check{\eta}_m(\rho,\rho u)$ are the fake mixed derivatives (*cf.* Definition 2.27). Integrating over a compact set $K\subset\mathbb{R}$ yields

$$\begin{aligned}&\int_0^T\int_K \check{q}(\rho,m)\,dx\,dt \\ &\quad= T\int_K \check{q}(\rho_-,m_-)\,dx + \int_K\int_{-\infty}^x \check{\eta}(\rho_0,m_0)\,dy\,dx \\ &\quad\quad - \int_K\int_{-\infty}^x \check{\eta}(\rho,m)(T,y)\,dy\,dx + \varepsilon\int_0^T\int_K u_x\check{\eta}_m\,dx\,dt \\ &\quad\quad - \varepsilon\int_0^T\int_K\int_{-\infty}^x \check{\eta}_{mu}|u_x|^2\,dy\,dx\,dt - \varepsilon\int_0^T\int_K\int_{-\infty}^x \rho_x u_x\check{\eta}_{m\rho}\,dy\,dx\,dt \\ &\quad= J_1 + \cdots + J_6.\end{aligned} \tag{2.86}$$

The first integral J_1 is manifestly bounded by a constant $M = M(|K|,\rho_-,m_-,T)$. The third integral J_3 is bounded by

$$|J_3| \le \sup_{t\in[0,T]}\left|\int_K\int_{-\infty}^x \check{\eta}(\rho(t,y),m(t,y))\,dy\,dx\right|,$$

and we will bound the second integral J_2 in the same fashion as the term above. For the fourth integral, we have, using Lemma 2.5, that $|J_4|$ is bounded by

$$\varepsilon\int_0^T\int_K |u_x\check{\eta}_m|\,dx\,dt \le \varepsilon M\int_0^T\int_K (|u| + \rho^\theta\mathbb{1}_{\rho<1} + (1+\sqrt{\log\rho})\mathbb{1}_{\rho\ge1})|u_x|\,dx\,dt.$$

Using the Hölder inequality and the uniform estimates of Lemmas 2.47 and 2.49, the right-hand side is bounded by $M\big(1+\varepsilon\int_0^T\int_K |u|^2\,dx\,dt\big)$.

Claim 2.52. *The term $\varepsilon\int_0^T\int_K |u|^2\,dx\,dt$ is bounded independently of ε.*

We postpone the proof of the claim for the time being. Meanwhile, from Lemma 2.5, the fifth integral J_5 is bounded in absolute value by

$$|J_5| \le \varepsilon \int_0^T \int_K \int_{-\infty}^x |\check{\eta}_{mu}||u_x|^2 \, dy \, dx \, dt \le \varepsilon M \int_0^T \int_K \int_{-\infty}^x |u_x|^2 \, dy \, dx \, dt \le M,$$

using Lemma 2.47. The sixth integral J_6 is bounded in absolute value by

$$\begin{aligned} |J_6| &\le \varepsilon \int_0^T \int_K \int_{-\infty}^x |\rho_x||u_x||\check{\eta}_{m\rho}| \, dy \, dx \, dt \\ &\le \varepsilon M \int_0^T \int_K \int_{-\infty}^x |\rho_x|^2 |\check{\eta}_{m\rho}|^2 \, dy \, dx \, dt + \varepsilon M \int_0^T \int_K \int_{-\infty}^x |u_x|^2 \, dy \, dx \, dt \le M, \end{aligned}$$

using Lemmas 2.5, 2.47, and 2.48. In summary, we have from (2.86)

$$\int_0^T \int_K \check{q}(\rho, m) \, dx \, dt \le M + 2 \sup_{t \in [0,T]} \left| \int_K \int_{-\infty}^x \check{\eta}(\rho(t,y), m(t,y)) \, dy \, dx \right|,$$

from which one obtains

$$\begin{aligned} \int_0^T \int_K \hat{q}(\rho, m - \rho u_-) \, dx \, dt &\le M + |u_-| \int_0^T \int_K |\hat{\eta}(\rho, m - \rho u_-)| \, dx \, dt \\ &\quad + 2 \sup_{t \in [0,T]} \left| \int_K \int_{-\infty}^x \check{\eta}(\rho(t,y), m(y,t)) \, dy \, dx \right| \\ &=: M + J_1' + J_2'. \end{aligned} \tag{2.87}$$

Using Lemmas 2.5, 2.24, and 2.38, the first term can be estimated as follows

$$\begin{aligned} J_1' &= |u_-| \int_0^T \int_K |\hat{\eta}(\rho, m - \rho u_-)| \, dx \, dt \\ &\le M \int_0^T \int_K \left(\rho |u - u_-|^2 + \rho e(\rho) \mathbb{1}_{\rho < 1} + (\rho + \rho \log \rho) \mathbb{1}_{\rho \ge 1} \right) \, dx \, dt \\ &\le M \left(1 + \int_0^T E[\rho, u](\tau) \, d\tau \right), \end{aligned}$$

which is uniformly bounded, by Lemma 2.47. Next, consider J_2'. We have

$$\begin{aligned} \left| \int_{-\infty}^x \check{\eta}(\rho, m) \, dy \right| &\le \left| \int_{-\infty}^x \check{\eta}(\rho, m) - \hat{\eta}_m(\rho, 0) \rho (u - u^-) \, dy \right| \\ &\quad + \left| \int_{-\infty}^x \hat{\eta}_m(\rho, 0) \rho (u - u^-) \, dy \right|. \end{aligned} \tag{2.88}$$

Using the Taylor expansion (2.83), the right-hand side is bounded above by

$$\begin{aligned} \int_K \bigg(\int_{-\infty}^x \rho |u - u_-|^2 \, dy &+ \int_{-\infty}^x \rho |\hat{\eta}_m(\rho, 0) - \hat{\eta}_m(\rho_-, 0)|^2 \, dy \\ &+ |\hat{\eta}_m(\rho_-, 0)| \left| \int_{-\infty}^x \rho (u - u_-) \, dy \right| \bigg) \, dx. \end{aligned} \tag{2.89}$$

The first term of (2.89) is manifestly controlled by $|K| \sup_{t\in[0,T]} E[\rho,u](t)$, which is bounded independently of ε. We bound the last two terms of (2.89). By integrating the mass and momentum conservation equations in time and space,

$$\begin{aligned}\int_{-\infty}^{x} \rho(t,y)(u(t,y)-u_-)\,dy = &\int_{-\infty}^{x} \rho_0(u_0-\bar{u})\,dy + \int_{-\infty}^{x} \rho_0(\bar{u}-u_-)\,dy\\ &- \int_0^t \big(\rho u^2 + p(\rho) - p(\rho_-) - \rho u u_-\big)\,d\tau + \varepsilon \int_0^t u_x\,d\tau.\end{aligned}$$

Now, we have, using the Cauchy–Schwarz inequality,

$$\varepsilon \int_K \left|\int_0^t u_x\,d\tau\right|\,dx \le \varepsilon \int_0^t \int_K |u_x|\,dx\,d\tau \le \sqrt{\varepsilon}\left(\varepsilon \int_0^t \int_{\mathbb{R}} |u_x|^2\,dx\,d\tau\right)^{1/2} \le \sqrt{\varepsilon} M,$$

and

$$\begin{aligned}&\int_K \left|\int_0^t \big(\rho u^2 + p - p(\rho_-) - \rho u u_-\big)\,d\tau\right| dx\\ =& \int_K \left|\int_0^t \big(\rho|u-u_-|^2 + p - p(\rho_-) + u_-\rho(u-u_-)\big)\,d\tau\right|\,dx\\ \le& M\int_0^t \int_K \rho|u-u_-|^2\,dx\,d\tau + M\int_0^t\int_K \rho\,dx\,d\tau + \int_K \left|\int_0^t (p(\rho)-p(\rho_-))\,d\tau\right|\,dx,\end{aligned}$$

which is manifestly bounded independently of ε using Lemmas 2.38 and 2.47. Clearly, we also have

$$\int_K \left|\int_{-\infty}^x \rho_0(u_0-\bar{u})\,dy\right|\,dx \le \int_K\int_{\mathbb{R}} \rho_0|u_0-\bar{u}|\,dy\,dx \le M_0|K|,$$

from the hypothesis (2.84). Assuming without loss of generality $\sup K \ge L_0$,

$$\begin{aligned}&\int_K \left|\int_{-\infty}^x \rho_0(\bar{u}-u_-)\,dy\right| dx\\ &\qquad \le |K| \int_{-\infty}^{\sup K} \rho_0|\bar{u}-u_-|\,dy\\ &\qquad = |K| \int_{-L_0}^{\min\{L_0,\sup K\}} \rho_0|\bar{u}-u_-|\,dy + |K|\int_{\min\{L_0,\sup K\}}^{\sup K} \rho_0|u_+-u_-|\,dy\\ &\qquad \le M\int_{-L_0}^{\sup K} \rho_0\,dy,\end{aligned}$$

which is finite, since ρ_0 was assumed to be locally integrable. In the above, $M = M(|K|,\rho_0,L_0,u_+,u_-)$, so the above bound is independent of ε.

Also, an application of the fundamental theorem of calculus and Lemma 2.5 gives

$$
\begin{aligned}
&\frac{1}{2}\int_K \int_{-\infty}^{x} \rho|\hat{\eta}_m(\rho,0)-\hat{\eta}_m(\rho_-,0)|^2\,dy\,dx \\
&\quad\le \int_K \int_{-\infty}^{x} \rho|\hat{\eta}_m(\rho,0)-\hat{\eta}_m(\bar{\rho},0)|^2\,dy\,dx + \int_K \int_{-\infty}^{x} \rho|\hat{\eta}_m(\bar{\rho},0)-\hat{\eta}_m(\rho_-,0)|^2\,dy\,dx \\
&\quad\le M\int_K \int_{-\infty}^{x} e^*(\rho,\bar{\rho})\,dy\,dx + |K|\int_{-L_0}^{\min\{L_0 \sup K\}} \rho|\hat{\eta}_m(\bar{\rho},0)-\hat{\eta}_m(\rho_-,0)|^2\,dy \\
&\qquad + |K|\int_{\min\{L_0,\sup K\}}^{\sup K} \rho|\hat{\eta}_m(\rho_+,0)-\hat{\eta}_m(\rho_-,0)|^2\,dy \\
&\quad\le M\bigg(\int_{-\infty}^{\sup K} \big(e^*(\rho(t,y),\bar{\rho}(t,y))+\rho(t,y)\big)\,dy\bigg).
\end{aligned}
$$

Appealing to Corollary 2.39, we get

$$
\bigg|\int_K \int_{-\infty}^{x} \rho|\hat{\eta}_m(\rho,0)-\hat{\eta}_m(\rho_-,0)|^2\,dy\,dx\bigg| \le M\bigg(1+\sup_{t\in[0,T]}\int_{\mathbb{R}} e^*(\rho(t,y),\bar{\rho}(t,y))\,dy\bigg).
$$

As a result, the middle term in (2.89) is bounded by $M\sup_{t\in[0,T]} E[\rho,u](t)$, which is bounded independently of ε, by Lemma 2.47. Hence, we have shown

$$
\bigg|\int_K \int_{-\infty}^{x} \check{\eta}(\rho(t,y),m(t,y))\,dy\,dx\bigg| \le M,
$$

and thus, by referring back to (2.87),

$$
\int_0^T \int_K \hat{q}(\rho, m-\rho u_-)\,dx\,dt \le M.
$$

In view of Lemma 2.5, the previous estimate yields

$$
\begin{aligned}
&\int_0^T \int_K \rho|u-u_-|^3\,dx\,dt \\
&\qquad\le M\bigg(1+\int_0^T \int_K \big(\rho|u-u_-|^2 + (\rho+\rho(\log\rho)^4)\mathbb{1}_{\rho\ge 1} + \rho^\gamma \mathbb{1}_{\rho<1}\big)\,dx\,dt\bigg),
\end{aligned}
$$

and the right-hand side is uniformly bounded, by Lemmas 2.47 and 2.49. □

Proof of Claim 2.52. We follow the strategy in [16, Lemma 3.4] to the letter. Begin by observing that, due to [16, Lemma 3.1],

$$
\sup_{\tau\in[0,t]} \int_{\mathbb{R}} e^*(\rho(\tau,x),\bar{\rho}(x))\,dx \le C(t).
$$

Hence, there is a strictly positive non-decreasing function of time $C(t)$ such that

$$
\int_{\{x\in\mathbb{R}:\rho(t,x)\le\bar{\rho}(x)/2\}} e^*(\rho(t,x),\bar{\rho}(x))\,dx \le C(t). \tag{2.90}
$$

Define the constant $\check{\rho} := \min\{\rho_-, \rho_+\}$. Due to the monotonicity of $\bar{\rho}$ we have that $\check{\rho} \le \bar{\rho}(x)$ for every $x \in \mathbb{R}$. Thus, for every x in the set $\{x \in \mathbb{R} : \rho(t,x) \le \check{\rho}/2\}$, we manifestly have $\rho(t,x) \le \bar{\rho}(x)/2$. In view of (2.58) (*cf.* Lemma 2.37), we therefore have

$$e^*(\check{\rho}/2, \check{\rho})\mathbb{1}_{\{\rho \le \check{\rho}/2\}} \le e^*(\rho(t,x), \bar{\rho}(x))\mathbb{1}_{\{\rho \le \check{\rho}/2\}}. \tag{2.91}$$

Integrating (2.91) in x and using (2.90), we obtain the Markov type inequality

$$|\{x \in \mathbb{R} : \rho(t,x) \le \check{\rho}/2\}| \le \frac{C(t)}{e^*(\check{\rho}/2, \check{\rho})}.$$

Suppose without loss of generality that K contains an interval $[a,b]$ of length at least $2C(t)/e^*(\check{\rho}/2, \check{\rho})$. Now define

$$A := [a,b] \cap \{x \in \mathbb{R} : \rho(t,x) > \check{\rho}/2\}, \qquad B := [a,b] \cap \{x \in \mathbb{R} : \rho(t,x) \le \check{\rho}/2\}.$$

It is clear that $|B| \le C(t)/e^*(\check{\rho}/2, \check{\rho})$, and thus, by taking complements within K, we deduce that $|A| \ge C(t)/e^*(\check{\rho}/2, \check{\rho})$. Since the latter quantity is positive, it is legitimate to define the mean value of u on A to be

$$u_A(t) := \frac{1}{|A|} \int_A u(t,y)\, dy.$$

Then, observe that for any $x \in [a,b]$, at this fixed t, we have

$$|u(t,x)| \le |u_A(t)| + \int_K |u_x|\, dy. \tag{2.92}$$

Remark 2.53. Indeed, for any $x \in K$ and $a \in A$, we may write $u(t,x) = u(t,a) + \int_a^x u_x(t,y)\, dy$. Integrating in a over A and dividing by $|A|$ gives (2.92).

It follows straightforwardly that

$$\begin{aligned}
|u_A(t)| &\le \frac{1}{|A|} \int_A |u(t,y)|\, dy \le \frac{1}{|A|}\sqrt{\frac{2}{\check{\rho}}} \int_A \sqrt{\rho(t,y)}|u(t,y)|\, dy \\
&\le \sqrt{\frac{2}{\check{\rho}|A|}} \left(\int_K \rho(t,y)|u(t,y)|^2\, dy \right)^{1/2} \\
&\le \sqrt{\frac{2e^*(\check{\rho}/2, \check{\rho})}{\check{\rho} C(t)}} C(E_0, K),
\end{aligned} \tag{2.93}$$

in view of Lemma 2.47. Using the previous inequalities and Lemma 2.47,

$$\varepsilon M \int_0^t \int_K |u|^2\, dx\, d\tau \le M \left(\varepsilon \int_0^t \int_{\mathbb{R}} |u_x|^2\, dx\, d\tau + \int_0^t |u_A(\tau)|^2\, d\tau \right) \le M.$$

□

Remark 2.54. The bound in (2.93) does not depend on the value $\check{\rho}$. Indeed,

$$2e^*(\check{\rho}/2,\check{\rho})/\check{\rho} = e(\check{\rho}/2) - e(\check{\rho}) + \check{\rho}e'(\check{\rho}) \to 0 \qquad \text{in the limit as } \check{\rho} \to 0.$$

In turn, we are free to choose initial data that achieve vacuum in the limit as $\varepsilon \to 0$.

Finally, we collect the assumptions on the regularized initial data $(\rho_0^\varepsilon, u_0^\varepsilon)$ for Lemmas 2.47-2.51, and the main theorem of Hoff [45], to hold (*cf.* [80, Remark 3.7]).

Definition 2.55. We say that a family of functions $\{(\rho_0^\varepsilon, u_0^\varepsilon)\}_{\varepsilon>0}$ is an *admissible sequence of initial data* if the following assumptions hold.

- The initial data must be of finite energy, i.e., $\sup_\varepsilon E[\rho_0^\varepsilon, u_0^\varepsilon] \le E_0 < \infty$.
- The initial density must satisfy a weighted derivative bound, i.e.,

$$\sup_\varepsilon \varepsilon^2 \int_{\mathbb{R}} \frac{|\rho_{0,x}^\varepsilon(x)|^2}{\rho_0^\varepsilon(x)^3}\, dx \le E_1 < \infty.$$

- The relative total initial momentum should be finite, i.e.,

$$\sup_\varepsilon \int_{\mathbb{R}} \rho_0^\varepsilon(x)|u_0^\varepsilon(x) - \bar{u}(x)|\, dx \le M_0 < \infty.$$

- The initial density stays strictly away from the vacuum, i.e.,

$$\exists c_0^\varepsilon > 0 \text{ such that } \rho_0^\varepsilon(x) \ge c_0^\varepsilon \text{ for all } x \in \mathbb{R}.$$

2.6 Compactness of the entropy dissipation measures

In this section, $(\rho^\varepsilon, u^\varepsilon)$ are the solutions of the Navier–Stokes equations (1.25) generated by an admissible sequence of initial data, as per Definition 2.55. In what follows, we show that these viscous approximate solutions converge weakly, and prove that the family of entropy dissipation measures generated by them is compact, in some sense. This enables us to derive the Tartar–Murat commutation relation in the next section. This relation is an essential tool in reducing the Young measure generated by the viscous solutions, which allows us to prove their convergence in measure.

Lemma 2.56 (Weak compactness of ρ^ε)**.** *There exists a function $\rho \in L^2_{loc}(\mathbb{R}^2_+)$ such that, on any bounded subset $U \subset \mathbb{R}^2_+$, there exists a subsequence $\{\rho^{\varepsilon'}\}_{\varepsilon'}$ such that*

$$\rho^{\varepsilon'} \rightharpoonup \rho \qquad \text{in } L^2(U).$$

Proof. Fix any compact set $K \subset \mathbb{R}$ and any $T > 0$. Then, an application of Lemma 2.49 on each of the sets $\{\rho^\varepsilon < 1\} \cap ([0,T] \times K)$ and $\{\rho^\varepsilon \geq 1\} \cap ([0,T] \times K)$ shows that there exists a positive constant $C_{K,T}$, independent of ε, such that

$$\int_0^T \int_K |\rho^\varepsilon(t,x)|^2 \, dx \, dt \leq C_{K,T}. \tag{2.94}$$

Let $V_n := [0,n] \times [-n,n]$, and observe that $\{V_n\}_{n\in\mathbb{N}}$ is a countable cover of nested sets that cover the plane $\mathbb{R}^2_+ = \bigcup_{n\in\mathbb{N}} V_n$. Starting with $n = 1$, the Banach–Alaoglu theorem implies that there exists a square-integrable function ρ_1 for which a subsequence $\rho^{\varepsilon^{(1)}} \rightharpoonup \rho_1$ in $L^2(V_1)$. Going down further subsequences, we find that for each $n \in \mathbb{N}$, there exists a subsequence $\rho^{\varepsilon^{(n)}}$ and a square-integrable function ρ_n such that $\rho^{\varepsilon^{(n)}} \rightharpoonup \rho_n$ in $L^2(V_n)$. So, in accordance with the Lebesgue differentiation theorem (*cf.* [35, Theorem 1.32]), we legitimately define the limit function $\rho \in L^2_{loc}(\mathbb{R}^2_+)$ by

$$\rho(t,x) := \lim_{\delta\to 0} \frac{1}{|B((t,x),\delta)|} \int_{B((t,x),\delta)} \rho_n(\tau,y) \, d\tau \, dy \qquad \text{if } (t,x) \in \operatorname{int} V_n,$$

i.e., $\rho(t,x) = \rho_n(t,x)$ for a.e. $(t,x) \in \operatorname{int} V_n$; in a sense, we require a further subsequence at each step n to ensure that we get convergence to the restriction of ρ to a larger set. Note that this limit is well-defined: suppose that (t,x) belongs to two such sets $\operatorname{int} V_n$ and $\operatorname{int} V_m$, and assume without loss of generality $V_n \subset V_m$. Then, going down the finer subsequence $\varepsilon^{(m)}$, we have $\rho^{\varepsilon^{(m)}} \rightharpoonup \rho_n$ and $\rho^{\varepsilon^{(m)}} \rightharpoonup \rho_m$ in $L^2(V_n)$. But also, since $\{\rho^{\varepsilon^{(m)}}\}$ is a subsequence of $\{\rho^{\varepsilon^{(n)}}\}$, we have that $\text{w-lim}_{\varepsilon^{(m)}} \rho^{\varepsilon^{(m)}} = \text{w-lim}_{\varepsilon^{(n)}} \rho^{\varepsilon^{(n)}} = \rho_n$, in the smaller set $L^2(V_n)$. Then, by subtracting the two sequences, $\rho_n - \rho_m = \text{w-lim}_{\varepsilon^{(m)}} \big(\rho^{\varepsilon^{(m)}} - \rho^{\varepsilon^{(m)}}\big) = 0$ in $L^2(V_n)$. Thus, $\rho_n = \rho_m$ in the sense of L^2 functions on the smaller set V_n. Hence, for δ chosen small enough such that $B(x,\delta) \subset V_n$,

$$\begin{aligned}
&\left| \frac{1}{|B((t,x),\delta)|} \int_{B((t,x),\delta)} \rho_n(\tau,y) \, d\tau \, dy - \frac{1}{|B((t,x),\delta)|} \int_{B((t,x),\delta)} \rho_m(\tau,y) \, d\tau \, dy \right| \\
&\qquad = \frac{1}{|B((t,x),\delta)|} \left| \int_{B((t,x),\delta)} \big(\rho_n(\tau,y) - \rho_m(\tau,y)\big) \, d\tau \, dy \right| \\
&\qquad \leq \frac{1}{|B((t,x),\delta)|} \int_{V_n} |\rho_n(\tau,y) - \rho_m(\tau,y)| \, d\tau \, dy.
\end{aligned}$$

Using the Cauchy–Schwarz inequality, the above is bounded by $\frac{\sqrt{|V_n|}\|\rho_n - \rho_m\|_{L^2(V_n)}}{|B((t,x),\delta)|}$, which is precisely zero for such chosen δ, as required. We have thus defined the limit function $\rho \in L^2_{loc}(\mathbb{R}^2_+)$ almost everywhere, as a locally square-integrable function. Then, given any bounded set $U \subset \mathbb{R}^2_+$, there exists an $n \in \mathbb{N}$ such that $U \subset V_n$, on which $\rho^{\varepsilon^{(n)}}$ converges weakly to ρ in $L^2(V_n) \supset L^2(U)$, as required. $\square$

Lemma 2.57. *There exists a subsequence* $\{\rho^{\varepsilon'}\}_{\varepsilon'}$ *and a function* $\rho \in L^2_{loc}(\mathbb{R}^2_+)$ *such that, on any bounded subset* $U \subset \mathbb{R}^2_+$*,*

$$\rho^{\varepsilon'} \rightharpoonup \rho \quad \textit{in } L^2(U).$$

Proof. As before, let $V_n = [0, n] \times [-n, n]$ be a family of subsets of $\mathbb{R}^2_+$ which exhaust the plane, in the sense that $\mathbb{R}^2_+ = \bigcup_{n\in\mathbb{N}} V_n$. Then, Lemma 2.56 shows that, for each fixed $n \in \mathbb{N}$, there exists a subsequence $\{\varepsilon_k^{(n)}\}_{k=1}^\infty$ such that $\rho^{\varepsilon_k^{(n)}} \rightharpoonup \rho$ in $L^2(V_n)$ as $k \to \infty$. Now, from the array $\{\varepsilon_k^{(n)}\}_{n,k\in\mathbb{N}}$, select a diagonal subsequence $\{\varepsilon_{\sigma(n)}\}_{n\in\mathbb{N}}$ such that $\rho^{\varepsilon_{\sigma(n)}} \rightharpoonup \rho$ in $L^2(V_m)$ for any m, as $n \to \infty$. Relabel this subsequence as ε'. This proves the result. □

Remark 2.58. Note that $\rho^\varepsilon \rightharpoonup \rho$ in $L^2_{loc}(\mathbb{R}^2_+)$ certainly does not imply that $\eta^\psi(\rho^\varepsilon, m^\varepsilon)$ converges weakly. Indeed, $\eta^\psi \in C^2(\mathbb{R}^2_+)$, but is not necessarily continuous with respect to the weak topology on $L^2(\mathbb{R}^2_+)$. In fact, we do not even have the weak convergence of m^ε to a locally square-integrable function, so we must proceed differently and use the entropy pairs directly.

Lemma 2.59 (Weak compactness of $\eta^\psi(\rho^\varepsilon, m^\varepsilon)$ and $q^\psi(\rho^\varepsilon, m^\varepsilon)$)**.** *Let* $\psi \in C^2_c(\mathbb{R})$*, and let* η^ψ *and* q^ψ *be the entropy and entropy-flux generated by* ψ *through* (1.13) *and* (1.15)*. Then, there exist two functions* $Z^\psi, W^\psi \in L^2_{loc}(\mathbb{R}^2_+)$*, such that, for any bounded subset* $U \subset \mathbb{R}^2_+$*, there exists a subsequence* $(\rho^{\varepsilon'}, m^{\varepsilon'})$*, such that*

$$\begin{aligned} \eta^\psi(\rho^{\varepsilon'}, m^{\varepsilon'}) \rightharpoonup Z^\psi \qquad & \textit{in } L^2(U), \\ q^\psi(\rho^{\varepsilon'}, m^{\varepsilon'}) \rightharpoonup W^\psi \qquad & \textit{in } L^2(U). \end{aligned}$$

Proof. We follow the same procedure as in Lemma 2.56. We consider, as before, the sets $V_n = [0, n] \times [-n, n]$ which form a nested cover of the whole of the plane $\mathbb{R}^2_+$. On each of those sets, we have, by Lemma 2.6,

$$|\eta^\psi(\rho^\varepsilon, m^\varepsilon)| + |q^\psi(\rho^\varepsilon, m^\varepsilon)| \le C_\psi \rho^\varepsilon,$$

which implies, using (2.94), that, for $n = 1$,

$$\|\eta^\psi(\rho^\varepsilon, m^\varepsilon)\|_{L^2(V_1)} + \|q^\psi(\rho^\varepsilon, m^\varepsilon)\|_{L^2(V_1)} \le C\|\rho^\varepsilon\|_{L^2(V_1)} \le C,$$

for a positive constant C independent of ε. Thus, by the theorem of Banach–Alaoglu, there exist a subsequence $(\rho^{\varepsilon^{(1)}}, m^{\varepsilon^{(1)}})$ and functions $Z^\psi_1, W^\psi_1 \in L^2(V_1)$ such that

$$\begin{aligned} \eta^\psi(\rho^{\varepsilon^{(1)}}, m^{\varepsilon^{(1)}}) \rightharpoonup Z^\psi_1 \qquad & \text{in } L^2(V_1), \\ q^\psi(\rho^{\varepsilon^{(1)}}, m^{\varepsilon^{(1)}}) \rightharpoonup W^\psi_1 \qquad & \text{in } L^2(V_1). \end{aligned}$$

As before, we go down a further subsequence at each step n, and obtain the existence of a subsequence $(\rho^{\varepsilon^{(n)}}, m^{\varepsilon^{(n)}})$ and functions $Z_n^\psi, W_n^\psi \in L^2(V_n)$ such that

$$\begin{aligned} \eta^\psi(\rho^{\varepsilon^{(n)}}, m^{\varepsilon^{(n)}}) &\rightharpoonup Z_n^\psi \qquad \text{in } L^2(V_n), \\ q^\psi(\rho^{\varepsilon^{(n)}}, m^{\varepsilon^{(n)}}) &\rightharpoonup W_n^\psi \qquad \text{in } L^2(V_n). \end{aligned}$$

Appealing to [35, Theorem 1.32], we may once again define

$$\begin{aligned} Z^\psi(t,x) &= \lim_{\delta\to 0} \frac{1}{|B((t,x),\delta)|} \int_{B((t,x),\delta)} Z_n^\psi(\tau,y)\,d\tau\,dy \qquad \text{if } (t,x) \in \operatorname{int} V_n, \\ W^\psi(t,x) &= \lim_{\delta\to 0} \frac{1}{|B((t,x),\delta)|} \int_{B((t,x),\delta)} W_n^\psi(\tau,y)\,d\tau\,dy \qquad \text{if } (t,x) \in \operatorname{int} V_n, \end{aligned}$$

so that $Z^\psi(t,x) = Z_n^\psi(t,x)$ and $W^\psi(t,x) = W_n^\psi(t,x)$ for a.e. $(t,x) \in \operatorname{int} V_n$; this guarantees equality as L^2 functions. The same procedure as in the proof of Lemma 2.56 shows that this limit is well-defined. We have thus defined the limit functions $Z^\psi, W^\psi \in L^2_{loc}(\mathbb{R}^2_+)$ as locally square-integrable functions. Then, given any bounded set $U \subset \mathbb{R}^2_+$, there exists an $n \in \mathbb{N}$ such that $U \subset V_n$, on which $\eta^\psi(\rho^{\varepsilon^{(n)}}, m^{\varepsilon^{(n)}})$ and $q^\psi(\rho^{\varepsilon^{(n)}}, m^{\varepsilon^{(n)}})$ converge weakly to Z^ψ, W^ψ, respectively, in $L^2(V_n) \supset L^2(U)$. □

Lemma 2.60. *Let $\psi \in C_c^2(\mathbb{R})$, and let η^ψ and q^ψ be the entropy and entropy-flux generated by ψ through* (1.13) *and* (1.15)*. Then, there exists a subsequence $(\rho^{\varepsilon'}, m^{\varepsilon'})$ and two functions $Z^\psi, W^\psi \in L^2_{loc}(\mathbb{R}^2_+)$, such that, for any bounded subset $U \subset \mathbb{R}^2_+$*

$$\begin{aligned} \eta^\psi(\rho^{\varepsilon'}, m^{\varepsilon'}) &\rightharpoonup Z^\psi \qquad \text{in } L^2(U), \\ q^\psi(\rho^{\varepsilon'}, m^{\varepsilon'}) &\rightharpoonup W^\psi \qquad \text{in } L^2(U). \end{aligned} \tag{2.95}$$

Proof. We first argue for $\eta^\psi(\rho^{\varepsilon'}, m^{\varepsilon'})$ using the same strategy as in Lemma 2.57. Now that we have found this subsequence $(\rho^{\varepsilon'}, m^{\varepsilon'})$, use Lemma 2.59 to obtain subsequences $\{\varepsilon_k^{(n)}\}_{k=1}^\infty$ of $q^\psi(\rho^{\varepsilon'}, m^{\varepsilon'})$ that converge in $L^2(V_n)$ for each fixed n. We once again select a diagonal subsequence from this array, denoted by $\{\varepsilon_{\sigma(n)}\}_{n\in\mathbb{N}}$, for which $q^\psi(\rho^{\varepsilon_{\sigma(n)}}, m^{\varepsilon_{\sigma(n)}})$ converges to W^ψ in $L^2(V_m)$ for any m, as $n \to \infty$. This final subsequence $\varepsilon_{\sigma(n)}$ was a subsequence of the original ε' sequence, so we also have that $\eta^\psi(\rho^{\varepsilon_{\sigma(n)}}, m^{\varepsilon_{\sigma(n)}}) \rightharpoonup Z^\psi$ in $L^2(V_m)$ for any m, as $n \to \infty$. We relabel this final subsequence as ε'. This proves the lemma. □

Lemma 2.61 (Propostion 3.9 of [80])**.** *Let $\psi \in C_c^2(\mathbb{R})$, and let η^ψ and q^ψ be the entropy and entropy-flux generated by ψ through* (1.13) *and* (1.15)*. Then, for the viscous approximate solutions $(\rho^\varepsilon, u^\varepsilon)$ associated with an admissible sequence of initial data, we have that the* entropy dissipation measures*,*

$$\eta^\psi(\rho^\varepsilon, m^\varepsilon)_t + q^\psi(\rho^\varepsilon, m^\varepsilon)_x \tag{2.96}$$

are confined to a compact subset of $W^{-1,\tilde{p}}_{loc}(\mathbb{R}^2_+)$ for any $\tilde{p} \in [1,2)$.

Proof. Fix any compact set $K \subset \mathbb{R}$ and $T > 0$. Multiplying the first equation of (1.25) by $\eta^\psi_\rho(\rho^\varepsilon, m^\varepsilon)$ and the second by $\eta^\psi_m(\rho^\varepsilon, m^\varepsilon)$, and adding them, we get

$$\begin{aligned} \eta^\psi(\rho^\varepsilon, m^\varepsilon)_t + q^\psi(\rho^\varepsilon, m^\varepsilon)_x = \varepsilon(\eta^\psi_m(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)u^\varepsilon_x)_x - \varepsilon\eta^\psi_{mu}(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)|u^\varepsilon_x|^2 \\ - \varepsilon\eta^\psi_{m\rho}(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)\rho^\varepsilon_x u^\varepsilon_x, \end{aligned} \tag{2.97}$$

where the mixed derivative quantities are those of Definition 2.27. Below, we bound the last two terms on the right-hand side of (2.97). Applying Lemma 2.6 yields

$$\left|\varepsilon\eta^\psi_{mu}(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)|u^\varepsilon_x|^2\right| \le \varepsilon C_\psi |u^\varepsilon_x|^2,$$

which is uniformly bounded in $L^1_{loc}(\mathbb{R}^2_+)$, independently of ε, by Lemma 2.47, i.e.,

$$\|\varepsilon\eta^\psi_{mu}(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)|u^\varepsilon_x|^2\|_{L^1([0,T]\times K)} \le C_\psi \|\sqrt{\varepsilon} u^\varepsilon_x\|^2_{L^2([0,T]\times K)} \le M.$$

Similarly,

$$\begin{aligned} \left|\varepsilon\eta^\psi_{m\rho}(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)\rho^\varepsilon_x u^\varepsilon_x\right| \le & \varepsilon C_\psi \left(\eta^\psi_{m\rho}(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)^2(\rho^\varepsilon_x)^2 + |u^\varepsilon_x|^2\right) \\ \le & C_\psi \left(|\sqrt{\varepsilon}\eta^\psi_{m\rho}(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)\rho^\varepsilon_x|^2 + |\sqrt{\varepsilon} u^\varepsilon_x|^2\right). \end{aligned}$$

By an application of Lemma 2.6, it follows that

$$\|\varepsilon\eta^\psi_{m\rho}(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)\rho^\varepsilon_x u^\varepsilon_x\|_{L^1([0,T]\times K)} \le C_\psi\Big(\Big\|\sqrt{\varepsilon}\frac{\sqrt{p'(\rho^\varepsilon)}}{\rho^\varepsilon}\rho^\varepsilon_x\Big\|^2_{L^2([0,T]\times K)} + \|\sqrt{\varepsilon}u^\varepsilon_x\|^2_{L^2([0,T]\times K)}\Big),$$

and the latter is uniformly bounded, by Lemmas 2.47-2.48. Hence, $\varepsilon\eta^\psi_{mu}(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)|u^\varepsilon_x|^2$ and $\varepsilon\eta^\psi_{m\rho}(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)\rho^\varepsilon_x u^\varepsilon_x$ are bounded independently of ε in $L^1_{loc}(\mathbb{R}^2_+)$, meaning that they are precompact in $W^{-1,\tilde{p}}_{loc}(\mathbb{R}^2_+)$, for $\tilde{p} \in (1, 2)$, by the Kondrachov embedding.

Also, Lemma 2.6 implies that $|\eta^\psi_m(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)| \le C_\psi$, from which we get

$$\|\varepsilon\eta^\psi_m(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)u^\varepsilon_x\|_{L^2([0,T]\times\mathbb{R})} \le C\sqrt{\varepsilon}\|\sqrt{\varepsilon}u^\varepsilon_x\|_{L^2([0,T]\times\mathbb{R})} \le C\sqrt{\varepsilon} \to 0.$$

Hence, $\varepsilon(\eta^\psi_m(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)u^\varepsilon_x)_x \to 0$ in $W^{-1,2}_{loc}(\mathbb{R}^2_+)$. We conclude $\eta^\psi(\rho^\varepsilon, m^\varepsilon)_t + q^\psi(\rho^\varepsilon, m^\varepsilon)_x$ belongs to a compact subset of $W^{-1,\tilde{p}}_{loc}(\mathbb{R}^2_+)$ for $\tilde{p} \in (1, 2)$.

Finally note that, on a bounded subset $U \subset \mathbb{R}^2_+$, the embedding $L^\infty(U) \hookrightarrow L^{\tilde{p}}(U)$ is continuous for $\tilde{p} \ge 1$. As a result, arguing by duality, compactness in $W^{-1,\tilde{p}}(U)$ for $\tilde{p} \in (1, 2)$ implies compactness in $W^{-1,1}(U) = (W^{1,\infty}_0(U))^*$ (*cf.* [24, Section 1]). This verifies the result for the end-point case $\tilde{p} = 1$, which concludes the proof. □

Lemma 2.62. *Let $\psi_1, \psi_2 \in C^2_c(\mathbb{R})$. Correspondingly, define (η_1, q_1) and (η_2, q_2) be the two entropy pairs generated by ψ_1 and ψ_2, respectively, via* (1.13) *and* (1.15)*. Then, there exist $Z^i, W^i \in L^2_{loc}(\mathbb{R}^2_+)$, for $i = 1, 2$, and, for any open and bounded domain $U \subset \mathbb{R}^2_+$ with Lipschitz boundary, there exists a subsequence $(\rho^{\varepsilon'}, m^{\varepsilon'})$ such that*

$$\begin{aligned} &(\eta_i(\rho^{\varepsilon'}, m^{\varepsilon'}), q_i(\rho^{\varepsilon'}, m^{\varepsilon'})) \rightharpoonup (Z^i, W^i) \quad \textit{weakly in } L^2(U), \textit{ for } i = 1, 2, \textit{ and} \\ &\eta_1(\rho^{\varepsilon'}, m^{\varepsilon'})q_2(\rho^{\varepsilon'}, m^{\varepsilon'}) - \eta_2(\rho^{\varepsilon'}, m^{\varepsilon'})q_1(\rho^{\varepsilon'}, m^{\varepsilon'}) \rightharpoonup Z^1W^2 - Z^2W^1 \quad \textit{weakly in } L^1(U). \end{aligned}$$

Proof. Begin by fixing the open and bounded domain $U \subset \mathbb{R}^2_+$. We define the vectors

$$v_1^\varepsilon := (\eta_1(\rho^\varepsilon, m^\varepsilon), q_1(\rho^\varepsilon, m^\varepsilon)), \qquad v_2^\varepsilon := (q_2(\rho^\varepsilon, m^\varepsilon), -\eta_2(\rho^\varepsilon, m^\varepsilon)).$$

Select ε' to be the subsequence along which the conclusion of Lemma 2.59 holds, so that there exists vector fields (Z^i, W^i) for which $v_i^{\varepsilon'} \rightharpoonup (Z^i, W^i)$ in $L^2(U)$ for $i = 1, 2$. Note that

$$\operatorname{div} v_1^{\varepsilon'} = \eta_1(\rho^{\varepsilon'}, m^{\varepsilon'})_t + q_1(\rho^{\varepsilon'}, m^{\varepsilon'})_x,$$

which is precompact in $W^{-1,1}(U)$ by Lemma 2.61. Direct calculation also yields

$$\operatorname{curl} v_2^{\varepsilon''} = \Big[\eta_2(\rho^{\varepsilon''}, m^{\varepsilon''})_t + q_2(\rho^{\varepsilon''}, m^{\varepsilon''})_x\Big]\, \mathbf{e}_3,$$

which is also precompact in $W^{-1,1}(U)$. Additionally,

$$v_1^{\varepsilon'} \cdot v_2^{\varepsilon'} = \eta_1(\rho^{\varepsilon'}, m^{\varepsilon'})q_2(\rho^{\varepsilon'}, m^{\varepsilon'}) - \eta_2(\rho^{\varepsilon'}, m^{\varepsilon'})q_1(\rho^{\varepsilon'}, m^{\varepsilon'}),$$

which is an equi-integrable family. Indeed, observe that, using Lemma 2.6,

$$|v_1^{\varepsilon'} \cdot v_2^{\varepsilon'}| \leq M\left(1 + \frac{(\rho^{\varepsilon'})^2}{1 + \sqrt{|\log \rho^{\varepsilon'}|}}\right), \tag{2.98}$$

where without loss of generality we may assume $M > 1$. Define the non-negative increasing function $G(t) := \mathbb{1}_{t\geq 1} t\sqrt{\log t}$. Then, it is straightforward to show directly from (2.98) that

$$G\big(|v_1^{\varepsilon'} \cdot v_2^{\varepsilon'}|\big) \leq M\big(1 + \rho^{\varepsilon'} p(\rho^{\varepsilon'})\big), \tag{2.99}$$

for some positive constant M. The right-hand side of (2.99) is locally integrable on $\mathbb{R}^2_+$, bounded independently of ε', by Lemma 2.49. Thus, the non-negative increasing function $G : \mathbb{R}_+ \to \mathbb{R}_+$ is such that $\lim_{t\to\infty} \frac{G(t)}{t} = \infty$, and $\int_U G(|v_1^{\varepsilon'} \cdot v_2^{\varepsilon'}|)\, dx\, dt \leq C_U$, where C_U is independent of ε'. Hence, by de la Vallée–Poussin's criterion (*cf.* [7, Theorem 4.5.9]), the family $\{v_1^{\varepsilon'} \cdot v_2^{\varepsilon'}\}_{\varepsilon'}$ is equi-integrable. Thus, all of the assumptions of the main theorem of [24] (div-curl lemma in $W^{-1,1}$) are verified, and we conclude that $v_1^{\varepsilon'} \cdot v_2^{\varepsilon'} \rightharpoonup (Z^1, W^1) \cdot (W^2, -Z^2)$ in $L^1(U)$. □

By exhausting $\mathbb{R}^2_+$ with the sets $\{V_n\}_{n\in\mathbb{N}}$, as done in the proof of Lemmas 2.57 and 2.60, we find one subsequence that achieves the desired compactness on all compact subsets of $\mathbb{R}^2_+$. In view of this, we have the next lemma, which concludes this section.

Lemma 2.63. *Let $\psi_1, \psi_2 \in C^2_c(\mathbb{R})$. Correspondingly, define (η_1, q_1) and (η_2, q_2) be the two entropy pairs generated by ψ_1 and ψ_2, respectively, via* (1.13) *and* (1.15)*. Then, there exist a subsequence $(\rho^{\varepsilon'}, m^{\varepsilon'})$ and $Z^i, W^i \in L^2_{loc}(\mathbb{R}^2_+)$, for $i = 1, 2$, such that, for any open and bounded domain $U \subset \mathbb{R}^2_+$ with Lipschitz boundary,*

$$(\eta_i(\rho^{\varepsilon'}, m^{\varepsilon'}), q_i(\rho^{\varepsilon'}, m^{\varepsilon'})) \rightharpoonup (Z^i, W^i) \quad \textit{weakly in } L^2(U), \ \textit{for } i = 1, 2, \ \textit{and}$$

$$\eta_1(\rho^{\varepsilon'}, m^{\varepsilon'})q_2(\rho^{\varepsilon'}, m^{\varepsilon'}) - \eta_2(\rho^{\varepsilon'}, m^{\varepsilon'})q_1(\rho^{\varepsilon'}, m^{\varepsilon'}) \rightharpoonup Z^1W^2 - Z^2W^1 \quad \textit{weakly in } L^1(U).$$

2.7 The Young measure and derivation of the commutation relation

2.7.1 Construction of the Young measure

In this section, we establish a framework in which we can apply the fundamental theorem of Young measures (*cf.* [1, 3]). We follow [80, Section 4] to the letter, while a more thorough treatment may be found in [79, Chapter 3].

We denote the upper half-plane by $\mathbb{H} := \{(\rho, u) \in \mathbb{R}^2 : \rho > 0\}$, and consider the following subset of continuous functions

$$\bar{C}(\mathbb{H}) := \left\{ \phi \in C(\overline{\mathbb{H}}) \middle| \begin{array}{l} \phi(\rho, u) \text{ is constant on } \{\rho = 0\} \text{ and the function} \\ (\rho, u) \mapsto \lim_{s \to \infty} \phi(s\rho, su) \text{ belongs to } C(\mathbb{S}^1 \cap \overline{\mathbb{H}}) \end{array} \right\},$$

where $\mathbb{S}^1 \subset \mathbb{R}^2$ is the unit circle. Since $\bar{C}(\mathbb{H})$ is a complete sub-ring of the continuous functions on $\mathbb{H}$ containing the constant functions, there exists a compactification $\overline{\mathcal{H}}$ of $\mathbb{H}$ such that $C(\overline{\mathcal{H}})$ is isometrically isomorphic to $\bar{C}(\mathbb{H})$ (*cf.* [77, Proposition 1.5.3]), written $C(\overline{\mathcal{H}}) \cong \bar{C}(\mathbb{H})$. We denote this isometric isomorphism by $\iota : \bar{C}(\mathbb{H}) \to C(\overline{\mathcal{H}})$.

Remark 2.64. The topology of $\overline{\mathcal{H}}$ is the weak-star topology induced by $C(\overline{\mathcal{H}})$, i.e., a sequence $(v_n)_{n\in\mathbb{N}}$ in $\overline{\mathcal{H}}$ converges to $v \in \overline{\mathcal{H}}$ if $|\varphi(v_n) - \varphi(v)| \to 0$ for all $\varphi \in C(\overline{\mathcal{H}})$. This topology is separable and metrizable (*cf.* [78, Section 3.8]). In view of the functions that lie in $\bar{C}(\mathbb{H})$, the topology of $\overline{\mathcal{H}}$ does not distinguish between points of the vacuum.

In view of the previous remark, since $\overline{\mathcal{H}}$ is homeomorphic to a compact metric space, we may apply the fundamental theorem of Young measures of Alberti–Müller (*cf.* [1, Theorem 2.4]) in the way described underneath, as is done in [16, 58, 80].

Lemma 2.65. *Given a sequence of functions* $(\rho^\varepsilon, u^\varepsilon) : \mathbb{R}^2_+ \to \mathbb{H}$*, there exists a subsequence generating a Young measure* $\nu_{(t,x)} \in \mathrm{Prob}(\overline{\mathcal{H}})$ *in the sense that, for any* $\phi \in \bar{C}(\mathbb{H})$*,*

$$\phi(\rho^{\varepsilon'}(\cdot,\cdot), u^{\varepsilon'}(\cdot,\cdot)) \overset{*}{\rightharpoonup} \int_{\overline{\mathcal{H}}} \iota(\phi)(r, v)\, d\nu_{(\cdot,\cdot)}(r, v) \quad \text{in } L^\infty(\mathbb{R}^2_+).$$

Armed with the previous result, we show that the Young measure is in fact only supported on the interior of $\mathcal{H}$, and $\mathcal{H} = \mathbb{H} \cup V$ with $V := \{\rho = 0\}$ being the *vacuum line* (*cf.* [79, Chapter 3]). This is encapsulated in the following lemma, which was originally proved by LeFloch–Westdickenberg in [58].

Lemma 2.66 (Proposition 2.3 of [58])**.** *Let* $\nu_{(t,x)}$ *be a Young measure generated by a sequence of viscous approximate solutions of* (1.25) *associated with an admissible sequence of initial data. Then, the measure* $\nu_{(t,x)} \in \mathrm{Prob}(\mathcal{H})$ *for a.e.* $(t, x) \in \mathbb{R}^2_+$.

Next, we extend the Young measure $\nu_{(t,x)}$ to a larger class of test functions than just $\bar{C}(\mathbb{H})$. We follow the strategy of proof presented in [16, Proposition 5.1] to the letter.

Lemma 2.67 (Proposition 4.1 of [80]). *The following statements hold:*

1. *For the Young measure $\nu_{(t,x)}$ of Lemmas 2.65 and 2.66,*

$$(t,x) \mapsto \int_{\mathbb{H}} \left(\rho p(\rho) + \rho|u|^3\right) \, d\nu_{(t,x)}(\rho,u) \in L^1_{loc}(\mathbb{R}^2_+). \tag{2.100}$$

2. *Let $\phi \in C(\overline{\mathbb{H}})$ be a continuous function with the additional properties:*

   - *$\phi = 0$ on $\partial\mathbb{H}$;*
   - *there exists an $a > 0$ such that $\operatorname{supp}\phi \subset \{u + k(\rho) \geq -a, u - k(\rho) \leq a\}$;*
   - *$\rho^{-2}|\phi(\rho,u)| \to 0$ uniformly in u as $\rho \to \infty$.*

   Then, ϕ is integrable with respect to the measure $\nu_{(t,x)}$ for almost every $(t,x) \in \mathbb{R}^2_+$, and, with ε' the subsequence of Lemma 2.65,

$$\phi(\rho^{\varepsilon'}(\cdot,\cdot), u^{\varepsilon'}(\cdot,\cdot)) \rightharpoonup \int_{\mathbb{H}} \phi(\rho,u) \, d\nu_{(\cdot,\cdot)}(\rho,u) \quad \text{in } L^1_{loc}(\mathbb{R}^2_+). \tag{2.101}$$

Proof. We begin by verifying the first statement. To this end, choose a non-negative cut-off function $\omega_j \in C(\overline{\mathbb{H}})$ such that

- $\omega_j = 1$ in the box $\{(\rho,u) : k(\rho) \in [j^{-1}, j], |u| \leq j\}$,
- $\omega_j = 0$ outside of the box $\{(\rho,u) : k(\rho) \in [(2j)^{-1}, 2j], |u| \leq 2j\}$.

Now define the functions $\tilde{g}_j(\rho,u) := (\rho p(\rho) + \rho|u|^3)\,\omega_j(\rho,u)$. Observe that these functions are manifestly elements of $\bar{C}(\mathbb{H})$, and, as such, we have that

$$\lim_{\varepsilon' \to 0} \int_0^T \int_K \tilde{g}_j(\rho^{\varepsilon'}(t,x), u^{\varepsilon'}(t,x)) \, dx \, dt = \int_0^T \int_K \left(\int_{\mathbb{H}} \tilde{g}_j(\rho,u) \, d\nu_{(t,x)}(\rho,u) \right) dx \, dt,$$

for any compact subset $K \subset \mathbb{R}$ and $T > 0$. In light of Lemmas 2.49 and 2.51 we get the following bound, where M is independent of both ε' and k,

$$\int_0^T \int_K \tilde{g}_j(\rho^{\varepsilon'}(t,x), u^{\varepsilon'}(t,x)) \, dx \, dt \leq M. \tag{2.102}$$

On a separate note, since the $\tilde{g}_j$ form a family of increasing non-negative functions, the monotone convergence theorem applied to the probability measure $\nu_{(t,x)}$ implies

$$\int_{\mathbb{H}} \left(\rho p(\rho) + \rho|u|^3\right) \, d\nu_{(t,x)}(\rho,u) = \lim_{j \to \infty} \int_{\mathbb{H}} \tilde{g}_j(\rho,u) \, d\nu_{(t,x)}(\rho,u) \leq M. \tag{2.103}$$

Hence, given (2.102), we deduce that $\int_{\mathbb{H}} (\rho p(\rho) + \rho|u|^3)\, d\nu_{(t,x)}(\rho, u)$ is (t,x)-locally integrable, as required. It follows, from Markov's inequality, that this quantity is finite for a.e. $(t,x) \in \mathbb{R}^2_+$. In summary, the mapping $(t,x) \mapsto \int_{\mathbb{H}} (\rho p(\rho) + \rho|u|^3)\, d\nu_{(t,x)}(\rho, u)$ exists as an $L^1_{loc}(\mathbb{R}^2_+)$ function.

We now turn to the second statement. With ω_j as before, we have that, given any ϕ satisfying the assumptions of this second statement, the function $\phi\omega_j$ lies in $\bar{C}(\mathbb{H})$. As such, $\int_{\mathbb{H}} \phi(\rho, u)\omega_j(\rho, u)\, d\nu_{(t,x)}(\rho, u)$ is well-defined for a.e. $(t,x) \in \mathbb{R}^2_+$. Also, given $\delta > 0$, there exists an $\tilde{R} = \tilde{R}(\delta)$ such that $\rho^{-2}|\phi(\rho, u)| \le \delta$ for $\rho \ge \tilde{R}$. Without loss of generality we may assume $\tilde{R} \ge 1$, which implies that $\rho p(\rho) = \rho^2$ for $\rho \ge \tilde{R}$. Hence,

$$\begin{aligned}\left|\frac{\phi(\rho, u)}{\rho^2}\right| \rho^2 &\le \sup\left\{|\phi(\rho, u)| : (\rho, u) \in [0, \tilde{R}] \times \mathbb{R}\right\} + \delta\rho p(\rho) \\ &\le c_\phi \left(1 + \rho p(\rho)\right),\end{aligned}$$

where the supremum is well-defined in view of the assumptions on ϕ. Using (2.103) and $\int_{\mathbb{H}} d\nu_{(t,x)} = 1$, we deduce that ϕ is $\nu_{(t,x)}$-integrable for a.e. $(t,x) \in \mathbb{R}^2_+$. Hence,

$$\lim_{j\to\infty} \int_{\mathbb{H}} \phi(\rho, u)\omega_j(\rho, u)\, d\nu_{(t,x)}(\rho, u) = \int_{\mathbb{H}} \phi(\rho, u)\, d\nu_{(t,x)}(\rho, u) \quad \text{a.e. } (t,x) \in \mathbb{R}^2_+,$$

by the Lebesgue dominated convergence theorem, and

$$\begin{aligned}\lim_{j\to\infty} \int_0^T \int_K \left(\int_{\mathbb{H}} \phi(\rho, u)\omega_j(\rho, u)\, d\nu_{(t,x)}(\rho, u)\right) dx\, dt& \\ = \int_0^T \int_K \left(\int_{\mathbb{H}} \phi(\rho, u)\, d\nu_{(t,x)}(\rho, u)\right) dx\, dt.&\end{aligned}$$

On the other hand, from the definition of the Young measure, we also know that

$$\begin{aligned}\lim_{j\to\infty}\lim_{\varepsilon'\to 0} \int_0^T \int_K \phi(\rho^{\varepsilon'}(t,x), u^{\varepsilon'}(t,x))\omega_j(\rho^{\varepsilon'}(t,x), u^{\varepsilon'}(t,x))\, dx\, dt& \\ = \lim_{j\to\infty} \int_0^T \int_K \left(\int_{\mathbb{H}} \phi(\rho, u)\omega_j(\rho, u)\, d\nu_{(t,x)}(\rho, u)\right) dx\, dt,&\end{aligned}$$

from which we straightforwardly deduce

$$\begin{aligned}\lim_{j\to\infty}\lim_{\varepsilon'\to 0} \int_0^T \int_K \phi(\rho^{\varepsilon'}(t,x), u^{\varepsilon'}(t,x))\omega_j(\rho^{\varepsilon'}(t,x), u^{\varepsilon'}(t,x))\, dx\, dt& \\ = \int_0^T \int_K \left(\int_{\mathbb{H}} \phi(\rho, u)\, d\nu_{(t,x)}(\rho, u)\right) dx\, dt.&\end{aligned} \tag{2.104}$$

Claim 2.68.

$$\begin{aligned}\int_0^T \int_K \phi(\rho^{\varepsilon'}(t,x), u^{\varepsilon'}(t,x))\omega_j(\rho^{\varepsilon'}(t,x), u^{\varepsilon'}(t,x))\, dx\, dt& \\ \to \int_0^T \int_K \phi(\rho^{\varepsilon'}(t,x), u^{\varepsilon'}(t,x))\, dx\, dt&\end{aligned}$$

uniformly in ε' for $\varepsilon' \in [0, \varepsilon_0)$.

Provided the claim holds, we may interchange the limits in (2.104) to obtain

$$\begin{aligned}
&\int_0^T \int_K \left(\int_{\mathbb{H}} \phi(\rho, u)\, d\nu_{(t,x)}(\rho, u) \right) dx\, dt \\
&\qquad = \lim_{j\to\infty} \lim_{\varepsilon'\to 0} \int_0^T \int_K \phi(\rho^{\varepsilon'}(t,x), u^{\varepsilon'}(t,x)) \omega_j(\rho^{\varepsilon'}(t,x), u^{\varepsilon'}(t,x))\, dx\, dt \\
&\qquad = \lim_{\varepsilon'\to 0} \lim_{j\to\infty} \int_0^T \int_K \phi(\rho^{\varepsilon'}(t,x), u^{\varepsilon'}(t,x)) \omega_j(\rho^{\varepsilon'}(t,x), u^{\varepsilon'}(t,x))\, dx\, dt \\
&\qquad = \lim_{\varepsilon'\to 0} \int_0^T \int_K \phi(\rho^{\varepsilon'}(t,x), u^{\varepsilon'}(t,x))\, dx\, dt,
\end{aligned}$$

which concludes the proof of the lemma. □

Proof of Claim 2.68. Fix $j_1 < j_2$ and consider

$$\begin{aligned}
&G(\varepsilon; j_1, j_2) := \\
&\quad \int_0^T \int_K \phi(\rho^\varepsilon(t,x), u^\varepsilon(t,x)) \Big[\omega_{j_1}(\rho^\varepsilon(t,x), u^\varepsilon(t,x)) - \omega_{j_2}(\rho^\varepsilon(t,x), u^\varepsilon(t,x)) \Big]\, dx\, dt.
\end{aligned}$$

Notice that $\operatorname{supp}\left(\omega_{j_1} - \omega_{j_2}\right) \subset \left\{(\rho, u) \in \mathbb{H} : j_1^{-1} \le k(\rho) \le j_1, |u| \le j_1\right\}^c$, and $\sup_{0\le k(\rho)\le 1/j_1} |\phi(\rho, u)| =: c_{j_1} \to 0$ as $j_1 \to \infty$ due to the continuity of ϕ and the requirement that $\phi = 0$ on $\{\rho = 0\}$. Additionally,

$$\left\{j_1^{-1} \le k(\rho) \le j_1, |u| \le j_1\right\}^c = \{k(\rho) < 1/j_1\} \cup \{k(\rho) > j_1\} \cup \{|u| > j_1\}.$$

Therefore, if $(\rho, u) \in \operatorname{supp}\phi \cap \left\{j_1^{-1} \le k(\rho) \le j_1, |u| \le j_1\right\}^c$, then (provided j_1 is sufficiently large, i.e., $j_1 \ge 2$) either $k(\rho) > \frac{j_1}{2}$ or $k(\rho) < 1/j_1$. Then,

$$|G(\varepsilon; j_1, j_2)| \le T|K| c_{j_1} + \int_0^T \int_K \phi(\rho^\varepsilon(t,x), u^\varepsilon(t,x)) \mathbb{1}_{k(\rho^\varepsilon(t,x)) > j_1/2}\, dx\, dt.$$

The last term is dealt with as follows,

$$\begin{aligned}
\left|\phi(\rho^\varepsilon(t,x), u^\varepsilon(t,x)) \mathbb{1}_{k(\rho^\varepsilon(t,x)) > j_1/2}\right| &= \left| \frac{\phi(\rho^\varepsilon(t,x), u^\varepsilon(t,x))}{(\rho^\varepsilon)^2} \right| (\rho^\varepsilon)^2 \mathbb{1}_{k(\rho^\varepsilon(t,x)) > j_1/2} \\
&= \left| \frac{\phi(\rho^\varepsilon(t,x), u^\varepsilon(t,x))}{(\rho^\varepsilon)^2} \right| \rho^\varepsilon p(\rho^\varepsilon) \mathbb{1}_{k(\rho^\varepsilon(t,x)) > j_1/2},
\end{aligned}$$

for j_1 sufficiently large, as k is strictly increasing.

