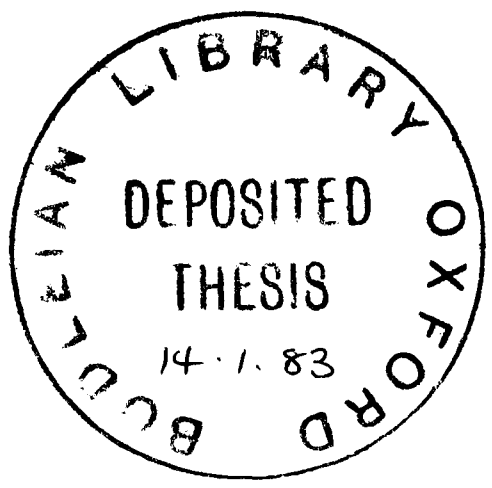
THEORY OF NUCLEAR STRUCTURE

by

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ABSTRACT

Inelastic scattering of heavy ions to low-lying collective states is studied. Symmetries of the scattering amplitude are discussed and, in particular, the conditions for the validity of an approximate symmetry property are derived. One condition is that the excitation should be 'adiabatic'. The consequences of this approximate symmetry are given: these include zero vector polarization and a gamma-ray distribution that is symmetric about the direction of recoil.

In the more general case, and under strong absorption conditions, the full inelastic scattering amplitude is decomposed into components corresponding to waves turning in opposite directions about the scattering centre. It is shown that this allows a simple interpretation of cross-section structure, particularly in the quasi-classical limit, which is derived and explained. The polarization and γ-ray symmetry angle are also discussed in these terms, and conditions of weaker absorption and Coulomb excitation are considered using the same method.

A new proof of the asymptotic form of certain Clebsch-Gordan coefficients is given, and a useful property of spherical harmonics derived.

To my parents

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"The important thing in science is not so much to obtain new facts as to discover new ways of thinking about them."

WL Bragg, 1890-1971.

"Nessuna humana investigazione si pio dimandara vera scienzia s'essa non passa per le matematiche dimonstrazione."

Leonardo da Vinci, 1452-1519.

CHAPTER I

INTRODUCTION AND OVERVIEW.

The study of heavy-ion reactions has attracted much attention in recent years, but a complete theory that explicitly deals with the many-body dynamics of nucleus-nucleus collisions is still unavailable and likely to remain so, at least for the foreseeable future. Any theory used to describe the interactions of nuclei must, therefore, involve some degree of approximation in order to make the problem manageable. The complexity of the problem has led people to propose simple, but in many ways reasonable, models to explain, or at least indicate, the origin of the features that characterize heavy-ion processes.

This thesis is concerned with an interpretation of heavy-ion inelastic scattering data which provides insight into some of these characteristics by dividing the scattering amplitude into components having readily identifiable physical origins. Before the substance of this work is outlined however, we review the field of heavy-ion direct reactions at energies above the Coulomb barrier.

The simplest process that can occur between two nuclei is elastic scattering and, in the absence of a full theory, one is forced initially to take a phenomenological view by seeking a potential that will reproduce available data. This is the optical-model approach, but unfortunately the potential is by no means unique (Igo 1958), because the scattering process is only sensitive to the potential in the region close to the nuclear surface. Data covering a wider angular range can reduce this

ambiguity to some extent. Coupling to other channels depletes the elastic scattering probability and can be described by introducing an absorptive imaginary part into the nucleus-nucleus potential. In principle one would like to relate the optical potential to the fundamental nucleon-nucleon interaction, but this is a rather complicated task. A compromise is to retain the most important features of a microscopic approach, such as the neutron and proton densities and a nucleon-nucleon effective interaction, but make the approach flexible enough to fit data by allowing phenomenological adjustments. This is the folding model (Dover and Vary 1974 and references therein) which has been reviewed by Satchler and Love (1979).

A microscopic approach using the Generator Co-ordinate method has been used for α-α and ^{16}O-^{16}O elastic scattering (Canto and Brink 1977, Baldock 1980).

Inelastic reactions and those involving rearrangement also require a description of the internal nuclear states and their coupling to the relative motion. Theoretical treatments of these processes make a variety of assumptions, but all treat the internal motion quantum mechanically: a simplifying feature of heavy-ion reactions is that in many cases the relative motion can be treated approximately by so-called semi-classical methods (see below). To date the most widely used approach to inelastic and simple rearrangement collisions is the distorted-wave Born approximation (DWBA). The first step in such a calculation is to compute the initial and final channel wavefunctions by solving the Schrödinger equation for a suitable potential. These wavefunctions are then folded with a transition potential, which in the collective model is the derivative of the optical

potential, to give the scattering amplitude. The simplicity of the calculation often outweighs its inaccuracies and, with modern computers, results for inelastic scattering can be obtained in a matter of seconds, even when integration to large radii is necessary for convergence. However, for rearrangement reactions accurate calculations require recoil effects and the finite range of the interaction to be taken into account (Hodgson 1978 and references therein) and the amount of computer time is then considerably larger. An exact treatment is only available for the transfer of a single nucleon, multi-nucleon transfer being treated as a sum of single nucleon transfers or as a cluster expansion. The success of such an approximation for the transfer of a few nucleons provides evidence for clustering phenomena in nuclei (Buck et al. 1975, Buck et al. 1979, Merchant 1979 and references therein).

The DWBA is only a single-step theory and in many cases intermediate states are important. These can be accounted for by multi-step theories such as the coupled-channels distorted-wave Born approximation (CCBA) (Tamura 1974).

Rather than following a full quantum mechanical treatment of heavy-ion scattering it is possible to make a 'semi-classical' approach. Whilst not intended to be perfectly accurate, these approaches do provide the physical insight that is lacking from large-scale computer calculations. Although use of the word 'semi-classical' is widespread, it is perhaps better to distinguish between 'semi-classical' and 'semi-quantal' theories. The former retain the axioms of classical mechanics whilst the latter are essentially classical interpretations of approximate quantal theories. Naturally however, the distinction is often vague.

Time-dependent semi-classical theories treat the projectile as moving along a trajectory according to classical laws and for each such trajectory time-dependent quantum theory is used to calculate the transition probability. This approach has been studied in detail by Broglia and Winther (1972), Broglia (1974) and Broglia et al. (1974).

Semi-quantal theories are based around some sort of expansion in powers of $\hbar$, which sets the scale for quantum effects. However, in the presence of strong absorption the earlier of such theories (for example that of Ford and Wheeler (1959)) are inadequate. (Note though that diffraction models (Frahn 1966 and references therein) place too much emphasis on the absorptive properties of the interaction.) The $\hbar$-expansion can be carried out using the Feynman path-integral formulation of quantum mechanics (Koeling and Malfliet 1975, Koonin 1981) or by extending the WKB approximation (Berry and Mount 1972) to the complex radial co-ordinate plane and complex potentials (Knoll and Schaeffer 1976 and 1977). In the latter one is led naturally to speak of complex trajectories and complex deflection functions, and the results are, perhaps not too surprisingly, fairly complicated.

An alternative, and physically more appealing semi-quantal approach is based around the structure of the elastic S-matrix. In the heavy-ion case the important values of angular momentum are large and so a series of approximations to the quantum description is possible and has been discussed and justified by Frahn and Gross (1976). Frahn and his co-workers do, however, make extensive use of the stationary phase approximation, which is sometimes unrealistic (Rowley and Marty 1976). Apart from this criticism, we note that this so-called 'closed formalism'

has been applied with varying degrees of success to elastic (Frahn 1973, Rowley and Marty 1976, Rowley 1980) and inelastic scattering (Frahn 1976, Frahn and Rehm 1978) of spinless 'light' heavy-ions and also to heavier systems (Frahn 1978a, 1978b, 1978c). Zero-range quasi-elastic transfer reactions (Frahn 1980c), coupled-channels effects (Frahn and Hussein 1980) and dynamic polarization (Frahn 1978c) have been treated, as have large-angle elastic and inelastic scattering (Frahn 1980a and 1980b, Frahn et al. 1981) and the scattering of particles with spin (Frahn and Hill 1980a and 1980b). An application to finite-range transfer reactions, including recoil effects, has been considered (Hill 1982). The connection between the optical model, which parametrizes the potential, and the closed formalism, which parametrizes at the level of the S matrix, has been sought both for inelastic scattering (Frahn 1978b, Shastry 1982) and transfer reactions (Hasan and Warke 1981).

All of the closed formalism work (except that of Rowley (1980)) assumes that the summation over angular momentum can be replaced by an integration over a continuous variable and that the (associated) Legendre polynomials can be approximated by one of various asymptotic forms. These are good approximations for heavy-ions at energies above the Coulomb barrier because the density of angular momentum states per unit impact parameter is large (Glendenning 1975). For lighter ions this is no longer the case, but some aspects of deuteron elastic scattering have been treated reasonably well in the large angular momentum limit (Johnson and Stephenson 1981).

The analytical parts of this thesis use the closed formalism, but for our calculations we invariably perform the partial-wave sum numerically. We study inelastic scattering to

low-lying collective states and divide the scattering amplitude into components that can be identified with waves 'turning around the nucleus'. This identification, arising from the asymptotic form of the associated Legendre polynomials, allows one to interpret cross section structure as interference between two relatively smoothly varying amplitudes $f^{\pm}$, such that, in the elastic case,

$$\sigma = \left| f^{+} + f^{-} \right|^{2} \qquad (1.1)$$

and gives insight into the reaction process. In Chapter III we review this approach for elastic scattering and apply it in Chapter IV to the inelastic scattering of light heavy-ions. We show that a decomposition similar to eq.(1.1) has as much validity in this case as for the elastic one, thus allowing physical interpretation of the two most simple reaction processes. Furthermore, the component amplitudes in both cases may have 'refractive' as well as 'diffractive' contributions and so our approach is 'unified' in the language of Frahn (1974).

Before discussing this decomposition however, we consider in Chapter II the question of general invariance principles and the constraints that these place on approximate theories. In particular we consider a much used approximate symmetry property and discuss its validity. Early inelastic scattering theory was often carried out with the simplifying approximation of zero excitation energy and, as this is how the above symmetry property arises, we deal almost exclusively in Chapter II with this 'adiabatic limit' of inelastic scattering.

The geometrical decomposition of the scattering amplitude into turning-wave components, the so called 'nearside-farside' decomposition, can be complemented by division of the amplitude

into parts having different dynamical origins. In the case of elastic scattering one such dynamical decomposition, into 'internal' and 'barrier' waves, has been used greatly in recent years, partly in an attempt to explain the so-called 'anomalous large-angle scattering' of nuclei.

The observation of 'anomalously' large cross sections close to 180° scattering angle cannot be explained on a model employing ordinary strongly absorbing optical potentials. Various approaches have been made, involving such phenomena as cluster exchange (Franey et al. 1979), parity-dependent potentials, Regge poles (Braun-Munzinger et al. 1977) and 'molecular' resonances (Barrette et al. 1978), but none is capable of consistently describing all the experimental data, which show that enhanced backward angle cross sections occur to a greater extent in α-like nuclei. As noted by Braun-Munzinger and Barrette (1982), excitation functions also behave 'anomalously' at backward angles and so any explanation should involve descriptions of both the angular and energy dependence of the cross section. It is clear (Braun- Munzinger and Barrette 1982) that an odd-even staggering in S-matrix elements is required to reproduce data. This can arise from interference between barrier and internal waves, that is, between waves reflected from the external potential barrier and waves that penetrate the barrier before re-emerging due to fairly weak absorption (Brink and Takigawa 1977, Lee and Takigawa 1978, Lee et al. 1978, Rowley et al. 1980).

Though a full explanation of backward angle phenomena is still lacking, it is interesting to calculate the effect that these internal waves have on elastic and inelastic cross sections. This we do in Chapter V. The nearside-farside

decomposition is also useful here, and has already been applied to elastic scattering (Doubre and Marty 1982).

Another dynamical decomposition involves splitting the elastic S-matrix into smooth and fluctuating parts (Strutinskii 1978, Rowley 1980, Mermaz 1981). The fluctuating component can also give 'anomalous' large-angle scattering and explain the 'fine structure' in backward angle excitation functions (Braun-Munzinger and Barrette 1982).

Also in Chapter V we consider the effect of Coulomb excitation on inelastic cross sections. This process is more important for heavier ions and it tends to result in a smoother angular dependence.

Inelastic cross sections are an incoherent sum of different magnetic-substate contributions, but these components contain useful information on the two nuclei and can be studied by polarization and particle-γ angular correlation experiments. General polarization studies provide a good test of nuclear properties and have been reviewed by Fick (1981). Cluster structure can dramatically alter the predictions for analyzing powers (Nishioka et al. 1982), as can break-up effects (see, for example, Nagarajan et al. 1982), but we only consider the scattering of spin-zero nuclei, which we regard as essentially structureless. As noted by Tanimura and Mosel (1982), various models can be used to describe inelastic cross sections of such nuclei reasonably well, though they often fail to describe polarization data, in particular the 'alignment' (Trombik 1981). In this respect such studies are of great use.

In Chapter VI we explain the form of the vector polarization in terms of the nearside-farside decomposition and show that it is dramatically altered by the presence of either an internal

wave or Coulomb excitation. This is also true of the γ-ray distribution that results from the decay of the excited state. In the adiabatic limit this distribution is symmetric about the recoil direction (Satchler 1960) and, in certain circumstances, remains symmetric, but about some other direction, when this limit is no longer valid. We discuss these circumstances in Chapter VI. The dependence of this symmetry angle on scattering angle is sensitive to magnetic-substate populations and hence to the contributions of different dynamical mechanisms. Therefore correlation studies are also a useful test of reaction models (see, for example, Balamuth et al. 1981).

Phenomenological optical-model potentials are used as examples throughout this thesis, though our qualitative conclusions should also be valid for potentials generated by a folding procedure. Angular distributions, polarizations and symmetry angles will generally be interpreted over the full angular range, although the potentials used were often only fitted to a more restricted range of angles. A fairly representative sample of reactions has been considered, but other interference phenomenon could certainly be obtained if a wider range of situations was studied. The main part of the discussion is concerned with light heavy-ions ($A<60$) in the limit of strong absorption, but we also give some indication of how Coulomb excitation and an internal wave affect our interpretations. These effects are generally more important for larger and smaller mass numbers respectively.

CHAPTER II

ADIABATIC SCATTERING.

1. Introduction.

Any physical theory must guarantee the observation of all symmetries imposed by invariance principles. Scattering amplitudes have such symmetries due to invariance of the interactions under rotations, the parity operation and time reversal, and these symmetries can be most concisely expressed in terms of the T-matrix elements $T^{J}_{\ell' I_\beta ; \ell I_\alpha}$ which may be introduced by making a partial-wave decomposition of the scattering amplitude.

In the notation of Goldberger and Watson (1964) we may write the amplitudes for scattering from an initial channel α with channel spin I_α to a final channel β with spin I_β in the form

$$f_{I_\beta m_\beta ; I_\alpha m_\alpha}(\hat{\underline{k}}', \hat{\underline{k}}) \equiv f_{\beta\alpha}(\hat{\underline{k}}', \hat{\underline{k}})$$

$$= -(2\pi)^2 \left[\frac{\rho_\epsilon \, \rho_{\epsilon'}}{k \, k'} \right]^{1/2} \langle \hat{\underline{k}}' \beta I_\beta m_\beta | T(k) | \hat{\underline{k}} \alpha I_\alpha m_\alpha \rangle \qquad (1.1)$$

where ρ_ϵ is the density of states and the co-ordinate system is entirely arbitrary. The partial-wave decomposition is

$$\langle \hat{\underline{k}}' \beta I_\beta m_\beta | T(k) | \hat{\underline{k}} \alpha I_\alpha m_\alpha \rangle =$$

$$\sum_{\substack{JM \\ \ell m \\ \ell' m'}} Y_{\ell' m'}(\hat{\underline{k}}') \langle \ell' I_\beta m' m_\beta | JM \rangle \langle \ell I_\alpha m m_\alpha | JM \rangle T^J_{\ell' I_\beta ; \ell I_\alpha}(k) \, Y^*_{\ell m}(\hat{\underline{k}}) \tag{1.2}$$

The initial and final momenta are denoted by $\underline{k}$ and $\underline{k}'$ respectively and, since we only need the on-shell T-matrix for the evaluation of $f_{\beta\alpha}$, we write

$$T_{\beta\alpha}(k) \equiv T_{\beta\alpha}(k', k) \tag{1.3}$$

where $\hbar^2 k'^2/2M_\beta = \hbar^2 k^2/2M_\alpha - (\epsilon_\beta - \epsilon_\alpha)$, ϵ_α and ϵ_β being the energies of the initial and final intrinsic states $|\alpha\rangle$ and $|\beta\rangle$. We are only considering elastic and inelastic processes so that the reduced masses in both channels are equal ($M_\alpha = M_\beta$) and henceforth we use units such that $\hbar = 1$ and $M_\alpha = M_\beta = 1/2$. Non-relativistically we may then write the density of states as

$$\rho_\epsilon \equiv k^2/(d\epsilon/dk) = \frac{1}{2} k \tag{1.4}$$

In these units the S-matrix elements are related to those of the T-matrix by

$$S^J_{\ell' I_\beta ; \ell I_\alpha}(k) = \delta_{\alpha\beta} \delta_{\ell\ell'} - i\pi (kk')^{1/2} T^J_{\ell' I_\beta ; \ell I_\alpha}(k) \tag{1.5}$$

and the differential cross section is

$$\sigma_{\beta\alpha} = \frac{k'}{k} \left| f_{\beta\alpha}(\hat{\underline{k}}', \hat{\underline{k}}) \right|^2 \tag{1.6}$$

The T-matrix elements are scalars since the interactions are rotationally invariant, and invariance under the parity operation means that they are zero unless

$$(-1)^{\ell-\ell'+\Delta\pi} = 1 \tag{1.7}$$

where $\Delta\pi$ is the change in parity between the states $|\alpha\rangle$ and $|\beta\rangle$. Finally, time-reversal invariance gives

$$\langle \underline{k}'\beta I_\beta m_\beta | T | \underline{k}\alpha I_\alpha m_\alpha \rangle = (-1)^{I_\alpha + m_\alpha - I_\beta - m_\beta} \langle -\underline{k}\alpha I_\alpha - m_\alpha | T | -\underline{k}'\beta I_\beta - m_\beta \rangle \tag{1.8}$$

or equivalently

$$T^J_{\ell' I_\beta; \ell I_\alpha}(k',k) = T^J_{\ell I_\alpha; \ell' I_\beta}(k,k') \tag{1.9}$$

All the above results are well known (Goldberger and Watson 1964), but there is another symmetry property which is very different from those above and which has often been discussed in the context of inelastic nuclear scattering, particularly in the so-called 'adiabatic' limit (Satchler 1960, Blair and Wilets 1961, Alder and Winther 1962). (There is actually some confusion in the literature as to the precise meaning of 'adiabatic' limit and its relationship to the 'sudden' limit. We only use 'adiabatic' to describe the nuclear collective motion, which is slow compared to the

relative motion and so the latter we describe as 'sudden'. In the semi-classical theory of Coulomb excitation however, (Alder and Winther 1960, dos Aidos 1982), 'adiabatic' is used to describe the relative motion as slow compared to the internal motion - the opposite of our usage of the word).

In terms of the full T matrix the 'adiabatic symmetry property' is

$$\langle -\underline{k}\beta I_\beta m_\beta | T | -\underline{k}'\alpha I_\alpha m_\alpha \rangle = \langle \underline{k}'\beta I_\beta m_\beta | T | \underline{k}\alpha I_\alpha m_\alpha \rangle \qquad (1.10)$$

for k=k'. The crucial difference between this equation and eq.(1.8) is that in the latter we interchange the initial and final channels as well as making the transformation

$$\underline{k} \rightarrow -\underline{k}' \; ; \; \underline{k}' \rightarrow -\underline{k} \qquad (1.11)$$

so that both the original and the transformed amplitudes are on the energy shell. However, in eq.(1.10) we simply make the transformation (1.11) and so the two amplitudes cannot both be on the energy shell unless k=k'. To keep all our T-matrix elements on-shell, so that they can describe physical scattering processes asymptotically, we rewrite eq.(1.10) in the form

$$\langle -\hat{\underline{k}}\beta I_\beta m_\beta | T(k') | -\hat{\underline{k}}'\alpha I_\alpha m_\alpha \rangle \rightarrow \langle \hat{\underline{k}}'\beta I_\beta m_\beta | T(k) | \hat{\underline{k}}\alpha I_\alpha m_\alpha \rangle \tag{1.12}$$

for $k' \rightarrow k$.

The validity of eq.(1.10) has been proved by Satchler (1960) within the distorted-wave Born approximation (DWBA) under certain restrictions. These are that k=k', which does not mean a zero Q-value since in this reference the reduced masses in the initial and final channels are not assumed to be equal; that $M_\alpha U_\alpha = M_\beta U_\beta$, where the U are the channel-diagonal optical potentials; and that the transition potential is both local and of zero range. This condition on the optical potentials implies that the initial and final distorted waves are the same: we will also obtain this condition, though our work is more general.

Blair and Wilets (1961) suggest that eq.(1.10) may be true outside the DWBA provided that the nucleus-nucleus interaction satisfies certain constraints. Alder and Winther (1962) assume the result on the grounds that the internal motion is frozen during the collision and so the excitation process cannot distinguish between the two directions of relative motion. This means that time-reversal can be applied to the intrinsic and relative motions separately. This argument assumes ('adiabatic') collective motion in the nucleus, and also neglects the possibility of certain forms of the inter-nuclear potential.

After investigating the conditions under which (1.12)

holds, we will discuss its consequences in terms of the T-matrix elements for both the exact theory and the DWBA.

2. The adiabatic symmetry property in elastic scattering.

Before discussing the general validity of (1.12) we consider elastic scattering since our technique in this case will be used again later. Our method is similar to, but more general than, that given by Glauber (1958) and earlier by Glauber and Schomaker (1953). In these references the condition k=k' is imposed from the outset, but we shall not do this in order to obtain some conditions on the validity of (1.12) in the elastic limit.

Consider two solutions $\Psi_{\underline{k}}(\underline{r})$ and $\Psi_{-\underline{k}'}(\underline{r})$ of the Schrödinger equation

$$(-\nabla^2 + V(\underline{r}))\psi(\underline{r}) = k^2\psi(\underline{r}) \qquad (2.1)$$

These solutions correspond to incident waves in the directions $\hat{\underline{k}}$ and $-\hat{\underline{k}}'$, and from eq.(2.1) we obtain

$$\psi_{-\underline{k}'}\nabla^2\psi_{\underline{k}} - \psi_{\underline{k}}\nabla^2\psi_{-\underline{k}'} = (k'^2 - k^2)\psi_{-\underline{k}'}\psi_{\underline{k}} + (\psi_{-\underline{k}'}V\psi_{\underline{k}} - \psi_{\underline{k}}V\psi_{-\underline{k}'}) \tag{2.2}$$

Glauber sets the right-hand side of this equation equal to zero, which, as well as assuming that k=k', assumes that

$$\psi_{-\underline{k}'}V\psi_{\underline{k}} - \psi_{\underline{k}}V\psi_{-\underline{k}'} = 0 \tag{2.3}$$

and hence that $V(\underline{r})$ is momentum independent. Later we shall show that the symmetry property resulting from this assumption is manifestly incorrect in a simple problem involving a spin-orbit potential and so (1.12) cannot be true in general if V does depend on momentum. Hence we assume the validity of eq.(2.3), but do not set k=k'.

Integrating eq.(2.2) over the volume of a large sphere of radius R and applying Green's theorem then gives

$$\int(\psi_{-\underline{k}'}\frac{\partial}{\partial r}\psi_{\underline{k}} - \psi_{\underline{k}}\frac{\partial}{\partial r}\psi_{-\underline{k}'})d\mathcal{S} = (k'^2 - k^2)\int\psi_{-\underline{k}'}\psi_{\underline{k}}d\mathcal{V} \tag{2.4}$$

where $d\mathcal{S}$ and $d\mathcal{V}$ are surface and volume elements of the sphere and the left-hand side could be abbreviated using the Wronskian

$$W(a,b) \equiv a \frac{\partial}{\partial r} b - b \frac{\partial}{\partial r} a \tag{2.5}$$

In the surface integral we may replace the wavefunctions by their asymptotic expressions, for example

$$\psi_{\underline{k}}(\underline{r}) \sim e^{i\underline{k}\cdot\underline{r}} + f(\hat{\underline{r}},\hat{\underline{k}}) \frac{e^{ikr}}{r} \tag{2.6}$$

where $f(\hat{\underline{r}},\hat{\underline{k}})$ is the elastic scattering amplitude. This equation is clearly inadequate in the presence of the long-range Coulomb field but, as we show in appendix B, this does not substantially change the following results.

We substitute eq.(2.6), and an analogous expression for the asymptotic $\psi_{-\underline{k}'}(\underline{r})$, into eq.(2.4) to obtain

$$\begin{aligned}
&\int f(\hat{\underline{r}},\hat{\underline{k}}) \left[e^{-i\underline{k}'\cdot\underline{r}} \frac{\partial}{\partial r} \frac{e^{ikr}}{r} - \frac{e^{ikr}}{r} \frac{\partial}{\partial r} e^{-i\underline{k}'\cdot\underline{r}} \right] d\mathcal{S} - \{\underline{k} \leftrightarrow -\underline{k}'\} \\
&= \int f(\hat{\underline{r}},\hat{\underline{k}}) f(\hat{\underline{r}},-\hat{\underline{k}}') \left[\frac{e^{ikr}}{r} \frac{\partial}{\partial r} \frac{e^{ik'r}}{r} - \frac{e^{ik'r}}{r} \frac{\partial}{\partial r} \frac{e^{ikr}}{r} \right] d\mathcal{S} \\
&+ (k'^2 - k^2) \int \left[\psi_{-\underline{k}'} \psi_{\underline{k}} - e^{i(\underline{k}-\underline{k}')\cdot\underline{r}} \right] d\mathcal{V}
\end{aligned} \tag{2.7}$$

where $\{\underline{k} \leftrightarrow -\underline{k}'\}$ indicates the same expression with both the directions and magnitudes of these vectors interchanged. In deriving eq.(2.7) we have used

$$\int W(e^{-i\underline{k}'\cdot\underline{r}}, e^{i\underline{k}\cdot\underline{r}})\, d\mathcal{S} = (k'^2 - k^2)\int e^{i(\underline{k}-\underline{k}')\cdot\underline{r}}\, d\mathcal{V}$$

(2.8)

which follows from Green's theorem, and, on evaluating the Wronskians, eq.(2.7) becomes

$$i\int f(\hat{\underline{r}},\hat{\underline{k}})[kr+\underline{k}'\cdot\underline{r}]e^{i(kr-\underline{k}'\cdot\underline{r})}d\Omega - \{\underline{k}\leftrightarrow -\underline{k}'\} = i(k'-k)e^{i(k+k')R}\int f(\hat{\underline{r}},\hat{\underline{k}})f(\hat{\underline{r}},-\hat{\underline{k}}')d\Omega$$

$$+ (k'^2-k^2)\int(\psi_{-\underline{k}},\psi_{\underline{k}} - e^{i(\underline{k}-\underline{k}')\cdot\underline{r}})d\mathcal{V} + \mathcal{O}((kR)^{-1}) + \mathcal{O}((k'R)^{-1})$$

(2.9)

$d\Omega = R^{-2}d\mathcal{S}$ being the element of solid angle. The terms $\mathcal{O}(1/kR)$ and $\mathcal{O}(1/k'R)$ arise from the left-hand side of eq.(2.7). The integral on the left-hand side of eq.(2.9) can be evaluated using contour techniques with the result

$$i\int f(\hat{\underline{r}},\hat{\underline{k}})[kr+\underline{k}'\cdot\underline{r}]e^{i(kr-\underline{k}'\cdot\underline{r})}\, d\Omega$$

$$= 2\pi e^{ikR}\left[f(-\hat{\underline{k}}',\hat{\underline{k}})\frac{(k-k')}{k'}e^{ik'R} - f(\hat{\underline{k}}',\hat{\underline{k}})\frac{(k+k')}{k'}e^{-ik'R}\right] + \mathcal{O}((k'R)^{-1}).$$

(2.10)

We present this derivation in appendix A.

Inserting eq.(2.10) into eq.(2.9) gives

$$e^{i(k'-k)R}\, k'f(-\hat{\underline{k}},-\hat{\underline{k}}') - e^{-i(k'-k)R}\, kf(\hat{\underline{k}}',\hat{\underline{k}}) = \frac{(k'-k)}{(k'+k)}\, C(R,\underline{k},\underline{k}')$$

$$+ \mathcal{O}((kR)^{-1}) + \mathcal{O}((k'R)^{-1})$$

(2.11)

with

$$C(R,\underline{k},\underline{k}') = \frac{e^{i(k+k')R}}{2\pi}\left\{2\pi[k'f(\hat{\underline{k}},-\hat{\underline{k}}') + kf(-\hat{\underline{k}}',\hat{\underline{k}})] + ikk'\int f(\hat{\underline{r}},-\hat{\underline{k}}')f(\hat{\underline{r}},\hat{\underline{k}})d\Omega\right.$$

$$\left. + (k+k')kk'e^{-i(k+k')R}\int[\psi_{-\underline{k}'}\psi_{\underline{k}} - e^{i(\underline{k}-\underline{k}')\cdot\underline{r}}]d\mathcal{V}\right\} . \qquad (2.12)$$

Both sides of eq.(2.11) are R-dependent, though the relationship between $f(\hat{\underline{k}}',\hat{\underline{k}})$ and $f(-\hat{\underline{k}},-\hat{\underline{k}}')$, which are asymptotic quantities, must clearly be independent of R. For sufficiently small (k'-k) we may obtain this relationship by choosing a value of R such that

$$\frac{1}{k} << R << \frac{1}{|k'-k|} \qquad (2.13)$$

whereupon we find

$$f(-\hat{\underline{k}},-\hat{\underline{k}}') \approx f(\hat{\underline{k}}',\hat{\underline{k}}) \qquad (2.14)$$

When k=k' this is an equality, usually known as the 'reciprocity' theorem (Glauber 1958, Schiff 1968), and it can be derived more simply by using time-reversal invariance with a momentum-independent potential. One might, therefore, wonder why we have bothered with the more complicated derivation presented here. There are two reasons: firstly we have related the two amplitudes $f(\hat{\underline{k}}',\hat{\underline{k}})$ and $f(-\hat{\underline{k}},-\hat{\underline{k}}')$ outside the limit of k=k',and so some idea of the likely errors in this approximate equality is obtained. Secondly this method is readily extendable to the case of inelastic scattering, where simple time reversal gives the result (1.8) and not the required symmetry of (1.10).

There is, however, a problem with the second inequality in eq.(2.13) since in order to use the asymptotic wavefunctions (2.6) we must choose a value of R much greater than the range, R_{int}, of the interaction. Thus eq.(2.11) yields the more correct expression for non-negligible $(k'-k)R_{int}$

$$f(-\hat{\underline{k}},-\hat{\underline{k}}') = f(\hat{\underline{k}}',\hat{\underline{k}}) + \mathcal{O}\left(\frac{k'-k}{k'+k}\right) + \mathcal{O}((k'-k)R_{int}) + \mathcal{O}((kR_{int})^{-1}) \qquad (2.15)$$

The quantity kR_{int} may be thought of as a grazing angular momentum λ_g for the scattering process, and so as well as requiring $|k'-k| \ll k'+k$ we also require λ_g to be large and the difference between the grazing angular momenta for k and k' to be small for the approximate equality of $f(\hat{\underline{k}}',\hat{\underline{k}})$ and $f(-\hat{\underline{k}},-\hat{\underline{k}}')$. At high energies $\lambda_g \gg 1$ and so the condition $|k'-k|R_{int} \ll 1$ will be more restrictive than $|k'-k| \ll k'+k$.

This section has served as an introduction to the methods that we use now in our consideration of the general vaildity of the adiabatic symmetry property of eq.(1.12).

3. An extension of Glauber's method to inelastic scattering.

In order to include processes other than elastic scattering we must take a Schrödinger equation with a more general Hamiltonian

$$(-\nabla^2 + H(\xi) + V(\underline{r},\xi))\psi = k^2\psi \tag{3.1}$$

where ξ represents <u>all</u> the intrinsic co-ordinates of the target and projectile and is not simply a collective variable. Considering two solutions of eq.(3.1), $\psi_{\underline{k}}(\underline{r},\xi)$ and $\psi_{-\underline{k}'}(\underline{r},\xi')$, the equivalent of eq.(2.4) is

$$\int\left[\psi_{-\underline{k}'}(\underline{r},\xi')\frac{\partial}{\partial r}\psi_{\underline{k}}(\underline{r},\xi) - \psi_{\underline{k}}(\underline{r},\xi)\frac{\partial}{\partial r}\psi_{-\underline{k}'}(\underline{r},\xi')\right]d\mathcal{S}$$

$$= \int\left\{\psi_{-\underline{k}'}(\underline{r},\xi')\left[H(\xi)+V(\underline{r},\xi)\right]\psi_{\underline{k}}(\underline{r},\xi) - \psi_{\underline{k}}(\underline{r},\xi)\left[H(\xi')+V(\underline{r},\xi')\right]\psi_{-\underline{k}'}(\underline{r},\xi')\right\}d\mathcal{V} \tag{3.2}$$

where we have already let k=k'. We expand the wavefunctions ψ in terms of the eigenstates $|\gamma(\xi)\rangle$ of $H(\xi)$, for example,

$$\psi_{\underline{k}}(\underline{r},\xi) = \sum_{\gamma}\phi_{\underline{k}\gamma}(\underline{r})\,|\gamma(\xi)\rangle \tag{3.3}$$

and so for an incident plane wave of energy k_α^2 in

channel α we may write the asymptotic wavefunction in the form

$$\psi_{\underline{k}}(\underline{r},\xi) \sim \left[e^{ik_\alpha \hat{\underline{k}}\cdot\underline{r}} + f_{\alpha\alpha}(\hat{\underline{r}},\hat{\underline{k}})\,\frac{e^{ik_\alpha r}}{r}\right]|\alpha(\xi)\rangle + \sum_{\gamma\neq\alpha} f_{\gamma\alpha}(\hat{\underline{r}},\hat{\underline{k}})\,\frac{e^{ik_\gamma r}}{r}\,|\gamma(\xi)\rangle \tag{3.4}$$

with a similar expression for $\psi_{-\underline{k}'}$. Here the $f_{\gamma\alpha}$ are the scattering amplitudes of eq.(1.1) and $k_\gamma^2+\epsilon_\gamma=k_\alpha^2+\epsilon_\alpha$.

We introduce the channel-coupling potentials through the expansion

$$V(\underline{r},\xi) = \sum_{\delta\gamma} |\delta\rangle\langle\delta|V(\underline{r},\xi)|\gamma\rangle\langle\gamma| = \sum_{\delta\gamma} |\delta\rangle V_{\delta\gamma}(\underline{r})\langle\gamma| \tag{3.5}$$

and so obtain a formula analogous to eq.(2.7) by equating the sums of the coefficients of $|\alpha(\xi')\rangle|\beta(\xi)\rangle$ and $|\alpha(\xi)\rangle|\beta(\xi')\rangle$ appearing on either side of the expression resulting from the substitution of eqs.(3.3), (3.4) and (3.5) into eq.(3.2). From the left-hand side we obtain

$$\int f_{\beta\alpha}(\hat{\underline{r}},\hat{\underline{k}})\left[\left(e^{-ik_\alpha \hat{\underline{k}}'\cdot\underline{r}} + f_{\alpha\alpha}(\hat{\underline{r}},-\hat{\underline{k}}')\,\frac{e^{ik_\alpha r}}{r}\right)\frac{\partial}{\partial r}\,\frac{e^{ik_\beta r}}{r}\right.$$
$$\left. - \frac{e^{ik_\beta r}}{r}\,\frac{\partial}{\partial r}\left(e^{-ik_\alpha \hat{\underline{k}}'\cdot\underline{r}} + f_{\alpha\alpha}(\hat{\underline{r}},-\hat{\underline{k}}')\,\frac{e^{ik_\alpha r}}{r}\right)\right] d\mathcal{S} - \{\hat{\underline{k}}\leftrightarrow -\hat{\underline{k}}'\} \tag{3.6}$$

which, on using the same techniques as for elastic scattering, becomes

$$4\pi\left[f_{\beta\alpha}(-\hat{\underline{k}},-\hat{\underline{k}}') - f_{\beta\alpha}(\hat{\underline{k}}',\hat{\underline{k}})\right] + \mathcal{O}((k_\beta-k_\alpha)/(k_\beta+k_\alpha)) + \mathcal{O}((k_\alpha-k_\beta)R_{int})$$

$$+ \mathcal{O}((k_\alpha R_{int})^{-1}) \qquad (3.7)$$

where we emphasize that only the directions of $\underline{k}$ and $-\underline{k}'$ are different, and that the different momenta k_α and k_β refer to the initial and final channels. The quantity $(k_\alpha-k_\beta)R_{int}$ can now be interpreted physically as the angular momentum 'mismatch' between the two channels, which is a useful parameter for specifying the degree of non-adiabaticity.

