SPACELIKE HYPERSURFACES
OF
CONSTANT MEAN CURVATURE

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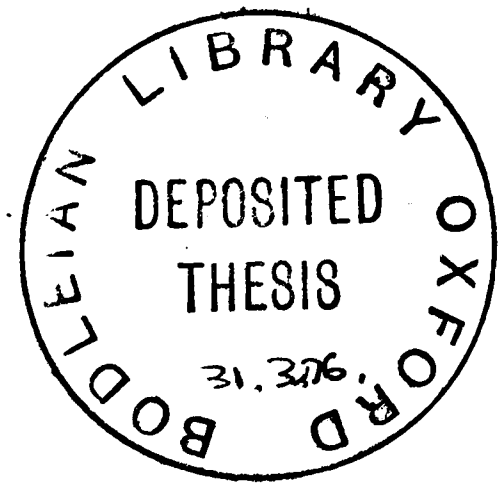

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I would like to thank Professor R. Penrose, my supervisor, for his guidance and also to express my gratitude to my colleagues at the Mathematical Institute for many helpful discussions. Thanks are also due to the Science Research Council for their partial financial support and to Jesus College for their grant to me from the Research and Advanced Study Fund.

ABSTRACT

The objects under discussion in this thesis are spacelike hypersurfaces on a four dimensional pseudo-Riemannian manifold. Various differentiability classes of such hypersurfaces are discussed and are made into topological spaces. The hypersurfaces* are taken to be topologically either $\mathcal{B}_3$ or S_3, in the latter case the manifold is taken to be topologically $S_3 \times \mathcal{I}$, $\mathcal{I} \subseteq \mathbb{R}$ an interval.

In chapter I the preliminary definitions are given together with some topological results on the existence and location of such hypersurfaces. These properties and the compactness of certain spaces of hypersurfaces are related to the underlying topological and causality properties of the pseudo-Riemannian manifold. In order to do this sufficient conditions are found for the compactness of the domain of dependence of a compact spacelike hypersurface. The main results of this chapter are theorems, (1.16), (1.28), (1.32), which state that a spacelike hypersurface with boundary in an asymptotically simple and empty, stably causal, manifold has a compact domain of dependence and that there is a neighbourhood of Minkowski space in the Whitney C^3 fine topology of manifolds with these properties, further the closure of the space of C_∞ spacelike hypersurfaces with a boundary $\mathcal{B}$ in the space of C_0 hypersurfaces with boundary $\mathcal{B}$ on such a manifold is compact with respect to the supremum topology.

In chapter II various functionals on the spaces of spacelike hypersurfaces are discussed with respect to their continuity properties and a relationship is found between the geometrical mean curvature of a hypersurface and its property of extremising a linear combination of the area and hypovolume functionals.

The area and hypovolume functionals are defined first for C_1 and C_0 hypersurfaces respectively and then their definitions are extended to the compact space of hypersurfaces discussed in chapter I.

* compact unless otherwise stated.

The principal results of this chapter are theorems (2.10), (2.15.3), (2.24), in which it is shown that a C_2 extremal of the functional $A + kV$ has constant mean curvature k and that both the area functional, A , and its extension, $\tilde{A}$, are upper semicontinuous with respect to the supremum topology. The variational result is then used to define the notion of generalised constant mean curvature and the compact spaces of spacelike hypersurfaces, discussed in chapter I, are shown to contain members of generalised constant mean curvature, by theorems (2.29), (2.30).

The concept of a foliation of a region, or a manifold, by spacelike hypersurfaces is introduced in chapter III and the properties of such foliations whose leaves are hypersurfaces of constant mean curvature are discussed. It is shown that the constant mean curvature π can be used as a natural time coordinate unless the manifold is static, for stably causal manifolds which are topologically $S^3 \times I$, provided that Hawking's timelike convergence condition holds. It is further shown that a C_1 spacelike hypersurface of constant mean curvature π is always a local maximum of the functional $A + \pi V$ and is a strict local maximum if the hypersurface is part of a foliation on which π is a strictly monotonic increasing function of time with $\frac{d\pi}{dt} > 0$. A relationship is found between the mean curvatures of two touching, spacelike hypersurfaces at their point of contact. This result is then used to show that any C_2 foliation of a manifold by spacelike hypersurfaces without boundary of constant mean curvature is unique and that there are no such hypersurfaces which are not members of the given foliation. The principal results of this chapter are theorems, (3.9), (3.23.1), (3.24), (3.26).

In chapter IV the properties of the one parameter families of spacelike hypersurfaces of generalised constant mean curvature which were shown to exist at the end of chapter II are discussed. It is found, with the additional assumptions that there is one and only one such spacelike hypersurface for each value of the generalised constant mean curvature k , and that for $k_1 \neq k_2$, $S_{k_1} \neq S_{k_2}$ (Where S_{k_i} is the spacelike hypersurface of generalised constant mean curvature k_i), that these families can be considered as the leaves of foliations with many of the properties of the foliations discussed in chapter III.

In particular, if the hypersurfaces S_{K_i} maximise the functionals $A + K_i V$ and if $K_1 > K_2$ then $S_{K_2} \subseteq J^-[S_{K_1}]$: essentially this is the result that K increases monotonically with time. It is also shown that the region between two such hypersurfaces is foliated by the hypersurfaces S_K with $K \in [K_1, K_2]$, in the sense that at least one such hypersurface passes through each point in this region and that every stably causal, topologically $S^3 \times \mathbb{R}$ manifold is foliated in this sense by its family of S_K, if it is presumed to have an S_K for each $K \in \mathbb{R}$. The principal results of this chapter are theorems, (4.6), (4.8), (4.10)).

Chapter V consists of a discussion of non-compact spacelike hypersurfaces of constant mean curvature. Bernstein's theorem and some related concepts are investigated; there follows a short discussion of some limited existence theorems for such hypersurfaces.

There is an appendix about some of the applications to physics of the results above. The possibility of using mean curvature as a naturally defined cosmic time is discussed as is the problem of positivity of gravitational energy.

CHAPTER I

SPACELIKE HYPERSURFACES

Throughout the following I shall be dealing with spaces of spacelike hypersurfaces defined with reference to a four dimensional, C^∞, pseudo-Riemannian, time orientable manifold of signature $\{+---\}$. Before giving a precise definition to the notion 'spacelike hypersurface' I shall give some examples which motivate this definition.

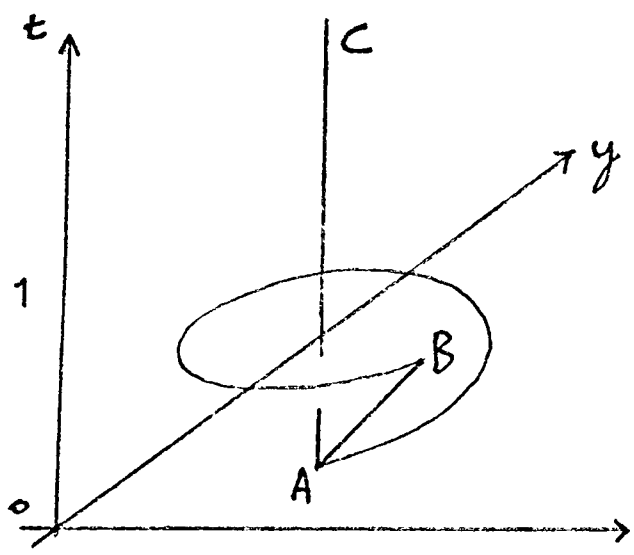


FIG. 1

1 + 2 dimensional Minkowski Space. The points A, B are joined by a spacelike geodesic segment and also by a section of a helical curve, the tangent vector to which lies in a spacelike direction AC is a timelike geodesic.

In figure one it is clear that if the given curve is the boundary of any two-dimensional submanifold of the given three-dimensional space, then the future light cone of A will intersect this submanifold and so there will be points on it which can be reached from A along a timelike geodesic, and also points on it reachable from A along null geodesics.

This situation appears to arise because the given 'boundary curve' is neither an acausal nor an achronal set. It will be shown below that any 'achronal boundary' can be spanned by an 'achronal surface' and that any 'acausal boundary' can be spanned by an 'acausal surface'.

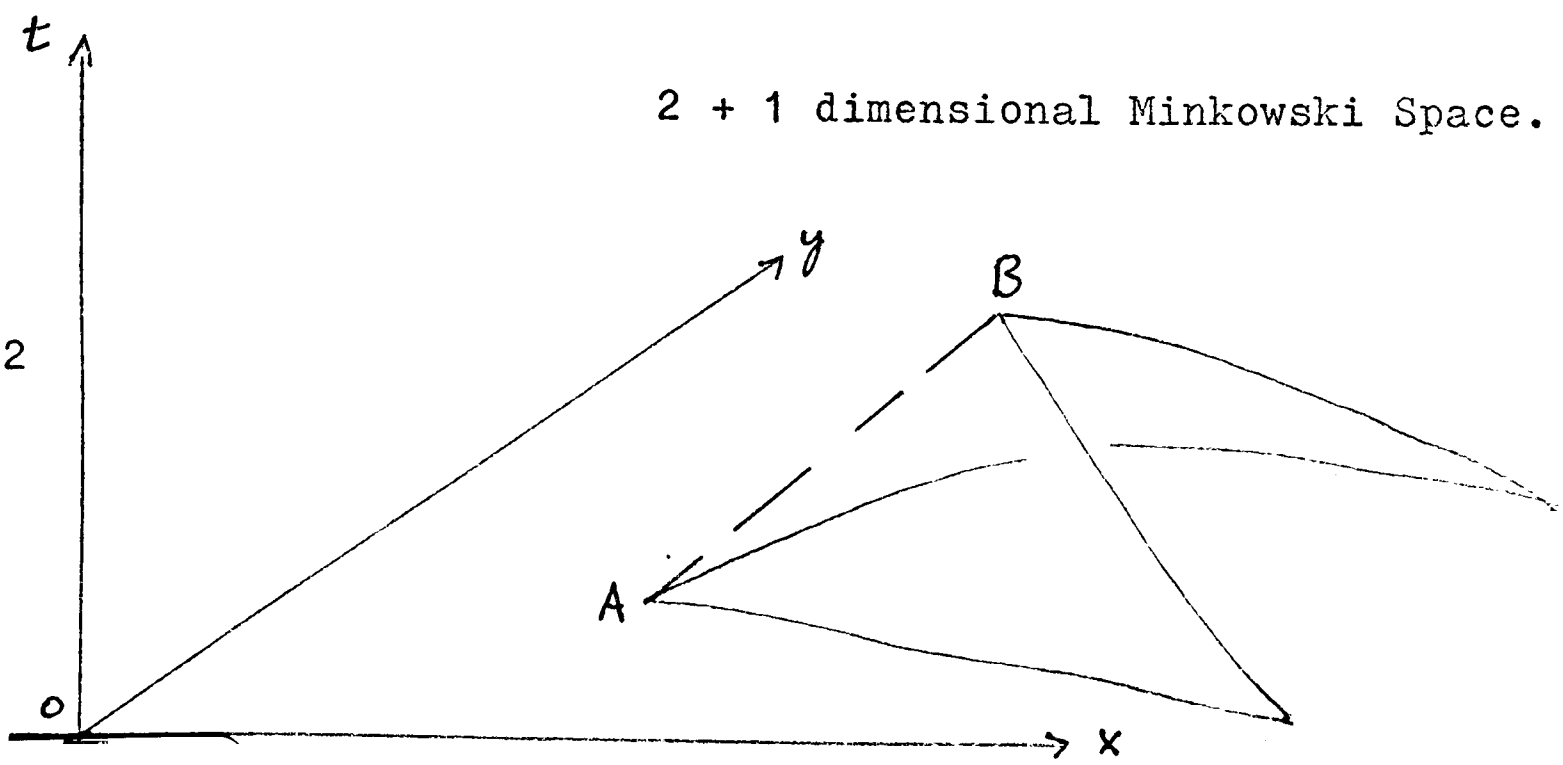


FIG. 2

2 + 1 dimensional Minkowski Space.

The points A and B can be joined by a null geodesic. It is clear that any achronal 'surface' spanning the curve in figure two must contain the null geodesic segment AB.

The definitions I shall give of a 'spacelike hypersurface with boundary' and of a 'spacelike hypersurface without boundary' are in a sense dual to usual definitions of a causal curve.

(1.1) Definition

If F is some connected interval, $F \subseteq \mathbb{R}$, a causal curve (timelike curve) is defined as the image under a continuously differentiable map from F to the pseudo-Riemannian manifold M of F .

$$\gamma : F \longrightarrow M$$
$$x \longmapsto \gamma^\alpha(x)$$

where $g_{\alpha\beta} \frac{d\gamma^\alpha}{dx} \frac{d\gamma^\beta}{dx} \geq 0$
$\left(g_{\alpha\beta} \frac{d\gamma^\alpha}{dx} \frac{d\gamma^\beta}{dx} > 0\right)$

γ^α admissible coordinates on M
$g_{\alpha\beta}$ is the metric tensor on M .

(1.2) Definition

A spacelike hypersurface with boundary is the image of the unit sphere in $\mathbb{R}^3$, $B^3 = \{(x,y,z) \mid x^2+y^2+z^2 \leq 1\}$, under a continuous map to M , such that each timelike or null (i.e. causal) curve in M intersects this image once only or not at all.

(1.3) Definition

A spacelike hypersurface without boundary is the image of the surface of the unit sphere in $\mathbb{R}^4$, that is the set $\{(\omega,x,y,z) \mid \omega^2+x^2+y^2+z^2 = 1\}$ under a continuous map to M, such that each causal curve in M intersects this image once only or not at all.

It might be thought that these definitions do not permit of spacelike hypersurfaces with boundary spanning a very wide range of boundaries, however:-

(1.4) Theorem

Let $\mathcal{B}$ be a two-dimensional acausal set in M , homeomorphic to the surface of the unit sphere in $\mathbb{R}^3$ such that there exists an open globally hyperbolic set $\mathcal{U}$ with $\tilde{H}^+(\mathcal{B}) \subseteq \mathcal{U}$, $\tilde{H}^-(\mathcal{B}) \subseteq \mathcal{U}$ and suppose that M is stably causal, then there is a spacelike hypersurface S , with boundary $\mathcal{B}$.

where: the terms of theorem (1.4) are defined as follows :-

(1.5) Definition

A pseudo-Riemannian manifold M is said to be stably causal if there exists a neighbourhood $\mathcal{U}$ of the metric in the C^0 open topology, such that if $h \in \mathcal{U}$ then the spacetime with the metric h contains no closed timelike curves.

(1.6) Definition (The C^0 open topology)

If we attach to each point of M the space of symmetric doubly covariant tensors (of signature $+---$) at that point, then we obtain a manifold, the bundle of such tensors over M

A Lorentz metric g on M is a cross-section:-

$$\hat{g} : M \longrightarrow T^0_{s2}(M)$$

$$\text{s.t.} \quad \pi \circ \hat{g} = 1$$

where $\pi(p, t^0_{s2}(p)) = p$ $\qquad$ π : Bundle $\to M$.

If $\mathcal{U}$ is an open set of $T^0_{s2}(M)$ which includes $\hat{g}(M)$ then a neighbourhood of $\hat{g}$ is defined to be all cross-sections $\hat{h}$ such that $\hat{h} \in \mathcal{U}$ and $\pi \circ \hat{h} = 1$.

These neighbourhoods of metrics form the basis for the C^0 open topology on the metrics on M .

(1.7) Definition (Strong Causality)

The strong causality condition holds at p iff every neighbourhood of p contains a neighbourhood of p which no causal curve intersects more than once.

(1.8) Definition (Global Hyperbolicity) 34

N is an open globally hyperbolic set if it is an open set with

(i) Strong causality holds in N .

(ii) $p, q \in N \Rightarrow J^+(p) \cap J^-(q) \subset N$ and is compact

where

(1.8.1) $J^+(p) = p \cup \{x \in M \mid$ there is a future directed causal curve from p to $x\}$

(1.8.2) $J^-(p) = p \cup \{x \in M \mid$ there is a past directed causal curve from p to $x\}$.

(1.9) Definition

$\tilde{H}^+(B)$ is the closure of that connected component of $\{\dot{J}^+(B) - B\}$, the closure of which is compact. It is ruled by null geodesics through B 48. $\tilde{H}^-(B)$ is defined similarily. ($\dot{X} \overset{def}{=} \overline{X} \cap \overline{(M-X)}$)

Proof (of theorem 1.4).

Since M is stably causal we can choose a coordinate system on M so that the metric form can be written:-

$$ds^2 = g_{00}\,dt^2 + g_{ij}\,dx^i dx^j \qquad i,j = 1,2,3$$

(1.4.1)

This is because M is stably causal iff there is a scaler field $f : M \to \mathbb{R}$ such that the gradient of f is everywhere timelike 29,54. We merely choose the level surfaces of f to be the t is constant surfaces and fix the x^i along each orthogonal trajectory of this family. Call this metric g .

Now define a one-parameter family of auxiliary metrics $\tilde{g}_\lambda$ on M by :-

$$ds^2 = \lambda^2 g_{00}\,dt^2 + g_{ij}\,dx^i dx^j \qquad \lambda \in [1, \infty)$$

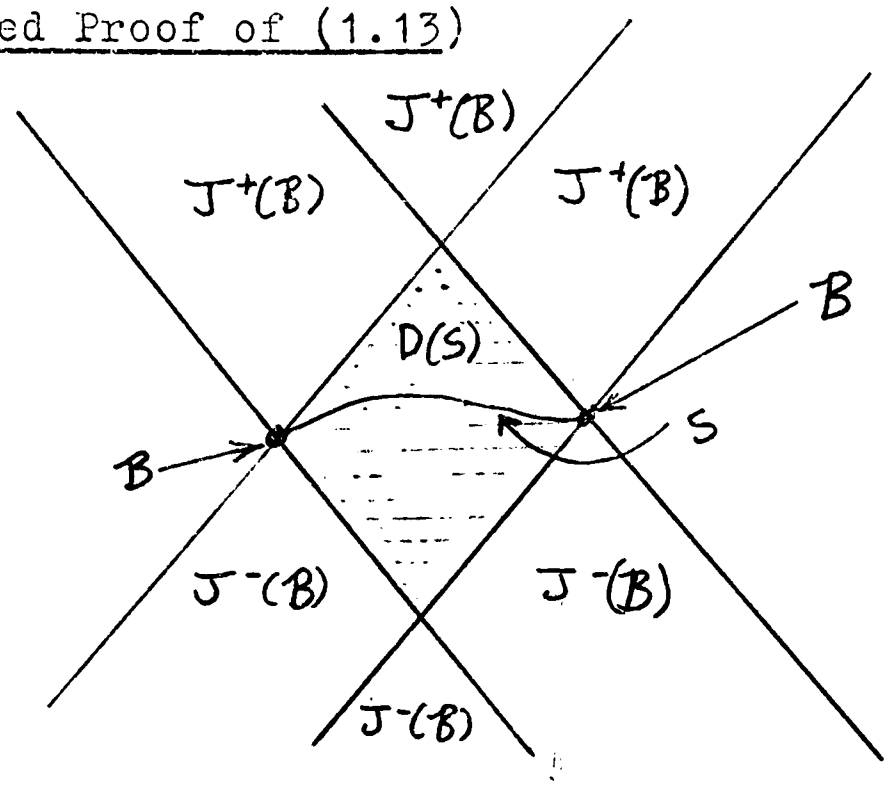
It is clear that $\tilde{g}_\lambda$ depends continuously on λ and that $\tilde{g}_1 = g$. Also that any curve which is causal in the g metric will be timelike in any of the $\tilde{g}_\lambda$ metrics with $\lambda > 1$.

Now consider a point $x \in B$ and the family of null cones from x , indexed by λ . For some interval $[1, 2\lambda_x)$ the null cone from x will not intersect B at any other point, since the null cone depends continuously on λ . Also there must be some neighbourhood U_x of x in the induced topology on B such that $y \in U_x$ implies that the λ_x cone drawn from y will not intersect B at any other point, since the null cone depends continuously on its vertex.

As x ranges over B , the U_x form an open cover for B , which is compact as it is the continuous image of a compact set. So there is a finite subcover $U_{x_1}, U_{x_2} \ldots U_{x_p}$. Choose $\lambda = \min_{i \in \{1,2 \ldots p\}} \lambda_i$ Then the λ cones drawn from each point of B will not intersect B at any other point, in other words B will be an acausal set with respect to the causality structure on M defined by $\tilde{g}_\lambda$.

Now consider $\tilde{H}^+_{\lambda/2}(B)$ (i.e. $\tilde{H}^+(B)$ defined w.r.t. the $\tilde{g}_{\lambda/2}$ metric). It is clear that $\tilde{H}^+_{\lambda/2}(B) \subseteq J^-(\tilde{H}^+(B)) \cap J^+(\tilde{H}^-(B))$ and so from the global hyperbolicity is contained in U . To show that $\tilde{H}^+_{\lambda/2}(B)$ is a spacelike hypersurface with boundary B , it is sufficient to prove that it is an achronal set w.r.t. the auxiliary metric $\tilde{g}_{\lambda/2}$ causality structure. It is merely necessary to note that $\tilde{H}^+_{\lambda/2}(B)$ is the future boundary of the domain of dependence (see below) (w.r.t. $\tilde{g}_{\lambda/2}$) of $\tilde{H}^+_\lambda(B)$ 4 3

Expanded Proof of (1.13)



Since M is asymptotically simple and empty $D(S)$ has boundary $H(S) = H^+(S) \cup H^-(S)$ and $H(S) \subseteq J^+(B) \cup J^-(B)$, for if $x \in H^+(S)$ there is a null geodesic from x to a point of $\mathcal{B}$ [48].

Suppose T is a spacelike hypersurface with boundary $\mathcal{B}$ s.t. $T \not\subseteq D(S)$. Suppose $t \in T$ and $t \notin D(S)$. There are two cases to consider.

(i) $T \cap D(S) = \mathcal{B}$

In this case, since $T \cap (J^+(B) \cup J^-(B)) = \mathcal{B}$, $S \cup T$ is a compact, achronal surface without boundary. But M has a Cauchy hypersurface, C, which is topologically R^3 [25,26].

Since $S \cup T$ is achronal there is a continuous $1-1$, into map $\pi : S \cup T \to C$.

(To obtain such a map choose a timelike congruence and project down it.) This is impossible.

(ii) $\exists u \in (T \cap D(S)) - \mathcal{B}$

In this case, since T is connected, there must be a curve from t to u contained in T. The curve cannot intersect $\mathcal{B}$, since $\mathcal{B}$ is the boundary of T, ~~and cannot intersect , since is the boundary of~~ , and cannot intersect $[(J^+(B) \cup J^-(B)) - \mathcal{B}]$ since T is acausal.

But since $J^+(B) \cup J^-(B)$ encloses $D(S)$ it must do one or the other.

The contradiction establishes the falsity of the premise $t \notin D(S)$ so $T \subseteq D(S)$.

I shall find it convenient to discuss spaces of spacelike hypersurface's of a particular differentiability class.

(1.10) Definition

$C_k^{slb}(\mathcal{B})$ is the set of all spacelike hypersurfaces with boundary $\mathcal{B}$ which are the image of the unit sphere in $\mathbb{R}^3$ under a C_k map to M .

Similarily we define C_k^{sl} the set of all C_k spacelike hypersurfaces without boundary.

(1.11) Remark

If the coordinates on M are y^α with $ds^2 = g_{\alpha\beta}\, dy^\alpha dy^\beta$ $\alpha, \beta = 0, 1, 2, 3$ and if a C_1 spacelike hypersurface is given locally by $y^\alpha = y^\alpha(x^i)$ $i = 1,2,3$ then if $\zeta^\alpha \neq 0$

Then $g_{\alpha\beta}\, y^\alpha_{,i}\, \zeta^\beta = 0$ $i = 1,2,3 \Rightarrow g_{\alpha\beta}\, \zeta^\alpha \zeta^\beta > 0$

That is the normal to the hypersurface is timelike and

$g_{\alpha\beta}\, y^\alpha_{,i}\, y^\beta_{,i} < 0$ not summed over i , each i.

It is of interest to discuss the region of M in which the spacelike hypersurfaces with boundary $\mathcal{B}$ lie:-

(1.12) Definition

The future domain of dependence of a hypersurface S , $D^+(S)$.

$$D^+(S) = \{p \in M \mid \text{every past directed timelike curve through } p \text{ without past end point intersects } S\}$$

The past domain of dependence of S , $D^-(S)$.

$$D^-(S) = \{p \in M \mid \text{every future directed timelike curve through } p \text{ without future end point intersects } S\}$$

The domain of dependence of S , $D(S) = D^+(S) \cup D^-(S)$.

where p is said to be the future (past) endpoint of a future (past) directed timelike curve $\gamma: F \to M$ if for every neighbourhood V in M of p $\exists\, t \in F$ s.t. $\gamma(t_1) \in V$ $\forall\, t_1 \in F$ with $t_1 \geq t$ $(t_1 \leq t)$.

A future (past) directed timelike curve without future (past) end point is said to be future (past) inextendible.

(1.13) Theorem

Suppose S , T are spacelike hypersurfaces with boundary $\mathcal{B}$, In an ~~spatially orientable~~ asymptotically simple & empty manifold M . Then $T \subset D(S)$.

Proof (see preceding insert)

Suppose $t \in T$ and $t \notin D(S)$, then by the orientability of space in M and the continuity of T .

$$T \cap \{I^+(\mathcal{B}) \cup I^-(\mathcal{B})\} \neq \emptyset$$

where $I^{(+)}(\mathcal{B}) = \{x \in M \mid$ there is a future (past) directed timelike curve from $\mathcal{B}$ to $x\}$.
This contradicts the acausality of T. (We need space orientability, suppose M was $S^3 \times \mathbb{R}$).

