

Finite axiomatizability for profinite groups

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ABSTRACT

A group is finitely axiomatizable (FA) in a class \mathcal{C} if it can be determined up to isomorphism within \mathcal{C} by a sentence in the first-order language of group theory. We show that profinite groups of various kinds are FA in the class of profinite groups, or in the class of pro- p groups for some prime p . Both algebraic and model-theoretic methods are developed for the purpose. Reasons why certain groups cannot be FA are also discussed.

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1. Introduction

Some properties of a group can be expressed by a sentence in the first-order language L_{gp} of group theory, and some cannot. If the group is assumed to be *finite*, a lot more can be said about it in first-order language than in the general case. We mention examples of these phenomena below.

The strongest property of a group G is that of ‘being isomorphic to G ’. If this can be expressed by a first-order sentence, G is said to be *finitely axiomatizable* (FA). It is obvious that every finite group is FA: if $|G| = n$, the fact that G has exactly n elements and that they satisfy the multiplication table of G is clearly a first-order property. An infinite group cannot be FA by the Löwenheim–Skolem theorem [38, Theorem 2.3.1]; to make the question interesting we have to limit the universe of groups under consideration. For example, the first author in [21] called a finitely generated, infinite group G *QFA* (for quasi-finitely axiomatizable) if some first-order sentence determines it up to isomorphism within the class of finitely generated groups (this is not the notion of ‘quasi-finite axiomatizability’ used in model theory, cf. [29, Chapter 3]).

He showed that several well-known groups, such as the restricted wreath product $C_p \wr \mathbb{Z}$, have this property (here C_p denotes the cyclic group of order p). The QFA nilpotent groups

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are completely characterized by Oger and Sabbagh in [27]. Further results were obtained by Lasserre [18]. Nies [22] contains a survey up to 2007.

In the present paper, we address the question of relative finite axiomatizability in the universe of *profinite groups*. These (unless finite) are necessarily uncountable, so cannot be finitely generated as groups; but from some points of view, they behave rather like finite groups. For example, Jarden and Lubotzky show in [12] that if G is a (topologically) finitely generated profinite group, then the elementary theory of G characterizes G up to isomorphism among all profinite groups (cf. [34, Theorem 4.2.3]); in this case one says that G is *quasi-axiomatizable*. This is very different from the situation in abstract groups: for example, a celebrated theorem of Sela [36] (see also [13]) shows that all finitely generated non-abelian free groups have the same elementary theory.

The elementary theory of G consists of *all* the sentences satisfied by G . We consider the question: which profinite groups can be characterized by a single sentence? To make this more precise, let us say that a group G is *FA (with respect to L) in \mathcal{C}* if \mathcal{C} is a class of groups containing G , L is a language and there is a sentence σ_G of L such that for any group H in \mathcal{C} ,

$$H \models \sigma_G \text{ if and only if } H \cong G.$$

For instance, QFA means: FA (with respect to L_{gp}) in the class of all f.g. groups. When \mathcal{C} is a class of profinite groups, isomorphisms are required to be *topological*. Usually, we will write ‘FA’ to mean ‘FA in the class of all profinite groups’.

1.1. Classes of groups and their theories

It is often the case that a natural class of (abstract) groups cannot be axiomatized in the first-order language L_{gp} of group theory. This holds for the class of simple groups (see [39]), the 2-generated groups, the finitely generated groups and classes such as nilpotent or soluble groups, none of which is closed under the formation of ultraproducts.

Since finite groups are FA, every class \mathcal{C} of finite groups can be axiomatized within the finite groups: a finite group H is in \mathcal{C} if and only if $H \models \neg\sigma_G$ for every finite group $G \notin \mathcal{C}$ (cf. [39, § 1]). Whether such a class can be *finitely* axiomatized within the finite groups is usually a much subtler question. For example, a theorem of Felgner shows that this holds for the class of non-abelian finite simple groups (see [39, Theorem 5.1]), and Wilson [41] shows that the same is true for the class of finite soluble groups. On the other hand, Cornulier and Wilson show in [4] that nilpotency cannot be characterized by a first-order sentence in the class of finite groups.

The main object of study in Nies [21] was the first-order separation of classes of groups $\mathcal{C} \subset \mathcal{D}$. Even if the classes are not axiomatizable, can we distinguish them using first-order logic, by showing that some sentence ϕ holds in all groups of \mathcal{C} but fails in some group in \mathcal{D} ? If this holds, one says that \mathcal{C} and \mathcal{D} are *first-order separated*. One way to establish this is to find a witness for separation: a group G not in \mathcal{C} that is FA in \mathcal{D} . Then one takes ϕ to be the negation of a sentence describing G within \mathcal{D} .

Some of our results serve to provide first-order separations of interesting classes of profinite groups:

- the finite rank profinite groups are first-order separated from the (topologically) finitely generated profinite groups by Proposition 5.5,
- similarly for pro- p groups, also by Proposition 5.5,
- the f.g. profinite groups are first-order separated from the class of all profinite groups by Corollary 1.5.

1.2. Obstructions to finite axiomatizability

We know of two obstructions to being FA for a profinite group: the centre may ‘stick out too much’, or the group may involve too many primes. The first is exemplified by the following result of Oger and Sabbagh, which generalizes work of Wanda Szmielew (see [11, Theorem A.2.7]) for infinite abelian groups; here $Z(G)$ denotes the centre of G and $\Delta(G)/G'$ the torsion subgroup of G/G' where G' is the derived group.

THEOREM 1.1 [27, Theorem 2]. *Let G be a group such that $Z(G) \not\subseteq \Delta(G)$. If ϕ is a sentence such that $G \models \phi$, then $G \times C_p \models \phi$ for almost all primes p .*

If, for example, G is a finitely generated profinite group, then $G \times C_p \not\cong G$ for every prime p , so G cannot be FA.

The second obstruction comes from a different direction. The Feferman–Vaught theorem controls first-order properties of Cartesian products, and in §7 we note its consequence.

PROPOSITION 1.2. *Let $(G_i)_{i \in I}$ be an infinite family of groups. If ϕ is a sentence such that $\prod_{i \in I} G_i \models \phi$, then there exist $q \neq r \in I$ such that*

$$G_r \times \prod_{i \in I \setminus \{q\}} G_i \models \phi.$$

We will deduce

PROPOSITION 1.3. (i) *If the pronilpotent group G is FA in the class of profinite, or pronilpotent, groups, then the set of primes p such that G has a nontrivial Sylow pro- p subgroup is finite.*

(ii) *Let \mathfrak{G} be a linear algebraic group of positive dimension defined over \mathbb{Q} . If $\mathfrak{G}(\mathbb{Z}_\pi)$ is FA in the class of profinite groups, then the set of primes π is finite.*

Here $\mathbb{Z}_\pi = \prod_{p \in \pi} \mathbb{Z}_p$.

Our main results tend to suggest that for a wide range of profinite groups these are the only obstructions. For example, the converse of Proposition 1.3(ii) holds when \mathfrak{G} is a simple Chevalley group; for $\mathfrak{G} = \mathrm{PSL}_n$ this is a consequence of Theorem 1.8, for other cases see [35, Theorem 1.5] (and Remark(ii), §4.3 below).

However, there are two caveats.

One: it is obvious that two groups that are isomorphic (as abstract groups) must satisfy the same first-order sentences; it is possible for non-isomorphic profinite groups to be isomorphic as abstract groups (cf. [15]), and such groups cannot be FA as profinite groups. In general, there is a strict hierarchy of implications for a profinite group G :

- $G \text{ is FA} \implies G \text{ is quasi-axiomatizable} \implies G \text{ is ‘algebraically rigid’},$

the third condition meaning: *any profinite group abstractly isomorphic to G is topologically isomorphic to G .*

The problem does not arise for groups that are ‘strongly complete’: this means that every subgroup of finite index is open. Every group homomorphism from such a group to any profinite group is continuous; in fact these groups are also quasi-axiomatizable (see [10]). Every finitely generated profinite group is strongly complete (see Theorem 2.1 below). Most of the profinite groups we consider in this paper are finitely generated (as topological groups), but not all (see Corollary 1.5).

Two: There are only countably many sentences, but uncountably many groups, even among those that avoid the above obstructions. We exhibit in §7 a family of such pro- p groups parametrized by the p -adic integers.

There are various ways around this problem. One may restrict attention to the groups that have a *strictly finite presentation*: a profinite (or pro- p) group G has this property if it has a finite presentation as a profinite (or pro- p) group in which the relators are finite group words; equivalently, if G is the completion of a finitely presented abstract group. In §5.3 we define a more general concept called *L -presentation*, which allows for groups like $C_p \wr \mathbb{Z}_p$, the pro- p completion of the aforementioned $C_p \wr \mathbb{Z}$: this is not strictly finitely presentable, but it is finitely presented within the class of metabelian pro- p groups (cf. [9] for abstract metabelian groups); Proposition 5.5 shows that it is FA in the class of all profinite groups. An L -presentation is like a finite presentation in which the usual relations may be replaced by any sentence in the language L .

Another way is to enlarge the first-order language: given a finite set of primes π , we take L_π to be the language L_{gp} augmented with extra unary function symbols P_λ , one for each $\lambda \in \mathbb{Z}_\pi = \prod_{p \in \pi} \mathbb{Z}_p$; for a group element g , $P_\lambda(g)$ is interpreted as the profinite power g^λ . We shall see that many pro- p groups are indeed FA (with respect to $L_{\{p\}}$) within the class of pro- p groups.

1.3. Bi-interpretation

We shall explore two different ways of showing that profinite groups are FA. The first is a model-theoretic procedure known as *bi-interpretation*, first used to show that certain groups are FA by Khelif [14]. This is defined, for example, in [29, Definition 3.1]; further applications of bi-interpretation are described in [22, § 7.7] and in [1, § 2].

If (a group) A is interpreted in (a ring) B and B is interpreted in A , we have an ‘avatar’ \tilde{A} of A in some $B^{(n)}$, and an avatar \tilde{B} of B in some $A^{(m)}$. Composing these procedures produces another avatar $\tilde{\tilde{A}}$ of A in $A^{(mn)}$, and an isomorphism from A to $\tilde{\tilde{A}}$. Similarly, one obtains an isomorphism from B to $\tilde{\tilde{B}} \subseteq B^{(nm)}$. If these two isomorphisms are definable (in the language of groups, respectively, rings), then A and B are said to be *bi-interpretable*.

When dealing with profinite groups and profinite rings, the definition has to be tweaked to take account of the topology. We postpone the precise definitions to §4, where the following result is established.

• *Let R be a profinite ring and G a profinite group. If G is ‘topologically bi-interpretable’ with R , then G is FA in profinite groups if and only if R is FA in profinite rings, assuming algebraic rigidity where appropriate.*

As an illustration of the method, we prove the following.

THEOREM 1.4. *Let R be a complete, unramified regular local ring with finite residue field κ . Then each of the profinite groups $\text{Af}_1(R)$, $\text{SL}_2(R)$ is FA in the class of profinite groups, assuming in the second case that $\text{char}(\kappa)$ is odd.*

Here $\text{Af}_1(R) = (R, +) \rtimes R^*$ denotes the 1-dimensional affine group over R . The theorem combines Theorems 4.5, 4.7 and 4.9, proved below. This result is extended in [35] to Chevalley groups of rank at least 2 over a more general class of rings.

Although we do not pursue this aspect, it may be of interest to mention that the proof of Theorem 1.4 can be adapted to show that the respective groups are uniformly bi-interpretable with the corresponding rings, that is, the defining formulae are independent of the ring.

In Theorem 1.4 the rings in question are the following:

- power series rings in finitely many variables over a finite field,
- power series rings in finitely many variables over an unramified p -adic ring $\mathbb{Z}_p[\zeta]$ (ζ a $(p^f - 1)$ th root of unity).

While the groups $\mathrm{SL}_2(R)$ (for these rings R) are finitely generated as profinite groups (see Proposition 4.11 below), the groups $\mathrm{Af}_1(R)$ are *not*, in most cases (see the remark following the proof of Proposition 4.8); this shows that a profinite group can be very far from finitely presented and still be FA. It also establishes the following.

COROLLARY 1.5. *The classes of f.g. profinite groups and all profinite groups are first-order separable, with witness group $\mathrm{Af}_1(\mathbb{F}_p[[t]])$.*

1.4. p -Adic analytic groups and more

The other approach to establishing that certain groups are FA is purely group-theoretic; as such, it is limited to groups that are ‘not very big’, in a sense about to be clarified. A pro- p group is an inverse limit of finite p -groups, where by convention p always denotes a prime. We observed above that ‘involving too many primes’ can be an obstruction to being FA. In fact all our positive results concern groups that are virtually pro- p (that is, pro- p up to finite index), or finite products of such groups.

The pro- p groups in question are compact p -adic analytic groups. This much-studied class of groups can alternatively be characterized as the *virtually pro- p groups of finite rank*; the profinite group G has finite rank r if every closed subgroup can be generated by r elements (‘generated’ will always mean: ‘generated topologically’). For all this, see the book [5], in particular Chapter 8.

The possibility of showing that (some of) these groups are FA rests on the fact that they have a finite *dimension*: this can be used rather like the order of a finite group, to control when a group has no proper quotients of the same ‘size’.

Let $\pi = \{p_1, \dots, p_k\}$ be a finite set of primes. A \mathcal{C}_π group is one of the form $G_1 \times \dots \times G_k$ where G_i is a pro- p_i group for each i . A \mathcal{C}_π group of finite rank need not be strictly finitely presented, but it always has an L_π presentation (see Subsection 5.3).

The first main result about \mathcal{C}_π applies in particular to all p -adic analytic pro- p groups, but limits the universe.

THEOREM 1.6. *Every \mathcal{C}_π group of finite rank is FA (with respect to L_π) in the class \mathcal{C}_π ; if it has an L_{gp} -presentation (for example, if it is strictly finitely presented), then it is FA (with respect to L_{gp}) in the class \mathcal{C}_π .*

This will be the key to several theorems, showing that groups in certain limited classes of \mathcal{C}_π groups are FA among all profinite groups. The first of these is a profinite analogue of [27, Theorem 10].

THEOREM 1.7. *Let G be a nilpotent \mathcal{C}_π group, and suppose that G has an L_{gp} -presentation. Then G is FA in the class of all profinite groups if and only if $\mathrm{Z}(G) \subseteq \Delta(G)$.*

(The hypothesis implies that G is f.g.; as a product of finitely many nilpotent pro- p groups, G then has finite rank.)

Note that by Proposition 1.3(i), both results would fail if π were an infinite set of primes.

In §5.6 we establish some results intermediate between the last two, characterizing those \mathcal{C}_π groups of finite rank that are FA in the class of all *pronilpotent* groups.

The final main result shows how these methods may be applied to p -adic analytic groups that are far from nilpotent.

THEOREM 1.8. *Let $n \geq 2$ and let p be an odd prime such that $p \nmid n$. Then each of the groups*

$$\mathrm{SL}_n^1(\mathbb{Z}_p), \mathrm{SL}_n(\mathbb{Z}_p), \mathrm{PSL}_n(\mathbb{Z}_p)$$

is FA in the class of profinite groups.

Here $\mathrm{SL}_n^1(\mathbb{Z}_p)$ denotes the principal congruence subgroup modulo p in $\mathrm{SL}_n(\mathbb{Z}_p)$.

The proof for $\mathrm{SL}_n^1(\mathbb{Z}_p)$ uses both Theorem 1.6 and Theorem 1.7, which can be applied to the upper unitriangular group (when $n \geq 3$). The extension to $\mathrm{SL}_n(\mathbb{Z}_p)$ depends on Theorem 3.1, proved in §3, which establishes some sufficient conditions for a finite extension of an FA group to be FA.

A different proof for $\mathrm{SL}_n(\mathbb{Z}_p)$ and $\mathrm{PSL}_n(\mathbb{Z}_p)$ with $n \geq 3$, via bi-interpretability, appears in [35].

1.5. Some problems

The Oger-Sabbagh theorem characterizing the nilpotent (abstract) groups that are QFA has been extended to polycyclic groups by Lasserre [18]: such a group G is QFA if and only if $Z(H) \subseteq \Delta(H)$ for each subgroup H of finite index. The analogous class of pro- p groups is the soluble pro- p groups of finite rank, suggesting the following.

PROBLEM 1. Let G be a soluble pro- p group of finite rank. Show that the following are equivalent:

- (a) G is FA in the class of profinite groups and
- (b) $Z(H) \subseteq \Delta(H)$ for each open subgroup H of G .

We do not know whether the hypothesis of finite rank in Theorem 1.6 is necessary, even if π only contains a single prime.

PROBLEM 2. (a) Is every finitely generated pro- p group G FA with respect to $L_{\{p\}}$ in \mathcal{C}_p ?
 (b) If G is strictly finitely presented as a pro- p group, is G FA (with respect to L_{gp}) in the class \mathcal{C}_p ?

The answer is probably ‘no’, to both. A possible candidate for a counterexample is a non-abelian free pro- p group $(\widehat{F_n})_p$, $n \geq 2$. As far as we know, even the following is open.

PROBLEM 3. Is the group $(\widehat{F_n})_p$ (where $n \geq 2$) FA in the class of profinite groups?

1.6. Organization of the paper

The next section introduces notation and presents some general results about definability in profinite groups. Section 3 is devoted to showing that under certain conditions, a finite extension of an FA group is again FA; this is useful in situations like that of Theorem 1.8, which deal with groups that are virtually pro- p but not actually pro- p . Section 4 deals with bi-interpretability and applications. The material about \mathcal{C}_π -groups occupies Sections 5 and 6. Some negative results are collected together in Section 7. The short Section 8 consists of a list of first-order formulas for lookup.

