Density of the set of probability measures with the martingale representation property

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Abstract

Let ψ be a multi-dimensional random variable. We show that the set of probability measures $\mathbb{Q}$ such that the $\mathbb{Q}$-martingale $S_t^{\mathbb{Q}} = \mathbb{E}^{\mathbb{Q}}\left[\psi \middle| \mathcal{F}_t\right]$ has the Martingale Representation Property (MRP) is either empty or dense in $\mathcal{L}_\infty$-norm. The proof is based on a related result involving analytic fields of terminal conditions $(\psi(x))_{x\in U}$ and probability measures $(\mathbb{Q}(x))_{x\in U}$ over an open set U. Namely, we show that the set of points $x \in U$ such that $S_t(x) = \mathbb{E}^{\mathbb{Q}(x)}\left[\psi(x)\middle| \mathcal{F}_t\right]$ does not have the MRP, either coincides with U or has Lebesgue measure zero. Our study is motivated by the problem of endogenous completeness in financial economics.

Keywords: martingale representation property, martingales, stochastic integrals, analytic fields, endogenous completeness, complete market, equilibrium.

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1 Introduction

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space, $\mathbb{Q}$ be an equivalent probability measure, and $S = (S_t^i)$ be a multi-dimensional martingale under $\mathbb{Q}$. It is often important to know whether S has the Martingale Representation Property (MRP), that is, whether every local martingale under $\mathbb{Q}$ is a stochastic integral with respect to S. For instance, in mathematical finance such MRP corresponds to the *completeness* of the market with stock prices S.

In many applications, S is defined in a *forward form*, as a solution of an SDE, and the verification of the MRP is quite straightforward. Suppose, for example, that S is an Itô process with drift vector-process $b = (b_t)$ and volatility matrix-process $\sigma = (\sigma_t)$. Then the MRP holds if and only if σ has full rank $d\mathbb{P} \times dt$ almost surely. The density process Z of the martingale measure $\mathbb{Q}$ is computed by Girsanov's theorem.

We are interested in the situation where both S and Z are described in a *backward form* through their terminal values:

$$\begin{aligned} Z_\infty &= \frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{\zeta}{\mathbb{E}[\zeta]}, \\ S_t &= \mathbb{E}^{\mathbb{Q}}[\psi|\mathcal{F}_t], \quad t \geq 0, \end{aligned} \tag{1}$$

where $\zeta > 0$ and $\psi = (\psi^i)$ are *given* random variables. Such setup naturally arises in the problem of *endogeneous completness* of financial economics, where the random variable ψ represents the terminal values of the traded securities and $\mathbb{Q}$ defines an equilibrium pricing measure. The term "endogenous" indicates that the stock prices $S = (S^i)$ are *computed* by (1) as part of the solution. The examples include the construction of Radner equilibrium [1, 4, 10, 6] and the verification of the completeness property for a market with options [2, 11].

The main focus of the existing literature has been on the case when the random variables ζ and ψ are defined in terms of a Markov diffusion in a form consistent with Feynman-Kac formula. The proofs have relied on PDE methods and, in particular, on the theory of analytic semigroups [7]. A key role has been played by the assumption that time-dependencies are analytic.

In this paper we do not impose any conditions on the form of the random variables ζ and ψ. Our main results are stated as Theorems 2.3 and 3.1. In

Theorem 2.3 we show that the set

$$\mathcal{Q}(\psi) \triangleq \left\{ \mathbb{Q} \sim \mathbb{P} : \; S_t^{\mathbb{Q}} \triangleq \mathbb{E}^{\mathbb{Q}}\left[\psi | \, \mathcal{F}_t \right] \text{ has the MRP} \right\}$$

is either empty or $\mathcal{L}_\infty$-dense in the set of all equivalent probability measures. In Theorem 3.1 we consider analytic fields of probability measures $(\mathbb{Q}(x))_{x\in U}$ and terminal conditions $(\psi(x))_{x\in U}$ over an open set U. We prove that the exception set

$$I \triangleq \left\{ x \in U : \; S_t(x) \triangleq \mathbb{E}^{\mathbb{Q}(x)}\left[\psi(x) | \, \mathcal{F}_t \right] \text{ does not have the MRP} \right\}$$

either coincides with U or has Lebesgue measure zero.

We expect the results of this paper to be useful in problems of financial economics involving the endogeneous completeness property. In particular, they play a key role in our work, in progress, on the problem of optimal investment in a "backward" model of price impact [3, 8], where stock prices and wealth processes solve a coupled system of quadratic BSDEs. Theorem 2.3 allows us to relax this apparently complex stochastic control problem into a simple *static* framework, where maximization is performed over a set of random variables. We show that the solution ξ of the static problem yields the optimal investment strategy provided that the stock prices S given by (1) have the MRP. Using Theorem 3.1 we prove such MRP for almost all values of the model parameters.

2 Density of the set of probability measures with the MRP

We work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions of completeness and right-continuity; the initial σ-algebra $\mathcal{F}_0$ is trivial and $\mathcal{F} = \mathcal{F}_\infty$. We denote by $\mathcal{L}_1 = \mathcal{L}_1(\mathbf{R}^d)$ and $\mathcal{L}_\infty = \mathcal{L}_\infty(\mathbf{R}^d)$ the Banach spaces of (equivalence classes of) d-dimensional random variables ξ with the norms $\|\xi\|_{\mathcal{L}_1} \triangleq \mathbb{E}\left[|\xi|\right]$ and $\|\xi\|_{\mathcal{L}_\infty} \triangleq \inf\left\{c > 0 : \; \mathbb{P}\left[|\xi| \leq c\right] = 1\right\}$. We use same notation $\mathcal{L}_1$ for the isometric Banach space of uniformly integrable martingales M with the norm $\|M\|_{\mathcal{L}_1} \triangleq \|M_\infty\|_{\mathcal{L}_1}$.

For a matrix $A = (A^{ij})$ we denote its transpose by A^* and define its norm as

$$|A| \triangleq \sqrt{\operatorname{tr} AA^*} = \sqrt{\sum_{i,j} |A^{ij}|^2}.$$

If X is a m-dimensional semimartingale and γ is a $m \times n$-dimensional X-integrable predictable process, then $\gamma \cdot X = \int \gamma^* dX$ denotes the n-dimensional stochastic integral of γ with respect to X. We recall that a $n \times k$-dimensional predictable process ζ is $(\gamma \cdot X)$-integrable if and only if $\gamma\zeta$ is X-integrable. In this case, $\zeta \cdot (\gamma \cdot X) = (\gamma\zeta) \cdot X$ is a k-dimensional semimartingale.