Now fix $\delta > 0$ arbitrarily. Since $\rho^{-2}\phi(\rho, u)$ vanishes uniformly in the limit as $\rho \to \infty$, there exists $\tilde{r} = \tilde{r}(\delta)$ such that $\left|\phi(\rho^\varepsilon(t,x), u^\varepsilon(t,x)) \mathbb{1}_{k(\rho^\varepsilon(t,x)) > j_1/2}\right| \le \delta \rho^\varepsilon p(\rho^\varepsilon)$ for $j_1 \ge \tilde{r}$. Hence, given any $\delta > 0$, there exists $\tilde{R} = \tilde{R}(\delta)$, independent of ε, such that $|G(\varepsilon; j_1, j_2)| \le C(T, K)\delta$ if $j_1 \ge \tilde{R}$. □

Having established Lemmas 2.65 and 2.66, we are in a position to prove the Tartar–Murat commutation relation for the entropy dissipation measures. This relation is precisely the statement that the limit function characterised by the Young measure is equal to the limit that was obtained in Lemma 2.63.

2.7.2 Derivation of the commutation relation

We begin by proving the Tartar–Murat commutation relation, which will later be used to show that the Young measure is supported at a single point. This latter observation will then enable us to rigorously pass to the limit in the viscous equations (1.25), thereby showing that the limiting object characterised by the Young measure is an entropy solution of the original Euler system (1.7).

Remark 2.69. We adopt the notation $\overline{f} = \int_{\mathcal{H}} f(\rho,u)\,d\nu_{(t,x)}(\rho,u)$, where there is no confusion over the point (t,x), with $\chi(s_j) = \chi(\cdot,\cdot - s_j)$ and $\sigma(s_j) = \sigma(\cdot,\cdot,s_j)$.

Lemma 2.70 (Proposition 4.2 of [80])**.** *Recall the kernels χ and σ (cf. Definitions 1.10 and 1.11). Let $\nu_{(t,x)}$ be a Young measure generated by the viscous approximate solutions of* (1.25) *associated with an admissible sequence of initial data. Then, for $s_1, s_2 \in \mathbb{R}$,*

$$\overline{\chi(s_1)\sigma(s_2) - \chi(s_2)\sigma(s_1)} = \overline{\chi(s_1)}\,\overline{\sigma(s_2)} - \overline{\chi(s_2)}\,\overline{\sigma(s_1)}. \tag{2.105}$$

Proof. In view of Lemma 2.63, given two compactly supported test functions $\psi_1, \psi_2 \in C_c^2(\mathbb{R})$ and correspondingly generated entropy pairs (η_1, q_1) and (η_2, q_2), there exists a subsequence $(\rho^{\varepsilon'}, m^{\varepsilon'})$ and L^2_{loc} pairs (Z^1, W^1) and (Z^2, W^2) such that

$$(\eta_i(\rho^{\varepsilon'}, m^{\varepsilon'}), q_i(\rho^{\varepsilon'}, m^{\varepsilon'})) \rightharpoonup (Z^i, W^i) \quad \text{weakly in } L^2(U), \text{ for } i = 1,2, \text{ and}$$

$$\eta_1(\rho^{\varepsilon'}, m^{\varepsilon'})q_2(\rho^{\varepsilon'}, m^{\varepsilon'}) - \eta_2(\rho^{\varepsilon'}, m^{\varepsilon'})q_1(\rho^{\varepsilon'}, m^{\varepsilon'}) \rightharpoonup Z^1W^2 - Z^2W^1 \quad \text{weakly in } L^1(U),$$

for any bounded open subset $U \subset \mathbb{R}^2_+$. However, note that, since the entropy pairs generated by the test functions $\psi_1, \psi_2 \in C_c^2(\mathbb{R})$ possess the properties required for ϕ to be admissible (according to Lemma 2.67), we have

$$(Z^i, W^i) = (\overline{\eta_i}, \overline{q_i}) \qquad \text{for } i = 1,2.$$

Hence, by the uniqueness of weak limits in $L^1_{loc}(\mathbb{R}^2_+)$, we deduce that

$$\overline{\eta_1 q_2 - \eta_2 q_1} = \overline{\eta_1}\,\overline{q_2} - \overline{\eta_2}\,\overline{q_1} \qquad \text{a.e. } (t,x) \in \mathbb{R}^2_+.$$

Using the density of the $C_c^2(\mathbb{R})$ test functions and the fundamental lemma of the calculus of variations (*cf.* [79, Proof of Proposition 5.4.3]), we deduce

$$\overline{\chi(s_1)\sigma(s_2) - \chi(s_2)\sigma(s_1)} = \overline{\chi(s_1)}\,\overline{\sigma(s_2)} - \overline{\chi(s_2)}\,\overline{\sigma(s_1)} \qquad \text{a.e. } (t,x) \in \mathbb{R}^2_+.$$

□

2.8 Reduction of the Young measure and proof of main result

2.8.1 Reduction of the Young measure

In what follows, we will consider the fractional derivatives of the kernels. To this end, we have the following definition (*cf.* [14, Section 2]).

Definition 2.71. Let $\delta > 0$ and g be a distribution of compact support. Then, with Γ the gamma function, we define the *δ-th fractional derivative* to be

$$\partial_s^\delta g(s) = \Gamma(-\delta) g * [s]_+^{-\delta-1} \quad \text{in the sense of distributions.} \tag{2.106}$$

With this in mind we prove the main result of this subsection, stated below, where the notation was explained in Remark 2.69. Throughout, we follow [79, Section 5.5].

Theorem 2.72. *Recall the kernels χ and σ (cf. Definitions 1.10 and 1.11). Let $\nu \in \mathrm{Prob}(\mathcal{H})$ be a probability measure such that the function $(\rho, u) \mapsto \rho^2 \in L^1(\mathcal{H}, \nu)$ and, for all $s_1, s_2 \in \mathbb{R}$,*

$$\overline{\chi(s_1)\sigma(s_2) - \chi(s_2)\sigma(s_1)} = \overline{\chi(s_1)}\,\overline{\sigma(s_2)} - \overline{\chi(s_2)}\,\overline{\sigma(s_1)}. \tag{2.107}$$

Then either ν is supported in V or the support of ν is a single point in $\mathbb{H}$.

Proof. Begin by taking $s_1, s_2, s_3 \in \mathbb{R}$. Multiplying the commutation relation (2.107) for s_1, s_2 by $\overline{\chi(s_3)}$, and cyclically permuting s_1, s_2, s_3 and summing, we get

$$\begin{aligned}\overline{\chi(s_1)}\,\overline{\chi(s_2)\sigma(s_3) - \chi(s_3)\sigma(s_2)} =& \overline{\chi(s_3)}\,\overline{\chi(s_2)\sigma(s_1) - \chi(s_1)\sigma(s_2)} \\ &- \overline{\chi(s_2)}\,\overline{\chi(s_3)\sigma(s_1) - \chi(s_1)\sigma(s_3)}.\end{aligned}$$

We then apply the fractional derivative operators $P_2 := \partial_{s_2}^{\lambda+1}$ and $P_3 := \partial_{s_3}^{\lambda+1}$ in the sense of distributions to obtain

$$\begin{aligned}\overline{\chi(s_1)}\,\overline{P_2\chi(s_2)P_3\sigma(s_3) - P_3\chi(s_3)P_2\sigma(s_2)} =& \overline{P_3\chi(s_3)}\,\overline{P_2\chi(s_2)\sigma(s_1) - \chi(s_1)P_2\sigma(s_2)} \\ &- \overline{P_2\chi(s_2)}\,\overline{P_3\chi(s_3)\sigma(s_1) - \chi(s_1)P_3\sigma(s_3)},\end{aligned} \tag{2.108}$$

where, for example, the distribution $\overline{P_2\chi(s_2)}$ acts on test functions $\psi \in \mathcal{D}(\mathbb{R})$ by

$$\langle \overline{P_2\chi(s_2)}, \psi \rangle = -\int_{\mathbb{R}} \overline{\partial_{s_2}^\lambda \chi(s_2)} \psi'(s_2)\, ds_2.$$

Define $\phi_2, \phi_3 \in \mathcal{D}(-1,1)$ such that $\int_{\mathbb{R}} \phi_j(y)\, dy = 1$ and $\phi_j \geq 0$ for $j = 2, 3$. Additionally, for $\tau > 0$, we define $\phi_j^\tau(y) := \tau^{-1}\phi_j(y/\tau)$. We choose ϕ_2, ϕ_3 such that

$$Y(\phi_2, \phi_3) := \int_{-\infty}^{\infty} \int_{-\infty}^{s_2} (\phi_2(s_2)\phi_3(s_3) - \phi_2(s_3)\phi_3(s_2))\, ds_3\, ds_2 > 0.$$

Now integrate (2.108) against $\phi_2^\tau(s_1-s_2)\phi_3^\tau(s_1-s_3)$ in s_2 and s_3 to get

$$\begin{aligned}\overline{\chi(s_1)}\,\overline{P_2\chi_2^\tau P_3\sigma_3^\tau - P_3\chi_3^\tau P_2\sigma_2^\tau} =& \overline{P_3\chi_3^\tau}\,\overline{P_2\chi_2^\tau\sigma_1 - \chi_1 P_2\sigma_2^\tau} \\ &- \overline{P_2\chi_2^\tau}\,\overline{P_3\chi_3^\tau\sigma_1 - \chi_1 P_3\sigma_3^\tau},\end{aligned} \tag{2.109}$$

where, for example, $\chi_j = \chi(s_j)$ for $j = 1, 2, 3$, and

$$\overline{P_2\chi_2^\tau} = \overline{P_2\chi_2} * \phi_2^\tau(s_1) = \int \overline{\partial_{s_2}^\lambda \chi(s_2)}\tau^{-2}\phi_2'\left(\frac{s_1-s_2}{\tau}\right)ds_2.$$

The next lemma considers the structure of singularities of the kernels. H denotes the Heavyside; PV the principal value; and Ci the cosine integral. We omit the proof.

Lemma 2.73 (Lemma 2.7 of [80])**.** *For $\rho \geq 1$, the fractional derivatives $\partial_s^{\lambda+1}\chi(\rho, u-s)$ and $\partial_s^{\lambda+1}\sigma(\rho,u,s)$ admit the expansions*

$$\begin{aligned}\partial_s^{\lambda+1}\chi(\rho,u-s) =& \sum_\pm \big(A_{1,\pm}(\rho)\delta(s-u\pm k(\rho)) + A_{2,\pm}(\rho)H(s-u\pm k(\rho)) \\ &\quad + A_{3,\pm}(\rho)\,\mathrm{PV}(s-u\pm k(\rho)) + A_{4,\pm}(\rho)\,\mathrm{Ci}(s-u\pm k(\rho))\big) \\ &+ r_\chi(\rho,u-s), \\ \partial_s^{\lambda+1}(\sigma-u\chi)(\rho,u-s) =& \\ &\sum_\pm (s-u)\big(B_{1,\pm}(\rho)\delta(s-u\pm k(\rho)) + B_{2,\pm}(\rho)H(s-u\pm k(\rho)) \\ &\quad + B_{3,\pm}(\rho)\,\mathrm{PV}(s-u\pm k(\rho)) + B_{4\pm}(\rho)\,\mathrm{Ci}(s-u\pm k(\rho))\big) \\ &\sum_\pm \big(B_{5,\pm}(\rho)H(s-u\pm k(\rho)) + B_{6,\pm}(\rho)\,\mathrm{Ci}(s-u\pm k(\rho))\big) + r_\sigma(\rho,u-s).\end{aligned} \tag{2.110}$$

Moreover, there exists a positive constant C independent of ρ, u, s such that

$$\sum_{j=1,\pm}^{4}|A_{j,\pm}(\rho)| + \sum_{j=1,\pm}^{6}|B_{j,\pm}(\rho)| \leq C\sqrt{\rho}\log\rho \qquad \text{for } \rho \geq 1. \tag{2.111}$$

Additionally, the remainders r_χ, r_σ are Hölder continuous, and there holds

$$\|r_\chi(\rho,\cdot)\|_{L^\infty(\mathbb{R})} + \|r_\sigma(\rho,\cdot)\|_{L^\infty(\mathbb{R})} \leq C\rho \qquad \text{for } \rho \geq 1. \tag{2.112}$$

Armed with Lemma 2.73, one shows the following two technical lemmas, the proofs of which can be found in [80, Section 5] or [79, Section 5.5] (*cf.* Section 3.6).

Lemma 2.74 (Lemma 5.2 of [80])**.** *For any test function $\psi \in \mathcal{D}(\mathbb{R})$,*

$$\begin{aligned}\lim_{\tau\to 0}\int_{\mathbb{R}} &\overline{\chi(s_1)}\,\overline{P_2\chi_2^\tau P_3\sigma_3^\tau - P_3\chi_3^\tau P_2\sigma_2^\tau}(s_1)\psi(s_1)\,ds_1 \\ &= \int_{\mathcal{H}} Y(\phi_2,\phi_3)Z(\rho)\sum_\pm (K^\pm)^2\overline{\chi(u\pm k(\rho))}\psi(u\pm k(\rho))\,d\nu(\rho,u),\end{aligned}$$

where $Z(\rho) = (\lambda+1)M_\lambda^{-2}k(\rho)^{2\lambda}D(\rho) > 0$ for $\rho > 0$, with $D(\rho)$ as in Lemma 1.13.

Note that the constants $M_\lambda > 0$ and $K^\pm \neq 0$ were introduced in [14, Section 2].

Lemma 2.75 (Lemma 5.3 of [80])**.** *For any test function* $\psi \in \mathcal{D}(\mathbb{R})$*,*

$$\lim_{\tau\to 0}\int_{\mathbb{R}} \overline{P_3\chi_3^\tau}\,\overline{P_2\chi_2^\tau\sigma_1 - \chi_1 P_2\sigma_2^\tau}\psi(s_1)\,ds_1 = \lim_{\tau\to 0}\int_{\mathbb{R}} \overline{P_2\chi_2^\tau}\,\overline{P_3\chi_3^\tau\sigma_1 - \chi_1 P_3\sigma_3^\tau}\psi(s_1)\,ds_1.$$

Multiplying (2.109) by $\psi(s_1)$, integrating in s_1 and passing to the limit $\tau \to 0$,

$$Y(\phi_2,\phi_3)\int_{\mathcal{H}} Z(\rho)\sum_{\pm}(K^\pm)^2\overline{\chi(u\pm k(\rho))}\psi(u\pm k(\rho))\,d\nu(\rho,u) = 0. \tag{2.113}$$

Note that $Y(\phi_2,\phi_3)$ is strictly positive, $Z(\rho) > 0$ provided $\rho > 0$, $\overline{\chi(s)} \geq 0$, and ψ is an arbitrary test function. Choosing ψ non-negative, we deduce from (2.113) that

$$\int_{\mathcal{H}} Z(\rho)\overline{\chi(u+k(\rho))}\,d\nu(\rho,u) = 0 \quad\text{and}\quad \int_{\mathcal{H}} Z(\rho)\overline{\chi(u-k(\rho))}\,d\nu(\rho,u) = 0.$$

In terms of the Riemann invariants $z(\rho,u) = u - k(\rho)$ and $w(\rho,u) = u + k(\rho)$, the previous line may be written as

$$\int_{\mathcal{H}} Z(\rho)\overline{\chi(w(\rho,u))}\,d\nu(\rho,u) = 0 \quad\text{and}\quad \int_{\mathcal{H}} Z(\rho)\overline{\chi(z(\rho,u))}\,d\nu(\rho,u) = 0. \tag{2.114}$$

Since the map $\chi(\rho,u,\cdot)$ is Hölder continuous, by Theorem 1.12 (*cf.* Lemma 3.17), one can straightforwardly show that $s \mapsto \overline{\chi(s)}$ is continuous (*cf.* [79, Lemma 5.5.2]). With this in mind we define, for χ and ν given in the statement of the theorem,

$$\mathbb{S} := \{s \in \mathbb{R} : \overline{\chi(s)} > 0\}. \tag{2.115}$$

Note that the set $\mathbb{S}$ is open in $\mathbb{R}$, by virtue of the continuity of $\overline{\chi(s)}$. The following result provides a useful representation for $\mathbb{S}$, the proof of which is contained within [79, Proof of Lemma 5.5.2].

Claim 2.76. *Let* χ *and* ν *be the entropy kernel and the probability measure given in the statement of Theorem 2.72. Then the set* $\mathbb{S}$*, cf.* (2.115)*, admits the representation*

$$\mathbb{S} = \bigcup_{(\rho,u)\in\operatorname{supp}\nu} (z(\rho,u), w(\rho,u)),$$

where $z(\rho,u) = u - k(\rho)$ *and* $w(\rho,u) = u + k(\rho)$ *are the Riemann invariants.*

If it were the case that $\mathbb{S}$ were empty, then $\overline{\chi(s)}$ would be identically zero. However, Theorem 1.12 implies that $\chi(\rho,u,s)$ is strictly positive on the interior of its support, which is non-empty for $\rho > 0$. Hence, $\overline{\chi(s)} \equiv 0$ implies that ν concentrates all of its mass on $\{\rho = 0\}$, thereby yielding $\operatorname{supp}\nu \subset V$.

Assume on the other hand that $\mathbb{S}$ is non-empty. Then, since $\mathbb{S}$ is open in $\mathbb{R}$, it is the union of at most countably many disjoint open intervals. We thereby write

$$\mathbb{S} = \bigcup_k (z_k, w_k), \tag{2.116}$$

for at most countably many numbers z_k, w_k in the extended real line such that, whenever $k \neq k'$, we either have $w_k \leq z_{k'}$ or $w_{k'} \leq z_k$. Using this observation we obtain the next claim, to be proved later.

Claim 2.77. *Let ν be the measure given in the statement of Theorem 2.72, and recall the representation of the set $\mathbb{S}$, cf.* (2.115)*, provided by* (2.116)*. We have*

$$\operatorname{supp} \nu \subset \bigcup_k \{(\rho, u) \in \mathbb{H} : [z(\rho, u), w(\rho, u)] \cap [z_k, w_k] \neq \emptyset\} \cup V. \tag{2.117}$$

With this in hand we get the following, the proof of which is postponed for the time being.

Claim 2.78. *Let ν be the measure given in the statement of Theorem 2.72, and recall the representation of the set $\mathbb{S}$, cf.* (2.115)*, provided by* (2.116)*. We have*

$$\operatorname{supp} \nu \cap \{(\rho, u) \in \mathbb{H} : w(\rho, u) \in (z_k, w_k) \ \textit{or}\ z(\rho, u) \in (z_k, w_k)\} = \emptyset.$$

The previous result implies that the support of the measure ν must be contained in the vacuum set V and an at most countable union of points $(\rho_k, u_k) = (\rho(w_k, z_k), u(w_k, z_k))$, i.e.,

$$\operatorname{supp} \nu \subset V \cup \bigcup_{k : \rho_k, u_k \in \mathbb{R}} (\rho_k, u_k).$$

Note that the ρ_k, u_k are real numbers because $\nu \in \operatorname{Prob}(\mathcal{H})$ in the statement of the theorem is assumed to be a measure on $\mathcal{H}$, not on the compactification $\overline{\mathcal{H}}$.

We therefore write, with $\alpha_k \in [0, 1]$ and the measure ν_V supported only in V,

$$\nu = \nu_V + \sum_k \alpha_k \delta_{(\rho_k, u_k)}.$$

The next claim will be used later to show that we do not need to consider interactions between the points $\{(\rho_k, u_k)\}_k$ when computing the duality product of the measure ν with particular functions.

Claim 2.79. *Fix $s \in \mathbb{R}$. Suppose there exist $k \neq k'$ such that*

$$(\rho_{k'}, u_{k'}), (\rho_k, u_k) \in \operatorname{supp} \chi(\cdot, \cdot, s).$$

Then $\chi(\rho_k, u_k, s) = \chi(\rho_{k'}, u_{k'}, s) = 0$, and neither point belongs to the interior of the support, $\operatorname{int}\big(\operatorname{supp} \chi(\cdot, \cdot, s)\big)$.

In other words, we have the following.

Claim 2.80. *Fix $s \in \mathbb{R}$, and suppose that there exists an index k such that $(\rho_k, u_k) \in \operatorname{int}\big(\operatorname{supp}\chi(\cdot,\cdot,s)\big)$. Then, $(\rho_{k'}, u_{k'}) \notin \operatorname{int}\big(\operatorname{supp}\chi(\cdot,\cdot,s)\big)$ for all $k' \neq k$.*

Now, fix the index k. Choose $s_1, s_2 \in \mathbb{R}$ such that the point (ρ_k, u_k) lies in the interior of $\operatorname{supp}\chi(\cdot,\cdot,s_1)\chi(\cdot,\cdot,s_2)$, and so that this latter set is non-empty. By Claim 2.80, this condition precludes any other point $(\rho_{k'}, u_{k'})$, with $k' \neq k$, from having a non-zero contribution when applying the measure ν to the functions $\chi(\cdot,\cdot,s_i)$ for $i = 1, 2$, and to the products of these functions with $\sigma(\cdot,\cdot,s_j)$ for $j = 1, 2$. We thereby obtain, from the commutation relation (2.107),

$$(\alpha_k - \alpha_k^2)\left(\chi(\rho_k, u_k, s_1)\sigma(\rho_k, u_k, s_2) - \chi(\rho_k, u_k, s_2)\sigma(\rho_k, u_k, s_1)\right) = 0.$$

Taking s_1 and s_2 such that the second factor in this expression is non-zero, we deduce that $\alpha_k \in \{0, 1\}$ for every k. This concludes the proof of the theorem. □

Proof of Claim 2.77. We prove the contrapositive by contradiction. Notice that

$$\bigcup_k \{(\rho, u) \in \mathbb{H} : [z(\rho, u), w(\rho, u)] \cap [z_k, w_k] \neq \emptyset\}$$
$$= \big\{(\rho, u) \in \mathbb{H} : [z(\rho, u), w(\rho, u)] \cap [z_k, w_k] \neq \emptyset \text{ for some index } k\big\}.$$

Now select the point (ρ, u), with $\rho > 0$ so that $(\rho, u) \in V^c$, to be in the complement of the set above, i.e.,

$$[z(\rho, u), w(\rho, u)] \cap [z_k, w_k] = \emptyset \quad \text{for every index } k.$$

Note that since $\rho > 0$, the open subset $(z(\rho, u), w(\rho, u))$ is non-empty. It follows that (ρ, u) does not belong to $\operatorname{supp}\nu$. Indeed, suppose for contradiction that we had $(\rho, u) \in \operatorname{supp}\nu$. Then, by Claim 2.76, there exists an $s \in \mathbb{S}$ such that $s \in (z(\rho, u), w(\rho, u))$. But then by (2.116), $s \in (z_k, w_k)$ for some k, and so the sets $[z(\rho, u), w(\rho, u)]$ and $[z_k, w_k]$ have non-empty intersection; a contradiction. □

Proof of Claim 2.78. Suppose for contradiction that there exists an index k such that $\operatorname{supp}\nu \cap \{(\rho, u) \in \mathbb{H} : w(\rho, u) \in (z_k, w_k)\} \neq \emptyset$. Then there exists a subset $A \subset \{(\rho, u) \in \mathbb{H} : w(\rho, u) \in (z_k, w_k)\}$ of positive measure, with respect to ν. Since $\overline{\chi(s)}$ is strictly positive for any $s \in \mathbb{S}$, and $(z_k, w_k) \subset \mathbb{S}$, it follows that

$$\int_{\mathcal{H}} Z(\rho)\overline{\chi(w(\rho, u))}\, d\nu(\rho, u) \geq \int_A Z(\rho)\overline{\chi(w(\rho, u))}\, d\nu(\rho, u) > 0,$$

as the integrand is strictly positive for $\rho > 0$, and the integral ranges over a portion of $\mathbb{H}$. This contradicts (2.114). The same reasoning proves the case where $\operatorname{supp}\nu \cap \{(\rho, u) \in \mathbb{H} : z(\rho, u) \in (z_k, w_k)\}$ is non-empty. □

Proof of Claim 2.79. Recall that $\operatorname{supp}\chi(\cdot,\cdot,s) = \{(\rho,u) : z(\rho,u) \le s \le w(\rho,u)\}$, and that $\chi(\rho,u,s) = 0$ whenever $|u-s| = k(\rho)$, i.e., whenever $s = z(\rho,u)$ or $s = w(\rho,u)$.

Recall also that $z_k = z(\rho_k,u_k)$ and $w_k = w(\rho_k,u_k)$. The hypothesis of the claim and the disjointness of the intervals $\{(z_j,w_j)\}_j$ then imply that we either have

$$z_{k'} \le s \le w_{k'} \le z_k \le s \le w_k,$$

in which case $w_{k'} = s = z_k$, or

$$z_k \le s \le w_k \le z_{k'} \le s \le w_{k'},$$

in which case $w_k = s = z_{k'}$. In either case, $\chi(\rho_k,u_k,s) = \chi(\rho_{k'},u_{k'},s) = 0$, and we conclude that neither point can belong to the interior of the support of $\chi(\cdot,\cdot,s)$. □

2.8.2 Proof of main result

Below, we present the proof of the main result of this chapter, namely Theorem 2.2, which follows from the results of the previous sections.

Proof of Theorem 2.2. Let $\tilde{\rho}_0^\varepsilon(x) := \max\{\rho_0(x), \sqrt{\varepsilon}\}$. We now mollify $\tilde{\rho}_0^\varepsilon$ and u_0 suitably, such that we obtain an admissible sequence of initial data $(\rho_0^\varepsilon, u_0^\varepsilon)$, in the sense of Definition 2.55. The sequence of smooth solutions $(\rho^\varepsilon, u^\varepsilon)$ of (1.25) corresponding to this initial data then generates a Young measure $\nu_{(t,x)}$, constrained by the Tartar–Murat commutation relation, according to Lemma 2.70. An application of Theorem 2.72 then yields that $\nu_{(t,x)}$ is either a point mass or is supported in the vacuum set V. In the coordinates (ρ,m), where $m = \rho u$, this measure is a Dirac mass. As such, we write $\nu_{(t,x)} = \delta_{(\rho(t,x),m(t,x))}$, where ρ and m are measurable functions. In view of this, we deduce that the convergence of the subsequence $(\rho^{\varepsilon'}, \rho^{\varepsilon'}u^{\varepsilon'}) \to (\rho,m)$ occurs in measure, and therefore also (up to a further subsequence) in the almost everywhere sense. The convergence in measure, in conjunction with the uniform estimates of Section 2.5, shows that the convergence also happens in $L^p_{loc}(\mathbb{R}^2_+) \times L^q_{loc}(\mathbb{R}^2_+)$ for $p \in [1,2)$ and $q \in [1,3/2)$. By passing to the limit in (1.25), it follows that (ρ,m) is a weak solution of (1.7). It remains to check that it is an entropy solution, in the precise sense of Definition 1.16.

Since the weak entropies are taken to be C^2 functions, the almost everywhere convergence guarantees that $\overline{\eta^*}(\rho^{\varepsilon'}, \rho^{\varepsilon'}u^{\varepsilon'}) \to \overline{\eta^*}(\rho,m)$ a.e. $(t,x) \in \mathbb{R}^2_+$. In turn, a direct application of Fatou's lemma yields

$$\int_{\mathbb{R}} \overline{\eta^*}(\rho,m)\,dx \le \liminf_{\varepsilon'} \int_{\mathbb{R}} \overline{\eta^*}(\rho^{\varepsilon'}, \rho^{\varepsilon'}u^{\varepsilon'})\,dx \text{ for almost every } t \ge 0.$$

In view of Lemma 2.47, the right-hand side of the previous line is bounded independently of ε'. We thereby deduce that there exists $M = M(E_0, t)$, monotonically increasing such that

$$\int_{\mathbb{R}} \overline{\eta^*}(\rho, m)\, dx \leq M(E_0, t) \text{ for almost every } t \geq 0.$$

This shows that our solution (ρ, m) is of finite relative energy.

Recall the equation for the entropy pairs generated by test functions in $C_c^2(\mathbb{R})$, (2.97). We see that, in the sense of distributions, for any $\psi \in C_c^2(\mathbb{R})$,

$$\begin{aligned}\int_{\mathbb{R}} \big(\chi(\rho^{\varepsilon'}, u^{\varepsilon'}, s)_t + \sigma(\rho^{\varepsilon'}, u^{\varepsilon'}, s)_x\big)\psi(s)\, ds &= \varepsilon'(\eta_m^{\psi}(\rho^{\varepsilon'}, \rho^{\varepsilon'} u^{\varepsilon'}) u_x^{\varepsilon'})_x \\ - \varepsilon' \int_{\mathbb{R}} \big(\frac{1}{\rho^{\varepsilon'}}\chi(\rho^{\varepsilon'}, u^{\varepsilon'}, s)\psi''(s)|u_x^{\varepsilon'}|^2 &+ \frac{1}{(\rho^{\varepsilon'})^2} F(\rho^{\varepsilon'}, u^{\varepsilon'}, s)\psi''(s)\rho_x^{\varepsilon'} u_x^{\varepsilon'}\big)\, ds,\end{aligned}$$

where the antiderivative $F(\rho, u, s)$ was defined in Corollary 2.11 and is such that $F_s(\rho, u, s) = \rho\chi_\rho(\rho, u, s) - \chi(\rho, u, s)$. For each $\varepsilon > 0$, define

$$\mu^\varepsilon(t, x, s) := -\varepsilon\left(\frac{1}{\rho^\varepsilon}\chi(\rho^\varepsilon, u^\varepsilon, s)|u_x^\varepsilon|^2 + \frac{1}{(\rho^\varepsilon)^2} F(\rho^\varepsilon, u^\varepsilon, s)\rho_x^\varepsilon u_x^\varepsilon\right)(t, x),$$

and fix any open subset $U \subset \mathbb{R}_+^2$. Applying the Tonelli–Fubini theorem and Corollary 2.11 to the second term in the right-hand side of the above gives, for each $\varepsilon' > 0$,

$$\int_{U\times\mathbb{R}} \mu^{\varepsilon'}(t, x, s)\, dx\, dt\, ds = -\varepsilon' \int_{U\times\mathbb{R}} \frac{1}{\rho^{\varepsilon'}}\chi(\rho^{\varepsilon'}, u^{\varepsilon'}, s)|u_x^{\varepsilon'}|^2\, dx\, dt\, ds \leq 0. \tag{2.118}$$

The convergence of $(\mu^{\varepsilon'})_{\varepsilon'>0}$ to some limit μ, a bounded Radon measure, follows from an application of the Banach–Alaoglu theorem in the context of compactly supported continuous functions, making use of the ε'-independent estimates provided by Lemmas 2.47-2.49. In view of (2.118), it follows that the limit measure μ satisfies the required sign condition in Definition 1.16. The proof is complete. □

Chapter 3

Vanishing viscosity of the Navier–Stokes equations for an asymptotically isothermal gas

3.1 Introduction

In Chapter 2, we showed the existence of a finite relative energy entropy solution of the one-dimensional barotropic Euler system (1.7), under the assumption of an approximately isothermal pressure law (*cf.* Definition 2.1), and this solution was obtained as the inviscid limit of solutions of the Navier–Stokes equations (1.25). Herein, we consider a more general pressure law, characterised by the following definition.

Definition 3.1. We say that a fluid behaves according to an *asymptotically isothermal pressure law* if it satisfies the following constitutive assumptions.

1. The pressure, $p \in C^1([0,\infty)) \cap C^4((0,\infty))$, is such that $p(\rho) > 0$ for all $\rho > 0$, and satisfies the assumptions of strict hyperbolicity and genuine nonlinearity, i.e.,
$$p'(\rho) > 0 \quad \text{and} \quad \rho p''(\rho) + 2p'(\rho) > 0 \qquad \text{for } \rho > 0. \tag{3.1}$$

2. There exist constants $\gamma \in (1,3)$ and $\kappa_1 > 0$, and a function $P \in C^4((0,\infty))$ such that
$$p(\rho) = \kappa_1 \rho^\gamma (1 + P(\rho)) \qquad \text{for } \rho \in [0,r), \tag{3.2}$$
for some fixed $r > 0$, and there exists a positive C_r such that $|P^{(j)}(\rho)| \leq C_r \rho^{2\theta - j}$ for $\rho \in [0,r)$, and $j \in \{0,\dots,4\}$.

3. There exists $\alpha > 0$, $\kappa_2 > 0$ and $C_p > 0$ such that, for some fixed $R > 0$,
$$\left| \left(\frac{p(\rho)}{\rho} - \kappa_2 \right)^{(j)} \right| \leq C_p \rho^{-\alpha - j} \qquad \text{for } \rho \in [R,\infty) \text{ and } j \in \{0,\dots,4\}. \tag{3.3}$$

Note that the criterion (3.2) is the one that appears in Theorem 1.12 (*cf.* [14, Section 2]), and in Definition 2.1 in Chapter 2. The main result of this chapter is the following, which is both the analogue of Theorem 2.2 for asymptotically isothermal gases and a precise version of Theorem 1.18.

Theorem 3.2. *Suppose that the initial data* $(\rho_0, u_0) \in L^1_{loc}(\mathbb{R}^2_+)$ *with* $\rho_0 \geq 0$ *and end-states* $(\rho_\pm, u_\pm)$ *is of finite relative energy, i.e.,*

$$E[\rho_0, u_0] = \int_{\mathbb{R}} \overline{\eta^*}(\rho_0, \rho_0 u_0)\, dx \leq E_0 < \infty,$$

and suppose that the pressure function $p(\rho)$ *satisfies the criteria for an asymptotically isothermal gas, in the precise sense of Definition 3.1. Then there exists a sequence of regularised initial data* $(\rho_0^\varepsilon, u_0^\varepsilon)$ *such that the sequence of unique, smooth solutions* $(\rho^\varepsilon, u^\varepsilon)$ *of* (1.25) *with this initial data converges as* $\varepsilon \to 0$, $(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon) \to (\rho, \rho u)$, *to a finite relative energy entropy solution of the Euler equations* (1.7) *with initial data* $(\rho_0, \rho_0 u_0)$, *in the precise sense of Definition 1.16. The convergence is almost everywhere and* $L^p_{loc}(\mathbb{R}^2_+) \times L^q_{loc}(\mathbb{R}^2_+)$ *for* $p \in [1, 2)$ *and* $q \in [1, 3/2)$.

Our approach is the following. Again, we consider the entropy equation for the Euler system (*cf.* Definition 1.10), i.e.,

$$\begin{cases} \chi_{\rho\rho} - k'(\rho)^2 \chi_{uu} = 0 \quad \text{for } (\rho, u) \in \mathbb{R}^2_+, \\ \chi(0, u) = 0, \\ \chi_\rho(0, u) = \delta_{u=0}, \end{cases} \tag{3.4}$$

where $k(\rho)$ was defined in Definition 1.10 of Chapter 2. The global existence of a unique such χ is guaranteed by Theorem 1.12 (*cf.* [14, Theorem 2.1]). However, in order to obtain improved estimates, we seek an explicit characterisation of the kernel, and so generate the solution of this linear wave equation in two steps. First, in order to deal with the singularity in $k'(\rho)^2$ near the vacuum, we solve (3.4) in the interval $(0, \rho_*]$, **for some** ρ_* **large to be chosen later**, using the results of Theorem 1.12. Then, as a second step, we solve

$$\begin{cases} \chi_{\rho\rho} - k'(\rho)^2 \chi_{uu} = 0 \qquad \text{for } (\rho, u) \in (\rho_*, \infty) \times \mathbb{R}, \\ \chi(\rho_*, u) = \chi(\rho_*, u), \\ \chi_\rho(\rho_*, u) = \chi_\rho(\rho_*, u). \end{cases} \tag{3.5}$$

In order to do this, we split the entropy kernel into two distinct quantities: a re-scaled version of the kernel obtained in Chapter 2, which we call χ^{iso}, and a perturbation, called χ^{error}. To this end, we have the following definitions.

Definition 3.3. Recall the kernels $\chi^\sharp$ and $\chi^\flat$, introduced in Chapter 2 (*cf.* Theorem 2.3). Accordingly, for $(\rho, u) \in [\rho_*, \infty) \times \mathbb{R}$, we define $g^\sharp$ and $g^\flat$ via the explicit formulas

$$g^\sharp(\rho, u) := \rho_* \chi^\sharp\left(\frac{\rho}{\rho_*}, \frac{u}{\sqrt{\kappa_2}}\right) \quad \text{and} \quad g^\flat(\rho, u) := \chi^\flat\left(\frac{\rho}{\rho_*}, \frac{u}{\sqrt{\kappa_2}}\right). \tag{3.6}$$

One can directly verify that these re-scaled kernels solve the problems

$$\left\{\begin{array}{cc} g^\sharp_{\rho\rho} - \frac{\kappa_2}{\rho^2} g^\sharp_{uu} = 0, & g^\flat_{\rho\rho} - \frac{\kappa_2}{\rho^2} g^\flat_{uu} = 0, \\ g^\sharp|_{\rho=\rho_*} = 0, & g^\flat|_{\rho=\rho_*} = \delta_{u=0}, \\ g^\sharp_\rho|_{\rho=\rho_*} = \delta_{u=0}, & g^\flat_\rho|_{\rho=\rho_*} = 0. \end{array}\right\} \tag{3.7}$$

We are now in a position to define χ^{iso}, the *re-scaled approximately isothermal kernel.*

Definition 3.4. We define, for $(\rho, u) \in [\rho_*, \infty) \times \mathbb{R}$,

$$\chi^{iso}(\rho, u) := \int_{\mathbb{R}} \chi_\rho(\rho_*, s) g^\sharp(\rho, u - s)\, ds + \int_{\mathbb{R}} \chi(\rho_*, s) g^\flat(\rho, u - s)\, ds. \tag{3.8}$$

We deduce from (3.7) and from Theorem 1.12 that χ^{iso} is the unique solution of

$$\begin{cases} \chi^{iso}_{\rho\rho} - \dfrac{\kappa_2}{\rho^2} \chi^{iso}_{uu} = 0 \quad \text{for } (\rho, u) \in (\rho_*, \infty) \times \mathbb{R}, \\ \chi^{iso}(\rho_*, u) = \chi(\rho_*, u), \\ \chi^{iso}_\rho(\rho_*, u) = \chi_\rho(\rho_*, u). \end{cases} \tag{3.9}$$

We now define the *perturbation kernel* χ^{error} as the difference between χ and χ^{iso}.

Definition 3.5. Having established the existence of χ and χ^{iso}, we define

$$\chi^{error}(\rho, u) := \chi(\rho, u) - \chi^{iso}(\rho, u) \qquad \text{for } (\rho, u) \in [\rho_*, \infty) \times \mathbb{R}. \tag{3.10}$$

Direct computation shows that χ^{error} satisfies the following linear wave equation,

$$\begin{cases} \chi^{error}_{\rho\rho} - k'(\rho)^2 \chi^{error}_{uu} = \big(k'(\rho)^2 - \dfrac{\kappa_2}{\rho^2}\big) \chi^{iso}_{uu} \quad \text{for } (\rho, u) \in (\rho_*, \infty) \times \mathbb{R}, \\ \chi^{error}(\rho_*, u) = 0, \\ \chi^{error}_\rho(\rho_*, u) = 0. \end{cases} \tag{3.11}$$

By estimating χ^{error} in terms of χ^{iso} using representation formulas akin to [14, Equation (3.38)], we establish estimates on the entropy pairs analogous to the ones of Lemmas 2.5 and 2.6. All of the uniform estimates required to establish the compactness of the entropy dissipation measures, and to reduce the support of the Young measure generated by the viscous approximates $(\rho^\varepsilon, u^\varepsilon)$, are then verified. These observations are enough to prove Theorem 3.2 rigorously.

This chapter is structured as follows. In Section 3.2, we introduce some elementary quantities linked to the pressure, and compute estimates on them that are essential in later sections. In Section 3.3, we obtain a representation formula for χ^{error}, which enables us to obtain a uniform L^∞ estimate via a Grönwall type argument. With this, in Section 3.4, we estimate the special entropy $\hat{\eta}$ and its flux $\hat{q}$ (*cf.* Lemma 3.52), and entropies generated by compactly supported test functions (*cf.* Lemma 3.60). Similar procedures give rise to estimates on the derivatives of these entropies, which we also outline in detail in Section 3.4. In Section 3.5, we calculate the structure of the singularities of entropy kernel, and prove a result akin to Lemma 2.73, which was used crucially in Subsection 2.8.1 (*cf.* Chapter 2). Having established this, we are able to reprove the technical lemmas of Chapter 2 (*cf.* Lemmas 2.74 and 2.75) for an asymptotically isothermal gas, which are key to the reduction of the Young measure. This is contained in Section 3.6, which ends with a proof of the main result.

3.2 Elementary quantities

In this section, we define elementary quantities related to the pressure, and make note of some of their properties which will be essential in the proof of the main result. To begin with, in accordance with the definition of $k(\rho)$ given in Definition 1.10, we have the following definition.

Definition 3.6. We define, for $\rho \geq \rho_*$, the quantity

$$k_*(\rho) := \int_{\rho_*}^{\rho} \frac{\sqrt{\kappa_2}}{y}\, dy + k(\rho_*) = \sqrt{\kappa_2}\log(\rho/\rho_*) + k(\rho_*). \tag{3.12}$$

Note that $k_*'(\rho)$ is the speed of propagation for a purely isothermal pressure law $p(\rho) = \kappa_2\rho$. Meanwhile, $k'(\rho)$ is the speed of propagation for the actual pressure law $p(\rho)$, i.e., the asymptotically isothermal gas.

Remark 3.7. Observe that $k_*'(\rho) = \frac{\sqrt{\kappa_2}}{\rho}$ and $k(\rho_*) = k_*(\rho_*)$. Thus,

$$k(\rho) - k_*(\rho) = \int_{\rho_*}^{\rho} \frac{\sqrt{p'(y)} - \sqrt{\kappa_2}}{y}\, dy \qquad \text{for all } \rho \geq \rho_*. \tag{3.13}$$

Since $\kappa_2 = \lim_{\rho\to\infty} p'(\rho)$, and $p \in C^4((0,\infty))$, we can rewrite the above as

$$k(\rho) - k_*(\rho) = -\int_{\rho_*}^{\rho} \frac{1}{y}\left(\int_y^\infty \frac{p''(z)}{2\sqrt{p'(z)}}\, dz\right) dy. \tag{3.14}$$

Definition 3.8. We define the quantities $d(\rho)$ and $d_*(\rho)$ by

$$d(\rho) := 2 + (\rho - \rho_*)\frac{k''(\rho)}{k'(\rho)}, \qquad d_*(\rho) := 2 + (\rho - \rho_*)\frac{k_*''(\rho)}{k_*'(\rho)} \qquad \text{for } \rho \geq \rho_*. \tag{3.15}$$

Note that both of these quantities are strictly positive on the interval $[\rho_*, \infty)$, due to the assumptions of strict hyperbolicity and genuine nonlinearity (3.1).

In the lemmas that follow we establish the asymptotic comportment of the pressure (and associated quantities) for large values of the density. These results seem somewhat out of context for the time being, but will play a crucial role in the estimates of Sections 3.3 and 3.4.

Lemma 3.9. *For all $\rho \geq R$, we have*

$$|p'(\rho) - \kappa_2| \leq 2C_p\rho^{-\alpha}, \qquad |p^{(j)}(\rho)| \leq (j+1)C_p\rho^{-\alpha-(j-1)} \qquad \textit{for } j = 2, 3, 4. \tag{3.16}$$

As such, choosing $\rho_* \geq \max\{R, (4C_p/\kappa_2)^{1/\alpha}\}$,

$$\rho^2 \leq \frac{4\kappa_2}{3}\rho p(\rho), \quad \frac{\kappa_2}{2} \leq p'(\rho) \leq \frac{3\kappa_2}{2}, \quad \sqrt{\frac{\kappa_2}{2}}\rho^{-1} \leq k'(\rho) \leq \sqrt{\frac{3\kappa_2}{2}}\rho^{-1} \quad \textit{for } \rho \geq \rho_*, \tag{3.17}$$

and

$$|k(\rho) - k_*(\rho)| \leq \frac{3C_p}{\alpha^2\sqrt{2\kappa_2}}\rho_*^{-\alpha} \qquad \textit{for } \rho \geq \rho_*. \tag{3.18}$$

Proof. Observe that, for $j \geq 1$,

$$\big(p - \rho\kappa_2\big)^{(j)}(\rho) = \rho\left(\frac{p(\rho)}{\rho} - \kappa_2\right)^{(j)} + j\left(\frac{p(\rho)}{\rho} - \kappa_2\right)^{(j-1)}.$$

Thus, using the bounds provided by (3.3), we obtain the result. Also, from (3.14),

$$|k(\rho) - k_*(\rho)| \leq \frac{3C_p}{\alpha\sqrt{2\kappa_2}}\int_{\rho_*}^{\rho} y^{-\alpha-1}\,dy \leq \frac{3C_p}{\alpha^2\sqrt{2\kappa_2}}\rho_*^{-\alpha}.$$

□

Corollary 3.10. *Assume that $\rho_* \geq \max\{R, (4C_p/\kappa_2)^{1/\alpha}\}$. Then, there exists an $M = M(\alpha, \kappa_2, C_p)$ such that*

$$\left|k'(\rho)^2 - \frac{\kappa_2}{\rho^2}\right| + \left|-\frac{\sqrt{\kappa_2}}{\rho^2} - k''(\rho)\right| \leq M\rho^{-\alpha-2} \qquad \textit{for } \rho \geq \rho_*. \tag{3.19}$$

Meanwhile,

$$\left|\frac{2\sqrt{\kappa_2}}{\rho^3} - k^{(3)}(\rho)\right| + \rho\left|-\frac{6\sqrt{\kappa_2}}{\rho^4} - k^{(4)}(\rho)\right| \leq M\rho^{-\alpha-3} \qquad \textit{for } \rho \geq \rho_*. \tag{3.20}$$

It follows that there exists a positive constant $M = M(\alpha, \kappa_2, C_p, \rho_*)$ such that

$$|k''(\rho)| + \rho|k^{(3)}(\rho)| + \rho^2|k^{(4)}(\rho)| \leq M\rho^{-2} \qquad \textit{for } \rho \geq \rho_*. \tag{3.21}$$

Proof. Since $k'(\rho) = \sqrt{p'(\rho)}/\rho$, we have

$$\left|k'(\rho)^2 - \frac{\kappa_2}{\rho^2}\right| = \left|\frac{p'(\rho) - \kappa_2}{\rho^2}\right| \le \frac{1}{\rho^2}\int_\rho^\infty |p''(y)|\, dy = \frac{3C_p}{\alpha}\rho^{-\alpha-2},$$

while

$$\left|k''(\rho) + \frac{\sqrt{\kappa_2}}{\rho^2}\right| = \left|\frac{p''(\rho)}{2\rho\sqrt{p'(\rho)}} + \frac{\sqrt{\kappa_2} - \sqrt{p'(\rho)}}{\rho^2}\right| \le \frac{3C_p\rho^{-\alpha-2}}{\sqrt{2\kappa_2}} + \frac{1}{\rho^2}\int_\rho^\infty \frac{|p''(y)|}{2\sqrt{p'(y)}}\, dy$$
$$\le \frac{3C_p}{\sqrt{2\kappa_2}}\left(1 + \alpha^{-1}\right)\rho^{-\alpha-2}.$$

For the third derivative,

$$k^{(3)}(\rho) - \frac{2\sqrt{\kappa_2}}{\rho^3} = \frac{p'''(\rho)}{2\rho\sqrt{p'(\rho)}} - \frac{p''(\rho)}{\rho^2\sqrt{p'(\rho)}} - \frac{p''(\rho)^2}{4\rho p'(\rho)^{3/2}} + \frac{2}{\rho^3}\big(\sqrt{p'(\rho)} - \sqrt{\kappa_2}\big),$$

from which we obtain

$$\left|k^{(3)}(\rho) - \frac{2\sqrt{\kappa_2}}{\rho^3}\right| \le M\rho^{-\alpha-3} + \frac{1}{\rho^3}\int_\rho^\infty \frac{|p''(y)|}{\sqrt{p'(y)}}\, dy.$$

In a similar vein, we have the following, from which the result is easily deduced,

$$\left|k^{(4)}(\rho) + \frac{6\sqrt{\kappa_2}}{\rho^4}\right| \le M\rho^{-\alpha-4} + \frac{3}{\rho^4}\int_\rho^\infty \frac{|p''(y)|}{\sqrt{p'(y)}}\, dy.$$

□

Corollary 3.11. *Assume that $\rho_* \ge \max\{R, (4C_p/\kappa_2)^{1/\alpha}\}$. Then, there exists a positive $M = M(k(\rho_*), \kappa_2)$ such that*

$$M^{-1}\big(1 + \log(\rho/\rho_*)\big) \le k(\rho) \le M\big(1 + \log(\rho/\rho_*)\big) \qquad \text{for } \rho \ge \rho_*. \tag{3.22}$$

Proof. Integrating (3.17), we find

$$k(\rho_*) + \sqrt{\frac{\kappa_2}{2}}\log(\rho/\rho_*) \le k(\rho) \le k(\rho_*) + \sqrt{\frac{3\kappa_2}{2}}\log(\rho/\rho_*) \qquad \text{for } \rho \ge \rho_*,$$

from which the result follows easily, choosing $M = \max\{k(\rho_*), \sqrt{3\kappa_2/2}\}$. □

Lemma 3.12. *Assume that $\rho_* \ge \max\{R, (4C_p/\kappa_2)^{1/\alpha}\}$. Then,*

$$0 < d_*(\rho) - 1 = \frac{\rho_*}{\rho} \qquad \text{and} \qquad |d(\rho) - d_*(\rho)| \le M\rho^{-\alpha} \qquad \text{for } \rho \ge \rho_*. \tag{3.23}$$

for some positive constant $M = M(C_p, \kappa_2)$ independent of ρ_*. Additionally, it follows that

$$|d(\rho) - 1| \le 2 \qquad \text{for } \rho \ge \rho_*. \tag{3.24}$$

Proof. Observe that

$$d(\rho) - d_*(\rho) = (\rho - \rho_*) \left[\frac{k''(\rho)}{k'(\rho)} - \frac{k_*''(\rho)}{k_*'(\rho)} \right] = (\rho - \rho_*) \frac{p''(\rho)}{2p'(\rho)}.$$

In turn, using the bounds provided by (3.3) and Lemma 3.9, we obtain $|d(\rho) - d_*(\rho)| \leq M\rho^{-\alpha}$, where $M = \frac{3C_p}{\kappa_2}$. Meanwhile, direct computation yields

$$d_*(\rho) - 1 = \frac{\rho_*}{\rho}.$$

Thus, $0 < d_*(\rho) - 1 \leq 1$ for any $\rho \geq \rho_*$. As such, applying the triangle inequality,

$$|d(\rho) - 1| \leq |d(\rho) - d_*(\rho)| + |d_*(\rho) - 1| \leq \frac{3C_p}{\kappa_2} \rho_*^{-\alpha} + 1 \leq 7/4.$$

□

Lemma 3.13. *We have the equality*

$$\frac{k'(\rho_0)}{k_*'(\rho_0)} \frac{k_*'(\rho)}{k'(\rho)} = \sqrt{\frac{p'(\rho_0)}{p'(\rho)}} \qquad \text{for } R \leq \rho \leq \rho_0. \tag{3.25}$$

As such, provided $\rho_* \geq \max\{R, (4C_p/\kappa_2)^{1/\alpha}\}$, *there is a positive* $M = M(\alpha, C_p, \kappa_2)$ *such that*

$$\left| 1 - \frac{k'(\rho_0)}{k_*'(\rho_0)} \frac{k_*'(\rho)}{k'(\rho)} \right| \leq M\rho^{-\alpha} \qquad \text{for } \rho_* \leq \rho \leq \rho_0. \tag{3.26}$$

In turn,

$$0 < \frac{k'(\rho_0)}{k_*'(\rho_0)} \frac{k_*'(\rho)}{k'(\rho)} \leq 2 \qquad \text{for } \rho_* \leq \rho \leq \rho_0. \tag{3.27}$$

Proof. The fundamental theorem of calculus yields

$$\left| \sqrt{\frac{p'(\rho_0)}{p'(\rho)}} - 1 \right| = \frac{1}{\sqrt{p'(\rho)}} \left| \int_\rho^{\rho_0} \frac{p''(y)}{2\sqrt{p'(y)}} \, dy \right| \leq \frac{3C_p}{\kappa_2} \int_\rho^\infty y^{-\alpha-1} \, dy = M\rho^{-\alpha},$$

where $M = \frac{3C_p}{\alpha\kappa_2}$, and note that we have only required $\alpha > 0$. The final result follows easily. □

Lemma 3.14. *Assume that* $\rho_* \geq \max\{R, (4C_p/\kappa_2)^{1/\alpha}\}$. *Then, there exists a positive constant* $M = M(\alpha, C_p, \kappa_2)$ *such that*

$$|(k(\rho_0) - k(\rho)) - (k_*(\rho_0) - k_*(\rho))| \leq M\rho^{-\alpha} \qquad \text{for } \rho_* \leq \rho \leq \rho_0. \tag{3.28}$$

Proof. Using (3.14) we get, for $\rho_* \leq \rho \leq \rho_0$, with $M = \frac{3C_p}{\alpha^2\sqrt{2\kappa_2}}$,

$$\left| (k(\rho_0) - k(\rho)) - (k_*(\rho_0) - k_*(\rho)) \right| = \left| \int_\rho^{\rho_0} \frac{1}{y} \left(\int_y^\infty \frac{p''(z)}{2\sqrt{p'(z)}} \, dz \right) dy \right| \leq M\rho^{-\alpha}.$$

□

Remark 3.15. One can also write the estimate

$$\left|\int_{\rho}^{\rho_0} \frac{1}{y}\left(\int_y^{\infty} \frac{p''(z)}{2\sqrt{p'(z)}}\,dz\right)\,dy\right| \leq \frac{3C_p}{\alpha\sqrt{2\kappa_2}}(\rho_0-\rho)\rho^{-\alpha-1}.$$

In turn,

$$\frac{1}{(\rho_0-\rho_*)}\left|(k(\rho_0)-k(\rho))-(k_*(\rho_0)-k_*(\rho))\right| \leq \frac{3C_p}{\alpha\sqrt{2\kappa_2}}\rho^{-\alpha-1}, \tag{3.29}$$

for all $\rho \geq \rho_*$, so there is no singularity near $\rho_0 = \rho_*$. This arises throughout this chapter for terms of this kind and, as such, any bounds involving $\frac{1}{\rho_0-\rho_*}$ can be replaced with $\frac{M}{\rho_0}$ or $\frac{M}{1+(\rho_0-\rho_*)}$ for some positive constant M depending on ρ_*.

Lemma 3.16. *Assume that* $\rho_* \geq \max\{R, (4C_p/\kappa_2)^{1/\alpha}\}$. *Then, there exists a positive constant* $M = M(\alpha, C_p, \kappa_2)$ *such that*

$$\left|1-\frac{k'(\rho_0)}{k_*'(\rho_0)}\right| \leq M\rho_0^{-\alpha} \qquad \textit{for } \rho_0 \geq \rho_*. \tag{3.30}$$

Proof. Notice that, by letting $M = \frac{3C_p}{\sqrt{2}\alpha\kappa_2}$,

$$\left|1-\frac{k'(\rho_0)}{k_*'(\rho_0)}\right| = \left|1-\frac{\sqrt{p'(\rho_0)}}{\sqrt{\kappa_2}}\right| = \frac{1}{\sqrt{\kappa_2}}\left|\int_{\rho_0}^{\infty}\frac{p''(y)}{2\sqrt{p'(y)}}\,dy\right| \leq \frac{3C_p}{\sqrt{2}\kappa_2}\int_{\rho_0}^{\infty} y^{-\alpha-1}\,dy$$
$$= M\rho_0^{-\alpha}.$$

□

3.3 Representation formulas for the perturbation

In this section, we derive two representation formulas for the kernel χ^{error}. We obtain the first representation by taking the difference of the representation formulas for χ and χ^{iso}, and the second by directly considering the linear wave equation for χ^{error}, namely (3.11). The first representation formula (*cf.* Lemma 3.20) is required to estimate $\|\chi^{error}(\rho,\cdot)\|_{L^\infty(\mathbb{R})}$ (*cf.* Lemma 3.23). Armed with this bound, we are able to compute the derivatives with respect to u of the entropies generated by the perturbation, which is required for the second representation to make sense (*cf.* Lemma 3.26).

At this point, we fix $\rho_* \geq \max\{R, (4C_p/\kappa_2)^{1/\alpha}\}$, such that all of the estimates of Section 3.2 hold. We also make note of the following lemma, which outlines some important properties of χ^{iso}. These follow directly from the analysis of the entropy kernel of Chapter 2, and will be indispensable for our later estimates.

Lemma 3.17. *By Lemma 2.10, we have the identity*

$$\int_{\mathbb{R}} \chi(\rho_*, s)\,ds = \rho_* \int_{\mathbb{R}} \chi_\rho(\rho_*, s)\,ds.$$

In view of this and the calculations of Section 2.4, we find that the estimates of Section 2.3 (namely Lemmas 2.5 and 2.6) also hold for $\hat{\eta}^{iso}$ and $\eta^{iso,\psi}$ with $\psi \in C_c^2(\mathbb{R})$, where

$$\hat{\eta}^{iso}(\rho, \rho u) = \frac{1}{2}\int_{\mathbb{R}} \chi^{iso}(\rho, u-s)s|s|\,ds, \qquad \eta^{iso,\psi}(\rho, \rho u) = \int_{\mathbb{R}} \chi^{iso}(\rho, u-s)\psi(s)\,ds,$$

for $\rho \geq \rho_*$. Note however that the constants in these lemmas may now depend on ρ_*. Also, there exists a positive constant M depending on ρ_* such that, with $\tilde{\lambda} := \min\{\lambda, 1\}$,

$$\|\chi^{iso}(\rho,\cdot)\|_{L^\infty(\mathbb{R})} + [\chi^{iso}(\rho,\cdot)]_{C^{\tilde{\lambda}}(\mathbb{R})} \leq M\frac{\rho}{\sqrt{k(\rho)}} \qquad \text{for all } \rho \geq \rho_*, \tag{3.31}$$

where $[\cdot]_{C^{\tilde{\lambda}}(\mathbb{R})}$ is the Hölder seminorm, $\lambda = \frac{3-\gamma}{2(\gamma-1)}$, and we recall $k(\rho) \geq k(\rho_*)$.