Unlike the case of elastic scattering in a potential $V(\underline{r})$ we may not set the expression (3.7) equal to zero since from the right-hand side of eq.(3.2) we obtain

$$(\varepsilon_\beta-\varepsilon_\alpha)\int(\phi_{\underline{k}\beta}\phi_{-\underline{k}'\alpha} - \phi_{\underline{k}\alpha}\phi_{-\underline{k}'\beta})d\mathscr{V}$$

$$+ \sum_\gamma \int\left[V_{\alpha\gamma}(\phi_{-\underline{k}'\beta}\phi_{\underline{k}\gamma} - \phi_{\underline{k}\beta}\phi_{-\underline{k}'\gamma}) + V_{\beta\gamma}(\phi_{-\underline{k}'\alpha}\phi_{\underline{k}\gamma} - \phi_{\underline{k}\alpha}\phi_{-\underline{k}'\gamma})\right]d\mathscr{V} \qquad (3.8)$$

where, as in the previous section, we have assumed that the coupling potentials $V_{\delta\gamma}(\underline{r})$ are momentum-independent.

Thus even in the perfectly adiabatic case, $\epsilon_\beta=\epsilon_\alpha$, we are left with the equation

$$4\pi\left[f_{\beta\alpha}(-\hat{\underline{k}},-\hat{\underline{k}}') - f_{\beta\alpha}(\hat{\underline{k}}',\hat{\underline{k}})\right] = \int(\phi_{-\underline{k}'\beta}\phi_{\underline{k}\alpha} - \phi_{\underline{k}\beta}\phi_{-\underline{k}'\alpha})(V_{\alpha\alpha}-V_{\beta\beta})d\mathscr{V}$$

$$+ \sum_{\gamma\neq\alpha,\beta}\int\left[V_{\alpha\gamma}(\phi_{-\underline{k}'\beta}\phi_{\underline{k}\gamma} - \phi_{\underline{k}\beta}\phi_{-\underline{k}'\gamma}) + V_{\beta\gamma}(\phi_{-\underline{k}'\alpha}\phi_{\underline{k}\gamma} - \phi_{\underline{k}\alpha}\phi_{-\underline{k}'\gamma})\right]d\mathscr{V} \tag{3.9}$$

Note that all terms containing $V_{\alpha\beta}$ have vanished from the right-hand side of this equation and, in addition, if the channel-diagonal potentials $V_{\alpha\alpha}$ and $V_{\beta\beta}$ are identical then the first term also vanishes. There is, however, no reason why the sum over γ should be zero in general. It does not vanish term by term and also there is no particular relationship between the individual channel-coupling potentials.

We see, therefore, that in addition to the adiabatic condition ($\epsilon_\beta=\epsilon_\alpha$), the expression (1.12) can only be valid under the following conditions:

(a) the potentials $V_{\delta\gamma}(r)$ should contain no momentum dependence, a point that we will discuss further in the next section of this chapter;

(b) the diagonal potentials $V_{\alpha\alpha}$ and $V_{\beta\beta}$ should be

identical. This is a non-trivial condition since, for example, if channel α has spin I_α=0 and β has spin I_β=2 there may be channel-diagonal tensor forces in β (re-orientation terms) which clearly cannot be present in α. We must, therefore, be able to ignore such re-orientation effects;

(c) the couplings from α and β to all other channels must be negligible, though there is no restriction on $V_{\alpha\beta}$ which can be arbitrarily strong.

These conditions may appear very restrictive, but are quite often satisfied in simpler theoretical calculations if not in the physical situations that these calculations try to represent. If a reaction fulfils the above conditions, we shall refer to it as being "kinematically matched".

In the DWBA it is not unusual to neglect spin-orbit forces, especially for heavier systems, and to use the same complex optical potential to generate the initial and final channel wavefunctions. Since the DWBA is a one-step theory it also implicitly ignores re-orientation terms and coupling to other channels. However, the DWBA is only a first-order theory in $V_{\alpha\beta}$ whereas our result is valid for a $V_{\alpha\beta}$ of arbitrary strength. Thus a coupled-channels calculation explicitly considering only the two channels α and β should also give the kinematically-matched result provided that the other conditions above are satisfied.

To consider the consequences of kinematic matching it is more convenient to express the result (1.12) in terms of the T-matrix elements defined by eq.(1.2), and this we now do.

4. Symmetry of T-matrix elements under kinematic matching.

We consider eqs.(1.1) and (1.2) which relate the scattering amplitudes to the T-matrix elements. By interchanging the dummy variables (ℓm) and ($\ell' m'$) in eq.(1.2) and using

$$Y_{\ell m}(-\hat{\underline{k}}) = (-1)^{\ell+m} Y^*_{\ell -m}(\hat{\underline{k}}) \tag{4.1}$$

we readily obtain

$$\begin{aligned}<-\hat{\underline{k}}\beta I_\beta m_\beta |T(k_\alpha)| - \hat{\underline{k}}'\alpha I_\alpha m_\alpha> = \sum_{\substack{\ell\ell' mm' \\ JM}} Y_{\ell' m'}(\hat{\underline{k}}') \{(-1)^{\ell'+\ell+m'+m} <\ell I_\beta -m m_\beta | JM> \\ <\ell' I_\alpha -m' m_\alpha | JM> T^J_{\ell I_\beta ; \ell' I_\alpha}(k_\alpha)\} Y^*_{\ell m}(\hat{\underline{k}}) \ .\end{aligned} \tag{4.2}$$

The product of the two Clebsch-Gordan coefficients in this

equation may be written in the form (Brink and Satchler 1968)

$$\langle \ell I_\beta -m m_\beta | JM\rangle \langle \ell' I_\alpha -m' m_\alpha | JM\rangle = (2J+1)(-1)^{I_\alpha+I_\beta+m+m'+2J'} \sum_{J'} W(\ell' I_\alpha I_\beta \ell; JJ') \langle \ell I_\alpha m m_\alpha | J'M\rangle \langle \ell' I_\beta m' m_\beta | J'M\rangle \tag{4.3}$$

so that on equating $f_{\beta\alpha}(\hat{\underline{k}}', \hat{\underline{k}})$ and $f_{\beta\alpha}(-\hat{\underline{k}}, -\hat{\underline{k}}')$ we immediately obtain

$$T^J_{\ell' I_\beta; \ell I_\alpha}(k_\alpha) = (-1)^{2J} \sum_{J'} (2J'+1) \begin{Bmatrix} \ell & I_\alpha & J \\ \ell' & I_\beta & J' \end{Bmatrix} T^{J'}_{\ell I_\beta; \ell' I_\alpha}(k_\alpha) \tag{4.4}$$

which is the adiabatic symmetry property for a kinematically-matched reaction expressed at the level of the T-matrix elements.

We may now prove our assertion that this result is inconsistent with the presence of momentum-dependent forces by considering the elastic scattering (clearly adiabatic) of a spin-half particle in a central and spin-orbit potential. Such potentials are diagonal in ℓ and the sum in eq.(4.4) runs over the two terms $J'=\ell\pm{}^1/_2$. Inserting the values of the 6-j symbols (Brink and Satchler 1968) gives

$$T^{\ell-1/2}_{\ell 1/2; \ell 1/2} = T^{\ell+1/2}_{\ell 1/2; \ell 1/2} \tag{4.5}$$

or, in terms of the S matrix of eq.(1.5)

$$S^{\ell-1/2}_{\ell 1/2;\ell 1/2} = S^{\ell+1/2}_{\ell 1/2;\ell 1/2}$$

(4.6)

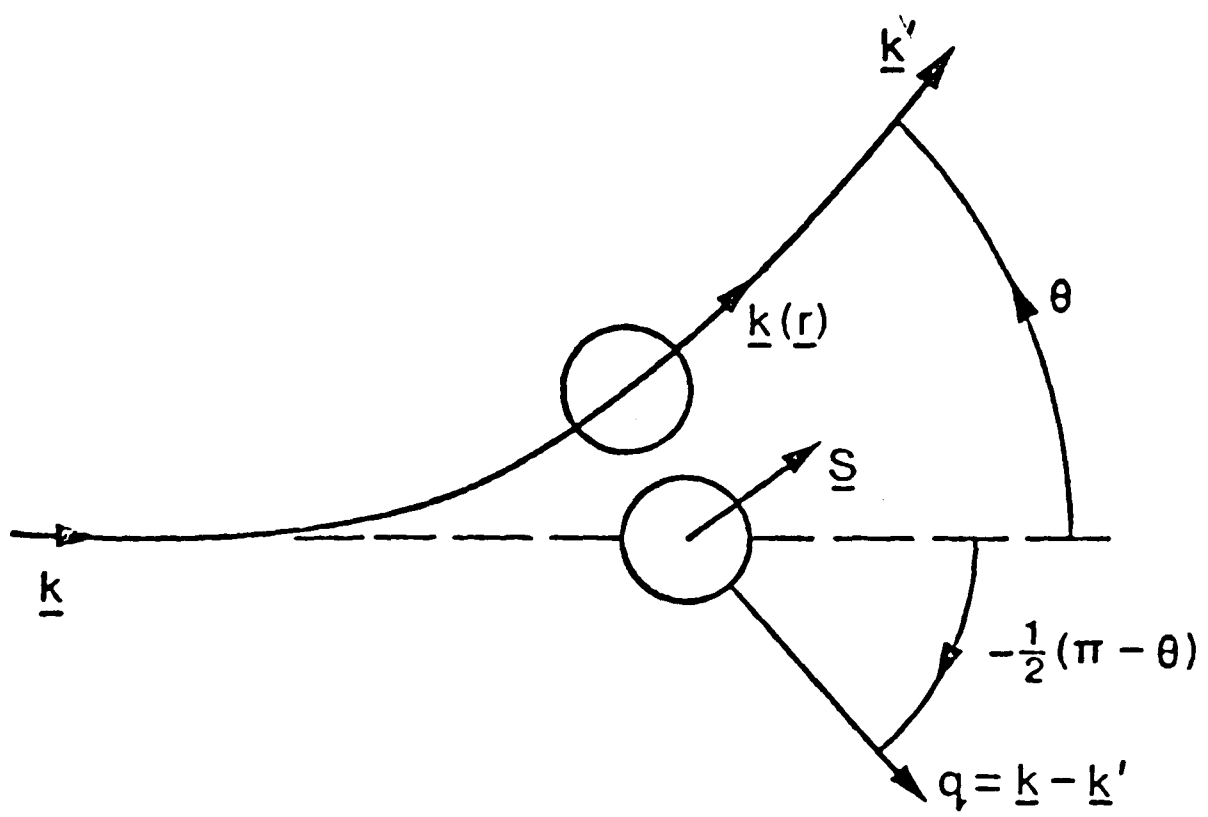
which is manifestly incorrect in the presence of a spin-orbit force. This simple counter-example proves that eqs.(4.4) and (1. 10) cannot be true in general with momentum-dependent interactions. The physical reason for this is clear: reversing the direction of the scattering without reversing the target and projectile spins corresponds to changing $\underline{\ell}.\underline{s}$ to $-\underline{\ell}.\underline{s}$ and thus effectively changes the sign of the spin-orbit force. (This is shown in fig.(II-1) where the (local) momentum is $\underline{k}(\underline{r})$ and the significance of the direction $\underline{q}$ will be explained later). Consequently the impulse given to the target is dependent on the projectile's direction of motion and the adiabatic symmetry result cannot hold.

Eq.(4.4) is trivially satisfied for $I_\alpha = I_\beta = 0$ since then J and J' must be equal. If only one of the channel spins (say I_α) is zero the 6-j coefficient restricts the value of J' to ℓ' and we obtain

$$(-1)^{I_\beta+\Delta\pi}\,(2\ell+1)^{1/2}\,T^{\ell}_{\ell' I_\beta;\,\ell 0}(k_\alpha) = (2\ell'+1)^{1/2}\,T^{\ell'}_{\ell I_\beta;\,\ell' 0}(k_\alpha)$$

(4.7)

where $\Delta\pi$ is the difference in parity between the initial and final states. The result of eq.(4.4), is very different from the corresponding time-reversal result of eq.(1.9), even when the latter is considered in the adiabatic limit, since we are now relating T-matrix elements of different total angular



(FIG. II-1): Schematic representation of the process $\underline{k} \rightarrow \underline{k}'$ showing the scattering angle θ and the recoil angle $-(\pi-\theta)/2$. Note that reversing the direction of $\underline{k}(\underline{r})$ without changing the spin direction changes the sign of $\underline{\ell}.\underline{s}$.

momentum.

We showed above that the conditions for kinematic matching are a little more general than those for the validity of the DWBA. It is, however, straightforward to show that the DWBA T-matrix elements do satisfy eq.(4.4) in the adiabatic limit. We use the usual co-ordinate system for DWBA calculations in which the z-axis lies along the beam direction $\hat{\underline{k}}$ and the y-axis is along $\hat{\underline{k}} \times \hat{\underline{k}}'$, perpendicular to the reaction plane.
For simplicity we consider only the excitation of a state of spin I, with magnetic quantum number M, from one of zero spin, although the more general case can be treated using more complicated Racah algebra (appendix C). In these axes we may write the DWBA amplitude with spin-independent forces in the form (Austern 1970, Jackson 1970)

$$f^{DWBA}_{IM}(\theta) \propto \sum_{\ell\ell'} i^{\ell-\ell'} \sqrt{(2\ell'+1)}\, Y_{\ell'-M}(\theta,0)\, \langle \ell' I -M M | \ell 0 \rangle\, \langle \ell' I 0 0 | \ell 0 \rangle \; R_{\ell'\ell} \exp[i(\sigma^{\beta}_{\ell'} + \sigma^{\alpha}_{\ell})] \tag{4.8}$$

where θ is the scattering angle and the radial integrals are defined as

$$R_{\ell'\ell} \equiv (k_\alpha k_\beta)^{-1} \int_0^\infty u^{\beta}_{\ell'}(r)\; V_{\alpha\beta}(r)\; u^{\alpha}_{\ell}(r)\, dr \tag{4.9}$$

Here u^{α} and u^{β} are distorted radial wavefunctions generated by the potentials $V_{\alpha\alpha}$ and $V_{\beta\beta}$ in the initial and final channels, and the quantities σ^{α} and σ^{β} are the corresponding Coulomb phases defined in terms of the Sommerfeld parameters η^{α} and η^{β} by (Abramowitz and Stegun 1965)

$$\sigma_{\ell}^{\alpha} \equiv \arg[\,\Gamma(\ell+1+i\eta^{\alpha})\,] \tag{4.10}$$

In our units $\eta^{\alpha}=Z_1 Z_2 e^2/2k_{\alpha}$ with a similar expression for η^{β}. We introduce

$$\delta_{mo} = \left(\frac{4\pi}{2\ell+1}\right)^{1/2} Y^{*}_{\ell m}(0,0) \tag{4.11}$$

to obtain

$$f^{DWBA}_{IM}(\theta) \propto \sum_{\substack{\ell\ell' \\ mm'}} Y_{\ell'm'}(\theta,0)\,\langle \ell' I m' M | \ell m\rangle\, Y^{*}_{\ell m}(0,0)$$
$$i^{\ell-\ell'} R_{\ell'\ell} \exp[i(\sigma^{\beta}_{\ell'} + \sigma^{\alpha}_{\ell})] \left[\frac{2\ell'+1}{2\ell+1}\right]^{1/2} \langle \ell' I 00 | \ell 0\rangle \tag{4.12}$$

Comparison with eq.(1.2) then yields

$$T^{\ell}_{\ell' I;\ell 0} \propto \sqrt{(2\ell'+1)}\; i^{\ell+\ell'} \begin{bmatrix} \ell' & I & \ell \\ 0 & 0 & 0 \end{bmatrix} R_{\ell'\ell} \exp[i(\sigma^{\beta}_{\ell'} + \sigma^{\alpha}_{\ell})] \tag{4.13}$$

The 3-j symbol expresses the fact that only natural parity states can be excited in one-step processes, and $\Delta\pi$ in eq.(4.7) is therefore restricted to being equal to I. The DWBA T-matrix elements of eq.(4.13) can only have the symmetry of eq.(4.7) if $R_{\ell'\ell}$ and the Coulomb phases are

invariant under the exchange of ℓ and ℓ'. This means that $\sigma^{\alpha}=\sigma^{\beta}$ and $u^{\alpha}=u^{\beta}$ for all angular momentum values. The former is satisfied in the adiabatic limit, since then $\eta^{\alpha}=\eta^{\beta}$, but the condition on the radial wavefunctions also requires that $V_{\alpha\alpha}=V_{\beta\beta}$, because only then are the initial and final channel distorted-waves generated by the same Schrödinger equation and are therefore identical. These conditions are just the ones for kinematic matching given above, and so under these conditions we see that the DWBA amplitude of eq.(4.8) has the adiabatic symmetry property. This is also true of the more general DWBA amplitude (appendix C). The condition that $V_{\alpha\alpha}=V_{\beta\beta}$ was also found by Satchler (1960).

5. Consequences of kinematic matching.

It is clear from fig.(II-1) that the transformation $\hat{\underline{k}}\leftrightarrow-\hat{\underline{k}}'$ required for the adiabatic symmetry property to hold can be effected by a rotation through π about the direction $\hat{\underline{q}}$, where $\underline{q}=\underline{k}-\underline{k}'$ is the (adiabatic) recoil direction. The effect of a general rotation through Ω about the direction $\hat{\underline{e}}$ on the scattering amplitude of eq.(1.1) is (Brink and Satchler 1968)

$$R(\Omega,\hat{\underline{e}})\ f_{I_\beta m_\beta ; I_\alpha m_\alpha}(\hat{\underline{k}} \to \hat{\underline{k}}') = \sum_{\mu\nu} D^{I_\alpha}_{\mu m_\alpha}(\Omega)\ D^{I_\beta *}_{\nu m_\beta}(\Omega)\ f_{I_\beta \nu ; I_\alpha \mu}(\hat{\underline{k}} \to \hat{\underline{k}}') \qquad (5.1)$$

which, if $\Omega=\pi$ and $\hat{\underline{e}}=\hat{\underline{q}}$ gives

$$R(\pi,\hat{\underline{q}})\ f_{I_\beta m_\beta ; I_\alpha m_\alpha}(\hat{\underline{k}} \to \hat{\underline{k}}') = f_{I_\beta m_\beta ; I_\alpha m_\alpha}(-\hat{\underline{k}}' \to -\hat{\underline{k}}) \qquad (5.2)$$

or, equivalently

$$R(\pi,\hat{\underline{q}})\ \langle \hat{\underline{k}}' I_\beta m_\beta | T(k_\alpha) | \hat{\underline{k}} I_\alpha m_\alpha \rangle = \langle -\hat{\underline{k}} I_\beta m_\beta | T(k_\alpha) | -\hat{\underline{k}}' I_\alpha m_\alpha \rangle \qquad (5.3)$$

We have not yet specified any particular set of axes, but if we quantize along $\hat{\underline{q}}$ the rotation matrix elements of eq.(5.1) become

$$D^{I_\alpha}_{\mu m_\alpha}(\Omega)\ D^{I_\beta *}_{\nu m_\beta}(\Omega) = \delta_{\mu m_\alpha}\ \delta_{\nu m_\beta}\ (-1)^{m_\beta - m_\alpha} \qquad (5.4)$$

Hence

$$R(\pi,\hat{\underline{q}})\ f_{I_\beta m_\beta ; I_\alpha m_\alpha}(\hat{\underline{k}} \to \hat{\underline{k}}') = (-1)^{m_\beta - m_\alpha}\ f_{I_\beta m_\beta ; I_\alpha m_\alpha}(\hat{\underline{k}} \to \hat{\underline{k}}') \qquad (5.5)$$

which combined with eq.(5.2) gives

$$f_{I_\beta m_\beta ; I_\alpha m_\alpha}(\hat{\underline{k}} \rightarrow \hat{\underline{k}}') = (-1)^{m_\beta - m_\alpha} f_{I_\beta m_\beta ; I_\alpha m_\alpha}(-\hat{\underline{k}}' \rightarrow -\hat{\underline{k}}) \tag{5.6}$$

or, equivalently

$$\langle -\hat{\underline{k}} I_\beta m_\beta | T(k_\alpha) | -\hat{\underline{k}}' I_\alpha m_\alpha \rangle = (-1)^{m_\beta - m_\alpha} \langle \hat{\underline{k}}' I_\beta m_\beta | T(k_\alpha) | \hat{\underline{k}} I_\alpha m_\alpha \rangle \tag{5.7}$$

Apart from the phase factor this expression is identical to the adiabatic symmetry property of eq.(1.10), but we stress that the above equation is true for <u>any</u> T matrix, but only because we have quantized along a particular direction. The property (1.10) relates T matrices evaluated in a totally arbitrary set of axes and so clearly a geometrical argument of the type leading to eq.(5.7) cannot prove the adiabatic symmetry property. However, combining eqs.(5.7) and (1.10) we obtain the 'kinematically-matched selection rule'

$$(-1)^{m_\beta - m_\alpha} = 1 \tag{5.8}$$

for quantization along the recoil direction. This rule has been given by Alder and Winther (1962) and subsequently used by them to discuss γ-ray angular distributions following an adiabatic excitation.

If the initial and final spins are both zero then the amplitude of eq.(1.1) is a scalar and so invariant under the rotation of eq.(5.1). The adiabatic symmetry property then

follows immediately, but this is not obvious from eq.(3.9). To check that the right-hand side of eq.(3.9) is indeed zero for $I_\alpha=I_\beta=0$ we must make a partial-wave decomposition of the wavefunction after removing the coupling potentials using the coupled equations

$$(\tau+\epsilon_\gamma-k_\gamma^2)\phi_{\underline{k}\gamma}(\underline{r}) = -\sum_\delta V_{\gamma\delta}(\underline{r})\phi_{\underline{k}\delta}(\underline{r}) \qquad (5.9)$$

where τ is the kinetic-energy operator. Straightforward Racah algebra then shows that the right-hand side of eq.(3.9) vanishes for $I_\alpha=I_\beta=0$ (as expected) and also for $(I_\alpha,m_\alpha)=(I_\beta,m_\beta)$ if the forces are spin independent. This we would expect from time-reversal arguments.

For the rest of this chapter we will use the co-ordinate system defined by the Basel convention. The initial beam direction now lies along the x-axis and the z-axis is perpendicular to the reaction plane, along $\hat{\underline{k}} \times \hat{\underline{k}}'$.

In these axes it can be shown (Bohr 1959) that only states satisfying

$$(-1)^{m_\beta-m_\alpha+\Delta\pi} = 1 \qquad (5.10)$$

are populated. This is simply a result of parity conservation and rotational invariance about the quantization axis and reduces to

$$(-1)^{I+M} = 1 \tag{5.11}$$

for the scattering of two spin-zero nuclei leading to a natural parity state of spin I, projection M. It is this case that we concentrate on now.

In the Basel axes we denote our amplitudes by $F_{IM}(\theta)$ and may write

$$F_{IM}(\theta) \propto \sum_{\substack{\ell\ell' \\ mm'}} Y_{\ell'm'}(\pi/2,\theta)\, \langle \ell' I m' M | \ell m \rangle\, Y^{*}_{\ell m}(\pi/2,0)\, T^{\ell}_{\ell' I;\ell 0} \tag{5.12}$$

If we interchange the dummy variables (ℓm) and $(\ell' m')$ and assume kinematic matching as expressed by eq.(4.7) we obtain

$$F_{IM}(\theta) = e^{-iM\theta} F_{I,-M}(\theta) \tag{5.13}$$

Consequently these two amplitudes have equal magnitude and the same relative phase as the spherical harmonic $Y^{*}_{IM}(\frac{1}{2}\pi, -\frac{1}{2}(\pi-\theta))$. The former implies that there is no vector polarization in the kinematically matched limit, since the final state populations are independent of

the sign of M.

We now consider the subsequent γ-decay of the excited state. The amplitude for emission of a photon of helicity $\sigma=\pm 1$ in the direction $\hat{\underline{k}}_\gamma$ in coincidence with the projectile scattered at an angle θ may be written as (Biedenharn and Rose 1953, Rose and Brink 1967, Rybicki et al. 1970)

$$A_\sigma^I(\theta,\hat{\underline{k}}_\gamma) \propto \sum_M \langle\overline{I\sigma}|H|IM\rangle \; F_{IM}(\theta) \tag{5.14}$$

where H is the electromagnetic interaction responsible for the decay and the state $|\overline{I\sigma}\rangle$ has the γ-ray quantized along its direction of propagation. Since H is a scalar and the final state has zero spin, we have

$$\langle\overline{I\sigma}|H|IM\rangle \propto \langle\overline{I\sigma}|IM\rangle = \langle I\sigma|D(-\phi_\gamma,-\theta_\gamma,0)|IM\rangle = D^{I*}_{M\sigma}(\phi_\gamma,\theta_\gamma,0) \tag{5.15}$$

and so

$$A_\sigma^I(\theta,\theta_\gamma,\phi_\gamma) \propto \sum_M e^{iM\phi_\gamma} \; d^I_{M\sigma}(\theta_\gamma) \; F_{IM}(\theta) \tag{5.16}$$

(Note that eq.(3) in one of the earlier papers on α-γ correlations (Cramer and Eidson 1964) is incorrect, the correct version being the above. Their subsequent discussion

is unaffected).

Since I+M must be even in these axes, we immediately obtain

$$A^I_\sigma(\theta,\theta_\gamma,\phi_\gamma \pm \pi) = (-1)^I A^I_\sigma(\theta,\theta_\gamma,\phi_\gamma) \tag{5.17}$$

and by letting M → -M in eq.(5.16) we can derive

$$A^I_\sigma(\theta,\theta_\gamma,\phi_\gamma) = (-1)^\sigma A^I_\sigma(\theta, \pi - \theta_\gamma, \theta - \phi_\gamma \pm \pi) \tag{5.18}$$

by using eq.(5.13). For in-plane γ-decay, where $\theta_\gamma = \pi/2$, we therefore see that the correlation amplitude for each helicity is antisymmetric about $\phi_\gamma = (\theta \pm \pi)/2$ and symmetric or antisymmetric about $\theta/2$ and $\pi+\theta/2$ depending on whether I is odd or even. The correlation function, which is proportional to the square modulus of A^I_σ, is therefore symmetric about all four of these angles for either helicity. It vanishes along $\phi_\gamma = (\theta \pm \pi)/2$ irrespective of I and along $\phi_\gamma = \theta/2$, $\pi+\theta/2$ for even I.

In the literature it is usual to define the adiabatic symmetry angle to be the recoil direction

$$\phi_{sym} = -\frac{1}{2}(\pi-\theta) \tag{5.19}$$

where the minus sign is sometimes (erroneously) omitted,

since this simply indicates that the symmetry axis is on the opposite side of the incident beam to the scattered particle. The recoil direction is shown in fig.(II-1). A γ-ray cannot propagate in this direction since eq.(5.8) shows that the change in the spin projection along this direction is even, whereas $\sigma=\pm 1$. This is reflected in the antisymmetry of A_σ^I about this direction.

Provided that the excitation process is kinematically matched, all these results for the symmetry axis are true irrespective of the spin I, but when the excitation is no longer adiabatic there is not necessarily a γ-ray symmetry axis. We investigate the existence of other symmetry axes and their dependence on the dominance of either nearside or farside scattering in Chapter VI.

CHAPTER III

THE NEARSIDE - FARSIDE DECOMPOSITION.

1. The meaning of the decomposition.

Much heavy-ion scattering data shows structure of an oscillatory nature in the angular distribution. In the case of elastic scattering, with a negligible Coulomb interaction, strong absorption gives rise to a diffractive angular distribution with marked oscillations in the forward direction. By analogy with the optical situation, this is often referred to as Fraunhofer diffraction. In the semi-classical approach to scattering these oscillations arise from interference between the positive and negative deflection angle contributions to the scattering amplitude. More physically, we identify the interfering contributions with waves which have been scattered from opposite sides - the 'nearside' and the 'farside' - of the interaction region. This, in essence, is the nearside-farside (NF) decomposition.

This decomposition remains valid when the Coulomb interaction is not negligible. In many cases the Fraunhofer

oscillations are then absent, implying that one only sees the contribution from one side of the interaction region. The angular distribution is then relatively smooth and resembles the Fresnel pattern familiar from optics. With the NF method any structure in the cross section is interpreted as the interference between relatively smoothly varying nearside and farside amplitudes.

The decomposition of the full scattering amplitude is straightforward when only peripheral partial waves participate, since then the Legendre polynomials, or more generally the associated Legendre polynomials, in the partial-wave series can be replaced by their asymptotic forms, which vary as $\exp(\pm i\ell\theta)$. These correspond to the far/near contributions respectively. Since elastic scattering and many inelastic processes are not entirely peripheral, we must seek a more general decomposition of the full amplitude which should, of course, reduce to this asymptotic form for peripheral partial waves. Such a decomposition has been widely used in nuclear physics (Fuller 1973, Fuller and Avishai 1974, Fuller 1975a and 1975b, Rowley and Marty 1976, Fuller and Moffa 1977), although it was introduced much earlier for other scattering problems by Nussenzveig (1965). In terms of the (associated) Legendre functions of the second kind $Q_{\ell m}(\cos\theta)$ (Abramowitz and Stegun 1965), the decomposition is made by replacing the (associated) Legendre functions in the partial-wave series by

$$P^{\pm}_{\ell m}(\cos\theta) = \frac{1}{2}\{P_{\ell m}(\cos\theta) \mp \frac{2i}{\pi} Q_{\ell m}(\cos\theta)\}$$

(1.1)

with the lower and upper signs being used for the near- and farside amplitudes respectively. We call this the 'Fuller decomposition'. Asymptotically

$$P^{\pm}_{\ell m}(\cos\theta) \approx \frac{\ell^m}{\sqrt{(2\pi\ell\sin\theta)}} \exp\pm i((\ell+1/2)\theta - \pi/4 - m\pi/2)$$

(1.2)

which is valid for $|m|/\ell \leq \theta \leq \pi - |m|/\ell$.

A suggested alternative to Fuller's decomposition involves writing the (associated) Legendre functions as (Marty and Rowley 1982)

$$P_{\ell m}(\cos\theta) = \sum_k a_k^{\ell m} \exp(ik\theta)$$

(1.3)

where $|k| \leq \ell$ and the coefficients are, in general, complex. The NF decomposition in this case takes $k > 0$ to be farside and $k < 0$ to be nearside. The term with $k=0$, should it be non-zero, is split equally between the near- and farside amplitudes. Unfortunately, the NF amplitudes arising from eq.(1.3) are not smooth, and the usefulness of the NF method relies on the underlying amplitudes being non-oscillatory.

One might ask why Fuller's decomposition is the most suitable in this respect since, for peripheral processes at least, we only need eq.(1.2) and many functions can be defined which have this asymptotic form. The success of eq.(1.1) is apparently due to $P^{+}_{\ell m}$ and $P^{-}_{\ell m}$ having the same analytic structure. In fact an alternative way of looking at the NF amplitudes for m=0 is to regard them as different continuations into the complex $\cos\theta$ plane of the same basic amplitude (Fuller 1975a and 1975b). The same is also true for the amplitudes with $m\neq 0$ since (Abramowitz and Stegun 1965)

$$-i\pi P_{\ell m}(x) = i^{m}Q_{\ell m}(x+i\epsilon) - i^{-m}Q_{\ell m}(x-i\epsilon) \tag{1.4}$$

and

$$2Q_{\ell m}(x) = i^{m}Q_{\ell m}(x+i\epsilon) + i^{-m}Q_{\ell m}(x-i\epsilon) \tag{1.5}$$

with $x=\cos\theta$. Consequently we can write eq.(1.1) in the form

$$P^{\pm}_{\ell m}(x) = \pi^{-1}\exp(\mp i(m+1)\pi/2)\ Q_{\ell m}(x\mp i\epsilon) \tag{1.6}$$

showing that if we define

$$f_Q^m(z) \equiv \sum B_{\ell m}^{\beta} \, Q_{\ell m}(z) \tag{1.7}$$

where β represents all the other quantum numbers that the coefficient B in general depends on, we may write the NF amplitudes as

$$f_m^{\pm}(x) = \pi^{-1} \exp(\mp i(m+1)\pi/2) \, f_Q^m(x \mp i\epsilon) \tag{1.8}$$

This analytical continuation is invaluable when studying Rutherford scattering, where a partial-wave decomposition is inappropriate (section 3 of this Chapter).

2. The local angular momentum.

The nearside-farside decomposition can be used, as shown above to interpret structure in angular distributions. However, we cannot explain the period of such structure unless we have some knowledge of the phases, or at least the relative phase, of the interfering amplitudes. A concept which is very useful in this respect is the local angular momentum, (LAM), introduced by Fuller (Fuller 1975b, Fuller and Moffa 1977). Consider the probability current $\underline{J}$ given by

$$\underline{J} = -i\ (\psi^* \nabla\psi - \psi \nabla\psi^*) \tag{2.1}$$

in units where $\hbar$=1 and 2M=1 as in Chapter II. Introducing the local density $\rho=\psi\psi^*$, and defining the local velocity $\underline{v}$ by $\underline{J}=\rho\underline{v}$, we obtain, with $\underline{v}=2\underline{k}_{loc}$,

$$\underline{k}_{loc} = -\frac{i}{2} \nabla \ell n(\psi/\psi^*) \tag{2.2}$$

for the local wave number. Writing $\psi=|\psi|\exp(i\phi)$ it follows that $\underline{k}_{loc}=\nabla\phi$ and the local <u>radial</u> momentum and the local <u>angular</u> momentum are then

$$p_{loc} = \frac{\partial \phi}{\partial r} \qquad \ell_{loc} = \frac{\partial \phi}{\partial \theta} \tag{2.3}$$

Hence, as the only θ-dependence in the asymptotic scattered wave is in the scattering amplitude, the LAM is simply the derivative of the phase of this amplitude

$$\ell_{loc} = \frac{\partial}{\partial \theta} \arg f(\theta) \tag{2.4}$$

The LAM has been used for elastic scattering (Fuller 1975b, Fuller and Moffa 1977) and also for transfer reactions (Fuller and Moffa 1977), and we will find the concept useful

in subsequent chapters for the discussion of inelastic scattering.

3. Elastic scattering.

When making a nearside-farside decomposition of the elastic scattering amplitude, one must also consider the Rutherford amplitude. Simple classical considerations would argue that repulsive Coulomb scattering has a nearside component only, but as Fuller showed (1975a, 1975b) there is a small farside component too. Using a different method, Rowley (1980) came to the same conclusion. Fuller's approach follows from eq.(1.8) with m=0 and leads to

$$\frac{f^{-}_{Ruth}(\theta)}{f_{Ruth}(\theta)} = (1-\zeta)^{-1} - i[\sin^2(\theta/2)]^{1+i\eta}\, S(\theta) \qquad (3.1)$$

$$\frac{f^{+}_{Ruth}(\theta)}{f_{Ruth}(\theta)} = -\zeta(1-\zeta)^{-1} + i[\sin^2(\theta/2)]^{1+i\eta}\, S(\theta) \qquad (3.2)$$

where $\zeta=\exp(-2\pi\eta)$ and

$$S(\theta)=(2\pi(1+i\eta))^{-1}\, F(1,1+i\eta,2+i\eta;\sin^2(\theta/2)) \qquad (3.3)$$

Here F(...) is a hypergeometric function and η is the

Sommerfeld parameter. The first term in eq.(3.1) dominates the full amplitude at forward angles, whilst the nearside and farside amplitudes merge at angles close to π. For an attractive Coulomb potential eqs.(3.1) and (3.2) should be interchanged. Using the above equations calculations show that the NF amplitudes are far smoother than if the Rutherford amplitude is assumed to be wholly nearsided. The smoothness of the NF amplitudes is the very essence of the technique.