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2. Definable subgroups

For a group G and a formula $\kappa(x)$ (possibly with parameters \bar{g} from G), we write

$$\kappa(G) = \kappa(\bar{g}; G) := \{x \in G \mid G \models \kappa(\bar{g}, x)\}.$$

(The notation will also be used, *mutatis mutandis*, for rings.) A subgroup is *definable* if it is of this form; unless otherwise stated, κ is supposed to be a formula of L_{gp} . Note that $\kappa(G)$ is a subgroup if and only if $G \models s(\kappa)$ where

$$s(\kappa) \equiv \exists x. \kappa(x) \wedge \forall x, y. (\kappa(x) \wedge \kappa(y) \rightarrow \kappa(x^{-1}y)),$$

and $\kappa(G)$ is a normal subgroup if and only if $G \models s_{\triangleleft}(\kappa)$ where

$$s_{\triangleleft}(\kappa) \equiv s(\kappa) \wedge \forall x, y. (\kappa(x) \rightarrow \kappa(y^{-1}xy)).$$

We will say that a subgroup H is *definably closed* if $H = \kappa(G)$ for a formula κ such that in any profinite group M , the subset $\kappa(M)$ is necessarily closed.

Suppose that $H = \kappa(G)$ is a definable subgroup of G . By the usual relativization process, for any formula $\varphi(y_1, \dots, y_k)$ there is a ‘restriction’ formula $\text{res}(\kappa, \varphi)(y_1, \dots, y_k)$ such that for each k -tuple $\bar{b} \in H^{(k)}$ we have

$$G \models \text{res}(\kappa, \varphi)(\bar{b}) \iff H \models \varphi(\bar{b}).$$

(Note that $\text{res}(\kappa, \varphi)$ is obtained from φ by relativizing the quantifiers of φ , that is, replacing any expression $\forall z. \psi(z)$ by $\forall z. (\kappa(z) \rightarrow \psi(z))$, and any expression $\exists z. \psi(z)$ by $\exists z. (\kappa(z) \wedge \psi(z))$. Clearly, if φ is quantifier-free, then $\text{res}(\kappa, \varphi)$ is just φ .)

Similarly, if $N = \kappa(G)$ is a definable *normal* subgroup, there is a ‘lifted’ formula $\text{lift}(\kappa, \varphi)$ such that

$$G \models \text{lift}(\kappa, \varphi)(\bar{b}) \iff G/N \models \varphi(\tilde{b}_1, \dots, \tilde{b}_k),$$

where \tilde{b} denotes the image of b modulo N . To obtain $\text{lift}(\kappa, \varphi)$ we replace each atomic formula $x = y$ in φ with $\kappa(x^{-1}y)$.

Suppose that $\kappa(G)$ is a definable subgroup, and let $n \in \mathbb{N}$. Then

$$|G : \kappa(G)| \leq n \iff G \models \text{ind}(\kappa; n),$$

$$|G : \kappa(G)| = n \iff G \models \text{ind}(\kappa; n) \wedge \neg \text{ind}(\kappa; n-1) := \text{ind}^*(\kappa; n),$$

where

$$\text{ind}(\kappa; n) \equiv \exists u_1, \dots, u_n. \forall x. \bigvee_j \kappa(x^{-1}u_j).$$

We define the frequently used formula

$$\text{com}(x, y) := (xy = yx).$$

For a profinite group G and $X \subseteq G$, the closure of X is denoted as \overline{X} . (This is not to be confused with \bar{x} , which stands for a tuple (x_1, \dots, x_n) .)

We write $X \leq_c G$, respectively, $X \leq_o G$, for ‘ X is a closed, respectively open, subgroup of G ’.

For any group G (abstract or profinite) and $Y \subseteq G$, the subgroup generated (algebraically) by Y is denoted as $\langle Y \rangle$. For $q \in \mathbb{N}$, $G^{\{q\}} = \{g^q \mid g \in G\}$ is the set of q -th powers and $G^q = \langle G^{\{q\}} \rangle$.

The derived group of G is $G' = \langle [x, y] \mid x, y \in G \rangle$. Note that

$$G'G^q = G'G^{\{q\}}.$$

The key fact that makes f.g. profinite groups accessible to first-order logic is the *definability of open subgroups*. We shall use the following without special mention.

THEOREM 2.1. (Nikolov and Segal) *Let G be a f.g. profinite group.*

- (i) *Every subgroup of finite index in G is both open and definably closed (with parameters).*
- (ii) *Each term $\gamma_n(G)$ of the lower central series of G is closed and definable (without parameters).*
- (iii) *Every group homomorphism from G to a profinite group is continuous.*

Proof. The definability of subgroups in a profinite group is related to the topology of the group through the concept of verbal width. A word w has *width* f in a group G if every product of w -values or their inverses is equal to such a product of length f . The verbal subgroup $w(G)$ generated by all w -values is closed in G if and only if w has finite width [34, Proposition 4.1.2]; in this case it is definable, by the formula $\kappa_{w,f}(x)$ which expresses that

$$x \in G_w \cdot \dots \cdot G_w \quad (f \text{ factors}), \quad (1)$$

where $w = w(x_1, \dots, x_k)$ has width f and $G_w = \{w(\bar{g})^{\pm 1} \mid \bar{g} \in G^{(k)}\}$. This formula defines a closed subset in every profinite group, since the verbal mapping $G^{(k)} \rightarrow G$ defined by w is continuous, hence has compact image.

In a finitely generated profinite group, each lower-central word and all power words have finite width [24, 25, 26]. (ii) follows at once.

For (i), suppose that H is a subgroup of finite index in G . Then $H \geq G^q = \langle g^q \mid g \in G \rangle$ for some q , and G^q is definably closed by the preceding remarks, because the word x^q has finite width. If $G^q \leq N \triangleleft_o G$, then G/N is a finite $d = d(G)$ -generator group of exponent dividing q , and hence has order bounded by a finite number $\beta(d, q)$ (by the positive solution of the Restricted Burnside Problem [42, 43]). As G^q is the intersection of all such N , it follows that G^q is open. Now (i) follows by the lemma below.

(iii) is an easy consequence of the fact that every subgroup of finite index open. \square

LEMMA 2.2. *Suppose that N is a definable subgroup in a group G . If $N \leq H \leq G$ and $|H : N|$ is finite, then H is definable, by a formula with parameters. If N is definably closed, then so is H . If $G = N\langle X \rangle$ for some subset X , we may choose the parameters in X .*

Proof. Say $N = \kappa(G)$ and $H = Ng_1 \cup \dots \cup Ng_n$. Then H is defined by $\bigvee_{i=1}^n \kappa(xg_i^{-1})$. The second claim is clear since the union of finitely many translates of a closed set is closed. For the final claim, we may replace each g_i by a suitable word on X . \square

REMARK. If every subgroup of finite index in G is open, then every subgroup of finite index contains a definable open subgroup, whether or not G is f.g.: this follows from [37, Theorem 2] in a similar way to the proof of (ii) above; it is implicit in the proof of [10, Theorem 3.11].

The special case of these results where G is a pro- p group is much easier, and suffices for most of our applications; see, for example, [5, Chapter 1, ex. 19] and [34, § 4.3].

Note that subgroups like $w(G)$ when w is a word of finite width are definable as in (1) without parameters.

When proving that a certain group G is FA in some class \mathcal{C} , we often establish a stronger property, namely: for some finite (usually generating) tuple \bar{g} in G , there is a formula σ_G such that for a group H in \mathcal{C} and a tuple \bar{h} in H , $H \models \sigma_G(\bar{h})$ if and only if there is an isomorphism from G to H mapping \bar{g} to \bar{h} , a situation denoted by $(G, \bar{g}) \cong (H, \bar{h})$. In this case we say that (G, \bar{g}) is FA in \mathcal{C} . Of course, this implies that G is FA in \mathcal{C} : indeed, for H in \mathcal{C} , we have $H \cong G$ if and only if $H \models \exists \bar{x}. \sigma_G(\bar{x})$.

3. Finite extensions

If a group G is FA, one would expect that (definable) subgroups of finite index in G and finite extension groups of G should inherit this property. In this section we establish the latter under some natural hypotheses.

Fix a class \mathcal{C} of profinite groups, and assume that \mathcal{C} is closed under taking open subgroups. $L \supseteq L_{\text{gp}}$ is a language. By ‘FA’ we mean FA (with respect to L) in \mathcal{C} .

Given a group N and elements $h_1, \dots, h_s \in N$, we say that an element g of N is \bar{h} -definable in N if there is a formula ϕ_g such that for $c \in N$,

$$N \models \phi_g(\bar{h}, c) \iff c = g. \quad (2)$$

This holds in particular if $g \in \langle h_1, \dots, h_s \rangle$.

REMARKS. (i) if $\theta : N \rightarrow M$ is an isomorphism and (2) holds, then $g\theta$ is the unique element b of M such that $N \models \phi_g(\bar{h}\theta, b)$.

(ii) If (N, \bar{h}) is FA and g is \bar{h} -definable in N , then $(N, (\bar{h}, g))$ is FA.

(iii) If g is \bar{h} -definable in N and $N = \kappa(G)$ is a definable subgroup of G , then g is \bar{h} -definable in G , by the formula

$$\kappa(y) \wedge \text{res}(\kappa, \phi_g).$$

THEOREM 3.1. *Let*

$$N = \overline{\langle h_1, \dots, h_s \rangle} \triangleleft_o G = \overline{\langle g_1, \dots, g_r \rangle} \in \mathcal{C},$$

and assume that (N, \bar{h}) is FA. Then G is FA provided one of the following holds:

- (a) $N \cap \langle g_1, \dots, g_r \rangle = \langle h_1, \dots, h_s \rangle$, in which case (G, \bar{g}) is FA; or
- (b) $Z(N) = 1$, $\{h_1, \dots, h_s\} \subseteq \langle g_1, \dots, g_r \rangle$, and $h_i^{g_j}$ is \bar{h} -definable in N for each i and j .

Proof. Say $|G : N| = m$. By Theorem 2.1 there is a formula κ such that $N = \kappa(G; \bar{g})$, and such that κ always defines a closed subset in any profinite group. Thus G satisfies

$$\Phi_1(\bar{g}) := \text{s}_{\triangleleft}(\kappa(\bar{g})) \wedge \text{ind}^*(\kappa(\bar{g}), m),$$

which asserts that $\kappa(G; \bar{g})$ is a closed normal subgroup of index m (and is therefore open).

By hypothesis, there is a formula ψ , where $N \models \psi(\bar{h})$, such that if $k_1, \dots, k_s \in M \in \mathcal{C}$ and $M \models \psi(\bar{k})$, then there is an isomorphism $N \rightarrow M$ sending \bar{h} to \bar{k} . For each i there is a word w_i such that $h_i = w_i(\bar{g})$; then G satisfies

$$\Phi_2(\bar{g}) := \bigwedge_{i=1}^s \kappa(\bar{g}, w_i) \wedge \text{res}(\kappa(\bar{g}), \psi(w_1, \dots, w_s)), \quad (3)$$

where for aesthetic reasons w_i is written in place of $w_i(\bar{g})$, a convention we keep throughout this proof.

Since \mathcal{C} is closed under taking open subgroups, $\Phi_1(\bar{g})$ implies that $\kappa(G; \bar{g}) \in \mathcal{C}$, and then $\Phi_2(\bar{g})$ ensures that $\kappa(G; \bar{g}) \cong N$. We set

$$\Phi := \Phi_1 \wedge \Phi_2.$$

To fix the isomorphism type of G , we need also to specify the conjugation action of G on N , the quotient G/N and the extension class. These are done in the following manner. To begin with, note that $G = N\langle g_1, \dots, g_r \rangle$ because N is open; hence, there exists a transversal $\{t_i(\bar{g}) \mid i = 1, \dots, m\}$ to the cosets of N in G , where each t_i is a word. There is a formula $\tau(\bar{g})$ (depending on κ) which asserts that $G = \bigcup_{i=1}^m N t_i(\bar{g})$.

Now we deal separately with cases (a) and (b).

Case (a): For each i and j we have $h_i^{g_j} = v_{ij}(\bar{h})$ for some word v_{ij} . Thus G satisfies

$$\text{conj}(\bar{g}) := \bigwedge_{i,j} [g_j^{-1} w_i g_j = v_{ij}(w_1, \dots, w_s)].$$

For each i there exist i^* and a word u_i such that $g_i = u_i(\bar{h})t_{i^*}(\bar{g})$. Then G satisfies

$$\begin{aligned} \rho(\bar{g}) &:= \bigwedge_{i=1}^r [g_i = u_i(w_1, \dots, w_s)t_{i^*}(\bar{g})] \\ \text{extn}(\bar{g}) &:= \bigwedge_{i,j} [t_i(\bar{g})t_j(\bar{g}) = c_{ij}(w_1, \dots, w_s)t_{s(i,j)}(\bar{g})] \end{aligned}$$

for suitable words c_{ij} ; here $(i, j) \mapsto s(i, j)$ describes the multiplication table of G/N , and $(i, j) \mapsto c_{ij}(\bar{h})$ represents the 2-cocycle defining the extension of N by G/N ; this takes values in $\langle h_1, \dots, h_s \rangle$ because of hypothesis (a).

Now suppose that $y_1, \dots, y_r \in H \in \mathcal{C}$ and that

$$H \models \Phi(\bar{y}) \wedge \tau(\bar{y}) \wedge \rho(\bar{y}) \wedge \text{conj}(\bar{y}) \wedge \text{extn}(\bar{y}). \quad (4)$$

Put $M = \kappa(H; \bar{y})$ and set $k_i = w_i(\bar{y})$ for $i = 1, \dots, s$.

The fact that $H \models \Phi(\bar{y})$ implies that each $k_i \in M$ and that the map sending \bar{h} to \bar{k} extends to an isomorphism $\theta_1 : N \rightarrow M$. Define $\theta : G \rightarrow H$ by

$$(at_i(\bar{g}))\theta = a\theta_1 \cdot t_i(\bar{y}) \quad (a \in N, 1 \leq i \leq m).$$

Then $\tau(\bar{y})$ ensures that θ is a bijection, and using $\text{conj}(\bar{y})$ and $\text{extn}(\bar{y})$, one verifies that θ is a homomorphism; the key point is that $\text{conj}(\bar{g})$ determines the conjugation action of each g_i on N because the h_j generate N topologically and inner automorphisms are continuous, and similarly $\text{conj}(\bar{y})$ determines the action of each y_i on m . This implies that for $b \in N$ and each j ,

$$t_j(\bar{y})^{-1} \cdot b\theta_1 \cdot t_j(\bar{y}) = (t_j(\bar{g})^{-1} \cdot b \cdot t_j(\bar{g}))\theta_1.$$

Finally, $\rho(\bar{y})$ implies that $g_i\theta = y_i$ for each i .

Thus (4) implies that there is an isomorphism $G \rightarrow H$ sending \bar{g} to \bar{y} .

Case (b): Assume now that $Z(N) = 1$. Given a group N with trivial centre, a group Q , and a homomorphism $\gamma : Q \rightarrow \text{Out}(N)$, there is (up to equivalence) at most one extension group G of N by Q such that conjugation in G induces the mapping $\gamma : Q \rightarrow \text{Out}(N)$ ([7, § 5.4, Theorem 2, Remark 1]). So in this case, it suffices to fix N , G/N and the action.

We fix the multiplication table of G/N with

$$\text{quot}(\bar{g}) := \bigwedge_{i,j} \kappa(\bar{g}, t_i t_j t_{s(i,j)}^{-1})$$

(writing t_i in place of $t_i(\bar{g})$ throughout). We redefine ρ as follows:

$$\rho(\bar{g}) := \bigwedge_{i=1}^r \kappa(\bar{g}, t_{i^*} g_i^{-1}).$$

where i^* is defined above. To fix the action, we now set

$$\text{conj}(\bar{g}) := \text{res} \left(\kappa(\bar{g}), \bigwedge_{i,j} \phi_{v(i,j)}(w_1, \dots, w_s, g_j^{-1} w_i g_j) \right),$$

where $v(i, j) = h_i^{g_j}$, and $\phi_{v(i,j)}$ defines $h_i^{g_j}$ in N in terms of \bar{h} .

Now suppose that $y_1, \dots, y_r \in H \in \mathcal{C}$ and that

$$H \models \Phi(\bar{y}) \wedge \tau(\bar{y}) \wedge \rho(\bar{y}) \wedge \text{quot}(\bar{y}) \wedge \text{conj}(\bar{y}).$$

Put $M = \kappa(H; \bar{y})$ and set $k_i = w_i(\bar{y})$ for $i = 1, \dots, s$. As before we have an isomorphism $\theta_1 : N \rightarrow M$ sending \bar{h} to \bar{k} . The map sending $t_i(\bar{g})$ to $t_i(\bar{y})$ for each i induces an isomorphism $\theta_2 : G/N \rightarrow H/M$. Thus we have a diagram of group extensions:

$$\begin{array}{ccccccccc} 1 & \rightarrow & N & \rightarrow & G & \rightarrow & G/N & \rightarrow & 1 \\ & & \downarrow & & & & \downarrow & & \\ 1 & \rightarrow & N & \xrightarrow{\alpha} & H & \xrightarrow{\beta} & G/N & \rightarrow & 1, \end{array}$$

where $\alpha : N \rightarrow M \hookrightarrow H$ and $\beta : H \twoheadrightarrow H/M \rightarrow G/N$ are induced, respectively, by $\theta_1 : N \rightarrow M$ and $\theta_2^{-1} : H/M \rightarrow G/N$, and the vertical arrows represent identity maps. Now $\rho(\bar{g})$ and $\rho(\bar{y})$ ensure that $(Ng_i)\theta_2 = My_i$ for each i . Then using $\text{conj}(\bar{g})$ and $\text{conj}(\bar{y})$ together with Remark (i), we can verify that the two mappings $G/N \rightarrow \text{Out}(N)$ induced by the top extension and the bottom extension are identical. Hence there exists a homomorphism $\theta : G \rightarrow H$ making the diagram commute, and then θ must be an isomorphism since the end maps are bijective.