Proof. We have, in view of Definition 3.4,

$$\chi^{iso}(\rho, u) = \int_{-k(\rho_*)}^{k(\rho_*)} \chi_\rho(\rho_*, s) g^\sharp(\rho, u-s)\,ds + \int_{-k(\rho_*)}^{k(\rho_*)} \chi(\rho_*, s) g^\flat(\rho, u-s)\,ds,$$

where the integrals are taken over the interval $[-k(\rho_*), k(\rho_*)]$ instead of all of $\mathbb{R}$ in view of the compact support of $\chi(\rho_*,\cdot)$ (*cf.* Remark 3.18). We now bound the right-hand side using the explicit form of the kernels $g^\sharp$ and $g^\flat$. For example, using $\chi_\rho(\rho_*,\cdot) \in L^1(\mathbb{R})$ (*cf.* Lemma 2.8), the first term on the right-hand side can be estimated as

$$\left|\int_{-k(\rho_*)}^{k(\rho_*)} \chi_\rho(\rho_*, s) g^\sharp(\rho, u-s)\,ds\right| \leq \|\chi_\rho(\rho_*,\cdot)\|_{L^1(\mathbb{R})} \cdot \sup_{|s|\leq k(\rho_*)} \rho_* \left|\chi^\sharp\left(\frac{\rho}{\rho_*}, \frac{u-s}{\sqrt{\kappa_2}}\right)\right|.$$

From Lemma A.6, we have the following bound on $\chi^\sharp$, for $\rho \geq 1$,

$$|\chi^\sharp(\rho, u)| = \left|\frac{\sqrt{\rho}}{2} I_0\left(\frac{\sqrt{(\log\rho)^2 - u^2}}{2}\right)\right| \mathbb{1}_{|u|<\log\rho} \leq \frac{\sqrt{\rho}}{2} I_0\left(\frac{\log\rho}{2}\right) \leq \frac{C\rho}{1+\sqrt{\log\rho}}, \tag{3.32}$$

where the constant C is independent of $\alpha, \kappa_2, C_p, \rho_*$. In view of this,

$$\left|\int_{-k(\rho_*)}^{k(\rho_*)} \chi_\rho(\rho_*, s) g^\sharp(\rho, u-s)\,ds\right| \leq C\|\chi_\rho(\rho_*,\cdot)\|_{L^1(\mathbb{R})} \frac{\rho}{1+\sqrt{\log(\rho/\rho_*)}},$$

provided $\rho \geq \rho_*$, from which the result follows using Corollary 3.11. The bound on the term involving $g^\flat$ is similar.

For $[\chi^{iso}(\rho,\cdot)]_{C^{\tilde\lambda}(\mathbb{R})}$, we have, for instance

$$\left|\int_{-k(\rho_*)}^{k(\rho_*)} \chi_\rho(\rho_*,s)\big(g^\sharp(\rho,u-s)-g^\sharp(\rho,v-s)\big)\,ds\right| \leq$$
$$\|\chi_\rho(\rho_*,\cdot)\|_{L^1(\mathbb{R})}\cdot \sup_{|s|\leq k(\rho_*)} \rho_* \left|\chi^\sharp\left(\frac{\rho}{\rho_*},\frac{u-s}{\sqrt{\kappa_2}}\right)-\chi^\sharp\left(\frac{\rho}{\rho_*},\frac{v-s}{\sqrt{\kappa_2}}\right)\right|.$$

And, for $\rho\geq 1$,

$$\begin{aligned}|\chi^\sharp(\rho,u)-\chi^\sharp(\rho,v)| &= \frac{\sqrt{\rho}}{2}\left|\int_u^v \frac{\partial}{\partial y} I_0\left(\frac{\sqrt{(\log\rho)^2-y^2}}{2}\right)\mathbb{1}_{|y|<\log\rho}\,dy\right| \\ &= \frac{\sqrt{\rho}}{4}\left|\int_u^v \frac{y}{\sqrt{(\log\rho)^2-y^2}} I_1\left(\frac{\sqrt{(\log\rho)^2-y^2}}{2}\right)\mathbb{1}_{|y|<\log\rho}\,dy\right| \\ &\leq \frac{\sqrt{\rho}\log\rho}{4}\int_u^v \frac{1}{\sqrt{(\log\rho)^2-y^2}} I_1\left(\frac{\sqrt{(\log\rho)^2-y^2}}{2}\right)\mathbb{1}_{|y|<\log\rho}\,dy.\end{aligned}$$

Since $(x^{-1}I_1(x))' = x^{-1}I_2(x)\geq 0$, the integrand is bounded by $\frac{1}{\log\rho}I_1\left(\frac{\log\rho}{2}\right)$. Thus,

$$|\chi^\sharp(\rho,u)-\chi^\sharp(\rho,v)| \leq \frac{\sqrt{\rho}}{4}I_1\left(\frac{\log\rho}{2}\right)|u-v| \leq \frac{C\rho}{1+\sqrt{\log\rho}}|u-v|.$$

where we used Lemma A.6, as we did for (3.32). The regular terms from the $g^\flat(\rho,\cdot)$ convolution (i.e. the ones containing Bessel functions) are dealt with similarly. The Dirac masses from $g^\flat(\rho,\cdot)$ produce the terms

$$\sqrt{\frac{\rho}{\rho_*}}\big|\chi(\rho_*,u\pm\sqrt{\kappa_2}\log(\rho/\rho_*))-\chi(\rho_*,v\pm\sqrt{\kappa_2}\log(\rho/\rho_*))\big|,$$

which, using the $\tilde\lambda$-Hölder bound on $\chi(\rho_*,\cdot)$ provided by [14, Proposition 2.4] (*cf.* [79, Remark 5.2.8]), are bounded above by $M\sqrt{\rho}|u-v|^{\tilde\lambda}$ for some M depending on ρ_*. □

Remark 3.18. Recall that [14, Theorem 2.1] showed that, as χ is the solution of the linear wave equation (3.4),

$$\operatorname{supp}\chi(\rho,\cdot) = [-k(\rho),k(\rho)] =: \mathcal{K}. \tag{3.33}$$

Similarly, for $\rho\geq\rho_*$,

$$\operatorname{supp}\chi^{iso}(\rho,\cdot) = [-k_*(\rho),k_*(\rho)] =: \mathcal{K}^{iso}. \tag{3.34}$$

We thereby deduce from (3.10) that

$$\operatorname{supp}\chi^{error}(\rho,\cdot)\subset[-\max\{k(\rho),k_*(\rho)\},\max\{k(\rho),k_*(\rho)\}] =: \mathcal{K}^{error}. \tag{3.35}$$

Lemma 3.19. *There exists a positive constant $M=M(\alpha,\kappa_2,C_p,\rho_*)$ such that*

$$\max\{k(\rho),k_*(\rho)\}\leq Mk(\rho) \qquad \textit{for } \rho\geq\rho_*. \tag{3.36}$$

Proof. The assertion is immediate from (3.12) and (3.22). □

3.3.1 First representation formula and uniform estimate on the perturbation

We begin with the analogue of (3.38) in [14] for an entropy kernel with initial data posed at ρ_*.

Lemma 3.20. *Given any* $(\rho_0, u_0) \in \mathcal{K}$ *and any* $0 < \rho_* < \rho_0$*, we have*

$$\begin{aligned}\chi(\rho_0,u_0) =& \frac{1}{2(\rho_0-\rho_*)k'(\rho_0)}\int_{\rho_*}^{\rho_0} k'(\rho)d(\rho)\chi(\rho,u_0+k(\rho_0)-k(\rho))\,d\rho\\ &+\frac{1}{2(\rho_0-\rho_*)k'(\rho_0)}\int_{\rho_*}^{\rho_0} k'(\rho)d(\rho)\chi(\rho,u_0-k(\rho_0)+k(\rho))\,d\rho\\ &-\frac{1}{2(\rho_0-\rho_*)k'(\rho_0)}\int_{-(k(\rho_0)-k(\rho_*))}^{k(\rho_0)-k(\rho_*)}\chi(\rho_*,u_0-y)\,dy.\end{aligned} \tag{3.37}$$

Proof. Fix any $(\rho_0, u_0) \in \mathcal{K}$ and an arbitrary $0 < \rho_* < \rho_0$. Then from applying the Fourier transform to the equation for the entropy kernel χ we see that

$$(\rho-\rho_*)k'(\rho)^2\mathcal{F}\chi(\rho,\xi) = -(\rho-\rho_*)\xi^{-2}\mathcal{F}\chi_{\rho\rho}(\rho,\xi).$$

Integrating this in the interval $\rho \in [\rho_*, \rho_0]$ we get

$$\begin{aligned}\int_{\rho_*}^{\rho_0}&(\rho-\rho_*)k'(\rho)^2\sin((k(\rho)-k(\rho_0))\xi)\mathcal{F}\chi(\rho,\xi)\,d\rho\\ =&-\xi^{-2}\int_{\rho_*}^{\rho_0}(\rho-\rho_*)\sin((k(\rho)-k(\rho_0))\xi)\mathcal{F}\chi_{\rho\rho}(\rho,\xi)\,d\rho,\\ =&\xi^{-2}\int_{\rho_*}^{\rho_0}\partial_\rho\big((\rho-\rho_*)\sin((k(\rho)-k(\rho_0))\xi)\big)\mathcal{F}\chi_\rho(\rho,\xi)\,d\rho\\ &-\xi^{-2}[(\rho-\rho_*)\sin((k(\rho)-k(\rho_0))\xi)\mathcal{F}\chi_\rho(\rho,\xi)]_{\rho_*}^{\rho_0},\\ =&-\xi^{-2}\int_{\rho_*}^{\rho_0}\partial_\rho^2\big[(\rho-\rho_*)\sin((k(\rho)-k(\rho_0))\xi)\big]\mathcal{F}\chi(\rho,\xi)\,d\rho\\ &+\xi^{-2}\Big[\partial_\rho\big[(\rho-\rho_*)\sin((k(\rho)-k(\rho_0))\xi)\big]\mathcal{F}\chi(\rho,\xi)\Big]_{\rho_*}^{\rho_0},\end{aligned}$$

and we find that the final line can be written as

$$\begin{aligned}-\xi^{-1}&\int_{\rho_*}^{\rho_0}\Big(k'+\big((\rho-\rho_*)k'\big)'\Big)\cos((k(\rho)-k(\rho_0))\xi)\mathcal{F}\chi(\rho,\xi)\,d\rho\\ &+\int_{\rho_*}^{\rho_0}(\rho-\rho_*)k'(\rho)^2\sin((k(\rho)-k(\rho_0))\xi)\mathcal{F}\chi(\rho,\xi)\,d\rho\\ &+\xi^{-1}(\rho_0-\rho_*)k'(\rho_0)\mathcal{F}\chi(\rho_0,\xi)-\xi^{-2}\sin((k(\rho_*)-k(\rho_0))\xi)\mathcal{F}\chi(\rho_*,\xi).\end{aligned}$$

Hence, we have

$$
\begin{aligned}
&(\rho_0-\rho_*)k'(\rho_0)\mathcal{F}\chi(\rho_0,\xi) = \\
&\qquad\qquad \xi^{-1}\sin((k(\rho_*)-k(\rho_0))\xi)\mathcal{F}\chi(\rho_*,\xi) \\
&\qquad\qquad +\int_{\rho_*}^{\rho_0}\left(k' + \big((\rho-\rho_*)k'\big)'\right)\cos((k(\rho)-k(\rho_0))\xi)\mathcal{F}\chi(\rho,\xi)\,d\rho.
\end{aligned}
$$

Now, observe that for $a>0$ and $g\in\mathcal{S}'(\mathbb{R})\cap C(\mathbb{R})$, we have

$$
\mathcal{F}^{-1}\big(\xi^{-1}\sin(a\xi)\mathcal{F}g(\xi)\big)(x) = \frac{1}{2}\int_{-a}^{a} g(x-y)\,dy,
$$

and

$$
\mathcal{F}^{-1}\left(\cos(a\xi)\mathcal{F}g(\xi)\right)(x) = \frac{1}{2}\left(g(x+a)+g(x-a)\right).
$$

So, applying the inverse Fourier transform we obtain the result as claimed. □

Remark 3.21. The same representation formula as (3.37) holds for χ^{iso}, except k should be replaced with k_*, d with d_*, and $\mathcal{K}$ with $\mathcal{K}^{iso}$.

In view of Lemma 3.20 and Remark 3.21, by subtracting $\chi^{iso}(\rho_0,u_0)$ from $\chi(\rho_0,u_0)$ and recalling that $\chi(\rho_*,\cdot)=\chi^{iso}(\rho_*,\cdot)$, we arrive at the first representation formula for the perturbation.

Lemma 3.22 (First representation formula)**.** *Given any* $(\rho_0,u_0)\in\mathcal{K}^{error}$*,*

$$
\begin{aligned}
&2(\rho_0-\rho_*)\chi^{error}(\rho_0,u_0) = \\
&\quad \int_{\rho_*}^{\rho_0}\frac{k'(\rho)}{k'(\rho_0)}d(\rho)\Big[\chi^{error}(\rho,u_0+k(\rho_0)-k(\rho)) + \chi^{error}(\rho,u_0-k(\rho_0)+k(\rho))\Big]\,d\rho \\
&\quad +\int_{\rho_*}^{\rho_0}\frac{k'(\rho)d(\rho)}{k'(\rho_0)}\Big[\chi^{iso}(\rho,u_0+k(\rho_0)-k(\rho)) - \chi^{iso}(\rho,u_0+k_*(\rho_0)-k_*(\rho))\Big]\,d\rho \\
&\quad +\int_{\rho_*}^{\rho_0}\left[\frac{k'(\rho)d(\rho)}{k'(\rho_0)} - \frac{k_*'(\rho)d_*(\rho)}{k_*'(\rho_0)}\right]\chi^{iso}(\rho,u_0+k_*(\rho_0)-k_*(\rho))\,d\rho \\
&\quad +\int_{\rho_*}^{\rho_0}\frac{k'(\rho)d(\rho)}{k'(\rho_0)}\Big[\chi^{iso}(\rho,u_0-k(\rho_0)+k(\rho)) - \chi^{iso}(\rho,u_0-k_*(\rho_0)+k_*(\rho))\Big]\,d\rho \\
&\quad +\int_{\rho_*}^{\rho_0}\left[\frac{k'(\rho)d(\rho)}{k'(\rho_0)} - \frac{k_*'(\rho)d_*(\rho)}{k_*'(\rho_0)}\right]\chi^{iso}(\rho,u_0-k_*(\rho_0)+k_*(\rho))\,d\rho \\
&\quad -\left[\int_{-(k(\rho_0)-k(\rho_*))}^{k(\rho_0)-k(\rho_*)}\frac{\chi^{iso}(\rho_*,u_0-y)}{k'(\rho_0)}\,dy - \int_{-(k_*(\rho_0)-k_*(\rho_*))}^{k_*(\rho_0)-k_*(\rho_*)}\frac{\chi^{iso}(\rho_*,u_0-y)}{k_*'(\rho_0)}\,dy\right].
\end{aligned}
\tag{3.38}
$$

Notice that, in view of what is known about χ^{iso} from Lemma 3.17, the formula (3.38) is perfectly set up to perform a Grönwall type argument on the quantity $\|k'(\rho_0)\chi^{error}(\rho_0,\cdot)\|_{L^\infty(\mathbb{R})}$. With this observation, we proceed to the most important result of this section, which is crucial to the rest of the chapter.

Lemma 3.23. *There exists a positive constant* $M = M(\alpha, C_p, \kappa_2, \rho_*)$ *such that*

$$\|\chi^{error}(\rho,\cdot)\|_{L^\infty(\mathbb{R})} \le M \max\{k(\rho), \rho^{1-\alpha\tilde{\lambda}}\} \qquad \text{for } \rho \ge \rho_*, \tag{3.39}$$

where we recall that $\tilde{\lambda} = \min\{\lambda, 1\}$.

Proof. We begin by multiplying (3.38) by $k'(\rho_0)$. We then bound the first line of (3.38) by

$$\int_{\rho_*}^{\rho_0} 2d(\rho)\|k'(\rho)\chi^{error}(\rho,\cdot)\|_{L^\infty(\mathbb{R})}\, d\rho.$$

The second line of (3.38) is bounded by

$$\int_{\rho_*}^{\rho_0} d(\rho)[k'(\rho)\chi^{iso}(\rho,\cdot)]_{C^{\tilde{\lambda}}(\mathbb{R})}\big|(k(\rho_0)-k(\rho)) - (k_*(\rho_0)-k_*(\rho))\big|^{\tilde{\lambda}}\, d\rho,$$

which, in view of Lemma 3.14 and the bound (3.24), is bounded by

$$M\int_{\rho_*}^{\rho_0} \rho^{-\alpha\tilde{\lambda}}[k'(\rho)\chi^{iso}(\rho,\cdot)]_{C^{\tilde{\lambda}}(\mathbb{R})}\, d\rho,$$

where $M = M(\alpha, \kappa_2, C_p)$. Notice that the fourth line of (3.38) can be bounded in exactly the same way. Next, the third line of (3.38) is bounded by

$$\int_{\rho_*}^{\rho} \left|d(\rho) - d_*(\rho)\frac{k'(\rho_0)}{k_*'(\rho_0)}\frac{k_*'(\rho)}{k'(\rho)}\right| \|k'(\rho)\chi^{iso}(\rho,\cdot)\|_{L^\infty(\mathbb{R})}\, d\rho,$$

and we bound the term in the absolute values by

$$d(\rho)\left|1 - \frac{k'(\rho_0)}{k_*'(\rho_0)}\frac{k_*'(\rho)}{k'(\rho)}\right| + |d(\rho) - d_*(\rho)|\,\frac{k'(\rho_0)}{k_*'(\rho_0)}\frac{k_*'(\rho)}{k'(\rho)}.$$

Using Lemma 3.12 for the right-hand term and Lemma 3.13 for the left-hand term, respectively, along with (3.24), we see that the whole of the third line of (3.38) may be bounded by

$$M\int_{\rho_*}^{\rho_0} \rho^{-\alpha}\|k'(\rho)\chi^{iso}(\rho,\cdot)\|_{L^\infty(\mathbb{R})}\, d\rho,$$

where $M = M(\alpha, \kappa_2, C_p)$. Once again, the fifth line of (3.38) may be bounded in exactly the same way. The very final line of (3.38) may be split into two terms, i.e.,

$$\begin{aligned} &\int_{-(k(\rho_0)-k(\rho_*))}^{k(\rho_0)-k(\rho_*)} \chi^{iso}(\rho_*, u_0 - y)\, dy - \int_{-(k_*(\rho_0)-k_*(\rho_*))}^{k_*(\rho_0)-k_*(\rho_*)} \chi^{iso}(\rho_*, u_0 - y)\, dy \\ &\qquad + \left(1 - \frac{k'(\rho_0)}{k_*'(\rho_0)}\right)\int_{-(k_*(\rho_0)-k_*(\rho_*))}^{k_*(\rho_0)-k_*(\rho_*)} \chi^{iso}(\rho_*, u_0 - y)\, dy. \end{aligned} \tag{3.40}$$

We concentrate on the first line of (3.40). Observe that the two intervals, $[-(k(\rho_0) - k(\rho_*)), k(\rho_0)-k(\rho_*)]$ and $[-(k_*(\rho_0)-k_*(\rho_*)), k_*(\rho_0)-k_*(\rho_*)]$, are always nested within one another; one interval is always entirely contained in the other. Hence, since the same quantity is being integrated, we may bound this line by

$$2\Big|(k(\rho_0)-k(\rho_*))-(k_*(\rho_0)-k_*(\rho_*))\Big|\cdot\|\chi^{iso}(\rho_*,\cdot)\|_{L^\infty(\mathbb{R})},$$

and, using Lemma 3.14, this is itself bounded by

$$M\rho_*^{-\alpha}\|\chi^{iso}(\rho_*,\cdot)\|_{L^\infty(\mathbb{R})},$$

where $M=M(\alpha,\kappa_2,C_p)$. On the other hand, the final term of (3.40) is bounded by

$$\left|1-\frac{k'(\rho_0)}{k'_*(\rho_0)}\right|\cdot 2\Big|k_*(\rho_0)-k_*(\rho_*)\Big|\cdot\|\chi^{iso}(\rho_*,\cdot)\|_{L^\infty(\mathbb{R})},$$

and, using Lemma 3.16 and (3.12), this is bounded by

$$M\rho_0^{-\alpha}\log\left(\frac{\rho_0}{\rho_*}\right)\|\chi^{iso}(\rho_*,\cdot)\|_{L^\infty(\mathbb{R})},$$

where $M=M(\alpha,\kappa_2,C_p)$. We emphasise that it has been sufficient to assume $\alpha>0$.

In total, we therefore have

$$\begin{aligned}\|k'(\rho_0)\chi^{error}(\rho_0,\cdot)\|_{L^\infty(\mathbb{R})}\leq&\frac{1}{(\rho_0-\rho_*)}\int_{\rho_*}^{\rho_0}d(\rho)\|k'(\rho)\chi^{error}(\rho,\cdot)\|_{L^\infty(\mathbb{R})}\,d\rho\\&+\frac{M}{(\rho_0-\rho_*)}\int_{\rho_*}^{\rho_0}\rho^{-\alpha\tilde\lambda}[k'(\rho)\chi^{iso}(\rho,\cdot)]_{C^{\tilde\lambda}(\mathbb{R})}\,d\rho\\&+\frac{M}{(\rho_0-\rho_*)}\int_{\rho_*}^{\rho_0}\rho^{-\alpha}\|k'(\rho)\chi^{iso}(\rho,\cdot)\|_{L^\infty(\mathbb{R})}\,d\rho\\&+\frac{M}{(\rho_0-\rho_*)}\rho_*^{-\alpha}\|\chi^{iso}(\rho_*,\cdot)\|_{L^\infty(\mathbb{R})}\\&+\frac{M}{2(\rho_0-\rho_*)}\rho_0^{-\alpha}\log\left(\frac{\rho_0}{\rho_*}\right)\|\chi^{iso}(\rho_*,\cdot)\|_{L^\infty(\mathbb{R})}.\end{aligned}$$

Although it appears that the right-hand side explodes when $\rho_0=\rho_*$, this is not the case (*cf.* Remark 3.15). As such, for some positive constant $M=M(\alpha,\kappa_2,C_p,\rho_*)$,

$$\begin{aligned}\|k'(\rho_0)\chi^{error}(\rho_0,\cdot)\|_{L^\infty(\mathbb{R})}\leq&\frac{1}{(\rho_0-\rho_*)}\int_{\rho_*}^{\rho_0}d(\rho)\|k'(\rho)\chi^{error}(\rho,\cdot)\|_{L^\infty(\mathbb{R})}\,d\rho\\&+\frac{M}{(\rho_0-\rho_*)}\int_{\rho_*}^{\rho_0}\rho^{-\alpha\tilde\lambda}[k'(\rho)\chi^{iso}(\rho,\cdot)]_{C^{\tilde\lambda}(\mathbb{R})}\,d\rho\\&+\frac{M}{(\rho_0-\rho_*)}\int_{\rho_*}^{\rho_0}\rho^{-\alpha}\|k'(\rho)\chi^{iso}(\rho,\cdot)\|_{L^\infty(\mathbb{R})}\,d\rho\\&+\frac{M}{1+(\rho_0-\rho_*)}\rho_*^{-\alpha}\|\chi^{iso}(\rho_*,\cdot)\|_{L^\infty(\mathbb{R})}\\&+\frac{M}{1+2(\rho_0-\rho_*)}\rho_0^{-\alpha}\log\left(1+\frac{\rho_0}{\rho_*}\right)\|\chi^{iso}(\rho_*,\cdot)\|_{L^\infty(\mathbb{R})}.\end{aligned}\tag{3.41}$$

We now apply Grönwall's lemma to (3.41) and divide by $k'(\rho_0)$, which yields

$$\begin{aligned}
&\|\chi^{error}(\rho_0,\cdot)\|_{L^\infty(\mathbb{R})} \leq \\
&\bigg\{\frac{M}{(\rho_0-\rho_*)k'(\rho_0)}\int_{\rho_*}^{\rho_0} \rho^{-\alpha\tilde{\lambda}}[k'(\rho)\chi^{iso}(\rho,\cdot)]_{C^{\tilde{\lambda}}(\mathbb{R})}\,d\rho \\
&\quad + \frac{M}{(\rho_0-\rho_*)k'(\rho_0)}\int_{\rho_*}^{\rho_0} \rho^{-\alpha}\|k'(\rho)\chi^{iso}(\rho,\cdot)\|_{L^\infty(\mathbb{R})}\,d\rho \\
&\quad + \frac{M}{k'(\rho_0)[1+(\rho_0-\rho_*)]}\rho_*^{-\alpha}\|\chi^{iso}(\rho_*,\cdot)\|_{L^\infty(\mathbb{R})} \\
&\quad + \frac{M}{k'(\rho_0)[1+2(\rho_0-\rho_*)]}\rho_0^{-\alpha}\log\left(1+\frac{\rho_0}{\rho_*}\right)\|\chi^{iso}(\rho_*,\cdot)\|_{L^\infty(\mathbb{R})}\bigg\} e^{\frac{1}{(\rho_0-\rho_*)}\int_{\rho_*}^{\rho_0} d(\rho)\,d\rho}.
\end{aligned} \tag{3.42}$$

It is clear that, in view of the bound on $d(\rho)$ provided by Lemma 3.12, the exponential on the right-hand side is bounded above by e^3. Now recall from Lemma 3.17 that

$$\|\chi^{iso}(\rho,\cdot)\|_{L^\infty(\mathbb{R})} + [\chi^{iso}(\rho,\cdot)]_{C^{\tilde{\lambda}}(\mathbb{R})} \leq M\rho \qquad \text{for all } \rho \geq \rho_*.$$

Then, using also Lemma 3.16, we have

$$\|k'(\rho)\chi^{iso}(\rho,\cdot)\|_{L^\infty(\mathbb{R})} + [k'(\rho)\chi^{iso}(\rho,\cdot)]_{C^{\tilde{\lambda}}(\mathbb{R})} \leq M \qquad \text{for all } \rho \geq \rho_*.$$

We are now in a position to bound each of the terms in the curly brackets of (3.42). We begin with the first term.

$$\frac{M}{(\rho_0-\rho_*)k'(\rho_0)}\int_{\rho_*}^{\rho_0} \rho^{-\alpha\tilde{\lambda}}[k'(\rho)\chi^{iso}(\rho,\cdot)]_{C^{\tilde{\lambda}}(\mathbb{R})}\,d\rho \leq \frac{M}{(\rho_0-\rho_*)k'(\rho_0)}\int_{\rho_*}^{\rho_0} \rho^{-\alpha\tilde{\lambda}}\,d\rho. \tag{3.43}$$

We easily verify that there is no singularity at $\rho_0 = \rho_*$, so the right-hand side of (3.43) is bounded by

$$\frac{M}{\rho_0 k'(\rho_0)}\int_{\rho_*}^{\rho_0} \rho^{-\alpha\tilde{\lambda}}\,d\rho \leq M\rho_0^{1-\alpha\tilde{\lambda}}, \tag{3.44}$$

if $\alpha\tilde{\lambda} \in (0,1)$. If $\alpha\tilde{\lambda} \geq 1$, then (3.43) is bounded by $M(1+\log(\rho_0/\rho_*))$.

Similarly, for the second term of (3.42), we have

$$\frac{M}{(\rho_0-\rho_*)k'(\rho_0)}\int_{\rho_*}^{\rho_0} \rho^{-\alpha}\|k'(\rho)\chi^{iso}(\rho,\cdot)\|_{L^\infty(\mathbb{R})}\,d\rho \leq \frac{M}{(\rho_0-\rho_*)k'(\rho_0)}\int_{\rho_*}^{\rho_0} \rho^{-\alpha}\,d\rho. \tag{3.45}$$

There is no singularity at $\rho_0 = \rho_*$, so the right-hand side of (3.45) is bounded by

$$\frac{M}{\rho_0 k'(\rho_0)}\int_{\rho_*}^{\rho_0} \rho^{-\alpha}\,d\rho \leq M\rho_0^{1-\alpha}, \tag{3.46}$$

provided $\alpha \in (0,1)$. If $\alpha \geq 1$, the right-hand side of (3.45) is again bounded by a constant multiple of $1+\log(\rho_0/\rho_*)$.

For the third and fourth terms of (3.42), observe that, by Lemma 3.9,

$$\frac{1}{k'(\rho_0)[1+2(\rho_0-\rho_*)]} \leq M. \tag{3.47}$$

Thus, collecting the results in (3.44)-(3.47), we have shown that: either $1-\alpha\tilde{\lambda} \leq 0$, in which case we also know that $1-\alpha \leq 0$ and then, using Corollary 3.11,

$$\|\chi^{error}(\rho_0,\cdot)\|_{L^\infty(\mathbb{R})} \leq Mk(\rho),$$

for some positive $M = M(\alpha, \kappa_2, C_p, \rho_*)$. Or, $\alpha\tilde{\lambda} \in (0,1)$, in which case

$$\|\chi^{error}(\rho_0,\cdot)\|_{L^\infty(\mathbb{R})} \leq M\left(\rho_0^{1-\alpha\tilde{\lambda}} + \rho_0^{1-\alpha} + 1 + \log(\rho_0/\rho_*)\right),$$

which, by appealing to Lemma A.8, straightforwardly yields the result. □

Remark 3.24. We note that, if $\alpha\tilde{\lambda} > 1$, the proof of Lemma 3.23 shows that we actually have the stronger estimate

$$\|\chi^{error}(\rho,\cdot)\|_{L^\infty(\mathbb{R})} \leq M \qquad \text{for } \rho \geq \rho_*.$$

3.3.2 Generating entropies and second representation formula

In what follows, we generate entropies by convolving with the entropy kernel. Specifically, given a suitably integrable test function ψ, we generate an entropy according to the formula

$$\begin{aligned}\eta^\psi(\rho,u) &:= \int_{\mathbb{R}} \chi(\rho,u-s)\psi(s)\,ds \\ &= \int_{\mathbb{R}} \chi^{iso}(\rho,u-s)\psi(s)\,ds + \int_{\mathbb{R}} \chi^{error}(\rho,u-s)\psi(s)\,ds \\ &=: \eta^{iso,\psi}(\rho,u) + \eta^{error,\psi}(\rho,u) \qquad \text{for } \rho \geq \rho_*.\end{aligned} \tag{3.48}$$

Remark 3.25. Since χ is given as a function of (ρ,u), we will also generate entropies as functions of (ρ,u), even though these are technically functions of (ρ,m). **We adopt this convention throughout this subsection and Section 3.4 only**, in order to simplify computations.

In particular, when we choose the special test function $\hat{\psi}(s) = \frac{1}{2}s|s|$, for $\rho \geq \rho_*$,

$$\hat{\eta}(\rho,u) := \eta^{\hat{\psi}}(\rho,u) = \eta^{iso,\hat{\psi}}(\rho,u) + \eta^{error,\hat{\psi}}(\rho,u) =: \hat{\eta}^{iso}(\rho,u) + \hat{\eta}^{error}(\rho,u). \tag{3.49}$$

Note that, provided $\psi \in C^1(\mathbb{R})$ and $\chi^{error}(\rho, u - \cdot)\psi'(\cdot)$ is dominated by some locally integrable function independent of u, Lebesgue's dominated convergence theorem yields

$$\eta_u^{error,\psi}(\rho, u) = \frac{\partial}{\partial u} \int_{\mathbb{R}} \chi^{error}(\rho, s)\psi(u - s)\, ds = \int_{\mathbb{R}} \chi^{error}(\rho, u - s)\psi'(s)\, ds.$$

In this case, the derivative is well-defined, and the same procedure yields

$$\eta_u^{iso}(\rho, u) = \int_{\mathbb{R}} \chi^{iso}(\rho, u - s)\psi'(s)\, ds,$$

provided the same integrability conditions hold. Asking for $\psi \in C^2(\mathbb{R})$ and similar integrability requirements on the convolution of the kernel with ψ'', one can show that, in this case, $\eta_{uu}^{error,\psi}$ and $\eta_{uu}^{iso,\psi}$ are well-defined. Now that we have seen how to make sense of partial derivatives of the entropy with respect to u, we derive the second representation formula for the perturbation.

Lemma 3.26 (Second representation formula)**.** *Let $\psi \in C^2(\mathbb{R})$ be such that $\eta_{uu}^{error,\psi}$, $\eta_u^{iso,\psi}$, $\eta_{uu}^{iso,\psi}$ are well-defined continuous functions. Then, for any $(\rho_0, u_0) \in \mathcal{K}^{error}$,*

$$\begin{aligned} 2(\rho_0 - \rho_*)k'(\rho_0)\eta^{error,\psi}(\rho_0, u_0) = \\ \int_{\rho_*}^{\rho_0} d(\rho)k'(\rho)\big\{\eta^{error,\psi}(\rho, u_0 + k(\rho_0) - k(\rho)) + \eta^{error,\psi}(\rho, u_0 - k(\rho_0) + k(\rho))\big\}\, d\rho \\ + \int_{\rho_*}^{\rho_0} (\rho - \rho_*)\big(k'(\rho)^2 - \frac{\kappa_2}{\rho^2}\big)\eta_u^{iso,\psi}(\rho, u_0 + k(\rho_0) - k(\rho))\, d\rho \\ - \int_{\rho_*}^{\rho_0} (\rho - \rho_*)\big(k'(\rho)^2 - \frac{\kappa_2}{\rho^2}\big)\eta_u^{iso,\psi}(\rho, u_0 - k(\rho_0) + k(\rho))\, d\rho. \end{aligned} \tag{3.50}$$

Proof. Convolving (3.11) with ψ, we obtain

$$\begin{cases} \eta_{\rho\rho}^{error,\psi} - k'(\rho)^2\eta_{uu}^{error,\psi} = \big(k'(\rho)^2 - \frac{\kappa_2}{\rho^2}\big)\eta_{uu}^{iso,\psi} & \text{for } (\rho, u) \in [\rho_*, \infty) \times \mathbb{R}, \\ \eta^{error,\psi}(\rho_*, u) = 0, \\ \eta_\rho^{error,\psi}(\rho_*, u) = 0, \end{cases}$$

and define, for the rest of this proof,

$$f(\rho, u) := \big(k'(\rho)^2 - \frac{\kappa_2}{\rho^2}\big)\eta_{uu}^{iso,\psi}(\rho, u).$$

Fix any $(\rho_0, u_0) \in \mathcal{K}^{error}$ and an arbitrary $0 < \rho_* < \rho_0$. Then from applying the Fourier transform to the previous wave equation, we see that

$$\begin{aligned} -\xi^{-2}(\rho - \rho_*)\sin((k(\rho) - k(\rho_0))\xi)\mathcal{F}\eta_{\rho\rho}^{error,\psi}(\rho, \xi) \\ + \xi^{-2}(\rho - \rho_*)\sin((k(\rho) - k(\rho_0))\xi)\mathcal{F}f(\rho, \xi) \\ = (\rho - \rho_*)k'(\rho)^2\sin((k(\rho) - k(\rho_0))\xi)\mathcal{F}\eta^{error,\psi}(\rho, \xi), \end{aligned}$$

where $\mathcal{F}$ is the Fourier transform, to avoid confusion with $\hat{\cdot}$ denoting quantities related to the special entropy. Next, integrating by parts twice with respect to ρ, we obtain

$$\begin{aligned}(\rho_0-\rho_*)k'(\rho_0)\mathcal{F}\eta^{error,\psi}(\rho_0,\xi) = &\int_{\rho_*}^{\rho_0} d(\rho)k'(\rho)\cos((k(\rho)-k(\rho_0))\xi)\mathcal{F}\eta^{error,\psi}(\rho,\xi)\,d\rho\\ &+\xi^{-1}\sin((k(\rho_*)-k(\rho_0))\xi)\mathcal{F}\eta^{error,\psi}(\rho_*,\xi)\\ &-\xi^{-1}\int_{\rho_*}^{\rho_0}(\rho-\rho_*)\sin((k(\rho)-k(\rho_0))\xi)\mathcal{F}f(\rho,\xi)\,d\rho.\end{aligned}$$

Note that

$$\mathcal{F}f(\rho,\xi) = -\big(k'(\rho)^2-\frac{\kappa_2}{\rho^2}\big)\xi^2\mathcal{F}\eta^{iso,\psi}(\rho,\xi).$$

Next, we note that, for $g\in\mathcal{S}'(\mathbb{R})\cap C^1(\mathbb{R})$,

$$\mathcal{F}^{-1}(\xi\sin(a\xi)\mathcal{F}g(\xi)) = -\left(\frac{g'(x+a)-g'(x-a)}{2}\right).$$

Thus, the inverse transform of the final term on the right-hand side is

$$\begin{aligned}&\mathcal{F}^{-1}\Big(-\xi^{-1}\int_{\rho_*}^{\rho_0}(\rho-\rho_*)\sin((k(\rho)-k(\rho_0))\xi)\mathcal{F}f(\rho,\xi)\,d\rho\Big)\\ &=\int_{\rho_*}^{\rho_0}(\rho-\rho_*)\big(k'(\rho)^2-\frac{\kappa_2}{\rho^2}\big)\mathcal{F}^{-1}\big(\xi\sin((k(\rho)-k(\rho_0))\xi)\mathcal{F}\eta^{iso,\psi}(\rho,\xi)\big)\,d\rho\\ &=\frac{1}{2}\int_{\rho_*}^{\rho_0}(\rho-\rho_*)\big(k'(\rho)^2-\frac{\kappa_2}{\rho^2}\big)\eta_u^{iso,\psi}(\rho,u_0+k(\rho_0)-k(\rho))\,d\rho\\ &\quad-\frac{1}{2}\int_{\rho_*}^{\rho_0}(\rho-\rho_*)\big(k'(\rho)^2-\frac{\kappa_2}{\rho^2}\big)\eta_u^{iso,\psi}(\rho,u_0-k(\rho_0)+k(\rho))\,d\rho.\end{aligned}$$

Applying the inverse Fourier transform, and noting that $\eta^{error,\psi}(\rho_*,\cdot)=0$, yields the result. □

3.4 Estimating entropy pairs

In this section, we use the representation formulas obtained in Section 3.3 and the estimate of Lemma 3.23 to estimate the entropy pairs generated by the special function $\hat{\psi}(s)=\frac{1}{2}s|s|$ and by compactly supported test functions. In particular, we prove results analogous to Lemmas 2.5 and 2.6 of Chapter 2.

3.4.1 Special entropy pair

We first consider the special entropy, $\hat{\eta}(\rho,u)$, generated by $\hat{\psi}(s)=\frac{1}{2}s|s|$, defined by the formula (3.49) (*cf.* Section 3.3.2),

$$\begin{aligned}\hat{\eta}(\rho,u) &= \frac{1}{2}\int_{\mathbb{R}}\chi^{iso}(\rho,u-s)s|s|\,ds+\frac{1}{2}\int_{\mathbb{R}}\chi^{error}(\rho,u-s)s|s|\,ds,\\ &=\hat{\eta}^{iso}(\rho,u)+\hat{\eta}^{error}(\rho,u)\qquad\text{for }\rho\geq\rho_*.\end{aligned}\tag{3.51}$$

Unexpectedly, it is the case that this entropy is three times continuously differentiable in its second variable, as demonstrated in the next lemma. This property will be useful when we come to estimate the entropy-flux (*cf.* Lemmas 3.41, 3.42, and 3.43).

Lemma 3.27. *The entropies $\hat{\eta}^{iso}$ and $\hat{\eta}^{error}$ are three times continuously differentiable in their second variable, i.e., $\hat{\eta}^{iso}(\rho,\cdot), \hat{\eta}^{error}(\rho,\cdot) \in C^3(\mathbb{R})$ for each $\rho \in [\rho_*, \infty)$. In fact,*

$$\hat{\eta}^{iso}_{uuu}(\rho,u) = 2\chi^{iso}(\rho,u), \quad \hat{\eta}^{error}_{uuu}(\rho,u) = 2\chi^{error}(\rho,u) \quad \text{for } (\rho,u) \in [\rho_*,\infty)\times\mathbb{R}. \tag{3.52}$$

Correspondingly, there exists a positive constant $M = M(\alpha, \kappa_2, C_p, \rho_*, k(\rho_*))$ such that

$$\|\hat{\eta}^{iso}_{uuu}(\rho,\cdot)\|_{L^\infty(\mathbb{R})} \leq \frac{M\rho}{\sqrt{k(\rho)}}, \quad \|\hat{\eta}^{error}_{uuu}(\rho,\cdot)\|_{L^\infty(\mathbb{R})} \leq M \max\{k(\rho), \rho^{1-\alpha\tilde{\lambda}}\} \quad \text{for } \rho \geq \rho_*. \tag{3.53}$$

Proof. Recall that, by definition,

$$\hat{\eta}^{error}(\rho,u) = \frac{1}{2}\int_{-\infty}^{u} \chi^{error}(\rho,s)(u-s)^2\,ds - \frac{1}{2}\int_{u}^{\infty} \chi^{error}(\rho,s)(u-s)^2\,ds, \tag{3.54}$$

where this integral is well-defined, since $\chi^{error}(\rho,\cdot)$ is Hölder continuous and compactly supported. In view of this, we can apply Lebesgue's dominated convergence theorem, and take derivatives under the integral. Differentiating repeatedly with respect to u, we thereby obtain

$$\begin{aligned}
\hat{\eta}^{error}_{u}(\rho,u) &= \int_{-\infty}^{u} \chi^{error}(\rho,s)(u-s)\,ds - \int_{u}^{\infty} \chi^{error}(\rho,s)(u-s)\,ds,\\
\hat{\eta}^{error}_{uu}(\rho,u) &= \int_{-\infty}^{u} \chi^{error}(\rho,s)\,ds - \int_{u}^{\infty} \chi^{error}(\rho,s)\,ds,\\
\hat{\eta}^{error}_{uuu}(\rho,u) &= 2\chi^{error}(\rho,u).
\end{aligned} \tag{3.55}$$

The same argument applies for χ^{iso}, which concludes the proof, by virtue of Lemmas 3.17 and 3.23. □

3.4.1.1 Bounds on the special entropy and its u derivatives

Below is the first estimate on the special entropy, which shows that it is controlled by the mechanical energy, η^*.

Lemma 3.28. *There exists a positive constant $M = M(\alpha, \kappa_2, C_p, \rho_*, k(\rho_*))$ such that*

$$|\hat{\eta}^{error}(\rho,u)| \leq M \max\{k(\rho), \rho^{1-\alpha\tilde{\lambda}}\}\big(k(\rho)u^2 + k(\rho)^3\big). \tag{3.56}$$

Proof. By Remark 3.18, $\operatorname{supp}\chi^{error}(\rho,\cdot) \subset [-\max\{k(\rho),k_*(\rho)\},\max\{k(\rho),k_*(\rho)\}]$. Using the bound on $\max\{k(\rho),k_*(\rho)\}$ provided by Lemma 3.19, we see directly from (3.54) and the Cauchy–Schwarz inequality that

$$|\hat{\eta}^{error}(\rho,u)| \leq Mk(\rho)u^2\|\chi^{error}(\rho,\cdot)\|_{L^\infty(\mathbb{R})} + Mk(\rho)^3\|\chi^{error}(\rho,\cdot)\|_{L^\infty(\mathbb{R})}, \tag{3.57}$$

from which the result follows easily, after an application of Lemma 3.23. □

Corollary 3.29. *There exists a positive constant* $M = M(\alpha,\kappa_2,C_p,\rho_*,k(\rho_*))$ *such that*

$$|\hat{\eta}^{error}(\rho,u)| \leq M\eta^*(\rho,u) \qquad \textit{for } (\rho,u)\in[\rho_*,\infty)\times\mathbb{R}. \tag{3.58}$$

Proof. Recall that the mechanical energy is given by $\eta^*(\rho,m) = \frac{1}{2}\frac{m^2}{\rho} + \rho e(\rho)$, where the internal energy is characterised by the formula $e(\rho) = \int_0^\rho \frac{p(y)}{y^2}\,dy$, as per Definition 1.9. When $\rho \geq \rho_*$ and ρ is assumed to be large, the internal energy scales as $e(\rho) = O(\log\rho)$. Therefore for large ρ, we indeed have that $\rho^{1-\alpha\tilde{\lambda}}(1+\log(\rho/\rho_*))^3$ is bounded by $\rho e(\rho)$, up to a multiplicative constant depending on ρ_*, α, and $\tilde{\lambda}$. In view of this, we manifestly obtain $|\hat{\eta}^{error}(\rho,u)| \leq M\eta^*(\rho,u)$. □

Additionally, recall from (3.55) that

$$\hat{\eta}^{error}_u(\rho,u) = \int_{-\infty}^u \chi^{error}(\rho,s)(u-s)\,ds - \int_u^\infty \chi^{error}(\rho,s)(u-s)\,ds.$$

Recall that, since $\hat{\eta}^{error}_m(\rho,u) = \rho^{-1}\hat{\eta}^{error}_u(\rho,u)$, it follows from the compact support of χ^{error} and Lemma 3.19 that

$$|\hat{\eta}^{error}_m(\rho,u)| \leq M\rho^{-1}k(\rho)\big(|u|+k(\rho)\big)\|\chi^{error}(\rho,\cdot)\|_{L^\infty(\mathbb{R})},$$

where $M = M(\alpha,C_p,\kappa_2,\rho_*)$. We deduce, using Lemma 3.23, that

$$|\hat{\eta}^{error}_m(\rho,u)| \leq M\big(\max\{\rho^{-1}k(\rho),\rho^{-\alpha\tilde{\lambda}}\}k(\rho)|u| + \max\{\rho^{-1}k(\rho),\rho^{-\alpha\tilde{\lambda}}\}k(\rho)^2\big),$$

from which it is evident that

$$|\hat{\eta}^{error}_m(\rho,u)| \leq \frac{M}{k(\rho)}\big(|u|+1\big) \qquad \text{for } (\rho,u)\in[\rho_*,\infty)\times\mathbb{R}. \tag{3.59}$$

Differentiating once more in u, we find, as we did in (3.55),

$$\hat{\eta}^{error}_{uu}(\rho,u) = \int_{-\infty}^u \chi^{error}(\rho,s)\,ds - \int_u^\infty \chi^{error}(\rho,s)\,ds.$$

In view of this, another application of Lemma 3.23 yields

$$|\hat{\eta}^{error}_{mm}(\rho,u)| \leq M\rho^{-2}k(\rho)\|\chi^{error}(\rho,\cdot)\|_{L^\infty(\mathbb{R})} \leq M\max\{\rho^{-2}k(\rho),\rho^{-1-\alpha\tilde{\lambda}}\}k(\rho), \tag{3.60}$$

and

$$|\hat{\eta}^{error}_{mu}(\rho,u)| \leq M\rho^{-1}k(\rho)\|\chi^{error}(\rho,\cdot)\|_{L^\infty(\mathbb{R})} \leq M\max\{\rho^{-1}k(\rho),\rho^{-\alpha\tilde{\lambda}}\}k(\rho), \quad (3.61)$$

where $M = M(\alpha, C_p, \kappa_2, \rho_*, k(\rho_*))$ and $\eta_{mu}(\rho,u) = \partial_u\eta_m(\rho,\rho u)$, corresponds to the fake mixed derivative (*cf.* Definition 2.27). We collect our results in the lemma underneath, which concludes our investigation of the derivatives with respect to u of $\hat{\eta}^{error}$.

Lemma 3.30. *There exists a positive constant $M = M(\alpha, \lambda, \kappa_2, C_p, \rho_*, k(\rho_*))$ such that*

$$\begin{aligned} &|\hat{\eta}^{error}_{m}(\rho,u)| \leq \frac{M}{1+\log(\rho/\rho_*)}\big(|u|+1\big) && \textit{for } (\rho,u) \in [\rho_*,\infty)\times\mathbb{R}, \\ &|\hat{\eta}^{error}_{mu}(\rho,u)| + |\rho\hat{\eta}^{error}_{mm}(\rho,u)| \leq \frac{M}{1+\log(\rho/\rho_*)} && \textit{for } (\rho,u) \in [\rho_*,\infty)\times\mathbb{R}. \end{aligned} \quad (3.62)$$

3.4.1.2 Lower bound for the special entropy-flux

Recall that we may decompose the entropy-flux kernel $\sigma(\rho,u,s)$ for our problem as

$$\sigma(\rho,u,s) = u\chi(\rho,u,s) + h(\rho,u,s), \quad (3.63)$$

as per Definition 1.11, where $h(\rho,u,s)$ satisfies (1.16), i.e.,

$$\begin{cases} h_{\rho\rho} - k'(\rho)^2 h_{uu} = \dfrac{p''(\rho)}{\rho}\chi_u, \\ h|_{\rho=0} = 0, \\ h_\rho|_{\rho=0} = 0. \end{cases} \quad (3.64)$$

Recall that, much like $\chi(\rho,u,s) = \chi(\rho,u-s,0)$, we have that $h(\rho,u,s) = h(\rho,u-s,0)$. Therefore, in a slight abuse of notation, we shall write $h = h(\rho, u-s)$. Recall that [14, Theorem 2.1] shows that there exists a unique globally-defined Hölder continuous function h solving (3.64), with $\operatorname{supp} h(\rho,\cdot) = [-k(\rho),k(\rho)]$, as noted in Theorem 1.12.

It is then the case that an entropy-flux q^ψ, generated by the test function ψ, and its corresponding entropy η^ψ are linked according to

$$\begin{aligned} q^\psi(\rho,u) = \int_{\mathbb{R}} \sigma(\rho,u,s)\psi(s)\,ds &= u\int_{\mathbb{R}}\chi(\rho,u-s)\psi(s)\,ds + \int_{\mathbb{R}} h(\rho,u-s)\psi(s)\,ds \\ &= u\eta^\psi(\rho,u) + \int_{\mathbb{R}} h(\rho,u-s)\psi(s)\,ds. \end{aligned}$$

In turn, defining

$$H^\psi(\rho,u) := \int_{\mathbb{R}} h(\rho,u-s)\psi(s)\,ds, \quad (3.65)$$

we see that $q^\psi(\rho,u) = u\eta^\psi(\rho,u) + H^\psi(\rho,u)$. As such, any entropy-flux can be generated from the kernels χ and h. Our aim is now to decompose the kernel h for our problem in terms of an isothermal kernel, and a perturbation. To this end, we have the following definition.

Definition 3.31. Define h^{iso} by

$$h^{iso}(\rho,u) := \int_{\mathbb{R}} h_\rho(\rho_*,s) g^\sharp(\rho,u-s)\,ds + \int_{\mathbb{R}} h(\rho_*,s) g^\flat(\rho,u-s)\,ds, \tag{3.66}$$

where $g^\sharp$ and $g^\flat$ were introduced in Definition 3.3. Let $\psi \in C_c^2(\mathbb{R})$ and $\hat{\psi}(s) = \frac{1}{2}s|s|$, and define, for $(\rho,u) \in [\rho_*,\infty)\times\mathbb{R}$,

$$H^{iso,\psi}(\rho,u) := \int_{\mathbb{R}} h^{iso}(\rho,u-s)\psi(s)\,ds, \quad \hat{H}^{iso}(\rho,u) := \int_{\mathbb{R}} h^{iso}(\rho,u-s)\hat{\psi}(s)\,ds. \tag{3.67}$$

Note that (3.66) and Theorem 1.12 imply that h^{iso} is the unique solution of

$$\begin{cases} h^{iso}_{\rho\rho} - \dfrac{\kappa_2}{\rho^2} h^{iso}_{uu} = 0 \quad \text{for } (\rho,u) \in (\rho_*,\infty)\times\mathbb{R}, \\ h^{iso}(\rho_*,u) = h(\rho_*,u), \\ h^{iso}_\rho(\rho_*,u) = h_\rho(\rho_*,u). \end{cases} \tag{3.68}$$

Remark 3.32. Recall from Theorem 1.12 that $h^{iso}(\rho,\cdot)$ is Hölder continuous, and

$$\operatorname{supp} h^{iso}(\rho,\cdot) \subset [-k_*(\rho),k_*(\rho)] \qquad \text{for } (\rho,u) \in [\rho_*,\infty)\times\mathbb{R}. \tag{3.69}$$

Remark 3.33. Note that h^{iso} solves (3.64) in the particular instance where $p(\rho) = \kappa_2\rho$. Hence, due to the uniqueness of solutions of (3.68) ensured by [14, Theorem 2.1],

$$\sigma^{iso}(\rho,u,s) = u\chi^{iso}(\rho,u-s) + h^{iso}(\rho,u-s),$$

where σ^{iso} is in fact the entropy-flux kernel considered in Subsection 2.4.4 of Chapter 2 (up to a re-scaling in terms of ρ_*).

In view of this, we have that the entropy-flux $\hat{q}^{iso}$, where

$$\begin{aligned} \hat{q}^{iso}(\rho,u) &:= \int_{\mathbb{R}} \sigma^{iso}(\rho,u,s)\hat{\psi}(s)\,ds \\ &= u\int_{\mathbb{R}} \chi^{iso}(\rho,u-s)\hat{\psi}(s)\,ds + \int_{\mathbb{R}} h^{iso}(\rho,u-s)\hat{\psi}(s)\,ds \\ &= u\hat{\eta}^{iso}(\rho,u) + \hat{H}^{iso}(\rho,u), \end{aligned} \tag{3.70}$$

satisfies the lower bound of Lemma 2.5, namely, for some M depending also on ρ_*,

$$q^{iso}(\rho,u) \geq M^{-1}\rho|u|^3 - M\big(\rho|u|^2 + \rho + \rho(\log(\rho/\rho_*))^4\big). \tag{3.71}$$

Similarly, for $\psi \in C_c^2(\mathbb{R})$, the entropy-flux $q^{iso,\psi}$, where

$$\begin{aligned} q^{iso,\psi}(\rho,u) &:= \int_{\mathbb{R}} \sigma^{iso}(\rho,u,s)\psi(s)\,ds \\ &= u\int_{\mathbb{R}} \chi^{iso}(\rho,u-s)\psi(s)\,ds + \int_{\mathbb{R}} h^{iso}(\rho,u-s)\psi(s)\,ds \\ &= u\eta^{iso,\psi}(\rho,u) + H^{iso,\psi}(\rho,u), \end{aligned} \tag{3.72}$$

satisfies the estimate of Lemma 2.6, namely, for some M_ψ depending also on ρ_*,

$$|q^{iso,\psi}(\rho,u)| \le M_\psi \rho. \tag{3.73}$$

Lemma 3.34. *There exists a positive constant M, depending also on ρ_*, such that*

$$\|h^{iso}(\rho,\cdot)\|_{L^\infty(\mathbb{R})} \le M\frac{\rho}{\sqrt{k(\rho)}} \qquad \textit{for } \rho \ge \rho_*. \tag{3.74}$$

Proof. Using the kernels $g^\sharp$ and $g^\flat$ (*cf.* Definition 3.3), we see that

$$h^{iso}(\rho,u) = \int_{\mathbb{R}} h_\rho(\rho_*,s)g^\sharp(\rho,u-s)\,ds + \int_{\mathbb{R}} h(\rho_*,s)g^\flat(\rho,u-s)\,ds. \tag{3.75}$$

The result is now easily deduced using the explicit forms of $g^\sharp$ and $g^\flat$, as was done for χ^{iso} in the proof of Lemma 3.17. □

Next, we record some elementary facts concerning $\hat{H}^{iso}$ (*cf.* Definition 3.31).

Lemma 3.35. *The function $\hat{H}^{iso}$ is three times continuously differentiable in its second variable, i.e., $\hat{H}^{iso}(\rho,\cdot) \in C^3(\mathbb{R})$. In fact,*

$$\hat{H}^{iso}_{uuu}(\rho,u) = 2h^{iso}(\rho,u) \qquad \textit{for } (\rho,u) \in [\rho_*,\infty)\times\mathbb{R}, \tag{3.76}$$

from which we see that there is a positive constant M depending on ρ_* such that

$$\|\hat{H}^{iso}_{uuu}(\rho,\cdot)\|_{L^\infty(\mathbb{R})} \le M\frac{\rho}{\sqrt{k(\rho)}} \qquad \textit{for } \rho \ge \rho_*. \tag{3.77}$$

Proof. Recall that

$$\begin{aligned} \hat{H}^{iso}(\rho,u) &= \frac{1}{2}\int_{\mathbb{R}} s|s|h^{iso}(\rho,u-s)\,ds \\ &= \frac{1}{2}\int_{-\infty}^{u} h^{iso}(\rho,s)(u-s)^2\,ds - \frac{1}{2}\int_{u}^{\infty} h^{iso}(\rho,s)(u-s)^2\,ds. \end{aligned}$$

Note that, since $h^{iso}(\rho, \cdot)$ is continuous and compactly supported, an application of Lebesgue's dominated convergence theorem enables us to differentiate under the integral. This yields

$$\begin{aligned}\hat{H}^{iso}_u(\rho, u) &= \int_{-\infty}^{u} h^{iso}(\rho, s)(u - s)\, ds - \int_{u}^{\infty} h^{iso}(\rho, s)(u - s)\, ds \\ &= \int_{\mathbb{R}} h^{iso}(\rho, s)|u - s|\, ds.\end{aligned}$$

Repeating this process, we find

$$\hat{H}^{iso}_{uu}(\rho, u) = \int_{\mathbb{R}} h^{iso}(\rho, s)\,\mathrm{sgn}(u - s)\, ds, \qquad \hat{H}^{iso}_{uuu}(\rho, u) = 2h^{iso}(\rho, u),$$

from which the result follows, in view of the bound on h^{iso} provided by Lemma 3.34. □

Remark 3.36. Recall from Remark 3.32 that $\mathrm{supp}\, h^{iso}(\rho, \cdot) \subset [-k_*(\rho), k_*(\rho)]$, and thus

$$\hat{H}^{iso}_{uu}(\rho, u) = \int_{-k_*(\rho)}^{k_*(\rho)} h^{iso}(\rho, s)\,\mathrm{sgn}(u - s)\, ds. \tag{3.78}$$

It now follows from the bound on h^{iso} provided by Lemmas 3.34 and 3.19 that there exists a positive constant $M = M(\alpha, \kappa_2, C_p, \rho_*, k(\rho_*))$ such that

$$\|\hat{H}^{iso}_{uu}(\rho, \cdot)\|_{L^\infty(\mathbb{R})} \leq M\rho\sqrt{k(\rho)} \qquad \text{for } \rho \geq \rho_*. \tag{3.79}$$

We note that this bound is very blunt and could improved, for instance, by following the procedures for the estimates on $h^\sharp$ and $h^\flat$ in Lemma 2.32 in Chapter 2.

