

### Abstract

In terms of linearized Gross-Pitaevskii equation we have studied the process of sound emission arises from a supersonic particle motion in a Bose-condensed gas. By analogy with the method used for description of Vavilov-Cherenkov phenomenon, we have found a friction work created by the particle generated condensate polarization. For comparison we have found radiation intensity of excitations. Both methods gives the same result.

# “Cherenkov radiation” of a sound in a Bose-condensed gas

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## 1 Introduction

The recent experiments [6] of Bose-Einstein condensation (BEC) in trapped alkali-metal atomic gases have stimulated a great interest on the problem [3]. One of the reasons of such an interest is the fact that nonlinear Schrödinger equation (Gross-Pitaevskii equation) is well suited for describing the fundamental characteristics of an interacting Bose-gas at zero temperature and also appears in other physical problems: theory of superconductivity, theory of elementary particles, nonlinear optics [10] etc.

Examination of fast particle's energy loss is one of the popular experimental technique in condensed matter physics. Such experiments also can give an additional information about properties of an ultracold Bose-gas.

It is known, when a particle moves in a matter with a velocity  $v$  above the supersonic speed  $c$  which equals to the speed of excitations, the matter begins to radiate. In electrodynamics this phenomenon names Vavilov-Cherenkov radiation [1], in aerodynamics such an effect appears from a supersonic motion of a body, in condensed matter physics — this is a polaron problem [4].

In the paper we use linearized Gross-Pitaevsky equation to study the process of radiation of the excitations in a Bose-condensate, arises from a supersonic particle motion. Under this consideration, the Bose-condensate excitations are described as the excitations of a classic nonrelativistic scalar field, same as radiation of a classic electromagnetic field in a continuous matter. As it was shown [8], the GPE can be reduced to the system of hydrodynamic equations. But this approach adequately describes only the long-wave part of the spectrum, which corresponds to the low energy levels of elementary excitations  $\omega \ll \mu$ . As it will be shown below, the maximum of the radiation is in the region  $\omega_{\max} = 2\mu(v/c)\sqrt{(v/c)^2 - 1}$ . So that, direct investigation of the GPE gives more detailed description of the problem. Note that to find an energy loss of a particle it is enough to solve the problem by the methods of quantum mechanics using “Fermi's golden rule.” The result will be the same.

## 2 Particle driven condensate polarization

The state of the condensate (with a fixed chemical potential  $\mu$ ), is described by the macroscopic wave-function  $\Psi(\mathbf{r}, t)$ , which satisfies the GPE equation [2] ( $\hbar = 1$ ),

$$i \frac{\partial}{\partial t} \Psi(\mathbf{r}, t) = \left( \frac{\hat{p}^2}{2m} + V(\mathbf{r}, t) - \mu \right) \Psi(\mathbf{r}, t) + \int d\mathbf{r}_1 U(\mathbf{r} - \mathbf{r}_1) \Psi^*(\mathbf{r}_1, t) \Psi(\mathbf{r}_1, t) \Psi(\mathbf{r}, t).$$

The interaction energy of Bose-gas atoms  $U(\mathbf{r} - \mathbf{r}_1) = U_0 \delta(\mathbf{r} - \mathbf{r}_1)$  and potential energy of its interaction with the straightly and uniformly moving particle  $V(\mathbf{r}, t) = V_0 e^{st} \delta(\mathbf{r} - \mathbf{v}t)$ ,  $s = +0$ . In the last expression we use an adiabatic switching of interaction to exclude automatically advanced solutions. Wave-function of the ground state of a static gas without a particle corresponding to zero energy (counted from the chemical potential) doesn't depend on the coordinates and equals  $\Psi_0 = \sqrt{\mu/U_0}$ . In the presence of a particle condensate becomes heterogeneous — “polarizes.” The “polarization”, in general, depends on time  $\psi(\mathbf{r}, t) = \Psi(\mathbf{r}, t) - \Psi_0$ , in linear approximation it satisfies inhomogeneous equation:

$$i \frac{\partial}{\partial t} \psi(\mathbf{r}, t) = \left( \frac{\hat{p}^2}{2m} + \mu \right) \psi(\mathbf{r}, t) + \mu \psi^*(\mathbf{r}, t) + b e^{st} \delta(\mathbf{r} - \mathbf{v}t), \quad (1)$$

$$b = V_0 \Psi_0, \quad [b] = [r^{3/2}/t], s = +0.$$

Let's solve the equation (1) using Fourier's method. We present the potential and the  $\delta$ -function as a plane waves decomposition,

$$\psi(\mathbf{r}, t) = \int \frac{d^3 k}{(2\pi)^3} e^{i\mathbf{k}\mathbf{r} - i\omega t + st} \psi(\mathbf{k}), \quad (2)$$

$$\delta(\mathbf{r} - \mathbf{v}t) e^{st} = \int \frac{d^3 \mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\mathbf{r} - i\omega t + st}, \quad \omega = \mathbf{k}\mathbf{v}. \quad (3)$$