Finally, because G is finitely generated, Theorem 2.1 (iii) ensures that any group isomorphism $G \rightarrow H$ is a topological isomorphism. \square

REMARK. This argument gives the same result for a class \mathcal{C} of abstract groups, if we add the hypothesis that N (has finite index and) is definable in G .

4. Bi-interpretation

Our definition of bi-interpretation will not be the most general one; instead, we introduce a few ad hoc definitions tailored to the purpose at hand. In particular, the ring is supposed to be *definable* as a *closed* subgroup of the group (rather than just ‘interpretable’ as a subgroup of a power of the group), and the group is supposed to be *definable* as a *closed* subset of some power of the ring.

The purpose is to show that for certain groups built out of rings, finite axiomatizability of the ring is equivalent to that of the group. Because we are interested in profinite objects, some topological tweaks are added. The precise result is Theorem 4.3 below; applications are given in §4.4.

4.1. Interpreting rings in groups

All rings are commutative, with identity. L_{rg} is the first-order language of rings. A ring is *profinite* if it is an inverse limit of finite rings. We also need a slightly weaker version: the ring R is *additively profinite* if its additive group $(R, +)$ is profinite as a group.

A profinite ring R is *FA* (respectively, *strongly FA*) if there is a formula σ of L_{rg} such that (i) $R \models \sigma$ and (ii) if S is a profinite (respectively, additively profinite) ring and $S \models \sigma$, then S is topologically isomorphic to R .

Let R be an additively profinite ring. We say that R is *topologically interpreted* in a profinite group G if there are formulae τ, μ and a tuple of parameters \bar{g} in G with the following property:

- for every profinite group H and tuple \bar{h} from H , the set $\tau(\bar{h}; H)$ is a closed subgroup of H ;
- $\tau(\bar{g}; G)$ becomes a ring $\hat{\tau}(G) = \hat{\tau}(\bar{g}; G)$ with ring addition given by the group operation, and ring multiplication defined by $\text{res}(\mu(\bar{g}), \tau(\bar{g}))$, in the sense that for $x, y, z \in \tau(\bar{g}; G)$,

$$x \cdot y = z \iff G \models \mu(\bar{g}, x, y, z).$$

- $\hat{\tau}(\bar{g}; G)$ is topologically isomorphic to R .

(Here $\hat{\tau}$ stands for (τ, μ) , and we will write $\hat{\tau}(G)$ for $\hat{\tau}(\bar{g}; G)$ when there is no risk of confusion.)

In this situation, there is a formula ρ (depending on τ and μ) such that (i) $G \models \rho(\bar{g})$ and (ii) for any profinite group H and tuple \bar{h} from H , if $H \models \rho(\bar{h})$ then the subgroup $\tau(\bar{h}; H)$ is a ring $S := \hat{\tau}(\bar{h}; H)$ with operations defined as above. (The formula ρ expresses the statements that μ defines a binary operation on S and that the axioms for a commutative ring with identity are satisfied). This ring S will be additively profinite, because $\tau(\bar{h}; H)$ is a profinite group.

We call such an interpretation *strongly topological* if it has the following additional property: for any profinite group H and tuple \bar{h} from H , if $H \models \rho(\bar{h})$, then the ring $S = \hat{\tau}(\bar{h}; H)$ is actually a profinite ring: that is, the multiplication map from $S \times S$ to S is continuous.

For each formula ϕ of L_{rg} there is a formula ϕ^* of L_{gp} such that

$$\hat{\tau}(\bar{h}; H) \models \phi \iff H \models \phi^*(\bar{h}),$$

obtained in the obvious way by translating each atomic L_{rg} subformula of ϕ into an equivalent L_{gp} formula.

LEMMA 4.1. *Let R be an FA profinite ring R . Suppose that R is topologically interpreted in a profinite group G , and assume further either that R is strongly FA, or that the interpretation is strongly topological. Then there is an L_{gp} formula $\psi(\bar{y})$ such that $G \models \psi(\bar{g})$ (where \bar{g} is as above), and for each profinite group H and tuple \bar{h} , if $H \models \psi(\bar{h})$, then $\hat{\tau}(\bar{h}; H)$ is a ring topologically isomorphic to R .*

Indeed, it suffices to set $\psi(\bar{y}) = \rho(\bar{y}) \wedge \sigma_R^*(\bar{y})$.

REMARK 4.2. The ring R has the property ‘2 is not a zero divisor’ if and only if $\hat{\tau}(G)$ satisfies a certain formula $\phi(\bar{g})$. In this case, we make the convention that ρ implies ϕ^* . If H as above now satisfies $\rho(\bar{h})$, then 2 is not a zero divisor in the ring $S = \hat{\tau}(\bar{h}; H)$, and then in S the identity

$$2xy = (x + y)^2 - x^2 - y^2 \tag{5}$$

determines xy . Since addition is continuous, by definition of the topology on S , to establish continuity of multiplication, it will suffice to show that the map $x \mapsto x^2$ on S is continuous.

Thus if 2 is not a zero divisor in R , for the interpretation to be topological, it suffices to have: whenever H as above satisfies $\rho(\bar{h})$, the squaring map from $S = \hat{\tau}(\bar{h}; H)$ to S is continuous.

4.2. Interpreting groups in rings

Let G be a profinite group. We say that G is *topologically interpreted* in a profinite ring R if, for some d , there are L_{rg} formulae α_1, α_2 such that

- for every profinite ring T , the subset $\alpha_1(T^{(d)})$ is closed in $T^{(d)}$;
- $\alpha_1(R^{(d)})$ is a group $\hat{\alpha}(R)$, with operation defined by

$$\bar{a} \cdot \bar{b} = \bar{c} \iff R \models \alpha_2(\bar{a}, \bar{b}, \bar{c});$$

- G is topologically isomorphic to $\hat{\alpha}(R)$ (with the subspace topology induced by $\bar{\alpha}(R) \subseteq R^{(d)}$).

As in the preceding subsection, there is a formula α_3 (depending on α_1, α_2) such that (i) $R \models \alpha_3$ and (ii) for any profinite ring T , if $T \models \alpha_3$, then $\alpha_1(T^{(d)})$ is a group $\hat{\alpha}(T)$ with the operation defined as above.

The interpretation is *strongly topological* if, in addition, for every profinite ring T , the group operation defined by α_2 on $\hat{\alpha}(T)$ is continuous; in this case, $\hat{\alpha}(T)$ will be a profinite group, as the topology it inherits from $T^{(d)}$ is compact and totally disconnected.

For example, if $\mathfrak{G} \leq \mathrm{SL}_n$ is an algebraic group defined over \mathbb{Z} , then $\mathfrak{G}(R)$ is interpreted in R for any ring R ; here $d = n^2$, α_1 expresses the defining equations of \mathfrak{G} , and α_2 is the formula for matrix multiplication (which is continuous when R is a topological ring).

Now let G be a profinite group and R a profinite ring. We say that G and R are *topologically bi-interpretable* in the following circumstances:

- R is topologically interpreted in G by τ and G is topologically interpreted in R by $\bar{\alpha}$, as above;
- identifying G with $\hat{\alpha}(R) \subseteq R^{(d)}$ and R with $\hat{\tau}(G) \subseteq G$ gives two mappings

$$\theta : G = \hat{\alpha}(R) \rightarrow \hat{\alpha}(\hat{\tau}(G)) \hookrightarrow G^{(d)},$$

$$\theta' : R = \hat{\tau}(G) \rightarrow \hat{\tau}(\hat{\alpha}(R)) \hookrightarrow R^{(d)};$$

then θ is L_{gp} -definable and θ' is L_{rg} -definable.

Note that in this situation, θ is a group isomorphism from G to $\hat{\alpha}(\hat{\tau}(G))$, and θ' is a ring isomorphism from R to $\hat{\tau}(\hat{\alpha}(R))$.

While first-order language may suffice to determine the algebraic structure of a group, it cannot say anything about the topology. Recall that the profinite group G is *algebraically rigid* if every profinite group abstractly isomorphic to G is topologically isomorphic to G . This holds in particular if G is strongly complete (that is, every subgroup of finite index is open), but the conditions are not equivalent; in §4.4 we will exhibit groups that are FA, and therefore algebraically rigid, but not strongly complete.

Analogously, a profinite ring is said to be algebraically rigid if its topology is uniquely determined by its algebraic structure.

THEOREM 4.3. *Let G be a profinite group, R a profinite ring and suppose that G and R are topologically bi-interpretable.*

- Suppose that R is FA. Assume that G is algebraically rigid, and that the interpretation of R in G is strongly topological or that R is strongly FA. Then G is FA in profinite groups.*
- Suppose that G is FA in profinite groups. Assume that R is algebraically rigid, and that the interpretation of G in R is strongly topological. Then R is FA.*

Proof. (i) Let $\psi(\bar{y})$ be the formula provided by Lemma 4.1. Given that θ is definable, the statement that θ is an isomorphism from G onto $\hat{\alpha}(\hat{\tau}(G))$ can be expressed by a certain L_{gp} formula $\Theta(\bar{y})$, depending on a straightforward way on $\bar{\alpha}$ and $\bar{\tau}$. Then $G \models \Sigma_G(\bar{y})$ where

$$\Sigma_G(\bar{y}) \equiv \psi(\bar{y}) \wedge \Theta(\bar{y}).$$

Now let H be a profinite group and suppose that $H \models \Sigma_G(\bar{h})$ for some tuple \bar{h} in H . As $H \models \psi(\bar{h})$, the ring $S = \hat{\tau}(\bar{h}; H)$ is topologically isomorphic to R . In particular, S is a profinite ring and $S \models \alpha_3$, so $\hat{\alpha}(S)$ is a group with operation defined by α_2 . As $H \models \Theta(\bar{h})$, the formula defining θ establishes a group isomorphism

$$\tilde{\theta} : H \rightarrow \hat{\alpha}(S) \cong \hat{\alpha}(R) \cong \hat{\alpha}(\hat{\tau}(\bar{y}; G)).$$

Then $\theta\tilde{\theta}^{-1}$ is a group isomorphism $G \rightarrow H$. As H is a profinite group and G is algebraically rigid, the groups are topologically isomorphic.

Thus $\exists \bar{y}. \Sigma_G(\bar{y})$ determines G as a profinite group.

- This is similar, swapping the roles of G and R . □

4.3. Some profinite rings

Familiar examples of profinite rings are the complete local rings with finite residue field: if R is one of these, with (finitely generated) maximal ideal \mathfrak{m} and finite residue field $R/\mathfrak{m} \cong \mathbb{F}_q$, then

R is the inverse limit of the finite rings R/\mathfrak{m}^n ($n \in \mathbb{N}$). We will keep this notation throughout this section, and set $p = \text{char}(R/\mathfrak{m})$, $q = p^f$.

The fundamental structure theorem of I. S. Cohen describes most of these rings quite explicitly. R is said to be *regular* if \mathfrak{m} can be generated by d elements where $d = \dim R$ is the Krull dimension of R . Also R is said to be *unramified* if either $pR = 0$ or $p \cdot 1_R \notin \mathfrak{m}^2$. (For background on regular local rings, see, for example, [6, § 10.3]. The ‘unramified’ condition serves to avoid complications in the case of unequal characteristic.)

A basic example is the complete discrete valuation ring

$$\mathfrak{o}_q = \mathbb{Z}_p[\zeta_{q-1}],$$

where ζ_{q-1} is a primitive $(q-1)$ th root of unity; this is the ring of integers in the unique unramified extension of degree f over \mathbb{Q}_p . According to [3, Theorem 11, Corollary 2], this is the only complete local domain R of characteristic 0 with maximal ideal pR and residue field \mathbb{F}_q .

THEOREM 4.4 [3, Theorem 15]. *Let R be a regular, unramified complete local ring with residue field \mathbb{F}_q and dimension $d \geq 1$. Then one of the following holds.*

- (a) $R \cong \mathbb{F}_q[[t_1, \dots, t_d]]$.
- (b) $R \cong \mathfrak{o}_q[[t_1, \dots, t_{d-1}]]$.

The point is that R is determined up to isomorphism by its characteristic and the parameters d, q ; it is then hardly surprising that such a ring is FA. (The same very likely holds also in the ramified case, when R is an ‘Eisenstein extension’ of the ring specified in (b); we shall not go into this here, but it can probably be covered by suitably extending the arguments below.)

THEOREM 4.5. *Let R be a regular, unramified complete local ring with finite residue field. Then R is FA.*

Proof. Until further notice S denotes an arbitrary ring. Each of the statements ‘ S is an integral domain’, ‘ $\text{char} S = 0$ ’, ‘ $\text{char} S = p$ ’ is easily expressible by a sentence of L_{rg} . There are formulae μ , φ_q and ρ such that

- $S \models \mu(a_1, \dots, a_d)$ if and only if $S \setminus \sum_1^d a_i S$ consists of units;
- $S \models \varphi_q(a_1, \dots, a_d)$ if and only if $|S / \sum_1^d a_i S| = q$;
- $S \models \rho(a_1, \dots, a_d)$ if and only if for each i , the element a_i is not a zero divisor modulo $\sum_{j=1}^{i-1} a_j S$ (the zero ideal when $i = 1$).

Put

$$\Sigma_q(\bar{x}) := \mu(\bar{x}) \wedge \varphi_q(\bar{x}) \wedge \rho(\bar{x}) \wedge \forall y, z. (yz = 0 \longrightarrow (y = 0 \vee z = 0)).$$

Now suppose that S satisfies $\Sigma_q(a_1, \dots, a_d)$, and set $I = \sum_1^d a_i S$. Then S is a local domain with maximal ideal I and residue field $S/I \cong \mathbb{F}_q$. The sentence $\rho(a_1, \dots, a_d)$ implies that $\dim S$ is at least d , and hence that $\dim S = d$ ([3, Theorem 14]), so S is regular.

Now we separate cases.

Case (a): $R \cong \mathbb{F}_q[[t_1, \dots, t_d]]$. Then R satisfies

$$\Sigma_{q,p}(t_1, \dots, t_d) \equiv \Sigma_q(t_1, \dots, t_d) \wedge \forall y. (py = 0).$$

Suppose that S is a profinite ring and that $S \models \Sigma_{q,p}(s_1, \dots, s_d)$ for some $s_1, \dots, s_d \in S$. Put $I = \sum_1^d s_i S$. Then S is a regular, unramified local domain of dimension d and characteristic p , with maximal ideal I and residue field \mathbb{F}_q .

Case (b): $R \cong \mathfrak{o}_q[[t_1, \dots, t_{d-1}]]$. Note that 1_R is a definable element, by the formula $\forall y.(xy = y)$. Now $\mathfrak{m} = pR + \sum_1^{d-1} t_i R$. The fact that $p1_R \notin \mathfrak{m}^2$ is expressible by

$$\tau_p(t_1, \dots, t_{d-1}) := \forall z_{ij}, y_i, x. \left(\sum_{i,j} t_i t_j z_{ij} + \sum_i p t_i y_i + p^2 x \neq p1_R \right).$$

Thus R satisfies

$$\Sigma_{q,0}(t_1, \dots, t_{d-1}) \equiv \Sigma_q(p1_R, t_1, \dots, t_{d-1}) \wedge \exists y.(py \neq 0) \wedge \tau_p(t_1, \dots, t_{d-1}).$$

Suppose now that S is profinite ring and that $S \models \Sigma_{q,0}(s_1, \dots, s_{d-1})$ for some $s_1, \dots, s_{d-1} \in S$. Put $I = pS + \sum_1^{d-1} s_i S$. Then again, S is a regular, unramified local domain of dimension d , with maximal ideal I and residue field \mathbb{F}_q , and S has characteristic 0.

Conclusion. Since ring multiplication is continuous, I is compact and therefore closed in S ; as it has finite index, I is open. The same argument shows that I^n is open for each n (each I^{n-1}/I^n is finite because it is finitely generated as a module for S/I). Now S is the inverse limit of finite rings S/J_α , where $\{J_\alpha\}$ is a family of open ideals that form a base for the neighbourhoods of 0. For each α the ring S/J_α is finite with Jacobson radical I/J_α , so for some n we have $I^n \subseteq J_\alpha$. Hence the system $\{I^n \mid n \in \mathbb{N}\}$ is also a base for the neighbourhoods of 0. Thus the given profinite topology is the I -adic topology, and as S is complete for the former, it is complete as a local ring.

Now Theorem 4.5 shows that $S \cong \mathbb{F}_q[[t_1, \dots, t_d]]$ (Case a) or $S \cong \mathfrak{o}_q[[t_1, \dots, t_{d-1}]]$ (Case b). \square

REMARKS. (i) If R is one-dimensional and of characteristic 0, that is, $R \cong \mathfrak{o}_q$ for some q , then in fact R is *strongly FA*. Indeed, in Case (b) above we merely need to assume that S is additively profinite, for S satisfies $\Sigma_{q,0}$ with $d = 1$, which asserts that $I = pS$. As multiplication by p is continuous on the profinite additive group $(S, +)$, the given argument shows that the powers of I define the topology, which again implies that S is complete.

(ii) We can deduce: *If π is a finite set of primes, then \mathbb{Z}_π is strongly FA.* Taking a set of pairwise orthogonal idempotents e_1, \dots, e_k as parameters, we find that R is determined by a first-order statement which asserts:

- $R = e_1 R \times \dots \times e_k R$ and $e_i R \cong \mathbb{Z}_{p_i}$ for $i = 1, \dots, k$.

(iii) We used the well-known fact that in a finite (more generally, Artinian) ring, the Jacobson radical is nilpotent. It is worth stating an immediate consequence.

LEMMA 4.6. *Let R be a complete local ring with finite residue field. Then every ideal of finite index in R is open.*

In other words, rings of this kind are ‘strongly complete’. In particular, they are algebraically rigid.

4.4. Some worked examples

Now we are ready to prove Theorem 1.4.

If R is a complete, unramified regular local ring with finite residue field then each of the profinite groups $\mathrm{Af}_1(R)$, $\mathrm{SL}_2(R)$ is FA in the class of profinite groups, assuming in the second case that the residue characteristic is odd.