(1.14) Remark

In a geodesically complete manifold M, if S is a spacelike hypersurface without boundary s.t. $D(S) = M$, the equivalent to the above theorem is trivially true.

I will want to be able to make the space of spacelike hypersurfaces into a compact topological space, for this reason and for others, it is useful to know when the domain of dependence of a spacelike hypersurface is compact.

First some definitions:-

(1.15) Definition

M, with metric g, is said to be asymptotically simple and empty if:-

(i) It is space and time orientable

(ii) It is strongly causal

(iii) $\exists (\tilde{M}, \tilde{g})$ and an imbedding $\theta : M \to \tilde{M}$ imbedding M as a manifold with smooth boundary ∂M in $\tilde{M}$ s.t.

(a) $\exists$ a C_3 function $\Omega : \tilde{M} \to \mathbb{R}$ such that on $\theta(M)$
$\Omega > 0$, $\Omega^2 g = \theta_*(\tilde{g})$, $\Omega|_{\partial M} = 0$, $d\Omega|_{\partial M} \neq 0$.

(b) Every null geodesic in M has two end points on ∂M.

(iv) $R_{ab} = 0$ on an open neighbourhood of ∂M in $\overline{M} = M \cup \partial M$.

(1.16) Theorem

A spacelike hypersurface S with boundary $\mathcal{B}$ in an asymptotically simple and empty, stably causal manifold M has a compact domain of dependence, $D(S)$.

The proof of this theorem is rather long and will be preceded by a number of lemmas.

First we note some results established by Hawking and Ellis, and Geroch [25,29]

asymptotically simple and empty

(i) M is causally simple.

(ii) M is globally hyperbolic.

(iii) M is topologically $\mathbb{R}^4$.

where

(1.17) Definition

M is said to be causally simple if $K \subseteq M$, K compact implies $J^+(K)$ and $J^-(K)$ are closed in M .

(1.18) Theorem [29]

If K_1 , K_2 are compact sets in N , an open globally hyperbolic set, then $J^+(K_1) \cap J^-(K_2)$ is compact.

(1.19) Definition

The future Cauchy horizon of a spacelike hypersurface S , $H^+(S)$.

$$H^+(S) = \{ x \in M \mid (x \cup I^+(x)) \cap D^+(S) = x \}$$

Similarily $H^-(S)$. $\quad H(S) \overset{def}{=} H^+(S) \cup H^-(S)$.

(1.20) Lemma $\quad D^+(S) \supseteq J^-[H^+(S)] \cap J^+(S)$

Proof(of lemma)

Suppose $p \in J^-[H^+(S)] \cap J^+(S)$ and suppose γ is a past directed timelike curve from p without past endpoint. Now there is a past directed causal curve $\tilde{c}$ from x to p , where x is some element of $H^+(S)$ and so [48] there is a causal geodesic c from x to p .

Case (i) The geodesic is timelike.

Then $c \cup \gamma$ is a piecewise continuously differentiable, past directed, timelike curve from x , without past endpoint. Choose a normal neighbourhood $\mathcal{U}$, s.t. $p \in \mathcal{U}$, $\mathcal{U} \cap S = \phi$. (This is possible unless $p \in S$, a trivial subcase). Then $c \cup \gamma$ can be 'smoothed' at p by replacing part of it within $\mathcal{U}$ by another timelike curve. Call the modified curve γ'.

From the definition of $H^+(S)$, γ' meets S , furthermore it does so only once, by the acausality of S .

This intersection must occur on γ as $(\mathcal{U} \cup c) \subseteq J^+(S)$ so an intersection in $\mathcal{U} \cup c$ would violate the acausality of S .

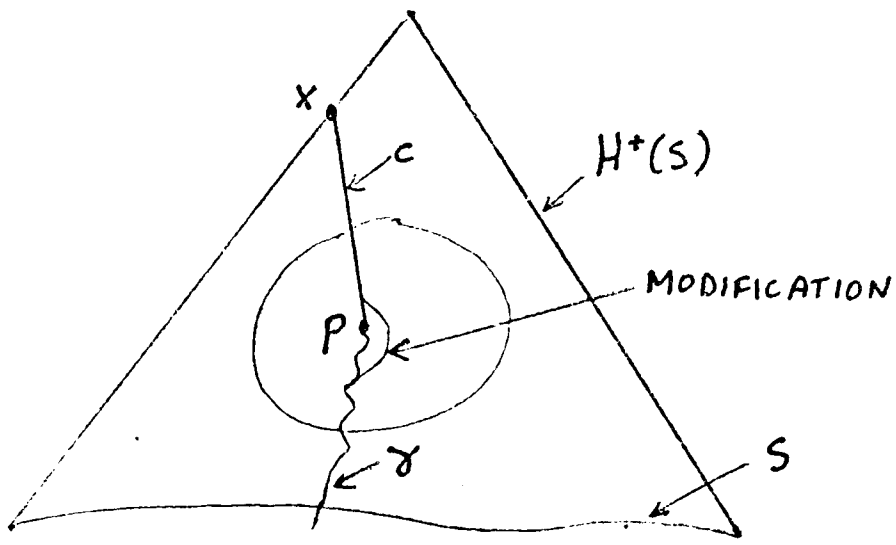
Case (ii) The geodesic is null.

We can choose a normal neighbourhood V , s.t. $p \in V$, $V \cap S = \phi$ and $v \in V \cap I^-(p)$ implies that there is a future directed timelike curve from v to x .

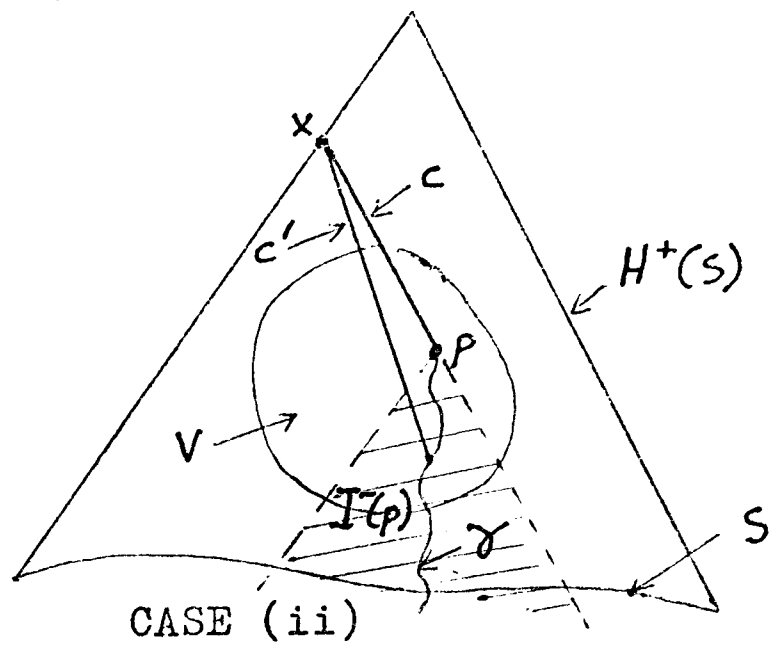
Let v be a point of $\gamma \cap V$, then we can use the future directed timelike geodesic c' from v to x to construct a piecewise continuously differentiable, past directed, timelike curve from x without past endpoint $c' \cup (I^-(v) \cap \gamma)$. We then

proceed as in case (i) to show that γ intersects S.

We have proved that $p \in D^+(S)$.



CASE (i)



CASE (ii)

(1.21) Definition

An Indecomposable past set $(I.P.)$, W, is a set with the properties:-

1) W is open.
2) $I^-(W) \subseteq W$
3) W cannot be expressed as the union of two proper subsets having properties 1) and 2).

An Indecomposable future set $(I.F.)$ is defined similarily.

(1.22) Definition

An $I.P.$ that is the past of a point in M is called a proper indecomposable past set, $P.I.P.$

An $I.P.$ that is not the past of any point in M is called a terminal indecomposable past set, $T.I.P.$

Proper and terminal indecomposable future sets are defined similarily.

(1.23) Theorem [27]

W is a $T.I.P.$ iff W is the past of a future inextendible timelike curve.

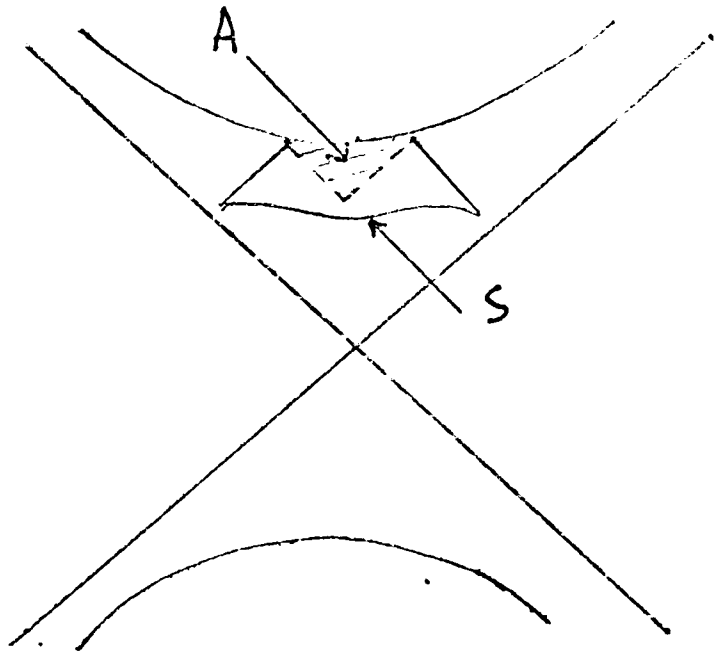
I am interested in circumstances under which

$$D^+(S) \subseteq J^-[H^+(S)] \cap J^+(S)$$

clearly we always have $D^+(S) \subseteq J^+(S)$

however it is not always true that $D^+(S) \subseteq J^-[H^+(S)]$

for example:-



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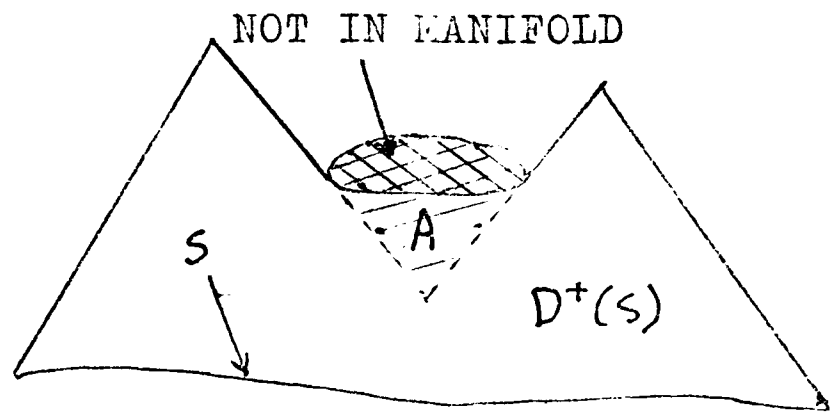


In both cases $A \not\subseteq J^-[H^+(S)]$ but $A \subseteq D^+(S)$
Also, in both cases, the trouble stems from the intrusion of the causal boundary of the manifold 'into' $D^+(S)$.

This problem is avoided if we know that every future inextendible timelike curve that intersects S must also intersect $H^+(S)$ or equivalently, since from the definition of $H^+(S)$ any future directed timelike curve from a point of $D^+(S)$ which emerges from $D^+(S)$ must do so at a point of $H^+(S)$, that every T.I.P. , W , such that $W \cap S \neq \phi$ also has $W \cap I^+[H^+(S)] \neq \phi$

For suppose that we know this property to hold, then, if $p \in D^+(S) - H^+(S)$, choose a future directed timelike geodesic from p . This geodesic is a future inextendible timelike curve, whose past, from the definition of $D^+(S)$ must intersect S . It follows that it must intersect $H^+(S)$ say at x . Then $p \in I^-(x)$. As p was any point in $D^+(S) - H^+(S)$ and as any point $p \in H^+(S)$ is also in $J^-[H^+(S)]$ it follows that $D^+(S) \subseteq J^-[H^+(S)]$.

(1.24) Lemma

If S is a spacelike hypersurface with boundary B in an asymptotically simple and empty manifold M then any future inextendible timelike curve that intersects S must also intersect $H^+(S)$.

Before proving this lemma the T.I.P.'s that can arise in such a manifold must be classified.

Following Penrose [49,50,51] it can be seen that the boundary ∂M of M in $\tilde{M}$ is a null hypersurface, that is its normal is everywhere a null vector. This means that locally M lies to the past or to the future of ∂M and so that ∂M must have two disconnected components:-

$\mathcal{I}^+$, where the null geodesics in M have their future endpoints.

$\mathcal{I}^-$, where the null geodesics in M have their past endpoints.

There cannot be more than two compoents since the set of future null directions (the set of past null directions) at each point $p \in M$ is connected.

Further [25] every point of $\mathcal{I}^+$ is the future endpoint of a null geodesic in M.

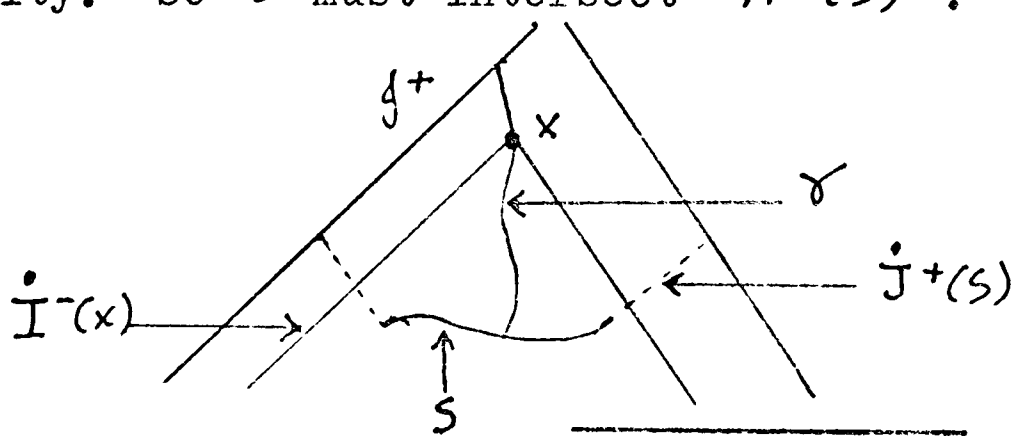
Now any point $p \in \mathcal{I}^+$ defines the T.I.P. $I^-(p, \tilde{M}) \cap M$ (where $I^-(p, \tilde{M})$ means $I^-(p)$ w.r.t. $\tilde{M}$).

Let λ be a future inextendible timelike curve in M. Then either λ has a future endpoint on $\mathcal{I}^+$ at a point p, in which case $I^-(\lambda) = I^-(p, \tilde{M}) \cap M$ and the T.I.P. defined by λ is the same as the T.I.P. defined by p, or it does not. If it does not $M - I^-(\lambda) = \phi$, as otherwise the null geodesic generators of $\dot{I}^-(\lambda) \equiv \overline{I^-(\lambda)} \cap \overline{M - I^-(\lambda)}$ would intersect $\mathcal{I}^+$ which is impossible as λ does not.

So the T.I.P.'s consist of one for every point of $\mathcal{I}^+$ and M itself, call this T.I.P., i^+.

Proof (of lemma (1.24)

Suppose γ is a future inextendible timelike curve that intersects S, but does not intersect $H^+(S)$.
Then $\gamma \cap J^+(S) \subseteq D^+(S)$ and γ 'intersects $\mathcal{I}^+$' or 'goes to i^+'. But $x \in \gamma \cap J^+(S) \Rightarrow I^-(x) \cap J^+(S) \subseteq D^+(S)$ and in fact since S is closed $\dot{I}^-(x) \cap J^+(S) \subseteq D^+(S)$
This means that there must be points of S arbitrarily close to $\mathcal{I}^+$. This contradicts S being bounded as ∂M is at infinity. So γ must intersect $H^+(S)$.



(1.24.1) Corollary

Together with the argument on page 12 this proves that

$$D^+(S) \subseteq J^-[H^+(S)] \cap J^+(S)$$

and so together with the lemma:-

$$D^+(S) = J^-[H^+(S)] \cap J^+(S)$$

and similarily

$$D^-(S) = J^+[H^-(S)] \cap J^-(S)$$

(1.25) Lemma

If S is a spacelike hypersurface with boundary $\mathcal{B}$ in an asymptotically simple and empty, stably causal manifold M, then $H^+(S)$, $H^-(S)$ are compact.

Proof

As in the proof of theorem (1.4), the stable causality implies the existence of a smooth congruence of timelike curves in M. As $D^+(S) \subseteq J^-[H^+(S)]$, the points of $S \subseteq D^+(S)$ may be put in a natural one to one correspondence with the points of $H^+(S)$, by following the member of the congruence that passes through $x \in S$ until it intersects $H^+(S)$, at $h(x)$ say. As $h: S \longrightarrow H^+(S)$ is a continuous map it follows that $H^+(S)$ is compact, because S is. Similarily $H^-(S)$ is compact.

Proof of theorem (1.16)

Apply theorem (1.18) with $K_1 = S$, $K_2 = H^+(S)$ to show that $J^+(S) \cap J^-[H^+(S)]$ is compact. By (1.25) this means that $D^+(S)$ is compact. Similarily $D^-(S)$ is compact and so $D(S) = D^+(S) \cup D^-(S)$ is compact.

An obvious question that now presents itself is what other asymptotically simple and empty, stably causal manifolds are there, apart from the obvious example, Minkowski space? A partial answer to this is provided by D. Lerner and J. R. Porter[35]. To discuss their results one needs the notion of a C^k Whitney fine topology on the metrics on a manifold[36,39], it is a generalisation of the C^0 open topology defined at (1.6).

(1.26) Definition Whitney C^k fine topology

Let $E \to M$ be any bundle (total space E, projection $\to$, base space M) and let $J^k(M,E) \to M \times E$ be the bundle of k jets of local C^k maps from M to E. (If f is such a map and p is in the domain of f, $j^k(f)(p)$ is the set of all local C^k maps g, k tangent to f at p, that is f and g agree at p up to an including their k th partial derivatives. $j^k(f)(p)$ is the k jet of f at p.) By restricting to local C^k sections of E we get $J^k(E)$, the bundle of k-jets of sections of E.

If a C^k section f is a geometric object field with components $f^{\dots}_{\dots}$ relative to local coordinates u^a on M, (Here E would be the bundle of geometric objects of type $t^{\dots}_{\dots}$ over M) the local coordinates of $j^k(f)(p)$ are:-

$$(u^a(p), f^{\dots}_{\dots}(p), f^{\dots}_{\dots,i_1}(p), f^{\dots}_{\dots,i_1,i_2}(p), \dots, f^{\dots}_{\dots,i_1,\dots,i_k}(p))$$

where the comma represents partial differentiation.

Let $\mathcal{U}$ be open in $J^k(E)$

Let $N(\mathcal{U}) = \{ f \in C^k(E) \mid j^k(f)(M) \subseteq \mathcal{U} \}$

Then the Whitney C^k fine topology on $C^k(E)$, W_k, is that topology defined by the basis $N(\mathcal{U})$, where $\mathcal{U}$ ranges over the open sets of $J^k(E)$.

If $k > r \geq 0$, then the W_r topology is well defined on $C^k(E)$ and is strictly coarser than the W_k topology. So if a set $\mathcal{U}$ is open in the W_r topology, then it is automatically open in the W_k topology.

If we take $E \to M$ in the above to be the bundle of symmetric doubly covariant tensors (of signature $+---$) over our pseudo-Riemannian manifold M then we have a definition of the W_k topologies on the metrics on M (see 1.6). W_0 is the C_0 open topology.

(1.27) Definition

A property P of the manifold M with metric g is k-stable at g if there is a neighbourhood $\mathcal{U}$ of g in the W_k topology such that if $h \in \mathcal{U}$ then the manifold M with metric h has the property P.

Lerner and Porter show that:-

Geodesic completeness is k -stable, $k \geqslant 2$
Time orientability is k -stable, $k \geqslant 2$
Asymptotic simplicity is k -stable, $k \geqslant 3$ at Minkowski space

---

But, by definition, stable causality is k -stable, $k \geqslant 0$, at any stably causal metric on a manifold M . By the remark (1.4.1) it is clear that Minkowski space is stably causal so for $k \geqslant 3$ there must exist neighbourhoods $\mathcal{U}_k$ of Minkowski space, η , in the W_k topologies such that any metric on M in $\mathcal{U}_k$ gives a stably causal, asymptotically simple manifold. Note that η is also asymptotically empty.

If V is any bounded open set in M , look at the bundle of metrics over V , call it B . Consider $J^k(B)$, it is a subbundle of $J^k(E)$ and if $\mathcal{U}$ is open in $J^k(B)$ it is open in $J^k(E)$. Also if K is open in $J^k(E)$, $K \cap J^k(B)$ is open in $J^k(B)$ and so open in $J^k(E)$.

Suppose $\mathcal{U}_k$, one of the 'stability open sets' is given by:- $\mathcal{U}_k = \bigcup_{\alpha \in A} N(K_\alpha)$ A some index set.

This must be the case by definition of the W_k topology.

Now consider $\bigcup_{\alpha \in A} N(K_\alpha \cap J^k(B)) \overset{def}{=} \tilde{\mathcal{U}}_k$

$\tilde{\mathcal{U}}_k$ is a neighbourhood of η in the W_k topology.

$\tilde{\mathcal{U}}_k \subseteq \mathcal{U}_k$, $\tilde{\mathcal{U}}_k \neq \emptyset$. $\tilde{\mathcal{U}}_k$ has the property that any metric on M in it gives a stably causal asymptotically simple and empty manifold.

(1.28) To summarize, there is a neighbourhood of Minkowski space in the W_k topology, $k \geqslant 3$, such that any metric in this neighbourhood gives rise to a stably causal, asymptotically simple and empty manifold.

Compactness of the space of surfaces

As before, let $C^{SLB}_k(B)$ denote the set of the C^k spacelike hypersurfaces with boundary B on M . Let any one of these hypersurfaces be S_0 , we shall assume $D(S_0)$ to be compact. Then by (1.13) above $C^{SLB}_k(B)(B^3) \subseteq D(S_0)$.

As on (1.4.1) we introduce a coordinate system y^0 , y^1 , y^2, y^3 onto M , using the timelike hypersurface orthogonal congruence which exists by virtue of the stable causality of M .

That is y^0 labels a family of spacelike hypersurfaces that slice M and y^i $(i = 1,2,3)$ label the trajectories of orthogonal congruence of timelike curves.

From the acausality of a spacelike hypersurface it is clear that any member of $C_k^{SLB}(B)^{(\text{or } C_k^{SL})}_{\lambda}$ can be written as a function

$$y^0 = y^0(y^i)$$

y^i ranges over some fixed compact set C. $C \subseteq \mathbb{R}^3$ (or C is S^3)

Using this representation we introduce on $C_k^{SLB}(B)^{(\text{or on } C_k^{SL})}_{\lambda}$ the metric ρ.

(1.29) Definition

$$\rho(y^0, z^0) = \sup_{y^i \in C} |y^0(y^i) - z^0(y^i)|$$

We can regard $C_k^{SLB}(B)^{(\text{or } C_k^{SL})}_{\lambda}$ as a subspace of $C_0(C)$, the space of all continuous functions $C \to \mathbb{R}$, endowed with the above metric. The topology induced by ρ is independent of the particular choice of 'stable causality' coordinates.

(1.30) Definition

$\overline{C_k^{SLB}(B)}^{(C_k^{SL})}_{\lambda}$ is the closure of $C_k^{SLB}(B)^{(C_k^{SL})}_{\lambda}$ in $C_0(C)$ with respect to the metric ρ.

Remark

$\overline{C_k^{SLB}(B)}$ is the space of all continuous maps from the unit sphere in $\mathbb{R}^3$ to M, with the property that the image in M is an achronal set.

(1.31) Theorem (Ascoli)$_3$

Let $F \subseteq C_0(C)$

If (i) F is closed in $C_0(C)$

(ii) $\overline{F(x)}$ is compact for each $x \in C$

(iii) F is equicontinuous on C

Then F is compact.

(1.32) Theorem asymptotically simple & empty

If M is stably causal$_{\lambda}$ and $D(S_0)$ is compact, $S_0 \in C_k^{SLB}(B)$ then $\overline{C_k^{SLB}(B)}$ is compact with respect to the ρ-topology.

Proof

$\overline{C_k^{SLB}(B)} \subseteq C_0(C)$, so condition (i) of theorem (1.31) is fulfilled by definition.

Range $\overline{C_k^{SLB}(B)} \subseteq \overline{D(S)} = D(S)$ as S is closed. So (ii) is fulfilled as $\overline{F(x)}$ is a closed subset of a compact set.

The metric on M is given by $ds^2 = g_{oo}\,dy^{o2} + g_{ij}\,dy^i dy^j$ so the slope of the null cone at y^o, y^i in the direction n^i is certainly less than $\max_{i,j}(|g_{ij}|/g_{oo}) \overset{def}{=} k(y^o, y^i)$

But g_{oo} is a continuous positive function on $D(S)$, as is $|g_{ij}|$ each i,j. So $k(y^o,y^i) \leq K \quad \forall (y^o,y^i) \in D(S)$, some K

This means that if $y^o = y^o(y^i)$ is an element of $C_k^{SLB}(B)$ then $|y^o(y^i) - y^o(z^i)| \leq K \| y^i - z^i \|$

$\| \quad \|$ is the Euclidean norm on $\mathbb{R}^3$.