Definition 2.1. Let $\mathbb{Q}$ be an equivalent probability measure ($\mathbb{Q} \sim \mathbb{P}$) and S be a d-dimensional local martingale under $\mathbb{Q}$. We say that S *has the Martingale Representation Property (MRP)* if every local martingale M under $\mathbb{Q}$ is a stochastic integral with respect to S, that is, there is a predictable S-integrable process γ with values in $\mathbf{R}^d$ such that

$$M = M_0 + \gamma \cdot S.$$

Remark 2.2. Jacod's theorem in [5, Section XI.1(a)] states that S has the MRP if and only if there is only one $\mathbb{Q} \sim \mathbb{P}$ such that S is a local martingale under $\mathbb{Q}$. Thus, there is no need to mention $\mathbb{Q}$ in the definition of the MRP.

Let $\psi = (\psi^i)_{i=1,\dots,d}$ be a d-dimensional random variable. We denote by $\mathcal{Q}(\psi)$ the family of probability measures $\mathbb{Q} \sim \mathbb{P}$ such that $\mathbb{E}^{\mathbb{Q}}\left[|\psi|\right] < \infty$ and the $\mathbb{Q}$-martingale

$$S^{\mathbb{Q}}_t = \mathbb{E}^{\mathbb{Q}}\left[\psi | \mathcal{F}_t\right], \quad t \geq 0,$$

has the MRP.

This is our first main result.

Theorem 2.3. *Suppose that $\psi \in \mathcal{L}_1(\mathbf{R}^d)$ and $\mathcal{Q}(\psi) \neq \emptyset$. Then for every $\epsilon > 0$ there is $\mathbb{Q} \in \mathcal{Q}(\psi)$ such that*

$$\| \frac{d\mathbb{Q}}{d\mathbb{P}} - 1 \|_{\mathcal{L}_\infty} \leq \epsilon.$$

The proof is based on Theorem 3.1 from Section 3 and on the following elementary lemma. We recall the definition of an analytic function with values in a Banach space at the beginning of Section 3.

Lemma 2.4. *Let ζ be a nonnegative random variable. Then the map $x \mapsto e^{-x\zeta}$ from $(0,\infty)$ to $\mathcal{L}_\infty$ is analytic.*

Proof. Fix $y > 0$. For every $\omega \in \Omega$ the function $x \mapsto e^{-x\zeta(\omega)}$ has a Taylor's expansion

$$e^{-x\zeta(\omega)} = \sum_{n=0}^{\infty} A_n(y)(\omega)(x-y)^n, \quad x \in \mathbf{R}, \tag{2}$$

where

$$A_n(y) = \frac{1}{n!}\frac{d^n}{dx^n}\left(e^{-x\zeta}\right)|_{x=y} = \frac{1}{n!}(-1)^n\zeta^n e^{-y\zeta}.$$

We deduce that

$$\|A_n(y)\|_{\mathcal{L}_\infty} \leq \frac{1}{n!}\max_{t\geq 0}(t^n e^{-yt}) = \frac{1}{n!}\left(\frac{n}{ey}\right)^n \leq K\frac{1}{\sqrt{n}}\left(\frac{1}{y}\right)^n,$$

where the existence of a constant $K > 0$ follows from Sterling's formula:

$$\lim_{n\to\infty}\frac{\sqrt{2\pi n}}{n!}\left(\frac{n}{e}\right)^n = 1.$$

It follows that the series in (2) converges in $\mathcal{L}_\infty$ provided that $|x-y| < y$. ☐

Proof of Theorem 2.3. We take $\mathbb{R} \in \mathcal{Q}(\psi)$, denote $\zeta \triangleq \frac{d\mathbb{R}}{d\mathbb{P}}$, and for $x > 0$ define the random variables

$$\zeta(x) \triangleq \frac{1-e^{-x\zeta}}{x} + \frac{x}{1+x},$$
$$\xi(x) \triangleq \zeta(x)\psi,$$

and a probability measure $\mathbb{Q}(x)$ such that

$$\frac{d\mathbb{Q}(x)}{d\mathbb{P}} = \frac{\zeta(x)}{\mathbb{E}\left[\zeta(x)\right]}.$$

We set $\zeta(0) \triangleq \zeta$, $\xi(0) \triangleq \zeta\psi$, and $\mathbb{Q}(0) \triangleq \mathbb{R}$ and observe that for every $\omega \in \Omega$ the functions $x \mapsto \zeta(x)(\omega)$ and $x \mapsto \xi(x)(\omega)$ on $[0,\infty)$ are continuous. Since

$$|\zeta(x)| \leq \zeta \sup_{t\geq 0}\frac{1-e^{-t}}{t} + \frac{x}{1+x} \leq \zeta + 1,$$

the dominated convergence theorem yields that $x \mapsto \zeta(x)$ and $x \mapsto \xi(x)$ are continuous maps from $[0,\infty)$ to $\mathcal{L}_1$. By Lemma 2.4, $x \mapsto \zeta(x)$ is an analytic map from $(0,\infty)$ to $\mathcal{L}_\infty$ and thus $x \mapsto \zeta(x)$ and $x \mapsto \xi(x)$ are analytic maps from $(0,\infty)$ to $\mathcal{L}_1$. Theorem 3.1 then implies that the exception set

$$I \triangleq \{x > 0 : \ \mathbb{Q}(x) \notin \mathcal{Q}(\psi)\}$$

is at most countable.