This will follow from Theorem 4.5 once we show that the hypotheses of Theorem 4.3(i) are satisfied.

Let R be a complete local domain, with maximal ideal \mathfrak{m} and finite residue field R/\mathfrak{m} . Then R is a profinite ring, a base for the neighbourhoods of 0 being the powers of \mathfrak{m} . In particular, $R^* = R \setminus \mathfrak{m}$ is an open subset, being the union of finitely many additive cosets of \mathfrak{m} .

The semi-direct product $(R, +) \rtimes R^*$ can be identified with the affine group

$$\mathrm{Af}_1(R) = \begin{pmatrix} 1 & R \\ 0 & R^* \end{pmatrix} < \mathrm{GL}_2(R).$$

THEOREM 4.7. *If R is FA, then the group $\mathrm{Af}_1(R)$ is FA in profinite groups.*

Proof. We will verify the hypotheses of Theorem 4.3(i). Write $G = \mathrm{Af}_1(R)$. We have a visible topological interpretation of G as a subset of $M_2(R) = R^{(4)}$.

Define the following elements of G , where $1 = 1_R$ and $\lambda \in R$, $\xi \in R^*$:

$$u(\lambda) = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \quad h(\xi) = \begin{pmatrix} 1 & 0 \\ 0 & \xi \end{pmatrix},$$

and fix the parameters $u := u(1)$, $h := h(r)$ for some $r \in R^*$, $r \neq 1$. Then $G = H \cdot U$ where

$$U := u(R) = C_G(u)$$

$$H := h(R^*) = C_G(h)$$

are both definable subgroups. For technical reasons, we want to encode the fact that H is abelian and normalizes U ; to this end, we define

$$\kappa(u, h, x) \equiv \forall y. ([\mathrm{com}(y, h) \rightarrow \mathrm{com}(y, x)] \wedge [\mathrm{com}(y, u) \longleftrightarrow \mathrm{com}(y^x, u)]);$$

and note that $H = \kappa(u, h; G)$.

We will frequently use the identity

$$u(\lambda)^{h(\xi)} = u(\xi\lambda).$$

All formulae are supposed to involve the parameters u , h , which we will sometimes omit for brevity.

Claim 1. *The ring R has a strongly topological interpretation in G , given by $u : R \rightarrow U = \mathrm{com}(u; G)$.*

Certainly $\mathrm{com}(u, -)$ defines a closed subgroup in any profinite group, as it defines a centralizer. The map $u : R \rightarrow U$ is a topological isomorphism from $(R, +)$ to U . It becomes a topological ring isomorphism if one defines

$$u(\alpha) \cdot u(\beta) = u(\alpha\beta).$$

We need to provide an L_{gp} formula μ such that for $x, y, z \in U$,

$$x \cdot y = z \iff G \models \mu(u, h; x, y, z).$$

Let $v = u(\beta) \in U$. If $\beta \in R^*$, then $v = u^{h(\beta)}$, while if $\beta \in \mathfrak{m}$, then $\beta + 1 \in R^*$ and $v = [u, h(\beta + 1)]$. Thus $v_1 \cdot v_2 = v_3$ if and only if there exist $x, y \in H$ such that one of the following holds:

$$\begin{aligned} v_1 &= u^x, \quad v_2 = u^y, \quad v_3 = u^{xy}, \quad \text{or} \\ v_1 &= u^x, \quad v_2 = [u, y], \quad v_3 = [u^x, y], \quad \text{or} \\ v_1 &= [u, x], \quad v_2 = u^y, \quad v_3 = [u^y, x], \quad \text{or} \\ v_1 &= [u, x], \quad v_2 = [u, y], \quad v_3 = [[u, y], x]. \end{aligned}$$

This can be expressed by a first-order formula since H is definable.

To say that the interpretation is strongly topological means the following: if a profinite group \tilde{G} satisfies the appropriate sentence ρ with parameters \tilde{u}, \tilde{h} , which in particular implies that $\mu(\tilde{u}, \tilde{h}; -)$ defines a binary operation \cdot on $\tilde{U} = \text{com}(\tilde{u}; \tilde{G})$, this operation is continuous.

We will write u in place of \tilde{u} for aesthetic reasons.

Let N be an open normal subgroup of \tilde{G} . If $u^x \equiv u^{x'} \pmod{N}$ and $u^y \equiv u^{y'} \pmod{N}$ with $x, x', y, y' \in \tilde{H}$, then

$$u^{xy} \equiv u^{x'y} = u^{yx'} \equiv u^{y'x'} = u^{x'y'} \pmod{N}$$

since \tilde{H} is abelian. Similar congruences hold if u^x is replaced by $[u, x]$ or u^y is replaced by $[u, y]$. Thus in all cases we see that if $v_i \equiv v'_i \pmod{N}$ for $i = 1, 2$ and $v_1 \cdot v_2 = v_3$, $v'_1 \cdot v'_2 = v'_3$, then $v_3 \equiv v'_3 \pmod{N}$. Thus the operation \cdot is continuous as required.

Claim 2. The map θ sending $g = \begin{pmatrix} 1 & \lambda \\ 0 & \xi \end{pmatrix} \in G$ to $(u(1), u(\lambda), u(0), u(\xi)) \in G^{(4)}$ is definable.

We have $g = \tilde{h}(g)\tilde{u}(g)$ where $\tilde{u}(g) = u(\lambda)$ and $\tilde{h}(g) = h(\xi)$. Also

$$\{\tilde{u}(g)\} = Hg \cap U$$

$$\{\tilde{h}(g)\} = gU \cap H.$$

As H and U are definable subsets of G , the mappings $\tilde{u} : G \rightarrow G$ and $\tilde{h} : G \rightarrow G$ are both definable. Hence so is θ , because $u(1)$ is the parameter u , $u(0) = 1_G$, $u(\lambda) = \tilde{u}(g)$ and

$$u(\xi) = u^{h(\xi)} = u^{\tilde{h}(g)}.$$

Claim 2bis. The map sending $\lambda \in R$ to $(1, \lambda, 0, 1) \in R^{(4)}$ is definable. Obviously.

Claim 3. G is algebraically rigid. This follows from the stronger result Proposition 4.8, below, and completes the proof of Theorem 4.7. \square

PROPOSITION 4.8. *Let R be a complete local domain with finite residue field. Then every group isomorphism from $\text{Af}_1(R)$ to a profinite group is continuous (and therefore a topological isomorphism).*

Proof. $G = \text{Af}_1(R) = U \rtimes H$ where $U = u(R)$ and $H = h(R^*)$ (notation as above). A base for the neighbourhoods of 1 in G is the family of subgroups

$$G(n) := H(n)U(n), \quad n \geq 1, \text{ where}$$

$$U(n) = u(\mathfrak{m}^n), \quad H(n) = h(1 + \mathfrak{m}^n).$$

Let $\theta : G \rightarrow \tilde{G}$ be a group isomorphism, where \tilde{G} is a profinite group. Set $\tilde{U} = U\theta$ and $\tilde{H} = H\theta$. As R is an integral domain, $H = C_G(H)$, and so $\tilde{H} = C_{\tilde{G}}(\tilde{H})$ is closed in \tilde{G} . Similarly, $\tilde{U} = C_{\tilde{G}}(\tilde{U})$ is closed. We will show that θ^{-1} is continuous (which, for an isomorphism of profinite groups, is equivalent to being a homeomorphism).

Suppose that $\tilde{N} \triangleleft_o \tilde{G}$. Then $N := \tilde{N}\theta^{-1}$ is a normal subgroup of finite index in G , so $N \cap U = u(B)$ for some additive subgroup B of finite index in R . If $r \in R^*$, then

$$u(Br) = u(B)^{h(r)} = u(B),$$

so $B = Br$, and as $R = R^* \cup (R^* - 1)$ it follows that B is an ideal of R ; therefore, $B \supseteq \mathfrak{m}^n$ for some n , by Lemma 4.6. Thus $U(n) \subseteq \tilde{N}\theta^{-1}$.

It follows that $\theta|_U : U \rightarrow \tilde{U}$ is a continuous isomorphism, and consequently a homeomorphism. This in turn implies that each $\tilde{U}(n) := U(n)\theta$ is open in \tilde{U} .

Now for each n ,

$$H(n)\theta = C_H(U/U(n))\theta = C_{\tilde{H}}(\tilde{U}/\tilde{U}(n))$$

is open in \tilde{H} since $\tilde{U}(n)$ is open in \tilde{U} (here $C_H(U/U(n))$ denotes the kernel of the conjugation action of H on the factor $U/U(n)$). Thus

$$G(n)\theta = H(n)\theta.U(n)\theta$$

is closed in \tilde{G} , hence open as it has finite index. It follows that θ^{-1} is continuous. \square

REMARK. It is not usually the case that $G = \text{Af}_1(R)$ is strongly complete. In fact, G is strongly complete if and only if its open subgroup $G(1)$ (the principal congruence subgroup $\text{mod } \mathfrak{m}$) is, and as $G(1)$ is a pro- p group this holds if and only if $G(1)$ is finitely generated (for example, by [37, Theorem 2]: a pro- p group that is not f.g. maps onto an infinite elementary abelian p -group, and so has uncountably many subgroups of index p). This in turn holds if and only if the multiplicative subgroup $T := 1 + \mathfrak{m}$ of R^* is finitely generated as a pro- p group. Now T is finitely generated if and only if T/T^p is finite; of the rings listed in Theorem 4.4, only the ones denoted \mathfrak{o}_q have this property.

For the next example, let R be as above, and **assume that $q = |R/\mathfrak{m}|$ is odd**. Every element of R/\mathfrak{m} is then a difference of two squares, and as R is complete and q is odd it follows that every element of R is of the form x^2 or $x^2 - y^2$ with $x, y \in R^*$. Fix $r \in R$ such that $r + \mathfrak{m}$ is a generator for $(R/\mathfrak{m})^*$ and $r^4 \neq 1$.

THEOREM 4.9. *If R is FA, then the group $\text{SL}_2(R)$ is FA in profinite groups.*

Proof. Put $G = \text{SL}_2(R)$. We will show that all hypotheses of Theorem 4.3 are satisfied, with a strongly topological interpretation of R in G .

Define the following elements of G , where $1 = 1_R$ and $\lambda \in R$:

$$\begin{aligned} u(\lambda) &= \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \quad v(\lambda) = \begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix}, \\ h(\lambda) &= \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix} \quad (\lambda \in R^*), \\ w &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned} \tag{6}$$

We take $u := u(1)$, $u' := u(r)$, $v := v(1)$, w and $h := h(r)$ as parameters and write

$$\bar{u} = (u, u', v, w, h).$$

‘Definable’ will mean definable with these parameters. All formulae below are supposed to include these parameters, which we mostly omit for brevity. We will use without special mention the identities

$$u(\lambda)^w = v(\lambda),$$

$$u(\lambda)^{h(\mu)} = u(\lambda\mu^2).$$

Write $U = u(R)$, $V = v(R)$, $H = h(R^*)$. Write $\pm U = \langle \pm 1 \rangle \cdot U$, etc. Then $\pm U = C_G(u)$, $\pm V = C_G(v)$ are definable subgroups of G .

To show that U is definable, observe first that no element of $-U$ is conjugate to an element of U . On the other hand, we shall see below that each element of U takes one of the forms u^x or $u^x u^{-y}$ ($x, y \in H$). Thus $U = \rho(G)$ where

$$\rho(s) := \text{com}(s, u) \wedge \exists x, y. (s = u^x \vee s = u^x u^{-y}).$$

Note that ρ will define a closed subset in any profinite group, since centralizers and conjugacy classes are closed. Using Lemma 4.10 below, we now adjust ρ to a new formula ρ^* , such that $U = \rho^*(G)$, and for any profinite group A , the subset $\rho^*(A)$ is a closed subgroup.

Then $V = U^w$ is also definable, as is the subgroup $H = C_G(h)$; for technical reasons, we want to encode the fact that H is abelian and normalizes U ; thus, $H = \eta(G)$ where

$$\eta(\bar{u}, x) \equiv \forall y. (\text{com}(h, y) \rightarrow \text{com}(x, y) \wedge \rho^*(y) \longleftrightarrow \rho^*(y^x)).$$

Claim 1. The map $u : R \rightarrow U = \rho^(G)$ gives a strongly topological interpretation of R in G .*

We adapt the method used in [17, proof of Theorem 3.2]. The map $u : R \rightarrow U$ is a topological isomorphism from $(R, +)$ to U . It becomes a ring isomorphism if one defines

$$u(\alpha) \cdot u(\beta) = u(\alpha\beta). \quad (7)$$

Now we need to provide an L_{gp} formula μ such that for $v_1, v_2, v_3 \in U$,

$$v_1 \cdot v_2 = v_3 \iff G \models \mu(v_1, v_2, v_3).$$

If $\alpha = \xi^2$ (respectively, $\xi^2 - \eta^2$) with $\xi, \eta \in R^*$, then $u(\alpha) = u^x$ (respectively, $u^x u^{-y}$) where $x = h(\xi)$, $y = h(\eta)$, and

$$u(\alpha^2) = u^{x^2}, \text{ respectively, } u^{x^2} u^{-2xy} u^{y^2}.$$

Let $\text{sq}(v_1, v_2)$ be the formula asserting that there exist x, y in H such that

$$\left[v_1 = u^x \wedge v_2 = u^{x^2} \right] \vee \left[v_1 = u^{-y} u^x \wedge v_2 = u^{x^2} u^{-2xy} u^{y^2} \right].$$

One verifies easily that this holds if and only if $v_2 = v_1 \cdot v_1$ in the sense of (7). Now in view of (5) we can take $\mu(v_1, v_2, v_3)$ to assert that there exist a and b such that

$$\text{sq}(v_1, a) \wedge \text{sq}(v_2, b) \wedge \text{sq}(v_1 v_2, a v_3^2 b).$$

To complete the proof of Claim 1, it remains to establish that the interpretation is strongly topological. In view of Remark 4.2, it will suffice to show that $\text{sq}(v_1, v_2)$ defines a continuous map — not just on U but also on any profinite group arising as $\rho^*(\bar{u}^*; G^*)$ where G^* satisfies the appropriate sentences (which include one asserting that $\text{sq}(v_1, v_2)$ does define a mapping). For simplicity (‘by abuse of notation’) we keep the notation attached to G , but will not use any special properties of G . The fact that $H = \eta(\bar{u}; G)$ is abelian and normalizes U is now implied by the definition of η .

If N is an open normal subgroup of G , $u \in U$ and $x, y, x', y' \in H$, then $u^x \equiv u^{x'} \pmod{N}$ implies

$$u^{x^2} \equiv u^{x'^2} = u^{xx'} \equiv u^{x'^2} \pmod{N}.$$

Similarly, $u^{-y} u^x \equiv u^{-y'} u^{x'} \pmod{N}$ implies

$$\begin{aligned} u^{x^2} u^{-2xy} u^{y^2} &= (u^{-y} u^x)^x (u^{-x} u^y)^y \\ &\equiv (u^{-y'} u^{x'})^x (u^{-x'} u^{y'})^y = (u^x u^{-y})^{x'} (u^{-x} u^y)^{y'} \\ &\equiv (u^{x'} u^{-y'})^{x'} (u^{-x'} u^{y'})^{y'} = u^{x'^2} u^{-2x'y'} u^{y'^2}. \end{aligned}$$

Given that $\text{sq}(v_1, v_2)$ defines a map, it follows that if $v_1 \equiv v'_1 \pmod{N}$ and $\text{sq}(v_1, v_2)$ and $\text{sq}(v'_1, v'_2)$ hold, then $v_2 \equiv v'_2 \pmod{N}$; this map is therefore continuous as required.

Claim 1 is now established.

Claim 2. The map θ from G to $G^{(4)}$ given by

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} u(a) & u(b) \\ u(c) & u(d) \end{pmatrix} \in \text{M}_2(U) \subseteq G^{(4)}$$

is definable.

To begin with, we partition G as $G_1 \dot{\cup} G_2$ where

$$G_1 = \{g \in G \mid g_{11} \in R^*\}$$

$$G_2 = \{g \in G \mid g_{11} \in \mathfrak{m}\}.$$

If $g \in G_1$, then $g = \tilde{v}(g)\tilde{h}(g)\tilde{u}(g)$ where

$$\tilde{v}(g) = v(-a^{-1}c) \in V,$$

$$\tilde{h}(g) = h(a^{-1}) \in H,$$

$$\tilde{u}(g) = u(a^{-1}b) \in U.$$

This calculation shows that in fact $G_1 = VHU$, so G_1 is definable; these three functions on G_1 are definable since

$$x = \tilde{v}(g) \iff x \in V \cap HUG,$$

$$y = \tilde{u}(g) \iff y \in U \cap HVg,$$

$$z = \tilde{h}(g) \iff z \in H \cap VgU.$$

If $g \in G_2$, then $gw \in G_1$ since a and b cannot both lie in \mathfrak{m} , and then

$$g = \tilde{v}(gw)\tilde{h}(gw)\tilde{u}(gw)w^{-1}.$$

Define $\alpha : V \rightarrow U$ by $x\alpha = x^w$. Then $v(\lambda)\alpha = u(\lambda)$.

Define $\beta : H \rightarrow U \times U$ as follows. Let $(x, y_1, y_2) \in H \times U \times U$. Then $x\beta = (y_1, y_2)$ if and only if

$$\exists t \in H. ((x = t^2 \wedge y_1^t = u \wedge y_2 = u^t) \vee (x = t^2 h \wedge y_1^{th} = u' \wedge y_2 = u'^t)).$$

This decodes as $h(\lambda)\beta = (u(\lambda^{-1}), u(\lambda))$.