That is the functions $\overline{C_k^{SLB}(B)}$ satisfy a uniform Lipschitz condition on C and so are equicontinuous on C. This is condition (iii). So by Ascoli's theorem, $\overline{C_k^{SLB}(B)}$ is compact.

Similarily we have

(1.33) Theorem

Let M be a compact, stably causal manifold with boundary, admitting stable causality coordinates, topologically $\bar{I} \times S^3$, $\bar{I}$ a closed interval in $\mathbb{R}$. Then C_k^{SL} (the closure of C_k^{SL} in $C_0(C)$) is compact with respect to the ρ_- topology.

(11.34) Remark

The conditions imposed in theorem (1.33) to ensure the compactness of $\overline{C_k^{SL}}$ may seem unreasonably restrictive. In fact $\overline{C_k^{SL}}$ is compact for a larger class of physically interesting space-times.

Consider a space-time which, as a manifold, is topologically $I \times S^3$, I an open interval in $\mathbb{R}$ and such that there exist global coordinates with the properties :-

(i) the metric can be put into the form :

$$ds^2 = g_{oo}\,dt^2 + g_{ij}\,dx^i dx^j \qquad i,j = 1,2,3 \qquad t \in (a,b) \qquad a,b \in \mathbb{R}$$

(ii) there exist positive constants $0 < \alpha \leq \beta$, K such that :

$$0 < \alpha \leq g_{oo} \leq \beta \quad \text{and} \quad |g_{ij}| \leq K$$

uniformly for all $(t, x^i) \in M$ and for each pair (i,j).

If these conditions are fulfilled then $\overline{C_k^{SL}}$ is compact as the conditions for theorem (1.31) are satisfied. It is convenient to think of the elements of $\overline{C_k^{SL}}$ as spacelike hypersurfaces in some $\tilde{M}$, where $\tilde{M}$ is any completion of M as a manifold. For example, since M is topologically $\mathbb{R} \times S^3$, $\tilde{M}$ could be taken to be the two point compactification of M, making $\tilde{M}$ topologically S^4. It is important to realise that $\tilde{M}$ is not a space time.

The Robertson-Walker universes, with initial and final singularities are of this type, the global coordinate system being simply the natural comoving coordinate system.

(1.35) Definition

I shal refer in the following to manifolds satisfying the above two conditions (1.34) as manifolds of type (R-W).

(1.36) Remark

Compact, stably causal manifolds with boundary (i.e. those to which (1.33) is applicable) are of direct physical interest, since any finite slice taken out of a cosmological model forms an object of this type. (By a finite slice I mean $J^+[S] \cap J^-[T]$ where $S, T \in C_0^{SL}$ and $S \cap T = \emptyset$)

From now on all spacelike hypersurfaces without boundary will be situated either in manifolds which are topologically $S^3 \times I$, I an open interval, or in manifolds with boundary which are topologically $S^3 \times \overline{I}$.

CHAPTER II

FUNCTIONALS ON THE SPACE OF HYPERSURFACES

In this chapter the area and hypovolume of a spacelike hypersurface will be defined and discussed, first for C_1, hypersurfaces and then for C_0 hypersurfaces, by an extension of the definitions. The properties of spacelike hypersurfaces extremising a linear combination of these two functionals will be examined.

The Area functional

Let $y^\alpha = y^\alpha(x^i)$ be a spacelike hypersurface, with or without boundary in M. Here y^α are the coordinates in M and x^i are those in the unit sphere in $\mathbb{R}^3$, or in S^3 respectively.

$$\alpha \in \{0,1,2,3\} \qquad i \in \{1,2,3\}$$

(2.1) Definition

If $y^\alpha = y^\alpha(x^i)$ is a C_1 spacelike hypersurface, S then the induced metric on S is defined by :-

$$ds^2 = a_{ij}\, dx^i\, dx^j$$

where $a_{ij} = g_{\alpha\beta}\, y^\alpha{}_{,i}\, y^\beta{}_{,j}$

and the comma stands for partial differentiation w.r.t. the x^i

Clearly a_{ij} is negative definite.

(2.2) Definition

The Area of S is defined by :-

$$A[S] = \int_S \sqrt{|-a_{ij}|}\; d^3x$$

So $A[S] > 0$

The area functional takes on a simplified form in a stably causal manifold, if we use the natural coordinates on M (see 1.4.1.).

Unless I explicitly specify to the contrary, I shall assume M to be stably causal from here on and that the natural coordinates are employed. And, in the case of spacelike hypersurfaces with boundary, that $D(S)$ is compact.

(2.2.1) Notation

Any spacelike hypersurface S, with or without boundary, can be written as a graph in the form $y^0 = y^0(y^i)$, in these natural coordinates, where the appropriate domain of the y^i is denoted by C. That is C denotes the y^i labelling those orthogonal trajectories of the family of spacelike hypersurfaces $y^0 = const$ which intersect S.

For the hypersurface S the functions $y^i = y^i(x^j)$ are one to one and onto C. So there exist inverse functions $x^i = x^i(y^j)$.

Furthermore the Jacobian matrix:-

$$\begin{pmatrix} y^1_{,1} & y^1_{,2} & y^1_{,3} \\ y^2_{,1} & y^2_{,2} & y^2_{,3} \\ y^3_{,1} & y^3_{,2} & y^3_{,3} \end{pmatrix}$$

is never singular, as this would imply the existence of a vector dx^i such that $y^\alpha_{,i}\, dx^i = 0$ for $\alpha = 1,2,3$. But since S is spacelike the vector $dy^\alpha = y^\alpha_{,i}\, dx^i$ $\alpha = 0,1,2,3$. is spacelike, that is :-

$$g_{\alpha\beta}\, y^\alpha_{,i}\, dx^i\, y^\beta_{,j}\, dx^j < 0$$

so

$$g_{00}(y^0_{,i}\, dx^i)^2 < 0$$

which is impossible as $g_{00} > 0$

So we may apply the inverse function theorem [3] to conclude that the functions $x^i = x^i(y^j)$ are C_1 on C.
This means that when the hypersurface is given in the form
$y^0 = y^0(y^i)$ (i.e. $y^0 = y^0(x^j(y^i))$).
we know that the function y^0 is C_1 on C.

So we may write

(2.2.2) $$a_{ij} = g_{00}\, y^0_{,i}\, y^0_{,j} + g_{ij}$$

where the comma now stands for partial differentiation w.r.t. the y^i, $i = (1,2,3)$,
and the y^i are now being used as the coordinates on S.
The area of S is now given by, changing the variables,

$$A[S] = \int_C \sqrt{|1 - g_{00}\, y^0_{,i}\, y^0_{,j} - g_{ij}|}\; d^3y$$

Now $|-a_{ij}| = -|a_{ij}|$

and $|a_{ij}| = \epsilon_{ijk}(g_{00}y^0_{,1}y^0_{,i} + g_{1i})(g_{00}y^0_{,2}y^0_{,j} + g_{2j})(g_{00}y^0_{,3}y^0_{,k} + g_{3k})$

$= |g_{ij}| + g_{00}[\epsilon_{ijk}\, y^0_{,1}y^0_{,i}\, g_{2j}\, g_{3k} + \quad +]$

$= |g_{ij}| + \frac{1}{2} g_{00}\, \epsilon_{ijk}\epsilon_{lmn}\, y^0_{,l}\, y^0_{,i}\, g_{mj}\, g_{nk}$

(2.2.3) $= |g_{ij}| + g_{00}\, y^0_{,i}\, y^0_{,j}\, G^{ij}$

where $G^{ij} = g^{ij}|g_{ij}|$ the cofactor of g_{ij}.

As g_{ij} is negative definite it follows that G^{ij} is positive definite.

So $0 < |-a_{ij}| \leq |-g_{ij}|$

And $A[S] = \int_C \sqrt{\left(-|g_{ij}| - g_{00}\, y^0_{,i}\, y^0_{,j}\, G^{ij}\right)}\, d^3y$

If we are looking at $C^{SCB}_1(\mathcal{B})$ and the domain of dependence $D(S)$ is compact, since $|-g_{ij}|$ is a continuous function on $D(S)$ there exists $m > 0$ such that :-

$0 < |-g_{ij}| \leq m$ at all points of $D(S)$.

So if T is any surface in $C^{SCB}_1(\mathcal{B})$ we know

(2.2.4) $0 < A[T] \leq m\mu(C)$ where $\mu(C)$ is the volume of C in $\mathbb{R}^3$. That is the areas of the C_1 spacelike hypersurfaces with boundary $\mathcal{B}$ are bounded above.

The Hypovolume functional

(2.3) Definition The hypovolume of a spacelike hypersurface S with boundary $\mathcal{B}$, $V[S]$, is the volume of the portion of M which lies to the past of S and to the future of $H^-(S)$.

That is :-

$$V[S] = \int_{J^-[S] \cap J^+[H^-(S)]} \sqrt{-|g_{\alpha\beta}|}\; d^4y$$

Since we are assuming $D(S)$ compact, and as $\sqrt{-|g_{\alpha\beta}|}$ is a continuous function from M to $\mathbb{R}^+$, this integral will exist for all $S \in C_0^{SLB}(\mathcal{B})$.
Clearly, if $S, T \in C_0^{SLB}(\mathcal{B})$ and $T \subseteq J^+[S]$ then

$$V[T] \geq V[S]$$

(2.4) Definition

If S, T are spacelike hypersurfaces without boundary, elements of C_0^{SL}, and if $S \subseteq I^+[T]$, then the hypovolume of S, with respect to T, $V[S;T]$ is given by:-

$$V[S;T] = \int_{J^-[S] \cap J^+[T]} \sqrt{-|g_{\alpha\beta}|}\, d^4y$$

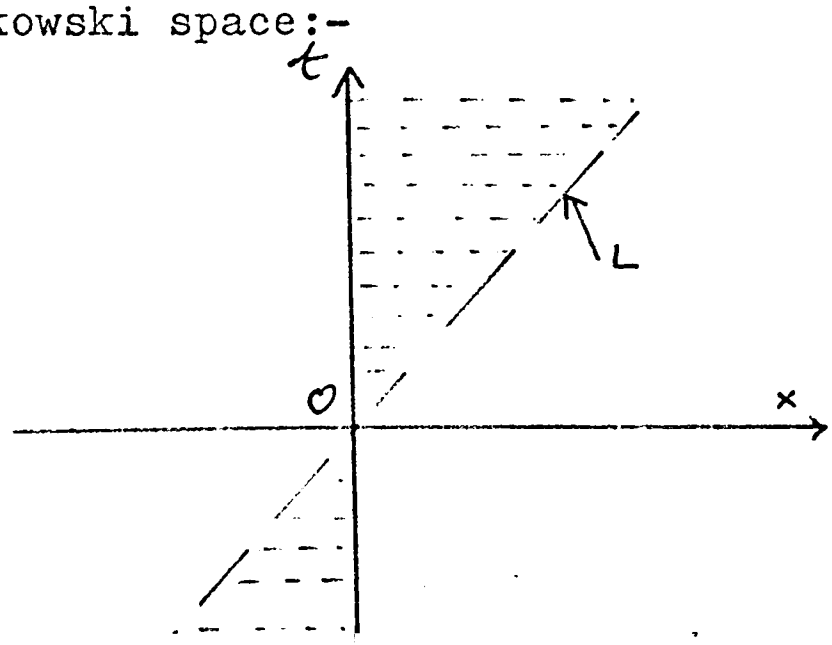
Since we are assuming M to be globally hyperbolic, $J^-[S] \cap J^+[T]$ is compact, by theorem (1.18) and the integral will exist for all such S and T.

(2.4.1) Evidently the hypovolumes of the C_0 spacelike hypersurfaces with boundary $\mathcal{B}$ are bounded above by $N\, m(D(S))$, where

$$N = \sup_{x \in D(S)} \sqrt{-|g_{\alpha\beta}|}\,(x)$$

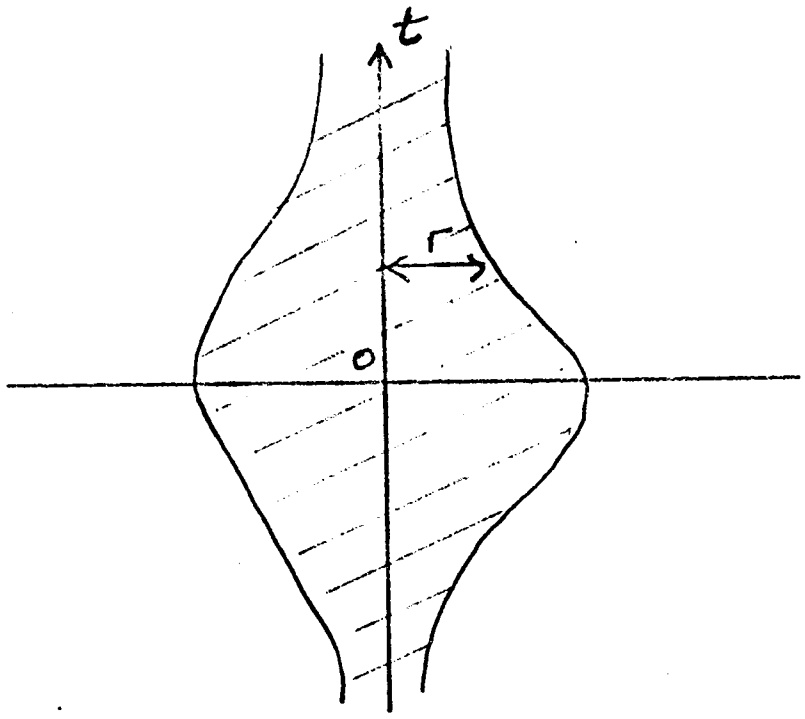
and $m(D(S))$ is the volume of $D(S)$ as measured in $\mathbb{R}^4$ coordinate space.

(2.4.2) In the case of spacelike hypersurfaces without boundary the area and hypovolume are not necessarily bounded above, for example they are unbounded for the flat space $S^1 \times \mathbb{R}$ obtained by making identifications on two dimensional Minkowski space:-



Consider the shaded portion with Ot and L identified at points of equal t.

However, they are bounded above for the Friedmann Universes with positive k , that is those whose space sections are diffeomorphic to S^3 . This is because such a manifold is of type $(R-W)$ (1.35). Clearly the manifold being compact is a sufficient condition for bounds on the area and hypovolume to exist, but it is not necessary. Consider the four dimensional submanifold of a four dimensional Minkowski space drawn below :-



$r \propto \frac{1}{|t|}$ for large $|t|$

It is not compact but clearly A is bounded and as r asymptotically behaves like $1/|t|$ it is evident that the volume of the manifold is finite. In fact A and V are bounded for any manifold of type $(R-W)$, see (1.35) and (2.24) and (2.41).

Area, hypovolume and mean curvature

In this section the geometrical properties of smooth local extrema of the one parameter family of functionals $A + \kappa V$, $\kappa \in \mathbb{R}$, are exhibited. Throughout it will be assumed that the hypersurfaces concerned are at least C_2.

(2.5) Definition

A spacelike hypersurface S is said to be a local extremum of the functional $A + \kappa V$ if the first variation of $A + \kappa V$ at S , where the variation is carried out through C_2 spacelike hypersurfaces, is zero.

Using the definitions (2.2), (2.3) or (2.4) it is seen that

$$\delta(A + \kappa V) = \delta A + \kappa \delta V$$

and $$\delta A = \int_S \left[\frac{\partial \mathcal{L}}{\partial y^\alpha} - \frac{\partial}{\partial x^i}\left(\frac{\partial \mathcal{L}}{\partial y^\alpha_{;i}}\right)\right] \delta y^\alpha \, d^3x$$

where $$\mathcal{L} = \sqrt{|-a_{ij}|}$$

(2.5.1) and $$\delta V = \int_S \mathcal{L}\, \mathfrak{Z}_\alpha \, \delta y^\alpha \, d^3x$$

where $\mathfrak{Z}_\alpha$ is the unit future directed normal to S.
with $g_{\alpha\beta}\, y^\alpha_{;i}\, \mathfrak{Z}^\beta = 0$ and $g_{\alpha\beta}\, \mathfrak{Z}^\alpha \mathfrak{Z}^\beta = 1$

So by the usual argument, the surface S, $y^\alpha = y^\alpha(x^i)$, must satisfy the associated Euler-Lagrange equations:-

$$\frac{\partial \mathcal{L}}{\partial y^\alpha} - \frac{\partial}{\partial x^i}\left(\frac{\partial \mathcal{L}}{\partial y^\alpha_{;i}}\right) + k\,\mathcal{L}\,\mathfrak{Z}_\alpha = 0 \qquad \alpha = 0, 1, 2, 3.$$

a straightforward calculation yields :-

(2.6)
$$-\sqrt{-|a_{ij}|}\, g_{\alpha\beta}\left[a^{ij}\left(y^\beta_{;ij} + \left(\Gamma^\beta_{\mu\nu}\right)_g y^\mu_{;i}\, y^\nu_{;j}\right)\right] + k\sqrt{-|a_{ij}|}\, \mathfrak{Z}_\alpha = 0$$

where the semi-colon indicates covariant differentiation with respect to the metric a_{ij}, of course $y^\mu_{;i} = y^\mu_{,i}$, and $\left(\Gamma^\beta_{\mu\nu}\right)_g$ is the Christoffel symbol associated with the metric $g_{\alpha\beta}$. That is :-

(2.7)
$$B^k_{;j} \overset{def}{=} B^k_{,j} + \left(\Gamma^k_{ij}\right)_a B^i$$

$$\left(\Gamma^\beta_{\mu\nu}\right)_g \overset{def}{=} \tfrac{1}{2} g^{\beta\tau}\left(g_{\nu\tau,\mu} + g_{\tau\mu,\nu} - g_{\mu\nu,\tau}\right)$$

(Greek indices are always associated with the coordinates y^α and run from 0 to 3, Latin indices run from 1 to 3 and are associated with the coordinates used in S, here the x^i)

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(2.8) Definition

The second fundamental form Ω_{ij} of a spacelike hypersurface S is defined by the equation :-

$$y^\alpha_{;ij} = -\left(\Gamma^\alpha_{\mu\nu}\right)_g y^\mu_{;i}\, y^\nu_{;j} + \Omega_{ij}\, \mathfrak{Z}^\alpha$$

(2.9) Definition

The mean curvature Ω of a spacelike hypersurface S is defined by :-

$$\Omega = a^{ij}\Omega_{ij}$$

Equation (2.6) may be written

$$\partial_\alpha \sqrt{-|a_{ij}|}\left(-a^{ij}\Omega_{ij} + K\right) = 0$$

which holds iff

(2.10)
$$a^{ij}\Omega_{ij} = K$$

So we have shown that the extremal S, has constant mean curvature K.

In the natural coordinate system in a stably causal manifold (1.4.1) this condition becomes, in the case where S is a hypersurface $y^0 = const$.

(2.10.1)
$$K = -\frac{g^{ij}g_{ij,0}}{2\sqrt{g_{00}}}$$

It may be of interest to note here a geometrical property of the second fundamental form [19].

Consider the C_2 hypersurface S, $y^\alpha = y^\alpha(x^i)$ $i=1,2,3$ $\alpha = 0,1,2,3$ Let P be the point in the hypersurface with coordinates x^i. and let P' be the point in the hypersurface with coordinates $x^i + dx^i$. C is the geodesic in S determined by these two points it has the equation :-

$$\frac{d^2x^i}{ds^2} + \left(\Gamma^{\ i}_{j\ k}\right)_a \frac{dx^j}{ds}\frac{dx^k}{ds} = 0$$

But the radius of first curvature R of a curve C in M $y^\alpha = y^\alpha(s)$ is defined by:-

$$\frac{d^2y^\alpha}{ds^2} + \left(\Gamma^{\ \alpha}_{\beta\gamma}\right)_g \frac{dy^\beta}{ds}\frac{dy^\gamma}{ds} = \mu^\alpha$$

$$1/R = \sqrt{|g_{\alpha\beta}\mu^\alpha\mu^\beta|}$$

the principal normal being $R\mu^\alpha$.

(That this is the generalisation of the corresponding concepts in flat space can be seen by choosing local coordinates so that the $\left(\Gamma^{\ \alpha}_{\beta\gamma}\right)_g$'s vanish).

Furthermore:-

$$\frac{d^2y^\alpha}{ds^2} + \left(\Gamma^\alpha_{\mu\nu}\right)_g \frac{dy^\mu}{ds}\frac{dy^\nu}{ds} = \Omega_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}\zeta^\alpha + y^\alpha_{,i}\left(\frac{d^2x^i}{ds^2} + \left(\Gamma^i_{jk}\right)_a \frac{dx^j}{ds}\frac{dx^k}{ds}\right)$$

from which it may be seen that

$$\frac{1}{R} = \Omega_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} \qquad \text{for the curve } C$$

Choosing local coordinates at P so that the $\left(\Gamma^\alpha_{\beta\gamma}\right)_g$'s vanish the equations of C are expressible in the form :-

$$y^\alpha(s) = \left(\frac{dy^\alpha}{ds}\right)_0 s + \frac{1}{2}\left(\frac{d^2y^\alpha}{ds^2}\right)_0 s^2 + \ldots.$$

where we measure s from P , and the equations of the geodesic in M tangent to C at P are :-

$$\overline{y^\alpha}(s) = \left(\frac{dy^\alpha}{ds}\right)_0 s$$

So the distance ρ from P' to the geodesic in M tangent to C at P is given by

(2.10.2) $$2\rho = \Omega_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}$$

to within terms of higher order.

(2.11) Associated problems and Lagrange multipliers

The above analysis incidentally provides the solutions to two associated problems.

Suppose we are looking for a C_2 spacelike hypersurface S which is a local extremal of area, subject to the constraint that the hypovolume is equal to some predetermined value, $\mathcal{U}$ say.

That is $\delta A = 0$ subject to $V = \mathcal{U}$

It is well known that the solutions to this problem are the same as the solutions to

$$\delta(A + \lambda V) = 0$$

where λ is a scalar constant, the lagrange multiplier, its value is determined by imposing the constraint $V = \mathcal{U}$, after the variation has been carried out. That is equations (2.3), (2.10) with $K = \lambda$ and $V = \mathcal{U}$ are solved together. We have shown that the solution to this problem is a hypersurface of constant mean curvature λ .

Similarily if we were looking for a C_2 spacelike hypersurface S which was to be a local extremal of hypovolume, subject to the constraint that the area is equal to some predetermined value, b say.

Then S would be a hypersurface of constant mean curvature $\frac{1}{\mu}$, unless S is also an unconstrained extremal. Where the problem was solved in the form.

$$\delta (V + \mu A) = 0$$
$$A = b$$

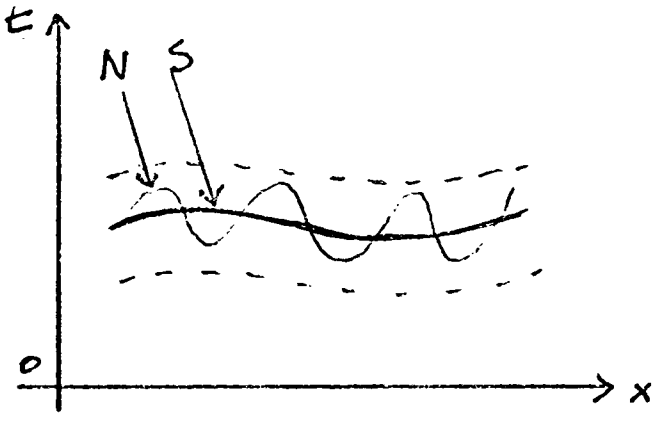
But should S be an unconstrained extremal of V we would have from (2.5.1)

$$\sqrt{|-a_{ij}|}\, \zeta_\alpha = 0 \qquad \alpha = 0, 1, 2, 3.$$

which is impossible as for a spacelike hypersurface $\sqrt{|-a_{ij}|} > 0$.

(2.12) Continuity properties of area and hypovolume

The area functional cannot be continuous with respect to the topology on $C^{1\,SLB}(\mathcal{B})$ and on $C^{1\,SL}$ defined by the metric ρ (1.29) as spacelike hypersurfaces of arbitrarily small area can be found in any ρ-neighbourhood of any spacelike hypersurface S .



By making the surface N 'almost' null its area can be made to approach zero, while it remains within the ρ-neighbourhood of S (indicated by dotted lines).

(2.13) Definition

With the notation of (1.29) :-

$$\rho_1(y^0, z^0) = \sup_{y^i \in C} |y^0(y^i) - z^0(y^i)| + \sum_{j=1}^{3} \sup_{y^i \in C} |y^0_{;j}(y^i) - z^0_{;j}(y^i)|$$

ρ_1 , is a metric on $C^{1\,SLB}(\mathcal{B})$ or on $C^{1\,SL}$ and so defines a topology which is strictly finer than the ρ topology.

(2.14) Lemma

The area functional on $C^{1\,SLB}(\mathcal{B})$ or on $C^{1\,SL}$ is continuous with respect to the ρ_1 topology.