Definition 3.37. Having already defined h and h^{iso}, we now define the perturbation

$$h^{error}(\rho, u) := h(\rho, u) - h^{iso}(\rho, u). \tag{3.80}$$

Let $\psi \in C^2_c(\mathbb{R})$ and $\hat{\psi}(s) = \frac{1}{2}s|s|$. We define, for $(\rho, u) \in [\rho_*, \infty) \times \mathbb{R}$,

$$H^{error,\psi}(\rho, u) := \int_{\mathbb{R}} h^{error}(\rho, u - s)\psi(s)\, ds, \quad \hat{H}^{error}(\rho, u) := \int_{\mathbb{R}} h^{error}(\rho, u - s)\hat{\psi}(s)\, ds. \tag{3.81}$$

Correspondingly, for $(\rho, u) \in [\rho_*, \infty) \times \mathbb{R}$, we define the fluxes

$$\begin{aligned} q^{error,\psi}(\rho, u) &:= u\eta^{error,\psi}(\rho, u) + H^{error,\psi}(\rho, u), \\ \hat{q}^{error}(\rho, u) &:= u\hat{\eta}^{error}(\rho, u) + \hat{H}^{error}(\rho, u). \end{aligned} \tag{3.82}$$

Lemma 3.38. *As a consequence of the decomposition* (3.80) *and the wave equation for* h^{iso} (3.68), *the perturbation solves, for* $(\rho,u)\in(\rho_*,\infty)\times\mathbb{R}$,

$$\begin{cases} h^{error}_{\rho\rho}-k'(\rho)^2h^{error}_{uu}=\big(k'(\rho)^2-\frac{\kappa_2}{\rho^2}\big)h^{iso}_{uu}(\rho,u)+\frac{p''(\rho)}{\rho}\chi_u(\rho,u),\\ h^{error}(\rho_*,u)=0,\\ h^{error}_\rho(\rho_*,u)=0.\end{cases} \tag{3.83}$$

It is then the case that, for our problem, the entropy-flux q^ψ generated by $\psi\in C^2_c(\mathbb{R})$ can be written as

$$\begin{aligned} q(\rho,u):=&\int_{\mathbb{R}}\sigma(\rho,u,s)\psi(s)\,ds=u\int_{\mathbb{R}}\chi(\rho,u-s)\psi(s)\,ds+\int_{\mathbb{R}}h(\rho,u-s)\psi(s)\,ds\\ =&\big(u\eta^{iso,\psi}(\rho,u)+H^{iso,\psi}(\rho,u)\big)+\big(u\eta^{error,\psi}(\rho,u)+H^{error,\psi}(\rho,u)\big)\\ =&q^{iso,\psi}(\rho,u)+q^{error,\psi}(\rho,u),\end{aligned}$$

where the third equality was deduced by expanding χ in terms of χ^{iso} and χ^{error}, and h in terms of h^{iso} and h^{error}. Analogously, we have

$$\hat{q}(\rho,u)=\hat{q}^{iso}(\rho,u)+\hat{q}^{error}(\rho,u). \tag{3.84}$$

We shall, at a later stage, be concerned with bounding $\hat{q}^{error}$ (*cf.* Corollary 3.47). In order to do this, we first need to estimate $\hat{H}^{error}$. To this end, we begin by convolving (3.83) with the special function $\hat{\psi}(s)=\frac{1}{2}s|s|$. In turn, we obtain, for $(\rho,u)\in[\rho_*,\infty)\times\mathbb{R}$,

$$\begin{cases} \hat{H}^{error}_{\rho\rho}-k'(\rho)^2\hat{H}^{error}_{uu}=\big(k'(\rho)^2-\frac{\kappa_2}{\rho^2}\big)\hat{H}^{iso}_{uu}(\rho,u)+\frac{p''(\rho)}{\rho}\hat{\eta}_u(\rho,u),\\ \hat{H}^{error}(\rho_*,u)=0,\\ \hat{H}^{error}_\rho(\rho_*,u)=0.\end{cases} \tag{3.85}$$

This wave equation will enable us to derive a representation formula of the type (3.50) for $\hat{H}^{error}$, which is the focus of the following lemma.

Lemma 3.39. *For any* $(\rho_0,u_0)\in\mathcal{K}^{error}$ *and for any* $0<\rho_*<\rho_0$, *we have*

$$\begin{aligned} &2(\rho_0-\rho_*)k'(\rho_0)\hat{H}^{error}(\rho_0,u_0)=\\ &\int_{\rho_*}^{\rho_0}d(\rho)k'(\rho)\big\{\hat{H}^{error}(\rho,u_0+k(\rho_0)-k(\rho))+\hat{H}^{error}(\rho,u_0-k(\rho_0)+k(\rho))\big\}\,d\rho\\ &+\int_{\rho_*}^{\rho_0}(\rho-\rho_*)\big(k'(\rho)^2-\frac{\kappa_2}{\rho^2}\big)\hat{H}^{iso}_u(\rho,u_0+k(\rho_0)-k(\rho))\,d\rho\\ &-\int_{\rho_*}^{\rho_0}(\rho-\rho_*)\big(k'(\rho)^2-\frac{\kappa_2}{\rho^2}\big)\hat{H}^{iso}_u(\rho,u_0-k(\rho_0)+k(\rho))\,d\rho\\ &+\int_{\rho_*}^{\rho_0}\frac{p''(\rho)}{\rho}(\rho-\rho_*)\big\{\hat{\eta}(\rho,u_0+k(\rho_0)-k(\rho))-\hat{\eta}(\rho,u_0-k(\rho_0)+k(\rho))\big\}\,d\rho.\end{aligned} \tag{3.86}$$

Proof. For the rest of this proof, let

$$f(\rho,u) := \big(k'(\rho)^2 - \frac{\kappa_2}{\rho^2}\big)\hat{H}^{iso}_{uu}(\rho,u) + \frac{p''(\rho)}{\rho}\hat{\eta}_u(\rho,u) \qquad \text{for } (\rho,u)\in[\rho_*,\infty)\times\mathbb{R}. \tag{3.87}$$

Fix any $(\rho_0,u_0)\in\mathcal{K}$ and any $\rho_*\in(0,\rho_0)$. Applying the Fourier transform to (3.85),

$$\begin{aligned}
-\xi^{-2}(\rho-\rho_*)\sin((k(\rho)-k(\rho_0))\xi)\mathcal{F}\hat{H}^{error}_{\rho\rho}(\rho,\xi)&\\
+\,\xi^{-2}(\rho-\rho_*)\sin((k(\rho)-k(\rho_0))\xi)\mathcal{F}f(\rho,\xi)&\\
=(\rho-\rho_*)k'(\rho)^2\sin((k(\rho)-k(\rho_0))\xi)\mathcal{F}\hat{H}^{error}(\rho,\xi)&,
\end{aligned}$$

where $\mathcal{F}$ denotes the Fourier transform, to avoid confusion with $\hat{\cdot}$ denoting quantities related to the special test function. Integrating by parts twice with respect to ρ,

$$\begin{aligned}
(\rho_0-\rho_*)k'(\rho_0)\mathcal{F}\hat{H}^{error}(\rho_0,\xi) = &\int_{\rho_*}^{\rho_0} d(\rho)k'(\rho)\cos((k(\rho)-k(\rho_0))\xi)\mathcal{F}\hat{H}^{error}(\rho,\xi)\,d\rho\\
&+\xi^{-1}\sin((k(\rho_*)-k(\rho_0))\xi)\mathcal{F}\hat{H}^{error}(\rho_*,\xi)\\
&-\xi^{-1}\int_{\rho_*}^{\rho_0}(\rho-\rho_*)\sin((k(\rho)-k(\rho_0))\xi)\mathcal{F}f(\rho,\xi)\,d\rho.
\end{aligned}$$

Observe that

$$\mathcal{F}f(\rho,\xi) = -\big(k'(\rho)^2-\frac{\kappa_2}{\rho^2}\big)\xi^2\hat{H}^{iso}(\rho,\xi) + i\xi\frac{p''(\rho)}{\rho}\hat{\chi}(\rho,\xi).$$

Next, we invert the Fourier transform and note that, for $g\in\mathcal{S}'(\mathbb{R})\cap C^1(\mathbb{R})$,

$$\begin{aligned}
\mathcal{F}^{-1}(i\sin(a\xi)\hat{g}(\xi))(x) &= \frac{g(x+a)-g(x-a)}{2},\\
\mathcal{F}^{-1}(\xi\sin(a\xi)\hat{g}(\xi)) &= -\left(\frac{g'(x+a)-g'(x-a)}{2}\right).
\end{aligned}$$

Thus, the inverse transform of the final term on the right-hand side is

$$\begin{aligned}
&\mathcal{F}^{-1}\Big(-\xi^{-1}\int_{\rho_*}^{\rho_0}(\rho-\rho_*)\sin((k(\rho)-k(\rho_0))\xi)\mathcal{F}f(\rho,\xi)\,d\rho\Big)\\
&=\int_{\rho_*}^{\rho_0}(\rho-\rho_*)\big(k'(\rho)^2-\frac{\kappa_2}{\rho^2}\big)\mathcal{F}^{-1}\big(\xi\sin((k(\rho)-k(\rho_0))\xi)\mathcal{F}\hat{H}^{iso}(\rho,\xi)\big)\,d\rho\\
&\quad-\int_{\rho_*}^{\rho_0}\frac{p''(\rho)}{\rho}(\rho-\rho_*)\mathcal{F}^{-1}\big(i\sin((k(\rho)-k(\rho_0))\xi)\mathcal{F}\hat{\eta}(\rho,\xi)\big)\,d\rho\\
&=\frac{1}{2}\int_{\rho_*}^{\rho_0}(\rho-\rho_*)\big(k'(\rho)^2-\frac{\kappa_2}{\rho^2}\big)\hat{H}^{iso}_u(\rho,u_0+k(\rho_0)-k(\rho))\,d\rho\\
&\quad-\frac{1}{2}\int_{\rho_*}^{\rho_0}(\rho-\rho_*)\big(k'(\rho)^2-\frac{\kappa_2}{\rho^2}\big)\hat{H}^{iso}_u(\rho,u_0-k(\rho_0)+k(\rho))\,d\rho\\
&\quad+\frac{1}{2}\int_{\rho_*}^{\rho_0}\frac{p''(\rho)}{\rho}(\rho-\rho_*)\big\{\hat{\eta}(\rho,u_0+k(\rho_0)-k(\rho))-\hat{\eta}(\rho,u_0-k(\rho_0)+k(\rho))\big\}\,d\rho.
\end{aligned}$$

Finally, making note of $\hat{H}^{error}(\rho_*,\cdot)=0$, due to (3.85), concludes the proof. □

Differentiating (3.86) with respect to u_0, we obtain the following result.

Corollary 3.40. *For any* $(\rho_0, u_0) \in \mathcal{K}^{error}$ *and* $0 < \rho_* < \rho_0$, *we have*

$$\begin{aligned}
&2(\rho_0-\rho_*)k'(\rho_0)\hat{H}_u^{error}(\rho_0,u_0) = \\
&\int_{\rho_*}^{\rho_0} d(\rho)k'(\rho)\big\{\hat{H}_u^{error}(\rho,u_0+k(\rho_0)-k(\rho)) + \hat{H}_u^{error}(\rho,u_0-k(\rho_0)+k(\rho))\big\}\,d\rho \\
&+\int_{\rho_*}^{\rho_0} (\rho-\rho_*)\big(k'(\rho)^2-\frac{\kappa_2}{\rho^2}\big)\hat{H}_{uu}^{iso}(\rho,u_0+k(\rho_0)-k(\rho))\,d\rho \\
&-\int_{\rho_*}^{\rho_0} (\rho-\rho_*)\big(k'(\rho)^2-\frac{\kappa_2}{\rho^2}\big)\hat{H}_{uu}^{iso}(\rho,u_0-k(\rho_0)+k(\rho))\,d\rho \\
&+\int_{\rho_*}^{\rho_0} \frac{p''(\rho)}{\rho}(\rho-\rho_*)\big\{\hat{\eta}_u(\rho,u_0+k(\rho_0)-k(\rho)) - \hat{\eta}_u(\rho,u_0-k(\rho_0)+k(\rho))\big\}\,d\rho.
\end{aligned} \tag{3.88}$$

Lemma 3.41. *There exists a positive constant* $M = M(\alpha, \kappa_2, C_p, \rho_*, k(\rho_*))$ *such that*

$$\|\hat{H}_u^{error}(\rho,\cdot)\|_{L^\infty(\mathbb{R})} \le M\max\{k(\rho)^2, \rho^{1-\alpha}k(\rho)\} \qquad \textit{for } \rho \ge \rho_*. \tag{3.89}$$

Proof. Observe that, in view of the bound on $\hat{H}_{uuu}^{iso}$ provided by Lemma 3.35,

$$\begin{aligned}
&\big|\hat{H}_{uu}^{iso}(\rho,u_0+k(\rho_0)-k(\rho)) - \hat{H}_{uu}^{iso}(\rho,u_0-k(\rho_0)+k(\rho))\big| \\
&\qquad = \left|\int_{u_0-k(\rho_0)+k(\rho)}^{u_0+k(\rho_0)-k(\rho)} \hat{H}_{uuu}^{iso}(\rho,y)\,dy\right| \le M\rho k(\rho_0).
\end{aligned}$$

Observe that, using Corollary 3.10, the sum of the second and third terms on the right-hand side of (3.88) is bounded by

$$\begin{aligned}
\frac{Mk(\rho_0)}{\rho_0-\rho_*}\int_{\rho_*}^{\rho_0}(\rho-\rho_*)\big|k'(\rho)^2-\frac{\kappa_2}{\rho^2}\big|\rho\,d\rho &\le \frac{Mk(\rho_0)}{\rho_0-\rho_*}\left(\int_{\rho_*}^{\rho_0}\rho^{-\alpha}\,d\rho\right) \\
&\le \frac{Mk(\rho_0)\rho_0^{1-\alpha}}{1+(\rho_0-\rho_*)},
\end{aligned}$$

provided $\alpha \in (0,1)$. If $\alpha \ge 1$, the final line would be replaced with $\frac{Mk(\rho_0)^2}{1+(\rho_0-\rho_*)}$, using Corollary 3.11. We also observe the following about the final term in (3.88),

$$\begin{aligned}
&\frac{1}{2(\rho_0-\rho_*)}\int_{\rho_*}^{\rho_0}\frac{p''(\rho)}{\rho}(\rho-\rho_*)\big\{\hat{\eta}_u(\rho,u_0+k(\rho_0)-k(\rho)) - \hat{\eta}_u(\rho,u_0-k(\rho_0)+k(\rho))\big\}\,d\rho \\
&\qquad = \frac{1}{2(\rho_0-\rho_*)}\int_{\rho_*}^{\rho_0}\frac{p''(\rho)}{\rho}(\rho-\rho_*)\left(\int_{u_0-k(\rho_0)+k(\rho)}^{u_0+k(\rho_0)-k(\rho)}\hat{\eta}_{uu}(\rho,y)\,dy\right)d\rho.
\end{aligned}$$

Recall that $|\hat{\eta}_{uu}(\rho,u)| = |\rho^2\hat{\eta}_{mm}(\rho,u)| \le M\rho$, from Lemma 2.5 of Chapter 2 and Lemma 3.30. As such,

$$\left|\int_{u_0-k(\rho_0)+k(\rho)}^{u_0+k(\rho_0)-k(\rho)}\hat{\eta}_{uu}(\rho,y)\,dy\right| \le M\rho k(\rho_0),$$

and so the final term on the right-hand side of (3.88) is bounded by

$$\frac{Mk(\rho_0)}{(\rho_0-\rho_*)}\int_{\rho_*}^{\rho_0}|p''(\rho)|(\rho-\rho_*)\,d\rho \le \frac{Mk(\rho_0)}{(\rho_0-\rho_*)}\left(\int_{\rho_*}^{\rho_0}\rho^{-\alpha}\,d\rho\right) \le \frac{Mk(\rho_0)\rho_0^{1-\alpha}}{1+(\rho_0-\rho_*)},$$

provided $\alpha \in (0,1)$. If $\alpha \ge 1$, then the final line would be replaced with $\frac{Mk(\rho_0)^2}{1+(\rho_0-\rho_*)}$. Hence, we have

$$\|k'(\rho_0)\hat{H}_u^{error}(\rho_0,\cdot)\|_{L^\infty(\mathbb{R})} \le \frac{1}{\rho_0-\rho_*}\int_{\rho_*}^{\rho_0} d(\rho)\|k'(\rho)\hat{H}_u^{error}(\rho,\cdot)\|_{L^\infty(\mathbb{R})}\,d\rho + M\max\{\rho_0^{-1}k(\rho_0),\rho_0^{-\alpha}\}k(\rho_0).$$

By applying Grönwall's lemma, we obtain

$$\|k'(\rho_0)\hat{H}_u^{error}(\rho_0,\cdot)\|_{L^\infty(\mathbb{R})} \le M\max\{\rho_0^{-1}k(\rho_0),\rho_0^{-\alpha}\}k(\rho_0)\exp\left(\frac{1}{\rho_0-\rho_*}\int_{\rho_*}^{\rho_0} d(\rho)\,d\rho\right), \tag{3.90}$$

and the exponential term is bounded by e^3. Using Lemma 3.9, we conclude. □

Arguing as before, by taking a derivative in u_0 to the representation formula (3.88) and applying Grönwall's lemma, we obtain the following result.

Lemma 3.42. *There exists a positive constant $M = M(\alpha,\kappa_2,C_p,\rho_*,k(\rho_*))$ such that*

$$\|k'(\rho)\hat{H}_{uu}^{error}(\rho,\cdot)\|_{L^\infty(\mathbb{R})} \le M\max\{\rho^{-1}k(\rho),\rho^{-\alpha}\} \qquad \text{for } \rho \ge \rho_*. \tag{3.91}$$

Lemma 3.43. *There exists a positive constant $M = M(\alpha,\kappa_2,C_p,\rho_*,k(\rho_*))$ such that*

$$|\hat{H}^{error}(\rho,u)| \le M\big(\rho|u| + \rho + \rho\sqrt{\log(\rho/\rho_*)}\big) \qquad \text{for } (\rho,u)\in[\rho_*,\infty)\times\mathbb{R}. \tag{3.92}$$

Proof. Convolving (3.83) with the special function $\hat{\psi}$, we obtain

$$\hat{H}_{\rho\rho}^{error}(\rho,u) = k'(\rho)^2\hat{H}_{uu}^{error}(\rho,u) + \big(k'(\rho)^2 - \frac{\kappa_2}{\rho^2}\big)\hat{H}_{uu}^{iso}(\rho,u) + \frac{p''(\rho)}{\rho}\hat{\eta}_u(\rho,u),$$

from which we may directly estimate, using (3.19) and (3.79), and the bound on $\hat{\eta}_u$ provided by Lemma 3.17 (*cf.* Lemma 2.5) and Lemma 3.30,

$$|\hat{H}_{\rho\rho}^{error}(\rho,u)| \le M\bigg(\max\{\rho^{-2}k(\rho),\rho^{-\alpha-1}\} + \rho^{-\alpha-1}\big(|u| + \sqrt{k(\rho)}\big)\bigg),$$

We now integrate in ρ, making use of the fact that $\hat{H}_\rho^{error}(\rho_*,\cdot) = 0$, and obtain

$$|\hat{H}_{\rho}^{error}(\rho,u)| \le M\bigg(\max\{\rho_*^{-1/2},\rho_*^{-\alpha}\} + \rho_*^{-\alpha}\big(|u| + \sqrt{\log(\rho/\rho_*)}\big)\bigg).$$

Integrating once more in ρ, and making use of the fact that $\hat{H}^{error}(\rho_*,\cdot) = 0$, gives the result. □

3.4.1.3 A fake mixed derivative of the special entropy

Before proceeding with the bound on $\hat{\eta}^{error}_{m\rho}$, we need an improved estimate on $\hat{\eta}^{error}_{m}$. This is the subject of the next result, Lemma 3.44. To this end, convolving (3.11) with the special test function $\hat{\psi} = \frac{1}{2}s|s|$, we find that $\hat{\eta}^{error}$ satisfies

$$\begin{cases} \hat{\eta}^{error}_{\rho\rho} - k'(\rho)^2\hat{\eta}^{error}_{uu} = \big(k'(\rho)^2 - \frac{\kappa_2}{\rho^2}\big)\hat{\eta}^{iso}_{uu} & \text{for } (\rho, u) \in [\rho_*, \infty) \times \mathbb{R}, \\ \hat{\eta}(\rho_*, u) = 0, \\ \hat{\eta}_\rho(\rho_*, u) = 0. \end{cases} \tag{3.93}$$

Proceeding exactly as was done in the proof of Lemma 3.26, we obtain

$$\begin{aligned} &2(\rho_0 - \rho_*)k'(\rho_0)\hat{\eta}^{error}(\rho_0, u_0) = \\ &\int_{\rho_*}^{\rho_0} d(\rho)k'(\rho)\big\{\hat{\eta}^{error}(\rho, u_0 + k(\rho_0) - k(\rho)) + \hat{\eta}^{error}(\rho, u_0 - k(\rho_0) + k(\rho))\big\}\, d\rho \\ &+ \int_{\rho_*}^{\rho_0} (\rho - \rho_*)\big(k'(\rho)^2 - \frac{\kappa_2}{\rho^2}\big)\hat{\eta}^{iso}_u(\rho, u_0 + k(\rho_0) - k(\rho))\, d\rho \\ &- \int_{\rho_*}^{\rho_0} (\rho - \rho_*)\big(k'(\rho)^2 - \frac{\kappa_2}{\rho^2}\big)\hat{\eta}^{iso}_u(\rho, u_0 - k(\rho_0) + k(\rho))\, d\rho. \end{aligned} \tag{3.94}$$

Thus, by taking a derivative with respect to u_0,

$$\begin{aligned} &2(\rho_0 - \rho_*)\rho_0 k'(\rho_0)\hat{\eta}^{error}_m(\rho_0, u_0) = \\ &\int_{\rho_*}^{\rho_0} d(\rho)\rho k'(\rho)\big\{\hat{\eta}^{error}_m(\rho, u_0 + k(\rho_0) - k(\rho)) + \hat{\eta}^{error}_m(\rho, u_0 - k(\rho_0) + k(\rho))\big\}\, d\rho \\ &+ \int_{\rho_*}^{\rho_0} (\rho - \rho_*)\rho\big(k'(\rho)^2 - \frac{\kappa_2}{\rho^2}\big)\hat{\eta}^{iso}_{mu}(\rho, u_0 + k(\rho_0) - k(\rho))\, d\rho \\ &- \int_{\rho_*}^{\rho_0} (\rho - \rho_*)\rho\big(k'(\rho)^2 - \frac{\kappa_2}{\rho^2}\big)\hat{\eta}^{iso}_{mu}(\rho, u_0 - k(\rho_0) + k(\rho))\, d\rho. \end{aligned} \tag{3.95}$$

Lemma 3.44. *There exists a positive constant $M = M(\alpha, \kappa_2, C_p, \rho_*, k(\rho_*))$ such that*

$$\|\hat{\eta}^{error}_m(\rho, \cdot)\|_{L^\infty(\mathbb{R})} \le M \max\{\rho^{-1}k(\rho), \rho^{-\alpha}\} \qquad \textit{for } \rho \ge \rho_*. \tag{3.96}$$

Proof. Observe that, after dividing by $2(\rho_0 - \rho_*)$, the first term on the right-hand side of (3.95) is bounded by

$$\frac{1}{(\rho_0 - \rho_*)}\int_{\rho_*}^{\rho_0} d(\rho)\|\rho k'(\rho)\hat{\eta}^{error}_m(\rho, \cdot)\|_{L^\infty(\mathbb{R})}\, d\rho.$$

On the other hand, the sum of the second and third terms on the right-hand side of (3.95) is bounded by

$$\frac{M}{(\rho_0 - \rho_*)}\int_{\rho_*}^{\rho_0} \big|k'(\rho)^2 - \frac{\kappa_2}{\rho^2}\big|(\rho - \rho_*)\rho\, d\rho,$$

where we made use of the bound on $\hat{\eta}^{iso}_{mu}$ provided by Lemma 3.17 (*cf.* Lemma 2.5). Thus, appealing to Corollary 3.10 to control the $k'(\rho)^2$ term, the sum of the second and third terms on the right-hand side of (3.95) is bounded by

$$\frac{M}{\rho_0-\rho_*}\int_{\rho_*}^{\rho_0}\rho^{-\alpha}\,d\rho\le M\rho_0^{-\alpha},$$

provided $\alpha\in(0,1)$, where we noted that there is no singularity where $\rho_0=\rho_*$. Note that if $\alpha\ge 1$, then this term would be bounded by $M\rho_0^{-1}k(\rho_0)$. In summary, we get

$$\begin{aligned}\|\rho_0k'(\rho_0)\hat{\eta}^{error}_m(\rho_0,\cdot)\|_{L^\infty(\mathbb{R})}\le& M\max\{\rho_0^{-1}k(\rho_0),\rho_0^{-\alpha}\}\\&+\frac{1}{(\rho_0-\rho_*)}\int_{\rho_*}^{\rho_0}d(\rho)\|\rho k'(\rho)\hat{\eta}^{error}_m(\rho,\cdot)\|_{L^\infty(\mathbb{R})}\,d\rho,\end{aligned}$$

from which an application of Grönwall's lemma yields

$$\|\rho_0k'(\rho_0)\hat{\eta}^{error}_m(\rho_0,\cdot)\|_{L^\infty(\mathbb{R})}\le M\max\{\rho_0^{-1}k(\rho_0),\rho_0^{-\alpha}\}\exp\Big(\frac{1}{\rho_0-\rho_*}\int_{\rho_*}^{\rho_0}d(\rho)\,d\rho\Big).$$

Using the bound on $d(\rho)$ provided by Lemma 3.12 concludes the proof. □

Armed with the previous result, we are in a position to prove the required estimate on the fake mixed derivative $\hat{\eta}^{error}_{m\rho}$, contained in the next lemma.

Lemma 3.45. *There exists a positive constant* $M=M(\alpha,\kappa_2,C_p,\rho_*,k(\rho_*))$ *such that*

$$\|\hat{\eta}^{error}_{m\rho}(\rho,\cdot)\|_{L^\infty(\mathbb{R})}\le M\max\{\rho^{-2}k(\rho)^2,\rho^{-1-\alpha\tilde{\lambda}}k(\rho)\}\qquad \textit{for } \rho\ge\rho_*. \tag{3.97}$$

Proof. We begin by differentiating (3.95) with respect to ρ_0. We get

$$\begin{aligned}&2(\rho_0-\rho_*)\rho_0k'(\rho_0)\hat{\eta}^{error}_{m\rho}(\rho_0,u_0)+\partial_{\rho_0}\big(2(\rho_0-\rho_*)\rho_0k'(\rho_0)\big)\hat{\eta}^{error}_m(\rho_0,u_0)=\\&\quad k'(\rho_0)\int_{\rho_*}^{\rho_0}d(\rho)\rho k'(\rho)\big\{\hat{\eta}^{error}_{mu}(\rho,u_0+k(\rho_0)-k(\rho))-\hat{\eta}^{error}_{mu}(\rho,u_0-k(\rho_0)+k(\rho))\big\}\,d\rho\\&+2d(\rho_0)\rho_0k'(\rho_0)\hat{\eta}^{error}_m(\rho_0,u_0)\\&+k'(\rho_0)\int_{\rho_*}^{\rho_0}(\rho-\rho_*)\rho\big(k'(\rho)^2-\frac{\kappa_2}{\rho^2}\big)\hat{\eta}^{iso}_{muu}(\rho,u_0+k(\rho_0)-k(\rho))\,d\rho\\&+k'(\rho_0)\int_{\rho_*}^{\rho_0}(\rho-\rho_*)\rho\big(k'(\rho)^2-\frac{\kappa_2}{\rho^2}\big)\hat{\eta}^{iso}_{muu}(\rho,u_0-k(\rho_0)+k(\rho))\,d\rho.\end{aligned}\tag{3.98}$$

The four terms on the right-hand side will be denoted by $\tilde{I}_j$, for $j=1,\dots,4$. Observe firstly that, using Lemma 3.9 and Corollary 3.10,

$$\begin{aligned}\big|\partial_{\rho_0}\big(2(\rho_0-\rho_*)\rho_0k'(\rho_0)\big)\big|&=\big|\rho_0k'(\rho_0)+(\rho_0-\rho_*)k'(\rho_0)+(\rho_0-\rho_*)\rho_0k''(\rho_0)\big|\\&\le M,\end{aligned}$$

for some positive $M = M(\alpha, \kappa_2, C_p, \rho_*, k(\rho_*))$. Thus, appealing to Lemma 3.44, , the second term on the left-hand side of (3.98) is bounded by $M \max\{\rho_0^{-1} k(\rho_0), \rho_0^{-\alpha}\}$. We now turn to the four terms on the right-hand side of (3.98). Firstly, if $\alpha\tilde{\lambda} \in (0,1)$,

$$|\tilde{I}_1| \leq M k'(\rho_0) \int_{\rho_*}^{\rho_0} \sqrt{p'(\rho)} \rho^{-\alpha\tilde{\lambda}} k(\rho)\, d\rho \leq M k'(\rho_0) k(\rho_0) \int_{\rho_*}^{\rho_0} \rho^{-\alpha\tilde{\lambda}}\, d\rho,$$

where we made use of (3.61) for the bound on $\hat{\eta}^{error}_{mu}$, $k'(\rho) > 0$ when $\rho > 0$, and that $\rho k'(\rho) = \sqrt{p'(\rho)}$. Evaluating the integral, we find, using also (3.17),

$$|\tilde{I}_1| \leq M k'(\rho_0) \rho_0^{1-\alpha\tilde{\lambda}} k(\rho_0) \leq M \rho_0^{-\alpha\tilde{\lambda}} k(\rho_0).$$

On the other hand, if $\alpha\tilde{\lambda} \geq 1$, using (3.61), Lemma 3.9, and Corollary 3.11,

$$|\tilde{I}_1| \leq M k'(\rho_0) \int_{\rho_*}^{\rho_0} \rho^{-1} k(\rho)^2\, d\rho \leq M \rho_0^{-1} k(\rho_0)^2.$$

Meanwhile, using (3.17) and $\rho k'(\rho) = \sqrt{p'(\rho)}$, and Lemma 3.44, we get $|\tilde{I}_2| \leq M \max\{\rho_0^{-1} k(\rho_0), \rho_0^{-\alpha}\}$. Finally,

$$|\tilde{I}_3| + |\tilde{I}_4| \leq M k'(\rho_0) \int_{\rho_*}^{\rho_0} (\rho - \rho_*) \rho \Big| k'(\rho)^2 - \frac{\kappa_2}{\rho^2} \Big|\, d\rho,$$

where we made use of the fact that $|\hat{\eta}^{iso}_{uuu}(\rho, u)| \leq M\rho$, and that $\hat{\eta}^{iso}_{muu}(\rho, u) = \rho^{-1} \hat{\eta}^{iso}_{uuu}(\rho, u)$. Using the bound obtained in (3.19), we therefore get

$$|\tilde{I}_3| + |\tilde{I}_4| \leq M k'(\rho_0) \int_{\rho_*}^{\rho_0} \rho^{-\alpha}\, d\rho.$$

Evaluating the integral and using the estimate provided by (3.17), we see that

$$|\tilde{I}_3| + |\tilde{I}_4| \leq M k'(\rho_0) \rho_0^{1-\alpha} \leq M \rho_0^{-\alpha},$$

provided $\alpha \in (0,1)$. If $\alpha \geq 1$, then $|\tilde{I}_3| + |\tilde{I}_4| \leq M \rho_0^{-1} k(\rho_0)$. Thus,

$$|(\rho_0 - \rho_*) \rho_0 k'(\rho_0) \hat{\eta}^{error}_{m\rho}(\rho_0, u_0)| \leq M \max\{\rho_0^{-1} k(\rho_0)^2, \rho_0^{-\alpha\tilde{\lambda}} k(\rho_0)\}.$$

Noting that no singularity was in fact present when $\rho_0 = \rho_*$, we therefore deduce the result. □

Having established this strong an estimate on the fake mixed derivative $\hat{\eta}_{m\rho}$, we prove a stronger estimate on $\hat{\eta}^{error}$.

Corollary 3.46. *There exists a positive constant $M = M(\alpha, \lambda, \kappa_2, C_p, \rho_*, k(\rho_*))$ such that*

$$|\hat{\eta}^{error}(\rho, u)| \leq M\rho|u| \qquad \text{for } (\rho, u) \in [\rho_*, \infty) \times \mathbb{R}. \tag{3.99}$$

Proof. Firstly, recall from the definition of $\hat{\eta}^{error}$ that

$$\hat{\eta}^{error}(\rho, 0) = \frac{1}{2}\int_{\mathbb{R}} \chi^{error}(\rho, 0-s)s|s|\,ds = 0 \qquad \text{for } \rho \geq \rho_*, \tag{3.100}$$

since the integrand is odd, as $\chi^{error}(\rho, \cdot)$ is even by Lemma 2.7. We also have

$$\hat{\eta}^{error}_m(\rho_*, u) = \rho_*^{-1}\frac{\partial}{\partial u}\hat{\eta}^{error}(\rho_*, u) = 0 \qquad \text{for all } u \in \mathbb{R}. \tag{3.101}$$

Additionally, since $k(\rho) \leq M(1 + \log(\rho/\rho_*))$ by Corollary 3.11, Lemma A.8 shows that there exists $M = M(\alpha, \tilde{\lambda}, \rho_*, k(\rho_*))$ such that

$$\rho^{-\alpha\tilde{\lambda}/2}k(\rho) + \rho^{-1/2}k(\rho)^2 \leq M \qquad \text{for } \rho \geq \rho_*.$$

The fundamental theorem of calculus and (3.101) show that

$$\hat{\eta}^{error}_m(\rho, u) - \hat{\eta}^{error}_m(\rho_*, u) = \int_{\rho_*}^{\rho} \frac{\partial}{\partial\rho}\bigg|_u \hat{\eta}^{error}_m(y, u)\,dy = \int_{\rho_*}^{\rho} \hat{\eta}^{error}_{m\rho}(y, u)\,dy,$$

where the final integrand is the fake mixed derivative (*cf.* Definition 2.27). Therefore, integrating in ρ and using Lemma 3.45 yields

$$\begin{aligned}
|\rho^{-1}\hat{\eta}^{error}_u(\rho, u)| = |\hat{\eta}^{error}_m(\rho, u)| &\leq \int_{\rho_*}^{\rho} |\hat{\eta}^{error}_{m\rho}(y, u)|\,dy \\
&\leq M\int_{\rho_*}^{\rho} \max\{y^{-2}k(y)^2, y^{-1-\alpha\tilde{\lambda}}k(y)\}\,dy \\
&\leq M\int_{\rho_*}^{\rho} \max\{y^{-3/2}, y^{-1-\alpha\tilde{\lambda}/2}\}\,dy,
\end{aligned}$$

so that $|\hat{\eta}^{error}_u(\rho, u)| \leq M\rho$ for all $\rho \geq \rho_*$. Integrating the above in u, using the fact that $\hat{\eta}^{error}(\rho, 0) = 0$ from (3.100), yields the result. ☐

Using this result, we are able to bound $\hat{q}^{error}$ in absolute value.

Corollary 3.47. *There exists a positive constant* $M = M(\alpha, \lambda, \kappa_2, C_p, \rho_*, k(\rho_*))$ *such that*

$$|\hat{q}^{error}(\rho, u)| \leq M\big(\rho|u|^2 + \rho + \rho\sqrt{\log(\rho/\rho_*)}\big) \qquad \text{for } (\rho, u) \in [\rho_*, \infty) \times \mathbb{R}. \tag{3.102}$$

Proof. Given the decomposition for $\hat{q}^{error}$ given in (3.82),

$$\begin{aligned}
|\hat{q}^{error}(\rho, u)| &\leq |u\hat{\eta}^{error}(\rho, u)| + |\hat{H}^{error}(\rho, u)| \\
&\leq M\big(\rho|u|^2 + \rho + \rho\sqrt{\log(\rho/\rho_*)}\big),
\end{aligned}$$

where we used the results of Corollary 3.46, Lemma 3.43, and the Cauchy–Schwarz inequality. ☐

We now state the lower bound on the entropy-flux, as promised earlier.

Corollary 3.48. *There exists a positive constant $M = M(\alpha, \lambda, \kappa_2, C_p, \rho_*, k(\rho_*))$ such that*

$$\hat{q}(\rho, u) \geq M^{-1}\rho|u|^3 - M\big(\rho|u|^2 + \rho + \rho(\log(\rho/\rho_*))^4\big) \qquad \text{for } (\rho, u) \in [\rho_*, \infty) \times \mathbb{R}. \tag{3.103}$$

Proof. Recall that, as we saw in (3.84), we may write $\hat{q}(\rho, u) = \hat{q}^{iso}(\rho, u) + \hat{q}^{error}(\rho, u)$, where $\hat{q}^{iso}$ satisfies the lower bound (3.71). The result follows from Corollary 3.47. □

3.4.1.4 Results regarding the relative internal energy

Lemma 3.49. *There exists a positive constant $M = M(\gamma, \kappa_1, \rho_*, k(\rho_*), \bar{\rho})$ such that*

$$\rho + \rho\log(\rho/\rho_*) \leq M\big(1 + e^*(\rho, \bar{\rho})\big) \qquad \text{for } \rho \geq \rho_*. \tag{3.104}$$

Proof. Recall from Lemma 2.37 that

$$e^*_{\rho\rho}(\rho, \bar{\rho}) = \frac{p'(\rho)}{\rho} = \frac{\kappa_2}{\rho} + \frac{1}{\rho}\left(p'(\rho) - \kappa_2\right) \qquad \text{for } \rho \geq \rho_*,$$

so that, using (3.17) and the value of ρ_*,

$$\frac{\kappa_2}{2}\rho^{-1} \leq e^*_{\rho\rho}(\rho, \bar{\rho}) \leq \frac{3\kappa_2}{2}\rho^{-1} \qquad \text{for } \rho \geq \rho_*.$$

Integrating in ρ twice yields

$$\begin{aligned}\frac{\kappa_2}{2}\big(\rho\log(\rho/\rho_*) - (\rho - \rho_*)\big) &\leq \big[e^*(\rho, \bar{\rho}) - (\rho - \rho_*)e^*_\rho(\rho_*, \bar{\rho}) - e^*(\rho_*, \bar{\rho})\big] \\ &\leq \frac{3\kappa_2}{2}\big(\rho\log(\rho/\rho_*) - (\rho - \rho_*)\big) \qquad \text{for } \rho \geq \rho_*.\end{aligned}$$

At this point, we recall that the reference function $\bar{\rho}$ was chosen to be smooth and constant outside of a compact set. Given that the functions $e(\rho)$ and $\rho e'(\rho)$ are continuous on $[0, \infty)$, there exists a positive constant $M = M(\bar{\rho}, \gamma, \kappa_1)$ such that

$$|e(\bar{\rho})| + |\bar{\rho}e'(\bar{\rho})| \leq M,$$

which implies that there exists a positive constant $M = M(\rho_*, \bar{\rho}, \gamma, \kappa_1)$ such that

$$|e^*_\rho(\rho_*, \bar{\rho})| + e^*(\rho_*, \bar{\rho}) \leq M.$$

Hence, $\frac{\kappa_2}{2}\big(\rho\log(\rho/\rho_*) - \rho\big) \leq e^*(\rho, \bar{\rho}) + M(\rho - \rho_*) + M$. Choosing M larger if necessary, we thereby obtain

$$\rho\log(\rho/\rho_*) \leq M\big(e^*(\rho, \bar{\rho}) + \rho + 1\big).$$

Additionally, since $\frac{\rho}{1+\rho\log(\rho/\rho_*)} \to 0$ as $\rho \to \infty$, there exists a positive $\tilde{R} = \tilde{R}(M, \rho_*) \geq \rho_*$ such that

$$0 < \frac{\rho}{1+\rho\log(\rho/\rho_*)} \leq \frac{1}{2M} \qquad \text{for all } \rho \geq \tilde{R}.$$

Hence, $\rho \leq (\tilde{R} + \frac{1}{2M}) + \frac{1}{2M}\rho\log(\rho/\rho_*)$ for all $\rho \geq \rho_*$. In view of this, there exists a positive constant $\tilde{M} = \tilde{M}(M, \gamma, \rho_*, \bar{\rho}, \kappa_1)$ such that

$$\rho\log(\rho/\rho_*) \leq \tilde{M}\big(e^*(\rho,\bar{\rho}) + 1\big) \qquad \text{for all } \rho \geq \rho_*.$$

Combining the above inequality with $\rho \leq (\tilde{R} + \frac{1}{2M}) + \frac{1}{2M}\rho\log(\rho/\rho_*)$ yields the result. □

Lemma 3.50. *There exists a positive constant $M = M(\gamma, \kappa_1, \rho_*, k(\rho_*), \bar{\rho})$, such that*

$$0 \leq \rho + p(\rho) \leq M\big(1 + e^*(\rho, \bar{\rho})\big) \qquad \textit{for } \rho \geq 0. \tag{3.105}$$

Proof. Consider the region $\rho \geq \rho_*$. In this case,

$$\rho + p(\rho) = (1+\kappa_2)\rho + \rho\Big(\frac{p(\rho)}{\rho} - \kappa_2\Big).$$

Note that, by (3.3),

$$\rho\left|\frac{p(\rho)}{\rho} - \kappa_2\right| \leq \rho C_p \rho_*^{-\alpha} \leq \frac{\kappa_2}{4}\rho.$$

Hence,

$$\rho \leq \rho + p(\rho) \leq \Big(1 + \frac{3}{2}\kappa_2\Big)\rho \qquad \text{for } \rho \geq \rho_*.$$

Hence, $\rho + p(\rho) \leq M\big(e^*(\rho,\bar{\rho}) + 1\big)$ for $\rho \geq \rho_*$. When $\rho \in [0, \rho_*]$, since $\rho + p(\rho)$ is a non-negative continuous function of ρ, we see that

$$\rho + p(\rho) \leq \sup_{\rho\in[0,\rho_*]} \big(\rho + p(\rho)\big),$$

and so the quantity is bounded by a constant over this complement region, which may be absorbed into M. This concludes the proof. □

Remark 3.51. Following the strategy in the proof of (2.12) (*cf.* Subsection 2.4.6) to the letter, while making use of Lemmas 3.49 and 3.50, we obtain

$$\rho\big|\hat{\eta}_m(\rho, 0) - \hat{\eta}_m(\bar{\rho}, 0)\big|^2 \leq M e^*(\rho, \bar{\rho}) \qquad \text{for all } \rho, \bar{\rho} \geq 0.$$

3.4.1.5 Compilation of results

To finish this section, we collect the results obtained in Sections 3.4.1.1, 3.4.1.2, and 3.4.1.3. Indeed, we have the following lemma, which is the analogue of Lemma 2.5 of Chapter 2.

Lemma 3.52. *Let $\hat{\psi}(s) = \frac{1}{2}s|s|$, and $(\hat{\eta}, \hat{q})$ its associated entropy pair via* (1.13) *and* (1.15)*. There exists a positive constant $M = M(\alpha, r, \kappa_2, C_p, \rho_*, k(\rho_*))$ such that*

$$\begin{aligned} &|\hat{\eta}(\rho, m)| \le M\eta^*(\rho, m), \qquad \hat{q}(\rho, \rho u) \ge M^{-1}\rho|u|^3 - M\big(\rho|u|^2 + \rho + \rho(\log(\rho/\rho_*))^4\big), \\ &|\hat{\eta}_m(\rho, \rho u)| \le M\big(|u| + \sqrt{\log(\rho/\rho_*)}\big), \qquad |\rho\hat{\eta}_{mm}(\rho, m)| \le M, \end{aligned} \tag{3.106}$$

for all $\rho \ge \rho_*$. Considering $\hat{\eta}_m(\rho, m)$ as a function of ρ and u, the fake mixed derivatives (cf. Definition 2.27) may be bounded by

$$|\hat{\eta}_{mu}(\rho, \rho u)| \le M\frac{1}{1 + \sqrt{\log(\rho/\rho_*)}}, \qquad |\hat{\eta}_{m\rho}(\rho, \rho u)| \le M\rho^{-1}. \tag{3.107}$$

Moreover, on the complement region $\rho \le \rho_*$, we have

$$\begin{aligned} &|\hat{\eta}(\rho, m)| \le M\eta^*(\rho, m), && \hat{q}(\rho, \rho u) \ge M^{-1}\big(\rho|u|^3 + \rho^{\gamma+\theta}\big) - M\big(\rho|u|^2 + \rho^\gamma\big), \\ &|\hat{\eta}_m(\rho, \rho u)| \le M\big(|u| + \rho^\theta\big), && |\rho\hat{\eta}_{mm}(\rho, m)| \le M, \\ &|\hat{\eta}_{mu}(\rho, \rho u)| \le M, && |\hat{\eta}_{m\rho}(\rho, \rho u)| \le M\rho^{\theta-1}, \end{aligned} \tag{3.108}$$

where in the final line we consider $\hat{\eta}_m(\rho, m)$ as a function of ρ and u. Finally,

$$\rho\big|\hat{\eta}_m(\rho, 0) - \hat{\eta}_m(\bar{\rho}, 0)\big|^2 \le Me^*(\rho, \bar{\rho}) \qquad \text{for } \rho, \bar{\rho} \ge 0. \tag{3.109}$$

Following the proofs of Lemmas 2.47-2.51 of Chapter 2 to the letter, we obtain the following result.

Lemma 3.53 (Uniform estimates)**.** *Assume that $\{(\rho_0^\varepsilon, u_0^\varepsilon)\}_{\varepsilon>0}$ is an admissible sequence of initial data, in the precise sense of Definition 2.55, and let $\{(\rho^\varepsilon, u^\varepsilon)\}_{\varepsilon>0}$ be the sequence of viscous solutions of* (1.25) *that it generates. Then,*

- *For any $T > 0$, there exists a positive constant M depending on $E_0, T, \bar{\rho}, \bar{u}$, but independent of ε, such that*

$$\sup_{t\in[0,T]} E[\rho^\varepsilon, u^\varepsilon](t) + \int_0^T \int_{\mathbb{R}} \varepsilon|u_x^\varepsilon|^2 \, dx \, dt \le M. \tag{3.110}$$

- *For any $T > 0$, there exists a positive constant M depending on $E_0, E_1, \bar{\rho}, \bar{u}, T$, but independent of ε, such that*

$$\varepsilon^2 \int_{\mathbb{R}} \frac{|\rho^\varepsilon(T, x)|^2}{\rho^\varepsilon(T, x)^3} \, dx + \varepsilon \int_0^T \int_{\mathbb{R}} \frac{p'(\rho^\varepsilon)}{(\rho^\varepsilon)^2}|\rho_x^\varepsilon|^2 \, dx \, dt \le M. \tag{3.111}$$

- *Let $K \subset \mathbb{R}$ be compact. Then, for any $T > 0$, there exists a positive constant $M = M(E_0, K, \bar{\rho}, \bar{u}, T)$, independent of ε, such that*

$$\int_0^T \int_K \rho^\varepsilon p(\rho^\varepsilon)\,dx\,dt \leq M. \tag{3.112}$$

- *Let $K \subset \mathbb{R}$ be compact. Then, for any $T > 0$, there exists a positive constant M, depending on K but not on ε, such that*

$$\int_0^T \int_K \rho^\varepsilon |u^\varepsilon|^3\,dx\,dt \leq M. \tag{3.113}$$

3.4.2 Entropies generated by compactly supported test functions

We now consider entropies generated by functions $\psi \in C_c^2(\mathbb{R})$. Recall from (3.48) that

$$\begin{aligned}\eta^\psi(\rho, u) &= \int_{\mathbb{R}} \chi^{iso}(\rho, u-s)\psi(s)\,ds + \int_{\mathbb{R}} \chi^{error}(\rho, u-s)\psi(s)\,ds \\ &= \eta^{iso,\psi}(\rho, u) + \eta^{error,\psi}(\rho, u) \qquad \text{for } \rho \geq \rho_*.\end{aligned}$$

In view of Lemma 3.23, we can bound the error as shown below,

$$\begin{aligned}|\eta^{error,\psi}(\rho, u)| &\leq M \max\{k(\rho), \rho^{1-\alpha\tilde{\lambda}}\} \int_{\mathbb{R}} |\psi(s)|ds \\ &\leq M \max\{k(\rho), \rho^{1-\alpha\tilde{\lambda}}\}.\end{aligned} \tag{3.114}$$

Notice also that, if $\operatorname{supp}\psi \subset [a, b]$, then, in view of (3.35),

$$\eta^{error}(\rho, u) = \int_{\mathbb{R}} \chi^{error}(\rho, s)\psi(u-s)\,ds = \int_{-\max\{k(\rho), k_*(\rho)\}}^{\max\{k(\rho), k_*(\rho)\}} \chi^{error}(\rho, s)\psi(u-s)\,ds,$$

and the right-hand side vanishes outside of the interval $[a - \max\{k(\rho), k_*(\rho)\}, b + \max\{k(\rho), k_*(\rho)\}]$. Additionally, making the change of variables $u \mapsto u + s$ then differentiating in u and returning to the original coordinates, we get

$$\eta_u^{error,\psi}(\rho, u) = \int_{\mathbb{R}} \chi^{error}(\rho, u-s)\psi'(s)\,ds.$$

Taking further derivatives and arguing as in (3.114), it easily follows from Lemma 3.23 that

$$|\eta_m^{error,\psi}(\rho, u)| + |\eta_{mu}^{error,\psi}(\rho, u)| + |\rho\eta_{mm}^{error,\psi}(\rho, u)| \leq M \max\{\rho^{-1}k(\rho), \rho^{-\alpha\tilde{\lambda}}\},$$

where $\eta_{mu}^{error,\psi}$ is the fake mixed derivative. Hence, we have proved the following lemma.

Lemma 3.54. *Let $\psi \in C_c^2(\mathbb{R})$ be such that* $\operatorname{supp}\psi \subset [a,b]$. *Then,*

$$\operatorname{supp}\eta^{error,\psi}(\rho,\cdot) \subset [a - \max\{k(\rho), k_*(\rho)\}, b + \max\{k(\rho), k_*(\rho)\}]. \tag{3.115}$$

Also, there exists a positive constant $M_\psi = M_\psi(\alpha, \lambda, \kappa_2, C_p, \rho_*, k(\rho_*))$ such that

$$\|\eta^{error,\psi}(\rho,\cdot)\|_{L^\infty(\mathbb{R})} + \|\rho\eta_m^{error,\psi}(\rho,\cdot)\|_{L^\infty(\mathbb{R})} \leq M_\psi \max\{k(\rho), \rho^{1-\alpha\tilde{\lambda}}\} \qquad \text{for } \rho \geq \rho_*, \tag{3.116}$$

and

$$\|\eta_{mu}^{error,\psi}(\rho,\cdot)\|_{L^\infty(\mathbb{R})} + \|\rho\eta_{mm}^{error,\psi}(\rho,\cdot)\|_{L^\infty(\mathbb{R})} \leq M_\psi \max\{\rho^{-1}k(\rho), \rho^{-\alpha\tilde{\lambda}}\} \qquad \text{for } \rho \geq \rho_*. \tag{3.117}$$

Corollary 3.55. *Let $\psi \in C_c^2(\mathbb{R})$. Then, there exists a positive constant $M_\psi = M_\psi(\alpha, \lambda, \kappa_2, C_p, \rho_*, k(\rho_*))$ such that*

$$|u\eta^{error,\psi}(\rho,u)| \leq M_\psi \max\{k(\rho), \rho^{1-\alpha\tilde{\lambda}}\}k(\rho) \qquad \text{for } (\rho,u) \in [\rho_*,\infty)\times\mathbb{R}. \tag{3.118}$$

Proof. The result now follows easily from Lemma 3.54, using the compact support of $\eta^{error,\psi}$ and the estimate (3.116), along with the bound on $\max\{k(\rho), k_*(\rho)\}$ provided by Lemma 3.19. □

On the other hand, Lemma 2.6 showed that

$$|\eta^{iso,\psi}(\rho,u)| + |\eta_u^{iso,\psi}(\rho,u)| \leq \frac{M_\psi \rho}{\sqrt{\log(\rho/\rho_* + 1)}} \qquad \text{for } \rho \geq \rho_*.$$

Thus, adding the contributions from $\eta^{iso,\psi}$ and $\eta^{error,\psi}$,

$$\|\eta^{\psi}(\rho,\cdot)\|_{L^\infty(\mathbb{R})} + \|\eta_u^{\psi}(\rho,\cdot)\|_{L^\infty(\mathbb{R})} \leq \frac{M_\psi \rho}{\sqrt{k(\rho)}} \qquad \text{for } \rho \geq \rho_*. \tag{3.119}$$

We now consider the fake mixed derivative $\eta_{m\rho}^{error}$, for which the following result holds.

Lemma 3.56. *Let $\psi \in C_c^2(\mathbb{R})$ and η^ψ be generated by* (1.15). *Then, there exists a positive constant $M_\psi = M_\psi(\alpha, \lambda, \kappa_2, C_p, \rho_*, k(\rho_*))$ such that*

$$\|\eta_{m\rho}^{\psi}(\rho,\cdot)\|_{L^\infty(\mathbb{R})} \leq M_\psi \frac{\sqrt{p'(\rho)}}{\rho} \qquad \text{for } \rho \geq \rho_*. \tag{3.120}$$

Proof. We first use the representation formula (3.37) to write down

$$\begin{aligned}
\eta_m^{\psi}(\rho_0,u_0) =& \frac{1}{2\rho_0(\rho_0-\rho_*)k'(\rho_0)} \int_{\rho_*}^{\rho_0} k'(\rho)d(\rho)\eta_u^{\psi}(\rho, u_0 + k(\rho_0) - k(\rho))\,d\rho \\
&+ \frac{1}{2\rho_0(\rho_0-\rho_*)k'(\rho_0)} \int_{\rho_*}^{\rho_0} k'(\rho)d(\rho)\eta_u^{\psi}(\rho, u_0 - k(\rho_0) + k(\rho))\,d\rho \\
&- \frac{1}{2\rho_0(\rho_0-\rho_*)k'(\rho_0)} \int_{-(k(\rho_0)-k(\rho_*))}^{k(\rho_0)-k(\rho_*)} \eta_u^{\psi}(\rho_*, u_0 - y)\,dy, \\
=:& I + II + III.
\end{aligned}$$

where η_u^ψ is given by $\eta_u^\psi(\rho,u) = \int_{\mathbb{R}} \chi(\rho,u-s)\psi'(s)\,ds$. Hence, the fake mixed derivative $\eta_{m\rho}^\psi$ is given by

$$\eta_{m\rho}^\psi(\rho_0,u_0) = \partial_{\rho_0} I + \partial_{\rho_0} II + \partial_{\rho_0} III.$$

We begin by controlling the term $\partial_{\rho_0} I$,

$$\begin{aligned}
\partial_{\rho_0} I =& \partial_{\rho_0}\left(\frac{1}{2\rho_0(\rho_0-\rho_*)k'(\rho_0)}\right)\int_{\rho_*}^{\rho_0} k'(\rho)d(\rho)\eta_u^\psi(\rho,u_0+k(\rho_0)-k(\rho))\,d\rho \\
&+ \frac{1}{2\rho_0(\rho_0-\rho_*)}d(\rho_0)\eta_u^\psi(\rho_0,u_0) \\
&+ \frac{1}{2\rho_0(\rho_0-\rho_*)}\int_{\rho_*}^{\rho_0} k'(\rho)d(\rho)\eta_{uu}^\psi(\rho,u_0+k(\rho_0)-k(\rho))\,d\rho, \\
=:& \tilde{I}_1 + \tilde{I}_2 + \tilde{I}_3,
\end{aligned}$$

where we write $\tilde{I}_j$ $(j=1,2,3)$ to avoid confusion with Bessel functions. Likewise, one expands $\partial_{\rho_0} II = II_1 + II_2 + II_3$. To begin with, using Lemma 2.6, the last term in the expansion of $\partial_{\rho_0} I$ and $\partial_{\rho_0} II$ are controlled as

$$|\tilde{I}_3| + |II_3| \leq \frac{M}{\rho_0(\rho_0-\rho_*)}\int_{\rho_*}^{\rho_0} d\rho = \frac{M}{\rho_0}.$$

Meanwhile, we expand the prefactor in the big brackets,

$$\begin{aligned}
\partial_{\rho_0}\left(\frac{1}{2\rho_0(\rho_0-\rho_*)k'(\rho_0)}\right) = -\frac{1}{2\rho_0^2(\rho_0-\rho_*)k'(\rho_0)} &- \frac{1}{\rho_0(\rho_0-\rho_*)^2k'(\rho_0)} \\
&- \frac{k''(\rho_0)}{2\rho_0(\rho_0-\rho_*)k'(\rho_0)^2},
\end{aligned}$$

and expand the final term,

$$\begin{aligned}
\partial_{\rho_0} III =& -\partial_{\rho_0}\left(\frac{1}{2\rho_0(\rho_0-\rho_*)k'(\rho_0)}\right)\int_{-(k(\rho_0)-k(\rho_*))}^{k(\rho_0)-k(\rho_*)} \eta_u^\psi(\rho_*,u_0-y)\,dy \\
&- \frac{1}{2\rho_0(\rho_0-\rho_*)}\big\{\eta_u^\psi(\rho_*,u_0-k(\rho_0)+k(\rho_*)) + \eta_u^\psi(\rho_*,u_0+k(\rho_0)-k(\rho_*))\big\}.
\end{aligned}$$

Now, using Lemma 3.9, $\left|\partial_{\rho_0}\left(\frac{1}{2\rho_0(\rho_0-\rho_*)k'(\rho_0)}\right)\right| \leq \frac{M}{(\rho_0-\rho_*)^2}$. Thus, using Lemmas 2.6 and 3.54,

$$|\tilde{I}_1| + |II_1| \leq \frac{M}{(\rho_0-\rho_*)^2}\int_{\rho_*}^{\rho_0} d\rho = \frac{M}{\rho_0-\rho_*}.$$

For $\tilde{I}_2$ and II_2, we have $\left|\frac{1}{2\rho_0(\rho_0-\rho_*)}d(\rho_0)\eta_u^\psi(\rho_0,u_0)\right| \leq \frac{M}{\rho_0-\rho_*}$. Similarly, one may check, using (3.119), that

$$|\partial_{\rho_0} III| \leq \frac{M\rho_0}{(\rho_0-\rho_*)^2}(k(\rho_0)-k(\rho_*)) + \frac{M\rho_0}{\rho_0(\rho_0-\rho_*)} \leq \frac{M}{\rho_0-\rho_*}.$$

As such, the terms $\partial_{\rho_0} I$, $\partial_{\rho_0} II$, and $\partial_{\rho_0} III$ are each bounded by $\frac{M}{\rho_0-\rho_*}$. One can also verify by direct calculation that $\lim_{\rho_0\to\rho_*} |\partial_{\rho_0} I + \partial_{\rho_0} II + \partial_{\rho_0} III|$ is finite. Hence, we deduce

$$|\eta^\psi_{m\rho}(\rho, m)| \le \frac{M}{1+(\rho_0-\rho_*)}.$$

Meanwhile, using Lemma 3.9, we have $\sqrt{p'(\rho)}\rho^{-1} \ge \sqrt{\frac{\kappa_2}{2}}\rho^{-1}$. The result follows at once. ☐

In fact, the same procedure can be followed so as to obtain the same estimates for $\eta^{iso,\psi}_{m\rho}$, from which we see that $|\eta^{error,\psi}_{m\rho}(\rho_0, u_0)| \le M\rho_0^{-1}$. In view of this, we have the following corollary.

Corollary 3.57. *There exists a positive constant* $M_\psi = M_\psi(\alpha, \lambda, \kappa_2, C_p, \rho_*, k(\rho_*))$ *such that*

$$\|\eta^{iso,\psi}_{m\rho}(\rho,\cdot)\|_{L^\infty(\mathbb{R})} + \|\eta^{error,\psi}_{m\rho}(\rho,\cdot)\|_{L^\infty(\mathbb{R})} \le M_\psi \frac{\sqrt{p'(\rho)}}{\rho} \qquad \textit{for } \rho \ge \rho_*. \tag{3.121}$$

We have yet to inspect the entropy-flux. To this end, we recall the equation for $H^{error,\psi}$, obtained by convolving (3.83) with $\psi \in C^2_c(\mathbb{R})$,

$$\begin{cases} H^{error,\psi}_{\rho\rho} - k'(\rho)^2 H^{error,\psi}_{uu} = f^\psi(\rho, u) \quad \text{for } (\rho, u) \in (\rho_*, \infty)\times\mathbb{R}, \\ H^{error,\psi}(\rho_*, u) = 0, \\ H^{error,\psi}_\rho(\rho_*, u) = 0, \end{cases} \tag{3.122}$$

where

$$f^\psi(\rho, u) := \big(k'(\rho)^2 - \frac{1}{\rho^2}\big) H^{iso,\psi}_{uu}(\rho, u) + \frac{p''(\rho)}{\rho}\eta^\psi_u(\rho, u) \qquad \text{for } (\rho, u) \in [\rho_*, \infty)\times\mathbb{R}. \tag{3.123}$$

Arguing as we did in Lemma 3.39, we thereby obtain the following result.