Substituting (2) and (3) into equation (1), we get

$$(\omega + is - \xi) \psi(\mathbf{k}) = \mu \psi^*(-\mathbf{k}) + b, \quad \xi = \frac{k^2}{2m} + \mu. \quad (4)$$

Combining this equation with complex conjugated one, we find

$$\psi(\mathbf{k}) = b \frac{S(\mathbf{k})}{(\omega + is)^2 - \varepsilon^2(k)}, \quad (5)$$

$$S(\mathbf{k}) = \frac{k^2}{2m} + \omega, \quad \varepsilon^2(k) = \xi^2 - \mu^2 = c^2 k^2 + \left( \frac{1}{2m} k^2 \right)^2, \quad c^2 = \frac{\mu}{m}. \quad (6)$$

The poles of (5) describes the Bogolyubov's oscillations spectrum  $\varepsilon(k)$  of the Bose-condensate. Substitution of (5) into (2) gives an important expression for

the field of condensate “polarization”,

$$\psi(\mathbf{r}, t) = b \int \frac{d^3 k}{(2\pi)^3} e^{i\mathbf{k}\mathbf{r} - i\omega t + st} \frac{S(\mathbf{k})}{(\omega + is)^2 - \varepsilon^2(k)}, \quad \omega = \mathbf{k}\mathbf{v}. \quad (7)$$

With a slow particle motion the field decays exponentially at a large distance from the particle. The poles of (5) can lead to radiation,

$$\mathbf{k}\mathbf{v} = \pm \varepsilon(k)$$

Thus, for appearance of radiation in a sound spectrum region ( $\varepsilon(k) = ck$ ) we have the condition,

$$kv \cos \theta_0 = ck \quad \cos \theta_0 = \frac{c}{v}.$$

It gives the equation of the front of sound radiation. With a particle motion parallel to  $z$  axis, we have

$$z = vt - a \sin \theta_0, \quad r = a \cos \theta_0, \quad 0 \leq a < \infty. \quad (8)$$

If we place a center of spherical coordinates at the point of the particle then there is no field at right of the front in the region  $0 < \theta < \theta_0 + \frac{1}{2}\pi$ . Properly speaking if we consider the dispersion of sound speed then high-frequency field in the region is not zero, but only the low-frequency part of the spectrum

$$c(k) = \frac{\varepsilon(k)}{k} = \sqrt{c^2 + \frac{1}{4m^2}k^2} < v$$

gives contribution to radiation. The dispersion of the sound phase velocity  $c(k)$  plays the same role as the dispersion of inductivity, cutting the radiation frequency on top,

$$c(k_{\max}) = v, \quad (9)$$

$$\omega_{\max} = k_{\max}v = \varepsilon(k_{\max}) = 2mv\sqrt{v^2 - c^2} \quad (10)$$

### 3 Fast particle energy loss

In first order of perturbation theory, the potential energy of the particle in the field of condensate “polarization” is

$$\begin{aligned} \delta E &= \int d\mathbf{r} \Psi^*(\mathbf{r}, t) V(\mathbf{r} - \mathbf{R}) \Psi(\mathbf{r}, t), \\ &= b\Psi_0 + 2b \operatorname{Re} \psi(\mathbf{R}), \end{aligned}$$

where  $\mathbf{R}(t)$  — is the radius vector of the moving particle. The energy produces the drag force which acts on the particle from its created field

$$\mathbf{F} = -\frac{d}{d\mathbf{R}} \delta E = -2b \operatorname{Re} \left[ \frac{d\psi}{d\mathbf{R}} \right]_{\mathbf{r}=\mathbf{v}t}.$$