Now we construe $M_2(U)$ as a ring by transferring the ring operations from R to U via Claim 1. As θ is a group homomorphism, we find that for $g \in G_1$,

$$g\theta = \begin{pmatrix} 1 & 0 \\ -\tilde{v}(g)\alpha & 1 \end{pmatrix} \cdot \begin{pmatrix} \tilde{h}(g)\beta_1 & 0 \\ 0 & \tilde{h}(g)\beta_2 \end{pmatrix} \cdot \begin{pmatrix} 1 & \tilde{u}(g) \\ 0 & 1 \end{pmatrix}; \quad (8)$$

while if $g \in G_2$, then

$$g\theta = (gw)\theta \cdot w^{-1}.$$

As matrix multiplication in $M_2(U)$ is L_{gp} definable in $G^{(4)}$, it follows that θ is definable.

This completes the proof of Claim 2.

Claim 2bis. The map sending $\lambda \in R$ to $(1, \lambda, 0, 1) \in R^{(4)}$ is definable. Obviously.

Claim 3. G is algebraically rigid because it is strongly complete. This is presumably well known, but we include a proof in Proposition 4.11 below.

The theorem follows. \square

LEMMA 4.10. *Let G be a group and ρ a formula such that $\rho(G)$ is a subgroup. Then there is a formula ρ^* such that $\rho^*(G) = \rho(G)$, and for any group H the subset $\rho^*(H)$ is a subgroup. If ρ defines a closed subset in every profinite group, then so does ρ^* .*

Proof. Put $\rho_1(x) = \rho(x) \vee (x = 1)$, and set

$$\rho^*(x) := \rho_1(x) \wedge \forall y (\rho_1(y) \longrightarrow (\rho_1(xy) \wedge \rho_1(x^{-1}y))).$$

One sees easily that this has the required properties. \square

PROPOSITION 4.11. *Let R be a complete local domain, with maximal ideal \mathfrak{m} and finite residue field R/\mathfrak{m} of odd characteristic. Then the profinite group $\mathrm{SL}_2(R)$ is finitely generated, hence strongly complete.*

Proof. Let $G(1)$ denote the principal congruence subgroup modulo \mathfrak{m} in $G = \mathrm{SL}_2(R)$. Then $G(1)$ is a so-called *R-perfect group*: that is, an R -analytic pro- p group whose associated Lie algebra is perfect (see [5, Exercise 13.10 on page 352]). Now Corollary 3.4 of [20] asserts that such pro- p groups are finitely generated (cf. [5, Proposition 13.29(i)]), hence strongly complete [5, Theorem 1.17].

As $G(1)$ is open in G it follows that G has both properties. \square

REMARKS. (i) The Lubotzky–Shalev results [20] hold for $\mathrm{SL}_n(R)$, any $n \geq 2$ (also when $\mathrm{char}(R/\mathfrak{m}) = 2$ provided $n \geq 3$).

(ii) The strong completeness can also be inferred directly from the fact that these groups have the *congruence subgroup property*: every subgroup of finite index contains a principal congruence subgroup modulo \mathfrak{m}^k for some k (this follows from [16, Satz 2]).

5. Profinite groups of finite rank

In the context of profinite groups, a ‘generating set’ will always mean a *topological* generating set. From a first-order point of view, the nicest profinite groups are the finitely generated pro- p groups, where p is a prime. These have the following special property: a finite generating set can be recognized in a definable finite quotient (as we shall see below). So for such groups, being generated by d elements is a first-order property. This is the key to most of our main results; it does not hold for f.g. profinite groups in general as we point out in Proposition 5.4, but it does hold for groups in the larger class \mathcal{C}_π of *pronilpotent pro- π groups*, where π is any finite set of primes: a group G is in \mathcal{C}_π if and only if it is a direct product

$$G = G_{p_1} \times \cdots \times G_{p_k},$$

where $\pi = \{p_1, \dots, p_k\}$ and G_{p_i} is a pro- p_i group for each i . To save repetition, we make the convention that π will always denote a finite set of primes.

5.1. Some preliminaries and an example

For basic facts about profinite groups, see [5, Chapter 1] and the earlier chapters of [40]. Besides the language L_{gp} of group theory, we will consider the language

- L_π : the language L_{gp} augmented by unary function symbols P_λ , one for each $\lambda \in \mathbb{Z}_\pi = \prod_{p \in \pi} \mathbb{Z}_p$; for $g \in G$, $P_\lambda(g)$ is interpreted as the profinite power g^λ .

For a profinite group G ,

- $d(G)$ is the minimal size of a (topological) generating set for G ,
- $\mathrm{rk}(G) = \sup\{d(H) \mid H \leq_c G\}$.

This is the *rank* of G (sometimes called *Prüfer rank*). The pro- p groups of finite rank are of particular interest, being just those that are *p-adic analytic*; see [5, Chapter 3] (which includes several equivalent definitions of rank). On the face of it, having a particular finite rank is not a first-order property (the definition involves quantifying over subgroups); the following result shows that the rank, if finite, can be more or less specified by a first-order sentence.

PROPOSITION 5.1. *For each positive integer r , there is a sentence ρ_r such that for a pro- p group G ,*

$$\mathrm{rk}(G) \leq r \implies G \models \rho_r \implies \mathrm{rk}(G) \leq r(2 + \log_2(r)).$$

We omit the proof, an application of the techniques described below.

We fix the notation

$$q(\pi) = p_1 \dots p_k,$$

$$q'(\pi) = 2^\varepsilon q(\pi),$$

where $\varepsilon = 0$ if $2 \notin \pi$, $\varepsilon = 1$ if $2 \in \pi$.

A sharper version of Theorem 2.1(ii) holds in some cases.

LEMMA 5.2. *Let $G = \overline{\langle a_1, \dots, a_d \rangle}$ be a pronilpotent group. Then*

$$G' = [a_1, G] \dots [a_d, G], \tag{9}$$

a closed subgroup of G .

Proof. The set $X = [a_1, G] \dots [a_d, G]$ is closed in G , so $X = \bigcap_{N \triangleleft_o G} XN$. If $N \triangleleft_o G$, then G/N is nilpotent and generated by $\{a_1, \dots, a_d\}$, which implies $XN/N = G'N/N$ (cf. [5, Lemma 1.23]), so $G' \leq XN$. Hence $G' \subseteq X$. \square

It is easy to see that (9) can be expressed by a first-order formula. If in a group G , every product of $d+1$ commutators belongs to the set X defined above, then (by an obvious induction) every product of commutators belongs to X . Hence there is a formula α such that

$$G \models \alpha(a_1, \dots, a_d) \iff G' = [a_1, G] \dots [a_d, G].$$

The *Frattini subgroup* $\Phi(G)$ of a profinite group G is the intersection of all maximal open subgroups of G . It follows from the definition that for $Y \subseteq G$,

$$\overline{\langle Y \rangle} = G \iff \overline{\langle Y \rangle} \Phi(G) = G. \tag{10}$$

If G is pronilpotent then every maximal open subgroup is normal of prime index; it follows that if $G \in \mathcal{C}_\pi$ then

$$\Phi(G) = \overline{G'} G^{q(\pi)}.$$

If also G is finitely generated, then this subgroup has finite index, so it is *open*, and then in (10) we have

$$\overline{\langle Y \rangle} \Phi(G) = \langle Y \rangle \Phi(G) = \langle Y \rangle G' G^{q(\pi)}.$$

Thus if $G = \overline{\langle a_1, \dots, a_d \rangle} \in \mathcal{C}_\pi$, then

$$\Phi(G) = \delta(\bar{a}; G),$$

where

$$\delta(\bar{a}, x) \equiv \exists z, y_1, \dots, y_d. x = [a_1, y_1] \dots [a_d, y_d] z^{q(\pi)}.$$

The definability of $\Phi(G)$ means that we can define generating sets in L_{gp} .

PROPOSITION 5.3. *For each $d \geq 1$ there is a formula β_d such that for $G \in \mathcal{C}_\pi$,*

$$G \models \beta_d(a_1, \dots, a_d) \iff G = \overline{\langle a_1, \dots, a_d \rangle}.$$

Proof. Set

$$\beta_d(u_1, \dots, u_d) \equiv \alpha(u_1, \dots, u_d) \wedge \forall x. \bigvee_{s(1), \dots, s(d) \in S} \delta(\bar{u}, x^{-1} u_1^{s(1)} \dots u_d^{s(d)}),$$

where $S = \{0, 1, \dots, q-1\}$. We have seen that if $G = \langle \overline{a_1, \dots, a_d} \rangle \in \mathcal{C}_\pi$, then $G \models \beta_d(a_1, \dots, a_d)$.

Conversely, if $G \in \mathcal{C}_\pi$ and $G \models \beta_d(a_1, \dots, a_d)$, then every element of G belongs to

$$\langle a_1, \dots, a_d \rangle [a_1, G] \dots [a_d, G] G^{q(\pi)} \subseteq \langle a_1, \dots, a_d \rangle \Phi(G).$$

□

Now setting

$$\tilde{\beta}_d \equiv \exists y_1, \dots, y_d. \beta_d(y_1, \dots, y_d)$$

we have

$$d(G) \leq d \iff G \models \tilde{\beta}_d,$$

$$d(G) = d \iff G \models \tilde{\beta}_d \wedge \neg \tilde{\beta}_{d-1} := \beta_d^*;$$

thus for groups in \mathcal{C}_π being d -generated can be expressed by a first-order sentence.

We note that the hypothesis that the group G be in \mathcal{C}_π is necessary.

PROPOSITION 5.4. *Within profinite groups, being d -generated cannot be expressed by a single first order sentence.*

Proof. According to Proposition 1.1, if ϕ is a sentence in the language of groups and a non-periodic abelian group G satisfies ϕ , then $G \times C_q$ satisfies ϕ for almost all primes q .

If ϕ expresses being d -generated, then $\hat{\mathbb{Z}}^d \models \phi$. Let q be a prime as above, then also $\hat{\mathbb{Z}}^d \times C_q \models \phi$. But this group needs $d+1$ generators. □

The same argument works for the category of abstract groups, using \mathbb{Z}^d in place of $\hat{\mathbb{Z}}^d$. A slightly more elaborate argument shows that being finitely generated, for profinite groups, also cannot be expressed by a single first-order sentence ϕ . One works with $\hat{\mathbb{Z}} \times (C_q)^{\aleph_0}$, and uses the fact that ϕ can be expressed as a Boolean combination of Szmielew invariant sentences: see [11, Theorem A.2.7].

To conclude this introductory section, we discuss a ‘small’ f.g. pro- p group of infinite rank,

$$G := C_p \hat{\wr} \mathbb{Z}_p = \varprojlim_{n \rightarrow \infty} C_p \wr C_{p^n}.$$

This group is the semidirect product of the ‘base group’ M by a procyclic group $T \cong \mathbb{Z}_p$; here

$$M \cong \mathbb{F}_p[[T]]$$

as a T -module, where $\mathbb{F}_p[[T]]$ is the completed group algebra of T (see [5, § 7.4]). Note that $\mathbb{F}_p[[T]]$ is a 1-dimensional complete local ring with residue field \mathbb{F}_p , whose non-zero closed ideals are just the powers of the maximal ideal, and therefore have finite index.

PROPOSITION 5.5. *The pro- p group $C_p \hat{\wr} \mathbb{Z}_p$ is FA within profinite groups.*

Proof. Let a be a generator of the T -module M , and let f be a generator of T . Then G has the pro- p presentation

$$\langle a, f; a^p = [w, a] = [u, w] = 1 \ (u, w \in F') \rangle,$$

where F denotes the free group on $\{a, f\}$.

Let $\sigma(a, f)$ be the formula saying for a pro- p group G that

- the elements a, f generate G , that is, $\beta_2(a, f)$ holds;
- the element a has order p and commutes with every commutator;
- all commutators commute;
- the centre of G is trivial.

Then $\sigma(a, f)$ holds in G .

Suppose to begin with that H is a pro- p group and that $H \models \sigma(b, h)$ for some $b, h \in H$. Then the map sending a to b and f to h extends to an epimorphism $\theta : G \rightarrow H$. Let $N = \ker \theta$. We aim to show that $N = 1$.

Now $G/N \cong H$ is a non-trivial pro- p group with trivial centre, so it is infinite (as every non-trivial finite p -group has non-trivial centre). Also $N \cap M$ corresponds to a closed ideal of $\mathbb{F}_p[[T]]$, so if $N \cap M \neq 1$, then $M/(N \cap M)$ is finite. As G/M is procyclic, this implies that the centre of G/N has finite index (if f^n centralizes $M/(N \cap M)$, then $[G, N\langle f^n \rangle] \leq N$). This contradicts $Z(H) = 1$, and we conclude that $N \cap M = 1$. But then $N \leq C_G(M) \cap N = M \cap N = 1$.

It follows that $H \cong G$. Thus $(G; a, f)$ is FA in pro- p groups.

To deal with the general case of profinite groups, we need a way to identify the prime p . Now we have

$$G' = [M, f] = \{[x, y] \mid x, y \in G\}$$

$$M = G' \langle a \rangle = G' \cup G'a \cup \dots \cup G'a^{p-1},$$

so M is definable by a formula $\mu(a, f)$ say. Since G is pro- p , $C_G(M) = M$ and $M^p = 1$, the following holds:

$$[M, x] \subseteq [M, x^p] \implies [M, x] = 1 \implies x \in M \implies x^p = 1.$$

So G satisfies a formula $\tau(a, f)$ that expresses

$$M \triangleleft G \text{ and } [M, M] = M^p = 1 \text{ and } [M, x] \subseteq [M, x^p] \implies x^p = 1.$$

Suppose now that H is a profinite group and that $H \models \sigma(b, h) \wedge \tau(b, h)$. Let $N = \mu(b, h; H)$, so N is an abelian normal subgroup of H , of exponent p . Suppose that x belongs to a Sylow pro- q subgroup of H where $q \neq p$. Then $x = x^{p^\lambda}$ for some $\lambda \in \mathbb{Z}_q$, and then $[N, x] = [N, x^{p^\lambda}] \subseteq [N, x^p]$. As $H \models \tau(b, h)$, this implies that $x^p = 1$, and hence that $x = 1$. It follows that H is a pro- p group, and then $H \models \sigma(b, h)$ implies that $H \cong G$. \square

COROLLARY 5.6. *The classes of profinite, respectively pro- p , groups of finite rank and of f.g. profinite, respectively, pro- p , groups are first-order separable, with witness group $C_p \hat{\mathbb{Z}}_p$.*

5.2. Powerful pro- p groups

Next we discuss a special class of pro- p groups, where p always denotes a prime. Fix

$$\varepsilon = 0 \text{ if } p \neq 2, \varepsilon = 1 \text{ if } p = 2.$$

A pro- p group G is *powerful* if $G/\overline{G^p}$ is abelian (replace p by 4 when $p = 2$). If G is also finitely generated, then

$$\overline{G^{p^n}} = G^{p^n} = G^{\{p^n\}}$$

for each $n \geq 1$. Thus for a f.g. pro- p group G , G is powerful if and only if

$$G \models \forall x, y \exists z. ([x, y] = z^p)$$

(replace p by 4 when $p = 2$). In this case we have

$$\Phi(G) = G^p \quad (\text{respectively, } G^4 \text{ if } p = 2).$$

For all this, see [5, Chapter 3]. A key result of Lazard [19] characterizes the compact p -adic analytic groups as the f.g. profinite groups that are virtually powerful (cf. [5, Chapter 8]).

The definition of a *uniform* pro- p group is given in [5, Chapter 4]. Rather than repeating it here, we use the simple characterization (*loc. cit.* Theorem 4.5): a pro- p group is uniform if and only if it is f.g., powerful and torsion-free.

The following theorem summarizes key facts established in [5, Chapters 3 and 4]. We set

$$\lambda(r) = \lceil \log_2 r \rceil + \varepsilon.$$

THEOREM 5.7. (i) *Let G be a pro- p group of finite rank r . Put $m = \lambda(r)$. Then G has open normal subgroups $W \geq W_0$, with*

$$W \geq \Phi^m G \geq G^{p^m},$$

such that every open normal subgroup of G contained in W is powerful, and every open normal subgroup of G contained in W_0 is uniform.

(ii) *If G is f.g. and powerful then $\text{rk}(G) = d(G)$.*

(iii) *If $G = \langle a_1, \dots, a_d \rangle$ is powerful, then*

$$G = \{a_1^{\mu_1} \dots a_d^{\mu_d} \mid \mu_1, \dots, \mu_d \in \mathbb{Z}_p\}.$$

(iv) *If G is f.g. and powerful, then G has a uniform open normal subgroup U , and*

$$d(V) = \text{rk}(V) = \text{rk}(U) = d(U)$$

for every uniform open subgroup V of G .

The common rank of open uniform subgroups of such a group G is denoted by $\dim(G)$; this is the dimension of G as a p -adic analytic group.

LEMMA 5.8. *Let G be a uniform pro- p group and let $N \triangleleft_c G$. If G/N is uniform, then N is uniform and*

$$\dim(G) = \dim(N) + \dim(G/N).$$

Proof. As explained in [5, Chapter 4], G has the structure of a \mathbb{Z}_p -Lie algebra $L(G)$, additively isomorphic to $\mathbb{Z}_p^{\dim(G)}$. Proposition 4.31 of [5] says that N is uniform, $L(N)$ is an ideal of $L(G)$, and the quotient mapping $G \rightarrow G/N$ induces an epimorphism $L(G) \rightarrow L(G/N)$. The claim follows from the additivity of dimension for free \mathbb{Z}_p -modules. \square

COROLLARY 5.9. *Let G be a pro- p group of finite rank and let $N \triangleleft_c G$. Then $\dim(G) = \dim(N) + \dim(G/N)$.*

Proof. Let H be a uniform open normal subgroup of G . Then $H/(H \cap N)$ is powerful, hence has a finite normal subgroup M/N such that H/M is uniform by [5, Theorem 4.20]. The claim follows on replacing G by H and N by M . \square

These results can be applied to \mathcal{C}_π groups of finite rank. Let $G \in \mathcal{C}_\pi$. Then $G = G_1 \times \dots \times G_k$ where each G_t is a pro- p_t group, the Sylow pro- p_t subgroup of G . If H is a closed subgroup of G , then $H = H_1 \times \dots \times H_k$ where $H_t = H \cap G_t$, notation we keep for the remainder of this subsection.