Choose now any $\epsilon > 0$. Since

$$-\frac{1}{1+x} \leq \zeta(x) - 1 \leq \frac{1}{x} - \frac{1}{1+x},$$

there is $x_0 = x_0(\epsilon)$ such that the assertion of the theorem holds for every $\mathbb{Q}(x)$ with $x \geq x_0$ and $x \notin I$. ☐

3 The MRP for analytic fields of martingales

Let $\mathbf{X}$ be a Banach space and U be an open connected set in $\mathbf{R}^d$. We recall that a map $x \mapsto X(x)$ from U to $\mathbf{X}$ is *analytic* if for every $y \in U$ there exist a number $\epsilon = \epsilon(y) > 0$ and elements $(Y_\alpha(y))$ in $\mathbf{X}$ such that the ϵ-neighborhood of y belongs to U and

$$X(x) = \sum_\alpha Y_\alpha(y)(x-y)^\alpha, \quad |y-x| < \epsilon.$$

Here the series converges in the norm $\|\cdot\|_{\mathbf{X}}$ of $\mathbf{X}$, the summation is taken with respect to multi-indices $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbf{Z}^l_+$ of non-negative integers, and if $x = (x_1, \dots, x_d) \in \mathbf{R}^d$, then $x^\alpha \triangleq \prod_{i=1}^d x_i^{\alpha_i}$.

This is our second main result.

Theorem 3.1. *Let U be an open connected set in $\mathbf{R}^l$ and suppose that the point $x_0 \in \mathbf{R}^l$ belongs to the closure of U. Let $x \mapsto \zeta(x)$ and $x \mapsto \xi(x)$ be continuous maps from $U \cup \{x_0\}$ to $\mathcal{L}_1(\mathbf{R})$ and $\mathcal{L}_1(\mathbf{R}^d)$, respectively, whose restrictions to U are analytic. For every $x \in U \cup \{x_0\}$, assume that $\zeta(x) > 0$ and define a probability measure $\mathbb{Q}(x)$ and a $\mathbb{Q}(x)$-martingale $S(x)$ by*

$$\frac{d\mathbb{Q}(x)}{d\mathbb{P}} = \frac{\zeta(x)}{\mathbb{E}\left[\zeta(x)\right]}, \quad S_t(x) = \mathbb{E}^{\mathbb{Q}(x)}\left[\left.\frac{\xi(x)}{\zeta(x)}\right| \mathcal{F}_t\right].$$

If $S(x_0)$ has the MRP, then the exception set

$$I \triangleq \{x \in U : \ S(x) \text{ does not have the MRP}\}$$

has Lebesgue measure zero. If, in addition, U is an interval in $\mathbf{R}$, then the set I is at most countable.

The following example shows that *any* countable set I in $\mathbf{R}$ can play the role of the exception set of Theorem 3.1. In this example we choose $\zeta(x) = 1$ (so that $\mathbb{Q}(x) = \mathbb{P}$) and take $x \mapsto \xi(x)$ to be a *linear* map from $\mathbf{R}$ to $\mathcal{L}_\infty(\mathbf{R})$.

Example 3.2. Let $(\Omega, \mathcal{F}, (\mathcal{F}_n), \mathbb{P})$ be a filtered probability space, where the filtration is generated by independent Bernoulli random variables (ϵ_n) with

$$\mathbb{P}\left[\epsilon_n = 1\right] = \mathbb{P}\left[\epsilon_n = -1\right] = \frac{1}{2}.$$

It is well-known that every martingale (N_n) admits the unique "integral" representation:

$$N_n = N_0 + \sum_{k=1}^{n} h_k(\epsilon_1, \dots, \epsilon_{k-1})\epsilon_k, \tag{3}$$

for some functions $h_k = h_k(x_1, \dots, x_{k-1})$, $k \geq 1$, where h_1 is just a constant.

Let $I = (x_n)$ be an arbitrary sequence in $\mathbf{R}$. We define a linear map $x \mapsto \xi(x)$ from $\mathbf{R}$ to $\mathcal{L}_\infty(\mathbf{R})$ by

$$\xi(x) = \sum_{n=1}^{\infty} \frac{(x - x_n)}{2^n(1 + |x_n|)}\epsilon_n = \psi_0 + \psi_1 x,$$

where ψ_0 and ψ_1 are bounded random variables:

$$\psi_0 = -\sum_{n=1}^{\infty} \frac{x_n}{2^n(1 + |x_n|)}\epsilon_n, \quad \psi_1 = \sum_{n=1}^{\infty} \frac{1}{2^n(1 + |x_n|)}\epsilon_n.$$

We have that

$$M_n(x) = \mathbb{E}\left[\xi(x)|\,\mathcal{F}_n\right] = \mathbb{E}\left[\xi(x)|\epsilon_1, \dots, \epsilon_n\right] = \sum_{k=1}^{n} \frac{(x - x_k)}{2^k(1 + |x_k|)}\epsilon_k$$

and thus

$$\Delta M_n(x) = M_n(x) - M_{n-1}(x) = \frac{(x - x_n)}{2^n(1 + |x_n|)}\epsilon_n.$$

If $x \notin I$, then the martingale (N_n) from (3) is a stochastic integral with respect to $M(x)$:

$$N_n = N_0 + \sum_{k=1}^{n} h_k(\epsilon_1, \dots, \epsilon_{k-1})\frac{2^k(1 + |x_k|)}{(x - x_k)}\Delta M_k(x).$$

However, if $x_m \in I$, then the martingales $M(x_m)$ and

$$L_n^{(m)} = \sum_{k=1}^{n} 1_{\{k=m\}}\epsilon_k = 1_{\{n \geq m\}}\epsilon_m, \quad n \geq 0,$$

are orthogonal. Hence, $L^{(m)}$ does not admit an integral representation with respect to $M(x_m)$.

The rest of the section is devoted to the proof of Theorem 3.1. It relies on Theorems A.1 and B.1 from the appendices and on the lemmas below.

