

Local and Global Well-posedness of Compressible Navier-Stokes Equations



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This thesis is dedicated to
my Parents, Ying Wang and Luxi Huang
for their unending support and encouragement

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Abstract

The well-posedness theory of the full compressible Navier-Stokes equations (**CNS** for short) has been an important subject in the development of mathematical physics. For the spherically symmetric flow, even though many favourable regularity properties may be expected for its similarity to the one-dimensional equations, there are still a large number of open questions remaining unsolved. Two central difficulties of spherically symmetric **CNS** are the coordinate singularity at the centre of symmetry, and the upper/lower bounds of density.

In Chapter 1, the Cauchy problems of **CNS** for the general multidimensional flows, both with and without spherical symmetry, are introduced. Then a heuristic analysis on the Lagrangian coordinate transformation is conducted, which gives insight to the strategies used in the later chapters. Finally, several previous progresses on the well-posedness theory of **CNS** are highlighted at the end of this chapter.

In Chapter 2, the global-in-time existence of a spherically symmetric weak solution to the Cauchy problem in Eulerian coordinate is obtained. The main strategy is to construct a sequence of approximation problems posed in finite annular domains. This circumvents the problem of coordinate singularity at the origin. Moreover, they are also formulated in the Lagrangian coordinates, so that one can obtain *a-priori* estimates which are uniform with respect to the dimension of annular domains. The weak solution is then constructed as the limit of approximate solutions in the original Eulerian coordinate. The main result of this chapter is produced in collaboration with Gui-Qiang G. Chen and Shengguo Zhu.

In Chapter 3, for the Lagrangian approximation problem introduced in Chapter 2, the existence and uniqueness of a local-in-time strong solution are shown. This is done via the Picard iteration scheme, however some major difficulties arise due to the fact that second order differential operators in momentum and energy equations is quasilinear under the Lagrangian formulation. To resolve this, a set of *a-priori* bounds on each iterative solutions is established so that the quasilinear operators are elliptic at each iterative step. Moreover, the time dependent high regularity estimates for linear iterative solution are carefully derived so that Banach Fixed-point theorem can be applied to obtain the desired solution.

In Chapter 4, it is proved that the local-in-time strong solution in Chapter 3 can be extended to all time by bootstrap argument. Moreover, global-in-time strong solutions in the unbounded Lagrangian domain are also obtained by taking the limit of outer and inner radii of annular domains. The main difference of this chapter compared to Chapter 2 is that the limit is taken in the Lagrangian coordinates, and the initial data are assumed to have higher regularities.

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Chapter 1

Introduction

1.1 Cauchy Problem for Full Navier-Stokes Equations

The motion of a viscous polytropic ideal gas in $\mathbb{R}^n \times [0, T]$, for some time $T > 0$ with $n = 2$ or 3 is described by the following system in Eulerian coordinates:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{U}) = 0, \\ \partial_t(\rho \mathbf{U}) + \operatorname{div}(\rho \mathbf{U} \otimes \mathbf{U}) + \nabla P = \operatorname{div} \mathbb{S}, \\ \partial_t(\rho E) + \operatorname{div}((\rho E + P)\mathbf{U}) = \operatorname{div}(\mathbb{S}(\nabla \mathbf{U}) \cdot \mathbf{U}) + \kappa \Delta e, \end{cases} \quad \text{in } (\mathbf{x}, t) \in \mathbb{R}^n \times [0, T], \quad (1.1.1)$$

where ρ , e , and $\mathbf{U} = (U^1, \dots, U^n)$ are respectively the density, the internal energy per unit mass, and the velocity. $E := \rho e + |\mathbf{U}|^2/2$ is the total energy per unit mass. The pressure function P is given by the constitutive relation for ideal polytropic gas:

$$P = (\gamma - 1)\rho e, \quad (1.1.2)$$

with constant $\gamma > 1$ being the heat capacity ratio. The viscous stress tensor $\mathbb{S}(\nabla \mathbf{U})$ takes the form:

$$\mathbb{S}(\nabla \mathbf{U}) := \mu(\nabla \mathbf{U} + \nabla \mathbf{U}^\top) + \lambda \mathbb{I}_{n \times n} \operatorname{div} \mathbf{U}, \quad (1.1.3)$$

where $\mathbb{I}_{n \times n}$ is the $n \times n$ identity matrix and $\mu > 0, \lambda$ are the viscosity coefficients satisfying the physical condition:

$$\mu > 0, \quad \frac{2}{n}\mu + \lambda > 0. \quad (1.1.4)$$

In Chapter 2, the global-in-time existence of a weak solution is considered. For this, we impose the initial condition:

$$(\rho, \mathbf{U}, e)(\mathbf{x}, t) = (\rho_0, \mathbf{U}_0, e_0)(\mathbf{x}) \quad \text{for } \mathbf{x} \in \mathbb{R}^n, \quad (1.1.5)$$

where $(\rho_0, \mathbf{U}_0, e_0)(\mathbf{x})$ belongs in some suitable Sobolev spaces.

In Chapter 3–4, we set $(\rho_0, \mathbf{U}_0, e_0)(\mathbf{x})$ to be a smooth initial data satisfying

$$(\rho_0, \mathbf{U}_0, e_0, \nabla e_0)(\mathbf{x}) \rightarrow (1, \mathbf{0}, 1, 0) \quad \text{as } |\mathbf{x}| \rightarrow \infty,$$

and we aim to obtain a strong solution to the problem (1.1.1)–(1.1.4) with the initial-boundary conditions:

$$\begin{cases} (\rho, \mathbf{U}, e)(\mathbf{x}, 0) = (\rho_0, \mathbf{U}_0, e_0)(\mathbf{x}) & \text{for } \mathbf{x} \in \mathbb{R}^n, \\ (\rho, \mathbf{U}, e, \nabla e)(\mathbf{x}, t) \rightarrow (1, \mathbf{0}, 1, 0) & \text{for } t \in [0, T] \text{ as } |\mathbf{x}| \rightarrow \infty. \end{cases} \quad (1.1.6)$$

Here the reference density and internal energy per unit mass are set to be $\tilde{\rho} = 1$ and $\tilde{e} = 1$.

1.2 Spherically Symmetric Problem

Given a spherical symmetric initial data:

$$(\rho_0(\mathbf{x}), \mathbf{U}_0(\mathbf{x}), e_0(\mathbf{x})) = (\rho_0(|\mathbf{x}|), u_0(|\mathbf{x}|) \frac{\mathbf{x}}{|\mathbf{x}|}, e_0(|\mathbf{x}|)),$$

our aim is to find spherically symmetric solution to the Cauchy Problem (1.1.1)–(1.1.6):

$$(\rho(\mathbf{x}, t), \mathbf{U}(\mathbf{x}, t), e(\mathbf{x}, t)) = (\rho(|\mathbf{x}|, t), u(|\mathbf{x}|, t) \frac{\mathbf{x}}{|\mathbf{x}|}, e(|\mathbf{x}|, t)).$$

Denoting the radial coordinate by $r \equiv |\mathbf{x}|$, and letting $m \equiv n - 1$, then under the symmetry assumption, the Cauchy problem (1.1.1)–(1.1.6) can be restated as:

$$\begin{cases} \partial_t \rho + \partial_r(\rho u) + m \frac{\rho u}{r} = 0 \\ \rho \partial_t u + \rho u \partial_r u + \partial_r P(\rho, e) = (2\mu + \lambda) \partial_r \left(\partial_r u + m \frac{u}{r} \right) \\ \rho \partial_t e + \rho u \partial_r e + P(\rho, e) \left(\partial_r u + m \frac{u}{r} \right) = 2\mu \left(|\partial_r u|^2 + m \frac{u^2}{r^2} \right) + \lambda \left(\partial_r u + m \frac{u}{r} \right)^2 + \kappa \left(\partial_r^2 e + m \frac{\partial_r e}{r} \right) \end{cases} \quad (1.2.1)$$

in the domain $(r, t) \in \Omega_{0,T} := [0, \infty) \times [0, T]$, and it is supplemented with the initial-boundary condition:

$$\begin{cases} u(0, t) = 0, \quad \partial_r e(0, t) = 0 & \text{for } t \in [0, T], \\ \lim_{r \rightarrow \infty} (\rho, u, e, \partial_r e)(r, t) = (1, 0, 1, 0) & \text{for } t \in [0, T], \\ (\rho, u, e)(r, 0) = (\rho_0, u_0, e_0)(r) & \text{for } r \in [0, \infty), \end{cases} \quad (1.2.2)$$

It is important to remark here that the following boundary conditions at the origin is a necessary consequence of the spherical coordinate structure:

$$u(0, t) = 0, \quad \partial_r e(0, t) = 0, \quad \text{for } t \in [0, T]. \quad (1.2.3)$$

The necessity of these two conditions are entirely mathematical motivated by the fact that vector fields $\mathbf{U}(\mathbf{x}, t), \nabla e(\mathbf{x}, t)$ satisfy the spherical symmetry:

$$\mathbf{U}(-\mathbf{x}, t) = -\mathbf{U}(\mathbf{x}, t) \quad \text{and} \quad \nabla e(-\mathbf{x}, t) = \nabla e(\mathbf{x}, t), \quad \text{for } (\mathbf{x}, t) \in \mathbb{R}^n \times [0, T]$$

Hence, if $\mathbf{x} \mapsto (\mathbf{U}, \nabla e)(\cdot, t)$ is continuous near the origin, then we must have

$$\lim_{|\mathbf{x}| \rightarrow 0^+} \mathbf{U}(\mathbf{x}, t) = \lim_{|\mathbf{x}| \rightarrow 0^+} \nabla e(\mathbf{x}, t) = 0.$$

1.3 The Exterior Problem and Lagrange Reformulation

The spherical coordinate transformation in previous section leads to the coordinate singularity at the origin, which poses a significant difficulty in the existence problem. To cope with this issue, we consider the problem in an exterior domain $\Omega_{a,T} := \{(r, t) : a \leq r < \infty, 0 \leq t \leq T\}$ for $a \in (0, 1)$, and find approximating solutions (ρ_a, u_a, e_a) to the equations

$$\begin{cases} \partial_t \rho_a + \partial_r(\rho_a u_a) + m \frac{\rho_a u_a}{r} = 0 \\ \rho_a \partial_t u_a + \rho_a u_a \partial_r u_a + \partial_r P(\rho_a, e_a) = \beta \partial_r \left(\partial_r u_a + m \frac{u_a}{r} \right) \\ \rho_a \partial_t e_a + \rho_a u_a \partial_r e_a + P(\rho_a, e_a) \left(\partial_r u_a + m \frac{u_a}{r} \right) \\ = 2\mu \left(|\partial_r u_a|^2 + m \frac{|u_a|^2}{r^2} \right) + \lambda \left(\partial_r u_a + m \frac{u_a}{r} \right)^2 + \kappa \left(\partial_r^2 e_a + m \frac{\partial_r e_a}{r} \right) \end{cases} \quad \text{for } (r, t) \in \Omega_{a,T}, \quad (1.3.1)$$

where $\beta \equiv 2\mu + \lambda$, and it is supplemented with the following initial boundary conditions:

$$\begin{cases} u_a(a, t) = 0, & \partial_r e_a(a, t) = 0 \\ \lim_{r \rightarrow \infty} (\rho_a, u_a, e_a, \partial_r e_a)(r, t) = (1, 0, 1, 0) & \text{for } t \in [0, T], \\ (\rho_a, u_a, e_a)(r, 0) = (\rho_a^0, u_a^0, e_a^0)(r) & \text{for } r \in [a, \infty), \end{cases} \quad (1.3.2)$$

where (ρ_a^0, u_a^0, e_a^0) is the approximating initial data modified with respect to (ρ_0, u_0, e_0) . The details of this construction will be discussed in Sections 2.2.2 and 4.5.1. The boundary conditions $u_a(a, t) = 0$, $\partial_r e_a(a, t) = 0$ for $t \in [0, T]$ can be interpreted physically as the presence of a solid insulating ball at the centre of symmetry, with no slip-boundary condition for the velocity field.

In order to obtain point-wise upper and lower bounds of density, it is advantageous to formulate the Exterior Problem (1.3.1)–(1.3.2) in the Lagrangian coordinate (x, t) . Here we give a heuristic study on the coordinate transformation law between Eulerian and Lagrangian under the spherical symmetry assumption. Suppose that $(\rho_a, u_a, e_a)(r, t)$ is a smooth solution such that $C^{-1} \leq \rho_a \leq C$ for some constant $C > 0$. For general multi-dimensional flow, the transformation law between the Eulerian coordinate variables, denoted as $(\mathbf{y}, t) \in \mathbb{R}^n \times [0, T]$, and Lagrangian coordinate variables, $(\mathbf{z}, t) \in \mathbb{R}^n \times [0, T]$, is given as:

$$(\mathbf{y}, t) = (\mathbf{X}(\mathbf{z}, t), t) \quad \text{where } \mathbf{X} \text{ is the solution of } \begin{cases} \frac{d\mathbf{X}}{dt}(\mathbf{z}, t) = \mathbf{U}_a(\mathbf{X}(\mathbf{z}, t), t) & \text{for } t \in [0, T], \\ \mathbf{X}(\mathbf{z}, 0) = \varphi_0(\mathbf{z}) & \text{for } \mathbf{z} \in \mathbb{R}^n, \end{cases} \quad (1.3.3)$$

where $\varphi_0 : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a given diffeomorphism, and $\mathbf{U}_a(\mathbf{y}, t) := u_a(|\mathbf{y}|, t) \frac{\mathbf{y}}{|\mathbf{y}|}$. Denote $r \equiv |\mathbf{y}|$ and $x \equiv |\mathbf{z}|$. Here, we set φ_0 to be the spherically symmetric map given by:

$$\varphi_0(\mathbf{z}) := \tilde{r}_a^0(|\mathbf{z}|) \frac{\mathbf{y}}{|\mathbf{y}|} \quad \text{where } \tilde{r}_a^0(\cdot) \text{ is implicitly defined by } |\mathbf{z}| = \int_a^{\tilde{r}_a^0(|\mathbf{z}|)} \rho_a^0(\zeta) \zeta^m d\zeta \quad \text{for all } |\mathbf{z}| \geq 0.$$

Since $0 < C_0^{-1} \leq \rho_a^0 \leq C_0 < \infty$ for some constant $C_0 > 0$, it follows that $\tilde{r}_a^0(x)$ is well-defined by Inverse Function theorem. This definition means that $|\mathbf{z}|$ amount of initial mass is contained in the annular region: $\{\mathbf{y} \in \mathbb{R}^n : a \leq |\mathbf{y}| \leq \tilde{r}_a^0(x)\}$. By the uniqueness of solution to (1.3.3), it follows that $\mathbf{z} \mapsto \mathbf{X}(\mathbf{z}, t)$ is also spherically symmetric for each $t \in [0, T]$, hence the function $\tilde{r}_a(x, t) := |\mathbf{X}(\mathbf{z}, t)|$, where $x \equiv |\mathbf{z}|$, is well-defined for $x \geq 0$. Suppose $\tilde{r}_a(x, t) \geq a$ for all $t \in [0, T]$. Then (1.3.3) implies that $\tilde{r}_a(x, t)$ satisfies

$$\begin{cases} \partial_t \tilde{r}_a(x, t) = u_a(\tilde{r}_a(x, t), t) & \text{for } t \in [0, T], \\ \tilde{r}_a(x, 0) = \tilde{r}_a^0(x) & \text{for } x \geq 0. \end{cases}$$

Using the continuity equation (1.3.1)₁ and Leibniz integral rule, one can compute to see that

$$\frac{d}{dt} \int_a^{\tilde{r}_a(x, t)} \rho_a(\zeta, t) \zeta^m d\zeta = 0 \quad \text{for all } t \in [0, T] \text{ and for each fixed } x \geq 0.$$

Integrating in time and using the construction of $\tilde{r}_a^0(x)$, we have that

$$\int_a^{\tilde{r}_a(x, t)} \rho_a(\zeta, t) \zeta^m d\zeta = \int_a^{\tilde{r}_a^0(x)} \rho_a^0(\zeta) \zeta^m d\zeta = x \quad \text{for all } (x, t) \in [0, \infty) \times [0, T]. \quad (1.3.4)$$

This states that x amount of mass is conserved within the annular region $\{\mathbf{y} \in \mathbb{R}^n : a \leq |\mathbf{y}| \leq \tilde{r}_a(x, t)\}$. Now fix a time $t \in [0, T]$. Implicitly differentiating (1.3.4) with respect to x , we also obtains:

$$\partial_x \tilde{r}_a(x, t) = \frac{(\tilde{r}_a(x, t))^{-m}}{\rho_a(\tilde{r}_a(x, t), t)} \quad \text{and} \quad \tilde{r}_a(x, t) = \left(a^n + \int_0^x \frac{n}{\rho_a(\tilde{r}_a(y, t), t)} dy \right)^{\frac{1}{n}} \quad \text{for all } x \geq 0.$$

If $C^{-1} \leq \rho_a \leq C$ for some constant $C > 0$, then the inverse mapping $(\tilde{x}_a(r, t), t)$ exists due to Inverse Function theorem. For $r = \tilde{r}_a(x, t)$ and $x = \tilde{x}_a(r, t)$, these differential relations for the coordinate transformation can be summarized into the Jacobian matrix as follows

$$\frac{\partial(r, t)}{\partial(x, t)} = \begin{pmatrix} \tilde{\rho}_a^{-1} \tilde{r}_a^{-m} & \tilde{u}_a \\ 0 & 1 \end{pmatrix}, \quad \frac{\partial(x, t)}{\partial(r, t)} = \begin{pmatrix} \rho_a r^m & -\rho_a u_a r^m \\ 0 & 1 \end{pmatrix}. \quad (1.3.5)$$

Now we set $\tilde{\rho}_a(x, t) := \rho_a(\tilde{r}_a(x, t), t)$, $\tilde{u}_a(x, t) := u_a(\tilde{r}_a(x, t), t)$, and $\tilde{e}_a(x, t) := e_a(\tilde{r}_a(x, t), t)$. We denote (D_t, D_x) as the partial derivative operators with respect to the Lagrangian coordinate variables and let $\tilde{v}_a := 1/\tilde{\rho}_a$ be the specific volume. Then, under Lagrange coordinates, equations (1.3.1) can be reformulated into the following form:

$$\begin{cases} D_t \tilde{v}_a - D_x(\tilde{r}_a^m \tilde{u}_a) = 0 \\ D_t \tilde{u}_a + \tilde{r}_a^m D_x p(\tilde{v}_a, \tilde{e}_a) = \beta \tilde{r}_a^m D_x \left(\frac{D_x(\tilde{r}_a^m \tilde{u}_a)}{\tilde{v}_a} \right) \\ D_t \tilde{e}_a + p(\tilde{v}_a, \tilde{e}_a) D_x(\tilde{r}_a^m \tilde{u}_a) = \beta \frac{|D_x(\tilde{r}_a^m \tilde{u}_a)|^2}{\tilde{v}_a} - 2m\mu D_x(\tilde{r}_a^{m-1} \tilde{u}_a^2) + \kappa D_x \left(\frac{\tilde{r}_a^{2m} D_x \tilde{e}_a}{\tilde{v}_a} \right) \\ \tilde{r}_a(x, t) = \left(a^n + n \int_0^x \tilde{v}_a(y, t) dy \right)^{\frac{1}{n}} \end{cases} \quad (1.3.6)$$

in $(x, t) \in [0, \infty) \times [0, T]$ where p denotes $p(v, e) \equiv P(v^{-1}, e) = (\gamma - 1)e/v$. In addition (1.3.6) is supplemented with the initial boundary conditions:

$$\begin{cases} \tilde{u}_a(0, t) = 0, \quad D_x \tilde{e}_a(0, t) = 0 & \text{for } t \in [0, T], \\ (\tilde{v}_a, \tilde{u}_a, \tilde{e}_a, D_x \tilde{e}_a)(x, t) \rightarrow (1, 0, 1, 0) & \text{for } t \in [0, T] \text{ as } x \rightarrow \infty, \\ (\tilde{v}_a, \tilde{u}_a, \tilde{e}_a)(x, 0) = (\tilde{v}_a^0, \tilde{u}_a^0, \tilde{e}_a^0)(x) & \text{for } x \in [0, \infty), \end{cases} \quad (1.3.7)$$

where $(\tilde{v}_a^0, \tilde{u}_a^0, \tilde{e}_a^0)(x) = (\rho_a^0, u_a^0, e_a^0)(\tilde{r}_a^0(x))$. In later chapters, we will often abbreviate $v \equiv \tilde{v}_a$, $u \equiv \tilde{u}_a$, and $e \equiv \tilde{e}_a$ for simplicity.

1.4 Previous Results and Recent Progress

The well-posedness problem to the equations (1.1.1)–(1.1.6) has been investigated by many authors. For large smooth initial data with $\inf \rho_0 > 0$ in one dimensional bounded domain, Kazhikhov and Shelukhin in 1977 [16] obtained the global classical solution to the initial-boundary value problem in Lagrangian coordinate. Kawashima and Nishida later extended this result to the unbounded domain in 1981 [15]. As for the three dimensional Cauchy Problem, Itaya in 1971 [12] proved the local-in-time existence and uniqueness of a Hölder continuous solution with Hölder initial data satisfying $0 < \inf \rho_0 < \sup \rho_0 < \infty$. Using energy methods in Sobolev spaces, Matsumura and Nishida in 1981 [20] showed the global existence of classical solution provided that the initial data is small in some energy norms, and initial density is away from vacuum. By imposing the compatibility condition on the initial data:

$$\operatorname{div} \mathbb{S}(\nabla \mathbf{U}_0) - \nabla P(\rho_0, e_0) = \sqrt{\rho_0} f_1, \quad \text{and} \quad \kappa \nabla e_0 + \frac{\mu}{2} |\nabla \mathbf{U}_0 + \nabla \mathbf{U}_0^\top|^2 + \lambda (\operatorname{div} \mathbf{U}_0)^2 = \sqrt{\rho_0} f_2,$$

for some $f_1, f_2 \in L^2(\mathbb{R}^3)$, Cho and Kim in 2006 [2] were able to obtain a unique local-in-time strong solution with initial density possibly containing vacuum. For multi-dimensional spherical symmetric solutions, Jiang in his 1998 paper [13] considered the exterior problem (See Section 1.3), where he replaced the boundary condition near the origin (1.2.3) with: $u(r = a, t) = \partial_r e(r = a, t) = 0$, for $t \in [0, \infty)$, for a fixed $a \in (0, 1)$. He proved the global-in-time existence of a unique spherically symmetric smooth solution to the Cauchy

Problem in Lagrangian coordinate with large smooth initial data such that $0 < \inf \rho_0 < \sup \rho_0 < \infty$. Improving upon the argument of Kazhikhov and Shelukhin in [16], Jiang in [13] obtained an upper and lower bound for density that includes the region near far field: $|x| \rightarrow \infty$, and this played a crucial role in the existence of solution to the Cauchy Problem. However the estimates in his proof heavily depend on the radius of exterior ball $a \in (0, 1)$ centred at the origin. In an attempt to deal with the problem near the origin, Hoff in 1992 [7] constructed a class of global-in-time spherical symmetric weak solutions to the Cauchy Problem of Compressible Navier-Stokes equations, with isothermal pressure $P = A\rho$, in the Eulerian coordinate. Moreover, the weak solutions obtained in [7] possess many additional properties. For instance, Hoff showed that solutions may possibly develop vacuum region about the origin bordered by a bounded Hölder continuous curve $t \mapsto r(t)$. In addition, away from this possible vacuum region, the density is bounded from above and below. Furthermore, he also showed that discontinuity in the initial velocity is smoothed out in positive time, while the discontinuity in initial density persist for all time, convecting along the particle trajectories. Following this work, Hoff and Jenssen in 2004 [11] developed the case for non-isentropic compressible Navier-Stokes equations on the bounded domains in \mathbb{R}^3 , and they obtained global-in-time spherically or cylindrically symmetric weak solutions to the corresponding initial-boundary value problem. The main result of Chapter 2 in this thesis extends the existence theorem of spherically symmetric solution in [11] to the whole domain \mathbb{R}^n , for $n = 2, 3$, hence obtaining a spherically symmetric weak solution to the Cauchy Problem of (1.1.1).

For the isentropic Compressible Navier-Stokes equations, where the pressure solely depends on density: $P = P(\rho)$, we also mention few notable results. The first known well-posedness theorem is obtained by Nash in his 1962 paper, where he showed the local-in-time existence and uniqueness of a classical solution to the multi-dimensional Cauchy problem, provided that $(\rho_0, u_0) \in C^2(\mathbb{R}^n)$, and the initial density satisfies $C_0^{-1} \leq \rho_0 \leq C_0$ for a fixed constant $C_0 > 0$. For a more general class of initial data, which only satisfies the finite energy condition:

$$\int_{\mathbb{R}^n} \left(\rho_0 \frac{|u_0|^2}{2} + \frac{A}{\gamma - 1} \rho_0^\gamma \right) (\mathbf{x}) d\mathbf{x} \leq C_0,$$

where $C_0 > 0$ is a given constant, Lions in 1993 [18] showed the global-in-time existence of finite energy weak solutions, under the assumption that pressure law obeys $P(\rho) = A\rho^\gamma$, where $A > 1$, $\gamma \geq 3/2$ when $n = 2$, and $\gamma \geq 9/5$ when $n = 3$. Subsequently, several improvements on this result has been made, for instance, in the spherically symmetric case, Jiang and Zhang generalised the assumption by allowing $\gamma > 1$ for both $n = 2, 3$, and for general 3 dimensional case, Feireisl, Novotný, and Petzeltová in 2001 [6] relaxed the condition to $\gamma > 3/2$ for $n = 3$. Moreover, under a different set of assumptions on initial data where $\rho_0 - 1$ is small in $L^2 \cap L^\infty$ and u_0 is small in L^2 , Hoff in his 1995 paper [9] also showed the global-in-time existence of weak solutions with certain numerical restriction on the viscosity coefficients μ, λ . Cho, Choe, and Kim in 2003 [3] investigated the problem for initial data with the regularity assumptions $\nabla u_0 \in L^2$, $\rho_0 \in H^1 \cap W^{1,6}$, and $\rho_0 \geq 0$, where initial density can possibly contain vacuum. They were able to obtain a local-in-time strong solution, given that the following initial compatibility condition is satisfied:

$$\rho_0^{-1/2} \{ \mu \Delta u_0 + (\mu + \lambda) \nabla \operatorname{div} u_0 + \nabla P(\rho_0) \} \in L^2.$$

One of the key ideas used in [9], [3], and [17] is the estimate on effective viscosity flux F which is defined by $F := (2\mu + \lambda) \operatorname{div} u + P(\rho)$. The usefulness of F was also illustrated by Desjardin in his 1997 paper [4], where he describes a gain of regularity enjoyed by the quantity F , despite the low regularity assumption on initial data.

Chapter 2

Global Existence of Weak Solutions in Eulerian Coordinates

The main aim of this chapter is to show the global-in-time existence of a weak solution to the Eulerian heat-conducting compressible Navier-Stokes equations (1.1.1) in the Cauchy Domain $(\mathbf{x}, t) \in \mathbb{R}^n \times [0, T]$ for any time $T > 0$, and $n = 2, 3$. In Section 2.1, the precise notion of the weak solutions is defined, and using this definition, we state the main result, Theorem 2.1.1. Then, we give a detailed outline for the proof in Section 2.2.

2.1 Main Results

Definition 2.1.1 (Weak Solutions). *The vector function $(\rho, \mathbf{U}, e)(\mathbf{x}, t)$ is called to be a weak solution of the Cauchy problem (1.1.1)–(1.1.5) in $\mathbb{R}^n \times [0, T]$ if they satisfy the following:*

- (i) *There exists a constant $C_0 = C_0(\rho_0, \mathbf{U}_0, e_0, \gamma, \mu, \lambda, \kappa, n) > 0$ and a upper semi-continuous function $\underline{r}(t) : [0, T] \rightarrow [0, \infty)$ such that*

$$\lim_{t \searrow 0} \underline{r}(t) = 0, \quad 0 \leq \underline{r}(t) \leq C_0 \text{ for all } t \in [0, T].$$

With this, define $\mathcal{F} := \{(\mathbf{x}, t) \in \mathbb{R}^n \times [0, T] : |\mathbf{x}| > \underline{r}(t) \text{ for } t \in [0, T]\}$.

- (ii) *$\rho \in L_{\text{loc}}^\infty(\mathcal{F})$, and $\rho(\mathbf{x}, t) = 0$ for $(\mathbf{x}, t) \in \mathbb{R}^n \times [0, T] \setminus \mathcal{F}$ almost everywhere, $(\mathbf{U}, e)(\mathbf{x}, t)$ is locally Hölder continuous in \mathcal{F} , and $\hat{\mathcal{F}} := \mathcal{F} \cap \{0 < t < T\}$ is an open set.*
- (iii) *For each $\Phi \in C^1([0, T]; C_c^1(\mathbb{R}^n))$,*

$$\int_{\mathbb{R}^n} \rho(\mathbf{x}, t) \Phi(\mathbf{x}, t) d\mathbf{x} - \int_{\mathbb{R}^n} \rho_0(\mathbf{x}) \Phi(\mathbf{x}, 0) d\mathbf{x} = \int_0^t \int_{\mathbb{R}^n} (\rho \partial_t \Phi + \rho \mathbf{U} \cdot \nabla \Phi)(\mathbf{x}, s) d\mathbf{x} ds.$$

- (iv) *$\nabla \mathbf{U} \in L_{\text{loc}}^2(\mathcal{F})$. Moreover, for any $\Psi \in C^1([0, T]; C_c^1(\mathbb{R}^n))$ with $\text{supp}(\Psi) \subset\subset \mathcal{F}$,*

$$\begin{aligned} & \int_{\mathbb{R}^n} \rho U^i(\mathbf{x}, t) \Psi(\mathbf{x}, t) d\mathbf{x} - \int_{\mathbb{R}^n} \rho_0 U_0^i(\mathbf{x}) \Psi(\mathbf{x}, 0) d\mathbf{x} \\ &= \int_0^t \int_{\mathbb{R}^n} (\rho U^i \partial_t \Psi + \rho U^i (\mathbf{U} \cdot \nabla) \Psi + (\gamma - 1) \rho e \partial_i \Psi)(\mathbf{x}, s) d\mathbf{x} ds \\ & \quad - \int_0^t \int_{\mathbb{R}^n} (\mu \nabla \Psi \cdot \nabla U^i + (\mu + \lambda) \partial_i \Psi \text{div} \mathbf{U})(\mathbf{x}, s) d\mathbf{x} ds \quad \text{for each } i = 1, \dots, n. \end{aligned}$$

(v) $\nabla e \in L^2_{\text{loc}}(\mathcal{F})$. Moreover, for any $\Phi \in C^1([0, T]; C^1_c(\mathbb{R}^n))$ with $\text{supp}(\Phi) \subset\subset \mathcal{F}$ then

$$\begin{aligned} & \int_{\mathbb{R}^n} \rho E(\mathbf{x}, t) \Phi(\mathbf{x}, t) \, d\mathbf{x} - \int_{\mathbb{R}^n} \rho_0 E_0(\mathbf{x}) \Phi(\mathbf{x}, 0) \, d\mathbf{x} \\ &= \int_0^t \int_{\mathbb{R}^n} \{ \rho E \partial_t \Phi + (\rho E + P)(\mathbf{U} \cdot \nabla) \Phi \}(\mathbf{x}, s) \, d\mathbf{x} ds \\ & \quad - \int_0^t \int_{\mathbb{R}^n} \left\{ \kappa \nabla e + \frac{\mu}{2} \nabla |\mathbf{U}|^2 + \lambda \mathbf{U} \operatorname{div} \mathbf{U} + \mu (\nabla \mathbf{U}) \cdot \mathbf{U} \right\} \cdot \nabla \Phi(\mathbf{x}, s) \, d\mathbf{x} ds. \end{aligned}$$

Theorem 2.1.1. Let $(\rho_0, \mathbf{U}_0, e_0)(\mathbf{x}) = (\rho_0(r), u_0(r) \frac{\mathbf{x}}{r}, e_0(r))$, with $r := |\mathbf{x}|$, be a spherically symmetric initial data defined in $\mathbf{x} \in \mathbb{R}^n$ satisfying

$$\begin{aligned} e_0(r) &\geq C_*^{-1}, \quad C_*^{-1} \leq \rho_0(r) \leq C_* \quad \text{for all } r \in [0, \infty), \\ \int_0^\infty \left\{ \frac{1}{2} \rho_0 |u_0|^2 + (\gamma - 1) G(\rho_0) + \rho_0 \psi(e_0) + |(\rho_0 - 1, u_0^2, e_0 - 1)|^2 \right\} (r) r^m \, dr &\leq C_*, \end{aligned} \quad (2.1.1)$$

for some given constant $C_* > 0$, where $G(\zeta) = 1 - \zeta + \zeta \log \zeta$ and $\psi(\zeta) = \zeta - 1 - \log \zeta$. Then, for each $T > 0$, there exists a global spherically symmetric weak solution $(\rho, \mathbf{U}, e)(\mathbf{x}, t)$ of the Cauchy problem (1.1.1)–(1.1.5) such that the following statements hold:

(i) There exists a continuous increasing function $g : [0, \infty) \rightarrow [0, \infty)$ with $g(y) \searrow 0$ as $y \searrow 0$ such that, for each bounded measurable set $E \subset \mathbb{R}^n$,

$$\begin{aligned} \operatorname{ess\,sup}_{t \in [0, T]} \int_{\mathbb{R}^n} \left\{ G(\rho) + \frac{\rho |\mathbf{U}|^2}{2} \right\}(\mathbf{x}, t) \, d\mathbf{x} &\leq C(T), \\ \operatorname{ess\,sup}_{t \in [0, T]} \int_E \rho e(\mathbf{x}, t) \, d\mathbf{x} &\leq C(T)(1 + g(|E|)), \end{aligned}$$

where $|E|$ denotes the Lebesgue measure of set E , and $C(T) = C(T, C_*, \gamma, \mu, \lambda, \kappa, n) > 0$ is a constant depending on $T > 0$.

(ii) There exists a continuous function $(y, t) \mapsto \tilde{r}(y, t) : [0, \infty) \times [0, T] \rightarrow [0, \infty)$ such that $y \mapsto \tilde{r}(y, t)$ is strictly increasing, and $\underline{r}(t) = \lim_{y \searrow 0} \tilde{r}(y, t)$ for $t \in [0, T]$. Moreover, there exists a constant $C_0 = C_0(C_*, \gamma, \mu, \lambda, \kappa, n) > 0$ such that

$$\begin{cases} \int_{\{\underline{r}(t) < |\mathbf{x}| < \tilde{r}(y, t)\}} \rho(\mathbf{x}, t) \, d\mathbf{x} = y & \text{for a.e. } (y, t) \in [0, \infty) \times [0, T], \\ \tilde{r}(y, t) = \tilde{r}_0(y) + \int_0^t u(\tilde{r}(y, s), s) \, ds & \text{for a.e. } (y, t) \in [0, \infty) \times [0, T], \\ n y \psi^{-1}\left(\frac{C_0}{y}\right) \leq (\tilde{r}(y, t))^n \leq C_0(1 + y) & \text{for all } (y, t) \in (0, \infty) \times [0, T], \end{cases}$$

where $u(|\mathbf{x}|, t) := \mathbf{U}(\mathbf{x}, t) \cdot \frac{\mathbf{x}}{|\mathbf{x}|}$, ψ^{-1} is the left inverse of $\psi(\zeta) = \zeta - 1 - \log \zeta$, and the function $y \mapsto \tilde{r}_0(y) : [0, \infty) \rightarrow [0, \infty)$ is implicitly defined as

$$y = \int_0^{\tilde{r}_0(y)} \rho_0(r) r^m \, dr \quad \text{for each } y \in [0, \infty).$$

(iii) For $\varepsilon \in (0, 1]$, there exists a constant $C(\varepsilon) = C(\varepsilon, T, C_*, \gamma, \mu, \lambda, \kappa, n) > 0$ such that

$$\begin{cases} |\tilde{r}(y_1, t) - \tilde{r}(y_2, t)| \leq C(\varepsilon) |y_1 - y_2| & \text{for all } (y_1, y_2, t) \in [\varepsilon, \infty)^2 \times [0, T], \\ |\tilde{r}(y, t_1) - \tilde{r}(y, t_2)| \leq C(\varepsilon) |t_1^{3/4} - t_2^{3/4}| & \text{for all } (y, t_1, t_2) \in [\varepsilon, \infty) \times [0, T]^2. \end{cases}$$

Moreover, let $\sigma(t) := \min\{1, t\}$ and $B^C(r) := \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| \geq r\}$. Then

$$C(\varepsilon)^{-1} \leq \rho(\mathbf{x}, t) \leq C(\varepsilon), \quad |\mathbf{U}(\mathbf{x}, t)| \leq \sigma(t)^{-\frac{1}{4}} C(\varepsilon), \quad 0 < e(\mathbf{x}, t) \leq \sigma(t)^{-\frac{1}{2}} C(\varepsilon)$$

for a.e. (\mathbf{x}, t) such that $t \in [0, T]$ and $\mathbf{x} \in B^C(\tilde{r}(\varepsilon, t))$, and

$$\begin{cases} \sup_{0 \leq t \leq T} \int_{B^C(\tilde{r}(\varepsilon, t))} |(\rho - 1, |\mathbf{U}|^2, e - 1, \sqrt{\sigma} \nabla \mathbf{U}, \sigma \nabla e)|^2(\mathbf{x}, t) d\mathbf{x} \leq C(\varepsilon), \\ \int_0^T \int_{B^C(\tilde{r}(\varepsilon, t))} |(\nabla \mathbf{U}, (\mathbf{U} \cdot \nabla) \mathbf{U}, \nabla e, \sqrt{\sigma} \partial_t \mathbf{U}, \sigma \partial_t e)|^2(\mathbf{x}, t) d\mathbf{x} dt \leq C(\varepsilon). \end{cases}$$

In addition, for all $t \in (0, T]$ and $\mathbf{x}, \mathbf{y} \in B^C(\tilde{r}(\varepsilon, t))$,

$$\sigma(t)^{\frac{1}{2}} |\mathbf{U}(\mathbf{x}, t) - \mathbf{U}(\mathbf{y}, t)| + \sigma(t) |e(\mathbf{x}, t) - e(\mathbf{y}, t)| \leq C(\varepsilon) |\mathbf{x} - \mathbf{y}|^{\frac{1}{2}},$$

and, for all $0 < t_1 < t_2 \leq T$ and $|\mathbf{x}| \geq \sup_{t_1 \leq t \leq t_2} \tilde{r}(\varepsilon, t)$,

$$\sigma(t_1)^{\frac{1}{2}} |\mathbf{U}(\mathbf{x}, t_1) - \mathbf{U}(\mathbf{x}, t_2)| + \sigma(t_1) |e(\mathbf{x}, t_1) - e(\mathbf{x}, t_2)| \leq C(\varepsilon) |t_2 - t_1|^{\frac{1}{4}}.$$

Furthermore, defining the function $\rho^{(\varepsilon)}(\mathbf{x}, t) := \rho(\mathbf{x}, t) \chi_\varepsilon(\mathbf{x}, t)$ with

$$\chi_\varepsilon(\mathbf{x}, t) := \begin{cases} 1 & \text{if } t \in [0, T] \text{ and } \mathbf{x} \in B^C(\tilde{r}(\varepsilon, t)), \\ 0 & \text{otherwise,} \end{cases}$$

then, for all $L \in \mathbb{N}$, $\rho^{(\varepsilon)} \in C^0([0, T]; H^{-1}(B_L))$ and

$$\|\rho^{(\varepsilon)}(\cdot, t_1) - \rho^{(\varepsilon)}(\cdot, t_2)\|_{H^{-1}(B_L)} \leq C(\varepsilon) |t_1 - t_2| \quad \text{for all } 0 \leq t_1 \leq t_2 \leq T,$$

where $C(\varepsilon)$ is independent of $L \in \mathbb{N}$, and $B_L := \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| < L\}$.

The domain for which the momentum equations hold weakly can be extended to the entire space-time domain. However, the viscosity term is interpreted as a non-standard limit distribution of approximate solutions.

Theorem 2.1.2. *Let $(\rho, \mathbf{U}, e)(\mathbf{x}, t)$ be a weak solution with initial data $(\rho_0, \mathbf{U}_0, e_0)(\mathbf{x})$ obtained in Theorem 2.1.1. Let $\xi : [0, \infty) \rightarrow [0, 1]$ be a fixed increasing C^1 function with $\xi \equiv 0$ on $[0, 1]$ and $\xi \equiv 1$ on $[2, \infty)$, and set $\xi^R(r) := \xi(\frac{r}{R})$. Then there exists a sequence of vector valued functions $\{\mathbf{U}_\alpha(\mathbf{x}, t)\}_{\alpha \in \mathbb{N}}$ such that $\mathbf{U}_\alpha^i \in L^2_{\text{loc}}(\mathbb{R}^n \times [0, T])$ for each $\alpha \in \mathbb{N}$ and $i = 1, \dots, n$. Moreover, for each $i = 1, \dots, n$ and for each $\Psi \in C^2([0, T]; C_c^2(\mathbb{R}^n))$ with $\Psi : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}$, defining the distribution $\mathbb{U}(\cdot, t) \in (C^2([0, T]; C_c^2(\mathbb{R}^n)))^*$ by*

$$\mathbb{U}^i(\Psi, t) := \lim_{R \searrow 0} \lim_{\alpha \rightarrow \infty} \int_0^t \int_{\{|\mathbf{x}| \geq 1/\alpha\}} \{(\mu + \lambda)(\mathbf{U}_\alpha \cdot \nabla) \partial_i \Psi^R + \mu \mathbf{U}_\alpha^i \Delta \Psi^R\} d\mathbf{x} ds,$$

with $\Psi^R(\mathbf{x}, t) := \xi^R(|\mathbf{x}|) \Psi(\mathbf{x}, t)$, then the following is true for each $\Psi \in C^2([0, T]; C_c^2(\mathbb{R}^n))$:

$$\begin{aligned} & \int_{\mathbb{R}^n} \rho U^i(\mathbf{x}, t) \Psi(\mathbf{x}, t) d\mathbf{x} - \int_{\mathbb{R}^n} \rho_0 U_0^i(\mathbf{x}) \Psi(\mathbf{x}, 0) d\mathbf{x} \\ &= \int_0^t \int_{\mathbb{R}^n} (\rho U^i \partial_t \Psi + \rho U^i (\mathbf{U} \cdot \nabla) \Psi + (\gamma - 1) \rho e \partial_{\mathbf{x}^i} \Psi)(\mathbf{x}, s) d\mathbf{x} ds + \mathbb{U}^i(\Psi, t), \end{aligned}$$

for each $i = 1, \dots, n$. Furthermore, $\mathbf{U}_\alpha(\mathbf{x}, t)$ is the velocity of solution to the exterior sphere problem, which satisfies the auxiliary boundary condition:

$$\mathbf{U}_\alpha(\mathbf{x}, t) = 0, \quad \nabla e_\alpha(\mathbf{x}, t) = 0 \quad \text{for } |\mathbf{x}| = 1/\alpha \text{ and } t \in [0, T].$$

The existence of such solution $(\rho_\alpha, \mathbf{U}_\alpha, e_\alpha)$ is stated in Theorem 2.2.1 of Section 2.2.3.

2.2 Main Strategy and Reformulation

Throughout the rest of this chapter, we use the following notations for five types of generic positive constants to indicate their specific dependence on various parameters:

$$\begin{aligned} C &\equiv C(\gamma, \mu, \lambda, \kappa, n), & C_0 &\equiv C_0(C, C_*), & C(T) &\equiv C(T, C_0), \\ C(a) &\equiv C(a, T, C_0), & C(\varepsilon) &\equiv C(\varepsilon, T, C_0) & \text{for each } a \in (0, 1) \text{ and } \varepsilon \in (0, 1], \end{aligned}$$

where $C_* = C_*(\gamma, \rho_0, u_0, e_0) > 0$ is the initial constant stated in (2.1.1) of Theorem 2.1.1.

2.2.1 Main strategy

Our approach for proving Theorem 2.1.1 is to split the original problem (1.2.1) into the following two parts:

- (I) For fixed $a \in (0, 1)$ and $T > 0$, we prove the global existence of weak solutions $\{(\rho_a, u_a, e_a)(r, t)\}_{a \in (0, 1)}$ to the initial-boundary value problem (IBVP) of equations (1.2.1)₁–(1.2.1)₃ in the exterior domain $(r, t) \in [a, \infty) \times [0, T]$ with the following initial and boundary conditions:

$$\begin{cases} (\rho_a, u_a, e_a)(r, 0) = (\rho_a^0, u_a^0, e_a^0)(r) & \text{for } r \in [a, \infty), \\ u(a, t) = \partial_r e(a, t) = 0 & \text{for } t \in [0, T], \end{cases}$$

where $(\rho_a^0, u_a^0, e_a^0)(r)$ is the approximate initial data constructed from the initial data $(\rho_0, u_0, e_0)(r)$ assumed in Theorem 2.1.1. The detailed construction can be found in Section 2.2.2, and (ρ_a^0, u_a^0, e_a^0) are shown to satisfy initial condition (2.1.1) with a constant $C_0 > 0$ independent of $a \in (0, 1)$ in Propositions 2.2.1. It is also worth pointing out that the corresponding solutions $\{(\rho_a, u_a, e_a)(r, t)\}_{a \in (0, 1)}$ satisfy certain high order estimates uniformly in $a \in (0, 1)$ (see Theorem 2.2.1 in Section 2.2.3).

- (II) It follows from the uniform estimates independent of a obtained in (I) and compactness arguments that there is a sub-sequence $a_j \searrow 0$ as $j \rightarrow \infty$ such that, the corresponding limit triple (ρ, u, e) is a weak solution described in Theorem 2.1.1.

Part (I) of our approach will be achieved via the following six steps:

- (I.1) We first modify the original initial data (ρ_0, u_0, e_0) into $(\rho_a^0, u_a^0, e_a^0)(r)$ in Eulerian coordinates, as constructed in Section 2.2.2.
- (I.2) The solution (ρ_a, u_a, e_a) in (I) is obtained as the limit of solutions of the approximate problems posed in bounded annular domains. For this purpose, we apply a truncation procedure on the mollified initial data (ρ_a^0, u_a^0, e_a^0) in Eulerian coordinates near the far-field region (*i.e.*, $r \geq 1$), which are parameterized by the integer $k \in \mathbb{N}$.
- (I.3) Next, we transform the IBVP in the exterior domain $(r, t) \in [a, \infty) \times [0, T]$ stated in (I) into a set of IBVPS shown in (2.2.16)–(2.2.17) in Lagrangian coordinates with the spatial domain being $[0, k]$. Indeed, the main point of the above reduction and modification of the initial data is to ensure that, for each $T > 0$, there exists a unique global-in-time smooth solution to the approximate IBVP (2.2.16)–(2.2.17) in $(x, t) \in [0, k] \times [0, T]$.
- (I.4) Moreover, we can derive the entropy inequalities on these solutions. Consequently, we obtain the upper and lower bounds of the density. These density bounds are independent of the approximation parameter $k \in \mathbb{N}$; in addition, they are independent of $a \in (0, 1)$ on the regions away from the origin in Lagrangian coordinates.

(I.5) Based on the bounds of the density obtained in (I.4), we seek for a set of high-order estimates on the regions away from the origin $x = 0$ in Lagrangian coordinates, which is independent of the approximate parameter $a \in (0, 1)$. In fact, these estimates can be obtained by introducing a set of C^1 cut-off functions $g_\varepsilon(x)$, for each $\varepsilon \in (0, 1)$, away from the origin (see (2.3.21)–(2.3.23)). However, the inclusion of these cut-off functions leads to difficulties in estimating certain integral terms near the boundary $x = \varepsilon$, which is described in the proof of Lemma 2.3.4. In order to overcome these difficulties, we make full use of the dissipation terms in the entropy estimates from step (I.4), which can be found in Lemma 2.3.3.

(I.6) Finally, we denote $(\tilde{v}_{a,k}, \tilde{u}_{a,k}, \tilde{e}_{a,k})(x, s)$ as the approximate solution of problem (2.2.16)–(2.2.17) in the Lagrangian domain $(x, s) \in [0, k] \times [0, T]$, and

$$\tilde{r}_{a,k}(x, s) = \left(a^n + n \int_0^x \tilde{v}_{a,k}(y, s) dy \right)^{1/n}$$

as the corresponding particle path function. For each $(a, k) \in (0, 1) \times \mathbb{N}$, we formulate coordinate transformations $(r, t) = \mathcal{T}_{a,k}(x, s) = (\mathcal{T}_{a,k}^{(1)}, \mathcal{T}_{a,k}^{(2)})(x, s)$ as

$$\mathcal{T}_{a,k}^{(1)}(x, s) = \tilde{r}_{a,k}(x, s), \quad \mathcal{T}_{a,k}^{(2)}(x, s) = s \quad \text{for } (x, s) \in [0, k] \times [0, T].$$

Owing to the density bounds from Step (I.4), the inverse function $\mathcal{T}_{a,k}^{-1}$ exists and, by these coordinate transformations, we define

$$(\bar{v}_{a,k}, \bar{u}_{a,k}, \bar{e}_{a,k})(r, t) := (\tilde{v}_{a,k}, \tilde{u}_{a,k}, \tilde{e}_{a,k})(\mathcal{T}_{a,k}^{-1}(r, t)) \quad \text{for } (r, t) \in R_{a,k},$$

where $R_{a,k} := \mathcal{T}_{a,k}([0, k] \times [0, T]) = \{(r, t) : t \in [0, T], r \in [a, \tilde{r}_{a,k}(k, t)]\}$. We then extend these functions into the entire exterior Eulerian domain $(r, t) \in [a, \infty) \times [0, T]$ by introducing the following test functions: Let $\xi : \mathbb{R} \rightarrow [0, 1]$ be a function with $\xi \in C^\infty$ such that $\xi(\zeta) = 1$ if $\zeta \leq 0$, $\xi(\zeta) = 0$ if $1 \leq \zeta$, and $|\xi'(\zeta)| \leq 2$ for all $\zeta \in \mathbb{R}$. With this, we define the following cut-off functions:

$$\varphi_{a,k}(r, t) = \xi\left(\frac{2r - \tilde{r}_{a,k}(k, t)}{\tilde{r}_{a,k}(k, t)}\right).$$

It is shown in Section 2.4.2 that $\text{supp}(\varphi_{a,k}) \subseteq R_{a,k}$, so that the approximate solutions in the Eulerian coordinates can be extended as follows:

$$\begin{aligned} \rho_{a,k}(r, t) &= \begin{cases} \varphi_{a,k}(r, t) \left(\frac{1}{\bar{v}_{a,k}(r, t)} - 1 \right) + 1 & \text{for } (r, t) \in R_{a,k}, \\ 1 & \text{for } (r, t) \in [a, \infty) \times [0, T] \setminus R_{a,k}, \end{cases} \\ u_{a,k}(r, t) &= \begin{cases} \varphi_{a,k}(r, t) \bar{u}_{a,k}(r, t) & \text{for } (r, t) \in R_{a,k}, \\ 0 & \text{for } (r, t) \in [a, \infty) \times [0, T] \setminus R_{a,k}, \end{cases} \\ e_{a,k}(r, t) &= \begin{cases} \varphi_{a,k}(r, t) (\bar{e}_{a,k}(r, t) - 1) + 1 & \text{for } (r, t) \in R_{a,k}, \\ 1 & \text{for } (r, t) \in [a, \infty) \times [0, T] \setminus R_{a,k}. \end{cases} \end{aligned} \quad (2.2.1)$$

Taking the advantages of the *a-priori* estimates derived in Steps (I.4)–(I.5), we find a sub-sequence, $(\rho_{a,k_j}, u_{a,k_j}, e_{a,k_j})_{j \in \mathbb{N}}$, such that the limit function (ρ_a, u_a, e_a) is a weak solution of the exterior problem stated in (I). Furthermore, it is also shown that such solutions continue to enjoy the uniform regularity estimates in $a \in (0, 1)$ attained in Steps (I.4)–(I.5). The detailed analysis of this procedure can be found in Section 2.4.

Part (II) of our approach can be achieved via the following four steps:

(II.1) For given approximate initial data (ρ_a^0, u_a^0, e_a^0) in (I), denote

$$\{(\rho_a, u_a, e_a)(r, t) : [a, \infty) \times [0, T] \rightarrow [0, \infty) \times \mathbb{R} \times [0, \infty)\}_{a \in (0,1)}$$

as the global weak solutions of the exterior problem obtained in Step (I). Based on the entropy estimate, we prove the uniform integrability of the approximate solutions $(\rho_a, u_a, e_a)_{a \in (0,1)}$ with respect to the approximate parameter $a \in (0, 1)$ in Section 2.5.1, which will play key roles in proving that the limit function of these approximate solutions are indeed weak solutions of the original problem (see Lemmas 2.5.9–2.5.11).

(II.2) The particle path functions $\tilde{r}_a(x, t) : [0, \infty) \times [0, T] \rightarrow [a, \infty)$ associated with the approximate density ρ_a for each $a \in (0, 1)$ are defined to be the functions satisfying:

$$x = \int_a^{\tilde{r}_a(x,t)} \rho_a(r, t) r^m dr \quad \text{for a.e. } (x, t) \in [0, \infty) \times [0, T],$$

which can be found in Lemma 2.4.3 in Section 2.4.3. Then, by the estimates obtained in Step (II.1) and compactness arguments, we show that, in Section 2.5.2, there exists a function $\tilde{r}(x, t) : [0, \infty) \times [0, T] \rightarrow [0, \infty)$ that is the limit function of the sequence $\{\tilde{r}_a(x, t)\}_{a \in (0,1)}$ when $a \searrow 0$. Moreover, we show that, for $t \in [0, T]$, the limit $\lim_{x \searrow 0} \tilde{r}(x, t)$ does not necessarily equal to zero, but a function of t instead in general: $\underline{r}(t) : [0, T] \rightarrow [0, \infty)$. In fact, $\tilde{r}(x, t)$ is the particle path of our original problem (1.2.1), and $\underline{r}(t)$ is the vacuum interface in (1.2.1) that separates the fluid region and the possible vacuum, which can be verified later. Furthermore, it is shown in Lemma 2.5.4(iii) that, for each $x > 0$, there exists $\delta(x) > 0$ depending only on $x > 0$ and initial constant $C_0 > 0$ such that

$$\inf_{t \in [0, T]} \tilde{r}(x, t) \geq \delta(x) > 0. \quad (2.2.2)$$

This lower bound will be crucial in the next step where we take limit $a \searrow 0$ in $\{(\rho_a, u_a, e_a)\}_{a \in (0,1)}$.

(II.3) Next, for each $\varepsilon \in (0, 1]$, we fix a domain away from the origin by

$$\mathcal{F}_\varepsilon := \{(r, t) \in [0, \infty) \times [0, T] : r \in [\tilde{r}(\varepsilon, t), \infty)\}, \quad (2.2.3)$$

where $\tilde{r}(x, t)$ is obtained in Step (II.2). Moreover, we set

$$\mathcal{F} := \{(r, t) \in [0, \infty) \times [0, T] : r > \underline{r}(t)\}.$$

Then, by the high order estimates derived in Step (I.5), we construct a limit function $(\rho, u, e)(r, t) : \mathcal{F} \rightarrow [0, \infty) \times \mathbb{R} \times [0, \infty)$ in Section 2.5.3 as follows: By an application of the Arzelá-Ascoli theorem, we first obtain $(\rho_\varepsilon, u_\varepsilon, e_\varepsilon)(r, t)$ defined in domain \mathcal{F}_ε for each $\varepsilon \in (0, 1]$; then the function (ρ, u, e) is constructed by first extending $(\rho_\varepsilon, u_\varepsilon, e_\varepsilon)$ with

$$(\rho_\varepsilon, u_\varepsilon, e_\varepsilon)(r, t) := \begin{cases} (0, 0, 0) & \text{if } (r, t) \in \mathcal{F} \setminus \mathcal{F}_\varepsilon, \\ (\rho_\varepsilon, u_\varepsilon, e_\varepsilon)(r, t) & \text{if } (r, t) \in \mathcal{F}_\varepsilon, \end{cases}$$

and next $\{(\rho_\varepsilon, u_\varepsilon, e_\varepsilon)(r, t)\}_{\varepsilon \in (0,1]}$ are glued together over the parameter $\varepsilon \in (0, 1]$ by the telescoping sum:

$$f = f_{\varepsilon_1} + \sum_{i=1}^{\infty} (f_{\varepsilon_{i+1}} - f_{\varepsilon_i}),$$

for $f = \rho, u, e$ respectively and $\varepsilon_i \equiv 1/i$. The detail of this procedure is given in Propositions A.0.1–A.0.2 in Appendix A. In addition, it is also shown in Lemma 2.5.6 that

$$x = \int_{\underline{r}(t)}^{\tilde{r}(x,t)} \rho(r,t)r^m dr \quad \text{for a.e. } (x,t) \in [0,\infty) \times [0,T],$$

which justifies that the function $\underline{r}(t)$ obtained in Step (II.2) is the vacuum interface curve of our original problem (1.2.1).

(II.4) Finally, by the estimates obtained in Step (II.1), and the convergences in Steps (II.2)–(II.3), we prove the weak forms listed in Theorem 2.1.1. The weak forms in Eulerian coordinates $(\mathbf{x}, t) \in \mathbb{R}^n \times [0, T]$ are shown in Section 2.5.4, and Theorem 2.1.2 is shown in Section 2.5.5.

Remark 2.2.1. *Before proceeding, we also give the following remarks:*

- (a) *In Steps (I.1)–(I.2), it is important to show that the modified initial data satisfy the entropy estimates uniformly with respect to $(a, k) \in (0, 1) \times \mathbb{N}$ and strongly converge to the original initial data $(\rho_0, u_0, e_0)(r)$ in proper functional spaces. These results will be needed later in the compactness argument in Step (I.6), which is detailed in Section 2.4.*
- (b) *The main purpose of the reformulation from Eulerian to Lagrangian coordinates in Step (I.3) is to obtain the crucial upper and lower bounds of the density (see Theorem 2.3.1(ii) in Section 2.3), which are independent of $k \in \mathbb{N}$. This is, as far as we are concerned, only possible in the Lagrangian formulation. Note that these bounds of the density are an improvement on the result of [13].*
- (c) *As mentioned in Step (I.5), the introduction of C^1 cut-off functions away from the origin leads to several hurdles in the estimates, and these issues have also been encountered in [11]. In this paper, we resolve these issues by making use of the dissipation terms that are present in the entropy estimates. This is detailed in Lemma 2.3.3 of Section 2.3.3.*
- (d) *The purpose of the coordinate transformation back into the Eulerian domain, introduced in Step (I.6) is due to the following consideration: In taking limit $k \rightarrow \infty$, a difficulty arises due to the nonlinear second-order spatial differential operators in Lagrangian forms of the momentum and internal energy equations. Since these operators are linear in Eulerian coordinates, this difficulty is resolved by converting the approximate solutions, originally obtained in Lagrangian coordinates, into the functions in the Eulerian domain. This allows us to take limit $k \rightarrow \infty$ and obtain the desired weak forms in Eulerian coordinates.*
- (e) *In Step (I.6), after taking limit $k \rightarrow \infty$, it requires to show that the limit solution still satisfies the entropy inequality, since it will be required in Step (II.1). This is achieved by an application of Mazur’s Lemma in Lemma 2.4.9 of Section 2.4.5.*
- (f) *In order to apply the compactness argument from the high order estimates obtained in (I.5), it is crucial that \mathcal{F}_ε defined in (2.2.3) is a set strictly bounded away from the origin $r = 0$ for each $\varepsilon > 0$. We obtain this by assertion (2.2.2) in Step (II.2).*

The rest of this section is organised as follows: In Section 2.2.3, we show our main result on the existence of weak solutions with large data of the exterior problem mentioned in (I). In Sections 2.2.2–2.2.4, we describe detailed procedures for the modification and truncation of the initial data listed in Steps (I.2)–(I.3) above. Finally, in Section 2.2.5, we reformulate the exterior problem in Eulerian coordinates into the approximate Lagrangian problems in domain $(x, t) \in [0, k] \times [0, T]$ as described in Step (I.4) and also establish the corresponding global-in-time well-posedness of the unique strong solution with regular initial data.

2.2.2 Mollification of the initial data in the Eulerian domain

Let $(\rho_0, u_0, e_0)(r)$ for $r \in [0, \infty)$ be the initial data in Theorem 2.1.1. First, we extend the initial data to \mathbb{R} by

$$(\hat{\rho}_0, \hat{u}_0, \hat{e}_0)(\zeta) := \begin{cases} (\rho_0(\zeta), u_0(\zeta), e_0(\zeta)) & \text{if } \zeta \in [0, \infty), \\ (\rho_0(-\zeta), -u_0(-\zeta), e_0(-\zeta)) & \text{if } \zeta \in (-\infty, 0). \end{cases} \quad (2.2.4)$$

Then, for each $a \in (0, 1)$, let $j_a \in C_c^\infty([-1, 1])$ be the standard 1-D mollifiers such that $\text{supp}(j_a) \subseteq [-a/2, a/2]$. We define $(\hat{\rho}_a^0, \hat{u}_a^0, \hat{e}_a^0) := (\hat{\rho}_0 * j_a, \hat{u}_0 * j_a, \hat{e}_0 * j_a)$. It follows that

$$\begin{aligned} (\hat{\rho}_a^0, \hat{u}_a^0, \hat{e}_a^0) &\in C^\infty(\mathbb{R}), \quad \hat{e}_a^0(\zeta) \geq C_*^{-1} \text{ and } C_*^{-1} \leq \hat{\rho}_a^0(\zeta) \leq C_* \text{ for a.e. } \zeta \in \mathbb{R}, \\ (\hat{\rho}_a^0, \hat{u}_a^0, \hat{e}_a^0)(\zeta) &\rightarrow (\rho_0, u_0, e_0)(r) \quad \text{as } a \searrow 0 \text{ for a.e. } \zeta \in (0, \infty). \end{aligned} \quad (2.2.5)$$

Next, let $\chi : [0, \infty) \rightarrow [0, 1]$ be such that $\chi \in C^\infty$, $\chi(\zeta) = 0$ if $\zeta \leq 1$, and $\chi(\zeta) = 1$ if $\zeta \geq 2$. Setting the cut-off functions $\chi_a(r) = \chi(\frac{r}{a})$ for each $a \in (0, 1)$, it follows that $\chi_a(r) = 0$ for $r \leq a$ and $\chi_a(r) = 1$ for $r \geq 2a$. With this, we define the approximate exterior initial data:

$$(\rho_a^0, u_a^0, e_a^0)(r) := ((\hat{\rho}_a^0 - 1)\chi_a + 1, \hat{u}_a^0\chi_a, (\hat{e}_a^0 - 1)\chi_a + 1)(r) \quad \text{for all } r \in (0, \infty). \quad (2.2.6)$$

Proposition 2.2.1. $(\rho_a^0, u_a^0, e_a^0) \in C^\infty([0, \infty))$ and $u_a^0(a) = \partial_r e_a^0(a) = 0$. Moreover, as $a \searrow 0$,

$$\begin{aligned} \|(\rho_a^0 - 1, |u_a^0|^2, e_a^0 - 1) - (\rho_0 - 1, |u_0|^2, e_0 - 1)\|_{L^2([0, \infty), r^m dr)} &\rightarrow 0, \\ (\rho_a^0, u_a^0, e_a^0)(r) &\rightarrow (\rho_0, u_0, e_0)(r) \quad \text{for a.e. } r \in [0, \infty). \end{aligned} \quad (2.2.7)$$

In addition, the following uniform estimates are satisfied for some constant $C_0 > 0$:

$$\begin{aligned} e_a^0(r) &\geq C_0^{-1} \quad \text{and} \quad C_0^{-1} \leq \rho_a^0(r) \leq C_0 \quad \text{for all } (a, r) \in (0, 1) \times [0, \infty), \\ \sup_{a \in (0, 1)} \int_a^\infty \{G(\rho_a^0) + \rho_a^0 \psi(e_a^0) + \rho_a^0 |u_a^0|^2 + |(\rho_a^0 - 1, |u_a^0|^2, e_a^0 - 1)|^2\}(r) r^m dr &\leq C_0. \end{aligned} \quad (2.2.8)$$

Proof. The statements $(\rho_a^0, u_a^0, e_a^0) \in C^\infty([0, \infty))$, $\min\{1, C_*^{-1}\} \leq \rho_a^0(r) \leq \max\{1, C_*\}$, $u_a^0(a) = 0 = \partial_r e_a^0(a)$, and $e_a^0(r) \geq \min\{1, C_*^{-1}\}$ for all $r \in [0, \infty)$ follow immediately from the construction (2.2.6), while the almost everywhere convergence (2.2.7)₂ follows from (2.2.5).

Next, by Minkowski's integration inequality and (2.2.6), we have

$$\begin{aligned} \|u_a^0\|_{L^4([0, \infty), r^m dr)} &\leq \int_{-a/2}^{a/2} \left(\int_0^\infty |\chi_a(r) \hat{u}_0(r - \zeta) j_a(\zeta)|^4 r^m dr \right)^{1/4} d\zeta \\ &\leq \int_{-a/2}^{a/2} j_a(\zeta) \left(\int_a^\infty |u_0(r - \zeta)|^4 r^m dr \right)^{1/4} d\zeta. \end{aligned}$$

Let $\xi \equiv r - \zeta$. Then $0 \leq 1 + \zeta/\xi \leq 2$ for all $(\zeta, \xi) \in [-a/2, a/2] \times [a/2, \infty)$. It follows that

$$\begin{aligned} \|u_a^0\|_{L^4([0, \infty), r^m dr)} &\leq \int_{-a/2}^{a/2} j_a(\zeta) \left(\int_{a-\zeta}^\infty |u_0(\xi)|^4 \xi^m (1 + \zeta/\xi)^m d\xi \right)^{1/4} d\zeta \\ &\leq 2^{m/4} \int_{-a/2}^{a/2} j_a(\zeta) \left(\int_{a/2}^\infty |u_0(\xi)|^4 \xi^m d\xi \right)^{1/4} d\zeta \leq 2^{m/4} C_*^{1/4} \int_{-a/2}^{a/2} j_a(\zeta) d\zeta = 2^{m/4} C_*^{1/4}. \end{aligned}$$

Applying the same argument to $\rho_a^0 - 1$ and $e_a^0 - 1$ yields (2.2.8)₂.

Next, by Hölder's inequality,

$$\| |u_a^0|^2 - |u_0|^2 \|_{L^2([0, \infty), r^m dr)} \leq (1 + 2^{m/4}) C_*^{1/4} \|u_a^0 - u_0\|_{L^4([0, \infty), r^m dr)}.$$

From this, we wish to show $\|u_a^0 - u_0\|_{L^4([0,\infty),r^m dr)} \rightarrow 0$ as $a \searrow 0$. By construction (2.2.6),

$$\begin{aligned} \|u_a^0 - u_0\|_{L^4([0,\infty),r^m dr)} &= \|\chi_a(\hat{u}_a^0 - u_0) + (\chi_a - 1)u_0\|_{L^4([0,\infty),r^m dr)} \\ &\leq \|\chi_a(\hat{u}_a^0 - u_0)\|_{L^4([0,\infty),r^m dr)} + \left(\int_0^{2a} (1 - \chi_a)^4 |u_0|^4(r) r^m dr \right)^{1/4}. \end{aligned}$$

Using the fact that $\|u_0\|_{L^4([0,\infty),r^m dr)} \leq C_*^{1/4}$, $\chi_a \rightarrow 1$ as $a \searrow 0$ almost everywhere, $0 \leq \chi_a \leq 1$, and the dominated convergence theorem, we have

$$\lim_{a \searrow 0} \left(\int_0^{2a} (1 - \chi_a)^4 |u_0|^4(r) r^m dr \right)^{1/4} = 0.$$

Therefore, we conclude

$$\lim_{a \searrow 0} \| |u_a^0|^2 - |u_0|^2 \|_{L^2([0,\infty),r^m dr)} \leq (1 + 2^{m/4}) C_* \lim_{a \searrow 0} \|\chi_a(\hat{u}_a^0 - u_0)\|_{L^4([0,\infty),r^m dr)}.$$

Hence, it is left to prove the limit: $\|\chi_a(\hat{u}_a^0 - u_0)\|_{L^4([0,\infty),r^m dr)} \rightarrow 0$ as $a \searrow 0$. To show this, it follows from Minkowski's integration inequality that

$$\begin{aligned} \|\chi_a(\hat{u}_a^0 - u_0)\|_{L^4([0,\infty),r^m dr)} &\leq \int_{-a/2}^{a/2} j_a(\zeta) \left(\int_a^\infty |u_0(r - \zeta) - u_0(r)|^4 r^m dr \right)^{1/4} d\zeta \\ &\leq \int_{-a/2}^{a/2} j_a(\zeta) \left(\int_{a-\zeta}^\infty |u_0(\xi) - u_0(\xi + \zeta)|^4 (1 + \zeta/\xi)^m \xi^m d\xi \right)^{1/4} d\zeta \\ &\leq 2^{m/4} \int_{-a/2}^{a/2} j_a(\zeta) \left(\int_{a-\zeta}^\infty |u_0(\xi) - u_0(\xi + \zeta)|^4 \xi^m d\xi \right)^{1/4} d\zeta, \end{aligned}$$

where the last line follows from $0 \leq 1 + \zeta/\xi \leq 2$ if $\zeta \in [-a/2, a/2]$ and $\xi \geq a - \zeta$. Now, by the separability of the space $L^4([0, \infty), r^m dr)$, for each $\delta > 0$, there exists $h_\delta \in C_c^\infty([0, \infty))$ such that $\|h_\delta - \hat{u}_0\|_{L^4([0,\infty),r^m dr)} \leq 3^{-m/4} \delta/4$. By the triangular inequality, it follows that

$$\begin{aligned} &2^{-m/4} \|\chi_a(\hat{u}_a^0 - u_0)\|_{L^4([0,\infty),r^m dr)} \\ &\leq \int_{-a/2}^{a/2} j_a(\zeta) \left(\int_{a/2}^\infty |u_0(\xi) - h_\delta(\xi)|^4 \xi^m d\xi \right)^{1/4} d\zeta + \int_{-a/2}^{a/2} j_a(\zeta) \left(\int_{a/2}^\infty |h_\delta(\xi) - h_\delta(\xi + \zeta)|^4 \xi^m d\xi \right)^{1/4} d\zeta \\ &+ \int_{-a/2}^{a/2} j_a(\zeta) \left(\int_{a-\zeta}^\infty |h_\delta(\xi + \zeta) - u_0(\xi + \zeta)|^4 \xi^m d\xi \right)^{1/4} d\zeta =: \sum_{i=1}^3 I_i. \end{aligned}$$

First, I_1 and I_3 are estimated as:

$$\begin{aligned} I_1 &:= \int_{-a/2}^{a/2} j_a(\zeta) \left(\int_{a/2}^\infty |u_0(\xi) - h_\delta(\xi)|^4 \xi^m d\xi \right)^{1/4} d\zeta \leq \frac{\delta}{4} \int_{-a/2}^{a/2} j_a(\zeta) d\zeta = \frac{\delta}{4}, \\ I_3 &:= \int_{-a/2}^{a/2} j_a(\zeta) \left(\int_a^\infty |h_\delta(r) - u_0(r)|^4 (1 - \zeta/r)^m r^m dr \right)^{1/4} d\zeta \\ &\leq 3^{m/4} \int_{-a/2}^{a/2} j_a(\zeta) \left(\int_a^\infty |h_\delta(r) - u_0(r)|^4 r^m dr \right)^{1/4} d\zeta \leq \frac{\delta}{4} \int_{-a/2}^{a/2} j_a(\zeta) d\zeta = \frac{\delta}{4}, \end{aligned}$$

where we have used the fact that $1/2 \leq 1 - \zeta/r \leq 3/2$ for all $(\zeta, r) \in [-a/2, a/2] \times [a, \infty)$. Since h_δ is compactly supported for each $\delta > 0$, there exists $a_\delta \in (0, 1)$ such that,

$$I_2 := \int_{-a/2}^{a/2} j_a(\zeta) \left(\int_{a/2}^\infty |h_\delta(\xi) - h_\delta(\xi + \zeta)|^4 \xi^m d\xi \right)^{1/4} d\zeta \leq \frac{\delta}{2} \int_{-a/2}^{a/2} j_a d\zeta = \frac{\delta}{2} \quad \text{if } a \in (0, a_\delta).$$

Thus, for given $\delta > 0$, there exists $a_\delta \in (0, 1)$ such that $2^{-m/4} \|\chi_a(\hat{u}_a^0 - u_0)\|_{L^4([0, \infty), r^m dr)} \leq \delta$ for all $a \in (0, a_\delta)$. This implies $\lim_{a \searrow 0} \| |u_a^0|^2 - |u_0|^2 \|_{L^2([0, \infty), r^m dr)} = 0$. The proof for ρ_a^0 and e_a^0 follows the same.

Next, by Jensen's inequality and the fact that $\zeta \mapsto G(\zeta)$, $\psi(\zeta)$, $|\zeta|^2$ are convex functions, it follows that $G(\rho_a^0) \leq G(\hat{\rho}_0) * j_a$, $\psi(e_a^0) \leq \psi(\hat{e}_0) * j_a$, and $|u_a^0| \leq |\hat{u}_0|^2 * j_a$ for $r \in (0, \infty)$. Using Fubini-Tonelli's theorem, it follows that

$$\begin{aligned} \int_a^\infty G(\rho_a^0)(r) r^m dr &\leq \int_a^\infty (G(\rho_0) * j_a)(r) r^m dr = \int_a^\infty \int_{-a/2}^{a/2} G(\rho_0(r - \zeta)) j_a(\zeta) d\zeta r^m dr \\ &= \int_{-a/2}^{a/2} \int_{a-\zeta}^\infty G(\rho_0(\xi)) (\zeta + \xi)^m d\xi j_a(\zeta) d\zeta \leq \int_{-a/2}^{a/2} \int_{a/2}^\infty G(\rho_0)(\xi) \xi^m (1 + \zeta/\xi)^m d\xi j_a(\zeta) d\zeta \\ &\leq 2^m \int_{-a/2}^{a/2} \int_0^\infty G(\rho_0)(\xi) \xi^m d\xi j_a(\zeta) d\zeta \leq 2^m \int_{-a/2}^{a/2} C_* j_a(\zeta) d\zeta = 2^m C_*, \end{aligned}$$

where we have used the initial condition (2.1.1)₂. In a similar way, we have

$$\begin{aligned} \int_a^\infty \rho_a^0 \psi(e_a^0)(r) r^m dr &\leq \max\{1, C_*\} \int_{-a/2}^{a/2} \int_{a-\zeta}^\infty \psi(e_0(\xi)) \xi^m (1 + \zeta/\xi)^m d\xi j_a(\zeta) d\zeta \\ &\leq 2^m C_* \max\{1, C_*\} \int_{-a/2}^{a/2} \int_0^\infty \rho_0 \psi(e_0)(\xi) \xi^m d\xi j_a(\zeta) d\zeta \leq 2^m C_*^2 \max\{1, C_*\}. \end{aligned}$$

Similarly, we also obtain $\|\rho_a^0 |u_a^0|^2\|_{L^1([0, \infty), r^m dr)} \leq 2^m C_*^2 \max\{1, C_*\}$. Finally, by setting

$$C_0 = \max\{2^m C_*, 2^m C_*^2 \max\{1, C_*\}, \min\{1, C_*^{-1}\}^{-1}\},$$

we complete the proof. \square

2.2.3 The approximation problem in the exterior Eulerian domain

To resolve the coordinate singularity at the origin, which poses a significant difficulty in the existence of solutions, we first consider the approximation problem in the exterior domain: $\Omega_{a,T} := [a, \infty) \times [0, T]$ for fixed $a \in (0, 1)$, and find solutions (ρ_a, u_a, e_a) to the exterior problem for system (1.3.1) over domain $\Omega_{a,T}$ with the following initial-boundary conditions:

$$\begin{cases} u_a(a, t) = 0, & \partial_r e_a(a, t) = 0 & \text{for } t \in [0, T], \\ (\rho_a, u_a, e_a)(r, 0) = (\rho_a^0, u_a^0, e_a^0)(r) & \text{for } r \in [a, \infty), \end{cases} \quad (2.2.9)$$

where the initial data are given by $(\rho_a^0, u_a^0, e_a^0)(r)$ constructed in Section 2.2.2 above.

To prove Theorem 2.1.1, the existence of weak solutions to the exterior problem (1.2.1) and (2.2.9) for each $a \in (0, 1)$ is first obtained, so that some compactness argument can be applied when $a \searrow 0$. First, the definition of weak solutions is given below.

Definition 2.2.1 (Spherically symmetric weak solutions in the exterior domain). *Let*

$$\begin{aligned} \mathcal{D}^a &:= \{\phi \in C^\infty([a, \infty) \times [0, T]) : \exists N \in \mathbb{N} \text{ s.t. } \phi(r, t) = 0 \text{ for } (r, t) \in [N, \infty) \times [0, T]\}, \\ \mathcal{D}_0^a &:= \{\phi \in \mathcal{D}^a : \phi(a, t) = 0 \text{ for all } t \in [0, T]\}. \end{aligned}$$

Then $(\rho_a, u_a, e_a)(r, t)$ is a weak solution of problem (1.2.1) and (2.2.9), provided that

(i) For each $\phi \in \mathcal{D}^a$,

$$\int_a^\infty \rho_a(r, t) \phi(r, t) r^m dr - \int_a^\infty \rho_a^0(r) \phi(r, 0) r^m dr = \int_0^t \int_a^\infty (\rho_a \partial_t \phi + \rho_a u_a \partial_r \phi) r^m dr ds.$$

(ii) For each $\phi \in \mathcal{D}_0^a$,

$$\begin{aligned} & \int_a^\infty \rho_a u_a \phi(r, t) r^m dr - \int_a^\infty \rho_a^0 u_a^0(r) \phi(r, 0) r^m dr - \int_0^t \int_a^\infty \rho_a u_a (\partial_t \phi + u_a \partial_r \phi) r^m dr ds \\ &= \int_0^t \int_a^\infty \left(P_a - \beta (\partial_r u_a + m \frac{u_a}{r}) \right) (\partial_r \phi + m \frac{\phi}{r}) r^m dr ds. \end{aligned}$$

(iii) For each $\phi \in \mathcal{D}^a$,

$$\begin{aligned} & \int_a^\infty \rho_a E_a \phi(r, t) r^m dr - \int_a^\infty \rho_a^0 E_a^0(r) \phi(r, 0) r^m dr \\ &= \int_0^t \int_a^\infty \{ \rho_a E_a \partial_t \phi + (\rho_a E_a + P_a) u_a \partial_r \phi \} (r, s) r^m dr ds \\ & \quad - \int_0^t \int_a^\infty \left\{ 2\mu u_a \partial_r u_a + \lambda u_a (\partial_r u_a + m \frac{u_a}{r}) + \kappa \partial_r e_a \right\} \partial_r \phi(r, s) r^m dr ds, \end{aligned}$$

where $P_a := (\gamma - 1)\rho_a e_a$, $E_a := |u_a|^2/2 + e_a$, and $E_a^0 := |u_a^0|^2/2 + e_a^0$.

From now on, we denote $\sigma = \sigma(t) := \min\{1, t\}$, and for all functions $y(t) : [0, T] \rightarrow [0, \infty)$, we denote

$$\begin{aligned} \mathcal{E}[\rho, u, e; y](T) &:= \sup_{t \in [0, T]} \int_{y(t)}^\infty |(\rho - 1, u^2, e - 1, \sqrt{\sigma} \partial_r u, \sigma \partial_r e)|^2(r, t) r^m dr \\ & \quad + \int_0^T \int_{y(t)}^\infty |(\partial_r u, \partial_r e, u \partial_r u, \sqrt{\sigma} \partial_t u, \sigma \partial_t e)|^2(r, t) r^m dr dt. \end{aligned}$$

Theorem 2.2.1. For each fixed $a \in (0, 1)$, let $(\rho_a^0, u_a^0, e_a^0)(r)$ be the initial data constructed in Proposition 2.2.1. Then, for each $T > 0$, there exists a weak solution $(\rho_a, u_a, e_a)(r, t)$ to the exterior problem (1.2.1) and (2.2.9). Moreover, the following statements hold:

(i) The entropy inequality holds:

$$\begin{aligned} & \operatorname{ess\,sup}_{t \in [0, T]} \int_a^\infty \left\{ (\gamma - 1)G(\rho_a) + \rho_a \psi(e_a) + \rho_a \frac{|u_a|^2}{2} \right\} r^m dr + \kappa \int_0^T \int_a^\infty \frac{|\partial_r e_a|^2}{e_a^2} r^m dr dt \\ & + \int_0^T \int_a^\infty \left\{ \left(\frac{2\mu}{n} + \lambda \right) \frac{|\partial_r u_a + m u_a / r|^2}{e_a} + \frac{2m\mu}{n} \frac{|\partial_r u_a - u_a / r|^2}{e_a} \right\} r^m dr dt \leq C(T). \end{aligned}$$

(ii) $\mathcal{E}[\rho_a, u_a, e_a; a](T) \leq C(a)$. Moreover, for a.e. $(r, t) \in [a, \infty) \times [0, T]$,

$$C(a)^{-1} \leq \rho_a(r, t) \leq C(a), \quad |u_a(r, t)| \leq \sigma(t)^{-\frac{1}{4}} C(a), \quad C(a)^{-1} \leq e_a(r, t) \leq \sigma(t)^{-\frac{1}{2}} C(a).$$

(iii) There exists a continuous function $(x, t) \mapsto \tilde{r}_a(x, t) : [0, \infty) \times [0, T] \rightarrow [a, \infty)$ such that $x \mapsto \tilde{r}_a(x, t)$ is strictly increasing for each $t \in [0, T]$ and

$$\begin{cases} \int_a^{\tilde{r}_a(x, t)} \rho_a(r, t) r^m dr = x & \text{for a.e. } (x, t) \in [0, \infty) \times [0, T], \\ \tilde{r}_a(x, t) = \tilde{r}_a^0(x) + \int_0^t u_a(\tilde{r}_a(x, s), s) ds & \text{for all } (x, t) \in [0, \infty) \times [0, T], \\ n x \psi_-^{-1} \left(\frac{C_0}{x} \right) \leq (\tilde{r}_a(x, t))^n \leq C_0(1 + x) & \text{for all } (x, t) \in [0, \infty) \times [0, T], \end{cases}$$

where the function $x \mapsto \tilde{r}_a^0(x) : [0, \infty) \rightarrow [a, \infty)$ is implicitly defined as

$$x = \int_a^{\tilde{r}_a^0(x)} \rho_a^0(r) r^m dr \quad \text{for each } x \in [0, \infty).$$

(iv) For each $\varepsilon \in (0, 1]$, $\mathcal{E}[\rho_a, u_a, e_a; \tilde{r}_a(\varepsilon, \cdot)](T) \leq C(\varepsilon)$ and

$$\begin{cases} |\tilde{r}_a(x_1, t) - \tilde{r}_a(x_2, t)| \leq C(\varepsilon)|x_1 - x_2| & \text{for } (x_1, x_2, t) \in [\varepsilon, \infty)^2 \times [0, T], \\ |\tilde{r}_a(x, t_1) - \tilde{r}_a(x, t_2)| \leq C(\varepsilon)|t_1^{3/4} - t_2^{3/4}| & \text{for } (x, t_1, t_2) \in [\varepsilon, \infty) \times [0, T]^2. \end{cases}$$

Furthermore, for a.e. (r, t) such that $t \in [0, T]$ and $r \in [\tilde{r}_a(\varepsilon, t), \infty)$:

$$C(\varepsilon)^{-1} \leq \rho_a(r, t) \leq C(\varepsilon), \quad |u_a(r, t)| \leq \sigma(t)^{-\frac{1}{4}}C(\varepsilon), \quad e_a(r, t) \leq \sigma(t)^{-\frac{1}{2}}C(\varepsilon).$$

In addition, for all (r_1, r_2, t) such that $t \in [0, T]$ and $r_1, r_2 \in [\tilde{r}_a(\varepsilon, t), \infty)$,

$$\sigma(t)^{\frac{1}{2}}|u_a(r_1, t) - u_a(r_2, t)| + \sigma(t)|e_a(r_1, t) - e_a(r_2, t)| \leq C(\varepsilon)|r_1 - r_2|^{\frac{1}{2}},$$

and, for all (r, t_1, t_2) such that $0 < t_1 < t_2 \leq T$ and $r \geq \sup_{t_1 \leq t \leq t_2} \tilde{r}_a(\varepsilon, t)$,

$$\sigma(t_1)^{\frac{1}{2}}|u_a(r, t_1) - u_a(r, t_2)| + \sigma(t_1)|e_a(r, t_1) - e_a(r, t_2)| \leq C(\varepsilon)|t_2 - t_1|^{\frac{1}{4}}.$$

(v) For each connected interval $I \subseteq [0, \infty)$, denote $\tilde{H}_0^1(I, r^m dr)$ as the closure of

$$\mathcal{D}_0(I) := \{\phi \in C^\infty(I) : \exists N > 0 \text{ s.t. } [0, N] \subset I \text{ and } \phi(r) = 0 \text{ for } r \in I \cap [N, \infty)\}$$

via the $H^1(I, r^m dr)$ -norm, and denote its dual space as $\tilde{H}^{-1}(I, r^m dr)$. For each $\varepsilon > 0$, define $\rho_a^{(\varepsilon)}(r, t) := \rho_a(r, t)\chi_\varepsilon^\alpha(r, t)$ where $\chi_\varepsilon^\alpha(r, t)$ is the indicator function:

$$\chi_\varepsilon^\alpha(r, t) := \begin{cases} 1 & \text{if } t \in [0, T] \text{ and } r \in [\tilde{r}_a(\varepsilon, t), \infty), \\ 0 & \text{otherwise.} \end{cases}$$

Then, for all $L \in \mathbb{N}$, $\rho_a^{(\varepsilon)} \in C^0([0, T]; \tilde{H}^{-1}([0, L], r^m dr))$ and

$$\|\rho_a^{(\varepsilon)}(\cdot, t_1) - \rho_a^{(\varepsilon)}(\cdot, t_2)\|_{\tilde{H}^{-1}([0, L], r^m dr)} \leq C(\varepsilon)|t_1 - t_2| \quad \text{for all } t_1, t_2 \in [0, T],$$

where $C(\varepsilon) > 0$ is independent of $L \in \mathbb{N}$.

(vi) For each $\eta > a$, the following estimates hold:

$$\begin{cases} \int_0^T \sup_{r \geq \eta} \frac{|u_a|}{\sqrt{e_a}}(r, t) dt \leq C(T)\eta^{(2-n)/2} + C(T)\eta^{2-n} & \text{if } n = 2, 3, \\ \int_0^T \sup_{r \geq \eta} \log(\max\{1, e_a(r, t)\}) dt \leq C(T)(1 + \sqrt{\log(\max\{1, C(T)/\eta\})}) & \text{if } n = 2, \\ \int_0^T \sup_{r \geq \eta} \log(\max\{1, e_a^{\pm 1}(r, t)\}) dt \leq C(T)\eta^{2-n} & \text{if } n = 3. \end{cases}$$

2.2.4 Truncation of the initial data in the exterior Eulerian domain

Since (ρ_a, u_a, e_a) in Theorem 2.2.1 is obtained as the limit of solutions to the approximate problems posed in bounded annular domains, we need to construct the corresponding approximate initial data, which is the main purpose of this subsection. In particular, we construct $(\rho_{a,k}^0, u_{a,k}^0, e_{a,k}^0)$ for $k \in \mathbb{N}$ by truncating (ρ_a^0, u_a^0, e_a^0) with suitable cut-off functions $\varphi_{a,k}^0$.

We first consider the function:

$$r \mapsto \int_a^r \rho_a^0(\zeta) \zeta^m d\zeta. \quad (2.2.10)$$

Since $C_0^{-1} \leq \rho_a^0(r) \leq C_0$, it follows that this is a strictly increasing continuous function. Hence the inverse function $\tilde{r}_a^0(x) : [0, \infty) \rightarrow [a, \infty)$ exists so that

$$x = \int_a^{\tilde{r}_a^0(x)} \rho_a^0(r) r^m dr \quad \text{for each } x \in [0, \infty). \quad (2.2.11)$$

Moreover, due to the regularity of ρ_a^0 , it follows from the inverse function theorem that $\tilde{r}_a^0(x) : [0, \infty) \rightarrow [a, \infty)$ is a smooth, bijective, strictly increasing map.

From now on, we use D_x to represent the derivative with respect to $x \in [0, \infty)$.

Proposition 2.2.2. *For all $x \in [0, \infty)$,*

$$D_x \tilde{r}_a^0(x) = \frac{1}{(\tilde{r}_a^0(x))^m \rho_a^0(\tilde{r}_a^0(x))} \quad \text{and} \quad \tilde{r}_a^0(x) = \left(a^n + n \int_0^x \frac{1}{\rho_a^0(\tilde{r}_a^0(y))} dy \right)^{1/n}. \quad (2.2.12)$$

Proof. Since $\rho_a^0(r) \in C^\infty$ and $C_0^{-1} \leq \rho_a^0(r) \leq C_0$ by construction (2.2.6), it follows that

$$\lim_{r \rightarrow \tilde{r}_a^0(x)} \frac{1}{r - \tilde{r}_a^0(x)} \int_{\tilde{r}_a^0(x)}^r \rho_a^0(\zeta) \zeta^m d\zeta = (\tilde{r}_a^0(x))^m \rho_a^0(\tilde{r}_a^0(x)) \quad \text{for each } x \in [0, \infty).$$

Since $y \mapsto \tilde{r}_a^0(y)$ is continuous, then for all $x \in [0, \infty)$,

$$\lim_{y \rightarrow x} \frac{1}{\tilde{r}_a^0(y) - \tilde{r}_a^0(x)} \int_{\tilde{r}_a^0(x)}^{\tilde{r}_a^0(y)} \rho_a^0(\zeta) \zeta^m d\zeta = (\tilde{r}_a^0(x))^m \rho_a^0(\tilde{r}_a^0(x)).$$

Notice that, by (2.2.11),

$$\frac{\tilde{r}_a^0(y) - \tilde{r}_a^0(x)}{y - x} \frac{1}{\tilde{r}_a^0(y) - \tilde{r}_a^0(x)} \int_{\tilde{r}_a^0(x)}^{\tilde{r}_a^0(y)} \rho_a^0(r) r^m dr = \frac{1}{y - x} \int_{\tilde{r}_a^0(x)}^{\tilde{r}_a^0(y)} \rho_a^0(r) r^m dr = 1.$$

Then, using the fact that $0 < C_0^{-1} a^m \leq \rho_0(\tilde{r}_0(y)) (\tilde{r}_0(y))^m \leq C_0 (\tilde{r}_0(y))^m < \infty$ for all $y \in [0, \infty)$, it follows that, for all $x \in [0, \infty)$,

$$\lim_{y \rightarrow x} \frac{\tilde{r}_a^0(y) - \tilde{r}_a^0(x)}{y - x} = \lim_{y \rightarrow x} \left(\frac{1}{\tilde{r}_a^0(y) - \tilde{r}_a^0(x)} \int_{\tilde{r}_a^0(x)}^{\tilde{r}_a^0(y)} \rho_a^0(r) r^m dr \right)^{-1} = \frac{1}{(\tilde{r}_a^0(x))^m \rho_a^0(\tilde{r}_a^0(x))}.$$

Thus, the derivative of $\tilde{r}_a^0(x)$ exists and (2.2.12)₁ holds.