The full elastic scattering amplitude has been decomposed into its nearside and farside components <u>using analytic methods</u> by several authors (Rowley and Marty 1976, Frahn and Gross 1976 and references therein, Rowley 1980). These papers emphasize the physical insight obtained with this sort of method, but we stress, however, that the NF technique is purely <u>interpretational</u> since the NF components of the cross section cannot be measured separately. The elastic amplitude is written as a partial-wave sum

$$f(\theta) = (2ik)^{-1} \sum_{\ell=0}^{\infty} (2\ell+1)\, P_\ell(\cos\theta)\, (S_\ell - 1) \qquad (3.4)$$

which can be put into integral form using the Poisson sum formula (Titchmarsh 1948). This formula seems to have been first used in a scattering problem by Rubinow (1961) who used it in his study of the scattering of a scalar plane wave from

a penetrable sphere in the short wavelength limit, and it was later applied to atomic and molecular collisions by Berry (1966 and 1969) and Nussenzveig (1969). Application of this formula to eq.(3.4) gives

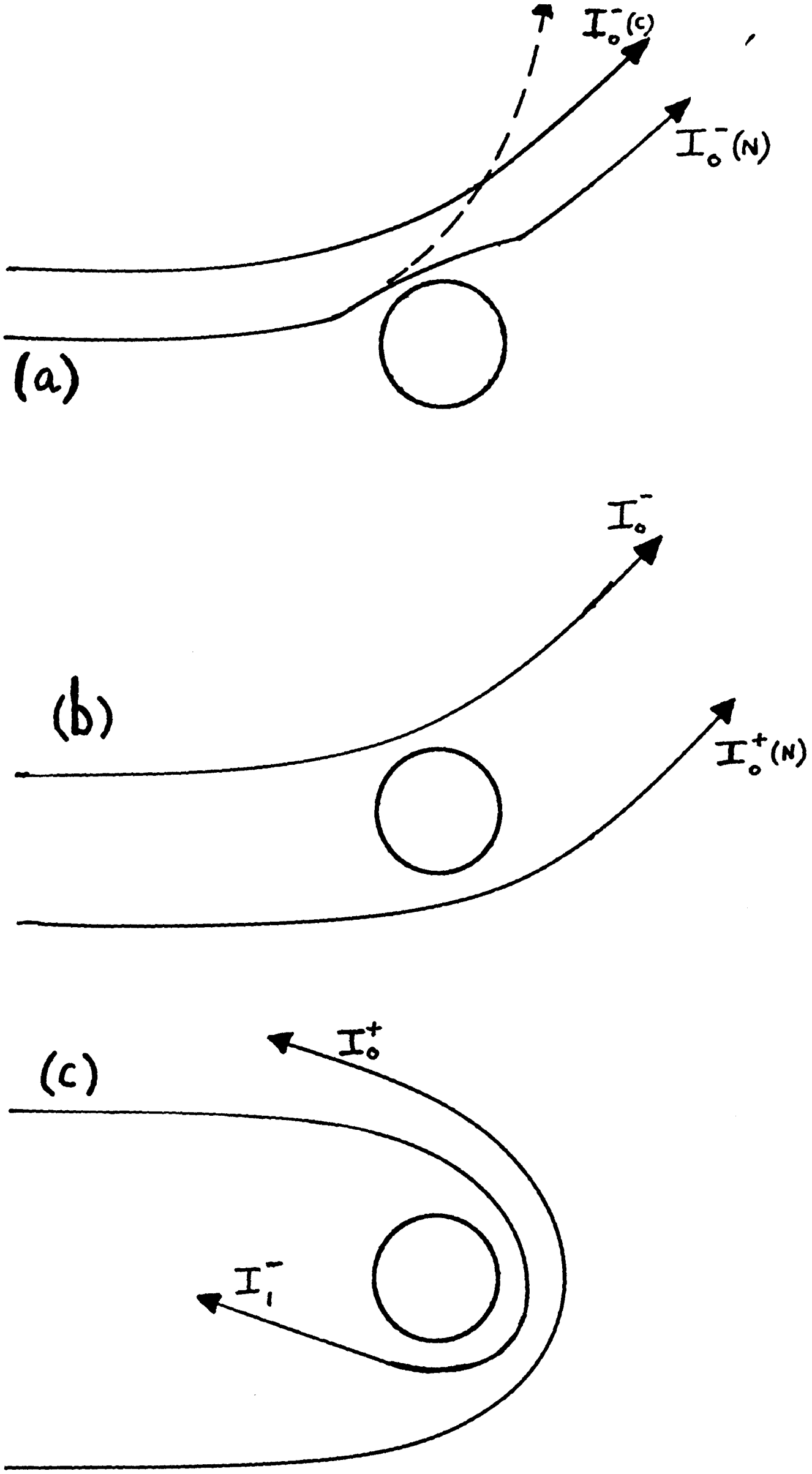
$$f(\theta) = \frac{(ik)^{-1}}{\sqrt{(2\pi\sin\theta)}} \sum_{m=-\infty}^{\infty} (-1)^m \{I_m^-(\theta) + I_m^+(\theta)\} \qquad (3.5)$$

where we have used the asymptotic form of the Legendre polynomials (eq.(1.2)) and defined

$$I_m^{\pm} = \exp(\mp i\pi/4) \int_0^{\infty} \sqrt{\lambda}\, \exp[i\lambda(\pm\theta+2\pi m)]\, S(\lambda)\, d\lambda \qquad (3.6)$$

with $\lambda=\ell^{+1}/2$. Generally very few of these integrals are important. The nearside and farside contributions to the full amplitude correspond to the terms I^- and I^+ respectively. Terms with $m < 0$ correspond to deflections greater than π and are therefore non-classical, but the other terms can be associated with equivalent (real) classical paths, and hence the NF interference pictured as interfering trajectories. We illustrate this in fig.(III-1).

Except at backward angles, where I_1^- is important, Rowley and Marty (1976) showed that the main contribution to the full amplitude comes from I_0^+ and I_0^-. Using a parametrized S matrix (McIntyre et al. 1960) and including a real nuclear phase by making the grazing angular momentum



(FIG. III-1): The real classical trajectories corresponding to (a) the forward angle oscillations, (b) the intermediate angle oscillations and (c) the backward angle oscillations. If the oscillations at large angles come from $I_0^-(\theta)$ and $I_{-1}^+(\theta)$ no such paths can be drawn. The broken curve in (a) indicates where the trajectory would lead to if only the Coulomb force were present. In none of these cases is it necessary to have such real trajectories in order to obtain oscillations. (N) and (C) refer to Nuclear and Coulomb amplitudes respectively.

complex (Ericson 1966), they showed that

$$|I_0^+/I_0^-| = \exp 2(\Lambda_I \theta - \pi\Delta\Theta) \qquad (3.7)$$

where Λ_I is the imaginary part of the grazing angular momentum, Θ is the grazing angle and Δ is a measure of the width of the S matrix in angular momentum space. It is therefore possible, for non-zero Λ_I, for these two terms to have the same magnitude, although whether this happens in the physical region ($0 \leqslant \theta \leqslant \pi$) or not depends on the actual values of the parameters in this equation. When these two terms do have similar magnitudes, the interference between them is strong and the angular distribution shows pronounced oscillations.

The actual evaluation of the integrals of eq.(3.6) has been the subject of much work (Frahn 1973, Frahn 1974, Rowley and Marty 1976, Frahn and Gross 1976, Rowley 1980). The details of the calculation depend on the physical situation under consideration, i.e. strong or weak absorption, strong or weak nuclear attraction etc., but essentially the main contributions to the integrals come from the regions near points where the phase of the integrand is stationary, and also from the regions of λ-space where the magnitude of $S(\lambda)$ varies rapidly. The former are 'refractive' since they are composed of angular momenta which are classically scattered into the region of the scattering angle and, in a similar analogy with optics, the latter are 'diffractive'

since they are characterized by the values of λ for which $|S(\lambda)|$ is rapidly varying. The analytical methods in the two cases are rather different, but we will not discuss them here since they are well covered in the literature.

An approach which avoids the mathematical complexities of the analytical method has been used by Fuller (1975a and 1975b). He simply takes the S-matrix elements generated from a standard optical-model code and evaluates the NF amplitudes using eq.(3.4) with the Legendre polynomials replaced by either eq.(1.2) or, more accurately, eq.(1.1). The Rutherford amplitude is treated using eqs.(3.1) and (3.2). The two methods are, however, complementary since the physics of the problem becomes more apparent through mathematics than through pure computation.

We now illustrate the use of the concept of the local angular momentum (LAM), introduced in the previous section, to explain the oscillations in the elastic cross section forward of the grazing angle ('quarter point'). When plotted as a ratio to the Rutherford cross section the angular distribution in this region consists of oscillations decreasing in frequency and increasing in magnitude followed by a precipitous fall beyond the grazing angle. At forward angles the Rutherford amplitude is virtually all nearsided and the nearside 'nuclear' amplitude is generally larger than the farside nuclear component. Assuming that the nuclear amplitude is characterized by a (Regge) pole at $\Lambda=\Lambda_R+i\Lambda_I$,

its phase will, from eq.(1.2), vary as $-\Lambda_R\theta$, provided that Λ_R is not too small. The Rutherford amplitude has the form

$$f_{Ruth}=-\eta/(2k\sin^2(\theta/2))\ \exp(2i\sigma_0-i\eta\ln(\sin^2(\theta/2))) \qquad (3.8)$$

and so has a LAM equal to $-\eta\cot(\theta/2)$, whereas that of the nuclear amplitude is simply $-\Lambda_R$. Hence the oscillations forward of the grazing angle are expected to have period $2\pi/(\Lambda_R-\eta\cot(\theta/2))$, which increases as θ decreases. Since the magnitude of the nuclear amplitude falls off towards zero angle the oscillations become damped as their period increases. The forward angle oscillations in heavy-ion elastic scattering can therefore be explained in terms of a nearside-nearside interference with the period given by the local angular momenta. If the farside nuclear amplitude is not too small, similar arguments show that rapid oscillations of period $2\pi/(\Lambda_R+\eta\cot(\theta/2))$ will be superimposed on the slower oscillations just discussed.

As an example of the ideas presented in this section, we show in fig.(III-2) the NF components of the ^{16}O-^{44}Ca elastic cross section at 60 MeV (Rehm et al. 1982). We used their optical potential (Table III-1) to generate the S-matrix elements using the standard DWBA code DWUCK (Kunz 1965), and we decomposed the Rutherford amplitude using Fuller's method. Pronounced oscillations are visible above about 90° scattering angle, where the nearside and farside

TABLE III-1

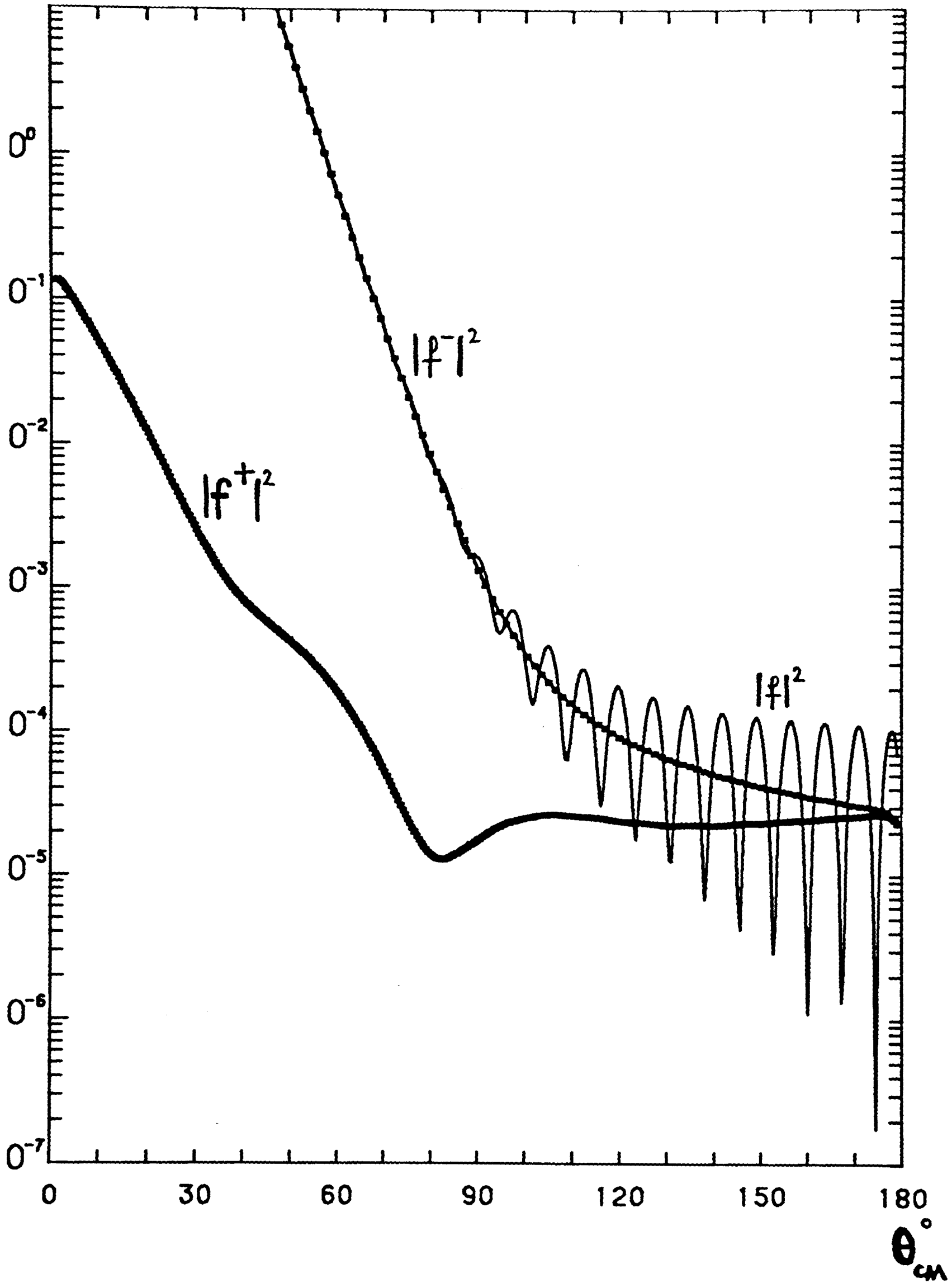
Optical model parameters for ^{16}O-^{44}Ca

(Rehm et al. 1982).

E_{lab} = 60 MeV; all potentials are standard Woods-Saxons.

Coulomb radius (uniform sphere) = 7.260 fm.

Potential	Depth (MeV)	Radius (fm)	Diffuseness (fm)
Real Nuclear	13.6	8.283	0.524
Imag Nuclear	7.1	8.216	0.541



(FIG. III-2): Decomposition of the elastic cross section int near- and farside components as discussed in the text. The ordinate scale is in arbitrary units.

components of the amplitude become very similar in magnitude. (These oscillations are not visible in the published data since measurements were only made for centre of mass angles less than 70°). The Sommerfeld parameter is about 13 for this process and so the first term in eq.(3.2) is extremely small, showing that the Rutherford amplitude is almost totally nearsided in this case.

Having discussed the NF decomposition and its application to elastic scattering, we now turn to inelastic scattering where similar ideas can be used.

CHAPTER IV

THE NEARSIDE-FARSIDE DECOMPOSITION IN INELASTIC SCATTERING.

1. Introduction and formalism.

As we have shown in the previous chapter, the decomposition of the full scattering amplitude into nearside and farside components gives considerable insight into the nature of heavy-ion elastic scattering. The situation for inelastic scattering is potentially more fruitful, though complicated by the incoherent sum over the magnetic substates which gives the total inelastic cross section. For the excitation of a spin-zero nucleus to a state of spin I we have

$$\sigma_I(\theta) = \sum_M \sigma_{IM}(\theta) \qquad (1.1)$$

where σ_{IM} is the cross section for excitation of a magnetic substate $|IM\rangle$ referred to some arbitrary quantization axis, and so the structure in σ_I is the sum of structures in the cross sections σ_{IM}. This makes the role of the underlying amplitudes more difficult to visualize, a further complication being that the σ_{IM} are not scalar quantities

and so depend on the particular choice of co-ordinate frame. The total inelastic cross section is, of course, a scalar.

As discussed in Chapter I, dynamical decompositions of the cross section are also possible. In this chapter we restrict ourselves to strong absorption conditions and also neglect, for the moment at least, Coulomb excitation. Deviations from the former and the importance of the latter for heavier systems will be discussed in a subsequent chapter.

In the notation of Goldberger and Watson (1964), the cross section for inelastic scattering of two spin-zero nuclei to a state of spin I can be expressed in the form

$$\sigma_I(\theta) = \sum_{\kappa}^{\underline{\kappa}'} \sum_{M} |f_{IM}(\theta)|^2 \qquad (1.2)$$

where θ is the scattering angle given (in terms of the initial and final momenta ($\underline{\kappa}$ and $\underline{\kappa}'$)) by $\cos\theta = \hat{\underline{\kappa}}' \cdot \hat{\underline{\kappa}}$ and

$$f_{IM}(\theta) = \sum_{\substack{\ell\ell' \\ mm'}} Y_{\ell'm'}(\hat{\underline{\kappa}}')\, Y^{*}_{\ell m}(\hat{\underline{\kappa}})\, \langle \ell' I m' M | \ell m \rangle\, T^{\ell}_{\ell' I;\ell 0}(K) \qquad (1.3)$$

Our discussion will be based on the distorted-wave Born

approximation, (DWBA), in which the T-matrix elements of the above equation may be written (Austern 1970, Jackson 1970)

$$T^{\ell}_{\ell' I;\ell 0} \propto \sqrt{(2\ell'+1)}\; i^{\ell+\ell'} \begin{pmatrix} \ell & \ell' & I \\ 0 & 0 & 0 \end{pmatrix} R_{\ell',\ell} \exp(i\tilde{\sigma}_{\ell'}+i\sigma_{\ell}) \qquad (1.4)$$

where $R_{\ell',\ell}$, the so-called DWBA radial integral, is

$$R_{\ell',\ell} = \int_0^{\infty} \mathcal{U}_{\ell'}(r)\, F(r)\, u_{\ell}(r)\, dr \qquad (1.5)$$

Here $\mathcal{U}_{\ell'}$ and u_{ℓ} are distorted radial wavefunctions for orbital angular momentum ℓ' and ℓ in the final and initial channels respectively, and F(r) is the transition potential, or form factor, for the given inelastic process. The quantities σ and $\tilde{\sigma}$ in eq.(1.4) are the Coulomb phases in the two channels: they are defined by eq.(II.4.10). (Note that this form of the radial integral is slightly different from that of Chapter II, but it is more convenient for present purposes). The Wigner coefficient expresses the fact that it is not possible to excite unnatural parity states in one-step processes and such states must, therefore, be excluded from our discussion. The axes we use initially are those usually employed in DWBA calculations: the incident beam direction $\underline{\kappa}$ defines the z-axis and the y-axis lies along $\underline{\kappa} \wedge \underline{\kappa}'$. Hence eq.(1.3) becomes

$$f_{IM}(\theta) = \sum_{\ell\ell'} i^{\ell-\ell'} \exp[i(\sigma_{\ell'}+\sigma_{\ell})] \sqrt{(2\ell'+1)}\, Y_{\ell',-M}(\theta,0) \langle \ell' I -M M | \ell 0 \rangle \langle \ell' I 0 0 | \ell 0 \rangle R_{\ell'\ell} \qquad (1.6)$$

Under strong absorption conditions the radial integral is large only if both ℓ and ℓ' lie in a relatively localized region around some 'critical' or 'grazing' angular momentum Λ associated with the interaction radius (Austern 1970, Jackson 1970). For $\ell,\ell' \gg \Lambda$ the higher angular momentum barrier means that the initial and final channel wavefunctions have a very small overlap with the transition potential which is, for direct reactions, strongest close to the nuclear surface. For $\ell,\ell' \ll \Lambda$ the radial integral is again small, but for a different reason. For these angular momenta the reflection coefficients are small under strong absorption conditions, and so the radial wavefunction is a purely ingoing wave. However, its wavelength and amplitude vary with r, particularly in the interior region where the latter gradually decreases due to absorption. Hence the integrand of the eq.(1.5) is essentially the product of two ingoing waves and so oscillates rapidly, tending to an average of zero. Indeed, in the lowest order WKB approximation the radial integral is, for low angular momenta, (Austern 1961, Rost 1962)

$$R_{\ell'\ell} \approx \int_0^{\infty} F(r) \exp(-i(\kappa_{\ell}+\kappa_{\ell'})r)\, dr \qquad (1.7)$$

where κ_ℓ is the local momentum. Since the range of the transition potential is large compared to $1/\kappa_\ell$ these integrals will average to zero.

At energies sufficiently above the Coulomb barrier we have $\Lambda \gg \Delta \sim 1$, where Δ is the width in ℓ-space of the radial integral, and therefore characterizes the number of 'important' partial waves. These points are concisely demonstrated in fig.(IV-1) which shows a 'three-dimensional' plot of the magnitude of the radial integral as a function of ℓ and ℓ'. These quantities were extracted from the standard DWBA code DWUCK (Kunz 1965), and correspond to the scattering of ^{16}O on ^{24}Mg at 60 MeV leading to the 1.37 MeV 2^+ state in ^{24}Mg. The figure was produced using the potential of Siwek-Wilczynska et al. (1974), which is given in Table (IV-1). Because of the Wigner coefficient in eq.(1.4) only values of $k \equiv \ell - \ell' = 0, \pm 2$ contribute to the scattering amplitude, and these are indicated in the figure. Though it is not immediately apparent from this figure, the radial integrals are not symmetric about some value of ℓ, if we regard them as a function of ℓ for fixed k. This is because different effects are responsible for the reduction in magnitude at small and large angular momenta as outined above. Many parametrizations of eq.(1.5) do, however, assume such a symmetry: an approximation which is less good for weaker absorption. For fixed $\ell'(\ell)$, however, the radial integral is largest for a value of $\ell(\ell')$ almost independent of $\ell'(\ell)$. This is part justification for Hahne's (1967)

TABLE IV-1.

Parameters of the potentials used in Chapter IV.

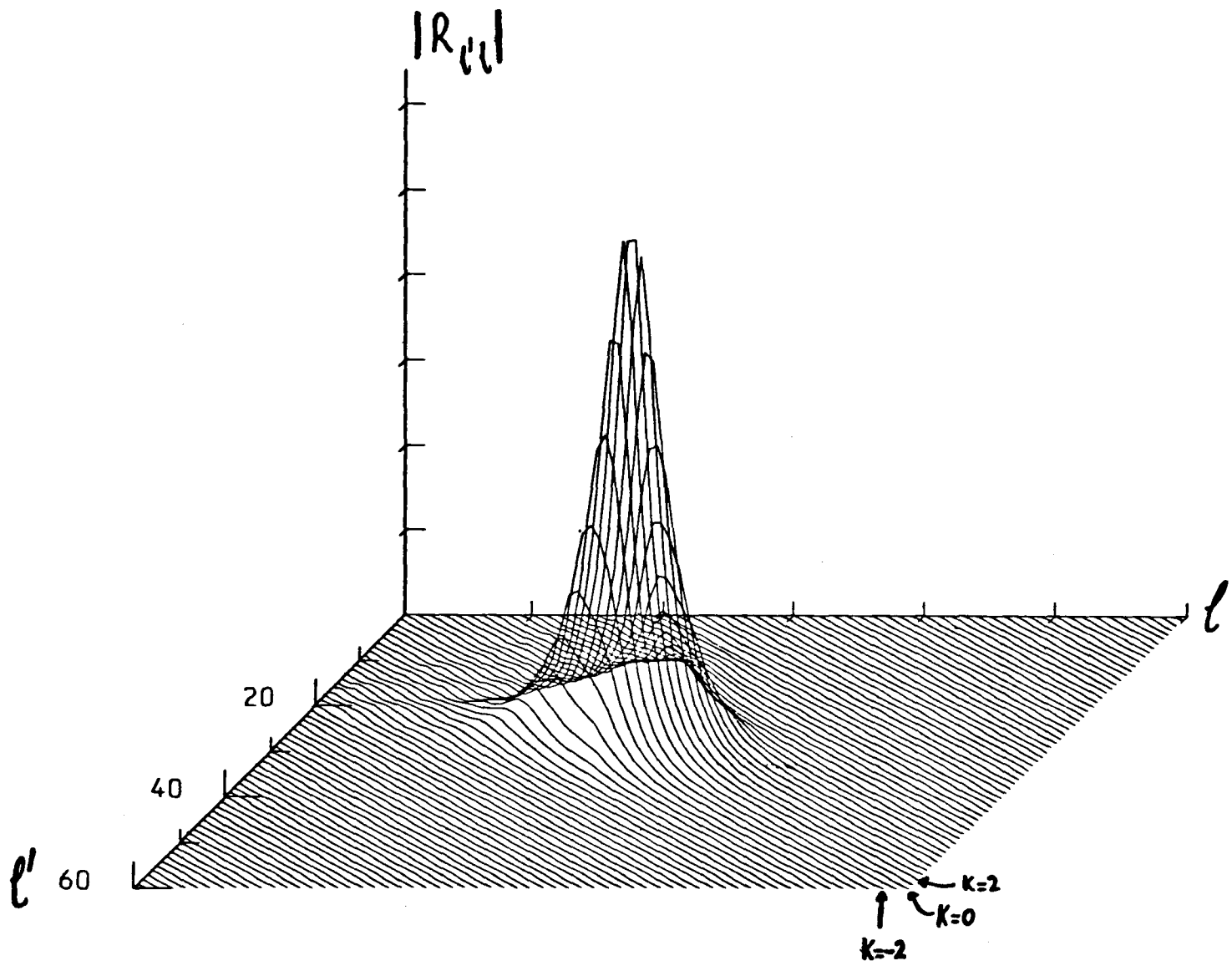
The potentials are all of Woods-Saxon form raised to the power ν_V or ν_W with (depth, radius, diffuseness) equal to (V, R_V, a_V) or (W, R_W, a_W) in MeV and fm.

	Real				Imaginary			
	V	R_V	a_V	ν_V	W	R_W	a_W	ν_W
^{16}O-^{24}Mg (1)	23.00	7.4	0.53	1	10.37	7.4	0.36	1
α-^{50}Ti (2)	126.15	5.64	1.00	2	21.35	5.56	1.14	2

Notes:

(1) : (Siwek-Wilczynska et al. 1974). The transition potential is the derivative of the listed potential multiplied by the deformation parameter β_2=0.30 (standard collective model).

(2) : (Gubler et al. 1981). The transition potential we took to be the derivative of a squared Woods-Saxon in shape (with a real part only) multiplied by the deformation parameter β_2=0.146 (Rebel et al. 1981).



(FIG. IV-1): The magnitude of the DWBA radial integral $R_{\ell'\ell}$ (in arbitrary units) is plotted as a function of ℓ and ℓ' for $^{16}O-^{24}Mg$ scattering (see text). The form is typical for strong absorption reactions. The lines are of constant $\ell-\ell'$, but only $k=\ell-\ell'=0,\pm2$ contribute to the scattering amplitude for quadrupole excitations.

approximation to eq.(1.5) which we will discuss shortly.

Under the above conditions we may expand the Coulomb phases as

$$\sigma_{\ell} = \sigma(\Lambda)+\tfrac{1}{2}(\ell-\Lambda)\theta(\Lambda)+\tfrac{1}{4}(\ell-\Lambda)^2\theta'(\Lambda)+\ldots$$

$$\tilde{\sigma}_{\ell'} = \tilde{\sigma}(\Lambda)+\tfrac{1}{2}(\ell'-\Lambda)\tilde{\theta}(\Lambda)+\tfrac{1}{4}(\ell'-\Lambda)^2\tilde{\theta}'(\Lambda)+\ldots \qquad (1.8)$$

where we have used the semi-classical approximation (Rowley and Marty 1976)

$$2i\sigma(\lambda) = (\lambda+i\eta)\ell n(\lambda+i\eta) - (\lambda-i\eta)\ell n(\lambda-i\eta) \qquad (1.9)$$

which gives

$$\theta(\lambda) = 2\arctan(\eta/\lambda) \qquad \theta'(\lambda) = -2\eta/(\eta^2+\lambda^2) \qquad (1.10)$$

η being the Sommerfeld parameter in the entrance channel. The corresponding exit channel functions are defined similarly in terms of η'. As long as $\eta\Delta^2 \ll 2(\eta^2+\lambda^2)$ we may neglect the second-order terms in eq.(1.8) and write

$$\tilde{\sigma}_{\ell'}+\sigma_{\ell} \approx (\ell'+k/2)\theta + (\tilde{\sigma}(\Lambda)+\sigma(\Lambda)-\Lambda\theta) \qquad (1.11)$$

where we have taken $\theta(\Lambda)\approx\tilde{\theta}(\Lambda)\equiv\theta$ which corresponds to

ignoring the difference between these quantities due to the energy loss in the scattering process. (Note that for elastic scattering it is necessary to retain the second-order terms of eq.(1.8) so that stationary points in the Coulomb contribution to the scattering amplitude can be properly accounted for. Similarly, for Coulomb excitation, where many partial waves are affected, these extra terms must also be kept).

We now introduce the nearside-farside (NF) decomposition by using the semi-classical approximation

$$Y^{\pm}_{\ell m}(\theta,0) \approx \frac{1}{2\pi\sqrt{\sin(\theta)}} \exp[\pm i((\ell+{}^1/_2)\theta - {}^{\pi}/_4 + {}^{m\pi}/_2) \tag{1.12}$$

which is entirely equivalent to eq.(III.1.2). All of this chapter is based upon this semi-classical decomposition rather than the exact Fuller decomposition of eq.(III.1.1). For large ℓ and ℓ' ($\gg$ I) it is also possible to use the asymptotic approximation (Brussaard and Tolhoek 1957, Brink and Satchler 1968)

$$\langle \ell' I -M M | \ell 0\rangle \langle \ell' I 0 0 | \ell 0\rangle \approx d^{I}_{M k}({}^{\pi}/_2)\, d^{I}_{0 k}({}^{\pi}/_2) \equiv C^{I}_{M k} \tag{1.13}$$

We give a new proof of this result in appendix E.

Inserting the above expressions into eq.(1.6) we obtain

$$f^{\pm}_{IM}(\theta) = \sum_{\ell' k} \exp[i(k(\pi+\theta)/2 + \ell'\theta)]\sqrt{(2\ell'+1)} \; C^{I}_{Mk} \; R_{\ell', \ell'+k} \; Y^{\pm}_{\ell', -M}(\theta, 0) \tag{1.14}$$

where we have dropped, as we shall also do in subsequent equations, an overall M-independent phase. Phases depending on M will be kept so that we can easily express our amplitude in different sets of axes.

As in the case of elastic scattering it is possible to use the Poisson sum formula (Titchmarsh 1948) to replace the sum over ℓ' in the above equation by a sum of integrals over $\lambda \equiv \ell' + {}^{1}/_{2}$

$$f^{\pm}_{IM} \approx \pi^{-1} \, i^{\mp M} \sqrt{(\Lambda/(2\sin\theta))} \exp(\mp i\pi/4) \sum_{n=-\infty}^{\infty} (-1)^{n} \sum_{k} C^{I}_{Mk} i^{k} \exp(ink\pi \mp ik\theta/2) \; \Omega_{k}(\theta \pm \theta + 2\pi n) \tag{1.15}$$

where the transform is defined by

$$\Omega_{k}(\omega) \equiv \exp(ik\omega/2) \int_{-\infty}^{\infty} R_{k}(\lambda) \exp(i\lambda\omega) \, d\lambda \tag{1.16}$$

and $R_{k}(\lambda)$ is a smooth function of λ equal to $R_{\ell', \ell'+k}$ at $\lambda \equiv \ell' + {}^{1}/_{2}$. In deriving eq.(1.15) we have assumed that the

factor $\sqrt{\lambda}$, which should strictly appear in the integrand, is a slowly varying function and so may be replaced by $\sqrt{\Lambda}$. The pre-multiplying phase factor in eq.(1.16) makes the overall phase of the transform almost independent of k in many cases: this we demonstrate in the next section, and also in appendix F.

If the radial integrals are reasonably smooth functions of λ, their transforms are small for large $|\omega|$ and so almost all of the terms with $n\neq 0$ in eq.(1.15) may be dropped. It is clear, however, that at scattering angles close to π

$$\begin{aligned} \Omega_k(\Theta-\theta+2\pi) &\approx \Omega_k(\Theta+\theta) \\ \Omega_k(\Theta+\theta-2\pi) &\approx \Omega_k(\Theta-\theta) \end{aligned} \qquad (1.17)$$

and so in these circumstances we should retain the $n=\pm 1$ terms too. We do not intend discussing this angular region in detail here, and so we write

$$f^{\pm}_{IM} \approx \pi^{-1}\, i^{\mp M} \sqrt{(\Lambda/(2\sin\theta))}\, \exp(\mp i\pi/4) \sum_k C^I_{Mk} \exp(ik(\pi\mp\theta)/2)\, \Omega_k(\Theta\pm\theta) \qquad (1.18)$$

and

$$f_{IM}(\theta) = f^{-}_{IM}(\theta) + f^{+}_{IM}(\theta) \qquad (1.19)$$

This is not a new result: indeed several authors (Hahne 1966 and 1967, Frahn 1976, Frahn and Rehm 1978) have re-expressed the scattering amplitude as a sum of such Fourier-like transforms. We shall, however, obtain many results directly from eqs.(1.18) and (1.19) without recourse to parametrizations. First we derive and explain the 'quasi-classical' limit of inelastic scattering.

2. The quasi-classical limit.

If we calculate the radial integral $R_{\ell'\ell}$ using the same distorting potential in the initial and final channels then, in the adiabatic limit the two wavefunctions of eq.(1.5) are identical since they are both solutions of the same Schrödinger equation. Thus

$$R_{\ell'\ell} = R_{\ell\ell'} = \int u_{\ell'+k}\, F(r)\, u_{\ell'}\, dr \qquad (2.1)$$

$$\approx \int u_{\ell'}\, F(r) \{ 1 + k\frac{\partial}{\partial \ell'} + \frac{1}{2}k^2\frac{\partial^2}{\partial \ell'^2} + \ldots \}\, u_{\ell'}\, dr$$

$$\approx R_{\ell'\ell'} + \frac{1}{2}k\frac{\partial}{\partial \ell'} \int u_{\ell'}\, F(r)\, u_{\ell'}\, dr + \ldots$$

$$\approx \left[1 + \frac{1}{2}k\frac{\partial}{\partial \ell'} \right] R_{\ell'\ell'} + \ldots\ldots \qquad (2.2)$$

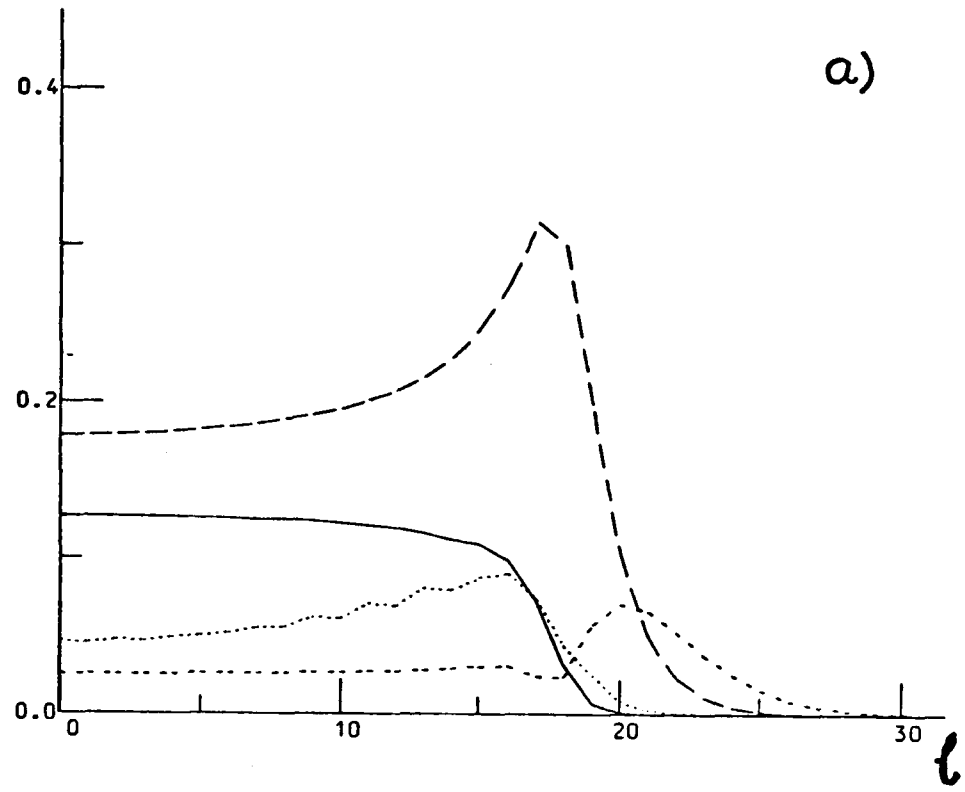
For the higher-order terms it is not possible to remove the differential operator from the integral, though if the radial wavefunctions are sufficiently smooth these extra terms will be small and can therefore be neglected. This expression then gives

a simple relationship between the off-diagonal and on-diagonal radial integrals. As the radial integral becomes more localized in ℓ-space this approximation becomes less good.

The smoothness of the radial wavefunction as a function of ℓ for fixed r can be seen in fig.(IV-2). We consider α-^{50}Ti scattering at 45 MeV and use the optical potential, given in Table (IV-1), that fits some recent elastic scattering data (Gubler et al. 1981). This potential has Woods-Saxon squared real and imaginary parts: indeed the use of such generalized Woods-Saxon potentials for α-particle scattering has become quite widespread, partly through the work of Michel and Vanderpoorten (1977). These potentials resemble those calculated by folding an effective α-nucleon interaction with a realistic nuclear shape and they can, therefore, be justified to some extent on semi-microscopic grounds (Michel 1976, Vinh Mau 1977, Majka et al. 1978).