Let X be a (uniformly) square integrable martingale taking values in $\mathbf{R}^m$. We denote by $\langle X \rangle = (\langle X^i, X^j \rangle)$ its predictable process of quadratic variation, which takes values in the cone $\mathcal{S}^m_+$ of symmetric nonnegative $m \times m$-matrices, and define the predictable increasing process

$$A^X \triangleq \operatorname{tr} \langle X \rangle = \sum_{i=1}^m \left\langle X^i, X^i \right\rangle .$$

Standard arguments show that there is a predictable process κ^X with values in $\mathcal{S}^m_+$ such that

$$\langle X \rangle = (\kappa^X)^2 \cdot A^X.$$

On the predictable σ-algebra $\mathcal{P}$ of $[0, \infty) \times \Omega$ we introduce a measure

$$\mu^X(dt, d\omega) \triangleq dA^X_t(\omega) \mathbb{P}\left[d\omega\right].$$

For a nonnegative predictable process γ the expectation under μ^X is given by

$$\mathbb{E}^{\mu^X}\left[\gamma\right] = \mathbb{E}\left[\int_0^\infty \gamma_t dA^X_t\right].$$

We observe that this measure is finite:

$$\mu^X([0, \infty) \times \Omega) = \mathbb{E}\left[A^X_\infty\right] = \mathbb{E}\left[|X_\infty - X_0|^2\right] < \infty.$$

For predictable processes (γ^n) and γ the notation $\gamma^n \xrightarrow{\mu^X} \gamma$ stands for the convergence in measure μ^X:

$$\forall \epsilon > 0 : \quad \mu^X\left[|\gamma^n - \gamma| > \epsilon\right] \to 0, \quad n \to \infty.$$

Lemma 3.3. *Let X be a square integrable martingale with values in $\mathbf{R}^m$ and γ be a predictable m-dimensional process. Then γ is X-integrable and $\gamma \cdot X = 0$ if and only if $\kappa^X \gamma = 0$, $\mu^X - a.s.$.*

Proof. Since $\gamma 1_{\{|\gamma| \leq n\}} \cdot X \to \gamma \cdot X$ as $n \to \infty$ in the semimartingale topology, we can assume without a loss in generality that γ is bounded. Then $\gamma \cdot X$ is a square integrable martingale with predictable quadratic variation

$$\langle \gamma \cdot X \rangle_t = \int_0^t \left|\kappa^X \gamma\right|^2 dA^X$$

and the result follows from the identity:

$$\mathbb{E}\left[(\gamma\cdot X)_\infty^2\right] = \mathbb{E}\left[\langle\gamma\cdot X\rangle_\infty\right] = \mathbb{E}\left[\int_0^\infty \left|\kappa^X\gamma\right|^2 dA^X\right] = \mathbb{E}^{\mu^X}\left[\left|\kappa^X\gamma\right|^2\right].$$

□

For every predictable process ζ taking values in $\mathcal{S}_+^m$ we can naturally define a $\mathcal{S}_+^m$-valued predictable process $\zeta^\oplus$ such that for all (ω,t) the matrix $\zeta_t^\oplus(\omega)$ is the pseudo-inverse to the matrix $\zeta_t(\omega)$.

From Lemma 3.3 we deduce that if α is an integrand for X then the predictable process

$$\beta \triangleq \kappa^{X\oplus}\kappa^X\alpha$$

is also X-integrable and $\alpha\cdot X = \beta\cdot X$. Moreover, $|\beta|\leq|\alpha|$, by the minimal norm property of the pseudo-inverse matrices. In view of this property, we call a predictable m-dimensional process γ a *minimal integrand* for X if γ is X-integrable and

$$\gamma = \kappa^{X\oplus}\kappa^X\gamma.$$

From the definition of a minimal integrand we immediately deduce that

$$\left|\kappa^X\gamma\right| \leq \left|\kappa^X\right|\left|\gamma\right|, \quad \left|\gamma\right| \leq \left|\kappa^{X\oplus}\right|\left|\kappa^X\gamma\right|. \tag{4}$$

We denote by $\mathcal{H}_1 = \mathcal{H}_1(\mathbf{R}^d)$ the Banach space of uniformly integrable d-dimensional martingales M with the norm:

$$\|M\|_{\mathcal{H}_1} \triangleq \mathbb{E}\left[\sup_{t\geq 0}|M_t|\right].$$

We say that a sequence (N^n) of local martingales converges to a local martingale N in $\mathcal{H}_{1,loc}$ if there are stopping times (τ^m) such that $\tau^m\uparrow\infty$ and $N^{n,\tau^m}\to N^{\tau^m}$ in $\mathcal{H}_1$. Here as usual, we write $Y^\tau \triangleq (Y_{\min(t,\tau)})$ for a semimartingale Y stopped at a stopping time τ.

Lemma 3.4. *Let X be a square integrable martingale with values in $\mathbf{R}^m$ and (γ^n) be a sequence of predictable m-dimensional X-integrable processes such that the stochastic integrals $(\gamma^n\cdot X)$ converge to 0 in $\mathcal{H}_{1,loc}$. Then $\kappa^X\gamma^n \stackrel{\mu^X}{\to} 0$. If, in addition, (γ^n) are minimal integrands then $\gamma^n \stackrel{\mu^X}{\to} 0$.*

Proof. It is sufficient to consider the case of minimal integrands. By localization, we can suppose that $\gamma^n \cdot X \to 0$ in $\mathcal{H}_1$, which by Davis' inequality is equivalent to the convergence of $([\gamma^n \cdot X]_\infty^{1/2})$ to 0 in $\mathcal{L}_1$.

Assume for a moment that $|\gamma^n| \leq 1$. Then $[\gamma^n \cdot X] \leq [X]$ and the theorem on dominated convergence yields that $[\gamma^n \cdot X]_\infty \to 0$ in $\mathcal{L}_1$. It follows that

$$\lim_{n\to\infty} \mathbb{E}\left[[\gamma^n \cdot X]_\infty\right] = \lim_{n\to\infty} \mathbb{E}^{\mu^X}\left[\left|\kappa^X \gamma^n\right|^2\right] = 0.$$

Hence, $\kappa^X \gamma^n \overset{\mu^X}{\to} 0$, which in view of (4), also implies that $\gamma^n \overset{\mu^X}{\to} 0$.