Next, it follows by the chain rule that

$$\frac{1}{n} D_x (\tilde{r}_a^0(x))^n = (\tilde{r}_a^0(x))^m D_x \tilde{r}_a^0(x) = \frac{1}{\rho_a^0(\tilde{r}_a^0(x))} \quad \text{for } x \in [0, \infty).$$

Integrating in $y \in [0, x]$ and using $\tilde{r}_a^0(0) = a$, we obtain (2.2.12)₂. The proof is complete. \square

Now, let $\chi : \mathbb{R} \rightarrow [0, 1]$ be such that $\chi \in C^\infty$, $\chi(\zeta) = 1$ if $\zeta \leq 0$, $\chi(\zeta) = 0$ if $\zeta \geq 1$, and $|\chi'(\zeta)| \leq 2$ for all $\zeta \in \mathbb{R}$. With this, the cut-off functions $\varphi_{a,k}^0$ is defined as

$$\varphi_{a,k}^0(r) = \chi\left(\frac{2r - \tilde{r}_a^0(k)}{\tilde{r}_a^0(k)}\right) \quad \text{for each } (a, k) \in (0, 1) \times \mathbb{N}.$$

It follows that $\varphi_{a,k}^0(r) \in C^\infty([a, \infty))$ and

$$\varphi_{a,k}^0(r) = 1 \quad \text{for } r \in [a, \tilde{r}_a^0(k)/2], \quad \varphi_{a,k}^0(r) = 0 \quad \text{for } r \in [\tilde{r}_a^0(k), \infty).$$

Using this, we define the truncated initial data as:

$$\begin{cases} \rho_{a,k}^0(r) := (\rho_a^0(r) - 1)\varphi_{a,k}^0(r) + 1, \\ u_{a,k}^0(r) := u_a^0(r)\varphi_{a,k}^0(r), \\ e_{a,k}^0(r) := (e_a^0(r) - 1)\varphi_{a,k}^0(r) + 1, \end{cases} \quad \text{for } r \in [a, \infty). \quad (2.2.13)$$

With above construction, it can be verified that

$$\begin{aligned} (\rho_{a,k}^0, u_{a,k}^0, e_{a,k}^0) &\in C^\infty([a, \infty)), \quad u_{a,k}^0(r) = \partial_r e_{a,k}^0(r) = 0 \quad \text{for } r \in \{a, \tilde{r}_a^0(k)\}, \\ (\rho_{a,k}^0, u_{a,k}^0, e_{a,k}^0)(r) &= \begin{cases} (\rho_a^0, u_a^0, e_a^0)(r) & \text{for all } r \in [a, \tilde{r}_a^0(k)/2), \\ (1, 0, 1) & \text{for all } r \in [\tilde{r}_a^0(k), \infty). \end{cases} \end{aligned} \quad (2.2.14)$$

Proposition 2.2.3. *For $G(z) = 1 - z + z \log z$ and $\psi(z) = z - 1 - \log z$, the following hold:*

$$\begin{aligned} C_0^{-1} \leq \rho_{a,k}^0(r) \leq C_0, \quad e_{a,k}^0(r) \geq C_0^{-1} \quad \text{for all } (r, k) \in [a, \infty) \times \mathbb{N}, \text{ and for all } a \in (0, 1), \\ \sup_{a \in (0,1)} \sup_{k \in \mathbb{N}} \int_a^\infty \left\{ \rho_{a,k}^0 \left(\frac{1}{2} |u_{a,k}^0|^2 + \psi(e_{a,k}^0) \right) + G(\rho_{a,k}^0) + |(\rho_{a,k}^0 - 1, |u_{a,k}^0|^2, e_{a,k}^0 - 1)|^2 \right\} r^m dr \leq C_0. \end{aligned}$$

Proof. By Proposition 2.2.1, we have

$$\begin{aligned} &\int_a^\infty \left\{ |\rho_{a,k}^0 - 1|^2 + |u_{a,k}^0|^4 + |e_{a,k}^0 - 1|^2 \right\} r^m dr \\ &= \int_a^\infty \left\{ |\varphi_{a,k}^0|^2 |\rho_a^0 - 1|^2 + |\varphi_{a,k}^0|^4 |u_a^0|^4 + |\varphi_{a,k}^0|^2 |e_a^0 - 1|^2 \right\} r^m dr \\ &\leq \int_a^\infty \left\{ |\rho_a^0 - 1|^2 + |u_a^0|^4 + |e_a^0 - 1|^2 \right\} r^m dr \leq C_0. \end{aligned}$$

By Proposition 2.2.1, $\min\{C_0^{-1}, 1\} \leq \rho_{a,k}^0(r) = \rho_a^0(r)\varphi_{a,k}^0(r) + 1 - \varphi_{a,k}^0(r) \leq \max\{C_0, 1\}$ for all $r \in [a, \infty)$.

By the same argument, we can show that $e_{a,k}^0(r) \geq \min\{C_0^{-1}, 1\}$ for all $r \in [a, \infty)$.

Finally, since G and ψ are convex, $G(1) = \psi(1) = 0$, and $0 \leq \varphi_{a,k}^0 \leq 1$, it follows that

$$\begin{aligned} G(\rho_{a,k}^0) &= G(\rho_a^0 \varphi_{a,k}^0 + 1 - \varphi_{a,k}^0) \leq \varphi_{a,k}^0 G(\rho_a^0) + (1 - \varphi_{a,k}^0) G(1) = \varphi_{a,k}^0 G(\rho_a^0), \\ \psi(e_{a,k}^0) &= \psi(e_a^0 \varphi_{a,k}^0 + 1 - \varphi_{a,k}^0) \leq \varphi_{a,k}^0 \psi(e_a^0) + (1 - \varphi_{a,k}^0) \psi(1) = \varphi_{a,k}^0 \psi(e_a^0). \end{aligned}$$

Then, using Proposition 2.2.1, we have

$$\begin{aligned} &\int_a^\infty \left\{ \rho_{a,k}^0 \left(\frac{1}{2} |u_{a,k}^0|^2 + \psi(e_{a,k}^0) \right) + G(\rho_{a,k}^0) \right\} (r) r^m dr \\ &\leq \int_a^\infty \left\{ \varphi_{a,k}^0 \left(\rho_a^0 (\psi(e_a^0) + \frac{1}{2} |u_a^0|^2) + G(\rho_a^0) \right) + (1 - \varphi_{a,k}^0) \left(\frac{1}{2} |u_a^0|^2 + \psi(e_a^0) \right) \right\} r^m dr \\ &\leq (1 + C_0^{-1}) \int_a^\infty \left\{ \rho_a^0 \left(\frac{1}{2} |u_a^0|^2 + \psi(e_a^0) \right) + G(\rho_a^0) \right\} r^m dr \leq C_0 + 1. \end{aligned}$$

This concludes the proof. \square

2.2.5 Reformulation in the bounded Lagrangian domain

For each $(a, k) \in (0, 1) \times \mathbb{N}$, we transform $(\rho_{a,k}^0, u_{a,k}^0, e_{a,k}^0)(r)$ into the Lagrangian domain as

$$(\tilde{v}_{a,k}^0, \tilde{u}_{a,k}^0, \tilde{e}_{a,k}^0)(x) := ((\rho_{a,k}^0)^{-1}, u_{a,k}^0, e_{a,k}^0)(\tilde{r}_a^0(x)) \quad \text{for } x \in [0, \infty). \quad (2.2.15)$$

Since $\tilde{r}_a^0(x) \in C^\infty([0, \infty))$ due to $\rho_a^0 \in C^\infty([a, \infty))$, Proposition 2.2.2 and (2.2.14) imply that

$$(\tilde{v}_{a,k}^0, \tilde{u}_{a,k}^0, \tilde{e}_{a,k}^0) \in C^\infty([0, \infty)), \quad \tilde{u}_{a,k}^0(x)|_{x=0,k} = D_x \tilde{e}_{a,k}^0(x)|_{x=0,k} = 0 \quad \text{for } (a, k) \in (0, 1) \times \mathbb{N}.$$

Proposition 2.2.4. *The following uniform estimate holds:*

$$\sup_{a \in (0,1)} \sup_{k \in \mathbb{N}} \int_0^k \left\{ \frac{1}{2} |\tilde{u}_{a,k}^0|^2 + \psi(\tilde{e}_{a,k}^0) + \psi(\tilde{v}_{a,k}^0) + |(\tilde{v}_{a,k}^0 - 1, |\tilde{u}_{a,k}^0|^2, \tilde{e}_{a,k}^0 - 1)|^2 \right\} (x) dx \leq C_0.$$

Proof. First, it follows from Proposition 2.2.2 that the map: $x \mapsto \tilde{r}_a^0(x)$ satisfies

$$0 < C_0^{-1} (\tilde{r}_a^0(x))^{-m} \leq |D_x \tilde{r}_a^0(x)| = (\tilde{r}_a^0(x))^{-m} \rho_a^0 (\tilde{r}_a^0(x))^{-1} \leq C_0 a^{-m} < \infty \text{ for } x \in [0, \infty),$$

so that the Jacobian of transformation $r = \tilde{r}_a^0(x)$ is bounded. Then, for any function $f(r) : [a, \infty) \rightarrow \mathbb{R}$ such that $f \in L^1_{\text{loc}}$, the following coordinate transformation is satisfied:

$$\int_a^{\tilde{r}_a^0(k)} f(r) \rho_a^0(r) r^m dr = \int_0^k f(\tilde{r}_a^0(x)) dx.$$

Using this, Propositions 2.2.1–2.2.3 and (2.2.14), it follows that, for each $(a, k) \in (0, 1) \times \mathbb{N}$,

$$\begin{aligned} \int_0^k |\tilde{v}_{a,k}^0 - 1|^2(x) dx &= \int_0^k |1 - \rho_{a,k}^0(\tilde{r}_a^0(x))|^2 |\rho_{a,k}^0(\tilde{r}_a^0(x))|^{-2} dx \\ &= \int_a^{\tilde{r}_a^0(k)} |1 - \rho_{a,k}^0(r)|^2 |\rho_{a,k}^0(r)|^{-2} \rho_a^0 r^m dr = \int_a^\infty |\rho_{a,k}^0 - 1|^2 |\rho_{a,k}^0|^{-2} \rho_a^0(r) r^m dr \\ &\leq C_0 (\min\{1, C_0^{-1}\})^{-2} \int_a^\infty |\rho_a^0(r) - 1|^2 r^m dr \leq C_0. \end{aligned}$$

By the same argument, we have $\|(|\tilde{u}_{a,k}^0|^2, \tilde{e}_{a,k}^0 - 1)\|_{L^2([a, \infty), r^m dr)} \leq C_0$.

Next, by Propositions 2.2.1–2.2.3 and the identity: $z\psi(z^{-1}) = 1 - z + z \log z = G(z)$, we have

$$\begin{aligned} \int_0^k \left\{ \frac{1}{2} |\tilde{u}_{a,k}^0|^2 + \psi(\tilde{e}_{a,k}^0) + (\gamma - 1) \psi(\tilde{v}_{a,k}^0) \right\} (x) dx \\ = \int_a^\infty \frac{\rho_a^0}{\rho_{a,k}^0} \left\{ \rho_{a,k}^0 \left(\frac{1}{2} |u_{a,k}^0|^2 + \psi(e_{a,k}^0) \right) + (\gamma - 1) G(\rho_{a,k}^0) \right\} r^m dr \leq C_0, \end{aligned}$$

for all $(a, k) \in (0, 1) \times \mathbb{N}$. This completes the proof. \square

The vector function $(\tilde{v}_{a,k}, \tilde{u}_{a,k}, \tilde{e}_{a,k})$ is called a solution of the (a, k) -approximate IBVP in Lagrangian coordinates provided that $(\tilde{v}_{a,k}, \tilde{u}_{a,k}, \tilde{e}_{a,k})$ satisfies the equations:

$$\begin{cases} D_t \tilde{v}_{a,k} - D_x(\tilde{r}_{a,k}^m \tilde{u}_{a,k}) = 0, \\ D_t \tilde{u}_{a,k} + \tilde{r}_{a,k}^m D_x p(\tilde{v}_{a,k}, \tilde{e}_{a,k}) = \beta \tilde{r}_{a,k}^m D_x \left(\frac{D_x(\tilde{r}_{a,k}^m \tilde{u}_{a,k})}{\tilde{v}_{a,k}} \right), \\ D_t \tilde{e}_{a,k} + p(\tilde{v}_{a,k}, \tilde{e}_{a,k}) D_x(\tilde{r}_{a,k}^m \tilde{u}_{a,k}) = \mathcal{G}_{a,k}, \\ \tilde{r}_{a,k}(x, t) = \left(a^n + n \int_0^x \tilde{v}_{a,k}(y, t) dy \right)^{1/n}, \end{cases} \quad (2.2.16)$$

in $[0, k] \times [0, T]$, where $p(v, e) := (\gamma - 1)e/v$ and

$$\mathcal{G}_{a,k} := \beta \frac{|D_x(\tilde{r}_{a,k}^m \tilde{u}_{a,k})|^2}{\tilde{v}_{a,k}} - 2m\mu D_x(\tilde{r}_{a,k}^{m-1} \tilde{u}_{a,k}^2) + \kappa D_x \left(\frac{r_{a,k}^{2m} D_x \tilde{e}_{a,k}}{\tilde{v}_{a,k}} \right),$$

and the initial-boundary conditions:

$$\begin{cases} \tilde{u}_{a,k}(x,t)|_{x=0,k} = D_x \tilde{e}_{a,k}(x,t)|_{x=0,k} = 0 & \text{for } t \in [0, T], \\ (\tilde{v}_{a,k}, \tilde{u}_{a,k}, \tilde{e}_{a,k})(x, 0) = (\tilde{v}_{a,k}^0, \tilde{u}_{a,k}^0, \tilde{e}_{a,k}^0)(x) & \text{for } x \in [0, k], \end{cases} \quad (2.2.17)$$

By virtue of the well-known global existence theorem for problem (2.2.16)–(2.2.17), the initial data constructed in (2.2.15) implies the following theorem (see [13, 23, 24]):

Theorem 2.2.2. *Given initial data $(\tilde{v}_{a,k}^0, \tilde{u}_{a,k}^0, \tilde{e}_{a,k}^0)(x)$ for $x \in [0, k]$, then, for each $T > 0$, there exists a unique classical solution $(v_{a,k}, u_{a,k}, e_{a,k})(x, t)$ of problem (2.2.16)–(2.2.17) satisfying*

$$v_{a,k}(x, t) \in C^{1+\alpha, 1+\alpha/2}([0, k] \times [0, T]), \quad (u_{a,k}, e_{a,k})(x, t) \in C^{2+\alpha, 1+\alpha/2}([0, k] \times [0, T]) \text{ for some } \alpha \in (0, 1).$$

By the above theorem, there exists a unique global classical solution of (2.2.16)–(2.2.17) for each fixed $(a, k) \in (0, 1) \times \mathbb{N}$. This allows us to derive the *a-priori* estimates in Section 2.3 below.

2.3 The A Priori Estimates

This section is devoted to deriving the *a-priori* estimates for solutions of the approximation problem (2.2.16)–(2.2.17). For simplicity, we suppress the approximation parameters $(a, k) \in (0, 1) \times \mathbb{N}$ and denote $(v, u, e, r) \equiv (\tilde{v}_{a,k}, \tilde{u}_{a,k}, \tilde{e}_{a,k}, \tilde{r}_{a,k})$ as the solution obtained from Theorem 2.2.2 in $[0, k] \times [0, T]$ for each $T > 0$. First, we state the main theorem of this section.

Theorem 2.3.1. *The following estimates are satisfied:*

(i) *The entropy inequality holds with $\psi(\zeta) := \zeta - 1 - \log \zeta$:*

$$\begin{aligned} & \sup_{t \in [0, T]} \int_0^k \left\{ \frac{1}{2} |u|^2 + \psi(e) + (\gamma - 1) \psi(v) \right\} (x, t) dx + \kappa \int_0^T \int_0^k \frac{r^{2m} |D_x e|^2}{v e^2} dx dt \\ & + \int_0^T \int_0^k \left\{ \left(\lambda + \frac{2\mu}{n} \right) \frac{|D_x(r^m u)|^2}{v e} + 2m\mu \frac{v}{e} \left(\frac{D_x(r^m u)}{v\sqrt{n}} - \sqrt{n} \frac{u}{r} \right)^2 \right\} dx dt \leq C_0. \end{aligned} \quad (2.3.1)$$

(ii) *For each given constant $\varepsilon \geq 0$,*

$$\underline{v}(\varepsilon, t, a) \leq v(x, t) \leq \bar{v}(\varepsilon, t, a) \quad \text{for all } (x, t) \in [\varepsilon, \infty) \times [0, T], \quad (2.3.2)$$

where $\bar{v} = \bar{v}(z, t, a)$ and $\underline{v} = \underline{v}(z, t, a)$ are explicitly given by

$$\begin{cases} \bar{v} = C_0(1+t)^3 f(z, a) h(z, a) \exp \{ C_0(1+t) h(z, a) f(z, a) \exp \{ C_0 t f(z, a) \} \}, \\ \underline{v} = C_0(1+t)^{-1} f(z, a)^{-1} \exp \{ -t C_0 f(z, a)^2 \} \Gamma(z, t, a), \end{cases} \quad (2.3.3)$$

and functions $h(z, a)$, $f(z, a)$ and $\Gamma(z, t, a)$ are defined by

$$\begin{aligned} f(z, a) &:= \exp \{ C_0 h(z, a) \}, \quad \Gamma(z, t, a) := \exp \left\{ -\frac{m C_0}{\beta} t h(z, a)^{n/(2m)} \right\}, \\ h(z, a) &:= (a^n + n z \psi_-^{-1}(C_0/z))^{-2m/n}, \end{aligned} \quad (2.3.4)$$

where $\psi_-^{-1}(y) : [0, \infty) \rightarrow (0, 1]$ is the left inverse of ψ . In particular, for each $\varepsilon > 0$,

$$\begin{cases} C(T)^{-1} \leq v(x, t) \leq C(T) & \text{for all } (x, t) \in [1, k] \times [0, T], \\ C(\varepsilon)^{-1} \leq v(x, t) \leq C(\varepsilon) & \text{for all } (x, t) \in [\varepsilon, k] \times [0, T], \\ C(a)^{-1} \leq v(x, t) \leq C(a) & \text{for all } (x, t) \in [0, k] \times [0, T]. \end{cases} \quad (2.3.5)$$

(iii) For each $a \in (0, 1)$, $\mathcal{L}_0[v, u, e](T) \leq C(a)$ and, for $(x, t) \in [0, k] \times [0, T]$,

$$C(a)^{-1} \leq v(x, t) \leq C(a), \quad |u(x, t)| \leq \sigma(t)^{-\frac{1}{4}} C(a), \quad C(a)^{-1} \leq e(x, t) \leq \sigma(t)^{-\frac{1}{2}} C(a);$$

and, for each $\varepsilon \in (0, 1]$, $\mathcal{L}_y[v, u, e](T) \leq C(\varepsilon)$ and, for $(x, t) \in [\varepsilon, k] \times [0, T]$,

$$C(\varepsilon)^{-1} \leq v(x, t) \leq C(\varepsilon), \quad |u(x, t)| \leq \sigma(t)^{-\frac{1}{4}} C(\varepsilon), \quad e(x, t) \leq \sigma(t)^{-\frac{1}{2}} C(\varepsilon),$$

where $\sigma(t) := \min\{1, t\}$ and, for $y \in [0, 1)$, the functional $\mathcal{L}_y[v, u, e](T)$ is defined by

$$\begin{aligned} \mathcal{L}_y[v, u, e](T) := & \sup_{t \in [0, T]} \int_y^k |(v - 1, u^2, e - 1, \sqrt{\sigma} r^m D_x u, \sigma r^m D_x e)|^2(x, t) dx \\ & + \int_0^T \int_y^k |(r^m D_x u, r^m D_x e, r^m u D_x u, \sqrt{\sigma} D_t u, \sigma D_t e)|^2(x, t) dx dt. \end{aligned}$$

2.3.1 Entropy Estimates

We now start the proof of Theorem 2.3.1 with the derivation of entropy estimate stated in (i), which is motivated from the second law of thermodynamics. It encapsulates the dissipative effect of viscosity and thermal diffusion.

Proof of Theorem 2.3.1(i). It follows from (2.2.16)₁–(2.2.16)₃ and direct calculations that

$$\begin{aligned} & \frac{d}{dt} \int_0^k \left\{ \frac{1}{2} |u|^2 + \psi(e) + (\gamma - 1) \psi(v) \right\} dx \\ & = -\beta \int_0^k \frac{|D_x(r^m u)|^2}{ev} dx + 2m\mu \int_0^k \frac{1}{e} D_x(r^{m-1} u^2) dx - \kappa \int_0^k \frac{r^{2m} |D_x e|^2}{ve^2} dx. \end{aligned} \quad (2.3.6)$$

According to (2.2.16)₁ and (2.2.16)₄, one can obtain

$$2m\mu \frac{1}{e} D_x(r^{m-1} u^2) - \beta \frac{|D_x(r^m u)|^2}{ev} = -\left(\frac{2\mu}{n} + \lambda\right) \frac{|D_x(r^m u)|^2}{ve} - 2m\mu \frac{v}{e} \left(\frac{D_x(r^m u)}{v\sqrt{n}} - \frac{u\sqrt{n}}{r}\right)^2,$$

which, along with (2.3.6), yields (2.3.1). \square

2.3.2 Upper and lower bounds of the density

Now we aim to prove (2.3.2)–(2.3.5). By (2.3.1), we first obtain some upper and lower bounds of the particle path function $r(x, t)$.

Lemma 2.3.1. *Let $\psi^{-1}(\cdot)$ be the left branch inverse of ψ . Then*

$$a^n + nx\psi^{-1}\left(\frac{C_0}{x}\right) \leq r(x, t)^n \leq C_0(1 + x) \quad \text{for } (x, t) \in [0, k] \times [0, \infty). \quad (2.3.7)$$

Proof. Since $\psi(\zeta) := \zeta - \log \zeta - 1$ is strictly convex in $(0, \infty)$ and $\psi(1) = 0$, then there exists the right inverse function $\psi^{-1} : [0, \infty) \rightarrow (0, 1]$ and the left inverse function $\psi_+^{-1} : [0, \infty) \rightarrow [1, \infty)$. In particular, it is direct to see that $y \mapsto \psi^{-1}(y)$ is strictly decreasing ($\psi^{-1}(0) = 1$ and $\psi^{-1}(y) \rightarrow 0$ as $y \rightarrow \infty$), while $y \mapsto \psi_+^{-1}(y)$ is strictly increasing ($\psi_+^{-1}(0) = 1$ and $\psi_+^{-1}(y) \rightarrow \infty$ as $y \rightarrow \infty$).

Now, from (2.2.16)₄, Theorem 2.3.1(i), and the Jensen inequality, it follows that, for all $(x, t) \in [0, k] \times [0, \infty)$,

$$\psi\left(\frac{r(x, t)^n - a^n}{nx}\right) = \psi\left(\frac{1}{x} \int_0^x v(y, t) dy\right) \leq \frac{1}{x} \int_0^x \psi(v)(y, t) dy \leq \frac{C_0}{x}. \quad (2.3.8)$$

Next, the proof is divided into two cases: $0 < \frac{r^n - a^n}{nx} \leq 1$ and $\frac{r^n - a^n}{nx} \geq 1$.

Case 1: $0 < \frac{r^n - a^n}{nx} \leq 1$. Then $\frac{r^n - a^n}{nx} = \psi_-^{-1}\left(\psi\left(\frac{r^n - a^n}{nx}\right)\right)$. Since $\psi_-^{-1}(\cdot)$ is monotone decreasing, it follows from (2.3.8) that

$$\frac{r^n - a^n}{nx} = \psi_-^{-1}\left(\psi\left(\frac{r^n - a^n}{nx}\right)\right) \geq \psi_-^{-1}\left(\frac{C_0}{x}\right).$$

On the other hand, it follows from $\psi_+^{-1}(y) \geq 1$ for all $y \in [0, \infty)$ that

$$(r(x, t))^n \leq a^n + nx \leq a^n + nx\psi_+^{-1}\left(\frac{C_0}{x}\right).$$

Case 2: $\frac{r^n - a^n}{nx} \geq 1$. Then $\frac{r^n - a^n}{nx} = \psi_+^{-1}\left(\psi\left(\frac{r^n - a^n}{nx}\right)\right)$. Since $\psi_+^{-1}(\cdot)$ is monotone increasing, it follows from (2.3.8) that

$$\frac{r^n - a^n}{nx} = \psi_+^{-1}\left(\psi\left(\frac{r^n - a^n}{nx}\right)\right) \leq \psi_+^{-1}\left(\frac{C_0}{x}\right).$$

On the other hand, it follows from $\psi_-^{-1}(y) \leq 1$ for all $y \in [0, \infty)$ that

$$(r(x, t))^n \geq a^n + nx \geq a^n + nx\psi_-^{-1}\left(\frac{C_0}{x}\right).$$

Finally, for both cases, it is verified in Proposition B.0.1 of Appendix B that $x \mapsto x\psi_+^{-1}(C_0/x)$ is monotone increasing. If $x \leq 1$, then $(r(x, t))^n \leq a^n + nx\psi_+^{-1}(C_0/x) \leq 1 + n\psi_+^{-1}(C_0) \leq C_0 + C_0x$. If $x > 1$, since $y \mapsto \psi_+^{-1}(y)$ is monotone increasing, then $(r(x, t))^n \leq a^n + nx\psi_+^{-1}(C_0/x) \leq 1 + nx\psi_+^{-1}(C_0) \leq C_0 + C_0x$. This concludes the proof. \square

Lemma 2.3.2. Fix $t \in [0, T]$. For each $i = 1, \dots, k$, there exist $A_i(t), B_i(t) \in (i-1, i)$ so that

$$C_0^{-1} \leq v(A_i(t), t), e(B_i(t), t) \leq C_0.$$

Proof. We give our proof only for $A_i(t)$, since the proof for $B_i(t)$ is the same. Fix an integer $i = 1, \dots, k$. Since $\psi(\cdot)$ is convex, it follows from Jensen's inequality and Theorem 2.3.1(i) that

$$\psi\left(\int_{i-1}^i v(x, t) dx\right) \leq \int_{i-1}^i \psi(v(x, t)) dx \leq \frac{C_0}{\gamma - 1}.$$

As in Lemma 2.3.1, denote $z \mapsto \psi_{\pm}^{-1}(z)$ to be the left and right branch of the convex function $\psi(\zeta)$. Since $\psi(1) = 0$, we have

$$0 < \psi_-^{-1}\left(\frac{C_0}{\gamma - 1}\right) \leq \int_{i-1}^i v(x, t) dx \leq \psi_+^{-1}\left(\frac{C_0}{\gamma - 1}\right) < \infty.$$

By the mean value theorem, there exists a point $A_i(t) \in (i-1, i)$ such that

$$\psi_-^{-1}\left(\frac{C_0}{\gamma - 1}\right) \leq v(A_i(t), t) \leq \psi_+^{-1}\left(\frac{C_0}{\gamma - 1}\right).$$

Since $e(x, t)$ satisfies the same estimate only differ by a constant $\gamma - 1$ in Theorem 2.3.1(i), by the same argument, there also exists $B_i(t) \in (i-1, i)$ such that $C_0^{-1} \leq e(B_i(t), t) \leq C_0$. \square

By Lemmas 2.3.1–2.3.2 and Theorem 2.3.1(i), we are ready to prove Theorem 2.3.1(ii).

Proof of Theorem 2.3.1(ii). Fix $i = 1, \dots, k-2$. Since we focus on obtaining an explicit bound for $v(x, t)$ near the origin $x \rightarrow 0$, the case for $i \geq k-1$ is not considered in the proof (but can be obtained by repeating the same argument). We divide the proof into six steps.

1. Applying Lemma 2.3.2, it follows that, for each $(x, t) \in [i-1, i+1] \times [0, T)$, there exist $A_i(t) \in (x, x+3)$ and $B_i(t) \in (i, i+1)$ such that

$$C_0^{-1} \leq v(A_i(t), t), \quad e(B_i(t), t) \leq C_0. \quad (2.3.9)$$

Now, from (2.2.16)₁ and (2.2.16)₂,

$$D_t u + r^m D_x p = \beta r^m D_x D_t \log(v). \quad (2.3.10)$$

Dividing both sides of (2.3.10) by βr^m and integrating over $[x, A_i(t)] \times [0, t]$ for some $(x, t) \in (i-1, i+1] \times [0, T)$, we have

$$\begin{aligned} & \log \frac{v(A_i(t), t)v_0(x)}{v_0(A_i(t))v(x, t)} \\ &= \frac{1}{\beta} \int_x^{A_i(t)} \frac{u}{r^m} \Big|_0^t dy + \frac{m}{\beta} \int_x^{A_i(t)} \int_0^t \frac{u^2}{r^n} ds dy + \frac{1}{\beta} \int_0^t (p(A_i(t), s) - p(x, s)) ds, \end{aligned}$$

where $p(x, s) := p(v(x, s), e(x, s))$. Taking the exponential on both sides yields

$$\frac{E(x, t)}{D(x, t)} Y(t) \frac{v_0(A_i(t))v(x, t)}{v_0(A_i(t), t)v_0(x)} = \exp\left(\frac{1}{\beta} \int_0^t p(x, s) ds\right), \quad (2.3.11)$$

where

$$\begin{aligned} E(x, t) &:= \exp\left(\frac{m}{\beta} \int_0^t \int_x^{A_i(t)} \frac{u^2}{r^n} dy ds\right), \quad Y(t) := \exp\left(\frac{1}{\beta} \int_0^t p(A_i(t), s) ds\right), \\ D(x, t) &:= \exp\left(\frac{1}{\beta} \int_x^{A_i(t)} \left(\frac{u_0(y)}{r_0^m(y)} - \frac{u(y, t)}{r^m(y, t)}\right) dy\right). \end{aligned}$$

Multiplying (2.3.11) with $\beta^{-1}p(x, t)$ and integrating in time, we have

$$1 + \frac{1}{\beta} \int_0^t \frac{E(x, s)}{D(x, s)} Y(s) p(x, s) \frac{v_0(A(x, s))v(x, s)}{v_0(A(x, s), s)v_0(x)} ds = \exp\left(\frac{1}{\beta} \int_0^t p(x, s) ds\right).$$

Substituting this back into (2.3.11), it follows that

$$\frac{E(x, t)}{D(x, t)} Y(x, t) \frac{v_0(A_i(t))v(x, t)}{v_0(A_i(t), t)v_0(x)} = 1 + \frac{1}{\beta} \int_0^t \frac{E(x, s)}{D(x, s)} Y(x, s) p(x, s) \frac{v_0(A(x, s))v(x, s)}{v_0(A(x, s), s)v_0(x)} ds. \quad (2.3.12)$$

2. First, for $D(x, t)$, it follows from Lemma 2.3.1–2.3.2 and Theorem 2.3.1(i) that

$$\begin{aligned} & \left| \int_x^{A_i(t)} \frac{u}{r^m}(y, t) dy \right| \leq \left(\int_x^{A_i(t)} r^{-2m} dy \right)^{1/2} \left(\int_x^{A_i(t)} |u|^2 dy \right)^{1/2} \\ & \leq \left(\int_x^{A_i(t)} (a^n + ny\psi_-^{-1}(C_0/y))^{-\frac{2m}{n}} dy \right)^{1/2} C_0^{1/2} \leq C_0 + C_0 \left(a^n + nx\psi_-^{-1}\left(\frac{C_0}{x}\right) \right)^{-\frac{2m}{n}}. \end{aligned}$$

Then, using the definition of $f(x, a)$ in (2.3.4) and Lemma 2.3.2, we have

$$e^{-C_0} f(x, a)^{-1} \leq D(x, t) \leq e^{C_0} f(x, a). \quad (2.3.13)$$

3. Now we estimate the terms $Y(t)$ and $E(x, t)$. In view of the definition of $\Gamma(z, t, a)$ in (2.3.4), Theorem 2.3.1(i), and Lemma 2.3.1, we see that, for $t \geq s \geq 0$,

$$1 \leq E(x, t) \leq \Gamma(x, t, a)^{-1}, \quad \Gamma(x, t - s, a) \leq \frac{E(x, s)}{E(x, t)} = \exp\left(-\frac{m}{\beta} \int_s^t \int_{i-1}^i \frac{u^2}{r^n} dy d\tau\right) \leq 1. \quad (2.3.14)$$

Putting (2.3.13)–(2.3.14) into (2.3.11) and using (2.3.9) yield that, for all $(x, t) \in (i - 1, i + 1) \times [0, T]$,

$$\begin{aligned} Y(t)v(x, t) &\leq C_0 \left\{ f(x, a) + \int_0^t f^2(x, a) \Gamma(x, t - s, a) Y(s) e(x, s) ds \right\} \\ &\leq C_0 \left\{ f(x, a) + \int_0^t f^2(x, a) Y(s) e(x, s) ds \right\}. \end{aligned} \quad (2.3.15)$$

Since the above argument is true for any $(x, t) \in (i - 1, i + 1) \times [0, T]$, for a fixed $x \in (i - 1, i]$, we can integrate both sides of the above equation in $y \in [x, x + 1] \subset (i - 1, i + 1)$. It then follows from the entropy inequality (2.3.1) and Jensen's inequality that

$$Y(t) \leq C_0 \left\{ f(x, a) + \int_0^t f^2(x, a) Y(s) ds \right\} \quad \text{for all } t \in [0, T],$$

where we have used that $x \mapsto f(x, a)$ is monotone decreasing. Thus, by Grönwall's inequality,

$$\begin{aligned} 1 \leq Y(t) &\leq C_0 \left(f(x, a) + \int_0^t C_0 f^3(x, a) \exp(C_0(t - s) f^2(x, a)) ds \right) \\ &= C_0(1 + t) f^3(x, a) \exp(C_0 t f^2(x, a)) \quad \text{for each } t \in (0, T]. \end{aligned} \quad (2.3.16)$$

4. Since $B_i(t) \in (i, i + 1)$ from (2.3.9), then $x < i < B_i(t)$ for all $t \in [0, T]$. Using Cauchy-Schwartz's inequality, Lemma 2.3.1, estimate (2.3.1), and Jensen's inequality, it follows that, for each $y \in [x, i + 1] \subseteq (i - 1, i + 1)$ and $s \in [0, T]$,

$$\begin{aligned} \left| \sqrt{e(B_i(s), s)} - \sqrt{e(y, s)} \right| &= \left| \frac{1}{2} \int_y^{B_i(s)} \frac{D_x e}{\sqrt{e}}(z, s) dz \right| \\ &\leq \frac{1}{2} \left(\int_0^k \frac{r^{2m} |D_x e|^2}{ve^2}(z, s) dz \right)^{\frac{1}{2}} \left(\sup_{z \in [x, i+1]} r^{-2m} v(z, s) \int_{i-1}^{i+1} e(z, s) dz \right)^{\frac{1}{2}} \\ &\leq C_0 \left(a^n + nx\psi^{-1}\left(\frac{C_0}{x}\right) \right)^{-m/n} (Q(s)V(x, s))^{\frac{1}{2}}, \end{aligned} \quad (2.3.17)$$

where

$$V(x, s) := \sup_{z \in [x, i+1]} v(z, s), \quad Q(s) := \int_0^k \frac{r^{2m} |D_x e|^2}{ve^2}(z, s) dz, \quad (2.3.18)$$

and $Q(t) \in L^1(0, T)$ due to the entropy estimate (2.3.1). According to the definition of $h(z, a)$ in (2.3.4), the construction in (2.3.9) implies that, for all $y \in [x, i + 1] \subseteq (i - 1, i + 1)$,

$$e(y, t) \leq C_0 + C_0 h(x, a) Q(s) V(x, s) \quad \text{for all } (y, s) \in [x, i + 1] \times [0, T]. \quad (2.3.19)$$

Substituting (2.3.19) and (2.3.16) into (2.3.15), it implies that, for all $y \in [x, i + 1] \subseteq (i - 1, i + 1)$,

$$\begin{aligned} v(y, t) &\leq v(y, t) Y(t) \leq C_0 \left\{ f(y, a) + \int_0^t f^2(y, a) Y(s) e(y, s) ds \right\} \\ &\leq C_0(1 + t) f^5(x, a) \exp(C_0 t f^2(x, a)) \left\{ 1 + t + \int_0^t h(x, a) Q(s) V(x, s) ds \right\}, \end{aligned}$$

where we used that $x \mapsto f(x, a)$ is decreasing. Taking supremum over $y \in [x, i + 1]$, we have

$$V(x, t) \leq C_0(1+t)f^5(x, a) \exp(C_0 t f^2(x, a)) \left\{ 1 + t + \int_0^t h(x, a) Q(s) V(x, s) ds \right\}. \quad (2.3.20)$$

Applying Grönwall's inequality to (2.3.20), we obtain that, for all $(x, t) \in [i - 1, i] \times [0, T]$,

$$v(x, t) \leq V(x, t) \leq C_0(1+t)^3 (f^{10} h)(x, a) \exp \{ C_0(1+t)(f^5 h)(x, a) \exp \{ C_0 t f^2(x, a) \} \}.$$

Since $f(x, a)$ takes the form: $f = \exp(C_0(\dots))$, we will denote $(f(x, a))^j \equiv f(x, a)$ for any given positive integer $j \in \mathbb{N}$ when no confusion arises.

In addition, for the lower bound of $v(x, t)$, it follows from (2.3.11) that

$$v(x, t) = \frac{D(x, t)}{Y(t)E(x, t)} \frac{v_0(x)v(A_i(t), t)}{v_0(A_i(t))} \exp \left(\int_0^t p(x, s) ds \right) \geq C_0^{-1} \frac{D(x, t)}{Y(t)E(x, t)}.$$

By (2.3.9), (2.3.13)–(2.3.14), and estimates (2.3.16), we find that, for all $(x, t) \in [i - 1, i] \times [0, T]$,

$$v(x, t) \geq C_0 f^{-1} Y^{-1} E^{-1} \geq C_0(1+t)^{-1} f(x, a)^{-1} \exp(-C_0 t f(x, a)) \Gamma(x, t, a).$$

Since this is true for each $i = 1, \dots, k - 2$, it follows that, for all $(x, t) \in [0, k - 2] \times [0, T]$,

$$\begin{cases} v(x, t) \leq C_0(1+t)^3 (fh)(x, a) \exp \{ C_0(1+t)(fh)(x, a) \exp(C_0 t f(x, a)) \}, \\ v(x, t) \geq C_0 f^{-2} \exp(-C_0 t f^2) \Gamma(x, t, a). \end{cases}$$

5. From definition (2.3.4), it follows that for all $x \in [1, k - 2]$,

$$\begin{aligned} f(x, a) &\leq \exp \left(C_0 (n\psi^{-1}(C_0))^{-\frac{2m}{n}} \right), & h(x, a) &\leq (n\psi^{-1}(C_0))^{-\frac{2m}{n}}, \\ \Gamma(x, t, a) &\geq \exp \left(-\frac{mC_0}{\beta} t (n\psi^{-1}(C_0))^{-1} \right), \end{aligned}$$

so that $C(T)^{-1} \leq v(x, t) \leq C(T)$ for all $(x, t) \in [1, k - 2] \times [0, T]$. Since $x \mapsto x\psi^{-1}(C_0/x)$ is positive and monotone increasing, then, for all $x \in [\varepsilon, k - 2]$,

$$\begin{aligned} f(x, a) &\leq \exp \left(C_0 (n\varepsilon\psi^{-1}(C_0/\varepsilon))^{-m/n} \right), & h(x, a) &\leq (n\varepsilon\psi^{-1}(C_0/\varepsilon))^{-m/n}, \\ \Gamma(x, t, a) &\geq \exp \left(-\frac{mC_0 t}{n\beta\varepsilon\psi^{-1}(C_0/\varepsilon)} \right), \end{aligned}$$

which yields that for each $\varepsilon \in (0, 1]$, $C(\varepsilon)^{-1} \leq v(x, t) \leq C(\varepsilon)$ for all $(x, t) \in [\varepsilon, k - 2] \times [0, T]$.

6. Finally, we remark that, for the case when $x \in (k - 2, k]$, we can choose $k \geq 3$ and $A_k(t), B_k(t) \in (k - 1, k)$ by Lemma 2.3.2, and then repeat the same argument presented above. From this, we conclude that $C(T)^{-1} \leq v(x, t) \leq C(T)$ for all $(x, t) \in (k - 2, k] \times [0, T]$. \square

2.3.3 Exterior $L^\infty(0, T; L^2)$ -Estimates

To obtain the estimates independent of $(a, k) \in (0, 1) \times \mathbb{N}$, the derivations are restricted on domain $[\varepsilon, k] \times [0, T]$. Thus, given fixed $\varepsilon > 0$, we introduce the spatial cut-off function $g_\varepsilon(x) \in C^1$ as follows:

$$g_\varepsilon(x) := \begin{cases} 0 & \text{if } x \leq \frac{\varepsilon}{2}, \\ \frac{8}{\varepsilon^2} (x - \frac{\varepsilon}{2})^2 & \text{if } \frac{\varepsilon}{2} \leq x \leq \frac{3\varepsilon}{4}, \\ 1 - \frac{8}{\varepsilon^2} (x - \varepsilon)^2 & \text{if } \frac{3\varepsilon}{4} \leq x \leq \varepsilon, \\ 1 & \text{if } x \geq \varepsilon. \end{cases} \quad (2.3.21)$$

It can directly be verified that $g_\varepsilon(x)$ satisfies the following properties: for all $x \in [0, \infty)$,

$$0 \leq g_\varepsilon \leq 1, \quad g'_\varepsilon(x) \geq 0, \quad \text{supp}(g'_\varepsilon) = \left[\frac{\varepsilon}{2}, \varepsilon\right], \quad |g'_\varepsilon(x)|^2 \leq \frac{32}{\varepsilon^2} g_\varepsilon(x). \quad (2.3.22)$$

Moreover, it is crucial that the estimates are independent of the approximation parameters $(a, k) \in (0, 1) \times \mathbb{N}$. In order to achieve this, the upper and lower bound of $r(x, t)$ must be given careful consideration. Recall that, from Lemma 2.3.1, $r(x, t)$ satisfies

$$(nx\psi_-^{-1}(C_0/x))^{1/n} \leq r(x, t) \leq C_0^{1/n}(1+x)^{1/n} \quad \text{for } (x, t) \in [0, k] \times [0, T].$$

It follows that, for each $\varepsilon \in (0, 1]$,

$$\sup_{x \in [\varepsilon/2, \varepsilon]} r(x, t) \leq C(\varepsilon), \quad \sup_{x \in [\varepsilon, k]} \frac{1}{r(x, t)} \leq C(\varepsilon). \quad (2.3.23)$$

Using $g_\varepsilon(x)$ and Theorem 2.3.1(i)–(ii), our aim is to derive the $L^\infty(0, T; L^2)$ estimates listed in Theorem 2.3.1(iii). In order to do this, we first prove the following lemma, which is necessary for resolving the problematic boundary integrals on domain $x \in [\frac{\varepsilon}{2}, \varepsilon]$.

Lemma 2.3.3 (Exterior $L^1(0, T; L^\infty)$ -Estimates of e and u^2). *For any fixed $\varepsilon > 0$,*

$$\int_0^T \sup_{y \in [\varepsilon, k]} e(y, s) ds \leq C(\varepsilon), \quad \int_0^T \sup_{y \in [\varepsilon, k]} u^2(y, s) ds \leq C(\varepsilon). \quad (2.3.24)$$

Proof. By Lemma 2.3.2, for each $(y, s) \in [\varepsilon, k] \times [0, T]$, there exists $B_i(s) \in [0, k]$ such that $|y - B_i(s)| \leq 1$ and $C_0^{-1} \leq e(B_i(s), s) \leq C_0$. By the same calculation in (2.3.17), we have

$$\begin{aligned} |\sqrt{e}(y, s) - \sqrt{e}(B_i(s), s)|^2 &\leq \sup_{z \in [\varepsilon, k]} \frac{v}{4r^{2m}}(z, s) \left| \int_{B_i(s)}^y e(x, s) dx \right| \left| \int_{B_i(s)}^y \frac{r^{2m}}{v} \frac{|D_x e|^2}{e^2}(x, s) dx \right| \\ &\leq C(\varepsilon) \int_0^k \frac{r^{2m}}{v} \frac{|D_x e|^2}{e^2}(x, s) dx, \end{aligned}$$

where, for the last inequality, we used Theorem 2.3.1(ii), (2.3.21)–(2.3.23), $\|\psi(e)\|_{L^1(0, k)} \leq C_0$ from Theorem 2.3.1(i) and Jensen's inequality. Taking supremum over $y \in [\varepsilon, k]$ on the above and integrating in $t \in [0, T]$, it follows from Lemma 2.3.2 and Theorem 2.3.1(i) that

$$\int_0^T \sup_{y \in [\varepsilon, k]} e(y, s) ds \leq C_0 T + C(\varepsilon) \int_0^T \int_0^k \frac{r^{2m}}{v} \frac{|D_x e|^2}{e^2}(x, s) dx ds \leq C(\varepsilon). \quad (2.3.25)$$

Next, notice that

$$|uD_x u| = |r^{-m} D_x(r^m u) - mr^{-n} v u|^2 \leq 2r^{-2m} |D_x(r^m u)|^2 + m^2 r^{-2n} v^2 |u|^2.$$

Then, by the Sobolev embedding theorem: $W^{1,1}(\varepsilon, k) \hookrightarrow C^0(\varepsilon, k)$, and Theorem 2.3.1(i)–(ii),

$$\begin{aligned} \sup_{y \in [\varepsilon, k]} u^2(y, t) &\leq C \int_\varepsilon^k u^2 dx + C \int_\varepsilon^k |uD_x u| dx \\ &= C \int_\varepsilon^k u^2 dx + C \int_\varepsilon^k |r^{-m} u D_x(r^m u) - mr^{-n} v u|^2 dx \\ &\leq C_0 + C \sup_{y \in [\varepsilon, k]} \left(\frac{v}{r^{2m}} e + \frac{v}{r^n} \right)(y, s) \int_\varepsilon^k u^2(x, t) dx + C \int_\varepsilon^k \frac{|D_x(r^m u)|^2}{ve} dx \\ &\leq C(\varepsilon) + C(\varepsilon) \sup_{y \in [\varepsilon, k]} e(y, t) + C \int_0^k \frac{|D_x(r^m u)|^2}{ve} dx. \end{aligned} \quad (2.3.26)$$

(2.3.24)₂ follows from integrating the above in $[0, T]$, then using (2.3.25) and Theorem 2.3.1(i). \square

Lemma 2.3.4 (Exterior $L^\infty(0, T; L^2)$ -estimate of total energy). *For any fixed $\varepsilon > 0$,*

$$\begin{aligned} & \sup_{t \in [0, T]} \int_\varepsilon^k \{|e - 1|^2 + u^4\}(x, t) dx + \int_0^T \sup_{x \in [\varepsilon, k]} e^2(x, s) ds \\ & + \int_0^T \int_\varepsilon^k \frac{r^{2m}}{v} \{|D_x e|^2 + |uD_x u|^2\}(x, s) dx ds \leq C(\varepsilon). \end{aligned} \quad (2.3.27)$$

Proof. We divide the proof into four steps.

1. Denote the total energy density function $w := e - 1 + u^2/2$. Then multiplying (2.2.16)₂ with u and adding it to (2.2.16)₃ yield

$$D_t w = D_x(r^m u \tilde{F}) - 2m\mu D_x(r^{m-1} u^2) + \kappa D_x\left(\frac{r^{2m}}{v} D_x e\right), \quad (2.3.28)$$

where $\tilde{F} := \beta v^{-1} D_x(r^m u) - p(v, e)$. Multiplying (2.3.28) with $g_\varepsilon w$ and then integrating in space, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_0^k g_\varepsilon w^2 dx \\ & = - \int_0^k g_\varepsilon r^m u \tilde{F} D_x w dx + 2m\mu \int_0^k g_\varepsilon r^{m-1} u^2 D_x w dx - \kappa \int_0^k g_\varepsilon \frac{r^{2m}}{v} D_x e D_x w dx \\ & \quad - \int_0^k g'_\varepsilon r^m u w \tilde{F} dx + 2m\mu \int_0^k g'_\varepsilon r^{m-1} u^2 w dx - \kappa \int_0^k g'_\varepsilon \frac{r^{2m}}{v} w D_x e dx =: \sum_{i=1}^3 I_i + \mathcal{O}_\varepsilon. \end{aligned}$$

2. The terms $I_j, j = 1, 2, 3$, can be estimated as follows:

$$\begin{aligned} I_1 & := - \int_0^k g_\varepsilon r^m u \tilde{F} D_x w dx \\ & = - \beta \int_0^k g_\varepsilon \left\{ \frac{r^{2m}}{v} u D_x u D_x e + m r^{m-1} u^3 D_x u + m r^{m-1} u^2 D_x e \right\} dx \\ & \quad + (\gamma - 1) \int_0^k g_\varepsilon \frac{e}{v} r^m \{u^2 D_x u + u D_x e\} dx - \beta \int_0^k g_\varepsilon \frac{r^{2m}}{v} |uD_x u|^2 dx \\ & \leq \left(\frac{5\beta^2}{2\kappa} - \frac{2\beta}{3} \right) \int_0^k g_\varepsilon \frac{|r^m u D_x u|^2}{v} dx + \frac{3\kappa}{10} \int_0^k g_\varepsilon \frac{|r^m D_x e|^2}{v} dx + C \int_0^k g_\varepsilon \left(\frac{v u^4}{r^2} + \frac{e^2 u^2}{v} \right) dx, \\ I_2 & := 2m\mu \int_0^k g_\varepsilon r^{m-1} u^2 D_x w dx = 2m\mu \int_0^k g_\varepsilon r^{m-1} (u^3 D_x u + u^2 D_x e) dx \\ & \leq \frac{\beta}{6} \int_0^k g_\varepsilon \frac{r^{2m}}{v} |uD_x u|^2 dx + \frac{\kappa}{10} \int_0^k g_\varepsilon \frac{r^{2m}}{v} |D_x e|^2 dx + C \int_0^k g_\varepsilon \frac{v}{r^2} u^4 dx, \\ I_3 & := - \kappa \int_0^k g_\varepsilon \frac{r^{2m}}{v} D_x e D_x w dx = - \kappa \int_0^k g_\varepsilon \frac{r^{2m}}{v} D_x e \{D_x e + u D_x u\} dx \\ & \leq - \frac{9}{10} \kappa \int_0^k g_\varepsilon \frac{r^{2m}}{v} |D_x e|^2 dx + \frac{5\kappa}{2} \int_0^k g_\varepsilon \frac{r^{2m}}{v} |uD_x u|^2 dx. \end{aligned}$$

Using Theorem 2.3.1(ii) and (2.3.21)–(2.3.23), the term \mathcal{O}_ε is estimated as follows:

$$\begin{aligned}
\mathcal{O}_\varepsilon &= \int_{\varepsilon/2}^\varepsilon g'_\varepsilon \left\{ -r^m u w \left(\beta \frac{r^m}{v} D_x u + \frac{m\beta}{r} u - (\gamma - 1) \frac{e}{v} \right) + 2m\mu r^{m-1} u^2 w - \kappa \frac{r^{2m}}{v} w D_x e \right\} dx \\
&\leq \int_{\varepsilon/2}^\varepsilon \frac{\varepsilon^2 |g'_\varepsilon|^2}{32} \frac{r^{2m}}{v} \left\{ \frac{\beta}{2} |u D_x u|^2 + \frac{\kappa}{4} |D_x e|^2 \right\} dx \\
&\quad + C \int_{\varepsilon/2}^\varepsilon \left\{ \frac{\varepsilon^2 |g'_\varepsilon|^2}{32} \left(\frac{v u^4}{r^2} + \frac{e^2 u^2}{v} \right) + \frac{32 r^{2m}}{\varepsilon^2} w^2 \right\} dx \\
&\leq \int_0^k g_\varepsilon \frac{r^{2m}}{v} \left\{ \frac{\beta}{2} |u D_x u|^2 + \frac{\kappa}{4} |D_x e|^2 \right\} dx + C \int_0^k g_\varepsilon \left\{ \frac{v}{r^2} u^4 + \frac{e^2 u^2}{v} \right\} dx + C(\varepsilon) \int_{\varepsilon/2}^\varepsilon w^2 dx.
\end{aligned}$$

Combining all above estimates, it follows that

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \int_0^k g_\varepsilon w^2 dx &\leq -\frac{\kappa}{4} \int_0^k g_\varepsilon \frac{r^{2m}}{v} |D_x e|^2 dx + \frac{5(\beta^2 + \kappa^2)}{2\kappa} \int_0^k g_\varepsilon \frac{r^{2m}}{v} |u D_x u|^2 dx \\
&\quad + C \int_0^k g_\varepsilon \frac{v}{r^2} u^4 dx + C \int_0^k g_\varepsilon \frac{e^2 u^2}{v} dx + C(\varepsilon) \mathcal{R}_\varepsilon(t),
\end{aligned} \tag{2.3.29}$$

where $\mathcal{R}_\varepsilon(t)$ is the problematic boundary integral near the origin, given by

$$\mathcal{R}_\varepsilon(t) := \int_{\varepsilon/2}^\varepsilon ((e-1)^2 + u^4)(x, t) dx. \tag{2.3.30}$$

3. Multiplying $g_\varepsilon u^3$ to the momentum equation (2.2.16)₂ and integrating in $x \in [0, k]$, we obtain the following L^4 -estimate for u :

$$\begin{aligned}
\frac{1}{4} \frac{d}{dt} \int_0^k g_\varepsilon |u|^4 dx &= -\beta \int_0^k g_\varepsilon \frac{1}{v} D_x(r^m u) D_x(r^m u^3) dx + (\gamma - 1) \int_0^k g_\varepsilon \frac{e}{v} D_x(r^m u^3) dx \\
&\quad + (\gamma - 1) \int_0^k g'_\varepsilon \frac{e}{v} r^m u^3 dx - \beta \int_0^k g'_\varepsilon \frac{1}{v} D_x(r^m u) r^m u^3 dx =: \sum_{i=1}^2 J_i + \tilde{\mathcal{O}}_\varepsilon.
\end{aligned}$$

The terms J_1 and J_2 can be estimated as follows:

$$\begin{aligned}
J_1 &:= -\beta \int_0^k g_\varepsilon \frac{1}{v} D_x(r^m u) D_x(r^m u^3) dx \\
&= -\beta \int_0^k g_\varepsilon \left(3 \frac{|r^m u D_x u|^2}{v} + 4m r^{m-1} u^3 D_x u + m^2 \frac{u^4 v}{r^2} \right) dx \\
&\leq -\beta \int_0^k g_\varepsilon \frac{r^{2m}}{v} |u D_x u|^2 dx + m^2 \beta \int_0^k g_\varepsilon \frac{v}{r^2} u^4 dx, \\
J_2 &:= (\gamma - 1) \int_0^k g_\varepsilon \frac{e}{v} D_x(r^m u^3) dx \leq \frac{\beta}{4} \int_0^k g_\varepsilon \frac{|r^m u D_x u|^2}{v} dx + C \int_0^k g_\varepsilon \left(\frac{e^2 u^2}{v} + \frac{v u^4}{r^2} \right) dx.
\end{aligned}$$

Using (2.3.21)–(2.3.23) and Theorem 2.3.1(ii), the term $\tilde{\mathcal{O}}_\varepsilon$ is estimated as follows.

$$\begin{aligned}\tilde{\mathcal{O}}_\varepsilon &:= (\gamma - 1) \int_0^k g'_\varepsilon \frac{e}{v} r^m u^3 dx - \beta \int_0^k g'_\varepsilon \frac{1}{v} D_x(r^m u) r^m u^3 dx \\ &\leq \frac{\gamma - 1}{2} \int_{\varepsilon/2}^\varepsilon \frac{\varepsilon^2}{32} |g'_\varepsilon|^2 \frac{e^2 u^2}{v} dx + \frac{\gamma - 1}{2} \int_{\varepsilon/2}^\varepsilon \frac{32 r^{2m}}{\varepsilon^2} u^4 dx + C \int_{\varepsilon/2}^\varepsilon \frac{1}{\varepsilon} r^{m-1} u^4 dx \\ &\quad + \frac{\beta}{4} \int_{\varepsilon/2}^\varepsilon \frac{\varepsilon^2}{32} |g'_\varepsilon|^2 \frac{r^{2m}}{v} |uD_x u|^2 dx + \beta \int_{\varepsilon/2}^\varepsilon \frac{32 r^{2m}}{\varepsilon^2} u^4 dx \\ &\leq \frac{\beta}{4} \int_0^k g_\varepsilon \frac{r^{2m}}{v} |uD_x u|^2 dx + \frac{\gamma - 1}{2} \int_0^k g_\varepsilon \frac{e^2 u^2}{v} dx + C(\varepsilon) \int_{\varepsilon/2}^\varepsilon u^4 dx.\end{aligned}$$

Combining all above estimates, it follows that

$$\frac{d}{dt} \int_0^k g_\varepsilon \frac{|u|^4}{4} dx \leq C \int_0^k g_\varepsilon \left(\frac{e^2 u^2}{v} + \frac{v u^4}{r^2} \right) dx - \frac{\beta}{2} \int_0^k g_\varepsilon \frac{|r^m u D_x u|^2}{v} dx + C(\varepsilon) \mathcal{R}_\varepsilon(t). \quad (2.3.31)$$

4. Multiplying (2.3.31) by $4\tilde{C} \equiv 10(\beta^2 + \kappa^2)/(\kappa\beta)$ and adding it to (2.3.29), one has from (2.3.1):

$$\begin{aligned}\tilde{C} \frac{d}{dt} \int_0^k g_\varepsilon u^4 dx + \frac{1}{2} \frac{d}{dt} \int_0^k g_\varepsilon w^2 dx + \int_0^k g_\varepsilon \frac{r^{2m}}{v} \left\{ \frac{\kappa}{4} |D_x e|^2 + \tilde{C} |uD_x u|^2 \right\} dx \\ \leq C(\varepsilon) \mathcal{R}_\varepsilon(t) + C \int_0^k g_\varepsilon \left(\frac{v}{r^2} u^4 + \frac{|ue|^2}{v} \right) dx.\end{aligned} \quad (2.3.32)$$

Using Theorem 2.3.1(i)–(ii) and (2.3.21)–(2.3.23), we have

$$\int_0^k g_\varepsilon \left(\frac{v}{r^2} u^4 + \frac{|ue|^2}{v} \right) dx \leq C(\varepsilon) \int_0^k g_\varepsilon u^4 dx + C(\varepsilon) \sup_{y \in [\varepsilon/2, k]} g_\varepsilon |e|^2(y, t). \quad (2.3.33)$$

By Lemma 2.3.2, for each $(x, t) \in [0, k] \times [0, T]$, there exist both an integer $i \in \{0, \dots, k-1\}$ and a point $B_i(t) \in [0, k]$ such that $x, B_i(t) \in [i, i+1]$ and $C_0^{-1} \leq e(B_i(t), t) \leq C_0$. Using Theorem 2.3.1(i), $|x - B_i(t)| \leq 1$, and (2.3.21)–(2.3.23), we have

$$\begin{aligned}|g_\varepsilon e(x, t) - g_\varepsilon e(B_i(t), t)|^2 &= \left(\int_{B_i(t)}^x \{g_\varepsilon D_x e + g'_\varepsilon e\}(y, t) dy \right)^2 \\ &\leq 2 \left(\int_{B_i(t)}^x g_\varepsilon \frac{r^{2m}}{v} \frac{|D_x e|^2}{e^2} dy \right) \left(\int_{B_i(t)}^x g_\varepsilon \frac{v}{r^{2m}} e^2 dy \right) + C(\varepsilon) \left(\int_{B_i(t)}^x e(y, t) dy \right)^2 \\ &\leq 4 \left(\int_0^k \frac{r^{2m}}{v} \frac{|D_x e|^2}{e^2} dy \right) \left\{ \left| \int_{B_i(t)}^x g_\varepsilon \frac{v}{r^{2m}} |e - 1|^2 dy \right| + \left| \int_{B_i(t)}^x \frac{g_\varepsilon v}{r^{2m}} dy \right| \right\} + C(\varepsilon) \left(\int_i^{i+1} e dy \right)^2 \\ &= C(\varepsilon) Q(t) \left\{ \left| \int_{B_i(t)}^x g_\varepsilon |e - 1|^2 dy \right| + 1 \right\} + C_0^2 C(\varepsilon),\end{aligned}$$

where $Q(t)$ is defined in (2.3.18). Taking supremum over $x \in [\varepsilon/2, k]$, it follows from Theorem 2.3.1(ii) and (2.3.21)–(2.3.23) that

$$\sup_{x \in [\varepsilon/2, k]} g_\varepsilon |e|^2(x, t) \leq C(\varepsilon) + C(\varepsilon) Q(t) \left(1 + \int_0^k g_\varepsilon |e - 1|^2 dx \right). \quad (2.3.34)$$

Substituting (2.3.34) into (2.3.33) yields

$$\int_0^k g_\varepsilon \left(\frac{v}{r^2} u^4 + \frac{|ue|^2}{v} \right) dx \leq C(\varepsilon) (1 + Q(t)) \left(1 + \int_0^k g_\varepsilon (u^4 + w^2) dx \right), \quad (2.3.35)$$

where the following inequality is used:

$$|u|^4 + w^2 = \frac{5}{4}|u|^4 + (e-1)^2 + (e-1)|u|^2 \geq \frac{3}{4}|u|^4 + \frac{1}{2}(e-1)^2.$$

Substituting (2.3.35) into (2.3.32), it follows that

$$\begin{aligned} & \frac{d}{dt} \int_0^k g_\varepsilon(u^4 + w^2) dx + \int_0^k g_\varepsilon \frac{r^{2m}}{v} |D_x e|^2 dx + \int_0^k g_\varepsilon \frac{r^{2m}}{v} |uD_x u|^2 dx \\ & \leq C(\varepsilon)(1 + Q(t)) \left(1 + \int_0^k g_\varepsilon(u^4 + w^2) dx\right) + C(\varepsilon)\mathcal{R}_\varepsilon(t). \end{aligned}$$

Therefore, applying Grönwall's inequality and using $Q(t) \in L^1(0, T)$, we have

$$\int_0^k (|u|^4 + |e-1|^2)(x, t) dx \leq C(\varepsilon) \int_0^k (u_0^4 + w_0^2) dx + C(\varepsilon) + C(\varepsilon) \int_0^t \mathcal{R}_\varepsilon ds. \quad (2.3.36)$$

4. Using Lemma 2.3.3, we claim that the term $\mathcal{R}_\varepsilon(t)$ defined in (2.3.30) satisfies:

$$\int_0^t \mathcal{R}_\varepsilon(s) ds = \int_0^t \int_{\varepsilon/2}^\varepsilon (|e-1|^2 + |u|^4) dx ds \leq C(\varepsilon). \quad (2.3.37)$$

To show this claim, we first use Lemma 2.3.3 to obtain

$$\begin{aligned} & \int_0^t \mathcal{R}_\varepsilon(s) ds = \int_0^t \int_{\varepsilon/2}^\varepsilon ((e-1)^2 + u^4) dx ds \\ & \leq C \int_0^t \sup_{y \in [\varepsilon/2, \varepsilon]} |e-1|(y, s) \int_{\varepsilon/2}^\varepsilon (e+1)(x, s) dx ds + C \int_0^t \sup_{y \in [\varepsilon/2, \varepsilon]} u^2(y, s) \int_{\varepsilon/2}^\varepsilon u^2(x, s) dx ds \\ & \leq C\varepsilon + \varepsilon C(\varepsilon) + C(\varepsilon) + C \int_0^t \sup_{y \in [\varepsilon/2, \varepsilon]} e(y, s) \int_{\varepsilon/2}^\varepsilon e(x, s) dx ds. \end{aligned}$$

It follows by Jensen's inequality and Theorem 2.3.1(i) that

$$\psi\left(\frac{2}{\varepsilon} \int_{\varepsilon/2}^\varepsilon e(x, s) dx\right) \leq \frac{2}{\varepsilon} \int_{\varepsilon/2}^\varepsilon \psi(e)(x, s) dx \leq \frac{2C_0}{\varepsilon},$$

so that

$$\int_{\varepsilon/2}^\varepsilon e(x, s) dx \leq \frac{\varepsilon}{2} \psi_+^{-1}\left(\frac{2C_0}{\varepsilon}\right) \leq C(\varepsilon).$$

Substituting this back into the estimate of \mathcal{R}_ε and using Lemma 2.3.3, it follows that

$$\begin{aligned} & \int_0^t \mathcal{R}_\varepsilon(s) ds \leq C\varepsilon + \varepsilon C(\varepsilon) + C(\varepsilon) + C \int_0^t \sup_{y \in [\varepsilon/2, \varepsilon]} e(y, s) \int_{\varepsilon/2}^\varepsilon e(x, s) dx ds \\ & \leq C(\varepsilon) + C(\varepsilon) \int_0^t \sup_{y \in [\varepsilon/2, \varepsilon]} e(y, s) ds \leq C(\varepsilon). \end{aligned}$$

This prove the claim (2.3.37).

5. Therefore, using Grönwall's inequality and (2.3.36)–(2.3.37), we have

$$\begin{aligned} & \int_0^k g_\varepsilon(u^4 + w^2)(x, t) dx + \int_0^t \int_0^k g_\varepsilon \frac{r^{2m}}{v} |D_x e|^2 dx ds + \int_0^t \int_0^k g_\varepsilon \frac{r^{2m}}{v} |uD_x u|^2 dx ds \\ & \leq \int_0^k (u_0^4 + w_0^2) dx + C(\varepsilon)(C_0 + C(\varepsilon)). \end{aligned}$$

By the inequality: $|u|^4 + w^2 \geq 3|u|^4/4 + (e-1)^2/2$, we see that $\|(e-1, u^2)\|_{L^\infty(0,T;L^2(\varepsilon,k))} \leq C(\varepsilon)$.

Finally, substituting the above estimate into (2.3.34), we obtain

$$\begin{aligned} \int_0^T \sup_{x \in [\varepsilon, k]} e^2(x, t) dt &\leq \int_0^T \sup_{x \in [\varepsilon/2, k]} g_\varepsilon e^2(x, t) dt \\ &\leq TC(\varepsilon) + C(\varepsilon) \int_0^T Q(t) \left(1 + \int_0^k g_\varepsilon |e-1|^2 dx\right) dt \leq C(\varepsilon). \end{aligned}$$

This completes the proof. \square

Lemma 2.3.5 (Exterior $L^\infty(0, T; L^2)$ -estimate of specific volume). *There exists a constant $C(\varepsilon) > 0$ depending only on $\varepsilon > 0$ such that*

$$\sup_{t \in [0, T]} \int_\varepsilon^k |v-1|^2(x, t) dx \leq C(\varepsilon).$$

Proof. Multiplying (2.2.16)₁ with $g_\varepsilon(v-1)$ and integrating in $[0, k]$, by Theorem 2.3.1(ii), we have

$$\begin{aligned} \frac{1}{2} D_t \int_0^k g_\varepsilon (v-1)^2 dx &= \int_0^k g_\varepsilon (v-1) D_x(r^m u) dx \\ &\leq C(\varepsilon) \int_0^k g_\varepsilon (v-1)^2 dx + \frac{1}{2} \int_0^k g_\varepsilon \frac{|D_x(r^m u)|^2}{v} dx. \end{aligned}$$

Applying Lemma 2.3.4 and Grönwall's inequality, the lemma is proved. \square

The following lemma is necessary for the high order estimates of (u, e) in Section 2.3.4 below.

Lemma 2.3.6 (Exterior L^2 -estimate of $D_x u$). *There exists a constant $C(\varepsilon) > 0$ depending only on $\varepsilon > 0$ such that*

$$\int_0^T \int_\varepsilon^k \left(\frac{|D_x(r^m u)|^2}{v} + \frac{r^{2m} |D_x u|^2}{v} \right) dx ds \leq C(\varepsilon). \quad (2.3.38)$$

Proof. Multiplying (2.2.16)₁ by g_ε and integrating over $[0, k]$, we have

$$\int_0^k g_\varepsilon(x) D_t(v(x, t) - 1) dx = - \int_0^k g'_\varepsilon(x) r^m u(x, t) dx. \quad (2.3.39)$$

Multiplying (2.2.16)₂ by $g_\varepsilon u$ and integrating over $x \in [0, k]$, using Theorem 2.3.1(i)–(ii), Theorem 2.3.4, and (2.3.39), it follows that

$$\begin{aligned} &\frac{d}{dt} \int_0^k g_\varepsilon(x) \frac{|u|^2}{2} dx + \beta \int_0^k g_\varepsilon(x) \frac{|D_x(r^m u)|^2}{v} dx \\ &= (\gamma - 1) \int_0^k g_\varepsilon \frac{D_x(r^m u)}{v} (e-1) dx + (\gamma - 1) \int_0^k g_\varepsilon \frac{D_t v}{v} dx \\ &\quad + (\gamma - 1) \int_0^k g'_\varepsilon(x) \frac{r^m}{v} u (e-1) dx + (\gamma - 1) \int_0^k g'_\varepsilon(x) \frac{r^m}{v} u dx \\ &\quad - \beta \int_0^k g'_\varepsilon(x) r^m u \frac{D_x(r^m u)}{v} dx =: \sum_{i=1}^2 I_i + \mathcal{O}_\varepsilon. \end{aligned} \quad (2.3.40)$$

The term I_1 is estimated by using Theorem 2.3.1(ii) and Lemma 2.3.4:

$$I_1 := (\gamma - 1) \int_0^k g_\varepsilon \frac{D_x(r^m u)}{v} (e-1) dx \leq \frac{\beta}{4} \int_0^k g_\varepsilon \frac{|D_x(r^m u)|^2}{v} dx + C(\varepsilon). \quad (2.3.41)$$

I_2 is estimated by using (2.2.16)₁, Theorem 2.3.1(i), and (2.3.21)–(2.3.23):

$$\begin{aligned}
\frac{I_2}{\gamma-1} &:= \int_0^k g_\varepsilon \frac{D_t v}{v} dx = \int_0^k g_\varepsilon D_t (\log v - v + 1) dx - \int_0^k g'_\varepsilon r^m u dx \\
&\leq \int_0^k g_\varepsilon D_t (\log v - v + 1) dx + \left(\int_{\varepsilon/2}^\varepsilon r^{2m} dx \right)^{1/2} \left(\int_{\varepsilon/2}^\varepsilon u^2 dx \right)^{1/2} \\
&\leq \int_0^k g_\varepsilon D_t (\log v - v + 1) dx + C(\varepsilon).
\end{aligned} \tag{2.3.42}$$

\mathcal{O}_ε is estimated by using Lemma 2.3.4 and (2.3.21)–(2.3.23):

$$\begin{aligned}
\mathcal{O}_\varepsilon &:= (\gamma-1) \int_0^k g'_\varepsilon(x) \frac{r^m}{v} u e dx - \beta \int_0^k g'_\varepsilon(x) r^m u \frac{D_x(r^m u)}{v} dx \\
&\leq \frac{(\gamma-1)^2}{2} \sup_{y \in [\varepsilon/2, \varepsilon]} v^{-2} r^{2m}(y, t) \int_{\varepsilon/2}^\varepsilon u^2 dx + \frac{1}{2} \int_{\varepsilon/2}^\varepsilon (e-1)^2 dx \\
&\quad + (\gamma-1) \left(\int_{\varepsilon/2}^\varepsilon v^{-2} r^{2m} dx \right)^{1/2} \left(\int_{\varepsilon/2}^\varepsilon u^2 dx \right)^{1/2} \\
&\quad + \frac{\beta}{4} \int_{\varepsilon/2}^\varepsilon \frac{\varepsilon^2}{32} |g'_\varepsilon|^2 \frac{|D_x(r^m u)|^2}{v} dx + \frac{32\beta}{\varepsilon^2} \sup_{y \in [\varepsilon/2, \varepsilon]} \frac{r^{2m}}{v}(y, t) \int_{\varepsilon/2}^\varepsilon u^2 dx \\
&\leq \frac{\beta}{4} \int_0^k g_\varepsilon \frac{|D_x(r^m u)|^2}{v} dx + C(\varepsilon).
\end{aligned} \tag{2.3.43}$$

According to (2.3.40)–(2.3.43), we have

$$\begin{aligned}
&\int_0^k g_\varepsilon \frac{u^2}{2}(x, t) dx + \frac{\beta}{2} \int_0^t \int_0^k g_\varepsilon(x) \frac{|D_x(r^m u)|^2}{v} dx ds \\
&\leq C(\varepsilon) + (\gamma-1) \int_0^k g_\varepsilon (\log v - v + 1) dx + (\gamma-1) \int_0^k g_\varepsilon (v_0 - \log v_0 - 1) dx.
\end{aligned}$$

Since $\log v - v + 1 \leq 0$ and $C_0 \leq v_0(x) \leq C_0^{-1}$, it implies from Taylor's theorem that

$$\frac{\beta}{2} \int_0^t \int_0^k g_\varepsilon(x) \frac{|D_x(r^m u)|^2}{v} dx ds \leq C(\varepsilon) + (\gamma-1) C_0 \int_0^k (v_0 - 1)^2 dx \leq C(\varepsilon). \tag{2.3.44}$$

The other term in (2.3.38) is estimated by rewriting (2.3.44) and using Theorem 2.3.1(i)–(ii). \square

2.3.4 Exterior $L^\infty(0, T; H^1)$ –Estimates on (u, e)

In this subsection, we derive the high regularity estimates for (u, e) in Theorem 2.3.1(iii). For simplicity, we define

$$\Lambda(v, e) \equiv (v-1)^2 + (e-1)^2, \quad \sigma(t) := \min\{1, t\}.$$

It follows from Lemma 2.3.4–2.3.5 that $\|\Lambda(v, e)\|_{L^\infty(0, T; L^1(\varepsilon, k))} \leq C(\varepsilon)$. Furthermore, we introduce the effective viscosity flux F , which is defined by

$$F := \beta \frac{D_x(r^m u)}{v} - p(v, e) + (\gamma-1) = \beta \frac{D_x(r^m u)}{v} - (\gamma-1) \frac{e-v}{v}.$$

Using Lemmas 2.3.4–2.3.6, we have the following observation on F :

Lemma 2.3.7 (Estimates on Effective Viscosity Flux).

$$\begin{aligned}
\int_0^k g_\varepsilon |F|^2(x, t) dx &\leq C(\varepsilon) + C(\varepsilon) \int_0^k g_\varepsilon \frac{|D_x(r^m u)|^2}{v} dx, \\
\int_0^T \int_0^k g_\varepsilon |F|^2(x, t) dx dt &\leq C(\varepsilon), \\
\int_0^T \int_0^k g_\varepsilon r^{2m} |D_x F|^2(x, t) dx dt &\leq \int_0^T \int_0^k g_\varepsilon |D_t u|^2(x, t) dx dt.
\end{aligned} \tag{2.3.45}$$

Proof. It follows from the definition of F and the momentum equation (2.2.16)₂ that

$$\begin{aligned}
F &= \beta \frac{D_x(r^m u)}{v} - (\gamma - 1) \frac{e - 1}{v} + (\gamma - 1) \frac{v - 1}{v}, \\
D_t u &= \beta r^m D_x \left(\frac{D_x(r^m u)}{v} \right) - r^m D_x(p - (\gamma - 1)) = r^m D_x F,
\end{aligned}$$

which, along with Theorem 2.3.1(ii) and Lemmas 2.3.4–2.3.6, yields that

$$\begin{aligned}
\int_0^k g_\varepsilon |F|^2(x, t) dx &\leq 2\beta^2 \int_0^k g_\varepsilon \frac{|D_x(r^m u)|^2}{v^2} dx + 4(\gamma - 1)^2 \int_0^k g_\varepsilon \frac{\Lambda(v, e)}{v^2} dx \\
&\leq C(\varepsilon) \int_0^k g_\varepsilon \frac{|D_x(r^m u)|^2}{v} dx + C(\varepsilon),
\end{aligned}$$

and (2.3.45)₂–(2.3.45)₃ obviously. □

Lemma 2.3.8 (Intermediate Step).

$$\begin{aligned}
&\int_0^T \int_0^k \sigma g_\varepsilon |D_t u|^2 dx ds + \sup_{0 \leq s \leq T} \int_0^k \sigma g_\varepsilon \frac{|D_x(r^m u)|^2}{v}(x, s) dx \\
&\leq C(\varepsilon) \left\{ 1 + \left(\int_0^T \int_0^k \sigma^2 g_\varepsilon^2 |D_t e|^2 dx ds \right)^{1/2} \right\}.
\end{aligned} \tag{2.3.46}$$

Proof. Multiplying equations (2.2.16)₂ with $g_\varepsilon \sigma D_t u$ and then integrating yield

$$\begin{aligned}
&\int_0^t \int_0^k \sigma g_\varepsilon |D_t u|^2 dx ds + \frac{\beta}{2} \int_0^k \sigma g_\varepsilon \frac{|D_x(r^m u)|^2}{v}(x, t) dx \\
&= \frac{\beta}{2} \int_0^t \int_0^k \sigma' g_\varepsilon \frac{|D_x(r^m u)|^2}{v} dx ds + m\beta \int_0^t \int_0^k \sigma g_\varepsilon D_x(r^{m-1} u^2) \frac{D_x(r^m u)}{v} dx ds \\
&\quad - \int_0^t \int_0^k \sigma g_\varepsilon r^m D_t u D_x p dx ds + \frac{\beta}{2} \int_0^t \int_0^k \sigma g_\varepsilon D_t(v^{-1}) |D_x(r^m u)|^2 dx ds \\
&\quad - \beta \int_0^t \int_0^k \sigma g_\varepsilon' r^m D_t u \frac{D_x(r^m u)}{v} dx ds =: \sum_{i=1}^4 I_i + \mathcal{O}_\varepsilon^{(*)}.
\end{aligned}$$

I_1 is estimated by using $\sigma(t) = \min\{1, t\}$ and Lemma 2.3.6 as

$$I_1 := \frac{\beta}{2} \int_0^t \int_0^k \sigma' g_\varepsilon \frac{|D_x(r^m u)|^2}{v} dx ds \leq \frac{\beta}{2} \int_0^1 \int_0^k g_\varepsilon \frac{|D_x(r^m u)|^2}{v} dx ds \leq C(\varepsilon).$$

I_2 is estimated by using Lemmas 2.3.4–2.3.6, Theorem 2.3.1(ii), and (2.3.21)–(2.3.23) as

$$\begin{aligned}
I_2 &:= m\beta \int_0^t \int_0^k \sigma g_\varepsilon D_x(r^{m-1}u^2) \frac{D_x(r^m u)}{v} dx ds \\
&= 2m\beta \int_0^t \int_0^k \sigma g_\varepsilon r^{m-1} u D_x u \frac{D_x(r^m u)}{v} dx ds + m(m-1)\beta \int_0^t \int_0^k \sigma g_\varepsilon \frac{v}{r^2} u^2 \frac{D_x(r^m u)}{v} dx ds \\
&\leq m\beta \int_0^t \sup_{y \in [\varepsilon/2, k]} r^{-2}(y, s) \int_0^k g_\varepsilon \frac{r^{2m}}{v} |u D_x u|^2 dx ds + m\beta \int_0^t \int_0^k g_\varepsilon \frac{|D_x(r^m u)|^2}{v} dx ds \\
&\quad + \frac{m(m-1)\beta}{2} \left\{ \int_0^t \sup_{y \in [\varepsilon/2, k]} \frac{v}{r^4}(y, t) \int_0^k g_\varepsilon u^4 dx ds + \int_0^t \int_0^k g_\varepsilon \frac{|D_x(r^m u)|^2}{v} \right\} \leq C(\varepsilon).
\end{aligned}$$

Using $p - \gamma + 1 = (\gamma - 1)(e - v)/v$, the term I_3 is separated into few parts as follows:

$$\begin{aligned}
\frac{I_3}{\gamma - 1} &:= -\frac{1}{\gamma - 1} \int_0^t \int_0^k \sigma g_\varepsilon r^m D_t u D_x p dx ds \\
&= \int_0^k \sigma g_\varepsilon D_x(r^m u) \frac{e - v}{v}(x, t) dx - \int_0^t \int_0^k \sigma' g_\varepsilon D_x(r^m u) \frac{e - v}{v} dx ds \\
&\quad - \int_0^t \int_0^k \sigma g_\varepsilon D_x(r^m u) D_t \left(\frac{e - v}{v} \right) dx ds - \int_0^t \int_0^k \sigma g_\varepsilon D_x(r^{m-1}u^2) \frac{e - v}{v} dx ds \\
&\quad + \int_0^t \int_0^k \sigma g'_\varepsilon D_t(r^m u) \frac{e - v}{v} dx ds - \int_0^t \int_0^k \sigma g'_\varepsilon r^{m-1} u^2 \frac{e - v}{v} dx ds \\
&=: \sum_{i=1}^4 I_3^{(i)} + \mathcal{O}_\varepsilon^{(1)} + \mathcal{O}_\varepsilon^{(2)}.
\end{aligned}$$

$I_3^{(1)}$ is estimated by using Theorem 2.3.1(ii) and Lemma 2.3.4–2.3.6:

$$\begin{aligned}
I_3^{(1)} &:= \int_0^k \sigma g_\varepsilon D_x(r^m u) \frac{e - v}{v} dx = \int_0^k \sigma g_\varepsilon D_x(r^m u) \left(\frac{e - 1}{v} + \frac{1 - v}{v} \right) dx \\
&\leq \frac{1}{\beta} \int_0^k \sigma g_\varepsilon \frac{\Lambda(v, e)}{v} dx + \frac{\beta}{4} \int_0^k \sigma g_\varepsilon \frac{|D_x(r^m u)|^2}{v} dx \leq C(\varepsilon) + \frac{\beta}{4} \int_0^k \sigma g_\varepsilon \frac{|D_x(r^m u)|^2}{v}(x, t) dx.
\end{aligned}$$

$I_3^{(2)}$ is estimated by using the definition of σ , Theorem 2.3.1(ii), and Lemma 2.3.4–2.3.6:

$$\begin{aligned}
-I_3^{(2)} &:= \int_0^t \int_0^k \sigma' g_\varepsilon D_x(r^m u) \frac{e - v}{v} dx ds = \int_0^t \int_0^k \sigma' g_\varepsilon D_x(r^m u) \left(\frac{e - 1}{v} + \frac{1 - v}{v} \right) dx ds \\
&\leq \int_0^1 \int_0^k g_\varepsilon \frac{|D_x(r^m u)|^2}{v} dx ds + \frac{1}{4} \int_0^1 \int_0^k g_\varepsilon \frac{\Lambda(v, e)}{v} dx ds \leq C(\varepsilon).
\end{aligned}$$

$I_3^{(3)}$ is estimated by using Theorem 2.3.1(ii) and Lemma 2.3.6:

$$\begin{aligned}
-I_3^{(3)} &:= \int_0^t \int_0^k \sigma g_\varepsilon D_x(r^m u) D_t \left(\frac{e-v}{v} \right) dx ds \\
&= \int_0^t \int_0^k \sigma g_\varepsilon D_x(r^m u) \left\{ \frac{D_t e}{v} - \frac{e}{v^2} D_x(r^m u) \right\} dx ds \\
&\leq C(\varepsilon) \left(\int_0^t \int_{\varepsilon/2}^k \frac{|D_x(r^m u)|^2}{v} dx ds \right)^{1/2} \left(\int_0^t \int_0^k \sigma^2 g_\varepsilon^2 |D_t e|^2 dx ds \right)^{1/2} \\
&\quad + C(\varepsilon) \int_0^t \sup_{y \in [\varepsilon/2, k]} e(y, s) \int_0^k \sigma g_\varepsilon \frac{|D_x(r^m u)|^2}{v} dx ds \\
&\leq C(\varepsilon) \left(\int_0^t \int_0^k \sigma^2 g_\varepsilon^2 |D_t e|^2 dx ds \right)^{1/2} + C(\varepsilon) \int_0^t \sup_{y \in [\varepsilon/2, k]} e(y, s) \int_0^k \sigma g_\varepsilon \frac{|D_x(r^m u)|^2}{v} dx ds.
\end{aligned}$$

$I_3^{(4)}$ is estimated by using Theorem 2.3.1(ii), Lemmas 2.3.4–2.3.6, and (2.3.21)–(2.3.23):

$$\begin{aligned}
I_3^{(4)} &:= - \int_0^t \int_0^k \sigma g_\varepsilon D_x(r^{m-1} u^2) \frac{e-v}{v} dx ds \\
&= - \int_0^t \int_0^k \sigma g_\varepsilon \left\{ 2r^{m-1} u D_x u + (n-2) \frac{v}{r^2} u^2 \right\} \left(\frac{e-1}{v} + \frac{1-v}{v} \right) dx ds \\
&\leq \int_0^t \int_0^k \sigma g_\varepsilon \frac{r^{2m}}{v} |u D_x u|^2 dx ds + C(\varepsilon) \int_0^t \int_0^k g_\varepsilon \{u^4 + \Lambda(v, e)\} dx ds \leq C(\varepsilon).
\end{aligned}$$

For $\mathcal{O}_\varepsilon^{(1)}$, we use Theorem 2.3.1(ii), Lemma 2.3.4, and (2.3.21)–(2.3.23) to obtain

$$\begin{aligned}
\mathcal{O}_\varepsilon^{(1)} &:= \int_0^t \int_0^k \sigma g'_\varepsilon D_t(r^m u) \frac{e-v}{v} dx ds \\
&= \int_0^t \int_0^k \sigma g'_\varepsilon (m r^{m-1} u^2 + r^m D_t u) \left(\frac{e-1}{v} + \frac{1-v}{v} \right) dx ds \\
&\leq C \left(\int_0^t \int_{\varepsilon/2}^\varepsilon \sigma \frac{\varepsilon^2}{32} |g'_\varepsilon|^2 |D_t u|^2 dx ds \right)^{1/2} \left(\int_0^t \sup_{y \in [\varepsilon/2, \varepsilon]} \frac{r^{2m}}{v^2}(y, s) \int_{\varepsilon/2}^\varepsilon \frac{32}{\varepsilon^2} \Lambda(v, e) dx ds \right)^{1/2} \\
&\quad + C \left(\int_0^t \sup_{y \in [\varepsilon/2, \varepsilon]} r^{-2}(y, s) \int_{\varepsilon/2}^\varepsilon u^4 dx ds \right)^{1/2} \left(\int_0^t \sup_{y \in [\varepsilon/2, \varepsilon]} \frac{r^{2m}}{v^2}(y, s) \int_{\varepsilon/2}^\varepsilon \Lambda(v, e) dx ds \right)^{1/2} \\
&\leq C(\varepsilon) + \frac{1}{8} \int_0^t \int_0^k \sigma g_\varepsilon |D_t u|^2 dx ds,
\end{aligned}$$

and $\mathcal{O}_\varepsilon^{(2)}$ is estimated by using Theorem 2.3.1(ii), Lemma 2.3.4–2.3.5, and (2.3.21)–(2.3.23).

$$\begin{aligned}
\mathcal{O}_\varepsilon^{(2)} &:= - \int_0^t \int_0^k \sigma g'_\varepsilon r^{m-1} u^2 \left(\frac{e-1}{v} + \frac{1-v}{v} \right) dx ds \\
&\leq \frac{1}{2} \int_0^t \sup_{y \in [\varepsilon/2, \varepsilon]} \frac{r^{2(m-1)}}{v^2}(y, s) \int_{\varepsilon/2}^\varepsilon |g'_\varepsilon| u^4 dx ds + \frac{1}{2} \int_0^t \int_{\varepsilon/2}^\varepsilon |g'_\varepsilon| \Lambda(v, e) dx ds \leq C(\varepsilon).
\end{aligned}$$

Combining the estimates of $I_3^{(1)}$ – $I_3^{(4)}$ and $\mathcal{O}_\varepsilon^{(1)}$ – $\mathcal{O}_\varepsilon^{(2)}$, it follows that

$$\begin{aligned}
I_3 &\leq C(\varepsilon) + \frac{\beta}{4} \int_0^k \sigma g_\varepsilon \frac{|D_x(r^m u)|^2}{v}(x, t) dx + C(\varepsilon) \left(\int_0^t \int_0^k \sigma g_\varepsilon^2 |D_t e|^2 dx ds \right)^{1/2} \\
&\quad + \frac{1}{8} \int_0^t \int_0^k \sigma g_\varepsilon |D_t u|^2 dx ds + C(\varepsilon) \int_0^t \sup_{y \in [\varepsilon/2, k]} e(y, s) \int_0^k \sigma g_\varepsilon \frac{|D_x(r^m u)|^2}{v}(x, s) dx ds.
\end{aligned}$$

I_4 is rewritten into three parts by using the continuity equation (2.2.16)₁ and the definition of F :

$$\begin{aligned} I_4 &:= \frac{\beta}{2} \int_0^t \int_0^k \sigma g_\varepsilon D_t(v^{-1}) |D_x(r^m u)|^2 dx ds \\ &= -\frac{1}{2\beta} \int_0^t \int_0^k \sigma g_\varepsilon F^2 D_x(r^m u) dx ds - \frac{\gamma-1}{\beta} \int_0^t \int_0^k \sigma g_\varepsilon F \frac{e-v}{v} D_x(r^m u) dx ds \\ &\quad - \frac{(\gamma-1)^2}{2\beta} \int_0^t \int_0^k \sigma g_\varepsilon \left(\frac{e-v}{v}\right)^2 D_x(r^m u) dx ds := \sum_{i=1}^3 I_4^{(i)}. \end{aligned}$$

$I_4^{(1)}$ is estimated by using integration by parts, Lemma 2.3.7, and (2.3.21)–(2.3.23).