In fig.(IV-2a) we have chosen r=7fm, which is close to the barrier radius, as well as r=4fm and r=9fm. We plot the magnitude of $\sqrt{(F(r))}u_\ell^b(r)$, and also that of $\sqrt{(F(r))}u_\ell^i(r)$ at 7fm only, u_ℓ^b and u_ℓ^i being the barrier and internal wavefunctions respectively (Chapter I). We see that the internal wave is rather small, but that both components vary fairly smoothly with ℓ. The total wavefunction for low angular momenta is, however, not a smooth function of ℓ because of interference between the barrier and internal waves, and consequently our smoothness assumptions are better for these components separately than for the full wavefunction.

The curves of fig.(IV-2b) are the magnitudes of the barrier part of the exit channel wavefunction, again weighted with $\sqrt{F(r)}$, at r=7fm for three different values of the outgoing



(FIG. IV-2): Magnitudes of wavefunctions weighted by $\sqrt{F(r)}$ as a function of ℓ at fixed r. (a) Entrance channel. Solid curve: r=4 fm, barrier wave; long-dash curve: r=7 fm, barrier wave; short-dash curve: r=9 fm, barrier wave; dotted curve: r=7 fm, internal wave x5. (b) Exit channel, at r=7 fm. Solid curve: zero excitation energy; long-dash curve: 1.55 MeV excitation energy; short-dash curve: 6.00 MeV excitation energy. Ordinate scale is in arbitrary units in both parts of the figure.

energy. Note that the scales on the abscissae of figs.(IV-2a) and (IV-2b) are different. We see that a change in energy of the wavefunction corresponds to a shift in the peak of the wavefunction, which is true for all values of the radius, but the shift is r-dependent. We shall see later that the most meaningful such shift to talk about, in view of the 'surface' nature of the reaction, is the one occurring at the interaction radius.

Introducing $R_k(\lambda)$, we can use eq.(2.2) to write the transform of eq.(1.16) as

$$\Omega_k(\omega) \approx \exp(ik\omega/2) \int_{-\infty}^{\infty} \left[1 + \frac{1}{2} k \frac{\partial}{\partial \lambda} \right] R_0(\lambda) \exp(i\omega\lambda)\, d\lambda$$

$$\approx \exp(ik\omega/2) \left(1 - \frac{1}{2} k\omega \right) \Omega_0(\omega) \qquad (2.3)$$

where we have assumed that $R_0(\lambda \leq 0) = R_0(\infty) = 0$. This is a good assumption for strong absorption conditions as can be seen from fig.(IV-1). If $|k\omega/2| \ll 1$ for all values of k, i.e. $|l\omega/2| \ll 1$, we may therefore write

$$\Omega_k(\omega) \approx \Omega_0(\omega) \qquad (2.4)$$

for all the transforms appearing in eq.(1.18).

We define the above conditions (adiabatic limit, smoothly peaked radial integral, smooth radial wavefunction, $|l\omega/2| \ll 1$) and the result eq.(2.4) as the 'quasi-classical (QC) limit'. This name is appropriate since the approximation works best for:

a) a large number of important partial waves, which is the classical situation. This number cannot become too large, however, since then the approximation (1.11) is invalid; and

b) scattering close to the classical scattering angle, since ω must be small.

Eq.(2.4) shows that in the QC-limit the phase of the

transform (and, indeed, its magnitude) is k-independent as claimed in the previous section. In appendix F we show that for particular forms of the radial integral the transforms continue to have k-independent phases even outside the QC-limit.

Substituting eq.(2.4) into eq.(1.18) we see that the sum over k is of the form

$$\sum_k \quad C^I_{Mk} \quad \exp(ik(\pi \mp \theta)/2) \tag{2.5}$$

which, using the result of appendix D, is simply

$$\left[\frac{4\pi}{2I+1} \right]^{1/2} \quad i^{-M} \quad Y_{IM}\left((\pi \mp \theta)/2,\ 0\right) \tag{2.6}$$

The result of appendix D, illustrated here, can be used to prove the asymptotic expression for the product of two Clebsch-Gordan coefficients. This proof is presented in appendix E. Hahne (1966) obtained a sum similar to (2.5), but did not rewrite it in the form (2.6). The two amplitudes $f_{IM}{}^{\pm}(\theta)$ of eq.(1.18) therefore vary as $Y_{IM}(\pi/2 \mp \theta/2, 0)$ and it is interesting to note from fig.(IV-3) that the arguments of these spherical harmonics correspond to points on the nuclear surface at which the 'trajectories' corresponding to $f^{\pm}$ have their distance of closest approach.

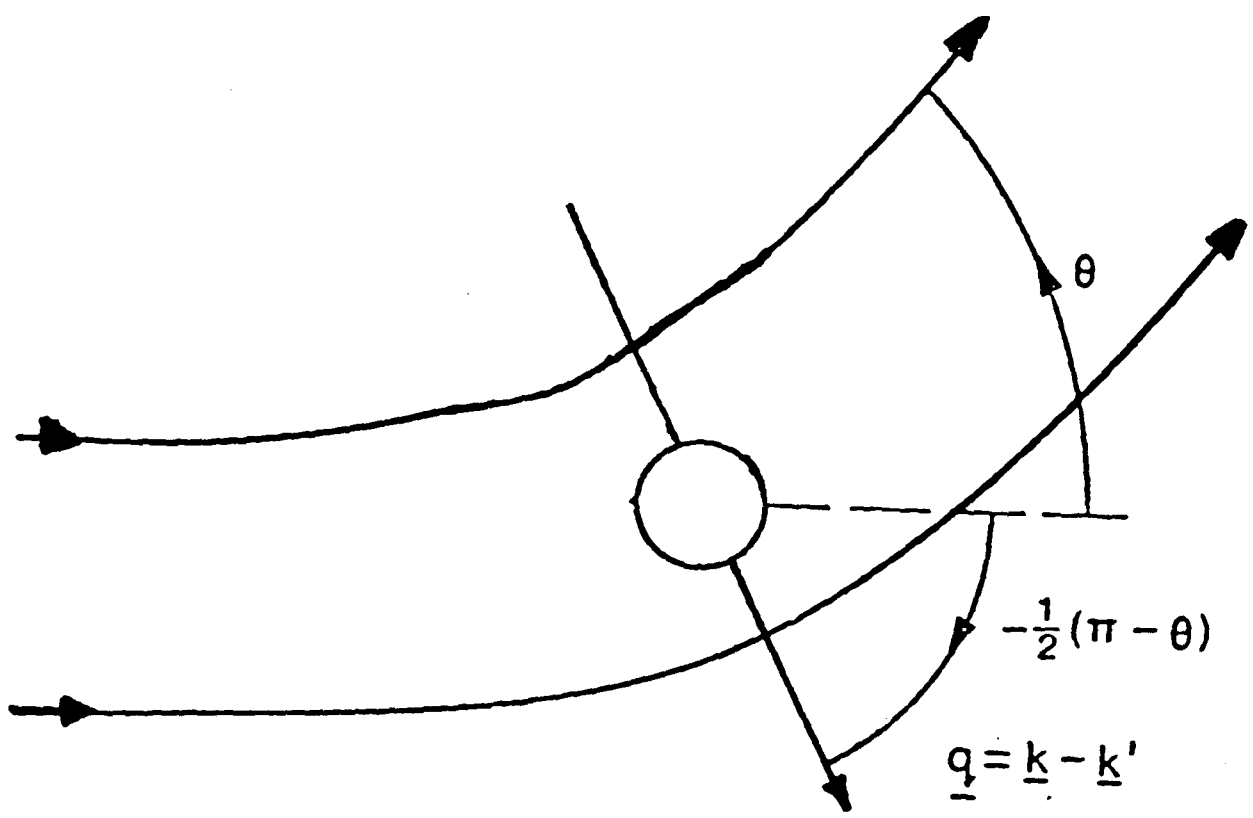
Using eqs.(2.6) and (1.18) we obtain for the full amplitude in the QC-limit

$$f_{IM}(\theta) \approx \left[\frac{2\Lambda}{(2I+1)\pi \sin\theta} \right]^{1/2} \{ \Omega_0(\Theta-\theta) - i(-1)^I \Omega_0(\Theta+\theta)\} \ Y_{IM}\ (\pi/2+\theta/2, 0) \tag{2.7}$$

where we have used

$$Y_{IM}(\pi/2 - \theta/2, 0) = (-1)^{I+M}\ Y_{IM}(\pi/2 + \theta/2, 0) \tag{2.8}$$

The entire M-dependence of eq.(2.7) lies in the spherical harmonic and so, using a sum rule for these quantities (Brink



(FIG. IV-3): Nearside and farside trajectories along with points of closest approach. $\underline{q}$ is the recoil direction of Satchler (1960).

and Satchler 1968), we obtain the quasi-classical cross section

$$\sigma_I(\theta) \propto | \Omega_0(\Theta-\theta) - i(-1)^I \Omega_0(\Theta+\theta) |^2 \qquad (2.9)$$

which is of exactly the same form as the elastic cross section of eq.(I.1.1). In this limit, therefore, the division of the cross section into nearside and farside components is just as simple as for the elastic case, since none of the complications associated with the magnetic-substate dependence of the inelastic amplitude persist in the full cross section.

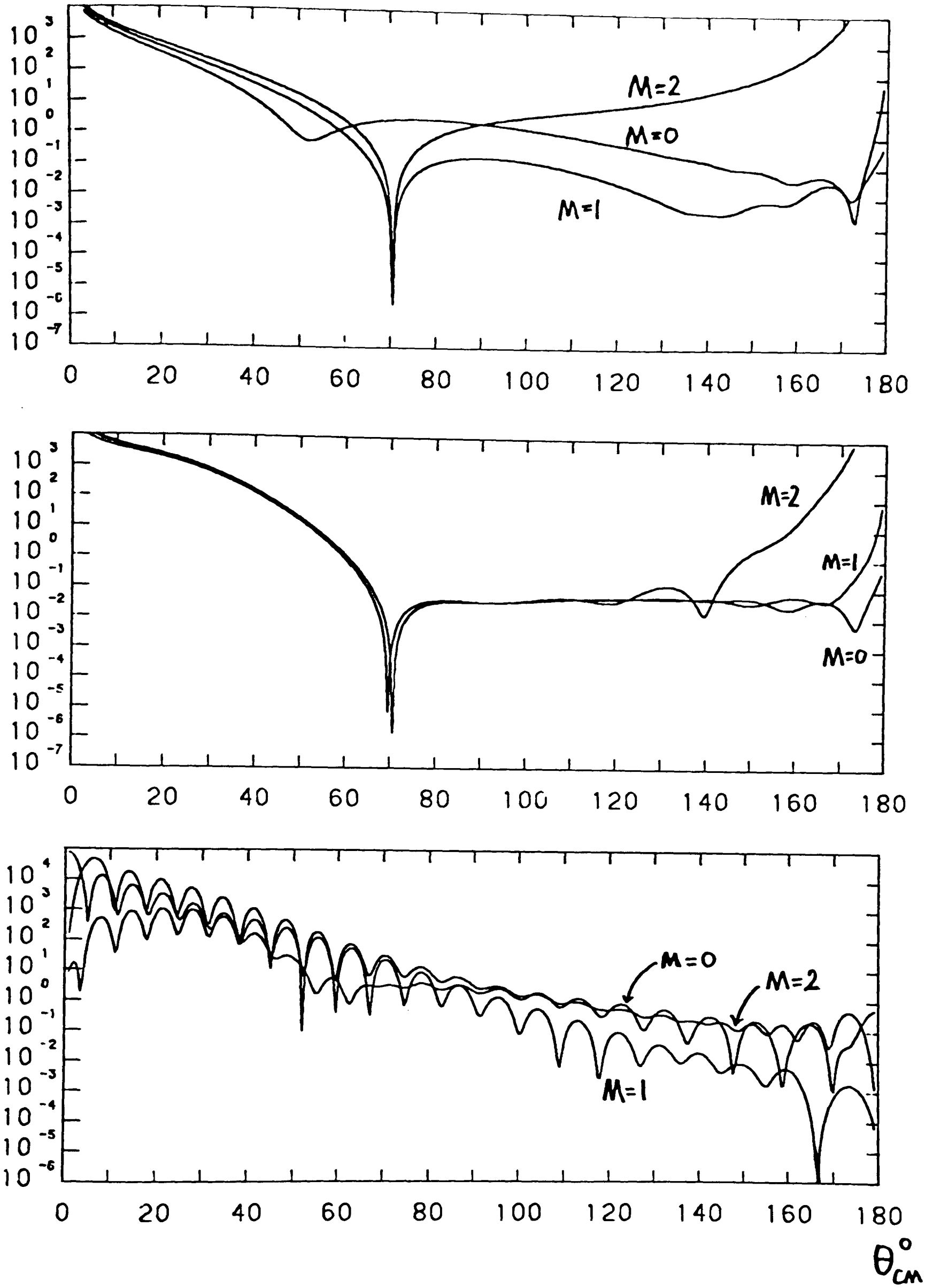
The $(-1)^I$ factor in eq.(2.9) is indicative of the well-known Blair phase rule (Blair 1959, Austern and Blair 1965) which says that the inelastic cross sections for excitation of natural parity states of even/odd spin I are out of/in phase with the elastic cross section and therefore out of phase with each other. We see that in the QC-limit the latter part of this rule holds, and we shall see later why the elastic/inelastic part of the rule is also valid.

In deriving eq.(2.9) we assumed that $|I\omega/2| \ll 1$ where $\omega=\Theta\pm\theta$ for the nearside and farside amplitudes respectively. Since $|\omega|$ is smaller in general for the nearside amplitude, we expect the QC-approximation to work better for this amplitude than for the farside one. Furthermore, if we divide the amplitudes $f_{IM}^{\pm}$ by the spherical harmonic of eq.(2.7), the resulting 'underlying' amplitude should be M-independent. Again this should work better for the nearside amplitude.

In fig.(IV-4) we plot the quantity

$$\left| f \frac{Y_{I0}\,(\pi/2 + \theta/2, 0)}{Y_{IM}\,(\pi/2 + \theta/2, 0)} \right|^2 \qquad (2.10)$$

where $f=f_{IM}^{\pm}$, and also the magnitude of the full amplitude f_{IM}.



(FIG. IV-4): ^{16}O-^{24}Mg at 60 MeV with zero excitation energy. The expression (2.10) is plotted for the farside (top) and nearside (middle) amplitudes. The magnetic quantum numbers are indicated, and the dips at ~70° are due to a zero in Y_{20} In the bottom figure we show the square modulus of the full amplitude. In all cases the ordinate scale is in arbitrary units.

These calculations again correspond to ^{16}O-^{24}Mg scattering (Siwek-Wilczynska et al. 1974), but we have set the excitation energy of the 2^+ state in ^{24}Mg to zero so that the reaction is truly adiabatic. (Note that in all our ^{16}O-^{24}Mg calculations the very small internal-wave contribution has been removed using the techniques of Chapter V.) We see that for the nearside amplitudes the curves are virtually identical, but the farside ones are rather different, especially for scattering angles much bigger than the grazing angle ($\Theta \approx 35°$). For a smaller grazing angle the QC-approximation would work better for the farside amplitude. We also see that in the case of the full amplitude the different magnetic substate cross sections are in phase with each other at smaller scattering angles, as expected from eq.(2.7).

It is interesting to calculate the QC-amplitudes in a different set of axes, where we quantize along the recoil direction defined by $\underline{q}=\underline{\kappa}-\underline{\kappa}'$, fig.(IV-3). In this co-ordinate system only states with even M can be populated (eq.(II.5.8)) and the amplitudes, $\bar{f}_{IM}(\theta)$, are related to those in the DWBA axes by (Brink and Satchler 1968)

$$\bar{f}_{IM}(\theta) = \sum_{M'} D^{I\,*}_{M'M}(0,-\pi/2+\theta/2,0)\; f_{IM'}(\theta) \qquad (2.11)$$

and so, using eq.(2.7),

$$\bar{f}_{IM}(\theta) = \pi^{-1}\left[\frac{\Lambda}{2\sin\theta}\right]^{1/2}\{\Omega_0(\Theta-\theta) - i(-1)^I\,\Omega_0(\Theta+\theta)\}\,\delta_{M0} \qquad (2.12)$$

and so only the M=0 substate is populated. Consequently, when this state decays the photon cannot propagate along the quantization axis and this direction is therefore an axis of symmetry for the γ-ray distribution. These distributions will be discussed further in Chapter VI.

3. Approximations to the radial integrals.

In their attempt to explain similarities between elastic and inelastic scattering cross sections, Austern and Blair (1965) found that

$$R_{\ell\ell} \propto \frac{\partial S^N(\ell)}{\partial \ell} \qquad (3.1a)$$

and assumed that

$$\langle \ell' I 00 | \ell 0 \rangle R_{\ell'\ell} \approx \langle \ell' I 00 | \ell 0 \rangle R_{t\,t} \propto \langle \ell' I 00 | \ell 0 \rangle \frac{\partial S^N(t)}{\partial t} \qquad (3.1b)$$

where $S^N(t)$ is the nuclear S matrix for elastic scattering evaluated at the entrance channel energy and $2t=\ell+\ell'$. The Clebsch-Gordan coefficients in this expression are rather important since if they are omitted two radial integrals with the same ℓ but wildly different ℓ' are apparently equal, which is not at all realistic. The approximation (3.1) is equivalent to taking only the first term of eq.(2.2) and gives an amplitude proportional to $Y_{IM}(\pi/2,0)$: it does not, therefore, allow for the population of states with I+M odd, in disagreement with the DWBA. However, the total inelastic cross section $\sigma_I(\theta)$ is identical to that obtained in the QC-limit (eq.(2.9)).

A significant improvement to eq.(3.1) has been given by Hahne (1967)

$$R_{\ell'\ell}(K',K) \approx \{ R_{\ell'\ell'}(K',K') R_{\ell\ell}(K,K) \}^{1/2} \propto \left[\frac{\partial S^N(\ell',K')}{\partial \ell'} \frac{\partial S^N(\ell,K)}{\partial \ell} \right]^{1/2} \qquad (3.2)$$

where we have explicitly indicated the energy dependence of the radial integrals and S matrices since they are to be evaluated in different channels. This approximation has been extensively tested (Hahne 1967, Gelbke and Cramer 1976) by comparison with both the Austern and Blair result (eq.(3.1)) and the exact DWBA radial integral. These tests show that (3.2) is a good approximation, especially under strong absorption conditions.

Hahne obtained eq.(3.2) by assuming a δ-function form for the transition potential, but, as can be seen from fig.(IV-2), the integrand of the radial integral is not strongly localized at the surface. Indeed, as was noted by Rost (1962), the surface region is quite wide and the replacement of the radial integral by the integrand at some 'surface' radius depends quite critically on the radius chosen. The exact radial integral is, in fact, more localized in ℓ-space than the ones obtained with either the δ-function approximation or a plane-wave approximation to the wavefunctions (Rost 1962). Austern (1961) pointed out that the localization of a reaction in ℓ-space is somewhat different to localization in co-ordinate space, mainly due to phase averaging for low partial waves.

In the QC-limit $\kappa=\kappa'$ and Hahne's approximation is then equivalent to eq.(2.2), but we stress that the latter was derived without assuming a δ-function interaction.

In fig.(IV-2b) we see that the effect of changing the energy is to shift the wavefunction in ℓ-space so that

$$u_\ell^b(r,\kappa) \approx u_{\ell-\zeta}^b(r,\kappa') \qquad (3.3)$$

The quantity ζ, which we call the 'mismatch', is greater than or equal to zero since $\kappa' \leq \kappa$. Calculations of the radial wavefunction for a wide range of energies and radii show that

the mismatch is actually a function of radius, but since $R_{\ell'\ell}$ is an integral over radius of the wavefunction weighted by a form factor, it is sensible to define the mismatch as $\zeta(r_0)$, r_0 being the radius for which the form factor is a maximum.

The approximation (3.3) can be arrived at using a semi-classical analysis of the wavefunction in the limit of strong absorption. The barrier wavefunction is a function of the quantity $\epsilon \equiv -iS_{21}/\pi$ (Brink and Takigawa 1977, Lee et al. 1978), known as the barrier penetrability, where S_{21} is the action integral

$$S_{21} \propto \int_{r_1}^{r_2} (E - V(r))^{1/2} \, dr \qquad (3.4)$$

Here r_1 and r_2 are the integrands' (complex) turning points that lie in the barrier region of the potential and V(r) is the total (complex) potential. By approximating the potential in the vicinity of the barrier by a parabola, it is straightforward to show that ϵ is, for ℓ not too different from L, a function of ℓ-L only, L being the grazing angular momentum. The initial and final wavefunctions are then functions of ℓ-L and ℓ'-L' respectively, where L' is the grazing angular momentum evaluated in the exit channel. Eq.(3.3) then follows immediately and our choice of the mismatch as $\zeta(r_0)$ is further justified.

If the mismatch is large, then the wavefunctions in the initial and final channels will not overlap very much and the corresponding radial integral will be small. This can be seen from eq.(3.2), and Austern (1970) has shown that the largest overlap occurs for $\ell-\ell' \approx \zeta$, which will be important subsequently. The value of the mismatch can be estimated from the expressions for L and L' (Nörenberg and Weidenmüller 1980), e.g.

$$L^2 = 2\mu r_0^2 \, (E - V(r_0)) \qquad (3.5)$$

Here μ is the reduced mass, $V(r_0)$ the barrier height and E the centre of mass energy. Consequently

$$\frac{\zeta}{L} \approx \frac{E^*}{2(E - V(r_0))} \qquad (3.6)$$

where E^* is the excitation energy. For the sort of reactions we are considering the mismatch is of order unity.

Similar ideas to those presented above have recently been published by Chu and Lee (1981).

In parametrizing the radial integral our guide is eq.(3.2). We can allow for a real nuclear phase shift by letting the grazing angular momentum become complex (Ericson 1966), with an imaginary part restricted by unitarity and the requirement that the nuclear deflection function is negative (Rowley and Marty 1976). To quantify the above arguments we use the well-known parametrization of the elastic S matrix introduced by McIntyre et al. (1960), with which $-\pi\Delta/2 \leqslant \mathrm{Im}(L,L') \leqslant 0$. Using Hahne's approximation this gives

$$R_{\ell'\ell} \propto \mathrm{sech}\left[\frac{\ell'-L'}{2\Delta}\right] \mathrm{sech}\left[\frac{\ell - L}{2\Delta}\right] \qquad (3.7)$$

where we have assumed for simplicity that Δ, the width of the S matrix, is the same in both channels. In reality, Δ is a slowly varying function of energy which can be estimated from the diffuseness of the nuclear surface or from barrier penetration arguments (Brink and Takigawa 1977, Lee et al. 1978). The mean of L and L' is essentially Λ, the parameter introduced earlier. The parametrization (3.7) is useful since the transforms of eq.(1.16) are easily evaluated using contour techniques with the result that

$$\Omega_k(\omega) \propto \frac{\sin[\omega(\zeta-k)/2] \quad \exp[i\omega(L+L'+1)/2]}{\sinh[(\zeta-k)/(2\Delta)] \sinh(\pi\omega\Delta)} \qquad (3.8)$$

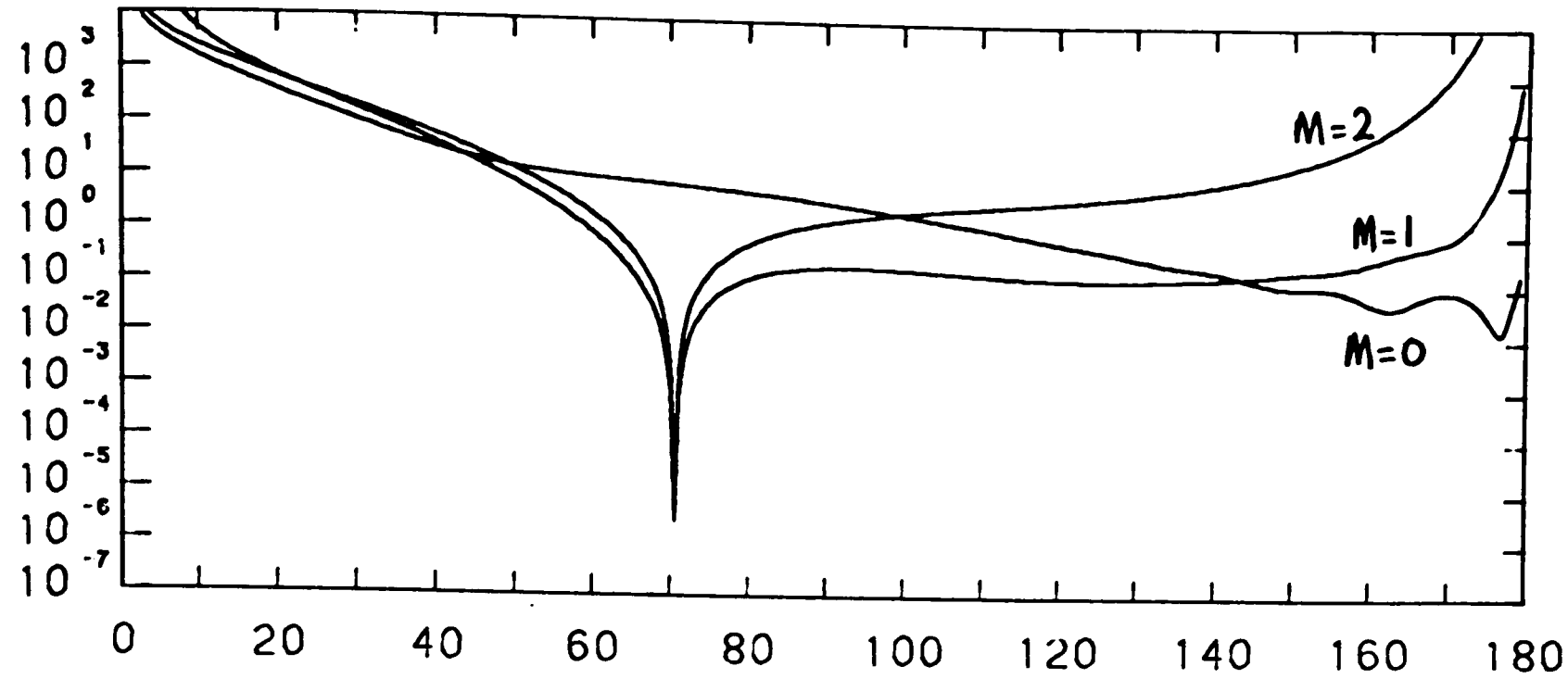
If $|\omega(\zeta+I)| \ll 2$ and $\zeta+I \ll 2\Delta$ then

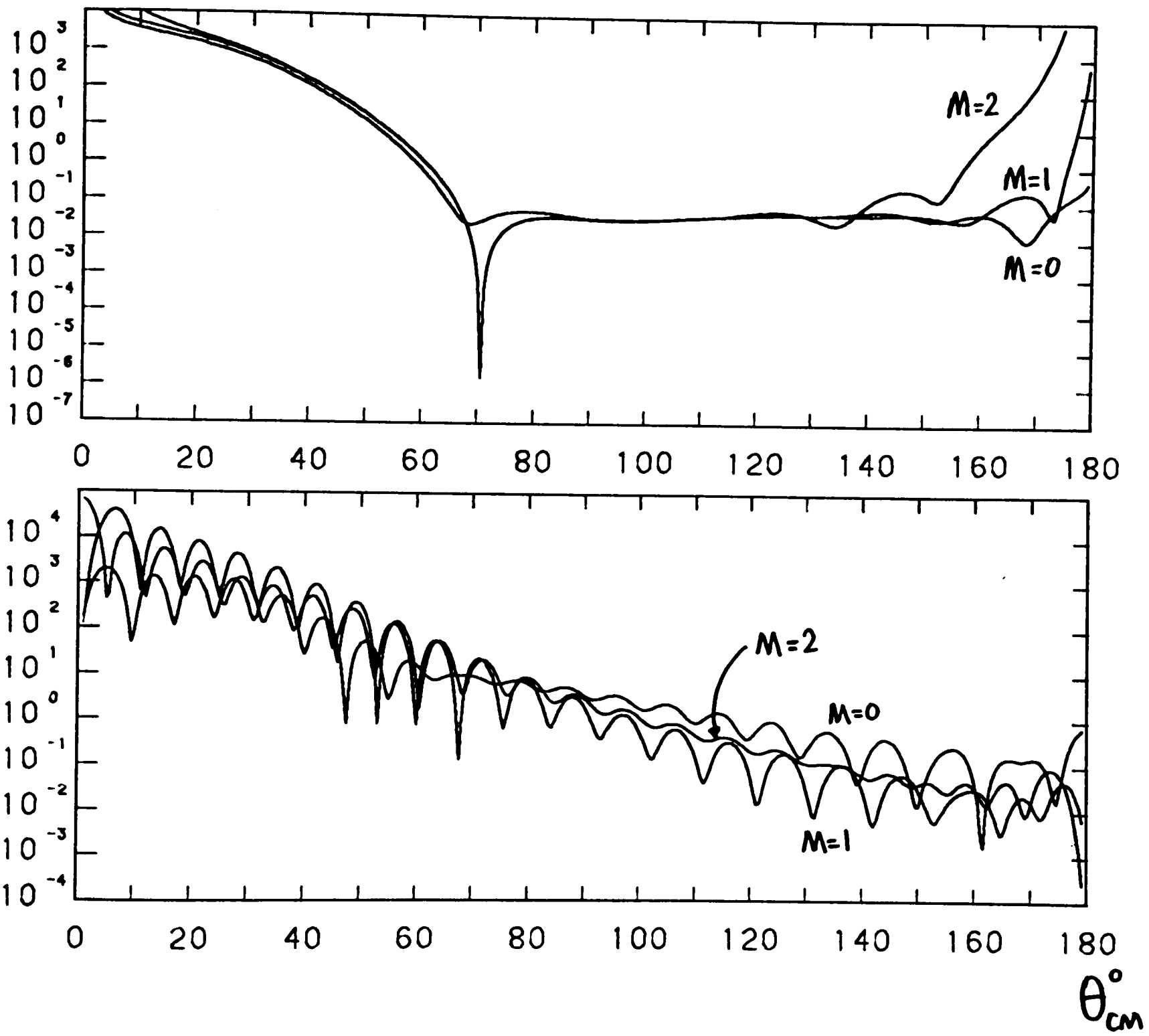
$$\Omega_k(\omega) \approx \frac{\omega\Delta \exp[i\omega(L+L'+1)/2]}{\sinh(\pi\omega\Delta)}$$

$$\approx \Omega_0(\omega) \qquad (3.9)$$

This is identical to eq.(2.4) and so we amend our definition of the QC-limit to take into account the mismatched situation. Since the number of contributing partial waves must be large, and ω be small, for the above conditions to hold, the reason for 'quasi-classical' as the name for this approximation is confirmed, but the conditions under which the limit holds are now more severe. In fig.(IV-5) we repeat the plots of fig.(IV-4), but with the natural excitation energy of 1.37 MeV ($\zeta \approx 0.9$): the 'reduced' amplitudes are now rather different. Again, however,the nearside amplitudes are nearer to satisfying the QC-limit than the farside ones. Since $\Delta \approx 1$ (section 6.1), the condition $\zeta+I \ll 2\Delta$ is not well satisfied for the adiabatic amplitudes, and even less so for the non-adiabatic ones, since $\zeta \approx 0.9$ for this reaction.

Other parametrizations of the DWBA radial integrals are, of course, possible: in particular a Gaussian dependence on angular momentum has been used extensively for transfer reactions, even with full finite-range effects (Strutinskii 1964, Kahana and Baltz 1977 and references therein, Bond 1980, Udagawa et al. 1980). One problem with the Gaussian parametrization is that a real nuclear phase cannot be introduced by making the grazing angular momentum complex since this results in an unrealistic





(FIG. IV-5): Same as fig.(IV-4) but with an excitation energy of 1.37 MeV.

deflection function. Consequently this phase must be dealt with separately, which means that analytical results are somewhat harder to obtain. The radial integrals for DWBA transfer reactions have also been considered by Hasan and Brink (1978).

4. An analysis of inelastic cross sections.

In the adiabatic limit we have seen (eq.(2.7)) that the cross sections for excitation of the different states $|IM\rangle$ are in phase, and this is borne out by DWBA calculations (fig.(IV-4)). When the reaction is no longer adiabatic, and if the stringent QC-conditions are not well satisfied, then we would not expect the differential cross sections to be in phase because of the now complicated sum of eq.(1.18). This is observed in fig.(IV-5). The Blair phase rule (Blair 1959, Austern and Blair 1965), however, does not refer to the magnetic-substate cross sections, but to the <u>total</u> inelastic cross section, and relates this to the total cross section in other (elastic or inelastic) channels. To calculate the total inelastic cross section it is actually more convenient to use the so-called Basel axes, defined in Chapter II, rather than the DWBA axes used thus far. The scattering amplitudes in this new co-ordinate system, $F_{IM}(\theta)$, are related to the ones in the DWBA axes by (Brink and Satchler 1968)

$$F_{IM}(\theta) = \sum_{M'} D^{I\,*}_{M'M}(-\pi/2,-\pi/2,0)\; f_{IM'}(\theta) \qquad (4.1)$$

with θ defined as before. Using eq.(1.18) this rotation is

easily performed giving

$$F_{IM}(\theta) = \left[\frac{2\Lambda}{(2I+1)\pi\sin\theta}\right]^{1/2} \{\Omega_{-M}(\Theta-\theta) - i(-1)^I \Omega_M(\Theta+\theta)\} Y_{IM}^*(\pi/2, \pi/2+\theta/2) \qquad (4.2)$$

The sum over k in eq.(1.18) has now collapsed giving a particularly simple form for the amplitude in these axes. The spherical harmonic in the above equation is indicative of Bohr's (1959) symmetry theorem. Note that eq.(4.2) is exact within our semi-quantal approximations, whereas eq.(2.7), which has a similar form, but for different axes, is only valid within the QC-limit.

Using eq.(4.2) the inelastic cross section becomes

$$\sigma_I(\theta) \propto \sum_M |d^I_{M0}(\pi/2)|^2 \, |\Omega_{-M}(\Theta-\theta) - i(-1)^I \Omega_M(\Theta+\theta)|^2 \qquad (4.3)$$

and since the phase of the transforms Ω_K is virtually k-independent (appendix F), we see that in these axes the magnetic-substate cross sections are in phase with each other, irrespective of whether the reaction is adiabatic or not. If the two terms $\Omega_{-M}(\Theta-\theta)$ and $\Omega_M(\Theta+\theta)$ are not too different in magnitude there can be strong interference between them, and hence oscillations in the angular distribution. The phases of these two components are essentially $\pm(L-\zeta/2)$ (appendix F) and so we expect oscillations of period $\pi/(L-\zeta/2)$: these have been observed many times.

There are two circumstances under which the Blair rule will not hold: if either the oscillations are at variance with the rule or there are no oscillations to which the rule can be applied. We have just seen that if there is structure in the cross section its period is ζ-dependent and so two reactions

having different mismatches will have different periods. Consequently, as the scattering angle increases the two cross sections will break the Blair rule to an increasing extent. It is, of course, possible for two inelastic channels to obey the Blair rule quite accurately if their relative mismatch is small, i.e. the two states have similar excitation energies.

The presence, or otherwise, of structure in a cross section depends on three quantities: the mismatch parameter, the grazing angle and the size of the real nuclear phase, since it is these parameters that determine the relative magnitude of the terms in eq.(4.2). If either the nearside or the farside amplitude dominates then there cannot be strong interference between these components and the cross section will be smooth. The nearside amplitude dominates if the nuclear attraction is weak relative to the Coulomb repulsion as measured by Θ. This can be seen from the parametrized radial integral, which gave eqn.(3.8) for $\Omega_k(\omega)$, since if L and L' are essentially real (weak nuclear attraction) the $\sinh(\pi\omega\Delta)$ term is the only one that has a strong ω-dependence and so determines which of the NF amplitudes is dominant. For large Θ, and not too small θ, $|\Theta-\theta| \ll |\Theta+\theta|$ and consequently the nearside amplitude is the greater. This amplitude will be particularly dominant for scattering angles close to Θ, if Θ is large. However, if the nuclear attraction is strong, L and L' have appreciable imaginary parts and so the farside component can dominate since the numerator of eq.(3.8) varies as

$$\exp[\ (\Theta \pm \theta)L_I\ ] \qquad (4.4)$$

where we have written $L=L_R-iL_I$. As L_I cannot be negative, the effect of the real nuclear phase is to damp the nearside amplitude with respect to the farside one as a function of

scattering angle. So, providing that the nuclear attraction is not too weak, the cross section at larger angles will be dominated by the farside amplitude.