In the general case, we observe that

$$\beta^n \triangleq \frac{1}{1+|\gamma^n|}\gamma^n$$

are minimal integrands for X such that $|\beta^n| \leq 1$ and $[\beta^n \cdot X] \leq [\gamma^n \cdot X]$. Hence, by what we have already proved, $\beta^n \overset{\mu^X}{\to} 0$, which clearly yields that $\gamma^n \overset{\mu^X}{\to} 0$ and then that $\kappa^X \gamma^n \overset{\mu^X}{\to} 0$. □

Lemma 3.5. *Let X be a square integrable m-dimensional martingale and $\gamma = (\gamma^{ij})$ be a predictable X-integrable process with values in $\mathbf{R}^{m\times d}$. Then X is a stochastic integral with respect to $Y \triangleq \gamma \cdot X$, that is $X = X_0 + \zeta \cdot Y$ for some predictable Y-integrable $d \times m$-dimensional process ζ, if and only if*

$$\operatorname{rank} \kappa^X \gamma = \operatorname{rank} \kappa^X, \quad \mu^X - a.s.. \tag{5}$$

Proof. We recall that a predictable process ζ is $Y = \gamma \cdot X$-integrable if and only if $\gamma\zeta$ is X-integrable. From Lemma 3.3 we deduce that ζ is Y-integrable and satisfies

$$X = X_0 + \zeta \cdot Y = X_0 + \zeta \cdot (\gamma \cdot X) = (\gamma\zeta) \cdot X$$

if and only if

$$\kappa^X \gamma\zeta = \kappa^X, \quad \mu^X - a.s..$$

However, the solvability of this linear equation with respect to ζ is equivalent to (5) by an elementary argument from linear algebra. □

Lemma 3.6. *Let U be an open connected set in $\mathbf{R}^d$ and $x \mapsto \sigma(x)$ be an analytic map with values in $k \times l$-matrices. Then there is a nonzero real-analytic function f on U such that*

$$E \triangleq \left\{x \in U : \operatorname{rank} \sigma(x) < \sup_{y \in U} \operatorname{rank} \sigma(y)\right\} = \{x \in U : f(x) = 0\}.$$

In particular, the set E has Lebesgue measure zero and if $d = 1$, then it consists of isolated points.

Proof. Let $m \triangleq \sup_{y \in U} \operatorname{rank} \sigma(y)$. If $m = 0$, then the set E is empty and we can take $f = 1$. If $m > 0$, then the result hods for

$$f(x) = \sum_{\alpha} \det \sigma_\alpha(x) \sigma^*_\alpha(x),$$

where (σ_α) is the family of all $m \times m$ sub-matrices of σ. The remaining assertions follow from the well-known properties of zero-sets of real-analytic functions. □

Proof of Theorem 3.1. Without restricting generality we can assume that $\zeta(x_0) = 1$ and, hence, $\mathbb{Q}(x_0) = \mathbb{P}$. Proposition 2 in [9] shows that if some multi-dimensional local martingale has the MRP, then there is a bounded, hence square integrable, m-dimensional martingale X that has the MRP. We fix such X and use for it the $\mathcal{S}^m_+$-valued predictable process κ^X and the finite measure μ^X on the predictable σ-algebra $\mathcal{P}$ introduced just before Lemma 3.3.

We define the martingales

$$Y_t(x) \triangleq \mathbb{E}\left[\zeta(x) | \mathcal{F}_t\right], \quad R_t(x) \triangleq \mathbb{E}\left[\xi(x) | \mathcal{F}_t\right],$$

and observe that $R(x) = S(x)Y(x)$. Let $\alpha(x)$ and $\beta(x)$ be integrands for X with values in $\mathbf{R}^m$ and $\mathbf{R}^{m \times d}$, respectively, such that

$$\begin{aligned} Y(x) &= Y_0(x) + Y_-(x)\alpha(x) \cdot X, \\ R(x) &= R_0(x) + Y_-(x)\beta(x) \cdot X. \end{aligned}$$

Integration by parts yields that

$$dR(x) - S_-(x)dY(x) = Y_-(x)d(S(x) + [S(x), \alpha(x) \cdot X]).$$

It follows that

$$S(x) + [S(x), \alpha(x) \cdot X] = S_0(x) + \sigma(x) \cdot X,$$

where

$$\sigma(x) = \beta(x) - \alpha(x)S^*_-(x).$$

From Theorem B.1 we deduce that $S(x)$ has the MRP (under $\mathbb{Q}(x)$) if and only if the stochastic integral $\sigma(x) \cdot X$ has the MRP. By Lemma 3.5 the latter property is equivalent to

$$\operatorname{rank} \kappa^X \sigma(x) = \operatorname{rank} \kappa^X, \quad \mu^X - a.s.,$$

and therefore, the exception set I admits the description:

$$I = \left\{ x \in U : \ \mu^X \left[D(x) \right] > 0 \right\},$$

where for $x \in U \cup \{x_0\}$ the predictable set $D(x)$ is given by

$$D(x) = \left\{ (\omega, t) : \ \operatorname{rank} \kappa^X_t(\omega) \sigma_t(x)(\omega) < \operatorname{rank} \kappa^X_t(\omega) \right\}.$$

From Theorem A.1 we deduce the existence of the integrands $\alpha(x)$ and $\beta(x)$ and of the modifications of the martingales $Y(x)$ and $R(x)$ such that for every $(\omega, t) \in \Omega \times [0, \infty)$ the function

$$x \mapsto \sigma_t(x)(\omega) = \beta_t(x)(\omega) - \alpha_t(x)(\omega) \frac{R^*_{t-}(x)(\omega)}{Y_{t-}(x)(\omega)},$$

taking values in the space of $m \times d$-matrices, is analytic on U. Hereafter, we shall use these versions.

Let λ be the Lebesgue measure on $\mathbf{R}^l$ and $\mathcal{B} = \mathcal{B}(U)$ be the Borel σ-algebra on U. Since for every (ω, t) the function $x \mapsto \sigma_t(x)(\omega)$ is continuous on U, the function $(\omega, t, x) \mapsto \sigma_t(x)(\omega)$ is $\mathcal{P} \times \mathcal{B}$-measurable. It follows that

$$E \triangleq \left\{ (\omega, t, x) : \ \operatorname{rank} \kappa^X_t(\omega) \sigma_t(x)(\omega) < \operatorname{rank} \kappa^X_t(\omega) \right\} \in \mathcal{P} \times \mathcal{B}.$$

From Fubini's theorem we deduce the equivalences:

$$(\mu^X \times \lambda) \left[E \right] = 0 \quad \Leftrightarrow \quad \mu^X \left[F \right] = 0 \quad \Leftrightarrow \quad \lambda \left[I \right] = 0,$$

where

$$F \triangleq \left\{ (\omega, t) : \ \lambda \left[\left\{ x \in U : \ \operatorname{rank} \kappa^X_t(\omega) \sigma_t(x)(\omega) < \operatorname{rank} \kappa^X_t(\omega) \right\} \right] > 0 \right\}.$$

Hence to obtain the multi-dimensional version of the theorem we need to show that $\mu^X(F) = 0$.