$$\begin{aligned} \beta I_4^{(1)} &:= -\frac{1}{2} \int_0^t \int_0^k \sigma g_\varepsilon F^2 D_x(r^m u) dx ds \\ &= \int_0^t \int_0^k \sigma g_\varepsilon u F r^m D_x F dx ds + \frac{1}{2} \int_0^t \int_0^k \sigma g'_\varepsilon F^2 r^m u dx ds \\ &\leq 2 \int_0^t \int_0^k \sigma g_\varepsilon u^2 F^2 dx ds + \frac{1}{8} \int_0^t \int_0^k \sigma g_\varepsilon r^{2m} |D_x F|^2 dx ds \\ &\quad + \frac{1}{4} \int_0^t \int_{\varepsilon/2}^\varepsilon \sigma \frac{\varepsilon^2}{32} |g'_\varepsilon|^2 |u|^2 F^2 dx ds + \frac{1}{4} \int_0^t \int_{\varepsilon/2}^\varepsilon \frac{32}{\varepsilon^2} r^{2m} F^2 dx ds \\ &\leq C(\varepsilon) + \frac{1}{8} \int_0^t \int_0^k \sigma g_\varepsilon |D_t u|^2 dx ds + C(\varepsilon) \int_0^t \sup_{y \in [\varepsilon/2, k]} |u|^2(y, s) \int_0^k \sigma g_\varepsilon \frac{|D_x(r^m u)|^2}{v} dx ds. \end{aligned}$$

$I_4^{(2)}$ is estimated by using Theorem 2.3.1(ii) and Lemmas 2.3.4–2.3.7:

$$\begin{aligned} \frac{\beta}{\gamma-1} I_4^{(2)} &:= -\int_0^t \int_0^k \sigma g_\varepsilon F \frac{e-v}{v} D_x(r^m u) dx ds \\ &\leq \int_0^t \int_0^k \sigma g_\varepsilon F^2 dx ds + \int_0^t \sup_{y \in [\varepsilon/2, k]} \frac{(e-v)^2}{v} \int_0^k \sigma g_\varepsilon \frac{|D_x(r^m u)|^2}{v} dx ds \\ &\leq C(\varepsilon) + C(\varepsilon) \int_0^t \sup_{y \in [\varepsilon/2, k]} |e(y, s)|^2 \int_0^k \sigma g_\varepsilon \frac{|D_x(r^m u)|^2}{v} dx ds. \end{aligned}$$

$I_4^{(3)}$ is estimated by using Theorem 2.3.1(ii) and Lemma 2.3.4–2.3.6:

$$\begin{aligned} -\frac{2\beta}{(\gamma-1)^2} I_4^{(3)} &:= \int_0^t \int_0^k \sigma g_\varepsilon \left\{ \frac{e-1}{v} + \frac{1-v}{v} \right\}^2 D_x(r^m u) dx ds \\ &\leq 2 \int_0^t \int_0^k \sigma g_\varepsilon \frac{\Lambda(v, e)}{v^2} |D_x(r^m u)| dx ds \\ &\leq \frac{(\gamma-1)^2}{2\beta} \int_0^t \int_0^k \sigma g_\varepsilon \frac{\Lambda(v, e)^2}{v^3} dx ds + \frac{(\gamma-1)^2}{2\beta} \int_0^t \int_0^k \sigma g_\varepsilon \frac{|D_x(r^m u)|^2}{v} dx ds \\ &\leq C(\varepsilon) + C(\varepsilon) \int_0^t \left(\sup_{y \in [\varepsilon/2, k]} |e-1|^2(y, s) + 1 \right) \int_0^k \sigma g_\varepsilon |e-1|^2 dx ds \\ &\leq C(\varepsilon) + C(\varepsilon) \int_0^t \sup_{y \in [\varepsilon/2, k]} e^2(y, s) ds \leq C(\varepsilon). \end{aligned}$$

Combining estimates for $I_4^{(1)}-I_4^{(3)}$, it follows that

$$I_4 \leq C(\varepsilon) + C(\varepsilon) \int_0^t \sup_{y \in [\varepsilon/2, k]} (e^2 + u^2)(y, s) \int_0^k \sigma g_\varepsilon \frac{|D_x(r^m u)|^2}{v} dx ds + \frac{1}{8} \int_0^t \int_0^k \sigma g_\varepsilon |D_t u|^2.$$

$\mathcal{O}_\varepsilon^{(*)}$ is estimated by using Theorem 2.3.1(ii), Lemmas 2.3.4–2.3.6, and (2.3.21)–(2.3.23):

$$\begin{aligned} \mathcal{O}_\varepsilon^{(*)} &:= -\beta \int_0^t \int_0^k \sigma g'_\varepsilon r^m D_t u \frac{D_x(r^m u)}{v} dx ds \\ &\leq \beta^2 \int_0^t \sup_{y \in [\varepsilon/2, \varepsilon]} \frac{r^{2m}}{v}(y, s) \int_{\varepsilon/2}^\varepsilon \frac{32 |D_x(r^m u)|^2}{\varepsilon^2 v} dx ds + \frac{1}{4} \int_0^t \int_{\varepsilon/2}^\varepsilon \sigma \frac{\varepsilon^2}{32} |g'_\varepsilon|^2 |D_t u|^2 dx dt \\ &\leq C(\varepsilon) + \frac{1}{4} \int_0^t \int_{\varepsilon/2}^\varepsilon \sigma g_\varepsilon |D_t u|^2 dx dt \end{aligned}$$

Putting the estimates of I_1-I_4 and $\mathcal{O}_\varepsilon^{(1)}$ together, it follows that

$$\begin{aligned} &\frac{1}{2} \int_0^t \int_0^k \sigma g_\varepsilon |D_t u|^2 dx ds + \frac{\beta}{4} \int_0^k \sigma g_\varepsilon \frac{|D_x(r^m u)|^2}{v}(x, t) dx \\ &\leq C(\varepsilon) + C(\varepsilon) \left(\int_0^t \int_0^k \sigma^2 g_\varepsilon^2 |D_t e|^2 dx ds \right)^{1/2} \\ &\quad + C(\varepsilon) \int_0^t \sup_{y \in [\varepsilon/2, k]} (e + e^2 + u^2)(y, s) \int_0^k \sigma g_\varepsilon \frac{|D_x(r^m u)|^2}{v}(x, s) dx ds. \end{aligned}$$

Applying Grönwall's inequality, we have

$$\begin{aligned} &\frac{1}{2} \int_0^t \int_0^k \sigma g_\varepsilon |D_t u|^2 dx ds + \frac{\beta}{4} \int_0^k \sigma g_\varepsilon \frac{|D_x(r^m u)|^2}{v}(x, t) dx \\ &\leq C(\varepsilon) \left\{ 1 + \left(\int_0^t \int_0^k \sigma^2 g_\varepsilon^2 |D_t e|^2 dx ds \right)^{1/2} \right\} \left\{ 1 + \left(\int_0^t h_\varepsilon(s) ds \right) \exp \left(\int_0^t h_\varepsilon(s) ds \right) \right\}, \end{aligned}$$

where $h_\varepsilon(s) := C(\varepsilon) \sup_{y \in [\varepsilon/2, k]} (e + e^2 + u^2)(y, s)$. By Lemmas 2.3.3-2.3.4, we obtain (2.3.46). \square

Lemma 2.3.9 ($L^\infty(0, T; L^2(\varepsilon, k))$ –Estimates of $D_x u$ and $D_x e$).

$$\begin{aligned} &\int_0^t \int_\varepsilon^k \sigma^2 |D_t e|^2 dx ds + \sup_{0 \leq s \leq t} \int_\varepsilon^k \sigma^2 \frac{r^{2m}}{v} |D_x e|^2(x, s) dx \leq C(\varepsilon), \\ &\int_0^t \int_\varepsilon^k \sigma g_\varepsilon |D_t u|^2 dx ds + \sup_{0 \leq s \leq t} \int_\varepsilon^k \sigma \frac{|D_x(r^m u)|^2}{v}(x, s) dx \leq C(\varepsilon). \end{aligned} \tag{2.3.47}$$

Proof. Rewriting the energy equation (2.2.16)₃ in terms of the effective viscosity flux F , then multiplying both sides by $\sigma^2(t)g_\varepsilon^2 D_t e$, and integrating by parts, it follows that

$$\begin{aligned} &\int_0^t \int_0^k \sigma^2 g_\varepsilon^2 |D_t e|^2 dx ds + \frac{\kappa}{2} \int_0^k \sigma^2 g_\varepsilon^2 \frac{r^{2m}}{v} |D_x e|^2(x, t) dx \\ &= \kappa \int_0^t \int_0^k \sigma \sigma'(s) g_\varepsilon^2 \frac{r^{2m}}{v} |D_x e|^2 dx ds + m\kappa \int_0^t \int_0^k \sigma^2 g_\varepsilon^2 \frac{r^{2m-1} u}{v} |D_x e|^2 dx ds \\ &\quad + \int_0^t \int_0^k \sigma^2 g_\varepsilon^2 A D_t e dx ds - \frac{\kappa}{2} \int_0^t \int_0^k \sigma^2 g_\varepsilon^2 \frac{r^{2m}}{v^2} D_x(r^m u) |D_x e|^2 dx ds \\ &\quad + \int_0^t \int_0^k \sigma^2 g_\varepsilon^2 F D_x(r^m u) D_t e dx ds - \kappa \int_0^t \int_0^k 2\sigma^2 g_\varepsilon g'_\varepsilon \frac{r^{2m}}{v} D_t e D_x e dx ds := \sum_{i=1}^5 I_i + \mathcal{O}_\varepsilon, \end{aligned}$$

where $\mathcal{A} := -(\gamma - 1)D_x(r^m u) - 4m\mu\frac{u}{r}D_x(r^m u) + 2mn\mu\frac{v}{r^2}u^2$. The term I_1 is estimated by Lemma 2.3.4:

$$I_1 := \kappa \int_0^t \int_0^k \sigma \sigma'(s) g_\varepsilon^2 \frac{r^{2m}}{v} |D_x e|^2 dx ds \leq \kappa \int_0^1 \int_{\varepsilon/2}^k \frac{r^{2m}}{v} |D_x e|^2 dx ds \leq C(\varepsilon).$$

I_2 is estimated by using (2.3.21)–(2.3.23) and Lemma 2.3.4:

$$\begin{aligned} I_2 &:= m\kappa \int_0^t \int_0^k \sigma^2 g_\varepsilon^2 \frac{r^{2m-1}u}{v} |D_x e|^2 dx ds \leq 2m\kappa \int_0^t \sup_{y \in [\varepsilon/2, k]} \frac{u}{r}(y, s) \int_0^k \sigma^2 g_\varepsilon^2 \frac{r^{2m}}{v} |D_x e|^2 dx ds \\ &\leq C(\varepsilon) + C(\varepsilon) \int_0^t \sup_{y \in [\varepsilon/2, k]} u^2(y, s) \int_0^k \sigma^2 g_\varepsilon^2 \frac{r^{2m}}{v} |D_x e|^2 dx ds. \end{aligned}$$

I_3 is estimated by using Theorem 2.3.1(ii) and Lemmas 2.3.4–2.3.6:

$$\begin{aligned} I_3 &:= \int_0^t \int_0^k \sigma^2 g_\varepsilon^2 \mathcal{A} D_t e dx ds \\ &= \int_0^t \int_0^k \sigma^2 g_\varepsilon^2 \left\{ -(\gamma - 1)D_x(r^m u) - 4m\mu\frac{u}{r}D_x(r^m u) + 2mn\mu\frac{v}{r^2}u^2 \right\} D_t e dx ds \\ &\leq C(\varepsilon) \int_0^t \sup_{y \in [\varepsilon/2, k]} (1 + u^2)(y, s) \int_0^k \sigma^2 g_\varepsilon^2 \frac{|D_x(r^m u)|^2}{v} dx ds \\ &\quad + \frac{1}{16} \int_0^t \int_0^k \sigma^2 g_\varepsilon^2 |D_t e|^2 dx ds + C(\varepsilon) \int_0^T \int_0^k \sigma^2 g_\varepsilon^2 u^4 dx dt \\ &\leq C(\varepsilon) + C(\varepsilon) \int_0^t \sup_{y \in [\varepsilon/2, k]} u^2(y, s) \int_0^k \sigma^2 g_\varepsilon^2 \frac{|D_x(r^m u)|^2}{v} dx ds + \frac{1}{16} \int_0^t \int_0^k \sigma^2 g_\varepsilon^2 |D_t e|^2 dx dt. \end{aligned} \tag{2.3.48}$$

In fact, by Lemmas 2.3.3 and 2.3.8, it follows that

$$\begin{aligned} &\int_0^t \sup_{y \in [\varepsilon/2, k]} u^2(y, s) \int_0^k \sigma^2 g_\varepsilon^2 \frac{|D_x(r^m u)|^2}{v} dx ds \\ &\leq \sup_{0 \leq s \leq t} \int_0^k \sigma g_\varepsilon \frac{|D_x(r^m u)|^2}{v}(x, s) dx \int_0^t \sup_{y \in [\varepsilon/2, k]} u^2(y, \tilde{t}) d\tilde{t} \\ &\leq C(\varepsilon) + C(\varepsilon) \left(\int_0^t \int_0^k \sigma^2 g_\varepsilon^2 |D_t e|^2 dx ds \right)^{1/2} \leq C(\varepsilon) + \frac{1}{16} \int_0^t \int_0^k \sigma^2 g_\varepsilon^2 |D_t e|^2 dx ds. \end{aligned}$$

Substituting the above estimate into (2.3.48) leads to

$$I_3 \leq C(\varepsilon) + \frac{1}{8} \int_0^t \int_0^k \sigma^2 g_\varepsilon^2 |D_t e|^2 dx ds.$$

I_4 is first rewritten in terms of F and then estimated by using Theorem 2.3.1(ii):

$$\begin{aligned} I_4 &:= -\frac{\kappa}{2} \int_0^t \int_0^k \sigma^2 g_\varepsilon^2 \frac{|r^m D_x e|^2}{v^2} D_x(r^m u) dx ds \\ &= -\frac{\kappa}{2\beta} \int_0^t \int_0^k \sigma^2 g_\varepsilon^2 \left\{ F + (\gamma - 1) \frac{e - v}{v} \right\} \frac{|r^m D_x e|^2}{v} dx ds \\ &\leq C(\varepsilon) + \frac{\kappa}{2\beta} \int_0^t \sup_{y \in [\varepsilon/2, k]} |g_\varepsilon^{1/2} F|(y, s) \int_0^k \sigma^2 g_\varepsilon^{3/2} \frac{r^{2m}}{v} |D_x e|^2 dx ds \\ &\quad + C(\varepsilon) \int_0^t \sup_{y \in [\varepsilon/2, k]} e(y, s) \int_0^k \sigma^2 g_\varepsilon^2 \frac{r^{2m}}{v} |D_x e|^2 dx ds. \end{aligned}$$

By the Sobolev embedding theorem: $W^{1,1}(\varepsilon, k) \hookrightarrow C^0(\varepsilon, k)$ and Lemma 2.3.7, it follows that

$$\begin{aligned} \sup_{y \in [\varepsilon/2, k]} |g_\varepsilon^{1/2} F|^2(y, s) &\leq C \int_{\varepsilon/2}^k \{g_\varepsilon F^2 + 2g_\varepsilon |F D_x F| + 2g'_\varepsilon F^2\}(x, s) dx \\ &\leq \int_{\varepsilon/2}^k \{C(\varepsilon) F^2 + g_\varepsilon r^{2m} |D_x F|^2\} \leq C(\varepsilon) + \int_{\varepsilon/2}^k \left\{ C(\varepsilon) \frac{|D_x(r^m u)|^2}{v} + g_\varepsilon |D_t u|^2 \right\} dx, \end{aligned} \quad (2.3.49)$$

so that, using Lemma 2.3.8 and Cauchy-Schwartz's inequality, we have

$$\begin{aligned} &\frac{\kappa}{\beta} \int_0^t \sup_{y \in [\varepsilon/2, k]} |g_\varepsilon^{1/2} F|(y, s) \int_0^k \sigma^2 g_\varepsilon^{3/2} \frac{r^{2m}}{v} |D_x e|^2 dx ds \\ &\leq \int_0^t \left\{ \sigma^2 C(\varepsilon) + C(\varepsilon) \int_{\varepsilon/2}^k \sigma^2 \frac{|D_x(r^m u)|^2}{v}(x, s) dx + \int_{\varepsilon/2}^k \sigma^2 g_\varepsilon |D_t u|^2(x, s) dx \right\} ds \\ &\quad + C(\varepsilon) \int_0^t \sigma^2 \left(\int_0^k g_\varepsilon^{3/2} \frac{r^{2m}}{v} |D_x e|^2(x, s) dx \right)^2 ds \\ &\leq C(\varepsilon) + \int_0^t \int_0^k \sigma g_\varepsilon |D_t u|^2 dx ds \\ &\quad + C(\varepsilon) \int_0^t \left(\int_0^k g_\varepsilon \frac{r^{2m}}{v} |D_x e|^2(x, s) dx \right) \left(\int_0^k \sigma^2 g_\varepsilon^2 \frac{r^{2m}}{v} |D_x e|^2(x, s) dx \right) ds. \end{aligned}$$

Using the above estimate, Lemma 2.3.8, and Cauchy-Schwartz's inequality, one has

$$\begin{aligned} I_4 &\leq C(\varepsilon) + \int_0^t \int_0^k \sigma g_\varepsilon |D_t u|^2 dx ds + C(\varepsilon) \int_0^t \sup_{y \in [\varepsilon/2, k]} e(y, s) \int_0^k \sigma^2 g_\varepsilon^2 \frac{r^{2m}}{v} |D_x e|^2 dx ds \\ &\quad + C(\varepsilon) \int_0^t \left(\int_0^k g_\varepsilon \frac{r^{2m}}{v} |D_x e|^2(x, s) dx \right) \left(\int_0^k \sigma^2 g_\varepsilon^2 \frac{r^{2m}}{v} |D_x e|^2(x, s) dx \right) ds, \\ &\leq C(\varepsilon) + \frac{1}{8} \int_0^t \int_0^k \sigma^2 g_\varepsilon^2 |D_t e|^2 dx ds + C(\varepsilon) \int_0^t \sup_{y \in [\varepsilon/2, k]} e(y, s) \int_0^k \sigma^2 g_\varepsilon^2 \frac{r^{2m}}{v} |D_x e|^2 dx ds \\ &\quad + C(\varepsilon) \int_0^t \left(\int_0^k g_\varepsilon \frac{r^{2m}}{v} |D_x e|^2(x, s) dx \right) \left(\int_0^k \sigma^2 g_\varepsilon^2 \frac{r^{2m}}{v} |D_x e|^2(x, s) dx \right) ds. \end{aligned}$$

I_5 is first estimated as follows:

$$\begin{aligned} I_5 &:= \int_0^t \int_0^k \sigma^2 g_\varepsilon^2 F D_x(r^m u) D_t e dx ds \\ &\leq 4 \int_0^t \sup_{y \in [\varepsilon/2, k]} g_\varepsilon v |F|^2(y, s) \int_0^k \sigma^2 g_\varepsilon \frac{|D_x(r^m u)|^2}{v} dx ds + \frac{1}{16} \int_0^t \int_0^k \sigma^2 g_\varepsilon^2 |D_t e|^2 dx ds. \end{aligned} \quad (2.3.50)$$

By the same argument as (2.3.49), we have

$$\begin{aligned} &\sup_{y \in [\varepsilon/2, k]} g_\varepsilon |F|^2(y, s) \\ &\leq C(\varepsilon) + C(\varepsilon) \int_{\varepsilon/2}^k \frac{|D_x(r^m u)|^2}{v} dx + C(\varepsilon) \left(\int_0^k g_\varepsilon F^2 dx \right)^{1/2} \left(\int_0^k g_\varepsilon |D_t u|^2 dx \right)^{1/2}, \end{aligned}$$

which, along with Theorem 2.3.1(ii) and Lemma 2.3.6, implies that

$$\begin{aligned}
& 4 \int_0^t \sup_{y \in [\varepsilon/2, k]} g_\varepsilon v |F|^2(y, s) \int_0^k \sigma^2 g_\varepsilon \frac{|D_x(r^m u)|^2}{v} dx ds \\
& \leq C(\varepsilon) + C(\varepsilon) \sup_{0 \leq s \leq t} \int_0^k \sigma g_\varepsilon \frac{|D_x(r^m u)|^2}{v}(x, s) dx \\
& \quad + C(\varepsilon) \left(\int_0^t \int_0^k \sigma g_\varepsilon F^2 \right)^{1/2} \left(\int_0^t \int_0^k \sigma g_\varepsilon |D_t u|^2 \right)^{1/2} \sup_{0 \leq s \leq t} \int_0^k \sigma g_\varepsilon \frac{|D_x(r^m u)|^2}{v}(x, s) dx.
\end{aligned}$$

Then applying Lemmas 2.3.7–2.3.8 on the above and by Cauchy-Schwartz's inequality yield

$$\begin{aligned}
& 4 \int_0^t \sup_{y \in [\varepsilon/2, k]} g_\varepsilon v |F|^2(y, s) \int_0^k \sigma^2 g_\varepsilon \frac{|D_x(r^m u)|^2}{v} dx ds \\
& \leq C(\varepsilon) + C(\varepsilon) \left(\int_0^t \int_0^k \sigma^2 g_\varepsilon^2 |D_t e|^2(x, s) dx ds \right)^{1/2} \\
& \quad + C(\varepsilon) \left\{ 1 + \left(\int_0^t \int_0^k \sigma^2 g_\varepsilon^2 |D_t e|^2 dx ds \right)^{1/2} \right\}^{1/2} \left\{ 1 + \left(\int_0^t \int_0^k \sigma^2 g_\varepsilon^2 |D_t e|^2 dx ds \right)^{1/2} \right\} \\
& \leq C(\varepsilon) + \frac{1}{16} \int_0^t \int_0^k \sigma^2 g_\varepsilon^2 |D_t e|^2(x, s) dx ds.
\end{aligned}$$

Substituting the above into (2.3.50), one obtains

$$I_5 \leq C(\varepsilon) + \frac{1}{8} \int_0^t \int_0^k \sigma^2 g_\varepsilon^2 |D_t e|^2(x, s) dx ds.$$

\mathcal{O}_ε is estimated by using Theorem 2.3.1(ii), Lemma 2.3.4, and (2.3.21)–(2.3.23):

$$\begin{aligned}
\mathcal{O}_\varepsilon & := -\kappa \int_0^t \int_0^k 2\sigma^2 g_\varepsilon g'_\varepsilon \frac{r^{2m}}{v} D_t e D_x e dx ds \\
& \leq \frac{1}{8} \int_0^t \int_{\varepsilon/2}^\varepsilon \sigma^2 g_\varepsilon^2 |D_t e|^2 dx ds + 8\kappa^2 \int_0^t \sup_{y \in [\varepsilon/2, \varepsilon]} \frac{r^{2m}}{v}(y, s) \int_{\varepsilon/2}^\varepsilon \sigma^2 |g'_\varepsilon|^2 \frac{r^{2m}}{v} |D_x e|^2 dx ds \\
& \leq C(\varepsilon) + \frac{1}{8} \int_0^t \int_0^k \sigma^2 g_\varepsilon^2 |D_t e|^2 dx ds.
\end{aligned}$$

Summarising the estimates for I_1 – I_5 and \mathcal{O}_ε , it follows that

$$\begin{aligned}
& \frac{1}{2} \int_0^t \int_0^k \sigma^2 g_\varepsilon^2 |D_t e|^2 dx ds + \frac{\kappa}{2} \int_0^k \sigma^2 g_\varepsilon^2 \frac{r^{2m}}{v} |D_x e|^2(x, t) dx \\
& \leq C(\varepsilon) + C(\varepsilon) \int_0^t \tilde{h}_\varepsilon(s) \int_0^k \sigma^2 g_\varepsilon^2 \frac{r^{2m}}{v} |D_x e|^2 dx ds,
\end{aligned} \tag{2.3.51}$$

where $\tilde{h}_\varepsilon(s) := \int_0^k g_\varepsilon \frac{r^{2m}}{v} |D_x e|^2(x, s) dx + \sup_{y \in [\varepsilon/2, k]} (e + u^2)(y, s)$. Then it follows from Lemmas 2.3.3–2.3.4 and Grönwall's inequality that

$$\frac{1}{2} \int_0^t \int_0^k \sigma^2 g_\varepsilon^2 |D_t e|^2 dx ds + \frac{\kappa}{2} \int_0^k \sigma^2 g_\varepsilon^2 \frac{r^{2m}}{v} |D_x e|^2(x, t) dx \leq C(\varepsilon),$$

which, along with Lemma 2.3.8, yields (2.3.47). This completes the proof. \square

Corollary 2.3.1. For all $(x, t) \in [\varepsilon, k] \times [0, T]$,

$$\sigma^{1/4}(t)|u(x, t)| + \sigma^{1/2}(t)e(x, t) \leq C(\varepsilon). \quad (2.3.52)$$

Proof. First, repeating the same calculation (2.3.26) in the proof of Lemma 2.3.3, one has

$$\sigma^{1/2}(t) \sup_{y \in [\varepsilon, k]} u^2(y, t) \leq C(\varepsilon) \int_{\varepsilon}^k u^2 dx + C \int_{\varepsilon}^k \sigma \frac{|D_x(r^m u)|^2}{v} dx \leq C(\varepsilon),$$

where we have used Lemma 2.3.9. Second, by the Sobolev embedding theorem, Theorem 2.3.1(ii), Lemmas 2.3.4 and 2.3.9, and (2.3.21)–(2.3.23), we have

$$\begin{aligned} \sigma(t) \sup_{y \in [\varepsilon, k]} (e-1)^2(y, t) &\leq C \int_{\varepsilon}^k \sigma(t)(e-1)^2(x, t) dx + C \int_{\varepsilon}^k \sigma(t)|e-1||D_x e|(x, t) dx \\ &\leq C \sup_{y \in [\varepsilon, k]} \left(\sigma(t) + \frac{v}{r^{2m}}(y, t) \right) \int_{\varepsilon}^k (e-1)^2(x, t) dx + \int_{\varepsilon}^k \sigma^2(t) \frac{r^{2m}}{v} |D_x e|^2(x, t) dx \leq C(\varepsilon), \end{aligned}$$

which implies (2.3.52)₂. □

2.3.5 Lower bound of e

We now estimate the lower bound of the internal energy e .

Lemma 2.3.10 (Lower Bound of e). *One has the lower bound*

$$e(x, t) \geq C(a)^{-1} \quad \text{for all } (x, t) \in [0, \infty) \times [0, T]. \quad (2.3.53)$$

Proof. Multiplying equation (2.2.16)₃ with $-e^{-2}$ to get

$$D_t e^{-1} = (\gamma - 1) \frac{D_x(r^m u)}{ev} - \kappa e^{-2} D_x \left(\frac{r^{2m} D_x e}{v} \right) + \frac{2\mu m}{e^2} D_x(r^{m-1} u^2) - \beta \frac{|D_x(r^m u)|^2}{e^2 v}, \quad (2.3.54)$$

where $\beta \equiv 2\mu + \lambda$. To further reduce the equation, we write:

$$-\kappa e^{-2} D_x \left(\frac{r^{2m}}{v} D_x e \right) = \kappa D_x \left(\frac{r^{2m}}{v} D_x e^{-1} \right) - 2\kappa \frac{r^{2m} |D_x e|^2}{v e^3}.$$

Moreover, using the relation $D_x(r^{m-1} u^2) = (2u/r) D_x(r^m u) - n v u^2 / r^2$, we have

$$\begin{aligned} &-\beta \frac{|D_x(r^m u)|^2}{e^2 v} + \frac{2\mu m}{e^2} D_x(r^{m-1} u^2) \\ &= -\left(\frac{2\mu}{n} + \lambda \right) \frac{|D_x(r^m u)|^2}{e^2 v} - 2m\mu \frac{v}{e^2} \left(\frac{D_x(r^m u)}{\sqrt{nv}} - \sqrt{n} \frac{u}{r} \right)^2 \leq -\left(\frac{2\mu}{n} + \lambda \right) \frac{|D_x(r^m u)|^2}{e^2 v}. \end{aligned}$$

Therefore, using the above two inequalities, (2.3.54) can be reduced to the inequality:

$$D_t e^{-1} + 2\kappa \frac{r^{2m} |D_x e|^2}{v e^3} + \left(\frac{2\mu}{n} + \lambda \right) \frac{|D_x(r^m u)|^2}{e^2 v} \leq \frac{(\gamma - 1) D_x(r^m u)}{ev} + \kappa D_x \left(\frac{r^{2m}}{v} D_x e^{-1} \right). \quad (2.3.55)$$

Multiplying (2.3.55) by $j e^{-j+1}$ for $j \geq 2$ integers and integrating in $x \in [0, k]$ yield

$$\begin{aligned} &\frac{d}{dt} \int_0^k e^{-j} dx + j \left(\frac{2\mu}{n} + \lambda \right) \int_0^k \frac{|D_x(r^m u)|^2}{e^{j+1} v} dx + 2j\kappa \int_0^k \frac{r^{2m} |D_x e|^2}{v e^{j+2}} dx \\ &\leq j(\gamma - 1) \int_0^k e^{-j} \frac{D_x(r^m u)}{v} dx - j(j-1)\kappa \int_0^k \frac{r^{2m} |D_x e^{-1}|^2}{v e^{j-2}} dx \\ &\leq C \int_0^k v^{-1} e^{1-j} dx + \frac{j}{2} \left(\frac{2\mu}{n} + \lambda \right) \int_0^k \frac{|D_x(r^m u)|^2}{e^{j+1} v} dx - j(j-1)\kappa \int_0^k \frac{r^{2m} |D_x e^{-1}|^2}{v e^{j-2}} dx. \end{aligned}$$

Rearranging the above, using Lemma 2.3.1 and Theorem 2.3.1(ii), we have

$$\frac{d}{dt} \int_0^k e^{-j} dx \leq C \int_0^k e^{-j} \frac{e}{v} dx \leq C(a) \sigma(t)^{-1/2} \int_0^k e^{-j} dx.$$

Integrating the above inequality in time, it follows that

$$\int_0^k e^{-j}(x, t) dx \leq \|e_0^{-1}\|_{L^j(0, k)}^j + C(a) \int_0^t \sigma(\tau)^{-1/2} \int_0^k e^{-j} dx d\tau.$$

Thus, by Grönwall's Inequality, we obtain that, for each integer $j \geq 2$,

$$\|e^{-1}(\cdot, t)\|_{L^j(0, k)}^j \leq C(a) \|e_0^{-1}\|_{L^j(0, k)}^j + C(a) \left(\int_0^t \sigma(\tau)^{-1/2} \right) \exp \left(C(a) \int_0^t \sigma(\tau)^{-1/2} \right) \leq C(a).$$

Letting $j \rightarrow \infty$ in the above inequality, we conclude that $\|e^{-1}(\cdot, t)\|_{L^\infty} \leq C(a) \|e_0^{-1}\|_{L^\infty}$. \square

2.4 Global Weak Solutions to the Exterior Problem in the Eulerian Coordinates

In this section, the approximate Lagrangian solutions are converted into the approximate solutions in the Eulerian coordinates by constructing a set of coordinate transformations. Then these functions are extended into the entire domain $(r, t) \in [a, \infty) \times [0, T]$ by using a set of smooth cut-off functions. Finally, a weak solution to problem (1.2.1) and (2.2.9) is obtained via the limit $k \rightarrow \infty$.

2.4.1 Coordinate transformation and Jacobian

Let $\{(\tilde{v}_{a,k}, \tilde{u}_{a,k}, \tilde{e}_{a,k}, \tilde{r}_{a,k})(x, s)\}_{k \in \mathbb{N}}$ be the solutions obtained in the bounded Lagrangian domain, $(x, s) \in [0, k] \times [0, T]$. For each fixed $(a, k) \in (0, 1) \times \mathbb{N}$, the coordinate transformation $(r, t) = \mathcal{T}_{a,k}(x, s)$ is defined as:

$$r = \mathcal{T}_{a,k}^{(1)}(x, s) = \tilde{r}_{a,k}(x, s), \quad t = \mathcal{T}_{a,k}^{(2)}(x, s) = s \quad \text{for } (x, s) \in [0, k] \times [0, T]. \quad (2.4.1)$$

Then we see that $\mathcal{T}_{a,k}$ has the image:

$$\mathcal{T}_{a,k}([0, k] \times [0, T]) = R_{a,k} := \{(r, t) : t \in [0, T], r \in [a, \tilde{r}_{a,k}(k, t)]\}.$$

Moreover, it follows from a direct calculation that

$$J := |D_x \mathcal{T}_{a,k} \quad D_s \mathcal{T}_{a,k}| = \begin{vmatrix} \tilde{v}_{a,k}(\tilde{r}_{a,k})^{-m} & \tilde{u}_{a,k} \\ 0 & 1 \end{vmatrix} = \tilde{v}_{a,k}(\tilde{r}_{a,k})^{-m},$$

which, along with Theorem 2.3.1 and Lemma 2.3.1, yields that

$$C(a)^{-1} (a^n + C(a)x)^{-m/n} \leq J(x, t) \leq a^{-m} C(a) \quad \text{for all } (x, t) \in [0, k] \times [0, T]. \quad (2.4.2)$$

Thus, the map $\mathcal{T}_{a,k} : [0, k] \times [0, T] \rightarrow R_{a,k}$ is a diffeomorphism so that the inverse map $\mathcal{T}_{a,k}^{-1}$ exists. Define $(x_{a,k}, s_{a,k})(r, t) := \mathcal{T}_{a,k}^{-1}(r, t)$. Then

$$s_{a,k}(r, t) = t, \quad \tilde{r}_{a,k}(x_{a,k}(r, t), t) = r \quad \text{for all } (r, t) \in R_{a,k}.$$

Let $\tilde{\theta}(x, t) : [0, k] \times [0, T] \rightarrow \mathbb{R}$ be such that $\tilde{\theta} \in C^1$. Defining $\theta(r, t) := \tilde{\theta}(x_{a,k}(r, t), t)$ for each $(r, t) \in R_{a,k}$, then, by the inverse function theorem, its Eulerian derivative can be computed:

$$\begin{cases} \partial_r \theta(r, t) = \frac{(\tilde{r}_{a,k})^m(x_{a,k}(r, t), t)}{\tilde{v}_{a,k}(x_{a,k}(r, t), t)} D_x \tilde{\theta}(x_{a,k}(r, t), t) = r^m \frac{D_x \tilde{\theta}}{\tilde{v}_{a,k}}(x_{a,k}(r, t), t), \\ \partial_t \theta(r, t) = D_t \tilde{\theta}(x_{a,k}(r, t), t) - r^m \left(\frac{\tilde{u}_{a,k}}{\tilde{v}_{a,k}} D_x \tilde{\theta} \right)(x_{a,k}(r, t), t), \end{cases} \quad (2.4.3)$$

for each $(r, t) \in R_{a,k}$. Now, for each (a, k) , the solution $(\tilde{v}_{a,k}, \tilde{u}_{a,k}, \tilde{e}_{a,k})(x, s)$ in $(x, t) \in [0, k] \times [0, T]$ is pulled back to $R_{a,k}$ as:

$$(\bar{v}_{a,k}, \bar{u}_{a,k}, \bar{e}_{a,k})(r, t) := (\tilde{v}_{a,k}, \tilde{u}_{a,k}, \tilde{e}_{a,k})(x_{a,k}(r, t), t) \quad \text{for } (r, t) \in R_{a,k}. \quad (2.4.4)$$

2.4.2 Extension in the Eulerian domain

Next, we extend $(\bar{v}_{a,k}, \bar{u}_{a,k}, \bar{e}_{a,k})$ into the whole Eulerian domain $[a, \infty) \times [0, T]$. First let $\chi \in C^\infty : \mathbb{R} \rightarrow [0, 1]$ be such that $\chi(\zeta) = 1$ if $\zeta \leq 0$, $\chi(\zeta) = 0$ if $1 \leq \zeta$, and $|\chi'| \leq 2$ in \mathbb{R} . With this, the following cut-off functions is defined:

$$\varphi_{a,k}(r, t) := \chi\left(\frac{2r - \tilde{r}_{a,k}(k, t)}{\tilde{r}_{a,k}(k, t)}\right). \quad (2.4.5)$$

Since $\tilde{u}_{a,k}(k, t) = 0$ for all $t \in [0, T]$, it follows that for all $(r, t) \in [a, \infty) \times [0, T]$,

$$\partial_r \varphi_{a,k}(r, t) = \frac{2}{\tilde{r}_{a,k}(k, t)} \chi'\left(\frac{2r - \tilde{r}_{a,k}(k, t)}{\tilde{r}_{a,k}(k, t)}\right), \quad \partial_t \varphi_{a,k}(r, t) = 0. \quad (2.4.6)$$

From these, one can check that $\varphi \in C^\infty([0, \infty) \times [0, T])$. Moreover, by construction:

$$\varphi_{a,k}(r, t) = 1 \quad \text{if } r \in [0, \tilde{r}_{a,k}(k, t)/2], \quad \varphi_{a,k}(r, t) = 0 \quad \text{if } r \in [\tilde{r}_{a,k}(k, t), \infty).$$

Now, by Theorem 2.3.1(ii), one has

$$\begin{aligned} \tilde{r}_{a,k}(k, t) &= \left(a^n + n \int_0^k v_{a,k}(y, t) dy \right)^{1/n} \\ &\geq \left(n \int_{1/2}^k v_{a,k}(y, t) dy \right)^{1/n} \geq \left(nC(T)^{-1} \int_{1/2}^k dy \right)^{1/n} \geq \left(\frac{nC(T)^{-1}}{2} \right)^{1/n} = C(T)^{-1}. \end{aligned}$$

Moreover, it also follows from Theorem 2.3.1 that

$$\tilde{r}_{a,k}(k, t) = \left(a^n + n \int_0^k v_{a,k}(y, t) dy \right)^{1/n} \geq C(a)^{-1} k^{1/n}.$$

Hence by (2.4.6), one has that for all $(r, t) \in [a, \infty) \times [0, T]$:

$$|\partial_r \varphi_{a,k}(r, t)| \leq \min\{C(T), C(a)k^{-1/n}\}, \quad \text{and } \partial_t \varphi_{a,k}(r, t) = 0. \quad (2.4.7)$$

Using (2.4.1)–(2.4.5), the extended approximate functions $(\rho_{a,k}, u_{a,k}, e_{a,k})(r, t)$ in the entire Eulerian domain $(r, t) \in [a, \infty) \times [0, T]$ is defined by (2.2.1) in Section 2.2.1.

The following lemmas show that the extended functions inherits the a-priori estimates derived in Section 2.3. As before, we still denote $\sigma := \min\{1, t\}$.

Lemma 2.4.1. *There exists an integer $N(a) = N(a, T, C_0) \in \mathbb{N}$ such that for all $k \geq N(a)$,*

$$\begin{aligned} & \sup_{t \in [0, T]} \int_a^\infty \left\{ G(\rho_{a,k}) + \rho_{a,k} \psi(e_{a,k}) + \rho_{a,k} \frac{|u_{a,k}|^2}{2} \right\} r^m dr + \kappa \int_0^T \int_a^\infty \frac{|\partial_r e_{a,k}|^2}{e_{a,k}^2} r^m dr dt \\ & + \int_0^T \int_a^\infty \left\{ \left(\frac{2\mu}{n} + \lambda \right) \frac{|\partial_r u_{a,k} + m u_{a,k}/r|^2}{e_{a,k}} + \frac{2m\mu}{n} \frac{|\partial_r u_{a,k} - u_{a,k}/r|^2}{e_{a,k}} \right\} r^m dr dt \leq C(T). \end{aligned} \quad (2.4.8)$$

Proof. Throughout the proof, we suppress (a, k) for simplicity and denote:

$$\begin{aligned} (\rho, u, e, \varphi)(r, t) &\equiv (\rho_{a,k}, u_{a,k}, e_{a,k}, \varphi_{a,k})(r, t) && \text{for } (r, t) \in [a, \infty) \times [0, T], \\ (\bar{v}, \bar{u}, \bar{e})(r, t) &\equiv (\bar{v}_{a,k}, \bar{u}_{a,k}, \bar{e}_{a,k})(r, t) && \text{for } (r, t) \in R_{a,k}, \\ (\tilde{v}, \tilde{u}, \tilde{e}, \tilde{r})(x, t) &\equiv (\tilde{v}_{a,k}, \tilde{u}_{a,k}, \tilde{e}_{a,k}, \tilde{r}_{a,k})(x, t) && \text{for } (x, t) \in [0, k] \times [0, T]. \end{aligned} \quad (2.4.9)$$

Since $G(\cdot)$ and $\psi(\cdot)$ are convex, and $0 \leq \varphi \leq 1$, it follows that $G(\rho) \leq \varphi G(\bar{v}^{-1})$ and $\psi(e) \leq \varphi \psi(\bar{e})$. Using these, and the fact that Jacobian is bounded from (2.4.2), one can convert the integral in Eulerian coordinate into Lagrangian as follow:

$$\begin{aligned} & \int_a^\infty \left\{ G(\rho) + \rho \psi(e) + \rho \frac{|u|^2}{2} \right\} (r, t) r^m dr \\ & \leq \int_a^{\tilde{r}(k,t)} \left(\bar{v} G(\bar{v}^{-1}) + \psi(\bar{e}) + \frac{|\bar{u}|^2}{2} \right) \bar{v}^{-1} r^m dr + \int_a^{\tilde{r}(k,t)} (1 - \varphi) \left(\psi(\bar{e}) + \frac{|\bar{u}|^2}{2} \right) r^m dr \\ & = \int_0^k \left(\psi(\tilde{v}) + \psi(\tilde{e}) + \frac{|\tilde{u}|^2}{2} \right) (x, t) dx + \int_0^k (1 - \varphi(\tilde{r}(x, t), t)) \left(\tilde{v} \psi(\tilde{e}) + \tilde{v} \frac{|\tilde{u}|^2}{2} \right) (x, t) dx, \end{aligned}$$

where in the last line, one used the fact that $zG(z^{-1}) = \psi(z)$. By Theorem 2.3.1(i) the first term of right hand side is bounded by $C_0 > 0$. By the definition of $\tilde{r}(x, t)$, one has

$$\left(a^n + nC(a)^{-1}x \right)^{1/n} \leq \tilde{r}(x, t) = \left(a^n + n \int_0^x \tilde{v}(y, t) dy \right)^{1/n} \leq \left(a^n + nC(a)x \right)^{1/n}.$$

Thus for a fixed $a > 0$, if $k \in \mathbb{N}$ is such that

$$k \geq M(a) := \frac{C(a)}{n} a^n (2^n - 1) + 2^n C(a)^2, \quad \text{then } \tilde{r}(k, t) \geq 2\tilde{r}(1, t). \quad (2.4.10)$$

Using this and the support of cut-off function φ , it follows that

$$\begin{aligned} & \int_0^k (1 - \varphi(\tilde{r}(x, t), t)) \left(\tilde{v} \psi(\tilde{e}) + \tilde{v} \frac{|\tilde{u}|^2}{2} \right) (x, t) dx \\ & = \int_a^{\tilde{r}(k,t)} (1 - \varphi(r, t)) \left(\psi(\bar{e}) + \frac{|\bar{u}|^2}{2} \right) (r, t) r^m dr \\ & \leq \int_{\tilde{r}(k,t)/2}^{\tilde{r}(k,t)} \left(\psi(\bar{e}) + \frac{|\bar{u}|^2}{2} \right) (r, t) r^m dr \leq \int_{\tilde{r}(1,t)}^{\tilde{r}(k,t)} \left(\psi(\bar{e}) + \frac{|\bar{u}|^2}{2} \right) (r, t) r^m dr \\ & = \int_1^k \tilde{v} \left(\psi(\tilde{e}) + \frac{|\tilde{u}|^2}{2} \right) (x, t) dx \leq \sup_{y \in [1, k]} \tilde{v}(y, t) \int_1^k \left(\psi(\tilde{e}) + \frac{|\tilde{u}|^2}{2} \right) (x, t) dx \leq C(T)C_0, \end{aligned}$$

where one has used Theorem 2.3.1(i)–(ii).

Next the dissipation terms in (2.4.8) are considered. Using (2.2.1), and the fact that $z \mapsto z^{-2}$ is convex in the domain $z \in (0, \infty)$, one obtains

$$\begin{aligned} \frac{|\partial_r e|^2}{2|e|^2} &= \frac{|\varphi \partial_r \bar{e} + (\bar{e} - 1) \partial_r \varphi|^2}{2|\varphi \bar{e} + (1 - \varphi)|^2} \leq 2^{-1}(\varphi \bar{e}^{-2} + (1 - \varphi)) |\varphi \partial_r \bar{e} + (\bar{e} - 1) \partial_r \varphi|^2 \\ &\leq \varphi^2 \frac{|\partial_r \bar{e}|^2}{\bar{e}^2} + \frac{|\bar{e} - 1|^2}{\bar{e}^2} \varphi |\partial_r \varphi|^2 + (1 - \varphi) \varphi^2 |\partial_r \bar{e}|^2 + (1 - \varphi) |\partial_r \varphi|^2 |\bar{e} - 1|^2 \end{aligned}$$

Multiplying both sides by r^m and integrating in $[a, \infty) \times [0, T]$, it follows from (2.4.7) and (2.4.10) that, if $k \geq M(a)$ then

$$\begin{aligned} &\frac{1}{2} \int_0^T \int_a^\infty \frac{|\partial_r e|^2}{e^2} r^m dr dt \\ &\leq \int_0^T \int_a^\infty \left\{ \varphi^2 \frac{|\partial_r \bar{e}|^2}{\bar{e}^2} + \frac{|\bar{e} - 1|^2}{\bar{e}^2} \varphi |\partial_r \varphi|^2 + (1 - \varphi) \left(\varphi^2 |\partial_r \bar{e}|^2 + |\partial_r \varphi|^2 |\bar{e} - 1|^2 \right) \right\} r^m dr dt \\ &\leq \int_0^T \int_a^{\tilde{r}(k,t)} \frac{|\partial_r \bar{e}|^2}{\bar{e}^2} r^m dr dt + \int_0^T \int_{\tilde{r}(1,t)}^{\tilde{r}(k,t)} \left\{ |\partial_r \bar{e}|^2 + C(T) |\bar{e} - 1|^2 + \frac{C(a)}{k^{2/n}} \frac{|\bar{e} - 1|^2}{\bar{e}^2} \right\} r^m dr dt. \end{aligned}$$

If $k \geq \max\{C(a)^{n/2} C(T)^{n/2}, M(a)\}$, then translating the above integral into Lagrangian coordinates and using (2.4.3), Theorem 2.3.1, one has

$$\begin{aligned} &\int_0^T \int_a^\infty \frac{|\partial_r e|^2}{e^2} r^m dr dt \\ &\leq \int_0^T \int_0^k \frac{\tilde{r}^{2m}}{\tilde{v}} \frac{|D_x \tilde{e}|^2}{\tilde{e}^2} dx dt + \int_0^T \int_1^k \left\{ \frac{\tilde{r}^{2m}}{\tilde{v}} |D_x \tilde{e}|^2 + C(T) \tilde{v} |\tilde{e} - 1|^2 + \frac{C(a)}{k^{2/n}} \frac{\tilde{v} |\tilde{e} - 1|^2}{\tilde{e}^2} \right\} dx dt \\ &\leq C_0 + C(T) + C(T) \sup \left\{ \left(C(T) \tilde{v} + \frac{C(a)}{k^{2/n}} \frac{\tilde{v}}{\tilde{e}^2} \right) (y, s) : (y, s) \in [1, k] \times [0, T] \right\} \\ &\leq C_0 + C(T) + C(T) C(a) k^{-2/n} \leq C_0 + C(T) + 1. \end{aligned}$$

Next, since $z \rightarrow z^{-1}$ is convex in $z \in (0, \infty)$ and $\partial_r u + mu/r = r^{-m} \partial_r (r^m u)$, one has

$$\begin{aligned} \frac{|\partial_r u + mu/r|^2}{2e} &\leq 2^{-1}(\varphi \bar{e}^{-1} + (1 - \varphi)) |\varphi \partial_r \bar{u} + \bar{u} \partial_r \varphi + m \varphi \bar{u}/r|^2 \\ &\leq \varphi^3 \bar{e}^{-1} |r^{-m} \partial_r (r^m \bar{u})|^2 + \varphi |\partial_r \varphi|^2 \bar{e}^{-1} |\bar{u}|^2 + (1 - \varphi) (\varphi^2 |r^{-m} \partial_r (r^m \bar{u})|^2 + |\bar{u} \partial_r \varphi|^2). \end{aligned}$$

By similar argument as above, one obtains that for all $k \geq \max\{C(a)^{n/2} C(T)^{n/2}, M(a)\}$,

$$\begin{aligned} &\frac{1}{2} \int_0^T \int_a^\infty \frac{|r^{-m} \partial_r (r^m \bar{u})|^2}{e} r^m dr dt \\ &\leq \int_0^T \int_a^{\tilde{r}(k,t)} \frac{|r^{-m} \partial_r (r^m \bar{u})|^2}{\bar{e}} r^m dr dt + \frac{C(a)}{k^{2/n}} \int_0^T \int_a^{\tilde{r}(k,t)} \frac{|\bar{u}|^2}{\bar{e}} r^m dr dt \\ &\quad + \int_0^T \int_{\tilde{r}(1,t)}^{\tilde{r}(k,t)} \left\{ |r^{-m} \partial_r (r^m \bar{u})|^2 + C(T) \bar{u}^2 \right\} r^m dr dt \\ &= \int_0^T \int_0^k \left\{ \frac{|D_x(\tilde{r}^m \tilde{u})|^2}{\tilde{v} \tilde{e}} + \frac{C(a)}{k^{2/n}} \frac{\tilde{v}}{\tilde{e}} |\tilde{u}|^2 \right\} dx dt + \int_0^T \int_1^k \left\{ \frac{|D_x(\tilde{r}^m \tilde{u})|^2}{\tilde{v}} + C(T) \tilde{v} |\tilde{u}|^2 \right\} dx dt \\ &\leq C_0 + k^{-2/n} C(a) C(T) + C(T) \leq C_0 + C(T) + 1, \end{aligned}$$

By the exact same analysis, we also have that for all $k \geq \max\{C(a)^{n/2}C(T)^{n/2}, M(a)\}$,

$$\int_0^T \int_a^\infty \frac{1}{e} \left| \partial_r u - \frac{u}{r} \right|^2 r^m dr dt \leq C_0 + C(T) + 1.$$

By setting $N(a) := \text{ceiling}(\max\{C(a)^{n/2}C(T)^{n/2}, M(a)\}) \in \mathbb{N}$, one concludes the proof. \square

Lemma 2.4.2. *For each $a \in (0, 1)$, one has*

$$\begin{aligned} \sup_{k \in \mathbb{N}} \sup_{0 \leq t \leq T} \int_a^\infty \left\{ |(\rho_{a,k} - 1, u_{a,k}^2, e_{a,k} - 1, \sqrt{\sigma} \partial_r u_{a,k}, \sigma \partial_r e_{a,k})|^2 \right\} (r, t) r^m dr &\leq C(a), \\ \sup_{k \in \mathbb{N}} \int_0^T \int_a^\infty \left\{ |(\partial_r u_{a,k}, \partial_r e_{a,k}, u_{a,k} \partial_r u_{a,k}, \sqrt{\sigma} \partial_t u_{a,k}, \sigma \partial_t e_{a,k})|^2 \right\} (r, t) r^m dr dt &\leq C(a), \end{aligned} \quad (2.4.11)$$

and for all $(k, r, t) \in \mathbb{N} \times [a, \infty) \times [0, T]$:

$$\begin{aligned} C(a)^{-1} \leq \rho_{a,k}(r, t) \leq C(a), \quad e_{a,k}(r, t) \geq C(a)^{-1}, \\ \sigma(t)^{\frac{1}{4}} |u_{a,k}(r, t)| \leq C(a), \quad \sigma(t)^{\frac{1}{2}} e_{a,k}(r, t) \leq C(a). \end{aligned} \quad (2.4.12)$$

Moreover, for each $\varepsilon \in (0, 1]$, one has that for all $(a, k) \in (0, 1) \times \mathbb{N}$,

$$\begin{aligned} \sup_{0 \leq t \leq T} \int_{\tilde{r}_{a,k}(\varepsilon, t)}^\infty \left\{ |(\rho_{a,k} - 1, u_{a,k}^2, e_{a,k} - 1, \sqrt{\sigma} \partial_r u_{a,k}, \sigma \partial_r e_{a,k})|^2 \right\} (r, t) r^m dr &\leq C(\varepsilon), \\ \int_0^T \int_{\tilde{r}_{a,k}(\varepsilon, t)}^\infty \left\{ |(\partial_r u_{a,k}, \partial_r e_{a,k}, u_{a,k} \partial_r u_{a,k}, \sqrt{\sigma} \partial_t u_{a,k}, \sigma \partial_t e_{a,k})|^2 \right\} (r, t) r^m dr dt &\leq C(\varepsilon), \end{aligned} \quad (2.4.13)$$

and for all $(a, k) \in \mathbb{N}$, if $t \in [0, T]$ and $r \in [\tilde{r}(\varepsilon, t), \infty)$, then

$$C(\varepsilon)^{-1} \leq \rho_{a,k}(r, t) \leq C(\varepsilon), \quad \sigma(t)^{\frac{1}{4}} |u_{a,k}(r, t)| \leq C(\varepsilon), \quad \sigma(t)^{\frac{1}{2}} e_{a,k}(r, t) \leq C(\varepsilon). \quad (2.4.14)$$

Proof. As before, we suppress (a, k) for simplicity and use the notation (2.4.9). Only (2.4.13)–(2.4.14) are shown, as (2.4.11)–(2.4.12) follow in the exact same manner. First, using the fact that Jacobian is bounded from (2.4.2) and the construction (2.4.5), the integration in Eulerian coordinate is converted to Lagrangian as follows:

$$\begin{aligned} \int_{\tilde{r}(\varepsilon, t)}^\infty |\rho - 1|^2 r^m dr &= \int_{\tilde{r}(\varepsilon, t)}^\infty |\bar{v}^{-1} - 1|^2 \varphi^2 r^m dr \\ &\leq \int_{\tilde{r}(\varepsilon, t)}^{\tilde{r}(k, t)} |\bar{v}^{-1} - 1|^2 r^m dr = \int_\varepsilon^k |\tilde{v}^{-1} - 1|^2 \tilde{v} dx = \int_\varepsilon^k |\tilde{v} - 1|^2 \frac{1}{\tilde{v}} dx \leq C(\varepsilon), \end{aligned}$$

where in the last line, one used Theorem 2.3.1. By the same method, one can also obtain the corresponding estimates for $|e - 1|^2$ and $|u|^4$ in (2.4.13)₁.

Next, using (2.4.3), (2.4.7), and Theorem 2.3.1, it follows that

$$\begin{aligned} \int_{\tilde{r}(\varepsilon, t)}^\infty \sigma(t) |\partial_r u|^2 (r, t) r^m dr &= \int_{\tilde{r}(\varepsilon, t)}^\infty \sigma(t) |\varphi \partial_r \bar{u} + \bar{u} \partial_r \varphi|^2 (r, t) r^m dr \\ &\leq \int_{\tilde{r}(\varepsilon, t)}^{\tilde{r}(k, t)} \sigma(t) |\partial_r \bar{u}|^2 r^m dr + C \int_{\tilde{r}(\varepsilon, t)}^{\tilde{r}(k, t)} |\bar{u}|^2 r^m dr = \int_\varepsilon^k \sigma \frac{\tilde{r}^{2m}}{\tilde{v}} |D_x \tilde{u}|^2 dx + \int_\varepsilon^k \tilde{v} |\tilde{u}|^2 dx \leq C(\varepsilon). \end{aligned}$$

In the same way, one can also obtain the estimates for $\sigma^2 |\partial_r e|^2$, $|(\partial_r u, \partial_r e, u \partial_r u)|^2$ in (2.4.13).

Next, by (2.4.3) and Theorem 2.3.1, it follows that:

$$\begin{aligned}
& \int_0^T \int_{\tilde{r}(\varepsilon, t)}^\infty \sigma^2(t) |\partial_t e|^2(r, t) r^m dr dt = \int_0^T \int_{\tilde{r}(\varepsilon, t)}^\infty \sigma^2(t) \varphi^2(r, t) |\partial_t \bar{e}|^2(r, t) r^m dr dt \\
& \leq \int_0^T \int_{\tilde{r}(\varepsilon, t)}^{\tilde{r}(k, t)} \sigma^2(t) |\partial_t \bar{e}|^2(r, t) r^m dr dt = \int_0^T \int_\varepsilon^k \sigma^2(t) \tilde{v} \left| D_t \tilde{e} - \tilde{r}^m \frac{\tilde{u}}{\tilde{v}} D_x \tilde{e} \right|^2(r, t) dx dt \\
& \leq \int_0^T \int_\varepsilon^k \sigma^2(t) \tilde{v} |D_t \tilde{e}|^2(r, t) dx dt + \int_0^T \int_\varepsilon^k \sigma^{3/2}(t) \left(\sigma^{1/4}(t) |\tilde{u}| \right)^2 \frac{\tilde{r}^{2m}}{\tilde{v}} |D_x \tilde{e}|^2(r, t) dx dt \leq C(\varepsilon).
\end{aligned}$$

By the same argument, one can also obtain the estimate for $\sigma |\partial_t e|^2$ in (2.4.13).

Next, for (r, t) such that $t \in [0, T]$ and $r \in [\tilde{r}(\varepsilon, t), \infty) \cap R_{a, k}$, there exists $(x, s) \in [\varepsilon, k] \times [0, T]$ such that $r = \tilde{r}(x, s)$ and $t = s$, hence for $(r, t) \in R_{a, k}$ one has

$$\begin{aligned}
\rho(r, t) &= \varphi(\tilde{r}(x, t), t) \bar{v}^{-1}(\tilde{r}(x, t), t) + 1 - \varphi(\tilde{r}(x, t), t) \\
&= \varphi(\tilde{r}(x, t), t) \tilde{v}^{-1}(x, t) + 1 - \varphi(\tilde{r}(x, t), t) \\
&\leq C(\varepsilon) \varphi(\tilde{r}(x, t), t) + 1 - \varphi(\tilde{r}(x, t), t) \leq \max\{1, C(\varepsilon)\}, \\
\text{and } \rho(r, t) &= \varphi(\tilde{r}(x, t), t) \tilde{v}^{-1}(x, t) + 1 - \varphi(\tilde{r}(x, t), t) \\
&\geq C(\varepsilon)^{-1} \varphi(\tilde{r}(x, t), t) + 1 - \varphi(\tilde{r}(x, t), t) \geq \min\{1, C(\varepsilon)^{-1}\}.
\end{aligned}$$

For (r, t) such that $t \in [0, T]$ and $r \in [\tilde{r}(\varepsilon, t), \infty) \times [0, T] \setminus R_{a, k}$, one has $r > \tilde{r}(k, t)$, hence $\varphi(r, t) = 0$ and $\rho(r, t) = \bar{v}^{-1}(r, t) \varphi(r, t) - \varphi(r, t) + 1 = 1$. This proves the upper and lower bound of the extended density. In similar manner, one can also obtain bounds for u, e listed in (2.4.14). \square

Throughout Sections 2.4.3–2.4.7, we will suppress the parameter $a \in (0, 1)$, and denote

$$\begin{aligned}
(\rho_k, u_k, e_k, \varphi_k) &\equiv (\rho_{a, k}, u_{a, k}, e_{a, k}, \varphi_{a, k}), & (\bar{v}_k, \bar{u}_k, \bar{e}_k) &\equiv (\bar{v}_{a, k}, \bar{u}_{a, k}, \bar{e}_{a, k}), \\
(\tilde{v}_k, \tilde{u}_k, \tilde{e}_k, \tilde{r}_k) &\equiv (\tilde{v}_{a, k}, \tilde{u}_{a, k}, \tilde{e}_{a, k}, \tilde{r}_{a, k})
\end{aligned} \tag{2.4.15}$$

Note that the main aim of these sections is to prove Theorem 2.2.1 via the limit $k \rightarrow \infty$.

2.4.3 Construction of particle path function: $\tilde{r}(x, t)$

Lemma 2.4.3. *There exists a continuous function $\tilde{r}(x, t) : [0, \infty) \times [0, T] \rightarrow [a, \infty)$ and a sub-sequence $k \rightarrow \infty$ such that*

(i) *for any compact subset $K \subset\subset [0, \infty) \times [0, T]$,*

$$\lim_{k \rightarrow \infty} \sup_{(x, t) \in K} |\bar{r}_k(x, t) - \tilde{r}(x, t)| = 0,$$

where $\bar{r}_k(x, t)$ is defined by:

$$\bar{r}_k(x, t) = \begin{cases} \tilde{r}_k(x, t) & \text{if } (x, t) \in [0, k] \times [0, T], \\ x - k + \tilde{r}_k(k, t) & \text{if } (x, t) \in [k, \infty) \times [0, T]. \end{cases} \tag{2.4.16}$$

(ii) *for each $\varepsilon > 0$, one has*

$$\begin{aligned}
|\tilde{r}(x_1, t) - \tilde{r}(x_2, t)| &\leq C(\varepsilon) |x_1 - x_2| & \text{for all } (x_1, x_2, t) \in [\varepsilon, \infty)^2 \times [0, T], \\
|\tilde{r}(x, t_1) - \tilde{r}(x, t_2)| &\leq C(\varepsilon) |t_1^{3/4} - t_2^{3/4}| & \text{for all } (x, t_1, t_2) \in [\varepsilon, \infty) \times [0, T]^2.
\end{aligned}$$

(iii) for each $t \in [0, T]$ the function $x \mapsto \tilde{r}(x, t)$ is strictly increasing and

$$nx\psi_-^{-1}\left(\frac{C_0}{x}\right) \leq (\tilde{r}(x, t))^n \leq C_0(1+x) \quad \text{for all } (x, t) \in [0, \infty) \times [0, T].$$

Proof. For each integer $l \in \mathbb{N}$, let $S_l := [0, l] \times [0, T] \subset [0, \infty) \times [0, T]$. Set $x_1, x_2, x \in [0, l]$ and $t_1, t_2, t \in [0, T]$. We consider two cases: $k \geq l$ or $k < l$.

Case 1: $k \geq l$. Then $\bar{r}_k(x_i, t) = \tilde{r}_k(x_i, t)$ for $i = 1, 2$. It follows from Theorem 2.3.1, and Mean Value theorem that

$$|\bar{r}_k(x_1, t) - \bar{r}_k(x_2, t)| \leq |x_1 - x_2| \sup_{x \in [0, k]} D_x \tilde{r}_k(x, t) \leq |x_1 - x_2| \sup_{x \in [0, k]} \frac{\tilde{v}_k}{\tilde{r}_k^m} \leq C(a)|x_1 - x_2|.$$

Case 2: $k < l$. For this case, we consider three sub-cases. First, if $x_1, x_2 \in [0, k]$, then one can get the same inequality as Case 1 by repeating the argument. If $x_1, x_2 \in [k, l]$ then by (2.4.16), one has $|\bar{r}_k(x_1, t) - \bar{r}_k(x_2, t)| = |x_1 - x_2|$. If $x_1 \in [0, k]$ and $x_2 \in [k, l]$ then by (2.4.16),

$$|\bar{r}_k(x_1, t) - \bar{r}_k(x_2, t)| \leq |\tilde{r}_k(x_1, t) - \tilde{r}_k(k, t)| + |x_2 - k| \leq C(a)(k - x_1) + x_2 - k \leq C^*(a)|x_1 - x_2|,$$

where $C^*(a) \equiv \max\{1, C(a)\}$. In summary, for both cases, one obtains that

$$\sup_{k \in \mathbb{N}} |\bar{r}_k(x_1, t) - \bar{r}_k(x_2, t)| \leq C^*(a)|x_1 - x_2|. \quad (2.4.17)$$

Moreover, for both cases $k \geq l$ and $k < l$, it can be shown that if $t_* = \min\{1, t_1, t_2\}$ then

$$|\bar{r}_k(x, t_1) - \bar{r}_k(x, t_2)| = \left| \int_{t_1}^{t_2} u_k(x, s) ds \right| \leq C(a) \left| \int_{t_1}^{t_2} s^{-\frac{1}{4}} ds \right| \leq C(a)t_*^{-\frac{1}{4}}|t_1 - t_2|. \quad (2.4.18)$$

Similarly, by (2.4.16), Theorem 2.3.1, and Lemma 2.3.1, one can also show that for each $\varepsilon \in (0, 1]$,

$$\begin{aligned} \sup_{k \in \mathbb{N}} |\bar{r}_k(x_1, t) - \bar{r}_k(x_2, t)| &\leq C(\varepsilon)|x_1 - x_2| \quad \text{for all } (x_1, x_2, t) \in [\varepsilon, l]^2 \times [0, T], \\ \sup_{k \in \mathbb{N}} |\bar{r}_k(x, t_1) - \bar{r}_k(x, t_2)| &\leq C(\varepsilon)|t_1^{3/4} - t_2^{3/4}| \quad \text{for all } (x, t_1, t_2) \in [\varepsilon, l] \times [0, T]^2. \end{aligned} \quad (2.4.19)$$

Furthermore, the following bound holds for all $k \in \mathbb{N}$,

$$\sup_{(x, t) \in S_l} \bar{r}_k(x, t) = \sup_{(x, t) \in S_l} \left(a^n + n \int_0^x \tilde{v}_k(y, t) dy \right)^{1/n} \leq (a^n + nC(a)l)^{1/n}. \quad (2.4.20)$$

By (2.4.17)–(2.4.18) and (2.4.20), one can apply Proposition A.0.1 to obtain a sub-sequence, still denoted as $k \rightarrow \infty$, and a continuous function $\tilde{r}(x, t) : [0, \infty) \times [0, T] \rightarrow [a, \infty)$ such that

$$\lim_{k \rightarrow \infty} \sup_{(x, t) \in K} |\tilde{r}(x, t) - \bar{r}_k(x, t)| = 0 \quad \text{for all } K \subset\subset [0, \infty) \times [0, T]. \quad (2.4.21)$$

Applying convergence (2.4.21) on (2.4.19), one obtains Lemma 2.4.3(ii).

Next, let $0 < x_1 < x_2 < \infty$. For all $k \geq 2^n x_2 C(a)^2 + C(a)(2^n - 1)/n$, one can verify $2\tilde{r}_k(x_1, t) \leq 2\tilde{r}_k(x_2, t) \leq \tilde{r}_k(k, t)$. It follows that

$$\int_{\tilde{r}_k(x_1, t)}^{\tilde{r}_k(x_2, t)} \rho_k(r, t) r^m dr = \int_{\tilde{r}_k(x_1, t)}^{\tilde{r}_k(x_2, t)} \bar{v}_k^{-1}(r, t) r^m dr = \int_{x_1}^{x_2} \tilde{v}_k^{-1} \tilde{v}_k(x, t) dx = x_2 - x_1.$$

Since $\rho_k(r, t) \leq C(x_1)$ for $r \in [\tilde{r}_k(x_1, t), \infty)$ from Lemma 2.4.2, one has

$$x_2 - x_1 = \int_{\tilde{r}_k(x_1, t)}^{\tilde{r}_k(x_2, t)} \rho_k(r, t) r^m dr \leq \frac{C(x_1)}{n} \{(\tilde{r}_k(x_1, t))^n - (\tilde{r}_k(x_2, t))^n\}.$$

Taking the limit $k \rightarrow \infty$ on the above, and using (2.4.21), it follows that

$$0 < x_2 - x_1 < \frac{C(x_1)}{n} \{(\tilde{r}(x_1, t))^n - (\tilde{r}(x_2, t))^n\}.$$

This shows that $x \mapsto \tilde{r}(x, t)$ is strictly increasing for each $t \in [0, T]$.

For a fixed point $(x, t) \in [0, \infty) \times [0, T]$, one has from Lemma 2.3.1 that,

$$nx\psi_-^{-1}(C_0x^{-1}) \leq (\tilde{r}_k(x, t))^n \leq C_0(1+x) \quad \text{for all } k \geq x.$$

Taking the limit $k \rightarrow \infty$, and using (2.4.21), one obtains Lemma 2.4.3(iii). \square

2.4.4 Compactness results

Lemma 2.4.4. *There exists a continuous function $(u, e)(r, t) : [a, \infty) \times (0, T] \rightarrow \mathbb{R} \times [0, \infty)$, and a subsequence $\{(u_k, e_k)\}_{k \in \mathbb{N}}$ such that for each compact subset $K \subset\subset [a, \infty) \times [0, T]$,*

$$\lim_{k \rightarrow \infty} \sup_{(r, t) \in K} |(u_k - u, e_k - e)(r, t)| = 0. \quad (2.4.22)$$

Moreover, for each fixed $\varepsilon > 0$, one has that if $t \in [0, T]$ and $r_1, r_2 \geq \tilde{r}(\varepsilon, t)$ then

$$\sigma(t)^{\frac{1}{2}} |u(r_1, t) - u(r_2, t)| + \sigma(t) |e(r_1, t) - e(r_2, t)| \leq C(\varepsilon) |r_1 - r_2|^{\frac{1}{2}}, \quad (2.4.23)$$

and if $0 < t_1 < t_2 \leq T$ and $r \geq \sup_{t_1 \leq t \leq t_2} \tilde{r}(\varepsilon, t)$ then

$$\sigma(t_1)^{\frac{1}{2}} |u(r, t_1) - u(r, t_2)| + \sigma(t_1) |e(r, t_1) - e(r, t_2)| \leq C(\varepsilon) |t_2 - t_1|^{\frac{1}{4}}. \quad (2.4.24)$$

In addition, one has

$$\begin{cases} |u(r, t)| \leq C(a)\sigma(t)^{-\frac{1}{4}}, & C(a)^{-1} \leq e(r, t) \leq C(a)\sigma(t)^{-\frac{1}{2}} & \text{if } (r, t) \in [a, \infty) \times [0, T], \\ |u(r, t)| \leq C(\varepsilon)\sigma(t)^{-\frac{1}{4}}, & e(r, t) \leq C(\varepsilon)\sigma(t)^{-\frac{1}{2}} & \text{if } t \in [0, T], r \geq \tilde{r}(\varepsilon, t). \end{cases} \quad (2.4.25)$$

Proof. Only the proof for $u(r, t)$ is presented since it is the same for $e(r, t)$. Let $(r_1, r_2, t) \in [a, \infty)^2 \times (0, T]$. By Fundamental Theorem of Calculus and Lemma 2.4.2, for all $k \in \mathbb{N}$,

$$\begin{aligned} |u_k(r_2, t) - u_k(r_1, t)| &= \left| \int_{r_1}^{r_2} \partial_r u_k(r, t) dr \right| \leq (r_2 - r_1)^{1/2} r_1^{-m/2} \left(\int_{r_1}^{r_2} |\partial_r u_k|^2(r, t) r^m dr \right)^{1/2} \\ &\leq \sigma(t)^{-1/2} a^{-m/2} \left(\int_a^\infty \sigma(t) |\partial_r u_k|^2(r, t) r^m dr \right)^{1/2} |r_2 - r_1|^{1/2} \leq C(a)\sigma(t)^{-1/2} |r_2 - r_1|^{1/2}. \end{aligned}$$

In addition, let $r \in [a, \infty)$ and $0 < t_1 < t_2 \leq T$. Set $h := \sqrt{t_2 - t_1}$. By Mean Value theorem,

$$\frac{1}{h} \int_r^{r+h} |u_k(\zeta, t_1) - u_k(\zeta, t_2)| d\zeta = |u_k(\xi, t_1) - u_k(\xi, t_2)| \quad \text{for some } \xi \in (r, r+h). \quad (2.4.26)$$

By triangular inequality, and the Fundamental Theorem of Calculus, one has for all $k \in \mathbb{N}$,

$$\begin{aligned}
& |u_k(r, t_1) - u_k(r, t_2)| - |u_k(\xi, t_1) - u_k(\xi, t_2)| \\
& \leq |u_k(r, t_1) - u_k(r, t_2) - (u_k(\xi, t_1) - u_k(\xi, t_2))| = \left| \int_r^\xi (\partial_r u_k(\zeta, t_1) - \partial_r u_k(\zeta, t_2)) d\zeta \right| \\
& \leq \int_r^\xi \{|\partial_r u_k(\zeta, t_1)| + |\partial_r u_k(\zeta, t_2)|\} d\zeta \leq |\xi - r|^{1/2} r^{-m/2} \sum_{i=1}^2 \left(\int_r^\xi |\partial_r u_k(\zeta, t_i)|^2 \zeta^m d\zeta \right)^{1/2} \\
& \leq a^{-m/2} \sigma(t)^{-1/2} \sqrt{h} \sum_{i=1}^2 \left(\int_a^\infty \sigma(t) |\partial_r u_k(\zeta, t_i)|^2 \zeta^m d\zeta \right)^{1/2} \leq C(a) \sigma(t)^{-1/2} \sqrt{h}.
\end{aligned}$$

Thus by (2.4.26), it follows that for all $k \in \mathbb{N}$,

$$\begin{aligned}
& |u_k(r, t_1) - u_k(r, t_2)| \leq |u_k(\xi, t_1) - u_k(\xi, t_2)| + C(a) \sigma(t)^{-1/2} \sqrt{h} \\
& \leq C(a) \sigma(t)^{-1/2} \sqrt{h} + \frac{1}{h} \int_r^{r+h} \int_{t_1}^{t_2} |\partial_t u_k(\zeta, t)| dt d\zeta \\
& \leq C(a) \sigma(t)^{-1/2} \sqrt{h} + \frac{\sigma(t_1)^{-1/2} a^{-m/2}}{h} \left(\int_r^{r+h} \int_{t_1}^{t_2} dt d\zeta \right)^{1/2} \left(\int_{t_1}^{t_2} \int_a^\infty \sigma |\partial_t u_k|^2 \zeta^m d\zeta dt \right)^{1/2} \\
& \leq C(a) \sigma(t)^{-1/2} \sqrt{h} + C(a) \sigma(t_1)^{-1/2} h^{-1/2} |t_2 - t_1|^{1/2} \leq C(a) \sigma(t_1)^{-1/2} |t_2 - t_1|^{1/4}.
\end{aligned}$$

In summary, one has

$$\begin{cases} \sup_{k \in \mathbb{N}} |u_k(r_2, t) - u_k(r_1, t)| \leq C(a) \sigma(t)^{-\frac{1}{2}} |r_2 - r_1|^{\frac{1}{2}} & \text{if } (r_1, r_2, t) \in [a, \infty)^2 \times (0, T], \\ \sup_{k \in \mathbb{N}} |u_k(r, t_2) - u_k(r, t_1)| \leq C(a) \sigma(t_*)^{-\frac{1}{2}} |t_2 - t_1|^{\frac{1}{4}} & \text{if } (r, t_1, t_2) \in [a, \infty) \times (0, T]^2, \end{cases} \quad (2.4.27)$$

where $t_* \equiv \min\{t_1, t_2\}$. Let $S_l = [0, \infty) \times [l^{-1}, T]$ for each $l \in \mathbb{N}$. Then one can apply Proposition A.0.1 with (2.4.27) and S_l to obtain a continuous function $u(r, t) : [a, \infty) \times (0, T] \rightarrow \mathbb{R}$, and a sub-sequence (still denoted as $\{u_k\}_{k \in \mathbb{N}}$) such that (2.4.22) holds true.

Next, let $\varepsilon \in (0, 1]$. Since $x \rightarrow \tilde{r}(x, t)$ is strictly increasing and $(x, t) \mapsto \tilde{r}(x, t)$ is continuous, it follows that $d := \inf_{t \in [0, T]} \{\tilde{r}(\varepsilon, t) - \tilde{r}(\varepsilon/2, t)\} > 0$. By Lemma 2.4.3, there exists $N_\varepsilon \in \mathbb{N}$ such that $\sup_{0 \leq t \leq T} |\tilde{r}_k(\varepsilon/2, t) - \tilde{r}(\varepsilon/2, t)| \leq d/2$ if $k \geq N_\varepsilon$, which implies

$$\tilde{r}_k(\varepsilon/2, t) < \tilde{r}(\varepsilon, t) \quad \text{for each } t \in [0, T] \quad \text{and } k \geq N_\varepsilon. \quad (2.4.28)$$

Let (r_1, r_2, t) be a triplet such that $t \in (0, T]$ and $r_1, r_2 \in [\tilde{r}(\varepsilon, t), \infty)$. Then (2.4.28) implies that $r_1, r_2 > \tilde{r}_k(\varepsilon/2, t)$ if $k \geq N_\varepsilon$. By Fundamental Theorem of Calculus, and Lemma 2.4.2, it follows that for all $k \geq N_\varepsilon$,

$$\begin{aligned}
& |u_k(r_2, t) - u_k(r_1, t)| = \left| \int_{r_1}^{r_2} \partial_r u_k(r, t) dr \right| \leq (r_2 - r_1)^{1/2} r_1^{-m/2} \left(\int_{r_1}^{r_2} |\partial_r u_k|^2(r, t) r^m dr \right)^{1/2} \\
& \leq \frac{\sigma(t)^{-1/2}}{(\tilde{r}_k(\varepsilon/2, t))^{m/2}} \left(\int_{\tilde{r}_k(\varepsilon/2, t)}^\infty \sigma |\partial_r u_k|^2(r, t) r^m dr \right)^{1/2} |r_2 - r_1|^{1/2} \leq C(\varepsilon) \sigma(t)^{-\frac{1}{2}} |r_2 - r_1|^{1/2}.
\end{aligned}$$

In addition, let $(r, t_1, t_2) \in [a, \infty) \times (0, T]^2$ be a triplet such that $0 < t_1 < t_2 \leq T$ and $r \geq \sup_{t_1 \leq t \leq t_2} \tilde{r}(\varepsilon, t)$. Then by (2.4.28), $r > \tilde{r}_k(\varepsilon/2, t) \geq C(\varepsilon)^{-1}$ for all $t \in [0, T]$ if $k \geq N_\varepsilon$. Now set $h := \sqrt{t_2 - t_1}$. Then by Mean Value theorem,

$$\frac{1}{h} \int_r^{r+h} |u_k(\zeta, t_1) - u_k(\zeta, t_2)| d\zeta = |u_k(\xi, t_1) - u_k(\xi, t_2)| \quad \text{for some } \xi \in (r, r+h).$$

By triangular inequality, and the Fundamental Theorem of Calculus, one has for all $k \geq N_\varepsilon$,

$$\begin{aligned}
& |u_k(r, t_1) - u_k(r, t_2)| - |u_k(\xi, t_1) - u_k(\xi, t_2)| \\
& \leq |u_k(r, t_1) - u_k(r, t_2) - (u_k(\xi, t_1) - u_k(\xi, t_2))| = \left| \int_r^\xi (\partial_r u_k(\zeta, t_1) - \partial_r u_k(\zeta, t_2)) d\zeta \right| \\
& \leq \int_r^\xi (|\partial_r u_k(\zeta, t_1)| + |\partial_r u_k(\zeta, t_2)|) d\zeta \leq |\xi - r|^{1/2} r^{-m/2} \sum_{i=1}^2 \left(\int_r^\xi |\partial_r u_k(\zeta, t_i)|^2 \zeta^m d\zeta \right)^{1/2} \\
& \leq C(\varepsilon) h^{1/2} \sigma(t_1)^{-1/2} \sum_{i=1}^2 \left(\int_{\tilde{r}_k(\varepsilon/2, t_i)}^\infty \sigma(t_i) |\partial_r u_k(\zeta, t_i)|^2 \zeta^m d\zeta \right)^{1/2} \leq C(\varepsilon) h^{1/2} \sigma(t_1)^{-1/2}.
\end{aligned}$$

It follows that for all $k \geq N_\varepsilon$,

$$\begin{aligned}
& |u_k(r, t_1) - u_k(r, t_2)| \leq |u_k(\xi, t_1) - u_k(\xi, t_2)| + C(\varepsilon) \sigma(t_1)^{-1/2} h^{1/2} \\
& \leq C(\varepsilon) h^{\frac{1}{2}} \sigma(t_1)^{-\frac{1}{2}} + \frac{1}{h} \int_r^{r+h} \int_{t_1}^{t_2} |\partial_t u_k(\zeta, t)| dt d\zeta \\
& \leq C(\varepsilon) \sqrt{\frac{h}{\sigma(t_1)}} + \frac{\sigma(t_1)^{-\frac{1}{2}} r^{-\frac{m}{2}}}{h} \left(\int_r^{r+h} \int_{t_1}^{t_2} dt d\zeta \right)^{1/2} \left(\int_{t_1}^{t_2} \int_{\tilde{r}_k(\varepsilon/2, t)}^\infty t |\partial_t u_k(\zeta, t)|^2 \zeta^m d\zeta dt \right)^{1/2} \\
& \leq C(\varepsilon) h^{\frac{1}{2}} \sigma(t_1)^{-\frac{1}{2}} + C(\varepsilon) h^{-\frac{1}{2}} |t_2 - t_1|^{\frac{1}{2}} \sigma(t_1)^{-\frac{1}{2}} \leq C(\varepsilon) \sigma(t_1)^{-\frac{1}{2}} |t_2 - t_1|^{\frac{1}{4}}.
\end{aligned}$$

In summary, if one defines $R(\varepsilon, t_1, t_2) := \sup_{t_1 \leq t \leq t_2} \tilde{r}(\varepsilon, t)$, then

$$\begin{cases} \sup_{k \geq N_\varepsilon} |u_k(r_2, t) - u_k(r_1, t)| \leq C(\varepsilon) t^{-1/2} |r_2 - r_1|^{1/2} & \text{if } t \in (0, T], r_1, r_2 \geq \tilde{r}(\varepsilon, t), \\ \sup_{k \geq N_\varepsilon} |u_k(r, t_2) - u_k(r, t_1)| \leq C(\varepsilon) \sigma(t_*)^{-\frac{1}{2}} |t_2 - t_1|^{1/4} & \text{if } 0 < t_1 < t_2 \leq T, r \geq R(\varepsilon, t_1, t_2). \end{cases}$$

Taking $k \rightarrow \infty$, and using (2.4.22), one obtains continuity estimates for u in (2.4.23)–(2.4.24).

Finally, by Lemma 2.4.2 and (2.4.22), one obtains (2.4.25)₁. In addition, for each $\varepsilon \in (0, 1]$, (2.4.28) implies that $r \geq \tilde{r}_k(\varepsilon/2, t)$ if $r \geq \tilde{r}(\varepsilon, t)$ and $k \geq N_\varepsilon$. Hence, $|u_k(r, t)| \leq C(\varepsilon) \sigma(t)^{-1/4}$ for all (r, t, k) such that $k \geq N_\varepsilon$, $t \in [0, T]$, and $r \geq \tilde{r}(\varepsilon, t)$. Taking limit $k \rightarrow \infty$, and using (2.4.22), one obtains (2.4.25)₂. \square

Proposition 2.4.1. *Let $\varepsilon \in (0, 1]$, and $(r, t) \in [a, \infty) \times (0, T]$ be such that $r > \tilde{r}(\varepsilon, t)$. Denote $D_\delta(r, t) := \{(\zeta, s) : \sqrt{(\zeta - r)^2 + (s - t)^2} < \delta\}$. Then there exists $N(\varepsilon, r, t) \in \mathbb{N}$, $\delta(\varepsilon, r, t) > 0$ such that for all $\delta \in (0, \delta(\varepsilon, r, t))$ and $k \geq N(\varepsilon, r, t)$,*

$$D_\delta(r, t) \subset \{(\zeta, s) \in [a, \infty) \times [0, T] : \zeta > \tilde{r}_k(\varepsilon, s)\}.$$

Proof. Let $\eta := r - \tilde{r}(\varepsilon, t) > 0$. Since $(\zeta, s) \mapsto \zeta - \tilde{r}(\varepsilon, s)$ is continuous, there exists $\delta(\varepsilon, r, t) > 0$ such that for all $\delta \leq \delta(\varepsilon, r, t)$, if $(\zeta, s) \in D_\delta(r, t)$ then $|\zeta - \tilde{r}(\varepsilon, s) - r + \tilde{r}(\varepsilon, t)| < \eta/2$ which implies $\zeta > \eta/2 + \tilde{r}(\varepsilon, s)$. Therefore,

$$D_\delta(r, t) \subset \{(\zeta, s) \in [a, \infty) \times [0, T] : \zeta > \eta/2 + \tilde{r}(\varepsilon, s)\}. \quad (2.4.29)$$

By Lemma 2.4.3, one has $\lim_{k \rightarrow \infty} \sup_{s \in [0, T]} |\tilde{r}_k(\varepsilon, s) - \tilde{r}(\varepsilon, s)| = 0$. Thus there exists $N(\varepsilon, r, t) \in \mathbb{N}$ such that $\tilde{r}(\varepsilon, s) + \eta/4 > \tilde{r}_k(\varepsilon, s)$ for all $s \in [0, T]$ and $k \geq N(\varepsilon, r, t)$. If (ζ, s) is such that $\zeta > \eta/2 + \tilde{r}(\varepsilon, s)$, then $\zeta > \eta/4 + \tilde{r}(\varepsilon, s) > \tilde{r}_k(\varepsilon, s)$ for all $k \geq N(\varepsilon, r, t)$. The proposition follows by combining this with (2.4.29). \square

Lemma 2.4.5. *There exists $\rho \geq 0$, and a sub-sequence $\{\rho_k\}_{k \in \mathbb{N}}$ such that*

$$(\rho_k - 1) \xrightarrow{*} (\rho - 1) \text{ in } L^\infty(0, T; L^2([a, \infty), r^m dr)). \quad (2.4.30)$$

Moreover, for each $\varepsilon > 0$, one has

$$\begin{cases} C(\varepsilon)^{-1} \leq \rho(r, t) \leq C(\varepsilon) & \text{for a.e. } (r, t) \in \{(\zeta, s) \in [a, \infty) \times [0, T] : \zeta \geq \tilde{r}(\varepsilon, t)\}, \\ C(a)^{-1} \leq \rho(r, t) \leq C(a) & \text{for a.e. } (r, t) \in [a, \infty) \times [0, T]. \end{cases} \quad (2.4.31)$$

Proof. From Lemma 2.4.2, one has

$$\sup_{k \in \mathbb{N}} \sup_{0 \leq t \leq T} \int_a^\infty |\rho_k - 1|^2(r, t) dr dt \leq a^{-m} \sup_{k \in \mathbb{N}} \sup_{0 \leq t \leq T} \int_a^\infty |\rho_k - 1|^2(r, t) r^m dr dt \leq C(a). \quad (2.4.32)$$

Thus there exists f , $(\rho - 1) \in L^\infty(0, T; L^2([a, \infty), dr))$, and a sub-sequence (still denoted $\{\rho_k\}_{k \in \mathbb{N}}$) such that as $k \rightarrow \infty$

$$(\rho_k - 1) \overset{*}{\rightharpoonup} (\rho - 1) \quad \text{and} \quad r^{m/2}(\rho_k - 1) \overset{*}{\rightharpoonup} f \quad \text{in} \quad L^\infty(0, T; L^2([a, \infty), dr)). \quad (2.4.33)$$

Let $\phi \in C_c^\infty([a, \infty) \times [0, T])$. Using (2.4.33), it follows that as $k \rightarrow \infty$

$$\int_0^T \int_a^\infty f \phi dr dt \leftarrow \int_0^T \int_a^\infty r^{m/2}(\rho_k - 1) \phi dr dt \rightarrow \int_0^T \int_a^\infty (\rho - 1) r^{m/2} \phi dr dt.$$

By Fundamental Lemma of Calculus of Variation, $f(r, t) = r^{m/2}(\rho(r, t) - 1)$ for a.e. $(r, t) \in [a, \infty) \times [0, T]$. By (2.4.33) again, the weak star convergence (2.4.30) is obtained.

Next, fix $\varepsilon \in (0, 1]$, and a point $(r, t) \in [a, \infty) \times (0, T]$ such that $r > \tilde{r}(\varepsilon, t)$. Let $\delta(\varepsilon, r, t) > 0$ and $N(\varepsilon, r, t) \in \mathbb{N}$ be obtained in Proposition 2.4.1. For $0 < \delta \leq \delta(\varepsilon, r, t)$, one sets:

$$\phi_\delta^{(r, t)}(\zeta, s) := \frac{\mathbb{1}_{D_\delta(r, t)}(\zeta, s)}{|D_\delta(r, t)|} = \begin{cases} |D_\delta(r, t)|^{-1} & \text{if } (\zeta, s) \in D_\delta(r, t), \\ 0 & \text{if } (\zeta, s) \in [a, \infty) \times [0, T] \setminus D_\delta(r, t), \end{cases} \quad (2.4.34)$$

where $D_\delta(r, t)$ is defined in Proposition 2.4.1. It follows that $\phi_\delta^{(r, t)} \in L^1(0, T; L^2([a, \infty), dr))$. Moreover by Proposition 2.4.1, one has that for all $k \geq N(\varepsilon, r, t)$ and $\delta \leq \delta(\varepsilon, r, t)$,

$$\text{supp}(\phi_\delta^{(r, t)}) \subset \{(\zeta, s) \in [a, \infty) \times (0, T] : \zeta > \tilde{r}_k(\varepsilon, s)\}. \quad (2.4.35)$$

For $0 < \delta \leq \delta(\varepsilon, r, t)$ and $\eta > 0$, there exists $N_{\eta, \delta} \in \mathbb{N}$ by (2.4.33) such that for all $k \geq N_{\eta, \delta}$,

$$-\eta + \int_0^T \int_a^\infty \rho_k \phi_\delta^{(r, t)} d\zeta ds \leq \int_0^T \int_a^\infty \rho \phi_\delta^{(r, t)} d\zeta ds \leq \eta + \int_0^T \int_a^\infty \rho_k \phi_\delta^{(r, t)} d\zeta dt. \quad (2.4.36)$$

Let $N_1 := \max\{N_{\eta, \delta}, N(\varepsilon, r, t)\}$. Then by (2.4.35) and Lemma 2.4.2, it follows that for all $k \geq N_1$, $C(\varepsilon)^{-1} \leq \rho_k(\zeta, s) \leq C(\varepsilon)$ if $(\zeta, s) \in \text{supp}(\phi_\delta^{(r, t)})$. Using this in (2.4.36), one has

$$C(\varepsilon)^{-1} - \eta \leq \frac{1}{|D_\delta(r, t)|} \iint_{D_\delta(r, t)} \rho(\zeta, s) d\zeta ds \leq C(\varepsilon) + \eta \quad \text{for all } \eta > 0 \text{ and } 0 < \delta \leq \delta(\varepsilon, r, t).$$

by Lebesgue differentiation theorem over δ , (2.4.31)₁ is proved.