Lemma 3.58. *For any* $(\rho_0, u_0) \in \mathcal{K}$ *and any* $0 < \rho_* < \rho_0$*, we have*

$$\begin{aligned} &2(\rho_0-\rho_*)k'(\rho_0)H^{error,\psi}(\rho_0, u_0) = \\ &\quad\int_{\rho_*}^{\rho_0} d(\rho)k'(\rho)\big\{H^{error,\psi}(\rho, u_0 + k(\rho_0) - k(\rho)) + H^{error,\psi}(\rho, u_0 - k(\rho_0) + k(\rho))\big\}\, d\rho \\ &+ \int_{\rho_*}^{\rho_0} (\rho-\rho_*)\big(k'(\rho)^2 - \frac{\kappa_2}{\rho^2}\big) H^{iso,\psi}_u(\rho, u_0 + k(\rho_0) - k(\rho))\, d\rho \\ &- \int_{\rho_*}^{\rho_0} (\rho-\rho_*)\big(k'(\rho)^2 - \frac{\kappa_2}{\rho^2}\big) H^{iso,\psi}_u(\rho, u_0 - k(\rho_0) + k(\rho))\, d\rho \\ &+ \int_{\rho_*}^{\rho_0} \frac{p''(\rho)}{\rho}(\rho-\rho_*)\big\{\eta^\psi(\rho, u_0 + k(\rho_0) - k(\rho)) - \eta^\psi(\rho, u_0 - k(\rho_0) + k(\rho))\big\}\, d\rho. \end{aligned} \tag{3.124}$$

We are now in a position to bound $H^{error,\psi}$, and we proceed using a Grönwall argument on the representation formula (3.124). This is contained in the next lemma.

Lemma 3.59. *There exists a positive constant $M_\psi = M_\psi(\alpha, \lambda, \kappa_2, C_p, \rho_*, k(\rho_*))$ such that*

$$\|H^{error,\psi}(\rho,\cdot)\|_{L^\infty(\mathbb{R})} \le M_\psi \max\{k(\rho)^2, \rho^{1-\alpha}k(\rho)\} \quad \textit{for } (\rho,u) \in [\rho_*,\infty)\times\mathbb{R}. \quad (3.125)$$

Proof. We begin by dividing (3.124) by $2(\rho_0 - \rho_*)$. Then, the first term on the right-hand side of (3.124) is bounded by

$$\frac{1}{\rho_0 - \rho_*}\int_{\rho_*}^{\rho_0} d(\rho)\|k'(\rho)H^{error,\psi}(\rho,\cdot)\|_{L^\infty(\mathbb{R})}\,d\rho.$$

We now consider the sum of the second and third terms on the right-hand side of (3.124), which can be rewritten as

$$\frac{1}{2(\rho_0-\rho_*)}\int_{\rho_*}^{\rho_0}(\rho-\rho_*)\big(k'(\rho)^2 - \frac{\kappa_2}{\rho^2}\big)\left(\int_{u_0-k(\rho_0)+k(\rho)}^{u_0+k(\rho_0)-k(\rho)} H_{uu}^{iso,\psi}(\rho,y)\,dy\right)d\rho.$$

Now, recall that

$$H^{iso,\psi}(\rho,u) = \int_{-k_*(\rho)}^{k_*(\rho)} h^{iso}(\rho,s)\psi(u-s)\,ds,$$

where we made use of the compact support of $h^{iso}(\rho,\cdot)$. An application of Lebesgue's dominated convergence theorem shows that we may take derivatives under the integral, from which it follows that

$$H_{uu}^{iso,\psi}(\rho,u) = \int_{-k_*(\rho)}^{k_*(\rho)} h^{iso}(\rho,s)\psi''(u-s)\,ds = \int_{u-k_*(\rho)}^{u+k_*(\rho)} h^{iso}(\rho,u-s)\psi''(s)\,ds.$$

Further, since ψ is itself compactly supported, we find, using the bound on h^{iso} provided by Lemma 3.34,

$$|H_{uu}^{iso,\psi}(\rho,u)| \le M\rho\left(\int_{\mathbb{R}}|\psi''(s)|\,ds\right),$$

and therefore $|H_{uu}^{iso,\psi}(\rho,u)| \le M_\psi\rho$. Hence,

$$\left|\int_{u_0-k(\rho_0)+k(\rho)}^{u_0+k(\rho_0)-k(\rho)} H_{uu}^{iso,\psi}(\rho,y)\,dy\right| \le M_\psi\rho k(\rho_0).$$

In light of this, along with the bound on $k'(\rho)^2 - \kappa_2/\rho^2$ provided by (3.19) in Corollary 3.10, the sum of the second and third terms on the right-hand side of (3.124) is bounded by

$$\frac{M_\psi k(\rho_0)}{(\rho_0-\rho_*)}\int_{\rho_*}^{\rho_0}\rho^{-\alpha}\,d\rho \le M_\psi k(\rho_0)\frac{\rho_0^{1-\alpha}-\rho_*^{1-\alpha}}{1+(\rho_0-\rho_*)} \le M_\psi k(\rho_0)\rho_0^{-\alpha},$$

provided $\alpha \in (0,1)$, where the second inequality holds since there is in fact no singularity when $\rho_0 = \rho_*$. If, on the other hand, $\alpha \geq 1$, this term would be bounded by $M_\psi k(\rho_0)^2 \rho_0^{-1}$. The final term on the right-hand side of (3.124) can be written as

$$\frac{1}{2(\rho_0 - \rho_*)} \int_{\rho_*}^{\rho_0} p''(\rho)(\rho - \rho_*) \left(\int_{u_0 - k(\rho_0) + k(\rho)}^{u_0 + k(\rho_0) - k(\rho)} \eta_m^\psi(\rho, y)\, dy \right) d\rho,$$

where we used the fact that $\rho^{-1}\eta_u^\psi(\rho, u) = \eta_m^\psi(\rho, u)$. We already know from adding the contributions of $\eta_m^{\psi,iso}$ and $\eta_m^{\psi,error}$ that

$$|\eta_m^\psi(\rho, u)| \leq M_\psi.$$

In turn, using Lemma 3.9, we find that the final term in (3.124) is bounded by

$$\frac{M_\psi k(\rho_0)}{(\rho_0 - \rho_*)} \int_{\rho_*}^{\rho_0} \rho^{-\alpha}\, d\rho \leq M_\psi k(\rho_0) \frac{\rho_0^{1-\alpha} - \rho_*^{1-\alpha}}{1 + (\rho_0 - \rho_*)} \leq M_\psi k(\rho_0) \rho_0^{-\alpha},$$

provided $\alpha \in (0,1)$, where once again, the second inequality holds since there is no singularity at $\rho_0 = \rho_*$. If we had $\alpha \geq 1$, then this term would be bounded by $M_\psi k(\rho_0)^2 \rho_0^{-1}$. Collecting our previous results, we therefore have

$$\begin{aligned} \|k'(\rho_0) H^{error,\psi}(\rho_0, \cdot)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\rho_0 - \rho_*} \int_{\rho_*}^{\rho_0} d(\rho) \|k'(\rho) H^{error,\psi}(\rho, \cdot)\|_{L^\infty(\mathbb{R})}\, d\rho \\ + M_\psi k(\rho_0) \max\{\rho_0^{-1} k(\rho_0), \rho_0^{-\alpha}\}. \end{aligned}$$

Grönwall's lemma then yields, using the fact that $d(\rho)$ is bounded according to (3.24),

$$\|k'(\rho_0) H^{error,\psi}(\rho_0, \cdot)\|_{L^\infty(\mathbb{R})} \leq M_\psi k(\rho_0) \max\{\rho_0^{-1} k(\rho_0), \rho_0^{-\alpha}\}.$$

The result now follows easily, using the estimates of Lemma 3.9 and (3.14). □

Adding the contributions from $\eta^{iso,\psi}$ and $\eta^{error,\psi}$, we arrive at the following lemma, which is the analogue of Lemma 2.6.

Lemma 3.60. *Let $\psi \in C_c^2(\mathbb{R})$ and (η^ψ, q^ψ) the associated entropy pair via* (1.13) *and* (1.15)*. There exists a positive constant $M_\psi = M_\psi(\alpha, \lambda, \kappa_2, C_p, \rho_*, k(\rho_*))$ such that*

$$|\eta^\psi(\rho, m)| \leq M_\psi \frac{\rho}{\sqrt{\log(\rho/\rho_* + 1)}}, \quad |q^\psi(\rho, m)| \leq M_\psi \rho \quad \text{for } (\rho, m) \in \mathbb{R}_+^2. \tag{3.126}$$

Also, with $\eta_{mu}^\psi(\rho, \rho u) = \partial_u \eta_m^\psi(\rho, \rho u)$ the fake mixed derivative,

$$|\eta_m^\psi(\rho, m)| + |\eta_{mu}^\psi(\rho, m)| + |\rho \eta_{mm}^\psi(\rho, m)| \leq M_\psi \frac{1}{\sqrt{\log(\rho/\rho_* + 1)}} \quad \text{for } (\rho, m) \in \mathbb{R}_+^2. \tag{3.127}$$

Finally, with $\eta_{m\rho}^\psi(\rho, \rho u)$ the fake mixed derivative,

$$|\eta_{m\rho}^\psi(\rho, m)| \leq M_\psi \frac{\sqrt{p'(\rho)}}{\rho} \qquad \text{for } (\rho, m) \in \mathbb{R}_+^2. \tag{3.128}$$

Following the proof of Lemma 2.61 of Chapter 2 to the letter, while appealing to Lemma 3.60, we obtain the following compactness result.

Lemma 3.61. *Let* $\psi \in C_c^2(\mathbb{R})$ *and* (η^ψ, q^ψ) *the associated entropy pair via* (1.13) *and* (1.15). *Additionally, let* $\{(\rho_0^\varepsilon, u_0^\varepsilon)\}_{\varepsilon>0}$ *be a sequence of admissible initial data, in the precise sense of Definition 2.55, and let* $\{(\rho^\varepsilon, u^\varepsilon)\}_{\varepsilon>0}$ *be the associated viscous solutions of* (1.25). *Then, we have that the entropy dissipation measures*

$$\eta^\psi(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)_t + q^\psi(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)_x \tag{3.129}$$

are confined to a compact subset of $W^{-1,\tilde{p}}_{loc}(\mathbb{R}^2_+)$ *for any* $\tilde{p} \in [1, 2)$.

3.5 Singularities of the entropy kernel

The main result of this section is Lemma 3.66, which is the analogue of Lemma 2.73 of Chapter 2 (*cf.* [80, Lemma 2.7]), and is concerned with the fractional derivatives of the kernels (*cf.* Definition 2.71). This result is indispensable for the proof of the reduction of the Young measure, which is contained in the next section. Prior to Lemma 3.66, we require global bounds on the vacuum expansion of the entropy kernel and the entropy-flux kernel (*cf.* Theorem 1.12, (1.20)-(1.21)), which are found in the following two lemmas.

Lemma 3.62. *Recall from Theorem 1.12 that the entropy kernel admits globally the expansion*

$$\chi(\rho, u) = a_\sharp(\rho) G_\lambda(\rho, u) + a_\flat(\rho) G_{\lambda+1}(\rho, u) + g_1(\rho, u) \qquad \text{for } (\rho, u) \in \mathbb{R}^2_+, \tag{3.130}$$

where $G_\lambda(\rho, u) = [k(\rho)^2 - u^2]_+^\lambda$, *and* $g_1(\rho, \cdot)$ *and its fractional derivative* $\partial_u^{\lambda+1} g_1(\rho, \cdot)$ *are Hölder continuous. There exists a constant* $M = M(\gamma, \alpha, \lambda, \kappa_2, C_p, \rho_*, k(\rho_*))$ *such that*

$$|a_\sharp(\rho)| + |a_\flat(\rho)| \le M\sqrt{\rho}\, k(\rho)^{-\lambda} \qquad \text{for } \rho \ge \rho_*, \tag{3.131}$$

and

$$\|\partial_u^{\lambda+1} g_1(\rho, \cdot)\|_{L^\infty(\mathbb{R})} \le M\sqrt{\rho}\, k(\rho)^2 \qquad \text{for } \rho \ge \rho_*. \tag{3.132}$$

Proof. Recall from [14, Proposition 2.1] that $a_\sharp(\rho) = \alpha_\sharp(\rho) k(\rho)^{-2\lambda-1}$, where

$$\alpha_\sharp(\rho) = M_\lambda k(\rho)^{\lambda+1} k'(\rho)^{-1/2}, \tag{3.133}$$

while $a_\flat(\rho) = \alpha_\flat(\rho) k(\rho)^{-2\lambda-3}$, with

$$\alpha_\flat(\rho) = -\frac{1}{4(\lambda+1)} k(\rho)^{\lambda+2} k'(\rho)^{-1/2} \int_0^\rho k(\tau)^{-(\lambda+1)} k'(\tau)^{-1/2} \alpha_\sharp''(\tau)\, d\tau. \tag{3.134}$$

Recall from [14, Lemma 3.1] that there exists a constant $M = M(\gamma, r)$ such that

$$\rho^{-1}|\alpha_\sharp(\rho)| + |\alpha_\sharp'(\rho)| + |\alpha_\sharp''(\rho)| + \rho|\alpha_\sharp^{(3)}(\rho)| \le M \qquad \text{for } \rho \in [0, r), \tag{3.135}$$

from which it follows by direct calculation that

$$\rho^{-2}|\alpha_\flat(\rho)| + \rho^{-1}|\alpha_\flat'(\rho)| + |\alpha_\flat''(\rho)| \le M \qquad \text{for } \rho \in [0, r). \tag{3.136}$$

We now concentrate on the large density region, where $\rho \ge \rho_*$. Note that since $k'(\rho) > 0$ in this region (by assumption of strict hyperbolicity), we have, for instance,

$$k(\rho)^{\lambda-1} = k(\rho_*)^{\lambda-1}(k(\rho)/k(\rho_*))^{\lambda-1} \le k(\rho_*)^{\lambda-1}(k(\rho)/k(\rho_*))^{\lambda+1} \le Mk(\rho)^{\lambda+1},$$

for some positive M depending on ρ_*. In view of this, calculating the derivatives explicitly and using the bounds on the derivatives of k provided by (3.21), we obtain

$$|\alpha_\sharp(\rho)| + \rho|\alpha_\sharp'(\rho)| + \rho^2|\alpha_\sharp''(\rho)| + \rho^3|\alpha_\sharp^{(3)}(\rho)| \le M\sqrt{\rho}k(\rho)^{\lambda+1} \qquad \text{for } \rho \ge \rho_*, \tag{3.137}$$

for some positive $M = M(\alpha, \kappa_2, C_p, \rho_*)$. From the bound on $\max\{k(\rho), k_*(\rho)\}$ supplied by Lemma 3.19, we thereby obtain the bound on $a_\sharp$ in (3.131). Note that, when $\rho \ge \rho_*$,

$$\begin{aligned}\left|\int_0^\rho k(\tau)^{-(\lambda+1)}k'(\tau)^{-1/2}\alpha_\sharp''(\tau)\,d\tau\right| &\le M\int_0^r \tau^{-\theta}\,d\tau + M\int_r^{\rho_*} d\tau + M\int_{\rho_*}^\rho \frac{d\tau}{\tau} \\ &\le M\left(1 + \log(\rho/\rho_*)\right) \le Mk(\rho),\end{aligned}$$

where we used the fact that $\theta \in (0,1)$ for $\gamma \in (1,3)$ along with Corollary 3.11, and that the middle interval only adds a bounded contribution, since the integrand is continuous away from the vacuum. Therefore, we get

$$|\alpha_\flat(\rho)| + \rho|\alpha_\flat'(\rho)| + \rho^2|\alpha_\flat''(\rho)| \le M\sqrt{\rho}k(\rho)^{\lambda+3} \qquad \text{for } \rho \ge \rho_*, \tag{3.138}$$

and therefore the bound on $a_\flat$ in (3.131) follows easily. For the remainder term, we know from (3.30) in [14, Proof of Theorems 2.1 and 2.2] that

$$\begin{cases} g_{1,\rho\rho} - k'(\rho)^2 g_{1,uu} = A(\rho)k(\rho)^{-1}f_{\lambda+1}\big(\dfrac{u}{k(\rho)}\big), \\ g_1(0,\cdot) = 0, \\ g_{1,\rho}(0,\cdot) = 0, \end{cases} \tag{3.139}$$

where $f_\lambda(y) = [1-y^2]_+^\lambda$ and

$$A(\rho) = -\left(\alpha_\flat''(\rho) + \frac{2\lambda+3}{2(\lambda+1)}\alpha_\sharp''(\rho)\right) \qquad \text{for } \rho \ge 0, \tag{3.140}$$

and we recall from (3.31) in [14, Proof of Theorems 2.1 and 2.2] that A is locally bounded, so that there exists $M = M(\gamma, \rho_*)$ such that $|A(\rho)| \leq M$ for $\rho \in [0, \rho_*)$. Arguing as in the proof of Lemma 3.26, we get the representation

$$\begin{aligned} &k'(\rho_0)g_1(\rho_0, u_0) = \\ &\qquad \frac{1}{2\rho_0}\int_0^{\rho_0} d(\rho)k'(\rho)\big\{g_1(\rho, u + k(\rho_0) - k(\rho)) + g_1(\rho, u - k(\rho_0) + k(\rho))\big\}\, d\rho \\ &\qquad + \frac{1}{2\rho_0}\int_0^{\rho_0} \rho A(\rho)k(\rho)^{-1}\Big(\int_{u_0-k(\rho_0)+k(\rho)}^{u_0+k(\rho_0)-k(\rho)} f_{\lambda+1}\big(\frac{s}{k(\rho)}\big)\, ds\Big)\, d\rho. \end{aligned}$$

We note that $\rho A(\rho)k(\rho)^{-1}$ is bounded on $[0, \rho_*)$, so that the second integral makes sense. So, taking derivatives in u_0, we get

$$\begin{aligned} &k'(\rho_0)\partial_u g_1(\rho_0, u_0) = \\ &\frac{1}{2\rho_0}\int_0^{\rho_0} d(\rho)k'(\rho)\big\{\partial_u g_1(\rho, u_0 + k(\rho_0) - k(\rho)) + \partial_u g_1(\rho, u_0 - k(\rho_0) + k(\rho))\big\}\, d\rho \\ &+ \frac{1}{2\rho_0}\int_0^{\rho_0} \rho A(\rho)k(\rho)^{-1}\big\{f_{\lambda+1}\big(\frac{u_0 + k(\rho_0) - k(\rho)}{k(\rho)}\big) - f_{\lambda+1}\big(\frac{u_0 - k(\rho_0) + k(\rho)}{k(\rho)}\big)\big\}\, d\rho. \end{aligned} \tag{3.141}$$

Recall the fractional derivative of Definition 2.71. By the fundamental theorem of calculus,

$$\partial_s^\lambda f_{\lambda+1}(s) - \partial_s^\lambda f_{\lambda+1}(-1) = \int_{-1}^s \partial_y^{\lambda+1} f_{\lambda+1}(y)\, dy. \tag{3.142}$$

Note that $\operatorname{supp} f_\lambda = [-1, 1]$ so that the second term on the left-hand side vanishes. Also, from [80, Section 2.2],

$$\partial_s^{\lambda+1} f_{\lambda+1}(s) = A_1^{\lambda+1}\big(H(s+1) + H(s-1)\big) + A_2^{\lambda+1}\big(\operatorname{Ci}(s+1) - \operatorname{Ci}(s-1)\big) + \tilde{r}^{\lambda+1}(s),$$

where $A_1^{\lambda+1}, A_2^{\lambda+1} \in \mathbb{C}$ and $\tilde{r}^{\lambda+1}$ is a compactly supported Hölder continuous function. Hence, substituting this equality into (3.142), we get

$$\begin{aligned} \partial_s^\lambda f_{\lambda+1}(s) =& A_1^{\lambda+1}\big([s+1]_+ + [s-1]_+\big) + \int_{-1}^s \tilde{r}^{\lambda+1}(y)\, dy \\ &+ A_2^{\lambda+1}\Big(\int_{-1}^s \operatorname{Ci}(y+1)\, dy - \int_{-1}^s \operatorname{Ci}(y-1)\, dy\Big). \end{aligned} \tag{3.143}$$

Given the terms in (3.143) and the support of f_λ, we see that $\partial_s^\lambda f_{\lambda+1}$ is a continuous function with compact support, and it is therefore uniformly bounded. In view of this, by applying the fractional derivative $\partial_{u_0}^\lambda$ to the representation formula (3.141), we get

$$\begin{aligned} \|k'(\rho_0)\partial_u^{\lambda+1} g_1(\rho_0, \cdot)\|_{L^\infty(\mathbb{R})} \leq& \frac{1}{\rho_0}\int_0^{\rho_0} d(\rho)\|k'(\rho)\partial_u^{\lambda+1} g_1(\rho, \cdot)\|_{L^\infty(\mathbb{R})}\, d\rho \\ &+ \frac{M}{\rho_0}\int_0^{\rho_0} \rho A(\rho)k(\rho)^{-(\lambda+1)}\, d\rho, \end{aligned} \tag{3.144}$$

where we have taken into account the homogeneity of the fractional derivative in the final term. We now bound this final term. Suppose that $\rho_0 \in (0, r)$, then

$$\int_0^{\rho_0} |\rho A(\rho) k(\rho)^{-(\lambda+1)}| \, d\rho \leq M \int_0^{\rho_0} \rho^{\frac{1}{2}-\frac{\theta}{2}} \, d\rho \leq M \rho_0^{\frac{3}{2}-\frac{\theta}{2}},$$

and since $\frac{3}{2} - \frac{\theta}{2} > \frac{1}{2}$, the right-hand side is dominated by $M\sqrt{\rho_0}$. If $\rho_0 \in [r, \rho_*)$, only a bounded contribution is added, since the integrand is continuous on this interval. Finally, if $\rho_0 \geq \rho_*$,

$$\begin{aligned}\int_0^{\rho_0} |\rho A(\rho) k(\rho)^{-(\lambda+1)}| \, d\rho &\leq M \int_0^{r} \rho^{\frac{1}{2}-\frac{\theta}{2}} \, d\rho + M \int_r^{\rho_*} d\rho + M \int_{\rho_*}^{\rho_0} k(\rho)^2 \rho^{-1/2} \, d\rho \\ &\leq M\big(1 + \sqrt{\rho_0} k(\rho_0)^2\big).\end{aligned}$$

Thus, applying Grönwall's lemma to (3.144), we obtain

$$\|k'(\rho_0)\partial_u^{\lambda+1} g_1(\rho_0, \cdot)\|_{L^\infty(\mathbb{R})} \leq M \rho_0^{\frac{1}{2}-\frac{\theta}{2}} \qquad \text{for } \rho_0 \in [0, r), \tag{3.145}$$

while

$$\|k'(\rho_0)\partial_u^{\lambda+1} g_1(\rho_0, \cdot)\|_{L^\infty(\mathbb{R})} \leq \frac{M k(\rho_0)^2}{\sqrt{\rho_0}} \qquad \text{for } \rho_0 \geq \rho_*,$$

from which the result is easily deduced using (3.19). □

Remark 3.63. Using the fact that $\|f_\lambda\|_{L^\infty} = 1$, we get directly from (3.141) that

$$\begin{aligned}\|k'(\rho_0)\partial_u g_1(\rho_0, \cdot)\|_{L^\infty(\mathbb{R})} \leq & \frac{1}{\rho_0} \int_0^{\rho_0} d(\rho) \|k'(\rho)\partial_u g_1(\rho, \cdot)\|_{L^\infty(\mathbb{R})} \, d\rho \\ & + \frac{M}{\rho_0} \int_0^{\rho_0} \rho A(\rho) k(\rho)^{-1} \, d\rho.\end{aligned}$$

Therefore, applying Grönwall's lemma and arguing as before gives

$$\|k'(\rho_0)\partial_u g_1(\rho_0, \cdot)\|_{L^\infty(\mathbb{R})} \leq M \rho_0^{1-\theta} \qquad \text{for } \rho_0 \in [0, \rho_*),$$

while, for $\rho \geq \rho_*$,

$$\begin{aligned}\left|\int_0^{\rho_0} \rho A(\rho) k(\rho)^{-1} \, d\rho\right| &\leq M \int_0^{\rho_*} \rho^{1-\theta} \, d\rho + M \int_r^{\rho_*} d\rho + M \int_{\rho_*}^{\rho_0} k(\rho)^{\lambda+2} \rho^{-1/2} \, d\rho \\ &\leq M\big(1 + \sqrt{\rho_0} k(\rho)^{\lambda+2}\big).\end{aligned}$$

In view of the previous remark, we easily obtain the following result.

Corollary 3.64. *There exists a positive constant $M = M(\alpha, \kappa_2, C_p, \rho_*)$ such that*

$$\|\partial_u g_1(\rho, \cdot)\|_{L^\infty(\mathbb{R})} \leq M\big(\rho^{2-2\theta} \mathbb{1}_{\rho<\rho_*} + \sqrt{\rho} k(\rho)^{\lambda+2} \mathbb{1}_{\rho\geq\rho_*}\big) \qquad \textit{for } \rho \geq 0. \tag{3.146}$$

We also have the equivalent results for the entropy-flux kernel.

Lemma 3.65. *Recall from Theorem 1.12 that $h(\rho, u-s) = \sigma(\rho,u,s) - u\chi(\rho,u-s)$ admits globally the expansion*

$$h(\rho,u) = -u\big(b_\sharp(\rho)G_\lambda(\rho,u) + b_\flat(\rho)G_{\lambda+1}(\rho,u)\big) + g_2(\rho,u) \qquad \text{for } (\rho,u)\in\mathbb{R}^2_+, \tag{3.147}$$

where $G_\lambda(\rho,u) = [k(\rho)^2-u^2]^\lambda_+$, and $g_2(\rho,\cdot)$ and its fractional derivative $\partial_u^{\lambda+1}g_2(\rho,\cdot)$ are Hölder continuous. There exists a constant $M = M(\gamma,\alpha,\lambda,\kappa_2,C_p,\rho_*,k(\rho_*))$ such that

$$|b_\sharp(\rho)| + |b_\flat(\rho)| \le M\sqrt{\rho}k(\rho)^{-\lambda-1} \qquad \text{for } \rho \ge \rho_*, \tag{3.148}$$

and

$$\|\partial_u^{\lambda+1}g_2(\rho,\cdot)\|_{L^\infty(\mathbb{R})} \le M\sqrt{\rho}k(\rho)^2 \qquad \text{for } \rho\ge\rho_*. \tag{3.149}$$

Proof. Recall from [14, Theorem 2.3] that

$$b_\sharp(\rho) = M_\lambda \rho k(\rho)^{-\lambda-1}k'(\rho)^{1/2}, \tag{3.150}$$

while

$$\begin{aligned} b_\flat(\rho) =& -\frac{1}{4(\lambda+1)}k(\rho)^{-(\lambda+2)}k'(\rho)^{-1/2}\int_0^\rho k(\tau)^{\lambda+1}k'(\tau)^{-1/2}b_\sharp''(\tau)\,d\tau \\ &+\frac{1}{4(\lambda+1)}k(\rho)^{-(\lambda+2)}k'(\rho)^{-1/2}\int_0^\rho \tau k(\tau)^{-(\lambda+1)}k'(\tau)^{1/2}\alpha_\sharp''(\tau)\,d\tau. \end{aligned} \tag{3.151}$$

Using (3.2), we find that there exists a positive $M = M(\gamma, r)$

$$|b_\sharp(\rho)| + |b_\sharp'(\rho)| + \rho|b_\sharp''(\rho)| + \rho^2|b_\sharp^{(3)}(\rho)| \le M \qquad \text{for } \rho\in[0,r). \tag{3.152}$$

Additionally, for $\rho\in[0,r)$,

$$\left|\int_0^\rho k(\tau)^{\lambda+1}k'(\tau)^{-1/2}b_\sharp''(\tau)\,d\tau\right| + \left|\int_0^\rho \tau k(\tau)^{-(\lambda+1)}k'(\tau)^{1/2}\alpha_\sharp''(\tau)\,d\tau\right| \le M\int_0^\rho d\rho,$$

which shows that the two integrals are bounded by $M\rho$ for $\rho\in[0,r)$. Using this estimate we get, by explicitly computing the derivatives of $b_\flat$,

$$|b_\flat(\rho)| + \rho|b_\flat'(\rho)| + \rho^2|b_\flat''(\rho)| \le M\rho^{1-2\theta} \qquad \text{for } \rho\in[0,r), \tag{3.153}$$

for some constant $M = M(\gamma,r)$. On the other hand, for the high density region, explicit calculation and (3.3) show that

$$|b_\sharp(\rho)| + \rho|b_\sharp'(\rho)| + \rho^2|b_\sharp''(\rho)| + \rho^3|b_\sharp^{(3)}(\rho)| \le M\sqrt{\rho}k(\rho)^{-\lambda-1} \qquad \text{for } \rho\ge\rho_*. \tag{3.154}$$

Also, for $\rho \geq \rho_*$,

$$\left| \int_0^\rho k(\tau)^{\lambda+1} k'(\tau)^{-1/2} b_\sharp''(\tau)\, d\tau \right| + \left| \int_0^\rho \tau k(\tau)^{-(\lambda+1)} k'(\tau)^{1/2} \alpha_\sharp''(\tau)\, d\tau \right|$$
$$\leq M \int_0^r d\rho + M \int_r^{\rho_*} d\rho + M \int_{\rho_*}^\rho \frac{d\tau}{\tau} \leq M(1 + \log(\rho/\rho_*)) \leq M k(\rho),$$

where we used the fact that the middle interval only adds a bounded contribution, since the integrand is continuous away from the vacuum, and Corollary 3.11. In view of this, by explicitly computing derivatives of $b_\flat$, one can use the previous bound to show that

$$|b_\flat(\rho)| + \rho |b_\flat'(\rho)| + \rho^2 |b_\flat''(\rho)| \leq M \sqrt{\rho} k(\rho)^{-\lambda-1} \qquad \text{for } \rho \geq \rho_*. \tag{3.155}$$

It remains to bound the remainder term, which, for $(\rho, u) \in \mathbb{R}^2_+$, satisfies the equation

$$\begin{cases} g_{2,\rho\rho} - k'(\rho)^2 g_{2,uu} = u b_\flat''(\rho) k(\rho)^{2\lambda+2} f_{\lambda+1}\big(\frac{u}{k(\rho)}\big) + \dfrac{p''(\rho)}{\rho} g_{1,u}(\rho, u), \\ g_2(0, \cdot) = 0, \\ g_{2,\rho}(0, \cdot) = 0, \end{cases}$$

where we recall that $f_\lambda(y) = [1 - y^2]_+^\lambda$. Arguing as in the proof of Lemma 3.26, we get the representation

$$\begin{aligned} &k'(\rho_0) g_2(\rho_0, u_0) = \\ &\quad \frac{1}{2\rho_0} \int_0^{\rho_0} d(\rho) k'(\rho) \big\{ g_2(\rho, u + k(\rho_0) - k(\rho)) + g_2(\rho, u - k(\rho_0) + k(\rho)) \big\}\, d\rho \\ &\quad + \frac{1}{2\rho_0} \int_0^{\rho_0} \rho b_\flat''(\rho) k(\rho)^{2\lambda+2} \bigg(\int_{u_0 - k(\rho_0) + k(\rho)}^{u_0 + k(\rho_0) - k(\rho)} s f_{\lambda+1}\big(\frac{s}{k(\rho)}\big)\, ds \bigg)\, d\rho \\ &\quad + \frac{1}{2\rho_0} \int_0^{\rho_0} p''(\rho) \big\{ g_1(\rho, u_0 + k(\rho_0) - k(\rho)) - g_1(\rho, u_0 - k(\rho_0) + k(\rho)) \big\}\, d\rho. \end{aligned}$$

Taking a derivative with respect to u_0 then yields

$$\begin{aligned} &k'(\rho_0) \partial_u g_2(\rho_0, u_0) = \\ &\quad \frac{1}{2\rho_0} \int_0^{\rho_0} d(\rho) k'(\rho) \big\{ \partial_u g_2(\rho, u + k(\rho_0) - k(\rho)) + \partial_u g_2(\rho, u - k(\rho_0) + k(\rho)) \big\}\, d\rho \\ &\quad + \frac{1}{2\rho_0} \int_0^{\rho_0} \rho b_\flat''(\rho) k(\rho)^{2\lambda+3} \big(\frac{u_0 + k(\rho_0) - k(\rho)}{k(\rho)}\big) f_{\lambda+1}\big(\frac{u_0 + k(\rho_0) - k(\rho)}{k(\rho)}\big)\, d\rho \\ &\quad - \frac{1}{2\rho_0} \int_0^{\rho_0} \rho b_\flat''(\rho) k(\rho)^{2\lambda+3} \big(\frac{u_0 - k(\rho_0) + k(\rho)}{k(\rho)}\big) f_{\lambda+1}\big(\frac{u_0 - k(\rho_0) + k(\rho)}{k(\rho)}\big)\, d\rho \\ &\quad + \frac{1}{2\rho_0} \int_0^{\rho_0} p''(\rho) \big\{ \partial_u g_1(\rho, u_0 + k(\rho_0) - k(\rho)) - \partial_u g_1(\rho, u_0 - k(\rho_0) + k(\rho)) \big\}\, d\rho, \end{aligned}$$

and we write $\tilde{f}(s) := s f_{\lambda+1}(s)$. We now take an additional fractional derivative, noting the homogeneity in the terms comprising $f_{\lambda+1}$, to obtain

$$
\begin{aligned}
&k'(\rho_0)\partial_u^{\lambda+1} g_2(\rho_0,u_0) = \\
&\quad \frac{1}{2\rho_0}\int_0^{\rho_0} d(\rho)k'(\rho)\big\{\partial_u^{\lambda+1} g_2(\rho,u+k(\rho_0)-k(\rho)) + \partial_u^{\lambda+1} g_2(\rho,u-k(\rho_0)+k(\rho))\big\}\,d\rho \\
&\quad +\frac{1}{2\rho_0}\int_0^{\rho_0} \rho b_\flat''(\rho)k(\rho)^{\lambda+3}\big\{\tilde{f}^{(\lambda)}\big(\frac{u_0+k(\rho_0)-k(\rho)}{k(\rho)}\big) - \tilde{f}^{(\lambda)}\big(\frac{u_0-k(\rho_0)+k(\rho)}{k(\rho)}\big)\big\}\,d\rho \\
&\quad +\frac{1}{2\rho_0}\int_0^{\rho_0} p''(\rho)\big\{\partial_u^{\lambda+1} g_1(\rho,u_0+k(\rho_0)-k(\rho)) - \partial_u^{\lambda+1} g_1(\rho,u_0-k(\rho_0)+k(\rho))\big\}\,d\rho,
\end{aligned}
$$

where $\tilde{f}^{(\lambda)}(s) = \partial_s^\lambda \tilde{f}(s)$. Using the chain rule along with (3.143), and exploiting the compact support of $f_{\lambda+1}$, we deduce that $\tilde{f}^{(\lambda)}$ is a compactly supported continuous function. It is thus uniformly bounded. Therefore,

$$
\begin{aligned}
\|k'(\rho_0)\partial_u^{\lambda+1} g_2(\rho_0,\cdot)\|_{L^\infty(\mathbb{R})} \leq & \frac{1}{\rho_0}\int_0^{\rho_0} d(\rho)\|k'(\rho)\partial_u^{\lambda+1} g_2(\rho,\cdot)\|_{L^\infty(\mathbb{R})}\,d\rho \\
&+\frac{M}{\rho_0}\int_0^{\rho_0} \rho k(\rho)^{\lambda+3}|b_\flat''(\rho)|\,d\rho \\
&+\frac{1}{\rho_0}\int_0^{\rho_0} |p''(\rho)|\cdot\|\partial_u^{\lambda+1} g_1(\rho,\cdot)\|_{L^\infty(\mathbb{R})}\,d\rho.
\end{aligned}
$$

Suppose $\rho_0 \in [0,r)$. Then, using (3.153),

$$
\frac{M}{\rho_0}\int_0^{\rho_0} \rho k(\rho)^{\lambda+3}|b_\flat''(\rho)|\,d\rho \leq \frac{M}{\rho_0}\int_0^{\rho_0} \rho^{\frac{1}{2}+\frac{\theta}{2}}\,d\rho \leq M\rho_0^{\frac{1}{2}+\frac{\theta}{2}},
$$

and, using (3.145),

$$
\frac{1}{\rho_0}\int_0^{\rho_0} |p''(\rho)|\cdot\|\partial_u^{\lambda+1} g_1(\rho,\cdot)\|_{L^\infty(\mathbb{R})}\,d\rho \leq \frac{M}{\rho_0}\int_0^{\rho_0} \rho^{\frac{1}{2}+\frac{3\theta}{2}}\,d\rho \leq M\rho_0^{\frac{1}{2}+\frac{3\theta}{2}}.
$$

On the other hand using (3.155), if $\rho_0 \geq \rho_*$, we get

$$
\begin{aligned}
\frac{M}{\rho_0}\int_0^{\rho_0} \rho k(\rho)^{\lambda+3}|b_\flat''(\rho)|\,d\rho &\leq M\bigg(1+\int_r^{\rho_*} d\rho\bigg) + \frac{M}{\rho_0}\int_{\rho_*}^{\rho_0} \frac{k(\rho)^2}{\sqrt{\rho}}\,d\rho \\
&\leq M\big(1+\rho_0^{-1/2}k(\rho_0)^2\big).
\end{aligned}
$$

and, using (3.132),

$$
\begin{aligned}
\frac{1}{\rho_0}\int_0^{\rho_0} |p''(\rho)|\cdot\|\partial_u^{\lambda+1} g_1(\rho,\cdot)\|_{L^\infty(\mathbb{R})}\,d\rho &\leq M\bigg(1+\int_r^{\rho_*} d\rho\bigg) + \frac{M}{\rho_0}\int_0^{\rho_0} \frac{\rho^{-\alpha}k(\rho)^2}{\sqrt{\rho}}\,d\rho \\
&\leq M\big(1+\rho_0^{-1/2}\big).
\end{aligned}
$$

Thus, applying Grönwall's lemma, we get

$$
\|k'(\rho_0)\partial_u^{\lambda+1} g_2(\rho_0,\cdot)\|_{L^\infty(\mathbb{R})} \leq \frac{Mk(\rho_0)^2}{\sqrt{\rho_0}} \qquad \text{for } \rho_0 \geq \rho_*,
$$

from which the result is easily deduced using (3.19). □

We now arrive at the main result of this section, which will be used in key steps of the Young measure reduction in Section 3.6.

Lemma 3.66. *The fractional derivatives* $\partial_u^{\lambda+1}\chi$ *and* $\partial_u^{\lambda+1}h$ *admit the expansions:*

$$\begin{aligned}&\partial_s^{\lambda+1}\chi(\rho,u-s)=\\&\quad\sum_{\pm}\Big(A_{1,\pm}(\rho)\delta(s-u\pm k(\rho))+A_{2,\pm}(\rho)H(s-u\pm k(\rho))\\&\qquad+A_{3,\pm}(\rho)\,\mathrm{PV}(s-u\pm k(\rho))+A_{4,\pm}(\rho)\,\mathrm{Ci}(s-u\pm k(\rho))\Big)+r_\chi(\rho,u-s),\end{aligned}\tag{3.156}$$

and

$$\begin{aligned}&\partial_s^{\lambda+1}h(\rho,u-s)=\\&\quad\sum_{\pm}(s-u)\Big(B_{1,\pm}(\rho)\delta(s-u\pm k(\rho))+B_{2,\pm}(\rho)H(s-u\pm k(\rho))\\&\qquad+B_{3,\pm}(\rho)\,\mathrm{PV}(s-u\pm k(\rho))+B_{4,\pm}(\rho)\,\mathrm{Ci}(s-u\pm k(\rho))\Big)\\&\quad+\sum_{\pm}\Big(B_{5,\pm}(\rho)H(s-u\pm k(\rho))+B_{6,\pm}(\rho)\,\mathrm{Ci}(s-u\pm k(\rho))\Big)+r_\sigma(\rho,u-s),\end{aligned}\tag{3.157}$$

where the remainder terms r_χ *and* r_σ *are Hölder continuous functions. Moreover, there exists a positive constant* $M=M(\alpha,\gamma,\kappa_2,C_p,\rho_*,k(\rho_*))$ *such that*

$$\sum_{j=1,\pm}^{4}|A_{j,\pm}(\rho)|+\sum_{j=1,\pm}^{6}|B_{j,\pm}(\rho)|\le M\sqrt{\rho}\big(1+\log(\rho/\rho_*)\big)\qquad \textit{for } \rho\ge\rho_*,\tag{3.158}$$

and

$$\|r_\chi(\rho,\cdot)\|_{L^\infty(\mathbb{R})}+\|r_\sigma(\rho,\cdot)\|_{L^\infty(\mathbb{R})}\le M\sqrt{\rho}\big(1+(\log(\rho/\rho_*))^2\big)\qquad \textit{for } \rho\ge\rho_*.\tag{3.159}$$

Proof. By taking fractional derivatives of (3.130), we have, as in [80, Section 2.2],

$$\partial_s^{\lambda+1}\chi(\rho,u-s)=a_\sharp(\rho)\partial_s^{\lambda+1}G_\lambda(\rho,u-s)+a_\flat(\rho)\partial_s^{\lambda+1}G_{\lambda+1}(\rho,u-s)+\partial_s^{\lambda+1}g_1(\rho,u-s),$$

where we know from [58, Proposition 3.4] that

$$\begin{aligned}\partial_s^{\lambda}G_\lambda(\rho,u-s)=&k(\rho)^\lambda A_1^\lambda\big(H(s-u+k(\rho))+H(s-u-k(\rho))\big)\\&+k(\rho)^\lambda A_2^\lambda\big(\mathrm{Ci}(s-u+k(\rho))-\mathrm{Ci}(s-u-k(\rho))\big)+k(\rho)^\lambda\tilde r\big(\frac{s-u}{k(\rho)}\big),\end{aligned}$$

and

$$\begin{aligned}\partial_s^{\lambda+1}G_\lambda(\rho,u-s) =& k(\rho)^\lambda\big[A_1^\lambda\big(\delta(s-u+k(\rho))+\delta(s-u-k(\rho))\big)\\ &\quad + A_2^\lambda\big(\operatorname{PV}(s-u+k(\rho))-\operatorname{PV}(s-u-k(\rho))\big)\big]\\ &+ k(\rho)^{\lambda-1}\big[A_3^\lambda\big(H(s-u+k(\rho))-H(s-u-k(\rho))\big)\\ &\quad + A_4^\lambda\big(\operatorname{Ci}(s-u+k(\rho))-\operatorname{Ci}(s-u-k(\rho))\big)\big]\\ &+ k(\rho)^{\lambda-1}\big(-A_4^\lambda(\log k(\rho))^2+\tilde{q}\big(\frac{s-u}{k(\rho)}\big)\big),\end{aligned}$$

where $A_i^\lambda\in\mathbb{C}$ for $i\in\{1,\dots,4\}$ are λ-dependent constants, and $\tilde{r}$ and $\tilde{q}$ are uniformly bounded Hölder continuous functions. We therefore deduce the form for $\partial_u^{\lambda+1}\chi$ given in (3.156), with

$$A_{1,\pm}(\rho)=a_\sharp(\rho)k(\rho)^\lambda A_1^\lambda,\qquad A_{2,\pm}(\rho)=\pm a_\sharp(\rho)k(\rho)^{\lambda-1}A_3^\lambda+a_\flat(\rho)k(\rho)^{\lambda+1}A_1^{\lambda+1},$$
$$A_{3,\pm}(\rho)=\pm a_\sharp(\rho)k(\rho)^\lambda A_2^\lambda,\qquad A_{4,\pm}(\rho)=\pm a_\sharp(\rho)k(\rho)^{\lambda-1}A_4^\lambda\pm a_\flat(\rho)k(\rho)^{\lambda+1}A_2^{\lambda+1},$$

and

$$\begin{aligned}r_\chi(\rho,u-s)=&\,a_\sharp(\rho)k(\rho)^{\lambda-1}\tilde{q}\big(\frac{s-u}{k(\rho)}\big)+a_\flat(\rho)k(\rho)^{\lambda+1}\tilde{r}\big(\frac{s-u}{k(\rho)}\big)\\ &-k(\rho)^{\lambda-1}A_4^\lambda(\log k(\rho))^2+\partial_s^{\lambda+1}g_1(\rho,u-s).\end{aligned}$$

The conclusion of the lemma now follows easily from Lemma 3.62, using the fact that $k(\rho)\le M(1+\log(\rho/\rho_*))$ for $\rho\ge\rho_*$, by Corollary 3.11. An identical procedure using the estimates of Lemma 3.65 yields the result for the entropy-flux kernel. □

3.6 The Young measure and proof of main result

Following the approach in Section 2.7 of Chapter 2 to the letter, we obtain results analogous to Lemmas 2.67 and 2.70. We thereby deduce the following lemma.

Lemma 3.67. *Recall the kernels χ and σ (cf. Definitions 1.10 and 1.11). Let $\{(\rho_0^\varepsilon,u_0^\varepsilon)\}_{\varepsilon>0}$ be an admissible sequence of initial data in the sense of Definition 2.55 and let $\{(\rho^\varepsilon,u^\varepsilon)\}_{\varepsilon>0}$ be the associated viscous solutions of* (1.25)*. Correspondingly, let $\nu_{(t,x)}$ be a Young measure generated by the family $\{(\rho^\varepsilon,u^\varepsilon)\}_{\varepsilon>0}$. Then,*

$$\overline{\chi(s_1)\sigma(s_2)-\chi(s_2)\sigma(s_1)}=\overline{\chi(s_1)}\,\overline{\sigma(s_2)}-\overline{\chi(s_2)}\,\overline{\sigma(s_1)},\tag{3.160}$$

for all $s_1,s_2\in\mathbb{R}$, where the notation $\overline{f}=\int f\,d\nu_{(t,x)}$ was explained in Remark 2.69.

In order to deduce that the support of the Young measure reduces to a point, we again prove two technical lemmas (*cf.* Lemmas 2.74 and 2.75 of Chapter 2). We begin with the following result, the content of which is contained in [58, Lemmas 3.8-3.9]. Throughout, we use the same notations as those introduced in Subsection 2.8.1.

Lemma 3.68 (Lemmas 3.8 and 3.9 of [58]). *Let $R \in C^{0,\alpha}_{loc}(\mathbb{R})$ for some $\alpha \in (0,1)$ be bounded, let $g \in C^{0,\alpha}_c(\mathbb{R})$, and take $L > 2$ such that* $\operatorname{supp} g \subset B_{L-2}(0)$.

1. *Consider any pair of distributions $T_2, T_3 \in \mathcal{D}'(\mathbb{R})$ from the following collection:*

$$(T_2, T_3) = (\delta, Q_3), \qquad (T_2, T_3) = (\mathrm{PV}, Q_3), \qquad (T_2, T_3) = (Q_2, Q_3),$$

where $Q_2, Q_3 \in \{H, \mathrm{Ci}, R\}$. Then there exists a constant $C > 0$ such that

$$\sup_{\tau \in (0,1)} \left| \int_{-\infty}^{\infty} g(s_1)\big(T_2(s_2 - u \pm k(\rho))T_3(s_3 - u \pm k(\rho))\big) * \phi_2^\tau * \phi_3^\tau(s_1)\, ds_1 \right| \\ \leq C\|g\|_{C^{0,\alpha}(\mathbb{R})}\big(1 + \|R\|_{C^{0,\alpha}(\overline{B_L(0)})}\big)^2.$$

2. *Consider now any pair of distributions from*

$$\begin{aligned} &(T_2, T_3) = (\delta, \delta), && (T_2, T_3) = (\mathrm{PV}, \mathrm{PV}), && (T_2, T_3) = (Q_2, Q_3), \\ &(T_2, T_3) = (\delta, \mathrm{PV}), && (T_2, T_3) = (\mathrm{PV}, Q_3), && (T_2, T_3) = (Q_2, Q_3), \end{aligned}$$

where $Q_2, Q_3 \in \{H, \mathrm{Ci}, R\}$. Then there exists a positive constant C such that

$$\sup_{\tau \in (0,1)} \left| \int_{-\infty}^{\infty} g(s_1)\big((s_2 - s_3)T_2(s_2 - u \pm k(\rho)) \cdot \right. \\ \left. T_3(s_3 - u \pm k(\rho))\big) * \phi_2^\tau * \phi_3^\tau(s_1)\, ds_1 \right| \leq C\|g\|_{C^{0,\alpha}(\mathbb{R})}\big(1 + \|R\|_{C^{0,\alpha}(\overline{B_L(0)})}\big)^2.$$

Now recall the cancellation of singularities proved in [14, Lemmas 4.2-4.3].

Lemma 3.69 (Lemmas 4.2 and 4.3 of [14]). *The mollified fractional derivatives of the entropy and entropy-flux kernels satisfy the following convergence properties:*

1. *On sets on which ρ is bounded,*

$$P_2\chi_2^\tau P_3\sigma_3^\tau - P_3\chi_3^\tau P_2\sigma_2^\tau \rightharpoonup Y(\phi_2, \phi_3)Z(\rho)\sum_{\pm}(K^\pm)^2\delta_{s_1 = u \pm k(\rho)} \tag{3.161}$$

as $\tau \to 0$ weakly-star in measures in s_1 and locally uniformly in (ρ, u), where

$$Y(\phi_2, \phi_3) = \int_{-\infty}^{\infty}\int_{-\infty}^{s_2} \phi_2(s_2)\phi_3(s_3) - \phi_2(s_3)\phi_3(s_2)\, ds_3\, ds_2$$

and $Z(\rho) = (\lambda + 1)M_\lambda^{-2}k(\rho)^{2\lambda}D(\rho)$, where $D(\rho)$ is as in Lemma 1.13.

2. *There exists a Hölder continuous function $X(\rho, u, s_1)$ such that, as $\tau \to 0$,*

$$\chi_1 P_j\sigma_j^\tau - P_j\chi_j^\tau\sigma_1 \to X(\rho, u, s_1) \qquad \text{for } j = 2, 3 \tag{3.162}$$

uniformly in (ρ, u, s_1) on sets on which ρ is bounded.

Armed with these two results, we prove the two technical lemmas required for the Young measure reduction framework. We follow [80, Lemmas 5.2-5.3] to the letter.

Lemma 3.70. *For any test function $\psi \in \mathcal{D}(\mathbb{R})$,*

$$\lim_{\tau\to 0}\int_{\mathbb{R}} \overline{\chi(s_1)}\,\overline{P_2\chi_2^\tau P_3\sigma_3^\tau - P_3\chi_3^\tau P_2\sigma_2^\tau}(s_1)\psi(s_1)\,ds_1$$
$$= \int_{\mathcal{H}} Y(\phi_2,\phi_3)Z(\rho)\sum_{\pm}(K^{\pm})^2\overline{\chi(u\pm k(\rho))}\psi(u\pm k(\rho))\,d\nu(\rho,u),$$

where $Z(\rho) = (\lambda+1)M_\lambda^{-2}k(\rho)^{2\lambda}D(\rho) > 0$ for $\rho > 0$, and $D(\rho)$ is as in Lemma 1.13.

Proof. Let $\psi \in \mathcal{D}(\mathbb{R})$. From Lemma 3.69, when ρ is bounded,

$$\lim_{\tau\to 0}\int_{-\infty}^{\infty} \overline{\chi(s_1)}(P_2\chi_2^\tau P_3\sigma_3^\tau - P_3\chi_3^\tau P_2\sigma_2^\tau)\psi(s_1)\,ds_1$$
$$= Y(\phi_2,\phi_3)Z(\rho)\sum_{\pm}(K^{\pm})^2\overline{\chi(u\pm k(\rho))}\psi(u\pm k(\rho))$$

locally uniformly in (ρ,u) and hence pointwise for all (ρ,u). Therefore, for any $\rho_* > 0$,

$$\lim_{\tau\to 0}\int_{-\infty}^{\infty} \overline{\chi(s_1)}\langle\nu,(P_2\chi_2^\tau P_3\sigma_3^\tau - P_3\chi_3^\tau P_2\sigma_2^\tau)\mathbb{1}_{\rho\le\rho_*}\rangle\psi(s_1)\,ds_1$$
$$= \lim_{\tau\to 0}\left\langle\nu,\int_{-\infty}^{\infty}\overline{\chi(s_1)}(P_2\chi_2^\tau P_3\sigma_3^\tau - P_3\chi_3^\tau P_2\sigma_2^\tau)\psi(s_1)\,ds_1\mathbb{1}_{\rho\le\rho_*}\right\rangle$$
$$= \left\langle\nu, Y(\phi_2,\phi_3)Z(\rho)\sum_{\pm}(K^{\pm})^2\overline{\chi(u\pm k(\rho))}\psi(u\pm k(\rho))\mathbb{1}_{\rho\le\rho_*}\right\rangle$$
$$= Y(\phi_2,\phi_3)\sum_{\pm}(K^{\pm})^2\langle\nu, Z(\rho)\overline{\chi(u\pm k(\rho))}\psi(u\pm k(\rho))\rangle.$$

It therefore suffices to show that

$$\left|\overline{\chi(s_1)}(P_2\chi_2^\tau P_3\sigma_3^\tau - P_3\chi_3^\tau P_2\sigma_2^\tau)\psi(s_1)\,ds_1\mathbb{1}_{\rho>\rho_*}\right| \le C(\rho^2+1),$$

for some $C > 0$ independent of ρ, u and τ. Since $\rho^2 \in L^1(\mathcal{H},\nu)$ (with $\mathcal{H}$ as in Subsection 2.7.1), an application of Lebesgue's dominated convergence theorem then allows us to pass the pointwise limit inside the Young measure. We also notice that

$$P_2\chi_2^\tau P_3\sigma_3^\tau - P_3\chi_3^\tau P_2\sigma_2^\tau = P_2\chi_2^\tau P_3(\sigma_3^\tau - u\chi_3^\tau) - P_3\chi_3^\tau P_2(\sigma_2^\tau - u\chi_2^\tau).$$

Using Lemma 3.66, we see that this product consists of a sum of terms of the form

$$A_{i,\pm(\rho)}B_{j,\pm}(\rho)T_2(s_2 - u \pm k(\rho))T_3(s_3 - u \pm k(\rho)),$$

where $T_2, T_3 \in \{\delta, \mathrm{PV}, H, \mathrm{Ci}\}$, or terms with the same structure but where $T_2 \in \{\delta, H, \mathrm{PV}, \mathrm{Ci}, r_\chi\}$ and $T_3 \in \{H, \mathrm{Ci}, r_\sigma\}$ and likewise with s_2 and s_3 reversed.

Applying Lemma 3.68 yields, for any pair $T_2, T_3 \in \{\delta, \mathrm{PV}, H, \mathrm{Ci}\}$,

$$\left| \int_{-\infty}^{\infty} \overline{\chi(s_1)} \psi(s_1) \big((s_2 - s_3) T_2(s_2 - u \pm k(\rho)) T_3(s_3 - u \pm k(\rho)) \big) * \phi_2^\tau * \phi_3^\tau * (s_1)\, ds_1 \right| \\ \leq C \|\overline{\chi}\psi\|_{C^{0,\alpha}(\mathbb{R})}, \tag{3.163}$$

where we note that $s \mapsto \overline{\chi(s)}$ is Hölder continuous (*cf.* [79, Lemma 5.5.2]). Likewise,

$$\left| \int_{-\infty}^{\infty} \overline{\chi(s_1)} \psi(s_1) \big(T_2(s_2 - u \pm k(\rho)) T_3(s_3 - u \pm k(\rho)) \big) * \phi_2^\tau * \phi_3^\tau * (s_1)\, ds_1 \right| \\ \leq C \|\overline{\chi}\psi\|_{C^{0,\alpha}(\mathbb{R})} \big(1 + \|r_\chi\|_{C^{0,\alpha}_{s_1}(\overline{B_R})} + \|r_\sigma\|_{C^{0,\alpha}_{s_1}(\overline{B_R})} \big),$$

for $T_2 \in \{\delta, H, \mathrm{PV}, \mathrm{Ci}, r_\chi\}$ and $T_3 \in \{H, \mathrm{Ci}, r_\sigma\}$. In this case, $R > 0$ is such that $\operatorname{supp} \psi \subset B_{R-2}(0)$ and the terms involving r_χ and r_σ occur only if one of $T_2, T_3 \in \{r_\chi, r_\sigma\}$. Hence, Lemma 3.66 gives

$$\left| \int_{-\infty}^{\infty} \overline{\chi(s_1)} (P_2 \chi_2^\tau P_3 \sigma_3^\tau - P_3 \chi_3^\tau P_2 \sigma_2^\tau) \psi(s_1)\, ds_1 \right| \\ \leq C \max_{j,k,\pm} \{ |A_{j,\pm} B_{k,\pm}|, |A_{j,\pm}| \cdot \|r_\chi\|_{C^{0,\alpha}_{s_1}(\overline{B_R})}, |B_{j,\pm}| \cdot \|r_\sigma\|_{C^{0,\alpha}_{s_1}(\overline{B_R})} \} \leq C(\rho^2 + 1).$$

□

Lemma 3.71. *For any test function $\psi \in \mathcal{D}(\mathbb{R})$,*

$$\lim_{\tau \to 0} \int_{\mathbb{R}} \overline{P_3 \chi_3^\tau}\, \overline{P_2 \chi_2^\tau \sigma_1 - \chi_1 P_2 \sigma_2^\tau} \psi(s_1)\, ds_1 = \lim_{\tau \to 0} \int_{\mathbb{R}} \overline{P_2 \chi_2^\tau}\, \overline{P_3 \chi_3^\tau \sigma_1 - \chi_1 P_3 \sigma_3^\tau} \psi(s_1)\, ds_1.$$

Proof. Let $\psi \in \mathcal{D}(\mathbb{R})$ and fix $(\rho, u) \in \mathbb{H}$. Then, Lemma 3.69 shows that

$$(\chi_1 P_3 \sigma_3^\tau - P_3 \chi_3^\tau \sigma_1)(\rho, u, s_1) \to X(\rho, u, s_1) \quad \text{uniformly in } s_1 \text{ as } \tau \to 0.$$

Hence, since

$$\int_{\mathbb{R}} \overline{P_2 \chi_2^\tau} (\chi_1 P_3 \sigma_3^\tau - P_3 \chi_3^\tau \sigma_1) \psi(s_1)\, ds_1 \\ = \int_{\mathcal{H}} \int_{\mathbb{R}} P_2 \chi_2^\tau(\tilde{\rho}, \tilde{u}, s_1) (\chi_1 P_3 \sigma_3^\tau - P_3 \chi_3^\tau \sigma_1)(\rho, u, s_1) \psi(s_1)\, ds_1\, d\nu(\tilde{\rho}, \tilde{u}),$$

we find that

$$\int_{\mathbb{R}} \overline{P_2 \chi_2^\tau} (\chi_1 P_3 \sigma_3^\tau - P_3 \chi_3^\tau \sigma_1) \psi(s_1)\, ds_1 \to \int_{\mathcal{H}} \langle P_1 \chi_1(\tilde{\rho}, \tilde{u}, \cdot), X(\rho, u, \cdot) \psi(\cdot) \rangle\, d\nu(\tilde{\rho}, \tilde{u})$$

pointwise in (ρ, u) as $\tau \to 0$.