If G has finite rank, we define

$$\text{Dim}(G) = \dim G_1 + \cdots + \dim G_k.$$

If $p_t \nmid m$, then every element of H_t is an m th power in H ; thus if $q = p_1^{e_1} \cdots p_k^{e_k}$, then

$$\begin{aligned} H^{\{q\}} &= H_1^{\{p_1^{e_1}\}} \times \cdots \times H_k^{\{p_k^{e_k}\}} \\ H^q &= H_1^{p_1^{e_1}} \times \cdots \times H_k^{p_k^{e_k}}. \end{aligned}$$

We call H *semi-powerful* if each H_t is a powerful pro- p_t group. If $H \in \mathcal{C}_\pi$ is finitely generated, then H is semi-powerful if and only if $H/H^{q'(\pi)}$ is abelian. This holds if and only if

$$H \models \text{pow} \equiv \forall x, y \exists z. ([x, y] = z^{q'(\pi)}).$$

H is *semi-uniform* if each H_t is uniform. In this case, the *dimension* of H is the k -tuple

$$\dim H = (\dim H_1, \dots, \dim H_k).$$

LEMMA 5.10. *Let H and K be semi-uniform \mathcal{C}_π groups, and $\theta : H \rightarrow K$ an epimorphism. If $\dim H = \dim K$, then θ is an isomorphism.*

Proof. Restricting to Sylow subgroups, we may suppose that H and K are uniform pro- p groups of the same dimension. Then Lemma 5.8 shows that $\ker \theta$ is a uniform group of dimension 0, that is, the trivial group. \square

COROLLARY 5.11. *Let $G \in \mathcal{C}_\pi$ have finite rank. If $N \triangleleft_c G$ and $\text{Dim}(G/N) = \text{Dim}(G)$, then N is finite.*

Proof. This follows likewise from Corollary 5.9. \square

For $q, f \in \mathbb{N}$ set

$$\mu_{f,q}(x) \equiv \exists y_1, \dots, y_f. (x = y_1^q \cdots y_f^q).$$

As before, we see that the word x^q has width f in a group H , that is,

$$H^q = (H^{\{q\}})^{*f} := \left\{ h_1^q \cdots h_f^q \mid h_i \in H \right\},$$

if and only if H satisfies

$$\mathfrak{m}_{f,q} \equiv \forall x. (\mu_{f+1,q}(x) \rightarrow \mu_{f,q}(x)).$$

Of course, this holds if and only if $H \models \text{s}(\mu_{f,q})$; we can use either formulation.

PROPOSITION 5.12. *Let G be a f.g. profinite group and let $q \in \mathbb{N}$. There exists $f \in \mathbb{N}$ such that $G \models \mathfrak{m}_{f,q}$ and*

$$G^q = (G^{\{q\}})^{*f} = \mu_{f,q}(G)$$

is a definable open normal subgroup of G . Both f and $|G : G^q|$ can be bounded in terms of q and $\text{d}(G)$.

This is part of Theorem 2.1; the second claim was not made explicit in the statement but is included in the proof.

If H is semi-uniform of dimension (d_1, \dots, d_k) , we have

$$\Phi(H) = H^{q(\pi)} = H^{\{q(\pi)\}} = \mu_{1,q(\pi)}(H),$$

$$|H : \Phi(H)| = p_1^{d_1} \dots p_k^{d_k}.$$

Thus for semi-uniform H , the dimension is determined by

$$H \models \partial_{d_1, \dots, d_k} \equiv \text{ind}(\mu_{1,q(\pi)}; p_1^{d_1} \dots p_k^{d_k}).$$

REMARK. The primary components of a \mathcal{C}_π group are not in general definable; this can be deduced from [38, Theorem 3.3.5]. If they were, our results about \mathcal{C}_π could be reduced to the slightly less messy case where π consists of a single prime; it is easy to see that a direct product of finitely many definable subgroups is FA if each of the factors is FA.

For f.g. groups with trivial centre such an approach is feasible, as each primary component is then a centralizer, and so definable (with parameters). A slightly more elaborate argument, using [26, Theorem 1.6], shows that the same holds for f.g. groups with finite abelianization.

5.3. Presentations

In the context of profinite groups, a ‘finite presentation’ may involve relators that are ‘profinite words’, that is, limits of a convergent sequence of group words. For present purposes we need to consider concepts of finite presentation that are both more and less restrictive.

Let \mathcal{C} be a class of groups and $L \supseteq L_{\text{gp}}$ a language. For a group $G \in \mathcal{C}$ and a formula $\psi(x_1, \dots, x_r)$ of L , we say that ψ is an L -presentation of G in \mathcal{C} if G has a generating set $\{g_1, \dots, g_r\}$ such that

- (i) $G \models \psi(g_1, \dots, g_r)$, and
- (ii) if $h_1, \dots, h_r \in H \in \mathcal{C}$ and $H \models \psi(h_1, \dots, h_r)$, then there is an epimorphism $\theta : G \rightarrow H$ with $g_i\theta = h_i$ for each i .

In this case, we say that ψ is an L -presentation on $\{g_1, \dots, g_r\}$.

The concept of L -presentation generalizes the familiar idea of a finite presentation in group theory. We mention two particular cases.

PROPOSITION 5.13. *A group $G \in \mathcal{C}_\pi$ has an L -presentation in \mathcal{C}_π in each of the following cases.*

(i) $L = L_{\text{gp}}$, and G is strictly f.p. in \mathcal{C}_π ; that is, G has a finite presentation as a \mathcal{C}_π -group in which the relators are finite group words, or equivalently, G is the \mathcal{C}_π -completion of a finitely presented (abstract) group.

(ii) $L = L_\pi$, and G has finite rank.

Proof. (i) We have an epimorphism $\phi : F \rightarrow G$ where F is the free \mathcal{C}_π -group on a finite generating set $X = \{x_1, \dots, x_r\}$ and $\ker \phi$ is the closed normal subgroup of F generated by a finite set R of ordinary group words on X . Set

$$\psi(\bar{x}) := \beta_r(\bar{x}) \wedge \bigwedge_{w \in R} w(\bar{x}) = 1$$

(recall that $G \in \mathcal{C}_\pi$ satisfies $\beta_r(\bar{a})$ if and only if $\{a_1, \dots, a_r\}$ generates G).

Now put $g_i = x_i\phi$ for each i . Then $G \models \psi(\bar{g})$. Suppose that $h_1, \dots, h_r \in H \in \mathcal{C}_\pi$ and $H \models \psi(\bar{h})$. Then h_1, \dots, h_r generate H , so the homomorphism $\mu : F \rightarrow H$ sending \bar{x} to \bar{h} is onto. Also for each $w \in R$ we have $w(\bar{x})\mu = w(\bar{h}) = 1$, so $\ker \phi \leq \ker \mu$. It follows that μ factors through an epimorphism $\theta : G \rightarrow H$ with $g_i\theta = h_i$ for each i . Thus ψ is an L_{gp} presentation for G in \mathcal{C}_π . \square

Before proving (ii) we need yet another definition.

• Let $G \in \mathcal{C}_\pi$. Then $\langle X; R \rangle$ is a \mathbb{Z}_π -finite \mathcal{C}_π presentation for G if $G \cong F/N$ where $F = F(X)$ is the free \mathcal{C}_π -group on a finite generating set X and N is the closed normal subgroup of F generated by a finite set R of elements of the form

$$w_1^{\mu_1} \dots w_n^{\mu_n}, \quad (11)$$

where each w_i is a group word on X and $\mu_1, \dots, \mu_n \in \mathbb{Z}_\pi$.

Note that the free \mathcal{C}_π -group on a set X is the direct product of its Sylow pro- p_t subgroups, which themselves are free pro- p_t groups on the p_t -components of X .

Expressions like (11) will be called π -words. For a subset Y of G , one says that $\langle X; R \rangle$ is a presentation on Y if the implied epimorphism $F(X) \rightarrow G$ maps X to Y .

LEMMA 5.14. *Let $G = \overline{\langle Y \rangle} \in \mathcal{C}_\pi$ have finite rank, where Y is finite. Then G has a \mathbb{Z}_π -finite \mathcal{C}_π presentation on Y .*

Proof. Suppose that U is a uniform pro- p group with generating set $X = \{x_1, \dots, x_d\}$. For simplicity, assume that $\{x_1, \dots, x_r\}$ is a basis for U , where $r = \dim U \leq d$. Then U has a pro- p presentation on X with relators

$$[x_i, x_j] x_1^{\lambda_1(i,j)} \dots x_r^{\lambda_r(i,j)}, \quad 1 \leq i < j \leq r, \quad (12)$$

$$x_k x_1^{\mu_1(k)} \dots x_r^{\mu_r(k)}, \quad r < k \leq d, \quad (13)$$

with $\lambda_l(i, j) \in p\mathbb{Z}_p$ and $\mu_l(k) \in \mathbb{Z}_p$; Proposition 4.32 of [5] gives the presentation (12), and (13) expresses the redundant generators using Theorem 5.7(iii).

We rewrite (13) (also adding some trivial relators) as

$$x_1^{\mu_1(k)} \dots x_d^{\mu_d(k)}, \quad 1 \leq k \leq d, \quad (14)$$

by setting $\mu_l(k) = \delta_{kl}$ for $k, l > r$ and $\mu_l(k) = 0$ for $k < r$ and all l . Next, we add redundant relators of the form (12) for all $j > r$ and all $i < j$, again using Theorem 5.7(iii). Setting $\lambda_l(i, j) = 0$ whenever $l > r$, we obtain a presentation for U on X with relators (14) together with

$$[x_i, x_j] x_1^{\lambda_1(i,j)} \dots x_d^{\lambda_d(i,j)}, \quad 1 \leq i < j \leq d. \quad (15)$$

Now consider a semi-uniform \mathcal{C}_π group $V = U_1 \times \dots \times U_k$ where each U_t is a uniform pro- p_t group of dimension $d_t \leq d$. A generating set $X = \{x_1, \dots, x_d\}$ for V projects to a generating set $X_t = \{x_1^{(t)}, \dots, x_d^{(t)}\}$ for U_t ; then U_t has a pro- p_t presentation on X_t like (14)∪(15), with exponents $\lambda_l^{(t)}(i, j)$, $\mu_l^{(t)}(k) \in \mathbb{Z}_{p_t}$.

Let $\lambda_l(i, j)$, $\mu_l(k) \in \mathbb{Z}_\pi$ have \mathbb{Z}_{p_t} -components $\lambda_l^{(t)}(i, j)$, $\mu_l^{(t)}(k)$ respectively for each t . Then (14)∪(15) gives a presentation for V on X .

Finally, we have $G = G_1 \times \dots \times G_k = \overline{\langle y_1, \dots, y_m \rangle}$ where each G_t is a pro- p_t group of finite rank and $Y = \{y_1, \dots, y_m\}$. Let $V = U_1 \times \dots \times U_k$ be a semi-uniform open normal subgroup of G . Using the Schreier process we obtain a finite generating set $X = \{x_1, \dots, x_d\}$ for V , each element of X being equal to a finite word on \bar{y} , say $x_i = w_i(\bar{y})$. Substitute $w_i(\bar{y})$ for x_i in (14)∪(15) to obtain a set of relators R on \bar{y} . Note that R consists of π -words.

By Theorem 5.7(iii), each element of V is a finite product of \mathbb{Z}_π -powers of elements of X . The conjugation action of G on V is determined by specifying, for $j = 1, \dots, m$ and for each $x_i \in X$,

$$y_j^{-1} x_i y_j = W_{ij}(X),$$

where each $W_{ij}(X)$ is a finite product of \mathbb{Z}_π -powers (for clarity, we keep w for finite group words and write W for π -words).

Let

$$S := \{y_j^{-1}w_i^{-1}y_j.W_{ij}(w_1, \dots, w_d) \mid j = 1, \dots, m, i = 1, \dots, d\}.$$

A standard argument (see, for example, [33, Chapter 8, Lemma 10]) now shows that $\langle Y; R \cup S \rangle$ is a \mathcal{C}_π presentation for G . \square

Now we can give the the following proof.

Proof of Proposition 5.13(ii). Let $G = \overline{\langle Y \rangle}$ be as in the preceding lemma. Set

$$\rho(\bar{y}) := \bigwedge_{w \in R \cup S} w(\bar{y}) = 1,$$

where R and S are given above. As these are finite sets of π -words, ρ is a formula of L_π . Now put

$$\psi(\bar{y}) := \beta_m(\bar{y}) \wedge \rho(\bar{y}).$$

If $h_1, \dots, h_m \in H \in \mathcal{C}_\pi$ and $H \models \psi(\bar{h})$, then h_1, \dots, h_m generate H and satisfy the relations $R \cup S = 1$; as $\langle Y; R \cup S \rangle$ is a \mathcal{C}_π presentation for G , it follows that the map sending \bar{h} to \bar{y} extends to an epimorphism from G to H . Thus ψ is an L_π presentation for G in \mathcal{C}_π .

5.4. Finite axiomatizability in \mathcal{C}_π

Until further notice, L stands for one of L_{gp} , L_π . Theorem 1.6 is included in the following theorem.

THEOREM 5.15. *Suppose that $G \in \mathcal{C}_\pi$ has finite rank, and that G has an L presentation on the generating tuple (a_1, \dots, a_r) . Then (G, \bar{a}) is FA in \mathcal{C}_π .*

Note that when $L = L_\pi$, the existence of an L presentation is guaranteed by Proposition 5.13(ii).

Proof. We have $G = G_1 \times \dots \times G_k$ where each G_t is a pro- p_t group of finite rank and dimension d_t . There is a formula ψ of L such that (i) $G \models \psi(\bar{a})$ and (ii) if $h_1, \dots, h_r \in H \in \mathcal{C}_\pi$ and $H \models \psi(\bar{h})$, then there is an epimorphism $\theta : G \rightarrow H$ sending \bar{a} to \bar{h} .

It follows from Theorem 5.7 that G has an open normal subgroup W_0 such that every open normal subgroup of G contained in W_0 is semi-uniform. Then $W_0 \geq G^q$ for some π -number q . Now Proposition 5.12 shows that for some f , G satisfies $\text{m}_{f,q}$ and

$$G^q = \mu_{f,q}(G)$$

is open in G , hence semi-uniform. Set

$$\text{tf} \equiv \forall x (x^{q(\pi)} = 1 \rightarrow x = 1).$$

As G^q is semi-powerful and torsion-free, G satisfies

$$\text{res}(\mu_{f,q}, \text{pow} \wedge \text{tf})$$

(see §5.2). Say $|G : G^q| = m$ and $\dim(G^q) = (d_1, \dots, d_k)$. Then G also satisfies

$$\text{ind}^*(\mu_{f,q}; m) \wedge \text{res}(\mu_{f,q}, \partial_{d_1, \dots, d_k}).$$

We have established that G satisfies

$$\sigma_G(\bar{a}) \equiv \psi(\bar{a}) \wedge \text{m}_{f,q} \wedge \text{res}(\mu_{f,q}, \text{pow} \wedge \text{tf}) \wedge \text{ind}^*(\mu_{f,q}; m) \wedge \text{res}(\mu_{f,q}, \partial_{d_1, \dots, d_k}).$$

Now suppose that $H \in \mathcal{C}_\pi$ satisfies $\sigma_G(\bar{h})$. Let $\theta : G \rightarrow H$ be the epimorphism specified above. To complete the proof it will suffice to show that $\ker \theta = 1$.

$\sigma_G(\bar{h})$ implies that $H^q = \mu_{f,q}(H)$ is semi-uniform, that $|H : H^q| = m = |G : G^q|$, and that $\dim(H^q) = \dim(G^q)$. Applying Lemma 5.10 to $\theta|_{G^q}$ we infer that $\ker \theta \cap G^q = 1$. As $|H : H^q| = |G : G^q|$ is finite, it follows that θ induces an isomorphism $G/G^q \rightarrow H/H^q$, whence $\ker \theta \leq G^q$. Thus $\ker \theta = 1$ as required. \square

5.5. Finite axiomatizability in profinite groups

The first case of Theorem 1.7 to be established was for the specific group $G = \text{UT}_3(\mathbb{Z}_p)$ ([23, § 10]). The key point of the proof is to recover the ring structure of \mathbb{Z}_p from the group structure of G : specifically, the commutator map in the group carries enough information to reconstruct multiplication in the ring. With Theorem 5.15 at our disposal, we shall see that it will suffice merely to identify the prime p , and this in turn is quite easy, provided that there are ‘enough’ commutators. The appropriate condition was identified by Oger and Sabbagh in [27], in the context of abstract nilpotent groups; fortunately for us, it transfers perfectly to the profinite context.

We will say that a group G satisfies the *O-S condition* if $Z(G)/(G' \cap Z(G))$ is periodic. Theorem 1.7 is included in the following.

THEOREM 5.16. *Let L be either L_{gp} or L_π . Suppose that $G \in \mathcal{C}_\pi$ is nilpotent, and that G has an L presentation as a \mathcal{C}_π group on the generating tuple (a_1, \dots, a_r) . Then the following are equivalent:*

- (a) (G, \bar{a}) is FA (with respect to L) in the class of all profinite groups;
- (b) G is FA (with respect to L) in the class of all f.g. nilpotent virtually pro- π groups;
- (c) G satisfies the O-S condition.

For the proof, we write

$$Z(x) \equiv \forall u. (xu = ux)$$

$$Z_2(x) \equiv \forall v. Z([x, v]);$$

these define the centre and second centre in a group. Let $S_p(x)$ be a formula asserting, for $x \in G$, that

$$[Z_2(G), x] \subseteq [Z_2(G), x^p].$$

Let ψ_π be a sentence asserting that $\bigwedge_{p \in \pi} S_p(x) \rightarrow [Z_2(G), x] = 1$ for each x .