Finally, we consider the proof for (2.4.31)₂. For $(r, t) \in [a, \infty) \times [0, T]$ and $\delta > 0$, one uses $\phi_\delta^{(r, t)}$ defined in (2.4.34). Then (2.4.31)₂ can be proved by repeating the same argument as before, and using $C(a)^{-1} \leq \rho_k(r, t) \leq C(a)$ for all $(k, r, t) \in \mathbb{N} \times [a, \infty) \times [0, T]$ from Lemma 2.4.2. \square

Before proceeding into the next lemma, we define the following function spaces: for each connected interval $I \subseteq [0, \infty)$, we define the space $\tilde{H}_0^1(I, r^m dr)$ to be the closure of

$$\mathcal{D}_0(I) := \{\phi \in C^\infty(I) : \exists N > 0 \text{ s.t. } [0, N] \subset I \text{ and } \phi(r) = 0 \text{ for } r \in I \cap [N, \infty)\} \quad (2.4.37)$$

via the $H^1(I, r^m dr)$ -norm. We also denote $\tilde{H}^{-1}(I, r^m dr)$ as its dual space.

Lemma 2.4.6. *Let (ρ, u, e) be the limit function obtained in Lemma 2.4.4–2.4.5. Then one has:*

- (i) *for all $L \in \mathbb{N}$, $\rho \in C^0([0, T]; \tilde{H}^{-1}([a, L], r^m dr))$, and there exists a further sub-sequence such that for all $L \in \mathbb{N}$,*

$$\lim_{k \rightarrow \infty} \sup_{t \in [0, T]} \|\rho_k(\cdot, t) - \rho(\cdot, t)\|_{\tilde{H}^{-1}([a, L], r^m dr)} = 0.$$

- (ii) *for each $\varepsilon \in (0, 1]$, if one defines $\rho^{(\varepsilon)}(r, t) := \rho(r, t)\chi_\varepsilon(r, t)$ where $\chi_\varepsilon(r, t)$ is given by*

$$\chi_\varepsilon(r, t) := \begin{cases} 1 & \text{if } t \in [0, T] \text{ and } r \in [\tilde{r}(\varepsilon, t), \infty), \\ 0 & \text{otherwise,} \end{cases} \quad (2.4.38)$$

then for all $L \in \mathbb{N}$,

$$\begin{cases} \rho^{(\varepsilon)} \in C^0([0, T]; \tilde{H}^{-1}([0, L], r^m dr)), \\ \|\rho^{(\varepsilon)}(\cdot, t_1) - \rho^{(\varepsilon)}(\cdot, t_2)\|_{\tilde{H}^{-1}([0, L], r^m dr)} \leq C(\varepsilon)|t_1 - t_2| \text{ for } t_1, t_2 \in [0, T]. \end{cases}$$

Proof. Let $\phi \in \tilde{H}_0^1([a, \infty), r^m dr)$. Then by (2.4.3)–(2.4.7) and (2.2.1), one has

$$\begin{aligned} & \int_a^\infty \{\rho_k(r, t_2) - \rho_k(r, t_1)\} \phi(r) r^m dr = \int_{t_1}^{t_2} \int_a^\infty \partial_t \rho_k(r, t) \phi(r) r^m dr dt \\ &= \int_{t_1}^{t_2} \int_a^\infty \varphi_k \partial_t \tilde{v}_k^{-1}(r, t) \phi(r) r^m dr dt = \int_{t_1}^{t_2} \int_a^{\tilde{r}_k(k, t)} \varphi_k \partial_t \tilde{v}_k^{-1}(r, t) \phi(r) r^m dr dt \\ &= \int_{t_1}^{t_2} \int_0^k (\varphi_k \cdot \phi)(\tilde{r}_k(x, t), t) \{ -\tilde{v}_k^{-1} D_t \tilde{v}_k - \tilde{r}_k^m \tilde{u}_k D_x \tilde{v}_k^{-1} \}(x, t) dx dt. \end{aligned}$$

Applying integration by parts, continuity equation (2.2.16)₁, the boundary condition $\tilde{u}_k(0, t) = \tilde{u}_k(k, t)$, and Theorem 2.3.1(i)–(ii), it follows that

$$\begin{aligned} & \int_a^\infty \{\rho_k(r, t_2) - \rho_k(r, t_1)\} \phi(r) r^m dr = \int_{t_1}^{t_2} \int_0^k \tilde{u}_k(x, t) (\varphi'_k \cdot \phi + \phi' \cdot \varphi_k)(\tilde{r}_k(x, t), t) dx dt \\ & \leq C \int_{t_1}^{t_2} \left(\int_0^k \frac{|\tilde{u}_k|^2}{\tilde{v}_k} dx \right)^{1/2} \left(\int_0^k \tilde{v}_k(x, t) \{|\phi|^2 + |\phi'|^2\}(\tilde{r}_k(x, t)) dx \right)^{1/2} dt \\ & \leq C(a) \|\phi(\cdot)\|_{H^1([a, \infty), r^m dr)} |t_2 - t_1|. \end{aligned}$$

Thus the following equi-continuity inequality holds:

$$\sup_{k \in \mathbb{N}} \|\rho_k(\cdot, t_2) - \rho_k(\cdot, t_1)\|_{\tilde{H}^{-1}([a, \infty), r^m dr)} \leq C(a) |t_2 - t_1|.$$

By the same calculation, one can also verify that $\rho_k \in L^\infty(0, T; \tilde{H}^{-1}([a, L], r^m dr))$ for all $(k, L) \in \mathbb{N}^2$. In addition, from Lemma 2.4.2, it follows that

$$\sup_{k \in \mathbb{N}} \sup_{t \in [0, T]} \int_a^\infty |\rho_k - 1|^2(r, t) r^m dr \leq C(a).$$

Thus one can apply Proposition A.0.2 of Appendix A to obtain f such that for all $L \in \mathbb{N}$,

$$\begin{cases} f \in C^0([0, T]; \tilde{H}^{-1}([a, L], r^m dr)), \quad f - 1 \in L^\infty(0, T; L^2([a, \infty), r^m dr)), \\ \lim_{k \rightarrow \infty} \sup_{t \in [0, T]} \|\rho_k(\cdot, t) - f(\cdot, t)\|_{\tilde{H}^{-1}([a, L], r^m dr)} = 0, \\ \|f(\cdot, t_1) - f(\cdot, t_2)\|_{\tilde{H}^{-1}([a, L], r^m dr)} \leq C(a) |t_1 - t_2| \text{ for all } t_1, t_2 \in [0, T]. \end{cases}$$

By Lemma 2.4.5 and Fundamental Lemma of Calculus of Variation, one can verify that $\rho = f$ for a.e. $(r, t) \in [a, \infty) \times [0, T]$. This proves (i) of this lemma.

For each $\varepsilon > 0$ and $k \in \mathbb{N}$, one defines $\rho_k^{(\varepsilon)}(r, t) := \rho_k(r, t)\chi_\varepsilon^k(r, t)$ where

$$\chi_\varepsilon^k(r, t) := \begin{cases} 1 & \text{if } t \in [0, T] \text{ and } r \in [\tilde{r}_k(\varepsilon, t), \infty), \\ 0 & \text{otherwise.} \end{cases} \quad (2.4.39)$$

If $k \geq M(\varepsilon, a) := \varepsilon C(a)^2 + (2^n - 1)a^n C(a)/n > 0$, then it follows from the construction (2.4.5) that $\varphi_k(\tilde{r}_k(\varepsilon, t), t) = 1$ for all $t \in [0, T]$. Thus by (2.2.1) and (2.4.4),

$$(\rho_k, u_k, e_k)(\tilde{r}_k(\varepsilon, t), t) = (\bar{v}_k^{-1}, \bar{u}_k, \bar{e}_k)(\tilde{r}_k(\varepsilon, t), t) = (\tilde{v}_k^{-1}, \tilde{u}_k, \tilde{e}_k)(\varepsilon, t). \quad (2.4.40)$$

Let $\phi \in \tilde{H}_0^1([0, \infty), r^m dr)$. Then by (2.4.40), and Leibniz Rule, for all $k \geq M(\varepsilon, a)$,

$$\begin{aligned} & \int_0^\infty (\rho_k^{(\varepsilon)}(r, t_2) - \rho_k^{(\varepsilon)}(r, t_1)) \phi(r) r^m dr = \int_{t_1}^{t_2} \frac{d}{dt} \int_{\tilde{r}_k(\varepsilon, t)}^\infty \rho_k(r, t) \phi(r) r^m dr dt \\ & = \int_{t_1}^{t_2} \int_{\tilde{r}_k(\varepsilon, t)}^\infty \phi \partial_t \rho_k r^m dr dt - \int_{t_1}^{t_2} (\tilde{r}_k(\varepsilon, t))^m \phi(\tilde{r}_k(\varepsilon, t)) \tilde{u}_k(\varepsilon, t) \tilde{v}_k^{-1}(\varepsilon, t) dt =: I_1 + I_2. \end{aligned} \quad (2.4.41)$$

Using (2.2.1) and (2.4.3)–(2.4.4), I_1 can be written as:

$$\begin{aligned} I_1 & := \int_{t_1}^{t_2} \int_{\tilde{r}_k(\varepsilon, t)}^\infty \partial_t \rho_k(r, t) \phi(r) r^m dr dt = \int_{t_1}^{t_2} \int_{\tilde{r}_k(\varepsilon, t)}^{\tilde{r}_k(k, t)} \varphi_k \partial_t \tilde{v}_k^{-1}(r, t) \phi(r) r^m dr dt \\ & = \int_{t_1}^{t_2} \int_\varepsilon^k (\varphi_k \cdot \phi)(\tilde{r}_k(x, t), t) \{ \tilde{v}_k D_t \tilde{v}_k^{-1} - \tilde{r}_k^m \tilde{u}_k D_x \tilde{v}_k^{-1} \}(x, t) dx dt. \end{aligned}$$

Applying integration by parts, and using continuity equation (2.2.16)₁, the boundary condition $\tilde{u}_k(k, t) = 0$, and Theorem 2.3.1(i)–(ii), it follows that

$$\begin{aligned} I_1 & = \int_{t_1}^{t_2} \int_\varepsilon^k (\varphi_k \cdot \phi)(\tilde{r}_k(x, t), t) \{ -\tilde{v}_k^{-1} D_x (\tilde{r}_k^m \tilde{u}_k) - \tilde{r}_k^m \tilde{u}_k D_x \tilde{v}_k^{-1} \} dx dt \\ & = \int_{t_1}^{t_2} \phi(\tilde{r}_k(\varepsilon, t)) (\tilde{r}_k^m \tilde{u}_k \tilde{v}_k^{-1})(\varepsilon, t) dt + \int_{t_1}^{t_2} \int_\varepsilon^k \tilde{u}_k(x, t) (\varphi_k' \cdot \phi + \phi' \cdot \varphi_k)(\tilde{r}_k(x, t), t) dx dt \\ & \leq -I_2 + C \int_{t_1}^{t_2} \left(\int_\varepsilon^k \frac{|\tilde{u}_k|^2}{\tilde{v}_k} dx \right)^{1/2} \left(\int_\varepsilon^k \tilde{v}_k(x, t) \{ |\phi|^2 + |\phi'|^2 \}(\tilde{r}_k(x, t)) dx \right)^{1/2} dt \\ & \leq -I_2 + C_0^{1/2} C(\varepsilon)^{1/2} \int_{t_1}^{t_2} \left(\int_{\tilde{r}_k(\varepsilon, t)}^{\tilde{r}_k(k, t)} \{ |\phi|^2 + |\phi'|^2 \}(r) r^m dr \right)^{1/2} dt \\ & \leq -I_2 + C(\varepsilon) \|\phi\|_{H^1([0, \infty), r^m dr)} |t_2 - t_1|. \end{aligned}$$

Substituting this back into (2.4.41), it follows that the following equi-continuity inequality holds:

$$\sup_{k \in \mathbb{N}} \|\rho_k^{(\varepsilon)}(\cdot, t_2) - \rho_k^{(\varepsilon)}(\cdot, t_1)\|_{\tilde{H}^{-1}([0, \infty), r^m dr)} \leq C(\varepsilon) |t_2 - t_1|. \quad (2.4.42)$$

Let $L \in \mathbb{N}$ be such that $\tilde{r}_k(1, t) < L$ for all $(k, t) \in \mathbb{N} \times [0, T]$. By the same derivation, one can also show that $\rho_k^{(\varepsilon)} \in L^\infty(0, T; \tilde{H}^{-1}([0, L], r^m dr))$ for all $(k, L) \in \mathbb{N}$. Moreover, set $r_\varepsilon = \inf_{t \in [0, T]} \tilde{r}(\varepsilon, t)/2 > 0$. Then

by Lemma 2.4.3, there exists $N_\varepsilon \in \mathbb{N}$ such that $r_\varepsilon < \tilde{r}_k(\varepsilon, t)$ for all $k \geq N_\varepsilon$ and $t \in [0, T]$. From Lemma 2.4.2 and (2.4.42), it follows that

$$\begin{cases} \sup_{k \geq N_\varepsilon} \sup_{t \in [0, T]} \int_{r_\varepsilon}^{\infty} |\rho_k^{(\varepsilon)} - 1|^2(r, t) r^m dr \leq C(\varepsilon), \\ \sup_{k \geq N_\varepsilon} \|\rho_k^{(\varepsilon)}(\cdot, t_2) - \rho_k^{(\varepsilon)}(\cdot, t_1)\|_{\tilde{H}^{-1}([r_\varepsilon, L], r^m dr)} \leq C(\varepsilon) |t_2 - t_1|. \end{cases}$$

Thus one can apply Proposition A.0.2 of Appendix A to obtain f_ε such that for all $L \in \mathbb{N}$,

$$\begin{cases} f_\varepsilon \in C^0([0, T]; \tilde{H}^{-1}([r_\varepsilon, L], r^m dr)), \quad f_\varepsilon - 1 \in L^\infty(0, T; L^2([r_\varepsilon, \infty), r^m dr)), \\ \lim_{k \rightarrow \infty} \sup_{t \in [0, T]} \|\rho_k^{(\varepsilon)}(\cdot, t) - f_\varepsilon(\cdot, t)\|_{\tilde{H}^{-1}([r_\varepsilon, L], r^m dr)} = 0, \\ \|f_\varepsilon(\cdot, t_1) - f_\varepsilon(\cdot, t_2)\|_{\tilde{H}^{-1}([r_\varepsilon, L], r^m dr)} \leq C(\varepsilon) |t_1 - t_2| \quad \text{for all } t_1, t_2 \in [0, T]. \end{cases} \quad (2.4.43)$$

Now let $\phi \in C_c^\infty([a, \infty) \times [0, T])$. One has

$$\begin{aligned} & \int_0^T \int_{r_\varepsilon}^{\infty} \phi(\rho^{(\varepsilon)} - f_\varepsilon) dr dt = \int_0^T \int_{r_\varepsilon}^{\infty} \phi(\rho^{(\varepsilon)} - \rho_k^{(\varepsilon)}) + \int_0^T \int_{r_\varepsilon}^{\infty} \phi(\rho_k^{(\varepsilon)} - f_\varepsilon) \\ &= \int_0^T \int_{r_\varepsilon}^{\infty} \phi \chi_\varepsilon(\rho - \rho_k) + \int_0^T \int_{r_\varepsilon}^{\infty} \phi \rho_k(\chi_\varepsilon - \chi_\varepsilon^k) + \int_0^T \int_{r_\varepsilon}^{\infty} \phi(\rho_k^{(\varepsilon)} - f_\varepsilon) = \sum_{i=1}^3 \Pi_i^k. \end{aligned}$$

By Lemma 2.4.5, $\Pi_1^k \rightarrow 0$ as $k \rightarrow \infty$. By (2.4.38)–(2.4.39), and Lemma 2.4.3, $\chi_\varepsilon^k \rightarrow \chi_\varepsilon$ a.e. as $k \rightarrow \infty$. Since $\rho_k \leq C(a)$ from Lemma 2.4.2, by Dominated Convergence theorem, $\Pi_2^k \rightarrow 0$ as $k \rightarrow \infty$. Finally $\Pi_3^k \rightarrow 0$ as $k \rightarrow \infty$ by (2.4.43). Therefore, $\rho^{(\varepsilon)} = f_\varepsilon$ for a.e. $(r, t) \in [r_\varepsilon, \infty) \times [0, T]$. Since $\rho^{(\varepsilon)} = \rho_k^{(\varepsilon)} = 0$ for $r \in [0, r_\varepsilon]$ and $k \geq N_\varepsilon$, Lemma 2.4.6(ii) is proved. \square

Lemma 2.4.7. *The following relations hold:*

$$\begin{aligned} x &= \int_a^{\tilde{r}(x, t)} \rho(r, t) r^m dr && \text{for a.e. } t \in [0, T] \text{ and for each } x > 0, \\ \tilde{r}(x, t) &= \tilde{r}_a^0(x) + \int_0^t u(\tilde{r}(x, s), s) ds && \text{for all } (x, t) \in [0, \infty) \times [0, T], \end{aligned} \quad (2.4.44)$$

where the initial data $\tilde{r}_a^0(x)$ is the function constructed in Section 2.2.4, which satisfies:

$$x = \int_a^{\tilde{r}_a^0(x)} \rho_a^0(r) r^m dr, \quad \text{for all } x \in [0, \infty).$$

Proof. Let $(x, t) \in [0, \infty) \times [0, T]$. Then for $k \geq M(a, x) := C(a)(2^n - 1)a^n/n + 2^n C(a)^2 x$, one can verify that $2\tilde{r}_k(x, t) \leq \tilde{r}_k(k, t)$. Thus by (2.2.1), if $k \geq M(a, x)$ and $r \in [a, \tilde{r}_k(x, t)]$, then $\rho_k(r, t) = \varphi_k \bar{v}_k^{-1}(r, t) + (1 - \varphi_k(r, t)) = \bar{v}_k^{-1}(r, t)$. Using this, and (2.4.2)–(2.4.4), one has

$$\int_a^{\tilde{r}_k(x, t)} \rho_k(r, t) r^m dr = \int_a^{\tilde{r}_k(x, t)} \bar{v}_k^{-1}(r, t) r^m dr = \int_0^x \tilde{v}_k^{-1}(y, t) \tilde{v}_k(y, t) dy = x.$$

Using the above identity, one has

$$x - \int_a^{\tilde{r}(x, t)} \rho(r, t) r^m dr = \int_{\tilde{r}(x, t)}^{\tilde{r}_k(x, t)} \rho_k(r, t) r^m dr + \int_a^{\tilde{r}(x, t)} (\rho_k - \rho)(r, t) r^m dr.$$

Let $\chi(t) \in C_c^\infty([0, T])$. Multiplying it to the above and integrating in time, it follows that:

$$\begin{aligned} & \int_0^T \left(x - \int_a^{\tilde{r}(x,t)} \rho(r,t) r^m dr \right) \chi(t) dt \\ &= \int_0^T \int_{\tilde{r}(x,t)}^{\tilde{r}_k(x,t)} \rho_k(r,t) \chi(t) r^m dr dt + \int_0^T \int_a^{\tilde{r}(x,t)} (\rho_k - \rho)(r,t) \chi(t) r^m dr dt = I_k^{(1)} + I_k^{(2)}. \end{aligned}$$

$I_k^{(1)}$ is estimated using Lemma 2.4.3 and the bounds $\rho_k \leq C(a)$ from Lemma 2.4.2.

$$\begin{aligned} |I_k^{(1)}| &:= \left| \int_0^T \int_{\tilde{r}(x,t)}^{\tilde{r}_k(x,t)} \rho_k(r,t) \chi(t) r^m dr dt \right| \leq C(a) \|\chi(\cdot)\|_{L^\infty} \left| \int_0^T \int_{\tilde{r}(x,t)}^{\tilde{r}_k(x,t)} r^m dr dt \right| \\ &\leq \frac{C(a)}{n} T \|\chi(\cdot)\|_{L^\infty} \sup_{0 \leq t \leq T} |\tilde{r}_k^n(x,t) - \tilde{r}^n(x,t)| \rightarrow 0 \text{ as } k \rightarrow \infty. \end{aligned}$$

For $I_k^{(2)}$, one first observes that $\chi(t) \cdot r^m \mathbb{1}_{[a, \tilde{r}(x,t)]}(r) \in L^1(0, T; L^2([a, \infty), dr))$, where $\mathbb{1}_E(r)$ denotes the indicator function for spatial set $E \subset [a, \infty)$. By Lemma 2.4.5, it follows that

$$\lim_{k \rightarrow \infty} I_k^{(2)} = \lim_{k \rightarrow \infty} \int_0^T \int_a^\infty (\rho_k - \rho)(r,t) \chi(t) \mathbb{1}_{[a, \tilde{r}(x,t)]}(r) r^m dr dt = 0.$$

Since $I_k^{(1)}, I_k^{(2)} \rightarrow 0$ as $k \rightarrow \infty$, one concludes that

$$\int_0^T \left(x - \int_a^{\tilde{r}(x,t)} \rho(r,t) r^m dr \right) \chi(t) dt = 0 \text{ for each } \chi(t) \in C_c^\infty([0, T]).$$

Then, (2.4.44)₁ is shown by Fundamental Lemma of Calculus of Variation.

Next, by (2.2.16)₁ and (2.2.16)₄, $\tilde{r}_k(x, t)$ satisfies:

$$\tilde{r}_k(x, t) = \tilde{r}_k(x, 0) + \int_0^t \tilde{u}_k(x, s) ds, \quad \text{for } (x, t) \in [0, k] \times [0, T],$$

where from (2.2.16)₄ and Section 2.2.4, $\tilde{r}_k(x, 0)$ is given by

$$\tilde{r}_k(x, 0) = \left(a^n + n \int_0^x \tilde{v}_{a,k}^0(y) dy \right)^{1/n} = \left(a^n + n \int_0^x \frac{1}{\rho_{a,k}^0(\tilde{r}_a^0(y))} dy \right)^{1/n}.$$

Fix $(x, t) \in [0, \infty) \times [0, T]$. If $k \geq M(a, x)$ then by (2.2.1), one has $u_k(\tilde{r}_k(x, t), t) = \tilde{u}_k(x, t)$,

$$\tilde{r}_k(x, t) = \tilde{r}_k(x, 0) + \int_0^t u_k(\tilde{r}_k(x, s), s) ds, \quad \text{for } (x, t) \in [0, k] \times [0, T]. \quad (2.4.45)$$

Moreover, by Proposition 2.2.2 in Section 2.2.4, if $k \geq N(x) := C_0(2^n - 1)a^n/n + 2^n C_0^2 x$, then

$$\begin{aligned} \frac{1}{2} \tilde{r}_a^0(k) &= \frac{1}{2} \left(a^n + n \int_0^k \frac{1}{\rho_{a,k}^0(\tilde{r}_a^0(y))} dy \right)^{1/n} \geq \frac{1}{2} \left(a^n + n C_0^{-1} k \right)^{1/n} \\ &\geq (a^n + n C_0 x)^{1/n} \geq \left(a^n + n \int_0^x \frac{1}{\rho_a^0(\tilde{r}_a^0(y))} dy \right)^{1/n} = \tilde{r}_a^0(x). \end{aligned}$$

By the construction of $\rho_k^0(r)$ in (2.2.13) of Section 2.2.4, it follows that if $k \geq N(x)$ then

$$\tilde{v}_{a,k}^0(x) = \frac{1}{\rho_{a,k}^0(\tilde{r}_a^0(x))} = \left(\rho_a^0(\tilde{r}_a^0(x)) \varphi_{a,k}^0(\tilde{r}_a^0(x)) + 1 - \varphi_{a,k}^0(\tilde{r}_a^0(x)) \right)^{-1} = \frac{1}{\rho_a^0(\tilde{r}_a^0(x))}.$$

Thus, by Proposition 2.2.2, one has for all $k \geq N(x)$,

$$\tilde{r}_k(x, 0) = \left(a^n + n \int_0^x \frac{1}{\rho_{a,k}^0(\tilde{r}_a^0(y))} dy \right)^{1/n} = \left(a^n + n \int_0^x \frac{1}{\rho_a^0(\tilde{r}_a^0(y))} dy \right)^{1/n} = \tilde{r}_a^0(x).$$

This shows that for each $x \in [0, \infty)$, $\tilde{r}_k(x, 0) \rightarrow \tilde{r}_a^0(x)$ as $k \rightarrow \infty$. Moreover, by Lemma 2.4.3 and the inequality (2.4.27) obtained in Lemma 2.4.4, it follows that for all $(x, s) \in [0, \infty) \times [0, T]$,

$$|u_k(\tilde{r}_k(x, s), s) - u_k(\tilde{r}(x, s), s)| \leq C(a) |\tilde{r}_k(x, s) - \tilde{r}(x, s)| \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

By Lemma 2.4.4, $|u_k(\tilde{r}(x, s), s) - u(\tilde{r}(x, s), s)| \rightarrow 0$ for all $(x, s) \in [0, \infty) \times [0, T]$ as $k \rightarrow \infty$. Combining these limits in (2.4.45), (2.4.44)₂ is shown by Dominated Convergence Theorem. \square

Lemma 2.4.8. *There exists a sub-sequence $(\rho_k, u_k, e_k)_{k \in \mathbb{N}}$ such that, as $k \rightarrow \infty$*

$$\begin{cases} (\sqrt{\sigma} \partial_r u_k, \sigma \partial_r e_k) \xrightarrow{*} (\sqrt{\sigma} \partial_r u, \sigma \partial_r e) & \text{in } L^\infty(0, T; L^2([a, \infty), r^m dr)), \\ (\partial_r u_k, \partial_r e_k, \sqrt{\sigma} \partial_t u_k, \sigma \partial_t e_k) \rightharpoonup (\partial_r u, \partial_r e, \sqrt{\sigma} \partial_t u, \sigma \partial_t e) & \text{in } L^2(0, T; L^2([a, \infty), r^m dr)), \end{cases}$$

where $\sigma = \min\{1, t\}$. In addition, one has

$$\begin{aligned} \sup_{t \in [0, T]} \int_a^\infty \{(\rho - 1)^2 + u^4 + (e - 1)^2 + \sigma |\partial_r u|^2 + \sigma^2 |\partial_r e|^2\}(r, t) r^m dr &\leq C(a), \\ \int_0^T \int_a^\infty \{|\partial_r u|^2 + |u \partial_r u|^2 + |\partial_r e|^2 + \sigma |\partial_t u|^2 + \sigma^2 |\partial_t e|^2\}(r, t) r^m dr dt &\leq C(a). \end{aligned} \quad (2.4.46)$$

Furthermore, for each $\varepsilon > 0$, one has

$$\begin{aligned} \sup_{0 \leq t \leq T} \int_{\tilde{r}(\varepsilon, t)}^\infty \{(\rho - 1)^2 + u^4 + (e - 1)^2 + \sigma |\partial_r u|^2 + \sigma^2 |\partial_r e|^2\} r^m dr &\leq C(\varepsilon), \\ \int_0^T \int_{\tilde{r}(\varepsilon, t)}^\infty \{|\partial_r u|^2 + |u \partial_r u|^2 + |\partial_r e|^2 + \sigma |\partial_t u|^2 + \sigma^2 |\partial_t e|^2\}(r, t) r^m dr dt &\leq C(\varepsilon). \end{aligned} \quad (2.4.47)$$

Proof. Only the proof for $\sqrt{\sigma} \partial_r u_k \xrightarrow{*} \sqrt{\sigma} \partial_r u$ is presented, as the other limits can be shown using the same argument. By Lemma 2.4.2, one has

$$\sup_{k \in \mathbb{N}} \sup_{t \in [0, T]} \int_a^\infty \sigma |\partial_r u_k|^2(r, t) dr \leq \frac{1}{a^m} \sup_{k \in \mathbb{N}} \sup_{t \in [0, T]} \int_a^\infty \sigma |\partial_r u_k|^2(r, t) r^m dr \leq \frac{C(a)}{a^m}.$$

Thus, there exists a sub-sequence, still denoted as $\{\partial_r u_k\}_{k \in \mathbb{N}}$, such that

$$\sqrt{\sigma} \partial_r u_k \xrightarrow{*} w_1 \quad \text{and} \quad r^{m/2} \sqrt{\sigma} \partial_r u_k \xrightarrow{*} w_2 \quad \text{in } L^\infty(0, T; L^2([a, \infty), dr)). \quad (2.4.48)$$

By (2.4.48), Lemma 2.4.4, and Dominated Convergence Theorem, one can show that the spatial weak derivative of $\sqrt{\sigma} u$ exists and $w_1 = \sqrt{\sigma} \partial_r u$. As a result, by (2.4.48) and Fundamental Lemma of Calculus of Variation, $w_2 = r^{m/2} \sqrt{\sigma} \partial_r u$ for a.e. $(r, t) \in [a, \infty) \times [0, T]$. Then (2.4.48) implies that $\sqrt{\sigma} \partial_r u_k \xrightarrow{*} \sqrt{\sigma} \partial_r u$ in $L^\infty(0, T; L^2([a, \infty), r^m dr))$. By weakly-star lower semi-continuity of the Sobolev norm, one obtains

$$\|\sqrt{\sigma} \partial_r u\|_{L^\infty(0, T; L^2([a, \infty), r^m dr))} \leq \liminf_{k \rightarrow \infty} \|\sqrt{\sigma} \partial_r u_k\|_{L^\infty(0, T; L^2([a, \infty), r^m dr))} \leq C(a).$$

The other terms in (2.4.46) can be proved by the same argument.

Next, we show (2.4.47). Only the proof for $\sqrt{\sigma}\partial_r u$ is proved, since other terms in (2.4.47) can be obtained in the same way. Let $\phi \in L^1(0, T; L^2([a, \infty), r^m dr))$. Then by Lemma 2.4.2,

$$\begin{aligned} & \left| \int_0^T \int_{\tilde{r}_k(\varepsilon, t)}^\infty \sqrt{\sigma} \partial_r u_k \phi r^m dr dt - \int_0^T \int_{\tilde{r}(\varepsilon, t)}^\infty \sqrt{\sigma} \partial_r u \phi r^m dr dt \right| \\ & \leq \left| \int_0^T \int_{\tilde{r}_k(\varepsilon, t)}^{\tilde{r}(\varepsilon, t)} \sqrt{\sigma} \partial_r u_k \phi r^m dr \right| + \left| \int_0^T \int_{\tilde{r}(\varepsilon, t)}^\infty \sqrt{\sigma} (\partial_r u_k - \partial_r u) \phi r^m dr \right| \\ & \leq C(a) \int_0^T \left(\int_{\tilde{r}_k(\varepsilon, t)}^{\tilde{r}(\varepsilon, t)} |\phi|^2 r^m dr \right)^{1/2} dt + \left| \int_0^T \int_{\tilde{r}(\varepsilon, t)}^\infty \sqrt{\sigma} (\partial_r u_k - \partial_r u) \phi r^m \right| = I_k^{(1)} + I_k^{(2)}. \end{aligned}$$

$I_k^{(2)} \rightarrow 0$ as $k \rightarrow \infty$ by (2.4.48). For $I_k^{(1)}$, one defines

$$\chi^{(k, \varepsilon)}(r, t) := \begin{cases} 1 & \text{if } r \in [\tilde{r}_k(\varepsilon, t), \tilde{r}(\varepsilon, t)] \text{ or } r \in [\tilde{r}(\varepsilon, t), \tilde{r}_k(\varepsilon, t)], \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 2.4.3, for each fixed $\varepsilon > 0$, $\chi^{(k, \varepsilon)}(r, t) \rightarrow 0$ as $k \rightarrow \infty$ for a.e. $(r, t) \in [a, \infty) \times [0, T]$. Moreover, since $|\chi^{(k, \varepsilon)} \phi| \leq |\phi|$, by Dominated Convergence theorem, as $k \rightarrow \infty$,

$$I_k^{(1)} := C(a) \int_0^T \left(\int_{\tilde{r}_k(\varepsilon, t)}^{\tilde{r}(\varepsilon, t)} |\phi|^2 r^m dr \right)^{1/2} dt = C(a) \int_0^T \left(\int_a^\infty \chi^{(\varepsilon, k)} |\phi|^2 r^m dr \right)^{1/2} dt \rightarrow 0.$$

Let $\chi_\varepsilon, \chi_\varepsilon^k$ be defined in (2.4.38)–(2.4.39). Then the limits $I_k^{(1)}, I_k^{(2)} \rightarrow 0$ imply the weak-star convergence $\chi_\varepsilon^k \sqrt{\sigma} \partial_r u_k \xrightarrow{*} \chi_\varepsilon \sqrt{\sigma} \partial_r u$ as $k \rightarrow \infty$ in $L^\infty(0, T; L^2([a, \infty), r^m dr))$. By the weakly-star lower semi-continuity of the Sobolev norm, it follows that

$$\begin{aligned} & \sup_{t \in [0, T]} \int_{\tilde{r}(\varepsilon, t)}^\infty \sigma |\partial_r u|^2(r, t) r^m dr = \|\chi_\varepsilon \sqrt{\sigma} \partial_r u\|_{L^\infty(0, T; L^2([a, \infty), r^m dr))} \\ & \leq \liminf_{k \rightarrow \infty} \|\chi_\varepsilon^k \sqrt{\sigma} \partial_r u_k\|_{L^\infty(0, T; L^2([a, \infty), r^m dr))} = \sup_{k \in \mathbb{N}} \sup_{t \in [0, T]} \int_{\tilde{r}_k(\varepsilon, t)}^\infty \sigma |\partial_r u_k|^2(r, t) r^m dr \leq C(\varepsilon). \end{aligned}$$

□

In following Sections 2.4.5–2.4.7, we will denote $(\rho, u, e)(r, t)$ in $[a, \infty) \times [0, T]$ as the limit functions obtained in Lemma 2.4.4–2.4.5, and we will also keep using the simplified notation (2.4.15) as previous Section.

2.4.5 Entropy estimate for the limit solution

Lemma 2.4.9. *The following estimates hold:*

$$\begin{aligned} & \operatorname{ess\,sup}_{t \in [0, T]} \int_a^\infty \left\{ G(\rho) + \rho \psi(e) + \frac{\rho |u|^2}{2} \right\}(r, t) r^m dr + \kappa \int_0^T \int_a^\infty \frac{|\partial_r e|^2}{e^2} r^m dr dt \\ & + \int_0^T \int_a^\infty \left\{ \left(\frac{2\mu}{n} + \lambda \right) \frac{1}{e} \left| \partial_r u + m \frac{u}{r} \right|^2 + \frac{2m\mu}{n} \frac{1}{e} \left| \partial_r u - \frac{u}{r} \right|^2 \right\} r^m dr dt \leq C(T). \end{aligned} \tag{2.4.49}$$

Proof. For each integer $L \geq 2$, let $\phi_L \in C^\infty([a, \infty))$ be such that $\phi_L(r) = 1$ if $r \in [a, L - 1]$ and $\phi_L(r) = 0$ if $r \in [L, \infty)$. Then $\phi_L \in \tilde{H}_0^1([a, L], r^m dr)$, where \tilde{H}_0^1 is the Sobolev space defined in (2.4.37). Moreover,

$\partial_r \psi(e_k) = (e_k - 1) \partial_r e_k / e_k$. Thus, for each fixed $t \in (0, T]$, $\phi_L \cdot \psi(e_k)(\cdot, t) \in \tilde{H}_0^1([a, L], r^m dr)$ for all $(k, t) \in \mathbb{N} \times (0, T]$, and one has

$$\begin{aligned} & \int_a^\infty \rho_k \phi_L \psi(e_k) r^m dr - \int_a^\infty \rho \phi_L \psi(e) r^m dr \\ & \leq \int_a^L (\rho_k - \rho) \phi_L \psi(e_k) r^m dr + \int_a^\infty \rho \phi_L (\psi(e_k) - \psi(e)) r^m dr =: I_k^{(1)} + I_k^{(2)}. \end{aligned}$$

For all $t \in (0, T]$, $I_k^{(1)}$ is estimated using Lemmas 2.4.2, 2.4.6 as:

$$\begin{aligned} I_k^{(1)} & := \int_a^L (\rho_k - \rho) \phi_L \psi(e_k) r^m dr \leq \|\rho_k - \rho\|_{\tilde{H}^{-1}([a, L], r^m dr)} \|\phi_L \psi(e_k)\|_{H^1([a, L], r^m dr)} \\ & = \|\rho_k - \rho\|_{\tilde{H}^{-1}} \left\{ \|\phi_L\|_{L^2} \|\psi(e_k)\|_{L^\infty} + \|\phi_L'\|_{L^2} \|\psi(e_k)\|_{L^\infty} + \left\| \frac{e_k - 1}{e_k} \phi_L \right\|_\infty \cdot \frac{1}{\sigma} \|\sigma \partial_r e_k\|_{L^2} \right\} \\ & \leq \|\rho_k - \rho\|_{\tilde{H}^{-1}} \{C(a) \|\phi_L(\cdot)\|_{H^1} + C(a) \sigma(t)^{-1}\} \rightarrow 0 \text{ as } k \rightarrow \infty. \end{aligned}$$

From Lemma 2.4.4, $e_k(r, t) \rightarrow e(r, t)$ for $(r, t) \in [a, \infty) \times [0, T]$ as $k \rightarrow \infty$, and from Lemmas 2.4.4–2.4.5, $C(a)^{-1} \leq \rho(r, t) \leq C(a)$, $C(a)^{-1} \leq e_k(r, t)$, $e(r, t) \leq C(a) \sigma(t)^{-1/2}$ for a.e. $(r, t) \in [a, \infty) \times (0, T]$. Using these and Dominated Convergence theorem, it follows that

$$\lim_{k \rightarrow \infty} I_k^{(2)} := \lim_{k \rightarrow \infty} \int_a^\infty \rho \phi_L (\psi(e_k) - \psi(e))(r, t) r^m dr = 0 \text{ for each } t \in (0, T].$$

Using Lemma 2.4.1, the limits $I_k^{(1)}, I_k^{(2)} \rightarrow 0$ then implies that for each $t > 0$,

$$\int_a^\infty \phi_L \rho \psi(e)(r, t) r^m dr = \lim_{k \rightarrow \infty} \int_a^\infty \phi_L \rho_k \psi(e_k)(r, t) r^m dr \leq C(T).$$

Now by construction one has $\phi_L(r) \rightarrow 1$ for all $r \in [a, \infty)$ as $L \rightarrow \infty$. Since $\rho \psi(e) > 0$, one can apply Monotone Convergence theorem to obtain that

$$\int_a^\infty \rho \psi(e)(r, t) r^m dr = \lim_{L \rightarrow \infty} \int_a^\infty \phi_L \rho \psi(e)(r, t) r^m dr \leq C(T), \text{ for each } t \in (0, T].$$

Next, from Lemma 2.4.2, one has $u_k^2 \in L^2([a, \infty), r^m dr)$, and for all $(k, t) \in \mathbb{N} \times (0, T]$,

$$\|\partial_r u_k^2(\cdot, t)\|_{L^2([a, \infty), r^m dr)} = \|u_k\|_{L^\infty} \|\partial_r u_k(\cdot, t)\|_{L^2([a, \infty), r^m dr)} \leq \sigma(t)^{-3/4} C(a).$$

From these one has that $\phi_L \cdot u_k \in \tilde{H}_0^1([a, L], r^m dr)$, and

$$\begin{aligned} & \left| \int_a^\infty \phi_L \rho_k u_k^2 r^m dr - \int_a^\infty \phi_L \rho u^2 r^m dr \right| = \left| \int_a^\infty \{ \phi_L (\rho_k - \rho) u_k^2 + \phi_L \rho (u_k^2 - u^2) \} r^m dr \right| \\ & \leq \|\rho_k - \rho\|_{\tilde{H}^{-1}([a, L], r^m dr)} \|\phi_L \cdot u_k^2\|_{H^1([a, L], r^m dr)} + \int_a^\infty \phi_L \rho (u_k^2 - u^2) r^m dr \\ & \leq C(a) (1 + \sigma(t)^{-3/4}) \|\rho_k - \rho\|_{\tilde{H}^{-1}([a, L], r^m dr)} + \int_a^\infty \phi_L \rho (u_k^2 - u^2) r^m dr. \end{aligned}$$

Since $u_k(r, t) \rightarrow u(r, t)$ from Lemma 2.4.4, $|u_k(r, t)|, |u(r, t)| \leq C(a) \sigma(t)^{-1/4}$ from Lemmas 2.4.2, 2.4.4, and $\rho_k \rightarrow \rho$ strongly in \tilde{H}^{-1} from Lemma 2.4.6, by Dominated Convergence theorem,

$$\int_a^\infty \phi_L \rho u^2 r^m dr = \lim_{k \rightarrow \infty} \int_a^\infty \phi_L \rho_k u_k^2 r^m dr \leq \limsup_{k \rightarrow \infty} \int_a^\infty \rho_k u_k^2 r^m dr \leq C(T).$$

Then $\|\rho u^2(\cdot, t)\|_{L^1([a, \infty), r^m dr)} \leq C(T)$ follows by Monotone Convergence theorem over $L \rightarrow \infty$.

Next, let $\phi \in L^2(0, T; L^2([a, \infty), r^m dr))$. By Lemma 2.4.2, one has

$$\begin{aligned} & \left| \int_0^T \int_a^\infty \frac{\partial_r e_k}{e_k} \phi r^m dr dt - \int_0^T \int_a^\infty \frac{\partial_r e}{e} \phi r^m dr dt \right| \\ & \leq C(a) \left(\int_0^T \int_a^\infty (e_k^{-1} - e^{-1})^2 |\phi|^2 r^m \right)^{1/2} + \left| \int_0^T \int_a^\infty (\partial_r e_k - \partial_r e) \frac{\phi}{e} r^m \right| =: \Pi_k^{(1)} + \Pi_k^{(2)}. \end{aligned}$$

Using $e_k(r, t) \rightarrow e(r, t)$ from Lemma 2.4.4, and bounds $C(a) \leq e_k(r, t), e(r, t) \leq C(a)\sigma(t)^{-1/2}$ from Lemmas 2.4.2, 2.4.4, one can use Dominated Convergence theorem to get:

$$\lim_{k \rightarrow \infty} \Pi_k^{(1)} := \lim_{k \rightarrow \infty} C(a) \left(\int_0^T \int_a^\infty (e_k^{-1} - e^{-1})^2 |\phi|^2 r^m dr dt \right)^{1/2} = 0.$$

For $\Pi_k^{(2)}$, since $e(r, t) \geq C(a)^{-1}$, one can use $\partial_r e_k \rightarrow \partial_r e$ in Lemma 2.4.8 to get:

$$\lim_{k \rightarrow \infty} \Pi_k^{(2)} := \lim_{k \rightarrow \infty} \left| \int_0^T \int_a^\infty (\partial_r e_k - \partial_r e) \frac{\phi}{e}(r, t) r^m dr dt \right| = 0.$$

The above limits implies that $\partial_r e_k / e_k \rightarrow \partial_r e / e$ as $k \rightarrow \infty$ weakly in $L^2(0, T; L^2([a, \infty), r^m dr))$. By the weak-star lower semi-continuity of the Sobolev norm, and Lemma 2.4.1, it follows that

$$\int_0^T \int_a^\infty \frac{|\partial_r e|^2}{e^2} r^m dr dt \leq \liminf_{k \rightarrow \infty} \int_0^T \int_a^\infty \frac{|\partial_r e_k|^2}{e_k^2} r^m dr dt \leq C(T).$$

By the same argument as above, one can also obtain estimates for $\|e^{-1}(\partial_r u + mu/r)\|_{L^2}$ and $\|e^{-1}(\partial_r u - u/r)\|_{L^2}$ in (2.4.49).

Finally, from Lemma 2.4.5 and Lemma 2.4.1–2.4.2, one has

$$\begin{cases} \rho_k \rightharpoonup \rho \text{ weakly in } L^2(0, T; L^2([a, \infty), r^m dr)) \text{ as } k \rightarrow \infty. \\ \sup_{k \in \mathbb{N}} \operatorname{ess\,sup}_{t \in [0, T]} \int_a^\infty G(\rho_k)(r, t) r^m dr \leq C(T), \\ C(a)^{-1} \leq \rho_k(r, t) \leq C(a) \text{ for a.e. } (r, t) \in [a, \infty) \times [0, T] \text{ and for each } k \in \mathbb{N}. \end{cases} \quad (2.4.50)$$

By Proposition B.0.2 of Appendix B, $\|G(\rho)(\cdot, t)\|_{L^1([a, \infty), r^m dr)} \leq C(T)$ for a.e. $t \in [0, T]$. \square

2.4.6 Weak Forms of the Exterior Problem

Lemma 2.4.10. *Let $(\rho, u, e)(r, t)$ be the limit function obtained in Lemmas 2.4.4–2.4.5. Then $(\rho, u, e)(r, t)$ is a weak solution to the problem (1.2.1) and (2.2.9) as stated in Definition 2.2.1.*

Proof. We start with the continuity equation, take $\phi \in \mathcal{D}^a$, then there exists an integer $N \in \mathbb{N}$ such that $\phi(r, t) = 0$ for $r \geq N$ and $t \in [0, T]$. If we take $k \in \mathbb{N}$ to be $k \geq M(\phi, a) := 2^n N^n \max\{C_0, C(a)\}$, then for all $t \in [0, T]$:

$$\begin{aligned} \tilde{r}_k(k, t) &= \left(a^n + \int_0^k \tilde{v}_k(x, t) dx \right)^{1/n} \geq (a^n + C(a)^{-1}k)^{1/n} \geq (a^n + 2^n N^n)^{1/n} > 2N, \\ \text{and } \tilde{r}_a^0(k) &= \left(a^n + \int_0^k \frac{1}{\rho_a^0(\tilde{r}_a^0(x))} dx \right)^{1/n} \geq (a^n + C_0^{-1}k)^{1/n} \geq (a^n + 2^n N^n)^{1/n} > 2N. \end{aligned}$$

Thus by construction, it follows that if $(r, t) \in \text{supp}(\phi)$ we have

$$\begin{aligned}\rho_k(r, t) &= \bar{v}_k^{-1} \varphi_k(r, t) + 1 - \varphi_k(r, t) = \bar{v}_k^{-1}(r, t), \quad u_k(r, t) = \bar{u}_k \varphi_k(r, t) = \bar{u}_k(r, t), \\ e_k(r, t) &= \bar{e}_k \varphi_k(r, t) + 1 - \varphi_k(r, t) = \bar{e}_k(r, t).\end{aligned}\tag{2.4.51}$$

Moreover, using that for $k \geq M(\phi, a) := 2^n N^n \max\{C_0, C(a)\}$, we have $\tilde{r}_a^0(k)/2 > N$, it follows from construction (2.2.13) in Section 2.2.4 that if $r \in \text{supp}(\phi(\cdot, 0))$ then,

$$\begin{aligned}\rho_{a,k}^0(r) &= \rho_a^0(r) \varphi_{a,k}^0(r) + 1 - \varphi_{a,k}^0(r) = \rho_a^0(r), \quad u_{a,k}^0(r) = u_a^0(r) \varphi_{a,k}^0(r) = u_a^0(r), \\ e_{a,k}^0(r) &= e_a^0(r) \varphi_{a,k}^0(r) + 1 - \varphi_{a,k}^0(r) = e_a^0(r).\end{aligned}\tag{2.4.52}$$

With this, we also have that if $x \in [0, \infty)$ is such that $\tilde{r}_a^0(x) \in \text{supp}(\phi(\cdot, 0))$, then by Proposition 2.2.2 from Section 2.2.4 we have that for all $k \geq M(a)$:

$$\tilde{r}_a^0(x) = \left(a^n + n \int_0^x \frac{1}{\rho_a^0(\tilde{r}_a^0(y))} dy \right)^{1/n} = \left(a^n + n \int_0^x \tilde{v}_{a,k}^0(y) dy \right)^{1/n} = \tilde{r}_k(x, 0).\tag{2.4.53}$$

Thus from the coordinate transformation (2.4.3) and the continuity equation for which the approximate Lagrangian functions satisfies, it follows that for all $k \geq M(\phi, a)$,

$$\begin{aligned}\int_a^\infty \left\{ \rho_k(r, t) \phi(r, t) - \rho_a^0(r) \phi(r, 0) \right\} r^m dr &= \int_a^{\tilde{r}_k(k, t)} \bar{v}_k^{-1} \phi(r, t) r^m dr - \int_a^{\tilde{r}_a^0(k)} \phi(r, 0) \rho_a^0(r) r^m dr \\ &= \int_0^k \phi(\tilde{r}_k(x, t), t) dx - \int_0^k \phi(\tilde{r}_k(x, 0), 0) dx = \int_0^t \int_0^k \{ \tilde{u}_k(\partial_r \phi) + (\partial_t \phi) \}(\tilde{r}_k(x, s), s) dx ds \\ &= \int_0^t \int_a^{\tilde{r}_k(k, t)} \{ \bar{v}_k^{-1} \bar{u}_k \partial_r \phi + \tilde{v}_k^{-1} \partial_t \phi \}(r, t) r^m dr ds = \int_0^t \int_a^\infty \{ \rho_k u_k \partial_r \phi + \rho_k \partial_t \phi \}(r, t) r^m dr ds,\end{aligned}$$

where in the last line, we used $\tilde{r}_k(k, t) \geq 2N$ and (2.4.51). Also, by Lemma 2.4.4–2.4.6, we have:

$$\begin{aligned}\lim_{k \rightarrow \infty} \sup_{t \in [0, T]} \|\rho_k(\cdot, t) - \rho(\cdot, t)\|_{\tilde{H}^{-1}([a, L], r^m dr)} &= 0 \quad \text{for all } L \in \mathbb{N}, \\ \rho_k - 1 &\xrightarrow{*} \rho - 1 \quad \text{in } L^\infty(0, T; L^2([a, \infty), r^m dr)), \\ u_k(r, t) &\rightarrow u(r, t) \quad \text{and } \rho_k(r, t), \rho(r, t) \leq C(a) \quad \text{for a.e. } (r, t) \in [a, \infty) \times [0, T].\end{aligned}\tag{2.4.54}$$

Using these, we can apply Dominated Convergence theorem to obtain that

$$\int_a^\infty \rho(r, t) \phi(r, t) r^m dr - \int_a^\infty \rho_a^0(r) \phi(r, 0) r^m dr = \int_0^t \int_a^\infty \{ \rho u \partial_r \phi + \rho \partial_t \phi \} r^m dr ds.$$

Next, we show the weak form for momentum equation. Let $\varphi \in \mathcal{D}_0^a$. Then there exists $N \in \mathbb{N}$ such that $\varphi(r, t) = 0$ for $r \geq N$ and $t \in [0, T]$. If $k \in \mathbb{N}$ is such that $k \geq M(\varphi, a)$, then as previously shown, we have $\tilde{r}_k(k, t) > 2N$ for all $t \in [0, T]$, and $\tilde{r}_a^0(k) > 2N$. By construction, if $(r, t) \in \text{supp}(\varphi)$ and $r \in \text{supp}(\varphi(\cdot, 0))$ then (2.4.51)–(2.4.52) hold, hence

$$\begin{aligned}\int_a^\infty \rho_k u_k(r, t) \varphi(r, t) r^m dr - \int_a^\infty \rho_a^0 u_a^0(r) \varphi(r, 0) r^m dr \\ &= \int_a^{\tilde{r}_k(k, t)} \bar{v}_k^{-1} \bar{u}_k(r, t) \varphi(r, t) r^m dr - \int_a^{\tilde{r}_a^0(k)} u_{a,k}^0(r) \varphi(r, 0) \rho_a^0(r) r^m dr \\ &= \int_0^k \tilde{u}_k(x, t) \varphi(\tilde{r}_k(x, t), t) dx - \int_0^k \tilde{u}_{a,k}^0(x) \varphi(\tilde{r}_k(x, 0), 0) dx \\ &= \int_0^t \int_0^k \{ \varphi(\tilde{r}_k(x, s), s) D_t \tilde{u}_k + (\partial_t \varphi)(\tilde{r}_k(x, s), s) \tilde{u}_k + (\partial_r \varphi)(\tilde{r}_k(x, s), s) \tilde{u}_k^2 \} dx ds.\end{aligned}$$

By (2.2.16)₂ and the boundary condition $\varphi(a, t) = 0$, $\tilde{u}_k(k, t) = 0$ for $t \in [0, T]$, it follows that

$$\begin{aligned}
& \int_a^\infty \rho_k u_k(r, t) \varphi(r, t) r^m dr - \int_a^\infty \rho_a^0 u_a^0(r) \varphi(r, 0) r^m dr \\
&= \int_0^t \int_0^k \{ \varphi(\tilde{r}_k(x, s), s) D_t \tilde{u}_k + (\partial_t \varphi)(\tilde{r}_k(x, s), s) \tilde{u}_k + (\partial_r \varphi)(\tilde{r}_k(x, s), s) \tilde{u}_k^2 \} dx ds \\
&= \int_0^t \int_0^k \{ \tilde{u}_k(x, s) (\partial_t \varphi)(\tilde{r}_k(x, s), s) + \tilde{u}_k^2(x, s) (\partial_r \varphi)(\tilde{r}_k(x, s), s) \} dx ds \\
&\quad + \int_0^t \int_0^k \tilde{r}_k^m(x, s) \varphi(\tilde{r}_k(x, s), s) D_x \left(\beta \frac{D_x(\tilde{r}_k^m \tilde{u}_k)}{\tilde{v}_k} - (\gamma - 1) \frac{\tilde{e}_k}{\tilde{v}_k} \right) dx ds \\
&= \int_0^t \int_0^k \{ \tilde{u}_k^2(x, s) (\partial_r \varphi)(\tilde{r}_k(x, s), s) + \tilde{u}_k(x, s) (\partial_t \varphi)(\tilde{r}_k(x, s), s) \} dx ds \\
&\quad + \int_0^t \int_0^k \left\{ (\gamma - 1) \tilde{e}_k - \beta \tilde{r}_k^m D_x \tilde{u}_k - m \beta \frac{\tilde{v}_k \tilde{u}_k}{\tilde{r}_k} \right\} \left(m \frac{\varphi(\tilde{r}_k(x, s), s)}{\tilde{r}_k(x, s)} + (\partial_r \varphi)(\tilde{r}_k(x, s), s) \right) dx ds \\
&= \int_0^t \int_a^{\tilde{r}_k(k, t)} \{ \bar{v}_k^{-1} \bar{u}_k^2 (\partial_r \varphi) + \bar{v}_k^{-1} \bar{u}_k (\partial_t \varphi) \} (r, s) r^m dr ds \\
&\quad + \int_0^t \int_a^{\tilde{r}_k(k, t)} \left\{ (\gamma - 1) \bar{v}_k^{-1} \bar{e}_k - \beta \partial_r \bar{u}_k - m \beta \frac{\bar{u}_k}{r} \right\} \left(\partial_r \varphi + m \frac{\varphi}{r} \right) (r, s) r^m dr ds \\
&= \int_0^t \int_a^\infty \{ \rho_k u_k^2 (\partial_r \varphi) + \rho_k u_k (\partial_t \varphi) \} (r, s) r^m dr ds \\
&\quad + \int_0^t \int_a^\infty \left\{ (\gamma - 1) \rho_k e_k - \beta \partial_r u_k - m \beta \frac{u_k}{r} \right\} \left(\partial_r \varphi + m \frac{\varphi}{r} \right) (r, s) r^m dr ds.
\end{aligned}$$

In addition from Lemma 2.4.2 and Lemma 2.4.4–2.4.8, one has,

$$\begin{aligned}
& \partial_r u_k \rightarrow \partial_r u && \text{in } L^2(0, T; L^2([a, \infty), r^m dr)), \\
& e_k(r, t) \rightarrow e(r, t) && \text{for } (r, t) \in [a, \infty) \times [0, T], \\
& \sigma(t)^{-\frac{1}{4}} |u(r, t)| + \sigma(t)^{-\frac{1}{2}} e(r, t) \leq C(a) && \text{for } (r, t) \in [a, \infty) \times [0, T], \\
& \sigma(t)^{-\frac{1}{4}} |u_k(r, t)| + \sigma(t)^{-\frac{1}{2}} e_k(r, t) \leq C(a) && \text{for } (k, r, t) \in \mathbb{N} \times [a, \infty) \times [0, T].
\end{aligned} \tag{2.4.55}$$

Using (2.4.54) and (2.4.55) in the previous equation, we prove (ii) of Definition 2.2.1.

Finally, we show the energy equation. Let $\phi \in \mathcal{D}^a$. Then for $k \geq M(\phi, a)$,

$$\begin{aligned}
& \int_a^\infty \rho_k e_k \phi(r, t) r^m dr - \int_a^\infty \rho_a^0 e_a^0(r) \phi(r, 0) r^m dr \\
&= \int_a^{\tilde{r}_k(k, t)} \bar{v}_k^{-1} \bar{e}_k \phi(r, t) r^m dr - \int_a^{\tilde{r}_a^0(k)} e_{a, k}^0(r) \phi(r, 0) \rho_a^0(r) r^m dr \\
&= \int_0^k \tilde{e}_k(x, t) \phi(\tilde{r}_k(x, t), t) dx - \int_0^k \tilde{e}_{a, k}^0(x) \phi(\tilde{r}_k(x, 0), 0) dx \\
&= \int_0^t \int_0^k \{ \phi(\tilde{r}_k(x, s), s) D_t \tilde{e}_k(x, s) + \tilde{e}_k(x, s) (\partial_t \phi + \tilde{u}_k(x, s) \partial_r \phi)(\tilde{r}_k(x, s), s) \} dx ds \\
&= \int_0^t \int_0^k \phi(\tilde{r}_k(x, s), s) D_t \tilde{e}_k dx ds + \int_0^t \int_a^\infty \rho_k e_k (\partial_t \phi + u_k \partial_r \phi) r^m dr ds =: I_1 + I_2.
\end{aligned} \tag{2.4.56}$$

Next, denoting $\tilde{F}_k := \beta \tilde{v}_k^{-1} D_x(\tilde{r}_k^m \tilde{u}_k) - (\gamma - 1) \tilde{v}_k^{-1} \tilde{e}_k$, it follows from momentum equation that $D_t \tilde{u}_k =$

$\tilde{r}_k^m D_x \tilde{F}_k$. Then substituting this into (2.2.16)₃, one obtains

$$D_t \tilde{e}_k = D_x (\tilde{r}_k^m \tilde{u}_k \tilde{F}_k) - \frac{1}{2} D_t |\tilde{u}_k|^2 - 2m\mu D_x (\tilde{r}_k^{m-1} \tilde{u}_k^2) + \kappa D_x \left(\frac{\tilde{r}_k^{2m}}{\tilde{v}_k} D_x \tilde{e}_k \right),$$

which, along with $D_x \tilde{e}_k(0, t) = D_x \tilde{e}_k(k, t) = 0$ for all $t \in [0, T]$, yields that

$$\begin{aligned} I_1 &= \int_0^t \int_0^k \phi(\tilde{r}_k(x, s), s) D_t \tilde{e}_k(x, s) dx ds \\ &= \int_0^t \int_0^k \phi(\tilde{r}_k(x, s), s) \left\{ D_x (\tilde{r}_k^m \tilde{u}_k \tilde{F}_k) - \frac{1}{2} D_t |\tilde{u}_k|^2 \right\} dx ds \\ &\quad + \int_0^t \int_0^k \phi(\tilde{r}_k(x, s), s) \left\{ -2m\mu D_x (\tilde{r}_k^{m-1} \tilde{u}_k^2) + \kappa D_x \left(\frac{\tilde{r}_k^{2m}}{\tilde{v}_k} D_x \tilde{e}_k \right) \right\} dx ds \\ &= \int_0^t \int_0^k \left\{ -m\beta \tilde{r}_k^{-1} \tilde{v}_k |\tilde{u}_k|^2 - \beta \tilde{r}_k^m \tilde{u}_k D_x \tilde{u}_k + (\gamma - 1) \tilde{u}_k \tilde{e}_k \right\} (\partial_r \phi)(\tilde{r}_k(x, s), s) dx ds \\ &\quad + \int_0^t \int_0^k \frac{|\tilde{u}_k|^2}{2} \left\{ (\partial_t \phi)(\tilde{r}_k(x, s), s) + \tilde{u}_k (\partial_r \phi)(\tilde{r}_k(x, s), s) \right\} dx ds \\ &\quad - \int_0^k \frac{|\tilde{u}_k|^2}{2}(x, t) \phi(\tilde{r}_k(x, t), t) dx + \int_0^k \frac{|\tilde{u}_{a,k}^0(x)|^2}{2} \phi(\tilde{r}_k(x, 0), 0) dx \\ &\quad + \int_0^t \int_0^k \left\{ 2m\mu \tilde{r}_k^{-1} \tilde{v}_k |\tilde{u}_k|^2 (\partial_r \phi)(\tilde{r}_k(x, s), s) - \kappa \tilde{r}_k^m D_x \tilde{e}_k (\partial_r \phi)(\tilde{r}_k(x, s), s) \right\} dx ds. \end{aligned}$$

From (2.4.53), we have $\tilde{r}_k(x, 0) = \tilde{r}_a^0(x)$ if $\tilde{r}_a^0(x) \in \text{supp}(\phi(\cdot, 0))$ and $k \geq M(\phi, a)$. By coordinate transformation (2.4.1)–(2.4.3), and (2.4.51)–(2.4.52) it follows that:

$$\begin{aligned} I_1 &= \int_0^t \int_a^{\tilde{r}_k(k,t)} \left\{ m(2\mu - \beta) \frac{|\bar{u}_k|^2}{r} - \beta \bar{u}_k \partial_r \bar{u}_k + (\gamma - 1) \bar{u}_k \bar{v}_k^{-1} \bar{e}_k - \kappa \partial_r \bar{e}_k \right\} \partial_r \phi(r, t) r^m dr ds \\ &\quad + \int_0^t \int_a^{\tilde{r}_k(k,t)} \frac{\bar{v}_k^{-1} |\bar{u}_k|^2}{2} \{ \partial_t \phi + \bar{u}_k \partial_r \phi \} (r, t) r^m dr ds \\ &\quad - \int_a^{\tilde{r}_k(k,t)} \frac{\bar{v}_k^{-1} |\bar{u}_k|^2}{2} \phi(r, t) r^m dr + \int_a^{\tilde{r}_a^0(k)} \rho_a^0 \frac{|u_a^0|^2}{2}(r) \phi(r, 0) r^m dr \\ &= \int_0^t \int_a^\infty \left\{ -m\lambda \frac{|u_k|^2}{r} - \beta u_k \partial_r u_k + (\gamma - 1) \rho_k u_k e_k - \kappa \partial_r e_k \right\} \partial_r \phi(r, t) r^m dr ds \\ &\quad + \int_0^t \int_a^\infty \rho_k \frac{|u_k|^2}{2} \{ \partial_t \phi + u_k \partial_r \phi \} r^m dr ds \\ &\quad - \int_a^\infty \rho_k \frac{|u_k|^2}{2} \phi(r, t) r^m dr + \int_a^\infty \rho_a^0 \frac{|u_a^0|^2}{2}(r) \phi(r, 0) r^m dr. \end{aligned}$$

If we define $E_k(r, t) := (|u_k|^2/2 + e_k)(r, t)$, then the above identity becomes:

$$\begin{aligned} I_1 &= \int_0^t \int_a^\infty \left\{ -m\lambda \frac{|u_k|^2}{r} - \beta u_k \partial_r u_k + (\gamma - 1) \rho_k u_k e_k - \kappa \partial_r e_k \right\} \partial_r \phi(r, t) r^m dr ds \\ &\quad + \int_0^t \int_a^\infty \left\{ \rho_k E_k (\partial_t \phi + u_k \partial_r \phi) - \rho_k e_k (\partial_t \phi + u_k \partial_r \phi) \right\} (r, t) r^m dr ds \\ &\quad - \int_a^\infty \rho_k \frac{|u_k|^2}{2}(r, t) \phi(r, t) r^m dr + \int_a^\infty \rho_a^0 \frac{|u_a^0|^2}{2}(r) \phi(r, 0) r^m dr. \end{aligned}$$

Denote $P_k := (\gamma - 1)\rho_k e_k$ and $E_a^0 := |u_a^0|^2/2 + e_a^0$. Substituting I_1 into (2.4.56), we have

$$\begin{aligned} & \int_a^\infty \rho_k E_k \phi(r, t) r^m dr - \int_a^\infty \rho_a^0 E_a^0(r) \phi(r, 0) r^m dr \\ &= \int_0^t \int_a^\infty \left\{ \left((\rho_k E_k + P_k) u_k - m\lambda \frac{|u_k|^2}{r} - \beta u_k \partial_r u_k - \kappa \partial_r e_k \right) \partial_r \phi + \rho_k E_k \partial_t \phi \right\} r^m dr ds. \end{aligned} \quad (2.4.57)$$

In addition to (2.4.55), we also uses the weak convergence from Lemma 2.4.8 that

$$\partial_r e_k \rightharpoonup \partial_r e \text{ in } L^2(0, T; L^2([a, \infty), r^m dr)).$$

Then (2.4.57) becomes:

$$\begin{aligned} & \int_a^\infty \rho E(r, t) \phi(r, t) r^m dr - \int_a^\infty \rho_a^0 E_a^0(r) \phi(r, 0) r^m dr \\ &= \int_0^t \int_a^\infty \left\{ \left((\rho E + P) u - m\lambda \frac{|u|^2}{r} - \beta u \partial_r u - \kappa \partial_r e \right) \partial_r \phi + \rho E \partial_t \phi \right\} (r, s) r^m dr ds, \end{aligned}$$

where $P := (\gamma - 1)\rho e$ and $E := |u|^2/2 + e$. □

2.4.7 Estimates away from origin, independent of particle path $\tilde{r}(x, t)$

The main aim of this subsection is to obtain $a \in (0, 1)$ -independent estimates which do not involve the particle path function $\tilde{r}(x, t)$.

Lemma 2.4.11. *For each $\eta > a$, it follows that*

$$\begin{cases} \int_0^T \sup_{r \geq \eta} \frac{|u|}{\sqrt{e}}(r, t) dt \leq C(T)\eta^{(2-n)/2} + C(T)\eta^{2-n}, & \text{if } n = 2, 3, \\ \int_0^T \sup_{r \geq \eta} \log(\max\{1, e^{\pm 1}(r, t)\}) dt \leq C(T)\eta^{2-n} & \text{if } n = 3, \\ \int_0^T \sup_{r \geq \eta} \log(\max\{1, e(r, t)\}) dt \leq C(T)(1 + \sqrt{\log(\max\{1, C(T)/\eta\})}) & \text{if } n = 2. \end{cases} \quad (2.4.58)$$

Proof. Fix $r \geq \eta$ and $0 < \delta < 1$. By the boundary condition: $\lim_{r \rightarrow \infty} u_k(r, t) = 0$ and $\lim_{r \rightarrow \infty} e_k(r, t) = 1$ for each $t \in [0, T]$ and $k \in \mathbb{N}$, one has

$$\begin{aligned} & \frac{u_k}{r^\delta \sqrt{e_k}}(r, t) = \lim_{r \rightarrow \infty} \frac{u_k}{r^\delta \sqrt{e_k}}(r, t) - \int_r^\infty \left\{ \zeta^{-\delta} \frac{\partial_r u_k}{\sqrt{e_k}} - \delta \zeta^{-\delta} \frac{u_k}{\zeta \sqrt{e_k}} - \frac{\zeta^{-\delta+1}}{2} \frac{u_k}{\zeta e_k^{3/2}} \partial_r e_k \right\} (\zeta, t) d\zeta \\ & \leq \left(\int_r^\infty \zeta^{-2\delta-m} d\zeta \right)^{1/2} \left\{ \delta \left(\int_r^\infty \frac{|u_k|^2}{\zeta^2 e_k} \zeta^m d\zeta \right)^{1/2} + \left(\int_r^\infty \frac{|\partial_r u_k|^2}{e_k} \zeta^m d\zeta \right)^{1/2} \right\} \\ & \quad + \frac{1}{2} \left(\int_r^\infty \frac{|u_k|^2}{\zeta^2 e_k} \zeta^{-\delta+1} d\zeta \right)^{1/2} \left(\int_r^\infty \frac{|\partial_r e_k|}{e_k^2} \zeta^{-\delta+1} d\zeta \right)^{1/2} \\ & \leq \frac{r^{-\delta} \eta^{(1-m)/2}}{\sqrt{2\delta+m-1}} \left\{ \delta \left(\int_\eta^\infty \frac{|u_k|^2}{\zeta^2 e_k} (\zeta, t) \zeta^m d\zeta \right)^{1/2} + \left(\int_\eta^\infty \frac{|\partial_r u_k|^2}{e_k} \zeta^m d\zeta \right)^{1/2} \right\} \\ & \quad + \frac{r^{-\delta} \eta^{1-m}}{2} \left(\int_\eta^\infty \frac{|u_k|^2}{\zeta^2 e_k} \zeta^m d\zeta \right)^{1/2} \left(\int_\eta^\infty \frac{|\partial_r e_k|}{e_k^2} \zeta^m d\zeta \right)^{1/2} =: r^{-\delta} g_k(\eta, t), \end{aligned}$$

where we used the fact that $-2\delta - m + 1 < 0$ and $-\delta + 1 - m < 0$ for $n = 2, 3$ since $0 < \delta < 1$. Multiplying both sides of the above inequality by r^δ , and taking limit inferior over $k \rightarrow \infty$, it follows by Lemma 2.4.4 that

$$\left| \frac{u}{\sqrt{e}} \right|(r, t) \leq \liminf_{k \rightarrow \infty} g_k(\eta, t) \quad \text{for all } (r, t) \in [\eta, \infty) \times [0, T].$$

Taking supremum in $r \in [\eta, \infty)$ and integrating in $t \in [0, T]$, we obtain:

$$\int_0^T \sup_{r \geq \eta} \left| \frac{u}{\sqrt{e}} \right|(r, t) dt \leq \int_0^T \liminf_{k \rightarrow \infty} g_k(\eta, t) dt \leq \liminf_{k \rightarrow \infty} \int_0^T g_k(\eta, t) dt, \quad (2.4.59)$$

where we used Fatou's Lemma since $g_k(\eta, t) > 0$. By definition of $g(\eta, t)$ and Lemma 2.4.1, we obtain that for all $k \in \mathbb{N}$:

$$\begin{aligned} \int_0^T g_k(\eta, t) dt &\leq C\eta^{(1-m)/2} \left\{ \int_0^T \left(\int_\eta^\infty \frac{|u|_k^2}{\zeta^2 e_k} \zeta^m d\zeta \right)^{1/2} dt + \int_0^T \left(\int_\eta^\infty \frac{|\partial_r u_k|^2}{e_k} \zeta^m d\zeta \right)^{1/2} dt \right\} \\ &+ C\eta^{1-m} \int_0^T \left(\int_\eta^\infty \frac{|u_k|^2}{\zeta^2 e_k} \zeta^m d\zeta \right)^{1/2} \left(\int_\eta^\infty \frac{|\partial_r e_k|^2}{e_k^2} \zeta^m d\zeta \right)^{1/2} dt \\ &\leq C\eta^{(1-m)/2} T^{1/2} \left\{ \left(\int_0^T \int_\eta^\infty \frac{|u|_k^2}{\zeta^2 e_k} \zeta^m d\zeta dt \right)^{1/2} + \left(\int_0^T \int_\eta^\infty \frac{|\partial_r u_k|^2}{e_k} \zeta^m d\zeta dt \right)^{1/2} \right\} \\ &+ C\eta^{1-m} \left(\int_0^T \int_\eta^\infty \frac{|u_k|^2}{\zeta^2 e_k} \zeta^m d\zeta dt \right)^{1/2} \left(\int_0^T \int_\eta^\infty \frac{|\partial_r e_k|^2}{e_k^2} \zeta^m d\zeta dt \right)^{1/2} \leq C(T)(\eta^{\frac{1-m}{2}} + \eta^{1-m}), \end{aligned}$$

which together with (2.4.59) proves (2.4.58)₁.

Next, consider the case $n = 3$. Let $(r, t) \in [\eta, \infty) \times [0, T]$. By $\lim_{r \rightarrow \infty} e_k(r, t) = 1$, we have

$$\begin{aligned} \log e_k(r, t) &= - \int_r^\infty \frac{\partial_r e_k}{e_k}(\zeta, t) d\zeta \leq \left(\int_r^\infty \zeta^{-m} d\zeta \right)^{1/2} \left(\int_r^\infty \frac{|\partial_r e_k|^2}{e_k^2}(\zeta, t) \zeta^m d\zeta \right)^{1/2} \\ &\leq \frac{\eta^{2-n}}{\sqrt{n-2}} \left(\int_\eta^\infty \frac{|\partial_r e_k|^2}{e_k^2}(\zeta, t) \zeta^m d\zeta \right)^{1/2} =: h_k(\eta, t). \end{aligned}$$

Thus $0 \leq \log(\max\{1, e_k\})(r, t) \leq h_k(\eta, t)$ for all $(r, t) \in [\eta, \infty) \times [0, T]$. Replacing the term $e_k(r, t)$ with $e_k^{-1}(r, t)$ in the above derivation, we also have $0 \leq \log(\max\{1, e_k^{-1}\})(r, t) \leq h_k(\eta, t)$ for all $(r, t) \in [\eta, \infty) \times [0, T]$. Taking limit inferior over $k \rightarrow \infty$ on both sides of these inequalities, we have by Lemma 2.4.4 that

$$\log(\max\{1, e^{\pm 1}\})(r, t) \leq \liminf_{k \rightarrow \infty} h_k(\eta, t) \quad \text{for all } (r, t) \in [\eta, \infty) \times [0, T].$$

Taking supremum in $r \geq \eta$ on the above inequality, and integrating in $t \in [0, T]$, we obtain:

$$\int_0^T \sup_{r \geq \eta} \log(\max\{1, e^{\pm 1}(r, t)\}) dt \leq \int_0^T \liminf_{k \rightarrow \infty} h_k(\eta, t) dt \leq \liminf_{k \rightarrow \infty} \int_0^T h_k(\eta, t) dt, \quad (2.4.60)$$

where we used Fatou's Lemma since $h_k(\eta, t) > 0$. By Lemma 2.4.1, we obtain for all $k \in \mathbb{N}$:

$$\int_0^T h_k(\eta, t) dt \leq C\eta^{2-n} T^{1/2} \left(\int_0^T \int_\eta^\infty \frac{|\partial_r e_k|^2}{e_k^2}(\zeta, t) \zeta^m d\zeta dt \right)^{1/2} \leq C(T)\eta^{2-n},$$

which together with (2.4.60) proves (2.4.58)₂.

Next, we consider the case $n = 2$. We set $M := \sup_{k \in \mathbb{N}} \sup_{t \in [0, T]} \tilde{r}_k(1, t)$, then by Lemma 2.3.1, we have $M \leq (2C_0)^{1/n}$. If $r \geq M$ then $r > \sup_{t \in [0, T]} \tilde{r}_k(1, t)$ for all $k \in \mathbb{N}$. By Lemma 2.4.2, we have

$e_k(r, t) \leq C(\varepsilon)\sigma(t)^{-1/2}$ for all $t \in [0, T]$ and $r \in [\tilde{r}_k(\varepsilon, t), \infty)$. It follows that there exists a constant $C(T) > 0$ such that

$$\log(\max\{1, e_k(r, t)\}) \leq \log(\max\{1, C(T)\sigma(t)^{-1/2}\}) \quad \text{for all } (r, t) \in [M, \infty) \times [0, T]. \quad (2.4.61)$$

By Fundamental Theorem of Calculus and (2.4.61), for all $r \in [\eta, M)$ one gets:

$$\begin{aligned} \log e_k(r, t) &= \log e_k(2M, t) - \int_r^{2M} \frac{\partial_r e_k}{e_k}(\zeta, t) d\zeta \\ &\leq \log(\max\{1, e_k(2M, t)\}) + \left(\int_r^{2M} \zeta^{-1} d\zeta \right)^{1/2} \left(\int_r^{2M} \frac{|\partial_r e_k|^2}{e_k^2}(\zeta, t) \zeta d\zeta \right)^{1/2} \\ &\leq \log(\max\{1, C(T)\sigma(t)^{-1/2}\}) + \sqrt{\log(C(T)/\eta)} \left(\int_\eta^\infty \frac{|\partial_r e_k|^2}{e_k^2}(\zeta, t) \zeta d\zeta \right)^{1/2}. \end{aligned} \quad (2.4.62)$$

Finally, in a similar way for the case $n = 3$, one has

$$\begin{aligned} \int_0^T \sup_{r \geq \eta} \log(\max\{1, e(r, t)\}) dt &\leq \frac{C(T)^2}{2} + \int_0^T \sup_{r \in [\eta, M)} \log(\max\{1, e(r, t)\}) dt \\ &\leq C(T) + \sqrt{\log(C(T)/\eta)} T^{\frac{1}{2}} \sup_{k \in \mathbb{N}} \left(\int_0^T \int_\eta^\infty \frac{|\partial_r e_k|^2}{e_k^2} \zeta^m d\zeta dt \right)^{\frac{1}{2}} \\ &\leq C(T) + C(T) \sqrt{\log(\max\{1, C(T)/\eta\})}. \end{aligned}$$

□

2.5 Weak Solution including the origin: $a \searrow 0$

Throughout this section, we will denote $(\rho_a, u_a, e_a)(r, t)$ in $[a, \infty) \times [0, T]$ as the solution to problem (1.2.1) and (2.2.9), which is guaranteed by Theorem 2.2.1, with the modified initial data $(\rho_a^0, u_a^0, e_a^0)(r)$ constructed in Section 2.2.2 for each $a \in (0, 1)$. We also denote $\tilde{r}_a(x, t)$ as the particle path function associated with ρ_a , satisfying

$$x = \int_a^{\tilde{r}_a(x, t)} \rho_a(y, t) r^m dr \quad \text{for a.e. } (x, t) \in [0, \infty) \times [0, T].$$

The main aim of this section is to take limit $a \searrow 0$, from which we can obtain spherically symmetric weak solution to the original Cauchy problem (1.1.1)–(1.1.5).

2.5.1 Uniform inequalities from entropy estimates

We obtain a set of inequalities involving measurable subsets of $[a, \infty)$ which are uniform in $a \in (0, 1)$. Throughout this section, we will denote $G(\cdot), \psi(\cdot), H(\cdot)$ as the convex functions:

$$G(\zeta) := 1 - \zeta + \zeta \log \zeta, \quad \psi(\zeta) := \zeta - 1 - \log \zeta, \quad H(\zeta) := \zeta \log \zeta. \quad (2.5.1)$$

Let $G_+^{-1}, \psi_+^{-1}, H_+^{-1}$ be the right branch inverses of G, ψ, H respectively. Then by Proposition B.0.1 of Appendix B, the following functions are well defined for $(y, z) \in (0, \infty)^2$:

$$f_1(y; z) := yG_+^{-1}\left(\frac{z}{y}\right), \quad f_2(y; z) := y\psi_+^{-1}\left(\frac{z}{y}\right) - z, \quad f_3(y; z) := yH_+^{-1}\left(\frac{z}{y}\right). \quad (2.5.2)$$

Some properties of $(y, z) \mapsto f_i(y; z)$ for $i = 1, 2, 3$ can be found in Proposition B.0.1.

Lemma 2.5.1. For any measurable set $E \subset [a, \infty)$, and for each $t \in [0, T]$,

$$\sup_{a \in (0,1)} \int_E \rho_a(r, t) r^m dr \leq \omega_1(E; C(T)), \quad \text{where } \omega_1(E; z) := f_1\left(\int_E r^m dr; z\right) \text{ for } z > 0.$$

Proof. For simplicity, we denote $\mathcal{L}_n(E) := \int_E r^m dr$ for each $E \subseteq [a, \infty)$. By Jensen's inequality and Theorem 2.2.1(i), it follows that

$$G\left(\frac{1}{\mathcal{L}_n(E)} \int_E \rho_a(r, t) r^m dr\right) \leq \frac{1}{\mathcal{L}_n(E)} \int_E G(\rho_a)(r, t) r^m dr \leq \frac{C(T)}{\mathcal{L}_n(E)}.$$

Taking the right branch inverse $G_+^{-1}(z)$ on both sides of above and using the fact that $z \mapsto G_+^{-1}(z)$ is monotone increasing, it follows that for all $a \in (0, 1)$:

$$\int_E \rho_a(r, t) r^m dr \leq \mathcal{L}_n(E) G_+^{-1}\left(\frac{C(T)}{\mathcal{L}_n(E)}\right) = f_1(\mathcal{L}_n(E); C(T)) =: \omega_1(E; C(T)).$$

□

Lemma 2.5.2. For any measurable set $E \subset [a, \infty)$, one has

$$\sup_{a \in (0,1)} \int_E \rho_a e_a(r, t) r^m dr \leq C(T) + \omega_2(E; C(T)) \text{ for all } t \in [0, T],$$

$$\text{where } \omega_2(E; z) := f_2(\omega_1(E; z); z) = f_2\left(f_1\left(\int_E r^m dr; z\right); z\right) \text{ for } z > 0.$$

Proof. Fixing an time $t \in [0, T]$, and setting $m_{a,t}(E) := \int_E \rho_a(r, t) r^m dr$. By Jensen's inequality and Theorem 2.2.1(i), it follows that

$$\psi\left(\frac{1}{m_{a,t}(E)} \int_E \rho_a e_a(r, t) r^m dr\right) \leq \frac{1}{m_{a,t}(E)} \int_E \rho_a \psi(e_a)(r, t) r^m dr \leq \frac{C(T)}{m_{a,t}(E)}.$$

Taking $\psi_+^{-1}(\cdot)$ on both sides of above, it follows that

$$\int_E \rho_a e_a(r, t) r^m dr \leq m_{a,t}(E) \psi_+^{-1}\left(\frac{C(T)}{m_{a,t}(E)}\right) - C(T) + C(T) = f_2(m_{a,t}(E); C(T)) + C(T).$$

By Lemma 2.5.1, one has $m_{a,t}(E) \leq \omega_1(E; C(T))$. In addition, since $y \mapsto f_2(y; z)$ is increasing for fixed $z > 0$ from Proposition B.0.1, it follows that for all $a \in (0, 1)$,

$$\int_E \rho_a e_a(r, t) r^m dr \leq C(T) + f_2(\omega_1(E; C(T)); C(T)) =: C(T) + \omega_2(E; C(T)).$$

□

Lemma 2.5.3. For each $\eta > a$, there exists a constant $C(\eta, T) > 0$, independent of $a \in (0, 1)$ such that if $E = \{E(t) \subseteq [\eta, \infty) : t \in [0, T]\}$ is a collection of measurable subsets satisfying

$$\mathcal{M}_E := \text{ess sup}_{t \in [0, T]} \int_{E(t)} r^m dr < \infty,$$

then the following estimates hold:

$$\begin{aligned} \int_0^T \int_{E(t)} \rho_a e_a(r, t) r^m dr dt &\leq f_3(V_a(E); C_E), \\ \int_0^T \int_{E(t)} \rho_a u_a^2(r, t) r^m dr dt &\leq C(\eta, T) \{f_3(V_a(E); C_E)\}^{1/4}, \end{aligned}$$

where $C_E > 0$ and $V_a(E)$ are given by:

$$C_E := C(\eta, T) \{C(T) + f_2(f_1(\mathcal{M}_E; C(T)); C(T))\}, \quad V_a(E) := \int_0^T \int_{E(t)} \rho_a(r, t) r^m dr dt.$$

Proof. By Jensen's inequality, Lemma 2.5.2, and Theorem 2.2.1(vi), one has

$$\begin{aligned} H\left(\frac{1}{V_a(E)} \int_0^T \int_{E(t)} \rho_a e_a(r, t) r^m dr dt\right) &\leq \frac{1}{V_a(E)} \int_0^T \int_{E(t)} \rho_a H(e_a)(r, t) r^m dr dt \\ &\leq \frac{1}{V_a(E)} \int_0^T \|\log(\max\{1, e_a(\cdot, t)\})\|_{L^\infty(\eta, \infty)} \left(\int_{E(t)} \rho_a e_a(r, t) r^m dr\right) dt \\ &\leq \frac{C(\eta, T)}{V_a(E)} (C(T) + \sup_{t \in [0, T]} \omega_2(E(t); C(T))). \end{aligned}$$

Since for each fixed $z > 0$, $y \rightarrow (f_1, f_2)(y; z)$ are increasing, for each $t \in [0, T]$ one has

$$\omega_2(E(t); C(T)) := f_2\left(f_1\left(\int_{E(t)} r^m dr; C(T)\right); C(T)\right) \leq f_2(f_1(\mathcal{M}_E; C(T)); C(T)).$$

If one defines $C_E := C(\eta, T) \{C(T) + f_2(f_1(\mathcal{M}_E; C(T)); C(T))\}$, then it follows that:

$$H\left(\frac{1}{V_a(E)} \int_0^T \int_{E(t)} \rho_a e_a(r, t) r^m dr dt\right) \leq \frac{C(\eta, T)}{V_a(E)} \{C + f_2(f_1(\mathcal{M}_E; C); C)\} = \frac{C_E}{V_a(E)}.$$

Taking $H_+^{-1}(\cdot)$ on both sides, one gets

$$\int_0^T \int_{E(t)} \rho_a e_a(r, t) r^m dr dt \leq V_a(E) H_+^{-1}\left(\frac{C_E}{V_a(E)}\right) =: f_3(V_a(E); C_E),$$

which along with Theorem 2.2.1(i), (vi) yield that

$$\begin{aligned} \int_0^T \int_{E(t)} \rho_a u_a^2 r^m dr dt &\leq \int_0^T \left\| \frac{u_a}{\sqrt{e_a}}(\cdot, t) \right\|_{L^\infty(\eta, \infty)}^{\frac{1}{2}} \left(\int_{E(t)} \rho_a u_a^2 r^m dr\right)^{\frac{3}{4}} \left(\int_{E(t)} \rho_a e_a r^m dr\right)^{\frac{1}{4}} dt \\ &\leq C(T) \left(\int_0^T \left\| \frac{u_a}{\sqrt{e_a}} \right\|_{L^\infty(\eta, \infty)} dt\right)^{\frac{1}{2}} \left(\int_0^T \int_{E(t)} \rho_a e_a r^m dr dt\right)^{\frac{1}{4}} \leq C(\eta, T) \{f_3(V_a(E); C_E)\}^{\frac{1}{4}}. \end{aligned}$$

□

2.5.2 Existence of Particle Path and Vacuum Interface

Lemma 2.5.4. *There exists a sub-sequence $\{a_j\}_{j \in \mathbb{N}}$ with $a_j \searrow 0$ as $j \rightarrow \infty$, and a continuous function $\tilde{r}(x, t) : [0, \infty) \times [0, T] \rightarrow [0, \infty)$ such that*

(i) for each compact subset $K \subset\subset (0, \infty) \times [0, T]$

$$\lim_{j \rightarrow \infty} \sup_{(x,t) \in K} |\tilde{r}_{a_j}(x, t) - \tilde{r}(x, t)| = 0.$$

(ii) for each fixed $\varepsilon > 0$,

$$\begin{aligned} |\tilde{r}(x_1, t) - \tilde{r}(x_2, t)| &\leq C(\varepsilon)|x_1 - x_2| \quad \text{for all } (x_1, x_2, t) \in [\varepsilon, \infty)^2 \times [0, T], \\ |\tilde{r}(x, t_1) - \tilde{r}(x, t_2)| &\leq C(\varepsilon)|t_1^{3/4} - t_2^{3/4}| \quad \text{for all } (x, t_1, t_2) \in [\varepsilon, \infty) \times [0, T]^2. \end{aligned}$$

(iii) for all $t \in [0, T]$ the function $x \mapsto \tilde{r}(x, t)$ is strictly increasing, hence the limit $\underline{r}(t) := \lim_{x \rightarrow 0^+} \tilde{r}(x, t)$ exists for all $t \in [0, T]$. In addition, $\underline{r}(t)$ satisfies:

$$\lim_{t \searrow 0} \underline{r}(t) = 0 \quad \text{and } t \mapsto \underline{r}(t) \text{ is upper semi-continuous,}$$

and if one defines $\mathring{F} := \{(r, t) \in (0, \infty) \times (0, T) : \underline{r}(t) < r\}$, then \mathring{F} is an open set.

(iv) $\tilde{r}(x, t)$ satisfies the point-wise bound:

$$nx\psi_-^{-1}\left(\frac{C_0}{x}\right) \leq (r(x, t))^n \leq C_0(1+x) \quad \text{for all } (x, t) \in [0, \infty) \times [0, T].$$

(v) if $\underline{r}(t) > 0$ for some $t \in (0, T]$ then

$$\lim_{j \rightarrow \infty} \int_{a_j}^{\underline{r}(t)} \rho_{a_j}(r, t) r^m dr = 0.$$

In particular, for such $t \in (0, T]$, one has

$$\liminf_{j \rightarrow \infty} \rho_{a_j}(r, t) = 0 \quad \text{for a.e. } r \in (0, \underline{r}(t)).$$

Proof. For each integer $i \in \mathbb{N}$, Let $S_i := [i^{-1}, i] \times [0, T]$. From Theorem 2.2.1(iii)–(iv), it follows that for all $(x, t) \in S_i$,

$$\tilde{r}_a(x, t) = \tilde{r}_a^0(x) + \int_0^t u_a(\tilde{r}(x, s), s) ds \leq \tilde{r}_a^0(x) + C(i) \int_0^T \sigma(s)^{-\frac{1}{4}} ds \leq C(i)(1+T), \quad (2.5.3)$$

where $C(i) > 0$ is a constant independent of $a \in (0, 1)$, and also,

$$\begin{aligned} |\tilde{r}_a(x_1, t) - \tilde{r}_a(x_2, t)| &\leq C(i)|x_1 - x_2| \quad \text{for all } (x_1, x_2, t) \in [i^{-1}, i]^2 \times [0, T], \\ |\tilde{r}_a(x, t_1) - \tilde{r}_a(x, t_2)| &\leq C(i)|t_1^{3/4} - t_2^{3/4}| \quad \text{for all } (x, t_1, t_2) \in [i^{-1}, i] \times [0, T]^2. \end{aligned} \quad (2.5.4)$$

In addition, one has $S_i \subset S_{i+1}$, $\text{Int}(S_i) \neq \emptyset$ for each $i \in \mathbb{N}$, and $(0, \infty) \times [0, T] = \cup_{i \in \mathbb{N}} S_i$. Thus by Proposition A.0.1, it follows that there exists a continuous function $\tilde{r}(x, t) : (0, \infty) \times [0, T] \rightarrow [0, \infty)$ such that for all compact subset $K \subset\subset (0, \infty) \times [0, T]$,

$$\lim_{j \rightarrow \infty} \sup_{(x,t) \in K} |\tilde{r}_{a_j}(x, t) - \tilde{r}(x, t)| = 0, \quad (2.5.5)$$

which, along with (2.5.4) and Theorem 2.2.1(iv), implies Lemma 2.5.4(ii).