From Lemma 3.6 and the analyticity of the function $x \mapsto \sigma_t(x)(\omega)$ we deduce that

$$F = \left\{ (\omega, t) : \ \operatorname{rank} \kappa^X_t(\omega) \sigma_t(x)(\omega) < \operatorname{rank} \kappa^X_t(\omega), \ \forall x \in U \right\}. \tag{6}$$

We recall now that if (x_n) is a sequence in U that converges to x_0, then the martingales $(R(x_n), Y(x_n))$ converge to the martingale $(R(x_0), Y(x_0)) = (S(x_0), 1)$ in $\mathcal{L}_1$. By Lemma A.3, passing to a subsequence, we can assume that $(R(x_n), Y(x_n)) \to (R(x_0), Y(x_0))$ in $\mathcal{H}_{1,loc}$. From Lemma 3.4 we deduce that

$$\kappa^X \alpha(x_n) \stackrel{\mu^X}{\to} 0,$$
$$\kappa^X \beta(x_n) \stackrel{\mu^X}{\to} \kappa^X \beta(x_0) = \kappa^X \sigma(x_0).$$

It follows that

$$\kappa^X \sigma(x_n) = \kappa^X (\beta(x_n) - \alpha(x_n) S^*_-(x_n)) \stackrel{\mu^X}{\to} \kappa^X \beta(x_0) = \kappa^X \sigma(x_0).$$

Passing to a subsequence we can choose the sequence (x_n) so that

$$\kappa^X \sigma(x_n) \to \kappa^X \sigma(x_0), \quad \mu^X - a.s..$$

As $a \mapsto \operatorname{rank} a$ is a lower-semicontinuous function on matrices, it follows that

$$\liminf_n \operatorname{rank} \kappa^X \sigma(x_n) \geq \operatorname{rank} \kappa^X \sigma(x_0), \quad \mu^X - a.s..$$

Accounting for (6) we obtain that

$$F \subset D(x_0), \quad \mu^X - a.s..$$

However, as $S(x_0)$ has the MRP, Lemma 3.5 yields that $\mu^X [D(x_0)] = 0$ and the multi-dimensional version of the theorem follows.

Assume now that U is an open interval in $\mathbf{R}$ and that contrary to the assertion of the theorem the exception set I is uncountable. Then there are $\epsilon > 0$, a closed interval $[a, b] \subset U$, and a sequence $(x_n) \subset [a, b]$ such that

$$\mu^X [D(x_n)] \geq \epsilon, \quad n \geq 1.$$

Since for every (ω, t) the function $x \mapsto \sigma_t(x)(\omega)$ is analytic, we deduce from Lemma 3.6 that on every closed interval the integer-valued function $x \mapsto \operatorname{rank}(\sigma_t(x)(\omega))$ has constant value except for a *finite* number of points, where its values are smaller. It follows that

$$\limsup_n D(x_n) \triangleq \cap_n \cup_{m \geq n} D(x_m) = F$$

and thus

$$\mu^X [F] \geq \limsup_n \mu^X [D(x_n)] \geq \epsilon.$$

However, as we have already shown, $\mu^X [F] = 0$ and we arrive to a contradiction. $\square$

A Analytic fields of martingales and stochastic integrals

We denote by $\mathbf{D}^\infty([0,\infty),\mathbf{R}^d)$ the Banach space of RCLL (right-continuous with left limits) functions $f: [0,\infty)\to\mathbf{R}^d$ equipped with the uniform norm: $\|f\|_\infty \triangleq \sup_{t\geq 0}|f(t)|$.

Theorem A.1. *Let U be an open connected set in $\mathbf{R}^l$ and $x\mapsto\xi(x)$ be an analytic map from U to $\mathcal{L}_1(\mathbf{R}^d)$. Then there are modifications of the accompanying d-dimensional martingales*

$$M_t(x) \triangleq \mathbb{E}\left[\xi(x)|\,\mathcal{F}_t\right],$$

such that for every $\omega\in\Omega$ the maps $x\mapsto M_\cdot(x)(\omega)$ taking values in $\mathbf{D}^\infty([0,\infty),\mathbf{R}^d)$ are analytic on U.

If in addition, the MRP holds for a local martingale X with values in $\mathbf{R}^m$, then there is a stochastic field $x\mapsto\sigma(x)$ of integrands for X such that

$$M(x) = M_0(x) + \sigma(x)\cdot X,$$

and for every $(\omega,t)\in\Omega\times[0,\infty)$ the function $x\mapsto\sigma_t(x)(\omega)$ taking values in $m\times d$-matrices is analytic on U.

The proof of the theorem is divided into a series of lemmas. For a multi-index $\alpha=(\alpha_1,\dots,\alpha_l)\in\mathbf{Z}^l_+$ we denote

$$|\alpha| \triangleq \alpha_1+\cdots+\alpha_l.$$

The space $\mathcal{H}_1$ has been introduced just before Lemma 3.4.

Lemma A.2. *Let $(M^\alpha)_{\alpha\in\mathbf{Z}^l_+}$ be uniformly integrable martingales with values in $\mathbf{R}^d$ such that*

$$\sum_\alpha 2^{|\alpha|}\|M^\alpha\|_{\mathcal{L}_1} < \infty.$$

Then there is an increasing sequence (τ_m) of stopping times such that $\{\tau_m=\infty\}\uparrow\Omega$ and

$$\sum_\alpha \|M^{\alpha,\tau_m}\|_{\mathcal{H}_1}<\infty, \quad m\geq 1.$$

Proof. We define the martingale

$$L_t \triangleq \mathbb{E}\left[\sum_\alpha 2^{|\alpha|}\left|M_\infty^\alpha\right|\middle|\mathcal{F}_t\right],\ t \geq 0,$$

and stopping times

$$\tau_m \triangleq \inf\left\{t \geq 0 :\ L_t \geq m\right\},\ m \geq 1.$$

Clearly, $\{\tau_m = \infty\} \uparrow \Omega$ and $|M^\alpha| \leq 2^{-|\alpha|}L$. Moreover,

$$\|L^{\tau_m}\|_{\mathcal{H}_1} = \mathbb{E}\left[\sup_{0\leq t\leq \tau_m} L_t\right] \leq m + \mathbb{E}\left[L_{\tau_m}\right] = m + L_0 < \infty.$$

