

# A New Method of Modelling Tuneable Lasers with Functional Composition

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**Abstract** A new nonlinear model is proposed for tuneable lasers. Using the generalized nonlinear Schrödinger equation as a starting point, expressions for the transformations undergone by the pulse are derived for each of the five components (gain, loss, dispersion, modulation, and nonlinearity) within the laser cavity. These transformations are then composed to give the overall effect of one trip around the cavity. This is in contrast to solving the generalized nonlinear Schrödinger equation which treats the processes as continuous.

## 1 Introduction

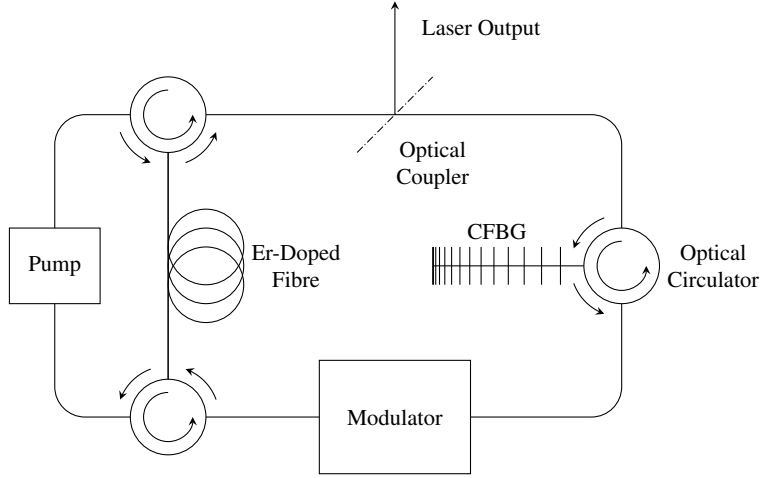
A tuneable laser has the ability to vary the frequency of its output by up to about 100 nanometres [5, 8, 38]. Tuneable lasers simultaneously lase at all frequencies within this bandwidth. This tuneability is quite useful and has applications in spectroscopy and high resolution imaging such as coherent anti-Stokes Raman spectroscopy and optical coherence tomography [5, 7, 38], as well as communications and diagnostics of ultra fast processes [31]. A typical tuneable laser cavity can be seen in Figure 1. In contrast to a standard laser, a tuneable laser contains two additional components, namely, a chirped fibre Bragg grating (CFBG), and a modulator.

A CFBG is a length of optical fibre where the refractive index oscillates along its length [10], and therefore, can act as a reflective filter [1, 3, 10, 32]. Due to the oscillatory nature, light with the corresponding wavelength will be reflected when the Bragg condition is satisfied [1, 3, 4, 10, 31, 32]. The spacial variation of the refractive index effectively creates a spacial dependence on the Bragg condition,

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**Fig. 1** Typical cavity of a fibre based tuneable laser. The laser pulses travel clockwise around each loop. The pulses iteratively pass through each component successively.

causing most wavelengths to be reflected by a CFBG, but with each wavelength satisfying the Bragg condition at a different spacial location<sup>1</sup>. A consequence of this is that a time delay is created between wavelengths—this causes the pulse to disperse and broaden.

The modulator serves the purpose of reshaping the pulse. Without it, the pulse will repeatedly widen due to the CFBG—the modulator ensures the pulse is band limited by altering the envelope.

## 2 Previous Modelling Efforts

The standard equation for studying nonlinear optics is the nonlinear Schrödinger equation (NLSE)<sup>2</sup>,

$$\frac{\partial A}{\partial z} = -i\frac{\beta_2}{2}\frac{\partial^2 A}{\partial T^2} + i\gamma|A|^2A. \quad (1)$$

Here  $A = A(T, z) : \mathbb{R}^2 \mapsto \mathbb{C}$  is the complex pulse amplitude,  $\beta_2 \in \mathbb{R}$  is the second order dispersion, and  $\gamma \in \mathbb{R}$  is the coefficient of nonlinearity. In practice, (1) lacks a few key terms, thus, it is often generalized by adding amplification, loss, and occasionally higher order terms. This gives the generalized nonlinear Schrödinger

<sup>1</sup> Note that a monotonic chirping ensures that the spacial dependence of the Bragg condition is continuous with respect to the frequency.