Next, let $0 < x_1 < x_2 < \infty$. It follows from Theorem 2.2.1(iii)–(iv) that for a.e. $t \in [0, T]$,

$$\begin{aligned} x_2 - x_1 &= \int_{\tilde{r}_a(x_1, t)}^{\tilde{r}_a(x_2, t)} \rho_a(r, t) r^m dr \\ &\leq C(x_1) \int_{\tilde{r}_a(x_1, t)}^{\tilde{r}_a(x_2, t)} r^m dr = \frac{C(x_1)}{n} \{(\tilde{r}_a(x_2, t))^n - (\tilde{r}_a(x_1, t))^n\}, \end{aligned}$$

where in the last line, one used $\rho_a(r, t) \leq C(x_1)$ for a.e. (r, t) such that $r \geq \tilde{r}_a(x_1, t)$. Taking $j \rightarrow \infty$ and using (2.5.5), one has

$$0 < x_2 - x_1 \leq \frac{C(x_1)}{n} \{(\tilde{r}(x_2, t))^n - (\tilde{r}(x_1, t))^n\}.$$

This shows that $x \mapsto \tilde{r}(x, t)$ is strictly increasing for a.e. $t \in [0, T]$. Since $(x, t) \mapsto \tilde{r}(x, t)$ is continuous, then $x \mapsto \tilde{r}(x, t)$ is strictly increasing for all $t \in [0, T]$. Due to $\tilde{r}(x, t) \geq 0$,

$$\underline{r}(t) := \lim_{x \rightarrow 0^+} \tilde{r}(x, t) \quad \text{exists for all } t \in [0, T]. \quad (2.5.6)$$

From Theorem 2.2.1(iii), it follows that $\tilde{r}_a(x, 0) = \tilde{r}_a^0(x)$ for each $a \in (0, 1)$ and $x \in [0, \infty)$, where $\tilde{r}_a^0(x)$ is the function constructed in Section 2.2.4, satisfying:

$$x = \int_a^{\tilde{r}_a^0(x)} \rho_a^0(r) r^m dr \quad \text{for all } x \in [0, \infty].$$

Then by the uniform convergence (2.5.5) and Proposition 2.2.1 in Section 2.2.2, one has

$$x = \int_0^{\tilde{r}(x, 0)} \rho_0(r) r^m dr \geq C_0^{-1} \frac{(\tilde{r}(x, 0))^n}{n} \quad \text{for all } x > 0.$$

This implies the inequality: $\tilde{r}(x, 0) \leq (nC_0 x)^{1/n}$. Given $\delta > 0$, let $x_\delta > 0$ be such that $\tilde{r}(x_\delta, 0) \leq (nC_0 x_\delta)^{1/n} \leq \delta/2$. Since $t \mapsto \tilde{r}(x_\delta, t)$ is continuous in $[0, T]$, there exists $t_\delta > 0$ such that if $t \in (0, t_\delta)$ then $\tilde{r}(x_\delta, t) < \tilde{r}(x_\delta, 0) + \delta/2 \leq \delta/2 + \delta/2 = \delta$. By (2.5.6), one has $\underline{r}(t) = \inf\{\tilde{r}(x, t) : x > 0\}$ for each $t \in [0, T]$. It follows that $\underline{r}(t) \leq \tilde{r}(x_\delta, t) < \delta$ if $t \in (0, t_\delta)$. This shows that $\lim_{t \searrow 0} \underline{r}(t) = 0$.

Next we show that $t \mapsto \underline{r}(t)$ is upper semi-continuous. To start, a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is upper semi-continuous if and only if the set $\{x \in \mathbb{R} : f(x) < y\}$ is open for all $y \in \mathbb{R}$. Hence one defines the sets:

$$\begin{aligned} U_\zeta &:= \{s \in (0, T) : \underline{r}(s) < \zeta\} \quad \text{for each } \zeta \in (0, \infty), \\ U_\zeta^x &:= \{s \in (0, T) : \tilde{r}(x, s) < \zeta\} \quad \text{for each } (x, \zeta) \in (0, \infty)^2. \end{aligned}$$

Suppose $t \in U_\zeta$. Since $\underline{r}(t) = \inf_{x>0} \tilde{r}(x, t)$, there exists $\delta > 0$ such that $\zeta > \tilde{r}(\delta, t) \geq \underline{r}(t)$. Thus $t \in U_\zeta^\delta \subseteq \cup_{x>0} U_\zeta^x$, which implies that $U_\zeta \subseteq \cup_{x>0} U_\zeta^x$. On the other hand, if $t \in U_\zeta^\delta$ for some $\delta > 0$ then $\zeta > \tilde{r}(\delta, t) \geq \underline{r}(t)$. Hence $t \in U_\zeta$, which implies $U_\zeta = \cup_{x>0} U_\zeta^x$. Since $t \mapsto \tilde{r}(x, t)$ is continuous U_ζ^x is an open set for each $x > 0$. Therefore U_ζ is also an open set as it is the union of open sets. Since this is true for arbitrary $\zeta > 0$, and $\{s \in [0, T] : \underline{r}(s) < \zeta\} = \emptyset$ for $\zeta \leq 0$, it implies that $t \mapsto \underline{r}(t)$ is upper semi-continuous.

Next, for each $\varepsilon \in (0, 1]$ one defines $\mathring{F}_\varepsilon := \{(r, t) \in (0, \infty) \times (0, T) : \tilde{r}(\varepsilon, t) < r\}$. Then \mathring{F}_ε is an open set since $t \mapsto \tilde{r}(\varepsilon, t)$ is continuous in $t \in [0, T]$. Since $\underline{r}(t) = \inf_{x>0} \tilde{r}(x, t)$ for all $t \in [0, T]$, it follows that $\mathring{F}_\varepsilon \subset \mathring{F}$ for each $\varepsilon > 0$. Furthermore, if $(r, t) \in \mathring{F}$ then there exists $\varepsilon > 0$ such that $\underline{r}(t) \leq \tilde{r}(\varepsilon, t) < r$, which implies $(r, t) \in \mathring{F}_\varepsilon$, hence $\mathring{F} = \cup_{\varepsilon>0} \mathring{F}_\varepsilon$. Thus \mathring{F} is open as it is a union of open sets. This concludes the proof for Lemma 2.5.4(iii).

Next, we show the upper and lower bound of $\tilde{r}(x, t)$. From Theorem 2.2.1(iii), we have

$$nx\psi_-^{-1}(C_0/x) \leq (\tilde{r}_a(x, t))^n \leq C_0(1+x) \quad \text{for all } (a, x, t) \in (0, 1) \times [0, \infty) \times [0, T],$$

which, along with the uniform convergence in (2.5.5), implies Lemma 2.5.4(iv).

Finally, we prove Lemma 2.5.4(v). If $\underline{r}(t) > 0$ for some $t \in (0, T]$, then for each $a \in (0, \underline{r}(t))$, Theorem 2.2.1(iii) and Lemma 2.5.1 imply that

$$\begin{aligned} \left| \int_a^{\underline{r}(t)} \rho_a(r, t) r^m dr \right| &\leq \left| \int_a^{\tilde{r}_a(x, t)} \rho_a(r, t) r^m dr \right| + \left| \int_{\tilde{r}_a(x, t)}^{\tilde{r}(x, t)} \rho_a(r, t) r^m dr \right| + \left| \int_{\tilde{r}(x, t)}^{\tilde{r}(t)} \rho_a(r, t) r^m dr \right| \\ &\leq x + \omega_1([\tilde{r}(x, t), \tilde{r}_a(x, t)]; C(T)) + \omega_1([\underline{r}(t), \tilde{r}(x, t)]; C(T)), \end{aligned}$$

$$\text{where } \omega_1(E; C) := f_1\left(\int_E r^m dr; C\right), \text{ with } f_1(y; C) := yG_+^{-1}\left(\frac{C}{y}\right).$$

By Proposition B.0.1, for each fixed $C > 0$, $y \mapsto f_1(y; C)$ is a positive, monotone increasing, continuous function with $\lim_{y \searrow 0} f_1(y; C) = 0$. Hence given $\varepsilon > 0$ there exists $\delta' > 0$ such that for $x \in (0, \delta')$, one has $f_1(|(\tilde{r}(x, t))^n - (\underline{r}(t))^n|/n; C) \leq \varepsilon/2$. Let $\delta := \min\{\delta', \varepsilon/2\}$. We have

$$\int_a^{\underline{r}(t)} \rho_a(r, t) r^m dr \leq \varepsilon + f_1\left(\frac{(\tilde{r}_a(x, t))^n - (\tilde{r}(x, t))^n}{n}; C(T)\right), \text{ if } x \in (0, \delta).$$

Now fix $x \in (0, \delta)$. It follows from (2.5.5) and $\lim_{y \searrow 0} f_1(y; C) = 0$ that

$$\limsup_{a_j \searrow 0} \int_{a_j}^{\underline{r}(t)} \rho_{a_j}(r, t) r^m dr = \varepsilon + \lim_{a_j \searrow 0} f_1\left(\frac{(\tilde{r}_{a_j}(x, t))^n - (\tilde{r}(x, t))^n}{n}; C(T)\right) = \varepsilon.$$

Since this is true for arbitrarily small $\varepsilon > 0$, taking $\varepsilon \searrow 0$, one has

$$\limsup_{j \rightarrow \infty} \int_{a_j}^{\underline{r}(t)} \rho_{a_j}(r, t) r^m dr = 0 \tag{2.5.7}$$

Since $\rho_a(r, t) \geq C(a)^{-1}$ for a.e. $(r, t) \in [a, \infty) \times [0, T]$ from Theorem 2.2.1(ii), one has

$$\int_a^{\underline{r}(t)} \rho_a(r, t) r^m dr \geq C(a)^{-1} \int_a^{\underline{r}(t)} r^m dr > 0, \text{ for all } a \in (0, \underline{r}(t)),$$

which together with 2.5.7 implies the first statement of Lemma 2.5.4(v). Since $\rho_a(r, t) > 0$ for each $a \in (0, 1)$, by Fatou's Lemma,

$$\int_0^\infty \liminf_{a_j \searrow 0} \rho_{a_j}(r, t) \mathbb{1}_{(a_j, \underline{r}(t))}(r) r^m dr \leq \liminf_{a_j \searrow 0} \int_{a_j}^{\underline{r}(t)} \rho_{a_j}(r, t) r^m dr = 0.$$

Thus $\liminf_{j \rightarrow \infty} \rho_{a_j}(r, t) \mathbb{1}_{(a_j, \underline{r}(t))}(r) = 0$ for a.e. $r \in (0, \underline{r}(t))$. Now fix one such $r \in (0, \underline{r}(t))$. It follows that $\rho_{a_j}(r, t) = \rho_{a_j}(r, t) \mathbb{1}_{(a_j, \underline{r}(t))}(r)$ for all $a_j \in (0, r)$. Taking limit inferior as $j \rightarrow \infty$ on both sides, we obtain the second statement of Lemma 2.5.4(v). \square

In what follows, we will denote the fluid region F as the set:

$$F := \{(r, t) \in (0, \infty) \times [0, T] : r > \underline{r}(t)\}.$$

2.5.3 Compactness Results in Exterior Domain

Lemma 2.5.5. *There exists a sub-sequence $a_j \searrow 0$ as $j \rightarrow \infty$, and Hölder continuous functions $(u, e)(r, t) : F \setminus \{t = 0\} \rightarrow \mathbb{R} \times [0, \infty)$ such that for any compact subset $K \subset\subset F \setminus \{t = 0\}$,*

$$\lim_{j \rightarrow \infty} \sup_{(r,t) \in K} |(u_{a_j} - u, e_{a_j} - e)(r, t)| = 0.$$

Moreover, for each $\varepsilon > 0$, if $t \in [0, T]$ and $r \in [\tilde{r}(\varepsilon, t), \infty)$ then

$$|u(r, t)| \leq C(\varepsilon)\sigma(t)^{-\frac{1}{4}} \quad \text{and} \quad e(r, t) \leq C(\varepsilon)\sigma(t)^{-\frac{1}{2}}; \quad (2.5.8)$$

if $t \in (0, T]$ and $r_1, r_2 \geq \tilde{r}(\varepsilon, t)$ then

$$\sigma(t)^{\frac{1}{2}}|u(r_1, t) - u(r_2, t)| + \sigma(t)|e(r_1, t) - e(r_2, t)| \leq C(\varepsilon)|r_1 - r_2|^{\frac{1}{2}}; \quad (2.5.9)$$

if $0 < t_1 < t_2 \leq T$ and $r \geq \sup_{t_1 \leq t \leq t_2} \tilde{r}(\varepsilon, t)$ then

$$\sigma(t_1)^{\frac{1}{2}}|u(r, t_1) - u(r, t_2)| + \sigma(t_1)|e(r, t_1) - e(r, t_2)| \leq C(\varepsilon)|t_2 - t_1|^{\frac{1}{4}}. \quad (2.5.10)$$

Proof. Only the proof for $u(r, t)$ is presented as the case for $e(r, t)$ follows in the exact same way. First, Let $\varepsilon \in (0, 1)$. We define the domain:

$$F_\varepsilon := \{(r, t) \in (0, \infty) \times [0, T] : t \in [\varepsilon, T], \text{ and } \tilde{r}(\varepsilon, t) \leq r \leq \varepsilon^{-1}\}.$$

By Lemma 2.5.4(iv), there exist $\varepsilon_0 > 0$ such that $\text{Int}(F_\varepsilon) \neq \emptyset$ for each $\varepsilon \in (0, \varepsilon_0)$. Using the fact that $x \mapsto \tilde{r}(x, t)$ is strictly increasing, one can verify that $F_{\varepsilon_1} \subset F_{\varepsilon_2}$ if $\varepsilon_1 > \varepsilon_2$. We claim:

$$F \setminus \{t = 0\} = \bigcup_{0 < \varepsilon < 1} F_\varepsilon. \quad (2.5.11)$$

If $(r, t) \in F_\varepsilon$ for some $\varepsilon \in (0, 1)$ then $r > \underline{r}(t)$ since $\tilde{r}(x, t) \searrow \underline{r}(t)$ as $x \searrow 0$ for each $t \in [0, T]$, hence $(r, t) \in F$. This shows the assertion " \supseteq ". If $(r, t) \in F \setminus \{t = 0\}$, then $\underline{r}(t) < r$ and there exists $\varepsilon_1 > 0$ for which $t > \varepsilon_1$. Since $\lim_{x \searrow 0} \tilde{r}(x, t) = \underline{r}(t)$ for all $t \in [0, T]$, there exists $\varepsilon_2 \in (0, 1)$ such that $\underline{r}(t) \leq \tilde{r}(\varepsilon_2, t) \leq r$. Thus $(r, t) \in F_\varepsilon$ for $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$. This proves the other inclusion " \subseteq ", hence (2.5.11) follows.

Next, since $x \mapsto \tilde{r}(x, t)$ is strictly increasing for each $t \in [0, T]$, and $(x, t) \mapsto \tilde{r}(x, t)$ is continuous, one has $d_\varepsilon := \inf_{t \in [0, T]} \{\tilde{r}(\varepsilon, t) - \tilde{r}(\varepsilon/2, t)\} > 0$ by the compactness of $[0, T]$. By Lemma 2.5.4, there exists $N_\varepsilon \in \mathbb{N}$ such that $\sup_{t \in [0, T]} |\tilde{r}_{a_j}(\varepsilon/2, t) - \tilde{r}(\varepsilon/2, t)| \leq d_\varepsilon/2$ for all $j \geq N_\varepsilon$. Hence one has $\tilde{r}_{a_j}(\varepsilon/2, t) < \tilde{r}(\varepsilon, t)$ for all $j \geq N_\varepsilon$ and $t \in [0, T]$, which implies that if $(r, t) \in F_\varepsilon$, then $\tilde{r}_{a_j}(\varepsilon/2, t) < r$ for all $j \geq N_\varepsilon$. Combining this with Theorem 2.2.1(iv), we have

$$\begin{cases} \sup_{j \geq N_\varepsilon} |u_{a_j}(r_1, t) - u_{a_j}(r_2, t)| \leq C(\varepsilon/2)\sigma(t)^{-\frac{1}{2}}|r_1 - r_2|^{\frac{1}{2}} & \text{for all } (r_1, t), (r_2, t) \in F_\varepsilon, \\ \sup_{j \geq N_\varepsilon} |u_{a_j}(r, t_1) - u_{a_j}(r, t_2)| \leq C(\varepsilon/2)\sigma(t_1)^{-\frac{1}{2}}|t_2 - t_1|^{\frac{1}{4}} & \text{for all } (r, t_1), (r, t_2) \in F_\varepsilon. \end{cases} \quad (2.5.12)$$

In addition, it also follows from Theorem 2.2.1(iv) that

$$\sup_{j \geq N_\varepsilon} |u_{a_j}(r, t)| \leq C(\varepsilon)\sigma(t)^{-\frac{1}{4}} \leq \varepsilon^{-\frac{1}{4}}C(\varepsilon) \quad \text{for all } (r, t) \in F_\varepsilon. \quad (2.5.13)$$

In light of (2.5.11)–(2.5.13), we can apply Proposition A.0.1 to obtain a further sub-sequence (still denoted as a_j) and a continuous function $u(r, t) : F \setminus \{t = 0\} \rightarrow \mathbb{R}$ such that

$$\lim_{j \rightarrow \infty} \sup_{(r,t) \in K} |u_{a_j}(r, t) - u(r, t)| = 0 \quad \text{for all compact subset } K \subset\subset F \setminus \{t = 0\}, \quad (2.5.14)$$

which together with Theorem 2.2.1(iv) implies (2.5.8)–(2.5.10). \square

Corollary 2.5.1. *Let $(u, e)(r, t)$ be the functions obtained in Lemma 2.5.5. Then for each $\varepsilon > 0$,*

$$\begin{aligned} & \sup_{0 \leq t \leq T} \int_{\tilde{r}(\varepsilon, t)}^{\infty} \{ |(u, e - 1)|^2 + |(\sqrt{\sigma} \partial_r u, \sigma \partial_r e)|^2 \} (r, t) r^m dr \leq C(\varepsilon), \\ & \int_0^T \int_{\tilde{r}(\varepsilon, t)}^{\infty} \{ |(\partial_r u, \partial_r e, u \partial_r u)|^2 + |(\sqrt{\sigma} \partial_t u, \sigma \partial_t e)|^2 \} (r, t) r^m dr dt \leq C(\varepsilon). \end{aligned}$$

Proof. The proof follows exactly the same as that of Lemma 2.4.8. \square

Lemma 2.5.6. *For each $\varepsilon > 0$, if one defines $\rho_a^{(\varepsilon)}$ and $f^{(\varepsilon)}$ to be $\rho_a^{(\varepsilon)}(r, t) := \rho_a(r, t) \chi_\varepsilon^a(r, t)$ and $f^{(\varepsilon)}(r, t) := f(r, t) \chi_\varepsilon(r, t)$, where $\chi_\varepsilon^a, \chi_\varepsilon$ are the indicator functions defined as*

$$\chi_\varepsilon(r, t) := \begin{cases} 1 & \text{if } r \in [\tilde{r}(\varepsilon, t), \infty), \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \chi_\varepsilon^a(r, t) := \begin{cases} 1 & \text{if } r \in [\tilde{r}_a(\varepsilon, t), \infty), \\ 0 & \text{otherwise.} \end{cases} \quad (2.5.15)$$

(i) *Then there exists a function $\rho(r, t) \in L_{loc}^2(\mathbb{F}, r^m dr dt)$ with $\rho \geq 0$, and a sub-sequence $a_j \searrow 0$ as $j \rightarrow \infty$ such that,*

$$\begin{cases} \rho_{a_j}^{(\varepsilon)} - 1 \xrightarrow{*} \rho^{(\varepsilon)} - 1 \text{ as } j \rightarrow \infty \text{ in } L^\infty(0, T; L^2([0, \infty), r^m dr)), \\ \sup_{t \in [0, T]} \int_{\tilde{r}(\varepsilon, t)}^{\infty} |\rho - 1|^2(r, t) r^m dr \leq C(\varepsilon). \end{cases}$$

(ii) *For all $L \in \mathbb{N}$, if one defines $X_L := \tilde{H}^{-1}([0, L], r^m dr)$ then*

$$\begin{cases} \lim_{j \rightarrow \infty} \sup_{t \in [0, T]} \|\rho_{a_j}^{(\varepsilon)}(\cdot, t) - \rho^{(\varepsilon)}(\cdot, t)\|_{X_L} = 0, \quad \rho^{(\varepsilon)} \in C^0([0, T]; X_L), \\ \|\rho^{(\varepsilon)}(\cdot, t_1) - \rho^{(\varepsilon)}(\cdot, t_2)\|_{X_L} \leq C(\varepsilon) |t_1 - t_2| \text{ for all } t_1, t_2 \in [0, T], \end{cases}$$

where $C(\varepsilon) > 0$ is independent of $L \in \mathbb{N}$.

(iii) *Let $\tilde{r}(x, t)$ and $\underline{r}(t)$ be the functions obtained in Lemma 2.5.4. Then for all $x > 0$,*

$$x = \int_{\underline{r}(t)}^{\tilde{r}(x, t)} \rho(r, t) r^m dr \text{ for } t \in [0, T] \text{ almost everywhere.}$$

Proof. We split the proof into 3 steps, showing Lemma 2.5.6(i)–(iii) in chronological order.

Step 1: proof of Lemma 2.5.6(i). From Theorem 2.2.1(iv), we have that for all $j \in \mathbb{N}$,

$$\sup_{t \in [0, T]} \int_0^{\infty} |\rho_{a_j}^{(\varepsilon)} - 1|^2 r^m dr \leq \frac{(\tilde{r}_{a_j}(\varepsilon, t))^n}{n} + \sup_{t \in [0, T]} \int_{\tilde{r}_{a_j}(\varepsilon, t)}^{\infty} |\rho_{a_j} - 1|^2 r^m dr \leq C(\varepsilon).$$

Hence there exists $\tilde{w}_\varepsilon - 1 \in L^\infty(0, T; L^2([0, \infty), r^m dr))$ such that

$$\rho_{a_j}^{(\varepsilon)} - 1 \xrightarrow{*} \tilde{w}_\varepsilon - 1 \text{ weakly star as } j \rightarrow \infty \text{ in } L^\infty(0, T; L^2([0, \infty), r^m dr)). \quad (2.5.16)$$

For each $l \in \mathbb{N}$, denote $\varepsilon_l := l^{-1}$ and $\tilde{w}^{(l)} := \tilde{w}_{\varepsilon_l}$. Fix $l, q \in \mathbb{N}$ such that $l < q$. Then $\tilde{r}_a(\varepsilon_q, t) < \tilde{r}_a(\varepsilon_l, t)$ for each $(a, t) \in (0, 1) \times [0, T]$. Thus by construction, it follows that for all $j \in \mathbb{N}$, $\chi_{\varepsilon_l}^{a_j}(r, t) = 1 = \chi_{\varepsilon_q}^{a_j}(r, t)$ if $r \in [\tilde{r}_{a_j}(\varepsilon_l, t), \infty)$ and one has

$$\rho_{a_j}^{(\varepsilon_q)}(r, t) = \rho_{a_j}^{(\varepsilon_l)}(r, t) \text{ for a.e. } t \in [0, T] \text{ and } r \in [\tilde{r}_{a_j}(\varepsilon_l, t), \infty) \text{ for each } j \in \mathbb{N}. \quad (2.5.17)$$

Let $\phi \in C_c^\infty(\mathbb{F}^{(l)})$ where $\mathbb{F}^{(l)} := \{(r, t) : t \in [0, T], r \in [\tilde{r}(\varepsilon_l, t), \infty)\}$. Then it follows that $\phi\chi_{\varepsilon_l} \in L^1(0, T; L^2([0, \infty), r^m dr))$, and one has

$$\begin{aligned} & \int_0^T \int_{\tilde{r}(\varepsilon_l, t)}^\infty \{\tilde{w}^{(q)} - \tilde{w}^{(l)}\} \phi(r, t) r^m dr dt \\ &= \int_0^T \int_0^\infty \{\tilde{w}^{(q)} - \rho_{a_j}^{(\varepsilon_q)}\} \chi_{\varepsilon_l} \phi r^m dr dt + \int_0^T \int_{\tilde{r}(\varepsilon_l, t)}^{\tilde{r}_{a_j}(\varepsilon_l, t)} \{\rho_{a_j}^{(\varepsilon_q)} - \rho_{a_j}^{(\varepsilon_l)}\} \phi r^m dr dt \\ & \quad + \int_0^T \int_{\tilde{r}_{a_j}(\varepsilon_l, t)}^\infty \{\rho_{a_j}^{(\varepsilon_q)} - \rho_{a_j}^{(\varepsilon_l)}\} \phi r^m dr dt + \int_0^T \int_0^\infty \{\rho_{a_j}^{(\varepsilon_l)} - \tilde{w}^{(l)}\} \chi_{\varepsilon_l} \phi r^m dr dt = \sum_{i=1}^4 I_i^{(j)}. \end{aligned}$$

The limit of $I_1^{(j)}$ and $I_4^{(j)}$ is evaluated by the weak star convergence (2.5.16). As $j \rightarrow \infty$,

$$\begin{aligned} & I_1^{(j)} + I_4^{(j)} \\ &:= \int_0^T \int_0^\infty \{\tilde{w}^{(q)} - \rho_{a_j}^{(\varepsilon_q)}\} \chi_{\varepsilon_l} \phi(r, t) r^m dr dt + \int_0^T \int_0^\infty \{\rho_{a_j}^{(\varepsilon_l)} - \tilde{w}^{(l)}\} \chi_{\varepsilon_l} \phi(r, t) r^m dr dt \rightarrow 0. \end{aligned}$$

Using (2.5.17), one also obtains

$$I_3^{(j)} := \int_0^T \int_{\tilde{r}_{a_j}(\varepsilon_l, t)}^\infty \{\rho_{a_j}^{(\varepsilon_q)} - \rho_{a_j}^{(\varepsilon_l)}\} \phi(r, t) r^m dr dt = 0.$$

For $I_2^{(j)}$, one first observes that

$$|\chi_{\varepsilon_l}^{a_j} - \chi_{\varepsilon_l}|(r, t) = \begin{cases} 1 & \text{if } r \in [\tilde{r}_{a_j}(\varepsilon_l, t), \tilde{r}(\varepsilon_l, t)] \text{ or } r \in [\tilde{r}(\varepsilon_l, t), \tilde{r}_{a_j}(\varepsilon_l, t)], \\ 0 & \text{otherwise.} \end{cases}$$

By the uniform convergence from Lemma 2.5.4, it follows that $|\chi_{\varepsilon_l}^{a_j} - \chi_{\varepsilon_l}|(r, t) \rightarrow 0$ as $j \rightarrow \infty$ for a.e. $(r, t) \in [0, \infty) \times [0, T]$. Using estimates from Theorem 2.2.1(iv), and Dominated Convergence theorem, one has as $j \rightarrow \infty$,

$$\begin{aligned} |I_2^{(j)}| &:= \left| \int_0^T \int_{\tilde{r}(\varepsilon_l, t)}^{\tilde{r}_{a_j}(\varepsilon_l, t)} \{\rho_{a_j}^{(\varepsilon_q)} - \rho_{a_j}^{(\varepsilon_l)}\} \phi r^m dr dt \right| \leq \int_0^T \int_0^\infty |\chi_{\varepsilon_l}^{a_j} - \chi_{\varepsilon_l}| |\rho_{a_j}^{(\varepsilon_q)} - \rho_{a_j}^{(\varepsilon_l)}| |\phi| r^m dr dt \\ &\leq \int_0^T \left(\int_0^\infty |\chi_{\varepsilon_l}^{a_j} - \chi_{\varepsilon_l}|^2 |\phi(r, t)|^2 r^m dr \right)^{1/2} \{ \|\rho_{a_j}^{(\varepsilon_q)}(\cdot, t) - 1\|_{L^2} + \|\rho_{a_j}^{(\varepsilon_l)}(\cdot, t) - 1\|_{L^2} \} dt \\ &\leq (C(\varepsilon_l) + C(\varepsilon_q)) \int_0^T \left(\int_0^\infty |\chi_{\varepsilon_l}^{a_j} - \chi_{\varepsilon_l}|^2 |\phi(r, t)|^2 r^m dr \right)^{1/2} dt \rightarrow 0. \end{aligned}$$

Summarising the convergence for $I_1^{(j)} - I_4^{(j)}$ as $j \rightarrow \infty$, it follows that

$$\int_0^T \int_{\tilde{r}(\varepsilon_l, t)}^\infty \{\tilde{w}^{(q)} - \tilde{w}^{(l)}\} \phi(r, t) r^m dr dt = 0 \quad \text{for all } \phi \in C_c^\infty(\mathbb{F}^{(l)}).$$

By Fundamental Theorem of Calculus of Variation, it follows that if $l \leq q \in \mathbb{N}$ then

$$\tilde{w}^{(q)}(r, t) = \tilde{w}^{(l)}(r, t) \quad \text{for almost every } t \in [0, T], \text{ and } r \in [\tilde{r}(\varepsilon_l, t), \infty). \quad (2.5.18)$$

In light of above, $\tilde{w}^{(q)}$ is truncated in the domain $(r, t) \in \mathbb{F}$ for each $q \in \mathbb{N}$ as:

$$w^{(q)}(r, t) := \begin{cases} \tilde{w}^{(q)}(r, t) & \text{if } t \in [0, T] \text{ and } r \in [\tilde{r}(\varepsilon_q, t), \infty), \\ 0 & \text{otherwise.} \end{cases}$$

Using this, one also defines for each $N \in \mathbb{N}$,

$$h_N(r, t) := w^{(1)}(r, t) + \sum_{i=1}^N \{w^{(i+1)}(r, t) - w^{(i)}(r, t)\} \quad \text{for } (r, t) \in \mathbb{F}.$$

Fix $(r, t) \in \mathbb{F}$. Lemma 2.5.4(iii)–(iv) imply that $\lim_{x \searrow 0} \tilde{r}(x, t) = \underline{r}(t)$, $\lim_{x \rightarrow \infty} \tilde{r}(x, t) = \infty$. Since $\underline{r}(t) < r$, there exists $q \in \mathbb{N}$ such that $\tilde{r}(\varepsilon_q, t) \leq r < \tilde{r}(\varepsilon_{q-1}, t)$. Thus (2.5.18) implies,

$$h_N(r, t) = w^{(1)}(r, t) + \sum_{i=1}^N \{w^{(i+1)}(r, t) - w^{(i)}(r, t)\} = w^{(q)}(r, t) = \tilde{w}^{(q)}(r, t) \quad \text{if } N \geq q + 1.$$

Thus for a.e. $(r, t) \in \mathbb{F}$, the limit $\lim_{N \rightarrow \infty} h_N(r, t) = \tilde{w}^{(q)}(r, t)$ exists for some $q \in \mathbb{N}$, and the following function is well-defined almost everywhere in $(0, \infty) \times [0, T]$:

$$\rho(r, t) := \begin{cases} \lim_{N \rightarrow \infty} h_N(r, t) & \text{for a.e. } (r, t) \in \mathbb{F}. \\ 0 & \text{for } (r, t) \in (0, \infty) \times [0, T] \setminus \mathbb{F}. \end{cases}$$

Now let $\varepsilon > 0$. Then there exists some integer $q \in \mathbb{N}$ such that $\varepsilon_q \leq \varepsilon$ and it follows from the above construction that:

$$\rho^{(\varepsilon)}(r, t) = \chi_\varepsilon(r, t)\rho(r, t) = \chi_\varepsilon(r, t)\tilde{w}^{(q)}(r, t) \quad \text{for a.e. } (r, t) \in (0, \infty) \times [0, T].$$

Let $\phi \in L^1(0, T; L^2([0, \infty), r^m dr))$. Using $\tilde{r}_{a_j}(\varepsilon_q, t) < \tilde{r}_{a_j}(\varepsilon, t)$ for each $j \in \mathbb{N}$, one has

$$\chi_\varepsilon^{(a_j)} = \chi_{\varepsilon_q}^{(a_j)} \chi_\varepsilon^{(a_j)} \quad \text{for all } (r, t) \in [0, \infty) \times [0, T] \text{ and } j \in \mathbb{N}. \quad (2.5.19)$$

Using this, it then follows that

$$\begin{aligned} & \int_0^T \int_0^\infty \{\rho^{(\varepsilon)} - \rho_{a_j}^{(\varepsilon)}\} \phi r^m dr dt = \int_0^T \int_0^\infty \{\chi_\varepsilon \tilde{w}^{(q)} - \chi_\varepsilon^{a_j} \rho_{a_j}\} \phi r^m dr dt \\ &= \int_0^T \int_0^\infty \{\chi_\varepsilon \tilde{w}^{(q)} - \chi_\varepsilon \chi_{\varepsilon_q}^{a_j} \rho_{a_j}\} \phi r^m dr dt + \int_0^T \int_0^\infty \{\chi_\varepsilon \chi_{\varepsilon_q}^{a_j} - \chi_\varepsilon^{a_j}\} \rho_{a_j} \phi r^m dr dt \\ &= \int_0^T \int_0^\infty \chi_\varepsilon \phi \{\tilde{w}^{(q)} - \rho_{a_j}^{(\varepsilon_q)}\} r^m dr dt + \int_0^T \int_0^\infty \{\chi_\varepsilon - \chi_\varepsilon^{a_j}\} \rho_{a_j}^{(\varepsilon_q)} \phi r^m dr dt =: J_1^{(j)} + J_2^{(j)}. \end{aligned}$$

By the weak-star convergence for each $q \in \mathbb{N}$ in (2.5.16), one has that $J_1^{(j)} \rightarrow 0$ as $j \rightarrow \infty$. For the term $J_2^{(j)}$, one uses the point-wise bound of ρ_a from Theorem 2.2.1(iv) to get:

$$|(\chi_\varepsilon - \chi_\varepsilon^{a_j}) \rho_{a_j}^{(\varepsilon_q)} \phi| \leq C(\varepsilon_q) |\phi| \in L^1(0, T; L^2([0, \infty), r^m dr)).$$

Moreover, due to the uniform convergence from Lemma 2.5.4, one has $(\chi_\varepsilon - \chi_\varepsilon^{(a_j)})(r, t) \rightarrow 0$ as $j \rightarrow \infty$ for a.e. $(r, t) \in [0, \infty) \times [0, T]$. Hence by Dominated Convergence theorem,

$$J_2^{(j)} := \int_0^T \int_0^\infty (\chi_\varepsilon - \chi_\varepsilon^{a_j}) \rho_{a_j}^{(\varepsilon_q)} \phi r^m dr dt \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

Combining the limit of $J_1^{(j)}$, $J_2^{(j)}$ as $j \rightarrow \infty$, it follows that there exists $\rho \in L_{\text{loc}}^2(\mathbb{F})$ such that

$$\rho_{a_j}^{(\varepsilon)} \xrightarrow{*} \rho^{(\varepsilon)} \quad \text{as } j \rightarrow \infty \text{ in } L^\infty(0, T; L^2([0, \infty), r^m dr)), \text{ for each } \varepsilon > 0. \quad (2.5.20)$$

This concludes the proof of Lemma 2.5.6(i).

Step 2: proof of Lemma 2.5.6(ii). By Lemma 2.5.4, if one sets $r_\varepsilon := \inf_{t \in [0, T]} \tilde{r}(\varepsilon, t)/2$, then there exists $N_\varepsilon \in \mathbb{N}$ such that $r_\varepsilon < \tilde{r}_{a_j}(\varepsilon, t)$ for all $j \geq N_\varepsilon$ and $t \in [0, T]$. By Theorem 2.2.1(v), we have for each $\varepsilon > 0$ and $t_1, t_2 \in [0, T]$,

$$\sup_{j \geq N_\varepsilon} \|\rho_{a_j}^{(\varepsilon)}(\cdot, t_1) - \rho_{a_j}^{(\varepsilon)}(\cdot, t_2)\|_{\tilde{H}^{-1}([r_\varepsilon, L], r^m dr)} \leq C(\varepsilon)|t_1 - t_2| \quad \text{for all } L \in \mathbb{N}.$$

Moreover it also follows that for all $j \geq N_\varepsilon$ and $L \in \mathbb{N}$,

$$\rho_{a_j}^{(\varepsilon)} \in C^0([0, T]; \tilde{H}^{-1}([r_\varepsilon, L], r^m dr)), \quad \sup_{j \geq N_\varepsilon} \|\rho_{a_j}^{(\varepsilon)} - 1\|_{L^\infty(0, T; L^2([r_\varepsilon, \infty), r^m dr))} \leq C(\varepsilon).$$

Therefore by Proposition A.0.2 from Appendix A, there exists \tilde{h}_ε such that for all $L \in \mathbb{N}$,

$$\left\{ \begin{array}{l} \tilde{h}_\varepsilon - 1 \in L^\infty(0, T; L^2([r_\varepsilon, \infty), r^m dr)), \quad \tilde{h}_\varepsilon \in C^0([0, T]; \tilde{H}^{-1}([r_\varepsilon, L], r^m dr)), \\ \lim_{j \rightarrow \infty} \sup_{t \in [0, T]} \|\rho_{a_j}^{(\varepsilon)}(\cdot, t) - \tilde{h}_\varepsilon(\cdot, t)\|_{H^{-1}([r_\varepsilon, L], r^m dr)} = 0, \\ \|\tilde{h}_\varepsilon(\cdot, t_1) - \tilde{h}_\varepsilon(\cdot, t_2)\|_{H^{-1}([r_\varepsilon, L], r^m dr)} \leq C(\varepsilon)|t_1 - t_2| \quad \text{for each } t_1, t_2 \in [0, T]. \end{array} \right. \quad (2.5.21)$$

We claim that $\rho^{(\varepsilon)} = \tilde{h}_\varepsilon$ for a.e. $(r, t) \in [r_\varepsilon, \infty) \times [0, T]$. Let $\phi \in C_c^\infty([r_\varepsilon, \infty) \times [0, T])$. Then there exists $L \in \mathbb{N}$ such that $\text{supp}(\phi) \subseteq [r_\varepsilon, L] \times [0, T]$. Using (2.5.20)–(2.5.21), we have as $j \rightarrow \infty$,

$$\begin{aligned} & \int_0^T \int_{r_\varepsilon}^\infty (\rho^{(\varepsilon)} - \tilde{h}_\varepsilon) \phi r^m dr dt = \int_0^T \int_{r_\varepsilon}^\infty (\rho^{(\varepsilon)} - \rho_{a_j}^{(\varepsilon)}) \phi r^m dr dt + \int_0^T \int_{r_\varepsilon}^L (\rho_{a_j}^{(\varepsilon)} - \tilde{h}_\varepsilon) \phi r^m dr dt \\ & \leq \int_0^T \int_{r_\varepsilon}^\infty \{\rho^{(\varepsilon)} - \rho_{a_j}^{(\varepsilon)}\} \phi r^m dr dt + \int_0^T \|\rho_{a_j}^{(\varepsilon)}(\cdot, t) - \tilde{h}_\varepsilon(\cdot, t)\|_{\tilde{H}^{-1}} \|\phi\|_{H^1} dt \rightarrow 0. \end{aligned}$$

By Fundamental Lemma of Calculus of Variation, $\rho^{(\varepsilon)} = \tilde{h}_\varepsilon$ for a.e. $(r, t) \in [r_\varepsilon, \infty) \times [0, T]$. Lemma 2.5.6(ii) then follows since $\text{supp}(\rho^{(\varepsilon)}(\cdot, t)) = [\tilde{r}(\varepsilon, t), \infty)$, and $r_\varepsilon < \tilde{r}(\varepsilon, t)$ for all $t \in [0, T]$.

Step 3: proof of Lemma 2.5.6(iii). Let $\xi \in C_c^\infty(0, T)$ and $0 < y < x$. By (2.5.20), we have,

$$\begin{aligned} & \int_0^T \xi(t) \int_{\tilde{r}_{a_j}(y, t)}^{\tilde{r}(x, t)} \rho_{a_j}(r, t) r^m dr dt - \int_0^T \xi(t) \int_{\tilde{r}(y, t)}^{\tilde{r}(x, t)} \rho(r, t) r^m dr dt \\ & = \int_0^T \int_0^{\tilde{r}(x, t)} \xi(\chi_y^{\rho_{a_j}} - \chi_y^\rho) r^m dr dt = \int_0^T \int_0^\infty (1 - \chi_x) \xi \{\rho_{a_j}^{(y)} - \rho^{(y)}\} r^m dr dt \rightarrow 0, \end{aligned}$$

as $j \rightarrow \infty$, where one used the fact that $(1 - \chi_x)\xi \in L^1(0, T; L^2([0, \infty), r^m dr))$. Hence,

$$\lim_{j \rightarrow \infty} \int_0^T \int_{\tilde{r}_{a_j}(y, t)}^{\tilde{r}(x, t)} \xi(t) \rho_{a_j}(r, t) r^m dr dt = \int_0^T \xi(t) \int_{\tilde{r}(y, t)}^{\tilde{r}(x, t)} \rho(r, t) r^m dr dt. \quad (2.5.22)$$

Moreover, by Lemma 2.5.1, $\lim_{y \searrow 0} \tilde{r}(y, t) = \underline{r}(t)$, and Dominated Convergence theorem,

$$\limsup_{y \searrow 0} \sup_{j \in \mathbb{N}} \left| \int_0^T \int_{\underline{r}(t)}^{\tilde{r}(y, t)} \xi \rho_{a_j} r^m dr dt \right| \leq \lim_{y \searrow 0} \int_0^T \omega_1([\underline{r}(t), \tilde{r}(y, t)]; C(T)) |\xi(t)| dt = 0.$$

For a given $\delta > 0$, the above limit implies that there exists $z_\delta > 0$ such that if $y \in (0, z_\delta)$ then

$$\sup_{j \in \mathbb{N}} \left| \int_0^T \int_{\underline{r}(t)}^{\tilde{r}(y, t)} \xi(t) \rho_{a_j}(r, t) r^m dr dt \right| \leq \delta. \quad (2.5.23)$$

Now fix $y \in (0, z_\delta)$. By Theorem 2.2.1(iii), we have for a.e. $t \in [0, T]$,

$$x = \int_{a_j}^{\tilde{r}_{a_j}(x,t)} \rho_{a_j}(r,t) r^m dr = \left(\int_{a_j}^{\underline{r}(t)} + \int_{\underline{r}(t)}^{\tilde{r}(y,t)} + \int_{\tilde{r}(y,t)}^{\tilde{r}_{a_j}(y,t)} + \int_{\tilde{r}_{a_j}(y,t)}^{\tilde{r}(x,t)} + \int_{\tilde{r}(x,t)}^{\tilde{r}_{a_j}(x,t)} \right) \rho_{a_j} r^m dr.$$

Multiplying the above equality with $\xi(t)$, then integrating in $t \in [0, T]$, and taking limit as $j \rightarrow \infty$, it follows from (2.5.22) and (2.5.23) that

$$-\delta \leq \lim_{j \rightarrow \infty} \sum_{i=1}^3 \mathcal{I}_i^{(j)} + \int_0^T \xi(t) \left\{ -x + \int_{\tilde{r}(y,t)}^{\tilde{r}(x,t)} \rho(r,t) r^m dr \right\} dt \leq \delta. \quad (2.5.24)$$

Repeating the same proof for assertion (v) of Lemma 2.5.4, one obtains

$$\lim_{j \rightarrow \infty} |\mathcal{I}_1^{(j)}| = \lim_{j \rightarrow \infty} \left| \int_0^T \xi(t) \int_{a_j}^{\underline{r}(t)} \rho_{a_j}(r,t) r^m dr dt \right| = 0.$$

Moreover by Lemma 2.5.1, 2.5.4, and Dominated Convergence Theorem, we also have that

$$\begin{aligned} \lim_{j \rightarrow \infty} |\mathcal{I}_2^{(j)}| + |\mathcal{I}_3^{(j)}| &\leq \lim_{j \rightarrow \infty} \int_0^T |\xi(t)| \left(\int_{\tilde{r}(y,t)}^{\tilde{r}_{a_j}(y,t)} + \int_{\tilde{r}(x,t)}^{\tilde{r}_{a_j}(x,t)} \right) \rho_{a_j}(r,t) r^m dr dt \\ &\leq \lim_{j \rightarrow \infty} \int_0^T |\xi(t)| \left\{ \omega_1([\tilde{r}(y,t), \tilde{r}_{a_j}(y,t)]; C(T)) + \omega_1([\tilde{r}(x,t), \tilde{r}_{a_j}(x,t)]; C(T)) \right\} dt = 0. \end{aligned}$$

It follows from (2.5.24) that, for each $\delta > 0$, there exists $z_\delta > 0$ such that if $y \in (0, z_\delta)$ then

$$-\delta \leq \int_0^T \xi(t) \left\{ -x + \int_{\tilde{r}(y,t)}^{\tilde{r}(x,t)} \rho(r,t) r^m dr \right\} dt \leq \delta.$$

Let $\{\xi_i\}_{i \in \mathbb{N}} \in C_c^\infty(0, T)$ be such that $\xi_i \geq 0$, $\|\xi_i\|_{L^1([0, T])} = 1$, and $\{\xi_i\}_{i \in \mathbb{N}}$ is a sequence of kernels approximating Dirac distribution. Then for all $(i, y) \in \mathbb{N} \times (0, z_\delta)$:

$$-\delta \leq -x + \int_0^T \xi_i(t) \int_{\tilde{r}(y,t)}^{\tilde{r}(x,t)} \rho(r,t) r^m dr dt \leq \delta.$$

Since $\rho \geq 0$, one can apply Monotone Convergence theorem with the limit $y \searrow 0$ to obtain

$$-\delta \leq -x + \int_0^T \xi_i(t) \int_{\underline{r}(t)}^{\tilde{r}(x,t)} \rho(r,t) r^m dr dt \leq \delta.$$

Since this is true for arbitrarily small $\delta > 0$, it follows that

$$x = \int_0^T \xi_i(t) \int_{\underline{r}(t)}^{\tilde{r}(x,t)} \rho(r,t) r^m dr dt, \quad \text{for each } i \in \mathbb{N}.$$

Finally taking $i \rightarrow \infty$, and since ξ_i converges to Dirac-delta in the sense of distributions, one concludes that

$$x = \int_{\underline{r}(t)}^{\tilde{r}(x,t)} \rho(r,t) r^m dr, \quad \text{for } t \in [0, T] \text{ almost everywhere.}$$

□

Corollary 2.5.2. *Let ρ be the function obtained in Lemma 2.5.6. Then for each $\varepsilon > 0$,*

$$C(\varepsilon)^{-1} \leq \rho(r, t) \leq C(\varepsilon) \quad \text{for almost every } t \in [0, T], \text{ and } r \in [\tilde{r}(\varepsilon, t), \infty).$$

Proof. The proof is exactly the same as that of Proposition 2.4.1 and Lemma 2.4.5. \square

In light of Lemma 2.5.4(v), and Lemma 2.5.6(iii), we extend $\rho \in L^2_{\text{loc}}(\mathbb{F})$ obtained above by setting $\rho(r, t) = 0$ for $(r, t) \in \mathbb{R}^n \times [0, T] \setminus \mathbb{F}$.

Lemma 2.5.7. *Let $(\rho, u, e)(r, t)$ be the limit function obtained in Lemma 2.5.5–2.5.6. Then*

$$\text{ess sup}_{t \in [0, T]} \int_E \rho e(r, t) r^m dr \leq C(T) + \omega_2(E; C(T)) \quad \text{for bounded measurable } E \subset [0, \infty),$$

where $\omega_2(E, z)$ for $z > 0$ is the set function given in Lemma 2.5.2, and

$$\text{ess sup}_{t \in (0, T)} \int_0^\infty \{G(\rho) + \rho|u|^2\}(r, t) r^m dr \leq C(T).$$

Proof. Let $\varepsilon \in (0, 1)$. Denote $\rho^{(\varepsilon)}$ and $\rho_{a_j}^{(\varepsilon)}$ as in Lemma 2.5.6. Let $E \subset [0, \infty)$ be a bounded measurable set and $\mathbb{1}_E(r)$ be the indicator function. Since $x \rightarrow \tilde{r}(x, t)$ is strictly increasing and $(x, t) \mapsto \tilde{r}(x, t)$ is a continuous function from Lemma 2.5.4, combining with the fact that $[0, T]$ is a compact set, one has $d_\varepsilon =: 2 \inf_{t \in [0, T]} \{\tilde{r}(\varepsilon, t) - \tilde{r}(\varepsilon/2, t)\} > 0$. By Lemma 2.5.4, there exists $N_\varepsilon \in \mathbb{N}$ such that if $j \geq N_\varepsilon$ then $\sup_{0 \leq t \leq T} |\tilde{r}_{a_j}(\varepsilon, t) - \tilde{r}(\varepsilon, t)| \leq d_\varepsilon/2$. Hence,

$$\tilde{r}_{a_j}(\varepsilon, t) > \tilde{r}(\varepsilon/2, t) \quad \text{for all } t \in [0, T] \text{ and } j \geq N_\varepsilon. \quad (2.5.25)$$

Let $[t_1, t_2] \subset (0, T]$. Then by (2.5.25), it follows that as $j \rightarrow \infty$,

$$\begin{aligned} & \int_{t_1}^{t_2} \int_{\tilde{r}_{a_j}(\varepsilon, t)}^\infty \mathbb{1}_E \rho_{a_j} |u_{a_j}|^2 r^m dr dt - \int_{t_1}^{t_2} \int_{\tilde{r}(\varepsilon, t)}^\infty \mathbb{1}_E \rho |u|^2 r^m dr dt \\ &= \int_{t_1}^{t_2} \int_{\tilde{r}(\varepsilon/2, t)}^\infty \mathbb{1}_E \rho_{a_j}^{(\varepsilon)} |u_{a_j}|^2 r^m dr dt - \int_{t_1}^{t_2} \int_{\tilde{r}(\varepsilon/2, t)}^\infty \mathbb{1}_E \rho^{(\varepsilon)} |u|^2 r^m dr dt \\ &= \int_{t_1}^{t_2} \int_{\tilde{r}(\varepsilon/2, t)}^\infty \mathbb{1}_E \rho_{a_j}^{(\varepsilon)} \{|u_{a_j}|^2 - |u|^2\} r^m dr dt + \int_{t_1}^{t_2} \int_{\tilde{r}(\varepsilon/2, t)}^\infty \mathbb{1}_E \{\rho_{a_j}^{(\varepsilon)} - \rho^{(\varepsilon)}\} |u|^2 r^m dr dt \rightarrow 0, \end{aligned}$$

where for the first convergence, one used Dominated Convergence theorem, Lemma 2.5.5 and point-wise bound for $|u_{a_j}|$ and $\rho_{a_j}^{(\varepsilon)}$ from Theorem 2.2.1(iv). For the second convergence one used Lemma 2.5.6 and Corollary 2.5.1. Let $L \in \mathbb{N}$ and set $E \equiv [0, L]$. By Theorem 2.2.1(i),

$$\begin{aligned} \int_{t_1}^{t_2} \int_{\tilde{r}(\varepsilon, t)}^L \rho |u|^2 r^m dr dt &= \lim_{j \rightarrow \infty} \int_{t_1}^{t_2} \int_{\tilde{r}_{a_j}(\varepsilon, t)}^L \rho_{a_j} |u_{a_j}|^2 r^m dr dt \\ &\leq |t_2 - t_1| \text{ess sup}_{t \in (0, T)} \int_{a_j}^\infty \rho_{a_j} |u_{a_j}|^2(r, t) r^m dr = C(T) |t_2 - t_1|, \end{aligned}$$

where $C(T) = C(T, C_0)$ is independent of $a \in (0, 1)$, $\varepsilon > 0$ and $L \in \mathbb{N}$. Thus by Lebesgue Differentiation theorem, it follows that

$$\text{ess sup}_{t \in (0, T)} \int_{\tilde{r}(\varepsilon, t)}^L \rho |u|^2(r, t) r^m dr \leq C(T).$$

Using Monotone Convergence theorem with the consecutive limits $L \rightarrow \infty$ then $\varepsilon \searrow 0$,

$$\operatorname{ess\,sup}_{t \in (0, T)} \int_{\underline{r}(t)}^{\infty} \rho |u|^2(r, t) r^m dr = \lim_{\substack{L \rightarrow \infty \\ \varepsilon \rightarrow 0^+}} \operatorname{ess\,sup}_{t \in (0, T)} \int_{\tilde{r}(\varepsilon, t)}^L \rho |u|^2(r, t) r^m dr \leq C(T).$$

Next, let $[t_1, t_2] \subseteq (0, T]$ and $E \subset [0, \infty)$ be a bounded measurable set. Using (2.5.25), convergence results from Lemma 2.5.5–2.5.6, and the bound $e(r, t) \leq C(\varepsilon)\sigma(t)^{-1/2}$ for $r \geq \tilde{r}(\varepsilon/2, t)$ from Lemma 2.5.5, one has that as $j \rightarrow \infty$

$$\begin{aligned} & \int_{t_1}^{t_2} \int_{\tilde{r}_{a_j}(\varepsilon, t)}^{\infty} \mathbb{1}_E \rho_{a_j} e_{a_j} r^m dr dt - \int_{t_1}^{t_2} \int_{\tilde{r}(\varepsilon, t)}^{\infty} \mathbb{1}_E \rho e r^m dr dt \\ &= \int_{t_1}^{t_2} \int_{\tilde{r}(\varepsilon/2, t)}^{\infty} \mathbb{1}_E \rho_{a_j}^{(\varepsilon)} e_{a_j} r^m dr dt - \int_{t_1}^{t_2} \int_{\tilde{r}(\varepsilon/2, t)}^{\infty} \mathbb{1}_E \rho^{(\varepsilon)} e r^m dr dt \\ &= \int_{t_1}^{t_2} \int_{\tilde{r}(\varepsilon/2, t)}^{\infty} \mathbb{1}_E \rho_{a_j}^{(\varepsilon)} (e_{a_j} - e) r^m dr dt + \int_{t_1}^{t_2} \int_{\tilde{r}(\varepsilon/2, t)}^{\infty} \mathbb{1}_E (\rho_{a_j}^{(\varepsilon)} - \rho^{(\varepsilon)}) e r^m dr dt \rightarrow 0. \end{aligned}$$

Let $E_j^\varepsilon(t) := E \cap [\tilde{r}_{a_j}(\varepsilon, t), \infty)$. Then by Lemma 2.5.2, one has

$$\begin{aligned} & \int_{t_1}^{t_2} \int_{\tilde{r}(\varepsilon, t)}^{\infty} \mathbb{1}_E \rho e r^m dr dt = \lim_{j \rightarrow \infty} \int_{t_1}^{t_2} \int_{\tilde{r}_{a_j}(\varepsilon, t)}^{\infty} \mathbb{1}_E \rho_{a_j} e_{a_j} r^m dr dt \\ & \leq \sup_{j \in \mathbb{N}} \int_{t_1}^{t_2} \int_{E_j^\varepsilon(t)} \rho_{a_j} e_{a_j} r^m dr dt \leq \int_{t_1}^{t_2} \{C(T) + \omega_2(E_j^\varepsilon(t); C(T))\} dt, \end{aligned}$$

$$\text{where } \omega_2(E_j^\varepsilon(t); C(T)) = f_2\left(f_1\left(\int_{E_j^\varepsilon(t)} r^m dr; C(T)\right); C(T)\right).$$

Since $E_j^\varepsilon(t) \subseteq E$ for all $(j, \varepsilon, t) \in \mathbb{N} \times (0, 1) \times [0, T]$, and $y \rightarrow f_1(y; C), f_2(y; C)$ are monotone increasing (See Proposition B.0.1), one has for all $(j, \varepsilon, t) \in \mathbb{N} \times (0, 1) \times [0, T]$,

$$\omega_2(E_j^\varepsilon(t); C(T)) \leq f_2\left(f_1\left(\int_E r^m dr; C(T)\right); C(T)\right) = \omega_2(E; C(T)).$$

Using this, (2.5.26) becomes

$$\int_{t_1}^{t_2} \int_{\tilde{r}(\varepsilon, t)}^{\infty} \mathbb{1}_E \rho e r^m dr dt \leq |t_2 - t_1| \{C(T) + \omega_2(E; C(T))\}.$$

Thus by Lebesgue Differentiation theorem, it follows that

$$\operatorname{ess\,sup}_{t \in [0, T]} \int_{\tilde{r}(\varepsilon, t)}^{\infty} \mathbb{1}_E \rho e(r, t) r^m dr \leq C(T) + \omega_2(E; C(T)).$$

Since $\rho = 0$ in $\{r < \underline{r}(t)\}$ and $\lim_{\varepsilon \searrow 0} \tilde{r}(\varepsilon, t) = \underline{r}(t)$, by Monotone Convergence theorem,

$$\operatorname{ess\,sup}_{t \in [0, T]} \int_E \rho e(r, t) r^m dr \leq C(T) + \omega_2(E; C(T)).$$

Let $\varepsilon \in (0, 1]$ and $r_\varepsilon := \inf_{t \in [0, T]} \tilde{r}(\varepsilon, t)/2$. By Lemma 2.5.4, there exists $M_\varepsilon \in \mathbb{N}$ such that

$$\inf_{t \in [0, T]} \tilde{r}_{a_j}(\varepsilon, t) > r_\varepsilon \text{ for all } j \geq M_\varepsilon. \quad (2.5.26)$$

Moreover, by Theorem 2.2.1(iii), $\tilde{r}_{a_j}(\varepsilon, t) \leq \tilde{r}_{a_j}(1, t) \leq C_0$ for all $(j, t) \in \mathbb{N} \times [0, T]$. By (2.5.26), Theorem 2.2.1(i), Lemma 2.5.6, Corollary 2.5.2, and the fact that $G(0) = 1$ one has

$$\begin{cases} \rho_{a_j}^{(\varepsilon)} - 1 \rightharpoonup \rho^{(\varepsilon)} - 1 \text{ as } j \rightarrow \infty \text{ weakly in } L^2(0, T; L^2([r_\varepsilon, \infty), r^m dr)), \\ \sup_{j \geq M_\varepsilon} \text{ess sup}_{t \in [0, T]} \int_{r_\varepsilon}^{\infty} G(\rho_{a_j}^{(\varepsilon)})(r, t) r^m dr \leq C(T), \\ 0 \leq \rho_{a_j}^{(\varepsilon)}(r, t) \leq C(\varepsilon) \text{ for a.e. } (r, t) \in [r_\varepsilon, \infty) \times [0, T] \text{ and for each } j \in \mathbb{N}. \end{cases}$$

Thus applying Proposition B.0.2 of Appendix B, it follows that

$$\text{ess sup}_{t \in (0, T)} \int_{r_\varepsilon}^{\infty} G(\rho^{(\varepsilon)})(r, t) r^m dr \leq C(T).$$

By (iv) of Lemma 2.5.4, $r_\varepsilon < \tilde{r}(\varepsilon, t) \leq C_0(1 + \varepsilon)^{1/n} \leq 2^{1/n} C_0$ for $t \in [0, T]$. Since $G(\rho^{(\varepsilon)}) = G(0) = 1$ for $r \in [r_\varepsilon, \tilde{r}(\varepsilon, t)]$ and $\rho^{(\varepsilon)} = \rho$ for a.e. $r \in [\tilde{r}(\varepsilon, t), \infty)$, one has for a.e. $t \in [0, T]$,

$$\int_{\tilde{r}(\varepsilon, t)}^{\infty} G(\rho)(r, t) r^m dr \leq \int_{r_\varepsilon}^{\tilde{r}(\varepsilon, t)} r^m dr + \int_{\tilde{r}(\varepsilon, t)}^{\infty} G(\rho)(r, t) r^m dr \leq C(T).$$

Since the above is true for all $\varepsilon \in (0, 1]$, using $\lim_{\varepsilon \searrow 0} \tilde{r}(\varepsilon, t) = \underline{r}(t)$, $\rho \equiv 0$ for $r < \underline{r}(t)$, and Monotone Convergence Theorem, it follows that

$$\text{ess sup}_{t \in [0, T]} \int_0^{\infty} G(\rho)(r, t) r^m dr = \lim_{\varepsilon \searrow 0} \text{ess sup}_{t \in [0, T]} \int_{\tilde{r}(\varepsilon, t)}^{\infty} G(\rho)(r, t) r^m dr \leq C(T).$$

□

2.5.4 Weak forms away from the vacuum

In this subsection, we denote $(\rho, u, e)(r, t)$ in F, and $\tilde{r}(x, t)$ in $[0, \infty) \times [0, T]$ as the limit functions obtained in Lemmas 2.5.4–2.5.6. The main aim is to show that they satisfies the weak forms described in Definition 2.1.1. Moreover, we also define a class of test functions \mathcal{D}_T as:

$$\mathcal{D}_T := \{\phi \in C^1([0, \infty) \times [0, T]) : \exists L \in \mathbb{N} \text{ s.t. } \phi(r, t) = 0 \text{ for } (r, t) \in [L, \infty) \times [0, T]\}.$$

Lemma 2.5.8. *Let $(\rho_0, u_0, e_0)(r)$ be the spherically symmetric initial data in Theorem 2.1.1, and let $(\rho_a^0, u_a^0, e_a^0)(r)$ be the modified initial data constructed in Section 2.2.2. Then for all $\phi \in \mathcal{D}_T$,*

$$\begin{aligned} \lim_{a \searrow 0} \int_a^{\infty} \rho_a^0(r) \phi(r, 0) r^m dr &= \int_0^{\infty} \rho_0(r) \phi(r, 0) r^m dr, \\ \lim_{a \searrow 0} \int_a^{\infty} (\rho_a^0 u_a^0)(r) \phi(r, 0) r^m dr &= \int_0^{\infty} (\rho_0 u_0)(r) \phi(r, 0) r^m dr, \\ \lim_{a \searrow 0} \int_a^{\infty} \rho_a^0 \left\{ \frac{|u_a^0|^2}{2} + e_a^0 \right\}(r) \phi(r, 0) r^m dr &= \int_0^{\infty} \rho_0 \left\{ \frac{|u_0|^2}{2} + e_0 \right\}(r) \phi(r, 0) r^m dr. \end{aligned}$$

Proof. By construction (2.2.6) in Section 2.2.2, $(\rho_a^0, u_a^0, e_a^0)(r) = (1, 0, 1)$ for $r \in [0, a]$. Thus,

$$\begin{aligned} & \left| \int_a^{\infty} \rho_a^0(r) \phi(r, 0) r^m dr - \int_0^{\infty} \rho_0(r) \phi(r, 0) r^m dr \right| \\ & \leq \int_0^a |\phi(r, 0)| r^m dr + \|\phi(\cdot, 0)\|_{L^2([0, \infty), r^m dr)} \|\rho_a^0 - \rho_0\|_{L^2([0, \infty), r^m dr)} \rightarrow 0 \text{ as } a \searrow 0. \end{aligned}$$

where one has used Proposition 2.2.1. Second, by (2.2.6) and Proposition 2.2.1 again, one gets

$$\begin{aligned}
& \left| \int_a^\infty (\rho_a^0 u_a^0)(r) \phi(r, 0) r^m dr - \int_0^\infty (\rho_0 u_0)(r) \phi(r, 0) r^m dr \right| \\
&= \left| \int_0^\infty \phi(r, 0) \{ \rho_a^0 (u_a^0 - u_0) + u_0 (\rho_a^0 - \rho_0) \}(r) r^m dr \right| \\
&\leq \| \rho_a^0 \|_{L^\infty} \| \phi(\cdot, 0) \|_{L^2([0, \infty), r^m dr)} \| u_a^0 - u_0 \|_{L^2([0, \infty), r^m dr)} + \\
&\quad + \| \phi(\cdot, 0) \|_{L^\infty} \| u_0 \|_{L^2([0, \infty), r^m dr)} \| \rho_a^0 - \rho_0 \|_{L^2([0, \infty), r^m dr)} \rightarrow 0 \text{ as } a \searrow 0.
\end{aligned}$$

Finally, it follows from (2.2.6) and Proposition 2.2.1 that

$$\begin{aligned}
& \left| \int_a^\infty \rho_a^0 |u_a^0|^2(r) \phi(r, 0) r^m dr - \int_0^\infty \rho_0 |u_0|^2(r) \phi(r, 0) r^m dr \right| \\
&= \left| \int_0^\infty \phi(r, 0) \{ \rho_a^0 (|u_a^0|^2 - |u_0|^2) + |u_0|^2 (\rho_a^0 - \rho_0) \}(r) r^m dr \right| \\
&\leq \| \rho_a^0 \|_{L^\infty} \| \phi(\cdot, 0) \|_{L^2([0, \infty), r^m dr)} \| |u_a^0|^2 - |u_0|^2 \|_{L^2([0, \infty), r^m dr)} + \\
&\quad + \| \phi(\cdot, 0) \|_{L^\infty} \| u_0 \|_{L^4([0, \infty), r^m dr)}^2 \| \rho_a^0 - \rho_0 \|_{L^2([0, \infty), r^m dr)} \rightarrow 0 \text{ as } a \searrow 0.
\end{aligned}$$

On the other hand, one also has that as $a \searrow 0$,

$$\begin{aligned}
& \left| \int_a^\infty \rho_a^0 e_a^0(r) \phi(r, 0) r^m dr - \int_0^\infty \rho_0 e_0(r) \phi(r, 0) r^m dr \right| \\
&\leq \left| \int_0^a \phi(r, 0) r^m dr \right| + \left| \int_0^\infty \phi(r, 0) \{ \rho_a^0 (e_a^0 - e_0) + e_0 (\rho_a^0 - \rho_0) \} r^m dr \right| \\
&\leq \| \phi(\cdot, 0) \|_{L^\infty} \cdot a^n / n + \| \rho_a^0 \|_{L^\infty} \| \phi(\cdot, 0) \|_{L^2([0, \infty), r^m dr)} \| e_a^0 - e_0 \|_{L^2([0, \infty), r^m dr)} + \\
&\quad + \| \phi(\cdot, 0) \|_{L^\infty} \| e_0 - 1 \|_{L^2([0, \infty), r^m dr)} + \| \phi(\cdot, 0) \|_{L^2([0, \infty), r^m dr)} \| \rho_a^0 - \rho_0 \|_{L^2([0, \infty), r^m dr)} \rightarrow 0.
\end{aligned}$$

This proves the last assertion, hence the lemma. \square

Lemma 2.5.9. *For all $\phi \in \mathcal{D}_T$, one has:*

$$\int_0^\infty \rho(r, t) \phi(r, t) r^m dr - \int_0^\infty \rho_0(r) \phi(r, 0) r^m dr = \int_0^t \int_0^\infty (\rho \partial_t \phi + \rho u \partial_r \phi)(r, s) r^m dr ds.$$

Proof. First from continuity equation of Theorem 2.2.1, we have for all $j \in \mathbb{N}$, $a_j \in (0, 1)$,

$$\int_{a_j}^\infty \rho_{a_j}(r, s) \phi(r, s) r^m dr \Big|_{s=0}^{s=t} = \int_0^t \int_{a_j}^\infty (\rho_{a_j} \partial_t \phi + \rho_{a_j} u_{a_j} \partial_r \phi)(r, t) r^m dr dt. \quad (2.5.27)$$

By Lemma 2.5.1 and 2.5.6, and $\rho \equiv 0$ in $\{r < \underline{r}(t)\}$ by construction, we have

$$\begin{aligned}
\mathcal{I}_1^j &:= \left| \int_{a_j}^\infty \rho_{a_j}(r, t) \phi(r, t) r^m dr - \int_{\underline{r}(t)}^\infty \rho(r, t) \phi(r, t) r^m dr \right| \\
&\leq \left| \int_{\tilde{r}_{a_j}(x, t)}^\infty \phi \rho_{a_j}(r, t) r^m dr - \int_{\tilde{r}(x, t)}^\infty \phi \rho(r, t) r^m dr \right| + \| \phi \|_\infty \int_{\underline{r}(t)}^{\tilde{r}(x, t)} \rho(r, t) r^m dr \\
&\quad + \| \phi \|_\infty \left(\int_{a_j}^{\underline{r}(t)} + \int_{\underline{r}(t)}^{\tilde{r}(x, t)} + \int_{\tilde{r}(x, t)}^{\tilde{r}_{a_j}(x, t)} \right) \rho_{a_j}(r, t) r^m dr \\
&\leq \left| \int_0^\infty \{ \rho_{a_j}^{(x)} - \rho^{(x)} \} \phi(r, t) r^m dr \right| + \| \phi \|_\infty \cdot x + \| \phi \|_\infty \int_{a_j}^{\underline{r}(t)} \rho_{a_j}(r, t) r^m dr \\
&\quad + \| \phi \|_\infty \omega_1([\underline{r}(t), \tilde{r}(x, t)]; C) + \| \phi \|_\infty \omega_1([\tilde{r}(x, t), \tilde{r}_{a_j}(x, t)]; C).
\end{aligned}$$

Since $\tilde{r}(x, t) \searrow \underline{r}(t)$ as $x \searrow 0$, for given $\delta > 0$, there exists $y \in (0, 1)$ such that

$$\|\phi\|_\infty x + \|\phi\|_\infty \omega_1([\underline{r}(t), \tilde{r}(x, t)]; C) \leq \delta, \quad \text{for all } 0 < x < y.$$

Hence it follows that for all $0 < x < y$,

$$\mathcal{I}_1^j \leq \delta + \left| \int_0^\infty \phi(r, t) \{ \rho_{a_j}^{(x)} - \rho^{(x)} \} (r, t) r^m dr \right| + \|\phi\|_\infty \int_{a_j}^{\underline{r}(t)} \rho_{a_j}(r, t) r^m dr.$$

Fix a point $x \in (0, y)$. Taking limit $j \rightarrow \infty$ on the above inequality, then using Lemma 2.5.6(i) and Lemma 2.5.4(v), we have that:

$$\lim_{j \rightarrow \infty} \left| \int_{a_j}^\infty \rho_{a_j}(r, t) \phi(r, t) r^m dr - \int_{\underline{r}(t)}^\infty \rho(r, t) \phi(r, t) r^m dr \right| \leq \delta.$$

Since this is true for arbitrarily small $\delta > 0$, it follows that $\lim_{j \rightarrow \infty} \mathcal{I}_1^j = 0$.

Next, for each $x > 0$, we rewrite the term:

$$\begin{aligned} \mathcal{I}_2^j &:= \left| \int_0^t \int_{a_j}^\infty \rho_{a_j} u_{a_j} \partial_r \phi(r, s) r^m dr ds - \int_0^t \int_{\underline{r}(s)}^\infty \rho u \partial_r \phi(r, s) r^m dr ds \right| \\ &\leq \|\partial_r \phi\|_\infty \int_0^t \int_{a_j}^{\tilde{r}_{a_j}(x, s)} |\rho_{a_j} u_{a_j}| r^m dr ds + \|\partial_r \phi\|_\infty \int_0^t \int_{\underline{r}(s)}^{\tilde{r}(x, s)} |\rho u| r^m dr ds \\ &\quad + \left| \int_0^t \int_{\tilde{r}_{a_j}(x, s)}^\infty \rho_{a_j} u_{a_j} \partial_r \phi r^m dr ds - \int_0^t \int_{\tilde{r}(x, s)}^\infty \rho u \partial_r \phi r^m dr ds \right| =: \mathcal{I}_{2,1}^j(x) + \mathcal{I}_{2,2}^j(x) + \mathcal{I}_{2,3}^j(x). \end{aligned}$$

Using Cauchy-Schwartz's inequality and Theorem 2.2.1(iii), we have

$$\begin{aligned} \mathcal{I}_{2,1}^j(x) &:= \|\partial_r \phi\|_\infty \int_0^t \int_{a_j}^{\tilde{r}_{a_j}(x, s)} |\rho_{a_j} u_{a_j}| r^m dr ds \\ &\leq \|\partial_r \phi\|_\infty \sqrt{x} \int_0^t \left(\int_{a_j}^{\tilde{r}_{a_j}(x, t)} \rho_{a_j} u_{a_j}^2 r^m dr \right)^{\frac{1}{2}} ds \leq \|\partial_r \phi\|_\infty C(T)^{\frac{1}{2}} t \sqrt{x}. \end{aligned}$$

Moreover, using Lemma 2.5.6(iii) and Lemma 2.5.7, we also have

$$\begin{aligned} \mathcal{I}_{2,2}^j(x) &:= \|\partial_r \phi\|_\infty \int_0^t \int_{\underline{r}(s)}^{\tilde{r}(x, s)} |\rho u| r^m dr ds \\ &\leq \|\partial_r \phi\|_\infty \int_0^t \left(\int_{\underline{r}(s)}^{\tilde{r}(x, s)} \rho |u|^2 r^m dr \right)^{\frac{1}{2}} \left(\int_{\underline{r}(s)}^{\tilde{r}(x, s)} \rho r^m dr \right)^{\frac{1}{2}} ds \leq \|\partial_r \phi\|_\infty C(T)^{\frac{1}{2}} t \sqrt{x}. \end{aligned}$$

Thus given $\delta > 0$, there exists $y_\delta \in (0, 1)$ such that if $0 < x < y_\delta$, then $\mathcal{I}_2^j(x) \leq \delta + \mathcal{I}_{2,3}^j(x)$. By Lemma 2.5.4, $x \rightarrow \tilde{r}(x, t)$ is increasing and $(x, t) \mapsto \tilde{r}(x, t)$ is continuous. Since $[0, T]$ is compact, we have $d = \inf_{t \in [0, T]} \{ \tilde{r}(x, t) - \tilde{r}(x/2, t) \}$ for some $d > 0$. By Lemma 2.5.4(i), there exists $N_x \in \mathbb{N}$ such that $\sup_{0 \leq t \leq T} |\tilde{r}_{a_j}(x, t) - \tilde{r}(x, t)| \leq d/2$ if $j \geq N_x$. Combining with the previous assertion, we get

$$\tilde{r}(x/2, t) < \tilde{r}_{a_j}(x, t) \quad \text{for each } t \in [0, T] \text{ and } j \geq N_x. \quad (2.5.28)$$

Hence $\chi_x^{a_j} = \chi_{x/2} \chi_x^{a_j}$ for all $j \geq N_x$, where $\chi_\varepsilon, \chi_\varepsilon^a$ are defined in (2.5.15). Thus for all $j \geq N_x$,

$$\begin{aligned} \mathcal{I}_{2,3}^j(x) &:= \left| \int_0^t \int_{\tilde{r}_{a_j}(x, s)}^\infty \rho_{a_j} u_{a_j} \partial_r \phi r^m dr ds - \int_0^t \int_{\tilde{r}(x, s)}^\infty \rho u \partial_r \phi r^m dr ds \right| \\ &= \left| \int_0^t \int_{\tilde{r}(x/2, s)}^\infty \rho_{a_j}^{(x)} u_{a_j} \partial_r \phi r^m dr ds - \int_0^t \int_{\tilde{r}(x/2, s)}^\infty \rho^{(x)} u \partial_r \phi r^m dr ds \right| \\ &\leq \left| \int_0^t \int_{\tilde{r}(x/2, s)}^\infty \rho_{a_j}^{(x)} (u_{a_j} - u) \partial_r \phi r^m dr ds \right| + \left| \int_0^t \int_{\tilde{r}(x/2, s)}^\infty \{ \rho_{a_j}^{(x)} - \rho^{(x)} \} u \partial_r \phi r^m dr ds \right|. \end{aligned}$$

By the point-wise bound: $\rho_a^{(x)}(r, t) \leq C(x)$ from Theorem 2.2.1(iv), and the uniform convergence from Lemma 2.5.5, it follows that as $j \rightarrow \infty$,

$$\left| \int_0^t \int_{\tilde{r}(x/2, s)}^\infty \rho_{a_j}^{(x)}(u_{a_j} - u) \partial_r \phi r^m dr ds \right| \leq C(x) \int_0^t \int_{\tilde{r}(x/2, s)}^\infty |u_{a_j} - u| \partial_r \phi r^m dr ds \rightarrow 0.$$

By Lemma 2.5.5, we have $\chi_{x/2} u \partial_r \phi \in L^1(0, T; L^2([0, \infty), r^m dr))$. Thus by Lemma 2.5.6(i),

$$\left| \int_0^t \int_{\tilde{r}(x/2, s)}^\infty \{\rho_{a_j}^{(x)} - \rho^{(x)}\} u \partial_r \phi r^m dr ds \right| \rightarrow 0 \text{ as } j \rightarrow \infty,$$

which implies $\lim_{j \rightarrow \infty} \mathcal{I}_{2,3}^j(x) = 0$ for each $x > 0$. Now for a given $\delta > 0$, we fix $x \in (0, y_\delta)$. Then $\mathcal{I}_2^j = \mathcal{I}_{2,1}^j(x) + \mathcal{I}_{2,2}^j(x) + \mathcal{I}_{2,3}^j(x) \leq \delta + \mathcal{I}_{2,3}^j(x)$. Taking limit $j \rightarrow \infty$ on both sides of this, we get $\lim_{j \rightarrow \infty} \mathcal{I}_2^j \leq \delta$. Since $\delta > 0$ is arbitrarily small, we conclude that $\lim_{j \rightarrow \infty} \mathcal{I}_2^j = 0$. By the exact same arguments, we can also show that

$$\mathcal{I}_3^j := \left| \int_0^t \int_{a_j}^\infty \rho_{a_j} \partial_t \phi(r, s) r^m dr ds - \int_0^t \int_{\underline{r}(s)}^\infty \rho \partial_t \phi(r, s) r^m dr ds \right| \rightarrow 0, \text{ as } j \rightarrow \infty.$$

Lemma 2.5.9 is proved by substituting these limits in (2.5.27), and using Lemma 2.5.8. \square

Lemma 2.5.10. *Let $\phi \in \mathcal{D}_T \cap C^1$ be such that $\text{supp}(\phi) \subset\subset F$. Then we have:*

$$\begin{aligned} & \int_0^\infty \rho(r, t) u(r, t) \varphi(r, t) r^m dr - \int_0^\infty \rho_0(r) u_0(r) \varphi(r, 0) r^m dr \\ &= \int_0^t \int_0^\infty \left\{ \rho u \partial_t \varphi + \rho u^2 \partial_r \varphi + (P - \beta(\partial_r u + m \frac{u}{r})) (\partial_r \varphi + m \frac{\varphi}{r}) \right\} r^m dr ds, \end{aligned} \quad (2.5.29)$$

$$\begin{aligned} & \int_0^\infty \rho E \phi(r, t) r^m dr - \int_0^\infty \rho_0 E_0(r) \phi(r, 0) r^m dr - \int_0^t \int_0^\infty \rho E \partial_t \phi r^m dr ds \\ &= \int_0^t \int_0^\infty \left\{ (\rho E + P) u - 2\mu u \partial_r u - \lambda u (\partial_r u + m \frac{u}{r}) - \kappa \partial_r e \right\} \partial_r \phi(r, s) r^m dr ds. \end{aligned} \quad (2.5.30)$$

Proof. We will only present the proof for (2.5.30) since (2.5.29) can be derived by the same way. First from energy equation of Theorem 2.2.1, we have the weak form:

$$\begin{aligned} & \int_{a_j}^\infty \rho_{a_j} E_{a_j}(r, t) \phi(r, t) r^m dr - \int_{a_j}^\infty \rho_{a_j}^0 \left\{ \frac{1}{2} |u_{a_j}^0|^2 + e_{a_j}^0 \right\} (r) \phi(r, 0) r^m dr \\ &= - \int_0^t \int_{a_j}^\infty \left\{ 2\mu u_{a_j} \partial_r u_{a_j} + \lambda u_{a_j} (\partial_r u_{a_j} + m \frac{u_{a_j}}{r}) + \kappa \partial_r e_{a_j} \right\} \partial_r \phi(r, s) r^m dr ds \\ & \quad + \int_0^t \int_{a_j}^\infty \left\{ \rho_{a_j} E_{a_j} \partial_t \phi + (\rho_{a_j} E_{a_j} + P_{a_j}) u_{a_j} \partial_r \phi \right\} (r, s) r^m dr ds, \end{aligned} \quad (2.5.31)$$

where $P_{a_j} := (\gamma - 1) \rho_{a_j} e_{a_j}$ and $E_{a_j} := |u_{a_j}|^2/2 + e_{a_j}$. Now for each $x \in (0, 1)$, we define the set $\tilde{F}_x := \{(r, t) : t \in [0, T], \tilde{r}(x, t) < r < x^{-1}\}$. Then $\tilde{F}_x \cap \{0 < t < T\}$ is open for each $x \in (0, 1)$ since $(x, t) \mapsto \tilde{r}(x, t)$ is continuous. Moreover, since $x \mapsto \tilde{r}(x, t)$ is strictly increasing for each $t \in [0, T]$, it follows that $\tilde{F}_x \subsetneq \tilde{F}_y$ if $y < x \in (0, 1)$. By the limit $\tilde{r}(x, t) \searrow \underline{r}(t)$ as $x \searrow 0$ for all $t \in [0, T]$, we also have that $F = \cup_{x \in (0, 1)} \tilde{F}_x$, hence $\{\tilde{F}_x \cap \{0 < t < T\}\}_{x \in (0, 1)}$ is an open covering of $F \cap \{0 < t < T\}$. Since $\text{supp}(\phi) \subset\subset F$ is a compact subset, there exists $\varepsilon > 0$ such that $\text{supp}(\phi) \subset \tilde{F}_{2\varepsilon}$. If we set $d = \inf_{t \in [0, T]} \tilde{r}(2\varepsilon, t)$ then $d > 0$ due to Lemma

2.5.4(iv), and there exists $N_1 \in \mathbb{N}$ such that $a_j < d$ for $j \geq N_1$. This implies that $a_j < \tilde{r}(2\varepsilon, t)$ if $t \in [0, T]$ and $j \geq N_1$. In addition, by the same argument (2.5.28) in the proof of Lemma 2.5.9, there exists $N_2 \in \mathbb{N}$ such that $\tilde{r}_{a_j}(\varepsilon, t) < \tilde{r}(2\varepsilon, t)$ if $j \geq N_2$ and $t \in [0, T]$. Let $N := \max\{N_1, N_2\}$. Then by Lemmas 2.5.5–2.5.6 and Dominated Convergence theorem, we have as $j \rightarrow \infty$:

$$\begin{aligned} & \left| \int_{a_j}^{\infty} \rho_{a_j} E_{a_j} \phi(r, t) r^m dr - \int_{\underline{r}(t)}^{\infty} \rho E \phi(r, t) r^m dr \right| \\ &= \left| \int_{\tilde{r}(2\varepsilon, t)}^{\infty} \left\{ \frac{1}{2} \rho_{a_j}^{(\varepsilon)} u_{a_j}^2 + \rho_{a_j}^{(\varepsilon)} e_{a_j} - \frac{1}{2} \rho^{(\varepsilon)} u^2 - \rho^{(\varepsilon)} e \right\} \phi(r, t) r^m dr \right| \\ &\leq \int_{\tilde{r}(2\varepsilon, t)}^{\infty} \rho_{a_j}^{(\varepsilon)} \left\{ \frac{|u_{a_j}^2 - u^2|}{2} + |e_{a_j} - e| \right\} \phi r^m dr + \left| \int_{\tilde{r}(2\varepsilon, t)}^{\infty} \{ \rho_{a_j}^{(\varepsilon)} - \rho^{(\varepsilon)} \} \left(\frac{u^2}{2} + e \right) \phi r^m dr \right| \rightarrow 0. \end{aligned}$$

By the exact same argument, we also obtain the convergence

$$\begin{aligned} & \lim_{j \rightarrow \infty} \int_0^t \int_{a_j}^{\infty} \{ \rho_{a_j} E_{a_j} \partial_t \phi + (\rho_{a_j} E_{a_j} + P_{a_j}) u_{a_j} \partial_r \phi \} (r, s) r^m dr ds \\ &= \int_0^t \int_{\underline{r}(s)}^{\infty} \{ \rho E \partial_t \phi + (\rho E + P) u \partial_r \phi \} (r, s) r^m dr ds. \end{aligned}$$

Next, integrating by parts and using Lemma 2.5.5, we have for all $j \geq N$,

$$\begin{aligned} & \int_0^t \int_{a_j}^{\infty} u_{a_j} \partial_r u_{a_j} \partial_r \phi(r, s) r^m dr ds = \int_0^t \int_{\tilde{r}(2\varepsilon, s)}^{\infty} u_{a_j} \partial_r u_{a_j} \partial_r \phi(r, s) r^m dr ds \\ &= - \int_0^t \int_{\tilde{r}(2\varepsilon, s)}^{\infty} \frac{1}{2} |u_{a_j}|^2 \partial_r^2 \phi(r, s) r^m dr ds \rightarrow - \int_0^t \int_{\tilde{r}(2\varepsilon, s)}^{\infty} \frac{1}{2} |u|^2 \partial_r^2 \phi(r, s) r^m dr ds \text{ as } j \rightarrow \infty. \end{aligned}$$

From Corollary 2.5.1, the spatial weak derivative of u exists in the space $\partial_r u \in L_{\text{loc}}^2(\text{F})$ and $\partial_r u \in L^2(\tilde{\text{F}}_\varepsilon)$ for each $\varepsilon \in (0, 1)$, it then follows that as $j \rightarrow \infty$

$$\begin{aligned} & \int_0^t \int_{a_j}^{\infty} u_{a_j} \partial_r u_{a_j} \partial_r \phi(r, s) r^m dr ds \rightarrow - \int_0^t \int_{\tilde{r}(2\varepsilon, s)}^{\infty} \frac{1}{2} |u|^2 \partial_r^2 \phi(r, s) r^m dr ds \\ &= \int_0^t \int_{\tilde{r}(2\varepsilon, s)}^{\infty} u \partial_r u \partial_r \phi(r, s) r^m dr ds = \int_0^t \int_{\underline{r}(s)}^{\infty} u \partial_r u \partial_r \phi(r, s) r^m dr ds, \end{aligned}$$

where in the last line we have used $\text{supp}(\phi) \subset\subset \text{F}$. By the exact same argument, we also have

$$\lim_{j \rightarrow \infty} \int_0^t \int_{a_j}^{\infty} \left\{ m\lambda \frac{|u_{a_j}|^2}{r} + \kappa \partial_r e_{a_j} \right\} \partial_r \phi r^m dr ds = \int_0^t \int_{\underline{r}(s)}^{\infty} \left\{ m\lambda \frac{|u|^2}{r} + \kappa \partial_r e \right\} \partial_r \phi r^m dr ds.$$

Putting all the limits proven above into (2.5.31), we obtain Lemma 2.5.10. \square

Finally, the weak forms obtained in Theorem 2.5.9–2.5.10 is translated into the domain $(\mathbf{x}, t) \in \mathbb{R}^n \times [0, T]$. Let $\Phi \in C^1([0, T]; C_c^1(\mathbb{R}^n))$. One defines

$$\phi(r, t) := \int_{S^{n-1}} \Phi(r\mathbf{y}, t) dS_{\mathbf{y}},$$

where S^{n-1} is the $(n-1)$ -dimensional sphere. Using this, we have

$$\begin{aligned} & \int_0^t \int_0^{\infty} \rho u \partial_r \phi(r, s) r^m dr ds = \int_0^t \int_0^{\infty} \int_{S^{n-1}} \rho u(r, s) \mathbf{y} \cdot \nabla \Phi(r\mathbf{y}, s) r^m dS_{\mathbf{y}} dr ds \\ &= \int_0^t \int_{\mathbb{R}^n} \rho u(|\mathbf{x}|, s) \frac{\mathbf{x}}{|\mathbf{x}|} \cdot \nabla \Phi(\mathbf{x}, s) d\mathbf{x} ds = \int_0^t \int_{\mathbb{R}^n} \rho(\mathbf{x}, s) \mathbf{U}(\mathbf{x}, s) \cdot \nabla \Phi(\mathbf{x}, s) d\mathbf{x} ds, \end{aligned}$$

where we defined $\rho(\mathbf{x}, t) := \rho(|\mathbf{x}|, t)$ and $\mathbf{U}(\mathbf{x}, t) := u(|\mathbf{x}|, t) \frac{\mathbf{x}}{r}$. By the same way, we also have

$$\int_0^\infty \rho(r, s) \phi(r, s) r^m dr \Big|_{s=0}^{s=t} = \int_{\mathbb{R}^n} \rho(\mathbf{x}, s) \Phi(\mathbf{x}, s) d\mathbf{x} - \int_{\mathbb{R}^n} \rho_0(\mathbf{x}) \Phi(\mathbf{x}, 0) d\mathbf{x},$$

and $\int_0^t \int_0^\infty \rho(r, s) \partial_t \phi(r, s) r^m dr ds = \int_0^t \int_{\mathbb{R}^n} \rho(\mathbf{x}, s) \partial_t \Phi(\mathbf{x}, s) d\mathbf{x} ds.$

Since $\phi(r, t)$ satisfies the assumption in Lemma 2.5.9, using the above identities in Lemma 2.5.9, we obtain the weak form of continuity equation.