Proof

(2.14.1) $$\sqrt{|-a_{ij}|} \overset{def}{=} \mathcal{L}(y^0(y^i), y^0_{;j}(y^i), y^j)$$
$$= \sqrt{(-|g_{ij}| - g_{00}\, y^0_{;i}\, y^0_{;j}\, G^{ij})}$$

Let S be the C_1 hypersurface $y^0 = z^0(y^i)$. Then $\mathcal{L}(S)$ is a continuous function from C to $\mathbb{R}$ (C is defined at (2.2.1.)) and so being positive is bounded away from zero as C is compact.

$$\mathcal{L}(S) \geq m > 0 \qquad \forall y^i \in C$$

Since $\mathcal{L}$ is a continuous function of y^α, $y^0_{,i}$, regarded as independent variables there exists an open set $\mathcal{U}(y^i)$ in $\mathbb{R}^7$ containing the point $(y^i, z^0(y^i), z^0_{,i}(y^i))$ such that $p \in \mathcal{U}(y^i)$ implies that $\mathcal{L}(p) > \frac{2}{3} m$. The natural map $\pi: \mathbb{R}^7 \longrightarrow C$, which projects onto the first three components, induces an open cover of C , as y^i ranges over C , by the open sets $\pi(\mathcal{U}(y^i))$. Since C is compact there is a finite subcover and so there exists an ϵ_1 , such that:-

$$\rho_1(x^0(y^i), z^0(y^i)) < \epsilon_1 \text{ implies}$$

$$\mathcal{L}(x^0(y^i), x^0_{,i}(y^i), y^i) > \frac{m}{2} \qquad \forall y^i \in C$$

Now consider $K = \{(y^0, y^i) \mid y^i \in C, |y^0 - z^0(y^i)| \leq \frac{\epsilon_1}{2}$

K is compact

Let $y^0 = y^0(y^i)$ be any spacelike hypersurface T s.t.

$$\rho_1(y^0(y^i), z^0(y^i)) < \frac{\epsilon_1}{2} .$$

We know $(y^0_{,i})^2 < \frac{-g_{ii}}{g_{00}}$ i not summed

But $T \subseteq K$ and $\exists\, \alpha$ s.t. $\frac{-g_{ii}}{g_{00}} \leq \alpha$ $\begin{cases} \forall (y^0, y^i) \in K \text{ as } K \text{ is compact.} \\ \forall\, i \in 1,2,3 \end{cases}$

From the fact that $\mathcal{L}$ is bounded uniformly away from zero and $|y^0_{,i}|$ is uniformly bounded for all hypersurfaces in $N(\rho_1, S, \frac{\epsilon_1}{2})$, the $\frac{\epsilon_1}{2}$ ρ_1-neighbourhood of S , together with the continuity of the metric and its first derivatives and the compactness of K we may conclude that there is a $\beta > 0$ s. t.

$$\left| \frac{\partial \mathcal{L}}{\partial y^\alpha}(y^0, y^0_{,i}, y^i) \right| < \beta$$

and $$\left| \frac{\partial \mathcal{L}}{\partial y^0_{,i}}(y^0, y^0_{,i}, y^i) \right| < \beta$$

uniformly for all surfaces in $N(\rho_1, S, \frac{\epsilon_1}{2})$

and so if $y^0 = f(y^i)$ and $y^0 = g(y^i)$ are two elements of $N(\rho_1, S, \frac{\epsilon_1}{2})$

$$| \mathcal{I}(f|_{y:}) - \mathcal{I}(g|_{y:}) | < 4\beta\rho_1(f,g)$$

So $$| A[f] - A[g] | = \left| \int_C [\mathcal{I}(f|_{y:}) - \mathcal{I}(g|_{y:})] d^3y \right|$$

$$\leq 4\beta\rho_1(f,g)\mu(C) \qquad \text{see (2.2.4)}$$

(2.15) Upper semi-continuity of the area functional

Although the area functional on $C_1^{SLB}(\mathcal{B})$ and C_1^{SL} is not continuous with respect to the ρ -topology, it is in fact upper semi-continuous.

(2.15.1) Definition

If T is a topological space $f : T \to \mathbb{R}$ is a functional, f is said to be upper semi-continuous if given $t \in T$ and $\epsilon > 0$, there exists an open set $\mathcal{U} \subseteq T$ such that $t \in \mathcal{U}$ and $u \in \mathcal{U} \Rightarrow f(u) - f(t) \leq \epsilon$

(2.15.2) Remark

It is equivalent to require that $\rho_n \to \rho$ in T implies that $f(\rho) \geq \limsup f(\rho_n)$ or $f^{-1}[a, \infty]$ closed $\forall$ $a \in \mathbb{R}$.

(2.15.3) Theorem

The area functional on the space $C_1^{SLB}(\mathcal{B})$ (or C_1^{SL}) is upper semi-continuous with respect to the ρ -topology.

Proof

Let S_0 be an element of $C_1^{SLB}(\mathcal{B})$ (or C_1^{SL}). Let the geodesic normal congruence to S_0 be labelled by x^i, where x^i are the coordinates in S_0, of the point in S_0 where the labelled member of the congruence intersects it. Let the interval from S_0, along each geodesic be called x^0 and let x^0 increase as one moves in the future sense along each geodesic. Since S_0 is compact there exists a $\delta > 0$ such that no geodesics will intersect for $|x^0| < \delta$ 1.

Let $n^\mu = (1,0,0,0)$ be the unit tangent vector to the congruence.

Suppose $S_1 \in C_1^{SLB}(\mathcal{B})$ (or C_1^{SL}) such that it can be described as a function $x^0 = y(x^i)$ s.t. $\sup_{x^i \in S_0} |y| = \tilde{\delta} < \frac{\delta}{2}$

Let $V = (J^+[S_0] \cap J^-[S_1]) \cup (J^-[S_0] \cap J^+[S_1])$

That is let V be the region between the two hypersurfaces. Let dS_μ be the future directed hypersurface element of area. Then we have, using Gauss' theorem :-

$$\int_{S_1} n^\mu dS_\mu - \int_{S_0} n^\mu dS_\mu \leq \int_V |n^\mu{}_{;\mu}| \, d\tau$$

as at all $x^i \in S_0$, V is to the future of one hypersurface and to the past of the other.

But $\int_{S_0} n^\mu dS_\mu = A[S_0]$ and $\int_{S_1} n^\mu dS_\mu \geq A[S_1]$

So $$A[S_1] \leq A[S_0] + \int_V |n^\mu{}_{;\mu}| \, d\tau$$

The metric on $S_0 \times [-\delta/2, \delta/2]$ may be written :-

$$ds^2 = dx^{0\,2} + g_{ij}\, dx^i dx^j$$

which means $n^\mu{}_{;\mu} = |g_{ij}|_{,0}/(2|g_{ij}|)$ this is a continuous function on $S_0 \times [-\delta/2, \delta/2]$, as is $\sqrt{-|g_{\mu\beta}|}$, which is compact so $\exists \mathcal{M}$ such that $\sqrt{-|g_{\mu\beta}|}\,|n^\mu{}_{;\mu}| \leq \mathcal{M}$ $\forall (x^i; x^0) \in S_0 \times [-\delta/2, \delta/2]$

and $$A[S_1] \leq A[S_0] + \mathcal{M} a \tilde{\delta}$$

where $$a = \int_{S_0} d^3x$$

(2.15.4)

There is homeomorphism between these coordinates (x^0, x^i) . and the corresponding 'stable causality' coordinates so as it has been shown that A is upper semi-continuous with respect to the ρ-topology defined using the (x^0, x^i) coordinates, it is equally true that A is *u.s.c. w.r.t.* the ρ-topology defined using the (y^0, y^i) coordinates since these ρ-topologies are the same.

(2.16) Definition

Let M be a stably causal manifold then $C_k^{PSLB}(B)$, the space of pieces of C^k spacelike hypersurfaces, defined with respect to the boundary B , is the collection of functions $y^0 = y^0(y^i)$, where y^α are the stable causality coordinates on M and the y^i range over C (2.2.1), such that $y^0|_{U_\alpha}$ is a C^k function and $\{(y^i, y^0(y^i)) \mid y^i \in U_\alpha\}$ is an acausal set for each U_α where the U_α are open sets, $U_\alpha \subseteq C$,

$U_\alpha \cap U_\beta = \phi$ if $\alpha \neq \beta$ and $\bigcup_\alpha \overline{U}_\alpha = C$, also
$\{(y^i, y^\circ(y^i)\} \subseteq \bigcup_\alpha \overline{\{(y^i; y^\circ(y^i)) \mid y^i \in U_\alpha\}}$

C^{PSL}_k is similarily defined for stably causal manifolds which are topologically $S^3 \times I$. (I an interval, $I \subseteq \mathbb{R}$).

(2.17) Definition

The area of an element S_1 of $C^{PSLB}_1(\mathcal{B})$ (or C^{PSL}_1) is defined in the obvious way as :-

$$A[S_1] = \sum_\alpha \int_{U_\alpha} \sqrt{-|a_{ij}|}\, d^3y$$

(and similarily for C^{PSL}_1)

(2.18) Lemma

Let $S_0 \in C^{SLB}_1(\mathcal{B})$ on a stably causal manifold M , endowed with the usual stable causality coordinates. Then given $\epsilon > 0$, $\exists\, \delta > 0$ s.t. if $S_1 \in C^{PSLB}_1(\mathcal{B})$ with $\sup_C |y^\circ_{S_1} - y^\circ_{S_0}| < \delta$, then $A[S_1] \leq A[S_0] + \epsilon$.

Proof

To prove this lemma the coordinate patch x°, x^i used in the proof of theorem (2.15.3) is used and the proof is very similar.

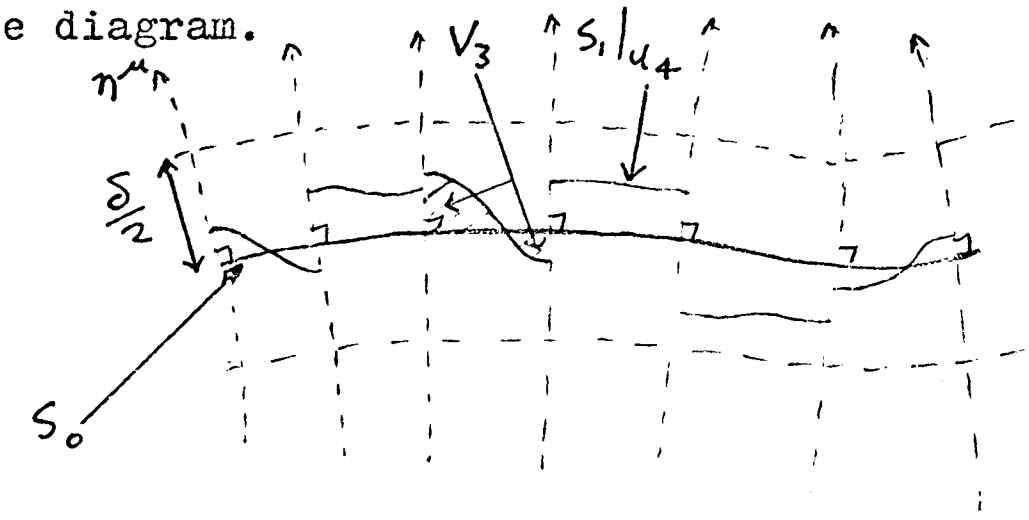
Let $V_\alpha = \{(x^\circ, x^i) \mid 0 \leq x^\circ \leq x^\circ_{S_1}(x^i)|_{U_\alpha}$ or $x^\circ_{S_1}(x^i)|_{U_\alpha} \leq x^\circ \leq 0\}$

where $x^\circ = 0$ is the equation of S_0 and $x^\circ = x^\circ_{S_1}(x^i)$ is the equation of S_1 and we have $|x^\circ_{S_1}| \leq \tilde{\delta} < \delta/2$, δ is the extent of the coordinate patch (cf. proof of (2.15.3)).

Then, with the notation of theorem (2.15.3) we have

$$\int_{S_1|_{U_\alpha}} n^\mu dS_\mu - \int_{S_0|_{U_\alpha}} n^\mu dS_\mu \leq \int_{V_\alpha} |n^\mu{}_{;\mu}|\, d\tau$$

as the congruence n^μ is timelike and not null there can be no surface integral contribution from the 'walls' of the tubes', see diagram.



It then follows that:-

$$\int_{S_1|_{U_\alpha}} n^\mu dS_\mu \leq \int_{S_0|_{U_\alpha}} n^\mu dS_\mu + m\tilde{\delta} \int_{U_\alpha} d^3x$$

where $\sqrt{-|g_{\alpha\beta}|}\,|n^{\mu}{}_{;\mu}| \leq \mathcal{M} \quad \forall (x^i, x^0) \in S_0 \times [-\delta/2, \delta/2]$

(c.f proof of (2.15.3))

if $a = \int_{S_0} d^3x$ using definition (2.17) it is clear that

$$A[S_1] \leq A[S_0] + \mathcal{M} a \tilde{\delta}$$

By the remark (2.15.4), this is enough to prove the lemma.

(2.19) Theorem

The hypovolume functional on $C_0^{SCB}(\mathcal{B})$ is continuous with respect to the ρ-topology. (Assuming $D(S)$ compact).

Proof

Let $S, T \in C_0^{SCB}(\mathcal{B})$

Then $$V[S] - V[T] = \int \sqrt{-|g_{\alpha\beta}|}\, d^4y - \int \sqrt{-|g_{\alpha\beta}|}\, d^4y$$

$J^-[S] \cap J^+[T] \overset{def}{\equiv} A \qquad J^+[S] \cap J^-[T] \overset{def}{\equiv} B$

$$|V[S] - V[T]| \leq \int_{A \cup B} \sqrt{-|g_{\alpha\beta}|}\, d^4y$$

Since $D(S)$ is compact $\exists\ b$ s.t. $\sqrt{-|g_{\alpha\beta}|} \leq b$ uniformly in $D(S)$

so $$|V[S] - V[T]| \leq b \int_{A \cup B} d^4y$$

$$\leq b\,\rho(S,T)\,\mu(S) \quad \text{where} \quad \mu(S) = \int_S d^3y$$

(2.20) Theorem

The hypovolume functional $V[S; \mathcal{U}]$ on C_0^{SL} is continuous with respect to the ρ-topology, assuming global hyperbolicity and stable causality.

Proof

First note that $\exists\ \epsilon_1 > 0$ such that $\rho(S,T) < \epsilon_1$ ensures that $T \subseteq I^+[\mathcal{U}]$, this is because of the compactness of S. It is then necessary to choose a $P \in C_0^{SL}$ such that $P \subseteq I^+[S]$, this can be done using the stable causality, if $\bar{y} = \sup_{y^i \in C} y^0(y^i)$ (S is $y^0(y^i)$) choose P to be the hypersurface $y^0 = \bar{y} + \frac{\epsilon_1}{2}$. Again one can choose an $\epsilon_2 > 0$ such that $\rho(S,T) < \epsilon_2$ ensures that $P \subseteq I^+(T)$. Now choose $\epsilon = \min\{\epsilon_1, \epsilon_2\}$ and copy the above proof restricting to T such that $\rho(S,T) < \epsilon$ and instead of using the compactness of $D(S)$ use the compactness of $J^-[P] \cap J^+[\mathcal{U}]$.

(2.21) Extension of Area and Hypovolume functionals to C_0 spacelike hypersurfaces

The area functional is extended to a functional on $\overline{C_0^{SLB}(\mathcal{B})}$ and C_0^{SL} in much the same way as Penrose defines the length of a causal curve from that of a timelike geodesic [48] and, as in that case, the extended functional is still upper semi-continuous. The hypovolume functional is simply extended by continuity.

(2.22) Definition

Using the stable causality coordinates, let S be a C_0 spacelike hypersurface : $y^0 = x^0(y^i) \quad y^i \in C$ (c.f 2.2.1)

Then $$T_\epsilon \stackrel{def}{=\!=} \{(y^0, y^i) \mid |y^0 - x^0(y^i)| < \epsilon\}$$

and $$S_\epsilon \stackrel{def}{=\!=} \sup_{y \in C_1^{PSL}[\epsilon]} A[y]$$

where $C_1^{PSL}[\epsilon]$ is the set of all C_1^{PSL} spacelike hypersurfaces, $y^0 = f(y^i)$ with $(f(y^i), y^i) \in T_\epsilon \quad \forall y^i \in C$.

(2.23) Definition

The value of the extended area functional on a C_0 hypersurface S, $\tilde{A}[S]$ is given by :-

$$\tilde{A}[S] = \lim_{\epsilon \to 0} S_\epsilon$$

First it must be checked that $\tilde{A}[S]$ is well defined. There are two cases to consider :-

(2.23.1)

Case (i) $\overline{C_0^{SLB}(\mathcal{B})}$

Let $S_0 \in \overline{C_0^{SLB}(\mathcal{B})}$, then since S_0 is compact there exist spacelike hypersurfaces $y^0 =$ constant P and Q say, such that $S_0 \subseteq I^-[P|_c] \cap I^+[Q|_c]$. Now $P|_c$ and $Q|_c$ are both compact so by theorem (1.18) (we assume throughout that M is globally hyperbolic) $J^-[P|_c] \cap J^+[Q|_c]$ is compact. Again using the compactness of S_0 , $\exists\ \tilde{\epsilon}$ such that $\epsilon \leq \tilde{\epsilon}$ implies $T_\epsilon \subseteq J^-[P|_c] \cap J^+[Q|_c]$ (c.f proof of (2.20)) and as this set is compact this means that $\exists\ K > 0$ such that $S_\epsilon \leq K \quad \forall \epsilon \leq \tilde{\epsilon}$, by a similar argument to that of (2.2.4).

(The sets $C_1^{PSLB}(\mathcal{B})[\epsilon]$ are never empty, for definition of $\overline{C_0^{SLB}(\mathcal{B})}$, any member of it can be uniformly approximated by members of $C_0^{SLB}(\mathcal{B})$ and $C_1^{PSLB}(\mathcal{B})$ is dense in $C_0^{SLB}(\mathcal{B})$.)

It is clear that if $\epsilon_1 < \epsilon_2$ then $T_{\epsilon_1} \subseteq T_{\epsilon_2}$ and so $S_{\epsilon_1} \leq S_{\epsilon_2}$, but $0 \leq S_\epsilon \ \forall \epsilon$. So $\lim_{\epsilon \to 0} S_\epsilon$ must exist.

Case (ii) $\overline{C_0^{SL}}$

The proof that $\tilde{A}$ is well defined is similar to that above except that here $S_0 \subseteq I^-(P) \cap I^+(Q) \subseteq J^-(P) \cap J^+(Q)$ which is compact.

(2.23.2)

Secondly, it must be checked that A and $\tilde{A}$ agree on their common domain of definition, the C_1 spacelike hypersurfaces.

So suppose S to be a C_1 spacelike hypersurface.

Then $S_\epsilon \geqslant A[S] \quad \forall \epsilon$

So $\tilde{A}[S] \geqslant A[S]$

Suppose $\tilde{A}[S] = A[S] + a \quad , \ a > 0$

Then $S_\epsilon \geqslant A[S] + a \quad \forall \epsilon$

So $\forall \epsilon > 0 \ , \exists y_\epsilon \in C_1^{PSL}[\epsilon] \ ,$ s.t. $A[y_\epsilon] > A[S] + a/2$

by definition of the supremum. This violates the upper semi-continuity of A proved in lemma (2.18). This forces $\tilde{A}[S] = A[S]$.

(2.24) Theorem

The extended area functional, $\tilde{A}$, on the space $\overline{C_0^{SLB}(\mathcal{B})}$, (or $\overline{C_0^{SL}}$) is upper semi-continuous with respect to the ρ-topology.

Proof

Given $S \in \overline{C_0^{SLB}(\mathcal{B})}$ (or $\overline{C_0^{SL}}$) and $\epsilon > 0$, by the definition of $\tilde{A}[S] \quad \exists \delta > 0$ s.t.

$0 \leq t < 4\delta \Rightarrow S_t(S) \leq \tilde{A}[S] + \epsilon$

Let $T \in \overline{C_0^{SLB}(\mathcal{B})}$ with $\rho(S, T) < \delta$

Then $\tilde{A}[T] \leq S_u(T) \quad \forall u \geqslant 0$ by definition of $\tilde{A}$.

and $\tilde{A}[S] \leq S_t(S) \quad \forall t \geqslant 0$

But for $t > \delta + u \quad T_t(S) \supseteq T_u(T) \Rightarrow S_t(S) \geqslant S_u(T)$

So taking $t = 2\delta \ , \ u = \delta/2$ we see that

$$\tilde{A}[T] \leq S_{\delta/2}(T) \leq S_{2\delta}(S) \leq \tilde{A}[S] + \epsilon$$

(2.25) Conjecture

Since C is compact and any $S \in \overline{C_0^{SLB}(\mathcal{B})}$ (or $\overline{C_0^{SL}}$) is continuous and satisfies a uniform Lipschitz condition on C, applying the theorem of Rajchman and Saks [53, 57], it is seen that S has a normal direction defined on it a. e. Using this normal direction it seems plausible that $\tilde{A}[S]$ is the Lebesgue integral of the area density (2.14.1).

(2.26) Definition

If $S \in \overline{C_0^{SLB}(\mathcal{B})}$ (or $\overline{C_0^{SL}}$) and if $\{S_n\}$ is a sequence of spacelike hypersurfaces in $C_0^{SLB}(\mathcal{B})$ (or C_0^{SL}) such that $S_n \to S$ in the ρ topology, then $\tilde{V}[S] = \lim_{n\to\infty} V[S_n]$ is the value of the extended hypovolume functional on S.

Since V is a continuous functional in the ρ topology it follows that $\tilde{V}$ is well defined and continuous.

(2.27) The Existence of a Maximising Hypersurface

An upper semi-continuous functional on a compact space, which is bounded above, achieves its upper bound. This well known topological result ensures the existence of maximising hypersurfaces in $\overline{C_0^{SLB}(\mathcal{B})}$ for the one parameter family of functionals $\tilde{A} + K\tilde{V}$, $K \in \mathbb{R}$, when we have $D(S)$ compact and in $\overline{C_0^{SL}}$, when M is compact.

This is because a continuous functional is always upper semi-continuous and of course the sum of two upper semi-continuous functionals is upper semi-continuous, this means that $\tilde{A} + K\tilde{V}$ is upper semi-continuous in both the above cases. Furthermore it has been shown above (2.2.4) (2.4.1) that A and V are bounded above for $C_1^{SLB}(\mathcal{B})$ and C_0^{SL}, (in the above cases) and so from the definitions of $\tilde{A}$ and $\tilde{V}$ it follows that they are bounded above on $\overline{C_0^{SLB}(\mathcal{B})}$ and $\overline{C_0^{SL}}$. And that $\tilde{V}$ is bounded below.

(2.28) Definition

The spacelike hypersurface $S \in \overline{C_0^{SLB}(\mathcal{B})}$ (or $\overline{C_0^{SL}}$) is called a spacelike hypersurface of generalised constant mean curvature K if it is a maximising hypersurface for the functional $\tilde{A} + K\tilde{V}$ on $\overline{C_0^{SLB}(\mathcal{B})}$ (or $\overline{C_0^{SL}}$).

(2.28.1) Remark

The above definition is motivated by the fact that if S, a spacelike hypersurface of generalised constant mean curvature

K is also at least C_2, then by (2.10) it is a spacelike hypersurface of constant mean curvature K.

We have proved :

(2.29) Theorem

If M is a stably causal manifold and if $D(S)$ is compact then there exists, for each $K \in \mathbb{R}$, a spacelike hypersurface S_K of generalised constant mean curvature K in $C_0^{SLB}(B)$.

And :

(2.30) Theorem

If M is a stably causal, compact manifold (admitting stable causality coordinates with boundary) topologically $S^3 \times [0,1]$ then there exists, for each $K \in \mathbb{R}$, a spacelike hypersurface S_K of generalised constant mean curvature K in C_0^{SL},

with the obvious generalisation :

(2.31) Theorem

If M is a ~~stably causal~~ manifold of type $(R-W)$ then there exists, for each $K \in \mathbb{R}$, a spacelike hypersurface S_K of generalised constant mean curvature K in C_0^{SL}.

(2.32) Note

Applying a theorem of Yamabe$_{58}$, York$_{60}$, has shown that a manifold with compact spacelike sections without boundary cannot have a spacelike hypersurface of zero mean curvature and negative scalar curvature, that is ${}^3R < 0$. He quotes the expanding Gowdy universes$_{28}$ as examples of this type of manifold, these manifolds are not compact or type $(R-W)$ and their existence illustrates the need for some such restriction, in order to obtain the above type of existence result.

CHAPTER III

CONSTANT MEAN CURVATURE FOLIATIONS

In this chapter I shall discuss foliations of two different types and their properties. For manifolds, such as that of the Friedmann universe with $K = +1$, which are topologically $I \times S^3$, where I is an interval, $I \subseteq R$, I shall discuss foliations, whose leaves are spacelike hypersurfaces without boundary, which fill the manifold. For spacelike hypersurfaces S_o with boundary $\mathcal{B}$, I shall discuss foliations whose leaves are spacelike hypersurfaces with boundary $\mathcal{B}$, which fill up $D(S_o)$. In what follows it will be assumed that the underlying manifold, M, is C^∞.

(3.1) Definition

A C^k foliation, F, of a manifold, M, by spacelike hypersurfaces without boundary is a map $F : M \to I$, where I is an interval on the real line with the properties:-

(i) F is a C^k map.

(ii) F is onto.

(iii) If $t \in I$, $F^{-1}(t)$ is a C^k spacelike hypersurface without boundary.

$F^{-1}(t)$ is called a leaf of F.

Definition

Given a spacelike hypersurface S_o with boundary $\mathcal{B}$, a C^k foliation, F, of $D(S_o)$, by spacelike hypersurfaces with boundary $\mathcal{B}$ is a map $F : Int\, D(S_o) \longrightarrow I$, an open interval on the real line with the properties:-

(i) F is a C^k map.

(ii) F is onto.

(iii) If $t \in I$, $F^{-1}(t) \cup \mathcal{B}$ is a C^k spacelike hypersurface with boundary $\mathcal{B}$.

$F^{-1}(t) \cup \mathcal{B}$ is called a leaf of F.