This results in particle energy loss

$$\dot{E} = \mathbf{F} \mathbf{v} = -2b\mathbf{v} \operatorname{Re} \left[ \frac{d\psi}{d\mathbf{R}} \right]_{\mathbf{r}=\mathbf{v}t},$$

that retires to infinity as a form of condensate excitations. Using the expression (7) we find the radiation intensity

$$I = -\dot{E} = 2b^2 \operatorname{Re} \int \frac{d^3k}{(2\pi)^3} (i\mathbf{k}\mathbf{v}) \frac{\frac{k^2}{2m} + \omega}{(\omega + is)^2 - \varepsilon^2(k)}.$$

To integrate the expression let's use Sokhotsky formula

$$\frac{1}{(\omega + is)^2 - \varepsilon^2} = P \frac{1}{\omega^2 - \varepsilon^2} - \frac{i\pi}{2\omega} [\delta(\omega - \varepsilon) + \delta(\omega + \varepsilon)]$$

The principal value of the integral gives no contribution into energy loss. Thus,

$$I = \frac{b^2}{4\pi} \int_0^\infty k^2 dk \int_{-1}^1 d\cos\theta \left( \frac{k^2}{2m} + \omega \right) [\delta(\omega - \varepsilon) + \delta(\omega + \varepsilon)]. \quad (11)$$

Transforming (11) using a new variable  $\omega = kv \cos\theta$ , we get the very simple formula,

$$I = \frac{b^2}{4\pi m v} \int_0^{k_{\max}} k^3 dk. \quad (12)$$

With the help of the formula which is inversed to (6),

$$k^2 = 2m(-\mu + \sqrt{\mu^2 + \varepsilon^2}),$$

we have an expression for the intensity in form of a spectrum decomposition

$$I = \frac{b^2 m}{2\pi v} \int_0^{\omega_{\max}} \varepsilon \left( 1 - \frac{\mu}{\sqrt{\mu^2 + \varepsilon^2}} \right) d\varepsilon. \quad (13)$$

## 4 Radiation intensity

Now, it is interesting to directly evaluate the radiation intensity, integrating the energy flow  $\mathbf{Q}$  by all directions [5],

$$\mathbf{Q} = \operatorname{Re} \left[ \left( \frac{\hat{\mathbf{P}}}{m} \psi \right)^* i \frac{\partial}{\partial t} \psi \right]$$

where  $\hat{\mathbf{P}} = -i\nabla$  — is the operator of momentum,  $i\frac{\partial}{\partial t}$  — the operator of energy. Whole energy flow through the surface of the cylinder with radius  $R \rightarrow \infty$  around the particle trajectory is

$$I = \int \mathbf{Q} d\mathbf{S} = 2\pi R \operatorname{Re} \int_{-\infty}^{\infty} dz \left( \frac{1}{im} \frac{\partial}{\partial R} \psi \right)^* i \frac{\partial}{\partial t} \psi \quad (14)$$

To evaluate the integral let's find form of the field (7) in the wave zone in cylindrical coordinates

$$\psi(\vec{r}, t) = \frac{b}{(2\pi)^3} \int d^2 k_{\perp} \int_{-\infty}^{\infty} dk_z e^{i\vec{k}_{\perp} \vec{R}} e^{ik_z z} e^{-i\omega t + st} \frac{\frac{1}{2m}(k_{\perp}^2 + k_z^2) + \omega}{(\omega + is)^2 - \varepsilon^2(k)}, \quad \omega = k_z v \quad (15)$$

Introducing the “retarded” time  $t' = t - \frac{z}{v}$ , and expressing the polar angle integral (by variable  $\varphi$ ) using the definition of Bessel function

$$\int_0^{2\pi} e^{ix \cos \varphi} d\varphi = 2\pi J_0(x),$$

we find

$$\psi(\vec{r}, t) = \frac{2mb}{(2\pi)^2 v} \int_0^{\infty} k_{\perp} dk_{\perp} J_0(k_{\perp} R) \int_{-\infty}^{\infty} d\omega e^{-i(\omega + is)t'} \frac{(k_{\perp}^2 + \frac{\omega^2}{v^2}) + 2m\omega}{4m^2(\omega + is)^2 - 4m\mu(k_{\perp}^2 + \frac{\omega^2}{v^2}) - (k_{\perp}^2 + \frac{\omega^2}{v^2})^2},$$

to get more compact expression let's factorize the denominator

$$\psi(\vec{r}, t) = -\frac{mb}{(2\pi)^2 v} \int_{-\infty}^{\infty} d\omega e^{-i(\omega + is)t'} F, \quad F = \int_0^{\infty} 2k_{\perp} dk_{\perp} J_0(k_{\perp} R) \frac{k_{\perp}^2 + \frac{\omega^2}{v^2} + 2m\omega}{(k_{\perp}^2 - k_1^2)(k_{\perp}^2 + k_2^2) - is\omega} \quad (16)$$

where

$$k_1^2 = -\left(2\mu m + \frac{\omega^2}{v^2}\right) + 2m\sqrt{\omega^2 + \mu^2}$$

$$k_2^2 = \left(2\mu m + \frac{\omega^2}{v^2}\right) + 2m\sqrt{\omega^2 + \mu^2}$$