Next, we show the momentum equations. Let $\Psi : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}$ be such that $\Psi \in C^2([0, T]; C_c^2(\mathbb{R}^n))$ and $\text{supp}(\Psi) \subset \mathcal{F} := \{(\mathbf{x}, t) \in \mathbb{R}^n \times [0, T] : \underline{r}(t) < |\mathbf{x}|\}$. We define:

$$\varphi^i(r, t) := \int_{S^{n-1}} \Psi(r\mathbf{y}, t) y^i dS_{\mathbf{y}} \quad \text{for } i = 1, \dots, n. \quad (2.5.32)$$

It can be verified that φ^i satisfies the assumption for Lemma 2.5.10, hence (2.5.29) holds with φ^i . The derivations for weak forms in Theorem 2.1.1 are similar to that occurring in the continuity equation, except for the term $\iint \{P - \beta(\partial_r u + mu/r)\} (\partial_r \varphi^i + m\varphi^i/r) r^m dr ds$. To deal with this, we use the following identity, which one can verify with few lines of calculation on spherical coordinate transformation:

$$\int_{|\mathbf{x}| \leq r} \partial_{x^i} f(\mathbf{x}) d\mathbf{x} = \int_{|\mathbf{x}|=r} f(\mathbf{x}) \frac{x^i}{|\mathbf{x}|} dS_{\mathbf{x}} \quad \text{for } i = 1, \dots, n, \text{ for } r > 0.$$

Using this identity, we then have that

$$\begin{aligned} \partial_r(r^m \varphi^i) &= \partial_r \left(\int_{S^{n-1}} y^i \Psi(r\mathbf{y}, t) r^m dS_{\mathbf{y}} \right) = \partial_r \left(\int_{|\mathbf{x}|=r} \frac{x^i}{|\mathbf{x}|} \Psi(\mathbf{x}, t) dS_{\mathbf{x}} \right) \\ &= \partial_r \left(\int_0^r \int_{S^{n-1}} \partial_{x^i} \Psi(\zeta\mathbf{y}, t) \zeta^m dS_{\mathbf{y}} d\zeta \right) = r^m \int_{S^{n-1}} \partial_{x^i} \Psi(r\mathbf{y}, t) dS_{\mathbf{y}}. \end{aligned}$$

Taking derivative on (2.5.32), we get $\partial_r \varphi^i(r, t) = \int_{S^{n-1}} y^i y^j \partial_{x^j} \Psi(r\mathbf{y}, t) dS_{\mathbf{y}}$. From this, we have

$$\begin{aligned} &\int_0^t \int_0^\infty P (\partial_r \varphi^i + \frac{m}{r} \varphi^i) r^m dr ds \\ &= \int_0^t \int_0^\infty \int_{S^{n-1}} P \partial_{x^i} \Psi(r\mathbf{y}, s) r^m dS_{\mathbf{y}} dr ds = \int_0^t \int_{\mathbb{R}^n} P \partial_{x^i} \Psi(\mathbf{x}, s) d\mathbf{x} ds. \end{aligned}$$

Moreover, integrating by parts, we have

$$\int_0^T \int_0^\infty (\partial_r u + m \frac{u}{r}) (\partial_r \varphi^i + m \frac{\varphi^i}{r}) r^m dr dt = \int_0^T \int_0^\infty \{ \partial_r u \partial_r \varphi^i + m \frac{\varphi^i u}{r^2} \} r^m dr dt.$$

Using the identity $\partial_{x^j} \mathbf{U}^i = \partial_r (\frac{u}{|\mathbf{x}|}) \frac{x^i x^j}{|\mathbf{x}|} + \frac{u}{|\mathbf{x}|} \delta^{ij}$, we have

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^n} \partial_{x^j} \mathbf{U}^i \partial_{x^j} \Psi d\mathbf{x} dt = \int_0^T \int_{\mathbb{R}^n} \left(\partial_r \left(\frac{u}{|\mathbf{x}|} \right) \frac{x^i x^j}{|\mathbf{x}|} + \frac{u}{|\mathbf{x}|} \delta^{ij} \right) \partial_{x^j} \Psi d\mathbf{x} dt \\ &= \int_0^T \int_0^\infty \int_{S^{n-1}} \left\{ r \partial_r \left(\frac{u}{r} \right) y^i y^j \partial_{x^j} \Psi(r\mathbf{y}, t) + \frac{u}{r} \partial_{x^i} \Psi(r\mathbf{y}, t) \right\} dS_{\mathbf{y}} r^m dr dt \\ &= \int_0^T \int_0^\infty \left\{ r^{m+1} \partial_r \left(\frac{u}{r} \right) \partial_r \varphi^i + \frac{u}{r} \partial_r (r^m \varphi^i) \right\} dr dt \\ &= \int_0^T \int_0^\infty \left\{ \partial_r u \partial_r \varphi^i + m \frac{\varphi^i u}{r^2} \right\} r^m dr dt = \int_0^T \int_0^\infty (\partial_r u + m \frac{u}{r}) (\partial_r \varphi^i + m \frac{\varphi^i}{r}) r^m dr dt. \end{aligned}$$

Similarly, since $\operatorname{div} \mathbf{U} = \partial_r u + mu/r$, we also have

$$\int_0^T \int_{\mathbb{R}^n} \operatorname{div} \mathbf{U} \partial_{x^i} \Psi d\mathbf{x} dt = \int_0^T \int_0^\infty (\partial_r u + m \frac{u}{r}) (\partial_r \varphi^i + m \frac{\varphi^i}{r}) r^m dr dt.$$

Finally, we show the weak form of energy equation. Let $\Phi \in C^2([0, T]; C_c^2(\mathbb{R}^n))$ be a test function such that $\operatorname{supp}(\Phi) \subset\subset \mathcal{F} = \{(\mathbf{x}, t) \in \mathbb{R}^n \times [0, T] : |\mathbf{x}| > \underline{r}(t)\}$. We set:

$$\phi(r, t) := \int_{S^{n-1}} \Phi(r\mathbf{y}, t) dS_{\mathbf{y}}.$$

It can be verified that ϕ satisfies the assumption of Lemma 2.5.10, hence we have (2.5.30) with ϕ . Most of the terms can be derived by the exact same procedure presented previously, except for the terms $u\partial_r u$, u^2/r and $\partial_r e$. Since $\operatorname{supp}(\phi) \subset\subset \{(r, t) : r > \underline{r}(t)\}$, it follows that:

$$\begin{aligned} & \int_0^t \int_{\underline{r}(s)}^\infty u \partial_r u \partial_r \phi(r, s) r^m dr ds = \int_0^t \int_{\underline{r}(s)}^\infty u \partial_r u \left(\int_{S^{n-1}} y^i \partial_{x^i} \Phi(r\mathbf{y}, s) dS_{\mathbf{y}} \right) r^m dr ds \\ &= \int_0^t \int_{|\mathbf{x}| > \underline{r}(s)} \partial_r \left(\frac{u^2}{2} \right) (\|\mathbf{x}\|, s) \frac{x^i}{|\mathbf{x}|} \partial_{x^i} \Phi(\mathbf{x}, s) d\mathbf{x} ds \\ &= \int_0^t \int_{|\mathbf{x}| > \underline{r}(s)} \partial_{x^i} \left(\frac{|\mathbf{U}|^2}{2} \right) \partial_{x^i} \Phi(\mathbf{x}, s) d\mathbf{x} ds = \frac{1}{2} \int_0^t \int_{|\mathbf{x}| > \underline{r}(s)} \nabla \Phi(\mathbf{x}, s) \cdot \nabla |\mathbf{U}|^2 d\mathbf{x} ds, \end{aligned}$$

and with few lines of derivations, we also have

$$\begin{aligned} & \int_0^t \int_{\underline{r}(s)}^\infty \{ \mu u \partial_r u + \lambda u (\partial_r u + m \frac{u}{r}) \} \partial_r \phi(r, s) r^m dr ds \\ &= \int_0^t \int_{\underline{r}(s)}^\infty \int_{S^{n-1}} \{ \mu u \partial_r u + \lambda u (\partial_r u + m \frac{u}{r}) \} y^i \partial_{x^i} \Phi(r\mathbf{y}, s) r^m dS_{\mathbf{y}} dr ds \\ &= \int_0^t \int_{|\mathbf{x}| > \underline{r}(s)} \{ \lambda (\operatorname{div} \mathbf{U}) \mathbf{U} + \mu (\nabla \mathbf{U}) \cdot \mathbf{U} \} \nabla \Phi(\mathbf{x}, s) d\mathbf{x} ds, \\ & \int_0^t \int_{\underline{r}(s)}^\infty \partial_r e \partial_r \phi(r, s) r^m dr ds = \int_0^t \int_{\underline{r}(s)}^\infty \int_{S^{n-1}} \partial_r e(r, s) y^i \partial_{x^i} \Phi(r\mathbf{y}, s) r^m dS_{\mathbf{y}} dr ds \\ &= \int_0^t \int_{|\mathbf{x}| > \underline{r}(s)} \nabla e \cdot \nabla \Phi d\mathbf{x} ds. \end{aligned}$$

With these, we conclude the proof for weak form of energy equation.

2.5.5 Proof of Theorem 2.1.2

In this subsection, we show Theorem 2.1.2. For each $t \in [0, T]$ and $\varphi \in C^1([0, \infty) \times [0, T])$ we define

$$\mathcal{U}_j(\varphi, t) := - \int_0^t \int_{a_j}^\infty u_{a_j} \partial_r (\partial_r \varphi + m \frac{\varphi}{r}) r^m dr dt.$$

Lemma 2.5.11. *Let $\varphi \in \mathcal{D}_T \cap C^2([0, \infty) \times [0, T])$ be such that there exists $\eta > 0$ for which $\varphi(r, t) = 0$ if $(r, t) \in [0, \eta] \times [0, T]$. Then $\lim_{j \rightarrow \infty} \mathcal{U}_j(\varphi)$ exists and one has:*

$$\begin{aligned} \beta \lim_{j \rightarrow \infty} \mathcal{U}_j(\varphi, t) &= \int_{\underline{r}(t)}^\infty \rho(r, t) u(r, t) \varphi(r, t) r^m dr - \int_0^\infty \rho_0(r) u_0(r) \varphi(r, 0) r^m dr \\ &\quad - \int_0^t \int_{\underline{r}(s)}^\infty \{ \rho u \partial_t \varphi + \rho u^2 \partial_r \varphi + (\gamma - 1) \rho e (\partial_r \varphi + m \frac{\varphi}{r}) \} r^m dr ds. \end{aligned} \tag{2.5.33}$$

Proof. Since $\eta > 0$ is fixed, there exists $N_1 \in \mathbb{N}$ such that if $j \geq N_1$ then $a_j \leq a_{N_1} < \eta$. Applying integration by parts on the momentum equation weak form of Theorem 2.2.1, we get

$$\begin{aligned} \beta \cdot \mathcal{U}_j(\varphi, t) &= \int_{a_j}^{\infty} \rho_{a_j} u_{a_j}(r, s) \varphi r^m dr \Big|_{s=0}^{s=t} - \int_0^t \int_{a_j}^{\infty} \rho_{a_j} u_{a_j} \partial_t \varphi r^m dr ds \\ &\quad - \int_0^t \int_{a_j}^{\infty} \left\{ \rho_{a_j} |u_{a_j}|^2 \partial_r \varphi + (\gamma - 1) \rho_{a_j} e_{a_j} (\partial_r \varphi + m \frac{\varphi}{r}) \right\} r^m dr ds. \end{aligned} \quad (2.5.34)$$

Next, using Theorem 2.2.1(i)–(iii), Lemma 2.5.6(iii), Lemma 2.5.7, we have

$$\begin{aligned} \mathcal{J}_1^j &:= \left| \int_{a_j}^{\infty} \rho_{a_j} u_{a_j} \varphi(r, t) r^m dr - \int_{\underline{r}(t)}^{\infty} \rho u \varphi(r, t) r^m dr \right| \\ &\leq \left| \int_{a_j}^{\tilde{r}_{a_j}(x, t)} \rho_{a_j} u_{a_j} \varphi r^m dr \right| + \left| \int_{\underline{r}(t)}^{\tilde{r}(x, t)} \rho u \varphi r^m dr \right| + \left| \int_{\tilde{r}_{a_j}(x, t)}^{\infty} \rho_{a_j} u_{a_j} \varphi r^m dr - \int_{\tilde{r}(x, t)}^{\infty} \rho u \varphi r^m dr \right| \\ &\leq \|\varphi\|_{\infty} \left(\int_{a_j}^{\tilde{r}_{a_j}(x, t)} \rho_{a_j} |u_{a_j}|^2 r^m dr \right)^{\frac{1}{2}} \left(\int_{a_j}^{\tilde{r}_{a_j}(x, t)} \rho_{a_j} r^m dr \right)^{\frac{1}{2}} \\ &\quad + \|\varphi\|_{\infty} \left(\int_{\underline{r}(t)}^{\tilde{r}(x, t)} \rho |u|^2 r^m dr \right)^{\frac{1}{2}} \left(\int_{\underline{r}(t)}^{\tilde{r}(x, t)} \rho r^m dr \right)^{\frac{1}{2}} + \left| \int_{\tilde{r}_{a_j}(x, t)}^{\infty} \rho_{a_j} u_{a_j} \varphi r^m - \int_{\tilde{r}(x, t)}^{\infty} \rho u \varphi r^m \right| \\ &\leq \|\varphi\|_{\infty} C(T) \sqrt{x} + \left| \int_{\tilde{r}_{a_j}(x, t)}^{\infty} \rho_{a_j} u_{a_j} \varphi r^m dr - \int_{\tilde{r}(x, t)}^{\infty} \rho u \varphi r^m dr \right| \end{aligned}$$

By (2.5.28) in the proof of Lemma 2.5.9, there exists an integer $N_2(x) \in \mathbb{N}$ such that if $j \geq N_2(x)$, then $\tilde{r}_{a_j}(x, t) > \tilde{r}(x/2, t)$ for all $t \in [0, T]$, and as $j \rightarrow \infty$ we have:

$$\begin{aligned} &\left| \int_{\tilde{r}_{a_j}(x, t)}^{\infty} \rho_{a_j} u_{a_j} \varphi r^m dr - \int_{\tilde{r}(x, t)}^{\infty} \rho u \varphi r^m dr \right| = \left| \int_{\tilde{r}(x/2, t)}^{\infty} \rho_{a_j}^{(x)} u_{a_j} \varphi r^m dr - \int_{\tilde{r}(x/2, t)}^{\infty} \rho^{(x)} u \varphi r^m dr \right| \\ &\leq \left| \int_{\tilde{r}(x/2, t)}^{\infty} \rho_{a_j}^{(x)} (u_{a_j} - u) \varphi r^m dr \right| + \left| \int_{\tilde{r}(x/2, t)}^{\infty} \{ \rho_{a_j}^{(x)} - \rho^{(x)} \} u \varphi r^m dr \right| \\ &\leq C(x/2) \left| \int_{\tilde{r}(x/2, t)}^{\infty} |u_{a_j} - u| |\varphi| r^m dr \right| + \left| \int_{\tilde{r}(x/2, t)}^{\infty} \{ \rho_{a_j}^{(x)} - \rho^{(x)} \} u \varphi r^m dr \right| \rightarrow 0. \end{aligned}$$

By choosing $x > 0$ small, we obtain $\lim_{j \rightarrow \infty} \mathcal{J}_1^j = 0$. In the exact same way, we also have

$$\mathcal{J}_2^j := \left| \int_0^t \int_{a_j}^{\infty} \rho_{a_j} u_{a_j} \partial_t \varphi r^m dr ds - \int_0^t \int_{\underline{r}(s)}^{\infty} \rho u \partial_t \varphi r^m dr ds \right| \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

Next, let $L \in \mathbb{N}$ be such that $\text{supp}(\phi(\cdot, t)) \subseteq [\eta, L]$. We consider the term:

$$\begin{aligned} \mathcal{J}_3^j &:= \left| \int_0^t \int_{a_j}^{\infty} \rho_{a_j} |u_{a_j}|^2(r, s) \partial_r \varphi r^m dr ds - \int_0^t \int_{\underline{r}(s)}^{\infty} \rho |u|^2 \partial_r \varphi(r, s) r^m dr ds \right| \\ &\leq \|\partial_r \varphi\|_{\infty} \left| \int_0^t \int_{E_j(x, s)} \rho_{a_j} |u_{a_j}|^2 r^m dr ds \right| + \|\partial_r \varphi\|_{\infty} \left| \int_0^t \int_{\underline{r}(s)}^{\tilde{r}(x, s)} \rho |u|^2 r^m dr ds \right| \\ &\quad + \left| \int_0^t \int_{\tilde{r}_{a_j}(x, s)}^{\infty} \rho_{a_j} |u_{a_j}|^2 \partial_r \varphi r^m dr ds - \int_0^t \int_{\tilde{r}(x, s)}^{\infty} \rho |u|^2 \partial_r \varphi r^m dr ds \right| =: \sum_{i=1}^3 \mathcal{J}_{3,i}^j, \end{aligned} \quad (2.5.35)$$

where $E_j^x(s) := [a_j, \tilde{r}_{a_j}(x, s)] \cap [\eta, L]$. Using Lemma 2.5.3, we have

$$\mathcal{J}_{3,1}^j := \|\partial_r \varphi\|_{\infty} \left| \int_0^t \int_{E_j^x(x, s)} \rho_{a_j} |u_{a_j}|^2 r^m dr ds \right| \leq \|\partial_r \varphi\|_{\infty} C(T) f_3(V_{a_j}(E_j^x); C_{E_j^x})^{\frac{1}{4}}, \quad (2.5.36)$$

where the constant $C_{E_j^x} > 0$ and the set function $V_{a_j}(E_j^x)$ are given by:

$$C_{E_j^x} := C(\eta, T) \{C(T) + f_2(f_1(\mathcal{M}_{E_j^x}; C(T)); C(T))\}, \quad \text{with } \mathcal{M}_{E_j^x} := \operatorname{ess\,sup}_{t \in [0, T]} \int_{E_j^x(t)} r^m dr,$$

$$V_{a_j}(E_j^x) := \int_0^T \int_{E_j^x(t)} \rho_{a_j}(r, t) r^m dr dt \leq \int_0^T \int_{a_j}^{\tilde{r}_{a_j}(x, s)} \rho_{a_j}(r, t) r^m dr dt = xT.$$

Using the fact that $y \mapsto f_i(y; C)$ is continuous increasing and $\lim_{y \searrow 0} f_i(y; C) = 0$ for each $i = 1, \dots, 3$ and $C \mapsto f_3(y; C)$ is increasing for each fixed $y \in (0, \infty)$, it follows that for a given $\delta > 0$ there exists $y_\delta \in (0, 1)$ such that for all $x \in (0, y_\delta)$

$$C_{E_j^x} = C(\eta, T) \{C(T) + f_2(f_1(\mathcal{M}_{E_j^x}; C(T)); C(T))\} \leq C(\eta, T)(1 + \delta) \leq C(\eta, T),$$

and $f_3(V_{a_j}(E_j^x); C_{E_j^x}) \leq f_3(V_{a_j}(E_j^x); C(\eta, T)) \leq f_3(xT; C(\eta, T)) \leq (C(T)^{-1} \|\partial_r \varphi\|_\infty^{-1} \frac{\delta}{2})^4$.

Hence (2.5.36) implies that $\mathcal{J}_{3,1}^j \leq \delta/2$ for all $j \geq \max\{N_1, N_2(x)\}$ and $x \in (0, y_\delta)$. Since $\tilde{r}(x, t) \searrow r(t)$ as $x \searrow 0$ for each $t \in [0, T]$, using Lemma 2.5.7, and Dominated Convergence theorem, we have $\lim_{x \searrow 0} \int_0^t \int_{r(s)}^{\tilde{r}(x, s)} \rho |u|^2 r^m dr ds = 0$. Hence for given $\delta > 0$, there exists $z_\delta \in (0, 1)$ such that for $x \in (0, z_\delta)$,

$$\mathcal{J}_{3,2}^j := \|\partial_r \varphi\|_\infty \int_0^t \int_{r(s)}^{\tilde{r}(x, s)} \rho |u|^2 r^m dr ds \leq \frac{\delta}{2}.$$

Putting estimates for $\mathcal{J}_{3,1}^j, \mathcal{J}_{3,2}^j$ into (2.5.35), we get $\mathcal{J}_3^j \leq \delta + \mathcal{J}_{3,3}^j$ if $j \geq \max\{N_1, N_2(x)\}$ and $x \in (0, \min\{y_\delta, z_\delta\})$. Fix $x \in (0, \min\{y_\delta, z_\delta\})$. We have from Lemmas 2.5.5–2.5.6 that as $j \rightarrow \infty$,

$$\begin{aligned} \mathcal{J}_{3,3}^j &:= \left| \int_0^t \int_{\tilde{r}_{a_j}(x, t)}^\infty \rho_{a_j} |u_{a_j}|^2 \partial_r \varphi r^m dr ds - \int_0^t \int_{\tilde{r}(x, s)}^\infty \rho |u|^2 \partial_r \varphi r^m dr ds \right| \\ &\leq \left| \int_0^t \int_{\tilde{r}(x/2, t)}^\infty \rho_{a_j}^{(x)} |u_{a_j}|^2 \partial_r \varphi r^m dr ds - \int_0^t \int_{\tilde{r}(x/2, s)}^\infty \rho^{(x)} |u|^2 \partial_r \varphi r^m dr ds \right| \\ &\leq C(x) \int_0^t \int_{\tilde{r}(x/2, t)}^\infty \left| |u_{a_j}|^2 - |u|^2 \right| |\partial_r \varphi| r^m + \left| \int_0^t \int_{\tilde{r}(x/2, s)}^\infty \{\rho_{a_j}^{(x)} - \rho^{(x)}\} |u|^2 \partial_r \varphi r^m \right| \rightarrow 0. \end{aligned}$$

Since $\delta > 0$ can be chosen arbitrarily small, we conclude that $\lim_{j \rightarrow \infty} \mathcal{J}_3^j = 0$.

Next, we consider the term:

$$\begin{aligned} \mathcal{J}_4^j &:= \left| \int_0^t \int_{a_j}^\infty \rho_{a_j} e_{a_j} (\partial_r \varphi + m \frac{\varphi}{r}) r^m dr ds - \int_0^t \int_{r(s)}^\infty \rho e (\partial_r \varphi + m \frac{\varphi}{r}) r^m dr ds \right| \\ &\leq \left\| \partial_r \varphi + m \frac{\varphi}{r} \right\|_\infty \left\{ \left| \int_0^t \int_{E_j^x(s)} \rho_{a_j} e_{a_j} r^m dr ds \right| + \left| \int_0^t \int_{E^x(s)} \rho e r^m dr ds \right| \right\} \\ &\quad + \left| \int_0^t \int_{\tilde{r}_{a_j}(x, s)}^\infty \rho_{a_j} e_{a_j} (\partial_r \varphi + m \frac{\varphi}{r}) r^m dr ds - \int_0^t \int_{\tilde{r}(x, s)}^\infty \rho e (\partial_r \varphi + m \frac{\varphi}{r}) r^m dr ds \right| = \sum_{i=1}^3 \mathcal{J}_{4,i}^j, \end{aligned}$$

where $E_j^x(s) := [a_j, \tilde{r}_{a_j}(x, s)] \cap [\eta, L]$ and $E^x(s) := [r(s), \tilde{r}(x, s)] \cap [\eta, L]$. By Lemma 2.5.3,

$$\left\| \partial_r \varphi + m \frac{\varphi}{r} \right\|_\infty^{-1} \mathcal{J}_{4,1}^j := \left| \int_0^t \int_{E_j^x(s)} \rho_{a_j} e_{a_j} r^m dr ds \right| \leq f_3(V_{a_j}(E_j^x), C_{E_j^x}),$$

where the constants and terms on the right hand side are exactly the same presented in the previous argument. Thus by the exact same procedure as before, for a given $\delta > 0$ it follows that there exists $y_\delta \in (0, 1)$ such that $\mathcal{J}_{4,1}^j \leq \delta/2$ for all $x \in (0, y_\delta)$ and $j \geq N_1$. By Lemma 2.5.7, it follows that for a.e. $t \in [0, T]$ we have

$$\int_{[\eta, L] \cap \{r > \underline{r}(t)\}} \rho e(r, t) r^m dr \leq C + \omega_1([\eta, L] \cap \{r > \underline{r}(t)\}; C) \leq C(\eta, L) < \infty,$$

where $C(\eta, L) > 0$ is a constant independent of $\{a_j\}_{j \in \mathbb{N}}$. Using the fact that $\tilde{r}(x, t) \searrow \underline{r}(t)$ as $x \searrow 0$, we have by Dominated Convergence theorem that $\lim_{x \searrow 0} \int_0^t \int_{E^x(s)} \rho e(r, s) r^m dr ds = 0$. Therefore there exists $z_\delta \in (0, 1)$ such that for all $x \in (0, z_\delta)$ we have

$$\mathcal{J}_{4,2}^j := \left\| \partial_r \varphi + m \frac{\varphi}{r} \right\|_\infty \left| \int_0^r \int_{E^x(s)} \rho e r^m dr ds \right| \leq \frac{\delta}{2}.$$

Hence we obtain that $\mathcal{J}_4^j \leq \delta + \mathcal{J}_{4,3}^j$ if $j \geq \max\{N_1, N_2\}$ and $x \in (0, \min\{y_\delta, z_\delta\})$. Using Lemmas 2.5.5–2.5.6 and the point-wise bound on ρ_a from Theorem 2.2.1(iv), we have as $j \rightarrow \infty$

$$\begin{aligned} \mathcal{J}_{4,3}^j &:= \left| \int_0^t \int_{\tilde{r}_{a_j}(x,s)}^\infty \rho_{a_j} e_{a_j} (\partial_r \varphi + m \frac{\varphi}{r}) r^m dr ds - \int_0^t \int_{\tilde{r}(x,s)}^\infty \rho e (\partial_r \varphi + m \frac{\varphi}{r}) r^m dr ds \right| \\ &= \left| \int_0^t \int_{\tilde{r}(x/2,s)}^\infty \rho_{a_j}^{(x)} e_{a_j} (\partial_r \varphi + m \frac{\varphi}{r}) r^m dr ds - \int_0^t \int_{\tilde{r}(x/2,s)}^\infty \rho^{(x)} e (\partial_r \varphi + m \frac{\varphi}{r}) r^m dr ds \right| \\ &\leq \int_0^t \int_{\tilde{r}(x/2,s)}^\infty \{C(x) |e_{a_j} - e| + e |\rho_{a_j}^{(x)} - \rho^{(x)}|\} |\partial_r \varphi + m \varphi / r| r^m dr ds \rightarrow 0. \end{aligned}$$

Since $\delta > 0$ is arbitrarily small, we get $\lim_{j \rightarrow \infty} \mathcal{J}_4^j = 0$. Using the limits for $\mathcal{J}_1^j - \mathcal{J}_4^j$ in (2.5.34), we obtain that $\lim_{j \rightarrow \infty} \mathcal{U}_j(\varphi, t)$ exists and (2.5.33) holds. \square

Lemma 2.5.12. *Let $\xi \in C^1([0, \infty))$ be an increasing function such that $\xi \equiv 0$ on $[0, 1]$ and $\xi \equiv 1$ on $[2, \infty)$, and $|\xi'(\cdot)| \leq 1$. Let $\xi^R(r) := \xi(r/R)$ for $R > 0$. If $\varphi \in \mathcal{D}_T \cap C^2([0, \infty) \times [0, T])$ is such that $\varphi(0, t) = 0$ for $t \in [0, T]$, then for $\varphi^R := \xi^R \varphi$, the following limit exists:*

$$\begin{aligned} \beta \lim_{R \searrow 0} \lim_{j \rightarrow \infty} \mathcal{U}_j(\varphi^R, t) &= \int_0^\infty \rho u(r, s) \varphi r^m dr \Big|_{s=0}^{s=t} - \int_0^t \int_0^\infty \{\rho u \partial_t \varphi + \rho u^2 \partial_r \varphi\} r^m dr ds \\ &\quad + (\gamma - 1) \int_0^t \int_0^\infty \rho e (\partial_r \varphi + m \frac{\varphi}{r}) r^m dr ds. \end{aligned} \tag{2.5.37}$$

Proof. Since φ^R satisfies the assumption for test function in Lemma 2.5.11, we have

$$\begin{aligned} \beta \lim_{j \rightarrow \infty} \mathcal{U}_j(\varphi^R, t) &= \int_0^\infty \rho u(r, s) \varphi^R r^m dr \Big|_{s=0}^{s=t} - \int_0^t \int_0^\infty \{\rho u \partial_t \varphi^R + \rho u^2 \partial_r \varphi^R\} r^m dr ds \\ &\quad - (\gamma - 1) \int_0^t \int_0^\infty \rho e (\partial_r \varphi^R + m \frac{\varphi^R}{r}) r^m dr ds. \end{aligned} \tag{2.5.38}$$

By Lemma 2.5.6–2.5.7, we have $\rho u \mathbb{1}_{\{r > \underline{r}(t)\}} \in L^1([0, \infty), r^m dr)$ for each $t \in [0, T]$. Thus by Dominated Convergence theorem, it follows that

$$\begin{aligned} \lim_{R \searrow 0} \int_0^\infty \rho u \varphi^R(r, s) r^m dr \Big|_{s=0}^{s=t} &= \int_0^\infty \rho u \varphi(r, s) r^m dr \Big|_{s=0}^{s=t}, \\ \lim_{R \searrow 0} \int_0^t \int_0^\infty \rho u \partial_t \varphi^R r^m dr ds &= \lim_{R \searrow 0} \int_0^t \int_0^\infty \rho u \xi^R \partial_t \varphi r^m dr ds = \int_0^t \int_0^\infty \rho u \partial_t \varphi r^m dr ds, \\ \lim_{R \searrow 0} \int_0^t \int_0^\infty \rho u^2 \xi^R \partial_r \varphi r^m dr ds &= \int_0^t \int_0^\infty \rho u^2 \partial_r \varphi r^m dr ds. \end{aligned}$$

Similarly, by Lemma 2.5.7, we also have $\rho e \mathbb{1}_{\{r > \underline{r}(t)\}} \in L^1([0, L], r^m dr)$ for each $t \in [0, T]$, where $L \in \mathbb{N}$ is such that $\text{supp}(\varphi(\cdot, t)) \subseteq [0, L]$. Hence by Dominated Convergence theorem,

$$\lim_{R \searrow 0} \int_0^t \int_0^\infty \rho e (\xi^R \partial_r \varphi + \xi^R m \frac{\varphi}{r}) r^m dr ds = \int_0^t \int_0^\infty \rho e (\partial_r \varphi + m \frac{\varphi}{r}) r^m dr ds.$$

Next, by the construction, we have $|\partial_r \xi^R(\cdot)| \leq R^{-1} |\xi'(\cdot)| \leq R^{-1}$. Moreover, since $\varphi \in C^2$ is such that $\varphi(0, t) = 0$ for all $t \in [0, T]$, there exists $\phi \in C^2([0, T]; C^1([0, \infty)))$ such that $\varphi = r\phi$. Therefore, by Lemma 2.5.7 we have:

$$\begin{aligned} & \left| \int_0^t \int_0^\infty \rho u^2 \varphi \partial_r \xi^R r^m dr ds \right| = \left| \int_0^t \int_{[R, 2R] \cap \{r > \underline{r}(s)\}} \rho u^2 \phi \frac{r}{R} \xi' \left(\frac{r}{R} \right) r^m dr ds \right| \\ & \leq \|\phi\|_\infty \int_0^t \int_{[R, 2R] \cap \{r > \underline{r}(s)\}} \rho u^2 r^m dr ds \rightarrow 0 \text{ as } R \searrow 0. \end{aligned}$$

In the exact same way, we also have from Lemma 2.5.7 that:

$$\begin{aligned} & \lim_{R \searrow 0} \left| \int_0^t \int_0^\infty \rho e \varphi \partial_r \xi^R r^m dr ds \right| = \lim_{R \searrow 0} \left| \int_0^t \int_{[R, 2R] \cap \{r > \underline{r}(s)\}} \rho e \phi \frac{r}{R} \xi' \left(\frac{r}{R} \right) r^m dr ds \right| \\ & \leq \|\phi\|_\infty \lim_{R \searrow 0} \int_0^t \int_{[R, 2R] \cap \{r > \underline{r}(s)\}} \rho e r^m dr ds \rightarrow 0. \end{aligned}$$

Putting all above limits into (2.5.38), it follows that $\lim_{R \searrow 0} \lim_{j \rightarrow \infty} \mathcal{U}_j(\varphi^R, t)$ exists and (2.5.37) holds. \square

Finally, we translate the term \mathcal{U}_j into \mathbb{U}^j as shown in Theorem 2.1.2. Let $\Psi : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}$ be such that $\Psi \in C^1([0, T]; C_c^2(\mathbb{R}^n))$. With $\varphi^R \equiv \xi^R \varphi$ (ξ^R is given in Lemma 2.5.12), we define

$$(\varphi^i)^R = \xi^R(r) \varphi^i(r, t) := \int_{S^{n-1}} \xi^R(r) \Psi(r\mathbf{y}, t) y^i dS_{\mathbf{y}} = \int_{S^{n-1}} \Psi^R(r\mathbf{y}, t) y^i dS_{\mathbf{y}}.$$

Then by construction, there exists $N \in \mathbb{N}$ such that for all $j \geq N$, $\varphi^R(a_j, t) = 0$ for all $t \in [0, T]$. Moreover, we also have $\partial_r(r^m(\varphi^R)^i) = r^m \int_{S^{n-1}} \partial_{\mathbf{x}^i} \Psi^R(r\mathbf{y}, t) dS_{\mathbf{y}}$. Using all these, it implies that for $i = 1, \dots, n$:

$$\begin{aligned} & \int_{a_j}^\infty u_{a_j} \partial_r (\partial_r (\varphi^R)^i + \frac{m}{r} (\varphi^R)^i) r^m dr \\ & = - \int_{a_j}^\infty (\partial_r u_{a_j} + m \frac{u_{a_j}}{r}) (\partial_r (\varphi^R)^i + m \frac{(\varphi^R)^i}{r}) r^m dr \\ & = - \int_{a_j}^\infty \left\{ \partial_r u_{a_j} \partial_r (\varphi^R)^i + \frac{m}{r} \partial_r (u_{a_j} (\varphi^R)^i) + \frac{m^2}{r^2} u_{a_j} (\varphi^R)^i \right\} r^m dr \\ & = - \int_{a_j}^\infty \left\{ r^{m+1} \partial_r \left(\frac{u_{a_j}}{r} \right) \partial_r (\varphi^R)^i + \frac{u_{a_j}}{r} \partial_r (r^m (\varphi^R)^i) \right\} dr \\ & = - \int_{a_j}^\infty \left\{ r \partial_r \left(\frac{u_{a_j}}{r} \right) \left(\int_{S^{n-1}} y^k y^i \partial_{x^k} \Psi^R(r\mathbf{y}, t) dS_{\mathbf{y}} \right) + \frac{u_{a_j}}{r} \left(\int_{S^{n-1}} \partial_{x^i} \Psi^R(r\mathbf{y}, t) dS_{\mathbf{y}} \right) \right\} r^m dr, \end{aligned}$$

where the repeated index $k \in \{1, \dots, n\}$ denotes the summation over that index. Now by the coordinate

transformation $\mathbf{x} = r\mathbf{y}$, we have

$$\begin{aligned}
& \int_{a_j}^{\infty} u_{a_j} \partial_r (\partial_r (\varphi^R)^i + \frac{m}{r} (\varphi^R)^i) r^m dr \\
&= - \int_{a_j}^{\infty} \int_{S^{n-1}} \left\{ r y^i y^k \partial_r \left(\frac{u_{a_j}}{r} \right) \partial_{x^k} \Psi^R(r\mathbf{y}, t) + \frac{u_{a_j}}{r} \partial_{x^i} \Psi^R(r\mathbf{y}, t) \right\} r^m dS_{\mathbf{y}} dr \\
&= - \int_{|\mathbf{x}| > a_j} \left\{ |\mathbf{x}| \frac{x^i}{|\mathbf{x}|} \frac{x^k}{|\mathbf{x}|} \partial_r \left(\frac{u_{a_j}}{|\mathbf{x}|} \right) \partial_{x^k} \Psi^R(\mathbf{x}, t) + \frac{u_{a_j}}{|\mathbf{x}|} \partial_{x^i} \Psi^R(\mathbf{x}, t) \right\} d\mathbf{x} \\
&= - \int_{|\mathbf{x}| > a_j} \left\{ \partial_{x^k} \left(\frac{x^i}{|\mathbf{x}|} u_{a_j} \right) \partial_{x^k} \Psi^R - \frac{u_{a_j}}{|\mathbf{x}|} \delta^{ik} \partial_{x^k} \Psi^R + \frac{u_{a_j}}{|\mathbf{x}|} \delta^{ik} \partial_{x^k} \Psi^R \right\}(\mathbf{x}, t) d\mathbf{x} \\
&= \int_{|\mathbf{x}| > a_j} \frac{x^i}{|\mathbf{x}|} u_{a_j} \Delta \Psi^R(\mathbf{x}, t) d\mathbf{x} = \int_{|\mathbf{x}| > a_j} U_{a_j}^i \Delta \Psi^R(\mathbf{x}, t) d\mathbf{x}.
\end{aligned}$$

Similarly, we also have

$$\begin{aligned}
& \int_{a_j}^{\infty} u_{a_j} \partial_r (\partial_r (\varphi^R)^i + \frac{m}{r} (\varphi^R)^i) r^m dr \\
&= - \int_{|\mathbf{x}| > a_j} \left\{ |\mathbf{x}| \frac{x^i}{|\mathbf{x}|} \frac{x^k}{|\mathbf{x}|} \partial_r \left(\frac{u_{a_j}}{|\mathbf{x}|} \right) \partial_{x^k} \Psi^R(\mathbf{x}, t) + \frac{u_{a_j}}{|\mathbf{x}|} \partial_{x^i} \Psi^R(\mathbf{x}, t) \right\} d\mathbf{x} \\
&= - \int_{|\mathbf{x}| > a_j} \left\{ \partial_{x^i} \left(\frac{x^k}{|\mathbf{x}|} u_{a_j} \right) \partial_{x^k} \Psi^R - \frac{u_{a_j}}{|\mathbf{x}|} \delta^{ik} \partial_{x^k} \Psi^R + \frac{u_{a_j}}{|\mathbf{x}|} \delta^{ik} \partial_{x^k} \Psi^R \right\}(\mathbf{x}, t) d\mathbf{x} \\
&= \int_{|\mathbf{x}| > a_j} \frac{x^k}{|\mathbf{x}|} u_{a_j} \partial_{x^i} \partial_{x^k} \Psi^R(\mathbf{x}, t) d\mathbf{x} = \int_{|\mathbf{x}| > a_j} \mathbf{U}_{a_j} \cdot \nabla \partial_{x^i} \Psi^R(\mathbf{x}, t) d\mathbf{x}.
\end{aligned}$$

Since $\beta = 2\mu + \lambda$, we have

$$\begin{aligned}
& \beta \lim_{R \searrow 0} \lim_{j \rightarrow \infty} \mathcal{U}_j(\varphi^R, t) = \lim_{R \searrow 0} \lim_{j \rightarrow \infty} (2\mu + \lambda) \int_0^t \int_{a_j}^{\infty} u_{a_j} \partial_r (\partial_r (\varphi^R)^i + \frac{m}{r} (\varphi^R)^i) r^m dr ds \\
&= \lim_{R \searrow 0} \lim_{j \rightarrow \infty} \int_0^t \int_{|\mathbf{x}| > a_j} \left\{ \mu U_{a_j}^i \Delta \Psi^R + (\mu + \lambda) \mathbf{U}_{a_j} \cdot \nabla \partial_{x^i} \Psi^R \right\}(\mathbf{x}, s) d\mathbf{x} ds = \mathbb{U}^i(\Psi, t).
\end{aligned}$$

Chapter 3

Local Existence of Strong Solutions in Bounded Lagrangian Domain

In this chapter, we prove the existence and uniqueness of a local-in-time strong solution to the exterior Lagrangian equations (1.3.6) in a bounded domain $(x, t) \in [0, k] \times [0, T]$ for each fixed $k \in \mathbb{N}$ and $a \in (0, 1)$. This is done via the Picard iteration scheme where the non-linear equations are first linearised into a collection of linear parabolic equations. Then the linear solution at each iterative step is obtained in Theorem 3.4.1 and 3.4.3, using the Galerkin's method. Consequently, we obtain high regularity energy estimates, Theorem 3.4.2, 3.4.4, and contraction inequality, Theorem 3.5.1, for this set of iterative solutions. Finally using Banach fixed-point theorem, we show there exists a time $T_* > 0$ such that a unique solution to the original non-linear equations (1.3.6) exists in the domain $(x, t) \in [0, k] \times [0, T_*]$. The main difficulty, in comparison to the case of Eulerian coordinate formulation, is that the second order differential operators for the momentum and internal energy equations in Lagrangian formulation has quasi-linear coefficients. To ensure ellipticity, one needs a careful analysis on the estimates for which each iterative solutions satisfy with respect to time. This point is illustrated in Lemma 3.3.1 and Proposition 3.3.1 in Section 3.3. Finally, we note that a good reference for the local-in-time well-posedness problem in Lagrangian formulation of the full compressible Navier-Stokes equations can be found in [19], where the authors deal with the general 3 dimensional case with a mixture of periodic and free boundary conditions.

3.1 Main Statement: Local-in-Time Well-Posedness

For given initial data $(v_0, u_0, e_0)(x)$ in $x \in [0, k]$ with a fixed parameter $k \in \mathbb{N}$, our goal is to find a $T > 0$ such that the classical solution exists in the domain: $(x, t) \in [0, k] \times [0, T]$. To do so, we first rewrite the system with a new set of unknowns:

$$\tau(x, t) := v(x, t)r(x, t)^{-m}, \quad u(x, t) = u(x, t), \quad e(x, t) = e(x, t).$$

The coefficient $r(x, t)$ is defined by:

$$r(x, t) := a + \int_0^x \tau(y, t)dy.$$

Given an initial data $(v_0, u_0, e_0)(x)$ in $x \in [0, k]$, then with these unknowns, the full compressible Navier-Stokes equations in bounded Lagrangian domain (2.2.16) is restated as the following Initial Boundary Value Problem (suppressing the approximation index $k \in \mathbb{N}$):

$$\left\{ \begin{array}{l}
D_t \tau - D_x u = 0 \\
D_t u - \beta D_x \left(\frac{r^m}{\tau} D_x u \right) + \beta \tau r^{m-2} u = m(\gamma - 1) \frac{e}{r} - (\gamma - 1) D_x \left(\frac{e}{\tau} \right) \\
D_t e - \kappa D_x \left(\frac{r^m}{\tau} D_x e \right) \\
= \beta \frac{|D_x(r^m u)|^2}{r^m \tau} - 2m\mu D_x(r^{m-1} u^2) - (\gamma - 1) e \frac{D_x(r^m u)}{r^m \tau} \\
\text{where } r(x, t) := a + \int_0^x \tau(y, t) dy \\
u(0, t) = u(k, t) = 0, \quad D_x e(0, t) = D_x e(k, t) = 0 \quad \text{for } t \in [0, T], \\
(\tau, u, e)(x, 0) = (v_0(r_0)^{-m}, u_0, e_0)(x) \\
\text{where } r_0(x) := \left(a^n + n \int_0^x v_0(y) dy \right)^{\frac{1}{n}} \quad \text{for } x \in [0, k].
\end{array} \right. \quad \text{in } [0, k] \times [0, T], \quad (3.1.1)$$

Theorem 3.1.1 (Local-in-time Existence of Bounded Lagrangian Problem). *Given initial data*

$$(v_0, u_0, e_0) \in H^2(0, k) \quad \text{with } C_0^{-1} \leq v_0 \leq C_0,$$

for some constant $C_0 > 0$. Then there exists a time $T > 0$ such that a unique classical solution (τ, u, e) to the problem (3.1.1) exists. Moreover, there exists a constant $C = C(C_0, T, a, k, \beta, \gamma, n) > 0$, such that the following regularities are satisfied:

$$\left\{ \begin{array}{l}
C^{-1} \leq r^m \tau \leq C, \quad \text{for } (x, t) \in [0, k] \times [0, T], \\
\tau \in C^0([0, T]; H^2(0, k)) \cap C^1([0, T]; H^1(0, k)), \\
(u, e) \in C^0([0, T]; H^2(0, k)) \cap C^1([0, T]; L^2(0, k)) \cap L^2(0, T; H^3(0, k)), \\
(D_x e, u) \in C^0([0, T]; H_0^1).
\end{array} \right.$$

Note that once the local-in-time solution (τ, u, e) is obtained, one can redefine:

$$v(x, t) := r(x, t)^m \tau(x, t) \quad \text{with } r(x, t) = a + \int_0^x \tau(y, t) dy,$$

and a unique local-in-time classical solution (v, u, e) to the original problem (2.2.16) is recovered. The proof for Theorem 3.1.1 is the main subject of this chapter. The outline is organised as follows: first, the iteration scheme is introduced to reduce the non-linear system (3.1.1) to a set of linear Initial Boundary Value Problems; next, we describe the suitable energy space for which each iterative solution belongs to; then, using Galerkin's method, the linear solutions are obtained and their higher regularity estimates are derived; finally, we show that the iterative linear solutions satisfy contraction estimates with certain distance function. As a consequence, the existence and uniqueness of a non-linear solution stated in Theorem 3.1.1 is obtained via the fixed-point theorem and its regularity is shown by the compactness argument.

3.2 Iteration Scheme

For the base case $j = 0$, we set: $(u^{(0)}, e^{(0)})(x, t)$ to be the unique solution to the linear equations:

$$\left\{ \begin{array}{l}
D_t u^{(0)} - \beta D_x^2 u^{(0)} = 0 \\
D_t e^{(0)} - \kappa D_x^2 e^{(0)} = 0 \\
u^{(0)}(0, t) = u^{(0)}(k, t) = 0, \quad D_x e^{(0)}(0, t) = D_x e^{(0)}(k, t) = 0 \quad \text{for } t \in [0, \infty), \\
(u^{(0)}, e^{(0)})(x, 0) = (u_0, e_0)(x) \quad \text{for } x \in [0, k].
\end{array} \right. \quad \text{in } (x, t) \in [0, k] \times [0, \infty), \quad (3.2.1)$$

Using base case solution $(u^{(0)}, e^{(0)})(x, t)$, we also define $\tau^{(0)}(x, t)$ to be:

$$\tau^{(0)}(x, t) := \tau_0(x) + \int_0^t D_x u^{(0)}(x, s) ds \quad \text{and} \quad r^{(0)}(x, t) := a + \int_0^x \tau^{(0)}(y, s) ds,$$

where

$$\tau_0(x) = \frac{v_0(x)}{(r_0(x))^m}, \quad \text{with} \quad r_0(x) := \left(a^n + n \int_0^x v_0(y) dy \right)^{1/n}.$$

Next, for the iteration step $j \in \mathbb{N}$: given functions

$$(\tilde{u}, \tilde{e}) := (u_{j-1}, e_{j-1}) \in C^0([0, T]; H^2(0, k)) \cap C^1([0, T]; L^2(0, k)) \cap L^2(0, T; H^3(0, k)),$$

for each $T > 0$, we define

$$\begin{aligned} \tilde{\tau}(x, t) &\equiv \tau_{j-1}(x, t) := \tau_0(x) + \int_0^t D_x \tilde{u}(x, s) ds, \\ \tilde{r}(x, t) &\equiv r_{j-1}(x, t) := a + \int_0^x \tilde{\tau}(y, t) dy. \end{aligned} \tag{3.2.2}$$

One can check that $\tilde{\tau} \in C^0([0, T]; H^2(0, k)) \cap C^1([0, T]; H^1(0, k))$, for each $T > 0$. A function $(u, e)(x, t)$ is said to be the next iterative step linear solution if it solves the following linear parabolic Boundary Value Problem:

$$\begin{cases} D_t u - \tilde{L}(u) = -(\gamma - 1) D_x \left(\frac{\tilde{e}}{\tilde{\tau}} \right) + m(\gamma - 1) \frac{\tilde{e}}{\tilde{\tau}} & \text{in } [0, k] \times [0, \infty), \\ D_t e - \tilde{K}(e) = \beta \frac{|D_x(\tilde{r}^m u)|^2}{\tilde{r}^m \tilde{\tau}} - 2\mu m D_x(\tilde{r}^{m-1} u^2) - (\gamma - 1) \tilde{e} \frac{D_x(\tilde{r}^m u)}{\tilde{r}^m \tilde{\tau}} & \\ (u, D_x e)(k, t) = (u, D_x e)(0, t) = (0, 0) & \text{for each } t \in [0, \infty), \\ (u, e)(x, 0) = (u_0, e_0)(x) & \text{for each } x \in [0, k], \end{cases} \tag{3.2.3}$$

where the operators \tilde{L} and \tilde{K} are given by

$$\begin{cases} \tilde{L}(w) := \beta D_x \left(\frac{\tilde{r}^m}{\tilde{\tau}} D_x w \right) - m \beta \tilde{r} \tilde{r}^{m-2} w \\ \tilde{K}(w) := \kappa D_x \left(\frac{\tilde{r}^m}{\tilde{\tau}} D_x w \right) \end{cases} \tag{3.2.4}$$

The solution (u, e) to the system (3.2.3) will be obtained in the following sections below using Galerkin's method. Once the solution $(u, e)(x, t)$ is given, we set $u_j(x, t) := u(x, t)$, $e_j(x, t) := e(x, t)$, and define:

$$\tau_j(x, t) := \tau_0(x) + \int_0^t D_x u_j(x, s) ds \tag{3.2.5}$$

In particular, $\tau_j(x, t)$ satisfies the continuity equation:

$$D_t \tau_j - D_x u_j = 0$$

3.3 Energy Space

We state the energy space, for which solutions of (3.2.2)–(3.2.5) belong to. Define the functional $E[u, e](t)$ to be:

$$\begin{aligned} E[u, e](t) &:= \sup_{0 \leq s \leq t} \{ \|(u, e)(\cdot, s)\|_{H^2}^2 + \|(D_t u, D_t e)(\cdot, s)\|_{L^2}^2 \} + \\ &+ \int_0^t \{ \|(D_t u, D_t e)(\cdot, s)\|_{H^1}^2 + \|(u, e)(\cdot, s)\|_{H^3}^2 \} ds. \end{aligned} \tag{3.3.1}$$

In what follows, we often use the notations E_0 and $\tilde{E}(t)$ to denote that $E_0 \equiv E[u_0, e_0]$ and $\tilde{E}(t) \equiv E[\tilde{u}, \tilde{e}](t)$ for convenience.

Lemma 3.3.1. *If (u, e) is such that $E[u, e](t) < \infty$ for each $t \in [0, \infty)$, then we have that for each $(x, t) \in [0, k] \times [0, \infty)$*

$$\begin{cases} \sup_{0 \leq s \leq t} |\tau(\cdot, s) - r_0^{-m} v_0(\cdot)|_\infty \leq Ct(E[u, e](t))^{\frac{1}{2}}, \\ \sup_{0 \leq s \leq t} \|D_x \tau(\cdot, s) - D_x \tau_0(\cdot)\|_{L^2} \leq t(E[u, e](t))^{\frac{1}{2}}, \\ \sup_{0 \leq s \leq t} |D_x \tau(\cdot, s) - D_x \tau_0(\cdot)|_{L^\infty} \leq Ct^{\frac{1}{2}}(E[u, e](t))^{\frac{1}{2}}, \\ \sup_{0 \leq s \leq t} \|D_x^2 \tau(\cdot, s) - D_x^2 \tau_0(\cdot)\|_{L^2} \leq t^{\frac{1}{2}}(E[u, e](t))^{\frac{1}{2}}. \end{cases}$$

where $C > 0$ is a generic constant from the Sobolev embedding theorem.

Proof. By definitions (3.2.5), we have:

$$\tau(x, s) - \tau_0(x) = \int_0^s D_t \tau(x, q) dq = \int_0^s D_x u(x, q) dq.$$

Thus, taking $L^\infty(0, k)$ norm and supremum over time $s \in [0, t]$, it follows that the Sobolev embedding Theorem $H^1(0, k) \hookrightarrow L^\infty(0, k)$, and definition (3.3.1) that

$$\begin{aligned} \sup_{0 \leq s \leq t} |\tau(\cdot, s) - \tau_0(\cdot)|_{L^\infty(0, k)} &\leq \int_0^t |D_x u(\cdot, q)|_{L^\infty(0, k)} dq \leq C \int_0^t \|D_x u(\cdot, q)\|_{H^1} dq \\ &\leq t \sup_{s \in [0, t]} \|u(\cdot, s)\|_{H^2} \leq t(E[u, e](t))^{1/2}. \end{aligned}$$

We also have:

$$D_x \tau(x, s) - D_x \tau_0(x) = \int_0^s D_x^2 u(x, q) dq. \quad (3.3.2)$$

Taking $L^2(0, k)$ norm on (3.3.2) and supremum over time $s \in [0, t]$, it follows that

$$\sup_{s \in [0, t]} \|D_x \tau(\cdot, s) - D_x \tau_0(\cdot)\|_{L^2(0, k)} \leq \int_0^t \|D_x^2 u(\cdot, q)\|_{L^2(0, k)} dq \leq t(E[u, e](t))^{1/2}.$$

Moreover, Taking $L^\infty(0, k)$ norm on (3.3.2) and supremum over time $s \in [0, t]$, we also get using Sobolev embedding theorem that

$$\begin{aligned} \sup_{s \in [0, t]} |D_x \tau(\cdot, s) - D_x \tau_0(\cdot)|_{L^\infty(0, k)} &\leq \int_0^t |D_x^2 u(\cdot, q)|_{L^\infty(0, k)} dq \leq C \int_0^t \|D_x^2 u(\cdot, q)\|_{H^1(0, k)} dq \\ &\leq C \int_0^t \|u(\cdot, q)\|_{H^3(0, k)} dq \leq Ct^{1/2} \left(\int_0^t \|u(\cdot, q)\|_{H^3(0, k)}^2 dq \right)^{1/2} \leq Ct^{1/2} (E[u, e](t))^{1/2}. \end{aligned}$$

Finally taking another D_x derivative on (3.3.2), we get:

$$D_x^2 \tau(x, s) - D_x^2 \tau_0(x) = \int_0^s D_x^3 u(x, q) dq.$$

Taking $L^2(0, k)$ norm and supremum over time $s \in [0, t]$, it follows that

$$\begin{aligned} & \sup_{s \in [0, t]} \|D_x^2 \tau(\cdot, s) - D_x^2 \tau_0(\cdot)\|_{L^2(0, k)} \leq \int_0^t \|D_x^3 u(\cdot, q)\|_{L^2(0, k)} dq \\ & \leq t^{1/2} \left(\int_0^t \|D_x^3 u(\cdot, q)\|_{L^2(0, k)}^2 dq \right)^{1/2} \leq t^{1/2} (E[u, e](t))^{1/2}. \end{aligned}$$

□

The implication of Lemma 3.3.1 is necessary to ensure the ellipticity of operators \tilde{L}, \tilde{K} defined in (3.2.4). We summarises this in the following proposition:

Proposition 3.3.1 (The *a-priori* bounds). *For a given constant $M_0 > 0$, define $T_0 = T_0(M_0) > 0$ by*

$$T_0 = T_0(M_0) := \min \left\{ 1, \frac{\bar{\tau}_0}{2CM_0^{1/2}}, \frac{\underline{\tau}_0}{2CM_0^{1/2}} \right\}, \quad (3.3.3)$$

where $\underline{\tau}_0 := \inf_{x \in [0, k]} \tau_0(x)$, $\bar{\tau}_0 := \sup_{x \in [0, k]} \tau_0(x)$, and $C > 0$ is the generic constant from the Sobolev embedding inequality in Lemma 3.3.1. Suppose there exists a time $T_* > 0$ such that $E[\tilde{u}, \tilde{\tau}](T_*) \leq M_0$, then

$$\begin{cases} \frac{1}{2}\underline{\tau}_0 \leq \tilde{\tau}(x, t) \leq \frac{3}{2}\bar{\tau}_0 \\ a \leq \tilde{r}(x, t) \leq a + \frac{3k}{2}\bar{\tau}_0 \end{cases} \quad \text{for all } (x, t) \in [0, k] \times [0, \min\{T_*, T_0\}],$$

and the operators \tilde{K}, \tilde{L} satisfies the ellipticity:

$$\begin{cases} -(w, \tilde{L}w) \geq \beta\nu_1 \|w\|_{H^1(0, k)}^2 \\ -(w, \tilde{K}w) \geq \kappa\nu_2 \|D_x w\|_{L^2(0, k)}^2 \end{cases} \quad \text{for all } w \in H_0^1(0, k) \quad \text{and } t \in [0, \min\{T_*, T_0\}],$$

where $\nu_1 = \nu_1(C_0, a, k, n)$ and $\nu_2 = \nu_2(C_0, a, k, n)$ are constants given by:

$$\nu_1 = \min \left\{ \frac{2a^m}{3\bar{\tau}_0}, \frac{m\underline{\tau}_0}{2} \left(\frac{3k\bar{\tau}_0}{2} \right)^{m-2} \right\} \quad \text{and} \quad \nu_2 = \frac{2a^m}{3\bar{\tau}_0}.$$

Proof. From Lemma 3.3.1, we have that for all $(x, t) \in [0, k] \times [0, \min\{T_0, T_*\}]$:

$$|\tilde{\tau}(x, t) - \tau_0(x)| \leq Ct(E[\tilde{u}, \tilde{e}](t))^{1/2} \leq CtM_0^{1/2}.$$

Thus it follows from the construction 3.3.3 that for all $(x, t) \in [0, k] \times [0, \min\{T_0, T_*\}]$:

$$\begin{aligned} 0 < \frac{1}{2}\underline{\tau}_0 &\leq \inf_{x \in [0, k]} \tau_0(x) - \frac{1}{2}\underline{\tau}_0 \leq \tau_0(x) - CM_0^{1/2} \frac{\underline{\tau}_0}{2CM_0^{1/2}} \leq \tau_0(x) - CM_0^{1/2} t \leq \tilde{\tau}(x, t), \\ \tilde{\tau}(x, t) &\leq \tau_0(x) + CM_0^{1/2} t \leq \tau_0(x) + CM_0^{1/2} \frac{\bar{\tau}_0}{2CM_0^{1/2}} \leq \sup_{x \in [0, k]} \tau_0(x) + \frac{\bar{\tau}_0}{2} \leq \frac{3}{2}\bar{\tau}_0. \end{aligned}$$

Next, $\tilde{r}(x, t)$ is defined in (3.2.2) as

$$\tilde{r}(x, t) = a + \int_0^x \tilde{\tau}(y, t) dy.$$

we have that $a \leq \tilde{r} \leq a + 3k\bar{\tau}_0/2$ for all $(x, t) \in [0, k] \times [0, \min\{T_*, T_0\}]$. Using these inequalities, it follows that for all $w \in H_0^1(0, k)$ and $t \in [0, \min\{T_*, T_0\}]$:

$$\begin{aligned} -(w, \tilde{L}w)_{L^2} &= \beta \int_0^k \frac{\tilde{r}^m}{\tilde{\tau}}(x, t) |D_x w|^2(x) dx + m\beta \int_0^k \tilde{r}^{m-2}(x, t) |w|^2(x) dx \\ &\geq \beta \min \left\{ \frac{2a^m}{3\bar{\tau}_0}, \frac{m\underline{\tau}_0}{2} \left(\frac{3k\bar{\tau}_0}{2} \right)^{m-2} \right\} \|w\|_{H^1}^2 =: \beta\nu_1(n, a, k, C_0) \|w\|_{H^1}^2. \end{aligned}$$

The ellipticity of \tilde{K} can be shown in the exact same way. □

3.4 Linear Solutions via Galerkin's Method

In this section, we consider existence result to the linear system (3.2.2)–(3.2.5). First we define the weak solution to the linear system (3.2.3) as follows:

Definition 3.4.1. *We say $(u, e)(x, t)$ is a weak solution to the system (3.2.2)–(3.2.5) on the domain $(x, t) \in [0, k] \times [0, T]$, if it satisfies the following:*

- $u \in L^2(0, T; H_0^1(0, k))$, $e \in L^2(0, T; H^1(0, k))$, $D_t u \in L^2(0, T; H^{-1}(0, k))$, $D_t e \in L^2(0, T; H^*(0, k))$ where $H^{-1}(0, k)$ and $H^*(0, k)$ are respectively the dual space of $H_0^1(0, k)$ and $H^1(0, k)$.
- For each $\varphi \in C^1([0, T]; H_0^1(0, k))$ with $\varphi(\cdot, T) = 0$, one has

$$\begin{aligned} & \int_0^k u_0(x)\varphi(x, 0)dx - \int_0^T \int_0^k u(x, t)D_t\varphi dxdt + \\ & + \beta \int_0^T \int_0^k \left\{ \frac{\tilde{r}^m}{\tilde{\tau}} D_x u D_x \varphi + \tilde{r}^{m-2} \tilde{\tau} u \varphi \right\} dxdt = (\gamma - 1) \int_0^T \int_0^k \left\{ m \frac{\tilde{e}}{\tilde{r}} - D_x \left(\frac{\tilde{e}}{\tilde{r}} \right) \right\} \varphi dxdt. \end{aligned}$$

- For each $\phi \in C^1([0, T]; H^1(0, k))$ with $\phi(\cdot, T) = 0$, one has

$$\begin{aligned} & \int_0^k e_0(x)\phi(x, 0)dx - \int_0^T \int_0^k e D_t \phi dxdt + \kappa \int_0^T \int_0^k \frac{\tilde{r}^m}{\tilde{\tau}} D_x e D_x \phi dxdt \\ & = \int_0^T \int_0^k \left\{ \beta \frac{|D_x(\tilde{r}^m u)|^2}{\tilde{r}^m \tilde{\tau}} - (\gamma - 1) \frac{D_x(\tilde{r}^m u)}{\tilde{r}^m \tilde{\tau}} \tilde{e} - 2m\mu D_x(\tilde{r}^{m-1} u^2) \right\} \phi dxdt \end{aligned}$$

3.4.1 Existence and higher regularity of u

In this section, we show the existence of a function u defined in Definition 3.4.1. This is done via the Galerkin method. In what follows we denote $C > 0$ as a constant which only depends on the parameters $C = C(C_0, \beta, \gamma, a, k, n) > 0$.

Theorem 3.4.1. *Given a fixed constant $M_0 > 0$ and let $T_0 = T_0(M_0) > 0$ be the time defined in Proposition 3.3.1 by:*

$$T_0(M_0) := \min \left\{ 1, \frac{\bar{\tau}_0}{2CM_0^{1/2}}, \frac{\underline{\tau}_0}{2CM_0^{1/2}} \right\},$$

where $\underline{\tau}_0 := \inf_{x \in [0, k]} \tau_0(x)$, $\bar{\tau}_0 := \sup_{x \in [0, k]} \tau_0(x)$, and $C > 0$ is some generic constant from the Sobolev embedding inequality. If there exists a time $T_* > 0$ such that $\tilde{E}(T_*) \equiv E[\tilde{u}, \tilde{e}](T_*) \leq M_0$, then for each given $T \in [0, \min\{T_*, T_0\}]$ there exists a function $u(x, t)$ which satisfies the weak form in Definition 3.4.1 on the domain $(x, t) \in [0, k] \times [0, T_1]$ with $T_1 \equiv \min\{T_*, T_0\}$. Moreover $u \in C(0, T_1; L^2(0, k)) \cap L^2(0, T_1; H_0^1(0, k))$ and $D_t u \in L^2(0, T_1; H^{-1}(0, k))$. In particular, the following estimate is satisfied:

$$\sup_{0 \leq t \leq T} \|u\|_{L^2}^2 + \int_0^T \{ \|D_x u\|_{L^2}^2 + \|u\|_{L^2}^2 + \|D_t u\|_{H^{-1}}^2 \} dt \leq C \|u_0\|_{L^2}^2 + CT \tilde{E}(T), \quad (3.4.1)$$

for each $T \in [0, \min\{T_*, T_0\}]$, where $H^{-1}(0, k)$ is the dual space of $H_0^1(0, k)$, and $C = C(n, \beta, \gamma, a, k, C_0) > 0$ is a constant independent of M_0 , T_0 , and T_* .

Proof. Let $\{\phi_i(x)\}_{i \in \mathbb{N}} \subset C^\infty(0, k)$ be an orthonormal basis of $L^2(0, k)$, which is also a basis for $H_0^1(0, k)$. Such basis can be obtained by solving the following eigenvalue problem:

$$\begin{cases} -D_x^2 \phi = \lambda \phi & \text{for } x \in (0, k), \\ \phi(0) = \phi(k) = 0. \end{cases}$$

Now we consider $a_N^i(t)$ to be the solution to the following system of ODE's:

$$\begin{cases} \frac{da_N^i}{dt} = -\beta \sum_{j=1}^N \left\{ \left(\frac{\tilde{r}^m}{\tilde{r}} \phi_j', \phi_i' \right)_{L^2} + (\tilde{r}^{m-2} \tilde{r} \phi_j, \phi_i)_{L^2} \right\} a_N^j(t) + (F, \phi_i)_{L^2}, \\ a_N^i(0) = (u_0, \phi_i)_{L^2}, \end{cases} \quad (3.4.2)$$

where F denotes:

$$F := (\gamma - 1) \left\{ m \frac{\tilde{e}}{\tilde{r}} - D_x \left(\frac{\tilde{e}}{\tilde{r}} \right) \right\}.$$

By the existence and uniqueness theorem of ordinary differential equations, there exists a unique solution $a_N^i(t) \in C^1([0, T_1])$ with $T_1 \equiv \min\{T_*, T_0\}$, solving (3.4.2). Next, we define:

$$u_N(x, t) := \sum_{i=1}^N a_N^i(t) \phi_i(x).$$

then by orthogonality of $\{\phi_i\}_{i \in \mathbb{N}}$, we have from (3.4.2) that

$$(D_t u_N, \phi_i)_{L^2} + \beta \left(\frac{\tilde{r}^m}{\tilde{r}} D_x u_N, \phi_i' \right)_{L^2} + \beta (\tilde{r}^{m-2} \tilde{r} u_N, \phi_i)_{L^2} = (F, \phi_i)_{L^2}. \quad (3.4.3)$$

Multiplying both sides by $a_N^i(t)$ and taking summation over $i = 1, \dots, N$, it follows that

$$(D_t u_N, u_N)_{L^2} + \beta \left(\frac{\tilde{r}^m}{\tilde{r}} D_x u_N, D_x u_N \right)_{L^2} + \beta (\tilde{r}^{m-2} \tilde{r} u_N, u_N)_{L^2} = (F, u_N)_{L^2}.$$

Hence using Hölder's inequality, Lemma 3.3.1, Proposition 3.3.1, integrating in time, and taking the supremum, we get

$$\begin{aligned} & \sup_{0 \leq t \leq T} \|u_N(\cdot, t)\|_{L^2}^2 + \frac{\beta a^m}{3 \bar{\tau}_0} \int_0^T \|D_x u_N(\cdot, t)\|_{L^2}^2 dt + \frac{\beta}{4} \mathcal{I}_0 (2k \bar{\tau}_0)^{m-2} \int_0^T \|u_N(\cdot, t)\|_{L^2}^2 dt \\ & \leq \|u_N(\cdot, 0)\|_{L^2}^2 + C \int_0^T \|\tilde{e}(\cdot, t)\|_{L^2}^2 dt \leq \|u_0\|_{L^2}^2 + CT \tilde{E}(T), \end{aligned} \quad (3.4.4)$$

where $C = C(n, \beta, \gamma, a, k, C_0) > 0$, and we have used the fact that $\{\phi_i\}_i$ is an orthonormal basis in H^1 , hence it satisfies the Bessel's inequality:

$$\|u_N(\cdot, 0)\|_{L^2}^2 = \left\| \sum_{i=1}^N (u_0, \phi_i)_{L^2} \phi_i \right\|_{L^2}^2 = \sum_{i=1}^N (u_0, \phi_i)_{L^2}^2 \leq \|u_0\|_{L^2}^2.$$

Now from the estimate (3.4.4), there exists $u \in L^2(0, T; H^1(0, k)) \cap L^\infty(0, T; L^2(0, k))$ such that:

$$u_N \rightharpoonup u \quad \text{in } L^2(0, T; H^1(0, k)), \quad \text{and} \quad u_N \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(0, T; L^2(0, k)). \quad (3.4.5)$$

Next we show that:

$$\sup_{N \in \mathbb{N}} \|D_t u_N\|_{L^2(0, T; H^{-1})}^2 \leq C (\|u_0\|_{H^1}^2 + CT \tilde{E}(T)),$$

where $C = C(n, \beta, \gamma, a, k, C_0) > 0$. Let $\varphi(x) \in H_0^1(0, k)$ be a test function. Without loss of generality, choose $\|\varphi\|_{H^1} = 1$. Since $\{\phi_i\}_{i \in \mathbb{N}}$ is an orthogonal basis in $H_0^1(0, k)$, we can approximate $\varphi(x)$ by

$$\lim_{M \rightarrow \infty} \sum_{i=1}^M \frac{(\phi_i, \varphi)_{H^1}}{1 + \lambda_i} \phi_i(x) = \varphi(x) \quad \text{in } H^1(0, k), \quad \text{where } (\phi_i, \varphi)_{H^1} = (\phi_i, \varphi)_{L^2} + (D_x \phi_i, D_x \varphi)_{L^2}.$$

Using this, the equation (3.4.3) becomes

$$(D_t u_N, \varphi)_{L^2} = -\beta \left(\frac{\tilde{r}^m}{\tilde{r}} D_x u_N, D_x \varphi \right)_{L^2} - \beta (\tilde{r}^{m-2} \tilde{r} u_N, \varphi)_{L^2} + (F, \varphi)_{L^2}. \quad (3.4.6)$$

Hence, we have:

$$(D_t u_N(\cdot, t), \varphi)_{L^2} \leq C \left(\|u_N\|_{H^1}^2 + \tilde{E}(T) \right)^{1/2} \|\varphi\|_{L^2(0, T; H^1)}.$$

Integrating the above equation in time, and using the estimate (3.4.4), we have

$$\|D_t u_N\|_{L^2(0, T; H^{-1})}^2 = \sup \left\{ \int_0^T (D_t u_N(\cdot, t), \varphi)_{L^2}^2 dt \mid \varphi \in H_0^1(0, k), \|\varphi\|_{H^1} = 1 \right\} \leq C \|u_0\|_{L^2}^2 + CT \tilde{E}(T).$$

Now combining this estimate with the convergence (3.4.5), there exists a subsequence $N \rightarrow \infty$ such that $D_t u_N \rightharpoonup D_t u$ in $L^2(0, T; H^{-1}(0, k))$ where here we have denoted $D_t u \in L^2(0, T; H^{-1}(0, k))$ as the weak temporal derivative of $u(\cdot, t)$ in the space $H^{-1}(0, k)$, in other words

$$\int_0^T \chi(t) (D_t u, \phi)_{H^{-1}, H_0^1} dt := - \int_0^T \chi'(t) (u, \phi)_{H^{-1}, H_0^1} dt := - \int_0^T \chi'(t) (u, \phi)_{L^2} dt,$$

for all $\chi \in C_c^1(0, T)$ and $\phi \in H_0^1(0, k)$. Combining with the fact that $u \in L^2(0, T; H_0^1(0, k))$, we have that $u \in C(0, T; L^2(0, k))$ as a consequence of interpolation Theorem.

Finally, we prove that u indeed satisfies the weak form in Definition 3.4.1. For $\varphi \in C^1(0, T; H_0^1(0, k))$ with $\varphi(\cdot, T) = 0$, approximating $\lim_{M \rightarrow \infty} \sum_{i=1}^M (\varphi(x, t), \psi_i)_{H^1} (1 + \lambda_i)^{-1} \phi_i(x) = \varphi(x, t)$ in $H^1(0, k)$ for each fixed $t \in [0, T]$, where λ_i being the i -th eigenvalue, we have:

$$\begin{aligned} (D_t u_N(\cdot, t), \varphi(\cdot, t))_{L^2} &= -\beta \left(D_x u_N(\cdot, t), \frac{\tilde{r}^m}{\tilde{r}} D_x \varphi(\cdot, t) \right)_{L^2} - \beta (u_N(\cdot, t), \tilde{r}^{m-2} \tilde{r} \varphi(\cdot, t))_{L^2} + \\ &\quad + (\gamma - 1) \left(m \frac{\tilde{e}}{\tilde{r}} - D_x \left(\frac{\tilde{e}}{\tilde{r}} \right), \varphi(\cdot, t) \right)_{L^2}. \end{aligned}$$

Using the convergence $\lim_{N \rightarrow \infty} u_N(x, 0) = u_0$ in $L^2(0, k)$, we have by integrating the above in time that:

$$\begin{aligned} &\int_0^k u_0(x) \varphi(x, 0) dx - \int_0^T \int_0^k u D_t \varphi dx dt = \lim_{N \rightarrow \infty} \int_0^T \int_0^k D_t u_N \varphi dx dt \\ &= -\beta \int_0^T \left(D_x u, \frac{\tilde{r}^m}{\tilde{r}} D_x \varphi \right)_{L^2} dt - \beta \int_0^T (u, \tilde{r}^{m-2} \tilde{r} \varphi)_{L^2} dt + \\ &\quad + m(\gamma - 1) \int_0^T \left(\frac{\tilde{e}}{\tilde{r}}, \varphi \right)_{L^2} dt - (\gamma - 1) \int_0^T \left(D_x \left(\frac{\tilde{e}}{\tilde{r}} \right), \varphi \right)_{L^2} dt, \end{aligned}$$

Hence we conclude the proof of this theorem. \square

Next we show that u satisfies the higher regularity in terms of the energy norm (3.3.1). This can be summarised into the following theorem, which will be proven with several lemmas.

Theorem 3.4.2. *Given a fixed constant $M_0 > 0$ and $T_0 = T_0(M_0) > 0$ as defined in Proposition 3.3.1. Suppose there exists a time $T_* > 0$ such that (\tilde{u}, \tilde{e}) satisfies $\tilde{E}(T_*) \equiv E[\tilde{u}, \tilde{e}](T_*) \leq M_0$, then the function $u(x, t)$ obtained in Theorem 3.4.1 satisfies the following estimates: there exists a polynomial $z \mapsto \mathcal{P}(z)$ such that, for all $T \in [0, \min\{T_*, T_0\}]$, we have*

$$\begin{aligned} &\sup_{0 \leq t \leq T} \{ \|u(\cdot, t)\|_{H^2}^2 + \|D_t u(\cdot, t)\|_{L^2}^2 \} + \int_0^T \{ \|u\|_{H^3}^2 + \|D_t u\|_{H^1}^2 + \|D_t^2 u\|_{H^{-1}}^2 \} (t) dt \\ &\leq C \mathcal{P}(E_0) + CT \mathcal{P}(\tilde{E}(T)) \exp(CT^2 \tilde{E}(T)), \end{aligned} \quad (3.4.7)$$

where $H^{-1}(0, k)$ is the dual space of $H_0^1(0, k)$, and $C = C(C_0, a, k, \beta, \gamma, n) > 0$ is a constant independent of $M_0 > 0$, $T_0 > 0$, and $T_* > 0$.

Lemma 3.4.1. *Suppose the assumptions of Theorem 3.4.2 hold, and let $u(x, t)$ be the weak solution obtained in Theorem 3.4.1. Then $D_t u \in L^\infty(0, T_1; L^2(0, k))$ and $D_t u \in L^2(0, T_1; H_0^1(0, k))$ with $T_1 \equiv \min\{T_*, T_0\}$. Moreover, there exists a constant $C = C(C_0, a, k, \beta, \gamma, n) > 0$ and a polynomial $z \mapsto \mathcal{P}(z)$ such that the following inequality holds:*

$$\sup_{0 \leq t \leq T} \|D_t u(\cdot, t)\|_{L^2}^2 + \int_0^T \|D_t u(\cdot, t)\|_{H^1}^2 dt \leq CE_0 + CTP(\tilde{E}(T)) \exp(CT^2 \tilde{E}(T)),$$

for all $T \in [0, \min\{T_*, T_0\}]$.

Proof. First, for each small $h > 0$, we denote the difference quotient in time as:

$$\Delta_t^h u(x) := \frac{u(x, t+h) - u(x, t)}{h}.$$

Let $u_N(x, t) := \sum_{i=1}^N a_i^N(t) \phi_i(x)$ be the Galerkin approximation obtained in the proof of Theorem 3.4.1, it follows that u_N satisfies:

$$(D_t u_N(\cdot, t), \phi_i)_{L^2} + \beta \left(\frac{\tilde{r}^m}{\tilde{\tau}} D_x u_N(\cdot, t), \phi_i' \right)_{L^2} + m\beta (\tilde{r}^{m-2} \tilde{\tau} u_N(\cdot, t), \phi_i)_{L^2} = (\gamma - 1) \left(m \frac{\tilde{e}}{\tilde{\tau}} - D_x \left(\frac{\tilde{e}}{\tilde{\tau}} \right), \phi_i \right)_{L^2}.$$

Operating Δ_t^h to the above equation, multiplying the both sides with $h^{-1}(a_i^N(t+h) - a_i^N(t))$, and taking summation in $i = 1, \dots, N$, we get:

$$\begin{aligned} & \left(D_t \Delta_t^h u_N, \Delta_t^h u_N \right)_{L^2} + \beta \left(\frac{\tilde{r}^m}{\tilde{\tau}}(\cdot, t+h) \Delta_t^h D_x u_N(\cdot), \Delta_t^h D_x u_N(\cdot) \right)_{L^2} \\ & + m\beta \left(\tilde{r}^{m-2} \tilde{\tau}(\cdot, t+h) \Delta_t^h u_N(\cdot), \Delta_t^h u_N(\cdot) \right)_{L^2} \\ & = -\beta \left(D_x u_N(\cdot, t) \Delta_t^h (\tilde{\tau}^{-1} \tilde{r}^m), \Delta_t^h D_x u_N(\cdot) \right)_{L^2} - m\beta \left(u_N(\cdot, t) \Delta_t^h (\tilde{r}^{m-2} \tilde{\tau}), \Delta_t^h u_N(\cdot) \right)_{L^2} \\ & + m(\gamma - 1) \left(\Delta_t^h (\tilde{\tau}^{-1} \tilde{e}), \Delta_t^h u_N(\cdot) \right)_{L^2} + (\gamma - 1) \left(\Delta_t^h (\tilde{\tau}^{-1} \tilde{e}), \Delta_t^h D_x u_N(\cdot) \right)_{L^2}. \end{aligned}$$

Integrating the above in $t \in [0, T-h]$ for each small $h > 0$, then by Lemma 3.3.1, Proposition 3.3.1, and Cauchy-Schwartz inequality we obtain the estimate:

$$\frac{1}{2} \|\Delta_t^h u_N(\cdot)\|_{L^2}^2 + \frac{\nu_1}{2} \int_0^t \|\Delta_t^h u_N(\cdot, s)\|_{H^1}^2 ds \leq C_h + A_{T,h} + B_{T,h} \int_0^t \|u_N(\cdot, s)\|_{H^1}^2 ds, \quad (3.4.8)$$

where $C = C(n, \beta, \gamma, a, k, C_0) > 0$ and $\nu_1 = \nu_1(C_0, a, k, n) > 0$ is the constant given in Proposition 3.3.1 and $A_{T,h}, B_{T,h}, C_{T,h} > 0$ are given by:

$$\begin{cases} A_{T,h} := C \int_0^{T-h} \left\{ \|\Delta_t^h (\tilde{\tau}^{-1} \tilde{e})\|_{L^2}^2 + \|\Delta_t^h (\tilde{\tau}^{-1} \tilde{e})\|_{L^2}^2 \right\} dt, \\ B_{T,h} := C \sup_{0 \leq s \leq T-h} \left\{ |\Delta_s^h (\tilde{\tau}^{-1} \tilde{r}^m)|_\infty^2 + |\Delta_s^h (\tilde{r}^{m-2} \tilde{\tau})|_\infty^2 \right\}, \quad C_h := \frac{1}{2} \|\Delta_0^h u_N(\cdot)\|_{L^2}^2. \end{cases}$$

We show that

$$\limsup_{h \rightarrow 0^+} A_{T,h} \leq CT \left(1 + \tilde{E}(T)^2 \right), \quad \limsup_{h \rightarrow 0^+} B_{T,h} \leq C\tilde{E}(T) \quad \text{and} \quad \limsup_{h \rightarrow 0^+} C_h \leq CE_0. \quad (3.4.9)$$

From the construction (3.2.2)–(3.2.5), we have:

$$\Delta_t^h(\tilde{\tau}^{-1}\tilde{r}^m)(x) = \frac{(\tilde{\tau}^{-1}\tilde{r}^m)(x, t+h) - (\tilde{\tau}^{-1}\tilde{r}^m)(x, t)}{h} = \frac{1}{h} \int_t^{t+h} \left(m \frac{\tilde{r}^{m-1}\tilde{u}}{\tilde{\tau}} - \frac{\tilde{r}^m}{\tilde{\tau}^2} D_x \tilde{u} \right)(x, s) ds.$$

Now using Lemma 3.3.1 and the Sobolev embedding theorem, it follows the estimate:

$$|\Delta_t^h(\tilde{\tau}^{-1}\tilde{r}^m)(\cdot)|_\infty \leq \frac{1}{h} \int_t^{t+h} C \|D_x \tilde{u}(\cdot, s)\|_{H^1} ds \leq C \sup_{0 \leq s \leq T} \|\tilde{u}(\cdot, s)\|_{H^2} \leq C \tilde{E}(T)^{1/2}. \quad (3.4.10)$$

In the similar fashion, we also obtain that $|\Delta_t^h(\tilde{r}^{m-2}\tilde{\tau})(\cdot)|_\infty \leq C \tilde{E}(T)^{1/2}$, hence taking the supremum over $t \in [0, T-h]$, we have

$$\limsup_{h \rightarrow 0^+} B_{T,h} = C \limsup_{h \rightarrow 0^+} \sup_{0 \leq s \leq T-h} \left\{ |\Delta_s^h(\tilde{\tau}^{-1}\tilde{r}^m)|_\infty^2 + |\Delta_s^h(\tilde{r}^{m-2}\tilde{\tau})|_\infty^2 \right\} \leq C \limsup_{h \rightarrow 0^+} \tilde{E}(T) \leq C \tilde{E}(T).$$

Next, we prove the statement for $A_{T,h}$. First, we write:

$$\Delta_t^h(\tilde{\tau}^{-1}\tilde{e}) = \frac{1}{h} \int_t^{t+h} D_s(\tilde{\tau}^{-1}\tilde{e}) ds = \frac{1}{h} \int_t^{t+h} \left\{ -\tilde{\tau}^{-2}\tilde{e} D_x \tilde{u} + \tilde{\tau}^{-1} D_t \tilde{e} \right\} ds,$$

hence by Sobolev embedding theorem, it follows that:

$$\int_0^{T-h} \|\Delta_t^h(\tilde{\tau}^{-1}\tilde{e})\|_{L^2}^2 dt \leq \int_0^{T-h} \left(\frac{C}{h} \int_t^{t+h} \{ \|\tilde{e}\|_{H^1} \|D_x \tilde{u}\|_{L^2} + \|D_t \tilde{e}\|_{L^2} \}(s) ds \right)^2 dt \leq CT(1 + \tilde{E}(T)^2).$$

In the same way, we also have :

$$\int_0^{T-h} \|\Delta_t^h(\tilde{r}^{-1}\tilde{e})\|_{L^2}^2 dt \leq CT(1 + \tilde{E}(T)^2).$$

It follows that $\limsup_{h \rightarrow 0^+} A_{T,h} \leq CT(1 + \tilde{E}(T))$. To bound the term C_h , we observe that:

$$\lim_{h \rightarrow 0^+} \|\Delta_0^h u_N(x)\|_{L^2}^2 := \lim_{h \rightarrow 0^+} \left\| \sum_{i=1}^N \phi_i(x) \Delta_0^h a_N^i \right\|_{L^2}^2 = \sum_{i=1}^N \lim_{h \rightarrow 0^+} |\Delta_0^h a_N^i|^2 = \sum_{i=1}^N \left| \frac{da_N^i}{dt}(0) \right|^2. \quad (3.4.11)$$

Recalling equation (3.4.6) in the proof of Theorem 3.4.1, and the fact that $a_N^i(t) \in C^1([0, T])$ it follows that:

$$\begin{aligned} \frac{da_N^i}{dt}(0) &= (D_t u_N(\cdot, 0), \phi_i)_{L^2} = \beta \left(\frac{r_0^m}{\tau_0} D_x^2 u_N(\cdot, 0), \phi_i \right)_{L^2} + \beta \left(\{ D_x(\tau_0^{-1} r_0^m) - r_0^{m-2} \tau_0 \} u_N(\cdot, 0), \phi_i \right)_{L^2} \\ &\quad + (\gamma - 1) \left(m \frac{e_0}{r_0} + D_x(\tau_0^{-1} e_0), \phi_i \right)_{L^2}. \end{aligned}$$

Substituting this into (3.4.11), and using the fact that $\{\phi_i\}_{i \in \mathbb{N}}$ is an orthonormal basis in L^2 , one can employ Bessel's inequality to obtain that:

$$\begin{aligned} \lim_{h \rightarrow 0^+} \|\Delta_0^h u_N(x)\|_{L^2}^2 &= \sum_{i=1}^N \left| \frac{da_N^i}{dt}(0) \right|^2 \\ &\leq C \left\| \frac{r_0^m}{\tau_0} D_x^2 u_N(\cdot, 0) \right\|_{L^2}^2 + C \left\| \{ D_x(\tau_0^{-1} r_0^m) - r_0^{m-2} \tau_0 \} u_N(\cdot, 0) \right\|_{L^2}^2 + C \left\| m \frac{e_0}{r_0} + D_x \left(\frac{e_0}{\tau_0} \right) \right\|_{L^2}^2 \\ &\leq C \|D_x^2 u_N(\cdot, 0)\|_{L^2}^2 + C \|u_N(\cdot, 0)\|_{L^2}^2 + C \|e_0\|_{H^1}^2. \end{aligned} \quad (3.4.12)$$

Now by the fact that $-D_x^2\phi_i = \lambda_i\phi_i$ and $\phi \in H_0^1(0, k)$, it implies $D_x^2\phi(0) = D_x^2\phi(k) = 0$. Integrating by parts twice, we get:

$$\|D_x^2u_N(\cdot, 0)\|_{L^2}^2 = (D_x^2u_N(\cdot, 0), D_x^2u_N(\cdot, 0))_{L^2} = (u_N(\cdot, 0), D_x^4u_N(\cdot, 0))_{L^2}.$$

Since ϕ_i is eigenfunction of $-D_x^2$, it follows that $D_x^4u_N \in \text{Span}\{\phi_i\}_{i \in \mathbb{N}}$. On the other hand, by the initial data of $t \mapsto a_N^i(t)$, we have $(u_N(\cdot, 0), \phi_i)_{L^2} = (u_0, \phi_i)_{L^2}$. Combining these facts, we obtain the following estimate:

$$\begin{aligned} \|D_x^2u_N(\cdot, 0)\|_{L^2}^2 &= (u_N(\cdot, 0), D_x^4u_N(\cdot, 0))_{L^2} = (u_0, D_x^4u_N(\cdot, 0))_{L^2} \\ &= (D_x^2u_0, D_x^2u_N(\cdot, 0))_{L^2} \leq \frac{1}{2}\|D_x^2u_0\|_{L^2}^2 + \frac{1}{2}\|D_x^2u_N(\cdot, 0)\|_{L^2}^2, \end{aligned}$$

hence

$$\|D_x^2u_N(\cdot, 0)\|_{L^2}^2 \leq \|D_x^2u_0\|_{L^2}^2 \quad (3.4.13)$$

Substituting this back to (3.4.12), we get

$$\lim_{h \rightarrow 0^+} C_h = \lim_{h \rightarrow 0^+} \|\Delta_0^h u_N(x)\|_{L^2}^2 \leq C\|u_0\|_{H^2}^2 + C\|e_0\|_{H^1}^2 \leq CE_0 \quad (3.4.14)$$

Thus we have proved $\lim_{h \rightarrow 0^+} C_h \leq CE_0$. With these estimates, (3.4.8) becomes:

$$\begin{aligned} &\frac{1}{2} \limsup_{h \rightarrow 0^+} \|\Delta_t^h u_N(\cdot)\|_{L^2}^2 + C^{-1} \limsup_{h \rightarrow 0^+} \int_0^t \left(\|\Delta_t^h D_x u_N(\cdot, s)\|_{L^2}^2 + \|\Delta_t^h u_N(\cdot, s)\|_{L^2}^2 \right) ds \\ &\leq CE_0 + CT \left(1 + \tilde{E}(T)^2 \right) + C\tilde{E}(T) \int_0^t \|u_N(\cdot, s)\|_{H^1}^2 ds < \infty, \end{aligned} \quad (3.4.15)$$

where the right hand side is finite due to the estimate in Theorem 3.4.1. Thus we obtain the weak convergence:

$$\Delta_t^h u_N \overset{*}{\rightharpoonup} D_t u_N \quad \text{in } L^\infty(0, T; L^2(0, k)) \quad \text{and} \quad \Delta_t^h u_N \rightharpoonup D_t u_N \quad \text{in } L^2(0, T; H_0^1(0, k)),$$

where $D_t u_N$ denotes the weak derivative of u_N in time. By the weakly (or weakly-*) lower semi-continuity of the Sobolev norm, it follows that

$$\frac{1}{2} \|D_t u_N(\cdot)\|_{L^2}^2 + C^{-1} \int_0^t \|D_t u_N(\cdot, s)\|_{H^1}^2 ds \leq CE_0 + CT \left(1 + \tilde{E}(T)^2 \right) + C\tilde{E}(T) \int_0^t \|u_N(\cdot, s)\|_{H^1}^2 ds,$$

Next, by fundamental theorem of calculus and the convexity of L^2 -norm we have

$$\begin{aligned} \|D_x u_N(\cdot, t)\|_{L^2} &\leq \|D_x u_N(\cdot, 0)\|_{L^2} + \int_0^t \|D_x D_t u_N(\cdot, s)\|_{L^2} ds \leq \|D_x u_0\|_{L^2} + \int_0^t \|D_x D_t u_N\|_{L^2} ds, \\ \|u_N(\cdot, t)\|_{L^2} &\leq \|u_N(\cdot, 0)\|_{L^2} + \int_0^t \|D_t u_N(\cdot, s)\|_{L^2} ds \leq \|u_0\|_{L^2} + \int_0^t \|D_t u_N(\cdot, s)\|_{L^2} ds, \end{aligned} \quad (3.4.16)$$

where we have used the fact that $D_x^2u_N \in \text{Span}\{\phi_i\}_{i \in \mathbb{N}}$ as ϕ_i is eigenfunction of $-D_x^2$, hence

$$\begin{aligned} \|D_x u_N(\cdot, 0)\|_{L^2}^2 &= (D_x u_N(\cdot, 0), D_x u_N(\cdot, 0))_{L^2} = (u_N(\cdot, 0), -D_x^2 u_N(\cdot, 0))_{L^2} \\ &= (u_0, -D_x^2 u_N(\cdot, 0))_{L^2} = (D_x u_0, D_x u_N(\cdot, 0))_{L^2} \leq \frac{1}{2}\|u_0\|_{L^2}^2 + \frac{1}{2}\|D_x u_N(\cdot, 0)\|_{L^2}^2. \end{aligned}$$

Combining (3.4.16) with (3.4.15):

$$\begin{aligned} & \|D_t u_N(\cdot, t)\|_{L^2}^2 + C^{-1} \int_0^t \|D_t u_N(\cdot, s)\|_{H^1}^2 ds \\ & \leq CE_0 + CT(1 + \tilde{E}(T)^2) + C\tilde{E}(T) \int_0^t s \int_0^s \|D_t u_N(\cdot, s')\|_{H^1}^2 ds' ds, \end{aligned}$$

Applying Grönwall inequality, we have:

$$\int_0^t \|D_t u_N(\cdot, s)\|_{H^1}^2 ds \leq C \left\{ E_0 + T(1 + \tilde{E}(T)^2) \right\} \exp \left(C \frac{T^2}{2} \tilde{E}(T) \right).$$

Substituting this back into equation (3.4.12), we obtain the estimate:

$$\sup_{0 \leq t \leq T} \|D_t u_N(\cdot, t)\|_{L^2}^2 + C^{-1} \int_0^T \|D_t u_N(\cdot, s)\|_{H^1}^2 ds \leq CE_0 + CTP(\tilde{E}(T)) \exp(CT^2 \tilde{E}(T)). \quad (3.4.17)$$

Hence by (3.4.17), convergence (3.4.5) in the proof of Theorem 3.4.1, and Fundamental Lemma of Calculus of Variation, we have the weak convergence:

$$D_t u_N \overset{*}{\rightharpoonup} D_t u \quad \text{in } L^\infty(0, T; L^2(0, k)) \quad \text{and} \quad D_t u_N \rightharpoonup D_t u \quad \text{in } L^2(0, T; H_0^1(0, k)),$$

where $D_t u$ is the temporal weak derivative of the weak solution u obtained in Theorem 3.4.1. Finally by the weakly-star lower semi-continuity of Sobolev norm, the main estimate of this lemma is obtained. \square

Lemma 3.4.2. *Suppose the assumptions of Theorem 3.4.2 hold, and let $u(x, t)$ be the weak solution obtained in Theorem 3.4.1. Then we have $u \in L^\infty(0, T_1; H^2(0, k))$ with $T_1 \equiv \min\{T_*, T_0\}$ and in particular, there exists a constant $C = C(C_0, a, k, \beta, \gamma, n) > 0$ and a polynomial $z \mapsto \mathcal{P}(z)$ such that the following inequality holds:*

$$\sup_{0 \leq t \leq T} \|u(\cdot, t)\|_{H^2}^2 \leq \mathcal{P}(E_0) + T\mathcal{P}(\tilde{E}(T)) \exp(CT^2 \tilde{E}(T)),$$

for all $T \in [0, \min\{T_*, T_0\}]$.