For the rest of this chapter I am concerned with foliations whose leaves are spacelike hypersurfaces of constant mean curvature. I shall need to impose Hawking's timelike convergence condition, or equivalently, the strong energy condition and Einstein's Equations on the manifold[29].

(3.4) Definition

The manifold M is said to satisfy the timelike convergence condition iff

$$g_{\alpha\beta}t^\alpha t^\beta > 0 \Rightarrow R_{\alpha\beta}t^\alpha t^\beta \leq 0 \quad \text{at all points in } M .$$

The sign conventions used here are ;-

(3.5) $$R^\alpha{}_{\beta\gamma\delta} = \Gamma^\alpha_{\epsilon\gamma}\Gamma^\epsilon_{\beta\delta} - \Gamma^\alpha_{\epsilon\delta}\Gamma^\epsilon_{\beta\gamma} + \frac{\partial \Gamma^\alpha_{\beta\delta}}{\partial y^\gamma} - \frac{\partial \Gamma^\alpha_{\beta\gamma}}{\partial y^\delta}$$

$$R_{\alpha\beta} = R^\gamma{}_{\alpha\beta\gamma}$$ for the Riemann and Ricci tensors

and for Einstein's Equations :-

$$-\kappa T_{\alpha\beta} = R_{\alpha\beta} - \tfrac{1}{2} R g_{\alpha\beta} \qquad \text{where } R = R^\alpha_\alpha$$

(3.6) Definition

The manifold M is said to satisfy the strong energy condition iff

$$g_{\alpha\beta}t^\alpha t^\beta > 0 \Rightarrow T_{\alpha\beta}t^\alpha t^\beta \geq \tfrac{1}{2} T g_{\alpha\beta} t^\alpha t^\beta$$

where $T = T^\alpha_\alpha$, at all points in M .

From Einstein's Equations it can easily be seen that the strong energy condition implies the timelike convergence condition on a manifold. The reason that the timelike convergence condition is so called will be explained below (3.24.5).

(3.7) Theorem

Suppose F is a C_2 foliation by spacelike hypersurfaces with boundary $\mathcal{B}$ of $D(S_0)$, the domain of dependence of a spacelike hypersurface with boundary $\mathcal{B}$. Suppose further that the leaves of F are spacelike hypersurfaces of constant mean curvature and that Hawking's timelike convergence condition holds in $D(S_0)$. Then if $\tilde{C}$ is any future directed timelike curve, with arc length parameter s , intersecting all leaves of F in $Int\, D(S_0)$, we have $\frac{d\Omega}{ds} > 0$ where $\Omega(s)$ is the mean curvature of the spacelike hypersurface intersected by $\tilde{C}$ at $\tilde{C}(s)$. (The mean curvature is a strictly monotonic increasing function of time through the foliation).

The proof of this theorem depends on the following computational lemma, a similar result to which was proved by Lichnerowicz [37].

(3.8) Lemma

Label the leaves of the foliation F, above, by the parameter s (defined by $\tilde{c}$). Label their orthogonal trajectories smoothly by x^i, $i = 1,2,3$. Then the metric in $D(S_0)$ can be written:-

$$d\tilde{s}^2 = l^2 ds^2 + g_{ij} dx^i dx^j \qquad l \geq 0$$

as in the case of the natural coordinate system on a stably causal manifold (2.10.1) this implies :-

$$\Omega(s) = -\frac{g^{ij} g_{ij,0}}{2l}$$ where we put $x^0 = s$

And we have :-

$$\nabla^2 l = \Omega_{,0} + l\,({}^4R^0_0 - \Omega^{jl}\Omega_{jl})$$

where $\nabla^2 l = a^{ij} l_{;ij}$ (as at (2.1) (2.7), a^{ij} is the induced metric and $;$ its covariant derivative operator).

and ${}^4R^0_0$ is a component of the Ricci tensor of M, this is indicated by the prefix 4. Similar tensors defined from the induced metric of a spacelike hypersurface will be given the prefix 3.

Proof (of lemma (3.8)

A straightforward calculation from the definition (3.5) yields :-

$$a^{ik}\,{}^4R_{jk} = {}^3R + \frac{\nabla^2 l}{l} + \Omega^2 - \frac{\Omega_{,0}}{l}$$

$$= {}^4R - {}^4R^0_0$$

(3.8.1) So $$\nabla^2 l = \Omega_{,0} + l\,({}^4R - {}^4R^0_0 - {}^3R - \Omega^2)$$

But the Gauss and Codazzi [19] equations for a spacelike hypersurface $y^\alpha = y^\alpha(x^i)$ are :-

$${}^3R_{ijkl} = (\Omega_{ik}\Omega_{jl} - \Omega_{il}\Omega_{jk}) + {}^4R_{\alpha\beta\gamma\delta}\, y^\alpha_{,i}\, y^\beta_{,j}\, y^\gamma_{,k}\, y^\delta_{,l}$$

[and $\Omega_{ij;k} - \Omega_{ik;j} = {}^4R_{\alpha\beta\gamma\delta}\, y^\alpha_{,i}\, y^\gamma_{,j}\, y^\delta_{,k}\, \zeta^\beta$]

(for notation c.f (2.5 - 2.7). So for the hypersurface $x^0 =$ constant (with internal coordinates the x^i) :-

$${}^3R_{ijkl} = {}^4R_{ijkl} + \Omega_{ik}\Omega_{jl} - \Omega_{il}\Omega_{jk}$$

So $${}^3R_{jk} = {}^4R_{jk} - {}^4R_{0jk0} + \Omega^l{}_k \Omega_{jl} - \Omega\,\Omega_{jk}$$

And $${}^3R = {}^4R - \frac{2}{l^2}\,{}^4R_{00} + \frac{1}{l^4}\,{}^4R_{0000} + \Omega^{jl}\Omega_{jl} - \Omega^2$$

But $^4R_{0000} = 0$ and $\frac{1}{l^2}\,{}^4R_{00} = {}^4R^0_0$

which in combination with equation (3.8.1) shows :-
(3.8.2)

$$\nabla^2 l = \Omega_{,0} + l\left({}^4R^0_0 - \Omega^{jl}\Omega_{j,l}\right)$$

Proof (of theorem (3.7)

Let S be one of the leaves of F, then Green's theorem implies :- $\int_{\mathcal{B}} l\,\nabla l.\,\underline{dS} = \int_S \left(l\,\nabla^2 l + \nabla l.\,\nabla l\right) d^3V$

where $d^3V = \sqrt{|-a_{ij}|}\; d^3x$ and $\underline{dS}$ is the two-dimensional surface element defined by the metric induced on $\mathcal{B}$ by that of S and the outward normal.

But $l = 0$ on $\mathcal{B}$, as all the hypersurfaces have the same boundary so

$$0 = \int_S \left(l\,\nabla^2 l + \nabla l.\,\nabla l\right) d^3V$$

Now assume

$$\Omega_{,0} \leq 0$$

Then, by the lemma (3.8) and the Hawking condition (3.4)

$$l\,\nabla^2 l \leq 0$$

But as $\nabla l.\,\nabla l \leq 0$ by the negative definiteness of a_{ij}
this forces $\nabla l \equiv 0$ and l is constant.
So $l \equiv 0$ This cannot be true, it would contradict the timelike character of the orthogonal trajectories of the leaves of the foliation. So the assumption $\Omega_{,0} \leq 0$ is false and $\Omega_{,0} > 0$.
(3.7.1)

A similar result holds true for foliations of manifolds by spacelike hypersurfaces without boundary.

(3.9) Theorem

Suppose F is a C_2 foliation, of a manifold M, by spacelike hypersurfaces without boundary. Suppose further that the leaves of F are spacelike hypersurfaces of constant mean curvature and that Hawking's timelike convergence condition holds in M. Then, in the sense of theorem (3.7), $\Omega_{,0} \geqslant 0$

and if $\Omega_{,o}(s) = 0$ then also ${}^4R^o_o(s) = 0$ and $\Omega_{ij}(s) = 0$ and $l(s)$ is a constant function of the x^i .

Proof

As in the proof of theorem (3.7), by an application of Green's theorem it may be deduced that :-

$$0 = \int_S (l \nabla^2 l + \nabla l . \nabla l)\, d^3V$$

if $\Omega_{,o} < 0$ then by the lemma (3.8) $l \nabla^2 l < 0$ which is impossible as $\nabla l . \nabla l \leq 0$. So we may conclude that $\Omega_{,o} \geq 0$. If $\Omega_{,o}(s) = 0$, a further application of the lemma gives $\nabla l \equiv 0$

and ${}^4R^o_o - \Omega^{;j} \Omega_{;j} \equiv 0$

That is $l(s)$ is constant and ${}^4R^o_o(s) \equiv \Omega_{;j}(s) \equiv 0$

(3.10) Corollary

$\Omega(t) = \Omega(s) \Leftrightarrow t = s$ unless the manifold M is static, such that its metric may be written in the form

(3.10.1) $ds^2 = d\tilde{t}^2 + \tilde{a}_{ij}\, dx^i dx^j$ with $\frac{\partial a_{ij}}{\partial \tilde{t}} = 0$

Proof

Since $\Omega_{,o} \geq 0$, $\Omega(t) = \Omega(s)$ implies that $\Omega_{,o}(u) = 0$ $\forall u \in [t,s]$. This interval is not trivial if and only if

$(\nabla l)|_u = 0$ and $\Omega_{ij}(u) = 0 \quad \forall u \in [t,s]$

But $\Omega_{ij} = 0 \Rightarrow g_{ij,o}(u) = 0$

and so, as $x^o = \frac{1}{2}(t+s)$ is a Cauchy surface for the manifold, that $g_{ij,o}(u) = 0 \quad \forall u$. The metric is then put in the form (3.10.1) by setting $\tilde{t} = \int l\, ds$.

(3.11)

As an example consider the Friedmann universe with $K = +1$, it has the metric :-

$$ds^2 = dt^2 - S^2(t)\, d\sigma^2$$

where $d\sigma^2 = d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\, d\phi^2)$

so $\Omega(t) = -(3S')/S$

$$\Omega'(t) = -(3S'')/S + 3\left(\frac{S'}{S}\right)^2$$

But the Einstein Equations yield 1

$$\frac{K}{2}\left(\rho + \frac{3p}{c^2}\right) = -\frac{3\ddot{S}}{S}$$

and the L.H.S. is positive by the Hawking convergence condition

So $\Omega' > 0$

At (2.10) it was seen that if a spacelike hypersurface S is a local extremum of the functional $A + \kappa V$, then it is a spacelike hypersurface of constant mean curvature κ. In the following section it will be shown that if S is a member of a foliation made up of constant mean curvature spacelike hypersurfaces with $\Omega_{,0}(S) > 0$ then it is in fact a strict local maximum of the functional $A + \kappa V$.

(3.12) Definition

Suppose S is a C_2 spacelike hypersurface. Then the ~~tangent~~ L-displacement space TS_L of S is the space of C_2 scalar fields defined on S (and which vanish on $\mathcal{B}$ in the case of hypersurfaces with boundary $\mathcal{B}$). An element $t \in TS_L$ is to be thought of as a field of displacements from S, measured along a congruence L of orthogonal trajectories of S.

(3.13) Definition

A one parameter variation of S is a function

$$\mathcal{I} : (-\epsilon, \epsilon) \longrightarrow C_2^{SL} \qquad (\text{or } C_2^{SLB}(\mathcal{B}))$$

such that $\mathcal{I}(0) = S$, $\mathcal{I}(t)$ is a C_2 spacelike hypersurface (with boundary $\mathcal{B}$, in the case $C_2^{SLB}(\mathcal{B})$).

(3.14) Claim

Given $W \in TS_L$, $\exists\, \mathcal{I}$ s.t. $\mathcal{I}(0) = S$, $\frac{\partial \mathcal{I}}{\partial u}(0) = W$

Consider, in the local coordinates in which S is $y^0 = 0$, $y^i = const$ are the congruence of orthogonal trajectories of S, L, and y^0 measures arc length along these trajectories (call this coordinate system F_{S_L}), the one parameter variation :-

$$\mathcal{I}(t)(y^i) = W(y^i)\, t$$

(3.15) Definition

A spacelike hypersurface S is said to be a local extremum for a functional $F : C_2^{SL}\ (\text{or } C_2^{SLB}(\mathcal{B})) \longrightarrow \mathbb{R}$ iff

$$\frac{d}{du}\left(F[\mathcal{I}(u)] \right)\Big|_{u=0} = 0 \qquad \forall \text{ one parameter variations } \mathcal{I}$$

(3.16) Note

This makes more precise the variation carried out at (2.5)

(3.17) Definition

A two parameter variation of S is a function

$\mathcal{I} : \mathcal{U} \longrightarrow C_2^{SL}$ (or $C_2^{SLB}(\mathcal{B})$) where $\mathcal{U}$ is an open set in $\mathbb{R}^2$ with $(0,0) \in \mathcal{U}$.

such that $\mathcal{I}(0,0) = S$, $\mathcal{I}(u,v)$ is a C_2 spacelike hypersurface (with boundary $\mathcal{B}$) $\forall (u,v) \in \mathcal{U}$.

(3.18) Claim

Given $W_1, W_2 \in TS_L$ $\exists\ \bar{\alpha}$ s.t. $\bar{\alpha}(0,0) = S$ and $\frac{\partial \bar{\alpha}}{\partial u_i}(0,0) = W_i \quad i = 1,2.$

Consider, using coordinates Fs_L

$$\bar{\alpha}(u_1,u_2)(y^i) = W_1(y^i)u_1 + W_2(y^i)u_2$$

(3.19) Definition

The L-Hessian of a functional $F: C_2^{sL}$ (or $C_2^{sLB}(B)) \longrightarrow \mathbb{R}$, F_{L**}, is defined by :-

$$F_{L**} : TS_L \times TS_L \longrightarrow \mathbb{R}$$

$$(W_1, W_2) \longmapsto \left.\frac{\partial^2 F[\bar{\alpha}(u_1,u_2)]}{\partial u_1 \partial u_2}\right|_{(0,0)}$$

It is necessary to show that this functional, F_{L**}, is well defined in the case in which it will be needed, that is for $F \equiv A + \kappa V$ and S a local extremum. This and the symmetricity of F_{L**} will be shown below.

(3.20) Remark

It is not strictly necessary to introduce the Hessian here to show that S is a local maximum of $A + \kappa V$, it is though perhaps of interest in itself.

(3.21) Lemma [40]

The diagonal terms $F_{L**}(W,W)$ (assuming this well defined) can be described in terms of a one parameter variation, $\bar{\alpha}(u)$.

In fact $F_{L**}(W,W) = \left.\frac{\partial^2 F[\bar{\alpha}(u)]}{\partial u^2}\right|_0$ where $\bar{\alpha}$ is any one parameter variation such that $\left.\frac{\partial \bar{\alpha}}{\partial u}\right|_{u=0} = W$

Proof (of lemma (3.21))

Introduce $\bar{\beta}(u_1,u_2) = \bar{\alpha}(u_1+u_2)$ and notice that

$$\left.\frac{\partial \bar{\beta}}{\partial u_i}\right|_{(0,0)} = \left.\frac{\partial \bar{\alpha}}{\partial u}\right|_0 \quad \text{and} \quad \left.\frac{\partial^2 F[\bar{\beta}(u_1,u_2)]}{\partial u_1 \partial u_2}\right|_{(0,0)} = \left.\frac{\partial^2 F[\bar{\alpha}(u_1+u_2)]}{\partial u^2}\right|_0$$

where $u = u_1 + u_2$

(3.22) Corollary

If $F_{L**}(W,W)$ is negative definite (assuming it to exist) then the second variation of F is negative definite.

(3.23)

An explicit calculation of the Hessian of the functional $A + \kappa V$

This calculation, carried out at a local extremum S of $A+\kappa V$ will prove that $(A+\kappa V)_{L}^{**}$ is well defined and symmetric. The calculation is performed in the coordinate system F_{S_L} (see (3.14)).

S is the hypersurface $y^0 = 0$, a local extremum of $A+\kappa V$. The metric takes the form $ds^2 = g_{00}\,dy^{0^2} + g_{ij}\,dy^i dy^j$, $\bar{\mathcal{I}}$ is a two parameter variation of S with $\frac{\partial \bar{\mathcal{I}}}{\partial u_i}\,|_{0,0}(y^j) = W_i(y^j)$

$$\text{So } (A+\kappa V)[\bar{\mathcal{I}}(u_1,u_2)] = \int_S \sqrt{-|g_{ij}|}\,(1+g_{00}\,y^0_{;i}\,y^0_{;j}\,g^{ij})\,d^3y + \kappa \int_{J^-[S]\cap J^+[P]} \sqrt{-|g_{\alpha\beta}|}\;d^4y$$

where P is $H^-[S]$ or T depending on the case involved (see (2.3) (2.4)) and $y^0(y^i) \equiv \bar{\mathcal{I}}(u_1,u_2,y^i)$.

Write $F \equiv (A+\kappa V)[\bar{\mathcal{I}}(u_1,u_2)]$

$$\text{So } \frac{\partial F}{\partial u_1} = \int_S d^3y \left\{ \frac{-1}{2\sqrt{-|a_{ij}|}} \left[\frac{\partial |g_{ij}|}{\partial u_1}(1+g_{00}\,y^0_{;i}\,y^0_{;j}\,g^{ij}) + |g_{ij}| \left\{ \frac{\partial g_{00}}{\partial u_1}\,y^0_{;i}\,y^0_{;j}\,g^{ij} + g_{00}\left(\frac{\partial g_{ij}}{\partial u_1}\,y^0_{;i}\,y^0_{;j} + 2g^{ij}\,y^0_{;i}\left(\frac{\partial y^0}{\partial u_1}\right)_{;j}\right)\right\}\right] + \kappa\sqrt{-|a_{ij}|}\;\zeta_0\,\frac{\partial y^0}{\partial u_1}\right\}$$

where ζ^α is the unit, future directed, normal to S and

$$a_{ij} = g_{00}\,y^0_{;i}\,y^0_{;j} + g_{ij} \qquad (2.2.2)\ \&\ (2.5.1)$$

$$\frac{\partial^2 F}{\partial u_1 \partial u_2} = \int_S d^3y \left\{ \frac{-1}{4(\sqrt{-|a_{ij}|})^3}\,\frac{\partial}{\partial u_1}(-|a_{ij}|)\,\frac{\partial}{\partial u_2}(-|a_{ij}|) \right.$$

$$\frac{-1}{2\sqrt{-|a_{ij}|}} \left[\frac{\partial^2 |g_{ij}|}{\partial u_1 \partial u_2}(1+g_{00}\,y^0_{;i}\,y^0_{;j}\,g^{ij}) + \frac{\partial |g_{ij}|}{\partial u_1}\frac{\partial}{\partial u_2}(g_{00}\,y^0_{;i}\,y^0_{;j}\,g^{ij}) + \right.$$

$$+ \frac{\partial |g_{ij}|}{\partial u_2}\frac{\partial}{\partial u_1}(g_{00}\,y^0_{;i}\,y^0_{;j}\,g^{ij}) + |g_{ij}|\frac{\partial}{\partial u_2}\left(\frac{\partial g_{00}}{\partial u_1}\,y^0_{;i}\,y^0_{;j}\,g^{ij} + g_{00}\frac{\partial g^{ij}}{\partial u_1}\,y^0_{;i}\,y^0_{;j}\right)$$

$$+ |g_{ij}|\frac{\partial}{\partial u_2}\left(2g_{00}\,g^{ij}\left(\frac{\partial y^0}{\partial u_1}\right)_{;j}\right)y^0_{;i} + 2|g_{ij}|\,g_{00}\,g^{ij}\left(\frac{\partial y^0}{\partial u_1}\right)_{;j}\left(\frac{\partial y^0}{\partial u_2}\right)_{;i}\Bigg]$$

$$\left. + \kappa\left[\frac{1}{2\sqrt{-|a_{ij}|}}\,\frac{\partial |-a_{ij}|}{\partial u_2}\,\zeta_0\,\frac{\partial y^0}{\partial u_1} + \sqrt{-|a_{ij}|}\left(\frac{\partial \zeta_0}{\partial u_2}\,\frac{\partial y^0}{\partial u_1} + \zeta_0\,\frac{\partial^2 y^0}{\partial u_1 \partial u_2}\right)\right]\right\}$$

$$\frac{\partial^2 F}{\partial u_1 \partial u_2}\Big|_{(0,0)} = \int_S d^3y \left\{ \frac{-1}{4(\sqrt{-1g_{ij}1})^3} (g^{ij} g_{ij,0})^2 W_1 W_2 - \frac{1}{2\sqrt{-1g_{ij}1}} \left[\frac{\partial^2 |g_{ij}|}{\partial u_1 \partial u_2} + \right.\right.$$

$$\left. + 2|g_{ij}| g_{00} g^{ij} W_{1,j} W_{2,i} \right] + K \left[\frac{-|g_{ij}| g^{ij} g_{ij,0} W_1 W_2 \sqrt{g_{00}}}{2\sqrt{-1g_{ij}1}} + \right.$$

$$\left.\left. + \sqrt{-1g_{ij}1} \left(\frac{\partial 3_0}{\partial u_2} W_1 + \sqrt{g_{00}} \frac{\partial^2 y^0}{\partial u_1 \partial u_2} \right) \right] \right\}$$

As $3_0|_{(0,0)} = \sqrt{g_{00}}$. Now $K = \frac{-g^{ij} g_{ij,0}}{2\sqrt{g_{00}}}$ as S is an extremum

and $(3_0)^2 (g^{00} + g^{ij} y^0_{,i} y^0_{,j}) = 1$ which gives

$$3_{0,0}\Big|_{y^0=0} = \frac{g_{00,0}}{2\sqrt{g_{00}}} \quad , \quad K_{,0}\Big|_{y^0=0} = \frac{-1}{2\sqrt{g_{00}}} \left(g^{ij}{}_{,0} g_{ij,0} + g^{ij} g_{ij,00} + \frac{K g_{00,0}}{\sqrt{g_{00}}} \right)$$

and $$\frac{\partial |g_{ij}|}{\partial u_1} = |g_{ij}| g^{ij} g_{ij,0} \frac{\partial y^0}{\partial u_1}$$ so

$$\frac{\partial^2 |g_{ij}|}{\partial u_1 \partial u_2} = |g_{ij}| \left[g^{ij}{}_{,0} g_{ij,0} + g^{ij} g_{ij,00} + (g^{ij} g_{ij,0})^2 \right] \frac{\partial y^0}{\partial u_1} \frac{\partial y^0}{\partial u_2} +$$

$$+ |g_{ij}| g^{ij} g_{ij,0} \frac{\partial^2 y^0}{\partial u_1 \partial u_2}$$

So

$$\frac{\partial^2 F}{\partial u_1 \partial u_2}\Big|_{(0,0)} = \int_S \sqrt{-1g_{ij}1} \left[-\sqrt{g_{00}}\, K_{,0} W_1 W_2 + g_{00} g^{ij} W_{1,i} W_{2,j} \right] d^3y$$

We have shown :-

(3.23.1) Theorem

$$(A + KV)_L{}^{**}(W_1, W_2) = \int_S \sqrt{-1g_{ij}1} \left[-\sqrt{g_{00}}\, K_{,0} W_1 W_2 + g_{00} g^{ij} W_{1,i} W_{2,j} \right] d^3y$$

(3.23.2) Corollary

If S is a leaf of a C_2 foliation by spacelike hypersurfaces of constant mean curvature and boundary $\mathcal{B}$ of $D(S_0)$, the domain of dependence of a spacelike hypersurface with boundary $\mathcal{B}$ and if Hawking's timelike convergence condition holds in $D(S_0)$ then if the mean curvature of S is K, S is a strict local maximum of the functional $A + KV$.

Proof

Apply theorems (3.7), (3.23.1) and lemmas(3.21), (3.22). A similar result holds for C_2 foliations by spacelike hypersurfaces of constant mean curvature without boundary of a manifold M in which Hawking's timelike convergence condition holds except that the local maximum is not strict unless there is no spacelike hypersurface of time-symmetry on which $^4R^0_0 = 0$.

However there is a rather more general result :-

(3.24) Theorem

If S is a C_2 spacelike hypersurface of constant mean curvature K which lies in an open region of the manifold M in which Hawking's timelike convergence condition holds then S is a local maximum, not necessarily strict, of the functional $A + \kappa V$.

Proof

Choose the congruence L to be the timelike geodesics orthogonal to S . Then by theorem (3.23.1) it is sufficient to show that $K_{,0} \geqslant 0$.

Consider the Landau - Raychaudhuri equation :-

(3.24.1) $$\frac{d\theta}{ds} = R_{\alpha\beta}V^\alpha V^\beta + 2\omega^2 - 2\sigma^2 - \tfrac{1}{3}\theta^2 + \dot{V}^\alpha{}_{;\alpha}$$

where V^α is the unit timelike tangent vector to a congruence of timelike curves, s is arc length along the curves and $h^\alpha{}_\beta = \delta^\alpha{}_\beta - V^\alpha V_\beta$ is the projection tensor onto those directions orthogonal to V^α .

$\theta_{\alpha\beta} = h_\alpha{}^\gamma h_\beta{}^\delta V_{(\gamma;\delta)}$ ( ) denotes symmetrisation is the expansion tensor of the congruence.