<sup>2</sup> The NLSE can be derived from the nonlinear wave equation for electric fields; this derivation is presented in detail in [2, 10].

equation (GNLSE) [2, 5, 11, 28, 29, 39],

$$\frac{\partial A}{\partial z} = -i\frac{\beta_2}{2}\frac{\partial^2 A}{\partial T^2} + i\gamma|A|^2A + \frac{1}{2}g(A)A - \alpha A, \quad (2)$$

where  $g(A)$  is an amplifying term due to the gain, and  $\alpha \in \mathbb{R}$  is the loss due to scattering and absorption.

The GNLSE has many applications in nonlinear optics and fibre optic communications, however, in the context of lasers we typically also add a modulation term. This yields the master equation of mode-locking [13, 14, 15, 16, 18, 20, 35, 37],

$$\frac{\partial A}{\partial z} = -i\frac{\beta_2}{2}\frac{\partial^2 A}{\partial T^2} + i\gamma|A|^2A + \frac{1}{2}g(A)A - \alpha A - M(T), \quad (3)$$

where  $M(T)$  is the modulation function. The solutions of three simplifications of (3) have been investigated:

- Omitting both modulation and nonlinearity [13, 15, 16].
- Omitting only modulation [19, 37].
- Omitting only nonlinearity [7, 13, 14, 17, 18, 20, 35, 37].

For a more comprehensive history see [18].

## 2.1 Discrete Component Models

While the derivation of (3) is sound mathematically, it is not representative of what happens within the laser cavity. The issue with (3) is that it has been assumed each process affects the pulse continuously within the cavity; for example, the pulse is amplified regardless of if it is in the Erbium-doped fibre. As highlighted by Figure 1, this is a rather poor assumption. Within the cavity each effect is localized to its corresponding component: almost all of the dispersion happens within the CFBG [1], the pulse is only amplified within the Erbium-doped fibre, etc. Thus, a better model is one where (3) is broken down into the individual components giving the effect of each ‘block’ of the cavity. Each of the blocks can then be functionally composed together to give an iterative map for the effect of one circuit around the cavity. This transforms the differential equation into an algebraic equation.

Such a method was first proposed in 1955 by Cutler [9] while analyzing a microwave regenerative pulse generator. This method was adapted for mode-locked lasers in 1969 by Siegman and Kuizenga [21, 30]. Kuizenga and Siegman also had success experimentally validating their model [22, 23]. The effects of the nonlinearity would not be considered until Martinez, Fork, and Gordon [26, 27] tried modelling passively mode-locked lasers. This issue has recently been readdressed by Burgoyne [7] in the literature for tuneable lasers. In these models the effect of each component is described by a transfer function.

These discrete component models differ from the often used split-step Fourier method (see [2, 33]). The split-step Fourier method is a numerical technique used for solving nonlinear partial differential equations, such as (1). The method considers the linear and nonlinear terms separately and has a half integration step for both parts—in a manner similar to the leap-frog algorithm<sup>3</sup>. Therefore, in the case of the NLSE, the dispersion and nonlinearity are still treated as continuous processes. However, in discrete component models the entire effect of each component is computed at once and the output of one component becomes the input for the following component, instead of alternating between the components in small integration steps. In this way, discrete component models are able to account for the geometry of a laser cavity, and indeed altering the permutation of the components gives rise to different dynamics.

Despite the development of discrete component models, several short-comings exist. The clearest is that none of these models have contained every block—either the nonlinearity or the modulation have been omitted. In the framework of tuneable lasers, each component plays a crucial role and the tuneable laser will not function correctly without the inclusion of all the components. Another key drawback is that the functional operations of some of the components used in their models are phenomenological. While these functions are chosen based on the observed output, they are not necessarily consistent with the underlying physics. Finally, none of these previous models have been able to exhibit a phenomenon called *modulation instability* in which the self-phase modulation of the pulse becomes too strong, distorting and damaging the wave until it ultimately becomes unstable and unsustainable.

### 3 A New Model

Using the ideas presented in the previous section of the prior functional models [7, 9, 21, 22, 23, 26, 27, 30] we shall derive a new model from (2)—with the exception of modulation in which we consider the exact functional form to be determined by the laser operator.

#### 3.1 Components

We shall determine the effect each component has on the pulse by solving (2) while only considering the dominant term within each section of the cavity, and neglecting the others.

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<sup>3</sup> Also known as velocity Verlet in molecular dynamics, the Störmer method in astronomy, and further names in other areas [12].