It follows that

$$\sum_\alpha \|M^{\alpha,\tau_m}\|_{\mathcal{H}_1} \leq \|L^{\tau_m}\|_{\mathcal{H}_1} \sum_\alpha 2^{-|\alpha|} < \infty.$$

□

Lemma A.3. *Let (M^n) and M be uniformly integrable martingales such that $M^n \to M$ in $\mathcal{L}_1$. Then there exists a subsequence of (M^n) that converges to M in $\mathcal{H}_{1,loc}$.*

Proof. Since $M^n \to M$ in $\mathcal{L}_1$ there exists a subsequence (M^{n_k}) such that

$$\sum_{k=1}^\infty \|M^{n_{k+1}} - M^{n_k}\|_{\mathcal{L}_1} 2^k < \infty.$$

Lemma A.2 implies that $M^{n_k} \to M$ in $\mathcal{H}_{1,loc}$. □

Let X be a square integrable martingale taking values in $\mathbf{R}^m$. As in Section 3 we associate with X the increasing predictable process $A^X \triangleq \operatorname{tr}\langle X\rangle$, the $\mathcal{S}_+^m$-valued predictable process κ^X such that $\langle X\rangle = (\kappa^X)^2 \cdot A^X$, and a finite measure $\mu^X(dt, d\omega) \triangleq dA_t^X(\omega)\mathbb{P}\left[d\omega\right]$ on the predictable σ-algebra $\mathcal{P}$ of $\Omega \times [0, \infty)$. We recall that an integrand γ for X is *minimal* if

$$\gamma = \kappa^{X\oplus}\kappa^X\gamma. \tag{7}$$

Lemma A.4. *Let X be a bounded martingale with values in $\mathbf{R}^m$ and $(\gamma^\alpha)_{\alpha\in\mathbf{Z}^l_+}$ be minimal integrands for X such that*

$$\sum_\alpha \|\gamma^\alpha \cdot X\|_{\mathcal{H}^1} < \infty. \tag{8}$$

Then

$$\sum_\alpha |\gamma^\alpha|^2 < \infty, \quad \mu^X - a.s.. \tag{9}$$

Proof. By Davis' inequality, (8) is equivalent to

$$\sum_\alpha \mathbb{E}\left[[\gamma^\alpha \cdot X]_\infty^{1/2}\right] < \infty.$$

By replacing if necessary γ^α with $\frac{1}{1+|\gamma^\alpha|}\gamma^\alpha$, we can assume without a loss of generality that $|\gamma^\alpha| \leq 1$. Let us show that in this case the increasing optional process

$$B_t \triangleq \sum_\alpha [\gamma^\alpha \cdot X]_t, \quad t \geq 0,$$

is locally integrable. Since

$$B_\infty = \sum_\alpha [\gamma^\alpha \cdot X]_\infty \leq \left(\sum_\alpha [\gamma^\alpha \cdot X]_\infty^{1/2}\right)^2 < \infty,$$

we only need to check that the positive jump process ΔB is locally integrable. Actually, we shall show that $\sup_{t\geq 0} \Delta B_t$ is integrable. Indeed, as X is bounded, there is a constant $c > 0$ such that $|(\gamma^\alpha)^*\Delta X| \leq c$. Hence,

$$\sup_{t\geq 0} \Delta B_t \leq \sum_\alpha ((\gamma^\alpha)^*\Delta X)^2 \leq c \sum_\alpha |(\gamma^\alpha)^*\Delta X| \leq c \sum_\alpha [\gamma^\alpha \cdot X]_\infty^{1/2},$$

where the right-hand side has finite expected value.

Since for every stopping time τ

$$\mathbb{E}[B_\tau] = \sum_\alpha \mathbb{E}[[\gamma^\alpha \cdot X]_\tau] = \sum_\alpha \mathbb{E}\left[\int_0^\tau \left|\kappa^X \gamma^\alpha\right|^2 dA^X\right],$$

the local integrability of B yields the existence of stopping times (τ^m) such that $\tau_m \uparrow \infty$ and

$$\sum_\alpha \mathbb{E}\left[\int_0^{\tau_m} \left|\kappa^X \gamma^\alpha\right|^2 dA^X\right] = \sum_\alpha \mathbb{E}^{\mu^X}\left[\left|\kappa^X \gamma^\alpha\right|^2 1_{[0,\tau^m]}\right] < \infty.$$

It follows that

$$\sum_{\alpha} \left|\kappa^X \gamma^\alpha\right|^2 < \infty, \quad \mu^X - a.s..$$

This convergence implies (9) in view of inequalities (4) for minimal integrands. $\square$

Lemma A.5. *Let X be a square integrable martingale taking values in $\mathbf{R}^m$ and (γ^n) be minimal integrands for X such that $(M^n \triangleq \gamma^n \cdot X)$ are uniformly integrable martingales. Suppose that there are a uniformly integrable martingale M and a predictable process γ such that $M^n \to M$ in $\mathcal{L}_1$ and $\gamma^n_t(\omega) \to \gamma_t(\omega)$ for every (ω, t). Then γ is a minimal integrand for X and $M = \gamma \cdot X$.*

Proof. In view of characterization (7) for minimal integrands, the minimality of every element of (γ^n) implies the minimality of γ provided that the latter is X-integrable. Thus we only need to show that γ is X-integrable and $M = \gamma \cdot X$.

By Lemma A.3, passing to subsequences, we can assume that $M^n = \gamma^n \cdot X \to M$ in $\mathcal{H}_{1,loc}$. Since the space of stochastic integrals is closed under the convergence in $\mathcal{H}_{1,loc}$, there is a X-integrable predictable process $\widetilde{\gamma}$ such that $M = \widetilde{\gamma} \cdot X$. From Lemma 3.4 we deduce that

$$\kappa^X(\gamma^n - \widetilde{\gamma}) \overset{\mu^X}{\to} 0.$$

It follows that

$$\kappa^X(\widetilde{\gamma} - \gamma) = 0, \quad \mu^X - a.s.,$$

and Lemma 3.3 yields the result. $\square$

Proof of Theorem A.1. It is sufficient to prove the existence of the required analytic versions only locally, in a neighborhood of every $y \in U$. Hereafter, we fix $y \in U$. There are $\epsilon = \epsilon(y) \in (0,1)$ and a family $(\zeta_\alpha = \zeta_\alpha(y))_{\alpha \in \mathbf{Z}^l_+}$ in $\mathcal{L}_1$ such that

$$\xi(x) = \xi(y) + \sum_{\alpha} \zeta_\alpha (x-y)^\alpha, \quad \max_i |x_i - y_i| < 2\epsilon,$$

$$\sum_{\alpha} \mathbb{E}\left[|\zeta_\alpha|\right](2\epsilon)^{|\alpha|} < \infty,$$

where the first series converges in $\mathcal{L}_1$.