(3.24.2)

$\theta = \theta_{\alpha\beta} h^{\alpha\beta}$ is the expansion

$\sigma_{\alpha\beta} = \theta_{\alpha\beta} - \frac{1}{3} h_{\alpha\beta}\theta$ is the shear tensor

$\omega_{\alpha\beta} = h_\alpha{}^\gamma h_\beta{}^\delta V_{[\gamma;\delta]}$ [ ] denotes antisymmetrisation is the rotation tensor of the congruence.

$$\omega^2 = \tfrac{1}{2}\omega_{\alpha\beta}\omega^{\alpha\beta} \geqslant 0$$

$$\sigma^2 = \tfrac{1}{2}\sigma_{\alpha\beta}\sigma^{\alpha\beta} \geqslant 0$$

and $\dot{V}^\alpha = V^\alpha{}_{;\beta} V^\beta$

applying this equation to the congruence L and remembering that an initially rotation free congruence stays rotation free as the propagation equation for $\omega_{\alpha\beta}$ has no source terms if the congruence is geodesic :-

$$\frac{d\omega_{\alpha\beta}}{ds} = 2\,\omega_{\gamma[\alpha}\theta_{\beta]}{}^{\gamma}$$

(3.24.3)

$$\frac{d\theta}{ds} = {}^{4}R^{0}_{0} - 2\sigma^{2} - \tfrac{1}{3}\theta^{2}$$
$$\leq 0$$

But $\theta = g^{ij}g_{ij,0}$ and $K = -\dfrac{g^{ij}g_{ij,0}}{2}$

So $K_{,0} \geqslant 0$

(3.24.4) Note

If we knew ${}^{4}R^{0}_{0} < 0$ then it would follow that the maximum was strict.

(3.24.5) Note

It is clear from equation (3.24.3) that if Hawking's timelike convergence condition (3.4) holds then $\frac{d\theta}{ds} \leq 0$ where θ is the expansion of the congruence of geodesics normal to the surface S , this is why the condition is so named.

(3.24.6) Note

In the case where S is a C_2 spacelike hypersurface of zero mean curvature in a manifold M in which Hawking's timelike convergence condition holds and is such that

$$(A + \kappa V)_{L}{}^{**}(W,W) = 0 \qquad \forall W$$

Brill [9] has shown that the fourth variation is always negative definite, unless M is flat with $R_{\alpha\beta\gamma\delta} \equiv 0$.
That is unless M is Minkowski Space.
(He has also calculated what is effectively (3.23.1) for the case $\kappa = 0$.)

(3.24.7) Corollary (to theorem (3.24))

If S is a C_2 spacelike hypersurface of constant mean curvature K which lies in an open region of the manifold M in which Hawking's timelike convergence condition holds then S is a local maximum or minimum of the functional V , subject to A being held constant at $A[S]$, according as K is positive or negative. If K is zero, then the constrained

second variation of V is also zero. Also S is a local maximum of the functional A, subject to V being held constant as $V[S]$.

Proof

Using the congruence L of theorem (3.24) and the notation of the preceeding calculation (3.23)

$$\left.\frac{\partial^2 V}{\partial u_1 \partial u_2}\right|_{(0,0)} = \int_S d^3y \sqrt{-|g_{ij}|}\left\{-g_{00} W_1 W_2 K + \frac{W_1 W_2 g_{00,0}}{2\sqrt{g_{00}}} + \sqrt{g_{00}}\frac{\partial^2 y^0}{\partial u_1 \partial u_2}\right\}$$

and

$$\left.\frac{\partial^2 A}{\partial u_1 \partial u_2}\right|_{(0,0)} = \int_S d^3y \sqrt{-|g_{ij}|}\left\{K^2 g_{00} W_1 W_2 + W_1 W_2\left(-K_{,0}\sqrt{g_{00}} - \frac{K g_{00,0}}{2\sqrt{g_{00}}}\right) + g_{00} g^{ij} W_{1,i} W_{2,j} - K\sqrt{g_{00}}\frac{\partial^2 y^0}{\partial u_1 \partial u_2}\right\}$$

But if A is constant $\left.\frac{\partial^2 A}{\partial u_1 \partial u_2}\right|_{(0,0)} = 0$

$$\Rightarrow K\left.\frac{\partial^2 V}{\partial u_1 \partial u_2}\right| = \int_S \sqrt{-|g_{ij}|}\left(-K_{,0}\sqrt{g_{00}} W_1 W_2 + g_{00} g^{ij} W_{1,i} W_{2,j}\right) d^3y$$

And if V is constant $\left.\frac{\partial^2 V}{\partial u_1 \partial u_2}\right|_{(0,0)} = 0$

$$\Rightarrow \left.\frac{\partial^2 A}{\partial u_1 \partial u_2}\right| = \int_S \sqrt{-|g_{ij}|}\left(-K_{,0}\sqrt{g_{00}} W_1 W_2 + g_{00} g^{ij} W_{1,i} W_{2,j}\right) d^3y$$

Hence the conclusions of the theorem.

In the next section it will be shown that if F is a C_2 foliation of a manifold M by spacelike hypersurfaces of constant mean curvature without boundary then F is complete, in the sense that if S is any C_2 spacelike hypersurface of constant mean curvature without boundary in M then it necessarily is a leaf of the foliation F.

(3.25) Touching Hypersurfaces and Mean Curvature

Suppose S and T are two C_2 spacelike hypersurfaces given by $y^0 = f(y^i)$ and $y^0 = g(y^i)$ respectively. Suppose that they touch at a point P.

That is $f|_P = g|_P$ and $f_{,i}|_P = g_{,i}|_P \quad i = 1,2,3$.

Then the induced metrics in S and T are the same at P and so are their normal vectors and if Ω and Π are the mean curvatures of S and T respectively.

$$\Omega = a^{ij} g_{\alpha\beta} \zeta^\beta y^\alpha_{;ij} + a^{ij} [\mu\nu,\beta]_g \, y^\mu_{;i} y^\nu_{;j} \zeta^\beta$$
$$= \zeta_\alpha a^{ij} \left[y^\alpha_{,ij} - y^\alpha_{,k} \left(\Gamma^k_{ij} \right)_{a_0} \right] + a^{ij} [\mu\nu,\beta]_g \, y^\mu_{;i} y^\nu_{;j} \zeta^\beta$$

$$(\Omega - \pi)|_p = a^{ij} \zeta_\alpha \left[{}^S y^\alpha_{;ij} - {}^T y^\alpha_{;ij} \right] \quad \text{as } \zeta_\alpha y^\alpha_{,k} = 0$$
$$= \zeta_0 a^{ij} \left[f_{,ij} - g_{,ij} \right]|_p$$

Now suppose S is the surface $f = c$, a constant and suppose further that $\exists$ a neighbourhood $\mathcal{U}$ of P s.t. $g(y^i) \leq f(y^i)$ if $(y^0, y^i) \in \mathcal{U}$. Then $\zeta_0|_p = \sqrt{g_{00}}|_p$ and $f_{,ij}|_p = 0$

$$\pi|_p = \Omega|_p + \sqrt{g_{00}} \, a^{ij} g_{,ij}|_p$$

But $a^{ij}|_p$ is a negative definite matrix and $g_{,ij}|_p$ is a negative semi-definite matrix so as $\sqrt{g_{00}}|_p > 0$

(3.25.1) $\pi|_p \geq \Omega|_p$

Similarily if we had $g(y^i) \geq f(y^i)$ for $(y^0, y^i) \in \mathcal{U}$ Then it would follow that

(3.25.2) $\pi|_p \leq \Omega|_p$

(3.26) Theorem

Suppose that F is a C_2 foliation of a manifold M by spacelike hypersurfaces of constant mean curvature without boundary such that if S and T are two leaves of the foliation $\Omega(S) = \Omega(T) \iff S \equiv T$. Suppose that L is a C_2 spacelike hypersurface without boundary of constant mean curvature π, then L is a leaf of the foliation F.

Proof

By theorem (3.9) the mean curvature Ω of the leaves of the foliation may be used as a time coordinate for M; for if S and T are any two leaves of F and if $S \subseteq I^+[T]$ then $\Omega(S) > \Omega(T)$. Label the orthogonal trajectories of the foliation smoothly by y^i. Then L has equation :-

$$\Omega = \Omega(y^i)$$

By the compactness of L, $\exists$ points $P_{MAX}, P_{MIN} \in L$ such that

$$Q \in L \Rightarrow \Omega(P_{MIN}) \leq \Omega(Q) \leq \Omega(P_{MAX})$$

Now applying (3.25.1), (3.25.2) :-

$$\pi \leq \Omega(P_{MIN}) \leq \Omega(P_{MAX}) \leq \pi$$

So $\Omega(P_{MIN}) = \Omega(P_{MAX}) = \pi$ and L is the hypersurface

$$\Omega = \pi .$$

(3.26.1) Corollary

If F_1 and F_2 are C_2 foliations of M by spacelike hypersurfaces without boundary and if F_1 has the property, if S , T leaves of F_1 then $\Omega(S)=\Omega(T) \Leftrightarrow S \equiv T$. Then any leaf of F_2 is a leaf of F_1 . In particular if M has no moments of time symmetry where ${}^4R^0_0 = 0$ and if Hawking's timelike convergence condition holds in M then any two foliations F_1 and F_2 of M have the same leaves. That is there is essentially only one such foliation.

Flaherty$_{21}$, has proved a similar theorem:-

(3.27) Theorem

Suppose S_0 is a Cauchy spacelike hypersurface of zero mean curvature in a universe with compact spacelike sections, topologically S^3 and suppose the Hawking timelike convergence condition holds in this manifold. Then if S_1 is a spacelike hypersurface of zero mean curvature it must be identical to S_0

Proof (sketch)

The idea of the proof is to erect a family of timelike geodesics from S_0 to S_1 , orthogonal to S_0 . One of these geodesics must have maximal length and so meet S_1 orthogonally. Calculating the second variation of the length of this maximising geodesic through the given family of timelike geodesics from S_0 to S_1 then achieves a contradiction as this variation turns out to be positive, whereas it should have been negative for a maximum. The proof does not go through in vacuum spacetimes, that is when ${}^4R_{\alpha\beta} \equiv 0$.

CHAPTER IV

THE FAMILIES OF SPACELIKE HYPERSURFACES DEFINED BY THE MAXIMA OF $\tilde{A} + \kappa\tilde{V}$

This chapter is concerned with the properties of the one parameter families of spacelike hypersurfaces of generalised constant mean curvature S_κ , which were defined at the end of chapter II (2.28 - 30). Throughout the chapter I make the assumption that there is one and only one such spacelike hypersurface for each κ .

It will be seen that the families can be considered as the leaves of foliations and that the generalised constant mean curvature κ is a monotonic increasing function of time on each timelike curve intersecting each such foliation.

Throughout the chapter, when dealing with hypersurfaces $S_\kappa \in \overline{C_0^{SLB}(\mathcal{B})}$ I shall assume that if S_0 is some spacelike hypersurface with boundary $\mathcal{B}$ then $D(S_0)$ is compact and when dealing with hypersurfaces $S_\kappa \in \overline{C_0^{SL}}$ I shall assume that M is compact (with boundary) or of type (R-W), unless explicitly stated to the contrary. I shall further assume, in both cases, that M is stably causal.

Inequalities concerning $\tilde{A}[S_\kappa], \tilde{V}[S_\kappa]$

(4.1) Lemma

If the spacelike hypersurfaces S_{κ_i} $i = 1, 2$ maximise the functionals $\tilde{A} + \kappa_i \tilde{V}$ in $C_0^{SLB}(\mathcal{B})$ and if $\kappa_1 > \kappa_2$, then $\tilde{V}[S_{\kappa_1}] > \tilde{V}[S_{\kappa_2}]$, unless $S_{\kappa_1} \equiv S_{\kappa_2}$.

Proof

Assume $S_{\kappa_1} \neq S_{\kappa_2}$

then $\tilde{A}[S_{\kappa_1}] + \kappa_1 \tilde{V}[S_{\kappa_1}] > \tilde{A}[S_{\kappa_2}] + \kappa_1 \tilde{V}[S_{\kappa_2}]$

and $\tilde{A}[S_{\kappa_2}] + \kappa_2 \tilde{V}[S_{\kappa_2}] > \tilde{A}[S_{\kappa_1}] + \kappa_2 \tilde{V}[S_{\kappa_1}]$

$\Rightarrow$

(4.1.1) $\kappa_2(\tilde{V}[S_\kappa] - \tilde{V}[S_{\kappa_2}]) < \tilde{A}[S_{\kappa_2}] - \tilde{A}[S_{\kappa_1}] < \kappa_1(\tilde{V}[S_{\kappa_1}] - \tilde{V}[S_{\kappa_2}])$

Clearly if $\tilde{V}[S_{\kappa_1}] < \tilde{V}[S_{\kappa_2}]$ this would contradict $\kappa_1 > \kappa_2$

and if $\tilde{V}[S_{\kappa_1}] = \tilde{V}[S_{\kappa_2}]$ this would contradict (4.1.1)

so we must have $\tilde{V}[S_{\kappa_1}] > \tilde{V}[S_{\kappa_2}]$

(4.2) Lemma

If the spacelike hypersurfaces S_κ maximise the functionals $\tilde{A} + \kappa_i \tilde{V}$ in $C_0^{SLB}(\mathcal{B})$ and if $\tilde{V}[S_{\kappa_1}] > \tilde{V}[S_{\kappa_2}]$ then $\kappa_1 > \kappa_2$.

Proof

From equation (4.1.1)

(4.3) Lemma

If the spacelike hypersurfaces S_{κ_i} maximise the functionals $\tilde{A}[\ ] + \kappa_i \tilde{V}[\ ; T]$ in $\overline{C_0^{SL}}$ and if $S_{\kappa_i} \leq I^+[T]$ and further $\kappa_1 > \kappa_2$, then $\tilde{V}[S_{\kappa_1}; T] > \tilde{V}[S_{\kappa_2}; T]$, unless $S_{\kappa_1} \equiv S_{\kappa_2}$.

Proof

Same as lemma (4.1).

(4.4) Remark

Since we are assuming M to be compact or $(R-W)$, if $S \in \overline{C_0^{SL}}$ then $\int_{J^-[S]} \sqrt{-|g_{\alpha\beta}|}\, d^4y$ will be finite so we could use $T = \phi$ with

$$\tilde{V}_\phi[S] \equiv \tilde{V}[S, \phi] \equiv \int_{J^-[S]} \sqrt{-|g_{\alpha\beta}|}\, d^4y$$

in the lemma above and dispense with the requirements $S_{\kappa_i} \subseteq I^+[T]$.

(4.5) Remark

If it were assumed that the S_{κ_i} are C_2 spacelike hypersurfaces then the possibility $\kappa_1 > \kappa_2$, $S_{\kappa_1} \equiv S_{\kappa_2}$ could not arise as the spacelike hypersurface S_{κ_i} would have constant mean curvature κ_i (2.28.1).

The next theorem orders the hypersurfaces S_κ.

(4.6) Theorem

If the hypersurfaces S_{κ_i} maximise the functionals $\tilde{A} + \kappa_i \tilde{V}$ in $\overline{C_0^{SLB}(\mathcal{B})}$, then, if $\kappa_1 > \kappa_2$, $S_{\kappa_2} \subseteq J^-[S_{\kappa_1}]$.

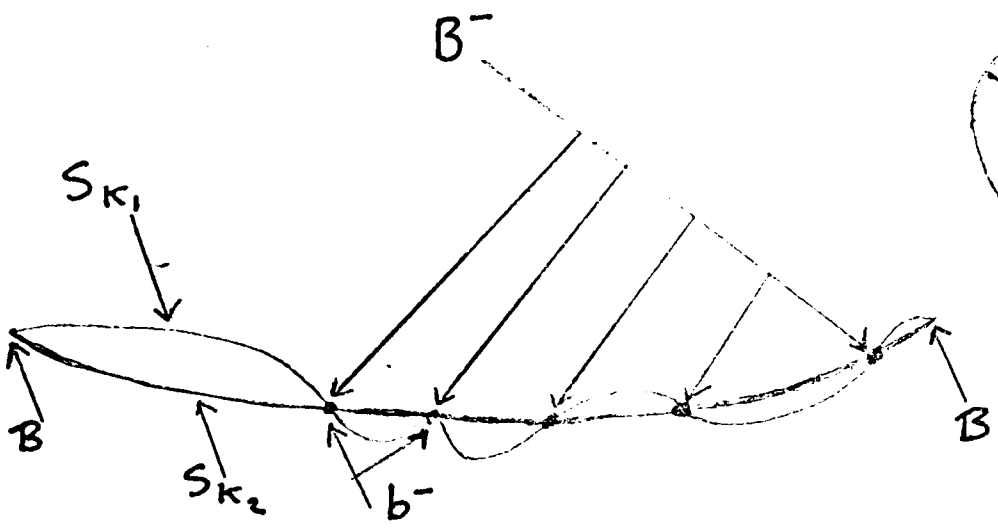
Proof

Suppose $S_{\kappa_2} \not\subseteq J^-[S_{\kappa_1}]$ then $S_{\kappa_1} \cap I^-[S_{\kappa_2}] \neq \phi$. Also it cannot be that $S_{\kappa_1} \cap I^+[S_{\kappa_2}] = \phi$ (unless $S_{\kappa_1} \equiv S_{\kappa_2}$ in which case the theorem holds, we assume we are not in this trivial sub-case) for then $\tilde{V}[S_{\kappa_1}] < \tilde{V}[S_{\kappa_2}]$ and $\kappa_1 < \kappa_2$, contrary to assumption.

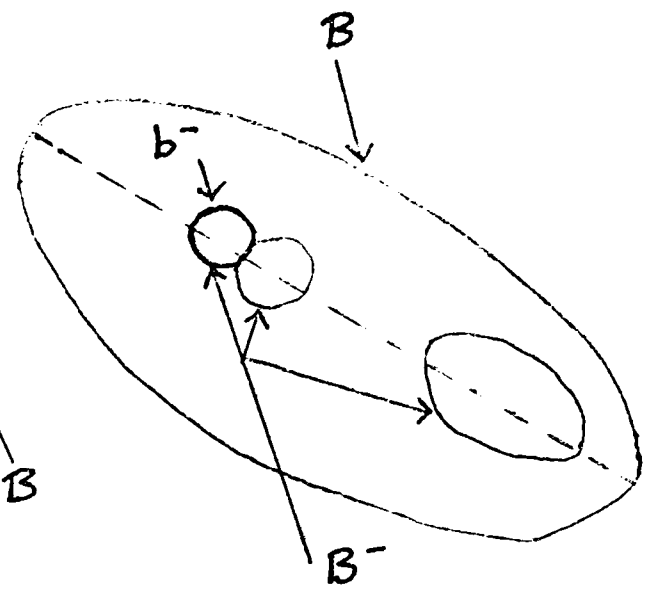
Let $B^+ = S_{\kappa_1} \cap \dot{I}^+[S_{\kappa_2}]$ where $\dot{A} \overset{def}{\equiv} \bar{A} \cap \overline{(M-A)}$

and $B^- = S_{\kappa_1} \cap \dot{I}^-[S_{\kappa_2}]$

Clearly B^+ and B^- are both closed submanifolds both of S_{κ_1} and S_{κ_2} of dimension two. B^+ and B^- can be regarded as the unions of the boundaries of spacelike hypersurfaces and each such boundary is spanned by parts of both S_{κ_1} and S_{κ_2} (see diagram).



SECTION THROUGH DASHED LINE ON PLAN



PLAN

Choose one of the 'component boundaries' of B^-, b^- say. Consider the space $\overline{C_0^{SLB}(b^-)}$. Call the parts of the S_{K_i} spanning b^-, $S_{K_i}(b^-)$. Now $S_{K_i}(b^-)$ maximises the functional $\tilde{A} + K_i \tilde{V}$ in $\overline{C_0^{SLB}(b^-)}$ for suppose $T \in \overline{C_0^{SLB}(b^-)}$, then $\tilde{T}_i \stackrel{def}{=} T \cup (S_{K_i} - S_{K_i}(b^-)) \cup b^- \in \overline{C_0^{SLB}(B)}$ which forces:-

$$\tilde{A}[\tilde{T}_i] + K_i \tilde{V}[\tilde{T}_i] < \tilde{A}[S_{K_i}] + K_i \tilde{V}[S_{K_i}]$$

we are assuming $T \not\equiv S_{K_i}(b^-)$

But $$\tilde{V}[S_{K_i}] - \tilde{V}[\tilde{T}_i] = \tilde{V}_{b^-}[S_{K_i}(b^-)] - \tilde{V}_{b^-}[T]$$

where $\tilde{V}_{b^-}$ is the hypovolume functional defined using the boundary b^-.

And $$\tilde{A}[S_{K_i}] = \tilde{A}[S_{K_i}(b^-)] + \tilde{A}[S_{K_i} - S_{K_i}(b^-)]$$

$$\tilde{A}[\tilde{T}_i] = \tilde{A}[T] + \tilde{A}[S_{K_i} - S_{K_i}(b^-)]$$

So $$\tilde{A}[T] + K_i \tilde{V}_{b^-}[T] < \tilde{A}[S_{K_i}(b^-)] + K_i \tilde{V}_{b^-}[S_{K_i}(b^-)]$$

showing that $S_{K_i}(b^-)$ is a hypersurface of generalised constant mean curvature K_i.

From the construction used $S_{K_2}(b^-) \subseteq I^+[S_{K_1}(b^-)] \cup b^-$

So $\tilde{V}_{b^-}[S_{K_2}(b^-)] > \tilde{V}_{b^-}[S_{K_1}(b^-)]$

So by lemma (4.2) $K_1 < K_2$.

(However by considering similarily a component boundary of B^+ it can be shown that $K_1 > K_2$.) This contradiction establishes the theorem.

Similarily it is true that :-

(4.7) Theorem

If the hypersurfaces S_{K_i} maximise the functionals $\tilde{A} + \tilde{V}_p$ in $\overline{C_0^{SL}}$, then, if $K_1 > K_2$, $S_{K_2} \subseteq J^-[S_{K_1}]$.

(4.7.1) Corollary

If one makes the further assumption that S_{K_i} are C_2, the conclusions of theorems (4.6), (4.7) can be strengthened to

$$S_{K_2} \subseteq I^-[S_{K_1}] \cup \mathcal{B} \qquad \text{and}$$

$$S_{K_2} \subseteq I^-[S_{K_1}] \qquad \text{respectively.}$$

Proof

Suppose, for example, that $S_{K_2} \subseteq J^-[S_{K_1}]$ but that $S_{K_2} \not\subseteq I^-[S_{K_1}] \cup \mathcal{B}$. Then by lemma (4.2) $K_1 > K_2$ but S_{K_1} and S_{K_2} must touch somewhere so by equation (3.25.1) $K_1 \leq K_2$. This contradiction establishes the result.

(4.8) Theorem

If the hypersurfaces S_{K_i} maximise the functionals $\tilde{A} + K_i \tilde{V}$ in $C_0^{sub}(\mathcal{B})$ on a stably causal manifold M and if $K_1 < K_2$ and if $K_i \neq K_j \Rightarrow S_{K_i} \not\equiv S_{K_j}$, then the region $D \equiv (J^-[S_{K_2}] \cap J^+[S_{K_1}]) - \mathcal{B}$ is foliated by the spacelike hypersurfaces $\{S_K \mid K_1 \leq K \leq K_2\}$ in the sense that there exists a one - many map $F : D \longrightarrow [K_1, K_2]$ which is onto and such that $t \in [K_1, K_2] \Rightarrow F^{-1}(t) \cup \mathcal{B}$ is a spacelike hypersurface with boundary $\mathcal{B}$ of constant generalised mean curvature t.

(4.8.1) Remark

This theorem states that there are no 'holes' in the region filled by the spacelike hypersurfaces S_K.

Proof

Consider $S = \bigcup_{K_1 \leq K \leq K_2} S_K$. By theorem (4.6) it is clear that

$S \subseteq J^-[S_{K_2}] \cap J^+[S_{K_1}] \equiv R$, say. Suppose that $R - S = \tilde{T} \neq \emptyset$ Then $\exists\, T \subseteq \tilde{T}$ and K_a, K_b with

(i) $K_1 \leq K_a \leq K_b \leq K_2$

(ii) $T \subseteq J^-[S_K] \;\; \forall\, K \geq K_b$ and $K < K_b \Rightarrow T \not\subseteq J^-[S_K]$

(iii) $T \subseteq J^+[S_K] \;\; \forall\, K \leq K_a$ and $K > K_a \Rightarrow T \not\subseteq J^+[S_K]$

(iv) T is connected.

To see this, let T be any connected component of $\tilde{T}$.
Let $A = \{K \mid T \subseteq J^-[S_K]\}$ $A \neq \emptyset$ as $T \subseteq J^-[S_{K_2}]$
A is bounded below as $T \not\subseteq J^-[S_{K_1}]$. Set $K_b = \inf A$
K_a is similarily found.
Let $K_{b_n} \searrow K_b$ with $K_{b_n} \leq K_2$ and let $K_{a_n} \nearrow K_a$ with $K_{a_n} \geq K_1$.
Then $S_{K_{a_n}} \subseteq J^-[T]$ and $S_{K_{b_n}} \subseteq J^+[T]$

so $S_{\kappa a_n} \subseteq J^-[S_{\kappa b_n}] \Rightarrow \kappa a_n \leq \kappa b_n$ by lemma (4.2) taking the limit $\kappa_a \leq \kappa b$.

Now consider the hypersurfaces $S_{\kappa a_n}$, employing the natural stable causality coordinates on M we may write them $y^o = f_n(y^i)$.

Since $m > n \Rightarrow S_{\kappa a_m} \subseteq J^+[S_{\kappa a_n}]$ it follows that

$$f_m(y^i) \geq f_n(y^i) \quad \forall\, y^i \quad \text{if} \quad m > n$$

That is the functions $f_n(y^i)$ form a monotonic increasing sequence, bounded above by $f(y^i)$ where S_{κ_2} has the equation $y^o = f(y^i)$. It may be concluded from this that the sequence $f_n(y^i)$ converges uniformly to a continuous limit function $f_a(y^i)$, which represents the spacelike hypersurface $S_A \in \overline{C_o^{SLB}(B)}$.