Proof. We let u_N be the Galerkin approximation constructed in the proof of Theorem 3.4.1. Since u_N satisfies the equation (3.4.3), The following can be attained by multiplying (3.4.3) with $a_N^i(t)$ and summing over $i = 1 \dots, N$:

$$\begin{aligned} & \beta \left(\frac{\tilde{r}^m}{\tilde{r}} D_x u_N, D_x u_N \right)_{L^2} + \beta (\tilde{r}^{m-2} \tilde{r} u_N, u_N)_{L^2} \\ & = - (D_t u_N, u_N)_{L^2} + m(\gamma - 1) \left(\frac{\tilde{e}}{\tilde{r}}, u_N \right)_{L^2} + (\gamma - 1) \left(\frac{\tilde{e}}{\tilde{r}}, D_x u_N \right)_{L^2} \end{aligned}$$

By Fundamental Theorem of Calculus in time, and Cauchy-Schwartz inequality we have:

$$\|\tilde{e}(\cdot, t)\|_{L^2}^2 = \|e_0(\cdot) + \int_0^t D_t \tilde{e}(\cdot, s) ds\|_{L^2}^2 \leq 2\|e_0\|_{L^2}^2 + 2T^2 \sup_{s \in [0, T]} \|D_t \tilde{e}(\cdot, s)\|_{L^2}^2 \leq 2E_0 + 2T^2 \tilde{E}(T).$$

Using the above estimate and Lemma 3.3.1, Proposition 3.3.1, we obtain that:

$$\begin{aligned} & \|D_x u_N(\cdot, t)\|_{L^2}^2 + \|u_N(\cdot, t)\|_{L^2}^2 \leq C \|D_t u_N\|_{L^2}^2 + C \|\tilde{e}\|_{L^2}^2 \\ & \leq \|D_t u_N\|_{L^2}^2 + CE_0 + CT^2 \tilde{E}(T) \leq C\mathcal{P}(E_0) + CTP(\tilde{E}(T)) \exp(CT^2 \tilde{E}(T)), \end{aligned} \quad (3.4.18)$$

where in the last line, we used estimate (3.4.17) from the proof of Lemma 3.4.1. Next, Performing integration by parts on (3.4.3), we also have

$$(D_t u_N, \phi_i)_{L^2} + \beta \left(D_x \left(\frac{\tilde{r}^m}{\tilde{r}} D_x u_N \right), \phi_i \right)_{L^2} + \beta (\tilde{r}^{m-2} \tilde{r} u_N, \phi_i)_{L^2} = (\gamma - 1) \left(m \frac{\tilde{e}}{\tilde{r}} - D_x (\tilde{r}^{-1} \tilde{e}), \phi_i \right)_{L^2}.$$

Next, we multiply the above equation with $-\lambda_i a_N^i(t)$ and sum over $i = 1, \dots, N$ for λ_i being the i -th eigenvalue of $-D_x^2$. It follows from Proposition 3.3.1 that:

$$\beta \frac{2}{3} \frac{a^m}{\bar{\tau}_0} \|D_x^2 u_N\|_{L^2}^2 \leq \beta \left(\frac{\tilde{r}^m}{\tilde{r}} D_x^2 u_N, D_x^2 u_N \right)_{L^2} \leq \frac{\beta}{3} \frac{a^m}{\bar{\tau}_0} \|D_x^2 u_N\|_{L^2}^2 + C \|D_t u_N\|_{L^2}^2 + C \|u_N\|_{H^1}^2 + \sum_{i=1}^3 I_i, \quad (3.4.19)$$

where $I_1 := C \|\tilde{e} D_x \tilde{r}\|_{L^2}^2$, $I_2 := C \|D_x \tilde{r} D_x u_N\|_{L^2}^2$, and $I_3 := C \|\tilde{e}\|_{H^1}^2$. We claim that

$$\sum_{i=1}^3 I_i \leq C \mathcal{P}(E_0) + C T \mathcal{P}(\tilde{E}(T)) \exp(CT^2 \tilde{E}(T)). \quad (3.4.20)$$

First, we estimate I_1 . Computing terms within the integral, by the construction (3.2.2)–(3.2.5) we have the identity: $D_t(\tilde{e} D_x \tilde{r}) = D_x \tilde{r} D_t \tilde{e} + \tilde{e} D_x^2 \tilde{u}$, hence it follows from Lemma 3.3.1 and Sobolev embedding theorem that:

$$\|D_t(\tilde{e} D_x \tilde{r})\|_{L^2} \leq (C_0 + t^{1/2} \tilde{E}(t)^{1/2}) \|D_t \tilde{e}\|_{L^2} + C \|\tilde{e}\|_{H^1} \|D_x^2 \tilde{u}\|_{L^2}.$$

thus, by Fundamental Theorem of Calculus and (3.2.2)–(3.2.5), we have:

$$\begin{aligned} I_1 &:= \|\tilde{e} D_x \tilde{r}(\cdot, t)\|_{L^2}^2 = \|C_0 e_0 + \int_0^t D_t(\tilde{e} D_x \tilde{r})(\cdot, s) ds\|_{L^2}^2 \leq C_0 E_0 + 2t \int_0^t \|D_t(\tilde{e} D_x \tilde{r})(\cdot, s)\|_{L^2}^2 ds \\ &\leq C_0 E_0 + t^2 C_0 \tilde{E}(t) + t^3 C \tilde{E}(t)^2 + t^2 C \tilde{E}(t)^2 =: C_0 E_0 + C T \mathcal{P}(\tilde{E}(T)). \end{aligned}$$

where $z \mapsto \mathcal{P}(z)$ denotes some explicitly known polynomial. Next, we estimate the term I_2 . Again, by (3.2.2)–(3.2.5), Lemma 3.3.1 and Sobolev embedding theorem, it follows that

$$\begin{aligned} \|D_t(D_x \tilde{r} D_x u_N)\|_{L^2}(t) &= \|D_x^2 \tilde{u} D_x u_N + D_x \tilde{r} D_x D_t u_N\|_{L^2}(t) \\ &\leq C \|D_x^2 \tilde{u}(\cdot, t)\|_{H^1} \|D_x u_N(\cdot, t)\|_{L^2} + (C_0 + t^{1/2} \tilde{E}(t)^{1/2}) \|D_x D_t u_N(\cdot, t)\|_{L^2}. \end{aligned}$$

Using Fundamental Theorem of Calculus, Cauchy-Schwartz inequality, estimate (3.4.18), and inequality (3.4.17) from the proof of Lemma 3.4.1, we obtain:

$$\begin{aligned} I_2 &:= \|D_x \tilde{r} D_x u_N(\cdot, t)\|_{L^2}^2 = \left\| C_0 D_x u_N(\cdot, 0) + \int_0^t D_t(D_x \tilde{r} D_x u_N)(\cdot, s) ds \right\|_{L^2}^2 \\ &\leq C_0 \|D_x u_N(\cdot, 0)\|_{L^2}^2 + C \left(\int_0^t \{ \|D_x^2 \tilde{u}\|_{H^1} \|D_x u_N\|_{L^2} + (C_0 + s^{1/2} \tilde{E}(s)^{1/2}) \|D_x D_t u_N\|_{L^2} \} ds \right)^2 \\ &\leq C_0 \|D_x u_0\|_{L^2}^2 + C T \mathcal{P}(\tilde{E}(T)) \exp(CT^2 \tilde{E}(T)), \end{aligned}$$

where in the last line, we used $\|D_x u_N(\cdot, 0)\|_{L^2} \leq \|D_x u_0\|_{L^2}$ obtained in (3.4.13) from proof of Lemma 3.4.1. Finally, we bound the terms I_3 . By Fundamental Theorem of Calculus, it follows that:

$$I_3 := \|\tilde{e}(\cdot, t)\|_{H^1}^2 = \|D_x e_0 + \int_0^t D_x D_t \tilde{e}(\cdot, s) ds\|_{L^2}^2 + \|e_0 + \int_0^t D_t \tilde{e}(\cdot, s) ds\|_{L^2}^2 \leq 2E_0 + 2T \tilde{E}(T).$$

This concludes the proof for claim (3.4.20). Finally substituting estimates (3.4.18), (3.4.20) and using (3.4.17) shown in the proof of Lemma 3.4.1, the inequality (3.4.19) becomes

$$\frac{\beta a^m}{3 \bar{\tau}_0} \|D_x^2 u_N(\cdot, t)\|_{L^2}^2 + \|u_N(\cdot, t)\|_{H^1}^2 \leq C\mathcal{P}(E_0) + CTP(\tilde{E}(T)) \exp(CT^2 \tilde{E}(T)).$$

Thus, using the weak convergence (3.4.5): $u_N \rightharpoonup u$ in $L^2(0, T; H^1(0, k))$ proven in Theorem 3.4.1, it follows the weak-star convergence: $u_N \overset{*}{\rightharpoonup} u$ in $L^\infty(0, T; H^2(0, k))$. Consequently, by the weakly-star lower semi-continuity of Sobolev norm, we obtain the main inequality of this lemma. \square

Corollary 3.4.1. *Suppose the assumptions of Theorem 3.4.2 hold, and let $u(x, t)$ be the weak solution obtained in Theorem 3.4.1. Then $u \in L^\infty(0, T_1; H^2(0, k))$, $D_t u \in L^\infty(0, T_1; L^2(0, k))$ with $T_1 \equiv \min\{T_*, T_0\}$, and we have for $(x, t) \in (0, k) \times [0, T_1]$ almost everywhere, the following equation is satisfied*

$$D_t u(x, t) = \beta D_x \left(\frac{\tilde{r}^m}{\tilde{\tau}} D_x u \right) (x, t) - \beta \tilde{r}^{m-2} \tilde{\tau} u(x, t) + m(\gamma - 1) \frac{\tilde{e}}{\tilde{r}}(x, t) - (\gamma - 1) D_x \left(\frac{\tilde{e}}{\tilde{\tau}} \right) (x, t).$$

Proof. Given test function $\varphi \in C_c^\infty([0, T_1] \times [0, k])$, since u satisfies the weak form as proven in Theorem 3.4.1, it follows from integration by parts that

$$-(\gamma - 1) \int_0^{T_1} \int_0^k \left\{ m \frac{\tilde{e}}{\tilde{r}} - D_x \left(\frac{\tilde{e}}{\tilde{\tau}} \right) \right\} \varphi dx dt = - \int_0^{T_1} \int_0^k \left\{ D_t u + \beta D_x \left(\frac{\tilde{r}^m}{\tilde{\tau}} D_x u \right) + \beta \tilde{r}^{m-2} \tilde{\tau} u \right\} \varphi dx dt,$$

hence by the Fundamental Theorem of Calculus, we conclude the proof of corollary. \square

Lemma 3.4.3. *Suppose the assumptions of Theorem 3.4.2 hold, and let $u(x, t)$ be the weak solution obtained in Theorem 3.4.1. Then $u \in L^2(0, T_1; H^3(0, k))$ with $T_1 \equiv \min\{T_*, T_0\}$, and in particular, there exists a constant $C = C(C_0, a, k, \gamma, \beta, n) > 0$ and a polynomial $z \mapsto \mathcal{P}(z)$ such that*

$$\int_0^T \|u(\cdot, t)\|_{H^3}^2 dt \leq \mathcal{P}(E_0) + CTP(\tilde{E}(T)) \exp(CT^2 \tilde{E}(T)),$$

for each $T \in [0, \min\{T_*, T_0\}]$.

Proof. First since $u \in L^\infty(0, T; H^2(0, k)) \cap L^\infty(0, T; H_0^1(0, k))$, by Sobolev embedding and trace theorem, $u \in L^\infty(0, T; C_0^1(0, k))$. For each fixed $t \in [0, T]$ we extend u to the domain $x \in \mathbb{R} \setminus [0, k]$ by setting $u(x, t) \equiv 0$. Denote Δ_x^h to be the difference quotient in space:

$$\Delta_x^h w(t) := \frac{w(x+h, t) - w(x, t)}{h}.$$

Then from Corollary 3.4.1, we have that for almost everywhere $(x, t) \in [0, k-h] \times [0, T]$

$$\begin{aligned} \beta \frac{\tilde{r}^m}{\tilde{\tau}} D_x^2 \Delta_x^h u &= \Delta_x^h D_t u - \beta D_x^2 u \Delta_x^h (\tilde{\tau}^{-1} \tilde{r}^m) - \beta D_x \Delta_x^h u D_x (\tilde{\tau}^{-1} \tilde{r}^m) - \beta D_x u D_x \Delta_x^h (\tilde{\tau}^{-1} \tilde{r}^m) \\ &\quad + \beta \tilde{r}^{m-2} \tilde{\tau} \Delta_x^h u + \beta u \Delta_x^h (\tilde{r}^{m-2} \tilde{\tau}) + m(\gamma - 1) \Delta_x^h \left(\frac{\tilde{e}}{\tilde{r}} \right) - (\gamma - 1) \Delta_x^h D_x \left(\frac{\tilde{e}}{\tilde{\tau}} \right) \end{aligned}$$

Taking L^2 -norm in $L^2(0, k-h)$ on both sides of the equation and integrating in time, it follows that:

$$\int_0^t \int_0^{k-h} |\tilde{\tau}^{-1} \tilde{r}^m \Delta_x^h D_x^2 u|^2 dx ds \leq \sum_{i=1}^8 \int_0^t I_i ds, \quad (3.4.21)$$

where we will indicate each term $\{I_i\}_{i=1, \dots, 8}$ as we proceed. First the following proposition is true:

Proposition 3.4.1. For $f(x) \in C^1(0, k)$, we have

$$\sup_{h>0} \int_0^{k-h} |\Delta_x^h f|^2 dx := \sup_{h>0} \int_0^{k-h} \left| \frac{f(x+h) - f(x)}{h} \right|^2 dx \leq \int_0^k |D_x f(x)|^2 dx$$

Proof. By Fundamental Theorem of Calculus, and Jensen's inequality, and Fubini's theorem, we get

$$\begin{aligned} \int_0^{k-h} |\Delta_x^h f|^2 dx &= \int_0^{k-h} \left| \frac{1}{h} \int_x^{x+h} D_x f(y) dy \right|^2 dx \leq \int_0^{k-h} \int_x^{x+h} \frac{1}{h} |D_x f(y)|^2 dy dx, \\ &= \int_0^h \frac{y}{h} |D_x f(y)|^2 dy + \int_h^{k-h} |D_x f(y)|^2 dy + \int_{k-h}^k \frac{k-y}{h} |D_x f(y)|^2 dy \leq \int_0^k |D_x f(y)|^2 dy \end{aligned}$$

□

Using Proposition 3.4.1, Lemma 3.3.1, Lemma 3.4.1 and Theorem 3.4.1, I_1, I_2, I_3 can be estimated as

$$\begin{aligned} I_1 &:= \int_0^t \int_0^{k-h} |\Delta_x^h D_t u|^2 dx ds \leq \int_0^t \int_0^k |D_x D_t u|^2 dx dt \leq \mathcal{P}(E_0) + CTP(\tilde{E}(T)) \exp(CT^2 \tilde{E}(T)^{1/2}), \\ I_2 &:= \int_0^t \int_0^{k-h} |\tilde{r}^{m-2} \tilde{\tau} \Delta_x^h u|^2 dx ds \leq C \int_0^t \int_0^k |D_x u|^2 dx ds \leq CTP(\tilde{E}(T)) \exp(CT^2 \tilde{E}(T)^{1/2}), \\ I_3 &:= \int_0^t \int_0^{k-h} |\Delta_x^h(\tilde{r}^{-1} \tilde{e})|^2 dx ds \leq \int_0^t \int_0^k |\tilde{r}^{-1} D_x \tilde{e} - \tilde{r}^{-2} \tilde{\tau} \tilde{e}|^2 dx ds \leq CT \tilde{E}(T). \end{aligned}$$

For I_4 , we first compute using Lemma 3.3.1 and equations (3.2.2)–(3.2.5):

$$\sup_{x \in [0, k-h]} |\Delta_x^h(\tilde{r}^{-1} \tilde{r}^m)(t)|^2 \leq 2m |\tilde{r}^{m-1}|_\infty^2 + 2|\tilde{r}^{-2} \tilde{\tau}^m D_x \tilde{\tau}|_\infty^2 \leq C(1 + C_0 + t \tilde{E}(t))$$

Using this and Lemma 3.4.2, we have

$$\begin{aligned} I_4 &:= \int_0^t \int_0^{k-h} |\Delta_x^h(\tilde{r}^{-1} \tilde{r}^m) D_x^2 u|^2 dx ds \leq \int_0^t |\Delta_x^h(\tilde{r}^{-1} \tilde{r}^m)(s)|_\infty^2 \|D_x^2 u(\cdot, s)\|_{L^2}^2 ds \\ &\leq CTP(\tilde{E}(T)) \exp(CT^2 \tilde{E}(T)) \end{aligned}$$

For I_5 , using Lemma 3.3.1 and (3.2.2)–(3.2.5), we first note that

$$\begin{aligned} |\Delta_x^h D_x(\tilde{r}^{-1} \tilde{r}^m)(t)| &= \left| \frac{1}{h} \int_x^{x+h} (m(m-1) \tilde{r}^{m-2} \tilde{\tau} - m \tilde{\tau}^{-1} \tilde{r}^{m-1} D_x \tilde{\tau} + 2 \tilde{r}^m \tilde{\tau}^{-3} |D_x \tilde{\tau}|^2 - \tilde{\tau}^{-2} \tilde{r}^m D_x^2 \tilde{\tau}) dy \right| \\ &\leq C \left(1 + t \tilde{E}(t) + \frac{1}{h} \int_x^{x+h} |D_x^2 \tilde{\tau}(y, t)| dy \right), \end{aligned}$$

Using this, and Lemma 3.4.2, I_5 is then estimated by

$$\begin{aligned} I_5 &:= \int_0^t \int_0^{k-h} |D_x u D_x \Delta_x^h(\tilde{r}^{-1} \tilde{r}^m)|^2 dx ds \\ &\leq CTP(\tilde{E}(T)) \exp(CT^2 \tilde{E}(T)) + \int_0^t |D_x u(\cdot, s)|_\infty^2 \int_0^{k-h} \left(\frac{1}{h} \int_x^{x+h} |D_x^2 \tilde{\tau}(y, s)| dy \right)^2 dx ds. \end{aligned}$$

By the exact same argument used in the proof of Proposition 3.4.1, We bound the second term on the right hand side of above inequality as:

$$\int_0^{k-h} \left(\frac{1}{h} \int_x^{x+h} |D_x^2 \tilde{\tau}(y, s)| dy \right)^2 dx \leq \frac{1}{h} \int_0^{k-h} \int_x^{x+h} |D_x^2 \tilde{\tau}(y, s)|^2 dy dx \leq \int_0^k |D_x^2 \tilde{\tau}(y, s)|^2 dy,$$

then using (3.2.2)–(3.2.5) and Jensen's inequality, we have

$$\int_0^k |D_x^2 \tilde{\tau}|^2 dx = \int_0^k |D_x^2 \tau_0(x) + \int_0^s D_x^3 \tilde{u}(x, q) dq|^2 dx \leq C_0 + 2 \int_0^k s \int_0^s |D_x^3 \tilde{u}(y, q)|^2 dq dy \leq C_0 + s \tilde{E}(s).$$

Therefore it implies from the above inequalities, Sobolev embedding theorem, and Lemma 3.4.2 that:

$$\begin{aligned} \int_0^t |D_x u(\cdot, s)|_\infty^2 \int_0^{k-h} \left(\frac{1}{h} \int_x^{x+h} |D_x^2 \tau(y, s)| dy \right)^2 dx ds &\leq C \sup_{s \in [0, T]} \|u(\cdot, s)\|_{H^2}^2 \int_0^t (C_0 + s \tilde{E}(s)) ds \\ &\leq CT \mathcal{P}(\tilde{E}(T)) \exp(CT^2 \tilde{E}(T)). \end{aligned}$$

Substituting this back into the estimate of I_5 , it follows that: $I_5 \leq CT \mathcal{P}(\tilde{E}(T)) \exp(CT^2 \tilde{E}(T))$. Next, we estimate the term I_6 and I_7 . Using Lemma 3.3.1, Proposition 3.4.1, and Lemma 3.4.2 we get

$$\begin{aligned} I_6 &:= \int_0^t \int_0^{k-h} |D_x \left(\frac{\tilde{\tau}^m}{\tilde{\tau}} \right) D_x \Delta_x^h u(x, s)|^2 dx ds \leq \int_0^t |m \tilde{\tau}^{m-1} - \tilde{\tau}^m \tilde{\tau}^{-2} D_x \tilde{\tau}|_\infty^2 \int_0^{k-h} |D_x \Delta_x^h u(x, s)|^2 dx ds \\ &\leq C \int_0^t (1 + C_0 + s \tilde{E}(s)) \int_0^k |D_x^2 u(x, s)|^2 dx ds \leq T \mathcal{P}(\tilde{E}(T)) \exp(CT^2 \tilde{E}(T)), \\ I_7 &:= \int_0^t \int_0^{k-h} |u \Delta_x^h (\tilde{\tau} \tilde{\tau}^{m-2})(x, s)|^2 dx ds \leq \int_0^t \|\{(m-2) \tilde{\tau}^{m-3} \tilde{\tau}^2 + \tilde{\tau}^{m-2} D_x \tilde{\tau}\}(\cdot, s)\|_\infty^2 \|u(\cdot, s)\|_{L^2}^2 ds \\ &\leq C \int_0^t \{1 + C_0 + s \tilde{E}(s)\} \|u(\cdot, s)\|_{L^2}^2 ds \leq T \mathcal{P}(\tilde{E}(T)) \exp(CT^2 \tilde{E}(T)). \end{aligned}$$

Finally we estimate I_8 using Proposition 3.4.1, Lemma 3.3.1

$$\begin{aligned} I_8 &:= \int_0^t \int_0^{k-h} |\Delta_x^h D_x (\tilde{\tau}^{-1} \tilde{e})|^2 dx ds \leq \int_0^t \int_0^k \left(\frac{D_x^2 \tilde{e}}{\tilde{\tau}} + 2 \frac{|D_x \tilde{\tau}|^2}{\tilde{\tau}^3} \tilde{e} - 2 \frac{D_x \tilde{\tau}}{\tilde{\tau}^2} D_x \tilde{e} - \tilde{\tau}^{-2} \tilde{e} D_x^2 \tilde{\tau} \right)^2 dx ds \\ &\leq T \mathcal{P}(\tilde{E}(T)) + C \int_0^t \tilde{E}(s) \|D_x^2 \tilde{\tau}(\cdot, s)\|_{L^2}^2 ds \end{aligned}$$

The last term on the right hand side of above equation is estimated by (3.2.2)–(3.2.5) as follows

$$\|D_x^2 \tilde{\tau}(\cdot, s)\|_{L^2}^2 = \|D_x^2 \tau_0(\cdot) + \int_0^s D_x^3 \tilde{u}(\cdot, q) dq\|_{L^2}^2 \leq 2 \|D_x \tau_0\|_{L^2}^2 + 2s \int_0^s \|D_x^3 \tilde{u}(\cdot, q)\|_{L^2}^2 dq \leq 2C_0 + 2s \tilde{E}(s).$$

Substituting this back into the estimate of I_8 , it follows that $I_8 \leq T \mathcal{P}(\tilde{E}(T))$. Summarising all the estimates of I_1 – I_8 into (3.4.21), using Lemma 3.3.1, it follows that:

$$\int_0^t \int_0^{k-h} |\Delta_x^h D_x^2 u|^2 dx ds \leq \mathcal{P}(E_0) + CT \mathcal{P}(\tilde{E}(T)) \exp(CT^2 \tilde{E}(T)),$$

hence we have the weak convergence: $\mathbb{1}_{[0, k-h]} \Delta_x^h D_x^2 u \rightharpoonup D_x^3 u$ in $L^2(0, T; L^2(0, k))$, where $\mathbb{1}_{[0, k-h]}(x)$ denotes indicator function on the spatial coordinate with the set $x \in [0, k-h]$ and $D_x^3 u \in L^2([0, T] \times [0, k])$ is the spatial weak derivative of $D_x^2 u$. By weakly lower semi-continuity of the Sobolev norm, we obtain the desired estimate of this Lemma. \square

Lemma 3.4.4. *Suppose the assumptions of Theorem 3.4.2 hold, and let u be the weak solution obtained in Theorem 3.4.1. Then $D_t^2 u \in L^2(0, T_1; H^{-1}(0, k))$ with $T_1 \equiv \min\{T_*, T_0\}$, and in particular, there exists a constant $C = C(C_0, a, k, \gamma, \beta, n) > 0$ and a polynomial $z \mapsto \mathcal{P}(z)$ such that*

$$\int_0^T \|D_t^2 u(\cdot, t)\|_{H^{-1}}^2 dt \leq C \mathcal{P}(E_0) + CT \mathcal{P}(\tilde{E}(T)) \exp(CT^2 \tilde{E}(T)),$$

for each $T \in [0, \min\{T_*, T_0\}]$, where $H^{-1}(0, k)$ is the dual space of $H_0^1(0, k)$.

Proof. Denoting Δ_t^h as the difference quotient in the temporal coordinate defined in the proof of Lemma 3.4.1. By Corollary 3.4.1, we can take difference quotient on the momentum equation to get that for almost everywhere $(x, t) \in [0, k] \times [0, T - h]$ and get:

$$\begin{aligned} \Delta_t^h D_t u &= \beta D_x \left(D_x u \Delta_t^h (\tilde{\tau}^{-1} \tilde{r}^m) \right) + \beta D_x \left(\frac{\tilde{r}^m}{\tilde{\tau}} \Delta_t^h D_x u \right) - \beta u \Delta_t^h (\tilde{r}^{m-2} \tilde{\tau}) \\ &\quad - \beta \tilde{r}^{m-2} \tilde{\tau} \Delta_t^h u + m(\gamma - 1) \Delta_t^h (\tilde{r}^{-1} \tilde{e}) + (\gamma - 1) \Delta_t^h D_x (\tilde{r}^{-1} \tilde{e}) \end{aligned}$$

Multiplying both sides of the above equation with $\varphi(x) \in H_0^1(0, k)$ such that $\|\varphi\|_{H^1}^2 = 1$, we have using Lemma 3.3.1 and integration by parts:

$$(\varphi, \Delta_t^h D_t u)_{L^2} \leq \sum_{i=1}^3 I_i. \quad (3.4.22)$$

By Lemma 3.3.1, equations (3.2.2)–(3.2.5) and Lemma 3.4.1–3.4.3, we have that:

$$\begin{aligned} I_1 &:= \beta |\tilde{\tau}^{-1} \tilde{r}^m|_\infty \|\Delta_t^h D_x u\|_{L^2} + \beta |\tilde{r}^{m-2} \tilde{\tau}|_\infty \|\Delta_t^h u\|_{L^2} \leq C \frac{1}{h} \int_t^{t+h} \|D_t u(\cdot, s)\|_{H^1} ds, \\ I_2 &:= |\Delta_t^h (\tilde{\tau}^{-1} \tilde{r}^m)(\cdot)|_\infty \|D_x u(\cdot, t)\|_{L^2} + |\Delta_t^h (\tilde{\tau} \tilde{r}^{m-2})(\cdot)|_\infty \|u(\cdot, t)\|_{L^2} \leq C \tilde{E}(T)^{1/2} \|u(\cdot, t)\|_{H^1}, \\ I_3 &:= \|\Delta_t^h (\tilde{r}^{-1} \tilde{e})\|_{L^2} + \|\Delta_t^h (\tilde{\tau}^{-1} \tilde{e})\|_{L^2} \leq C \tilde{E}(T)^{1/2} + C \tilde{E}(T). \end{aligned}$$

Substituting the estiamtes for I_1 – I_3 into (3.4.22), using Minkowski's inequality, Cauchy-Schwartz inequality, Lemma 3.4.1–3.4.2, we get:

$$\begin{aligned} \int_0^{T-h} \|\Delta_t^h D_t u(\cdot, t)\|_{H^{-1}}^2 dt &= \sup \left\{ \int_0^{T-h} (\varphi, \Delta_t^h D_t u(\cdot, t))_{L^2}^2 dt \mid \varphi \in H_0^1, \|\varphi\|_{H^1} = 1 \right\} \\ &\leq \mathcal{P}(E_0) + CTP(\tilde{E}(T)) \exp(CT^2 \tilde{E}(T)). \end{aligned}$$

Hence, we get the weak convergence: $\mathbb{1}_{[0, T-h]} \Delta_t^h D_t u \rightharpoonup D_t^2 u$ in $L^2(0, T; H^{-1}(0, k))$ as $h \searrow 0$, where $D_t^2 u$ is the temporal weak derivative of $D_t u$ in the space $L^2(0, T; H^{-1}(0, k))$. Finally, by weakly lower semi-continuity of the Sobolev norm, we obtain the main inequality of the lemma. \square

3.4.2 Existence and higher regularity of e

Using the function $u(x, t)$ obtained in the previous section, we prove the existence of weak solution $e(x, t)$ as well as its higher regularity estimates.

Theorem 3.4.3. *Given a fixed constant $M_0 > 0$ and let $T_0 = T_0(M_0) > 0$ be the time defined in Proposition 3.3.1 by:*

$$T_0(M_0) := \min \left\{ 1, \frac{\bar{\tau}_0}{2CM_0^{1/2}}, \frac{\underline{\tau}_0}{2CM_0^{1/2}} \right\},$$

where $\underline{\tau}_0 := \inf_{x \in [0, k]} \tau_0(x)$, $\bar{\tau}_0 := \sup_{x \in [0, k]} \tau_0(x)$, and $C > 0$ is the generic constant from the Sobolev embedding inequality in Proposition 3.3.1. If there exists a time $T_* > 0$ such that (\tilde{u}, \tilde{e}) satisfies $\tilde{E}(T_*) \equiv E[\tilde{u}, \tilde{e}](T_*) \leq M_0$, then there exists a function $e(x, t)$ which satisfies the weak form in Definition 3.4.1 on the domain $(x, t) \in [0, k] \times [0, T_1]$ with $T_1 \equiv \min\{T_*, T_0\}$. Moreover $e \in C(0, T_1; L^2(0, k)) \cap L^2(0, T_1; H^1(0, k))$, and $D_t e \in L^2(0, T_1; H^*(0, k))$ where $H^*(0, k)$ is the dual space of $H^1(0, k)$. In particular, there exists a polynomial $z \mapsto \mathcal{P}(z)$ such that the following estimate is satisfied:

$$\begin{aligned} &\sup_{0 \leq t \leq T} \|e(\cdot, t)\|_{L^2}^2 + \int_0^T \{ \|D_x e\|_{L^2}^2 + \|D_t e\|_{H^*}^2 \}(t) dt \\ &\leq \|e_0\|_{L^2}^2 + CTP(\tilde{E}(T)) \exp(CT(1 + T\tilde{E}(T))), \end{aligned} \quad (3.4.23)$$

for each $T \in [0, \min\{T_*, T_0\}]$, where $H^*(0, k)$ is the dual space of $H^1(0, k)$, and $C = C(C_0, a, k, \beta, \gamma, n) > 0$ is a constant independent of $M_0 > 0$, $T_0 > 0$, and $T_* > 0$.

Proof. Let $\{\psi_i(x)\}_{i \in \mathbb{N}} \subset C^\infty(0, k)$ be an orthonormal basis of $L^2(0, k)$ that it is also an orthogonal basis of $H^1(0, k)$, and in addition it satisfies the boundary condition $\psi'_i(0) = \psi'_i(k) = 0$ for all $i \in \mathbb{N}$. Such basis can be obtained by solving the following eigenvalue problem:

$$\begin{cases} -D_x^2 \psi = \lambda \psi & \text{for } x \in (0, k), \\ D_x \psi(0) = D_x \psi(k) = 0. \end{cases}$$

We consider $a_N^i(t)$ to be the solution to following system of ODE's:

$$\begin{cases} \frac{da_N^i}{dt} = -\kappa \sum_{j=1}^N \left(\frac{\tilde{r}^m}{\tilde{r}} \psi'_j, \psi'_i \right)_{L^2} a_N^j(t) + (G, \psi_i)_{L^2}, \\ a_N^i(0) = (e_0, \psi_i)_{L^2}, \end{cases} \quad (3.4.24)$$

where G is defined by

$$G := \beta \frac{|D_x(\tilde{r}^m u)|^2}{\tilde{r}^m \tilde{r}} - (\gamma - 1) \frac{\tilde{e}}{\tilde{r}^m \tilde{r}} D_x(\tilde{r}^m u) - 2m\mu D_x(\tilde{r}^{m-1} u^2). \quad (3.4.25)$$

By Proposition 3.3.1 and Theorem 3.4.2 from the previous section, we have that for each $T \in [0, \max\{T_0, T_*\}]$ and $t \in [0, T]$:

$$\left| \left(\frac{\tilde{r}^m}{\tilde{r}} \psi'_j, \psi'_i \right)_{L^2} \right| \leq C |\tilde{r}^{-1} \tilde{r}^m(\cdot, t)|_\infty \leq C \frac{(a + 3k\bar{\tau}_0/2)^m}{T_0} \leq C < \infty.$$

Hence, by the existence and uniqueness theorem of ordinary differential equations, there exists a unique solution $a_N^i(t) \in C^1([0, T_1])$ with $T_1 \equiv \min\{T_0, T_*\}$. Now we define:

$$e_N(x, t) := \sum_{i=1}^N a_N^i(t) \psi_i(x),$$

then by the orthogonality of $\{\psi_i\}_{i \in \mathbb{N}}$, we have from (3.4.24) that

$$(D_t e_N, \psi_i)_{L^2} + \kappa \left(\frac{\tilde{r}^m}{\tilde{r}} D_x e_N, \psi'_i \right)_{L^2} = (G, \psi_i)_{L^2} \quad (3.4.26)$$

Multiplying both sides by $a_N^i(t)$ and taking summation over $i = 1, \dots, N$, it follows that

$$(D_t e_N, e_N)_{L^2} + \kappa \left(\frac{\tilde{r}^m}{\tilde{r}} D_x e_N, D_x e_N \right)_{L^2} = (G, e_N)_{L^2},$$

hence by Proposition 3.3.1, we have:

$$\frac{1}{2} \frac{d}{dt} \|e_N(\cdot, t)\|_{L^2}^2 + \kappa \frac{2a^m}{T_0} \|D_x e_N(\cdot, t)\|_{L^2}^2 \leq \frac{1}{2} \|G(\cdot, t)\|_{L^2}^2 + \frac{1}{2} \|e_N(\cdot, t)\|_{L^2}^2.$$

Applying Grönwall Inequality, and Bessel's inequality on $\|e_N(\cdot, 0)\|_{L^2}^2 \leq \|e_0\|_{L^2}^2$, we get:

$$\sup_{0 \leq t \leq T} \|e_N(\cdot, t)\|_{L^2}^2 \leq \exp(T/2) \left\{ \|e_0\|_{L^2}^2 + T \sup_{0 \leq t \leq T} \|G(\cdot, t)\|_{L^2}^2 \right\}.$$

Integrating (3.4.26) and substituting the above estimate, we have

$$\|e_N(\cdot, t)\|_{L^2}^2 + \kappa \frac{4a^m}{\tau_0} \int_0^T \|D_x e_N\|_{L^2}^2 dt \leq (1 + T \exp(T/2)) \left\{ \|e_0\|_{L^2}^2 + T \sup_{0 \leq t \leq T} \|G(\cdot, t)\|_{L^2}^2 \right\}, \quad (3.4.27)$$

To estimate $\|G(\cdot, t)\|_{L^\infty(0, T; L^2(0, k))}$, we apply Sobolev embedding theorem, Lemma 3.3.1, and Theorem 3.4.2, to get

$$\|G\|_{L^2} \leq C \{ |u|_\infty \|u\|_{L^2} + |u|_\infty \|D_x u\|_{L^2} + |D_x u|_\infty \|D_x u\|_{L^2} + |\tilde{e}|_\infty \|u\|_{H^1} \} \leq \mathcal{P}(\tilde{E}(T)) \exp(CT^2 \tilde{E}(T)),$$

where $C = C(C_0, a, k, \beta, \gamma, n) > 0$ is a constant. Thus substituting this estimate into (3.4.27), it follows that

$$\sup_{0 \leq t \leq T} \|e_N(\cdot, t)\|_{L^2}^2 + \kappa \frac{4a^m}{\tau_0} \int_0^T \|D_x e_N(\cdot, t)\|_{L^2}^2 dt \leq \|e_0\|_{L^2}^2 + T \mathcal{P}(\tilde{E}(T)) \exp(CT(1 + T\tilde{E}(T))). \quad (3.4.28)$$

Thus for a fixed $T_1 = \min\{T_*, T_0\}$, there exists a subsequence $\{e_N\}_{N \in \mathbb{N}}$ such that

$$e_N \rightharpoonup^* e \quad \text{in } L^\infty(0, T_1; L^2(0, k)), \quad D_x e_N \rightharpoonup D_x e \quad \text{in } L^2(0, T_1; L^2(0, k)). \quad (3.4.29)$$

By weakly (and weakly-*) lower semi-continuity of Sobolev norm, we have:

$$\sup_{0 \leq t \leq T} \|e(\cdot, t)\|_{L^2}^2 + \kappa \frac{4a^m}{\tau_0} \int_0^T \|D_x e(\cdot, t)\|_{L^2}^2 dt \leq \|e_0\|_{L^2}^2 + T \mathcal{P}(\tilde{E}(T)) \exp(CT(1 + T\tilde{E}(T))),$$

for all $T \in [0, \min\{T_*, T_0\}]$. Next we show that:

$$\sup_{N \in \mathbb{N}} \|D_t e_N\|_{L^2(0, T; H^*)}^2 \leq C \|e_0\|_{L^2}^2 + CT \mathcal{P}(\tilde{E}(T)) \exp(CT^2 \tilde{E}(T)),$$

where $C = C(C_0, a, k, \beta, \gamma, n) > 0$. Let $\varphi(x) \in H^1(0, k)$ be a test function. Without loss of generality, choose $\|\varphi\|_{H^1} = 1$. Since $\{\psi_i\}_{i \in \mathbb{N}}$ is an orthogonal basis in $H^1(0, k)$, we can approximate $\varphi(x)$ by

$$\lim_{M \rightarrow \infty} \sum_{i=1}^M \frac{(\psi_i, \varphi)_{H^1}}{1 + \lambda_i} \psi_i(x) = \varphi(x) \quad \text{in } H^1(0, k), \quad \text{where } (\psi_i, \varphi)_{H^1} = (\psi_i, \varphi)_{L^2} + (D_x \psi_i, D_x \varphi)_{L^2}.$$

Using this, the equation (3.4.26) becomes

$$(D_t e_N, \varphi)_{L^2} = -\kappa \left(\frac{\tilde{r}^m}{\tilde{\tau}} D_x e_N, D_x \varphi \right)_{L^2} + (G, \varphi)_{L^2}. \quad (3.4.30)$$

Hence, with estimate (3.4.28) we have:

$$(D_t e_N(\cdot, t), \varphi)_{L^2} \leq C \|D_x e_N\|_{L^2} + C \sqrt{\mathcal{P}(\tilde{E}(T)) \exp(CT^2 \tilde{E}(T))}.$$

Integrating the above equation in time, and using the estimate (3.4.28), we have

$$\begin{aligned} \|D_t e_N\|_{L^2(0, T; H^*)}^2 &= \sup \left\{ \int_0^T (D_x e_N(\cdot, t), \varphi)_{L^2}^2 dt \mid \varphi \in H^1(0, k), \|\varphi\|_{H^1} = 1 \right\} \\ &\leq C \int_0^T \left\{ \|D_x e_N\|_{L^2}^2 + \mathcal{P}(\tilde{E}(T)) \exp(CT^2 \tilde{E}(T)) \right\} dt \leq C \|e_0\|_{L^2}^2 + CT \mathcal{P}(\tilde{E}(T)) \exp(CT^2 \tilde{E}(T)). \end{aligned}$$

Now combining this estimate with the convergence (3.4.29), we have $D_t e_N \rightharpoonup D_t e$ in $L^2(0, T_1; H^*(0, k))$, where here we have denoted $D_t e \in L^2(0, T_1; H^*(0, k))$ as the weak temporal derivative of $e(\cdot, t)$ in the

space $H^*(0, k)$. Finally, we prove that e indeed satisfies the weak form in Definition 3.4.1. For $\varphi \in C^1(0, T_1; H^1(0, k))$ with $\varphi(\cdot, T_1) = 0$. Approximating $\lim_{M \rightarrow \infty} \sum_{i=1}^M (\varphi(x, t), \psi_i)_{H^1} \varphi(x) = \varphi(x, t)$ in $H^1(0, k)$ for each fixed $t \in [0, T_1]$, we have:

$$(D_t e_N(\cdot, t), \varphi(\cdot, t))_{L^2} = -\kappa \left(D_x e_N(\cdot, t), \frac{\tilde{r}^m}{\tilde{\tau}} D_x \varphi(\cdot, t) \right)_{L^2} + (G(\cdot, t), \varphi(\cdot, t))_{L^2}.$$

Using the convergence $e_N(\cdot, 0) \rightarrow e_0(\cdot)$ in $L^2(0, k)$ we have by integrating the above in time that:

$$\begin{aligned} & \int_0^k e_0(x) \varphi(x, 0) dx - \int_0^{T_1} \int_0^k e D_t \varphi dx dt = \lim_{N \rightarrow \infty} \left\{ \int_0^k e_N(0, x) \varphi(0, x) dx - \int_0^{T_1} \int_0^k e_N D_t \varphi dx dt \right\} \\ & = \lim_{N \rightarrow \infty} \int_0^{T_1} \int_0^k D_t e_N \varphi dx dt = -\kappa \int_0^{T_1} \left(D_x e, \frac{\tilde{r}^m}{\tilde{\tau}} D_x \varphi \right)_{L^2} dt + \int_0^{T_1} (G, \varphi)_{L^2} dt. \end{aligned}$$

Hence we conclude the proof of this lemma. \square

Next we show that e satisfies the higher regularity in terms of the energy norm (3.3.1). This can be summarised into the following theorem, which will be again proven with several lemmas.

Theorem 3.4.4. *Given a fixed constant $M_0 > 0$ and $T_0 = T_0(M_0) > 0$ defined in Proposition 3.3.1. Suppose there exists a time $T_* > 0$ such that (\tilde{u}, \tilde{e}) satisfies $\tilde{E}(T_*) \equiv E[\tilde{u}, \tilde{e}](T_*) \leq M_0$, then the function $e(x, t)$ obtained in Theorem 3.4.3 satisfies the following estimate: there exists a polynomial $z \mapsto \mathcal{P}(z)$ such that for all $T \in [0, \min\{T_0, T_*\}]$, we have*

$$\begin{aligned} & \sup_{0 \leq t \leq T} \{ \|e(\cdot, t)\|_{H^2}^2 + \|D_t e(\cdot, t)\|_{L^2}^2 \} + \int_0^T \{ \|e\|_{H^3}^2 + \|D_t e\|_{H^1}^2 + \|D_t^2 e\|_{H^*}^2 \} (t) dt \\ & \leq C \mathcal{P}(E_0) + C T \mathcal{P}(\tilde{E}(T)) \exp \left(C T \left(1 + T \tilde{E}(T) + T^2 \exp(T) \tilde{E}(T) \right) \right), \end{aligned} \quad (3.4.31)$$

where $H^*(0, k)$ is the dual space of $H^1(0, k)$, and $C = C(C_0, a, k, \beta, \gamma, n) > 0$ is a constant independent of $M_0 > 0$, $T_0 > 0$, and $T_* > 0$.

Lemma 3.4.5. *Suppose the assumptions of Theorem 3.4.4 hold, then we have $D_t e \in L^\infty(0, T_1; L^2(0, k))$ and $D_x D_t e \in L^2(0, T_1; L^2(0, k))$ with $T_1 \equiv \min\{T_0, T_*\}$. In particular, there exists a constant $C = C(C_0, a, k, \beta, \gamma, n) > 0$ and a polynomial $z \mapsto \mathcal{P}(z)$ such that the following inequality holds:*

$$\begin{aligned} & \sup_{0 \leq t \leq T} \|D_t e_N(\cdot, t)\|_{L^2}^2 + C^{-1} \int_0^T \|D_x D_t e_N(\cdot, s)\|_{L^2}^2 ds \\ & \leq C \mathcal{P}(E_0) + T \mathcal{P}(\tilde{E}(T)) \exp \left(C T \left(1 + T \tilde{E}(T) + T^2 \exp(T) \tilde{E}(T) \right) \right), \end{aligned}$$

for each $T \in [0, \min\{T_0, T_*\}]$.

Proof. First, for each small $h > 0$, we denote Δ_t^h as the difference quotient in time. Let $e_N(x, t) := \sum_{i=1}^N a_i^N(t) \psi_i(x)$ be the Galerkin approximation obtained in the proof of Theorem 3.4.3. Operating Δ_t^h to the equation (3.4.26) for which e_N satisfies, multiplying both sides with $h^{-1}(a_i^N(t+h) - a_i^N(t))$, and taking summation in $i = 1, \dots, N$ we have:

$$\begin{aligned} & \left(D_t \Delta_t^h e_N, \Delta_t^h e_N \right)_{L^2} + \kappa \left(\frac{\tilde{r}^m}{\tilde{\tau}}(\cdot, t+h) \Delta_t^h D_x e_N(\cdot), \Delta_t^h D_x e_N(\cdot) \right)_{L^2} \\ & = -\kappa \left(D_x e_N(\cdot, t) \Delta_t^h(\tilde{\tau}^{-1} \tilde{r}^m), \Delta_t^h D_x e_N(\cdot) \right)_{L^2} + \left(\Delta_t^h G, \Delta_t^h e_N(\cdot) \right)_{L^2}. \end{aligned}$$

By Lemma 3.3.1, and Proposition 3.3.1 for $t \in [0, T - h]$, we obtain the estimate:

$$\frac{1}{2} \frac{d}{dt} \|\Delta_t^h e_N(\cdot)\|_{L^2}^2 + \frac{\kappa a^m}{3 \bar{\tau}_0} \|\Delta_t^h D_x e_N(\cdot, t)\|_{L^2}^2 \leq C \tilde{E}(T) \|D_x e_N\|_{L^2}^2 + \left(\Delta_t^h G, \Delta_t^h e_N \right)_{L^2}, \quad (3.4.32)$$

where we have used the estimate $|\Delta_t^h(\tilde{r}^m \tilde{\tau}^{-1})|_\infty \leq C \tilde{E}(T)^{1/2}$ which was obtained as (3.4.10) in the proof of Lemma 3.4.1. Next, we aim to show that.

$$\left(\Delta_t^h G, \Delta_t^h e_N \right)_{L^2} \leq C \mathcal{P}(\tilde{E}(T)) \exp(CT^2 \tilde{E}(T)) + \frac{\kappa a^m}{6 \bar{\tau}_0} \|\Delta_t^h D_x e\|_{L^2}^2 + \frac{1}{2} \|\Delta_t^h e\|_{L^2}^2$$

First, recalling the definition of G in (3.4.25), applying Δ_t^h to it, we have $\Delta_t^h G = \sum_{i=1}^4 I_i$, where

$$\begin{aligned} I_1 &:= \beta |D_x(\tilde{r}^m u)|^2(x, t) \Delta_t^h(\tilde{r}^{-m} \tilde{\tau}^{-1}) - (\gamma - 1) D_x(\tilde{r}^m u)(x, t) (\tilde{r}^{-m} \tilde{\tau}^{-1})(x, t + h) \Delta_t^h \tilde{e} \\ &\quad - (\gamma - 1) \tilde{e}(x, t) D_x(\tilde{r}^m u)(x, t) \Delta_t^h(\tilde{r}^{-m} \tilde{\tau}^{-1}), \\ I_2 &:= \beta \frac{D_x(\tilde{r}^m u)}{\tilde{r}^m \tilde{\tau}}(x, t + h) D_x \Delta_t^h(\tilde{r}^m u) + \beta D_x(\tilde{r}^m u)(x, t) \frac{1}{\tilde{r}^m \tilde{\tau}}(x, t + h) D_x \Delta_t^h(\tilde{r}^m u), \\ I_3 &:= -(\gamma - 1) (\tilde{e} \tilde{r}^{-m} \tilde{\tau}^{-1})(x, t + h) D_x \Delta_t^h(\tilde{r}^m u), \\ I_4 &:= -2m\mu D_x \Delta_t^h(\tilde{r}^{m-1} u^2) \end{aligned}$$

For I_1 , using Sobolev embedding theorem, Lemma 3.3.1, and (3.2.2)–(3.2.5), it follows that

$$\begin{aligned} |\Delta_t^h(\tilde{r}^{-m} \tilde{\tau}^{-1})|_\infty &\leq \frac{1}{h} \int_0^t |-m\tilde{r}^{-m-1} \tilde{\tau}^{-1} \tilde{u} - \tilde{r}^{-m} \tilde{\tau}^{-2} D_x \tilde{u}|_\infty(s) ds \leq C \tilde{E}(T)^{1/2}, \\ |D_x(\tilde{r}^m u)|_\infty &= |m\tilde{r}^{m-1} \tilde{\tau} u + \tilde{r}^m D_x u|_\infty \leq C \|u\|_{H^2}, \\ \|D_x(\tilde{r}^m u)\|_{L^2} &\leq \|m\tilde{r}^{m-1} \tilde{\tau} u + \tilde{r}^m D_x u\|_{L^2} \leq C \|u\|_{H^1}, \\ \|\Delta_t^h \tilde{e}\|_{L^2} &\leq \frac{1}{h} \int_t^{t+h} \|D_t \tilde{e}(\cdot, s)\|_{L^2} ds \leq C \tilde{E}(T)^{1/2}, \end{aligned} \quad (3.4.33)$$

Using these to estimate each terms in $\|I_1\|_{L^2}^2$, and applying Theorem 3.4.2, we have:

$$(I_1, \Delta_t^h e)_{L^2} \leq 2\|I_1\|_{L^2}^2 + \frac{1}{8} \|\Delta_t^h e\|_{L^2}^2 \leq C \mathcal{P}(\tilde{E}(T)) \exp(CT^2 \tilde{E}(T)) + \frac{1}{8} \|\Delta_t^h e\|_{L^2}^2.$$

Next, we estimate I_2 . Only the second term in I_2 will be considered in the following analysis as the first term only differs by a translation in time. First, we note that using Theorem 3.4.2, Lemma 3.3.1, relations (3.2.2)–(3.2.5), and the Sobolev embedding theorem, we get:

$$\begin{aligned} \|D_x^2(\tilde{r}^m u)\|_{L^2} &\leq C(C_0 + T^{1/2} \tilde{E}(T)^{1/2}) \|u\|_{H^2}, \\ \|\Delta_t^h(\tilde{r}^m u)\|_{L^2} &\leq C(\tilde{E}(T)^{1/2} \|u\|_{L^2} + \|D_t u\|_{L^2}), \\ |D_x(\tilde{r}^{-m} \tilde{\tau}^{-1})|_\infty &= |-m\tilde{r}^{-m-1} + \tilde{r}^{-m-1} \tilde{\tau}^{-2} D_x \tilde{\tau}|_\infty \leq C(1 + C_0 + T^{1/2} \tilde{E}(T)^{1/2}), \\ |\Delta_t^h e|_\infty &\leq C \|\Delta_t^h e\|_{H^1}. \end{aligned} \quad (3.4.34)$$

Next integrating by parts in space on I_2 , using (3.4.33), (3.4.34), and the boundary condition $\Delta_t^h u(0, t) = \Delta_t^h u(k, t) = 0$, $u(0, t) = u(k, t) = 0$, it follows from Theorem 3.4.2 that:

$$(I_2, \Delta_t^h e)_{L^2} \leq C \mathcal{P}(\tilde{E}(T)) \exp(CT^2 \tilde{E}(T)) + \frac{\kappa a^m}{18 \bar{\tau}_0} \|\Delta_t^h D_x e\|_{L^2}^2 + \frac{1}{8} \|\Delta_t^h e\|_{L^2}^2.$$

By the similar token, using $u(0, t) = u(k, t) = 0$ and $\Delta_t^h u(0, t) = \Delta_t^h u(k, t) = 0$, I_3 can be estimated by:

$$(I_3, \Delta_t^h e)_{L^2} \leq \mathcal{P}(\tilde{E}(T)) \exp(CT^2 \tilde{E}(T)) + \frac{\kappa}{18} \frac{a^m}{\bar{\tau}_0} \|\Delta_t^h D_x e\|_{L^2}^2 + \frac{1}{8} \|\Delta_t^h e\|_{L^2}^2,$$

where we have used estimates (3.4.34) and Theorem 3.4.2. Finally I_4 can be estimated similarly as follows:

$$\begin{aligned} (I_4, \Delta_t^h e)_{L^2} &= 2m\mu \int_0^k \Delta_t^h(\tilde{r}^{m-1} u^2) D_x \Delta_t^h e dx \leq C \|\Delta_t^h(\tilde{r}^{m-1} u^2)\|_{L^2}^2 + \frac{\kappa}{18} \frac{a^m}{\bar{\tau}_0} \|D_x \Delta_t^h e\|_{L^2}^2 \\ &\leq \mathcal{P}(\tilde{E}(T)) \exp(CT^2 \tilde{E}(T)) + \frac{\kappa}{18} \frac{a^m}{\bar{\tau}_0} \|\Delta_t^h D_x e\|_{L^2}^2 + \frac{1}{8} \|\Delta_t^h e\|_{L^2}^2, \end{aligned}$$

where we used the estimate:

$$\|\Delta_t^h(\tilde{r}^{m-1} u^2)\|_{L^2} \leq \frac{1}{h} \int_t^{t+h} \|(m-1)\tilde{r}^{m-2} \tilde{u} u^2 + 2\tilde{r}^{m-1} u D_t u\|_{L^2} ds \leq C \tilde{E}(T)^{1/2} \|u\|_{H^1}^2 + C \|u\|_{H^1} \|D_t u\|_{L^2}$$

Now summarising the estimates for I_1 – I_4 , we have:

$$(\Delta_t^h G, \Delta_t^h e_N)_{L^2} = \sum_{i=1}^4 (I_i, \Delta_t^h e_N)_{L^2} \leq C \mathcal{P}(\tilde{E}(T)) \exp(CT^2 \tilde{E}(T)) + \frac{\kappa}{6} \frac{a^m}{\bar{\tau}_0} \|\Delta_t^h D_x e\|_{L^2}^2 + \frac{1}{2} \|\Delta_t^h e\|_{L^2}^2.$$

Hence substituting this back into estimate 3.4.32, it follows that

$$\frac{1}{2} \frac{d}{dt} \|\Delta_t^h e_N(\cdot)\|_{L^2}^2 + \frac{\kappa}{6} \frac{a^m}{\bar{\tau}_0} \|\Delta_t^h D_x e_N(\cdot, t)\|_{L^2}^2 \leq C \mathcal{P}(\tilde{E}(T)) \exp(CT^2 \tilde{E}(T)) + C \tilde{E}(T) \|D_x e_N\|_{L^2}^2 + \frac{1}{2} \|\Delta_t^h e\|_{L^2}^2,$$

By Grönwall lemma, it follows that

$$\|\Delta_t^h e\|_{L^2}^2 \leq \exp(T) \left\{ \|\Delta_0^h e_N\|_{L^2}^2 + C \mathcal{P}(\tilde{E}(T)) \exp(CT^2 \tilde{E}(T)) + C \tilde{E}(T) \int_0^t \|D_x e_N(\cdot, s)\|_{L^2}^2 ds \right\}.$$

Now integrating (3.4.32) in time and substituting the above into it, we obtain:

$$\begin{aligned} \sup_{0 \leq t \leq T} \|\Delta_t^h e_N\|_{L^2}^2 + \int_0^T \|\Delta_t^h D_x e_N\|_{L^2}^2 dt &\leq C(1 + T \exp(T)) \left\{ \|\Delta_0^h e_N\|_{L^2}^2 + \tilde{E}(T) \int_0^t \|D_x e_N\|_{L^2}^2 ds \right\} \\ &\quad + C \mathcal{P}(\tilde{E}(T)) \exp(CT(1 + T \tilde{E}(T))). \end{aligned} \quad (3.4.35)$$

Next, we show that $\limsup_{h \rightarrow 0^+} \|\Delta_0^h e_N\|_{L^2}^2 \leq C \mathcal{P}(E_0)$. First, we observe that since $\{\psi_i\}_i$ is an orthonormal basis in $L^2(0, k)$:

$$\lim_{h \rightarrow 0^+} \|\Delta_0^h e_N(x)\|_{L^2}^2 := \lim_{h \rightarrow 0^+} \left\| \sum_{i=1}^N \psi_i(x) \Delta_0^h a_N^i \right\|_{L^2}^2 = \sum_{i=1}^N \lim_{h \rightarrow 0^+} |\Delta_0^h a_N^i|^2 = \sum_{i=1}^N \left| \frac{da_N^i}{dt}(0) \right|^2. \quad (3.4.36)$$

Recalling equation (3.4.26) in the proof of Theorem 3.4.3, and the fact that $a_N^i(t) \in C^1([0, T])$ it follows from integrating by parts, and the boundary condition $\psi'(0) = \psi'(k) = 0$ that:

$$\frac{da_N^i}{dt}(0) = (D_t e_N(\cdot, 0), \phi_i)_{L^2} = \kappa \left(\frac{r_0^m}{\tau_0} D_x^2 e_N(\cdot, 0), \psi_i \right)_{L^2} + \kappa (D_x(\tau_0^{-1} r_0^m) e_N(\cdot, 0), \psi_i)_{L^2} + (G(\cdot, 0), \psi_i)_{L^2}.$$

Substituting this into (3.4.36), and using the fact that $\{\psi_i\}_{i \in \mathbb{N}}$ is an orthonormal basis in L^2 , hence one can employ Bessel's inequality to get that:

$$\lim_{h \rightarrow 0^+} \|\Delta_0^h e_N\|_{L^2}^2 \leq C \|D_x^2 e_N(\cdot, 0)\|_{L^2}^2 + C \|\tau_0\|_{H^2}^2 \|e_N(\cdot, 0)\|_{L^2}^2 + C(\|u_0\|_{H^1}^2 + \|e_0\|_{H^1}^2) \|u_0\|_{H^1}^2, \quad (3.4.37)$$

where we have used the estimate

$$\begin{aligned} \|G(\cdot, 0)\|_{L^2} &= \beta \|m^2 r_0^{m-2} \tau_0 u_0^2 + 2mr_0^{m-1} u_0 D_x u_0 + \frac{r_0^m}{\tau_0} |D_x u_0|^2\|_{L^2} + (\gamma - 1) \left\| \frac{e_0}{\tau_0} D_x u_0 + m \frac{e_0}{\tau_0} u_0 \right\|_{L^2} + \\ &\quad + 2m\mu \|(m-1)r_0^{m-2} \tau_0 u_0^2 + 2r_0^{m-1} u_0 D_x u_0\|_{L^2} \leq C(\|u_0\|_{H^1}^2 + \|e_0\|_{H^1}^2) \|u_0\|_{H^1}^2. \end{aligned}$$

Now by construction, we have that $-D_x^2 \psi_i = \lambda_i \psi_i$ and $D_x \psi_i(0) = D_x \psi_i(k) = 0$. Integrating by parts twice, we get:

$$\|D_x^2 e_N(\cdot, 0)\|_{L^2}^2 = (D_x^2 e_N(\cdot, 0), D_x^2 e_N(\cdot, 0))_{L^2} = \left(e_N(\cdot, 0), \sum_{i=1}^N \lambda_i^2 (e_0, \psi_i)_{L^2} \psi_i \right)_{L^2}.$$

By the initial data of $t \mapsto a_N^i(t)$, we have $(e_N(\cdot, 0), \psi_i)_{L^2} = a_N^i(0) = (e_0, \psi_i)_{L^2}$. Thus reversing the above argument with the boundary condition $D_x e_0(0) = D_x e_0(k) = 0$, we obtain the following estimate:

$$\|D_x^2 e_N(\cdot, 0)\|_{L^2}^2 = \left(e_0, \sum_{i=1}^N \lambda_i^2 (e_0, \psi_i)_{L^2} \psi_i \right)_{L^2} = (D_x^2 e_0, D_x^2 e_N(\cdot, 0))_{L^2} \leq \frac{1}{2} \|D_x^2 e_0\|_{L^2}^2 + \frac{1}{2} \|D_x^2 e_N(\cdot, 0)\|_{L^2}^2,$$

hence $\|D_x^2 e_N(\cdot, 0)\|_{L^2}^2 \leq \|D_x^2 e_0\|_{L^2}^2$. Furthermore we also get $\|e_N(\cdot, 0)\|_{L^2}^2 \leq \|e_0\|_{L^2}^2$ by Bessel's inequality. Thus substituting these back to (3.4.37), it follows that $\lim_{h \rightarrow 0^+} \|\Delta_0^h e_N(x)\|_{L^2}^2 \leq C\mathcal{P}(E_0)$. In summary, (3.4.35) becomes:

$$\begin{aligned} &\sup_{0 \leq t \leq T-h} \|\Delta_t^h e_N\|_{L^2}^2 + C^{-1} \int_0^{T-h} \|\Delta_t^h D_x e_N\|_{L^2}^2 dt \\ &\leq C\mathcal{P}(E_0) + T\mathcal{P}(\tilde{E}(T)) \exp(CT(1 + T\tilde{E}(T))) + C(1 + T \exp(T)) \tilde{E}(T) \int_0^t \|D_x e_N(\cdot, s)\|_{L^2}^2 ds < \infty. \end{aligned}$$

The right hand side of above inequality is uniformly bounded for $T \in [0, \min\{T_*, T_0\}]$ and $h \in (0, 1)$ due to the estimate obtained in Theorem 3.4.3. Therefore, by the same argument as in the proof of Lemma 3.4.1, there exists a subsequence $h \searrow 0$ for $T_1 \equiv \min\{T_*, T_0\}$

$$\begin{aligned} \mathbb{1}_{[0, T_1-h]} \Delta_t^h e_N &\overset{*}{\rightharpoonup} D_t e_N && \text{in } L^\infty(0, T_1; L^2(0, k)), \\ \mathbb{1}_{[0, T_1-h]} \Delta_t^h D_x e_N &\rightharpoonup D_x D_t e_N && \text{in } L^2(0, T_1; L^2(0, k)). \end{aligned}$$

where $D_t e_N$ and $D_x D_t e_N$ are respectively the temporal weak derivative of e_N and $D_x e_N$. Moreover by the weakly (and weakly-*) lower semi-continuity of Sobolev norm, we get:

$$\begin{aligned} &\|D_t e_N(\cdot, t)\|_{L^2}^2 + C^{-1} \int_0^t \|D_x D_t e_N(\cdot, s)\|_{L^2}^2 ds \\ &\leq C\mathcal{P}(E_0) + T\mathcal{P}(\tilde{E}(T)) \exp(CT(1 + T\tilde{E}(T))) + C(1 + T \exp(T)) \tilde{E}(T) \int_0^t \|D_x e_N(\cdot, s)\|_{L^2}^2 ds. \end{aligned} \tag{3.4.38}$$

Now by Fundamental Theorem of Calculus, it follows that:

$$\|D_x e_N(\cdot, s)\|_{L^2}^2 \leq \left(\|D_x e_N(\cdot, 0)\|_{L^2} + \int_0^s \|D_x D_t e_N(\cdot, q)\|_{L^2} dq \right)^2 \leq 2\|D_x e_0\|_{L^2}^2 + 2s \int_0^s \|D_x D_t e_N\|_{L^2}^2 dq,$$

where in the last line we have used the following estimate: by the facts that basis ψ_i satisfies the boundary condition $D_x \psi_i(0) = D_x \psi_i(k) = 0$, eigenvalue equation $-D_x^2 \psi_i = \lambda_i \psi_i$, the initial data satisfies the boundary condition $D_x e_0(0) = D_x e_0(k) = 0$, and the construction $(e_N(\cdot, 0), \psi_i)_{L^2} = (e_0, \psi_i)_{L^2}$, it follows that

$$\|D_x e_N(\cdot, 0)\|_{L^2}^2 = (e_N(\cdot, 0), -D_x^2 e_N(\cdot, 0))_{L^2} = (D_x e_0, D_x e_N(\cdot, 0))_{L^2} \leq \frac{1}{2} \|D_x e_0\|_{L^2}^2 + \frac{1}{2} \|D_x e_N(\cdot, 0)\|_{L^2}^2.$$

Substituting this back into (3.4.38), we have the inequality:

$$\begin{aligned} & \|D_t e_N(\cdot, t)\|_{L^2}^2 + C^{-1} \int_0^t \|D_x D_t e_N(\cdot, s)\|_{L^2}^2 ds \\ & \leq C\mathcal{P}(E_0) + C\mathcal{TP}(\tilde{E}(T)) \exp(CT(1 + T\tilde{E}(T))) + C(1 + T \exp(T))\tilde{E}(T) \int_0^t s \int_0^s \|D_t D_x e_N(\cdot, q)\|_{L^2}^2 dq ds. \end{aligned}$$

Therefore by Grönwall Lemma for integral inequality, it follows that:

$$\begin{aligned} & \sup_{0 \leq t \leq T} \|D_t e_N(\cdot, t)\|_{L^2}^2 + C^{-1} \int_0^T \|D_x D_t e_N(\cdot, t)\|_{L^2}^2 dt \\ & \leq C\mathcal{P}(E_0) + T\mathcal{P}(\tilde{E}(T)) \exp\left(CT\left(1 + T\tilde{E}(T) + T^2 \exp(T)\tilde{E}(T)\right)\right). \end{aligned} \quad (3.4.39)$$

By the same argument as in the proof of Lemma 3.4.1, there exists a subsequence in $N \rightarrow \infty$ such that

$$D_t e_N \xrightarrow{*} D_t e \quad \text{in } L^\infty(0, T; L^2(0, k)) \quad \text{and} \quad D_x D_t e_N \rightharpoonup D_x D_t e \quad \text{in } L^2(0, T; L^2(0, k)),$$

where e and $D_x e$ are the weak solutions obtained in Theorem 3.4.3. Finally, by the weakly and weakly-star lower semi-continuity, we obtain the desired estimate for e and $D_x e$ stated in the lemma. \square

Lemma 3.4.6. *Suppose the assumptions of Theorem 3.4.4 hold, then we have $e \in L^\infty(0, T_1; H^2(0, k))$ with $T_1 \equiv \min\{T_*, T_0\}$. In particular, there exists a constant $C = C(C_0, a, k, \beta, \gamma, n) > 0$ and a polynomial $z \mapsto \mathcal{P}(z)$ such that the following inequality holds:*

$$\sup_{0 \leq t \leq T} \|e(\cdot, t)\|_{H^2}^2 \leq \mathcal{P}(E_0) + T\mathcal{P}(\tilde{E}(T)) \exp\left(CT\left(1 + T\tilde{E}(T) + T^2 \exp(T)\tilde{E}(T)\right)\right),$$

for each $T \in [0, \min\{T_0, T_*\}]$.

Proof. Let $e_N(x, t)$ be the Galerkin approximation obtained in the proof of Theorem 3.4.3, then we recall that $e_N(x, t)$ satisfies (3.4.26). Multiplying (3.4.26) with $a_N^i(t)$ and sum over $i = 1, \dots, N$ it follows by Lemma 3.3.1 that:

$$\begin{aligned} & \frac{4\kappa^2 a^m}{9 \bar{\tau}_0} \|D_x e_N\|_{L^2}^2 \leq \kappa \left(\frac{\tilde{r}^m}{\tilde{\tau}} D_x e_N, D_x e_N \right)_{L^2} = (-D_t e_N + G, e_N)_{L^2} \leq \| -D_t e_N + G \|_{L^2}^2 + \|e_N\|_{L^2}^2 \\ & \leq \|G\|_{L^2}^2 + \mathcal{P}(E_0) + T\mathcal{P}(\tilde{E}(T)) \exp\left(CT\left(1 + T\tilde{E}(T) + T^2 \exp(T)\tilde{E}(T)\right)\right), \end{aligned} \quad (3.4.40)$$

where in the last line we have used Lemma 3.4.5 and Theorem 3.4.3. Next integrating by parts in space and using the boundary condition $\psi_i'(0) = \psi_i'(k) = 0$ for each $i \in \mathbb{N}$, hence $D_x e_N(0, t) = D_x e_N(k, t) = 0$, equation (3.4.26) from the proof of Theorem 3.4.3 becomes

$$(D_t e_N, \psi_i)_{L^2} = \kappa \left(D_x \left(\frac{\tilde{r}^m}{\tilde{\tau}} D_x e_N \right), \psi_i \right)_{L^2} + (G, \psi_i)_{L^2}.$$

Since $-D_x^2 \psi_i = \lambda_i \psi_i$, multiplying the above with $\lambda_i a_N^i(t)$ and summing over $i = 1, \dots, N$, we have by Proposition 3.3.1 and Lemma 3.4.5:

$$\begin{aligned} & \frac{2\kappa a^m}{3 \bar{\tau}_0} \|D_x^2 e_N\|_{L^2}^2 \leq \kappa \left(\frac{\tilde{r}^m}{\tilde{\tau}} D_x^2 e_N, D_x^2 e_N \right)_{L^2} = -\kappa \left(D_x (\tilde{\tau}^{-1} \tilde{r}^m) D_x e_N, D_x^2 e_N \right)_{L^2} + (D_t e_N - G, D_x^2 e_N) \\ & \leq C \|D_x (\tilde{r}^m \tilde{\tau}^{-1}) D_x e_N\|_{L^2}^2 + C \|D_t e_N\|_{L^2}^2 + C \|G\|_{L^2}^2 + \frac{\kappa a^m}{3 \bar{\tau}_0} \|D_x^2 e_N\|_{L^2}^2. \end{aligned}$$

Thus reordering the terms, and by Lemma 3.4.5, we get:

$$\frac{\kappa a^m}{3 \bar{\tau}_0} \|D_x^2 e_N\|_{L^2}^2 \leq \mathcal{P}(E_0) + T\mathcal{P}(\tilde{E}(T)) \exp\left(CT\left(1 + T\tilde{E}(T) + T^2 \exp(T)\tilde{E}(T)\right)\right) + I_1 + I_2, \quad (3.4.41)$$

where $I_1 := \|G(\cdot, t)\|_{L^2}^2$ and $I_2 := \|D_x(\tilde{r}^m \tilde{\tau}^{-1})D_x e_N\|_{L^2}^2$. We estimate the term I_1 first. By the Fundamental Theorem of Calculus and Cauchy Schwartz Inequality, we have:

$$I_1 := \|G(\cdot, t)\|_{L^2}^2 = 2\|G(\cdot, 0)\|_{L^2}^2 + 2t \int_0^t \|D_t G(\cdot, s)\|_{L^2}^2 ds. \quad (3.4.42)$$

Next, using relations (3.2.2)–(3.2.5) and Lemma 3.3.1, we have:

$$\begin{aligned} |D_x(\tilde{r}^m u)|_\infty &= |m\tilde{r}^{m-1}\tilde{\tau}u + \tilde{r}^m D_x u|_\infty \leq C\|u\|_{H^2}, \\ \|D_t(\tilde{r}^m u)\|_{L^2} &= \|m\tilde{r}^{m-1}\tilde{u}u + \tilde{r}^m D_t u\|_{L^2} \leq C\tilde{E}(T)^{1/2}\|u\|_{L^2} + C\|D_t u\|_{L^2}, \\ \|D_x D_t(\tilde{r}^m u)\|_{L^2} &\lesssim \|m(m-1)\tilde{r}^{m-2}\tilde{\tau}\tilde{u}u\|_{L^2} + \|m\tilde{r}^{m-1}(u D_x \tilde{u})\|_{L^2} + \|m\tilde{r}^{m-1}(\tilde{u} D_x u)\|_{L^2} \\ &\quad + \|m\tilde{r}^{m-1}\tilde{\tau} D_t u\|_{L^2} + \|\tilde{r}^m D_x D_t u\|_{L^2} \leq C(\tilde{E}(T)^{1/2}\|u\|_{H^1} + \|D_t u\|_{H^1}), \\ \|D_t(\tilde{r}^{-m}\tilde{\tau}^{-1})\|_{L^2} &= \|-m\tilde{r}^{-m-1}\tilde{\tau}^{-1}\tilde{u} - \tilde{r}^m \tilde{\tau}^{-2} D_x \tilde{u}\|_{L^2} \leq C\|\tilde{u}\|_{H^1} \leq C\tilde{E}(T)^{1/2}, \\ \|D_x D_t(\tilde{r}^{m-1}u^2)\|_{L^2} &\lesssim C\tilde{E}(T)^{1/2}\|u\|_{H^1}^2 + C\|u\|_{H^1}\|D_t u\|_{H^1}. \end{aligned} \quad (3.4.43)$$

Now computing $D_t G$ according to the definition (3.4.25), and taking $L^2(0, k)$ norm, it follows that from (3.4.43) and Lemmas 3.4.1–3.4.2 that

$$\int_0^t \|D_t G\|_{L^2}^2 ds \leq C(1 + \|D_t u\|_{H^1}^2)\mathcal{P}(\tilde{E}(T)) \exp(CT^2\tilde{E}(T)) \leq C\mathcal{P}(\tilde{E}(T)) \exp(CT^2\tilde{E}(T)).$$

where we used Theorem 3.4.2. Hence substituting the above inequality to (3.4.42)

$$I_1 := \|G(\cdot, t)\|_{L^2}^2 \leq 2\|G(\cdot, 0)\|_{L^2}^2 + 2t \int_0^t \|D_t G(\cdot, s)\|_{L^2}^2 ds \leq \mathcal{P}(E_0) + T\mathcal{P}(\tilde{E}(T)) \exp(CT^2\tilde{E}(T)). \quad (3.4.44)$$

As an immediate consequence of this estimate we get from (3.4.40) that:

$$\sup_{0 \leq t \leq T} \|D_x e_N(\cdot, t)\|_{L^2}^2 \leq C\mathcal{P}(E_0) + T\mathcal{P}(\tilde{E}(T)) \exp\left(CT\left(1 + T\tilde{E}(T) + T^2 \exp(T)\tilde{E}(T)\right)\right). \quad (3.4.45)$$

Next, we estimate $I_2 := \|D_x(\tilde{r}^m \tilde{\tau}^{-1})D_x e_N\|_{L^2}^2$. By Fundamental Theorem of Calculus in time, we have:

$$I_2 \leq 2\|D_x(r_0^m \tau_0^{-1})D_x e_0\|_{L^2}^2 + 2t \int_0^t \|D_t(D_x(\tilde{r}^m \tilde{\tau}^{-1})D_x e_N)\|_{L^2}^2 ds \leq \mathcal{P}(E_0) + 2t \int_0^t \sum_{i=1}^2 I_2^{(i)} ds, \quad (3.4.46)$$

The term $I_2^{(1)}$ can be estimated by using Lemma 3.3.1 as:

$$\begin{aligned} I_2^{(1)} &:= \|D_t D_x(\tilde{r}^m \tilde{\tau}^{-1})D_x e_N\|_{L^2} \lesssim \|\tilde{u} D_x e_N\|_{L^2} + \|\tilde{u} D_x \tilde{\tau} D_x e_N\|_{L^2} + \|D_x \tilde{u} D_x \tilde{\tau} D_x e_N\|_{L^2} + \|D_x^2 \tilde{u} D_x e_N\|_{L^2} \\ &\leq C\tilde{E}(T)^{1/2}(C_0 + T^{1/2}\tilde{E}(T)^{1/2})\|D_x e_N\|_{L^2} + C\|D_x e_N\|_{L^2}\|\tilde{u}\|_{H^3}, \end{aligned}$$

and similarly, $I_2^{(2)}$ can be also estimated as

$$I_2^{(2)} = \|(m\tilde{r}^{m-1} - \tilde{\tau}^{-2}\tilde{r}^m D_x \tilde{\tau})D_x D_t e_N\|_{L^2} \leq C(C_0 + T^{1/2}\tilde{E}(T)^{1/2})\|D_x D_t e_N\|_{L^2},$$

Substituting $I_2^{(1)}, I_2^{(2)}$ into (3.4.46), applying Lemma 3.4.5 and (3.4.45), it follows that:

$$I_2 := \|D_x(\tilde{r}^m \tilde{\tau}^{-1})D_x e_N\|_{L^2}^2 \leq C\mathcal{P}(E_0) + T\mathcal{P}(\tilde{E}(T)) \exp\left(CT\left(1 + T\tilde{E}(T) + T^2 \exp(T)\tilde{E}(T)\right)\right).$$

Finally substituting estimate (3.4.44) for I_1 and the above estimate for I_2 into (3.4.41), it follows that

$$\sup_{0 \leq t \leq T} \|D_x^2 e_N(\cdot, t)\|_{L^2}^2 \leq C\mathcal{P}(E_0) + CT\mathcal{P}(\tilde{E}(T)) \exp\left(CT\left(1 + T\tilde{E}(T) + T^2 \exp(T)\tilde{E}(T)\right)\right).$$

With (3.4.45), and Theorem 3.4.3, we have for all $T \in [0, T_1]$ with $T_1 \equiv \min\{T_*, T_0\}$:

$$\sup_{0 \leq t \leq T} \|e_N(\cdot, t)\|_{H^2}^2 \leq C\mathcal{P}(E_0) + CT\mathcal{P}(\tilde{E}(T)) \exp\left(CT\left(1 + T\tilde{E}(T) + T^2 \exp(T)\tilde{E}(T)\right)\right).$$

Recalling from the proof of Theorem 3.4.3, we have the convergences $D_x e_N \rightharpoonup D_x e$ in $L^2(0, T; L^2(0, k))$, $e_N \overset{*}{\rightharpoonup} e$ in $L^\infty(0, T; L^2(0, k))$. Hence by the uniqueness of weak derivative, there exists a subsequence $N \rightarrow \infty$ such that $(D_x^2 e_N, D_x e_N) \overset{*}{\rightharpoonup} (D_x^2 e, D_x e)$ in $L^\infty(0, T_1; L^2(0, k))$ where $D_x^2 e$ denotes the spatial weak derivative of $D_x e$. By the weakly-star lower semi-continuity, the desired estimate of the lemma is obtained. \square

Corollary 3.4.2. *Suppose the assumptions of Theorem 3.4.2 hold, and let $e(x, t)$ be the weak solution obtained in Theorem 3.4.3. Then $e \in L^\infty(0, T_1; H^2(0, k))$, $D_t u \in L^\infty(0, T_1; L^2(0, k))$ with $T_1 \equiv \min\{T_*, T_0\}$, and we have for $(x, t) \in [0, k] \times [0, T_1]$ almost everywhere, the linear energy equation (3.2.3)₂ is satisfied:*

$$D_t e = \kappa D_x \left(\frac{\tilde{r}^m}{\tilde{\tau}} D_x e \right) + \beta \frac{|D_x(\tilde{r}^m u)|^2}{\tilde{r}^m \tilde{\tau}} - (\gamma - 1) \frac{D_x(\tilde{r}^m u)}{\tilde{r}^m \tilde{\tau}} \tilde{e} - 2m\mu D_x(\tilde{r}^{m-1} u^2).$$

Moreover for almost all $t \in [0, T_1]$, the trace function of $x \mapsto D_x e(x, t)$ exists in $C^0([0, k])$ and

$$D_x e(0, t) = D_x e(k, t) = 0 \quad \text{for almost all } t \in [0, T_1].$$

Proof. The first statement can be proven by letting a test function $\varphi \in C_c^\infty([0, T_1] \times [0, k])$ and using the exact same argument as in the proof of Corollary 3.4.1. So we focus on the proof of $D_x e(0, t) = D_x e(k, t) = 0$. First, let $e_N(x, t) := \sum_{i=1}^N a_N^i(t) \psi(x)$ be the Galerkin approximation in Theorem 3.4.3. Then by construction, we have $D_x e_N(0, t) = D_x e_N(k, t) = 0$ for each $t \in [0, T_1]$ and $N \in \mathbb{N}$. In addition, from the proof of Lemmas 3.4.5–3.4.6, there exists a subsequence $N \rightarrow \infty$ such that we have the convergences: $D_x D_x e_N \overset{*}{\rightharpoonup} D_x D_x e$ in $L^\infty(0, T_1; L^2(0, k))$ and $D_x e_N \rightharpoonup D_x e$ in $L^2(0, T_1; L^2(0, k))$. Thus picking a test function $\phi \in C_c^1(0, T_1; C^1(0, k))$, in light of the aforementioned remarks, we have that

$$\begin{aligned} \int_0^{T_1} \int_0^k D_x D_x e(x, t) \phi(x, t) dx dt &= \lim_{N \rightarrow \infty} \int_0^{T_1} \int_0^k D_x D_x e_N(x, t) \phi(x, t) dx dt \\ &= - \lim_{N \rightarrow \infty} \int_0^{T_1} \int_0^k D_x e_N(x, t) D_x \phi(x, t) dx dt = - \int_0^{T_1} \int_0^k D_x e(x, t) D_x \phi(x, t) dx dt \end{aligned}$$

Next, since $e \in L^\infty(0, T_1; H^2(0, k))$ from the previous lemma, it implies by Sobolev embedding theorem that $D_x e \in L^\infty(0, T_1; C^0[0, k])$, hence for almost all $t \in [0, T_1]$, the traces $D_x e(0, t) := \lim_{x \rightarrow 0^+} D_x e(x, t)$ and $D_x e(k, t) := \lim_{x \rightarrow k^-} D_x e(x, t)$ exist. Integrating by parts of the above identity, we obtain that

$$\int_0^{T_1} D_x e(k, t) \phi(k, t) dt = \int_0^{T_1} D_x e(0, t) \phi(0, t) dt.$$

Now for any given test function in temporal coordinate $\chi(t) \in C_c^1([0, T_1])$, we pick $\varphi(x) \in C^1(0, k)$ satisfying $\varphi(0) = 1$ and $\varphi(k) = 0$. Letting $\phi(x, t) = \chi(t)\varphi(x)$, in the above equation, we have:

$$\int_0^{T_1} D_x e(0, t) \chi(t) dt = 0.$$

Thus, by Fundamental Theorem of Calculus of Variation, it follows that $D_x e(0, t) = 0$ for almost all $t \in [0, T_1]$. By setting instead $\varphi(0) = 0$ and $\varphi(k) = 1$, we also get $D_x e(k, t) = 0$ for almost all $t \in [0, T_1]$, this proves the Corollary. \square

Lemma 3.4.7. *Suppose the assumptions of Theorem 3.4.4 hold, and let $e(x, t)$ be the weak solution obtained in Theorem 3.4.3. Then we have $e \in L^2(0, T_1; H^3(0, k))$ with $T_1 \equiv \min\{T_*, T_0\}$, and in particular there exists a constant $C = C(C_0, a, k, \gamma, \beta, n) > 0$ such that for each $T \in [0, \min\{T_0, T_*\}]$:*

$$\int_0^T \|e(\cdot, t)\|_{H^3}^2 dt \leq C\mathcal{P}(E_0) + CT\mathcal{P}(\tilde{E}(T)) \exp\left(CT\left(1 + T\tilde{E}(T) + T^2 \exp(T)\tilde{E}(T)\right)\right).$$

Proof. Denote Δ_x^h as the difference quotient on spatial coordinate as defined in the proof of Lemma 3.4.3. Taking Δ_x^h on the internal energy equation shown in Corollary 3.4.2, then taking L^2 -norm with the spatial domain $x \in [0, k-h]$ on both sides of the above equation, integrating in time, and with the help of Proposition 3.3.1, we have:

$$\kappa^2 \frac{4a^{2m}}{9\bar{\tau}_0} \int_0^t \int_0^{k-h} |\Delta_x^h D_x^2 e(\cdot, s)|^2 dx ds \lesssim \sum_{i=1}^5 \int_0^t I_i ds. \quad (3.4.47)$$

The terms I_1 – I_2 are estimated using Lemma 3.3.1, Lemmas 3.4.5–3.4.6, and Proposition 3.4.1 stated in the proof of Lemma 3.4.3:

$$\begin{aligned} I_1 &:= \int_0^t \int_0^{k-h} |\Delta_x^h D_t e|^2 dx dt \leq C\mathcal{P}(E_0) + CT\mathcal{P}(\tilde{E}(T)) \exp\left(CT\left(1 + T\tilde{E}(T) + T^2 \exp(T)\tilde{E}(T)\right)\right), \\ I_2 &:= \int_0^t \int_0^{k-h} |D_x(\tilde{\tau}^{-1}\tilde{r}^m)\Delta_x^h D_x e|^2 dx dt \leq \int_0^t |m\tilde{r}^{m-1} - \tilde{r}^m\tilde{\tau}^{-2}D_x\tilde{\tau}|_\infty^2 \|D_x^2 e(\cdot, s)\|_{L^2}^2 ds \\ &\leq C \int_0^t (C_0 + T\tilde{E}(T)) \|D_x^2 e\|_{L^2}^2 ds \leq CT\mathcal{P}(\tilde{E}(T)) \exp\left(CT\left(1 + T\tilde{E}(T) + T^2 \exp(T)\tilde{E}(T)\right)\right). \end{aligned}$$

For the term I_3 , we start by first using Sobolev embedding theorem and Proposition 3.4.1 to obtain:

$$I_3 := \kappa^2 \int_0^t \int_0^{k-h} |D_x e \Delta_x^h D_x(\tilde{\tau}^{-1}\tilde{r}^m)|^2 dx ds \leq C \int_0^t \|e(\cdot, s)\|_{H^2}^2 \|D_x^2(\tilde{\tau}^{-1}\tilde{r}^m)(\cdot, s)\|_{L^2}^2 ds.$$

By relation (3.2.2)–(3.2.5), Jensen's inequality, Cauchy-Schwartz inequality, and Lemma 3.3.1 it follows that:

$$\|D_x^2(\tilde{\tau}^{-1}\tilde{r}^m)(\cdot, s)\|_{L^2}^2 \leq C(1 + (T + T^2)\tilde{E}(T) + T^3\tilde{E}(T)^2).$$

Substituting this back into the estimate for I_3 and using Lemma 3.4.6, we have

$$I_3 \leq C \int_0^t \|e(\cdot, s)\|_{H^2}^2 \|D_x^2(\tilde{\tau}^{-1}\tilde{r}^m)(\cdot, s)\|_{L^2}^2 ds \leq CT\mathcal{P}(\tilde{E}(T)) \exp\left(CT\left(1 + T\tilde{E}(T) + T^2 \exp(T)\tilde{E}(T)\right)\right).$$

To estimate the term I_4 , we start by computing using Lemma 3.3.1:

$$|\Delta_x^h(\tilde{\tau}^{-1}\tilde{r}^m)(\cdot, s)|_\infty \leq |D_x(\tilde{\tau}^{-1}\tilde{r}^m)(\cdot, s)|_\infty = |(-\tilde{\tau}^{-2}\tilde{r}^m D_x\tilde{\tau} + m\tilde{r}^{m-1})(\cdot, s)|_\infty \leq C(C_0 + T^{1/2}\tilde{E}(T)^{1/2}).$$

Hence by Lemma 3.4.6 we obtain:

$$\begin{aligned} I_4 &:= \kappa^2 \int_0^t \int_0^{k-h} |D_x^2 e \Delta_x^h(\tilde{\tau}^{-1}\tilde{r}^m)|^2 dx ds \leq C \int_0^t (C_0 + T\tilde{E}(T)) \|D_x^2 e\|_{L^2}^2 ds \\ &\leq CT\mathcal{P}(\tilde{E}(T)) \exp\left(CT\left(1 + T\tilde{E}(T) + T^2 \exp(T)\tilde{E}(T)\right)\right). \end{aligned}$$

Finally, we estimate the term I_5 . First, recalling the definition for G in (3.4.25), we compute $D_x G$ and take L^2 -norm on both sides. It follows from using Lemma 3.3.1 and Sobolev embedding inequality, and Theorem 3.4.2 that:

$$\|D_x G\|_{L^2}^2 \leq C \left\{ (1 + T\tilde{E}(T)) \|u\|_{H^2}^4 + \tilde{E}(T)(1 + T\tilde{E}(T)) \|u\|_{H^2}^2 \right\} \leq \mathcal{P}(\tilde{E}(T)) \exp(CT^2\tilde{E}(T)).$$

It follows that:

$$I_5 := \int_0^t \int_0^{k-h} |\Delta_x^h G|^2 dx ds \leq \int_0^t \|D_x G(\cdot, s)\|_{L^2}^2 ds \leq T\mathcal{P}(\tilde{E}(T)) \exp(CT^2\tilde{E}(T)).$$

Therefore substituting the estimates for I_1 – I_5 into (3.4.47), we get for all $T \in [0, T_1]$ with $T_1 \equiv \min\{T_*, T_0\}$

$$\int_0^T \|\mathbb{1}_{[0, k-h]} \Delta_x^h D_x^2 e\|_{L^2}^2 ds \leq C\mathcal{P}(E_0) + CT\mathcal{P}(\tilde{E}(T)) \exp\left(CT\left(1 + T\tilde{E}(T) + T^2 \exp(T)\tilde{E}(T)\right)\right),$$

where $\mathbb{1}_{[0, k-h]}$ denotes the indicator function on spatial coordinate with the set $x \in [0, k-h]$. Therefore there is a subsequence $h \searrow 0$ such that $\mathbb{1}_{[0, k-h]} \Delta_x^h D_x^2 e \rightharpoonup D_x^3 e$ in $L^2(0, T_1; L^2(0, k))$, where $D_x^3 e$ denotes the spatial weak derivative of $D_x^2 e \in L^\infty(0, T; H^2(0, k))$. Finally by the weakly lower semi-continuity, we obtain the main statement of this lemma. \square