With small frequencies the positive value of  $k_1^2 \sim \omega^2$  have a maximum at  $\omega = \sqrt{(mv^2)^2 - \mu^2}$  and becomes zero at  $\omega_{\max} = 2mv\sqrt{v^2 - c}$ . The integral  $F(\omega)$  can be represented in a following form

$$F(\omega) = \int_0^{\infty} dk_{\perp} J_0(k_{\perp} R) \frac{k_{\perp}^2 + \frac{\omega^2}{v^2} + 2m\omega}{4m\sqrt{\omega^2 + \mu^2}} \left[ \frac{2k_{\perp}}{k_{\perp}^2 - (k_1 + i\omega s)^2} - \frac{2k_{\perp}}{k_{\perp}^2 + k_2^2} \right] \quad (17)$$

The last term in the square brackets has imaginary poles, it gives the exponentially small contribution to the integral at large distances from the particle trajectory ( $R \rightarrow \infty$ ) and can be neglected. The expression (17) can be transformed to integral by whole real axis [7].

Let's examine the integral along the closed contour

$$J = \frac{1}{2} \int_C f(z^2) H_0^{(1)}(z) z dz \quad (18)$$

where  $H_0^{(1)}(zR)$  — is Hankel function, analytical in an upper half-plane of complex variable  $z$  which have the following asymptotic behavior on the infinity

$$H_0^{(1)}(z) \approx \sqrt{\frac{2}{\pi z}} e^{iz - i\pi/4}. \quad (19)$$

The path  $C$  consists of the real axis and the infinite semicircle in an upper-halfplane. The integral along the last contour is exponentially small. So we can write

$$J = J_- + J_+, \quad J_- = \frac{1}{2} \int_{-\infty}^0 dz H_0^{(1)}(zR) z f(z^2), \quad J_+ = \frac{1}{2} \int_0^{\infty} dz H_0^{(1)}(zR) z f(z^2) \quad (20)$$

Let's transform first integral to a new variable  $z = \rho e^{i\varphi}$

$$J_- = -\frac{1}{2} \int_0^{\infty} d\rho H_0^{(1)}(e^{i\pi} \rho R) \rho f(\rho^2). \quad (21)$$

Taking into account (20) we have for sum (21)

$$J = \frac{1}{2} \int_0^{\infty} d\rho \rho f(\rho^2) \left[ \sqrt{\frac{2}{\pi \rho}} e^{i\rho R - i\pi/4} + \sqrt{\frac{2}{\pi \rho}} e^{-i\rho R + i\pi/4} \right] = \int_0^{\infty} d\rho \rho f(\rho^2) J_0(\rho R).$$

On the other hand the integral on the closed contour (18) equals the sum of residues of the poles  $z_n$  in upper-halfplane

$$J = 2\pi i \sum_{res} \frac{1}{2} H_0^{(1)}(z_n R) z_n f(z_n^2) \quad (22)$$

Applying (22) to integral (17) and remembering that integrand have one pole at  $|\omega| < \omega_{\max}$  close to the real axis in the upper-halfplane equal to  $k_{\perp} = (k_1 \text{sign } \omega + is)$  we have

$$F(\omega) = 2\pi i H_0^{(1)}(p) \frac{k_1^2 + \frac{\omega^2}{v^2} + 2m\omega}{4m\sqrt{\omega^2 + \mu^2}} = \pi i H_0^{(1)}(p) D(\omega), \quad (23)$$

$$D(\omega) = \frac{k_1^2 + \frac{\omega^2}{v^2} + 2m\omega}{2m\sqrt{\omega^2 + \mu^2}} = 1 + \frac{\omega - \mu}{\sqrt{\omega^2 + \mu^2}}, \quad p = k_1 R \operatorname{sign} \omega + is. \quad (24)$$