Furthermore as $S_A \subseteq J^+[S_{\kappa a_n}]$ $\forall n$ it follows that

$\tilde{V}[S_{\kappa a_n}] \nearrow \tilde{V}[S_A]$, as $\tilde{V}$ is continuous in the ρ-topology (see(2.26)).

Now suppose $\kappa_a > 0$, then $\exists\, N$ s.t. $n > N \Rightarrow \kappa a_n > 0$ and then by (4.1.1) $n > m > N \Rightarrow \tilde{A}[S_{\kappa a_n}] \leq \tilde{A}[S_{\kappa a_m}]$ and the sequence $\tilde{A}[S_{\kappa a_n}]$ is eventually monotonic decreasing and so tends to a limit, B say, as it is bounded below by zero. If, on the other hand, $\kappa_a \leq 0$ then there is either a subsequence with $\kappa a_n < 0$ $\forall n$, in which case by (4.1.1) the sequence $\tilde{A}[S_{\kappa a_n}]$ is monotonic increasing and so tends to a limit, B say, as it is bounded above (as slways we assume $D(S)$ compact c.f (2.2.4)) or there is a constant subsequence $\kappa a_n = 0$, in which case by the uniqueness assumption the sequence $f_n(y^i)$ is also a constant sequence and so $f_a(y^i) = f_n(y^i)$ $\forall n$ and $\tilde{A}[S_{\kappa a_n}] = B$, say, $\forall n$.

So in all cases we may conclude that $\tilde{A}[S_{\kappa a_n}] \longrightarrow B$ monotonically as $n \longrightarrow \infty$ where we assume a subsequence has been chosen if necessary.

Since $\tilde{A}$ is upper semi-continuous and $S_{\kappa a_n} \longrightarrow S_A$ uniformly (i.e. $S_{\kappa a_n} \longrightarrow S_A$ in the ρ-topology) it follows that :-

$$\tilde{A}[S_A] \geq \limsup \tilde{A}[S_{\kappa a_n}] = B$$

Let $T \in \overline{C_o^{SLB}(B)} - S_A$

then $$\tilde{A}[S_{\kappa a_n}] + \kappa a_n \tilde{V}[S_{\kappa a_n}] \geq \tilde{A}[T] + \kappa a_n \tilde{V}[T]$$

taking limits $$B + \kappa_a \tilde{V}[S_A] \geq \tilde{A}[T] + \kappa_a \tilde{V}[T]$$

and so $\tilde{A}[S_A] + K_a \tilde{V}[S_A] \geqslant \tilde{A}[T] + K_a \tilde{V}[T]$

but this is true $\forall\ T \in \overline{C_0^{SLB}(\mathcal{B})}$ and so by the uniqueness assumption S_A is a spacelike hypersurface of generalised constant mean curvature K_a and

$$J^-[T] \cap S = \bigcup_{K_1 \leq K \leq K_a} S_K$$

similarily $J^+[T] \cap S = \bigcup_{K_b \leq K \leq K_2} S_K$

and $K_b > K_a$ by the uniqueness assumption as it is assumed that $T \neq \phi$. Consider the hypersurface $S_{\frac{K_a + K_b}{2}}$. By theorem (4.6)

$$S_{\frac{K_a + K_b}{2}} \subseteq J^+[S_{K_a}] \cap J^-[S_{K_b}]$$

From the definition of K_a and K_b this forces

$S_{\frac{K_a+K_b}{2}} \equiv S_{K_a}$ or $S_{\frac{K_a+K_b}{2}} \equiv S_{K_b}$ contrary to supposition. This contradiction shows that $\tilde{T} = \phi$.
Let $\underline{x} \in D$. Since $\tilde{T} = \phi$, $\underline{x} \in S_K$ some $K \in [K_1, K_2]$. One can define $F : D \to [K_1, K_2]$ by $\underline{x} \longmapsto K$.

Similarily :-

(4.9) Theorem

If the hypersurfaces S_{K_i} maximise the functionals $\tilde{A} + K_i \tilde{V}_\phi$ in C_0^{SL} on a stably causal manifold M and if $K_1 < K_2$ and $K_i \neq K_j \Rightarrow S_{K_i} \not\equiv S_{K_j}$, then the region $D \equiv J^+[S_{K_1}] \cap J^-[S_{K_2}]$ is foliated by the spacelike hypersurfaces $\{S_K \mid K_1 \leq K \leq K_2\}$ in the sense that there exists a one - many map $F : D \to [K_1, K_2]$ which is onto and such that $t \in [K_1, K_2] \Rightarrow F^{-1}(t)$ is a spacelike hypersurface of generalised constant mean curvature t .

It is now only necessary to show that one can choose S_{K_1} sufficiently 'in the past' and S_{K_2} sufficiently 'in the future' that the region $J^+[S_{K_1}] \cap J^-[S_{K_2}]$ fills an 'arbitarily large part of' M and it will have been established that the maximising hypersurfaces of the functionals $\tilde{A} + K\tilde{V}$ foliate all of M , in the sense specified in the above theorems. Then, of course, theorem (4.6) tells us that K , the generalised constant mean curvature, is a monotonic increasing function of time, through the foliation.

(4.10) Theorem

Suppose M is type $(R-W)$ and suppose the hypersurfaces S_κ uniquely maximise the functionals $\tilde{A} + \kappa \tilde{V}_\phi$ in C_o^{SL} and $S_\kappa \subset M$ for each $\kappa \in \mathbb{R}$ then if M is stably causal it is foliated by the S_κ in the sense of theorem (4.9). (M is not compact here but $\tilde{A}$ and $\tilde{V}$ are presumed to be bounded on it (c.f (2.4.2))).

Proof

Suppose that $\exists\, x \in M$. with $x \in I^+[S_\kappa]$ $\forall \kappa \in \mathbb{R}$. Then if S_κ is given by the function $y^o = f_\kappa(y^i)$, using stable causality coordinates, the f_κ are all bounded above by f , if $y^o = f(y^i)$ is the equation of the past boundary of the future of x . So as in the proof of theorem (4.8), one can deduce that $f_\kappa(y^i)$ tends uniformly and monotonically to $g(y^i)$ as $\kappa \to \infty$. Where $y^o = g(y^i)$ is the equation of some spacelike hypersurface in C_o^{SL} , call it T .
We have $\tilde{V}_\phi[S_\kappa] \nearrow \tilde{V}_\phi[T]$ as $\kappa \to \infty$ and

$$\tilde{A}[S_\kappa] \searrow B \quad \text{as } \kappa \to \infty \quad \text{with } \tilde{A}[T] \geq B \ .$$

Let H be a spacelike hypersurface such that $H \subset I^+[T]$, for example choose one of the surfaces $y^o = const$ which lies to the future of the past boundary of the future of x .
Then

$$\tilde{A}[S_\kappa] + \kappa \tilde{V}_\phi[S_\kappa] > \tilde{A}[H] + \kappa \tilde{V}_\phi[H]$$

$$\tilde{V}_\phi[H] - \tilde{V}_\phi[S_\kappa] < \frac{\tilde{A}[S_\kappa] - \tilde{A}[H]}{\kappa}$$

and as S_o maximises $\tilde{A}$. $$< \frac{\tilde{A}[S_o]}{\kappa} \qquad \forall \kappa$$

But $$\tilde{V}_\phi[H] - \tilde{V}_\phi[T] < \tilde{V}_\phi[H] - \tilde{V}_\phi[S_\kappa] \quad \forall \kappa$$

So $$0 < \tilde{V}_\phi[H] - \tilde{V}_\phi[T] < \frac{\tilde{A}[S_o]}{\kappa} \quad \forall \kappa$$

which is impossible, so it can be immediately deduced that $\exists\, \kappa_2$ such that $x \in \bar{J}^-[S_{\kappa_2}]$. Similarily it can be deduced that $\exists\, \kappa_1$ s.t. $x \in J^+[S_{\kappa_1}]$. Applying Theorem (4.9) it follows that $\exists\, \kappa$ s.t. $x \in S_\kappa$. This is true for all $x \in M$ and so the conclusion of the theorem follows.

CHAPTER V

NON-COMPACT SPACELIKE HYPERSURFACES WITHOUT BOUNDARY

(5.1) The positive - definite case

The problem of finding complete minimal graphs in $\mathbb{R}^n$ has been investigated by many different approaches. The problem is to find a hypersurface $x^n = f(x^1, x^2, \ldots, x^{n-1})$, the x^i being Cartesian coordinates, satisfying the minimal surface equation :-

(5.1.1) $\nabla^2 x^n = 0$ where ∇^2 is the Lapacian operator associated with the induced metric on the hypersurface, for all $(x^1, x^2, \ldots, x^{n-1})$. The principle result is the theorem due to Bernstein [7] and generalised from the case $\mathbb{R}^3$ by Fleming [22], De Giorgi [17], Almgren [2] and Simons [56] to $\mathbb{R}^n$ for $n \leq 8$.

(5.1.2) Theorem (Bernstein)

The only complete minimal graphs in $\mathbb{R}^n$ for $n \leq 8$ are the hyperplanes.

Bombieri et al [8] have shown that for $n \geqslant 9$ the above result is false, thereby disposing completely of the 'Bernstein problem' for Euclidean space.

The techniques used to prove Bernstein's theorem are complex variable arguments for $n = 3$ and Hausdorff measure theoretic arguments for larger values of n. Neither of which techniques extend obviously to the case of a four dimensional curved Lorentz manifold.

(5.2) The indefinite case

I shall present some arguments which appear highly suggestive of Bernstein's theorem like results for spacelike hypersurfaces of constant mean curvature in Minkowski Space and De-Sitter Space.

Minkowski Space

The generalisation of Bernstein's theorem that I have in mind is that the only spacelike hypersurfaces of constant mean curvature zero should be hyperplanes and that the only spacelike hypersurfaces of constant mean curvature K should be the hyperboloid:

(5.2.1) $$t^2 - r^2 = \frac{1}{\pi^2} \quad (K = 3\pi)$$

and those hyperboloids which are translations of this one. (Each such hyperboloid has two branches, the one that lies to the future of the other has positive constant mean

curvature K $(\pi > 0)$; the other has constant mean curvature $-K$ $(\pi > 0)$.)

An elegant proof of the theorem for three dimensional Minkowski space and the case of zero constant mean curvature was presented by Schucking [55] based on a theorem of Jörgens [31]. I reproduce it below :-

(5.2.2) Proof

Let $t = f(x,y)$ be the C_2 equation of a maximal spacelike hypersurface in three dimensional Minkowski space :-

$$ds^2 = dt^2 - dx^2 - dy^2$$

then f satisfies :-

$$f_{xx}(1-f_y^2) + 2f_{xy}f_xf_y + f_{yy}(1-f_x^2) = 0$$

This equation may be written equivalently :-

$$\left(\frac{1-f_x^2}{\sqrt{1-f_x^2-f_y^2}}\right)_y = \left(\frac{-f_xf_y}{\sqrt{1-f_x^2-f_y^2}}\right)_x$$

which implies the existence of a function $F(x,y)$ such that :-

$$F_x = \frac{1-f_x^2}{\sqrt{1-f_x^2-f_y^2}} \quad \text{and} \quad F_y = \frac{-f_xf_y}{\sqrt{1-f_x^2-f_y^2}}$$

similarily $\exists\, G(x,y)$ such that

$$G_x = \frac{-f_xf_y}{\sqrt{1-f_x^2-f_y^2}} \quad \text{and} \quad G_y = \frac{1-f_y^2}{\sqrt{1-f_x^2-f_y^2}}$$

Now $G_x = F_y$ and so $\exists\, E(x,y)$ such that :-

$$E_x = F \quad \text{and} \quad E_y = G$$

this implies that :-

$$E_{xx}E_{yy} - (E_{xy})^2 = F_xG_y - F_yG_x = 1$$

But Jörgen proved that if $E(x,y)$ is a C_2 function on $\mathbb{R}^2$ and if $E_{xx}E_{yy} - (E_{xy})^2 = 1$, then

$$E(x,y) = \alpha x^2 + 2\beta xy + \gamma y^2 + 2\delta x + 2\epsilon y + \zeta$$
$$\alpha, \beta, \gamma, \delta, \epsilon, \zeta \in \mathbb{R}$$

and so it is forced that f_x and f_y should be constant functions. Calabi [12] has recently been able to establish the analogous result for four dimensional Minkowski Space.

(5.3) The Perturbation calculations

I shall now outline some perturbation calculations first for Minkowski Space and then for De-Sitter Space. The idea in these calculations is to take a known spacelike hypersurface of constant mean curvature, K say, and to look at the linearised condition for a neighbouring hypersurface to have the same constant mean curvature.

Minkowski Space

$$ds^2 = dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

Without loss of generality we may consider the typical hyperboloid of constant mean curvature 3π to be given by :-

$$t^2 - r^2 = 1/\pi^2$$

Introducing the advanced and retarded time coordinates

$u = t - r$; $v = t + r$ and the notation $c \equiv 1/\pi^2$

this may be written :-

(5.3.1) $$u = \frac{c}{v}$$

I shall consider the linearised condition that the perturbation

$\delta u = \delta u(v, \theta, \phi)$ be such that the hypersurface

$u = \frac{c}{v} + \delta u$ is of constant mean curvature 3π .

This is obtained by a somewhat lengthly calculation by varying the equation (5.3.1) about the hypersurface $u = \frac{c}{v}$, one obtains :-

$$0 = \frac{c}{4v^2}\left(1 - \frac{c}{v^2}\right)\delta u + (v-u)\left[\frac{1}{8}\left(1 - \frac{c}{v^2}\right) + \frac{3(v-u)}{16v} + \frac{c}{4v^2}\right](\delta u)_{,v} +$$

$$+ \frac{c\cot\theta}{4v^2}(\delta u)_{,\theta} + \frac{(v-u)^2}{16}(\delta u)_{,vv} + \frac{c}{4v^2}(\delta u)_{,\theta\theta} + \frac{c}{4v^2\sin^2\theta}(\delta u)_{,\phi\phi}$$

Writing $\delta u(v, \theta, \phi) \equiv Q(v)\,\Phi(\phi)\,\text{Ⓗ}(\theta)$ and separating variables one obtains :-

(5.3.2) $$\ddot{\Phi}/\Phi = -m^2$$ m^2 a separation constant

(5.3.3) $$(1-\mu^2)f'' - 2\mu f' + \left[n(n+1) - \frac{m^2}{1-\mu^2}\right]f = 0$$ n a separation constant

and $\mu = \cos\theta$, $\text{Ⓗ}(\theta) = f(\mu)$.

also

(5.3.4) $$\frac{c}{4}\left(1 - \frac{c}{v^2}\right) + v^3\left(1 - \frac{c}{v^2}\right)\left[\frac{5}{16}\left(1 - \frac{c}{v^2}\right) + \frac{c}{4v^2}\right]\frac{\dot{Q}}{Q} + \frac{v^4}{16}\left(1 - \frac{c}{v^2}\right)^2\frac{\ddot{Q}}{Q} +$$

$$- \frac{c}{4}n(n+1) = 0$$

It is well known that in order that the solutions to (5.3.2) and (5.3.3) be singled valued n and m must be positive integers and that the solutions to (5.3.2) and (5.3.3) in these cases are simply the spherical harmonic functions[20].

We are looking for solutions of (5.3.4) bounded on $r \in [0, \infty)$, or correspondingly $v \in [\sqrt{c}, \infty)$.

Making the substitutions $y = \left(1 - \frac{c}{v^2}\right)^{-1}$ and $Q = R y$ in (5.3.4) we obtain the following differential equation for $R = R(y)$

$$(5.3.5) \quad y(1-y)\ddot{R} + (3-2y)\dot{R} + n(n+1)R = 0$$

then, putting $z = 1-2y$ we obtain the following differential equation for $R(z)$:-

$$(1-z^2)\ddot{R} - 2(z+2)\dot{R} + n(n+1)R = 0$$

This is a particular case of the equation satisfied by the Jacobi polynomials[38] with general solution :-

$$(5.3.6) \quad R(z) = A P_n^{(2,-2)}(z) + \tilde{B} Q_n^{(2,-2)}(z) \quad \text{with } A, \tilde{B} \text{ arbitrary constants.}$$

$P_n^{(2,-2)}(z)$ is a polynomial, the jacobi polynomial, of degree n in z and

$$(5.3.7) \quad Q_n^{(2,-2)}(z) = \alpha \left(\frac{z+1}{z-1}\right)^2 P_n^{(2,-2)}(-z) \quad \text{where } \alpha \text{ is a constant.}$$

So we may write the general solution :-

$$Q(y) = A y P_n^{(2,-2)}(1-2y) + B \frac{(1-y)^2}{y} P_n^{(2,-2)}(2y-1)$$

This must be bounded for $y \in [1, \infty)$. As $P_n^{(2,-2)}(1-2y)$ is a polynomial of degree n in $(1-2y)$ it follows that Q may be written in the form :-

$$Q(y) = a_{-1} y^{-1} + a_0 + a_1 y + \ldots\ldots + a_{n+1} y^{n+1}$$

for some $a_{-1}, a_0, a_1 \ldots a_{n+1} \in \mathbb{R}$

It is only bounded if $a_1 = a_2 = \ldots = a_{n+1} = 0$

Substituting the form $Q = a_{-1} y^{-1} + a_0$ into the appropriate differential equation one is lead to the solutions :-

$$a_0 = 0 \quad \text{and} \quad n(n+1) = 2 \quad \Rightarrow n = 1$$

or $$2a_{-1} + a_0 = 0 \quad \text{and} \quad n(n+1) = 0 \quad \Rightarrow n = 0$$

This means that the only possible perturbation solutions are given by :-

(5.3.8)

$$\delta u(v,\theta,\phi) = A\left(1+\frac{c}{v^2}\right) + B\left(1-\frac{c}{v^2}\right)\left[P_{1,0}(\cos\theta) + D\,P_{1,1}(\cos\theta)\cos(\phi+\epsilon)\right]$$

A , B , D , ϵ arbitrary constants.

Let us now consider perturbations arising from translations of the entire hyperboloid.

Consider the family of hyperboloids which arises in this way :- $\tilde{t}^2 - \tilde{r}^2 = c^2$

where $\tilde{t} = t + b$ and $\tilde{r}$ is radial distance measured from $(r,\theta,\phi) \equiv (a,\theta_o,\phi_o)$.

In u , v , θ , ϕ coordinates we may write this four parameter family $u = u(b,a,\theta_o,\phi_o;\, v,\theta,\phi)$. Apart from the trivial perturbations which rotate the hyperboloid into itself ($a = b = 0$) the general perturbation, arising from translation, may be written :-

$$\delta u = \left.\frac{\partial u}{\partial b}\right|_{\substack{a=0\\b=0}} \delta b + \left.\frac{\partial u}{\partial a}\right|_{\substack{a=0\\b=0}} \delta a$$

a straightforward calculation shows this to be exactly the same as the set of allowable perturbations (5.3.8).

Although this is not a rigourous proof, it is highly suggestive that the generalised Bernstein conjecture holds good for Minkowski Space.

(5.4) De-Sitter Space

De-Sitter space may be regarded [29] as the four-dimensional hyperboloid :-

(5.4.1) $$c^2 = -V^2 + W^2 + X^2 + Y^2 + Z^2$$

embedded in the five-dimensional Minkowski Space :-

(5.4.2) $$ds^2 = dV^2 - dW^2 - dX^2 - dY^2 - dZ^2$$

The four-dimensional hyperplanes :-

(5.4.3) $$0 = \alpha V + \beta W + \gamma X + \delta Y + \epsilon Z$$

intersect the hyperboloid, De-Sitter space, in what may be regarded as three-dimensional hypersurfaces of De-Sitter space, these three-dimensional hypersurfaces have constant mean curvature in De-Sitter space.

According as the original hyperplane (5.4.3) is spacelike, null, or timelike, with respect to the metric (5.4.2), by a rotation of coordinates in W , X , Y , Z space followed by

a Lorentz transformation in V , W , X , Y , Z space, both of which preserve the form of (5.4.1) and (5.4.2) it can be put into the form :-

(i) V = constant if (5.4.3) was spacelike in (5.4.2)
(ii) $V+W$ = constant if (5.4.3) was null in (5.4.2)
(iii) W = constant if (5.4.3) was timelike in (5.4.2)

So, without loss of generality, these are the only hyperplanes one need consider. Furthermore, although in cases (i) and (ii) the three dimensional hypersurface in De-Sitter space is spacelike, in case (iii) it is only spacelike if $W=d$, $d>c$. (If $d=c$ the hypersurface is null and if $d<c$ it has a spacelike normal).

The generalised Bernstein conjecture for De-Sitter space is simply that these three classes of spacelike hypersurface in De-Sitter space are the only spacelike hypersurfaces of constant mean curvature.

Case (i) (The hypersurfaces considered are, in this case only, compact)

If the coordinates t, χ, θ, ϕ are introduced by :-

(5.4.4)

$$V = c\, sh\left(\frac{t}{c}\right)$$
$$W = c\, ch\,(t/c)\cos\chi$$
$$X = c\, ch\,(t/c)\sin\chi\cos\theta$$
$$Y = c\, ch\,(t/c)\sin\chi\sin\theta\cos\phi$$
$$Z = c\, ch\,(t/c)\sin\chi\sin\theta\sin\phi$$

$$t\in(-\infty,\infty),\ \chi\in[0,\pi]\ ,\ \theta\in[0,\pi]\ ,\ \phi\in[0,2\pi]$$

then they serve as coordinates for De-Sitter space and the metric induced on it by that of the five dimensional Minkowski Space is :-

$$ds^2 = dt^2 - c^2\, ch^2\left(\frac{t}{c}\right)\left\{ d\chi^2 + \sin^2\chi\left(d\theta^2+\sin^2\theta\, d\phi^2\right)\right\}$$

The hypersurface in De-Sitter space formed by the intersection of the hyperplane V = constant with (5.4.1) is simply the spacelike hypersurface

$$t = h \qquad\qquad h\in\mathbb{R}$$

It has constant mean curvature $\Omega = \frac{3}{c}\, th\left(\frac{h}{c}\right)$

The linearised condition that the perturbation $\delta t = \delta t(\chi, \theta, \phi)$ is such that the hypersurface $t = h + \delta t$ has constant mean curvature $\frac{3}{c}\,th(\frac{h}{c})$ is :-

(5.4.5)

$$0 = (\delta t)_{,\chi\chi} + \frac{(\delta t)_{,\theta\theta}}{\sin^2\chi} + \frac{(\delta t)_{,\phi\phi}}{\sin^2\chi \sin^2\theta} + 3\,\delta t + 2(\delta t)_{,\chi}\cot\chi + (\delta t)_{,\theta}\frac{\cot\theta}{\sin^2\chi}$$

writing $\delta t = A(\chi)\,\textcircled{H}(\theta)\,\Phi(\phi)$ and separating variables one obtains the equations :-

(5.4.6) $\Phi''/\Phi = -m^2$ $\quad m^2$, a separation constant

(5.4.7) $$\frac{\textcircled{H}''}{\textcircled{H}} + \frac{\textcircled{H}'}{\textcircled{H}}\cot\theta - \frac{m^2}{\sin^2\theta} = -n(n+1)$$ $\quad n$, a separation constant

as before the condition that $\textcircled{H}$ and Φ be single valued functions imposes the restriction that m and n be positive integers and forces $\textcircled{H}$ and Φ to be the spherical harmonic functions.

The Third equation is :-

(5.4.8) $$0 = A'' + \left(3 - \frac{n(n+1)}{\sin^2\chi}\right)A + 2A'\cot\chi \qquad \chi \in [0, \pi]$$

Putting $A = (\sin^n\chi)\,B$ and $\cos\chi = \mu$ one obtains the differential equation for $B(\mu)$

(5.4.9) $$0 = (1-\mu^2)B'' - \mu B' - 2(n+1)\mu B' + \left[-n(n+1) + 3 - n\right]B$$

substituting $z = \frac{1}{2}(1-\mu)$ one obtains for $B(z)$

(5.4.10) $$0 = z(1-z)B'' + \frac{2n+3}{2}(1-2z)B' + \left[-n(n+1) + 3 - n\right]B$$

This is the hypergeometric equation[20]

with $\alpha = n+1 \mp 2$

$\beta = n+1 \pm 2$

$\gamma = n + 3/2$

So $B(z) = {}_2F_1(\alpha, \beta, \gamma; z)$ satisfies (5.4.10).

But the Legendre function

$$P^a_b(z) = (z^2-1)^{a/2}\, {}_2F_1(\alpha, \beta, \gamma; \tfrac{1}{2}(1-z))$$

up to a constant factor

where $\alpha = a-b$, $\beta = a+b+1$, $\gamma = a+1$

so $a = n+\frac{1}{2}$, $b = -\frac{1}{2} \pm 2$

And $A(\mu) = \eta(1-\mu^2)^{-1/4} P^{n+\frac{1}{2}}_{3/2}(\mu) + \zeta(1-\mu^2)^{-1/4} Q^{n+\frac{1}{2}}_{3/2}(\mu)$

η, ζ arbitrary constants. It is necessary that this function be bounded on $\mu \in [-1, 1]$, however :-

(5.4.11)
$$P^{n+\frac{1}{2}}_{3/2}(\mu), Q^{n+\frac{1}{2}}_{3/2}(\mu) \sim (1 \mp \mu)^{-n/2-1/4} \quad \text{as } \mu \to \pm 1$$

$$\lim_{\mu \to +1} P^{n+\frac{1}{2}}_{3/2}(\mu)/Q^{n+\frac{1}{2}}_{3/2}(\mu) \neq \lim_{\mu \to -1} P^{n+\frac{1}{2}}_{3/2}(\mu)/Q^{n+\frac{1}{2}}_{3/2}(\mu)$$

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so this is impossible for any positive (or zero) integer n. The conclusion is that there are no permissible perturbations.