The extent to which a reaction is mismatched also determines the amount of structure in the angular distribution since this affects the relative weighting of the nearside and farside amplitudes. Austern's (1970) argument shows that the preferentially populated substates have $M = \mp\zeta$ for the nearside and farside respectively, and so if the mismatch is large those substates with $\pm M$ very different from ζ will have rather smooth cross sections, due to the predominance of one of the NF amplitudes, and so the structure in the total inelastic angular distribution will also be washed out.

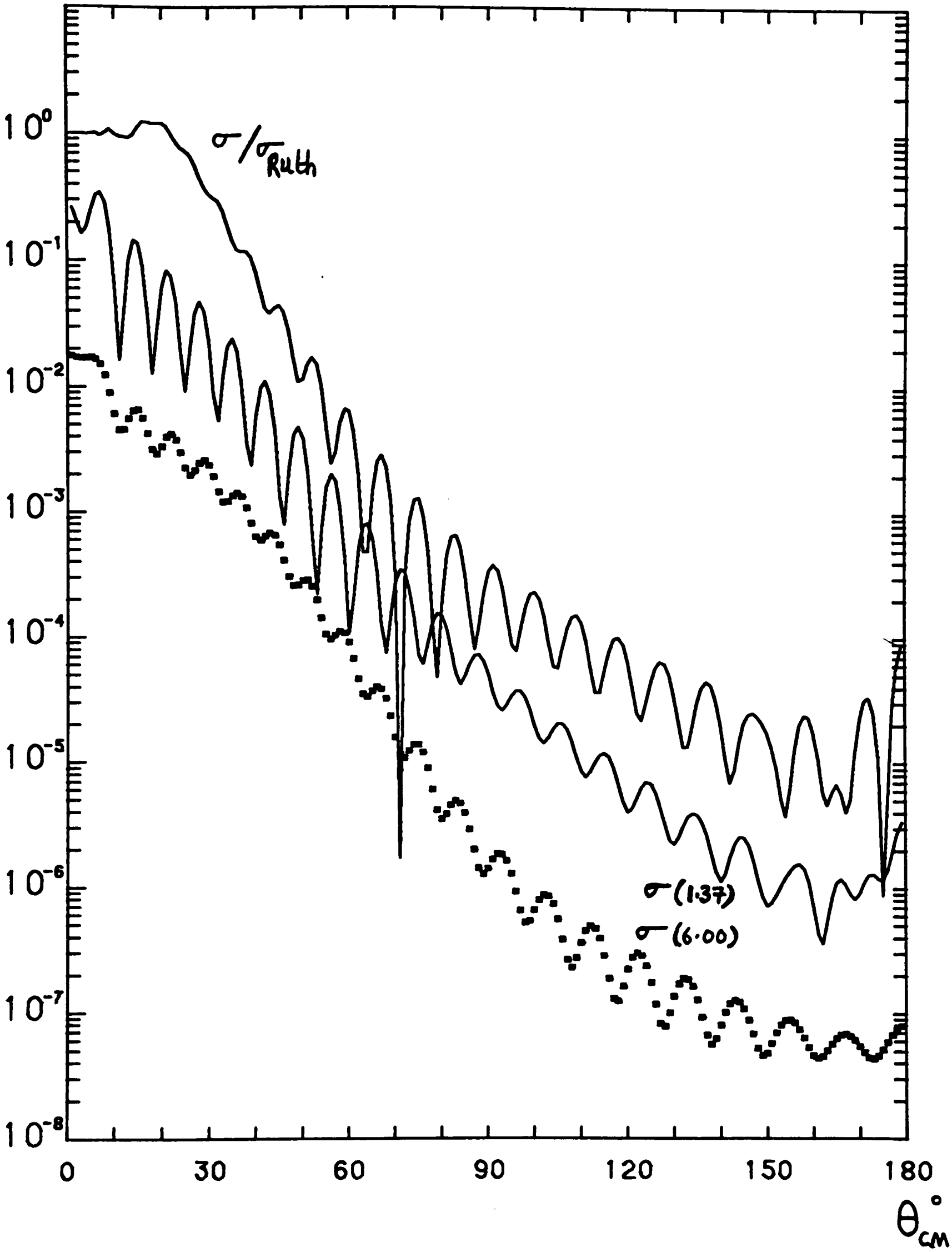
So far we have not really considered the part of the Blair rule which relates the inelastic and elastic cross sections. As we found in the previous chapter, the oscillations in the elastic cross section forward of the grazing angle can be explained as interference between the (predominantly nearside) Rutherford amplitude and the nearside nuclear amplitude. This gives oscillations with a θ-dependent period, but at larger scattering angles ($\theta > \Theta$) 'Fraunhofer' oscillations sometimes re-appear. These have a constant period, and so can be compared more easily with the oscillations in the inelastic cross section. The inelastic amplitudes are calculated by summing over a quantity, the radial integral, which behaves very like the derivative of the elastic S-matrix. Therefore, use of the Poisson sum formula and an integration by parts changes the radial integral into something similar to the S matrix itself, and the cosine that approximates the spherical harmonic into a sine. Because only magnetic substates with even/odd M contribute

to the full amplitude for even/odd I, the oscillations in the inelastic cross sections for these spins are out of/in phase with the elastic angular distribution. This is the Blair rule.

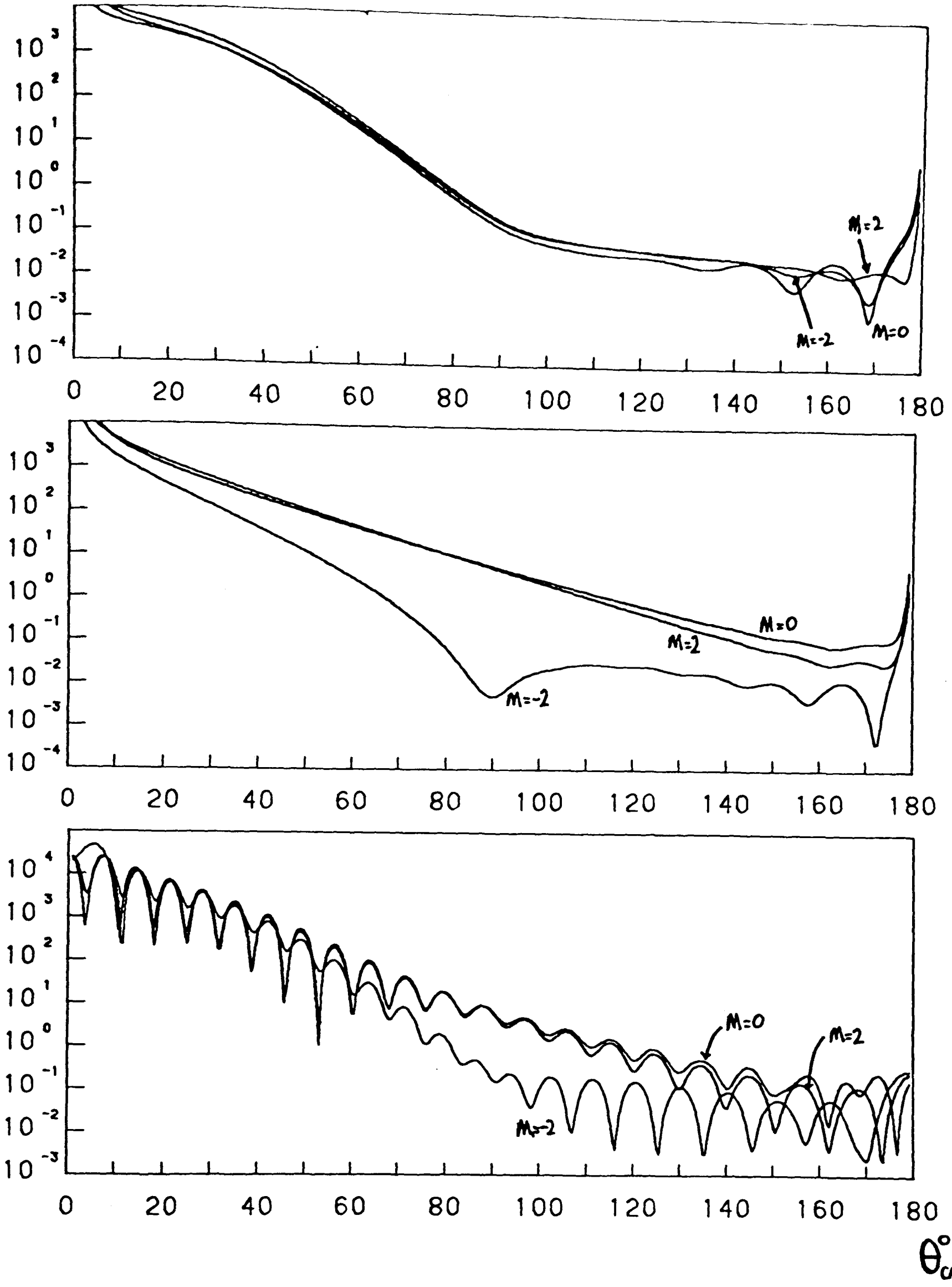
In figs.(IV-6), (IV-7) and (IV-8) we illustrate the points made in this section. Fig.(IV-6) compares the elastic and inelastic cross sections for the ^{16}O-^{24}Mg system introduced earlier. We show the latter for two excitation energies, the true value (1.37 MeV) and an artificially high value of 6 MeV, for which the mismatch parameters are about 0.9 and 3.8 respectively. The same transition potential was used in each case. The elastic and 'true' inelastic distributions are out of phase for most of the angular range as the Blair rule would predict for this I=2 excitation. However, for the artificial excitation energy the Blair rule is clearly violated, especially towards forward angles where the two inelastic distributions are almost out of phase. We also see in this figure the difference in form of the elastic cross section for $\theta \gtrsim \Theta \approx 35°$, and that the Blair rule is only applicable in the region of the 'shadow', as expected.

In fig.(IV-7) we show the magnetic-substate scattering amplitudes evaluated in the Basel axes for the same reaction. The top part of the figure shows the nearside amplitudes, which vary smoothly over virtually the whole angular range; this being the essence of the NF decomposition.

We stress that this figure was produced using radial integrals calculated with DWUCK, but some features, such as the dip in $|F_{2,-2}^{+}|$ close to 90° can be explained within the parametrization of eqs.(3.7) and (3.8). This dip is due to the sine part of the parametrized transform being zero at about 90°



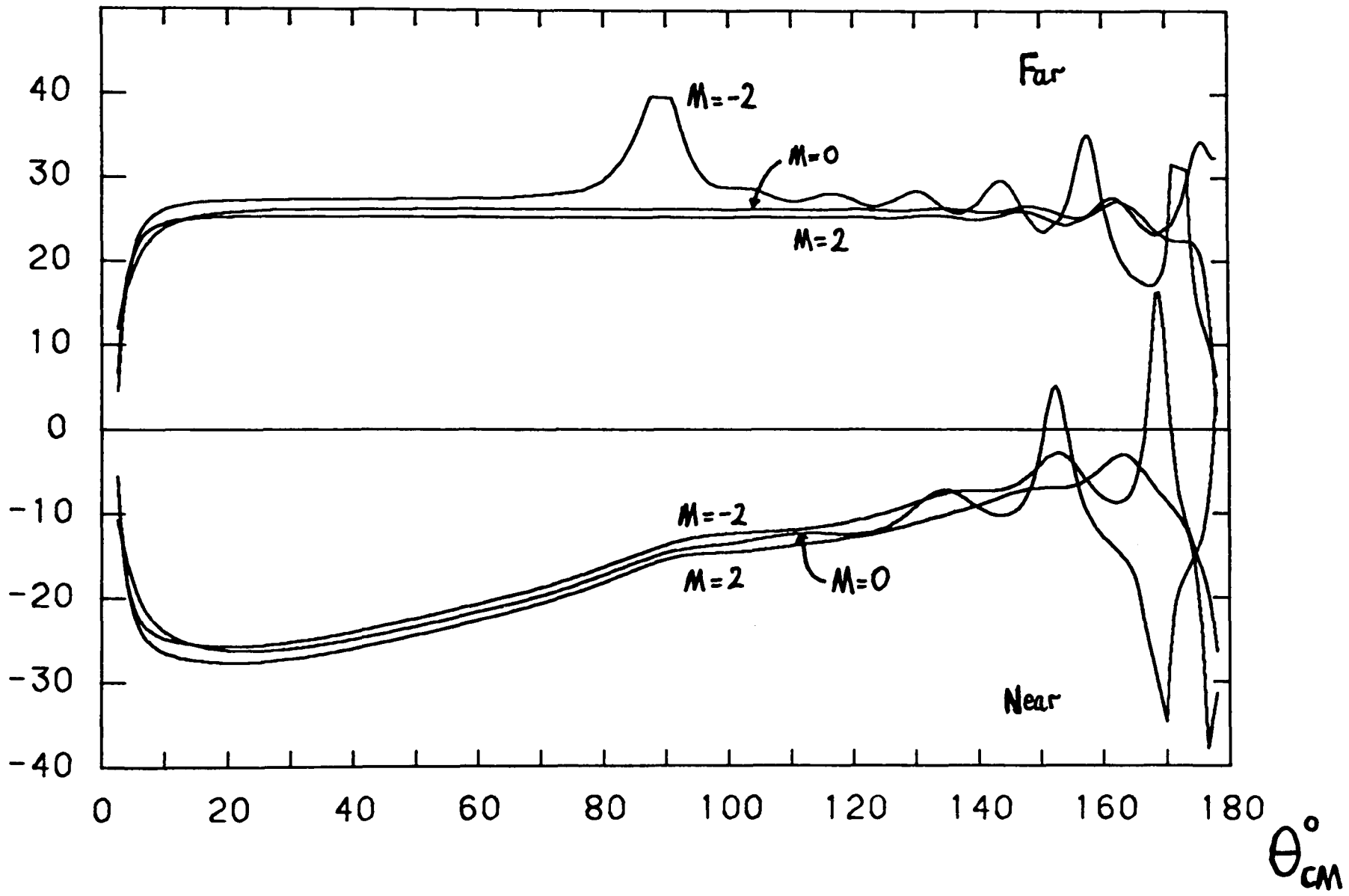
(FIG. IV-6): Elastic and inelastic cross sections for ^{16}O-^{24}Mg. The elastic is expressed as a ratio to the Rutherford value, whilst the 1.37 MeV (solid) and 6.00 MeV (dotted) inelastic cross sections are in arbitrary units.



(FIG. IV-7): Inelastic cross sections in Basel axes for ^{16}O-^{24}Mg. Square modulus of F_{2M^-} (top), F_{2M^+} (middle) and F_{2M} (bottom). The ordinate scale is the same in each case, but is expressed in arbitrary units.

for the parameters appropriate to this reaction: there is no such zero for any of the other transforms, and so we would not expect to observe such dips in any of the other amplitudes.

The full amplitudes $|F_{IM}(\theta)|$ are in phase at forward angles as expected, but the oscillations are damped towards 90° as the relative magnitude of the nearside and farside contributions changes. Beyond 90° the M=-2 amplitude is out of phase with the M=0 and M=2 amplitudes and, what is more, the period of the oscillations is somewhat different to that seen at forward angles (about 10° as opposed to about 7°). These two features have a common explanation in terms of the local angular momentum, which we calculated numerically using a five-point formula and show in fig.(IV-8). We see that for each magnetic substate the LAM varies fairly smoothly, except for that of $F_{2,-2}^{+}$ which deviates strongly near to 90°. This effect is related to the above-mentioned zero in the parametrized radial integral, since at this point it is no longer a good approximation to neglect the n≠0 terms in the Poisson sum (eq.(1.15)). Whilst all these terms have the same LAM, that of the total amplitude depends on how the different terms in eq.(1.15) are weighted. Once the n=0 term is dominant again the LAM returns to its original value. The amplitude calculated numerically using the partial-wave sum (eq.(1.6)) and the parametrization eq.(3.7) (with suitable values for the parameters) agrees with the analytic expressions of eq.(3.8) and (4.2) except for $F_{2,-2}^{+}$ around 90° and for all the farside amplitudes at angles beyond about 150°. The latter deviation is the effect of the n≠0 Poisson terms, whilst the former illustrates the importance of the terms neglected in eq.(1.15) and also the breakdown of our other



(FIG. IV-8): The local angular momentum for ^{16}O-^{24}Mg. (Top) the farside LAM; (bottom) the nearside LAM. Ordinate scale is angular momentum in units of $\hbar$.

assumptions

Since the phase of $F_{2,-2}^{+}$ is varying more rapidly than that of the other magnetic substates between about 80 and 100°, at angles beyond this region the former has a rather different phase relative to the latter, and so the $\underset{\sim}{M}$=-2 cross section becomes out of phase with the other substate cross sections. (In terms of the parametrization the sign of the sine term changes as we pass the zero, corresponding to a phase change of π).

From fig.(IV-8) we also see that the LAM of the farside amplitudes is constant at around 25, apart from the deviation discussed above. The LAM of the nearside amplitudes, however, varies from about -25 at forward angles to about -10 close to 140°. When these two amplitudes interfere we therefore expect oscillations with period depending on the scattering angle. Using these figures we expect the period to be about 7° and 10° for small and large θ respectively, in complete agreement with fig.(IV-7).

The behaviour of the farside LAM shows that this amplitude is purely diffractive, the large angle deviations from the constant value being due to the presence of low partial waves (see below), and n≠0 Poisson sum terms. The nearside LAM varies much more with angle which is due to several effects: the partly refractive nature of the amplitude, the presence of low partial waves, and the contribution of the n≠0 terms (n=1 in particular) to the Poisson sum. Simple considerations suggest that the last of these will be important above about 140° for this reaction. The refractive part of the amplitude can be obtained from an expansion of the Coulomb phases to higher than the linear term, which may give a point of stationary phase in the amplitude. However, for this particular process, (and 'light heavy-ions' in

general), comparison of the LAM for linear and quadratic expansions of these phases shows that the point of stationary phase has a small effect on the LAM and so our retention of only the linear term in eq.(1.11) is justified. The main part of the deviation is due, therefore, to the low partial waves. This is not an internal wave effect, but is due to the reflection of small ℓ partial waves from the barrier. In this respect our 'cosh' parametrization breaks down, since if L,L' and Δ are chosen to fit the DWBA radial integral close to the grazing angular momentum we underestimate the true radial integral by several orders of magnitude for $\ell,\ell' \ll L,L'$. Other parametrizations might be better in this respect.

The three figures just discussed show that the nearside-farside decomposition is useful for describing inelastic scattering data, especially when used in conjunction with the local angular momentum. We will show in Chapter VI that these ideas are also useful when considering vector polarization and angular correlation data.

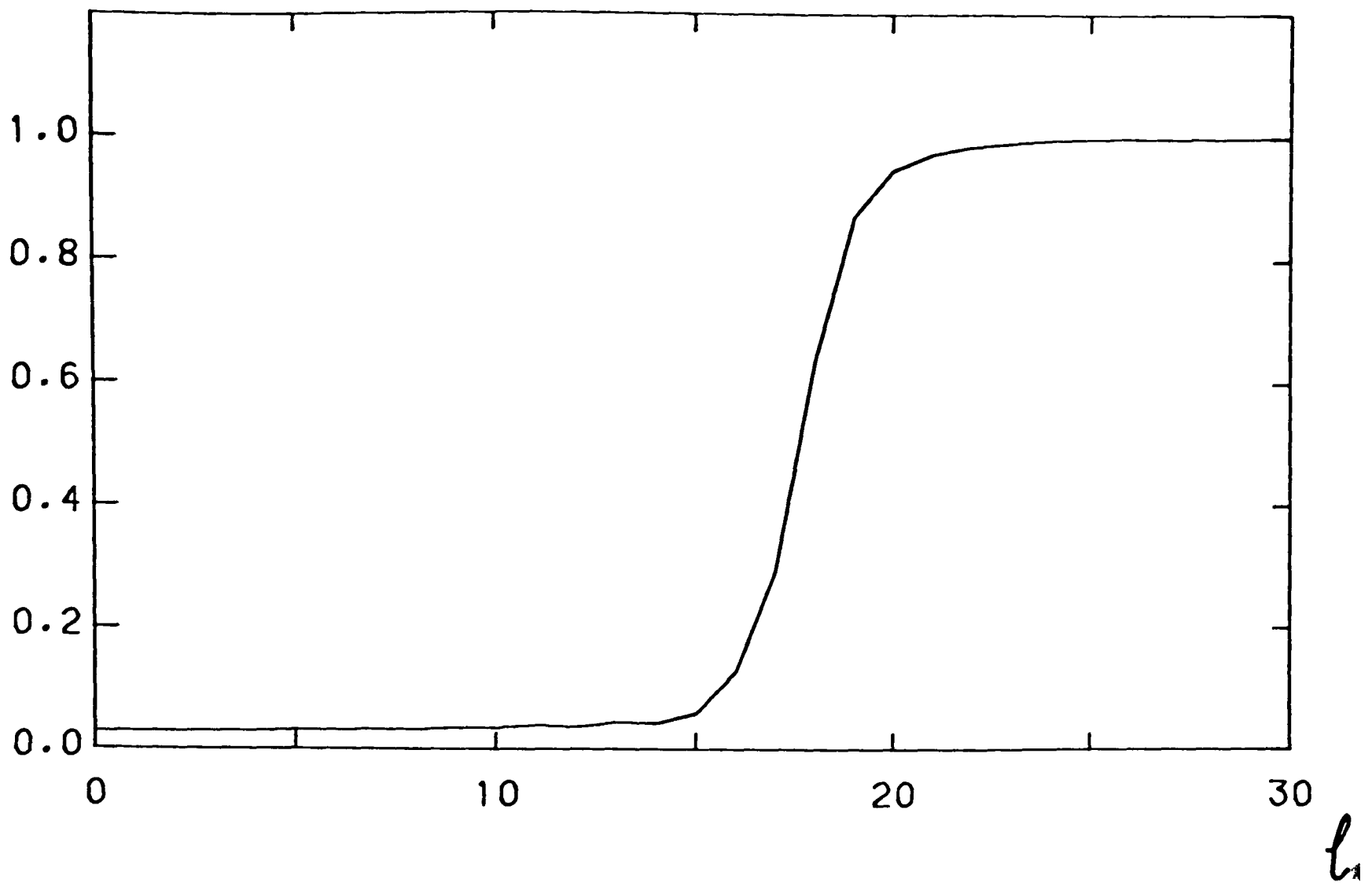
CHAPTER V.

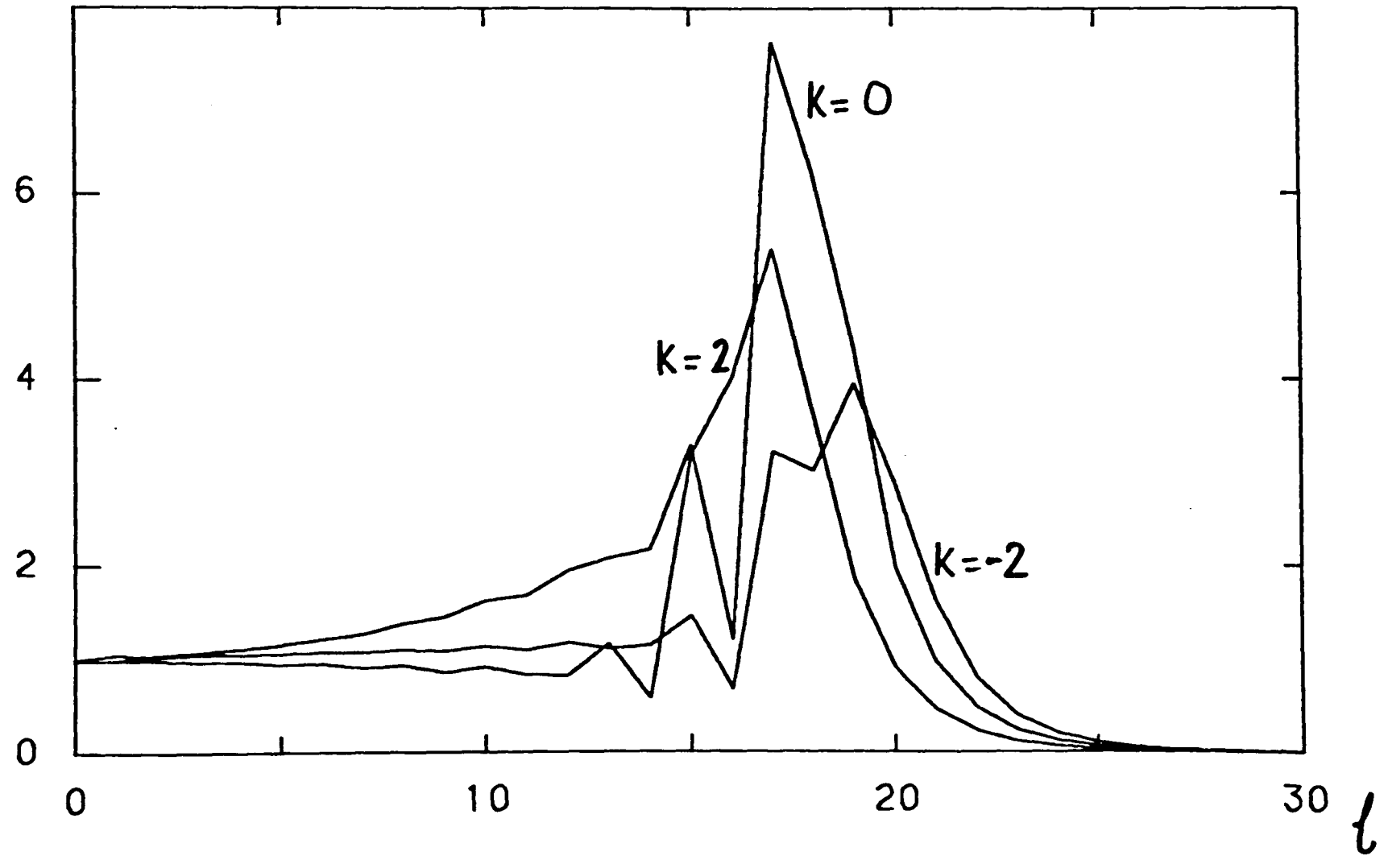
THE NEARSIDE-FARSIDE DECOMPOSITION FOR THE INTERNAL WAVE AND COULOMB EXCITATION.

1. The internal wave.

It is well known (Chapter I) that incomplete absorption of partial waves with low angular momentum can affect elastic cross sections substantially, particularly at larger scattering angles. Our aim here is to show that inelastic cross sections can be similarly affected, and to interpret the structure therein using the nearside-farside technique that proved useful under strong absorption conditions.

In the previous chapter we presented an argument to explain the small magnitude of the DWBA radial integral for low angular momenta and the same argument can be used to show that for weaker absorption the radial integral is larger in this region of ℓ-space. In these circumstances the initial and final channel wavefunctions are no longer purely ingoing waves and so the integrand of $R_{\ell'\ell}$ oscillates much less rapidly than before, giving a non-negligible result. This is illustrated in fig.(V-1) where we show the magnitude of the elastic S-matrix and that of the radial integral for $\ell-\ell'=0,\pm2$. These quantities correspond to α-^{50}Ti scattering at 45 MeV and were calculated with the potential of Gubler et al. (1981) (Table IV-1). Although $|S_\ell|$ is only about 0.028 for low ℓ, the presence of these partial waves does drastically alter the elastic cross section. (We note, however, that 'anomalous large-angle scattering' has not been





(FIG. V-1): (Upper) $|S_\ell|$ for α-^{50}Ti (see text); (lower) $|R_{\ell'\ell}|$ in arbitrary units, with $\ell'=\ell-k$, for the same reaction (see text).

observed for this system (Gubler et al. 1981) as it has for, say, α-^{40}Ca at 29 MeV where $|S_\ell|$ is about 0.07 for low partial waves (Brink and Takigawa 1977)). The figure shows that $|S_\ell|$ is constant for low ℓ, as one would expect from the form of the real potential, which is deep. 'Transit times' for the various partial waves are therefore roughly equal and this implies a constant magnitude S-matrix (Lee et al. 1978). For a shallower real potential the internal S-matrix is expected to peak near the grazing angular momentum for the barrier wave, a behaviour which occurs for the (small) internal wave in ^{16}O-^{24}Mg scattering.

The lower part of fig.(V-1) shows that the radial integral for this reaction has a very different form from that associated with strong absorption, and so rather different substate cross sections are to be expected in the two cases. Note, however, that the dominant value of $k=\ell-\ell'$ is zero, as mismatch arguments suggest since $\zeta \approx 0.4$.

We argued in Chapter I that the distorted wavefunction in both the entrance and exit channels, and hence the elastic S-matrix, could be split into 'barrier' and 'internal' parts, where the latter is only large for small angular momenta. In a similar way we can write the DWBA radial integral as

$$R_{\ell'\ell} = R^{b}_{\ell'\ell} + R^{i}_{\ell'\ell} \qquad (1.1)$$

where the barrier part is defined wholly in terms of barrier wavefunctions by eq.(IV.1.5) and the internal part consists of three terms, two involving products of barrier and internal wavefunctions and the third simply involving internal wavefunctions. Since the elastic S-matrix is rather small for low ℓ, we expect this third term to be negligible. The barrier

radial integral has all the properties associated with strong absorption (Chapter IV). It is here that a major difference occurs between elastic and inelastic scattering in the presence of an internal wave. In the elastic case the internal contribution is given simply by the asymptotic form of the internal wave, but for inelastic scattering this contribution arises from the overlap of the internal wave with a transition potential and a barrier wave. The radial integral wave cannot therefore be wholly separated into contributions from an internal wave and a barrier wave. In spite of this we will continue to call R^i the 'internal radial integral'.

Of course an optical-model code does not produce separate 'internal' and 'barrier' components of the wavefunctions or the S matrix and so these contributions must either be calculated semi-classically (Brink and Takigawa 1977, Lee and Takigawa 1978, Lee et al. 1978), or by using the optical-model code in a modified fashion. The advantage of the latter method is that the potential need not be available in an analytical form, which is a major drawback of the semi-classical treatment.

To extract the internal and barrier components of the radial integral we extend the method of Rowley et al. (1977) to inelastic scattering. (A similar but slightly more involved version of Rowley's technique has been suggested by Albinski and Michel (1982) and used for both light- and heavy-ion elastic scattering.) The essentials of Rowley's method are as follows: first a calculation is made with the full potential, which is weakly to moderately absorbing, and then the calculation is repeated with an additional (imaginary) potential introduced to absorb that part of the wavefunction which penetrates the barrier. This extra potential must be chosen with some care,

although the validity of the method relies on it being non-unique. Its radius must be smaller than that of the imaginary potential already present and its diffuseness must not be too small to prevent extra reflections being produced. The second calculation produces the barrier component and on subtraction from the results of the first we obtain the internal component: this usually agrees with that obtained semi-classically. The agreement is better if the technique is modified (Albinski and Michel 1982), but in this case several (three or four) optical-model calculations are required. It should be possible to extend this newer method to inelastic scattering, but we have not done so because of the greater simplicity of the former method. Both methods assume that the barrier wave is independent of the extra absorption.

We therefore write

$$R^{b}_{\ell'\ell} \equiv R_{\ell'\ell}(W_1) \qquad (1.2)$$

$$R^{i}_{\ell'\ell} \equiv R_{\ell'\ell}(W_1=0) - R_{\ell'\ell}(W_1) \qquad (1.3)$$

where W_1 is the additional imaginary potential. This method has also been used by Chu and Lee (1981) and it should be a good approximation provided that the absorption is not too weak.

Using a semi-classical argument, Lee (1977) showed that for spin-zero excitations the internal part of the radial integral behaves essentially like the internal wave elastic S-matrix, and so if the internal wave is dominant at backward angles we expect the elastic cross section, in contrast to the Blair rule, to be <u>in phase</u> with a $0^+ \rightarrow 0^+$ inelastic distribution. This has been observed for α-^{40}Ca scattering by Michel and Vanderpoorten (1977), whose forward angle data is in keeping with the Blair

rule. For a general spin excitation, similar semi-classical arguments show that (Chu and Lee 1981)

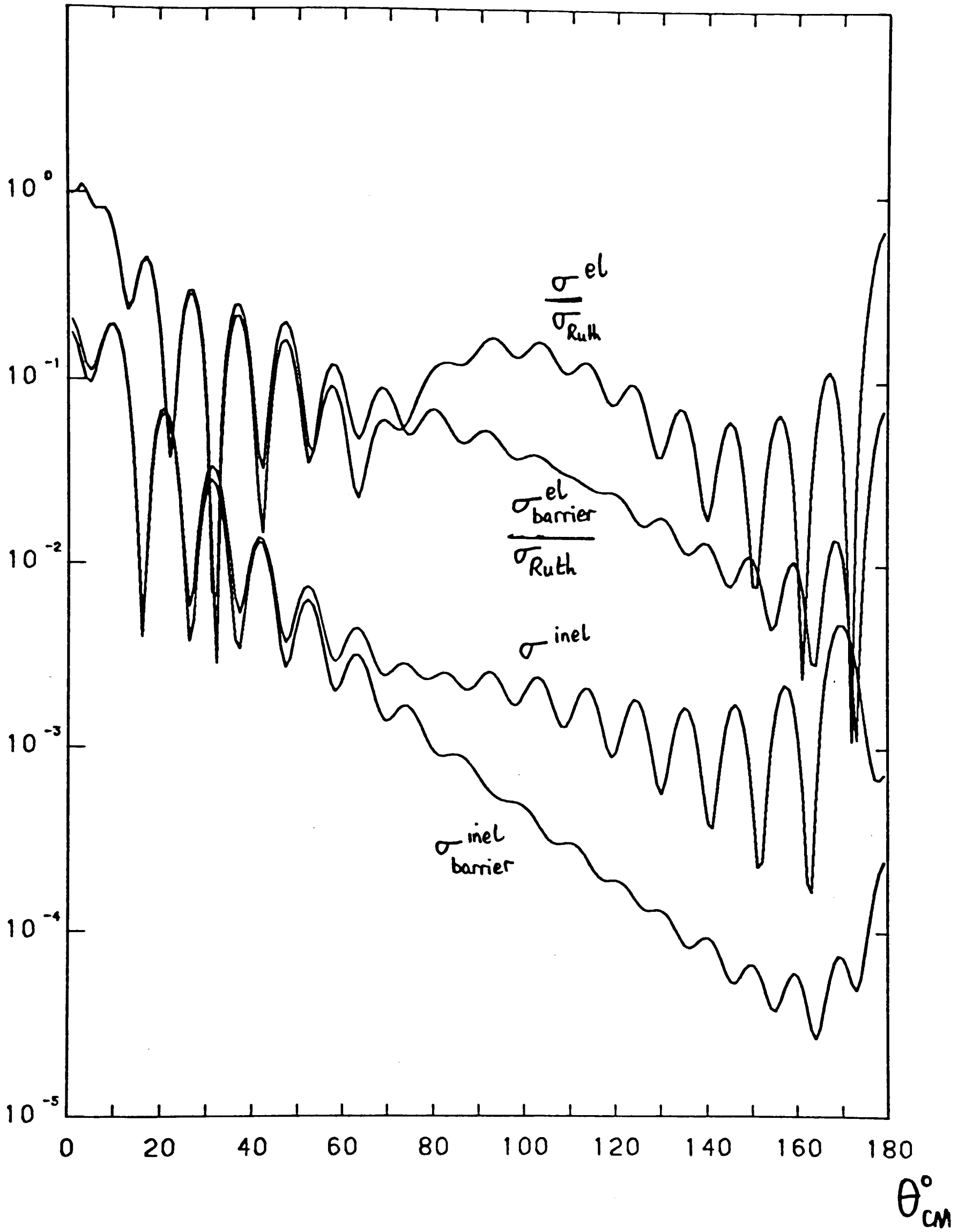
$$R^{i}_{\ell'\ell} \sim c_1 S^{i}_{\ell'} + c_2 S^{i}_{\ell} \qquad (1.4)$$

where S^i is the internal wave S-matrix and c_1 and c_2 are slowly varying functions of ℓ and ℓ'. Consequently the inelastic cross sections for general excitations will not obey the Blair rule since this relies on the radial integral behaving like the derivative of the elastic S-matrix.

However, Frahn (1980b) predicted that the large angle oscillations <u>should</u> obey the Blair rule by assuming that the so-called 'anomalous' part of the radial integral behaves like the <u>derivative</u> of the 'anomalous' part of the elastic S-matrix, which is the prescription generally used for strong absorption (Chapter IV). It is likely, therefore, that his model is more suited to dynamical mechanisms other than internal-barrier interference (Chapter I).

In fig.(V-2) we compare the elastic and total inelastic cross sections for the scattering of α-^{50}Ti at 45 MeV (Gubler et al. 1981) both with and without the internal wave. (The potential W_1 has a Woods-Saxon squared form with (depth, radius, diffuseness) equal to (70 MeV, 3.68fm, 1.1fm)). We see that the internal wave dramatically alters both the elastic and inelastic cross sections. At forward angles the Blair phase rule holds, but at larger angles this is no longer true <u>even for the barrier cross section</u>. This is because the interference is now between the n=0 farside and n=1 nearside amplitudes in the Poisson sum. The full cross section is almost <u>in</u> phase with the elastic cross section in agreement with the argument of Chu and Lee (1981).