LEMMA 5.17. *Let G be a nilpotent profinite group.*

- (i) *If $Z(G)$ is pro- π , then $G \models \psi_\pi$.*
- (ii) *If $G \models \psi_\pi$, then $G/Z(G)$ is a pro- π group.*

Proof. Write $Z = Z(G)$ and $Z_2 = Z_2(G)$. We use the facts that for each $x \in G$ the map $y \mapsto [x, y]$ is a continuous homomorphism from Z_2 to Z , with kernel containing Z , and that $(x, y) \mapsto [x, y]$ induces a bilinear map from $G^{\text{ab}} \times Z_2/Z$ into Z .

- (i) Suppose that Z is a pro- π group. Let $x \in G$. If $S_p(x)$ holds for each $p \in \pi$, then

$$[x, Z_2] = [x^p, Z_2] = [x, Z_2]^p$$

for each $p \in \pi$. But $[x, Z_2]$ is a closed subgroup of the abelian pro- π group Z , and so $[x, Z_2] = 1$. Thus $G \models \psi_\pi$.

(ii) Let $q \notin \pi$ be a prime, and let Q be a Sylow pro- q subgroup of G . Let $x \in Q$ and $p \in \pi$. Then $x = x^{\lambda p}$ where $\lambda \in \mathbb{Z}_q$ satisfies $\lambda p = 1$. Then for any $u \in Z_2$ we have

$$[x, u] = [x^p, u^\lambda].$$

Thus $S_p(x)$ holds.

Now suppose that $G \models \psi_p$. Then $[x, Z_2] = 1$ holds for each $x \in Q$, so $Q \cap Z_2 \leq Z(Q)$. Now $Q \cap Z_2 = Z_2(Q)$, so $Z_2(Q) = Z(Q)$. As Q is nilpotent this forces $Q = Z(Q) \leq Z$.

It follows that G/Z is a pro- π group. \square

To prove Theorem 5.16, we have to show that the following are equivalent:

- (a) (G, \bar{a}) is FA with respect to L in the class of all profinite groups;
- (b) G is FA with respect to L in the class of all f.g. nilpotent virtually pro- π groups;
- (c) G satisfies the O-S condition.

Note to begin with that G has finite rank (a familiar property of f.g. nilpotent groups).

Proposition 1.1 shows that if G does not satisfy the O-S condition, then G cannot be f.a. in the class of groups $\{G \times C_q, q \text{ prime}\}$. Thus (a) \implies (b) \implies (c), since (b) is formally weaker than (a).

Now suppose that G does satisfy the O-S condition. Given the hypotheses, Theorem 5.15 gives us a formula σ_G of L such that for $b_1, \dots, b_r \in H \in \mathcal{C}_\pi$, we have

$$H \models \sigma_G(\bar{b}) \iff (H, \bar{b}) \cong (G, \bar{a}).$$

Also $Z(G)/(G' \cap Z(G))$ is a periodic pro- π group of finite rank, so it is finite, of exponent q say; here q is a π -number. Recall (Lemma 5.2) that every element of G' is a product of d commutators, where $d = d(G)$. Therefore G satisfies

$$\theta_q \equiv \forall y. \left(Z(y) \longrightarrow \exists u_1, v_1, \dots, u_d, v_d. \left(y^q = \prod_{i=1}^d [u_i, v_i] \right) \right). \quad (16)$$

Say that G is nilpotent of class c . This is expressed by a sentence Γ_c (all simple commutators of weight $c+1$ are equal to 1). Now define

$$\Sigma_G \equiv \psi_\pi \wedge \sigma_G \wedge \theta_q \wedge \Gamma_c.$$

Then G satisfies $\Sigma_G(\bar{a})$. Suppose that H is a profinite group and that $H \models \Sigma_G(\bar{b})$ for some $\bar{b} \in H^{(r)}$. Then H is nilpotent, so by Proposition 5.17(ii) $H/Z(H)$ is a pro- π group. Also $Z(H)H'/H'$ has exponent dividing q , so H/H' is a pro- π group. As H is nilpotent this implies that H is a pro- π group (to see this, note that each finite continuous quotient \tilde{H} of H is the direct product of its Sylow subgroups, and its abelianization is the direct product of their respective abelianizations. So, if \tilde{H}/\tilde{H}' is a π -group, then the Sylow q -subgroups of \tilde{H} for $q \notin \pi$ have trivial abelianization, and as they are nilpotent, this means that they are trivial. Therefore \tilde{H} is a π -group). Thus $H \in \mathcal{C}_\pi$.

As $H \models \sigma_G(\bar{b})$, it follows that $(H, \bar{b}) \cong (G, \bar{a})$. Thus (c) \implies (a).

5.6. Finite axiomatizability in pronilpotent groups

The nilpotency hypothesis in Theorem 5.16 is very restrictive. Without it, we can prove a weaker result, giving finite axiomatizability in the class of all *pronilpotent* groups; this is strictly intermediate between \mathcal{C}_π and the class of all profinite groups, so the following results ‘interpolate’ the two preceding theorems.

THEOREM 5.18. *Let $G \in \mathcal{C}_\pi$ have finite rank, and assume that G has an L -presentation in \mathcal{C}_π . Then the following are equivalent:*

- (a) G is FA (with respect to L) in the class of all pronilpotent groups;
- (b) G is FA (with respect to L) in the class of all pronilpotent virtually pro- π groups of finite rank;
- (c) G satisfies the O-S condition.

THEOREM 5.19. *Let $G \in \mathcal{C}_\pi$ have finite rank, and assume that G has an L -presentation in \mathcal{C}_π . If $G/\gamma_m(G)$ satisfies the O-S condition for some $m \geq 2$, then G is FA (with respect to L) in the class of all pronilpotent groups.*

Here, $\gamma_m(G)$ denotes the m th term of the lower central series (a closed normal subgroup when G is a f.g. profinite group). Note that when L is L_π , Proposition 5.13(ii) makes the assumption of an L -presentation redundant.

We will use the fact that for each d and m , there exists $f = f(d, m)$ such that in any d -generator pronilpotent group G , we have

$$\gamma_m(G) = X^{*f} := \{x_1 \dots x_f \mid x_1, \dots, x_f \in X\},$$

where

$$X = \{[y_1, \dots, y_m] \mid y_1, \dots, y_m \in G\}$$

([34], Lemma 4.3.1). Define a formula

$$\Gamma_{m,f}(x) \equiv \exists y_{11}, \dots, y_{fm} \cdot \left(x = \prod_{i=1}^f [y_{i1}, \dots, y_{im}] \right);$$

this asserts that x is a product of f simple left-normed commutators of weight m . For any group G , $\Gamma_{m,f}(G)$ is a subset of $\gamma_m(G)$; and $\Gamma_{m,f}(G) = \gamma_m(G)$ if and only if $\Gamma_{m,f}(G)$ is a subgroup, that is, if and only if

$$G \models s(\Gamma_{m,f})$$

(see §2). In particular, this holds if G is pronilpotent and $f \geq f(d(G), m)$.

LEMMA 5.20. *Let H be a f.g. pronilpotent group. Put $H_n = \gamma_n(H)$ and $Z_n/H_n = Z(H/H_n)$ for each n . Suppose that $H_s^q \leq H_{s+1}$ for some $s \geq 1$, where q is a π -number. Then $Z_s/Z(H)$ is a pro- π group.*

Proof. We begin with some properties of the series (H_n) .

(i) $H_n^q \leq H_{n+1}$ for each $n \geq s$. Proof by induction on n . Let $n \geq s$ and suppose that $H_n^q \leq H_{n+1}$. Now H_{n+1} is generated by elements $[x, h]$ with $x \in H_n$ and $h \in H$. These satisfy

$$[x, h]^q \equiv [x^q, h] \equiv 1 \pmod{H_{n+2}}.$$

As H_{n+1}/H_{n+2} is abelian, it follows that $H_{n+1}^q \leq H_{n+2}$.

(ii) $Z_n^q \leq Z_{n+1}$ for each $n \geq s$. To see this, let $z \in Z_n$ and $h \in H$. Then

$$[z^q, h] \equiv [z, h]^q \equiv 1 \pmod{H_{n+1}}$$

by (i) so $z^q \in Z_{n+1}$.

(iii) $\bigcap_{n=s}^\infty Z_n = Z(H)$. This is immediate from the fact that $\bigcap_{n=s}^\infty H_n = 1$, which holds because H is pronilpotent.

To conclude the proof, observe that the subgroups H_n , and therefore also Z_n , are closed in H . Let $n > s$. Then H/Z_n is a f.g. nilpotent profinite group and Z_s/Z_n has exponent dividing q^{n-s} , so Z_s/Z_n is a finite π -group, and Z_n is open in Z_s . The claim now follows from (iii). \square

LEMMA 5.21. *Let $G \in \mathcal{C}_\pi$ have finite rank, and let $(G_n)_{n \in \mathbb{N}}$ be a descending chain of closed normal subgroups of G . Then there exists s such that G_n/G_{n+1} is finite for each $n \geq s$.*

Proof. The sequence $\text{Dim}(G/G_n)$ is non-decreasing and bounded by $\text{Dim}(G)$, so it becomes stationary at some point $n = s$. Then G_s/G_n is finite for all $n \geq s$, by Corollary 5.9. \square

Now let π be a finite set of primes, let $G \in \mathcal{C}_\pi$ have finite rank and assume that G has an L -presentation in \mathcal{C}_π . For Theorem 5.18 we have to establish the equivalence of

- (a) G is FA (with respect to L) in the class of all pronilpotent groups;
- (b) G is FA (with respect to L) in the class of all pronilpotent virtually pro- π groups of finite rank;
- (c) G satisfies the O-S condition.

Theorem 5.19 asserts that these follow from

- (d) $G/\gamma_m(G)$ satisfies the O-S condition for some $m \geq 2$.

The proof that **(a)** \implies **(b)** \implies **(c)** is the same as in the preceding subsection (proof of Theorem 5.16).

Fix $d = d(G)$. By Theorem 5.15, there is a sentence σ_G such that for any $H \in \mathcal{C}_\pi$, $H \models \sigma_G$ if and only if $H \cong G$.

Suppose that **(d)** holds. We have $\gamma_m(G) = \Gamma_{m,f}(G)$ where $f = f(d, m)$. By Theorem 5.16, there is a sentence Σ such that a profinite group L satisfies Σ if and only if $L \cong G/\gamma_m(G)$.

Let

$$\xi \equiv s(\Gamma_{m,f}) \wedge \text{lift}(\Gamma_{m,f}, \Sigma) \wedge \sigma_G$$

(note that in any group H satisfying $s(\Gamma_{m,f})$, the formula $\Gamma_{m,f}$ defines the normal subgroup $\gamma_m(H)$, so $\text{lift}(\Gamma_{m,f}, \Sigma)$ makes sense). Now suppose that H is a pronilpotent group and that $H \models \xi$. Then $\Gamma_{m,f}(H) = \gamma_m(H)$ is a closed normal subgroup of H , and $H/\gamma_m(H) \cong G/\gamma_m(G)$ is a pro- π group. As $m \geq 2$, this now implies that H is a pro- π group (as in the proof of Theorem 5.16, above). Thus $H \in \mathcal{C}_\pi$ and so $H \cong G$. Thus ξ determines G among all pronilpotent groups, and **(a)** holds.

Suppose now that **(c)** holds. According to Lemma 5.21, there exists $s \geq 2$ such that $\gamma_s(G)/\gamma_{s+1}(G)$ is finite. Then $\gamma_s(G)^q \leq \gamma_{s+1}(G)$ for some π -number q . Putting $f = f(d, s)$ and $f' = f(d, s+1)$, we see that G satisfies

$$\eta \equiv \forall x. (\Gamma_{s,f}(x) \rightarrow \Gamma_{s+1,f'}(x^q)).$$

Condition **(c)** implies that G satisfies $\theta_{q'}$, defined in (16), for some π -number q' . Let ψ_π be as in Lemma 5.17. Then $G/\gamma_s(G) \models \psi_\pi$ so $G \models \text{lift}(\Gamma_{s,f}, \psi_\pi)$.

Now put

$$\Sigma \equiv s(\Gamma_{s,f}) \wedge \eta \wedge \theta_{q'} \wedge \text{lift}(\Gamma_{s,f}, \psi_\pi) \wedge \sigma_G.$$

Let H be a pronilpotent group and define $Z_n \geq H_n = \gamma_n(H)$ as in Lemma 5.20. Suppose that H satisfies Σ . Then $H_s = \Gamma_{s,f}(H)$ and $H/H_s \models \psi_\pi$. It follows by Lemma 5.17 that H/Z_s is a pro- π group.

As $H \models \eta$, we have $H_s^q \subseteq \Gamma_{s+1,f'}(H) \subseteq H_{s+1}$, so $Z_s/Z(H)$ is a pro- π group, by Lemma 5.20. As $H \models \theta_{q'}$, we have $Z(H)^{q'} \leq H'$. It follows that H/H' is a pro- π group, and hence (as before) that H is pro- π . As $H \models \sigma_G$ it follows that $H \cong G$.

Thus Σ determines G , and so **(c)** implies **(a)**.

6. Special linear groups

Here we show how Theorem 1.8 may be deduced, quite neatly, from Theorems 5.16 and 5.15. The result for $\mathrm{SL}_n(\mathbb{Z}_p)$ with $n \geq 3$ (and all p) was subsequently established in [35] using bi-interpretability; the present relatively simple group-theoretic proof may be adaptable to other algebraic groups, where bi-interpretability is not known.

For the proof of Theorem 1.8, we start with the congruence subgroup

$$G = \mathrm{SL}_n^1(\mathbb{Z}_p) = \ker(\mathrm{SL}_n(\mathbb{Z}_p) \rightarrow \mathrm{SL}_n(\mathbb{F}_p)).$$

Until further notice we assume that $n \geq 3$. We also assume for convenience that $p \nmid 2n$.

Note that G is a uniform pro- p group; this is easy to see directly, or from [5, Theorem 5.2] and Lemma 5.8 above.

Set $u_{ij} = 1 + pe_{ij}$, $v_{ij} = 1 - pe_{ji}$ for the elementary upper and lower triangular matrices ($1 \leq i < j \leq n$). The u_{ij} (respectively, v_{ij}) form a basis for the full upper (respectively lower) unitriangular group U (respectively, V) in G , each of which is a nilpotent uniform pro- p group that satisfies the O-S condition (Subsection 5.5).

Put $T_p = 1 + p\mathbb{Z}_p$. Let $\eta \in T_p \setminus T_p^p$ satisfy $\eta^{2p} = (1 + p^2)^{-1}$, and $\zeta_i \in T_p$ satisfy $\zeta_i^n \eta^{n-2i} = 1$. Set

$$h_i = \zeta_i 1_n \cdot \mathrm{diag}(\eta^{-1}, \dots, \eta^{-1}, \eta, \dots, \eta) \text{ for } i = 1, \dots, n-1,$$

where the last η^{-1} occurs in the i th place and the first η in the $(i+1)$ th place. Then h_1, \dots, h_{n-1} form a basis for the diagonal group H in G , which is a free abelian pro- p group. Note that

$$G = V \cdot H \cdot U. \tag{17}$$

We take $\bar{u} = (u_{ij})$, $\bar{v} = (v_{ij})$ and $\bar{h} = (h_i)$ as parameters.

THEOREM 6.1. *$(G, (\bar{u}, \bar{v}, \bar{h}))$ is FA in the class of all profinite groups.*

Proof. For brevity, we will say ‘formula’ to mean ‘formula of L_{gp} with parameters $\bar{u}, \bar{v}, \bar{h}$ ’, except where parameters are explicitly mentioned. We make the following *claims*.

(1) H is definable, in fact $H = \chi(\bar{h}; G)$ for some formula χ , such that χ always defines a closed subgroup in any profinite group. Indeed, $H = \bigcap_{i=1}^{n-1} C_G(h_i)$.

(2) U and V are definable, in fact, $U = \varphi_1(\bar{u}, \bar{h}; G)$, $V = \varphi_2(\bar{v}, \bar{h}; G)$ for some formulae φ_1 , φ_2 , which always define closed subsets in any profinite group. This follows from the fact that U is the product (in a suitable order) of the ‘elementary’ subgroups $U_{ij} = \langle u_{ij} \rangle = Z(C_G(u_{ij}))$, and similarly for V .

(3) U has an L_{gp} presentation on \bar{u} as a pro- p group; V has an L_{gp} presentation on \bar{v} as a pro- p group. This follows from [5, Proposition 4.32] and the usual commutator relations.

(4) G has an L_{gp} presentation on $(\bar{u}, \bar{v}, \bar{h})$ as a pro- p group. This is not quite immediate. The usual commutator relations provide a pro- p presentation for the uniform pro- p group G . However, some of them cannot be expressed in L_{gp} as they involve non-integral powers, and a roundabout argument is required; this is indicated below.

Given these claims, the proof is concluded as follows.

By Claims 1 and 2, we can construct a formula $\Phi(\bar{x}, \bar{y}, \bar{z})$ such that $\Phi(\bar{u}, \bar{v}, \bar{h})$ expresses the conjunction of the facts (which are true in G):

- (a) multiplication maps $V \times H \times U$ bijectively to G ,
- (b) $[V, H] \leq V^p$ and $C_H(V) = 1$,
- (c) $[U, H] \leq U^p$ and $C_H(U) = 1$,
- (d) H is abelian and has no p -torsion.