### 3.1.1 Gain

Within the Er-doped gain fibre, the gain term is dominant, and equation (2) reduces to

$$\frac{\partial A}{\partial z} = \frac{1}{2}g(A)A, \quad (4)$$

where  $g(A)$  takes the form [5, 7, 13, 14, 16, 18, 19, 20, 28, 29, 31, 37, 39]

$$g(A) = \frac{g_0}{1 + E/E_{\text{sat}}}, \quad E = \int_{-\infty}^{\infty} |A|^2 dT, \quad (5)$$

where  $g_0$  is a small signal gain,  $E$  is the energy of the pulse, and  $E_{\text{sat}}$  is the energy at which the gain begins to saturate. Without much difficulty this differential equation can be solved, and the effect on an incident pulse is

$$G(A; E) = \left( \frac{E_{\text{out}}}{E} \right)^{1/2} A = \left( \frac{E_{\text{sat}}}{E} W \left( \frac{E}{E_{\text{sat}}} e^{E/E_{\text{sat}}} e^{g_0 L_g} \right) \right)^{1/2} A, \quad (6)$$

where  $L_g$  is the length of the gain fibre.

### 3.1.2 Nonlinearity

The nonlinearity of the fibre arises from the parameter  $\gamma$ ; in regions where this effect is dominant expression (2) becomes

$$\frac{\partial A}{\partial z} - i\gamma|A|^2 A = 0. \quad (7)$$

Using a similar method as with the gain, the effect of the nonlinearity can be shown to be

$$F(A) = A e^{i\gamma|A|^2 L_f}, \quad (8)$$

where  $L_f$  is the length of fibre.

### 3.1.3 Loss

Expression (2) leads to exponential decay due to the scattering and absorption of the fibre. However, a majority of the signal is removed from the cavity by the optical coupler. Combining these two effects yields

$$L(A) = (1 - R) e^{-\alpha L_T} A, \quad (9)$$

where  $R$  is the reflectivity of the output coupler, and  $L_T$  is the total length of the laser circuit, as the effect of the losses<sup>4</sup>.

### 3.1.4 Dispersion

Considering only the dispersive terms of (2), one obtains

$$\frac{\partial A}{\partial z} = -i \frac{\beta_2}{2} \frac{\partial^2 A}{\partial T^2}. \quad (10)$$

The effect of dispersion is then given by the map

$$D(A) = \mathcal{F}^{-1} \left\{ e^{i\omega^2 L_D \beta_2 / 2} \mathcal{F}\{A\} \right\}, \quad (11)$$

where  $L_D$  is the characteristic length of the dispersive medium, and  $\mathcal{F}$  denotes the Fourier transform.

### 3.1.5 Modulation

In this model, the modulation is considered to be applied externally in which ever way the operator sees fit. For simplicity the representation is taken as the Gaussian

$$M(A) = e^{-T^2 / 2T_M^2} A, \quad (12)$$

where  $T_M$  is the characteristic width of the modulation.

## 3.2 Non-Dimensionalization

The structure of each process of the laser can be better understood by re-scaling the time, energy, and amplitude. Nominal values for tuneable lasers are shown in Table 1. Knowing experimental durations and energies, the table suggests the convenient scalings:

$$T = T_M \tilde{T}, \quad E = E_{\text{sat}} \tilde{E}, \quad A = \left( \frac{E_{\text{sat}}}{T_M} \right)^{1/2} \tilde{A}. \quad (13)$$

Revisiting each process map shows each process has a characteristic non-dimensional parameter. The new mappings—after dropping the tildes—are

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<sup>4</sup> Depending on the layout of the laser cavity the loss may take the form  $L(A) = R e^{-\alpha L_T} A$  instead.