Note that the inner product $\langle\cdot,\cdot\rangle$ is, in a slight abuse of notation, the duality pairing of distributions and continuous functions. To pass to the limit, we used that $\{P_j\chi_j^\tau\}_{j=2}^3$ are measures in s_1 such that $|P_j\chi_j^\tau(\rho,u,\cdot)|_{\mathcal{M},\alpha} \le C_\alpha\rho$ to pass the limit inside the Young measure, where $|\mu|_{\mathcal{M},\alpha} = \sup\{|\langle\mu,f\rangle| : f \in C^{0,\alpha}(\mathbb{R}) \text{ and } \|f\|_{C^{0,\alpha}(\mathbb{R})} \le 1\}$, for any $\alpha \in (0,1)$.

The rest of the proof requires the following.

Claim 3.72. *There exists a positive constant C independent of τ such that*

$$\left| \int_{\mathbb{R}} \overline{P_2\chi_2^\tau}(\chi_1 P_3\sigma_3^\tau - P_3\chi_3^\tau\sigma_1)\psi(s_1)\,ds_1 \right| \le C(\rho^2+1).$$

Assuming the validity of the previous claim, an application of Lebesgue's dominated convergence theorem yields

$$\begin{aligned}
&\lim_{\tau\to0}\int_{\mathbb{R}} \overline{P_2\chi_2^\tau(s_1)(\chi_1 P_3\sigma_3^\tau - P_3\chi_3^\tau\sigma_1)(s_1)}\psi(s_1)\,ds_1 \\
&\qquad = \lim_{\tau\to0}\int_{\mathcal{H}}\int_{\mathbb{R}} \overline{P_2\chi_2^\tau(s_1)}(\chi_1 P_3\sigma_3^\tau - P_3\chi_3^\tau\sigma_1)(\rho,u,s_1)\psi(s_1)\,ds_1\,d\nu(\rho,u) \\
&\qquad = \int_{\mathcal{H}}\int_{\mathcal{H}} \langle P_1\chi_1(\tilde\rho,\tilde u,\cdot), X(\rho,u,\cdot)\psi(\cdot)\rangle\,d\nu(\tilde\rho,\tilde u)\,d\nu(\rho,u).
\end{aligned}$$

As the limit is independent of the choice of mollifying functions ϕ_2^τ and ϕ_3^τ, we may interchange the roles of s_2 and s_3, which concludes the proof. □

Proof of Claim 3.72. The result is straightforwardly verified by following [80, Proof of Claim 5.6] to the letter, this time making use of the estimates provided by Lemma 3.66 instead of [80, Theorem 2.6 and Lemma 2.7]. □

Following the proof of Theorem 2.72 to the letter, while making use of Lemmas 3.70 and 3.71 instead of Lemmas 2.74 and 2.75, we obtain the following.

Theorem 3.73. *Recall the kernels χ and σ (cf. Definitions 1.10 and 1.11). Let $\nu \in \mathrm{Prob}(\mathcal{H})$ be a probability measure such that the function $(\rho,u) \mapsto \rho^2 \in L^1(\mathcal{H},\nu)$ and, for all $s_1, s_2 \in \mathbb{R}$,*

$$\overline{\chi(s_1)\sigma(s_2) - \chi(s_2)\sigma(s_1)} = \overline{\chi(s_1)}\,\overline{\sigma(s_2)} - \overline{\chi(s_2)}\,\overline{\sigma(s_1)}. \tag{3.164}$$

Then either ν is supported in V or the support of ν is a single point in $\mathbb{H}$.

Finally, we conclude this chapter with a proof of the main theorem.

Proof of Theorem 3.2. The proof of Theorem 3.2 follows the proof of Theorem 2.2 of Chapter 2 to the letter, with the exception that we make use of Lemma 3.53 for the uniform estimates, Lemma 3.67 for the Tartar–Murat commutation relation, and Theorem 3.73 for the Young measure reduction. □

Chapter 4

Entropy methods for steady transonic flow

4.1 Introduction

This chapter focuses on the entropy equation arising in the Morawetz problem, introduced in Sections 1.4 and 1.5, which appears when considering the equations of steady compressible planar potential flow. Our objective is to employ ideas from the general pressure law formulation of the Euler equations (*cf.* [14, 80] and Chapters 2 and 3) to generate entropies of the potential flow system (1.40) that are compact in H^{-1}. This is a first step towards proving the existence of an entropy solution of the Morawetz problem, subject to the polytropic pressure law $p(\rho) \propto \rho^{\gamma}$ for $\gamma \geq 3$. As mentioned in Section 1.5, the goal of this chapter is to prove Theorem 1.21.

The rest of the chapter is as follows. To start with, in Section 4.2, we consider an artificial viscous system for the polar formulation of (1.40), and describe the invariant regions of this viscous system. Specifically, we discuss how these differ from those in [20, Section 5] and how this affects the analysis of the entropy equation. We then introduce the entropy equation for our problem in Section 4.3, and study the asymptotic behaviour of the coefficient $(M^2 - 1)/\rho^2$ in the vicinity of the vacuum, where M is the Mach number defined in (1.38). Next, we prove the main dissipation estimate for the problem under consideration, in Section 4.4. Following on from this, we prove the existence of a large family of Lax entropies using techniques from the theory of general pressure laws [14, 15] (*cf.* Section 4.7), and prove the local H^{-1}-compactness of the entropy dissipation measures corresponding to these entropies, in Sections 4.5 and 4.6. Hence, we confirm the validity of Theorem 1.21.

4.2 Invariant regions in polar coordinates

As mentioned in Section 1.4, [20, Section 3] introduces an artificial viscous perturbation of the polar formulation (1.41) of the potential flow equations, i.e.,

$$A^\varepsilon \begin{pmatrix} q^\varepsilon \\ t^\varepsilon \end{pmatrix}_x + B^\varepsilon \begin{pmatrix} q^\varepsilon \\ t^\varepsilon \end{pmatrix}_y = \begin{pmatrix} -R_1^\varepsilon \\ \frac{1}{\rho(q^\varepsilon)q^\varepsilon} R_2^\varepsilon \end{pmatrix}, \tag{4.1}$$

where

$$A^\varepsilon = \begin{pmatrix} -\sin t^\varepsilon & -q^\varepsilon \cos t^\varepsilon \\ \frac{c^2-q^2}{c^2 q} \cos t^\varepsilon & -\sin t^\varepsilon \end{pmatrix}, \qquad B^\varepsilon = \begin{pmatrix} \cos t^\varepsilon & -q^\varepsilon \sin t^\varepsilon \\ \frac{c^2-q^2}{c^2 q} \sin t^\varepsilon & \cos t^\varepsilon \end{pmatrix}. \tag{4.2}$$

Note that the matrices A^ε and B^ε commute. It is therefore possible to compute their common eigenvectors, from which we deduce that the equations for the Riemann invariants of (1.41), which we denote by $W_\pm$, are

$$\frac{\partial W_\pm}{\partial t} = 1, \qquad \frac{\partial W_\pm}{\partial q} = \mp \frac{\sqrt{q^2 - c^2}}{qc}, \tag{4.3}$$

in the region $q > c$, the solution of which are given by [20, Theorem 5.1], i.e.,

$$W_\pm(t, q) = t \mp \left(W(q) - W(\sqrt{2} q_{cr}) \right), \tag{4.4}$$

where the critical speed q_{cr} is given by (1.39), and, as in [53, Section 117],

$$W(q) = \sqrt{\frac{\gamma+1}{\gamma-1}} \arcsin \sqrt{\frac{\gamma-1}{2} \left(\frac{q^2}{q_{cr}^2} - 1 \right)} - \arcsin \sqrt{\frac{\gamma+1}{2} \left(1 - \frac{q_{cr}^2}{q^2} \right)}. \tag{4.5}$$

Using the Riemann invariants we symmetrize (4.1), and rewrite it as two scalar elliptic equations. This is the core of the next lemma, the proof of which is by explicit computation (*cf.* [20, Proposition 4.1]).

Lemma 4.1 (Proposition 4.1 of [20]). *Assume that*

$$R_1^\varepsilon = \varepsilon \operatorname{div}(\sigma_1(\rho^\varepsilon) \nabla t^\varepsilon), \qquad R_2^\varepsilon = \varepsilon \operatorname{div}(\sigma_2(\rho^\varepsilon) \nabla \rho^\varepsilon),$$

where $\sigma_1(\rho) = 1$ and $\sigma_2(\rho) = 1 - c^2/q^2$ for $q > c$. Then, the Riemann invariants $W_\pm$ satisfy

$$\frac{qc}{\sqrt{q^2-c^2}} \left(\lambda_\pm \frac{\partial W_\pm}{\partial x} + \mu_\pm \frac{\partial W_\pm}{\partial y} \right) \mp \varepsilon \Delta W_\pm = -\varepsilon c \frac{(\gamma-3)q^2 + 4c^2}{2\rho^2 q^2 \sqrt{q^2-c^2}} |\nabla \rho|^2, \tag{4.6}$$

where, in a slight abuse of notation, the Riemann invariants are viewed as functions of (x, y), i.e., $W_\pm = W_\pm(t(x, y), q(x, y))$, and we have dropped the ε superscripts.

When $\gamma \geq 3$, the right-hand side of (4.6) is always non-positive; the opposite of the case $\gamma \in [1, 3)$ studied in [20, Section 4]. In turn, the roles of W_+ and W_- are interchanged, and solutions are now longer constrained to live inside the area trapped by the level sets of $W_\pm$; instead they must stay **outside** (*cf.* the apple-shaped regions in [20, Figures 1-5]). This therefore implies that the scalar speed q^ε is bounded from below by some q_*, and we may select boundary data such that $q_* > q_{cr}$. As such, we obtain

$$q^\varepsilon \geq q_*, \quad \rho(q^\varepsilon) \leq \rho(q_*) \quad \text{for all } \varepsilon > 0. \tag{4.7}$$

Remark 4.2. Since our solutions have speed bounded from below by a positive constant, there will be no stagnation points. Hence, if the domain Ω is as in Domain (b) (*cf.* Section 1.4), we will necessarily violate the Kutta condition; a fundamental criterion in aerodynamics, which states that there is a stagnation point both at the leading edge and at the trailing edge of any solid body immersed in a flow. For this reason, in this chapter, we only consider Ω as in Domain (a).

We summarise the findings of our previous observations in the following lemma, and the method of proof is identical to that of [20, Theorem 5.1].

Lemma 4.3. *Assume there exists $C^2(\overline{\Omega})$ solutions $\mathbf{u}^\varepsilon = (u^\varepsilon, v^\varepsilon)$ of the viscous problem*

$$\begin{cases} v^\varepsilon_x - u^\varepsilon_y = \varepsilon \operatorname{div}\left(\sigma_1(\rho^\varepsilon)\nabla t^\varepsilon\right), \\ (\rho^\varepsilon u^\varepsilon)_x + (\rho^\varepsilon v^\varepsilon)_y = \varepsilon \operatorname{div}\left(\sigma_2(\rho^\varepsilon)\nabla \rho^\varepsilon\right), \end{cases} \tag{4.8}$$

in Ω depicted by Domain (a) (cf. Section 1.4) such that $q^\varepsilon \leq q_{cav}$, where $\rho^\varepsilon = \rho(q^\varepsilon)$ is prescribed by the Bernoulli law (1.35) *with $\gamma \geq 3$, where*

$$\sigma_1(\rho) = 1, \quad \sigma_2(\rho) = \left(1 - \frac{c^2}{q^2}\right) \qquad \text{for } q > q_{cr},$$

and the boundary conditions:

$$\begin{cases} \nabla t^\varepsilon \cdot \mathbf{n} = 0 & \text{on } \partial\Omega_1, \\ \varepsilon\sigma_2(\rho^\varepsilon)\nabla\rho^\varepsilon \cdot \mathbf{n} = |\rho^\varepsilon(u^\varepsilon, v^\varepsilon)\cdot \mathbf{n}| & \text{on } \partial\Omega_1, \\ (u^\varepsilon, v^\varepsilon) - (u_\infty, v_\infty) = 0 & \text{on } \partial\Omega_2 \text{ with } q_{cr} < q_\infty < q_{cav}. \end{cases} \tag{4.9}$$

Then, all solutions of (4.8)-(4.9) *are bounded below, uniformly in ε, away from q_{cr}. In detail, we have that there exists a positive constant $q_* \in (q_{cr}, \infty)$ such that $q_* \leq q^\varepsilon \leq q_{cav}$ for all $\varepsilon > 0$. In turn, this imposes that $0 \leq \rho(q^\varepsilon) \leq \rho(q_*)$ for all $\varepsilon > 0$, and $\rho(q_*) < \rho(q_{cr})$.*

Remark 4.4. Showing the existence of solutions to (4.8)-(4.9) with $q^\varepsilon \leq q_{cav}$ for each $\varepsilon > 0$ is an open problem. Notice that the sign in the second boundary condition in (4.9) is different from the one in [20, Theorem 5.1].

4.3 The entropy equation

To begin with, we define a new substitute variable for the density, which was originally defined by Morawetz in [69, Section 3], and subsequently in [20, Section 7].

Definition 4.5. Define $\mu = \mu(\rho)$ to be

$$\mu(\rho) := -\theta \int_{\rho}^{\rho(q_{cr})} \frac{x^{2\theta}}{1 - x^{2\theta}} \, dx \qquad \text{for } \rho \in [0, 1), \tag{4.10}$$

where q_{cr} is given by (1.39). As such, μ is the unique solution of the equation

$$\begin{cases} \mu'(\rho) = 1/M^2, \\ \mu(\rho(q_{cr})) = 0. \end{cases} \tag{4.11}$$

Remark 4.6. Note that the above equation implies that $\mu < 0$ when the flow is supersonic (which corresponds to (1.40) being hyperbolic), and $\mu > 0$ when the flow is subsonic (which corresponds to (1.40) being elliptic). In view of this, μ is a more appropriate variable to work with than ρ when one wants to divide the phase-space into the hyperbolic and elliptic subdomains.

In what follows, we shall denote entropy pairs of the potential flow system (1.40) by (Q_1, Q_2), so that Q_1 is the entropy and Q_2 its flux. Note that these entropies are associated to the phase space variables (u, v).

As already mentioned in Remark 1.6, requiring equality of the mixed partial derivatives of the entropy-flux gives rise to the entropy equation (*cf.* [20, Section 7]). Solutions of this equation are most easily generated using the new variable μ, and a recipe for constructing them can be found in the following lemma.

Lemma 4.7 (Lemma 7.1 of [20]). *Let H be a solution of the linear equation*

$$H_{\mu\mu} - \left(\frac{M^2 - 1}{\rho^2} \right) H_{tt} = 0. \tag{4.12}$$

Then, the quantities

$$Q_1 := \rho q H_\mu \cos t - q H_t \sin t, \qquad Q_2 := \rho q H_\mu \sin t + q H_t \cos t, \tag{4.13}$$

define entropy pairs of system (1.40) *in the* (u, v)*-plane.*

Definition 4.8. We define the *special generator*, which is a straightforward solution of (4.12), to be

$$H^*(\mu, t) := \frac{t^2}{2} + \int^{\mu} \int^{\mu'} \frac{M^2 - 1}{\rho^2} \, d\mu'' \, d\mu', \tag{4.14}$$

and we define (Q_1^*, Q_2^*) to be the corresponding entropy pair generated by this H^* via (4.13), which we call the *special entropy pair*.

By Lemma 4.3, in the case $\gamma \geq 3$, the system is hyperbolic with $q \geq q_* > q_{cr}$ throughout Ω. As such, the formula (4.10) shows that μ takes values in the interval $[-\mu_{cr}, -\mu_{cr} + \mu_*]$, where μ_{cr} and $\mu_* \in (0, \mu_{cr})$ are defined by

$$\mu_{cr} := \theta \int_0^{\rho(q_{cr})} \frac{x^{2\theta}}{1 - x^{2\theta}}\, dx, \qquad \mu_* := \theta \int_0^{\rho(q_*)} \frac{x^{2\theta}}{1 - x^{2\theta}}\, dx.$$

Remark 4.9. We note that μ_{cr} is a positive constant inferior to 1. Indeed, a very blunt bound using (1.35)-(1.39) shows that

$$0 < \mu_{cr} \leq \theta \rho(q_{cr}) \cdot \frac{\rho(q_{cr})^{2\theta}}{1 - \rho(q_{cr})^{2\theta}} = \rho(q_{cr}) < 1.$$

Note that $\mu = -\mu_{cr}$ when $\rho = 0$, which is the only singular point of the entropy equation. As such it is more suitable to use the translated variable ν, defined below.

Definition 4.10. We define the variable $\nu = \nu(\rho)$ by the explicit formula

$$\nu(\rho) := \theta \int_0^{\rho} \frac{x^{2\theta}}{1 - x^{2\theta}}\, dx \qquad \text{for } \rho \in [0, 1). \tag{4.15}$$

Note that

$$\begin{cases} \nu'(\rho) = 1/M^2, \\ \nu(0) = 0, \end{cases} \tag{4.16}$$

and $\nu(\rho) = \mu(\rho) + \mu_{cr}$. As such, ν takes values in the interval $[0, \mu_*]$.

Remark 4.11. Observe that the change of coordinates of (ρ, t) to $(\nu(\rho), t)$ has vanishing Jacobian at the point of cavitation, and hence (a priori) this hodograph transformation is not rigorously justified when cavitation occurs.

Provided this change of variable can be performed, the entropy equation can be written as

$$H_{\nu\nu} - \left(\frac{M^2 - 1}{\rho^2} \right) H_{tt} = 0. \tag{4.17}$$

Lemma 4.3 showed that $M > 1$ throughout the phase-space occupied by the viscous solutions $\mathbf{u}^\varepsilon$. In view of this, the coefficient $(M^2 - 1)/\rho^2$ is always non-negative. Inspired by the work of Chen–LeFloch on general pressure laws (*cf.* [14]), we have the following definition.

Definition 4.12. For $\rho \in [0, \rho(q_{cr})]$, define the coefficient k to be

$$k(\nu(\rho)) := \int_0^{\rho} \frac{\sqrt{M^2(\rho') - 1}}{\rho'} \cdot \frac{1}{M^2(\rho')}\, d\rho'. \tag{4.18}$$

A priori, there is no reason to believe that this integral converges, but this is justified in Appendix C. Assuming that this function is well-defined, taking a derivative with respect to ρ and applying the chain rule yields

$$k'(\nu(\rho)) = \frac{\sqrt{M^2-1}}{\rho} \qquad \text{for } \rho \in (0, \rho(q_{cr})],$$

and $k \in C^1((0,\mu_{cr}]) \cap C^\infty((0,\mu_{cr}))$. Hence, the entropy equation adopts the familiar form

$$H_{\nu\nu} - k'(\nu)^2 H_{tt} = 0 \qquad \text{for } (\nu, t) \in (0, \mu_*] \times [0, 2\pi], \tag{4.19}$$

with initial data prescribed at $\nu = 0$, which is of course reminiscent of the equation for the entropy kernel in the general pressure law formulation of the compressible Euler equations, *cf.* (1.12). Our objective is now to describe the asymptotic behaviour of the coefficient k solely in terms of ν. To this end, we have the following theorem, the proof of which is contained in Appendix C.

Theorem 4.13. *The quantity $k(\nu)$ admits the decomposition*

$$k(\nu) = a_\sharp \nu^{\theta/\gamma} + a_\flat \nu^{3\theta/\gamma} + L(\nu) \qquad \textit{for } \nu \in [0, \mu_*], \tag{4.20}$$

where

$$a_\sharp = \frac{1}{\sqrt{\theta}}\left(\frac{\gamma}{\theta}\right)^{\theta/\gamma}, \qquad a_\flat = -\frac{1}{\sqrt{\theta}}\frac{(4\theta-1)(\theta+1)}{6(4\theta+1)}\left(\frac{\gamma}{\theta}\right)^{3\theta/\gamma}. \tag{4.21}$$

Moreover, there exists a positive constant $C = C(\gamma, \mu_*)$ such that

$$|L^{(j)}(\nu)| \le C\nu^{5\theta/\gamma - j} \qquad \textit{for } \nu \in (0, \mu_*], \quad \textit{for } j \in \{0,1,2,3\}. \tag{4.22}$$

The next result follows from manipulating the expansion (4.20) and using (4.22).

Corollary 4.14. *The quantity $k'(\nu)^2$ admits the decomposition*

$$k'(\nu)^2 = \tilde{a}_\sharp \nu^{2\theta/\gamma-2} + \tilde{a}_\flat \nu^{4\theta/\gamma-2} + \tilde{L}(\nu) \qquad \textit{for } \nu \in (0, \mu_*], \tag{4.23}$$

where

$$\tilde{a}_\sharp = (a_\sharp\theta/\gamma)^2, \qquad \tilde{a}_\flat = 3a_\sharp a_\flat(\theta/\gamma)^2. \tag{4.24}$$

Moreover, $\tilde{L}$ is bounded by a constant, depending solely on γ and μ_*.

Remark 4.15. At this point it is worth noting that our situation is no more singular than that of the general pressure law for the compressible Euler equations (*cf.* [14]). Indeed, in our case, $k'(\nu)^2 = O(\nu^{-1-1/\gamma})$, where $-1-1/\gamma \in [-4/3, -1)$ for $\gamma \ge 3$. Analogously, in the case considered by Chen–LeFloch [14], $k'(\rho)^2 = O(\rho^{2\theta-2})$, where $2\theta - 2 \in (-2, 0)$. In either case, the singularity at the vacuum is of degree strictly less than two. However, one cannot define a pressure $p(\rho) = \int_0^\rho y^2 k'(y)^2\, dy$ that reframes (4.19) into the class of entropy equations considered in [14].

In what follows, we will need to exploit cancellations in derivatives of $k'(\nu)^2$. To this end, we have the following the lemma.

Lemma 4.16. *There exists a positive constant $C = C(\gamma, \mu_*)$ such that*

$$\left|k''(\nu)k(\nu) + (\gamma/\theta - 1)k'(\nu)^2\right| \leq C\nu^{4\theta/\gamma - 2} \qquad \text{for } \nu \in (0, \mu_*]. \tag{4.25}$$

Proof. Observe that we have

$$k''(\nu)k(\nu) = \left(a_\sharp \frac{\theta}{\gamma}\left(\frac{\theta}{\gamma} - 1\right)\nu^{\theta/\gamma - 2} + a_\flat \frac{3\theta}{\gamma}\left(\frac{3\theta}{\gamma} - 1\right)\nu^{3\theta/\gamma - 2} + L''(\nu)\right) \cdot \left(a_\sharp \nu^{\theta/\gamma} + a_\flat \nu^{3\theta/\gamma} + L(\nu)\right),$$

which we can expand as

$$k''(\nu)k(\nu) = a_\sharp^2 \frac{\theta}{\gamma}\left(\frac{\theta}{\gamma} - 1\right)\nu^{2\theta/\gamma - 2} + M(\nu),$$

where there exists a positive constant C, independent of ε, such that $|M(\nu)| \leq C\nu^{4\theta/\gamma - 2}$. Hence,

$$k''(\nu)k(\nu) + (\gamma/\theta - 1)k'(\nu)^2 = a_\sharp^2 \frac{\theta}{\gamma}\left(\frac{\theta}{\gamma} - 1\right)\nu^{2\theta/\gamma - 2} + (\gamma/\theta - 1)\tilde{a}_\sharp \nu^{2\theta/\gamma - 2} + \tilde{M}(\nu),$$

where $|\tilde{M}(\nu)| \leq C\nu^{4\theta/\gamma - 2}$, for some C is independent of ε. However, we see that

$$\begin{aligned} a_\sharp^2 \frac{\theta}{\gamma}\left(\frac{\theta}{\gamma} - 1\right) + (\gamma/\theta - 1)\tilde{a}_\sharp &= a_\sharp^2 \left(\frac{\theta}{\gamma}\left(\frac{\theta}{\gamma} - 1\right) + \left(\frac{\gamma}{\theta} - 1\right)\left(\frac{\theta}{\gamma}\right)^2\right) \\ &= 0. \end{aligned}$$

In fact, after explicitly computing the term of the next order we find that there exists a positive constant C, independent of ε, such that

$$k''(\nu)k(\nu) + (\gamma/\theta - 1)k'(\nu)^2 = 2a_\sharp a_\flat \frac{\theta}{\gamma}\left(\frac{5\theta}{\gamma} - 2\right)\nu^{4\theta/\gamma - 2} + \tilde{\tilde{M}}(\nu),$$

with $|\tilde{\tilde{M}}(\nu)| \leq C\nu^{6\theta/\gamma - 2}$. □

Armed with these results, we are in a position to show existence of the Lax entropies for the hyperbolic region, $g_n(\nu)e^{\pm nt}$. However, we must first take an in-depth look at what estimates are needed to guarantee the H^{-1}-compactness of the entropy dissipation measures, and what L^2 type estimates are straightforwardly available.

4.4 Main dissipation estimate

In this section, we verify that the dissipation estimate [20, Proposition 8.1] is still valid for the solutions generated by the viscous problems (4.8)-(4.9).

Lemma 4.17. *Let* $\mathbf{u}^\varepsilon = (u^\varepsilon, v^\varepsilon)$ *be a solution of the viscous approximate system* (4.8)-(4.9) *with* $u_\infty > 0$ *and* $v_\infty = 0$ *on* Ω, *satisfying* $q_* \leq q^\varepsilon \leq q_{cav}$ *for some* $q_* > q_{cr}$ *and* t^ε *uniformly bounded in* ε. *Then the integral*

$$\varepsilon \int_\Omega \left(\sigma_1(\rho^\varepsilon)|\nabla t^\varepsilon|^2 + \sigma_2(\rho^\varepsilon)\frac{c^2(\rho^\varepsilon)}{(\rho^\varepsilon q^\varepsilon)^2}|\nabla \rho^\varepsilon|^2 \right) dx\, dy \tag{4.26}$$

is bounded uniformly for all ε.

Proof. Fix $\bar{q}$ such that $q_* < \bar{q} < q_{cav}$, and let $\bar{\nu}$ be uniquely determined by this speed, i.e., $\bar{\nu} = \nu(\rho(\bar{q}))$. In view of the monotonicity provided by (4.16), we necessarily have $\bar{\nu} > 0$, and $0 < \rho(\bar{q}) < \rho(q_*)$. Consider the special generator of Definition 4.8,

$$H^*(\nu, t) = \frac{t^2}{2} + \int_0^\nu \int_{\bar{\nu}}^{\tilde{\nu}} k'(\tilde{\tilde{\nu}})^2\, d\tilde{\tilde{\nu}}\, d\tilde{\nu},$$

which manifestly solves the entropy equation (4.17). Note that this integral is well-defined, since $\bar{\nu} > 0$ and near the vacuum we have for the inner integral

$$\begin{aligned} \left| \int_{\bar{\nu}}^{\tilde{\nu}} k'(\tilde{\tilde{\nu}})^2\, d\tilde{\tilde{\nu}} \right| &\leq C \int_{\bar{\nu}}^{\tilde{\nu}} (\tilde{\nu})^{\frac{2\theta}{\gamma}-2}\, d\tilde{\tilde{\nu}} \\ &= C\left((\tilde{\nu})^{2\theta/\gamma-1} + \bar{\nu}^{2\theta/\gamma-1} \right), \end{aligned}$$

using Corollary 4.14. It follows that

$$\left| \int_0^\nu \int_{\bar{\nu}}^{\tilde{\nu}} k'(\tilde{\tilde{\nu}})^2\, d\tilde{\tilde{\nu}}\, d\tilde{\nu} \right| \leq C\left(\nu + \nu^{2\theta/\gamma} \right),$$

where $C = C(\bar{q}, q_*, \gamma)$. As is done in [20, Section 8], we produce the quantity V^* via the relations

$$\rho H^*_{\nu t} - H^*_t = -V^*_t, \qquad H^*_\nu + \frac{1}{\rho}H^*_{tt} = \frac{q^2}{c^2 - q^2}V^*_\rho,$$

from which we infer that $V^* = \frac{t^2}{2} + P(\rho)$, where

$$P'(\rho) = \frac{c^2 - q^2}{q^2} \int_{\bar{q}}^q \frac{ds}{\rho(s)s}.$$

Note that, despite the fact that cavitation is achievable, the above integral converges and is uniformly bounded independently of ε, provided $\gamma \geq 3$. Indeed, let

$$I^\varepsilon := \int_{\bar{q}}^{q^\varepsilon} \frac{ds}{\rho(s)s} = -\int_{\rho(\bar{q})}^{\rho(q^\varepsilon)} \frac{c^2}{\rho^2 q^2}\, d\rho \;= -\theta \int_{\rho(\bar{q})}^{\rho(q^\varepsilon)} \rho^{\gamma-3}\left(1 - \rho^{2\theta}\right)^{-1} d\rho,$$

where the final equality follows from the Bernoulli law (1.35). Hence, we may split the integral into two parts, as shown below

$$-I^\varepsilon = \left(\theta\int_{\rho(q^\varepsilon)}^{\rho(\bar q)} \rho^{\gamma-3}(1-\rho^{2\theta})^{-1}\,d\rho\right)\mathbb{1}_{q^\varepsilon>\bar q} + \left(\theta\int_{\rho(\bar q)}^{\rho(q^\varepsilon)} \rho^{\gamma-3}(1-\rho^{2\theta})^{-1}\,d\rho\right)\mathbb{1}_{q^\varepsilon\le\bar q}.$$

We infer that $\theta^{-1}|I^\varepsilon| \le \int_0^{\rho(\bar q)} \rho^{\gamma-3}(1-\rho^{2\theta})^{-1}\,d\rho + \int_{\rho(\bar q)}^{\rho(q_*)} \rho^{\gamma-3}(1-\rho^{2\theta})^{-1}\,d\rho$. Since $\gamma\ge 3$ and $\rho(\bar q)<\rho(q_*)<1$, we therefore conclude that

$$|I^\varepsilon| \le C\theta\left(\int_0^{\rho(q_*)} d\rho\right),$$

for some positive constant $C=C(\bar q, q_*, \gamma)$ independent of ε, as required. In turn, we obtain the special entropy pair

$$Q_1^*(\mathbf{u}^\varepsilon) = -q^\varepsilon(t^\varepsilon\sin t^\varepsilon + \cos t^\varepsilon) + \left(\int_{\bar q}^{q^\varepsilon}\frac{ds}{\rho(s)s}\right)\rho(q^\varepsilon)q^\varepsilon\cos t^\varepsilon,$$
$$Q_2^*(\mathbf{u}^\varepsilon) = q^\varepsilon(t^\varepsilon\cos t^\varepsilon - \sin t^\varepsilon) + \left(\int_{\bar q}^{q^\varepsilon}\frac{ds}{\rho(s)s}\right)\rho(q^\varepsilon)q^\varepsilon\sin t^\varepsilon,$$

and both quantities are uniformly bounded in ε. Note that the relation

$$Q_{1x}+Q_{2y} = -V_tR_1 + \frac{q^2}{c^2-q^2}V_\rho R_2,$$

(*cf.* [20, Section 7]) gives rise to

$$\begin{aligned} Q_{1x}^*(\mathbf{u}^\varepsilon)+Q_{2y}^*(\mathbf{u}^\varepsilon) =& -\varepsilon t^\varepsilon\,\mathrm{div}\,(\sigma_1(\rho^\varepsilon)\nabla t^\varepsilon) + \varepsilon\left(\int_{\bar q}^{q(\rho^\varepsilon)}\frac{dq}{\rho q}\right)\mathrm{div}\,(\sigma_2(\rho^\varepsilon)\nabla\rho^\varepsilon)\\ =&\,\mathrm{div}\,(-\sigma_1(\rho^\varepsilon)t^\varepsilon\nabla t^\varepsilon + \sigma_2(\rho^\varepsilon)I^\varepsilon\nabla\rho^\varepsilon)\\ &+\varepsilon\sigma_1(\rho^\varepsilon)|\nabla t^\varepsilon|^2 + \varepsilon\sigma_2(\rho^\varepsilon)\frac{c^2(\rho^\varepsilon)}{(\rho^\varepsilon q^\varepsilon)^2}|\nabla\rho^\varepsilon|^2. \end{aligned}$$

Integrating the above over Ω and applying the divergence theorem in the plane,

$$\begin{aligned} \int_{\partial\Omega}(Q_1^*(\mathbf{u}^\varepsilon),Q_2^*(\mathbf{u}^\varepsilon))\cdot\mathbf{n}\,ds =& \varepsilon\int_{\partial\Omega}(-\sigma_1(\rho^\varepsilon)t^\varepsilon(\nabla t^\varepsilon\cdot\mathbf{n}) + \sigma_2(\rho^\varepsilon)I^\varepsilon(\nabla\rho^\varepsilon\cdot\mathbf{n}))\,ds\\ &+\varepsilon\int_\Omega\left(\sigma_1(\rho^\varepsilon)|\nabla t^\varepsilon|^2 + \sigma_2(\rho^\varepsilon)\frac{c^2(\rho^\varepsilon)}{(\rho^\varepsilon q^\varepsilon)^2}|\nabla\rho^\varepsilon|^2\right)dx\,dy\\ =&:J_1^\varepsilon + J_2^\varepsilon. \end{aligned} \tag{4.27}$$

As previously discussed, the left-hand side is uniformly bounded in ε. Using the boundary conditions, we have that

$$J_1^\varepsilon = \varepsilon\int_{\partial\Omega_1}|\rho^\varepsilon(u^\varepsilon,v^\varepsilon)\cdot\mathbf{n}|I^\varepsilon\,ds,$$

which is also uniformly bounded in ε. We thereby conclude that the second integral J_2^ε is bounded, independently of ε, which completes the proof of the lemma. □

4.5 The search for the Lax entropies

Recall that we generate entropy pairs (Q_1, Q_2) via a generator H using (4.13). By direct calculation, following [20, Section 7], we have

$$\begin{aligned}
&\partial_x Q_1(\mathbf{u}^\varepsilon) + \partial_y Q_2(\mathbf{u}^\varepsilon)\\
&\quad =\varepsilon \operatorname{div}\left(\sigma_1(\rho^\varepsilon)\nabla t^\varepsilon \left(\rho^\varepsilon H_{\nu t} - H_t\right) + \sigma_2(\rho^\varepsilon)\nabla\rho^\varepsilon\left(H_\nu + \frac{1}{\rho^\varepsilon}H_{tt}\right)\right)\\
&\qquad - \varepsilon\sigma_1(\rho^\varepsilon)\nabla t^\varepsilon\cdot\nabla\left(\rho^\varepsilon H_{\nu t} - H_t\right) - \varepsilon\sigma_2(\rho^\varepsilon)\nabla\rho^\varepsilon\cdot\nabla\left(H_\nu + \frac{1}{\rho^\varepsilon}H_{tt}\right)\\
&\quad =: K_1^\varepsilon + K_2^\varepsilon.
\end{aligned} \tag{4.28}$$

We focus on the term K_2^ε and omit the ε superscripts. Observe that

$$\begin{aligned}
-\frac{1}{\varepsilon}K_2^\varepsilon =& \left[\sigma_1\left(1-\frac{c^2}{q^2}\right)H_{\nu t} + \sigma_1\frac{c^2}{q^2}\rho H_{\nu\nu t} + \sigma_2 H_{\nu t} + \sigma_2\frac{1}{\rho}H_{ttt}\right](\nabla t\cdot\nabla\rho)\\
&+ \left[\rho H_{\nu tt} - H_{tt}\right]\sigma_1|\nabla t|^2 + \left[\rho^2 H_{\nu\nu} + \rho H_{\nu tt} - \frac{q^2}{c^2}H_{tt}\right]\sigma_2\frac{c^2}{(\rho q)^2}|\nabla\rho|^2,
\end{aligned}$$

which, using the entropy equation, can be simplified to

$$-\varepsilon^{-1}K_2^\varepsilon = \left[\rho H_{\nu tt} - H_{tt}\right]\left(\sigma_1|\nabla t|^2 + \sigma_2\frac{c^2}{(\rho q)^2}|\nabla\rho|^2\right) + \left[H_{ttt} + \rho H_{\nu t}\right]\frac{2\sigma_2}{\rho}(\nabla t\cdot\nabla\rho).$$

By the dissipation estimate of Lemma 4.17, we know that the first term on the right-hand side is controlled if there exists a constant C independent of ε such that

$$|\rho(\nu^\varepsilon)H_{\nu tt}(\nu^\varepsilon, t^\varepsilon) - H_{tt}(\nu^\varepsilon, t^\varepsilon)| \le C.$$

Looking for a separable entropy $H_n = g_n(\nu)e^{\pm nt}$, the previous estimate imposes

$$|\rho(\nu^\varepsilon)\dot g_n(\nu^\varepsilon) - g_n(\nu^\varepsilon)| \le C.$$

The second term on the right-hand side is more challenging to control. Indeed,

$$\begin{aligned}
\left|\left(H_{ttt} + \rho H_{\nu t}\right)\frac{2\sigma_2}{\rho}(\nabla t\cdot\nabla\rho)\right| &= 2\left|H_{ttt} + \rho H_{\nu t}\right|\sqrt{\frac{\sigma_2}{\sigma_1}}\cdot\frac{q}{c}\left|\sqrt{\sigma_1}\nabla t\cdot\sqrt{\sigma_2}\frac{c}{\rho q}\nabla\rho\right|\\
&\le C\rho^{-\theta}(\nu)\left|H_{ttt} + \rho H_{\nu t}\right|\left(\sigma_1|\nabla t|^2 + \sigma_2\frac{c^2}{(\rho q)^2}|\nabla\rho|^2\right),
\end{aligned}$$

where we used the Cauchy–Schwarz inequality. This is controlled provided

$$|H_{ttt}(\nu^\varepsilon, t^\varepsilon) + \rho(\nu^\varepsilon)H_{\nu t}(\nu^\varepsilon, t^\varepsilon)| \le C\rho^\theta(\nu^\varepsilon).$$

Assuming a separable entropy, this estimate imposes that we require

$$\left|\rho(\nu)\dot g_n(\nu) + n^2 g_n(\nu)\right| \le C\rho^\theta(\nu).$$

In summary, when generating the Lax entropies (*cf.* Section 1.5), we look for separable solutions $H_n(\nu, t) = g_n(\nu)e^{\pm nt}$ such that, for some positive $C = C(n, \gamma, \mu_*)$,

$$|\rho(\nu)\dot g_n(\nu) - g_n(\nu)| \le C, \qquad \left|\rho(\nu)\dot g_n(\nu) + n^2 g_n(\nu)\right| \le C\rho^\theta(\nu). \tag{4.29}$$

4.6 Compactness of the Lax entropies

Throughout this section, we use the standard notation $f(x) \sim g(x)$ as $x \to 0$ to mean $\lim_{x\to 0} \frac{f(x)}{g(x)} = 1$ (*cf.* [31]).

The following lemma is essential in showing the existence of the Lax entropies.

Lemma 4.18. *Fix* $\alpha \in [2\theta/\gamma, 1]$. *Let* $r \in C([0, \mu_*])$ *be such that* $r(0) = 0$ *and*

$$|r(\nu_1) - r(\nu_2)| \leq C_0 |\nu_1 - \nu_2|^\alpha \qquad \text{for all } \nu_1, \nu_2 \text{ in the interval } [0, \mu_*],$$

for some $C_0 > 0$. *Let* g_n^ε *be the unique* C^2 *solution of*

$$\begin{cases} \ddot{g}_n^\varepsilon(\nu) - n^2 k'(\nu)^2 g_n^\varepsilon(\nu) = r(\nu) & \text{in } [\varepsilon, \mu_*], \\ g_n^\varepsilon(\varepsilon) = 0, \\ \dot{g}_n^\varepsilon(\varepsilon) = 0. \end{cases} \tag{4.30}$$

where $\varepsilon > 0$ *is constant. Then, there exists* $g_n \in C^2([0, \mu_*])$ *such that* $g_n^{\varepsilon'} \to g_n$ *in* $C^2([0, \mu_*])$, *for some subsequence* ε', *and the limit function solves the Cauchy problem*

$$\begin{cases} \ddot{g}_n(\nu) - n^2 k'(\nu)^2 g_n(\nu) = r(\nu) & \text{in } [0, \mu_*], \\ g_n(0) = 0, \\ \dot{g}_n(0) = 0, \end{cases} \tag{4.31}$$

as an equality between continuous functions in the interval $[0, \mu_*]$. *Further, there exists a positive constant* $C = C(n, \gamma, C_0, \mu_*)$ *such that the limit function satisfies*

$$|\nu^{-(\alpha+2)} g_n(\nu)| + |\nu^{-(\alpha+1)} \dot{g}_n(\nu)| + |\nu^{-\alpha} \ddot{g}_n(\nu)| \leq C \qquad \text{for } \nu \in [0, \mu_*]. \tag{4.32}$$

For a proof, we refer the reader to Section 4.7. Since $k'(\nu)^2 \sim \tilde{a}_\sharp^2 \nu^{2\theta/\gamma - 2}$ and $2\theta/\gamma - 2 \notin \mathbb{Z}$, the point $\nu = 0$ is not a regular singular point of (4.31). Hence, one cannot simply appeal to the method of Frobenius (*cf.* [6, Chapter 9]) to solve the ordinary differential equation (4.31). To this end, we require the following definition.

Definition 4.19. For $l \in \mathbb{R}$, define the function $F_l : [0, \infty) \to \mathbb{R}$ by

$$F_l(z) := z^l I_{-l}(z), \tag{4.33}$$

where I_{-l} is the modified Bessel function of the first kind of order $-l$.

Note that F_l is related to the Fourier transform of G_λ (*cf.* [38]), which was introduced in Theorem 1.12. Using the recurrence relations and the asymptotic expansions for the modified Bessel functions in [41], we obtain the following lemma.

Lemma 4.20. *For any $l \in \mathbb{R}$, the functions F_l are locally bounded on $[0, \infty)$, and*

$$F_l'(z) = zF_{l-1}(z) \quad \text{and} \quad F_l''(z) - F_l(z) = -(2l-1)F_{l-1}(z). \tag{4.34}$$

In the next lemma, we generate a family of Lax entropies. We then show, in Corollary 4.26, that this family of solutions satisfies the crucial estimates (4.29).

Lemma 4.21. *Consider the under-determined Cauchy problem*

$$\begin{cases} \ddot{g}_n(\nu) - n^2 k'(\nu)^2 g_n(\nu) = 0 \quad \text{in } (0, \mu_*], \\ g_n(0) = g_0, \end{cases} \tag{4.35}$$

where $g_0 \in \mathbb{R}$ is a constant. There exists a solution $g_n \in C([0, \mu_*]) \cap C^2((0, \mu_*])$ of this problem,

$$g_n(\nu) = \sum_{j=0}^{3} b_j(\nu) F_{\frac{\gamma}{2\theta}-j}(nk(\nu)) + h_n(\nu), \tag{4.36}$$

where the first coefficient is a constant given explicitly by $b_0 = g_0 \Gamma(1 - \gamma/2\theta) 2^{-\frac{\gamma}{2\theta}}$, while the others admit, for some positive $C = C(n, \gamma, \mu_*)$, the bounds

$$|b_j(\nu)| \leq C\nu^{2(j+1)\theta/\gamma}, \qquad |b_j'(\nu)| \leq C\nu^{2(j+1)\theta/\gamma - 1} \qquad \text{for } \nu \in (0, \mu_*], \tag{4.37}$$

for $j = 1, 2, 3$. The remainder term h_n lies in the space $C^2([0, \mu_*])$ and admits, for some positive $C = C(n, \gamma, \mu_*)$, the estimates

$$|\nu^{-(2\theta/\gamma+2)} h_n(\nu)| + |\nu^{-(2\theta/\gamma+1)} \dot{h}_n(\nu)| + |\nu^{-2\theta/\gamma} \ddot{h}_n(\nu)| \leq C \qquad \text{for } \nu \in [0, \mu_*]. \tag{4.38}$$

Corollary 4.22. *Moreover, direct calculation using (4.37) and (4.38) shows that the derivative of the solution given by (4.36) admits, for some positive $C = C(n, \gamma, \mu_*)$,*

$$|\dot{g}_n(\nu)| \leq C\nu^{-(1-2\theta/\gamma)} \qquad \text{for } \nu \in (0, \mu_*]. \tag{4.39}$$

Remark 4.23. Direct calculation yields

$$g_n(\nu) \sim g_0, \qquad \dot{g}_n(\nu) \sim -n^2 g_0 \left(\frac{\gamma}{\theta}\right)^{\frac{2\theta}{\gamma}-1} \nu^{\frac{2\theta}{\gamma}-1} \qquad \text{near } \nu = 0. \tag{4.40}$$

Using $\rho(\nu) \sim (\gamma/\theta)^{1/\gamma} \nu^{1/\gamma}$ (*cf.* Lemma C.8), we get $\rho(\nu)\dot{g}_n(\nu) \sim -n^2 g_0$ near $\nu = 0$.

Proof of Lemma 4.21. We define

$$b_1(\nu) := -b_0 n^2 (\gamma/\theta - 1) \int_0^\nu \left(\int_0^z \left(k''(y)k(y) + (\gamma/\theta - 1)k'(y)^2 \right) dy \right) dz.$$

Lemma 4.16 shows that there exists a positive constant C such that $|b_1(\nu)| \le C\nu^{4\theta/\gamma}$ and $|b_1'(\nu)| \le C\nu^{4\theta/\gamma-1}$ for $\nu \in (0, \mu_*]$. In turn, we define

$$b_2(\nu) := -n^2(\gamma/\theta - 3)\int_0^\nu \left(\int_0^z \left(2b_1'(\nu)k'(\nu)k(\nu) + b_1(\nu)k''(\nu)k(\nu) \right.\right.$$
$$\left.\left. + (\gamma/\theta - 3)b_1(\nu)k'(\nu)^2 \right) dy \right) dz.$$

We observe that, in view of the previous bounds, there exists a positive constant C such that $|b_2(\nu)| \le C\nu^{6\theta/\gamma}$ and $|b_2'(\nu)| \le C\nu^{6\theta/\gamma-1}$ for $\nu \in [0, \mu_*]$. Finally, define

$$b_3(\nu) := -n^2(\gamma/\theta - 5)\int_0^\nu \left(\int_0^z \left(2b_2'(\nu)k'(\nu)k(\nu) + b_2(\nu)k''(\nu)k(\nu) \right.\right.$$
$$\left.\left. + (\gamma/\theta - 3)b_2(\nu)k'(\nu)^2 \right) dy \right) dz,$$

and note that there exists a positive constant C such that

$$|b_3(\nu)| \le C\nu^{8\theta/\gamma}, \qquad |b_3'(\nu)| \le C\nu^{8\theta/\gamma-1} \qquad \text{for all } \nu \in [0, \mu_*]. \tag{4.41}$$

We therefore ask that the remainder term solves the Cauchy problem

$$\begin{cases} \ddot{h}_n(\nu) - n^2k'(\nu)^2h_n(\nu) = r(\nu), \\ h_n(0) = 0, \\ \dot{h}_n(0) = 0, \end{cases}$$

where

$$r(\nu) := -n^2(\gamma/\theta - 7)\Big(2b_3'(\nu)k'(\nu)k(\nu) + b_3(\nu)k''(\nu)k(\nu)$$
$$+ (\gamma/\theta - 7)b_3(\nu)k'(\nu)^2 \Big) F_{\frac{\gamma}{2\theta}-4}(nk(\nu)).$$

Claim 4.24. *The remainder r is $2\theta/\gamma$-Hölder continuous and vanishes at the origin.*

Finally, Lemma 4.18 yields the existence of such a remainder $h \in C^2([0, \mu_*])$. $\square$

Proof of Claim 4.24. We see from (4.41) that $|r(\nu)| \le C\nu^{10\theta/\gamma-2}$, which shows that r vanishes at the origin. Additionally, $|\dot{r}(\nu)| \le C\nu^{10\theta/\gamma-3}$ for all $\nu \in (0, \mu_*]$. In turn,

$$|r(\nu_1) - r(\nu_2)| \le C\int_{\nu_1}^{\nu_2} y^{\frac{10\theta}{\gamma}-3}\, dy.$$

Since $1 - 2\theta/\gamma \in (0, 1)$, Jensen's inequality for concave functions yields

$$(\nu_2 - \nu_1) ⨍_{\nu_1}^{\nu_2} y^{\frac{10\theta}{\gamma}-3}\, dy \le (\nu_2 - \nu_1)\left(⨍_{\nu_1}^{\nu_2} y^\omega\, dy \right)^{1-\frac{2\theta}{\gamma}},$$

with $\omega := \left(\frac{10\theta}{\gamma} - 3\right)\left(1 - \frac{2\theta}{\gamma}\right)^{-1} > -1$ for $\gamma \ge 3$. So, $\int_{\nu_1}^{\nu_2} y^{\frac{10\theta}{\gamma}-3}\, dy \le C|\nu_2 - \nu_1|^{\frac{2\theta}{\gamma}}$. $\square$

We now observe cancellations in the first few terms of the asymptotic expansions for g_n and its derivative near the vacuum.

Corollary 4.25. *Let $g_n \in C([0,\mu_*]) \cap C^2((0,\mu_*])$ be the solution of the problem* (4.35) *provided by* (4.36) *(cf. Lemma 4.21). Then, there exists a positive constant $C = C(n,\gamma,\mu_*)$ such that*

$$\left|g_n(\nu) - b_0 F_{\frac{\gamma}{2\theta}}(nk(\nu))\right| + \nu \left|\dot{g}_n(\nu) - \frac{d}{d\nu}\Big(b_0 F_{\frac{\gamma}{2\theta}}(nk(\nu))\Big)\right| \leq C\nu^{4\theta/\gamma} \qquad \text{for } \nu \in [0,\mu_*]. \tag{4.42}$$

Proof. Begin by observing that

$$\begin{aligned}\left|g_n(\nu) - b_0 F_{\frac{\gamma}{2\theta}}(nk(\nu))\right| &\leq \sum_{j=1}^{3} \left|b_j(\nu) F_{\frac{\gamma}{2\theta}-j}(nk(\nu))\right| + |h_n(\nu)| \\ &\leq C\big(\nu^{4\theta/\gamma} + \nu^{2+2\theta/\gamma}\big).\end{aligned}$$

Note that $4\theta/\gamma < 2$, and the first estimate follows. For the derivative bound,

$$\left|\dot{g}_n(\nu) - \frac{d}{d\nu}\Big(b_0 F_{\frac{\gamma}{2\theta}}(nk(\nu))\Big)\right| \leq \sum_{j=1}^{3} \left|\frac{d}{d\nu}\Big(b_j(\nu) F_{\frac{\gamma}{2\theta}-j}(nk(\nu))\Big)\right| + |\dot{h}_n(\nu)|.$$

Now,

$$\frac{d}{d\nu}\Big(b_j(\nu) F_{\frac{\gamma}{2\theta}-j}(nk(\nu))\Big) = b_j'(\nu) F_{\frac{\gamma}{2\theta}-j}(nk(\nu)) + b_j(\nu) n^2 k'(\nu) k(\nu) F_{\frac{\gamma}{2\theta}-j-1}(nk(\nu)),$$

from which it follows that, using (4.37) (*cf.* Lemma 4.21),

$$\left|\frac{d}{d\nu}\Big(b_j(\nu) F_{\frac{\gamma}{2\theta}-j}(nk(\nu))\Big)\right| \leq C\nu^{4\theta/\gamma-1} \qquad \text{for any } j = 1,2,3.$$

In light of this, we have $\left|\dot{g}_n(\nu) - \frac{d}{d\nu}\Big(b_0 F_{\frac{\gamma}{2\theta}}(nk(\nu))\Big)\right| \leq C\big(\nu^{4\theta/\gamma-1} + \nu^{1+2\theta/\gamma}\big)$. Since $4\theta/\gamma - 1 = 1 - 2/\gamma < 1$, we see that the second estimate holds. □

Thus, we obtain the following result, which concludes this section.

Corollary 4.26. *Let $g_n \in C([0,\mu_*]) \cap C^2((0,\mu_*])$ be the solution of the problem* (4.35) *provided by* (4.36)*. Then, there exists a positive constant $C = C(n,\gamma,\mu_*)$ such that*

$$\big|\rho(\nu)\dot{g}_n(\nu) + n^2 g_n(\nu)\big| \leq C\nu^{2\theta/\gamma} \qquad \text{for } \nu \in [0,\mu_*]. \tag{4.43}$$

Proof. Observe firstly that

$$\rho(\nu)\dot{g}_n(\nu) + n^2 g_n(\nu) =$$
$$\rho(\nu)\frac{d}{d\nu}\left(b_0 k(\nu)^{\frac{\gamma}{2\theta}} I_{-\frac{\gamma}{2\theta}}(nk(\nu))\right) + n^2 b_0 k(\nu)^{\frac{\gamma}{2\theta}} I_{-\frac{\gamma}{2\theta}}(nk(\nu)) + R_0(\nu),$$

where the remainder term $R_0(\nu)$ is such that

$$|R_0(\nu)| \le C\big(\rho(\nu)\nu^{4\theta/\gamma - 1} + \nu^{4\theta/\gamma}\big) \le C\nu^{2\theta/\gamma},$$

as required. It therefore suffices to inspect the term

$$\rho(\nu)\frac{d}{d\nu}\left(b_0 k(\nu)^{\frac{\gamma}{2\theta}} I_{-\frac{\gamma}{2\theta}}(nk(\nu))\right) + n^2 b_0 k(\nu)^{\frac{\gamma}{2\theta}} I_{-\frac{\gamma}{2\theta}}(nk(\nu))$$

near the vacuum. Expanding the term in brackets, using the recurrence relations for the derivatives of the Bessel functions, we find that the above is equal to

$$b_0 n\rho(\nu)k'(\nu)k(\nu)^{\gamma/2\theta} I_{1-\frac{\gamma}{2\theta}}(nk(\nu)) + n^2 b_0 k(\nu)^{\frac{\gamma}{2\theta}} I_{-\frac{\gamma}{2\theta}}(nk(\nu)). \tag{4.44}$$

Recall, for any $l \in \mathbb{R}$, the series representation of the Bessel functions,

$$I_l(z) = (z/2)^l \sum_{j=0}^{\infty} \frac{(z^2/4)^j}{j!\Gamma(l+j+1)},$$

with infinite radius of convergence, from which we deduce that (4.44) can be written

$$\left(b_0 n\rho(\nu)k'(\nu)k(\nu)^{\gamma/2\theta}\frac{(nk(\nu)/2)^{1-\gamma/2\theta}}{\Gamma(2-\gamma/2\theta)} + n^2 b_0 k(\nu)^{\gamma/2\theta}\frac{(nk(\nu)/2)^{-\gamma/2\theta}}{\Gamma(1-\gamma/2\theta)}\right) + R_1(\nu),$$

where the remainder $R_1(\nu)$ is such that $|R_1(\nu)| \le C\nu^{2\theta/\gamma}$. It remains to appropriately control the term in the brackets, which may be rewritten as

$$b_0 n^2 k(\nu)^{\gamma/2\theta}\frac{(nk(\nu)/2)^{-\gamma/2\theta}}{\Gamma(1-\gamma/2\theta)}\left[\frac{1}{2(1-\gamma/2\theta)}\rho(\nu)k'(\nu)k(\nu) + 1\right].$$

In view of Theorem 4.13, the term in the square brackets is equal to

$$-\theta\rho(\nu)\left[a_\sharp \nu^{\theta/\gamma} + a_\flat \nu^{3\theta/\gamma} + L(\nu)\right]\cdot\left[\frac{\theta a_\sharp}{\gamma}\nu^{\theta/\gamma-1} + \frac{3\theta a_\flat}{\gamma}\nu^{3\theta/\gamma-1} + L'(\nu)\right] + 1$$
$$= -\theta\rho(\nu)\frac{\theta a_\sharp^2}{\gamma}\nu^{2\theta/\gamma-1} + R_2(\nu) + 1,$$

where, in light of our asymptotic expansions, we have that $|R_2(\nu)| \le C\rho(\nu)\nu^{4\theta/\gamma-1}$, i.e., there is a positive $C = C(n,\gamma,\mu_*)$ such that $|R_2(\nu)| \le C\nu^{2\theta/\gamma}$ for any $\nu \in [0,\mu_*]$. In fact, the first term on the right-hand side can be rewritten in terms of ρ as

$$-\left(\frac{\gamma}{\theta}\right)^{2\theta/\gamma-1}\rho\nu(\rho)^{2\theta/\gamma-1} = -1 + R_3(\nu(\rho)),$$

where there is a positive constant C for which $|R_3(\nu(\rho))| \le C\rho^{2\theta}$, *cf.* (C.16) in Lemma C.10. Thus, $|\rho(\nu)\dot{g}_n(\nu) + n^2 g_n(\nu)| \le \sum_{j=0}^{3}|R_j(\nu)| \le C\nu^{2\theta/\gamma}$. □

4.7 Existence of the remainder term for the Lax entropies

We now prove a sequence of lemmas to show the existence of the remainder term for the Lax entropies. To begin with, we have the following comparison principle.

Remark 4.27. **In this section only**, since the local sound speed $c(\rho)$ does not appear anywhere, there are positive constants c which bear no relation to $c(\rho)$.

Lemma 4.28. *Let $g_n \in C^2([\varepsilon, \mu_*])$ satisfy*

$$\begin{cases} \ddot{g}_n(\nu) - n^2 k'(\nu)^2 g_n(\nu) \geq 0, \\ g_n(\varepsilon) = g_0, \\ \dot{g}_n(\varepsilon) = g_1, \end{cases} \tag{4.45}$$

where $\varepsilon > 0$ and $g_0, g_1 \geq 0$ are constants. We assume that either $g_0 > 0$ or $g_1 > 0$. Then, g_n and its first two derivatives are non-negative for any $\nu \in [\varepsilon, \mu_*]$; meaning that g_n is a convex increasing function. In fact, we even have $g_n(\nu) \geq g_0$ and $\dot{g}_n(\nu) \geq g_1$ on this interval.

Proof. Suppose that $g_1 > 0$. By continuity, there exists a small neighbourhood of the initial point on which this positivity is conserved. If this small neighbourhood covers the whole interval $[\varepsilon, \mu_*]$, then we are done. Indeed, if this is the case, then

$$g_n(\nu) - g_0 = \int_\varepsilon^\nu \dot{g}_n(y)\, dy > 0 \qquad \text{for any } \nu \in (\varepsilon, \mu_*],$$

since the integral of a strictly positive integrand is itself strictly positive. Thus, $g_n(\nu) > g_0$ for any $\nu \in (\varepsilon, \mu_*]$. The equation then implies that $\ddot{g}_n(\nu) > 0$ for all $\nu \in [\varepsilon, \mu_*]$, which shows, upon integrating, that $\dot{g}_n(\nu) \geq g_1$ on the whole interval.