Case (ii)

If we choose coordinates :-

$$\hat{t} = c \log\left(\frac{W+V}{c}\right), \quad \hat{x} = \frac{cx}{W+V}, \quad \hat{y} = \frac{cy}{W+V}, \quad \hat{z} = \frac{cz}{W+V}$$

$$(\hat{t}, \hat{x}, \hat{y}, \hat{z}) \in \mathbb{R}^4$$

we find De-Sitter space, or rather that part of De-Sitter space with $W+V > 0$, that is Hoyle's steady state universe[30], with the metric :-

$$ds^2 = d\hat{t}^2 - \exp\left(\frac{2}{c}\hat{t}\right)\left[d\hat{x}^2 + d\hat{y}^2 + d\hat{z}^2\right]$$

The hypersurface in De-Sitter space formed by the intersection of the hypersurface $W+V =$ constant with (5.4.1) is simply the spacelike hypersurface :-

$$\hat{t} = h \qquad h \in \mathbb{R}$$

It has constant mean curvature $\Omega = -3/c$

The linearised condition on the perturbation $\delta\hat{t}$ in order that the hypersurface $\hat{t} = h + \delta\hat{t}$ shall have constant mean curvature $-3/c$ is simply :-

$$\nabla^2(\delta\hat{t}) = 0$$

where ∇^2 is the flat space Laplacian.

and so the only bounded solution is $\delta\hat{t} =$ constant. But this perturbation is simply that which arises by moving through the family of hypersurfaces $\hat{t} =$ constant.

Note

The above includes the maximal hypersurfaces $t =$ constant in Minkowski Space, if one goes to the limit $c \to \infty$.

Case (iii)

This case is best examined in the following coordinates :-

$$W = c\,ch(t/c)$$

$$V = c\,sh(t/c)\,ch\chi$$

$$X = c\,sh(t/c)\,sh\chi\,\cos\theta$$

$$Y = c\,sh(t/c)\,sh\chi\,\sin\theta\,\cos\phi$$

$$Z = c\,sh(t/c)\,sh\chi\,\sin\theta\,\sin\phi$$

$$t \in (0,\infty),\ \chi \in (0,\infty),\ \theta \in [0,\pi],\ \phi \in [0,2\pi]$$

This gives De-Sitter space, or rather that part of it which has $W > c$ and $V > 0$, with the metric :-

$$ds^2 = dt^2 - sh^2 t\left[d\chi^2 + sh^2\chi\,(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

The hypersurface in De-Sitter space formed by the intersection of the hyperplane $W = \tilde{h}$, $\tilde{h} > c$ and (5.4.1) is the spacelike hypersurface :-

$$t = h \qquad h \in \mathbb{R}^+$$

it has constant mean curvature $-3\coth h$.

The linearised condition on the perturbation δt in this case is :-

(5.4.12) $$0 = 3\,\delta t - 2\coth\chi\,(\delta t)_{,\chi} - \frac{\cot\theta}{sh^2\chi}(\delta t)_{,\theta} - (\delta t)_{,\chi\chi} +$$

$$- \frac{(\delta t)_{,\theta\theta}}{sh^2\chi} - \frac{(\delta t)_{,\phi\phi}}{sh^2\chi\sin^2\theta}$$

(Note the marked similarity to (5.4.5))

writing $\delta t = R(\chi)\,\Theta(\theta)\,\Phi(\phi)$ and separating variables we find :-

Θ and Φ are spherical harmonics with associated m, n

and $$0 = 3 - 2\coth\chi\,\frac{R'}{R} + \frac{n(n+1)}{sh^2\chi} - \frac{R''}{R}$$

writing $R = P\,sh^n\chi$ and $\mu = ch\,\chi$ we obtain for $P(\mu)$

$$0 = (\mu^2-1)\,P'' + \mu P' + 2(n+1)\mu P' + P\left[(n-3) + n(n+1)\right]$$

for $\mu \in (1,\infty)$ But this is the same equation as (5.4.9) and so $P(\mu) = (\mu^2-1)^{-\frac{a}{2}} P^a_b(\mu)$

where $a = n+\frac{1}{2}$, $b = -\frac{1}{2} \pm 2$

and so $$R(\mu) = \eta\,(\mu^2-1)^{-1/4} P^{n+\frac{1}{2}}_{-\frac{1}{2}+2}(\mu) + \zeta\,(\mu^2-1)^{-1/4} Q^{n+\frac{1}{2}}_{-\frac{1}{2}+2}(\mu)$$

η and ζ arbitrary constants.

By (5.4.11) it follows that $R(\mu)$ inevitably has a pole at $\mu = 1$ and it follows that there are no admissible perturbations.

As for the Minkowski Space calculations, though not a proof, this is highly suggestive that the generalised Bernstein conjecture for De-Sitter Space holds good.

(5.5) Minkowski Space. Hypersurfaces of constant mean curvature and $\mathcal{I}$.

The definitions of asymptotic simplicity and of future and past null infinity ($\mathcal{I}^+$ and $\mathcal{I}^-$) have been given above (1.15, 1.24). Minkowski Space is the simplest example of such a manifold, following Penrose[52], I shall construct $\mathcal{I}$ for Minkowski Space and then discuss the spacelike hyperboloids of constant mean curvature in relation to it.

The metric of Minkowski Space may be written :-

$$ds^2 = dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

or changing to advanced and retarded radial time coordinates :-

$$u = t - r$$
$$v = t + r$$

$$(5.5.1) \quad ds^2 = du\,dv - \tfrac{1}{4}(v-u)^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

Taking the conformal factor $\Omega = (1+u^2)^{-\frac{1}{2}}(1+v^2)^{-\frac{1}{2}}$ we have as metric for the non-physical conformally rescaled manifold $\tilde{M}$:-

(5.5.2)

$$d\tilde{s}^2 = \Omega^2 ds^2 = \frac{du\,dv}{(1+u^2)(1+v^2)} - \frac{(v-u)^2}{4(1+u^2)(1+v^2)}(d\theta^2 + \sin^2\theta\, d\phi^2)$$

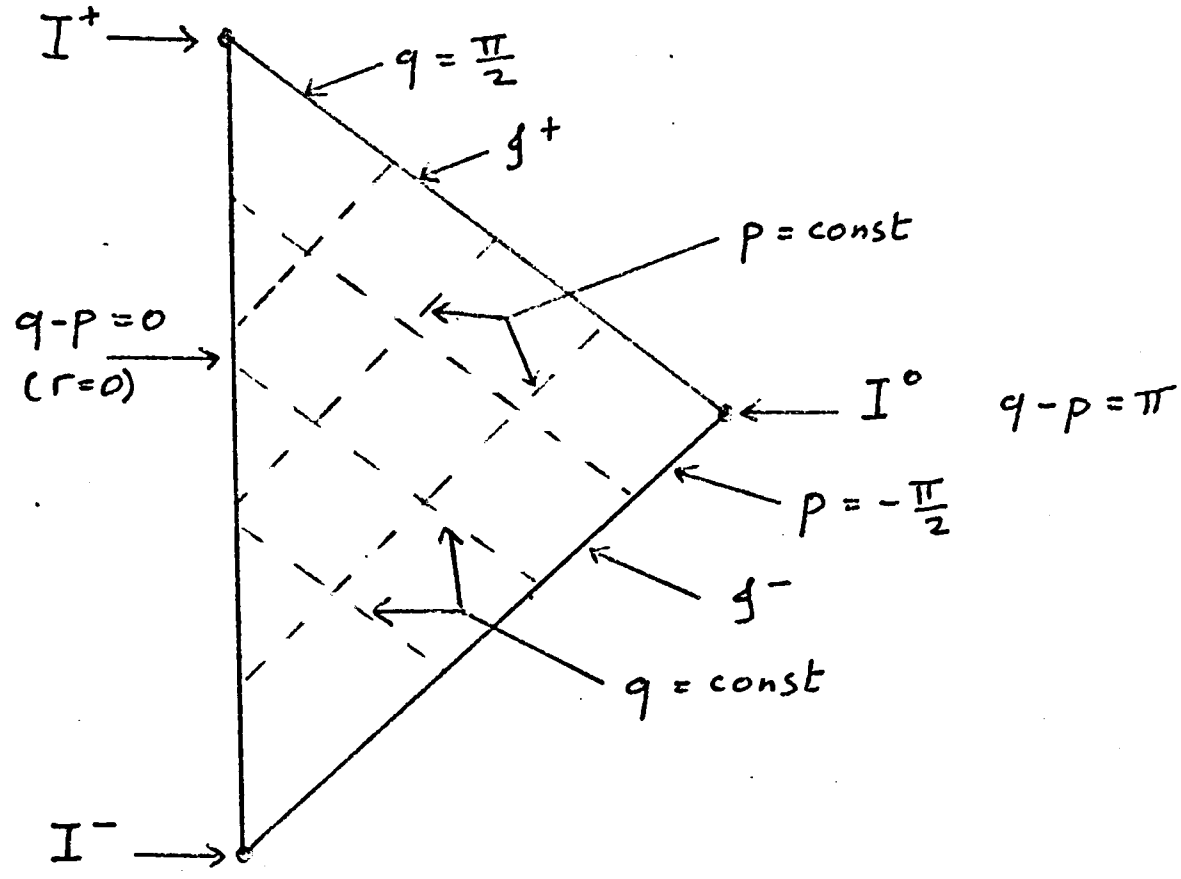
or making the change of coordinates :- $u = \tan p$; $v = \tan q$

$$(5.5.3) \quad d\tilde{s}^2 = dp\,dq - \tfrac{1}{4}\sin^2(q-p)(d\theta^2 + \sin^2\theta\, d\phi^2)$$

and $q = \frac{\pi}{2}$, $p \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $\theta \in [0, \pi]$, $\phi \in [0, 2\pi]$ is $\mathcal{I}^+$ (corresponding to $v = +\infty$)

while $p = -\frac{\pi}{2}$, $q \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $\theta \in [0, \pi]$, $\phi \in [0, 2\pi]$ is $\mathcal{I}^-$ (corresponding to $u = -\infty$)

The appropriate Penrose diagram is :-



Now consider the typical hyperboloid of constant mean curvature $\pm 3\pi$

$$t^2 - r^2 = \frac{1}{\pi^2}$$

i.e. $$uv = 1/\pi^2$$

Clearly as $v \to +\infty$, $u \to 0$ and as $u \to -\infty$, $v \to 0$ so this hypersurface (it is a two sheeted hypersurface) is asymptotically null and, drawn on $\tilde{M}$, intersects $\mathcal{I}^+$ in $p=0$ and $\mathcal{I}^-$ in $q=0$. However the future and past directed null cones of the origin $(p=q=0)$ have precisely the same intersection with $\mathcal{I}$ and the origin is in no way a distinguished point geometrically.

This means that corresponding to the intersection of the future (past) directed null cone of any point in Minkowski Space with $\mathcal{I}^+(\mathcal{I}^-)$ there is a spacelike hypersurface of arbitrary constant mean curvature, the top (bottom) sheet of one of the hyperboloids, spanning this intersection. These 'cuts of $\mathcal{I}$' are known as the good cuts[52]. If the Bernstein conjecture holds then, these good cuts are the only possible appropriate 'boundaries' for global spacelike hypersurfaces of constant mean curvature in Minkowski Space.

Together with the observation that one may regard the future (past) directed null cone of an event as a hypersurface of constant mean curvature $+\infty$ $(-\infty)$, in a limiting sense, and since in a general asymptotically flat manifold such a null hypersurface always develops caustics[52], this leads me to

conjecture that the class of asymptotically simple manifolds containing a global spacelike hypersurface of constant mean curvature may be considerably smaller than the general class of asymptotically simple manifolds. It is not possible to find good cuts of $\mathcal{I}$ for a general asymptotically simple manifold in either the sense of being the intersection with $\mathcal{I}$ of a non-self intersecting future (past) directed null cone or the more general sense of being the intersection with $\mathcal{I}$ of an asymptotically null hypersurface, whose asymptotic null generators form an asymptotically shear free congruence[52].

(5.6) Known existence results for spacelike hypersurfaces of constant mean curvature in asymptotically simple manifolds.

In the preceding section restrictions on global spacelike hypersurfaces of constant mean curvature and on their existence were discussed for asymptotically simple manifolds. In this section I shall discuss the existence results of Choquet-Bruhat[15] and Brill[9].

Mme Choquet's idea is to apply the implicit function theorem for Banach Spaces to prove the stability of the property of a manifold of possessing a spacelike hypersurface of given constant mean curvature, under certain conditions. Brill has presented a direct argument in the spherical symmetry case. Direct variational methods cannot be applied in the asymptotically simple manifold case since the spacelike hypersurfaces are non-compact and cannot be made into a compact topological space.

Local Stability

The following is a short exposition of Mme Choquet's results. She considers separately the cases when the manifold has compact spacelike hypersurfaces and when the manifold is Minkowski Space.

(5.6.1) Theorem

Let $(V_4, \hat{g})$ be a space-time, that is a four dimensional C_2 manifold V_4 endowed with a Lorentzian metric $\hat{g}$, with a compact spacelike hypersurface S_o of constant mean curvature K, which is not a moment of time symmetry for a vacuum space-time. Then there exists a neighbourhood X of $\hat{g}$ in E such that for all $g \in X$ the space-time (V_4, g) has a spacelike hypersurface S_ϕ, $\phi \in F$ of constant mean curvature

K . Where E , F are either :-

a) E , the space of C_2 , two-tensors on Ω , made into the Hölder space $C^{2,\alpha}$.

F , the space of $C^{2,\alpha}$ scalars on S_o .

or

b) E , two-tensors in the Sobolev space $H_4(\Omega)$

F , scalars in $H_4(S_o)$.

and Ω is an open set, $\Omega \subseteq S_o \times \mathbb{R}$ identified with a neighbourhood of S_o in V_4 with the aid of a congruence of curves passing through S_o in V_4 . One defines a family of hypersurfaces $S_\phi \subset V_4$ as the subvarieties of V_4 with equation

$$t + \phi(x) = 0 \qquad x \in S_o \,, \quad (x,t) \in \Omega$$

in the above identification, (which is so chosen that S_o is $t=0$)

(5.6.2) Theorem

Let $(V_4, \hat{g})$ be a vacuum space-time with a compact spacelike hypersurface S_o of zero mean curvature. Then there exists a neighbourhood X of $\hat{g}$ in E such that for all $g \in X$ (V_4, g) has a spacelike hypersurface S_ϕ, $\phi \in F$, of constant mean curvature.

((E, F) as above).

(5.6.3) Theorem

Let S_o be a spacelike hypersurface of constant mean curvature K in Minkowski Space. Then there exists a neighbourhood X of η (the Minkowski metric) in E such that for each $g \in X$, $(\mathbb{R}^4, g)$ has a spacelike hypersurface S_ϕ, $\phi \in F$ with constant mean curvature K . Where E, F are either :-

a) Hölder spaces with special asymptotic behaviour [16].

B) Weighted Sobolev spaces $H_{s,\delta}$ [14].

The idea in both cases being to weight the norm or inner product with a factor $(1+|x^2|)^{\frac{\delta}{2}+|\sigma|}$, where $|\sigma|$ is the order of the derivative involved and $|x|^2$ is the square of the Euclidean distance from an arbitrary, fixed, origin of Euclidean coordinates in Minkowski Space.

(5.6.4) Proof (sketch)

The idea in the above is to consider the map

$P_{S_\phi} : E \times F \longrightarrow G$ which takes a surface and a choice of metric on that surface to the scalar function on S_o , the mean curvature of S_ϕ (G is an appropriate Banach or Hilbert Space of C_o functions on S_o).

Further Remarks on the work of Avez and of Cantor et al.

Avez' work is incorrect in that he applies the Morrey regularity theory for variational problems in order to obtain a spacelike hypersurface of zero mean curvature. In fact all that he is justified in saying is that the spacelike hypersurface he obtains is of generalised zero mean curvature because Morrey's results are only applicable to variational problems in which the integrands satisfy certain inequalities. The area integrand here does not satisfy these inequalities as it cannot be uniformly bounded away from zero on the relevant space of spacelike hypersurfaces.

The result of Cantor et al is :-

Theorem Let $(g, \pi) \in \mathcal{C}_o$. Then in the Cauchy development of (g, π) there is a slice on which the trace of the second fundamental form is zero.

Where (g, π) is an initial data set on $\mathbb{R}^3$ which is asymptotically Euclidean and satisfies the constraint equations and $g = \delta + h$ with δ the flat metric and $h \in M^p_{s,\delta}, \pi \in M^p_{s-1,\delta+1}$

Definition $M^p_{s,\delta}(\mathbb{R}^n, \mathbb{R}^m)$ is the completion of $C^\infty(\mathbb{R}^n, \mathbb{R}^m)$ with respect to the norm

$$|f|_{p,s,\delta} = \sum_{|\alpha| \le s} \left(|(D^\alpha f)\sigma^{\delta+|\alpha|}|_{L^p} \right)$$

where $\sigma(x) = (1 + |x|^2)^{\frac{1}{2}}$, $p \in [1, \infty)$, $s \in \mathbb{N}$, $\delta \in \mathbb{R}$

Their proof depends on the uniform ellipticity of the Laplacian operator associated with the given metric g on $\mathbb{R}^3$ for any continuous curve of metrics $g(\alpha)$ with $(g(\alpha), \pi(\alpha)) \in \mathcal{C}^0$, $(g_0), \pi(0))$ being the flat initial data and $(g(\alpha), \pi(\alpha))$ being such that its Cauchy development contains a maximal (i.e. zero mean curvature) slice. This uniform ellipticity is equivalent to the initial hypersurfaces involved being uniformly bounded away from null directions. This, in my mind, is a statement requiring proof, being far from obvious and possibly incorrect.

In the above cases one can show that $P_{s\phi}$, (or in theorem (5.6.2) a map related to it) is C_1 on a neighbourhood of $(\hat{g},0)$. One then examines the partial functional derivative of $P_{s\phi}$ at $(\hat{g},0)$ with respect to ϕ . One finds :-

(5.6.5) $$P'_{s\phi}|_{(\hat{g},0)} \cdot \psi \equiv \nabla_0^2(V\psi) + V\left(\hat{K}_{s_0}^2 + {}^{+}Ric(\hat{g})(n_{s_0}, n_{s_0})\right)\psi$$

where

∇_0^2 is the Laplacian operator associated with the metric induced on S_0 by $\hat{g}$.

$\hat{K}_{s_0}^2$ is the covariant square of the extrinsic curvature of S_0 with respect to $\hat{g}$. $V = \sqrt{\hat{g}_{00}}$.

n_{s_0} is the future directed unit normal to S_0 with respect to $\hat{g}$.

Since S_0 is spacelike, rather than achronal, ∇_0^2 is an elliptic operator and $\psi \longmapsto P'_{s\phi}|_{(\hat{g},0)} \cdot \psi$ is an isomorphism. Applying the implicit function theorem the results follow.

Various attempts have been made to extend the above theory to an existence theory for maximal, that is zero mean curvature, spacelike hypersurfaces in general asymptotically flat spacetimes (in a recent preprint by Cantor, Fischer, Marsden, Murchadha and York [13]) and for periodic spacetimes (Avez [6]). All of these attempts require the extra hypothesis that all the spacelike hypersurfaces involved are uniformly bounded away from null directions. Cantor et al require it for the uniform ellipticity of the induced Laplacian operator and Avez for the applicability of the Morrey[41,42], regularity theory for variational problems. This is a very strong simplifying hypothesis, if one could assume it true one could directly apply Palais-Smale generalised Morse theory[46,47] to the problem, for example It is still not clear, however, that the hypothesis is reasonable, it would certainly require verification.

In a recent paper Brill [9] has shown that in the case of spherically symmetric static spacetimes, if one assumes that the maximal hypersurfaces are also spherically symmetric then they can be described by the following set of first order ordinary differential equations :-

Assuming (5.6.6) $$ds^2 = -B(r)\,dt^2 + C(r)\,dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

then $\phi \equiv \frac{B^{\frac{1}{2}}}{C^{\frac{1}{2}}} \frac{dt}{dr}$ for a maximal hypersurface is given by :-

$$\phi = 0 \quad \text{or} \quad \phi = \pm\left(1 + \alpha r + B\right)^{-\frac{1}{2}} \qquad \alpha \in \mathbb{R}$$

APPENDIX

NON-COMPACT SPACELIKE HYPERSURFACES AND PHYSICAL APPLICATIONS

Spacelike hypersurfaces of constant mean curvature have found many applications in relativity theory. York has shown that the constraint equations of the initial value problem take a particularly simple form on such a surface[43,44,59] and has discussed the role of mean curvature as a natural time coordinate as has Kuchař[32,33]. The total gravitational energy of a space-time, defined by Arnowitt, Deser, Misner[4], Brill[10,11] and Geroch[24] has also been discussed with relation to such hypersurfaces by Murchadha and York[45] and also by Geroch[23]. I shall give a brief summary of the above work.

York noticed that the Landau-Raychaudhuri equation (3.24.1) implies that the constant mean curvature Ω would be an increasing function of time with $\Omega_{,0} > 0$ for a foliation of constant mean curvature hypersurfaces with an orthogonal trajectory that was a timelike geodesic, in vacuum. He also obtained equation (3.8.1) in a slightly different form for the vacuum case and showed that it had a uniquely determined solution l.

Kuchař has pointed out that Ω increases through the moment of maximum expansion of the universe and that its sign could be used as a method of distinguishing between the epochs of expansion and contraction. It has been shown, in this connection, by York, that if one can use Ω as a globally defined natural time-coordinate then the Hamiltonian density for the A.D.M. Hamiltonian for relativity theory becomes simply $\sqrt{-|a_{ij}|}$, the area density and so the Hamiltonian is just the spatial volume of the universe.

In vacuum on a spacelike hypersurface S, the constraint equations of general relativity read :-

$$\pi^{ij}{}_{;j} = 0$$

$$\sqrt{-|a_{ij}|}\,{}^{3}R = \frac{1}{\sqrt{-|a_{ij}|}}\left(\pi^{ij}\pi_{ij} - \tfrac{1}{2}\pi^{2}\right)$$

where $$\pi^{ij} = \sqrt{-|a_{ij}|}\left(a^{ij}\Omega - \Omega^{ij}\right)$$, the semi-colon

stands for covariant differentiation with respect to the metric a_{ij} and the other notation is as above (Chapter III). If S has constant mean curvature the first of these is automatically satisfied. The initial value problem has been studied in this form in terms of the conformal geometry of the hypersurface S by York and Murchadha and York (for the matter case).

The Arnowitt, Deser and Misner energy is given by

$$E = \frac{1}{16\pi} \oint_{\infty} \sqrt{-|g_{ij}|}\; g^{ij} g^{kl} (g_{ik,j} - g_{ij,k})\, dS_l$$

where the metric is given in asymptotically Cartesian coordinates, that is

$$g_{\mu\nu} - \eta_{\mu\nu} = O(\tfrac{1}{r})$$

and $$g_{\mu\nu,\alpha} = O(\tfrac{1}{r^{2+\epsilon}}), \text{with} \quad \epsilon \geq 0.$$

$\eta_{\mu\nu}$ is the tensor $diag(1,-1,-1,-1)$.

Murchadha and York prove two theorems :-

1) The intrinsic geometry of any set of initial data, satisfying the constraint equations with positive local energy density, on a maximal spacelike hypersurface (one with $\Omega = 0$) is conformal to a unique time symmetric geometry (that is one with ${}^3R = 0$).
2) Among all such initial data sets the unique time symmetric geometry has minimal energy.

Together with a result of Brill's this establishes that every axisymmetric asymptotically flat space time with an $\Omega = 0$ spacelike hypersurface has non-negative energy.

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CORRIGENDA

Page 7 In the proof of (1.4) the sets $[1, 2\lambda_x)$ should be replaced by the sets $[1, 1+2\lambda_x)$ etc.

Page 9 After "Geroch[25,29]" the next line should be "$\tilde{M}$ asymptotically simple and empty $\Rightarrow$ ".

Page 16 $\tilde{\mathcal{U}}_k$ is in fact $\emptyset$ in the W_k topology. However it is true that all $g \in \mathcal{U}_k$ s.t. $\exists$ a bounded open set $V \subset M$ with $g|_{M-V} = \eta$ are asymptotically simple and empty.
Alternatively, defining the $\tilde{W}_k$ topology by the basis:-

$$\tilde{N}(\mathcal{U}) = \{ f \in C^k(E) \,|\, j^k(f)(M \cap \pi(\mathcal{U})) \subseteq \mathcal{U}, f|_{M-\pi(\mathcal{U})} = \eta \}$$

for $\mathcal{U}$ open, $\mathcal{U} \subseteq J^k(E)$.
It is clear that the W_k topology is finer than the W_k topology and that $\tilde{\mathcal{U}}_k$ is open and non-empty in the $\tilde{W}_k$ topology.

Page 24 Replace (2.24) by (2.2.4)
Replace (2.41) by (2.4.1)

Page 27 (2.10.2) should read $2p = \Omega_{ij}\, dx^i\, dx^j$

Page 29 Insert } at end of line 16.

Page 50 (3.26) and many subsequent theorems implicitly assume the Hawking convergence condition to be satisfied, where this is the case it is clear from the proof.

Page 53 Definition of $B^{+(-)}$ should read:-

$$B^{+(-)} = \{ S_{K_1} \cap I^{+(-)}[S_{K_2}] \}$$

Page 58 Replace $H \subset I^+[T]$ by $H \subset \{J^+[T]\}, T \neq H$.

Page 61 Line 20. Replace (5.3.1) by (2.9).