Having made the division into internal and barrier



(FIG. V-2): The effect of the internal wave on the elastic and inelastic scattering of α-^{50}Ti (see text). The ordinate is in arbitrary units for the inelastic cross sections and the elastic cross sections are expressed as ratios to the Rutherford value.

components (eqs.(1.1) and (1.2)) we make the nearside-farside decomposition by using the 'exact' expression (eq.(III.1.1)) rather than the semi-classical alternative used in the previous chapter. This we must do since low partial waves are now being considered. By analogy with eq.(IV.1.19) we write

$$F_{IM} = F_{IM,b} + F_{IM,i} \qquad (1.5)$$

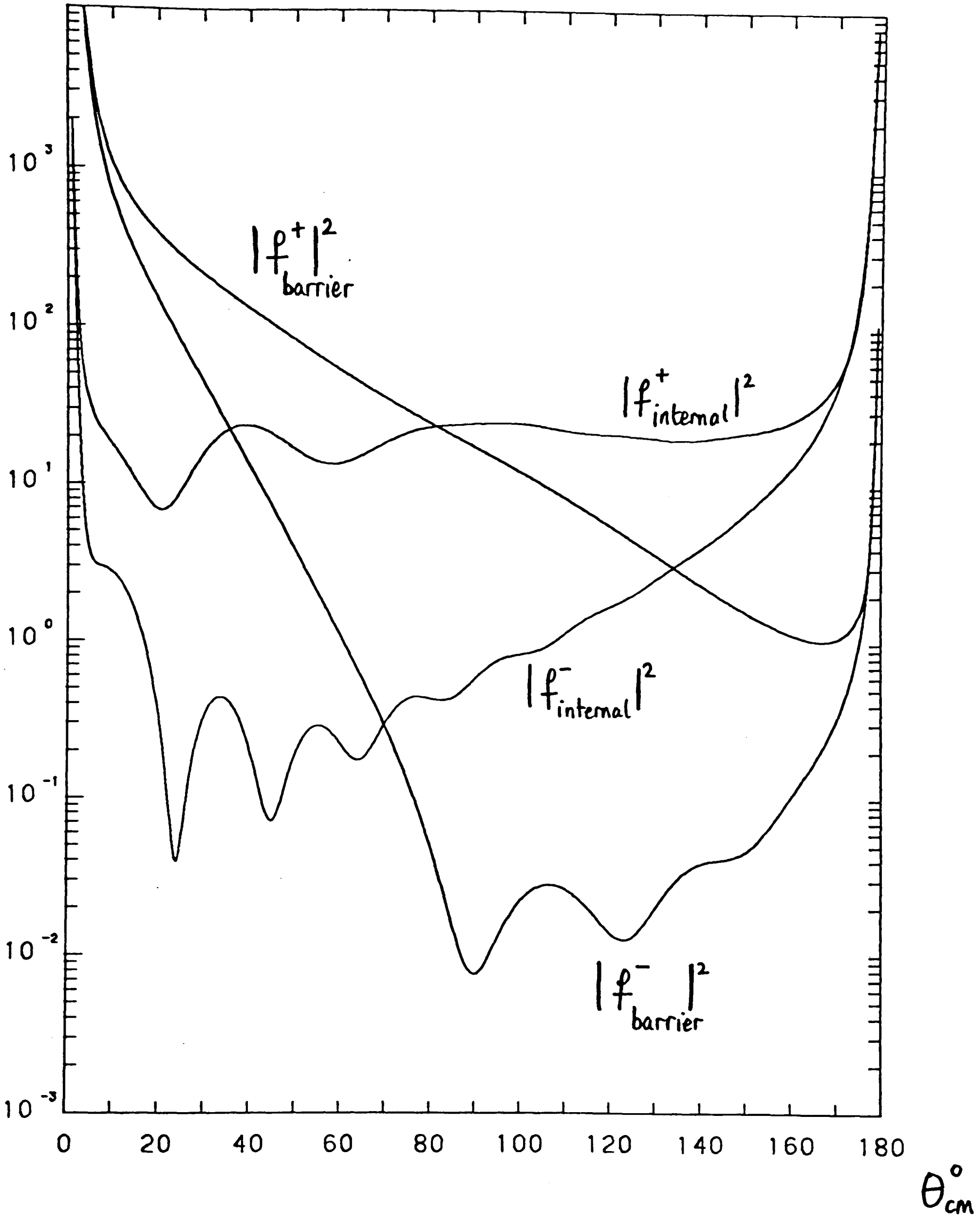
$$F_{IM,b} = F^{+}_{IM,b} + F^{-}_{IM,b} \qquad (1.6)$$

$$F_{IM,i} = F^{+}_{IM,i} + F^{-}_{IM,i} \qquad (1.7)$$

with all amplitudes calculated in the Basel axes. We do not use the DWBA axes since, as we showed in the previous chapter, angular distributions have a far simpler interpretation in the Basel system unless the quasi-classical limit is valid.

The number of different interference phenomena is now larger since as well as the NF (barrier) interference of Chapter IV, NF (internal), nearside-nearside and farside-farside interference can occur. To illustrate these we consider the process $^{50}Ti(\alpha,\alpha)^{50}Ti^{*}(2^{+},1.55)$ at 45 MeV (Gubler et al. 1981) and decompose the full radial integral of fig.(V-1) into internal and barrier components using eqs.(1.2) and (1.3) with the potential W_1 given above. The various amplitudes of eqs.(1.5)-(1.7) are then straightforward to calculate, and are illustrated in figs.(V-3) and (V-4). (We also tried to separate the internal and barrier amplitudes in this way for the potential of Rebel et al. (1981) but with less success, probably due to the large difference in radii between the real and imaginary parts of the potential.)

In fig.(V-3) we show the square modulus of the near- and



(FIG. V-3): Barrier and internal amplitudes decomposed into nearside and farside components. Only the M=0 substate (Basel axes) is shown for clarity (see text). Ordinate scale is in arbitrary units.

farside components of both the barrier and internal amplitudes as a function of scattering angle. These were calculated in the Basel axes and for clarity we show the M=0 amplitude only. The other substates behave similarly. At forward angles the barrier components dominate, being several orders of magnitude larger than their internal counterparts. The farside barrier term is greater than the corresponding nearside component since Coulomb repulsion is rather weak in this case ($\Theta \approx 12°$). Both barrier terms, and in particular the nearside one, fall off rapidly with increasing angle as expected from the arguments of the previous chapter. Above about 90° the nearside barrier amplitude becomes oscillatory due to other Poisson terms, in particular n=1. The local angular momenta of the n=0 and n=1 terms are different since the latter is, for realistic optical potentials, purely diffractive, whereas the former also has a refractive part and so interference can occur between them. Using the parametrization of Chapter IV the ratio of the n=1 to n=0 terms for the nearside amplitude is approximately

$$\exp\{ 2\pi[L_I + \Delta(\theta-\Theta-\pi)]\} \qquad (1.8)$$

for $\theta \gg \Theta$. Estimating Δ and L_I from the slopes of the nearside and farside amplitudes in the figure ($\Delta \approx 1.6$, $L_I \approx 2.3$), eq.(1.8) shows that the n=0 and n=1 terms are equal at about 110°. However, if we write down an analogous equation comparing the n=-1 and n=0 Poisson terms for the <u>farside</u> amplitude we find that they do not become equal within the physical region. Consequently refractive-diffractive interference gives structure at larger angles in the nearside barrier amplitude but not in its farside counterpart.

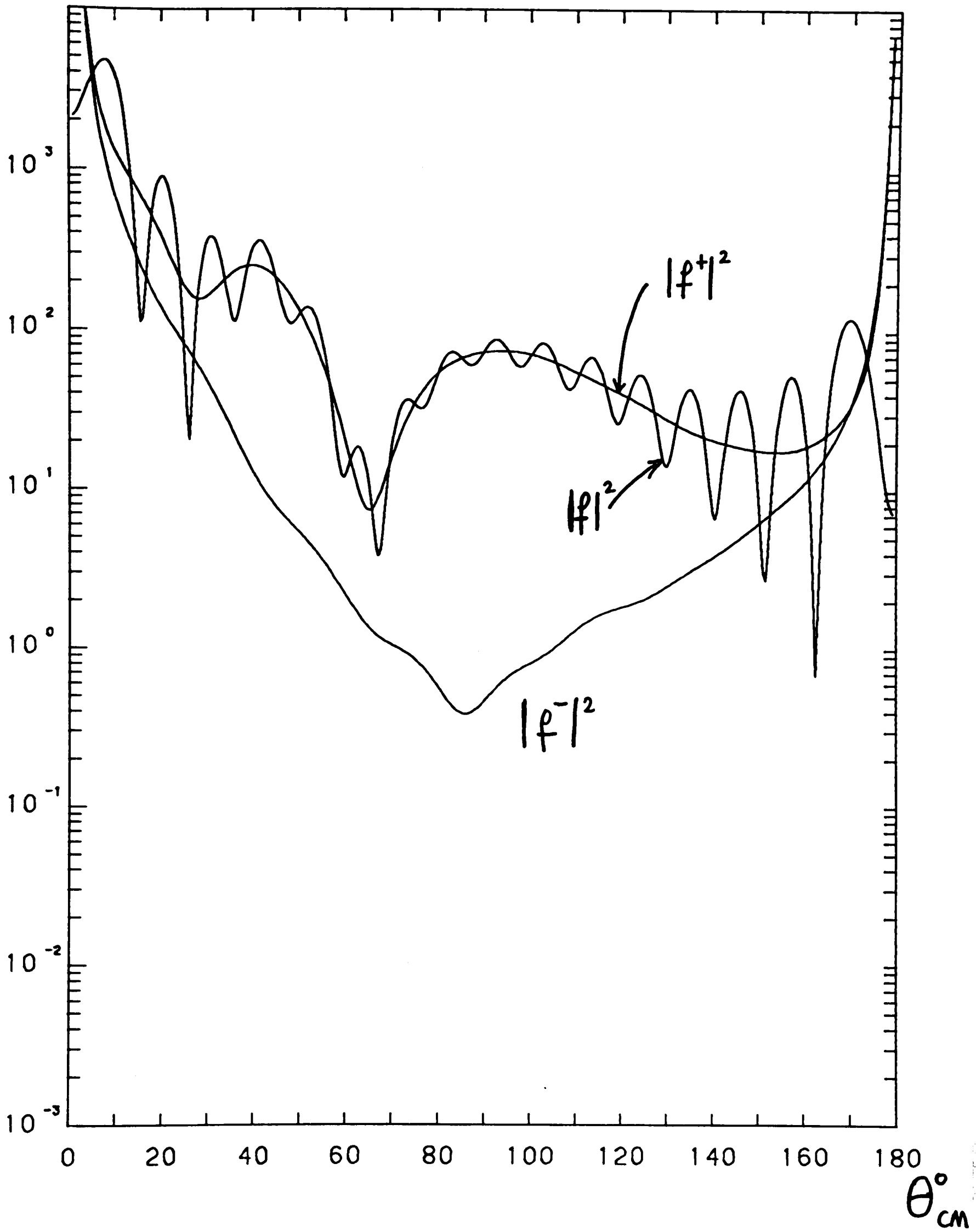
Consider now the internal wave amplitudes in fig.(V-3). Both

are very small at forward angles but they become dominant at backward angles. Both components of the internal wave show structure at forward angles, indicating that they are composed of more than one contributory amplitude. The deflection function for the internal radial integral, which is defined as

$$\Theta_i(\ell,\ell') \equiv 2\{\arg R^{i}_{\ell'\ell} - \arg R^{i}_{\ell'-1,\ell-1}\} \qquad (1.9)$$

is negative for all important values of ℓ,ℓ' and the maximum of $|\Theta_i|$ is just greater than π. Consequently the farside amplitude has a refractive contribution (from a point of stationary phase) for all scattering angles. The diffractive component of both the nearside and the farside internal amplitudes is due to the rapidly changing magnitude of the radial integral for angular momenta just less than the grazing angular momentum and for lower angular momenta too. It is the interference of these two components that gives the structure in the nearside amplitude, but for the farside amplitude refraction is more important and so this diffractive contribution is not seen.

Fig.(V-4) shows the decomposition of the full (barrier + internal) M=0 amplitude into its nearside and farside components, as well as the total (M=0) amplitude. More structure is now apparent, and can be interpreted with reference to the previous figure. Consider first the nearside amplitude, which varies smoothly except for angles around 80° where the barrier and internal components are similar. Below this angle the barrier component dominates, but above 80° the internal wave contribution is the greater. The observed (slow) oscillations are therefore nearside-nearside barrier-internal interference and have a period given by the local angular momenta of the two contributions.



(FIG. V-4): Full M=0 amplitude decomposed into near- and farside contributions for α-^{50}Ti scattering (see text). The ordinate scale is in arbitrary units.

The full farside amplitude also shows some structure. At less than 80° there are oscillations of period 40°, corresponding to interference of internal and barrier amplitudes with LAMs of about 9 and 18 respectively. These oscillations disappear at higher angles because the LAMs of the two components are now very similar (≈ 18 and ≈ 20) and so the oscillatory structure in their interference has a very large period.

For the case under consideration, fig.(V-4) shows that nearside-farside interference occurs at most angles, it being between the barrier and internal components at forward and backward angles respectively. At forward angles the barrier components have LAMs of ∓18 respectively and so oscillations of period about 10° would be expected and are observed. Similarly, close to π the internal components have LAMs of ∓14, also giving the period observed.

As for strong absorption conditions the structure in the total inelastic cross section depends on that in each of the magnetic substates.

This simple, but realistic, example shows how the nearside-farside method can be extended to systems where the absorption is weaker. The interference phenomena are now a little more complex, but are readily understandable in terms of the local angular momentum, which also indicates the refractive and diffractive components of the amplitudes. We discuss the effect of the internal wave on both the vector polarization and the γ-ray symmetry angle in the following chapter.

2. Coulomb excitation.

The theory of pure, single-step Coulomb excitation is both well known and well developed (Alder et al. 1956), but in the presence of the nuclear field the problem becomes more complicated. Because of the long range of the interaction a large number of partial waves (several hundred) must be considered, and, in numerical work, the radial integration carried out to large radii. This is particularly true for quadrupole excitations.

In the DWBA formalism the radial integrals are defined by eq.(IV.1.5), but the transition potential now has the form

$$F_I(r) = F_I^N(r) + F_I^C(r) \qquad (2.1)$$

where the Coulomb component is

$$F_I^C(r) = \frac{4\pi Z_p e \sqrt{[B(EI)\uparrow]}}{2I+1} \begin{cases} r^I/(R_C)^{2I+1} & r \leqslant R_C \\ 1/r^{I+1} & r \geqslant R_C \end{cases} \qquad (2.2)$$

and in the collective model the nuclear component of eq.(2.1) is simply proportional to the derivative of the optical potential. Here Z_p is the projectile charge, R_C the radius of the (uniform) charge distribution, and $[B(EI)\uparrow]$ the electromagnetic transition rate. The Coulomb part of the radial integrals can be approximated by factoring out the nuclear distortion of the radial wavefunctions and writing (Sopkovich 1962)

$$R_{\ell'\ell}^{\text{Coul}}(\kappa',\kappa) \approx \sqrt{(S_{\ell'}^N(\kappa'))}\ \bar{R}_{\ell'\ell}(\kappa'\kappa)\ \sqrt{(S_{\ell}^N(\kappa))} \qquad (2.3)$$

where S^N is the nuclear S matrix and $\bar{R}_{\ell'\ell}$, which is proportional to a standard Coulomb excitation integral, is defined in exactly the same way as $R_{\ell'\ell}$, but with Coulomb wavefunctions. For large angular momenta $\bar{R}_{\ell'\ell}$ can be evaluated quite accurately using the WKB approximation with the result (Alder et al. 1956)

$$\bar{R}_{\ell'\ell} \propto (\eta'-\eta)^{k/2} \left[\frac{\eta+\eta'}{\ell+\ell'} \right]^{(I+1-k)/2} \frac{1}{\Gamma[(I+1+k)/2]} \exp\{-(\eta'-\eta)(\ell+\ell')/(\eta+\eta')\} \qquad (2.4)$$

with all symbols defined as in Chapter IV. For lower excitation energies ($\eta' \rightarrow \eta$) these quantities decrease quite slowly with increasing angular momentum.

From eq.(2.4) we see that positive values of k have the largest radial integrals and so if we use the Basel axes it is clear that the nearside 'trajectory' preferentially populates negative M magnetic substates, the opposite being true for the farside component. For large $\eta'-\eta$ this bias is considerable.

Classical arguments would suggest that pure Coulomb excitation has a nearside component only, but, as we found for Rutherford scattering, such arguments are not strictly valid. If we evaluate the Coulomb excitation amplitude using the method of stationary phase we must, because of the large number of important partial waves, expand the Coulomb phases of eq.(IV.1.6) to at least second order and this will again only give a nearside contribution. However, in saddle-point terms (Rowley and Marty 1976) the farside amplitude can exist in such circumstances, although its magnitude is small. Consequently we expect the total cross section to be dominated by a negative M nearside amplitude.

These arguments are altered slightly by absorption, since

the radial integrals generated by the second term of eq.(2.1) involve wavefunctions distorted by the nuclear potential (eq.(2.3)), and there may be contributions to the amplitude from both stationary values of the nuclear phase and regions where the magnitude of the nuclear S-matrix is changing rapidly. By computing the radial integrals with transition potentials of $F(r)$ and $F^N(r)$ we are able to extract this 'absorption modified' Coulomb excitation contribution to the scattering amplitudes. Since the nuclear and Coulomb contributions are easily isolated the decomposition of the radial integrals into these components is somewhat easier than for the barrier-internal decomposition discussed earlier.

The nearside-farside decomposition in the presence of Coulomb excitation can be carried out entirely analogously to eqs.(1.5), (1.6) and (1.7):

$$F_{IM} = F_{IM,C} + F_{IM,N} \qquad (2.5)$$

$$F_{IM,N} = F^{+}_{IM,N} + F^{-}_{IM,N} \qquad (2.6)$$

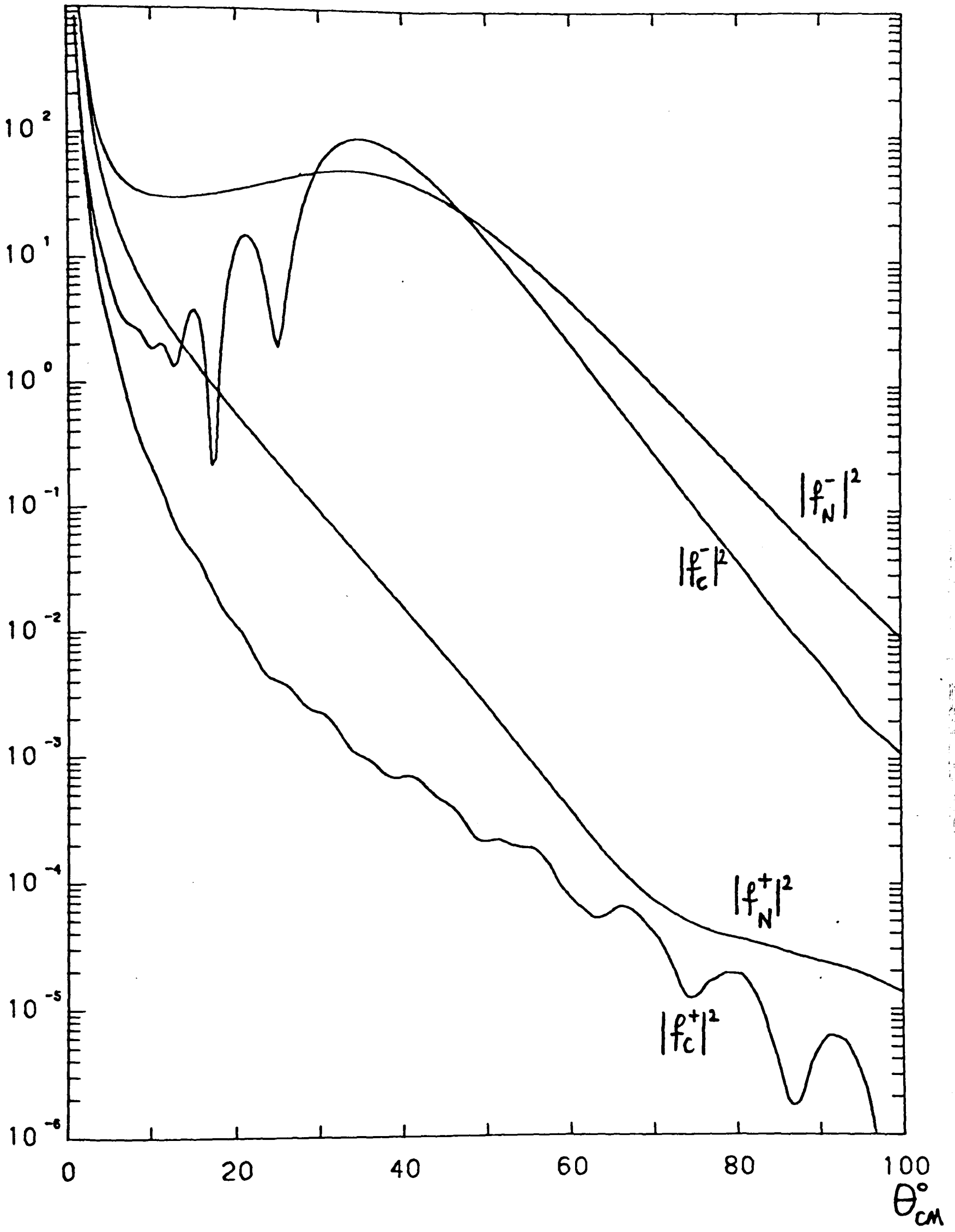
$$F_{IM,C} = F^{+}_{IM,C} + F^{-}_{IM,C} \qquad (2.7)$$

The long range of the Coulomb transition potential introduces some computational difficulties. Samuel and Smilansky (1971) exploited the recurrence relations that exist for the pure Coulomb radial integrals (Biedenharn et al. 1955, Alder et al. 1956) to generate these quantities for quadrupole excitations. Their subprogram (ITER) has been incorporated into a version of the DWBA code DWUCK (Kunz 1965) known as DWIS (Videbaek 1973). This code calculates the full radial integrals

for lower angular momenta by numerical integration and assumes that those for higher ℓ and ℓ' are equal to $\bar{R}_{\ell',\ell}$ (eq.(2.3)). For more general spin excitations the code PATIWEN (Feng and Barnett 1975) calculates each radial integral separately and independently and is therefore rather expensive in terms of computer time. Recurrence relations for pure Coulomb radial integrals involving any combination of regular and irregular Coulomb wavefunctions have been derived, even for the case where the lower integration limit in eq.(IV.1.5) is non-zero (Wright 1979, Rhoades-Brown et al. 1980, Raynal 1981). These relations, which hold for general spin excitations, have been used both in DWBA and coupled-channels codes and, even though they are rather involved, their use will become more necessary as the versatility of such codes increases.

As we are only considering quadrupole excitations we calculated the nuclear and Coulomb excitation contributions to the inelastic scattering of ^{16}O-$^{48}Ca(2^+, 3.833)$ at 56 MeV using DWIS (Videbaek 1973). The potential used was suggested by Videbaek (1982) and has a Woods-Saxon geometry with (depth, radius, diffuseness) equal to (28 MeV, 7.69 fm, 0.65 fm) for the real part and (11 MeV, 7.88 fm, 0.51 fm) for the imaginary part. A Coulomb radius of 6.15 fm was used, and the deformation parameter and B(E2) value of the excited state are 0.09 and 0.008 e^2b^2 respectively. A slightly different potential was used recently for the same reaction (Rehm et al. 1982).

In figs.(V-5) and (V-6) we show the decomposition of the amplitude for this reaction into its nuclear and Coulomb excitation and its nearside and farside components. Fig.(V-5) shows the near- and farside amplitudes in the Basel axes for each excitation process separately. (For clarity only the M=0

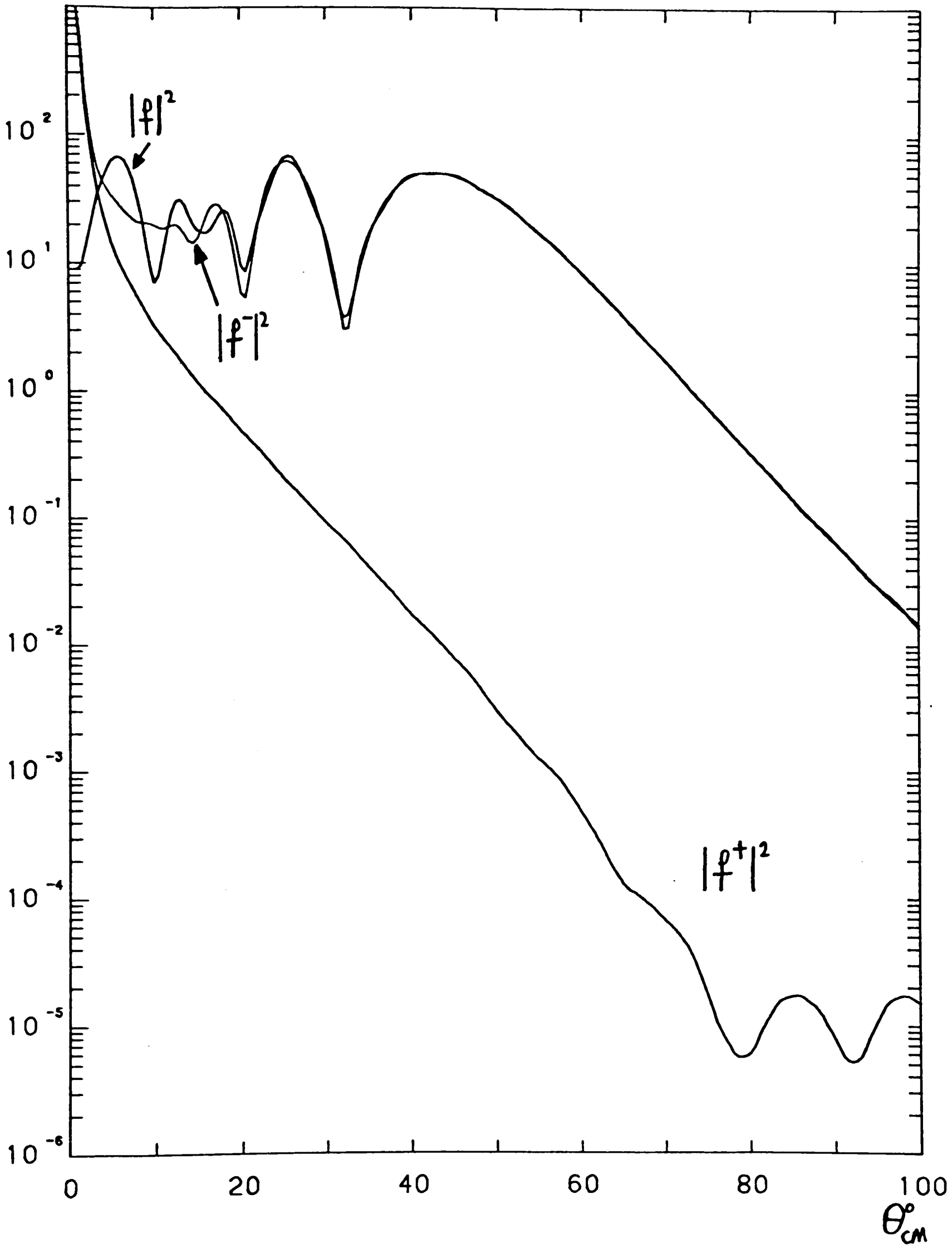


(FIG. V-5): Decomposition of the nuclear and absorption modified Coulomb excitation amplitudes into nearside and farside components for ^{16}O-^{48}Ca. Only the M=0 substate (Basel axes) is shown (see text). The ordinate scale is in arbitrary units.

amplitudes are shown). The nuclear components are smooth and the nearside amplitude dominates as one would expect for this reaction since the charges are fairly high and the nuclear potential fairly shallow. The nearside amplitude in fact dominates over the whole angular range, which is expected to be a common feature of reactions between heavier ions at all but very high energies.

The nearside branch of the Coulomb excitation amplitude is also larger than its farside counterpart, as expected from stationary-phase arguments. However, fig.(V-5) shows that both components of this amplitude have structure and so are themselves made up of more than one contribution. This is easy to understand since what is actually plotted are the components of the 'absorption modified' Coulomb excitation amplitude and eq.(2.3) shows how the radial integrals for this type of excitation can be approximated. As well as contributions from those stationary points due to $\bar{R}_{\ell',\ell}$ and the elastic S matrix, the amplitude receives a contribution from around the grazing angular momentum, and so Coulomb excitation is a refractive -diffractive phenomenon. Consideration of the local angular momenta of these effects gives agreement with the (varying) period of oscillation in the figure.

Fig.(V-6) illustrates the decomposition of the <u>full</u> amplitude into nearside and farside components. As one would expect from the previous figure, the full amplitude is almost entirely nearsided, except at the most forward of angles. The 'hump' in the cross section is reminiscent of the form of the elastic cross section, and indeed both can be interpreted in terms of Fresnel diffraction (Frahn and Rehm 1978). The grazing angle for this reaction is about 53°. The structure in the



(FIG. V-6): The full M=0 amplitude decomposed into near- and farside contributions for ^{16}O-^{48}Ca (see text). The ordinate scale is in arbitrary units.

forward angle cross section therefore arises partly from interference between the nearside and farside amplitudes, but mainly from interference between the Coulomb and nuclear excitation contributions to the nearside amplitude. This is 'justification' for the almost total neglect of the farside amplitude by some authors (Frahn 1976, Frahn and Rehm 1978).

Naturally the form of the total inelastic cross section depends on the structure in the different magnetic substates. However, since the nearside amplitude is almost completely dominant at these energies, the Blair phase rule will not hold since it arises essentially from nearside-farside interference. This rule can therefore only be valid if the contribution of Coulomb excitation is small or the centre of mass energy high.

Figs.(V-5) and (V-6) show that the nearside-farside decomposition is perhaps less useful when Coulomb excitation is important, but the method does, however, provide an interpretation of the forward angle oscillations. At higher energies the method becomes more useful over a wider angular range as the importance of nuclear excitation increases.

CHAPTER VI.

POLARIZATIONS AND SYMMETRY ANGLES IN HEAVY-ION REACTIONS.

1. The polarization under strong absorption.

The definition of the vector polarization, P, in terms of the substate cross sections is

$$P(\theta) \equiv \sum_{M} M\,\sigma_{IM}(\theta) \,/\, I\,\sigma_{I}(\theta) \qquad (1.1)$$

where the quantization axis is perpendicular to the reaction plane (Basel axes). This quantity is only one of many Cartesian or spherical tensors that can be used to describe a polarized beam, the general spherical tensor t_{k0} (k=1,2..) being defined in terms of the $\sigma_{IM}(\theta)$ by

$$t_{k0} = \sqrt{(2I+1)} \sum_{M} (-1)^{I-M} \langle I\,I\,M-M\,|\,k\,0\rangle\, \sigma_{IM} \,/\, \sigma_{I} \qquad (1.2)$$

Even though we have neglected spin-orbit effects and only dealt with spin-zero nuclei in the initial channel, there can still be a polarization in the final channel since different magnetic substates are populated to different extents. This is essentially due to lack of adiabaticity and the distortion of the wave-functions in the two channels. If we use a plane-wave approximation to the wavefunctions, the radial integral of eq.(IV.1. 5) becomes independent of the value of $\ell-\ell'$ and so the magnetic substates are equally populated. In Frahn's (1976) formalism the preferential population of a certain substate arises entirely from the Coulomb excitation contribution to the

amplitude, which is essentially due to his use of the Austern-Blair relation of eq.(IV.3.1b) rather than Hahne's generalization (eq.(IV.3.2). In this respect our results are better.

General polarization studies provide a good test of reaction mechanisms (Chapter I), but our interest is in the sensitivity of the vector polarization to the relative importance of the nearside and farside amplitudes.

As we saw in Chapter II, the amplitudes in the Basel axes, F_{IM} and $F_{I,-M}$, have the same magnitude in the kinematically matched limit. In this case the polarization is therefore zero, and we need only consider the non-adiabatic situation. We also restrict ourselves, for simplicity, to I=2, but our arguments are readily extendable to other spins.

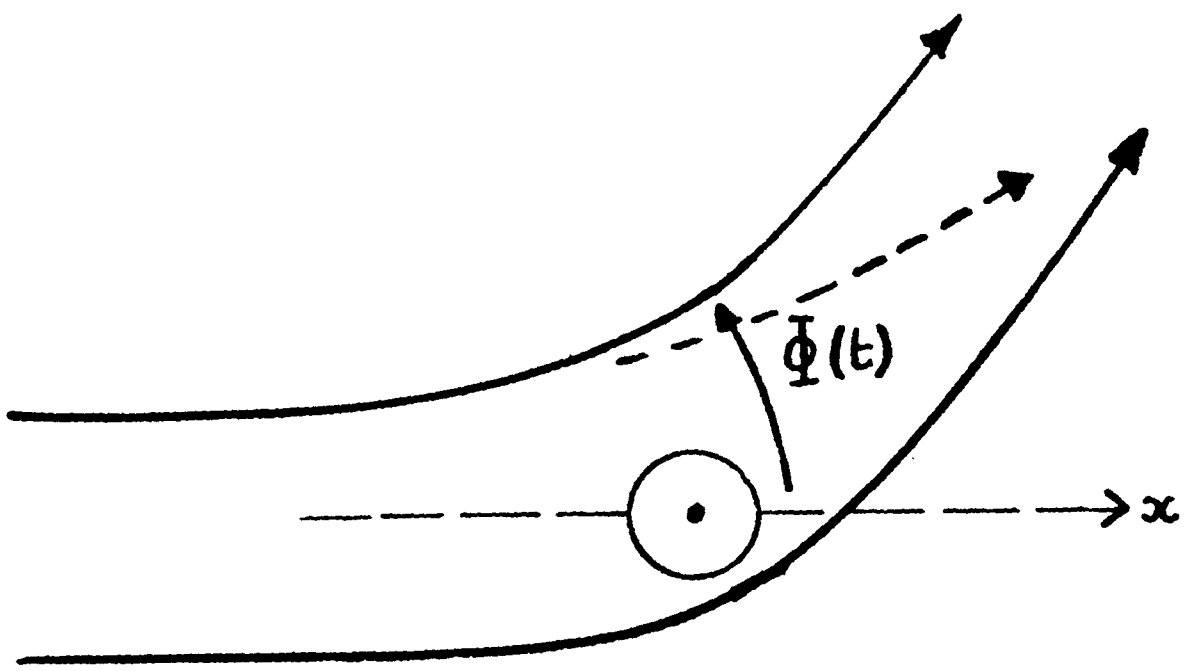
When the nuclear attraction is weak, we have shown that the nearside amplitude dominates and for this component the M=-2 substate is populated to a greater extent than the M=2 substate. We therefore expect a negative polarization. In the farside dominated case, occurring for stronger nuclear attraction, the opposite is true and a positive polarization results. Earlier (Chapter IV) we explained the dominance of some negative or positive value of M (for near- or farside scattering respectively) in terms of the overlap of wavefunctions, but the same result can be obtained from the semi-classical approach of Broglia (1974). (This approach is somewhat different from ours since the concept of a classical trajectory is retained). His result shows that the transition probability is greatest when the following condition holds:

$$E^{*} - M \frac{d}{dt} \Phi(t) = 0 \qquad (1.3)$$

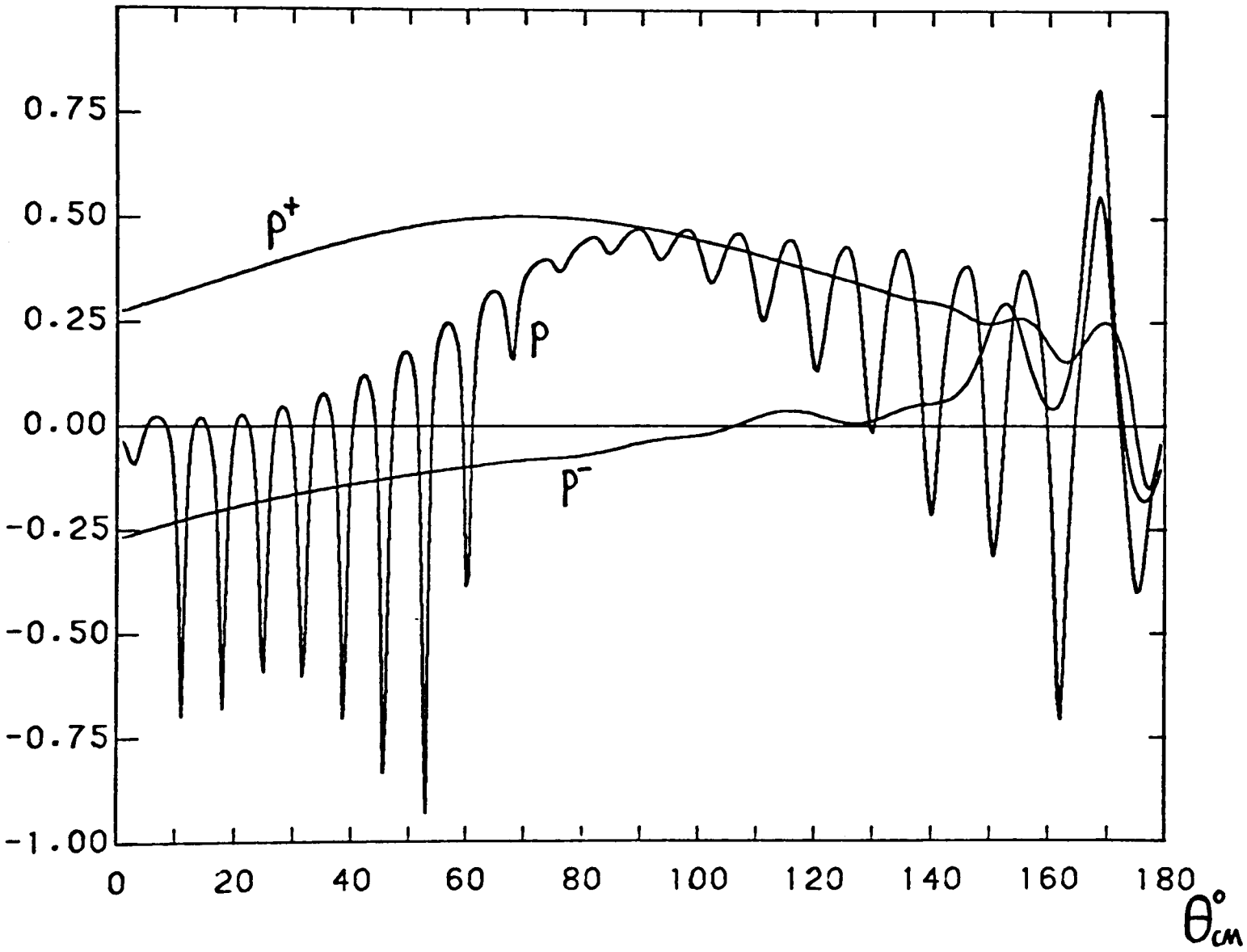
Here E^* is the excitation energy, $\Phi(t)$ is the instantaneous azimuthal angle of the projectile, and in our co-ordinate system $\Phi(t=-\infty)=\pi$ and $\Phi(t=\infty)=\theta$, the scattering angle. From fig.(VI-1) we see that for the nearside component Φ is always decreasing, and consequently eq.(1.3) has a negative value of M for its solution. On the farside Φ is always increasing and M is then positive. In fig.(VI-1) we also illustrate where the nearside 'trajectory' would have gone to in the presence of stronger nuclear attraction. For this case Φ decreases at a greater rate than before and a smaller value of |M|, and of the polarization, is expected. The nuclear attraction is effectively stronger at higher projectile energies, where the mismatch is smaller for a given E^*, and so a smaller polarization would also be expected from the arguments of Chapter IV, which also give the same signs for $P(\theta)$ as the above.