So, suppose that this small neighbourhood does not cover all of $[\varepsilon, \mu_*]$. Instead, suppose that there is a first point $\xi_0 \in (\varepsilon, \mu_*)$ such that $\dot{g}_n$ is strictly positive in $[\varepsilon, \xi_0)$, and $\dot{g}_n(\xi_0) = 0$. Then,

$$g_n(\nu) - g_0 = \int_\varepsilon^\nu \dot{g}_n(y)\, dy > 0 \qquad \text{for all } \nu \in [\varepsilon, \xi_0],$$

since, once again, the integral of a strictly positive integrand is itself strictly positive. Using the equation, $\ddot{g}_n(\nu) > 0$ for all $\nu \in [\varepsilon, \xi_0]$. Integrating, we find

$$\dot{g}_n(\xi_0) = \int_\varepsilon^{\xi_0} \ddot{g}_n(y)\, dy > 0,$$

which is a contradiction. The case $g_0 > 0$ is very similar. □

Lemma 4.29. *Let $R \in C([0,\mu_*])$ and $c > 0$. Then, the Cauchy problem*

$$\begin{cases} \ddot{w}_n(\nu) - n^2c^2\nu^{\frac{2\theta}{\gamma}-2}w_n(\nu) = R(\nu) \quad \text{in } [0,\mu_*], \\ w_n(0) = 0, \\ \dot{w}_n(0) = 0, \end{cases} \tag{4.46}$$

admits a solution $w_n \in C^1([0,\mu_*]) \cap C^2((0,\mu_*])$ given by the representation formula

$$w_n(\nu) = \int_0^\nu G(\nu;\tau)R(\tau)\,d\tau, \tag{4.47}$$

and with derivative

$$\dot{w}_n(\nu) = \int_0^\nu G_\nu(\nu;\tau)R(\tau)\,d\tau, \tag{4.48}$$

where the kernel $G : [0,\mu_*] \times [0,\mu_*] \to \mathbb{R}$ is given by

$$G(\nu;\tau) = \frac{\gamma}{\theta}\sqrt{\nu\tau}\Big[I_{\frac{\gamma}{2\theta}}\Big(\frac{nc\gamma}{\theta}\nu^{\theta/\gamma}\Big)K_{\frac{\gamma}{2\theta}}\Big(\frac{nc\gamma}{\theta}\tau^{\theta/\gamma}\Big) - K_{\frac{\gamma}{2\theta}}\Big(\frac{nc\gamma}{\theta}\nu^{\theta/\gamma}\Big)I_{\frac{\gamma}{2\theta}}\Big(\frac{nc\gamma}{\theta}\tau^{\theta/\gamma}\Big)\Big]. \tag{4.49}$$

Proof. In view of the asymptotic relations for the modified Bessel functions of the first and second kind $I_{\frac{\gamma}{2\theta}}, K_{\frac{\gamma}{2\theta}}$ (*cf.* [41, 73]),

$$|G(\nu;\tau)| \le C\,(\tau+\nu) \qquad \text{for all } \nu,\tau \in [0,\mu_*],$$

for some positive constant $C = C(n,\gamma,c)$. Hence, the integral is well-defined, and we also manifestly have $G(\nu;\nu) = 0$ for all $\nu \ge 0$. Additionally, for any $\nu,\tau \in (0,\mu_*]$,

$$\begin{aligned} G_\nu(\nu;\tau) = \frac{nc\gamma}{\theta}\tau^{\frac{1}{2}}\nu^{\frac{\theta}{\gamma}-\frac{1}{2}}\Big[&I_{\frac{\gamma}{2\theta}-1}\Big(\frac{nc\gamma}{\theta}\nu^{\theta/\gamma}\Big)K_{\frac{\gamma}{2\theta}}\Big(\frac{nc\gamma}{\theta}\tau^{\theta/\gamma}\Big) \\ &+ K_{\frac{\gamma}{2\theta}-1}\Big(\frac{nc\gamma}{\theta}\nu^{\theta/\gamma}\Big)I_{\frac{\gamma}{2\theta}}\Big(\frac{nc\gamma}{\theta}\tau^{\theta/\gamma}\Big)\Big]. \end{aligned}$$

Since the functions I and K are positive functions away from the origin, the above directly implies that $G_\nu(\nu;\tau) > 0$ for all $\nu,\tau \in (0,\mu_*]$, and that, referring to [73],

$$G_\nu(\nu;\nu) = \frac{nc\gamma}{\theta}\nu^{\frac{\theta}{\gamma}}\Big(I_{\frac{\gamma}{2\theta}-1}K_{\frac{\gamma}{2\theta}} + K_{\frac{\gamma}{2\theta}-1}I_{\frac{\gamma}{2\theta}}\Big)\Big(\frac{nc\gamma}{\theta}\nu^{\theta/\gamma}\Big) = 1 \qquad \text{for all } \nu \ge 0.$$

Additionally, using the asymptotic relations for the modified Bessel functions, we find

$$|G_\nu(\nu;\tau)| \le C\Big(1 + \tau\nu^{\frac{2\theta}{\gamma}-1}\Big) \qquad \text{for all } \nu,\tau \in (0,\mu_*].$$

So, given any $\nu_0 > 0$, we see that $|G_\nu(\nu;\tau)| \le C(1 + \tau\nu_0^{2\theta/\gamma-1})$ on the interval $\nu \in [\nu_0,\mu_*]$. Since this latter bound is independent of ν and is integrable with respect to τ over $[0, \mu_*]$, we deduce from the Lebesgue dominated convergence theorem that w_n is differentiable on the interval $[\nu_0, \mu_*]$, with derivative given by

$$\dot{w}_n(\nu) = G(\nu;\nu)R(\nu) + \int_0^\nu G_\nu(\nu;\tau)R(\tau)\,d\tau \quad \text{for } \nu \in [\nu_0, \mu_*].$$

Since $\nu_0 > 0$ was arbitrary, we deduce that

$$\dot{w}_n(\nu) = G(\nu;\nu)R(\nu) + \int_0^\nu G_\nu(\nu;\tau)R(\tau)\,d\tau \quad \text{for all } \nu \in (0, \mu_*],$$

and the first term on the right-hand side has already been verified to be null, which yields (4.48). We move to the second derivative, and calculate

$$G_{\nu\nu}(\nu;\tau) = n^2c^2\nu^{\frac{2\theta}{\gamma}-2}G(\nu;\tau) \qquad \text{for all } \nu, \tau \in (0, \mu_*],$$

which shows that

$$|G_{\nu\nu}(\nu;\tau)| \leq C\left(\tau\nu^{\frac{2\theta}{\gamma}-2} + \nu^{\frac{2\theta}{\gamma}-1}\right) \qquad \text{for all } \nu, \tau \in (0, \mu_*].$$

Noting that the right-hand side of the above line is integrable in τ, we use the same argument as for $\dot{w}_n$ to deduce that

$$\ddot{w}_n(\nu) = G_\nu(\nu;\nu)R(\nu) + \int_0^\nu G_{\nu\nu}(\nu;\tau)R(\tau)\,d\tau \qquad \text{for all } \nu \in (0, \mu_*].$$

As such, we do indeed recover

$$\ddot{w}_n(\nu) - n^2c^2\nu^{\frac{2\theta}{\gamma}-2}w_n(\nu) = R(\nu),$$

as a pointwise equality between continuous functions in $(0, \mu_*]$. Note also that, in view of the bounds on G and G_ν, we have

$$|w_n(\nu)| \leq C\nu^2, \qquad |\dot{w}_n(\nu)| \leq C\nu \quad \text{for all } \nu \in [0, \mu_*], \tag{4.50}$$

for some positive constant $C = C(n, \gamma, c, \mu_*)$. This shows that both initial data assumptions are satisfied and $w_n \in C^1([0, \mu_*])$. □

Remark 4.30. The requirement that $R \in C([0, \mu_*])$ can be weakened. For example, requiring the pointwise bound $|R(\tau)| \leq C\tau^{\delta-1}$ is enough to make the previous argument hold, with $\delta > 0$.

Corollary 4.31. *Suppose that the forcing term R is non-negative. Then, the solution w_n and its derivative are also non-negative on the whole interval $[0, \mu_*]$.*

Proof. Since $G_\nu(\nu;\tau) > 0$ for all $\nu, \tau \in (0, \mu_*]$, the representation formula for the derivative shows that $\dot{w}_n(\nu) \geq 0$ for all $\nu \in [0, \mu_*]$. Integrating, we find that

$$w_n(\nu) = \int_0^\nu \dot{w}_n(y)\, dy \geq 0 \quad \text{for all } \nu \in [0, \mu_*].$$

□

Corollary 4.32. *Fix $\alpha \in [2\theta/\gamma, 1]$ and suppose that $R \in C([0, \mu_*])$ satisfies, for some $C_0 > 0$, the pointwise bound*

$$|R(\nu)| \leq C_0 \nu^\alpha \qquad \text{for all } \nu \in [0, \mu_*]. \tag{4.51}$$

Correspondingly, let w_n be the solution of the problem (4.46) *given by* (4.47). *Then, there exists a positive constant $C = C(n, \gamma, c, C_0, \mu_*)$ such that*

$$|\nu^{-(\alpha+2)} w_n(\nu)| + |\nu^{-(\alpha+1)} \dot{w}_n(\nu)| + |\nu^{-\alpha} \ddot{w}_n(\nu)| \leq C \qquad \text{for all } \nu \in [0, \mu_*]. \tag{4.52}$$

Additionally, if R is chosen to be non-negative, then w_n and its derivative are also non-negative.

Proof. We already have the bound $|w_n(\nu)| \leq C\nu^2$ from (4.50) in the proof of Lemma 4.29. Then, using the equation and the pointwise bound on the forcing term (4.51), we deduce that

$$|\ddot{w}_n(\nu)| \leq C\nu^{\frac{2\theta}{\gamma}} \qquad \text{for all } \nu \in [0, \mu_*].$$

Integrating twice implies that $|w_n(\nu)| \leq C\nu^{\frac{2\theta}{\gamma}+2}$, which shows that $|\nu^{\frac{2\theta}{\gamma}-2} w_n(\nu)| \leq C\nu^{\frac{4\theta}{\gamma}}$. We can now use the equation to deduce that

$$|\ddot{w}_n(\nu)| \leq C\left(\nu^{\frac{4\theta}{\gamma}} + \nu^\alpha\right) \qquad \text{for all } \nu \in [0, \mu_*].$$

If $4\theta/\gamma \geq \alpha$, we are done. If not, we iterate this procedure m times, until $2(m+1)\theta/\gamma \geq \alpha$. □

Lemma 4.33. *Fix $\alpha \in [2\theta/\gamma, 1]$ and suppose that $r \in C([0, \mu_*])$ satisfies, for some $C_0 > 0$, the pointwise bound*

$$|r(\nu)| \leq C_0 \nu^\alpha \qquad \text{for all } \nu \in [0, \mu_*].$$

Let w_n be the function defined by

$$w_n(\nu) = \int_0^\nu G(\nu;\tau) \left(|r(\tau)| + 2C_0\tau^\alpha\right) d\tau, \tag{4.53}$$

so that it solves the Cauchy problem

$$\begin{cases} \ddot{w}_n(\nu) - n^2 C_*^2 \nu^{\frac{2\theta}{\gamma}-2} w_n(\nu) = |r(\nu)| + 2C_0\nu^\alpha & \text{in } [0,\mu_*], \\ w_n(0) = 0, \\ \dot{w}_n(0) = 0, \end{cases} \tag{4.54}$$

where $C_* = C_*(\gamma)$ is a constant chosen such that $k'(\nu)^2 \leq C_*^2 \nu^{\frac{2\theta}{\gamma}-2}$ for all $\nu \in [0,\mu_*]$ (cf. Corollary 4.14). Meanwhile, let $g_n^\varepsilon \in C^2([\varepsilon,\mu_*])$ be the solution of

$$\begin{cases} \ddot{g}_n^\varepsilon(\nu) - n^2 k'(\nu)^2 g_n^\varepsilon(\nu) = r(\nu) & \text{in } [\varepsilon,\mu_*], \\ g_n^\varepsilon(\varepsilon) = 0, \\ \dot{g}_n^\varepsilon(\varepsilon) = 0, \end{cases} \tag{4.55}$$

which exists and is unique, by virtue of the theorem of Cauchy–Lipschitz. Then,

$$0 \leq |g_n^\varepsilon(\nu)| \leq w_n(\nu) \quad \text{and} \quad 0 \leq |\dot{g}_n^\varepsilon(\nu)| \leq \dot{w}_n(\nu) \quad \text{for all } \nu \in [\varepsilon,\mu_*]. \tag{4.56}$$

In turn, there exists a positive constant $C = C(n,\gamma,c,C_0,\mu_*)$, independent of ε, such that

$$|\nu^{-(\alpha+2)} g_n^\varepsilon(\nu)| + |\nu^{-(\alpha+1)} \dot{g}_n^\varepsilon(\nu)| + |\nu^{-\alpha} \ddot{g}^\varepsilon(\nu)| \leq C \qquad \text{for all } \nu \in [\varepsilon,\mu_*]. \tag{4.57}$$

Proof. Firstly, recall from Corollary 4.31 that the solution of (4.54) is non-negative with non-negative derivative. Further,

$$\dot{w}_n(\varepsilon) = \int_0^\varepsilon G_\nu(\nu;\tau)\left(|r(\tau)| + 2C_0\tau^\alpha\right) d\tau \geq 2C_0 \int_0^\varepsilon G_\nu(\nu;\tau)\tau^\alpha \, d\tau > 0,$$

where the final inequality holds since we consider the integral of a strictly positive integrand. Now, let $\phi(\nu) := w_n(\nu) - g_n^\varepsilon(\nu)$, which satisfies the equation

$$\ddot{\phi}(\nu) = n^2 C_*^2 \nu^{\frac{2\theta}{\gamma}} w_n(\nu) + |r(\nu)| + 2C_0\nu^\alpha - n^2 k'(\nu)^2 g_n(\nu) - r(\nu),$$

which shows that

$$\begin{aligned} \ddot{\phi}(\nu) &\geq n^2 k'(\nu)^2 w_n(\nu) - n^2 k'(\nu)^2 g_n^\varepsilon(\nu) + |r(\nu)| - r(\nu) + 2C_0\nu^\alpha \\ &\geq n^2 k'(\nu)^2 \phi(\nu) + \big(|r(\nu)| - r(\nu)\big). \end{aligned}$$

As such, we have that ϕ satisfies

$$\begin{cases} \ddot{\phi}(\nu) - n^2 k'(\nu)^2 \phi(\nu) \geq 0, \\ \phi(\varepsilon) = w_n(\varepsilon), \\ \dot{\phi}(\varepsilon) = \dot{w}_n(\varepsilon), \end{cases}$$

which shows, using Lemma 4.28, that ϕ is non-negative with non-negative derivative. This automatically yields

$$g_n^\varepsilon(\nu) \le w_n(\nu) \quad \text{and} \quad \dot{g}_n^\varepsilon(\nu) \le \dot{w}_n(\nu) \quad \text{for all } \nu \in [\varepsilon, \mu_*].$$

We get the other side of the inequalities in (4.56) by following the same reasoning with $-w_n$. Then, using the bounds from Corollary 4.32 gives the uniform bounds at the end of the statement. □

We will require the following result to obtain the Hölder estimates.

Claim 4.34. *There exists a positive constant $C = C(\mu_*, \gamma)$ such that*

$$\int_{\nu_1}^{\nu_2} y^{2\theta/\gamma-1}\, dy \le C\,(\nu_2 - \nu_1)^{1-2\theta/\gamma}.$$

Proof. Observe that, since $2\theta/\gamma < 1$, Jensen's inequality shows that

$$\begin{aligned}\frac{1}{\nu_2-\nu_1}\int_{\nu_1}^{\nu_2} y^{2\theta/\gamma-1}\, dy = \fint_{\nu_1}^{\nu_2} y^{2\theta/\gamma-1}\, dy &= \fint_{\nu_1}^{\nu_2} \left(y^{1-\gamma/2\theta}\right)^{2\theta/\gamma}\, dy \\ &\le \left(\fint_{\nu_1}^{\nu_2} y^{1-\gamma/2\theta}\, dy\right)^{2\theta/\gamma}.\end{aligned}$$

The term on the right-hand side is equal to

$$\begin{aligned}\left[\frac{1}{\nu_2-\nu_1}\left(\frac{\nu_2^{2-\gamma/2\theta}}{2-\gamma/2\theta} - \frac{\nu_1^{2-\gamma/2\theta}}{2-\gamma/2\theta}\right)\right]^{2\theta/\gamma} &= C_\theta(\nu_2-\nu_1)^{-2\theta/\gamma}\left(\nu_2^{2-\gamma/2\theta} - \nu_1^{2-\gamma/2\theta}\right)^{2\theta/\gamma} \\ &\le C_\theta(\nu_2-\nu_1)^{-2\theta/\gamma}(2\mu_*^{2-\gamma/2\theta})^{2\theta/\gamma},\end{aligned}$$

since both $2\theta/\gamma$ and $2-\gamma/2\theta$ are strictly positive. Hence, letting $C := 2^{2\theta/\gamma}C_\theta\mu_*^{4\theta/\gamma-1}$,

$$\int_{\nu_1}^{\nu_2} y^{2\theta/\gamma-1}\, dy \le C(\nu_2-\nu_1)^{1-2\theta/\gamma}.$$

□

Lemma 4.35. *Fix $\alpha \in [2\theta/\gamma, 1]$. Suppose that $r \in C([0,\mu_*])$ is such that $r(0) = 0$ and, for some $C_0 > 0$,*

$$|r(\nu_1) - r(\nu_2)| \le C_0|\nu_1-\nu_2|^\alpha \qquad \textit{for all } \nu_1, \nu_2 \textit{ in the interval } [0,\mu_*].$$

Let $g_n^\varepsilon \in C^2([\varepsilon,\mu_*])$ be the solution of

$$\begin{cases} \ddot{g}_n^\varepsilon(\nu) - n^2k'(\nu)^2 g_n^\varepsilon(\nu) = r(\nu) & \textit{in } [\varepsilon,\mu_*], \\ g_n^\varepsilon(\varepsilon) = 0, \\ \dot{g}_n^\varepsilon(\varepsilon) = 0, \end{cases} \tag{4.58}$$

where $\varepsilon > 0$ is constant, which exists and is unique by virtue of the theorem of Cauchy–Lipschitz. Then, there exists a positive constant $C = C(n, \gamma, C_0, \mu_*)$, independent of ε, such that

$$|g_n^\varepsilon(\nu)| + |\dot{g}_n^\varepsilon(\nu)| + |\ddot{g}_n^\varepsilon(\nu)| \leq C \qquad \text{for all } \nu \in [\varepsilon, \mu_*], \tag{4.59}$$

and

$$|\ddot{g}_n^\varepsilon(\nu_1) - \ddot{g}_n^\varepsilon(\nu_2)| \leq C|\nu_1 - \nu_2|^{1-\frac{2\theta}{\gamma}} \qquad \text{for all } \nu_1, \nu_2 \text{ in the interval } [\varepsilon, \mu_*]. \tag{4.60}$$

Proof. Note that the requirements on $r(\nu)$ automatically implies that it satisfies the pointwise bound of Corollary 4.32. The uniform bounds thereby follow directly from Lemma 4.33. For the Hölder estimate, the equation shows that

$$|\ddot{g}_n^\varepsilon(\nu_1) - \ddot{g}_n^\varepsilon(\nu_2)| \leq n^2|k'(\nu_1)^2 g_n^\varepsilon(\nu_1) - k'(\nu_2)^2 g_n^\varepsilon(\nu_2)| + |r(\nu_1) - r(\nu_2)|.$$

Observe that, by the fundamental theorem of calculus,

$$|k'(\nu_1)^2 g_n^\varepsilon(\nu_1) - k'(\nu_2)^2 g_n^\varepsilon(\nu_2)| = \left| \int_{\nu_1}^{\nu_2} \frac{d}{dy}\left(k'(y)^2 g^\varepsilon(y)\right) dy \right|.$$

Now, we have

$$\left| \frac{d}{dy}\left(k'(y)^2 g_n^\varepsilon(y)\right) \right| = \left|2k'(y)k''(y)g_n^\varepsilon(y) + k'(y)^2 \dot{g}_n^\varepsilon(y)\right| \leq C y^{\alpha + \frac{2\theta}{\gamma} - 1},$$

where we used the estimate (4.57). Hence,

$$|k'(\nu_1)^2 g_n^\varepsilon(\nu_1) - k'(\nu_2)^2 g_n^\varepsilon(\nu_2)| \leq C \int_{\nu_1}^{\nu_2} y^{\alpha + \frac{2\theta}{\gamma} - 1}\, dy,$$

and we deduce the Hölder bound on $\ddot{g}_n^\varepsilon$ immediately from Claim 4.34.

□

Remark 4.36. The same Hölder bound also holds for g_n^ε and $\dot{g}_n^\varepsilon$. In light of this, we are now in a position to prove Lemma 4.18.

Proof of Lemma 4.18. The sequential compactness of the sequence $(g_n^\varepsilon)_{\varepsilon>0}$ follows directly from an application of the Arzelà–Ascoli theorem, using the Hölder bounds and uniform estimates from the previous lemma. Since the convergence happens in the strong $C^2([0, \mu_*])$ sense, we are able to pass to the limit in the equation and in the initial data. □

4.8 Proof of main result

Having established the existence of a solution g_n which satisfies the estimates (4.29) of Section 4.5, we now demonstrate the local compactness in $H^{-1}(\Omega)$ of the entropy dissipation measures generated by this Lax entropy.

Lemma 4.37. *Let $H(\nu, t) = g_n(\nu)e^{\pm nt}$, where g_n is the solution of the problem* (4.35) *provided by* (4.36) *(cf. Lemma 4.21). Then, H solves the entropy equation* (4.17)*. Let (Q_1, Q_2) be the corresponding entropy pair generated by H, via the relations*

$$Q_1(\mathbf{u}) = \rho q H_\nu \cos t - qH_t \sin t \qquad \textit{and} \qquad Q_2(\mathbf{u}) = \rho q H_\nu \sin t + qH_t \cos t, \quad (4.61)$$

as in Lemma 4.7. Accordingly, with $\mathbf{u}^\varepsilon$ the solution of the viscous problem (4.8)-(4.9) *(cf. Lemma 4.3), the sequence of entropy dissipation measures*

$$\partial_x Q_1(\mathbf{u}^\varepsilon) + \partial_y Q_2(\mathbf{u}^\varepsilon),$$

is confined to a compact subset of $H^{-1}(\Omega)$.

Proof. Recall that for such an entropy pair (Q_1, Q_2) generated by H, we have

$$\begin{aligned}
&\partial_x Q_1(\mathbf{u}^\varepsilon) + \partial_y Q_2(\mathbf{u}^\varepsilon) \\
&\qquad =\varepsilon \operatorname{div}\left(\sigma_1(\rho^\varepsilon)\nabla t^\varepsilon \left(\rho^\varepsilon H_{\nu t} - H_t\right) + \sigma_2(\rho^\varepsilon)\nabla\rho^\varepsilon \left(H_\nu + \frac{1}{\rho^\varepsilon}H_{tt}\right)\right) \\
&\qquad\quad - \varepsilon\sigma_1(\rho^\varepsilon)\nabla t^\varepsilon \cdot \nabla\left(\rho^\varepsilon H_{\nu t} - H_t\right) - \varepsilon\sigma_2(\rho^\varepsilon)\nabla\rho^\varepsilon \cdot \nabla\left(H_\nu + \frac{1}{\rho^\varepsilon}H_{tt}\right) \\
&\qquad =: K_1^\varepsilon + K_2^\varepsilon.
\end{aligned} \quad (4.62)$$

Now, dropping the ε superscript as before, we have

$$-\varepsilon^{-1}K_2^\varepsilon = \Big[\rho H_{\nu tt} - H_{tt}\Big]\left(\sigma_1|\nabla t|^2 + \sigma_2\frac{c^2}{(\rho q)^2}|\nabla\rho|^2\right) + \Big[H_{ttt} + \rho H_{\nu t}\Big]\frac{2\sigma_2}{\rho}(\nabla t\cdot\nabla\rho). \quad (4.63)$$

Note that, the coefficient in front of the first term of (4.63) is bounded as follows,

$$\begin{aligned}
|\rho(\nu)H_{\nu tt}(\nu,t) - H_{tt}(\nu,t)| &= n^2 e^{\pm nt}\left|\rho(\nu)\dot g_n(\nu) - g_n(\nu)\right| \\
&\le C\big(\rho(\nu)|\dot g_n(\nu)| + |g_n(\nu)|\big) \le C,
\end{aligned}$$

in view of Lemma 4.21, where C is independent of ε. Additionally, the coefficient in front of the second term of (4.63) can be written as

$$|H_{ttt}(\nu,t) + \rho(\nu)H_{\nu t}(\nu,t)| = ne^{\pm nt}\left|\rho(\nu)\dot g_n(\nu) + n^2 g_n(\nu)\right| \le C\nu^{2\theta/\gamma},$$

using Corollary 4.26 and Lemma C.8 . Hence, we write

$$\begin{aligned}\left|(H_{ttt}+\rho H_{\nu t})\frac{2\sigma_2}{\rho}(\nabla t\cdot\nabla\rho)\right| &\le C\sqrt{\frac{\sigma_2}{\sigma_1}}\nu^{2\theta/\gamma}\frac{q}{c}\left|\sqrt{\sigma_1}\nabla t\cdot\sqrt{\sigma_2}\frac{c}{\rho q}\nabla\rho\right|\\ &\le C\nu^{2\theta/\gamma}\rho(\nu)^{-\theta}\bigg(\sigma_1|\nabla t|^2+\sigma_2\frac{c^2}{(\rho q)^2}|\nabla\rho|^2\bigg)\\ &\le C\bigg(\sigma_1|\nabla t|^2+\sigma_2\frac{c^2}{(\rho q)^2}|\nabla\rho|^2\bigg).\end{aligned}$$

With C independent of ε, we obtain

$$|K_2^\varepsilon|\le C\varepsilon\bigg(\sigma_1|\nabla t|^2+\sigma_2\frac{c^2}{(\rho q)^2}|\nabla\rho|^2\bigg).$$

We deduce with the help of Lemma 4.17 that K_2^ε is bounded in $L^1(\Omega)$ uniformly in ε, which induces compactness in $W^{-1,\tilde p}(\Omega)$ for any $\tilde p\in(1,2)$ by the Rellich theorem, as $\partial\Omega$ is Lipschitz. For K_1^ε, the term inside the divergence is bounded as follows,

$$\begin{aligned}&\left|\sigma_1(\rho^\varepsilon)\nabla t^\varepsilon\left(\rho^\varepsilon H_{\nu t}-H_t\right)+\sigma_2(\rho^\varepsilon)\nabla\rho^\varepsilon\left(H_\nu+\frac{1}{\rho^\varepsilon}H_{tt}\right)\right|\\ &\qquad\le|\rho^\varepsilon H_{\nu t}-H_t|\,\sigma_1(\rho^\varepsilon)|\nabla t^\varepsilon|+|\rho^\varepsilon H_\nu+H_{tt}|\,\frac{q^\varepsilon}{c(\rho^\varepsilon)}\sigma_2(\rho^\varepsilon)\frac{c(\rho^\varepsilon)}{\rho^\varepsilon q^\varepsilon}|\nabla\rho^\varepsilon|\\ &\qquad\le C\bigg(\sigma_1|\nabla t|+\sigma_2\frac{c}{\rho q}|\nabla\rho|\bigg),\end{aligned}$$

where we appealed to the fact that $|\rho(\nu)\dot g_n(\nu)|+|g_n(\nu)|\le C$, and

$$|\rho^\varepsilon H_\nu+H_{tt}|\,\frac{q^\varepsilon}{c(\rho^\varepsilon)}\le C\big|\rho(\nu)\dot g_n(\nu)+n^2g_n(\nu)\big|\rho^{-\theta}(\nu)\le C. \tag{4.64}$$

Also, $\sigma_i\le\sqrt{\sigma_i}$ for $i=1,2$, by bluntly bounding the expressions given in Lemma 4.3. Thus, making use of (4.64) again,

$$\begin{aligned}&\varepsilon\left\|\sigma_1(\rho^\varepsilon)\nabla t^\varepsilon\left(\rho^\varepsilon H_{\nu t}-H_t\right)+\sigma_2(\rho^\varepsilon)\nabla\rho^\varepsilon\left(H_\nu+\frac{1}{\rho^\varepsilon}H_{tt}\right)\right\|_{L^2(\Omega)}\\ &\qquad\le C\varepsilon\bigg(\int_\Omega\Big(\sqrt{\sigma_1}|\nabla t|+\sqrt{\sigma_2}\frac{c}{\rho q}|\nabla\rho|\Big)^2\,dx\,dy\bigg)^{1/2}\\ &\qquad\le C\sqrt{\varepsilon}\bigg(\varepsilon\int_\Omega\Big(\sigma_1(\rho^\varepsilon)|\nabla t^\varepsilon|^2+\sigma_2(\rho^\varepsilon)\frac{c^2(\rho^\varepsilon)}{(\rho^\varepsilon q^\varepsilon)^2}|\nabla\rho^\varepsilon|^2\Big)\,dx\,dy\bigg)^{1/2}\le C\sqrt{\varepsilon},\end{aligned}$$

using Lemma 4.17. As such, we deduce that $\|K_1^\varepsilon\|_{H^{-1}(\Omega)}\le C\sqrt{\varepsilon}$, which vanishes in the limit as $\varepsilon\to0$. Additionally, we see directly from (4.61) that (Q_1,Q_2) are uniformly bounded, which yields that $K_1^\varepsilon+K_2^\varepsilon$ is bounded in $W^{-1,\infty}(\Omega)$. The interpolation compactness lemma of Ding, Chen, and Luo [28, Chapter 4] (*cf.* Murat's lemma [71]) now implies that $K_1^\varepsilon+K_2^\varepsilon$ is confined to a compact set of $H^{-1}(\Omega)$. □

In turn, by collating the results of Lemmas 4.3 and 4.37, Theorem 1.21 is proved.

Appendix A

Elementary results on Bessel functions

In this appendix, we present a multitude of elementary results that are used in Chapters 2, 3 and 4. As they stand, these are somewhat out of context, but they appear in crucial estimates in the main body of the thesis.

A.1 Some useful integral identities

Lemma A.1. *For any $R \geq 0$, we have that*

$$\begin{aligned}
&\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\, d\theta = \frac{2}{R}\sinh\left(\frac{R}{2}\right), \\
&\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\sin\theta\, d\theta = \frac{2}{R}I_1\left(\frac{R}{2}\right), \\
&\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\sin^2\theta\, d\theta = \frac{4}{R^2}\cosh\left(\frac{R}{2}\right) - \frac{8}{R^3}\sinh\left(\frac{R}{2}\right).
\end{aligned} \tag{A.1}$$

Proof. The above is contained in [41]. □

Lemma A.2. *For any $R \geq 0$, we have that*

$$\begin{aligned}
&\int_0^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right) d\theta = \frac{2}{R}\cosh\left(\frac{R}{2}\right) - \frac{2}{R}, \\
&\int_0^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right)\sin^2\theta\, d\theta = \frac{4}{R^2}\sinh\left(\frac{R}{2}\right) - \frac{2}{R}, \\
&\int_0^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right)\cos^2\theta\, d\theta = \frac{2}{R}\cosh\left(\frac{R}{2}\right) - \frac{4}{R^2}\sinh\left(\frac{R}{2}\right).
\end{aligned} \tag{A.2}$$

Proof. Let

$$\tilde{g}_1(R) := \int_0^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right) d\theta.$$

Then, explicit calculation shows that

$$\begin{aligned}\tilde{g}_1'(R) &= \frac{1}{2}\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\, d\theta - \frac{1}{R}\int_0^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right)\, d\theta, \\ &= \frac{1}{R}\sinh\left(\frac{R}{2}\right) - \frac{1}{R}g_1(R),\end{aligned}$$

thereby implying that $\tilde{g}_1'(R) + \frac{1}{R}\tilde{g}_1(R) = \frac{\sinh\left(\frac{R}{2}\right)}{R}$, which can easily be solved by multiplying by the integrating factor R. This yields

$$\int_0^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right)\, d\theta = \frac{2}{R}\cosh\left(\frac{R}{2}\right) - \frac{2}{R}. \tag{A.3}$$

Using the same idea for the second inner integral we find that, by letting $\tilde{g}_2(R) := \int_0^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right)\sin^2\theta\, d\theta$, we obtain

$$(R\tilde{g}_2(R))' = \frac{2}{R}\cosh\left(\frac{R}{2}\right) - \frac{4}{R^2}\sinh\left(\frac{R}{2}\right).$$

Despite appearances, a Taylor expansion shows that the right-hand side of the above is locally integrable near $R = 0$. In fact, by computing the integral explicitly, we arrive at

$$\int_0^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right)\sin^2\theta\, d\theta = \frac{4}{R^2}\sinh\left(\frac{R}{2}\right) - \frac{2}{R}.$$

The final integral is easily obtained from the previous two, noting that $\cos^2\theta = 1 - \sin^2\theta$. □

Lemma A.3. *For any $R \geq 0$, we have that*

$$\int_0^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right)\sin\theta\, d\theta = \frac{2}{R}\left(I_0\left(\frac{R}{2}\right) - 1\right). \tag{A.4}$$

Proof. Let $\tilde{g}_3(R) := \int_0^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right)\sin\theta\, d\theta$, and observe that differentiation under the integral yields the equation $\tilde{g}_3'(R) + \frac{1}{R}\tilde{g}_3(R) = \frac{1}{2}\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\sin\theta\cos\theta\, d\theta$. Using the result of Lemma A.1 and integrating establishes the result. □

Lemma A.4. *For any $R \geq 0$ we have that*

$$\int_0^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right)\sin^4\theta\, d\theta = \frac{24}{R^4}\left(R\cosh\left(\frac{R}{2}\right) - 2\sinh\left(\frac{R}{2}\right)\right) - \frac{2}{R}. \tag{A.5}$$

Proof. Let $\tilde{g}_4(R) := \int_0^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right)\sin^4\theta\, d\theta$. Then, observe that

$$\tilde{g}_4'(R) + \frac{1}{R}\tilde{g}_4(R) = \frac{1}{2}\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\sin^4\theta\, d\theta,$$

and so

$$(R\tilde{g}_4(R))' = \frac{12}{R^4}\left((R^2+12)\sinh\left(\frac{R}{2}\right) - 6R\cosh\left(\frac{R}{2}\right)\right).$$

Integrating and using the condition $g_3(0) = 0$ determines the constant of integration, thereby giving

$$\tilde{g}_4(R) = \frac{24}{R^4}\left(R\cosh\left(\frac{R}{2}\right) - 2\sinh\left(\frac{R}{2}\right)\right) - \frac{2}{R}.$$

□

Lemma A.5. *For any $R \geq 0$ we have that*

$$\begin{aligned}
\int_0^{\pi/2} I_2\left(\frac{R}{2}\cos\theta\right)\sec\theta\, d\theta &= \frac{2}{R}\sinh\left(\frac{R}{2}\right) - \frac{4}{R^2}\cosh\left(\frac{R}{2}\right) + \frac{4}{R^2} - \frac{1}{2},\\
\int_0^{\pi/2} I_2\left(\frac{R}{2}\cos\theta\right)\cos\theta\, d\theta &= \frac{2}{R}\sinh\left(\frac{R}{2}\right) - \frac{8}{R^2}\cosh\left(\frac{R}{2}\right) + \frac{8}{R^2}, \qquad \text{(A.6)}\\
\int_0^{\pi/2} I_2\left(\frac{R}{2}\cos\theta\right)\sin^2\theta\sec\theta\, d\theta &= \frac{4}{R^2}\cosh\left(\frac{R}{2}\right) - \frac{4}{R^2} - \frac{1}{2}.
\end{aligned}$$

Proof. The first two integrals can be found in [41]. The identity $\sin^2\theta + \cos^2\theta = 1$ yields the third integral. □

A.2 Some useful pointwise bounds

We begin with a very straightforward upper bound on the growth of the modified Bessel functions in the large.

Lemma A.6. *For any $\nu \geq 0$, there exists a positive constant $C = C(\nu)$ such that the inequality*

$$I_\nu(x) \leq \frac{Ce^x}{1+\sqrt{x}}, \qquad \text{(A.7)}$$

holds for any $x \geq 0$.

Proof. The result follows from the asymptotics $I_\nu(z) \sim \frac{(z/2)^\nu}{\Gamma(\nu+1)}$ as $z \to 0$, and $I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}}$ as $z \to \infty$ (*cf.* [41]). The notation $\sim$ was introduced in Section 4.6. □

Lemma A.7. *Given any $\delta > 0$, there exists a positive constant $C = C(\delta)$ such that*

$$e^{\sqrt{\log x}} \leq Cx^\delta \qquad \textit{for } x \geq 1. \qquad \text{(A.8)}$$

Proof. We achieve equality with $C = 1$ at $x = 1$. For $x > 1$, let $x = e^y$ for $y > 0$. The inequality then reads $e^{\sqrt{y}} \leq Ce^{\delta y}$,i.e., $\sqrt{y} \leq \log C + \delta y$. Since the curves $\sqrt{y}$ and δy intersect at 0 and at $y = \delta^{-2}$, at which point $\sqrt{y} = \delta y = \delta^{-1}$, the inequality is satisfied provided we choose $\log C \geq \delta^{-1}$, i.e., $C \geq \exp(\delta^{-1})$. □

Lemma A.8. *Given any $\delta > 0$, there exists a positive constant $C = C(\delta)$ such that*

$$\log x \le C x^{\delta} \qquad \text{for } x \ge 1. \tag{A.9}$$

Proof. The inequality is trivially satisfied at $x = 1$. For $x > 1$, write $x = e^{\tilde{y}/\delta}$ for $\tilde{y} > 0$. The inequality then reads $\frac{\tilde{y}}{C\delta} \le e^{\tilde{y}}$, which is true provided $C \ge \delta^{-1}$. ☐

Lemma A.9. *There exists a positive constant C such that, for any $R \ge 0$,*

$$\begin{aligned} \sup_{v\in[0,R]} \left| v I_0\left(\frac{\sqrt{R^2-v^2}}{2}\right)\right| &\le e^{R/2}, \\ \sup_{v\in[0,R]} \left| R I_0\left(\frac{\sqrt{R^2-v^2}}{2}\right) - \sqrt{R^2-v^2}\, I_1\left(\frac{\sqrt{R^2-v^2}}{2}\right)\right| &\le C e^{R/2}. \end{aligned} \tag{A.10}$$

Proof. For $(R, v) \in [0,\infty) \times [0, R]$, define $\tilde{g}_1(R, v) := v I_0\left(\frac{R^2-v^2}{2}\right) \ge 0$. Then, the fundamental theorem of calculus yields

$$\tilde{g}_1(R, v) = \int_0^v \partial_y \left(y I_0\left(\frac{R^2-y^2}{2}\right)\right) dy.$$

Explicit computation shows that $\partial_y \left(y I_0\left(\frac{R^2-y^2}{2}\right)\right)$ is equal to

$$I_0\left(\frac{\sqrt{R^2-y^2}}{2}\right) - \frac{y^2}{2\sqrt{R^2-y^2}} I_1\left(\frac{\sqrt{R^2-y^2}}{2}\right) \le I_0\left(\frac{\sqrt{R^2-y^2}}{2}\right),$$

where the inequality follows from the non-negativity of the second term on the right-hand side. As such, since $v \in [0, R]$, we see that

$$\tilde{g}_1(R, v) \le \int_0^R I_0\left(\frac{\sqrt{R^2-y^2}}{2}\right) dy = R\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\, d\theta = 2\sinh\left(\frac{R}{2}\right),$$

and the first estimate of (A.10) follows.

Define $\tilde{g}_2(R, v) := R I_0\left(\frac{\sqrt{R^2-v^2}}{2}\right) - \sqrt{R^2-v^2}\, I_1\left(\frac{\sqrt{R^2-v^2}}{2}\right) \ge 0$, and fix $R \ge 0$. We compute

$$\partial_v \tilde{g}_2(R, v) = \frac{v}{2} I_0\left(\frac{\sqrt{R^2-v^2}}{2}\right) - \frac{v}{2}\frac{R}{\sqrt{R^2-v^2}} I_1\left(\frac{\sqrt{R^2-v^2}}{2}\right),$$

from which we get $\tilde{g}_2(R, \cdot) \in C^1([0, R])$. Note that $|\tilde{g}_2(R, 0)| + |\tilde{g}_2(R, R)| \le CR$ for some positive constant C. Suppose $v_* = v_*(R) \in (0, R)$ is an interior maximum point for $\tilde{g}_2(R, \cdot)$. Since the function is C^1, we must have $\partial_v \tilde{g}_2(R, v_*) = 0$. Hence, at this specific point,

$$\tilde{g}_2(R, v_*) = \frac{v_*^2}{R} I_0\left(\frac{\sqrt{R^2-v_*^2}}{2}\right) \le \sup_{w\in[0,R]} w I_0\left(\frac{\sqrt{R^2-w^2}}{2}\right) \le e^{R/2},$$

where the final inequality follows from the first estimate of (A.10). Thus, $\tilde{g}_2(R, v) \le C\big(R + e^{R/2}\big)$, from which the second estimate of (A.10) follows easily. ☐

Appendix B

Computations for the pointwise estimates of Section 2.4

B.1 The first term, $\hat{\mathcal{J}}_1$

The term $\hat{\mathcal{J}}_1$ may be rewritten as

$$\hat{\mathcal{J}}_1 = \frac{\sqrt{\rho}}{4} \int_{\mathbb{R}} Y(t) \mathcal{K}_1(\rho, u; t)\, dt,$$

where

$$\mathcal{K}_1(\rho, u; t) := \int_{-\log\rho}^{\log\rho} I_0\left(\frac{\sqrt{(\log\rho)^2 - z^2}}{2}\right)(z + u - t)|z + u - t|\, dz.$$

Lemma B.1. *The kernel $\mathcal{K}_1$ can be written as*

$$\mathcal{K}_1(\rho, u; t) := L_1^+(\rho, u; t)\mathbb{1}_{t-u\leq -R} + L_1^-(\rho, u; t)\mathbb{1}_{t-u\geq R} + K_1(\rho, u; t)\mathbb{1}_{|t-u|<R}, \tag{B.1}$$

where

$$L_1^+(\rho, u; t) := \int_{-R}^{R} I_0\left(\frac{\sqrt{R^2 - z^2}}{2}\right)(z + u - t)^2\, dz, \tag{B.2}$$

$$L_1^-(\rho, u; t) := -\int_{-R}^{R} I_0\left(\frac{\sqrt{R^2 - z^2}}{2}\right)(z + u - t)^2\, dz, \tag{B.3}$$

and

$$K_1(\rho, u; t) := \int_{t-u}^{R} I_0\left(\frac{\sqrt{R^2 - z^2}}{2}\right)(z + u - t)^2\, dz - \int_{-R}^{t-u} I_0\left(\frac{\sqrt{R^2 - z^2}}{2}\right)(z + u - t)^2\, dz. \tag{B.4}$$

Thus, we can write the whole term $\hat{\mathcal{J}}_1$ *as*

$$\hat{\mathcal{J}}_1 = \frac{\sqrt{\rho}}{4}\int_{-\infty}^{u-R} Y(t)L_1^+(\rho,u;t)\,dt + \frac{\sqrt{\rho}}{4}\int_{u-R}^{u+R} Y(t)K_1(\rho,u;t)\,dt \\ + \frac{\sqrt{\rho}}{4}\int_{u+R}^{\infty} Y(t)L_1^-(\rho,u;t)\,dt. \tag{B.5}$$

Lemma B.2. *The terms* $L_1^{\pm}$ *can be simplified to be*

$$L_1^{\pm}(\rho,u;t) = \pm\, 2R^3\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\sin^2\theta\,d\theta \\ \pm\, 2R(u-t)^2\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\,d\theta. \tag{B.6}$$

Lemma B.3. *The term* $K_1(\rho,u;t)$ *can be decomposed explicitly into*

$$K_1(\rho,u;t) = K_1^-(\rho,u;t)\mathbb{1}_{\{t-u>0\}} + K_1^+(\rho,u;t)\mathbb{1}_{\{u-t>0\}}, \tag{B.7}$$

where

$$K_1^-(\rho,u;t) = -4(t-u)R^2\int_{\arcsin((t-u)/R)}^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\sin\theta\,d\theta \\ -\, 2R^3\int_0^{\arcsin((t-u)/R)} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\sin^2\theta\,d\theta \\ -\, 2(t-u)^2R\int_0^{\arcsin((t-u)/R)} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\,d\theta, \tag{B.8}$$

and

$$K_1^+(\rho,u;t) = 4(u-t)R^2\int_{\arcsin((u-t)/R)}^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\sin\theta\,d\theta \\ +\, 2R^3\int_0^{\arcsin((u-t)/R)} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\sin^2\theta\,d\theta \\ +\, 2(u-t)^2R\int_0^{\arcsin((u-t)/R)} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\,d\theta. \tag{B.9}$$

Remark B.4. The first term of K_1^- and K_1^+ can be computed explicitly as

$$-8(t-u)\sqrt{R^2-(t-u)^2}\,I_1\left(\frac{\sqrt{R^2-(t-u)^2}}{2}\right), \tag{B.10}$$

and

$$8(u-t)\sqrt{R^2-(u-t)^2}\,I_1\left(\frac{\sqrt{R^2-(u-t)^2}}{2}\right), \tag{B.11}$$

respectively.

Lemma B.5. *The term K_1 has derivative*

$$\partial_u K_1(\rho, u; t) = \partial_u K_1^-(\rho, u; t)\mathbb{1}_{t-u>0} + \partial_u K_1^+(\rho, u; t)\mathbb{1}_{u-t>0}, \tag{B.12}$$

where

$$\begin{aligned}\partial_u K_1^-(\rho, u; t) = 4R^2 &\int_{\arcsin\left(\frac{t-u}{R}\right)}^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\sin\theta\, d\theta \\ &+ 4(t-u)R\int_0^{\arcsin((t-u)/R)} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\, d\theta,\end{aligned}$$

and

$$\begin{aligned}\partial_u K_1^+(\rho, u; t) = 4R^2 &\int_{\arcsin\left(\frac{u-t}{R}\right)}^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\sin\theta\, d\theta \\ &+ 4(u-t)R\int_0^{\arcsin((u-t)/R)} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\, d\theta.\end{aligned}$$

Proof. Observe that all terms arising as Dirac masses are pre-multiplied by zero. The rest is acquired by explicit computation. □

Remark B.6. The first term of $\partial_u K_1^-$ and $\partial_u K_1^+$ can be computed explicitly as

$$8\sqrt{R^2-(t-u)^2}\, I_1\left(\frac{\sqrt{R^2-(t-u)^2}}{2}\right), \tag{B.13}$$

and

$$8\sqrt{R^2-(u-t)^2}\, I_1\left(\frac{\sqrt{R^2-(u-t)^2}}{2}\right) \tag{B.14}$$

Lemma B.7. *The terms $L_1^\pm$ have derivatives*

$$\begin{aligned}\partial_u L_1^+(\rho, u; t) &= 4R(u-t)\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\, d\theta, \\ \partial_u L_1^-(\rho, u; t) &= 4R(t-u)\int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\, d\theta.\end{aligned} \tag{B.15}$$

Lemma B.8. *The kernel $\mathcal{K}_1$ has derivative*

$$\begin{aligned}\partial_u \mathcal{K}_1(\rho, u; t) = \partial_u L_1^+(\rho, u; t)\mathbb{1}_{t-u\leq -R} &+ \partial_u L_1^-(\rho, u; t)\mathbb{1}_{t-u\geq R} \\ &+ \partial_u K_1(\rho, u; t)\mathbb{1}_{|t-u|<R}.\end{aligned} \tag{B.16}$$

Proof. Observe that all terms arising as Dirac masses cancel. □

Lemma B.9. *The term K_1 has second derivative*

$$\partial_{uu} K_1(\rho, u; t) = \partial_{uu} K_1^-(\rho, u; t)\mathbb{1}_{t-u>0} + \partial_{uu} K_1^+(\rho, u; t)\mathbb{1}_{u-t>0},$$

where

$$\partial_{uu} K_1^-(\rho, u; t) = -4R \int_0^{\arcsin((t-u)/R)} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\, d\theta,$$
$$\partial_{uu} K_1^+(\rho, u; t) = 4R \int_0^{\arcsin((u-t)/R)} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\, d\theta.$$

Proof. Observe that all terms arising as Dirac masses are pre-multiplied by zero. The rest is acquired by explicit computation. □

Lemma B.10. *The terms $L_1^\pm$ have derivatives*

$$\begin{aligned}\partial_{uu} L_1^+(\rho, u; t) &= 4R \int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\, d\theta,\\ \partial_{uu} L_1^-(\rho, u; t) &= -4R \int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\, d\theta.\end{aligned} \tag{B.17}$$

Lemma B.11. *The term K_1 has mixed derivative*

$$\partial_{\rho u} K_1(\rho, u; t) = \partial_{\rho u} K_1^-(\rho, u; t)\mathbb{1}_{t-u>0} + \partial_{\rho u} K_1^+(\rho, u; t)\mathbb{1}_{u-t>0},$$

where

$$\begin{aligned}\partial_{\rho u} K_1^-(\rho, u; t) =& \frac{8R}{\rho}\int_{\arcsin\left(\frac{t-u}{R}\right)}^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\sin\theta\, d\theta\\ &+ \frac{2R^2}{\rho}\int_{\arcsin\left(\frac{t-u}{R}\right)}^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right)\cos^2\theta\sin\theta\, d\theta,\\ &+ \frac{4(t-u)}{\rho}\int_0^{\arcsin\left(\frac{t-u}{R}\right)} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\, d\theta\\ &+ \frac{2R(t-u)}{\rho}\int_0^{\arcsin\left(\frac{t-u}{R}\right)} I_1\left(\frac{R}{2}\cos\theta\right)\cos^2\theta\, d\theta,\end{aligned} \tag{B.18}$$

and

$$\begin{aligned}\partial_{\rho u} K_1^+(\rho, u; t) =& \frac{8R}{\rho}\int_{\arcsin\left(\frac{u-t}{R}\right)}^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\sin\theta\, d\theta\\ &+ \frac{2R^2}{\rho}\int_{\arcsin\left(\frac{u-t}{R}\right)}^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right)\cos^2\theta\sin\theta\, d\theta\\ &+ \frac{4(u-t)}{\rho}\int_0^{\arcsin\left(\frac{u-t}{R}\right)} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\, d\theta\\ &+ \frac{2R(u-t)}{\rho}\int_0^{\arcsin\left(\frac{u-t}{R}\right)} I_1\left(\frac{R}{2}\cos\theta\right)\cos^2\theta\, d\theta.\end{aligned} \tag{B.19}$$

Remark B.12. The first two lines of $\partial_{\rho u} K_1^-$ and $\partial_{\rho u} K_1^+$ can be computed explicitly as

$$\frac{4R}{\rho} I_0\left(\frac{\sqrt{R^2-(t-u)^2}}{2}\right) - \frac{4(t-u)^2}{\rho R} I_0\left(\frac{\sqrt{R^2-(t-u)^2}}{2}\right), \tag{B.20}$$

and

$$\frac{4R}{\rho} I_0\left(\frac{\sqrt{R^2-(u-t)^2}}{2}\right) - \frac{4(u-t)^2}{\rho R} I_0\left(\frac{\sqrt{R^2-(u-t)^2}}{2}\right). \tag{B.21}$$

Lemma B.13. *The terms $L_1^{\pm}$ have mixed derivatives*

$$\begin{aligned}\partial_{\rho u} L_1^+(\rho, u; t) =& \frac{4(u-t)}{\rho} \int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta \, d\theta \\ &+ \frac{2R(u-t)}{\rho} \int_0^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right)\cos^2\theta \, d\theta, \\ \partial_{\rho u} L_1^-(\rho, u; t) =& \frac{4(t-u)}{\rho} \int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta \, d\theta \\ &+ \frac{2R(t-u)}{\rho} \int_0^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right)\cos^2\theta \, d\theta.\end{aligned} \tag{B.22}$$

Lemma B.14. *The kernel $\mathcal{K}_1$ has second derivatives*

$$\begin{aligned}\partial_{uu}\mathcal{K}_1(\rho, u; t) = \partial_{uu} L_1^+(\rho, u; t)\mathbb{1}_{t-u\le -R} &+ \partial_{uu} L_1^-(\rho, u; t)\mathbb{1}_{t-u\ge R} \\ &+ \partial_{uu} K_1(\rho, u; t)\mathbb{1}_{|t-u|<R}, \\ \partial_{\rho u}\mathcal{K}_1(\rho, u; t) = \partial_{\rho u} L_1^+(\rho, u; t)\mathbb{1}_{t-u\le -R} &+ \partial_{\rho u} L_1^-(\rho, u; t)\mathbb{1}_{t-u\ge R} \\ &+ \partial_{\rho u} K_1(\rho, u; t)\mathbb{1}_{|t-u|<R}.\end{aligned} \tag{B.23}$$

Proof. Observe that all terms arising as Dirac masses cancel. □

B.2 The second term, $\hat{\mathcal{J}}_2$

Lemma B.15. *The term $\hat{\mathcal{J}}_2$ can be split-up in the following way,*

$$\hat{\mathcal{J}}_2 = \hat{\mathcal{J}}_2^- + \hat{\mathcal{J}}_2^+, \tag{B.24}$$

where

$$\hat{\mathcal{J}}_2^- = -\frac{\sqrt{\rho}}{4} \int_u^{\infty} \big[X(t-R) + X(t+R)\big](t-u)^2 \, dt, \tag{B.25}$$

and

$$\hat{\mathcal{J}}_2^+ = \frac{\sqrt{\rho}}{4} \int_{-\infty}^{u} \big[X(t-R) + X(t+R)\big](u-t)^2 \, dt. \tag{B.26}$$

Lemma B.16. *By explicit computation, we have that* $\partial_u \hat{\mathcal{J}}_2 = \partial_u \hat{\mathcal{J}}_2^- + \partial_u \hat{\mathcal{J}}_2^+$*, where*

$$\partial_u \hat{\mathcal{J}}_2^- = \frac{\sqrt{\rho}}{2} \int_u^\infty [X(t-R) + X(t+R)]\,(t-u)\,dt,$$
$$\partial_u \hat{\mathcal{J}}_2^+ = \frac{\sqrt{\rho}}{2} \int_{-\infty}^u [X(t-R) + X(t+R)]\,(u-t)\,dt,$$

which may be rewritten as

$$\partial_u \hat{\mathcal{J}}_2^- = \frac{\sqrt{\rho}}{2} \int_{u-R}^\infty X(t)(t+R-u)\,dt + \frac{\sqrt{\rho}}{2} \int_{u+R}^\infty X(t)(t-R-u)\,dt,$$
$$\partial_u \hat{\mathcal{J}}_2^+ = \frac{\sqrt{\rho}}{2} \int_{-\infty}^{u-R} X(t)(u-R-t)\,dt + \frac{\sqrt{\rho}}{2} \int_{-\infty}^{u+R} X(t)(u+R-t)\,dt.$$

Lemma B.17. *By explicit computation, we have that* $\partial_{uu} \hat{\mathcal{J}}_2 = \partial_{uu} \hat{\mathcal{J}}_2^- + \partial_{uu} \hat{\mathcal{J}}_2^+$*, where*

$$\begin{aligned} \partial_{uu} \hat{\mathcal{J}}_2^- &= -\frac{\sqrt{\rho}}{2} \int_{u-R}^\infty X(t)\,dt - \frac{\sqrt{\rho}}{2} \int_{u+R}^\infty X(t)\,dt, \\ \partial_{uu} \hat{\mathcal{J}}_2^+ &= \frac{\sqrt{\rho}}{2} \int_{-\infty}^{u-R} X(t)\,dt + \frac{\sqrt{\rho}}{2} \int_{-\infty}^{u+R} X(t)\,dt, \end{aligned} \tag{B.27}$$

thereby giving

$$\partial_{uu} \hat{\mathcal{J}}_2 = \sqrt{\rho} \int_{-\infty}^{u-R} X(t)\,dt - \sqrt{\rho} \int_{u+R}^\infty X(t)\,dt. \tag{B.28}$$

Lemma B.18. *We have by explicit computation that*

$$\begin{aligned} \partial_{\rho u} \hat{\mathcal{J}}_2 =& \frac{1}{4\sqrt{\rho}} \int_{u-R}^\infty X(t)(t+R-u)\,dt + \frac{1}{2\sqrt{\rho}} \int_{u-R}^\infty X(t)\,dt \\ &+ \frac{1}{4\sqrt{\rho}} \int_{u+R}^\infty X(t)(t-R-u)\,dt - \frac{1}{2\sqrt{\rho}} \int_{u+R}^\infty X(t)\,dt \\ &+ \frac{1}{4\sqrt{\rho}} \int_{-\infty}^{u-R} X(t)(u-R-t)\,dt - \frac{1}{2\sqrt{\rho}} \int_{-\infty}^{u-R} X(t)\,dt \\ &+ \frac{1}{4\sqrt{\rho}} \int_{-\infty}^{u+R} X(t)(u+R-t)\,dt + \frac{1}{2\sqrt{\rho}} \int_{-\infty}^{u+R} X(t)\,dt. \end{aligned}$$

Lemma B.19. *Using the fake derivative* $\partial_{\rho m} \hat{\mathcal{J}}_2 = \rho^{-1} \partial_{\rho u} \hat{\mathcal{J}}_2 - \rho^{-2} \partial_u \hat{\mathcal{J}}_2$*, we get*

$$\begin{aligned} \partial_{m\rho} \hat{\mathcal{J}}_2 =& -\frac{1}{4\rho^{3/2}} \int_{u-R}^\infty X(t)(t+R-u)\,dt + \frac{1}{2\rho^{3/2}} \int_{u-R}^\infty X(t)\,dt \\ &- \frac{1}{4\rho^{3/2}} \int_{u+R}^\infty X(t)(t-R-u)\,dt - \frac{1}{2\rho^{3/2}} \int_{u+R}^\infty X(t)\,dt \\ &- \frac{1}{4\rho^{3/2}} \int_{-\infty}^{u-R} X(t)(u-R-t)\,dt - \frac{1}{2\rho^{3/2}} \int_{-\infty}^{u-R} X(t)\,dt \\ &- \frac{1}{4\rho^{3/2}} \int_{-\infty}^{u+R} X(t)(u+R-t)\,dt + \frac{1}{2\rho^{3/2}} \int_{-\infty}^{u+R} X(t)\,dt, \end{aligned}$$

which can be simplified to

$$\partial_{m\rho}\hat{\mathcal{J}}_2 = -\frac{1}{2\rho^{3/2}}\int_{-\infty}^{u-R} X(t)(u-t)\,dt - \frac{1}{2\rho^{3/2}}\int_{u+R}^{\infty} X(t)(t-u)\,dt \\ -\frac{R}{2\rho^{3/2}}\int_{u-R}^{u+R} X(t)\,dt + \frac{1}{\rho^{3/2}}\int_{u-R}^{u+R} X(t)\,dt.$$

B.3 The third term, $\hat{\mathcal{J}}_3$

Lemma B.20. *Consider the following integral*

$$\int_{\mathbb{R}} \frac{R}{\sqrt{R^2-(u-s-t)^2}} I_1\left(\frac{\sqrt{R^2-(u-s-t)^2}}{2}\right) \mathbb{1}_{|u-s-t|<R} s|s|\,ds, \tag{B.29}$$

which can be rewritten as

$$\int_{-R}^{R} \frac{R}{\sqrt{R^2-z^2}} I_1\left(\frac{\sqrt{R^2-z^2}}{2}\right)(z+u-t)|z+u-t|\,dz. \tag{B.30}$$

Then we can decompose the above explicitly into

$$\mathcal{K}_2(\rho,u;t) := L_2^+(\rho,u;t)\mathbb{1}_{t-u\le -R} + L_2^-(\rho,u;t)\mathbb{1}_{t-u\ge R} + K_2(\rho,u;t)\mathbb{1}_{|t-u|<R}, \tag{B.31}$$

where

$$\begin{aligned} L_2^+(\rho,u;t) &:= \int_{-R}^{R} \frac{R}{\sqrt{R^2-z^2}} I_1\left(\frac{\sqrt{R^2-z^2}}{2}\right)(z+u-t)^2\,dz, \\ L_2^-(\rho,u;t) &:= -\int_{-R}^{R} \frac{R}{\sqrt{R^2-z^2}} I_1\left(\frac{\sqrt{R^2-z^2}}{2}\right)(z+u-t)^2\,dz, \end{aligned} \tag{B.32}$$

and

$$\begin{aligned} K_2(\rho,u;t) := &\int_{t-u}^{R} \frac{R}{\sqrt{R^2-z^2}} I_1\left(\frac{\sqrt{R^2-z^2}}{2}\right)(z+u-t)^2\,dz \\ &- \int_{-R}^{t-u} \frac{R}{\sqrt{R^2-z^2}} I_1\left(\frac{\sqrt{R^2-z^2}}{2}\right)(z+u-t)^2\,dz. \end{aligned} \tag{B.33}$$

Lemma B.21. *The terms $L_2^{\pm}$ can be simplified into*

$$\begin{aligned} L_2^{\pm}(\rho,u;t) = &\pm 2R^3 \int_0^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right)\sin^2\theta\,d\theta \\ &\pm 2R(u-t)^2 \int_0^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right)\,d\theta. \end{aligned} \tag{B.34}$$

Lemma B.22. *The term $K_2(\rho, u; t)$ can be decomposed explicitly into*

$$K_2(\rho, u; t) = K_2^-(\rho, u; t)\mathbb{1}_{\{t-u>0\}} + K_2^+(\rho, u; t)\mathbb{1}_{\{u-t>0\}}, \tag{B.35}$$

where

$$\begin{aligned} K_2^-(\rho, u; t) = -4(t-u)R^2 &\int_{\arcsin((t-u)/R)}^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right)\sin\theta\, d\theta \\ &- 2R^3 \int_0^{\arcsin((t-u)/R)} I_1\left(\frac{R}{2}\cos\theta\right)\sin^2\theta\, d\theta \\ &- 2(t-u)^2 R \int_0^{\arcsin((t-u)/R)} I_1\left(\frac{R}{2}\cos\theta\right) d\theta, \end{aligned} \tag{B.36}$$

and

$$\begin{aligned} K_2^+(\rho, u; t) = 4(u-t)R^2 &\int_{\arcsin((u-t)/R)}^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right)\sin\theta\, d\theta \\ &+ 2R^3 \int_0^{\arcsin((u-t)/R)} I_1\left(\frac{R}{2}\cos\theta\right)\sin^2\theta\, d\theta \\ &+ 2(u-t)^2 R \int_0^{\arcsin((u-t)/R)} I_1\left(\frac{R}{2}\cos\theta\right) d\theta. \end{aligned} \tag{B.37}$$

Remark B.23. The first term of K_2^- and K_2^+ can be computed explicitly as

$$-8(t-u)R\left[I_0\left(\frac{\sqrt{R^2-(t-u)^2}}{2}\right) - 1\right], \tag{B.38}$$

and

$$8(u-t)R\left[I_0\left(\frac{\sqrt{R^2-(u-t)^2}}{2}\right) - 1\right]. \tag{B.39}$$

Lemma B.24. *The term K_2 has derivative*

$$\partial_u K_2(\rho, u; t) = \partial_u K_2^-(\rho, u; t)\mathbb{1}_{t-u>0} + \partial_u K_2^+(\rho, u; t)\mathbb{1}_{u-t>0}, \tag{B.40}$$

where

$$\begin{aligned} \partial_u K_2^-(\rho, u; t) =& 4R^2 \int_{\arcsin\left(\frac{t-u}{R}\right)}^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right)\sin\theta\, d\theta \\ &+ 4(t-u)R \int_0^{\arcsin((t-u)/R)} I_1\left(\frac{R}{2}\cos\theta\right) d\theta, \end{aligned}$$

and

$$\begin{aligned} \partial_u K_2^+(\rho, u; t) =& 4R^2 \int_{\arcsin\left(\frac{u-t}{R}\right)}^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right)\sin\theta\, d\theta \\ &+ 4(u-t)R \int_0^{\arcsin((u-t)/R)} I_1\left(\frac{R}{2}\cos\theta\right) d\theta. \end{aligned}$$

Proof. Observe that all terms arising as Dirac masses cancel. The rest is acquired by explicit computation. □

Remark B.25. The first term of $\partial_u K_2^-$ and $\partial_u K_2^+$ can be computed explicitly as

$$8R\left[I_0\left(\frac{\sqrt{R^2-(t-u)^2}}{2}\right)-1\right], \tag{B.41}$$

and

$$8R\left[I_0\left(\frac{\sqrt{R^2-(u-t)^2}}{2}\right)-1\right], \tag{B.42}$$

respectively.