**Table 1** Range of variation of various parameters.

Parameter	Symbol	Value	Sources
Absorption of Fibre <sup>a</sup>	$\alpha$	$10^{-4}$ – $0.3 \text{ m}^{-1}$	[6, 29, 36, 37, 39]
Fibre Dispersion	$\beta_2^f$	$-50$ – $50 \text{ ps}^2/\text{km}$	[1, 2, 7, 25, 28, 39]
Fibre Nonlinearity	$\gamma$	$0.001$ – $0.01 \text{ W}^{-1}\text{m}^{-1}$	[2, 11, 37, 39]
Grating Dispersion	$\beta_2^g L_D$	$10$ – $2000 \text{ ps}^2$	[1, 2, 7, 24]
Length of Cavity	$L_T$	$10$ – $100 \text{ m}$	[6, 28, 35]
Length of Fibre	$L_f$	$0.15$ – $1 \text{ m}$	[6]
Length of Gain Fibre	$L_g$	$2$ – $3 \text{ m}$	[7, 28, 29, 34, 39]
Modulation Time	$T_M$	$15$ – $150 \text{ ps}$	[5, 6, 7]
Reflectivity of Optical Coupler	$R$	$0.1$ – $0.9$	[6, 24, 28, 34, 35, 38]
Saturation Energy	$E_{\text{sat}}$	$10^3$ – $10^4 \text{ pJ}$	[6, 37, 39]
Small Signal Gain	$g_0$	$1$ – $10 \text{ m}^{-1}$	[6, 39]

<sup>a</sup> Fibre loss is typically reported as  $\sim 0.5 \text{ dB/km}$ .

$$\begin{aligned} G(A) &= (E^{-1}W(aEe^E))^{1/2}A, & F(A) &= Ae^{ib|A|^2}, & L(A) &= hA, \\ D(A) &= \mathcal{F}^{-1}\left\{e^{is^2\omega^2}\mathcal{F}\{A\}\right\}, & M(A) &= e^{-T^2/2}A, \end{aligned} \quad (14)$$

with the four dimensionless parameters, as defined by the values in Table 1,

$$\begin{aligned} a &= e^{g_0 L_g} \sim 8 \times 10^3, & h &= (1-R)e^{-\alpha L} \sim 0.04, \\ b &= \gamma L_f \frac{E_{\text{sat}}}{T_M} \sim 1, & s &= \sqrt{\frac{\beta_2 L_D}{2T_M^2}} \sim 0.2, \end{aligned} \quad (15)$$

which characterize the behaviour of the laser. Notice that the modulation is only characterized by  $T_M$ , and each other process has its own independent non-dimensional parameter.

### 3.3 Combining the Effects of Each Block of the Model

In this model the pulse is iteratively passed through each process, the order of which must now be considered. We are most interested in the output of the laser cavity, and so we shall start with the loss component. Next the pulse is passed through the CFBG, as well as the modulator. Finally, the pulse travels through the gain fibre to be amplified, and then we consider the effect of the nonlinearity since this is the region where the power is maximal. Note that in general the functional operators of the components do not commute, and therefore the order of the components is indeed important—in contrast to previous models. This is especially the case of dispersion as realized through the Fourier transform. Functionally this can be denoted as

$$\mathcal{L}(A) = F(G(M(D(L(A))))) \quad (16)$$

where  $\mathcal{L}$  denotes one loop of the laser. The pulse after one complete circuit of the laser cavity is then passed back in to restart the process. A steady solution to this model is one in which the envelope and chirp are unchanged after traversing every component in the cavity—we are uninterested in the phase. That is, such that  $\mathcal{L}(A) = Ae^{i\phi}$ —for some  $\phi \in \mathbb{R}$ .

## 4 Conclusion

Within this paper we developed a nonlinear model for tuneable lasers. In order to better represent the underlying physics within the laser cavity the nonlinear Schrödinger equation was reduced to simpler differential equations for each component of the laser. This led to a functional map that defines the effect of each component on a particular input pulse. These processes were then composed together to give an iterative mapping of the whole laser cavity. In a future publication we shall show the results obtained by this iterative mapping as well as discuss the dynamics exhibited by this model—including modulation instability—to predict the conditions under which the pulse is stable and sustainable.