All these arguments refer, in some sense, to the <u>average</u> polarization, since $P(\theta)$ is actually oscillatory due to the structure inherent in the substate cross sections. Since the relative importance of the farside amplitude is greater at larger scattering angles, one might expect a sign-change in this 'average' polarization at some angle. Using the parametrized radial integral of eqs.(IV.3.7) and (IV.3.8) we can estimate where this will occur, since $|F_{22}^{+}|\approx|F_{2,-2}^{-}|$ when $\theta=\Theta_{cross}\approx\pi\Delta/2L_I$. For the ^{16}O-^{24}Mg reaction we can estimate L_I and Δ, as defined in Chapter IV, from the slopes of $|F^{\pm}|$ in fig.(IV-7) since these amplitudes vary as $\exp(-\pi\Delta\Theta\pm L_I\theta)$ for $\theta\gg\Theta$. We obtain $L_I\approx1$ and $\Delta\approx1$, giving $\Theta_{cross}\approx$ 55°, as observed in fig.(VI-2).

Also in fig.(VI-2) we show the polarizations resulting from either the nearside or the farside amplitude alone. At forward



(FIG. VI-1): Variation of the azimuthal angle $\Phi(t)$ along near- and farside trajectories. The z-axis is along a line out of the plane of the paper through the scattering centre. The dotted line indicates the nearside trajectory for stronger nuclear attraction.



(FIG. VI-2): Polarizations for ^{16}O-^{24}Mg. P^+ and P^- are the polarizations resulting from suppression of the nearside and farside amplitudes respectively, and P is the full polarization.

angles $P(\theta)$ has the expected sign, but for the nearside amplitude we observe a sign-change at about 110°. This is due to the reflection of low partial waves from the barrier and if these are artificially suppressed, or the 'cosh' parametrization used with values of L, L' and Δ that fit $R_{\ell'\ell}$ near the grazing angular momentum, the nearside polarization is wholly negative. At more backward angles the other Poisson sum terms affect the polarization.

Magnetic substate populations, and so polarizations, can be measured using a variety of correlation experiments, but another method is also possible since the γ-decay in flight of an excited nucleus will broaden the peak in the energy spectrum in a way that depends on the substate populations. This is essentially a Doppler shift and, if the experimental conditions are favourable, the energy spectrum can be unfolded to yield these populations (Beene and DeVries 1976, Bohlen and von Oertzen 1978, Bohlen et al. 1978, Beene et al. 1980). Using this method for the ^{12}C-^{16}O reaction Bohlen et al. (1978) studied the importance of transfer channels in inelastic scattering, but it has not been used to extract M-state populations in the Basel axes, and hence determine vector polarizations.

The more traditional method involves detecting the emitted γ-ray in coincidence with the scattered particle. Theoretical expressions for the correlation function are well-known (see below), but experimentally one must be able to measure the γ-ray helicity in order to remove the ambiguity between whether the excited state was M=2 or M=-2. The M=0 substate does not contribute to the in-plane correlation function, but because of the finite size of detectors some 'contamination' of this function occurs and should be allowed for (Chait 1976).

Using the more traditional methods Hayward and Schmidt (1970) measured the polarization of the excited carbon nucleus in α-^{12}C(2^+,4.44) scattering at 22.75 MeV. They found that $P(\theta)$ was negative for $\theta<70°$, and positive for larger angles, and so, as $\Theta\approx 11°$, our arguments suggest that the imaginary part of the grazing angular momentum is about 15% of the maximum allowed by unitarity, corresponding to a fairly small real nuclear phase.

Chait and Sinclair (1977) measured the substate populations of the excited magnesium nucleus in the reaction ^{16}O-^{24}Mg(2^+,1.37) at 42 MeV. Their measurements show that $P(\theta)$ always has the same sign, which is thought to be negative. Measurements were only made for θ less than 70° and the γ-helicities were not determined. Assuming their sign for the polarization, which is reasonable in view of the above discussion, and taking as a first approximation the values of Δ and L_I for this reaction at 60 MeV incident energy (see above), then we would expect $P(\theta)$ to become positive for angles greater than about 85°, unfortunately above where the data were taken.

This discussion illustrates how the vector polarization reflects the dominance of either nearside or farside scattering. The situation is altered somewhat in the presence of an internal-wave contribution or Coulomb excitation, and it is these effects that we now discuss.

2. The polarization with an internal-wave or Coulomb excitation contribution.

We first consider the internal wave which, as we have seen, affects inelastic cross sections mainly at larger angles. It is also predominantly farsided, although interference between nearside and farside components can give the cross section structure in the backward angle region.

We take as our example α-^{50}Ti scattering at 45 MeV and use the potential of Gubler et al. (1981) shown in Table (IV-1). Calculations show that, at angles where the internal contribution is dominant, the M=-2 substate is always populated to a greater extent than the M=2 substate <u>irrespective</u> of whether we are looking at the nearside or the farside amplitude. This is in contrast to the situation for strong absorption and Coulomb excitation where negative and positive values of M are favoured by the nearside and farside amplitudes respectively. In these cases however, semi-classical arguments can be used to account for the preferential population of a particular substate whereas for the internal wave, which corresponds to quite considerable interpenetration of the two nuclei, such arguments are rather harder to make. However, the generality of this result for the internal amplitudes is difficult to assess since for the internal wave in ^{16}O-^{24}Mg scattering the preferentially populated substate on both the near- and farside agrees with that for strong absorption conditions.

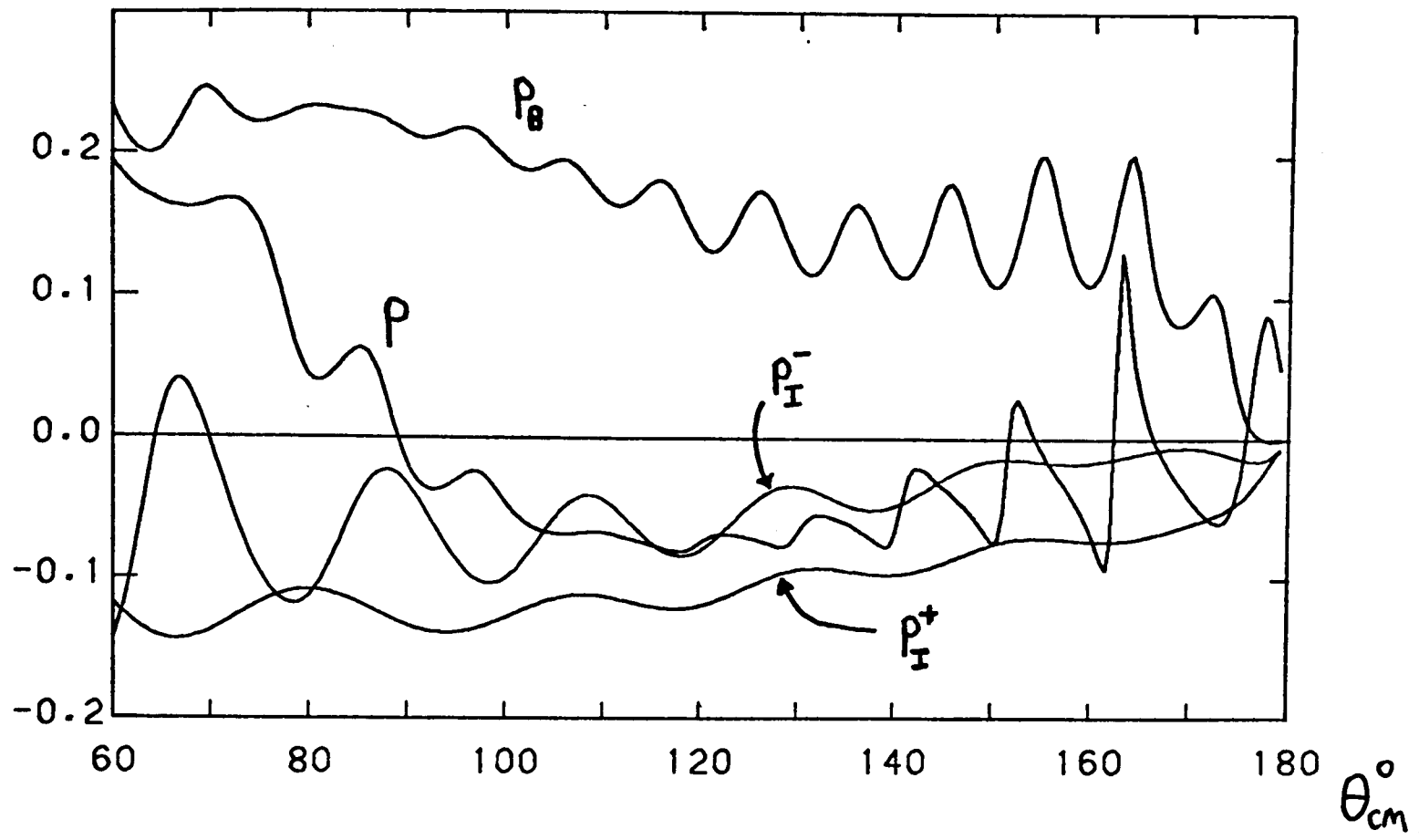
Consequently the polarization arising from the internal wave is, in this example, negative for both the NF components at larger angles. Since nuclear attraction dominates the barrier amplitudes in this reaction and gives a positive polarization

except at the most forward of angles, the onset of the internal wave contribution is therefore indicated by a sign change (from positive to negative) in $P(\theta)$. (Again we are talking about the 'average' polarization). These points are illustrated in fig.(VI-3) where we show, for angles where the internal wave is most important, the polarization resulting from the nearside and farside internal wave amplitudes, and also the full polarization, both with and without the internal wave. The latter is particularly oscillatory close to π where the nearside and farside amplitudes are similar, and the abovementioned sign change occurs at about 90°.

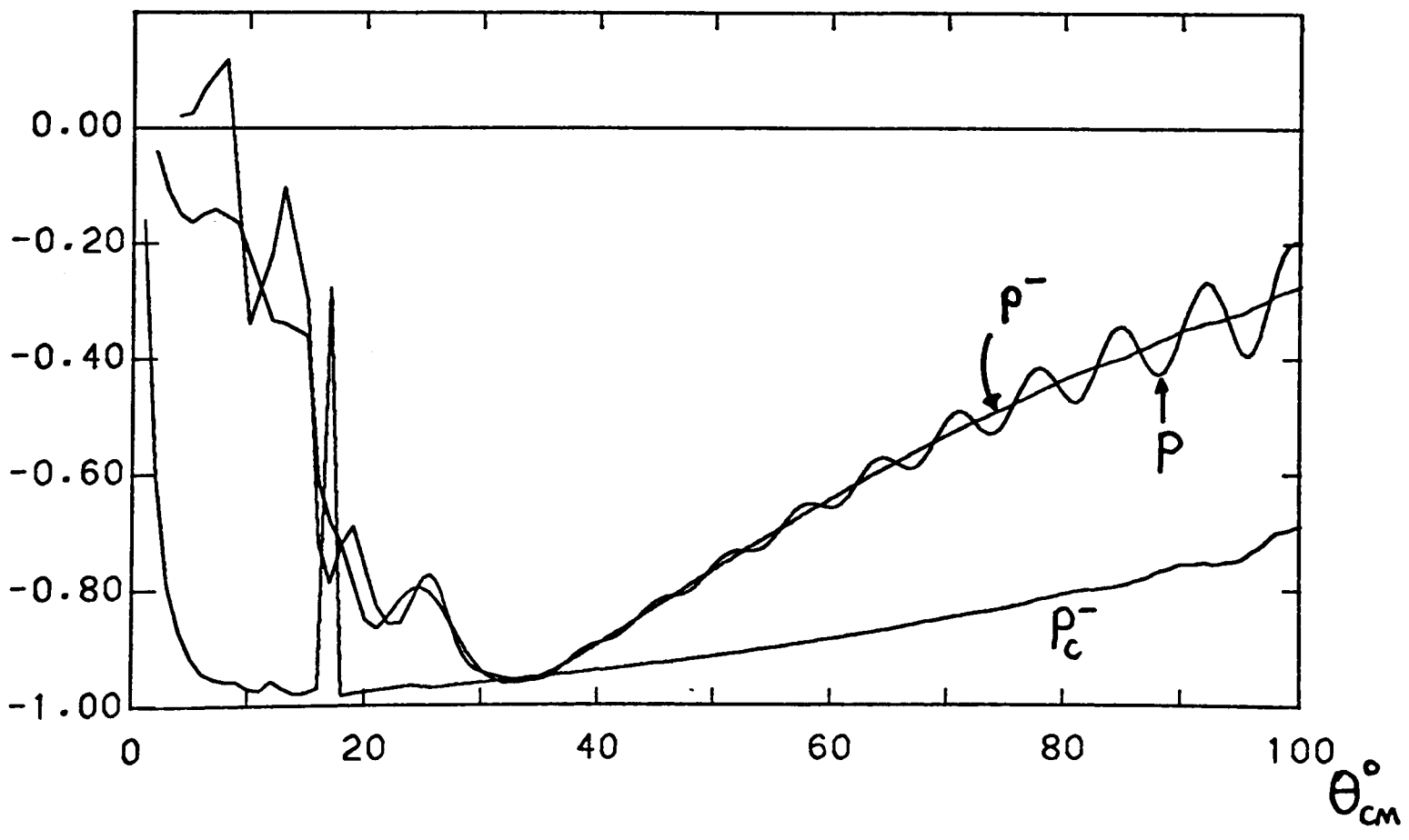
We see therefore that the presence of an internal wave can dramatically alter the form of the vector polarization and so its measurement could be a further test of the importance of this contribution.

We now consider the effect that Coulomb excitation has on the vector polarization. As we saw in the previous chapter, both pure and absorption modified Coulomb excitation are essentially nearside processes, and so the arguments presented above show that some negative M magnetic substate will be preferentially populated and a negative polarization result. The cross sections for different substates differ considerably in magnitude for not too small excitation energies and so the polarization will be close to its minimum value of -1, although its magnitude will decrease towards larger angles where the farside contribution becomes relatively more important.

We have seen that the average polarization resulting from nuclear excitation can change sign at some angle if the farside amplitude becomes sufficiently important. If the importance of Coulomb excitation is small, this sign change will still occur,



(FIG. VI-3): Polarizations for α-^{50}Ti. P_B is the polarization due to the barrier wave alone, P_I^+ and P_I^- are the polarizations due to the internal wave with suppression of the nearside and farside amplitudes respectively, and P is the full polarization.



(FIG. VI-4): Polarizations for ^{16}O-^{48}Ca. P_C^- is the polarization due to the nearside Coulomb excitation amplitude, P^- is that due to the nearside part of the full amplitude, and P is the full polarization.

but at a larger angle since Coulomb excitation contributes predominantly to the nearside amplitude. In the case of more important Coulomb excitation such a sign change will not take place and the average polarization will always be negative.

In fig.(VI-4) we compare the polarizations resulting from the nearside amplitude with and without the nuclear excitation contribution, and also show the polarization with all amplitudes included. The chosen reaction is ^{16}O-^{48}Ca at 56 MeV using Videbaek's (1982) potential described in the previous chapter. For this particular reaction the nearside component of the nuclear excitation is dominant over almost the entire angular range. As expected the nearside component of Coulomb excitation taken alone gives a large negative polarization whose magnitude decreases slowly with scattering angle. One very noticeable feature of this polarization is the spike at about 18°, which is due to a deep minimum in σ_{22} not occurring in the M=-2 cross section. Inclusion of nuclear excitation on the nearside gives a negative polarization of generally smaller magnitude since, for this type of process, the difference in population between positive and negative M magnetic substates is less marked. The contribution of the farside amplitudes is small since the full polarization is, apart from its oscillations, well approximated by the nearside contribution. The main effect of the farside amplitude is to give the oscillatory structure in $P(\theta)$.

Measurements of polarizations therefore provide not just a test of the dominance of either the nearside or the farside amplitude, but also tests the importance of different contributions to the full amplitude since each of these alters the relative population of the different magnetic substates. This is also true for other processes, such as α-transfer

(Bohlen et al. 1978), that contribute to a given inelastic reaction, but we do not consider these components here.

3. The symmetry angle under strong absorption.

The theory of angular correlations between two successive nuclear processes is well known and we have already given (in Chapter II) all the background necessary for the present discussion. The important result is that the correlation amplitude has the form of eq.(II.5.16).

It is also well known that for I=2 excitations there is always a symmetry angle for in-plane γ-decay, though its value deviates from the adiabatic value (eq.(II.5.19)) if the reaction is not well matched. There is actually always a symmetry angle if only two amplitudes contribute to the correlation amplitude, irrespective of the spin of the excited state. Let these amplitudes be

$$F_{IM_j} = a_j \exp(i\delta_j) \qquad (j=1,2) \qquad (3.1)$$

and so eq.(II.5.16) becomes

$$A^{I}_{\sigma}(\theta,\theta_\gamma,\phi_\gamma) = (a'_1 e^{i\psi} + a'_2) e^{i(\delta_2 + M_2\phi_\gamma)} \qquad (3.2)$$

where $a_j' = a_j d^{I}_{M_j\sigma}(\theta_\gamma)$, $\sigma=\pm1$ (the γ-helicity) and

$$\psi = (\delta_1 - \delta_2) + (M_1 - M_2)\phi_\gamma \qquad (3.3)$$

Consequently we have

$$|A^{I}_{\sigma}(\theta_{\gamma},\phi_{\gamma})|^{2} = a_1'^{2} + a_2'^{2} + 2a_1' a_2' \cos\psi \qquad (3.4)$$

which is symmetric about $\psi=n\pi$ or

$$\phi_{\gamma} = (n\pi - \delta_1 + \delta_2) / (M_1 - M_2) \qquad (3.5)$$

with $n=1,2,\ldots,2|M_1-M_2|$. This is valid for any θ_γ. For I=2 only M=0,±2 contribute to the correlation amplitude in general (Bohr 1959), but since $d^{2}_{0\pm1}(\pi/2)=0$ the M=0 component does not contribute for in-plane γ-decay, where $\theta_\gamma=\pi/2$. Consequently the correlation function is symmetric about the angles given by eq.(3.5), with $M_1-M_2=4$, as was found in Chapter II.

For higher I Bohr's (1959) theorem shows that the number of amplitudes contributing to eq.(II.5.16) will always be greater than two, and so one might think that the above result is of purely academic interest. However, it does show that outside the adiabatic limit approximate symmetry angles might exist for excitations of arbitrary spin if these are poorly matched, since then certain substates are strongly suppressed. This suppression also depends on θ_γ because the reduced rotation matrix element of eq.(II.5.16) alters the relative weighting of the magnetic substates. (For I=2, as we have seen, $\theta_\gamma=\pi/2$ totally suppresses the M=0 component and allows a symmetry angle for arbitrary mismatch. Out of the reaction plane this is no longer true.)

Consider the case I=3. If we detect the photon in plane the amplitudes F_{31} and $F_{3,-1}$ are somewhat suppressed since

$$|d^{3}_{1\sigma}(\pi/2)| = |d^{3}_{-1\sigma}(\pi/2)| \ll |d^{3}_{3\sigma}(\pi/2)| = |d^{3}_{-3\sigma}(\pi/2)| \qquad (3.6)$$

Furthermore, if we make helicity-sensitive measurements then at $\theta_\gamma \approx 41°$ and 139° the γ-distribution will be approximately symmetric for $\sigma=\pm 1$ separately since inequalities of the form

$$0 = d^{3}_{-11} \approx |d^{3}_{31}| \ll |d^{3}_{11}| \approx |d^{3}_{-31}| \quad (\theta = 41°,\ \sigma=1) \tag{3.7}$$

then hold. (If we do not measure the helicity then the distributions are no longer symmetric.) If the observed γ-distribution was not roughly symmetric for each helicity then this indicates that certain magnetic substates are being populated to a far greater extent than others. To our knowledge out-of-plane γ-correlation experiments have never been performed to determine whether certain substates dominate.

We now consider in detail the I=2 in-plane γ-ray distribution. This has a four-leaf clover shape and, if the reaction is adiabatic, is symmetric about the angles given in Chapter II. From eq.(3.5) we can write the general symmetry angle as

$$4\,\phi_{sym} = \arg F_{2-2} - \arg F_{22} \tag{3.8}$$

Assuming the result of appendix F, we can write the deviation of the symmetry angle from its adiabatic value as

$$\Delta\phi = \arctan\left[\frac{c \cos 2\Lambda_R\theta}{a + b \sin 2\Lambda_R\theta}\right] \tag{3.9}$$

where Λ_R is the average of the real parts of the grazing angular momenta in the initial and final channels, and

$$a \equiv \left|\Omega_{2}^{-}\ \Omega_{-2}^{-}\right| + \left|\Omega_{-2}^{+}\ \Omega_{2}^{+}\right| \qquad (3.10a)$$

$$b \equiv \left|\Omega_{2}^{-}\ \Omega_{2}^{+}\right| + \left|\Omega_{-2}^{+}\ \Omega_{-2}^{-}\right| \qquad (3.10b)$$

$$c \equiv \left|\Omega_{2}^{-}\ \Omega_{2}^{+}\right| - \left|\Omega_{-2}^{+}\ \Omega_{-2}^{-}\right| \qquad (3.10c)$$

with $\Omega_k^{\pm}$ being an abbreviation for $\Omega_k(\theta\pm\theta)$. The symmetry angle is very sensitive to the relative phases (through the result of appendix F) as well as the relative magnitudes of the contributing amplitudes and so it provides a good test of our assumptions.

Since the total inelastic cross section can be written in the form (from eq.(IV.4.3))

$$\sigma_I(\theta) \propto A + 2B \sin 2\Lambda_R\theta \qquad (3.11)$$

with

$$A \equiv \sum_{k=0,\pm2} \left|\Omega_{k}^{-}\right|^2 + \left|\Omega_{k}^{+}\right|^2 \qquad (3.12)$$

$$B \equiv \sum_{k=0,\pm2} \left|\Omega_{k}^{-}\ \Omega_{-k}^{+}\right|$$

we see that at maxima and minima in the cross section the symmetry angle always has its adiabatic value since then $\cos(2\Lambda_R\theta)=0$. For scattering angles just greater than a cross section maximum/minimum the cosine in eq.(3.9) is negative/positive and so the symmetry angle is smaller/larger than its adiabatic value. The actual form of the deviation for general θ depends on the relative magnitudes of a, b and c, which in turn depend on the mismatch. If this is small the deviation oscillates smoothly with a small magnitude, but if the

denominator of eq.(3.9) becomes small the phenomenon of 'rapid reverse rotation' (Eidson et al. 1964, Inglis 1966a and 1966b) can occur since $\Delta\phi$ is then a rapidly varying function of scattering angle near to minima in the cross section. This rapid variation has been explained as a beating effect by Inglis (1966a and 1966b), who obtains a similar expression to eq.(3.9) using a far less general argument than ours.

We can estimate where rapid reverse rotation will start by first assuming that $\theta \gg \Theta$ so that $b \approx c \approx |\Omega_2^- \Omega_2^+|$ for a farside dominated process. Then $a \approx |\Omega_{-2}^+ \Omega_2^+|$ and

$$\Delta\phi \approx \arctan\left[\frac{|\Omega^-_2| \cos 2\Lambda_R\theta}{|\Omega^+_{-2}| + |\Omega^-_2| \sin 2\Lambda_R\theta}\right] \qquad (3.13)$$

Rapid reverse rotation will therefore occur if

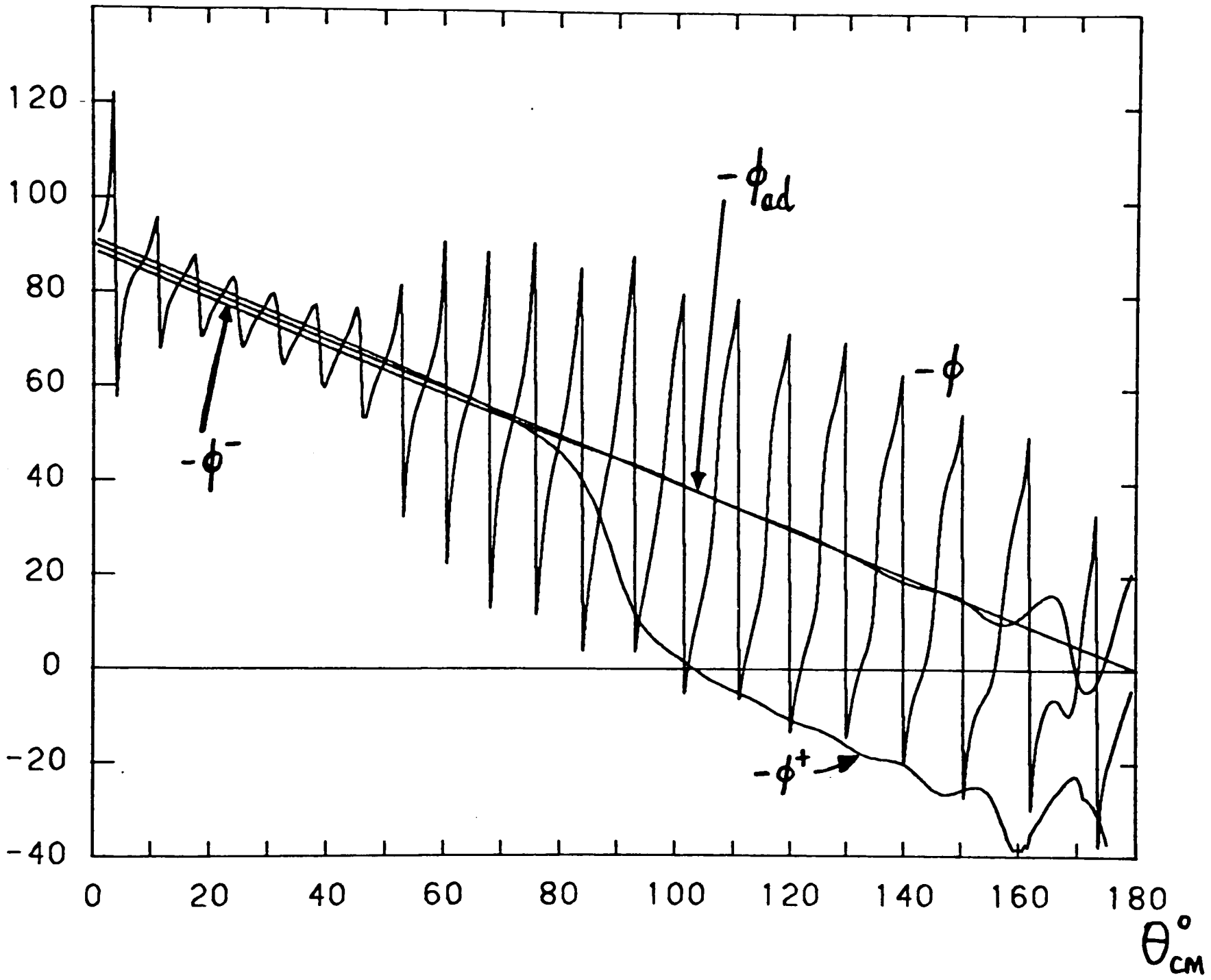
$$|\Omega^+_{-2}| \lesssim |\Omega^-_2| \qquad (3.14)$$

Similarly, for a nearside dominated reaction and $\theta \gg \Theta$ we obtain

$$|\Omega^-_{-2}| \lesssim |\Omega^+_2| \qquad (3.15)$$

Using the parametrized transform of eq.(IV.3.8) for the ^{16}O-^{24}Mg reaction (Siwek-Wilczynska et al. 1974) this suggests that rapid reverse rotation should start at around 60°, which agrees with the calculated symmetry angle shown in fig.(VI-5). The increased deviation for very small scattering angles can also be explained using this sort of argument.

If we assume that the full amplitude is well approximated by either the nearside or the farside amplitude alone then eq.(3.9) shows that the symmetry angle should have its adiabatic value.



(FIG. VI-5): γ-ray symmetry angle for ^{16}O-^{24}Mg. $-\phi_{sym}$ is plotted against the scattering angle. $-\phi_{ad}$ is the adiabatic value ($-\phi_{ad}(\theta=0°)=90°$; $-\phi_{ad}(\theta=180°)=0°$), $-\phi^{+}$ and $-\phi^{-}$ are the symmetry angles resulting from the suppression of the nearside and farside amplitudes respectively, and $-\phi$ is the full symmetry angle.

That this is not observed in fig.(VI-5) indicates that the approximation to $R_k(\lambda)$ in appendix F is not always good. Realistic radial integrals give transforms with k-dependent phases and in this case the symmetry angle of eq.(3.9) is incorrect, but its definition (eq.(3.8)) remains unaltered. In the figure the nearside symmetry angle lies very close to the adiabatic value, showing that the relative phase of F_{22}^- and F_{2-2}^- is roughly constant, as can also be seen from fig.(IV-8). The deviations at larger angles are due to the effects of low partial waves and the $n\neq0$ terms in eq.(IV.1.15). The farside symmetry angle, however, behaves far more spectacularly than its nearside counterpart due to the behaviour of the local angular momentum, illustrated in fig.(IV-8), which means that the relative phase of F_{22}^+ and F_{2-2}^+ is <u>not</u> constant around 90° scattering angle. This effect was explained in Chapter IV. Whether this dramatic behaviour could be observed is open to question since, as we have already stressed, the nearside and farside amplitudes cannot be measured separately. Observation would therefore require almost total farside domination at an angle small enough for internal wave effects to be negligible and, in addition, at this angle the parametrized transform of eq.(IV.3.8) must be close to zero, which relies on the mismatch and grazing angle to have a particular relationship to each other.

4. The symmetry angle with an internal-wave or Coulomb excitation contribution.

The effect on the γ-ray symmetry angle by either an internal wave or a Coulomb excitation contribution is essentially a sharp variation fairly localized in scattering angle. This is to be contrasted with the more regular variation observed under strong absorption conditions and arises because, at some scattering angle, the main contribution to each amplitude changes. Since the symmetry angle is essentially the difference in phase between the M=±2 substate amplitudes (eq.(3.8)), if these amplitudes are composite the symmetry angle behaves anomalously.

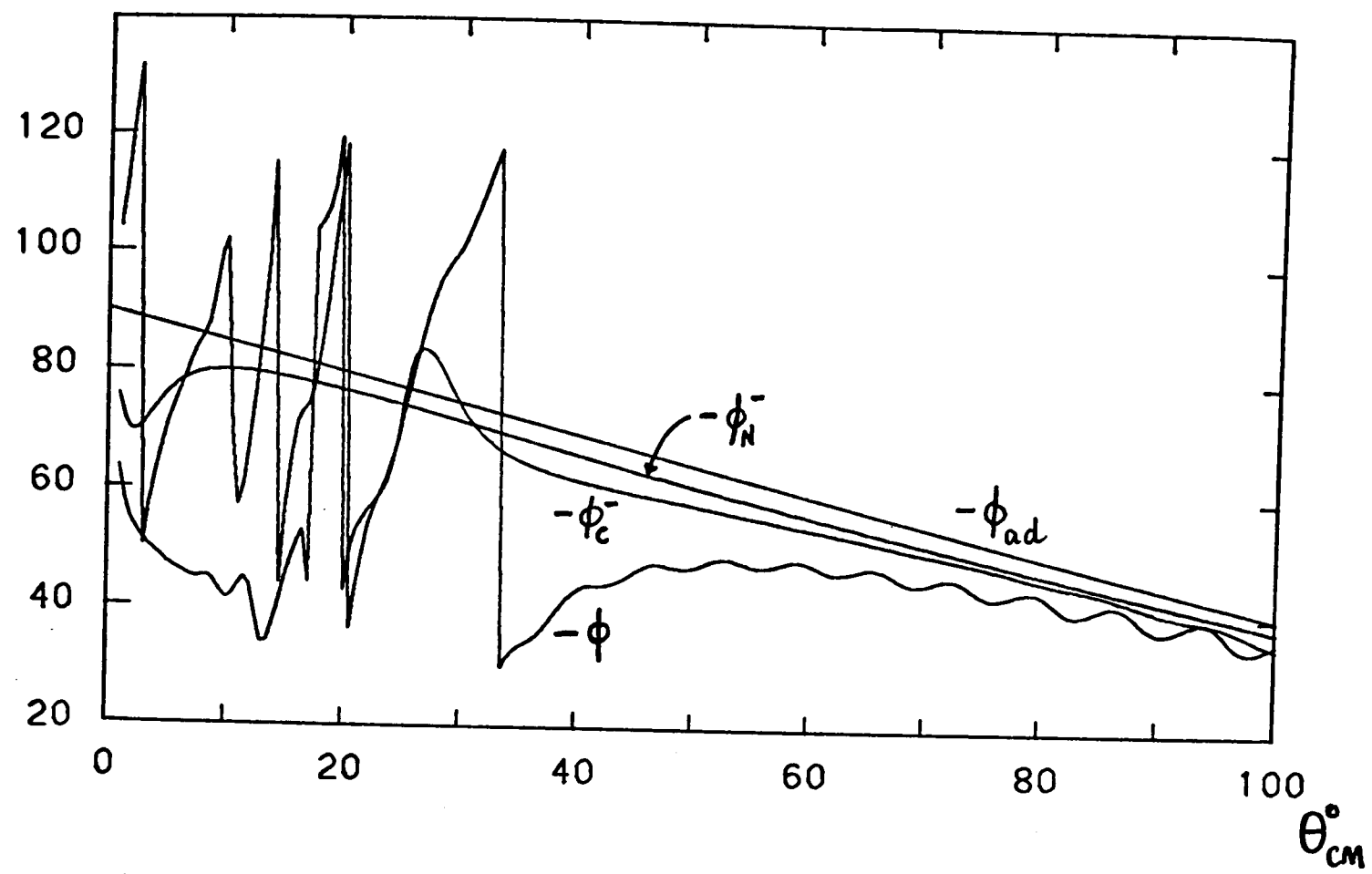
In the case of Coulomb excitation such behaviour has been observed (see, for example, Videbaek 1973, Steadman et al. 1974) and discussed in some detail by Frahn (1976) and Frahn and Rehm (1978). These authors showed that backward of the grazing angle the symmetry angle tends to its adiabatic value and at forward angles shows oscillations. These are due to the interference of the nearside nuclear and Coulomb excitation contributions, whereas for $\theta > \Theta$ the dominance of either of these ensures a smoothly varying symmetry angle.