By taking conditional expectations with respect to $\mathcal{F}_t$ we obtain that

$$M_t(x) = M_t(y) + \sum_\alpha L^\alpha_t (x-y)^\alpha, \quad \max_i |x_i - y_i| < 2\epsilon, \tag{10}$$

where $L^\alpha_t \triangleq \mathbb{E}\left[\zeta_\alpha | \mathcal{F}_t\right]$ and the series converges in $\mathcal{L}_1$. Lemma A.2 yields an increasing sequence (τ_m) of stopping times such that $\{\tau_m = \infty\} \uparrow \Omega$ and

$$\sum_\alpha \|L^{\alpha,\tau_m}\|_{\mathcal{H}_1} \epsilon^{|\alpha|} < \infty, \quad m \geq 1.$$

It follows that

$$\sum_\alpha \sup_{t\geq 0} |L^\alpha_t(\omega)| \, \epsilon^{|\alpha|} < \infty, \quad \mathbb{P} - a.s.$$

and we can modify the martingales (L^α) so that the above convergence holds true for every $\omega \in \Omega$. Then the series in (10) converges uniformly in t for every $\omega \in \Omega$ and every x such that $\max_i |x_i - y_i| < \epsilon$. Thus, it defines the modifications of $M(x)$ for such x with the required analytic properties.

For the second part of the theorem we observe that the statement is invariant with respect to the choice of the local martingale X that has the MRP. Proposition 2 in [9] shows that we can choose X to be a bounded m-dimensional martingale.

As X has the MRP, there are minimal integrands $\sigma(y)$ and (γ^α) such that

$$\begin{aligned} M(y) &= M_0(y) + \sigma(y) \cdot X, \\ L^\alpha &= L^\alpha_0 + \gamma^\alpha \cdot X, \quad \alpha \in \mathbf{Z}^l_+. \end{aligned}$$

From Lemma A.4 we deduce that

$$\sum_\alpha |\gamma^\alpha_t(\omega)|^2 \, \epsilon^{2|\alpha|} < \infty$$

for all (ω, t) except a predictable set of μ^X-measure 0. By Lemma 3.3 we can set $\gamma^\alpha = 0$ on this set without changing $\gamma^\alpha \cdot X$. Then the series converges for every (ω, t). As $\epsilon \in (0,1)$, we deduce that

$$\sum_\alpha |\gamma^\alpha_t(\omega)| \, \epsilon^{2|\alpha|} < \infty$$

and thus for $x = (x_1, \dots, x_l)$ such that $\max_i |x_i - y_i| < \epsilon^2$ and every (ω, t) we can define

$$\sigma_t(x)(\omega) \triangleq \sigma_t(y)(\omega) + \sum_\alpha \gamma^\alpha_t(\omega)(x-y)^\alpha.$$

By construction, the function $x \to \sigma_t(x)(\omega)$ is analytic in a neighborhood of y. By Lemma A.5, for every x such that $\max_i |x_i - y_i| < \epsilon^2$ the predictable process $\sigma(x)$ is an integrand for X and

$$\begin{aligned} M(x) &= M(y) + \sum_{\alpha} L^{\alpha}(x-y)^{\alpha} \\ &= M_0(x) + \sigma(y) \cdot X + \sum_{\alpha} (\gamma^{\alpha} \cdot X)(x-y)^{\alpha} \\ &= M_0(x) + \sigma(x) \cdot X. \end{aligned}$$

□

B The MRP under the change of measure

Let X be a d-dimensional local martingale and $Z > 0$ be the density process of $\widetilde{\mathbb{P}} \sim \mathbb{P}$. We denote by $\widetilde{Z} \triangleq 1/Z$ the density process of $\mathbb{P}$ under $\widetilde{\mathbb{P}}$ and set $L \triangleq \widetilde{Z}_- \cdot Z$ and $\widetilde{L} \triangleq Z_- \cdot \widetilde{Z}$. Using integration by parts we deduce that

$$d(\widetilde{Z}X) = X_- d\widetilde{Z} + \widetilde{Z}_- d\widetilde{X},$$

where

$$\widetilde{X} = X + \left[X, \widetilde{L}\right].$$

It follows that $\widetilde{X}$ is a d-dimensional local martingale under $\widetilde{\mathbb{P}}$. Of course, this is just a version of Girsanov's theorem.

We observe that the relations between X and $\widetilde{X}$ are symmetric in the sense that

$$X = \widetilde{X} + \left[\widetilde{X}, L\right].$$

Indeed, as we have already shown, $Y \triangleq \widetilde{X} + \left[\widetilde{X}, L\right]$ is a d-dimensional local martingale. Clearly, the local martingales X and Y have the same initial values and the same continuous martingale parts. Finally, they have identical jumps:

$$\begin{aligned} \Delta(Y - X) &= \Delta(\left[X, \widetilde{L}\right] + \left[\widetilde{X}, L\right]) = \Delta X(\Delta \widetilde{L} + \Delta L + \Delta \widetilde{L} \Delta L) \\ &= \Delta X \Delta(Z\widetilde{Z}) = 0. \end{aligned}$$

Theorem B.1. *The MRP holds under X if and only if it holds under $\widetilde{X}$.*

Proof. By symmetry, it is sufficient to prove only one of the implications. We assume that X has the MRP. Let $\widetilde{M}$ be a local martingale under $\widetilde{\mathbb{P}}$. The arguments before the statement of the theorem yield the unique local martingale M such that

$$\widetilde{M} = M + \left[M, \widetilde{L}\right].$$

If now H is an integrand for X such that $M = M_0 + H \cdot X$, then

$$\widetilde{M} = \widetilde{M}_0 + H \cdot (X + \left[X, \widetilde{L}\right]) = \widetilde{M}_0 + H \cdot \widetilde{X}.$$

□

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