As U and V are nilpotent pro- p groups satisfying the O-S condition, and given Claim 3, Theorem 5.16 provides formulae $\sigma_U(\bar{x})$, $\sigma_V(\bar{x})$ that determine (U, \bar{u}) and (V, \bar{v}) among all profinite groups. Let $\Psi(\bar{x}, \bar{y}, \bar{z})$ express the conjunction of the following:

- (e) $s(\varphi_1(\bar{x}, \bar{z})) \wedge s(\varphi_2(\bar{y}, \bar{z}))$, that is, $\varphi_1(\bar{x}, \bar{z})$ and $\varphi_2(\bar{x}, \bar{z})$ define subgroups,
- (f) $\text{res}(\varphi_1(\bar{x}, \bar{z}), \sigma_U(\bar{x}))$, that is, the subgroup defined by $\varphi_1(\bar{x}, \bar{z})$ satisfies $\sigma_U(\bar{x})$,
- (g) $\text{res}(\varphi_2(\bar{y}, \bar{z}), \sigma_V(\bar{y}))$, that is, the subgroup defined by $\varphi_2(\bar{y}, \bar{z})$ satisfies $\sigma_V(\bar{y})$.

Now suppose that \tilde{G} is a profinite group and $\bar{u}^\sim, \bar{v}^\sim, \bar{h}^\sim$ are tuples in \tilde{G} of the appropriate lengths such that

$$\tilde{G} \models \Phi(\bar{u}^\sim, \bar{v}^\sim, \bar{h}^\sim) \wedge \Psi(\bar{u}^\sim, \bar{v}^\sim, \bar{h}^\sim).$$

Let \tilde{U} , \tilde{V} , \tilde{H} be the subsets of \tilde{G} defined by $\varphi_1(\bar{u}^\sim, \bar{h}^\sim)$, $\varphi_2(\bar{v}^\sim, \bar{h}^\sim)$, $\chi(\bar{h}^\sim)$. Then Ψ ensures that $\tilde{U} \cong U$ and $\tilde{V} \cong V$ are pro- p groups, generated respectively by \bar{u}^\sim , \bar{v}^\sim . Also Φ ensures that \tilde{H} is closed, normalizes \tilde{V} , acting faithfully by conjugation, and that $[\tilde{V}, \tilde{H}] \subseteq \tilde{V}^p$; this now implies that \tilde{H} is a pro- p group, and hence that $\tilde{V} \cdot \tilde{H}$ is a pro- p group. Φ also ensures that $\tilde{G} = \tilde{V} \cdot \tilde{H} \cdot \tilde{U}$. Since a product of two pro- p subgroups is again pro- p , it now follows that \tilde{G} is a pro- p group.

Finally, Claim 4 with Theorem 5.15 provides a formula $\sigma_G(\bar{x}, \bar{y}, \bar{z})$ that determines $(G, \bar{u}, \bar{v}, \bar{h})$ among pro- p groups. It follows that

$$\Phi(\bar{x}, \bar{y}, \bar{z}) \wedge \Psi(\bar{x}, \bar{y}, \bar{z}) \wedge \sigma_G(\bar{x}, \bar{y}, \bar{z})$$

determines $(G, \bar{u}, \bar{v}, \bar{h})$ among all profinite groups.

Proof of Claim 4. Proposition 4.32 in [5] shows that G has a presentation as a pro- p group with relations that express the commutators of basis elements v_{ij} , u_{ij} , h_k in canonical form, as products of p -adic powers in a fixed order (see the proof of Lemma 5.14). In most cases, this canonical form is either 1 or $x^{\pm p}$ where x is a basis element; let us call the corresponding set of relations \mathcal{R}_1 . The exceptions are those which express $[v_{ij}, u_{ij}]$, $[u_{ij}, h_k]$, $[v_{ij}, h_k]$, where the canonical form involves non-integral powers of basis elements; let us call this set of relations \mathcal{R}_2 .

Relations in \mathcal{R}_2 of the first kind can be re-written as

$$[v_{ij}, u_{ij}] = h_{i-1}^p h_i^{-p} v_{ij}^{-p^2} u_{ij}^{p^2} h_{j-1}^{-p} h_j^p. \quad (18)$$

For the others, one verifies that

$$u_{ij}^{h_k^{-p}} = u_{ij}^{1+p^2} \quad (i \leq k < j), \quad (19)$$

$$v_{ij}^{h_k^p} = v_{ij}^{1+p^2} \quad (i \leq k < j). \quad (20)$$

Now we have shown above that given Claims 1–3, there are formulae Φ and Ψ such that for any profinite group \tilde{G} , if $\tilde{G} \models \Phi(\bar{u}^\sim, \bar{v}^\sim, \bar{h}^\sim) \wedge \Psi(\bar{u}^\sim, \bar{v}^\sim, \bar{h}^\sim)$, then \tilde{H} is abelian, has no p -torsion and acts faithfully on \tilde{U} . In this situation, the action of $h \in \tilde{H}$ on \tilde{U} is determined by the action of h^p , and similarly for the action on \tilde{V} . This now implies that (19), (20) are equivalent to the ‘canonical’ relations specifying $[u_{ij}, h_k]$ and $[v_{ij}, h_k]$.

Let $\Delta(\bar{u}, \bar{v}, \bar{h})$ be a formula that expresses the relations \mathcal{R}_1 together with (18)–(20). The preceding argument shows that $\Delta \wedge \Phi \wedge \Psi$ is equivalent to the conjunction of $\Phi \wedge \Psi$ with the original set of relations $\mathcal{R}_1 \cup \mathcal{R}_2$. As the latter give a pro- p presentation of G , it follows that $\Delta \wedge \Phi \wedge \Psi$ is an L_{gp} presentation of G as a pro- p group. \square

The case $n = 2$. In the above argument, the hypothesis $n \geq 3$ is only essential to ensure that U and V satisfy the O-S condition, which in turn is only used to establish that $\tilde{U} \cong U$ and

$\tilde{V} \cong V$ are pro- p groups. If $n = 2$, this has to be established by a different route. The idea is to show that the ring \mathbb{Z}_p can be interpreted in G by the definable subgroup U , and then use the fact that \mathbb{Z}_p is FA in the class of rings whose additive group is profinite (see §4.3 and the proof of Theorem 4.9). Together, these allow us to express the fact that $U \cong \mathbb{Z}_p$ by a suitable formula with parameters u, h . The same applies with V, v in place of U, u . With these formulae in place of $\text{res}(\varphi_1(\bar{x}, \bar{z}), \sigma_U(\bar{x}))$ and $\text{res}(\varphi_2(\bar{y}, \bar{z}), \sigma_V(\bar{y}))$ in the definition of $\Psi(\bar{x}, \bar{y}, \bar{z})$, one finds that $\tilde{U} \cong \tilde{V} \cong \mathbb{Z}_p$, and the argument then proceeds as before.

Finally, we go from G to the full linear group; here n may be any integer ≥ 2 .

Set $u_0 = 1 + e_{12}$, and let $w = e_{12} + \cdots + e_{n-1,n} \pm e_{n,1}$ be the permutation matrix for the n -cycle $(12 \dots n)$, adjusted to have determinant equal to 1. We need two easy lemmas.

LEMMA 6.2. *Let $m \in \mathbb{Z}$ with $m \equiv 1 \pmod{p}$. There is a formula $\chi_m(\bar{y}, \bar{z}, x)$ such that*

$$G \models \chi_m(\bar{v}, \bar{h}, x) \iff x = h(m^{-1}) := \text{diag}(m, m^{-1}, 1, \dots, 1).$$

LEMMA 6.3. *Put $m = 1 + p$. Then*

$$u_0^{-1} v_{12} u_0 = v_{12}^{m^{-1}} h(m^{-1}) u_{12}^{m^{-1}}, \quad (u_0^{-1} h_1 u_0)^p = u_{12}^p h_1^p.$$

Now we can deduce

THEOREM 6.4. *The groups $\text{SL}_n(\mathbb{Z}_p)$ and $\text{PSL}_n(\mathbb{Z}_p)$ are FA in the class of all profinite groups.*

Proof. Write $\gamma : \Gamma = \text{SL}_n(\mathbb{Z}_p) \rightarrow \tilde{\Gamma} = \text{PSL}_n(\mathbb{Z}_p)$ for the quotient map. As γ restricts to an injective map on G , we may consider both Γ and $\tilde{\Gamma}$ as finite extensions of G . As $Z(G) = 1$ we may apply Theorem 3.1(ii). The following argument deals with $\tilde{\Gamma}$; the same argument with applied to everything will give the result for Γ .

For convenience, we shall allow $\bar{u}, \bar{v}, \bar{h}$ to denote the sets $\{u_{ij} \dots\}$, etc., as well ordered tuples. By conjugating with w and forming commutators and inverses, we can obtain every elementary matrix from u_0 . It follows that

$$G = \langle \bar{u}, \bar{v}, \bar{h} \rangle \triangleleft \Gamma = \langle u_0, w, \bar{h} \rangle, \\ \langle \bar{u}, \bar{v}, \bar{h} \rangle \subseteq \langle u_0, w, \bar{h} \rangle.$$

Thus it will suffice to verify that x^y is $(\bar{u}, \bar{v}, \bar{h})$ -definable in G for each $x \in \bar{u} \cup \bar{v} \cup \bar{h}$ and $y \in \{u_0, w, \bar{h}\}$. This is obvious for $y \in \bar{h}$; for $y = w$ it follows from the relations $h_i^w = h_i^{-1} h_{i+1} h_{i-1}^w$.

Now u_0 commutes with every u_{ij} and every h_k for $k \geq 2$, and conjugates each v_{ij} with $(i, j) \neq (1, 2)$ into the group $\langle \bar{u}, \bar{v} \rangle$. So it remains to deal with $v_{12}^{u_0}$ and $h_1^{u_0}$.

We use the two preceding lemmas, and keep their notation. Set $m = 1 + p$,

$$\alpha(\bar{u}, \bar{v}, \bar{h}, x, y, z) := (x^m = v_{12}) \wedge \chi_m(\bar{v}, \bar{h}, y) \wedge (z^m = u_{12}),$$

$$\varphi(\bar{u}, \bar{v}, \bar{h}, t) := \exists x, y, z. (t = xyz \wedge \alpha(\bar{u}, \bar{v}, \bar{h}, x, y, z)).$$

Then $G \models \varphi(\bar{u}, \bar{v}, \bar{h}, v_{12}^{u_0})$, and this defines $v_{12}^{u_0}$ in G , using a ‘uniqueness of expression’ property in the identity (17).

Put

$$\psi(\bar{u}, \bar{h}, t) := (t^p = u_{12}^p h_1^p).$$

Then $\psi(\bar{u}, \bar{h}, t)$ defines $h_1^{u_0}$ in G , because extraction of p th roots is unique in the uniform pro- p group G .

The result follows from Theorem 3.1(ii). \square

7. Some negative results

7.1. Infinitely many primes

The Feferman–Vaught theorem [2, Proposition 6.3.2] implies the following: for each sentence ϕ of L_{gp} there exist finitely many sentences ψ_1, \dots, ψ_n of L_{gp} and a formula $\theta(x_1, \dots, x_n)$ in the language of Boolean algebras such that for any family of groups $\{G_i \mid i \in I\}$, setting $X_j = \{i \in I \mid G_i \models \psi_j\}$ we have

$$G(I) := \prod_{i \in I} G_i \models \phi \iff \mathcal{P}(I) \models \theta(X_1, \dots, X_n)$$

($\mathcal{P}(I)$ denotes the power set of I). To deduce Proposition 1.2, note that by the pigeonhole principle, provided $|I| > 2^n$, there exist $r \neq q \in I$ such that the characteristic functions of X_1, \dots, X_n each take the same values on r and q . Then for each j we have $G_r \models \psi_j \iff G_q \models \psi_j$, so setting $H_i = G_i$ ($i \neq q$) and $H_q = G_r$ we see that $H(I) \models \phi \iff G(I) \models \phi$. The proposition follows.

Now we may rephrase Proposition 1.3 as follows.

PROPOSITION 7.1. *Let π be an infinite set of primes.*

(i) *If $G_p \neq 1$ is a pro- p group for each $p \in \pi$, then $\prod_{p \in \pi} G_p$ is not FA in the class of pronilpotent groups.*

(ii) *Let \mathfrak{G} be a linear algebraic group of positive dimension defined over \mathbb{Q} . Then $\mathfrak{G}(\mathbb{Z}_\pi)$ is not FA in the class of profinite groups.*

Proof. (i) Suppose that ϕ is a sentence satisfied by $G := \prod_{p \in \pi} G_p$. For some $r \neq q \in \pi$ we have

$$H := G_r \times \prod_{p \in \pi \setminus \{q\}} G_p \models \phi.$$

But $H = H^q$ and $G \neq G^q$, so $H \not\cong G$ and so ϕ does not determine G .

(ii) Similarly, if ϕ determines $\mathfrak{G}(\mathbb{Z}_\pi) = \prod_{p \in \pi} \mathfrak{G}(\mathbb{Z}_p)$ among profinite groups, it follows that for some $r \neq q \in \pi$,

$$H := \mathfrak{G}(\mathbb{Z}_r) \times \mathfrak{G}(\mathbb{Z}_{\pi \setminus \{q\}}) \cong \mathfrak{G}(\mathbb{Z}_\pi) \quad (21)$$

as profinite groups. The projection $\mathfrak{G}(\mathbb{Z}_\pi) \rightarrow \mathfrak{G}(\mathbb{Z}_q)$ then induces a surjective continuous homomorphism $\mu : H \rightarrow \mathfrak{G}(\mathbb{Z}_q)$.

Write $\mathfrak{G}^1(\mathbb{Z}_p)$ for the congruence subgroup modulo p (or modulo 4 if $p = 2$). This is a pro- p group, it is torsion-free and has finite index in $\mathfrak{G}(\mathbb{Z}_p)$ (see [5, § 5.1]). Hence for $p \in \pi \setminus \{q\}$ we have $\mathfrak{G}^1(\mathbb{Z}_p)\mu \cap \mathfrak{G}^1(\mathbb{Z}_q) = 1$, and it follows that $\mathfrak{G}(\mathbb{Z}_p)\mu$ is finite, as well as normal in $\mathfrak{G}(\mathbb{Z}_q)$ (here $\mathfrak{G}(\mathbb{Z}_r)$ is appearing twice, as it does in H).

Thus writing K for the maximal finite normal subgroup of $\mathfrak{G}(\mathbb{Z}_q)$ we deduce:

$$\mathfrak{G}(\mathbb{Z}_p)\mu \leq K \text{ for all } p \in \pi \setminus \{q\}$$

(this applies in particular to both copies of $\mathfrak{G}(\mathbb{Z}_r)$ in H). As the subgroups $\mathfrak{G}(\mathbb{Z}_p)$ generate H topologically, this implies that $\mathfrak{G}(\mathbb{Z}_q) = K$ is finite. This is not the case, because $\mathfrak{G}(\mathbb{Z}_q)$ is open in $\mathfrak{G}(\mathbb{Q}_q)$, a q -adic manifold of positive dimension $\dim(\mathfrak{G})$ (see [30, § 3.1]). \square

REMARK. If the reductive part of \mathfrak{G} is semisimple, (21) cannot even be an abstract group isomorphism; this requires a little more argument. When \mathfrak{G} is connected, semisimple and simply connected, the profinite group H is finitely generated [31, Theorem 2], so μ is continuous by Theorem 2.1. In general, one can show that the image in H of the simply connected cover of the connected component of \mathfrak{G} contains H^m for some m ; the argument then goes as before,

noting that $\mathfrak{S}(\mathbb{Z}_q)^m$ has finite index in $\mathfrak{S}(\mathbb{Z}_q)$. We are grateful to Andrei Rapinchuk for help with this question.

7.2. Uncountably many pro- p groups

For $\lambda \in \mathbb{Z}_p$ let $T(\lambda)$ be the class-2 nilpotent pro- p group with pro- p presentation on generators $x_1, \dots, x_4, y_1, \dots, y_4, e, f$ and relations

$$\begin{aligned} [x_i, x_j] &= [y_i, y_j] = 1 \quad (\text{all } i, j) \\ [x_i, y_j] &= 1 \quad (\text{all } i \neq j) \\ [x_1, y_1] &= e, \quad [x_2, y_2] = ef^{-1} \\ [x_3, y_3] &= f, \quad [x_4, y_4] = ef^{-\lambda}. \\ e, f &\text{ central} \end{aligned}$$

This is clearly a pro- p group of rank 10 (with centre \mathbb{Z}_p^2 and central quotient \mathbb{Z}_p^8) and so contains an open normal uniform subgroup $T^*(\lambda)$ (for example, the subgroup generated by e, f and the p th powers of the x_i and y_j). It is proved in [8, § 6, page 153] that the groups $T(\lambda)$ are pairwise non-commensurable. It follows that the groups $T^*(\lambda)$ are pairwise non-isomorphic.

Note that $T(\lambda)$ is strictly f.p. when λ is a rational p -adic integer a/b ($a, b \in \mathbb{Z}$, $p \nmid b$): the relation involving λ is equivalent to

$$[y_4, x_4]^b e^b = f^a, \quad [x_4, y_4] \text{ central}$$

(because we have unique extraction of b th roots).

8. List of formulae

$$\begin{aligned} s(\kappa), \quad s_{\triangleleft}(\kappa), \quad \text{res}(\kappa, \varphi), \quad \text{lift}(\kappa, \varphi), \quad \text{ind}(\kappa; n), \quad \text{ind}^*(\kappa; n), \quad \text{com}(x, y) : & \quad \S 2 \\ \delta(\bar{a}, x), \quad \beta_d(a_1, \dots, a_d), \quad \tilde{\beta}_d, \quad \beta_d^* : & \quad \S 5.1 \\ \text{pow}, \mu_{f,q}(x), \quad m_{f,q}, \quad \partial_{d_1, \dots, d_k} : & \quad \S 5.2 \\ \text{tf} : & \quad \S 5.4 \\ Z(x), \quad Z_2(x), \quad S_p(x), \quad \psi_\pi, \quad \Gamma_c : & \quad \S 5.5 \\ \Gamma_{m,f} : & \quad \S 5.6 \end{aligned}$$

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