To our knowledge no experiments have been performed to investigate the behaviour of the γ-ray symmetry angle in the presence of an internal wave, although such data would be interesting. As the internal amplitude becomes more important the regular oscillations in the symmetry angle are distorted, especially in the region where the barrier and internal amplitudes are similar. Once the latter dominates, the deviations in ϕ_γ become more regular since the LAMs of the M=±2

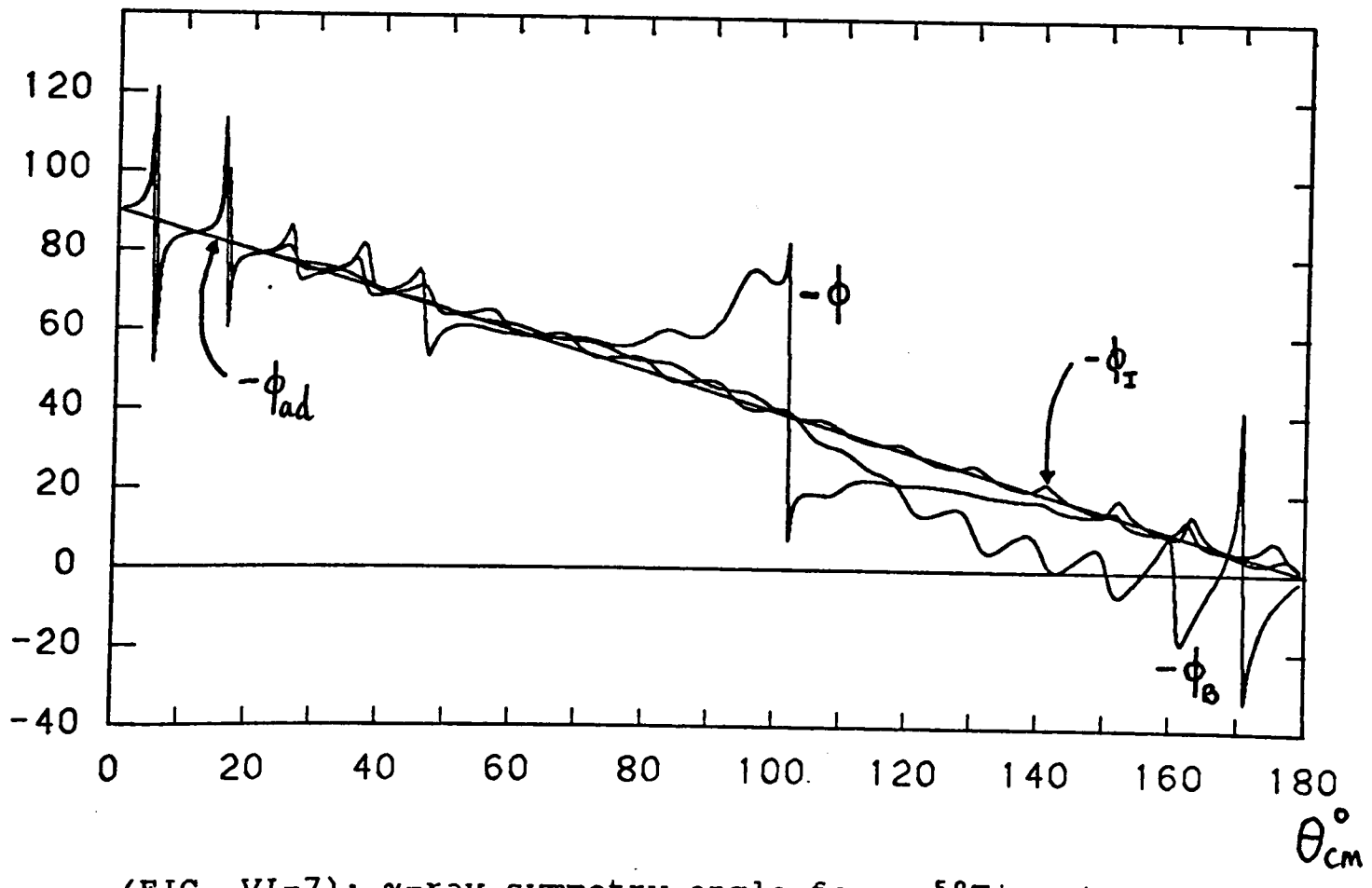
amplitudes become similar on both the near- and farside.

We illustrate these points in figs.(VI-6) and (VI-7), taking as our examples α-^{50}Ti (Gubler et al. 1981) and ^{16}O-^{48}Ca (Videbaek 1982). In fig.(VI-6) we show the symmetry angle for the latter system, which we have already shown to be a nearside-dominated reaction. We also show the symmetry angles resulting from the nearside Coulomb and nearside nuclear amplitudes alone. It is clear that the symmetry angle tends towards the adiabatic value as the scattering angle increases and our discussion of the previous chapter tells us that the oscillations at these larger angles are due to the interference of the nearsided Coulomb and nuclear amplitudes.

In fig.(VI-7) we compare the symmetry angle resulting from the full (barrier + internal) amplitude for α-^{50}Ti scattering with that due to the barrier and internal amplitudes alone. Clearly the behaviour of ϕ_γ at larger angles indicates that the barrier contribution is not the most important, whereas at forward angles ϕ_γ behaves entirely as one would expect for a barrier-dominated process.



(FIG. VI-6): γ-ray symmetry angle for ^{16}O-^{48}Ca. $-\phi_{sym}$ is plotted against the scattering angle. $-\phi_{ad}$ is the adiabatic value, $-\phi_N^-$ and $-\phi_C^-$ are the symmetry angles resulting from suppression of the farside amplitude and consideration of only nuclear or Coulomb excitation respectively, and $-\phi$ is the full symmetry angle.



(FIG. VI-7): γ-ray symmetry angle for α-^{50}Ti. $-\phi_{sym}$ is plotted against the scattering angle. $-\phi_{ad}$ is the adiabatic value, $-\phi_I$ and $-\phi_B$ are the symmetry angles resulting from suppression of the barrier and internal waves respectively and $-\phi$ is the full symmetry angle.

5. Conclusions.

We have shown that the polarization and γ-ray symmetry angle have a θ-dependence that is drastically altered by the presence of either an internal wave or Coulomb excitation contribution. The latter, normally referred to as an example of 'Coulomb-nuclear' interference, is well known. Since the cross section is also affected by these extra amplitudes, a systematic study of these three quantities can determine a more realistic nuclear potential than could be obtained from the total inelastic cross section alone. This is because $\sigma_I(\theta)$, $P(\theta)$ and the symmetry angle are all defined in terms of <u>different</u> combinations of the amplitudes $F_{IM}(\theta)$. Furthermore, the physical interpretation of these quantities becomes more transparent if the relevant amplitudeas are decomposed into their nearside and farside components.

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APPENDIX A

EVALUATION OF THE GLAUBER SURFACE INTEGRAL.

We wish to evaluate integrals of the form

$$ie^{ikR}\int f(\hat{\underline{r}},\hat{\underline{k}})[kr + \underline{k}'.\underline{r}]e^{-i\underline{k}'.\underline{r}}\, d\Omega \qquad (A.1)$$

which Glauber (1958) evaluates only in the case of k=k'

Putting $\mu=\hat{\underline{k}}'.\hat{\underline{r}}$ we obtain

$$Re^{ikR}\int_0^{2\pi} d\phi \int_{-1}^{1} f(\hat{\underline{r}}(\mu,\phi),\hat{\underline{k}})[k+k'\mu]e^{-ik'R\mu}d\mu \qquad (A.2)$$

Glauber then performs the μ-integral by parts assuming that the amplitude f is "sufficiently smooth" as a function of μ. We evaluate the integral slightly more rigorously, and also more generally, by using contour techniques. The main problem is that $f(\hat{\underline{r}},\hat{\underline{k}})$ is not an analytic function of μ since, as can be seen from eq.(II.1.2), its $\hat{\underline{r}}$-dependence is in a spherical harmonic and its μ-dependence is therefore contained in the associated Legendre function $P_{\ell m}(\mu)$. (A.2) may be considered as a sum of integrals of the form

$$I_{\ell m} = \int_{-1}^{1} P_{\ell m}(\mu)[k + k'\mu]\, e^{-ik'R\mu} d\mu \qquad (A.3)$$

The associated Legengdre function is an analytic function of its argument except on the cut along the real axis $-\infty < \mu \leq 1$, its value on the cut being given by (Abramowitz and Stegun 1965)

$$P_{\ell m}(\mu) = \frac{1}{2}\left(i^{m} P_{\ell m}(\mu + i\varepsilon) + i^{-m} P_{\ell m}(\mu - i\varepsilon)\right) \qquad (A.4)$$

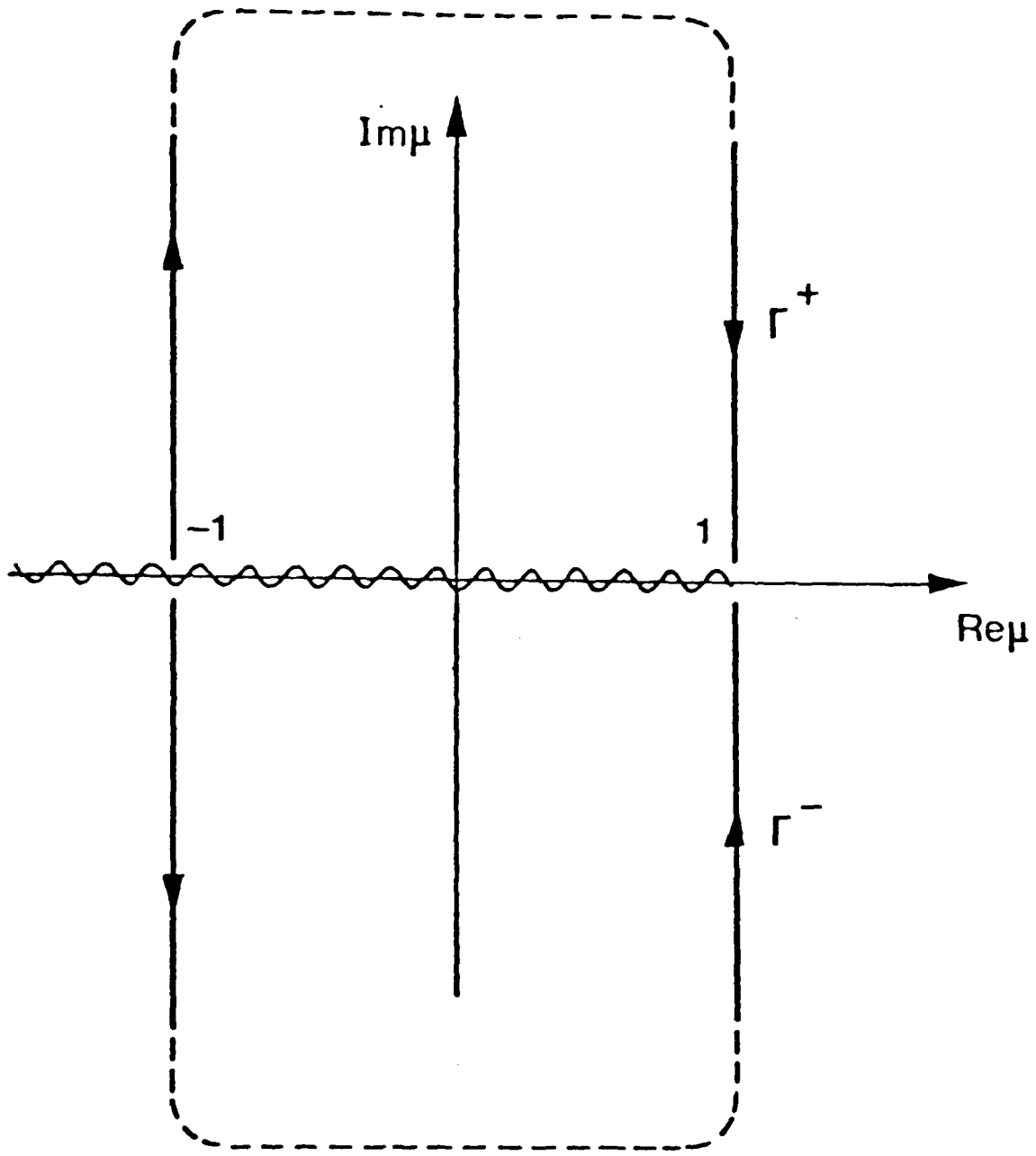
It is along part of this cut that the integral (A.3) is to be performed, and so it may be written

$$I_{\ell m} = I^{-}_{\ell m} + I^{+}_{\ell m}$$

$$= \frac{1}{2}\int_{-1}^{1} i^{-m} P_{\ell m}(\mu - i\varepsilon)[k + k'\mu] e^{-ik'R\mu} d\mu + \frac{1}{2}\int_{-1}^{1} i^{m} P_{\ell m}(\mu + i\varepsilon)[k + k'\mu] e^{-ik'R\mu} d\mu$$

$$= I^{-}_{\ell m} + \frac{1}{2}\int_{-1}^{1} i^{m} P_{\ell m}(-\mu + i\varepsilon)[k - k'\mu] e^{ik'R\mu} d\mu \,. \qquad (A.5)$$

We then evaluate $I^{\pm}_{\ell m}$ along the contours $\Gamma^{\pm}$ shown in fig.(A-1) without crossing the cut. Since $P_{\ell m}$ has no singularities off the real axis we obtain in the limit of large k'R



(FIG. A-1): Complex μ-plane showing the cut in the associated Legendre functions $-\infty < \mu \leq 1$ and the contours $\Gamma^{\pm}$ along which the integrals in eq.(A.5) may be evaluated.

$$I_{\ell m} = \frac{i}{2k'R}\{i^{-m}[P_{\ell m}(1-i\varepsilon)(k+k')e^{-ik'R} - P_{\ell m}(-1-i\varepsilon)(k-k')e^{ik'R}]$$

$$+ i^{m}[P_{\ell m}(1+i\varepsilon)(k+k')e^{-ik'R} - P_{\ell m}(-1+i\varepsilon)(k-k')e^{ik'R}]\} + \mathcal{O}[(k'R)^{-2}]$$

$$= \frac{i}{k'R}\{(k+k')P_{\ell m}(1)e^{-ik'R} - (k-k')P_{\ell m}(-1)e^{ik'R}\} + \mathcal{O}[(k'R)^{-2}] \quad (A.6)$$

which using eq.(A.4) gives

$$I_{\ell m} = \frac{i}{k'R}\left\{(k+k')\,P_{\ell m}(1)\,e^{-ik'R} - (k-k')\,P_{\ell m}(-1)\,e^{ik'R}\right\} + \mathcal{O}[(k'R)^{-2}] \quad (A.7)$$

Since $P_{\ell m}(\pm 1)$ is zero unless m=0, the ϕ-integral in eq.(A.2) gives simply 2π. Also, $\mu=\pm 1$ corresponds to $\hat{\underline{r}}=\pm\hat{\underline{k}}'$ and so we obtain

$$i\,e^{ikR}\int f(\hat{\underline{r}},\hat{\underline{k}})\,[kr+\underline{k}'.\underline{r}]\,e^{-i\underline{k}'.\underline{r}}\,d\Omega$$

$$= \frac{2\pi}{k'}e^{ikR}\left[(k-k')\,f(-\hat{\underline{k}}',\hat{\underline{k}})\,e^{ik'R} - (k+k')\,f(\hat{\underline{k}}',\hat{\underline{k}})\,e^{-ik'R}\right] + \mathcal{O}(1/k'R) \quad (A.8)$$

For k=k' the right-hand side becomes

$$-4\pi f(\hat{\underline{k}}',\hat{\underline{k}}) + \mathcal{O}(1/k'R) \quad (A.9)$$

In evaluating (A.1) we have considered the partial-wave decomposition of the scattering amplitude, integrated term by term and then re-summed the resulting expressions. This is legitimate provided that the sum converges for all values of μ, which is not the case for elastic scattering in the

presence of a point Coulomb potential for which $f(\hat{r},\hat{k})$ diverges at μ=1. In addition, as noted in section (II.2), the asymptotic form of the Coulomb wavefunction is not a plane wave. Clearly a little more attention is required here and we consider this somewhat pathological case in appendix B.

APPENDIX B

THE ADIABATIC SYMMETRY PROPERTY WITH A COULOMB POTENTIAL.

Let ψ^C be the solution of the Schrödinger equation with a point Coulomb potential and let ψ be the solution of the same equation in the presence of both a point Coulomb and a short-range nuclear potential. Since both ψ^C and ψ are finite for finite r we may write down an equation analogous to eq.(II.2.4) for each of these wavefunctions. Subtracting these equations we obtain

$$\int W(\psi^C_{-\underline{k}'} , \psi^N_{\underline{k}}) \, d\mathcal{S} - \{ \underline{k} \leftrightarrow -\underline{k}' \}$$
$$= \int W(\psi^N_{\underline{k}} , \psi^N_{-\underline{k}'}) \, d\mathcal{S} + (k'^2 - k^2) \int (\psi^N_{\underline{k}} \psi^C_{-\underline{k}'} + \psi^N_{-\underline{k}'} \psi^C_{\underline{k}} + \psi^N_{\underline{k}} \psi^N_{-\underline{k}'}) \, dV \qquad \text{(B.1)}$$

where we have introduced $\psi^N = \psi - \psi^C$ which we may think of as the 'nuclear part of the full wavefunction', although it is not the solution of Schrödinger's equation with simply a nuclear potential. Asymptotically (Schiff 1968)

$$\psi^C_{\underline{k}}(\underline{r}) \underset{r\to\infty}{\sim} \exp(i\underline{k}.\underline{r} + i\eta \ln(1-\hat{\underline{k}}.\hat{\underline{r}})) \left[1 + \frac{\eta^2}{ikr(1-\hat{\underline{k}}.\hat{\underline{r}})} \right]$$
$$+ \frac{1}{r} f^C(\hat{\underline{r}},\hat{\underline{k}}) \exp(ikr - i\eta \ln(2kr)) \qquad \text{(B.2)}$$

where f^C is the Coulomb amplitude

$$f^C(\hat{\underline{r}},\hat{\underline{k}}) \underset{r\to\infty}{\sim} \frac{-\eta}{k(1-\hat{\underline{k}}.\hat{\underline{r}})} \exp(-i\eta\ell n((1-\hat{\underline{k}}.\hat{\underline{r}})/2) + 2i\sigma_0) \qquad (B.3)$$

with $\sigma_0=\arg(\Gamma(1+i\eta))$, η being the Sommerfeld parameter. A similar expression may be written for ψ if we simply replace f^C by $f=f^C+f^N$, where f^N is the 'nuclear part of the scattering amplitude'. The asymptotic form of ψ^N is then a purely outgoing wave. The singularities present in eq.(B.2) arise from the infinite range of the Coulomb field which scatters an infinite number of partial waves at forward angles. Consequently we have problems when substituting eq.(B.2) in eq.(B.1) if we try to take the limit $r \to \infty$. At finite r however, the centrifugal potential ensures that only a finite number of partial waves contribute to ψ^C and ψ and so these wavefunctions are well defined. For large, finite r we write

$$\psi^C_{\underline{k}}(r) \approx \exp(i\underline{k}.\underline{r} + i\eta\ell nkr + i\eta h(\mu)) \left(1 + \frac{n^2}{ikr} g(\mu)\right)$$

$$+ r^{-1} f^C(\mu)\exp(ikr - i\eta\ell n2kr) \qquad (B.4)$$

where $\mu=\hat{\underline{k}}.\hat{\underline{r}}$ and the functions f^C, g and h are all well defined for finite r, but tend to their corresponding asymptotic values in eq.(B.2) in the limit $r \to \infty$.

We now evaluate eq.(B.1) using the technique of appendix A, but <u>excluding</u> the two regions $\hat{\underline{r}}\approx\hat{\underline{k}}$ and $\hat{\underline{r}}\approx-\hat{\underline{k}}'$ from the surface and volume integrals containing the cross products $\psi^N\psi^C$. The terms in $\psi^N\psi^N$ contain no singularities and may

be treated entirely as before to give a contribution of $\delta(k-k')$. We therefore obtain an equation analogous to eq.(II.2.11) but containing on the left-hand side the term

$$e^{i\phi}\, k' f^N(-\hat{\underline{k}},-\hat{\underline{k}}') - e^{-i\phi}\, k f^N(\hat{\underline{k}}',\hat{\underline{k}}) \tag{B.5}$$

where $\phi=(k'-k)R-\eta'\ell n(2k'R)+\eta\ell n(2kR)$ instead of simply $(k'-k)R$. Hence f^N satisfies the adiabatic symmetry property in the presence of a Coulomb field provided that those parts of the integrals we have cut out can, in some way, be neglected. The Coulomb amplitude alone satisfies the adiabatic symmetry property as can be seen directly from eq.(B.3)

$$f^C(-\hat{\underline{k}},-\hat{\underline{k}}') = f^C(\hat{\underline{k}}',\hat{\underline{k}}) + \mathcal{O}((k'-k)/(k'+k)) \tag{B.6}$$

although the coefficient of $(k'-k)$ is clearly large for $\hat{\underline{k}}\approx\hat{\underline{k}}'$.

Let us now consider the two parts of the integrals that we have excluded. These contribute

$$\int(k'^2-k^2)(\psi^N_{\underline{k}}\psi^C_{-\underline{k}'} + \psi^N_{-\underline{k}'}\psi^C_{\underline{k}})d\mathscr{V} - \left[\int(\psi^C_{-\underline{k}'}\frac{\partial}{\partial r}\psi^N_{\underline{k}} - \psi^N_{\underline{k}}\frac{\partial}{\partial r}\psi^C_{-\underline{k}'})d\mathscr{S} - \{\underline{k}\leftrightarrow-\underline{k}'\}\right] \tag{B.7}$$

at $\hat{\underline{r}} \approx \hat{\underline{k}}$ and $\hat{\underline{r}} \approx -\hat{\underline{k}}'$.

Since

$$\lim_{r \to 0} (\psi^C_{-\underline{k}'} \frac{\partial}{\partial r} \psi^N_{\underline{k}} - \psi^N_{\underline{k}} \frac{\partial}{\partial r} \psi^C_{-\underline{k}'}) r^2 = 0 \tag{B.8}$$

we may replace the above surface integral by a volume integral

$$\int [\frac{\partial}{\partial r} (r^2 (\psi^C_{-\underline{k}'} \frac{\partial}{\partial r} \psi^N_{\underline{k}} - \psi^N_{\underline{k}} \frac{\partial}{\partial r} \psi^C_{-\underline{k}'})) - \{\underline{k} \leftrightarrow -\underline{k}'\}] dr d\Omega \tag{B.9}$$

and so we can write the contribution to (B.7) from $\hat{\underline{r}} \approx -\hat{\underline{k}}'$ as

$$\int [(k'^2 - k^2) \psi^N_{\underline{k}} \psi^C_{-\underline{k}'} - r^{-2} \frac{\partial}{\partial r} (r^2 (\psi^C_{-\underline{k}'} \frac{\partial}{\partial r} \psi^N_{\underline{k}} - \psi^N_{\underline{k}} \frac{\partial}{\partial r} \psi^C_{-\underline{k}'}))] d\mathscr{V} - \{\underline{k} \leftrightarrow -\underline{k}'\} \tag{B.10}$$

with a similar expression for the other region. For any finite radius sphere this volume integral causes no problems, but we must consider it in the limit $r \to \infty$. Using eq.(B.4) we can write (B.10) in the form, valid for large r and to $\mathcal{O}(1/r)$

$$\int \psi^{N}_{\underline{k}}(\underline{r})\,(1-(\hat{\underline{k}}'.\hat{\underline{r}})^2)\; e^{[-i(\underline{k}'.\underline{r}\;-\;\eta'\ell n(k'r)\;-\;\eta'\ell n(1+\hat{\underline{k}}'.\hat{\underline{r}}))]}\; d\mathcal{V} \tag{B.11}$$

with a similar expression for the other region. The 'nuclear' wavefunction is finite for all $\mu=\hat{\underline{r}}.\hat{\underline{k}}'$, and so as $\mu \to -1$ the integrand in (B.11) behaves as

$$(1+\mu)\exp(i\eta'\ell n(1+\mu)) = (1+\mu)^{(1+i\eta)} \to 0 \tag{B.12}$$

and, mutatis mutandis, a similar expression for the other contribution. The divergent terms vanish identically from (B.7) and so both regions can be neglected and the adiabatic symmetry property takes the form

$$k'\, e^{i\phi} f(-\hat{\underline{k}},-\hat{\underline{k}}') \;-\; k\, e^{-i\phi} f(\hat{\underline{k}}',\hat{\underline{k}}) = \mathcal{O}(k-k') \tag{B.13}$$

with

$$\phi=(k'-k)R-\eta'\ell n(2k'R)+\eta\ell n(2kR) \tag{B.14}$$

in the presence of a point Coulomb potential.

APPENDIX C

THE ADIABATIC SYMMETRY PROPERTY FOR THE FULL DWBA S-MATRIX.

The distorted-waves theory (Satchler 1964) generally uses a different angular momentum coupling scheme to that of Goldberger and Watson (1964) which we use in Chapter II. In the latter the initial and final channel spins $\underline{S}$ and $\underline{S}'$ are formed from the spins of the two nuclei in each channel, and then the total angular momentum is given by

$$\underline{J} = \underline{\ell} + \underline{S} = \underline{\ell}' + \underline{S}' \qquad \text{(C.1)}$$

where $\underline{\ell}$ and $\underline{\ell}'$ are the orbital angular momenta in the initial and final channels respectively. The more usual scheme for the DWBA couples the spin and orbital angular momenta of the projectile first

$$\underline{J}_a = \underline{\ell} + \underline{s}_a$$

$$\underline{J}_b = \underline{\ell}' + \underline{s}_b$$

$$\underline{J} = \underline{J}_A + \underline{J}_a = \underline{J}_B + \underline{J}_b \qquad \text{(C.2)}$$

where $\underline{s}_a$ and $\underline{s}_b$ are the spins of the projectile and ejectile respectively, and the spins of the target and residual nucleus are $\underline{J}_A$ and $\underline{J}_B$. We define the spin, orbital and total angular momenta transferred as

$$\underline{s}=\underline{s}_a-\underline{s}_b$$

$$\underline{j}=\underline{J}_B-\underline{J}_A$$

$$\underline{L}=\underline{j}-\underline{s} \qquad \text{(C.3)}$$

The S matrix (or, equivalently, the T matrix) in the coupling scheme (C.1) is given in terms of that of (C.2) by a simple unitary transformation (Satchler 1964)

$$S^J_{\ell'S';\ell S} = \sum_{J_aJ_b} \hat{S}\hat{S}'\hat{J}_a\hat{J}_b(-1)^{J_b+J_a-\ell-\ell'-s_b-s_a}$$

$$W(J_Bs_bJ\ell';S'J_b)\ W(J_As_aJ\ell;SJ_a)\ U^J_{J_b\ell';J_a\ell} \qquad \text{(C.4)}$$

where

$$U^J_{J_b\ell';J_a\ell} = \hat{J}_a\hat{J}_b\hat{J}_B \sum_{Lsj} i^{L-\ell-\ell'-1}\ A_{Lsj}(-1)^{J_B+J_b-J}$$

$$W(J_aJ_bJ_AJ_B;jJ)\ X(jLs,J_a\ell s_a,J_b\ell's_b)\ I^{Lsj}_{\ell'J_b\ell J_a} \qquad \text{(C.5)}$$

We have used the abbreviation $\hat{\tau}=\sqrt{(2\tau+1)}$, the quantities I^{Lsj} are the generalized DWBA radial integrals and A_{Lsj} is proportional to the spectroscopic factor. If there are no spin-orbit forces the radial integrals become independent of J_a and J_b and so

$$I^{Lsj}_{\ell' J_b \ell J_b} \rightarrow I^{\beta}_{\ell' \ell} \qquad (C.6)$$

where β represents the three quantities L, s and j. On using the symmetries of the X-coefficient (9-j symbol) (Brink and Satchler 1968) we can perform the sum over J_a and J_b in eq.(C.4) giving

$$S^{J}_{\ell' S';\ell S} = \hat{S}\hat{S}'\hat{J}_B(-1)^{2s_a+2s_b+J_B-J_a-s+s'}$$

$$\sum_{\beta} i^{-L-\ell-\ell'-1} A_{\beta} I^{\beta}_{\ell' \ell}$$

$$W(LS'\ell J;S\ell')\ X(ss_b s_a, jJ_B J_A, LS'S)$$

$$\equiv \sum_{\beta} i^{L-\ell-\ell'} Z(\beta) I^{\beta}_{\ell',\ell}\ W(LS'\ell J;S\ell') \qquad (C.7)$$

where $Z(\beta)$ also depends on S, S', s_a, s_b, J_A and J_B. When eq.(C.7) (with ℓ and ℓ' interchanged) is substituted in the adiabatic symmetry property equation (eq.(II.4.4)) we obtain an expression involving sums over β and J'. The latter is easily performed giving

$$S^{J}_{\ell' S';\ell S} = \sum_{\beta} i^{L-\ell-\ell'} Z(\beta) I^{\beta}_{\ell\ell'}\ W(LS'\ell J;S\ell') \qquad (C.8)$$

This is identical to eq.(C.7) if $I^{\beta}_{\ell\ell'} = I^{\beta}_{\ell'\ell}$, which can only be true if the distorted waves in both channels are

computed at the same energy and with the same optical potential. Since we also assumed (C.6), for the general DWBA S-matrix elements to satisfy the adiabatic symmetry property the kinematically-matched conditions must apply.

APPENDIX D.

RELATION OF SPHERICAL HARMONICS TO SUMS OF PRODUCTS OF REDUCED ROTATION MATRIX ELEMENTS.

This relationship can be derived using the closure property of rotation matrix elements (Brink and Satchler 1968), but our proof is slightly different. We use the definitions and conventions of this reference and write

$$\left[\frac{4\pi}{2I+1}\right]^{1/2} Y_{IM}(\theta,0) \equiv D^{I}_{M0}(0,\theta,0)$$

$$\equiv \langle IM|\ D(0,\theta,0)\ |I0\rangle$$

$$= i^{M} \langle IM|\ D(\pi/2,\theta,\pi/2)\ |I0\rangle \qquad (D.1)$$

The rotation D can be re-expressed using a similarity transformation

$$D(\pi/2,\theta,\pi/2) = D(0,\pi/2,-\theta)\ D(\pi,\pi/2,0) \qquad (D.2)$$

which can be proved either by considering each rotation in turn or by considering the two rotations in the spin-half representation of the angular momentum operators. If the result holds for spin-half then it must also hold for other spins since the Pauli matrices used for the spin-half case are a faithful representation of the rotation group. Either way we obtain

$$\left[\frac{4\pi}{2I+1}\right]^{1/2} Y_{IM}(\theta,0) = i^{M} \sum_{K} \langle IM|\ D(0,\pi/2,-\theta)\ |IK\rangle \langle IK|\ D(\pi,\pi/2,0)\ |I0\rangle \qquad (D.3)$$

on insertion of a complete set of states $|IK\rangle$. Introducing the reduced rotation matrix elements we obtain the desired result

$$\left[\frac{4\pi}{2I+1}\right]^{1/2} Y_{IM}(\theta,0) = i^{M} \sum_{K} e^{iK\theta} d^{I}_{MK}(\pi/2)\, d^{I}_{0K}(\pi/2) \qquad (D.4)$$

APPENDIX E.

THE ASYMPTOTIC FORM OF CLEBSCH-GORDAN COEFFICIENTS.

We present here an alternative proof of the form of the product of two Clebsch-Gordan coefficients evaluated in the 'classical' limit of large angular momentum. The particular product that we consider is the one occuring in the DWBA, i.e. $\langle \ell' I 00 | \ell 0 \rangle \langle \ell' I -MM | \ell 0 \rangle$, for which asymptotic forms (in the sense $\ell, \ell' \gg I$) have already been given by Brink and Satchler (1968) and earlier by Brussaard and Tolhoek (1957). Our method, which is based upon the result of appendix D, is, however, slightly less general since it does not work for coefficients of the form $\langle ab\alpha\beta | c\gamma \rangle$ with $\gamma \neq 0$.

The above product of two Clebsch-Gordan coefficients can be written as

$$(-1)^{M} \left[\frac{4\pi\ (2\ell+1)}{(2\ell'+1)\ (2I+1)} \right]^{1/2} \int_0^{\pi} Y_{IM}(\theta,0)\ Y_{\ell'M}(\theta,0)\ Y_{\ell 0}(\theta,0)\ d\Omega \qquad (E.1)$$

and for large ℓ, ℓ' we can use the asymptotic form of the spherical harmonics (Abramowitz and Stegun 1965). The remaining spherical harmonic can be replaced by the exact expression from appendix D. Performing the ϕ-integral we obtain

$$\frac{2}{\pi} \left[\frac{2\ell+1}{2\ell'+1} \right]^{1/2} i^{|M|} \sum_{K'} d^{I}_{MK'}(\pi/2)\ d^{I}_{0K'}(\pi/2) \int_0^{\pi} e^{iK'\theta} \cos[(\ell'+1/2)\theta-(|M|+1/2)\pi/2] \cos[(\ell+1/2)\theta-\pi/4]\ d\theta \qquad (E.2)$$

$$+ \theta(1/\ell') + \theta(1/\ell)$$

To do the remaining integral we write the cosines as sums of exponentials and define $K=\ell-\ell'$ to get

$$\frac{1}{2}(-1)^{I}\sum_{K'} d^{I}_{MK'}(\pi/2)\, d^{I}_{0K'}(\pi/2)\left[(-1)^{M}\delta_{K',-K}+\delta_{K',K}\right] + \mathcal{O}(1/\ell)+\mathcal{O}(1/\ell') \qquad (E.3)$$

which, using the properties of d-matrices, gives

$$\langle \ell' I 0 0 | \ell 0\rangle \langle \ell' I -M M | \ell 0\rangle \approx d^{I}_{MK}(\pi/2)\, d^{I}_{0K}(\pi/2) \qquad (E.4)$$

as required.

APPENDIX F.

THE PHASE OF THE RADIAL INTEGRAL TRANSFORMS.

In Chapter IV we showed that the phase of the transforms $\Omega_k(\omega)$ is k-independent in the quasi-classical limit. Here we find an expression for the phase of this quantity when the radial integral $R_{\ell'\ell}$ is separable into a product of functions of ℓ and ℓ'. This is a good approximation under strong absorption conditions. We assume that these functions are identical, but have different arguments, that they are real for real arguments and also symmetric about zero argument. Such assumptions are usually made for simple parametrizations of $R_{\ell'\ell}$. We therefore write

$$R_k(\lambda) = g(\lambda-\lambda_0+k+i\Lambda_I)\, g(\lambda-\lambda_0+i\Lambda_I+\zeta) \qquad (F.1)$$

where $i\Lambda_I$ is included to account for the possibility of a real nuclear phase, ζ is the mismatch parameter, $k=\ell-\ell'$ and $\lambda=\ell'+1/2$. The grazing angular momentum in the initial channel is $\lambda_0-1/2$. Using the definition of the transform and letting $x'=\lambda-\lambda_0+i\Lambda_I+(k+\zeta)/2$ and $x=x'-i\Lambda_I$ we obtain

$$\Omega_k(\omega) = \exp[i\omega(\lambda_0-i\Lambda_I-\zeta/2)] \int_{-\infty+i\Lambda_I}^{\infty+i\Lambda_I} dx\, \exp(i\omega x)\, g(x+(k-\zeta)/2)\, g(x-(k-\zeta)/2) \qquad (F.2)$$

On taking the complex conjugate of this expression, replacing x by -x and using the assumed symmetry of the function g, this becomes

$$\Omega_k^*(\omega) = \exp[-i\omega(\lambda_0 + i\Lambda_I - \zeta/2)] \int_{-\infty+i\Lambda_I}^{\infty+i\Lambda_I} dx\, \exp(i\omega x)\, g(x+(k-\zeta)/2)\, g(x-(k-\zeta)/2) \quad (F.3)$$

Comparing eqs.(F.2) and (F.3) we then obtain

$$\arg \Omega_k(\omega) = \omega(\lambda_0 - \zeta/2) \quad (F.4)$$

which is independent of k. If the function $R_k(\lambda)$ cannot be written in the form of (F.1) then this result will not necessarily hold. In particular, any asymmetry in the radial integral will destroy this result.

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