Lemma B.26. *The terms $L_2^\pm$ have derivatives*

$$\begin{aligned}\partial_u L_2^+(\rho,u;t) &= 4R(u-t)\int_0^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right)\,d\theta,\\ \partial_u L_2^-(\rho,u;t) &= 4R(t-u)\int_0^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right)\,d\theta.\end{aligned} \tag{B.43}$$

Lemma B.27. *The kernel $\mathcal{K}_2$ has derivative*

$$\partial_u\mathcal{K}_2(\rho,u;t) := \partial_u L_2^+(\rho,u;t)\mathbb{1}_{t-u\leq -R} + \partial_u L_2^-(\rho,u;t)\mathbb{1}_{t-u\geq R} + \partial_u K_2(\rho,u;t)\mathbb{1}_{|t-u|<R}. \tag{B.44}$$

Proof. Observe that all terms arising as Dirac masses cancel. □

Lemma B.28. *The term K_2 has second derivative*

$$\partial_{uu}K_2(\rho,u;t) = \partial_{uu}K_2^-(\rho,u;t)\mathbb{1}_{t-u>0} + \partial_{uu}K_2^+(\rho,u;t)\mathbb{1}_{u-t>0},$$

where

$$\begin{aligned}\partial_{uu}K_2^-(\rho,u;t) &= -4R\int_0^{\arcsin((t-u)/R)} I_1\left(\frac{R}{2}\cos\theta\right)\,d\theta,\\ \partial_{uu}K_2^+(\rho,u;t) &= 4R\int_0^{\arcsin((u-t)/R)} I_1\left(\frac{R}{2}\cos\theta\right)\,d\theta.\end{aligned}$$

Proof. Observe that all terms arising as Dirac masses cancel. The rest is acquired by explicit computation. □

Lemma B.29. *The terms $L_2^\pm$ have second derivatives*

$$\begin{aligned}\partial_{uu} L_2^+(\rho,u;t) &= 4R\int_0^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right)\,d\theta,\\ \partial_{uu} L_2^-(\rho,u;t) &= -4R\int_0^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right)\,d\theta.\end{aligned} \tag{B.45}$$

Lemma B.30. *The term K_2 has mixed derivative*

$$\partial_{\rho u} K_2(\rho, u; t) = \partial_{\rho u} K_2^-(\rho, u; t)\mathbb{1}_{t-u>0} + \partial_{\rho u} K_2^+(\rho, u; t)\mathbb{1}_{u-t>0},$$

where

$$\begin{aligned}\partial_{\rho u} K_2^-(\rho, u; t) =& \frac{4R}{\rho} \int_{\arcsin\left(\frac{t-u}{R}\right)}^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right)\sin\theta \, d\theta \\ &+ \frac{2R^2}{\rho} \int_{\arcsin\left(\frac{t-u}{R}\right)}^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\sin\theta \, d\theta \\ &+ \frac{2R(t-u)}{\rho} \int_0^{\arcsin\left(\frac{t-u}{R}\right)} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta \, d\theta,\end{aligned}$$

and

$$\begin{aligned}\partial_{\rho u} K_2^+(\rho, u; t) =& \frac{4R}{\rho} \int_{\arcsin\left(\frac{u-t}{R}\right)}^{\pi/2} I_1\left(\frac{R}{2}\cos\theta\right)\sin\theta \, d\theta \\ &+ \frac{2R^2}{\rho} \int_{\arcsin\left(\frac{u-t}{R}\right)}^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\sin\theta \, d\theta \\ &+ \frac{2R(u-t)}{\rho} \int_0^{\arcsin\left(\frac{u-t}{R}\right)} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta \, d\theta.\end{aligned}$$

Remark B.31. The first two terms of $\partial_{\rho u} K_2^-$ and $\partial_{\rho u} K_2^+$ can be computed explicitly as

$$\frac{8}{\rho}\left[I_0\left(\frac{\sqrt{R^2-(t-u)^2}}{2}\right) - 1\right] + \frac{4}{\rho}\sqrt{R^2-(t-u)^2}\, I_1\left(\frac{\sqrt{R^2-(t-u)^2}}{2}\right), \tag{B.46}$$

and

$$\frac{8}{\rho}\left[I_0\left(\frac{\sqrt{R^2-(u-t)^2}}{2}\right) - 1\right] + \frac{4}{\rho}\sqrt{R^2-(u-t)^2}\, I_1\left(\frac{\sqrt{R^2-(u-t)^2}}{2}\right), \tag{B.47}$$

respectively.

Lemma B.32. *The terms $L_2^\pm$ have mixed derivatives*

$$\begin{aligned}\partial_{\rho u} L_2^-(\rho, u; t) &= \frac{2R(t-u)}{\rho} \int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta \, d\theta, \\ \partial_{\rho u} L_2^+(\rho, u; t) &= \frac{2R(u-t)}{\rho} \int_0^{\pi/2} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta \, d\theta.\end{aligned} \tag{B.48}$$

Lemma B.33. *The kernel $\mathcal{K}_2$ has second derivatives*

$$\begin{aligned}\partial_{uu}\mathcal{K}_2(\rho, u; t) := & \partial_{uu} L_2^+(\rho, u; t)\mathbb{1}_{t-u\le -R} + \partial_{uu} L_2^-(\rho, u; t)\mathbb{1}_{t-u\ge R} \\ &+ \partial_{uu} K_2(\rho, u; t)\mathbb{1}_{|t-u|<R}, \\ \partial_{\rho u}\mathcal{K}_2(\rho, u; t) := & \partial_{\rho u} L_2^+(\rho, u; t)\mathbb{1}_{t-u\le -R} + \partial_{\rho u} L_2^-(\rho, u; t)\mathbb{1}_{t-u\ge R} \\ &+ \partial_{\rho u} K_2(\rho, u; t)\mathbb{1}_{|t-u|<R}.\end{aligned} \tag{B.49}$$

Proof. Observe that all terms arising as Dirac masses cancel. □

B.4 Computing mixed derivatives

Lemma B.34. *In terms of the kernels $\mathcal{K}_1$ and $\mathcal{K}_2$, we can express the entropy integrals $\hat{\mathcal{J}}_1$ and $\hat{\mathcal{J}}_3$ as*

$$\begin{aligned}\hat{\mathcal{J}}_1(\rho,u) &= \frac{\sqrt{\rho}}{4}\int_{\mathbb{R}} \mathcal{K}_1(\rho,u;t)Y(t)\,dt,\\ \hat{\mathcal{J}}_3(\rho,u) &= \frac{\sqrt{\rho}}{8}\int_{\mathbb{R}} \Big[\mathcal{K}_2(\rho,u;t)-\mathcal{K}_1(\rho,u;t)\Big]X(t)\,dt,\end{aligned}\tag{B.50}$$

and have corresponding derivatives given by

$$\begin{aligned}\partial_u\hat{\mathcal{J}}_1(\rho,u) &= \frac{\sqrt{\rho}}{4}\int_{\mathbb{R}} \partial_u\mathcal{K}_1(\rho,u;t)Y(t)\,dt,\\ \partial_u\hat{\mathcal{J}}_3(\rho,u) &= \frac{\sqrt{\rho}}{8}\int_{\mathbb{R}} \Big[\partial_u\mathcal{K}_2(\rho,u;t)-\partial_u\mathcal{K}_1(\rho,u;t)\Big]X(t)\,dt,\end{aligned}\tag{B.51}$$

and

$$\begin{aligned}\partial_{uu}\hat{\mathcal{J}}_1(\rho,u) &= \frac{\sqrt{\rho}}{4}\int_{\mathbb{R}} \partial_{uu}\mathcal{K}_1(\rho,u;t)Y(t)\,dt,\\ \partial_{uu}\hat{\mathcal{J}}_3(\rho,u) &= \frac{\sqrt{\rho}}{8}\int_{\mathbb{R}} \Big[\partial_{uu}\mathcal{K}_2(\rho,u;t)-\partial_{uu}\mathcal{K}_1(\rho,u;t)\Big]X(t)\,dt.\end{aligned}\tag{B.52}$$

Note also that the mixed derivatives are given by

$$\begin{aligned}\partial_{\rho u}\hat{\mathcal{J}}_1(\rho,u) =& \frac{1}{8\sqrt{\rho}}\int_{\mathbb{R}} \partial_u\mathcal{K}_1(\rho,u;t)Y(t)\,dt + \frac{\sqrt{\rho}}{4}\int_{\mathbb{R}} \partial_{\rho u}\mathcal{K}_1(\rho,u;t)Y(t)\,dt,\\ \partial_{\rho u}\hat{\mathcal{J}}_3(\rho,u) =& \frac{1}{16\sqrt{\rho}}\int_{\mathbb{R}} \Big[\partial_u\mathcal{K}_2(\rho,u;t)-\partial_u\mathcal{K}_1(\rho,u;t)\Big]X(t)\,dt\\ &+\frac{\sqrt{\rho}}{8}\int_{\mathbb{R}} \Big[\partial_{\rho u}\mathcal{K}_2(\rho,u;t)-\partial_{\rho u}\mathcal{K}_1(\rho,u;t)\Big]X(t)\,dt,\end{aligned}\tag{B.53}$$

so that the fake derivatives $\partial_{m\rho}\hat{\mathcal{J}}_1$ and $\partial_{m\rho}\hat{\mathcal{J}}_3$ are given by

$$\begin{aligned}\partial_{m\rho}\hat{\mathcal{J}}_1(\rho,u) =& -\frac{1}{8\rho^{3/2}}\int_{\mathbb{R}} \partial_u\mathcal{K}_1(\rho,u;t)Y(t)\,dt + \frac{1}{4\sqrt{\rho}}\int_{\mathbb{R}} \partial_{\rho u}\mathcal{K}_1(\rho,u;t)Y(t)\,dt,\\ \partial_{m\rho}\hat{\mathcal{J}}_3(\rho,u) =& -\frac{1}{16\rho^{3/2}}\int_{\mathbb{R}} \Big[\partial_u\mathcal{K}_2(\rho,u;t)-\partial_u\mathcal{K}_1(\rho,u;t)\Big]X(t)\,dt\\ &+\frac{1}{8\sqrt{\rho}}\int_{\mathbb{R}} \Big[\partial_{\rho u}\mathcal{K}_2(\rho,u;t)-\partial_{\rho u}\mathcal{K}_1(\rho,u;t)\Big]X(t)\,dt.\end{aligned}\tag{B.54}$$

It is therefore helpful to compute the quantities underneath.

Lemma B.35. *By explicit computation, we have that*

$$\begin{aligned}\frac{\partial_{\rho u}L_1^+(\rho,u;t)}{4\sqrt{\rho}} - \frac{\partial_u L_1^+(\rho,u;t)}{8\rho^{3/2}} &= \frac{(u-t)}{\rho^2},\\ \frac{\partial_{\rho u}L_1^-(\rho,u;t)}{4\sqrt{\rho}} - \frac{\partial_u L_1^-(\rho,u;t)}{8\rho^{3/2}} &= \frac{(t-u)}{\rho^2}.\end{aligned}\tag{B.55}$$

Lemma B.36. *By explicit computation, we have that*

$$\frac{\partial_{\rho u} K_1^-(\rho,u;t)}{4\sqrt{\rho}} - \frac{\partial_u K_1^-(\rho,u;t)}{8\rho^{3/2}} =$$
$$\frac{(t-u)}{\rho^{3/2}}\left\{ \left(1-\frac{R}{2}\right)\int_0^{\arcsin\left(\frac{t-u}{R}\right)} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\, d\theta \right.$$
$$\left. + \frac{R}{2}\int_0^{\arcsin\left(\frac{t-u}{R}\right)} I_1\left(\frac{R}{2}\cos\theta\right)\cos^2\theta\, d\theta \right\}$$
$$+\frac{R}{\rho^{3/2}} I_0\left(\frac{\sqrt{R^2-(t-u)^2}}{2}\right) - \frac{\sqrt{R^2-(t-u)^2}}{\rho^{3/2}} I_1\left(\frac{\sqrt{R^2-(t-u)^2}}{2}\right)$$
$$-\frac{(t-u)^2}{R\rho^{3/2}} I_0\left(\frac{\sqrt{R^2-(t-u)^2}}{2}\right),$$

and

$$\frac{\partial_{\rho u} K_1^+(\rho,u;t)}{4\sqrt{\rho}} - \frac{\partial_u K_1^+(\rho,u;t)}{8\rho^{3/2}} =$$
$$\frac{(u-t)}{\rho^{3/2}}\left\{ \left(1-\frac{R}{2}\right)\int_0^{\arcsin\left(\frac{u-t}{R}\right)} I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta\, d\theta \right.$$
$$\left. + \frac{R}{2}\int_0^{\arcsin\left(\frac{u-t}{R}\right)} I_1\left(\frac{R}{2}\cos\theta\right)\cos^2\theta\, d\theta \right\}$$
$$+\frac{R}{\rho^{3/2}} I_0\left(\frac{\sqrt{R^2-(u-t)^2}}{2}\right) - \frac{\sqrt{R^2-(u-t)^2}}{\rho^{3/2}} I_1\left(\frac{\sqrt{R^2-(u-t)^2}}{2}\right)$$
$$-\frac{(u-t)^2}{R\rho^{3/2}} I_0\left(\frac{\sqrt{R^2-(u-t)^2}}{2}\right).$$

Lemma B.37. *By explicit computation, we have that*

$$\begin{aligned} \frac{\partial_{\rho u} L_2^+(\rho,u;t)}{8\sqrt{\rho}} - \frac{\partial_u L_2^+(\rho,u;t)}{16\rho^{3/2}} &= \frac{(u-t)}{2}\left(\frac{1}{\rho^{3/2}} - \frac{1}{\rho^2}\right), \\ \frac{\partial_{\rho u} L_2^-(\rho,u;t)}{8\sqrt{\rho}} - \frac{\partial_u L_2^-(\rho,u;t)}{16\rho^{3/2}} &= \frac{(t-u)}{2}\left(\frac{1}{\rho^{3/2}} - \frac{1}{\rho^2}\right). \end{aligned} \tag{B.56}$$

Lemma B.38. *By explicit computation, we have that*

$$\frac{\partial_{\rho u} K_2^-(\rho,u;t)}{8\sqrt{\rho}} - \frac{\partial_u K_2^-(\rho,u;t)}{16\rho^{3/2}} =$$

$$\frac{1}{\rho^{3/2}}\left[I_0\left(\frac{\sqrt{R^2-(t-u)^2}}{2}\right)-1\right]$$

$$+\frac{\sqrt{R^2-(t-u)^2}}{2\rho^{3/2}} I_1\left(\frac{\sqrt{R^2-(t-u)^2}}{2}\right)$$

$$-\frac{R}{2\rho^{3/2}}\left[I_0\left(\frac{\sqrt{R^2-(t-u)^2}}{2}\right)-1\right]$$

$$+\frac{R(t-u)}{4\rho^{3/2}}\int_0^{\arcsin\left(\frac{t-u}{R}\right)}\left[I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta - I_1\left(\frac{R}{2}\cos\theta\right)\right]\,d\theta,$$

and

$$\frac{\partial_{\rho u} K_2^+(\rho,u;t)}{8\sqrt{\rho}} - \frac{\partial_u K_2^+(\rho,u;t)}{16\rho^{3/2}} =$$

$$\frac{1}{\rho^{3/2}}\left[I_0\left(\frac{\sqrt{R^2-(u-t)^2}}{2}\right)-1\right]$$

$$+\frac{\sqrt{R^2-(u-t)^2}}{2\rho^{3/2}} I_1\left(\frac{\sqrt{R^2-(u-t)^2}}{2}\right)$$

$$-\frac{R}{2\rho^{3/2}}\left[I_0\left(\frac{\sqrt{R^2-(u-t)^2}}{2}\right)-1\right]$$

$$+\frac{R(u-t)}{4\rho^{3/2}}\int_0^{\arcsin\left(\frac{u-t}{R}\right)}\left[I_0\left(\frac{R}{2}\cos\theta\right)\cos\theta - I_1\left(\frac{R}{2}\cos\theta\right)\right]\,d\theta.$$

Appendix C

Asymptotic expansion of $k(\nu)$

Herein, we give full details of the asymptotic expansion of $k(\nu)$ in the vicinity of the vacuum, where $\nu = 0$.

Lemma C.1. *There exists a positive constant $C = C(\gamma, \mu_*)$ such that, for any $\rho \in (0, \rho(q_*)]$, we have*

$$C^{-1}\rho^{\theta} \leq k(\nu(\rho)) \leq C\rho^{\theta}, \qquad C^{-1}\rho^{-(\theta+1)} \leq k'(\nu(\rho)) \leq C\rho^{-(\theta+1)}. \tag{C.1}$$

Proof. By virtue of the chain rule,

$$k'(\nu(\rho)) = \frac{1}{\nu'(\rho)} \frac{d}{d\rho} k(\nu(\rho)) \quad \text{for } \rho > 0.$$

Recall that $k(\nu(\rho)) = \int_0^\rho \frac{\sqrt{M^2(\rho')-1}}{\rho'} \cdot \frac{1}{M^2(\rho')}\, d\rho'$ (*cf.* Definition 4.12). Using the Bernoulli law (1.35), we therefore have

$$k(\nu(\rho)) = \sqrt{\theta} \int_0^\rho y^{\theta-1} \left(1 - \frac{\gamma+1}{2} y^{2\theta}\right)^{1/2} \left(1 - y^{2\theta}\right)^{-1} dy, \tag{C.2}$$

from which it is evident that

$$\frac{d}{d\rho} k(\nu(\rho)) = \sqrt{\theta}\rho^{\theta-1} \left(1 - \frac{\gamma+1}{2}\rho^{2\theta}\right)^{1/2} \left(1 - \rho^{2\theta}\right)^{-1}.$$

In view of the invariant regions provided by Lemma 4.3, we have the uniform bound $\rho \leq \rho(q_*)$ for some $\rho(q_*) < q_{cr}$, as in (4.7). We therefore obtain

$$\sqrt{\theta} \left(1 - \frac{\gamma+1}{2} (\rho(q_*))^{2\theta}\right)^{1/2} \rho^{\theta-1} \leq \frac{d}{d\rho} k(\nu(\rho)) \leq \sqrt{\theta} \left(1 - (\rho(q_*))^{2\theta}\right)^{-1} \rho^{\theta-1}.$$

Integrating the above inequality in ρ readily yields (C.4). Meanwhile, $\nu'(\rho) = c^2/q^2$, which can be rewritten in terms of ρ as

$$\nu'(\rho) = \frac{\theta\rho^{2\theta}}{1 - \rho^{2\theta}}.$$

Hence,

$$k'(\nu(\rho)) = \frac{1}{\sqrt{\theta}}\rho^{-(\theta+1)}\left(1 - \frac{\gamma+1}{2}\rho^{2\theta}\right)^{1/2},$$

from which it follows that

$$\frac{1}{\sqrt{\theta}}\left(1 - \frac{\gamma+1}{2}(\rho(q_*))^{2\theta}\right)^{1/2}\rho^{-(\theta+1)} \leq k'(\nu(\rho)) \leq \frac{1}{\sqrt{\theta}}\rho^{-(\theta+1)}.$$

□

Lemma C.2. *There exists a positive constant $C = C(\gamma)$ such that, for all $\rho \in [0, \rho_{cr}]$,*

$$C^{-1}\rho^{\gamma} \leq \nu(\rho) \leq C\rho^{\gamma}. \tag{C.3}$$

Proof. The result follows immediately from the definition of ν and the pointwise bound

$$1 \leq \frac{1}{1 - x^{2\theta}} \leq \frac{1}{1 - (\rho_{cr})^{2\theta}},$$

after noting that $\int_0^\rho x^{2\theta}\, d\theta = \frac{\rho^{2\theta+1}}{2\theta+1}$, and $2\theta + 1 = \gamma$. □

Lemma C.3. *Using $-(\theta+1)/\gamma = \theta/\gamma - 1$ and the previous two lemmas, we see that there exists a positive constant $C = C(\gamma, \mu_*)$ such that, for all $\nu \in (0, \mu_*]$,*

$$C^{-1}\nu^{\theta/\gamma} \leq k(\nu) \leq C\nu^{\theta/\gamma}, \qquad C^{-1}\nu^{\theta/\gamma-1} \leq k'(\nu) \leq C\nu^{\theta/\gamma-1}. \tag{C.4}$$

Lemma C.4. *Provided $q > q_{cr}$, the quantity k admits the following bound on its derivatives,*

$$k'(\nu)k''(\nu) < 0. \tag{C.5}$$

Proof. Observe that

$$k'(\nu(\rho))^2 = \frac{M^2 - 1}{\rho^2},$$

which implies that, in terms of the variable ρ,

$$\begin{aligned} 2k'(\nu(\rho))k''(\nu(\rho)) &= \frac{d\rho}{d\nu}\frac{d}{d\rho}\left(\frac{M^2-1}{\rho^2}\right) \\ &= M^2\left(\frac{d(M^2)/d\rho}{\rho^2} - \frac{2(M^2-1)}{\rho^3}\right). \end{aligned} \tag{C.6}$$

Additionally, we have $M^2 = \frac{2}{\gamma-1}\left(\rho^{1-\gamma} - 1\right)$, from which we straightforwardly compute

$$\frac{dM^2}{d\rho} = -2\rho^{-\gamma}.$$

Hence,

$$k'(\nu(\rho))k''(\nu(\rho)) = -\frac{M^2}{\rho^{\gamma+2}} - \frac{M^2(M^2-1)}{\rho^3},$$

and the right-hand side is manifestly strictly negative in the supersonic region. □

Corollary C.5. *Provided $q > q_{cr}$, we have $k''(\nu) < 0$.*

Lemma C.6. *Let $\lambda_1 := \min\{2^{-1/\theta}, (\gamma+1)^{-1/2\theta}, \rho(q_*)\}$. In the vicinity of the vacuum, the quantity k admits the representation*

$$k(\nu(\rho)) = \frac{\rho^\theta}{\sqrt{\theta}} \sum_{n=0}^{\infty} \frac{c_n \rho^{2n\theta}}{2n+1}, \tag{C.7}$$

and this series converges absolutely and uniformly in the interval $\rho \in [0, \lambda_1]$. The coefficients $(c_n)_{n\in\mathbb{N}}$ are given by

$$c_n = \sum_{j=0}^{n} (-1)^j \binom{1/2}{j} \left(\frac{\gamma+1}{2}\right)^j \quad \textit{for each } n \in \mathbb{N}. \tag{C.8}$$

Remark C.7. As a result, near the vacuum, the quantity k scales in the following way

$$k(\nu(\rho)) = \frac{\rho^\theta}{\sqrt{\theta}} + O(\rho^{3\theta}) \quad \text{in the vicinity of } \rho = 0. \tag{C.9}$$

Proof. We begin by expanding the integrand in terms of ρ. Notice that $M^2 = \frac{2}{\gamma-1}\left(\rho^{1-\gamma} - 1\right)$, so $M^2 - 1 = \frac{2}{\gamma-1}\rho^{1-\gamma} - \frac{\gamma+1}{\gamma-1}$. Thus,

$$\begin{aligned} k(\nu(\rho)) &= \int_0^\rho \frac{\sqrt{\frac{2}{\gamma-1}y^{1-\gamma} - \frac{\gamma+1}{\gamma-1}}}{y} \cdot \frac{1}{\frac{2}{\gamma-1}(y^{1-\gamma} - 1)}\, dy \\ &= \frac{\sqrt{\gamma-1}}{2} \int_0^\rho \left(\frac{2y^{\gamma-1} - (\gamma+1)y^{2(\gamma-1)}}{y^2(1-y^{\gamma-1})^2}\right)^{1/2} dy, \end{aligned}$$

which can be rewritten more legibly as

$$k(\nu(\rho)) = \sqrt{\theta} \int_0^\rho y^{\theta-1} \left(1 - \frac{\gamma+1}{2} y^{2\theta}\right)^{1/2} \left(1 - y^{2\theta}\right)^{-1} dy. \tag{C.10}$$

Since we keep ρ very small, we can expand both parentheses as power series (which will converge absolutely and uniformly by the Weierstraß M-test following a judicious choice of upper bound for ρ, for example $\rho \leq 2^{-1/2\theta}$). Specifically, we have

$$\begin{aligned} k(\nu(\rho)) &= \sqrt{\theta} \int_0^\rho y^{\theta-1} \left(\sum_{n=0}^{\infty} (-1)^n \binom{1/2}{n} \left(\frac{\gamma+1}{2}\right)^n y^{2n\theta}\right) \left(\sum_{n=0}^{\infty} y^{2n\theta}\right) dy \\ &= \sqrt{\theta} \int_0^\rho y^{\theta-1} \left(\sum_{n=0}^{\infty} c_n y^{2n\theta}\right) dy, \end{aligned}$$

where

$$c_n = \sum_{j=0}^{n} (-1)^j \binom{1/2}{j} \left(\frac{\gamma+1}{2}\right)^j \quad \text{for each } n \in \mathbb{N}.$$

Note that $c_0 = 1$ and $c_1 = -\frac{\theta-1}{2}$. Additionally, whenever $j \geq 2$,

$$\begin{pmatrix} 1/2 \\ j \end{pmatrix} = \frac{1/2(1/2-1)(1/2-2)\dots(1/2-j+1)}{j!}$$
$$= \frac{1/2(-1/2)(-3/2)\dots(3/2-j)}{j!}$$
$$= \frac{(-1)^{j-1}}{2}\frac{(1/2)(3/2)\dots(j-3/2)}{j!},$$

from which it is apparent that

$$\left|\begin{pmatrix} 1/2 \\ j \end{pmatrix}\right| \leq \frac{1\cdot 2\dots(j-1)}{2\cdot j!} = \frac{1}{2j} \qquad \text{for each } j \geq 2.$$

So, in fact, the above binomial coefficient is also bounded above by 1 for all $n \geq 0$. Hence, $|c_n| \leq (n+1)2^{-n}(\gamma+1)^n$ for each $n \in \mathbb{N}$, so the series $\sum_{n=0}^{\infty} c_n y^{2n\theta}$ converges absolutely and uniformly by the Weierstraß M-test provided that we restrict to $\rho \leq (\gamma+1)^{-\frac{1}{2\theta}}$, as shown below.

$$\sum_{n=0}^{\infty} |c_n y^{2n\theta}| \leq \sum_{n=0}^{\infty} (n+1)2^{-n} = 4.$$

Hence, we can integrate the series term-by-term, and thereby obtain

$$k(\nu(\rho)) = \frac{\rho^\theta}{\sqrt{\theta}} \sum_{n=0}^{\infty} \frac{c_n \rho^{2n\theta}}{2n+1} \quad \text{near the vacuum.}$$

□

However, we do not yet know how to relate this asymptotic dependence in ρ to one in ν. Before we can show this, we need several other results, including an expansion for ν near the vacuum.

Lemma C.8. *In the vicinity of the vacuum, the quantity ν admits the representation*

$$\nu(\rho) = \frac{\theta}{\gamma}\rho^\gamma \left(1 + \sum_{n=1}^{\infty} \frac{\rho^{2n\theta}}{1 + 2n\theta/\gamma}\right), \tag{C.11}$$

and this series converges absolutely and uniformly in the interval $\rho \in [0, \lambda_1]$. Additionally, whenever $\rho \in [0, \lambda_1]$, the derivative ν' is bounded as follows

$$\theta\rho^{2\theta} \leq \nu'(\rho) \leq 2\theta\rho^{2\theta}. \tag{C.12}$$

Remark C.9. The above shows that

$$\nu(\rho) = \frac{\theta}{\gamma}\rho^\gamma + O(\rho^{2\gamma-1}) \quad \text{in the vicinity of } \rho = 0. \tag{C.13}$$

Proof. For $\rho \in [0, 1)$,

$$\nu(\rho) = \theta \int_0^\rho \frac{x^{2\theta}}{1 - x^{2\theta}}\, dx.$$

So, near the vacuum, we can expand the integrand as

$$\frac{x^{2\theta}}{1 - x^{2\theta}} = x^{2\theta}(1 - x^{2\theta})^{-1} = \sum_{n=1}^{\infty} x^{2n\theta},$$

where this series converges uniformly and absolutely provided we have $\rho \leq 2^{-1/2\theta}$, for instance. Therefore, integrating the series term-by-term, we have

$$\nu(\rho) = \theta \int_0^\rho \sum_{n=1}^{\infty} x^{2n\theta}\, dx = \theta \sum_{n=1}^{\infty} \frac{\rho^{1+2n\theta}}{1 + 2n\theta}.$$

That is,

$$\nu(\rho) = \frac{\theta}{\gamma}\rho^\gamma \left(1 + \sum_{n=1}^{\infty} \frac{\rho^{2n\theta}}{1 + 2n\theta/\gamma}\right), \tag{C.14}$$

near the vacuum. As for the derivative, we have

$$\nu'(\rho) = \theta\rho^{2\theta}(1 - \rho^{2\theta})^{-1} = \theta\rho^{2\theta}\left(\sum_{n=0}^{\infty} \rho^{2n\theta}\right) \geq \theta\rho^{2\theta},$$

which shows the first inequality. Expanding the derivative as a power series, we see that

$$\nu'(\rho) = \theta\rho^{2\theta}\left(\sum_{n=0}^{\infty} \rho^{2n\theta}\right) \leq \theta\rho^{2\theta}\left(\sum_{n=0}^{\infty} 2^{-n}\right),$$

provided we have $\rho \leq 2^{-1/2\theta}$, which concludes the proof of the lemma. □

Lemma C.10. *Define* $K(\rho) := \frac{\gamma}{\theta}\rho^{-\gamma}\nu(\rho) - 1$, *i.e.,*

$$K(\rho) = \sum_{n=1}^{\infty} \frac{\rho^{2n\theta}}{1 + 2n\theta/\gamma}. \tag{C.15}$$

Then, the above series converges absolutely and uniformly for $\rho \in [0, \lambda_1]$. *We deduce that there exists a positive constant* $C_1 = C_1(\gamma, \mu_*)$ *such that, whenever* $\rho \in [0, \lambda_1]$,

$$C_1^{-1}\rho^{2\theta} \leq K(\rho) \leq C_1\rho^{2\theta}. \tag{C.16}$$

Moreover, its derivative is given by term-by-term differentiation, i.e.,

$$K'(\rho) = \sum_{n=1}^{\infty} \frac{2n\theta}{1 + 2n\theta/\gamma}\rho^{2n\theta - 1}, \tag{C.17}$$

and the above also converges absolutely and uniformly for $\rho \in [0, \lambda_1]$. *Correspondingly, there exists a positive constant* $C_2 = C_2(\gamma, \mu_*)$ *such that, whenever* $\rho \in [0, \lambda_1]$,

$$C_2^{-1}\rho^{2\theta - 1} \leq K'(\rho) \leq C_2\rho^{2\theta - 1}. \tag{C.18}$$

Proof. Observe that, if $\rho \le 2^{-1/2\theta}$, then

$$\frac{\rho^{2n\theta}}{1+2n\theta/\gamma} \le 2^{-n},$$

for every $n \ge 1$, and the right-hand side is summable. Hence, by the Weierstraß M-test, the series given by $K(\rho)$ converges absolutely and uniformly. Further, for $\rho \le 2^{-1/2\theta}$,

$$\frac{2n\theta}{1+2n\theta/\gamma}\rho^{2n\theta-1} \le 2^{1/2\theta}\cdot 2^{-n},$$

and the right-hand side is also summable. Thus, by the Weierstraß M-test, the series given by the term-by-term derivatives of $K(\rho)$ converges absolutely and uniformly in the interval $\rho \in [0,\lambda_1]$. Additionally,

$$\begin{aligned} K(\rho) = \frac{\rho^{2\theta}}{1+2\theta/\gamma} + \sum_{n=2}^{\infty}\frac{\rho^{2n\theta}}{1+2n\theta/\gamma} &\ge \frac{\gamma\rho^{2\theta}}{2\gamma-1} - \rho^{2\theta}\sum_{n=2}^{\infty}(2^{-1/\theta})^{2(n-1)\theta} \\ &\ge \frac{1}{2}\rho^{2\theta} - \rho^{2\theta}\left(\sum_{n=1}^{\infty}2^{-2n}\right) = \frac{1}{6}\rho^{2\theta}, \end{aligned}$$

as required. The same strategy of proof yields the upper bound on K and (C.18). □

Lemma C.11. *Let $\lambda_2 := \min\{\lambda_1, (2C_1)^{-1/2\theta}\}$. Then, the series*

$$\sum_{n=1}^{\infty}\binom{\theta/\gamma}{n}K(\rho)^n \tag{C.19}$$

converges absolutely and uniformly for $\rho \in [0,\lambda_2]$. Further, in this interval, the above series can be differentiated term-by-term, i.e.,

$$\frac{d}{d\rho}\sum_{n=1}^{\infty}\binom{\theta/\gamma}{n}K(\rho)^n = \sum_{n=1}^{\infty}\binom{\theta/\gamma}{n}nK'(\rho)K(\rho)^{n-1}, \tag{C.20}$$

and the series on the right-hand side also converges absolutely and uniformly for $\rho \in [0,\lambda_2]$.

Remark C.12. In light of the above, we deduce that, for each $j \ge 1$, there exists a positive constant $C_3 = C_3(j,\gamma,\mu_*)$ such that

$$\left|\sum_{n=j}^{\infty}\binom{\theta/\gamma}{n}K(\rho)^n\right| \le C_3\rho^{2j\theta}, \tag{C.21}$$

provided $\rho \in [0,\lambda_2]$. Likewise, for each $j \ge 1$, there exists a positive constant $C_4 = C_4(j,\gamma,\mu_*)$ such that

$$\left|\sum_{n=j}^{\infty}\binom{\theta/\gamma}{n}nK'(\rho)K(\rho)^{n-1}\right| \le C_4\rho^{2j\theta-1}, \tag{C.22}$$

provided $\rho \in [0,\lambda_2]$.

Proof. The binomial coefficient can be expanded as

$$\begin{aligned}\binom{\theta/\gamma}{n} &= \frac{\theta/\gamma(\theta/\gamma-1)\dots(\theta/\gamma-n+1)}{n!} \\ &= \theta/\gamma\frac{(-1)^{n-1}(1-\theta/\gamma)\dots(n-1-\theta/\gamma)}{n!}.\end{aligned}$$

It is then apparent that, by adding θ/γ to each factor in the numerator,

$$\left|\binom{\theta/\gamma}{n}\right| \leq \theta/\gamma\frac{(n-1)!}{n!} = \frac{\theta}{n\gamma}. \tag{C.23}$$

Thus, by restricting the density to the interval $\rho \in [0, (2C_1)^{-1/2\theta}]$, we see that

$$\left|\binom{\theta/\gamma}{n} K(\rho)^n\right| \leq \frac{\theta}{n\gamma}2^{-n},$$

which is manifestly summable. Accordingly, by the Weierstraß M-test, the series given by (C.19) converges absolutely and uniformly whenever $\rho \in [0, \lambda_2]$. Observe also that the term-by-term derivatives are dealt with as shown underneath:

$$\begin{aligned}\sum_{n=1}^{\infty}\left|\binom{\theta/\gamma}{n} nK'(\rho)K(\rho)^{n-1}\right| &\leq \sum_{n=1}^{\infty}\left|\binom{\theta/\gamma}{n}\right| nC_2\rho^{2\theta-1}\cdot(C_1\rho^{2\theta})^{n-1} \\ &\leq \frac{C_2\theta}{\gamma}\rho^{2\theta-1}\sum_{n=0}^{\infty}(C_1\rho^{2\theta})^n,\end{aligned}$$

which is bounded above by $2C_2(\theta/\gamma)\rho^{2\theta-1}$, provided $\rho \leq (2C_1)^{-1/2\theta}$. Thus, by the Weierstraß M-test, the series of term-by-term derivatives converges absolutely and uniformly in the stated interval for ρ, which concludes the proof of the lemma. □

Lemma C.13. *The series*

$$\sum_{n=1}^{\infty}\binom{3\theta/\gamma}{n} K(\rho)^n \tag{C.24}$$

converges uniformly for $\rho \in [0, \lambda_2]$. Further, in this interval, the above series can be differentiated term-by-term, i.e.,

$$\frac{d}{d\rho}\sum_{n=1}^{\infty}\binom{3\theta/\gamma}{n} K(\rho)^n = \sum_{n=1}^{\infty}\binom{3\theta/\gamma}{n} nK'(\rho)K(\rho)^{n-1}, \tag{C.25}$$

and the series on the right-hand side also converges absolutely and uniformly for $\rho \in [0, \lambda_2]$.

Remark C.14. In light of the above, we deduce that, for each $j \geq 1$, there exists a positive constant $C_5 = C_5(j, \gamma, \mu_*)$ such that

$$\left| \sum_{n=j}^{\infty} \binom{3\theta/\gamma}{n} K(\rho)^n \right| \leq C_5 \rho^{2j\theta}, \tag{C.26}$$

provided $\rho \in [0, \lambda_2]$. Likewise, for each $j \geq 1$, there exists a positive constant $C_6 = C_6(j, \gamma, \mu_*)$ such that

$$\left| \sum_{n=j}^{\infty} \binom{3\theta/\gamma}{n} nK'(\rho)K(\rho)^{n-1} \right| \leq C_6 \rho^{2j\theta - 1}, \tag{C.27}$$

provided $\rho \in [0, \lambda_2]$.

Proof. As before, we begin by bounding the binomial coefficient,

$$\begin{aligned} \binom{3\theta/\gamma}{n} &= \frac{3\theta/\gamma(3\theta/\gamma - 1)(3\theta/\gamma - 2)\dots(3\theta/\gamma - n + 1)}{n!} \\ &= 3\theta/\gamma(3\theta/\gamma - 1)\frac{(-1)^{n-2}(2 - 3\theta/\gamma)\dots(n - 1 - 3\theta/\gamma)}{n!}, \end{aligned}$$

from which it is apparent that (by adding $3\theta/\gamma$ to each factor in the numerator)

$$\left| \binom{3\theta/\gamma}{n} \right| \leq 3\theta/\gamma|3\theta/\gamma - 1|\frac{(n-1)!}{n!} = \frac{3\theta/\gamma|3\theta/\gamma - 1|}{n}.$$

Hence, we have that

$$\left| \binom{3\theta/\gamma}{n} K(\rho)^n \right| \leq \frac{3\theta/\gamma|3\theta/\gamma - 1|}{n}(C_1\rho^{2\theta})^n \qquad \text{for each } n \geq 1,$$

and the right-hand term is bounded by $\frac{3}{2n} \cdot 2^{-n}$ in the stated interval, which is summable. Thus, by the Weierstraß M-test, our series converges uniformly in the stated interval, as required. We now look at the series of term-by-term derivatives, which is bounded in the following way

$$\begin{aligned} \sum_{n=1}^{\infty} \left| \binom{3\theta/\gamma}{n} nK'(\rho)K(\rho)^{n-1} \right| &\leq 3\theta/\gamma|3\theta/\gamma - 1| \sum_{n=1}^{\infty} C_2\rho^{2\theta-1} \cdot (C_1\rho^{2\theta})^{n-1} \\ &= 3\theta/\gamma|3\theta/\gamma - 1|C_2 \left(\sum_{n=0}^{\infty} (C_1\rho^{2\theta})^n \right) \rho^{2\theta-1}. \end{aligned}$$

It is clear that the latter term is bounded by $6(\theta/\gamma)|3\theta/\gamma - 1|C_2\rho^{2\theta-1}$, provided $\rho \leq (2C_1)^{-1/2\theta}$. As before, an application of the Weierstraß M-test yields the result. □

Lemma C.15. *Define the remainder term to be*

$$L(\nu) := k(\nu) - \frac{1}{\sqrt{\theta}}\left(\frac{\gamma}{\theta}\right)^{\theta/\gamma}\nu^{\theta/\gamma} - \frac{1}{\sqrt{\theta}}\left(\frac{c_1}{3} - \binom{\theta/\gamma}{1}\frac{1}{1+2\theta/\gamma}\right)\left(\frac{\gamma}{\theta}\right)^{3\theta/\gamma}\nu^{3\theta/\gamma}. \tag{C.28}$$

Then, provided $\rho \in [0, \lambda_2]$, there exists positive constants $C_7 = C_7(\gamma, \mu_*)$ and $C_8 = C_8(\gamma, \mu_*)$ such that

$$|L(\nu)| \le C_7 \nu^{5\theta/\gamma}, \tag{C.29}$$

and

$$|L'(\nu)| \le C_8 \nu^{(3\theta-1)/\gamma}. \tag{C.30}$$

Proof. Following on from (C.14), we write

$$\frac{1}{\sqrt{\theta}}\left(\frac{\gamma}{\theta}\right)^{\theta/\gamma}\nu^{\theta/\gamma} = \frac{\rho^\theta}{\sqrt{\theta}}\left(1+K(\rho)\right)^{\theta/\gamma}.$$

Restricting ρ to be small enough, we may assume that $K(\rho) < 1$, and thus we can once again binomially expand the right-hand side of the above expression. This yields

$$\frac{1}{\sqrt{\theta}}\left(\frac{\gamma}{\theta}\right)^{\theta/\gamma}\nu^{\theta/\gamma} = \frac{\rho^\theta}{\sqrt{\theta}}\left(1 + \sum_{n=1}^{\infty}\binom{\theta/\gamma}{n}K(\rho)^n\right).$$

Thus,

$$k(\nu(\rho)) - \frac{1}{\sqrt{\theta}}\left(\frac{\gamma}{\theta}\right)^{\theta/\gamma}\nu^{\theta/\gamma} = \frac{\rho^\theta}{\sqrt{\theta}}\sum_{n=1}^{\infty}\left[\frac{c_n\rho^{2n\theta}}{2n+1} - \binom{\theta/\gamma}{n}K(\rho)^n\right]. \tag{C.31}$$

By expanding the right-hand term of (C.31) a little more, we have that

$$\sum_{n=1}^{\infty}\left[\frac{c_n\rho^{2n\theta}}{2n+1} - \binom{\theta/\gamma}{n}K(\rho)^n\right] = \left(\frac{c_1}{3} - \binom{\theta/\gamma}{1}\frac{1}{1+2\theta/\gamma}\right)\rho^{2\theta} + L_1(\rho),$$

where we define the remainder term by

$$L_1(\rho) := \sum_{n=2}^{\infty}\left[\frac{c_n\rho^{2n\theta}}{2n+1} - \binom{\theta/\gamma}{n}K(\rho)^n\right] - \binom{\theta/\gamma}{1}\sum_{n=2}^{\infty}\frac{\rho^{2n\theta}}{1+2n\theta/\gamma}.$$

Claim C.16. *For $\rho \in [0, \lambda_2]$, there exists a positive $C_9 = C_9(\gamma, \mu_*)$ such that*

$$|L_1(\rho)| \le C_9 \rho^{4\theta}. \tag{C.32}$$

Proof of Claim C.16. Observe that

$$\begin{aligned} L_1(\rho) = &\left(\frac{c_2}{5} - \frac{1}{1+4\theta/\gamma}\binom{\theta/\gamma}{1}\right)\rho^{4\theta} - \binom{\theta/\gamma}{2}K(\rho)^2 \\ &+ \sum_{n=3}^{\infty}\left[\frac{c_n\rho^{2n\theta}}{2n+1} - \binom{\theta/\gamma}{n}K(\rho)^n\right] - \binom{\theta/\gamma}{1}\sum_{n=3}^{\infty}\frac{\rho^{2n\theta}}{1+2n\theta/\gamma}. \end{aligned} \tag{C.33}$$

It then follows that

$$\begin{aligned} L_1(\rho) \geq \quad & -\left|\frac{c_2}{5}-\frac{\theta/\gamma}{1+4\theta/\gamma}\right|\rho^{4\theta}-\left|\binom{\theta/\gamma}{2}\right|C_1^2\rho^{4\theta} \\ & -\rho^{4\theta}\frac{\theta}{\gamma}\sum_{n=1}^{\infty}\left[\frac{|c_{n+2}|\rho^{2n\theta}}{2n+5}+\left|\binom{\theta/\gamma}{n+2}\right|\rho^{-4\theta}K(\rho)^{n+2}\right] \\ & -\rho^{4\theta}\frac{\theta}{\gamma}\sum_{n=1}^{\infty}\frac{\rho^{2n\theta}}{1+4\theta/\gamma+2n\theta/\gamma}, \end{aligned}$$

from which the lower bound of (C.32) is apparent, after evaluating the series terms with ρ replaced by $(2C_1)^{-1/2\theta}$. Indeed we first have that, for each $n \geq 1$,

$$\begin{aligned} \frac{|c_{n+2}|\rho^{2n\theta}}{2n+5} &\leq \frac{(n+3)2^{-(n+2)}(\gamma+1)^{n+2}}{2n+5}\rho^{2n\theta} \\ &\leq \frac{n+3}{2n+5}\frac{(\gamma+1)^2}{4}2^{-n}\left(\frac{\gamma+1}{2}\rho^{2\theta}\right)^n, \end{aligned}$$

and the right-hand side is summable if we pick $\rho \leq (\gamma+1)^{-1/2\theta}$. Second,

$$\left|\binom{\theta/\gamma}{n+2}\right|\rho^{-4\theta}K(\rho)^{n+2} \leq \frac{C_1^2\theta}{\gamma(n+2)}(C_1\rho^{2\theta})^{2n} \qquad \text{for each } n \geq 1,$$

where we used the bound (C.23), and the right-hand side is summable for $\rho \leq (2C_1)^{-1/2\theta}$. Finally,

$$\frac{\rho^{2n\theta}}{1+4\theta/\gamma+2n\theta/\gamma} \leq \frac{\rho^{2n\theta}}{1+4\theta/\gamma},$$

and the right-hand side is again summable for $\rho \leq 2^{-1/2\theta}$. Also,

$$\begin{aligned} L_1(\rho) \leq & \left|\frac{c_2}{5}-\frac{\theta/\gamma}{1+4\theta/\gamma}\right|\rho^{4\theta}+\left|\binom{\theta/\gamma}{2}\right|C_1^2\rho^{4\theta} \\ & +\rho^{4\theta}\sum_{n=3}^{\infty}\left[\frac{|c_n|\rho^{2(n-2)\theta}}{2n+1}+\left|\binom{\theta/\gamma}{n}\right|\rho^{-4\theta}K(\rho)^n\right]+\rho^{4\theta}\frac{\theta}{\gamma}\sum_{n=3}^{\infty}\frac{\rho^{2(n-2)\theta}}{1+2n\theta/\gamma}, \end{aligned}$$

from which we see that the same bounding procedure gives the other inequality. □

Claim C.17. *For $\rho \in [0, \lambda_2]$, there exists a positive $C_{10} = C_{10}(\gamma, \mu_*)$ such that*

$$|L_1'(\rho)| \leq C_{10}\rho^{4\theta-1}. \tag{C.34}$$

Proof of Claim C.17. It is clear from the previous lemmas that L_1 is differentiable and that the derivative admits a series representation whose terms agree with the term-by-term derivatives of L_1. Specifically,

$$L_1'(\rho) = \sum_{n=2}^{\infty}\left[\frac{2n\theta c_n\rho^{2n\theta-1}}{2n+1}-\binom{\theta/\gamma}{n}nK'(\rho)K(\rho)^{n-1}\right]-\frac{\theta}{\gamma}\sum_{n=2}^{\infty}\frac{2n\theta\rho^{2n\theta-1}}{1+2n\theta/\gamma},$$

whenever $\rho \in [0, \lambda_2]$. Observe now that

$$\sum_{n=2}^{\infty} \frac{2n\theta |c_n| \rho^{2n\theta-1}}{2n+1} = \rho^{4\theta-1} \sum_{n=0}^{\infty} \frac{(2n+4)|c_{n+2}|\rho^{2n\theta}}{2n+5},$$

and

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(2n+4)|c_{n+2}|\rho^{2n\theta}}{2n+5} &\le \sum_{n=0}^{\infty} \frac{(2n+4)(n+3)}{2n+5} \frac{(\gamma+1)^2}{4} 2^{-n} \left(\frac{\gamma+1}{2}\rho^{2\theta}\right)^n \\
&\le \frac{(\gamma+1)^2}{4} \sum_{n=0}^{\infty} 2^{-n} \left[n \left(\frac{\gamma+1}{2}\rho^{2\theta}\right)^n + 3 \left(\frac{\gamma+1}{2}\rho^{2\theta}\right)^n \right] \\
&\le \frac{(\gamma+1)^2}{4} \left(\sum_{n=0}^{\infty} n 2^{-2n} + 3 \sum_{n=0}^{\infty} 2^{-2n} \right) \\
&\le \frac{1}{2}(\gamma+1)^2,
\end{aligned}$$

provided $\rho \le (\gamma+1)^{-1/2\theta}$. Similarly,

$$\begin{aligned}
\sum_{n=2}^{\infty} \left| \binom{\theta/\gamma}{n} n K'(\rho) K(\rho)^{n-1} \right| &\le \frac{C_2\theta}{\gamma} \rho^{2\theta-1} \sum_{n=2}^{\infty} (C_1 \rho^{2\theta})^{n-1} \\
&= \frac{C_1 C_2 \theta}{\gamma} \rho^{4\theta-1} \sum_{n=0}^{\infty} (C_1\rho^{2\theta})^n \le \frac{2C_1C_2\theta}{\gamma} \rho^{4\theta-1},
\end{aligned}$$

provided $\rho \le (2C_1)^{-1/2\theta}$. Finally,

$$\begin{aligned}
\frac{\theta}{\gamma} \sum_{n=2}^{\infty} \frac{2n\theta\rho^{2n\theta-1}}{1+2n\theta/\gamma} &= \rho^{4\theta-1} \frac{\theta}{\gamma} \sum_{n=0}^{\infty} \frac{2n\theta+4\theta}{1+4\theta/\gamma+2n\theta/\gamma} \rho^{2n\theta} \\
&\le \theta \rho^{4\theta-1} \left(\sum_{n=0}^{\infty} \rho^{2n\theta} + \sum_{n=0}^{\infty} \rho^{2n\theta} \right) \\
&\le 4\theta\rho^{4\theta-1},
\end{aligned}$$

provided that $\rho \le 2^{-1/2\theta}$. Now, let $C_{10} := \frac{1}{2}(\gamma+1)^2 + 2C_1C_2\theta/\gamma + 4\theta$. Hence,

$$|L_1'(\rho)| \le C_{10}\rho^{4\theta-1},$$

in the stated interval. This concludes the proof of the claim. □

Hence, we have shown that

$$k(\nu(\rho)) - \frac{1}{\sqrt{\theta}} \left(\frac{\gamma}{\theta}\right)^{\theta/\gamma} \nu^{\theta/\gamma} = \frac{1}{\sqrt{\theta}} \left(\frac{c_1}{3} - \binom{\theta/\gamma}{1} \frac{1}{1+2\theta/\gamma} \right) \rho^{3\theta} + \frac{\rho^\theta}{\sqrt{\theta}} L_1(\rho),$$

which implies that

$$L(\nu(\rho)) = \frac{1}{\sqrt{\theta}}\left(\frac{c_1}{3} - \begin{pmatrix} \theta/\gamma \\ 1 \end{pmatrix}\frac{1}{1+2\theta/\gamma}\right)\rho^{3\theta}\left(1-(1+K(\rho))^{3\theta/\gamma}\right) + \frac{\rho^\theta}{\sqrt{\theta}}L_1(\rho)$$

$$= -\frac{1}{\sqrt{\theta}}\left(\frac{c_1}{3} - \begin{pmatrix} \theta/\gamma \\ 1 \end{pmatrix}\frac{1}{1+2\theta/\gamma}\right)\rho^{3\theta}\sum_{n=1}^{\infty}\begin{pmatrix} 3\theta/\gamma \\ n \end{pmatrix}K(\rho)^n + \frac{\rho^\theta}{\sqrt{\theta}}L_1(\rho).$$

Our previous results immediately yield the claim underneath.

Claim C.18. *There exist positive constants $C_{11} = C_{11}(\gamma, \mu_*)$ and $C_{12} = C_{12}(\gamma, \mu_*)$ such that, provided $\rho \in [0, \lambda_2]$, we have*

$$|L(\nu(\rho))| \le C_{11}\rho^{5\theta}, \qquad \left|\frac{d}{d\rho}L(\nu(\rho))\right| \le C_{12}\rho^{5\theta-1}. \tag{C.35}$$

Now, in order to estimate the true derivative of L with respect to the variable ν, we use the chain rule. Indeed,

$$|L'(\nu)| = \left|\frac{d}{d\rho}L(\nu(\rho))\right| \cdot \frac{1}{|\nu'(\rho)|} \le C_{12}\rho^{5\theta-1}/\rho^{2\theta} = C_{12}\rho^{3\theta-1} \le C_{12}\left(\frac{\gamma}{\theta}\right)^{1/\gamma}\nu^{(3\theta-1)/\gamma},$$

since $\nu(\rho) \ge (\theta/\gamma)\rho^\gamma$, due to (C.14). Analogously,

$$|L(\nu(\rho))| \le C_{11}\rho^{5\theta} \le C_{11}\left(\frac{\gamma}{\theta}\right)^{1/\gamma}\nu^{5\theta/\gamma},$$

which concludes the proof of the lemma. □

Remark C.19. By repeating this chain rule procedure, we see that, for each $j \in \mathbb{N}$, there exists a positive constant $C = C(j, \gamma, \mu_*)$ such that, for ρ sufficiently small,

$$|L^{(j)}(\nu)| \le C\nu^{5\theta/\gamma - j}. \tag{C.36}$$

Remark C.20. We can write the coefficient of the second term as

$$\left(\frac{c_1}{3} - \begin{pmatrix} \theta/\gamma \\ 1 \end{pmatrix}\frac{1}{1+2\theta/\gamma}\right) = -\frac{(4\theta-1)(\theta+1)}{6(4\theta+1)}.$$

Proof of Theorem 4.13. Since $\nu(\rho) \ge \frac{\theta}{\gamma}\rho^\gamma$, by setting $\lambda_* = (\theta/\gamma)\lambda_2^\gamma$ or smaller if necessary, we have proved Theorem 4.13 for $\nu \in [0, \lambda_*]$. But since $k \in C^\infty((0, \mu_{cr}))$ (*cf.* Definition 4.12), k and its derivatives achieve their bounds on $[\lambda_*, \mu_*]$. Hence, there exists a (possibly larger) constant $C = C(\gamma, \mu_*)$ such that (C.36) holds on the entire interval $(0, \mu_*]$. The proof is complete. □

Finally, we have the following corollary.

Corollary C.21. *The quantity $k(\nu)$ can be written as*

$$k(\nu) = a_\sharp \nu^{\theta/\gamma}\left(1 + H(\nu)\right), \tag{C.37}$$

and there exists a positive constant $C = C(\gamma, \mu_*)$ such that

$$|H^{(j)}(\nu)| \le C\nu^{2\theta/\gamma - j} \qquad \text{for } \nu \in (0, \mu_*], \quad \text{for } j \in \{0,1,2,3\}. \tag{C.38}$$

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