## References

1. Agrawal, G.: Fiber-Optic Communication Systems, 3 edn. John Wiley & Sons, Inc. (2002)
2. Agrawal, G.: Nonlinear Fiber Optics, 5 edn. Academic Press (2013)
3. Al-Azzawi, A.: Fiber Optics: Principles and Advanced Practices, 2 edn. CRC Press (2017)
4. Becker, P.C., Olsson, N.A., Simpson, J.R.: Erbium-Doped Fiber Amplifiers Fundamentals and Technology, 1 edn. Academic Press (1999)
5. Bohun, C.S., Cher, Y., Cummings, L.J., Howell, P., Mitre, T., Monasse, L., Mueller, J., Rouillon, S.: Modelling and Specifying Dispersive Laser Cavities. In: Sixth Montréal Industrial Problem Solving Workshop, pp. 11–25 (2015)
6. Burgoyne, B.: Private Communication (2018)
7. Burgoyne, B., Dupuis, A., Villeneuve, A.: An Experimentally Validated Discrete Model for Dispersion-Tuned Actively Mode-Locked Lasers. IEEE Journal of Selected Topics in Quantum Electronics **20**(5), 390–398 (2014). DOI 10.1109/JSTQE.2014.2303794
8. Burgoyne, B., Villeneuve, A.: Programmable Lasers: Design and Applications. In: Proc.SPIE, vol. 7580 (2010). DOI 10.1117/12.841277
9. Cutler, C.C.: The Regenerative Pulse Generator. In: Proceedings of the IRE. IEEE (1955). DOI 10.1109/JRPROC.1955.278070
10. Ferreira, M.F.S.: Nonlinear Effects in Optical Fibers. John Wiley & Sons, Inc. (2011)
11. Finot, C., Kibler, B., Provost, L., Wabnitz, S.: Beneficial Impact of Wave-Breaking for Coherent Continuum Formation in Normally Dispersive Nonlinear Fibers. J. Opt. Soc. Am. B **25**(11), 1938–1948 (2008). DOI 10.1364/JOSAB.25.001938
12. Hairer, E., Lubich, C., Wanner, G.: Geometric Numerical Integration: Structure-Preserving Algorithms for Ordinary Differential Equations, 2nd edn. No. 31 in Springer Series in Computational Mathematics. Springer, Berlin; New York (2006)
13. Haus, H.A.: A Theory of Forced Mode Locking. IEEE Journal of Quantum Electronics **11**(7), 323–330 (1975). DOI 10.1109/JQE.1975.1068636
14. Haus, H.A.: Waves and Fields in Optoelectronics. Prentice-Hall, Inc. (1984)