Substituting (23) into (16) and taking into account that at  $|\omega| > \omega_{\max}$  parameter  $k_1$  becomes imaginary and Hankel function is exponentially small, we find  $\psi(\mathbf{r}, t)$  in the wave zone as a form of single integral by frequency

$$\psi(\mathbf{r}, t) = -\frac{imb}{8\pi v} \int_{-\omega_{\max}}^{\omega_{\max}} d\omega e^{-i(\omega+is)t'} H_0^{(1)}(p) D(\omega). \quad (25)$$

Let's use this expression to find (14). Using (25), (19), (24) we have

$$-\frac{i}{m} \frac{\partial}{\partial R} \psi = -\frac{imb}{8\pi v} \int_{-\omega_{\max}}^{\omega_{\max}} d\omega \left( \frac{k_1}{m} \operatorname{sign} \omega \right) e^{-i(\omega+is)t'} H_0^{(1)}(p) D(\omega) \quad (26)$$

and

$$i \frac{\partial \psi}{\partial t} = -\frac{imb}{8\pi v} \int_{-\omega_{\max}}^{\omega_{\max}} d\omega (\omega) e^{-i(\omega+is)t'} H_0^{(1)}(p) D(\omega) \quad (27)$$

Substituting this expressions into (14) and integrating by  $z$  with a formula

$$\int_{-\infty}^{\infty} e^{-i(\omega-\omega')(t-z/v)} dz = 2\pi v \delta(\omega - \omega')$$

and using explicit form (19) we have

$$\begin{aligned} I &= 4\pi^2 R v \left( \frac{mb}{8\pi v} \right)^2 \int_{-\omega_{\max}}^{\omega_{\max}} d\omega \frac{|\omega|}{m} k_1 \frac{2}{\pi |p|} D(\omega)^2 \\ &= \frac{mb^2}{8\pi v} \int_{-\omega_{\max}}^{\omega_{\max}} d\omega |\omega| D(\omega)^2 = \frac{mb^2}{2\pi v} \int_0^{\omega_{\max}} \omega d\omega \left( 1 - \frac{\mu}{\sqrt{\mu^2 + \omega^2}} \right) \end{aligned}$$

The last expression is identically coincides with (13). We see that radiation intensity is monotonically grows with the frequency. To evaluate integral radiation intensity it is the simplest to integrate expression (12):

$$I = \frac{b^2 k_{\max}^4}{16\pi m v} = \frac{b^2 m^3 (v^2 - c^2)^2}{\pi v}.$$



So, we have the exact agreement of the expressions for the radiation spectrum of the bose-condensate excitations generated by the fast particle received by the drag method and the method of the radiated energy flow evaluation. It is possible to show the “Fermi’s golden rule” gives the same result. Thus near the threshold the radiation intensity grows as  $(v^2 - c^2)^2$ , but at high velocity — it is as  $v^3$ . If in the case of Cherenkov effect the particle velocity is limited from the top, then for the nonrelativistic Bose-gas there is no such a limit and with  $v \gg c$  the radiation intensity can be very big.

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## References

- [1] L.D.Landau and E.M. Lifshitz Theoretical physics vol.VIII. Electrodynamics of Continuous Media Moscow, Nauka 1992 p.588
- [2] E.M.Lifshitz and L.P.Pitaevskii Theoretical physics vol. IX. Statistical physics part.2 Theory of condensed state. Moscow, Nauka 1978. p145
- [3] Lev P. Pitaevskii et. al Theory of Bose-Einstein condensation in trapped gases. Review of Modern Physics, vol. 71, No.3, April 1999 p.463
- [4] R.Feinman Statistical mechanics 1972 p.252
- [5] D.N.Zubarev Non-equilibrium statistical thermodynamics. Nauka, Moscow 1971. p.222
- [6] C.C. Bradley, C.A. Sackett, J.J. Tollett, and R.Hulet. Evidence of bose-einstein condensation in an atomic gas with attractive interaction. Physical Review Letters, 75(9), August 1995.
- [7] D.Ivanenko, A.Sokolov Field theory, Moscow 1951
- [8] S.Stringari Collective Excitations of a trapped Bose-condensed gas PRL 77(12) 2360
- [9] R.Onofrio, C.Raman, J.M.Vogels, et.al Observation of Superfluid Flow in a Bose-Einstein condensed gas. Physical Review Letters, 85(11), 2228 September 2000
- [10] V.I. Talanov, Pis'ma Zh. Eksp. Teor. Fiz. 2,31 (1965) [JETP Lett]