15. Haus, H.A.: Laser Mode Locking with Addition of Nonlinear Index. *IEEE Journal of Quantum Electronics* **22**(2), 325–331 (1986). DOI 10.1109/JQE.1986.1072944
16. Haus, H.A.: Analytic Theory of Additive Pulse and Kerr Lens Mode Locking. *IEEE Journal of Quantum Electronics* **28**(10), 2086–2096 (1992). DOI 10.1109/3.159519
17. Haus, H.A.: Theory of Soliton Stability in Asynchronous Modelocking. *Journal of Lightwave Technology* **14**(4), 622–627 (1996). DOI 10.1109/50.491401
18. Haus, H.A.: Mode-Locking of Lasers. *IEEE Journal of Selected Topics in Quantum Electronics* **6**(6), 1173–1185 (2000). DOI 10.1109/2944.902165
19. Haus, H.A., Fujimoto, J.G., Ippen, E.P.: Structures for Additive Pulse Mode Locking. *J. Opt. Soc. Am. B* **8**(10), 2068–2076 (1991). DOI 10.1364/JOSAB.8.002068
20. Kärtner, F.: Lecture Notes in Ultrafast Optics. Online (2005). Massachusetts Institute of Technology: MIT OpenCourseWare
21. Kuizenga, D.J., Siegman, A.E.: FM and AM Mode Locking of the Homogeneous Laser - Part I: Theory. *IEEE Journal of Quantum Electronics* **6**(11), 694–708 (1970). DOI 10.1109/JQE.1970.1076343
22. Kuizenga, D.J., Siegman, A.E.: FM and AM Mode Locking of the Homogeneous Laser - Part II: Experimental Results in a Nd:YAG Laser with Internal FM Modulation. *IEEE Journal of Quantum Electronics* **6**(11), 709–715 (1970). DOI 10.1109/JQE.1970.1076344
23. Kuizenga, D.J., Siegman, A.E.: FM-Laser Operation of the Nd:YAG Laser. *IEEE Journal of Quantum Electronics* **6**(11), 673–677 (1970). DOI 10.1109/JQE.1970.1076348
24. Li, S., Chan, K.T.: Electrical Wavelength Tunable and Multiwavelength Actively Mode-Locked Fiber Ring Laser. *Applied Physics Letters* **72**(16), 1954–1956 (1998). DOI 10.1063/1.121263. URL <https://doi.org/10.1063/1.121263>
25. Litchinitser, N.M., Eggleton, B.J., Patterson, D.B.: Fiber Bragg Gratings for Dispersion Compensation in Transmission: Theoretical Model and Design Criteria for Nearly Ideal Pulse Recompression. *Journal of Lightwave Technology* **15**(8), 1303–1313 (1997). DOI 10.1109/50.618327
26. Martinez, O.E., Fork, R.L., Gordon, J.P.: Theory of Passively Mode-Locked Lasers Including Self-Phase Modulation and Group-Velocity Dispersion. *Opt. Lett.* **9**(5), 156–158 (1984). DOI 10.1364/OL.9.000156
27. Martinez, O.E., Fork, R.L., Gordon, J.P.: Theory of Passively Mode-Locked Lasers for the Case of a Nonlinear Complex-Propagation Coefficient. *J. Opt. Soc. Am. B* **2**(5), 753–760 (1985). DOI 10.1364/JOSAB.2.000753
28. Peng, J., Luo, H., Zhan, L.: In-Cavity Soliton Self-Frequency Shift Ultrafast Fiber Lasers. *Opt. Lett.* **43**(24), 5913–5916 (2018). DOI 10.1364/OL.43.005913. URL <http://ol.osa.org/abstract.cfm?URI=ol-43-24-5913>
29. Shtyrina, O.V., Ivanenko, A.V., Yarutkina, I.A., Kemmer, A.V., Skidin, A.S., Kobtsev, S.M., Fedoruk, M.P.: Experimental Measurement and Analytical Estimation of the Signal Gain in an Er-Doped Fiber. *J. Opt. Soc. Am. B* **34**(2), 227–231 (2017). DOI 10.1364/JOSAB.34.000227
30. Siegman, A.E., Kuizenga, D.J.: Simple Analytic Expressions for AM and FM Mode-locked Pulses in Homogenous Lasers. *Appl. Phys. Lett.* (6), 181–182 (1969). DOI <https://doi.org/10.1063/1.1652765>
31. Silfvast, W.T.: *Laser Fundamentals*, 2 edn. Cambridge University Press (2004)
32. Starodoumov, A.N.: Optical Fibers and Accessories. In: D. Malacara-Hernández, B.J. Thompson (eds.) *Advanced Optical Instruments and Techniques, Handbook of Optical Engineering*, vol. 2, 2 edn., pp. 633–676. CRC Press (2018). Ch. 18. 2018
33. Taha, T.R., Ablowitz, M.J.: Analytical and Numerical Aspects of Certain Nonlinear Evolution Equations. II. Numerical, Nonlinear Schrödinger Equation. *Journal of Computational Physics* **55**(2), 203–230 (1984). DOI [https://doi.org/10.1016/0021-9991\(84\)90003-2](https://doi.org/10.1016/0021-9991(84)90003-2)
34. Tamura, K., Ippen, E.P., Haus, H.A., Nelson, L.E.: 77-fs Pulse Generation from a Stretched-Pulse Mode-Locked All-Fiber Ring Laser. *Opt. Lett.* **18**(13), 1080–1082 (1993). DOI 10.1364/OL.18.001080
35. Tamura, K., Nakazawa, M.: Dispersion-Tuned Harmonically Mode-Locked Fiber Ring Laser for Self-Synchronization to an External Clock. *Opt. Lett.* **21**(24), 1984–1986 (1996). DOI 10.1364/OL.21.001984

36. Tomlinson, W.J., Stolen, R.H., Johnson, A.M.: Optical Wave Breaking of Pulses in Nonlinear Optical Fibers. *Opt. Lett.* **10**(9), 457–459 (1985). DOI 10.1364/OL.10.000457
37. Usechak, N.G., Agrawal, G.P.: Rate-Equation Approach for Frequency-Modulation Mode Locking using the Moment Method. *J. Opt. Soc. Am. B* **22**(12), 2570–2580 (2005). DOI 10.1364/JOSAB.22.002570
38. Yamashita, S., Nakazaki, Y., Konishi, R., Kusakari, O.: Wide and Fast Wavelength-Swept Fiber Laser Based on Dispersion Tuning for Dynamic Sensing. *Journal of Sensors* **2009** (2009). DOI 10.1155/2009/572835
39. Yarutkina, I., Shtyrina, O., Fedoruk, M., Turitsyn, S.: Numerical Modeling of Fiber Lasers with Long and Ultra-long Ring Cavity. *Opt. Express* **21**(10), 12,942–12,950 (2013). DOI 10.1364/OE.21.012942