Lemma 3.4.8. *Suppose the assumptions of Theorem 3.4.4 hold, and let $e(x, t)$ be the weak solution obtained in Theorem 3.4.3. Then we have $D_t^2 e \in L^2(0, T_1; H^*(0, k))$ with $T_1 \equiv \min\{T_*, T_0\}$, where $H^*(0, k)$ is the dual space of $H^1(0, k)$. In particular, there exists a constant $C = C(C_0, a, k, \gamma, \beta, n) > 0$ and a polynomial $z \mapsto \mathcal{P}(z)$ such that for each $T \in [0, \min\{T_*, T_0\}]$:*

$$\int_0^t \|D_t^2 e(\cdot, s)\|_{H^*}^2 ds \leq C\mathcal{P}(E_0) + CT\mathcal{P}(\tilde{E}(T)) \exp\left(CT\left(1 + T\tilde{E}(T)^{1/2} + T^2 \exp(T)\tilde{E}(T)^{1/2}\right)\right).$$

Proof. Denoting Δ_t^h as the difference quotient in the temporal coordinate defined in the proof of Lemma 3.4.1. Applying Δ_t^h on the energy equation obtained in Corollary 3.4.2, it follows that for almost everywhere $(x, t) \in [0, k] \times [0, T-h]$:

$$\Delta_t^h D_t e_N(x) = \kappa D_x \left(\frac{\tilde{r}^m}{\tilde{\tau}}(x, t+h) \Delta_t^h D_x e_N(x) \right) + \kappa D_x \left(D_x e_N(x, t) \Delta_t^h (\tilde{\tau}^{-1} \tilde{r}^m)(x) \right) + \Delta_t^h G(\cdot),$$

with G defined in (3.4.25). Letting $\phi(x) \in H^1(0, k)$ be a test function such that $\|\phi\|_{H^1} = 1$, then by boundary condition $D_x e(0, t) = D_x e(k, t) = 0$ from Corollary 3.4.2, we have

$$(\Delta_t^h D_t e, \phi)_{L^2} = -\kappa \left(\frac{\tilde{r}^m}{\tilde{\tau}}(\cdot, t+h) \Delta_t^h D_x e + D_x e(\cdot, t) \Delta_t^h (\tilde{\tau}^{-1} \tilde{r}^m), D_x \phi \right)_{L^2} + (\Delta_t^h G, \phi)_{L^2}.$$

Integrating the above in time, using Lemma 3.3.1, Cauchy-Schwartz inequality, Lemma 3.4.6, and the estimate $|\Delta_t^h (\tilde{r}^m \tilde{\tau}^{-1})|_\infty \leq C\tilde{E}(T)^{1/2}$ which is derived on (3.4.10) in the proof of Lemma 3.4.1, it implies

$$\begin{aligned} \int_0^{T-h} \|\Delta_t^h D_t e(\cdot, t)\|_{H^*}^2 dt &= \sup \left\{ \int_0^T (\Delta_t^h D_t e(\cdot, t), \phi(\cdot))_{L^2}^2 dt \mid \phi \in H^1(0, k), \|\phi\|_{H^1} = 1 \right\} \\ &\leq C\mathcal{P}(E_0) + T\mathcal{P}(\tilde{E}(T)) \exp\left(CT\left(1 + T\tilde{E}(T) + T^2 \exp(T)\tilde{E}(T)\right)\right) + \int_0^T (\Delta_t^h G(\cdot), \phi)_{L^2}^2 dt, \end{aligned} \quad (3.4.48)$$

Now we are left with estimating the term with $\Delta_t G$. Recalling the definition of G in (3.4.25), we write $(\Delta_t^h G, \phi)_{L^2} = \sum_{i=1}^3 I_i$, where

$$\begin{aligned} I_1 &:= \left(\beta |D_x(\tilde{r}^m u)|^2(\cdot, t+h) \Delta_t^h \left(\frac{1}{\tilde{r}^m \tilde{\tau}} \right), \phi \right)_{L^2} - \left((\gamma-1) \frac{D_x(\tilde{r}^m u)}{\tilde{r}^m \tilde{\tau}}(\cdot, t) \Delta_t^h \tilde{e}, \phi \right)_{L^2} \\ &\quad - \left((\gamma-1) \tilde{e}(\cdot, t+h) D_x(\tilde{r}^m u)(\cdot, t) \Delta_t^h \left(\frac{1}{\tilde{r}^m \tilde{\tau}} \right), \phi \right)_{L^2}, \\ I_2 &:= - \left((\gamma-1) \frac{\tilde{e}}{\tilde{r}^m \tilde{\tau}}(\cdot, t+h) D_x \Delta_t^h(\tilde{r}^m u), \phi \right)_{L^2} - \left(2m\mu D_x \Delta_t^h(\tilde{r}^{m-1} u^2), \phi \right), \\ I_3 &:= \left(\beta \frac{D_x(\tilde{r}^m u)}{\tilde{r}^m \tilde{\tau}}(\cdot, t) \Delta_t^h D_x(\tilde{r}^m u), \phi \right)_{L^2} + \left(\beta \frac{1}{\tilde{r}^m \tilde{\tau}}(\cdot, t) D_x(\tilde{r}^m u)(\cdot, t+h) \Delta_t^h D_x(\tilde{r}^m u), \phi \right)_{L^2}. \end{aligned}$$

First, I_1 is estimated as:

$$\begin{aligned} I_1^2 &\lesssim \left| \Delta_t^h \left(\frac{1}{\tilde{r}^m \tilde{\tau}} \right) \right|_{\infty}^2 \|D_x(\tilde{r}^m u)\|_{\infty}^2 \|D_x(\tilde{r}^m u)\|_{L^2}^2 + \left| \frac{D_x(\tilde{r}^m u)}{\tilde{r}^m \tilde{\tau}} \right|_{\infty}^2 \|\Delta_t^h \tilde{e}\|_{L^2}^2 + |\tilde{e} D_x(\tilde{r}^m u)|_{\infty}^2 \|\Delta_t^h(\tilde{r}^{-m} \tilde{\tau}^{-1})\|_{L^2}^2 \\ &\leq C \tilde{E}(T) \|u\|_{H^2}^2 \|u\|_{H^1}^2 + C \tilde{E}(T) \|u\|_{H^2}^2 + C \tilde{E}(T)^2 \|u\|_{H^2}^2, \end{aligned}$$

where we have used estimates $|\Delta_t^h(\tilde{r}^{-m} \tilde{\tau}^{-1})|_{\infty} \leq C \tilde{E}(T)^{1/2}$ from (3.4.33) in the proof of Lemma 3.4.5, estimates $|D_x(\tilde{r}^m u)|_{\infty} \leq C \|u\|_{H^2}$, $\|\Delta_t^h(\tilde{r}^{-m} \tilde{\tau}^{-1})\|_{L^2} \leq C \tilde{E}(T)^{1/2}$ from (3.4.43) in the proof of Lemma 3.4.6, and $\|\Delta_t^h \tilde{e}\|_{L^2} \leq \|D_t \tilde{e}\|_{L^2} \leq \tilde{E}(T)^{1/2}$ by Fundamental Theorem of Calculus and Jensen's inequality. Integrating in time and using Lemma 3.4.2, it then follows that:

$$\int_0^T I_1^2 dt \leq T \mathcal{P}(\tilde{E}(T)) \exp(CT^2 \tilde{E}(T)).$$

On the other hand, using the boundary condition that $u(0, t) = u(k, t) = 0$, Sobolev embedding theorem, relations (3.2.2)–(3.2.5), and Lemma 3.3.1, we obtain after integrating by parts that

$$\begin{aligned} I_2 &= (\gamma-1) \left(\frac{D_x \tilde{e}}{\tilde{r}^m \tilde{\tau}} \Delta_t^h(\tilde{r}^m u), \phi \right)_{L^2} + (\gamma-1) \left(\tilde{e}(-m\tilde{r}^{-n} - \tilde{r}^{-m} \tilde{\tau}^{-2} D_x \tilde{\tau}) \Delta_t^h(\tilde{r}^m u), \phi \right)_{L^2} \\ &\quad + (\gamma-1) \left(\frac{\tilde{e}}{\tilde{r}^m \tilde{\tau}} \Delta_t^h(\tilde{r}^m u), D_x \phi \right)_{L^2} + 2m\mu \left(\Delta_t^h(\tilde{r}^{m-1} u^2), D_x \phi \right)_{L^2} \\ &\leq C \|\tilde{e}\|_{H^2} \|\Delta_t^h(\tilde{r}^m u)\|_{L^2} + C \|\tilde{e}\|_{\infty} \|D_x \tilde{\tau}\|_{\infty} \|\Delta_t^h(\tilde{r}^m u)\|_{L^2} + C \|\Delta_t^h(\tilde{r}^{m-1} u^2)\|_{L^2} \end{aligned}$$

Now using Jensen's inequality and relations (3.2.2)–(3.2.5) one can derive the following inequalities:

$$\begin{aligned} \|\Delta_t^h(\tilde{r}^m u)\|_{L^2} &\leq \sup_{0 \leq t \leq T} \|D_t(\tilde{r}^m u)(\cdot, t)\|_{L^2} \leq C \sup_{0 \leq t \leq T} (\tilde{E}(T)^{1/2} \|u\|_{L^2} + \|D_t u\|_{L^2}), \\ \|\Delta_t^h(\tilde{r}^{m-1} u^2)\|_{L^2} &\leq C \sup_{0 \leq t \leq T} (\tilde{E}(T)^{1/2} \|u\|_{H^1} \|u\|_{L^2} + \|u\|_{H^1} \|D_t u\|_{L^2}), \\ \|\Delta_t^h D_x(\tilde{r}^m u)\|_{L^2} &\leq \|D_t D_x(\tilde{r}^m u)(\cdot, t)\|_{L^2} \lesssim \tilde{E}(T)^{1/2} \|u\|_{H^1} + \|D_t u\|_{H^1}, \\ |D_x(\tilde{r}^m u)|_{\infty} &\leq |\tilde{r}^m D_x u + m\tilde{r}^{m-1} \tilde{\tau} u|_{\infty} \leq C(|D_x u|_{\infty} + |u|_{\infty}) \leq C \|u\|_{H^2}, \\ |D_x \tilde{\tau}|_{\infty} &\leq C \|D_x \tau_0\|_{H^1} + Ct^{1/2} \left(\int_0^t \|D_x^2 \tilde{u}(\cdot, s)\|_{H^1}^2 ds \right)^{1/2} \leq CE_0^{1/2} + CT^{1/2} \tilde{E}(T)^{1/2}, \\ |\tilde{e}|_{\infty} &= |e_0(\cdot) + \int_0^t D_t \tilde{e}(\cdot, s) ds|_{\infty} \lesssim \|e_0\|_{H^1} + t^{1/2} \left(\int_0^t \|D_t \tilde{e}\|_{H^1}^2 ds \right)^{1/2} \leq E_0^{1/2} + T^{1/2} \tilde{E}(T)^{1/2}. \end{aligned} \tag{3.4.49}$$

Summarising these, and applying Theorem 3.4.2 on the terms involving u , we have that:

$$\int_0^{T-h} I_2^2(t) dt \leq CT \mathcal{P}(\tilde{E}(T)) \exp(CT^2 \tilde{E}(T)).$$

Finally, we estimate I_3 . Only one of the term in I_3 is considered in the following analysis as the other term differs by a time translation of $h > 0$. First, using (3.4.49), we have the following estimate:

$$\begin{aligned} \int_0^{T-h} I_3^2 dt &= \int_0^{T-h} \left(\frac{D_x(\tilde{r}^m u)}{\tilde{r}^m \tilde{\tau}} \Delta_t^h D_x(\tilde{r}^m u), \phi \right)_{L^2}^2 dt \leq C \int_0^{T-h} |D_x(\tilde{r}^m u)(\cdot, t)|_\infty^2 \|\Delta_t^h D_x(\tilde{r}^m u)\|_{L^2}^2 dt \\ &\leq CT \tilde{E}(T) \sup_{0 \leq s \leq T} \|u(\cdot, s)\|_{H^2}^2 \cdot \|u(\cdot, s)\|_{H^1}^2 + C \sup_{0 \leq s \leq T} \|u(\cdot, s)\|_{H^2}^2 \int_0^{T-h} \|D_t u(\cdot, t)\|_{H^1}^2 dt \end{aligned}$$

Next, applying Theorem 3.4.2 for the terms involving u , it follows that

$$\int_0^{T-h} I_3^2 dt \leq C\mathcal{P}(E_0) + CTP(\tilde{E}(T)) \exp(CT^2 \tilde{E}(T)).$$

In conclusion, we have

$$\int_0^{T-h} (\Delta_t^h G, \phi)_{L^2}^2 dt \leq 4 \sum_{i=1}^3 \int_0^{T-h} I_i^2 dt \leq C\mathcal{P}(E_0) + CTP(\tilde{E}(T)) \exp(CT^2 \tilde{E}(T)).$$

Substituting this into (3.4.48), we have for all $T \in [0, T_1]$ with $T_1 \equiv \min\{T_0, T_*\}$:

$$\int_0^{T-h} \|\Delta_t^h D_t e(\cdot, t)\|_{H^*}^2 dt \leq C\mathcal{P}(E_0) + TP(\tilde{E}(T)) \exp\left(CT\left(1 + T\tilde{E}(T) + T^2 \exp(T)\tilde{E}(T)\right)\right).$$

Therefore there is a subsequence $h \searrow 0$ such that $\mathbb{1}_{\{0 \leq t \leq T-h\}} \Delta_t^h D_t e \rightharpoonup D_t^2 e$ in $L^2(0, T_1; H^*(0, k))$, where $D_t^2 e$ denotes the temporal weak derivative of $D_t e \in L^\infty(0, T; L^2(0, k))$. Finally by the weakly lower semi-continuity, we conclude that $D_t^2 e$ satisfies the main estimate of this lemma. \square

3.4.3 Invariance of energy space under solution operator

Let $(\tilde{u}, \tilde{e}) \equiv (u_j, e_j)$ for some $j \in \mathbb{N} \cup \{0\}$, we denote $(u, e) \equiv (u_{j+1}, e_{j+1})$ as its next iterative step linear solution obtained in Sections 3.4.1–3.4.2. Our aim in this subsection is to show that there exists a time and a constant:

$$T_* = T_*(C_0, a, k, \beta, \gamma, n) > 0 \quad \text{and} \quad M_0 = M_0(C_0, a, k, \beta, \gamma, n) > 0,$$

depending only on the parameters $n \in \{2, 3\}$, $a \in (0, 1)$, $k \in \mathbb{N}$, $\beta > 0$, $\gamma > 0$ and the initial data $C_0 > 0$ such that if $E[\tilde{u}, \tilde{e}](T_*) \leq M_0$, then $E[u, e](T_*) \leq M_0$. In other words, the following theorem:

Theorem 3.4.5. *For given constant $T > 0$ and $M > 0$, we define the space:*

$$\begin{aligned} X_{M,T} := & \left\{ (w, \theta) \in L^\infty(0, T; H^2(0, k)) : (w, \theta)(x, 0) = (u_0, e_0)(x) \quad \forall x \in [0, k], \right. \\ & \left. (w, D_x \theta)(0, t) = (w, D_x \theta)(k, t) = (0, 0) \quad \forall t \in [0, T], \quad \text{and} \quad E[w, \theta](T) \leq M. \right\}, \end{aligned}$$

then there exists $T_* = T_*(C_0, a, k, \beta, \gamma, n) > 0$ and $M_0 = M_0(C_0, a, k, \beta, \gamma, n) > 0$ such that

$$S(X_{M_0, T_*}) \subseteq X_{M_0, T_*},$$

where $S((\tilde{u}, \tilde{e})) = (u, e)$ is the solution operator obtained in the previous Sections 3.4.1–3.4.2.

Proof. We prove by induction principle that such $T_* > 0$ and $M_0 > 0$ exists and they are given by:

$$\begin{cases} M_0 := 2\mathcal{P}(E_0), \\ T_* := \min \left\{ T_0, \frac{\mathcal{P}(E_0)}{C \exp(1)\mathcal{P}(2\mathcal{P}(E_0))}, \frac{1}{3C}, \frac{1}{(6C\mathcal{P}(E_0))^{1/2}}, \frac{1}{(6C \exp(1)\mathcal{P}(E_0))^{1/3}} \right\}, \end{cases} \quad (3.4.50)$$

where $T_0 = T_0(M_0) = T_0(2\mathcal{P}(E_0))$ is defined in Proposition 3.3.1 as:

$$T_0 = T(M_0) = T(2\mathcal{P}(E_0)) := \min \left\{ 1, \frac{\bar{\tau}_0}{2C(2\mathcal{P}(E_0))^{1/2}}, \frac{\underline{\tau}_0}{2C(2\mathcal{P}(E_0))^{1/2}} \right\},$$

where $\underline{\tau}_0 := \inf_{x \in [0, k]} \tau_0(x)$, $\bar{\tau}_0 := \sup_{x \in [0, k]} \tau_0(x)$, $C > 0$ is some generic constant from the Sobolev embedding inequality, and $z \mapsto \mathcal{P}(z)$ is the polynomial obtained in Theorems 3.4.2 and 3.4.4.

First for the base case we consider $(u^{(0)}, e^{(0)})$ to be the solution to the parabolic system:

$$\begin{cases} D_t u^{(0)} - \beta D_x^2 u^{(0)} = 0 \\ D_t e^{(0)} - \kappa D_x^2 e^{(0)} = 0 \\ (u^{(0)}, D_x e^{(0)})(k, t) = (u^{(0)}, D_x e^{(0)})(0, t) = (0, 0) \text{ for each } t \in [0, \infty), \\ (u^{(0)}, e^{(0)})(x, 0) = (u_0, e_0)(x) \text{ for each } x \in [0, k]. \end{cases} \quad \text{in } (x, t) \in [0, k] \times [0, \infty),$$

Then by the linear parabolic theory (refer to Chapter 7 of [5]), it follows that $E[u^{(0)}, e^{(0)}](T) \leq E_0 \equiv E[u_0, e_0]$ for all $T > 0$, and one can define the coefficients $r^{(0)}(x, t)$ and $\tau^{(0)}(x, t)$ for the next iterative solution $(u^{(1)}, e^{(1)})$ via

$$\begin{cases} \tau^{(0)}(x, t) := \tau_0(x) + \int_0^t D_x u^{(0)}(x, s) ds, \\ r^{(0)}(x, t) := a + \int_0^x \tau^{(0)}(y, t) dy. \end{cases}$$

Next we prove the induction step. Assume For $T_* > 0$ and $M_0 > 0$ defined in (3.4.50), we have $\tilde{E}(T_*) \equiv E[\tilde{u}, \tilde{e}](T_*) \leq M_0$. Then from Theorems 3.4.2 and 3.4.2, we have the following statement: if there exists $T_* > 0$, such that $\tilde{E}(T_*) \equiv E[\tilde{u}, \tilde{e}](T_*) \leq M_0$, then for all $T \in [0, \min\{T_0, T_*\}] = [0, T_*]$, the following estimate holds:

$$E[u, e](T) \leq \mathcal{P}(E_0) + CTP(\tilde{E}(T)) \exp \left(CT \left(1 + T\tilde{E}(T) + T^2 \exp(T)\tilde{E}(T) \right) \right),$$

where $z \mapsto \mathcal{P}(z)$ is a given polynomial and a constant $C = C(C_0, a, k, \beta, \gamma, n) > 0$ independent of T_* and M_0 . Now by the construction (3.4.50) of $M_0 > 0$ and $T_* > 0$, it follows that for each $T \in [0, T_*]$:

$$\begin{aligned} E[u, e](T) &\leq \mathcal{P}(E_0) + CTP(2\mathcal{P}(E_0)) \exp \left(\frac{C}{3C} + \frac{C \cdot (2\mathcal{P}(E_0))}{6C\mathcal{P}(E_0)} + \frac{C \exp(1) \cdot (2\mathcal{P}(E_0))}{6C \exp(1)\mathcal{P}(E_0)} \right) \\ &\leq \mathcal{P}(E_0) + CTP(2\mathcal{P}(E_0)) \exp(1) \\ &= \mathcal{P}(E_0) + C \exp(1)\mathcal{P}(2\mathcal{P}(E_0)) \cdot \frac{\mathcal{P}(E_0)}{C \exp(1)\mathcal{P}(2\mathcal{P}(E_0))} = 2\mathcal{P}(E_0) \equiv M_0. \end{aligned}$$

In other words, $(\tilde{u}, \tilde{e}) \in X_{M, T_*}$ implies $(u, e) \in X_{M, T_*}$. Therefore, with such choice of $M > 0$ and $T_* > 0$, the energy space X_{M, T_*} is invariant under the solution operator. \square

Corollary 3.4.3. *Let $T_* > 0$ and $M_0 > 0$ be the constant in Theorem 3.4.5. Moreover for $l \in \mathbb{N}$, we also denote $(u_l, e_l)(x, t)$ as the l -th solution and*

$$r_l(x, t) = a + \int_0^x \pi_l(y, t) dy, \quad \pi_l(x, t) = \tau_0(x) + \int_0^t D_x u_l(x, s) ds,$$

as the l -th coefficients. Then we have, for all $l \in \mathbb{N}$:

$$\begin{cases} \frac{1}{2}\underline{\tau}_0 \leq \tau_l(x, t) \leq \frac{3}{2}\bar{\tau}_0 \quad \text{and} \quad a \leq r_l(x, t) \leq a + \frac{3k}{2}\bar{\tau}_0 \\ r_l(x, t) = r_0(x) + \int_0^t u_l(x, t) dx \quad \text{where} \quad r_0(x) = a + \int_0^x \tau_0(y) dy \\ E[u_l, e_l](t) \leq M_0 \end{cases} \quad \begin{array}{l} \text{for all } (x, t) \in [0, k] \times [0, T_*], \\ \\ \text{for all } t \in [0, T_*]. \end{array}$$

Moreover, for each $i, j \in \mathbb{N}$, we also have:

$$\begin{cases} \sup_{0 \leq s \leq t} \int_0^k |\tau_i - \tau_j|^2(x, s) dx \leq t \int_0^t \|D_x(u_i - u_j)\|_{L^2}^2(s) ds, \\ \sup_{0 \leq s \leq t} |(r_i - r_j)(\cdot, s)|_\infty^2 \leq Ct \int_0^t \|u_i - u_j\|_{H^1}^2(s) ds, \end{cases}$$

where $C > 0$ is the constant from the Sobolev embedding $H^1 \hookrightarrow C^0$.

3.5 Contraction Estimates

In this section, we derive the contraction estimate. For convenience of notation, we denote $M \equiv M_0$ in the following analysis. The main theorem of this section is given as following:

Theorem 3.5.1. *Let $T_* > 0$ be the time obtained in Theorem 3.4.5. For each $t > 0$, we define the distance function to be:*

$$d_t(u, e; \tilde{u}, \tilde{e}) := \sqrt{\sup_{0 \leq s \leq t} \|(u - \tilde{u}, e - \tilde{e})(\cdot, s)\|_{L^2(0, k)}^2 + \int_0^t \|(u - \tilde{u}, e - \tilde{e})(\cdot, s)\|_{H^1(0, k)}^2 ds},$$

and we also denote $S : X_{M_0, T_*} \rightarrow X_{M_0, T_*}$ as the iterative solution operator obtained in the previous sections. Then, there exists a time $0 < T \leq T_*$ and a constant $0 < \alpha_T < 1$ such that for all $t \in [0, T]$:

$$d_t(S(u_i, e_i); S(u_j, e_j)) \leq \alpha_T d_t(u_i, e_i; u_j, e_j) \quad \text{for all } (u_i, e_i), (u_j, e_j) \in X_{M_0, T_*}.$$

In what follows we will denote for simplicity, $(u_i, e_i) \equiv S(\tilde{u}_i, \tilde{e}_i)$ and

$$\tilde{r}_i(x, t) \equiv a + \int_0^x \tilde{\tau}_i(y, t) dy, \quad \tilde{\tau}_i(x, t) \equiv \tau_0(x) + \int_0^t D_x \tilde{u}_i(x, s) ds \quad \text{where} \quad \tau_0(x) = \frac{v_0}{r_0^m}(x),$$

for each given $(\tilde{u}_i, \tilde{e}_i) \in X_{M_0, T_*}$.

3.5.1 Contraction inequality for u

Lemma 3.5.1. *Let $T_* > 0$ be the time obtained in Theorem 3.4.5, then we have the following estimate:*

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_0^k |u_i - u_j|^2 dx + \frac{\beta}{2} \int_0^k \left(\frac{\tilde{r}_j^m}{\tilde{\tau}_j} |D_x(u_i - u_j)|^2 + \tilde{r}_j^{m-2} \tilde{\tau}_j |u_i - u_j|^2 \right) dx \\ & \leq CMt \int_0^t \|\tilde{u}_i - \tilde{u}_j\|_{H^1}^2 ds + C \|\tilde{e}_i - \tilde{e}_j\|_{L^2}^2, \end{aligned}$$

for all $t \in [0, T_*]$, where $C = C(C_0, a, k, \beta, \gamma, n) > 0$ is a constant independent of time or iterative index $i, j \in \mathbb{N}$

Proof. From Corollary 3.4.1 in the previous section, we had:

$$D_t u_l - \beta D_x \left(\frac{\tilde{r}_l^m}{\tilde{\tau}_l} D_x u_l \right) + \beta \tilde{r}_l^{m-2} \tilde{\tau}_l u_l = (\gamma - 1) \left(m \frac{\tilde{e}_l}{\tilde{\tau}_l} - D_x \left(\frac{\tilde{e}_l}{\tilde{\tau}_l} \right) \right) \quad \text{for } l = i \text{ or } j.$$

Taking the difference of two equations, we have:

$$\begin{aligned} & D_t(u_i - u_j) - \beta D_x \left(\frac{\tilde{r}_j^m}{\tilde{\tau}_j} D_x(u_i - u_j) \right) + \beta \tilde{r}_j^{m-2} \tilde{\tau}_j (u_i - u_j) \\ &= \beta D_x \left(\frac{D_x u_i}{\tilde{\tau}_i} (\tilde{r}_i^m - \tilde{r}_j^m) \right) + \beta D_x \left(\frac{\tilde{r}_j^m D_x u_i}{\tilde{\tau}_i \tilde{\tau}_j} (\tilde{\tau}_j - \tilde{\tau}_i) \right) - \beta \tilde{\tau}_i u_i (\tilde{r}_i^{m-2} - \tilde{r}_j^{m-2}) - \beta \tilde{r}_j^{m-2} u_i (\tilde{\tau}_i - \tilde{\tau}_j) \\ & \quad + m(\gamma - 1) \frac{\tilde{e}_i - \tilde{e}_j}{\tilde{r}_i} + m(\gamma - 1) \frac{\tilde{e}_j}{\tilde{r}_i \tilde{r}_j} (\tilde{r}_j - \tilde{r}_i) - (\gamma - 1) D_x \left(\frac{\tilde{e}_i - \tilde{e}_j}{\tilde{\tau}_i} \right) - (\gamma - 1) D_x \left(\frac{\tilde{e}_j}{\tilde{\tau}_i \tilde{\tau}_j} (\tilde{\tau}_j - \tilde{\tau}_i) \right). \end{aligned}$$

Now multiplying both sides of above equation by $u_i - u_j$, integrating in spatial coordinate and using the boundary condition of u_i and u_j , we have

$$\frac{1}{2} \frac{d}{dt} \int_0^k |u_i - u_j|^2 dx + \beta \int_0^k \left(\frac{\tilde{r}_j^m}{\tilde{\tau}_j} |D_x(u_i - u_j)|^2 + \tilde{r}_j^{m-2} \tilde{\tau}_j |u_i - u_j|^2 \right) dx = \sum_{i=1}^8 I_i.$$

Next, using Theorem 3.4.5 and Corollary 3.4.3, we estimate I_i for $i = 1$ to $i = 8$ as follows:

$$\begin{aligned} I_1 &:= -\beta \int_0^k \frac{D_x u_i}{\tilde{\tau}_i} (\tilde{r}_i^m - \tilde{r}_j^m) D_x(u_i - u_j) dx \leq \frac{\beta}{16} \int_0^k \frac{\tilde{r}_j^m}{\tilde{\tau}_j} |D_x(u_i - u_j)|^2 dx + CMt \int_0^t \|\tilde{u}_i - \tilde{u}_j\|_{H^1}^2 ds, \\ I_2 &:= -\beta \int_0^k \frac{\tilde{r}_j^m D_x u_i}{\tilde{\tau}_i \tilde{\tau}_j} (\tilde{\tau}_j - \tilde{\tau}_i) D_x(u_i - u_j) dx \leq \frac{\beta}{16} \int_0^k \frac{\tilde{r}_j^m}{\tilde{\tau}_j} |D_x(u_i - u_j)|^2 dx + CMt \int_0^t \|D_x(\tilde{u}_i - \tilde{u}_j)\|_{L^2}^2 ds, \\ I_3 &:= -\beta \int_0^k \tilde{\tau}_i u_i (\tilde{r}_i^{m-2} - \tilde{r}_j^{m-2}) (u_i - u_j) dx \leq \frac{\beta}{16} \int_0^k \tilde{\tau}_j \tilde{r}_j^{m-2} |u_i - u_j|^2 dx + CMt \int_0^t \|\tilde{u}_i - \tilde{u}_j\|_{H^1}^2 ds, \\ I_4 &:= -\beta \int_0^k \tilde{r}_j^{m-2} u_i (\tilde{\tau}_i - \tilde{\tau}_j) (u_i - u_j) dx \leq \frac{\beta}{16} \int_0^k \tilde{\tau}_j \tilde{r}_j^{m-2} |u_i - u_j|^2 dx + CMt \int_0^t \|D_x(\tilde{u}_i - \tilde{u}_j)\|_{L^2}^2 ds, \\ I_5 &:= m(\gamma - 1) \int_0^k \frac{\tilde{e}_i - \tilde{e}_j}{\tilde{r}_i} (u_i - u_j) dx \leq \frac{\beta}{16} \int_0^k \tilde{\tau}_j \tilde{r}_j^{m-2} |u_i - u_j|^2 dx + C \|\tilde{e}_i - \tilde{e}_j\|_{L^2}^2, \\ I_6 &:= m(\gamma - 1) \int_0^k \frac{\tilde{e}_j}{\tilde{r}_i \tilde{r}_j} (\tilde{r}_j - \tilde{r}_i) (u_i - u_j) dx \leq \frac{\beta}{16} \int_0^k \tilde{\tau}_j \tilde{r}_j^{m-2} |u_i - u_j|^2 dx + CMt \int_0^t \|\tilde{u}_i - \tilde{u}_j\|_{H^1}^2 dx, \\ I_7 &:= (\gamma - 1) \int_0^k \frac{\tilde{e}_i - \tilde{e}_j}{\tilde{\tau}_i} D_x(u_i - u_j) dx \leq \frac{\beta}{16} \int_0^k \frac{\tilde{r}_j^m}{\tilde{\tau}_j} |D_x(u_i - u_j)|^2 dx + C \|\tilde{e}_i - \tilde{e}_j\|_{L^2}^2, \\ I_8 &:= (\gamma - 1) \int_0^k \frac{\tilde{e}_j}{\tilde{\tau}_i \tilde{\tau}_j} (\tilde{\tau}_j - \tilde{\tau}_i) D_x(u_i - u_j) dx \leq \frac{\beta}{16} \int_0^k \frac{\tilde{r}_j^m}{\tilde{\tau}_j} |D_x(u_i - u_j)|^2 dx + CMt \int_0^t \|D_x(\tilde{u}_i - \tilde{u}_j)\|_{L^2}^2 ds. \end{aligned}$$

Summarising all above estimates, we have that:

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_0^k |u_i - u_j|^2 dx + \frac{\beta}{2} \int_0^k \left(\frac{\tilde{r}_j^m}{\tilde{\tau}_j} |D_x(u_i - u_j)|^2 + \tilde{r}_j^{m-2} \tilde{\tau}_j |u_i - u_j|^2 \right) dx \\ & \leq CMt \int_0^t \|\tilde{u}_i - \tilde{u}_j\|_{H^1}^2 ds + C \|\tilde{e}_i - \tilde{e}_j\|_{L^2}^2. \end{aligned}$$

□

3.5.2 Contraction inequality for e

Lemma 3.5.2. *Let $T_* > 0$ be the time obtained in Theorem 3.4.5, then we have the following estimate:*

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_0^k |e_i - e_j|^2 dx + \frac{\kappa}{2} \int_0^k \frac{\tilde{r}_j^m}{\tilde{\tau}_j} |D_x(e_i - e_j)|^2 dx \\ & \leq \frac{\beta}{4} \int_0^k \frac{\tilde{r}_j^m}{\tilde{\tau}_j} |D_x(u_i - u_j)|^2 dx + CM_0 \{ \|u_i - u_j\|_{L^2}^2 + \|e_i - e_j\|_{L^2}^2 \} \\ & \quad + CM_0^2 t \int_0^t \|\tilde{u}_i - \tilde{u}_j\|_{H^1}^2 ds + CM_0 \|\tilde{e}_i - \tilde{e}_j\|_{L^2}^2, \end{aligned}$$

for all $t \in [0, T_*]$, where $C = C(C_0, a, k, \beta, \gamma, n) > 0$ is a constant independent of $\tilde{u}_i, \tilde{u}_j \in X_{M_0, T_*}$.

Proof. From the previous section we had:

$$D_t e_i - \kappa D_x \left(\frac{\tilde{r}_i^m}{\tilde{\tau}_i} D_x e_i \right) = G_i \quad \text{and} \quad D_t e_j - \kappa D_x \left(\frac{\tilde{r}_j^m}{\tilde{\tau}_j} D_x e_j \right) = G_j,$$

where G_i is defined by:

$$G_i := \beta \frac{|D_x(\tilde{r}_i^m u_i)|^2}{\tilde{r}_i^m \tilde{\tau}_i} - (\gamma - 1) \frac{D_x(\tilde{r}_i^m u_i)}{\tilde{r}_i^m \tilde{\tau}_i} \tilde{e}_i - 2\mu m D_x(\tilde{r}_i^{m-1} u_i^2),$$

and likewise for G_j . Taking the difference of above equations, we have:

$$D_t(e_i - e_j) - \kappa D_x \left(\frac{\tilde{r}_j^m}{\tilde{\tau}_j} D_x(e_i - e_j) \right) = \kappa D_x \left(\frac{D_x e_i}{\tilde{\tau}_i} (\tilde{r}_i^m - \tilde{r}_j^m) \right) + \kappa D_x \left(\frac{\tilde{r}_j^m D_x e_i}{\tilde{\tau}_i \tilde{\tau}_j} (\tilde{\tau}_j - \tilde{\tau}_i) \right) + (G_i - G_j).$$

Multiplying both sides by $(e_i - e_j)$, integrating in space, using Cauchy-Schwartz inequality and the boundary condition of $D_x e_l$ and $D_x e_{l-1}$, we get:

$$\frac{1}{2} \frac{d}{dt} \|e_i - e_j\|_{L^2}^2 + \frac{3\kappa}{4} \int_0^k \frac{\tilde{r}_j^m}{\tilde{\tau}_j} |D_x(e_i - e_j)|^2 dx \leq CM_0 t \int_0^t \|\tilde{u}_i - \tilde{u}_j\|_{H^1}^2 ds + (G_i - G_j, e_i - e_j)_{L^2}, \quad (3.5.1)$$

where in the last line, we used Theorem 3.4.5 and Corollary 3.4.3. Next, we write Computing the term $G_i - G_j = \text{I} + \text{II} + \text{III}$, where

$$\begin{aligned} \text{I} & := \beta \left\{ \frac{|D_x u_i|^2}{\tilde{\tau}_i} (\tilde{r}_i^m - \tilde{r}_j^m) + \frac{\tilde{r}_j^m |D_x u_i|^2}{\tilde{\tau}_i \tilde{\tau}_j} (\tilde{\tau}_j - \tilde{\tau}_i) + \frac{\tilde{r}_j^m}{\tilde{\tau}_j} (D_x u_i + D_x u_j) (D_x u_i - D_x u_j) \right\} \\ & \quad + 2m\beta \left\{ u_i D_x u_i (\tilde{r}_i^{m-1} - \tilde{r}_j^{m-1}) + \tilde{r}_j^{m-1} D_x u_i (u_i - u_j) + \tilde{r}_j^{m-1} u_j D_x (u_i - u_j) \right\} \\ & \quad + m^2 \beta \left\{ \tilde{\tau}_i u_i^2 (\tilde{r}_i^{m-2} - \tilde{r}_j^{m-2}) + \tilde{r}_j^{m-2} u_i^2 (\tilde{\tau}_i - \tilde{\tau}_j) + \tilde{r}_j^{m-2} \tilde{\tau}_j (u_i + u_j) (u_i - u_j) \right\}, \\ \text{II} & := -(\gamma - 1) \left\{ \frac{D_x u_i}{\tilde{\tau}_i} + m \frac{u_i}{\tilde{r}_i} \right\} (\tilde{e}_i - \tilde{e}_j) + (\gamma - 1) \frac{\tilde{e}_j u_i}{\tilde{r}_i \tilde{\tau}_j} (\tilde{r}_j - \tilde{r}_i) - (\gamma - 1) \frac{\tilde{e}_j D_x u_i}{\tilde{\tau}_i \tilde{\tau}_j} (\tilde{\tau}_j - \tilde{\tau}_i) \\ & \quad + (\gamma - 1) \frac{\tilde{e}_j}{\tilde{r}_j} (u_i - u_j) - \frac{\tilde{e}_j}{\tilde{\tau}_j} D_x (u_i - u_j), \\ \text{III} & := -2m\mu D_x \left((\tilde{r}_i^{m-1} - \tilde{r}_j^{m-1}) u_i^2 \right) - 2m\mu D_x \left(\tilde{r}_j^{m-1} (u_i + u_j) (u_i - u_j) \right). \end{aligned}$$

By Theorem 3.4.5, Corollary 3.4.3 and integrating by parts, we have the estimates:

$$\begin{aligned}
\int_0^k \text{I}(e_i - e_j) dx &\leq \frac{\beta}{8} \int_0^k \frac{\tilde{r}_j^m}{\tilde{\tau}_j} |D_x(u_i - u_j)|^2 dx + CM_0 \|u_i - u_j\|_{L^2}^2 + \\
&\quad + CM_0^2 t \int_0^t \|\tilde{u}_i - \tilde{u}_j\|_{H^1}^2 ds + CM_0 \|e_i - e_j\|_{L^2}^2, \\
\int_0^k \text{II}(e_l - e_{l-1}) dx &\leq \frac{\beta}{8} \int_0^k \frac{\tilde{r}_j^m}{\tilde{\tau}_j} |D_x(u_i - u_j)|^2 dx + CM_0 \|\tilde{e}_i - \tilde{e}_j\|_{L^2}^2 + CM_0 \|e_i - e_j\|_{L^2}^2 \\
&\quad + CM_0^2 t \int_0^t \|\tilde{u}_i - \tilde{u}_j\|_{H^1}^2 ds + CM_0 \|u_i - u_j\|_{L^2}^2, \\
\int_0^k \text{III}(e_i - e_j) dx &\leq \frac{\kappa}{4} \int_0^k \frac{\tilde{r}_j^m}{\tilde{\tau}_j} |D_x(e_i - e_j)|^2 dx + CM_0^2 t \int_0^t \|\tilde{u}_i - \tilde{u}_j\|_{H^1}^2 ds + CM_0 \|u_i - u_j\|_{L^2}^2.
\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}
\int_0^k (G_i - G_j)(e_i - e_j) dx &= \int_0^k (\text{I} + \text{II} + \text{III})(e_i - e_j) dx \\
&\leq \frac{\beta}{4} \int_0^k \frac{\tilde{r}_j^m}{\tilde{\tau}_j} |D_x(u_i - u_j)|^2 dx + \frac{\kappa}{4} \int_0^k \frac{\tilde{r}_j^m}{\tilde{\tau}_j} |D_x(e_i - e_j)|^2 dx + CM_0 \{ \|u_i - u_j\|_{L^2}^2 + \|e_i - e_j\|_{L^2}^2 \} \\
&\quad + CM_0^2 t \int_0^t \|\tilde{u}_i - \tilde{u}_j\|_{H^1}^2 ds + CM_0 \|\tilde{e}_i - \tilde{e}_j\|_{L^2}^2.
\end{aligned}$$

Substituting this back into (3.5.1), we have

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int_0^k |e_i - e_j|^2 dx + \frac{\kappa}{2} \int_0^k \frac{\tilde{r}_j^m}{\tilde{\tau}_j} |D_x(e_i - e_j)|^2 dx \\
&\leq \frac{\beta}{4} \int_0^k \frac{\tilde{r}_j^m}{\tilde{\tau}_j} |D_x(u_i - u_j)|^2 dx + CM_0 \{ \|u_i - u_j\|_{L^2}^2 + \|e_i - e_j\|_{L^2}^2 \} \\
&\quad + CM_0^2 t \int_0^t \|\tilde{u}_i - \tilde{u}_j\|_{H^1}^2 ds + CM_0 \|\tilde{e}_i - \tilde{e}_j\|_{L^2}^2.
\end{aligned}$$

□

3.5.3 Proof for contraction estimate, Theorem 3.5.1

Proof. Combining Lemma 3.5.1 and 3.5.2, we have that

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \{ \|u_i - u_j\|_{L^2}^2(t) + \|e_i - e_j\|_{L^2}^2(t) \} \\
&\quad + \frac{\beta}{4} \int_0^k \left\{ \frac{\tilde{r}_j^m}{\tilde{\tau}_j} |D_x(u_i - u_j)|^2 + \tilde{r}_j^{m-2} \tilde{\tau}_j |u_i - u_j|^2 \right\} dx + \frac{\kappa}{2} \int_0^k \frac{\tilde{r}_j^m}{\tilde{\tau}_j} |D_x(e_i - e_j)|^2 dx \\
&\leq CM_0 \{ \|u_i - u_j\|_{L^2}^2 + \|e_i - e_j\|_{L^2}^2 \} + CM_0^2 t \int_0^t \|\tilde{u}_i - \tilde{u}_j\|_{H^1}^2 ds + CM \|\tilde{e}_i - \tilde{e}_j\|_{L^2}^2.
\end{aligned}$$

Using Lemma 3.3.1 and Proposition 3.3.1, it follows that:

$$\begin{aligned}
&\frac{d}{dt} \{ \|u_i - u_j\|_{L^2}^2 + \|e_i - e_j\|_{L^2}^2 \}(t) + \|u_i - u_j(\cdot, t)\|_{H^1}^2 + \|D_x(e_i - e_j)(\cdot, t)\|_{L^2}^2 \\
&\leq CM_0 \{ \|u_i - u_j\|_{L^2}^2 + \|e_i - e_j\|_{L^2}^2 \} + CM_0^2 t \int_0^t \|\tilde{u}_i - \tilde{u}_j\|_{H^1}^2 ds + CM \|\tilde{e}_i - \tilde{e}_j\|_{L^2}^2.
\end{aligned} \tag{3.5.2}$$

By Grönwall inequality and the initial data: $u_l(x, 0) = u_{l-1}(x, 0) = u_0(x)$ and $e_l(x, 0) = e_{l-1}(x, 0) = e_0(x)$, it implies that

$$\begin{aligned} & \sup_{0 \leq s \leq t} \left\{ \| (u_i - u_j)(\cdot, s) \|_{L^2}^2 + \| (e_i - e_j)(\cdot, s) \|_{L^2}^2 \right\} \\ & \leq \left\{ CM_0 t \sup_{0 \leq s \leq t} \| (\tilde{e}_i - \tilde{e}_j)(\cdot, s) \|_{L^2}^2 + CM_0^2 t^2 \int_0^t \| \tilde{u}_i - \tilde{u}_j \|_{H^1}^2 ds \right\} \exp(CMt). \end{aligned} \quad (3.5.3)$$

Now integrating (3.5.2) in time, substituting (3.5.3) and taking supremum in time, we get that:

$$\begin{aligned} & \sup_{0 \leq s \leq t} \left\{ \| u_i - u_j \|_{L^2}^2 + \| e_i - e_j \|_{L^2}^2 \right\}(s) + \int_0^t \left\{ \| u_i - u_j \|_{H^1}^2 + \| D_x(e_i - e_j) \|_{L^2}^2 \right\} ds \\ & \leq C \exp(CM_0 t) \left\{ M_0^2 t^2 \sup_{0 \leq s \leq t} \| (\tilde{e}_i - \tilde{e}_j)(\cdot, s) \|_{L^2}^2 + (M_0^2 t^2 + M_0^3 t^3) \int_0^t \| \tilde{u}_i - \tilde{u}_j \|_{H^1}^2 ds \right\}. \end{aligned}$$

In addition it also follows that

$$\begin{aligned} & \int_0^t \| e_i - e_j \|_{L^2}^2(s) ds \leq t \sup_{0 \leq s \leq t} \| (e_i - e_j)(\cdot, s) \|_{L^2}^2 \\ & \leq C \exp(CM_0 t) \left\{ M_0^2 t^3 \sup_{0 \leq s \leq t} \| (\tilde{e}_i - \tilde{e}_j)(\cdot, s) \|_{L^2}^2 + (M_0^2 t^3 + M_0^3 t^4) \int_0^t \| \tilde{u}_i - \tilde{u}_j \|_{H^1}^2 ds \right\}. \end{aligned}$$

Hence combing above estimates and recalling the definition of distance function $d_t(u, e; \tilde{u}, \tilde{e})$ as defined in Theorem 3.5.1, we finally obtain:

$$d_t(u_i, e_i; u_j, e_j)^2 \leq C \exp(CM_0 t) \left\{ M_0^2 t^2 (1+t) + M_0^3 t^3 (1+t) \right\} d_t(\tilde{u}_i, \tilde{e}_j; \tilde{u}_j, \tilde{e}_j)^2.$$

Thus, one can choose $T \in [0, T_*]$ such that $C \exp(CM_0 t) \left\{ M_0^2 t^2 (1+t) + M_0^3 t^3 (1+t) \right\} =: \alpha_T^2 < 1$, which proves the theorem. \square

3.6 The Proof of Theorem 3.1.1

3.6.1 Existence and uniqueness of non-linear weak solution

The existence and uniqueness of a weak solution $(u, e)(x, t)$ to the non-linear system (3.1.1) in the time interval $[0, T_*]$ is an immediate consequence of contraction inequality, Theorem 3.5.1, and Banach's fixed point theorem.

Corollary 3.6.1. *Let $T_* > 0$ be the time obtained in Theorem 3.5.1, and let $S : X_{M_0, T_*} \rightarrow X_{M_0, T_*}$ be the solution operator defined in Theorem 3.4.5 constructed in Section 3.4, then there exists a unique function $(u, e) \in X_{M_0, T_*}$ such that (u, e) is the fixed point of S in the sense that $S((u, e)) = (u, e)$. Moreover, one also has*

$$\begin{cases} E[u, e](T_*) \leq M_0, \\ (u, e)(x, 0) = (u_0, e_0)(x) & \text{for all } x \in [0, k], \\ (u, D_x e)(0, t) = (u, D_x e)(k, t) = 0 & \text{for all } t \in [0, T_*], \end{cases}$$

and the following convergence holds:

$$(u_j, e_j) \rightarrow (u, e) \text{ strongly in } C^0(0, T_*; L^2(0, k)) \text{ and } L^2(0, T_*; H^1(0, k)),$$

where for each $j \in \mathbb{N}$,

$$(u_j, e_j) := S^j((u^{(0)}, e^{(0)})) = \underbrace{S \cdots S}_{j\text{-times}}((u^{(0)}, e^{(0)})),$$

with $(u^{(0)}, e^{(0)})$ being the base case solution constructed in (3.2.1).

Proof. First, we recall the space X_{T_*, M_0} defined in Theorem 3.4.5 as:

$$X_{M_0, T_*} := \left\{ (w, \theta) \in L^\infty(0, T_*; H^2(0, k)) : (w, \theta)(x, 0) = (u_0, e_0)(x) \quad \forall x \in [0, k], \right. \\ \left. (w, D_x \theta)(0, t) = (w, D_x \theta)(k, t) = (0, 0) \quad \forall t \in [0, T_*], \quad \text{and} \quad E[w, \theta](T_*) \leq M_0. \right\},$$

where

$$E[u, e](T) := \sup_{t \in [0, T]} \left\{ \|(u, e)(\cdot, t)\|_{H^2(0, k)}^2 + \|(D_t u, D_t e)(\cdot, t)\|_{L^2(0, k)}^2 \right\} \\ + \int_0^T \left\{ \|(u, v)(\cdot, t)\|_{H^3(0, k)}^2 + \|D_t u(\cdot, t)\|_{H^1(0, k)}^2 \right\} dt,$$

and the distance function $d_{T_*} : X_{T_*, M_0} \times X_{T_*, M_0} \rightarrow [0, \infty)$ defined in Section 3.5 as

$$d_{T_*}(u, e; \tilde{u}, \tilde{e}) := \sqrt{\sup_{0 \leq s \leq T_*} \|(u - \tilde{u}, e - \tilde{e})(\cdot, s)\|_{L^2(0, k)}^2 + \int_0^{T_*} \|(u - \tilde{u}, e - \tilde{e})(\cdot, s)\|_{H^1(0, k)}^2 ds}.$$

We claim that the above space equipped with the distance function: (X_{M_0, T_*}, d_{T_*}) is a complete metric space. First, it follows from interpolation inequality that $X_{M_0, T_*} \subseteq C^0([0, T_*]; L^2(0, k)) \cap L^2(0, T_*; H^1(0, k))$, hence d_{T_*} is well defined on X_{M_0, T_*} . One can also verify that $d_{T_*}(\cdot, \cdot)$ is indeed a metric function, hence (X_{M_0, T_*}, d_{T_*}) is a metric space. So in order to apply Banach fixed point theorem, we are left to show that it is also complete. So suppose there is a sequence $\{(w_i, \theta_i)\}_{i \in \mathbb{N}} \subseteq X_{M_0, T_*}$ such that $\lim_{i, j \rightarrow \infty} d_{T_*}(w_i, \theta_i; w_j, \theta_j) = 0$.

Then since $C^0([0, T_*]; L^2(0, k))$ and $L^2(0, T_*; H^1(0, k))$ are both Banach spaces, it follows that there exists $(w, \theta) \in C^0([0, T_*]; L^2(0, k)) \cap L^2(0, T_*; H^1(0, k))$ such that $(w_i, \theta_i) \rightarrow (w, \theta)$ strongly in both Banach spaces $C^0([0, T_*]; L^2(0, k))$ and $L^2(0, T_*; H^1(0, k))$ as $i \rightarrow \infty$. In particular, there exists a subsequence $i_j \rightarrow \infty$ as $j \rightarrow \infty$ such that $(w_{i_j}, \theta_{i_j})(x, t) \rightarrow (w, \theta)(x, t)$ for a.e. $(x, t) \in (0, k) \times [0, T_*]$ as $j \rightarrow \infty$. Now since $\{(w_{i_j}, \theta_{i_j})\}_{j \in \mathbb{N}} \subseteq X_{M_0, T_*}$, it follows that $E[w_{i_j}, \theta_{i_j}](T_*) \leq M_0$ for each $j \in \mathbb{N}$. Thus there exists a further subsequence, which we will still denote as i_j for simplicity, such that $(D_t w_{i_j}, D_t \theta_{i_j}) \overset{*}{\rightharpoonup} (f, g)$ in $L^\infty(0, T_*; L^2(0, k))$ as $j \rightarrow \infty$. Using the previously proven point-wise almost everywhere convergence, it follows that the weak temporal derivative of w, θ exists and $D_t w = f, D_t \theta = g$ almost everywhere. Moreover, by the weakly-star lower semi-continuity, it follows that $\|(D_t w, D_t \theta)\|_{L^\infty(0, T_*; L^2(0, k))} \leq \liminf_{j \rightarrow \infty} \|(D_t w_{i_j}, D_t \theta_{i_j})\|_{L^\infty(0, T_*; L^2(0, k))} \leq M_0$. By the same argument, it can be shown that the weak derivatives $(D_x^2, D_x^3, D_x D_t)(w, \theta)$ exists and they satisfies the estimate $E[w, \theta](T_*) \leq M_0$. Next, by construction, for each $j \in \mathbb{N}$, we have that $(w_{i_j}, \theta_{i_j})(x, 0) = (u_0, e_0)(x)$ for $x \in [0, k]$ and $(w_{i_j}, D_x \theta_{i_j})(0, t) = (w_{i_j}, D_x \theta_{i_j})(k, t)$ for $t \in [0, T_*]$. Thus for any $\chi \in C_c^\infty([0, T_*])$, it follows from the previously proven weak convergence that:

$$\int_0^{T_*} D_x \theta(0, t) dt = \int_0^{T_*} \int_0^k (D_x^2 \theta - D_x^2 \theta_{i_j}) \frac{x - k}{k} dx dt + \frac{1}{k} \int_0^{T_*} \int_0^k (D_x \theta - D_x \theta_{i_j}) dx dt \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

Hence, by Fundamental Lemma of Calculus of Variation, we conclude $D_x \theta(0, t) = 0$ for almost all $t \in [0, T_*]$ and by the same argument, one can also verify that $w(0, t) = w(k, t) = D_x \theta(k, t) = 0$ for almost all $t \in [0, T_*]$. Moreover, for given function $\varphi \in C_c^\infty([0, k])$, it also follows that as $j \rightarrow \infty$:

$$\int_0^k (w(x, 0) - u_0(x)) \varphi(x) dx = - \int_0^{T_*} \int_0^k \frac{t - T_*}{T_*} (D_t w - D_t w_{i_j}) dx dt - \frac{1}{T_*} \int_0^{T_*} \int_0^k (w - w_{i_j}) dx dt \rightarrow 0.$$

Therefore $w(x, 0) = u_0(x)$ for almost every $x \in [0, k]$ and by the exact same argument, we also get $\theta(x, 0) = e_0(x)$ for almost every $x \in [0, k]$. In conclusion, we have shown that the limit function $(w, \theta) \in X_{M_0, T_*}$, which

implies that (X_{M_0, T_*}, d_{T_*}) is a complete metric space. Hence, using the contraction inequality Theorem 3.5.1, one can apply Banach fixed point theorem to obtain a unique function $(u, e) \in X_{M_0, T_*}$ such that $S((u, e)) = (u, e)$. Denoting $(u_j, e_j) := S^j((u^{(0)}, e^{(0)}))$, then we also have the convergence $d_{T_*}(u_j, e_j; u, e) \rightarrow 0$ as $j \rightarrow \infty$, which implies that $(u_j, e_j) \rightarrow (u, e)$ strongly in both $C^0([0, T_*]; L^2(0, k))$ and $L^2(0, T_*; H^1(0, k))$. \square

Next, combining the relation (3.2.2) and (3.2.5) with Corollary 3.6.1, we have the following convergence:

Corollary 3.6.2. *Let $T_* > 0$ be the time obtained in Theorem 3.5.1, then*

$$\tau_l \rightarrow \tau \text{ in } C(0, T_*; L^2(0, k)), \text{ and } r_l \rightarrow r \text{ in } C(0, T_*; C(0, k)),$$

where $\tau(x, t)$, $r(x, t)$ are defined by:

$$\tau(x, t) := \tau_0(x) + \int_0^t D_x u(x, s) ds \text{ and } r(x, t) := a + \int_0^x \tau(y, t) dy.$$

3.6.2 Higher regularity of non-linear solution

Finally we show the higher regularity of the non-linear solutions $(u, e)(x, t)$.

Theorem 3.6.1. *Let $T_* > 0$ be the time obtained in Theorem 3.5.1, and let (u, e) be the unique non-linear solution obtained in Corollary 3.6.1. Then*

$$(u, e) \in C^0(0, T_*; H^2(0, k)) \cap L^2(0, T_*; H^3(0, k)), \quad (D_t u, D_t e) \in C^0(0, T_*; L^2(0, k)), \\ (D_t u, D_t e) \in L^2(0, T_*; H^1(0, k)), \quad (u, D_x e) \in C^0(0, T_*; H_0^1(0, k)).$$

Proof. By Corollary 3.6.1, we have $E[u, e](T_*) \leq M_0 < \infty$, which implies in particular that $(D_t u, D_t e) \in L^2(0, T_*; H^1(0, k))$ and $(u, e) \in L^2(0, T_*; H^3(0, k))$. Hence one can apply interpolation inequality to get that $(u, e) \in C^0([0, T_*]; H^2(0, k))$. Now we are left to prove $(D_t u, D_t e) \in C^0([0, T_*]; L^2(0, k))$. As before we denote $(u_j, e_j) := S^j((u^{(0)}, e^{(0)}))$ be the j -th iterative linear solution. Then combining Theorems 3.4.2–3.4.4 and Theorem 3.4.5, the following additional inequality holds:

$$\sup_{i \in \mathbb{N}} \int_0^{T_*} \{ \|D_t^2 u_i(\cdot, t)\|_{H^{-1}(0, k)}^2 + \|D_t^2 e_i(\cdot, t)\|_{H^*(0, k)}^2 \} dt \leq M_0, \quad (3.6.1)$$

where $H^{-1}(0, k)$ and $H^*(0, k)$ are respectively the dual space of $H_0^1(0, k)$ and $H^1(0, k)$. Hence, there exists a subsequence $i_j \rightarrow \infty$ as $j \rightarrow \infty$ such that $D_t^2 u_{i_j} \rightharpoonup D_t^2 u$ weakly in $L^2(0, T_*; H^{-1}(0, k))$ and $D_t^2 e_{i_j} \rightharpoonup D_t^2 e$ weakly in $L^2(0, T_*; H^*(0, k))$, where $D_t^2 u$ and $D_t^2 e$ are respectively the unique temporal weak derivative of $D_t u$ and $D_t e$ in the respective spaces $L^2(0, T_*; H^{-1}(0, k))$ and $L^2(0, T_*; H^*(0, k))$. Moreover, we claim that $D_t u(0, t) = D_t u(k, t) = 0$ for almost all $t \in [0, T_*]$. First, from the Sobolev embedding theorem, it follows that $D_t u \in L^2(0, T_*; C^0([0, k]))$, hence the trace $t \rightarrow (D_t u(0, t), D_t u(k, t))$ exists in $L^2(0, T_*)$. For each functions $\chi \in C^1([0, T_*])$, integrating by parts, we get:

$$\int_0^{T_*} \chi(t) D_t u(0, t) dt = \frac{1}{k} \int_0^{T_*} \int_0^k \chi(t) D_t u(x, t) dx dt + \int_0^{T_*} \int_0^k \frac{x-k}{k} \chi(t) D_x D_t u(x, t) dx dt \\ = -\frac{1}{k} \int_0^{T_*} \int_0^k \chi'(t) u(x, t) dx dt - \int_0^{T_*} \int_0^k \frac{x-k}{k} \chi'(t) D_x u(x, t) dx dt = -\int_0^{T_*} \chi'(t) u(0, t) dt = 0,$$

where we used the boundary condition $u(0, t) = u(k, t) = 0$ obtained in Corollary 3.6.1. Thus $D_t u(0, t) = 0$ for almost all $t \in [0, T_*]$. Similarly, we can also show that $D_t u(k, t) = 0$ for almost all $t \in [0, T_*]$. Using this, we conclude that $D_t u \in L^2(0, T_*; H_0^1(0, k))$. In summary we have the regularities $D_t u \in L^2(0, T_*; H_0^1(0, k))$ and $D_t^2 u \in L^2(0, T_*; H^{-1}(0, k))$. By interpolation theorem again, we have $D_t u \in C^0([0, T_*]; L^2(0, k))$. In the exact same way, using the regularities $D_t e \in L^2(0, T_*; H^1(0, k))$ and $D_t^2 e \in L^2(0, T_*; H^*(0, k))$ and interpolation theorem, we have $D_t e \in C^0([0, T_*]; L^2(0, k))$. \square

Chapter 4

Global Existence of Solutions in Lagrangian Coordinates

In this chapter, we prove two global-in-time existence results for the Lagrangian heat-conducting compressible Navier-Stokes equations (1.3.6) in the unbounded domain $(x, t) \in [0, \infty) \times [0, T]$. The first Theorem 4.1.1 states the existence and uniqueness of a global-in-time strong solution to the exterior Lagrangian equations (1.3.6) with each fixed $a \in (0, 1)$. This result is obtained by the following procedure: first, we approximate the original problem (1.3.6) by solving the equations in the bounded Lagrangian domain $(x, t) \in [0, k] \times [0, T]$, for $k \in \mathbb{N}$. The original initial data are truncated with some cut-off functions near the far-field $x \rightarrow \infty$, and these are parametrised by $k \in \mathbb{N}$. Then by the main result of Chapter 3, Theorem 3.1.1, a strong solution is guaranteed to exist locally in time, and we derive a set of regularity estimates on this solution. These estimates are uniform in the parameter $k \in \mathbb{N}$, and in addition, they are independent of parameter $a \in (0, 1)$ restricted to the domain $(x, t) \in [\varepsilon, k] \times [0, T]$ for each fixed $\varepsilon \in (0, 1]$. Consequently one can apply the bootstrap arguments to extend the maximum time of existence to any chosen $T > 0$. Finally, by compactness argument in the limit $k \rightarrow \infty$, solution to the original problem 1.3.6 in the whole domain $(x, t) \in [0, \infty) \times [0, T]$ is obtained, and its uniqueness is also shown by deriving contraction inequality with respect to the initial data. The second Theorem 4.1.2 in Chapter 4 states the global-in-time existence of a certain class of solutions independent of the parameter $a \in (0, 1)$. The precise notion of such solution is stated in Definition 4.1.1. From the previous result, Theorem 4.1.1, we can obtain a set of solutions $\{(v_a, u_a, e_a, r_a)(x, t)\}_{a \in (0, 1)}$. They satisfy some estimates which are restricted to the domain $(x, t) \in [\varepsilon, \infty) \times [0, T]$, and are independent of parameter $a \in (0, 1)$. Using these, we prove Theorem 4.1.2 by applying compactness argument in the limit $a \searrow 0$.

4.1 Main Statements

4.1.1 Strong solution in the exterior domain

Before stating the main theorem, we denote the term \mathcal{H}_1 to be:

$$\begin{aligned} \mathcal{H}_1[v, u, e, r](x, t) = & \{|v - 1|^2 + r^{2m}|D_x v|^2 + |u|^4 + r^{2m}|D_x u|^2 + \sigma r^{4m}|D_x^2 u|^2\}(x, t) \\ & + \{|e - 1|^2 + r^{2m}|D_x e|^2 + \sigma r^{4m}|D_x^2 e|^2\}(x, t), \end{aligned}$$

where $\sigma = \sigma(t)$ is the weight defined by $\sigma(t) = \min\{1, t\}$ for $t \in [0, \infty)$. Moreover, we also define the term $\mathcal{H}_2[v, u, e, r](x, t)$ as

$$\begin{aligned} \mathcal{H}_2[v, u, e, \zeta](x, t) := & \{|v - 1|^2 + \zeta^{2m}|D_x v|^2 + \zeta^{4m}|D_x^2 v|^2 + |u|^4 + \zeta^{2m}|D_x u|^2 + \zeta^{4m}|D_x^2 u|^2\}(x, t) \\ & + \{|e - 1|^2 + \zeta^{2m}|D_x e|^2 + \zeta^{4m}|D_x^2 e|^2\}(x, t). \end{aligned}$$

Theorem 4.1.1 (Global Existence in the Exterior Domain). *For a given constant $C_0 > 0$, assume that the initial data $(v_0, u_0, e_0)(x)$ satisfies:*

$$\begin{aligned} C_0^{-1} &\leq v_0(x), \quad e_0(x) \leq C_0 \quad \text{for each } x \in [0, \infty), \\ \int_0^\infty &\left\{ \frac{1}{2}|u_0|^2 + (\gamma - 1)\psi(v_0) + \psi(e_0) + |v_0 - 1|^2 + |u_0|^4 + |e_0 - 1|^2 \right\}(x)dx \leq C_0, \\ \int_0^\infty &r_0^{2m} \{ |D_x v_0|^2 + |D_x u_0|^2 + |D_x e_0|^2 \}(x)dx \leq C_0 \quad \text{where } r_0(x) := \left(a^n + n \int_0^x v_0(y)dy \right)^{\frac{1}{n}}, \end{aligned}$$

where $\psi(zeta) := z - 1 - \log z$ and the following boundary condition is assumed:

$$u_0(0) = 0 \quad \text{and} \quad \lim_{x \rightarrow \infty} (v_0, u_0, e_0)(x) = (1, 0, 1).$$

(i) *There exists a unique solution $(v, u, e, r)(x, t) : [0, \infty) \times [0, \infty) \rightarrow [0, \infty) \times \mathbb{R} \times [0, \infty) \times [a, \infty)$ such that the equations (1.3.6)₁–(1.3.6)₃ are satisfied in $(x, t) \in [0, \infty) \times [0, \infty)$ point-wise almost everywhere, and the following boundary conditions hold:*

$$u(0, t) = 0 = D_x e(0, t) \quad \text{and} \quad \lim_{x \rightarrow \infty} (v, u, e, D_x e)(x, t) = (1, 0, 1, 0) \quad \text{for } t \in [0, \infty) \quad \text{a.e.},$$

and the function $r(x, t) : [0, \infty) \times [0, \infty) \rightarrow [a, \infty)$ satisfies:

$$r(x, t) = \left(a^n + n \int_0^x v(y, t)dy \right)^{\frac{1}{n}} = r_0(x) + \int_0^t u(x, s)ds \quad \text{for } (x, t) \in [0, \infty) \times [0, \infty).$$

In addition, there exists a constant $C_0 > 0$ depending only on the initial data such that

$$\left\{ \begin{array}{l} \sup_{0 \leq t \leq \infty} \int_0^\infty \left\{ \frac{1}{2}|u|^2 + (\gamma - 1)\psi(v) + \psi(e) \right\}(x, t)dx \leq C_0, \\ \int_0^\infty \int_0^\infty \left\{ \kappa \frac{r^{2m}|D_x e|^2}{ve^2} + \left(\lambda + \frac{2\mu}{n} \right) \frac{|D_x(r^m u)|^2}{ve} + 2m\mu \frac{v}{e} \left(\frac{D_x(r^m u)}{v\sqrt{n}} - \sqrt{n} \frac{u}{r} \right)^2 \right\} dxdt \leq C_0. \end{array} \right.$$

Moreover, for each $T > 0$, there exists a constant $C_1(a) = C_1(a, T, C_0) > 0$ such that:

$$\left\{ \begin{array}{l} C_1(a) \leq e(x, t) \quad \text{and} \quad C_1(a)^{-1} \leq v(x, t) \leq C_1(a) \quad \text{for all } (x, t) \in [0, \infty) \times [0, T], \\ \sup_{0 \leq t \leq T} \int_0^\infty \mathcal{H}_1[v, u, e, r](x, t)dx \leq C_1(a), \\ \sup_{t \in [0, T]} \int_0^\infty \{ |D_t v|^2 + \sigma(t)|D_t u|^2 + \sigma(t)|D_t e|^2 \}(x, t)dx \leq C_1(a), \\ \int_0^T \int_0^\infty r^{2m} \{ |D_t D_x v|^2 + \sigma(t)|D_t D_x u|^2 + \sigma(t)|D_t D_x e|^2 \}(x, t)dxdt \leq C_1(a), \\ |u(x, t)| \leq C_1(a) \quad \text{and} \quad e(x, t) \leq C_1(a) \quad \text{for all } (x, t) \in [0, \infty) \times [0, T], \end{array} \right.$$

where $\sigma(t) := \min\{1, t\}$. Furthermore, for each $\varepsilon \in (0, 1]$ and $T > 0$, there exists a constant $C_2(\varepsilon) = C_2(\varepsilon, T, C_0) > 0$ independent of $a \in (0, 1)$ such that:

$$\left\{ \begin{array}{l} C_2(\varepsilon)^{-1} \leq v(x, t) \leq C_2(\varepsilon) \quad \text{for all } (x, t) \in [\varepsilon, \infty) \times [0, T], \\ \sup_{0 \leq t \leq T} \int_\varepsilon^\infty \mathcal{H}_1[v, u, e, r](x, t)dx \leq C_2(\varepsilon), \\ \sup_{t \in [0, T]} \int_\varepsilon^\infty \{ |D_t v|^2 + \sigma(t)|D_t u|^2 + \sigma(t)|D_t e|^2 \} dx \leq C_2(\varepsilon), \\ \int_0^T \int_\varepsilon^\infty r^{2m} \{ |D_t D_x v|^2 + \sigma(t)|D_t D_x u|^2 + \sigma(t)|D_t D_x e|^2 \}(x, t)dxdt \leq C_2(\varepsilon), \\ |u(x, t)| \leq C_2(\varepsilon) \quad \text{and} \quad e(x, t) \leq C_2(\varepsilon) \quad \text{for all } (x, t) \in [\varepsilon, \infty) \times [0, T], \end{array} \right.$$

(ii) If the initial data $(v_0, u_0, e_0)(x)$ satisfies the additional condition:

$$\int_0^\infty r_0^{4m} \{|D_x^2 v_0|^2 + |D_x^2 u_0|^2 + |D_x^2 e_0|^2\}(x) dx \leq C_0,$$

for some given initial constant $C_0 > 0$, and the following additional boundary condition is assumed:

$$D_x e_0(0) = 0 \quad \text{and} \quad \lim_{x \rightarrow \infty} D_x e_0(x) = 0.$$

Then, the solution $(v, u, e)(x, t)$ obtained in the previous statement (i) satisfies the additional regularities: for each $T > 0$, there exists a constant $C_1(a) = C_1(a, T, C_0) > 0$ such that

$$\left\{ \begin{array}{l} \sup_{0 \leq t \leq T} \int_0^\infty \{\mathcal{H}_2[v, u, e, r] + |D_t u|^2 + |D_t e|^2\}(x, t) dx \leq C_1(a), \\ \int_0^T \int_0^\infty r^{2m} \{|D_t D_x u|^2 + |D_t D_x e|^2\}(x, t) dx dt \leq C_1(a), \\ \int_0^T \int_0^\infty r^{6m} \{|D_x^3 u|^2 + |D_x^3 e|^2\}(x, t) dx dt \leq C_1(a). \end{array} \right.$$

Moreover, for each $\varepsilon \in (0, 1]$ and $T > 0$, there exists $C_2(\varepsilon) = C_2(\varepsilon, T, C_0) > 0$ such that:

$$\left\{ \begin{array}{l} \sup_{0 \leq t \leq T} \int_\varepsilon^\infty \{\mathcal{H}_2[v, u, e, r] + |D_t u|^2 + |D_t e|^2\}(x, t) dx \leq C_2(\varepsilon), \\ \int_0^T \int_\varepsilon^\infty r^{2m} \{|D_t D_x u|^2 + |D_t D_x e|^2\}(x, t) dx dt \leq C_2(\varepsilon), \\ \int_0^T \int_\varepsilon^\infty r^{6m} \{|D_x^3 u|^2 + |D_x^3 e|^2\}(x, t) dx dt \leq C_2(\varepsilon). \end{array} \right.$$

4.1.2 Lagrangian solution away from origin

Definition 4.1.1. Given initial data $(v_0, u_0, e_0)(x)$ and time $T > 0$, then $(v, u, e, r)(x, t)$ is said to be a Lagrangian solution away from origin to the Full Compressible Navier-Stokes Equations in the domain $(x, t) \in [0, \infty) \times [0, T]$, with initial data $(v_0, u_0, e_0)(x)$, if the following equations are satisfied for $(x, t) \in (0, \infty) \times (0, T)$ almost everywhere:

$$\left\{ \begin{array}{l} r(x, t) = r_0(x) + \int_0^t u(x, s) ds \quad \text{where} \quad r_0(x) := \left(n \int_0^x v_0(y) dy \right)^{\frac{1}{n}}, \\ D_t v = D_x(r^m u), \\ D_t u = \beta r^m D_x \left(\frac{D_x(r^m u)}{v} \right) - (\gamma - 1) r^m D_x \left(\frac{e}{v} \right), \\ D_t e = \frac{D_x(r^m u)}{v} \{ \beta D_x(r^m u) - (\gamma - 1) e \} - 2m\mu D_x(r^{m-1} u^2) + \kappa D_x \left(\frac{r^{2m}}{v} D_x e \right). \end{array} \right.$$

Theorem 4.1.2 (Global existence of Lagrangian solution away from origin). Suppose that the initial data $(v_0, u_0, e_0)(x)$ satisfies:

$$\left\{ \begin{array}{l} C_0^{-1} \leq v_0(x), e_0(x) \leq C_0 \quad \text{and} \quad |u_0(x)| \leq C_0 \quad \text{for each } x \in [0, \infty), \\ \int_0^\infty \left\{ \frac{1}{2} |u_0|^2 + (\gamma - 1) \psi(v_0) + \psi(e_0) + |v_0 - 1|^2 + |u_0|^4 + |e_0 - 1|^2 \right\}(x) dx \leq C_0, \\ \int_0^\infty r_0^{2m} \{|D_x v_0|^2 + |D_x u_0|^2 + |D_x e_0|^2\}(x) dx \leq C_0 \quad \text{where} \quad r_0(x) := \left(n \int_0^x v_0(y) dy \right)^{\frac{1}{n}}, \end{array} \right.$$

for some given constant $C_0 > 0$. Then, for each $T > 0$, there exists a Lagrangian solution away from origin, $(v, u, e, r)(x, t)$ in the domain $(x, t) \in (0, \infty) \times [0, T]$ with $r(x, t)$ given by

$$r(x, t) = r_0(x) + \int_0^t u(x, s) ds.$$

Furthermore, the following properties are satisfied:

(i) There exists a constant $C_0 > 0$ depending only on the initial data such that

$$\sup_{0 \leq t \leq T} \int_0^\infty \left\{ \frac{1}{2} |u|^2 + (\gamma - 1) \psi(v) \right\} (x, t) dx \leq C_0.$$

(ii) The spatial weak derivative of $r(x, t)$ exists and it is given by $D_x r(x, t) = (r(x, t))^{-m} v(x, t)$ for $(x, t) \in (0, \infty) \times [0, T]$. Moreover, $x \mapsto r(x, t)$ is monotone increasing for each $t \in [0, T]$, and one has:

$$nx\psi_-^{-1}\left(\frac{C_0}{x}\right) \leq (r(x, t))^n \leq nx\psi_+^{-1}\left(\frac{C_0}{x}\right) \quad \text{for all } (x, t) \in (0, \infty) \times [0, T].$$

In particular the limit $\underline{r}(t) := \lim_{x \rightarrow 0^+} r(x, t)$ exists and $\underline{r}(t) \in [0, \infty)$ for each $t \in [0, T]$. As a consequence, $r(x, t)$ also takes the form:

$$r(x, t) = \left((\underline{r}(t))^n + n \int_0^x v(y, t) dy \right)^{\frac{1}{n}} \quad \text{for } (x, t) \in [0, \infty) \times [0, T].$$

(iii) For each $\varepsilon \in (0, 1]$, there exists $C(\varepsilon) = C(\varepsilon, T, C_0) > 0$ such that:

$$\left\{ \begin{array}{l} C_2(\varepsilon)^{-1} \leq v(x, t) \leq C_2(\varepsilon) \quad \text{for all } (x, t) \in [\varepsilon, \infty) \times [0, T], \\ |u(x, t)| \leq C_2(\varepsilon) \quad \text{and} \quad e(x, t) \leq C_2(\varepsilon) \quad \text{for all } (x, t) \in [\varepsilon, \infty) \times [0, T], \\ \sup_{0 \leq t \leq T} \int_\varepsilon^\infty \left\{ \mathcal{H}_1[v, u, e, r] + |D_t v|^2 + \sigma |D_t u|^2 + \sigma |D_t e|^2 \right\} (x, t) dx \leq C_2(\varepsilon), \\ \int_0^T \int_\varepsilon^\infty r^{2m} \left\{ |D_t D_x v|^2 + \sigma(t) |D_t D_x u|^2 + \sigma(t) |D_t D_x e|^2 \right\} dx dt \leq C_2(\varepsilon), \end{array} \right.$$

where $\sigma(t) \equiv \min\{1, t\}$.

4.2 Approximation Scheme and Local-in-Time Existence

In order to obtain the solutions stated in Theorem 4.1.1, a collection of approximated initial data is constructed such that one can apply Theorem 3.1.1 to guarantee a local-in-time strong solution, with which one can derive a set of high regularity estimates for bootstrap argument. We note that the mollification procedure for initial data detailed in Section 4.2.1 is only required for Theorem 4.1.1(i), since the additional initial regularity condition described in Theorem 4.1.1(ii) is sufficient for directly applying the local-in-time existence Theorem 3.1.1.

4.2.1 Mollification of initial data

Let $(v_0, u_0, e_0)(x)$, for $x \in [0, \infty)$, be initial data stated in Theorem 4.1.1(i). In view of the boundary condition $u_0(0) = D_x e_0(0) = 0$ for when $(u_0, e_0)(x)$ is smooth, we extend each functions as follows

$$\tilde{u}_0(x) := \begin{cases} u_0(x) & \text{if } x \in [0, \infty), \\ -u_0(-x) & \text{if } x \in (-\infty, 0), \end{cases}$$

and

$$(\tilde{v}_0, \tilde{e}_0, \tilde{r}_0)(x) := (v_0, e_0, r_0)(|x|) \text{ for all } x \in \mathbb{R}.$$

With this extension, we remark that $D_x \tilde{e}_0(x) = D_x e_0(x)$ if $x \in [0, \infty)$ and $D_x \tilde{e}_0(x) = -D_x e_0(-x)$ if $x \in (-\infty, 0)$. Now we let $\{\eta_\delta\}_{\delta \in (0,1)} \subset C_c^\infty([-1, 1])$ be the standard mollifiers such that $\eta_\delta(x) = \eta_\delta(-x)$ for each $\delta \in (0, 1)$. We set for $x \in \mathbb{R}$

$$(v_\delta^0, u_\delta^0, e_\delta^0)(x) := \int_{\mathbb{R}} (\tilde{v}_0, \tilde{u}_0, \tilde{e}_0)(y) \eta_\delta(x-y) dy, \quad r_\delta^0(x) := \left(a^n + n \int_0^{|x|} v_\delta^0(y) dy \right)^{\frac{1}{n}}. \quad (4.2.1)$$

Let $C_0 > 0$ be the initial constant given in the assumption of Theorem 4.1.1, which is independent of the mollification parameter $\delta \in (0, 1)$, then it follows that

$$(v_\delta^0, u_\delta^0, e_\delta^0)(x) \in C^\infty(-\infty, \infty) \text{ and } C_0^{-1} \leq v_\delta^0(x), e_\delta^0(x) \leq C_0 \text{ for all } x \in \mathbb{R},$$

and as $\delta \rightarrow 0^+$:

$$\begin{cases} (v_\delta^0, u_\delta^0, e_\delta^0, r_\delta^0)(x) \rightarrow (\tilde{v}_0, \tilde{u}_0, \tilde{e}_0, \tilde{r}_0)(x) \text{ for all } x \in \mathbb{R}, \\ (D_x v_\delta^0, D_x u_\delta^0, D_x e_\delta^0)(x) \rightarrow (D_x \tilde{v}_0, D_x \tilde{u}_0, D_x \tilde{e}_0)(x) \text{ for a.e. } x \in \mathbb{R}, \\ (v_\delta^0 - 1, |u_\delta^0|^2, e_\delta^0 - 1) \rightarrow (\tilde{v}_0 - 1, \tilde{u}_0, \tilde{e}_0 - 1) \text{ strongly in } L^2(\mathbb{R}), \\ (D_x v_\delta^0, D_x u_\delta^0, D_x e_\delta^0) \rightarrow (D_x \tilde{v}_0, D_x \tilde{u}_0, D_x \tilde{e}_0) \text{ strongly in } L^2(\mathbb{R}). \end{cases} \quad (4.2.2)$$

Moreover, we also have that for $x \in (-\infty, \infty)$

$$\underline{h}(|x|) \equiv \left(\frac{a^n + n C_0^{-1} |x|}{a^n + n C_0 |x|} \right)^{\frac{1}{n}} \leq \frac{r_\delta^0(x)}{\tilde{r}_0(x)} = \left(\frac{a^n + n \int_0^{|x|} v_\delta^0(y) dy}{a^n + n \int_0^{|x|} v_0(y) dy} \right)^{\frac{1}{n}} \leq \left(\frac{a^n + n C_0 |x|}{a^n + n C_0^{-1} |x|} \right)^{\frac{1}{n}} \equiv \bar{h}(|x|).$$

One can verify by taking derivatives that $\bar{h}(r)$ and $\underline{h}(r)$ are respectively monotone increasing and decreasing in $r \in [0, \infty)$. In addition, the following limits holds:

$$\lim_{r \rightarrow \infty} \bar{h}(r) = C_0^{2/n}, \quad \lim_{r \searrow 0} \bar{h}(r) = 1, \quad \lim_{r \rightarrow \infty} \underline{h}(r) = C_0^{-2/n}, \quad \lim_{r \searrow 0} \underline{h}(r) = 1.$$

Hence we conclude that

$$C_0^{-2/n} \leq \frac{r_\delta^0(x)}{\tilde{r}_0(x)} \leq C_0^{2/n} \text{ for all } x \in \mathbb{R} \quad (4.2.3)$$

Combining these results we claim that

Proposition 4.2.1. *Let $(v_0, u_0, e_0)(x)$ and*

$$r_0(x) := \left(a^n + n \int_0^x v_0(y) dy \right)^{1/n},$$

be the initial data given in Theorem 4.1.1 with the given initial constant $C_0 > 0$. Then the mollified initial data $(v_\delta^0, u_\delta^0, e_\delta^0, r_\delta^0)(x)$ constructed in (4.2.1) satisfies the convergence:

$$\begin{aligned} & (v_\delta^0, |u_\delta^0|^2, e_\delta^0, (r_\delta^0)^m D_x v_\delta^0, (r_\delta^0)^m D_x u_\delta^0, (r_\delta^0)^m D_x e_\delta^0) \\ & \rightarrow (\tilde{v}_0, |\tilde{u}_0|^2, \tilde{e}_0, (\tilde{r}_0)^m D_x \tilde{v}_0, (\tilde{r}_0)^m D_x \tilde{u}_0, (\tilde{r}_0)^m D_x \tilde{e}_0) \text{ strongly in } L^2(\mathbb{R}) \text{ as } \delta \searrow 0. \end{aligned}$$

Moreover the following estimates holds:

$$\sup_{\delta \in (0,1)} \int_0^\infty \left\{ (\gamma - 1) \psi(v_\delta^0) + \frac{1}{2} |u_\delta^0|^2 + \psi(e_\delta^0) \right\} (x) dx \leq C_0,$$

and there exists a constant $\delta_0 \in (0, 1)$ such that

$$\sup_{\delta \in (0, \delta_0)} \|(v_\delta^0 - 1, |u_\delta^0|^2, e_\delta^0 - 1, (r_\delta^0)^m D_x v_\delta^0, (r_\delta^0)^m D_x u_\delta^0, (r_\delta^0)^m D_x e_\delta^0)\|_{L^2([0, \infty))}^2 \leq 2C_0.$$

Finally, the mollified initial data satisfies the boundary condition at $x = 0$:

$$u_\delta^0(0) = 0 = D_x e_\delta^0(0).$$

Proof. First, we prove the entropy condition. Since $\psi(\zeta) = \zeta - 1 - \log \zeta$ is convex, it follows that $\psi(e_\delta^0) = \psi(\tilde{e}_0 * \eta_\delta) \leq \psi(\tilde{e}_0) * \eta_\delta$. Using this, and the construction above, we have

$$\int_0^\infty \psi(e_\delta^0(x)) dx \leq \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \psi(\tilde{e}_0(y)) \eta_\delta(x - y) dy dx = \int_0^\infty \psi(e_0(x)) dx \leq C_0.$$

In the similar fashion, one can also show:

$$\int_0^\infty \{\psi(v_\delta^0) + |u_\delta^0|^2\}(x) dx \leq \int_0^\infty \{\psi(v_0) + |u_0|^2\}(x) dx \leq C_0.$$

Next, we show the strong convergence in $L^2(\mathbb{R})$. We start by writing:

$$\int_{\mathbb{R}} |(r_\delta^0)^m D_x v_\delta^0 - (\tilde{r}_0)^m D_x \tilde{v}_0|^2 \leq 2 \int_{\mathbb{R}} (r_\delta^0)^{2m} |D_x v_\delta^0 - D_x \tilde{v}_0|^2 + 2 \int_{\mathbb{R}} |(r_\delta^0)^m - (\tilde{r}_0)^m|^2 |D_x \tilde{v}_0|^2. \quad (4.2.4)$$

From (4.2.2), we have $|(r_\delta^0)^m - (\tilde{r}_0)^m|^2 |D_x \tilde{v}_0|^2(x) \rightarrow 0$ as $\delta \searrow 0$. In addition, using (4.2.3) we also have $|(r_\delta^0)^m - (\tilde{r}_0)^m|^2 |D_x \tilde{v}_0|^2(x) \leq (C_0^{2m/n} + 1)^2 \tilde{r}_0^{2m} |D_x \tilde{v}_0|^2(x) \in L^\infty(\mathbb{R})$ for all $x \in \mathbb{R}$. Thus by Dominated Convergence theorem, it follows that

$$\lim_{\delta \searrow 0} \int_{\mathbb{R}} |(r_\delta^0)^m - (\tilde{r}_0)^m|^2 |D_x \tilde{v}_0|^2 dx = 0. \quad (4.2.5)$$

Next, we deal with the other term using (4.2.3):

$$\int_{\mathbb{R}} (r_\delta^0)^{2m} |D_x v_\delta^0 - D_x \tilde{v}_0|^2 \leq C_0^{4m/n} \int_{\mathbb{R}} (\tilde{r}_0)^{2m} |D_x v_\delta^0 - D_x \tilde{v}_0|^2 = C_0^{4m/n} \int_{\mathbb{R}} |(\tilde{r}_0)^m D_x v_\delta^0 - (\tilde{r}_0)^m D_x \tilde{v}_0|^2.$$

Using the definition of mollification, we have

$$\begin{aligned} \tilde{r}_0^m D_x v_\delta^0(x) - \tilde{r}_0^m D_x \tilde{v}_0(x) &= \int_{\mathbb{R}} \{\tilde{r}_0^m(x) (D_x \tilde{v}_0)(x - y) - (\tilde{r}_0^m D_x \tilde{v}_0)(x)\} \eta_\delta(y) dy \\ &= \int_{\mathbb{R}} \left\{ \frac{\tilde{r}_0^m(x)}{\tilde{r}_0^m(x - y)} - 1 \right\} (\tilde{r}_0^m D_x \tilde{v}_0)(x - y) \eta_\delta(y) dy + \int_{\mathbb{R}} \{(\tilde{r}_0^m D_x \tilde{v}_0)(x - y) - (\tilde{r}_0^m D_x \tilde{v}_0)(x)\} \eta_\delta(y) dy. \end{aligned}$$

Hence, we have

$$\begin{aligned} \int_{\mathbb{R}} (r_\delta^0)^{2m} |D_x v_\delta^0 - D_x \tilde{v}_0|^2 dx &\leq C_0^{4m/n} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left\{ \frac{\tilde{r}_0^m(x)}{\tilde{r}_0^m(x - y)} - 1 \right\} (\tilde{r}_0^m D_x \tilde{v}_0)(x - y) \eta_\delta(y) dy \right)^2 dx \\ &\quad + C_0^{4m/n} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \{(\tilde{r}_0^m D_x \tilde{v}_0)(x - y) - (\tilde{r}_0^m D_x \tilde{v}_0)(x)\} \eta_\delta(y) dy \right)^2 dx \end{aligned} \quad (4.2.6)$$

Next, by Minkowski's inequality, the translation invariance of L^2 -norm, $\|\eta_\delta\|_{L^1(\mathbb{R})} = 1$, and the fact that $\tilde{r}_0^m D_x \tilde{v}_0 \in L^2(\mathbb{R})$ we first get the limit

$$\lim_{\delta \searrow 0} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \{(\tilde{r}_0^m D_x \tilde{v}_0)(x - y) - (\tilde{r}_0^m D_x \tilde{v}_0)(x)\} \eta_\delta(y) dy \right)^2 dx = 0. \quad (4.2.7)$$

For the other term, we start by using Minkowski's inequality once again to get

$$\begin{aligned} & \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left\{ \frac{\tilde{r}_0^m(x)}{\tilde{r}_0^m(x-y)} - 1 \right\} (\tilde{r}_0^m D_x \tilde{v}_0)(x-y) \eta_\delta(y) dy \right)^2 dx \right)^{1/2} \\ & \leq \int_{\mathbb{R}} \eta_\delta(y) \left(\int_{\mathbb{R}} \left| \frac{\tilde{r}_0^m(x)}{\tilde{r}_0^m(x-y)} - 1 \right|^2 |(\tilde{r}_0^m D_x \tilde{v}_0)(x-y)|^2 dx \right)^{1/2} dy. \end{aligned} \quad (4.2.8)$$

Next, setting $z = x - y$, it follows that

$$\int_{\mathbb{R}} \left| \frac{\tilde{r}_0^m(x)}{\tilde{r}_0^m(x-y)} - 1 \right|^2 |(\tilde{r}_0^m D_x \tilde{v}_0)(x-y)|^2 dx = \int_{\mathbb{R}} \left| \frac{\tilde{r}_0^m(z+y)}{\tilde{r}_0^m(z)} - 1 \right|^2 |(\tilde{r}_0^m D_x \tilde{v}_0)(z)|^2 dz$$

Next, for each $y \in (-1, 1)$ and $z \in \mathbb{R}$ one can show that

$$\frac{\tilde{r}_0(z+y)}{\tilde{r}_0(z)} \leq \left(\frac{a^n + nC_0|z+y|}{a^n + nC_0^{-1}|z|} \right)^{1/n} \leq \left(1 + C_0^2 + nC_0 \frac{1}{a^n} \right)^{1/n} \equiv C(C_0, a),$$

with $C(C_0, a) > 0$ a constant independent of $y \in (-1, 1)$. In addition, since $\tilde{r}_0^m D_x \tilde{v}_0 \in L^2(\mathbb{R})$ and $\tilde{r}_0(z+y) \rightarrow \tilde{r}_0(z)$ as $y \rightarrow 0$, we can use Dominated Convergence theorem to obtain the limit

$$\lim_{y \rightarrow 0} \int_{\mathbb{R}} \left| \frac{\tilde{r}_0^m(x)}{\tilde{r}_0^m(x-y)} - 1 \right|^2 |(\tilde{r}_0^m D_x \tilde{v}_0)(x-y)|^2 dx = \lim_{y \rightarrow 0} \int_{\mathbb{R}} \left| \frac{\tilde{r}_0^m(z+y)}{\tilde{r}_0^m(z)} - 1 \right|^2 |(\tilde{r}_0^m D_x \tilde{v}_0)(z)|^2 dz = 0$$

Substituting this into (4.2.8), we have

$$\lim_{\delta \searrow 0} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left\{ \frac{\tilde{r}_0^m(x)}{\tilde{r}_0^m(x-y)} - 1 \right\} (\tilde{r}_0^m D_x \tilde{v}_0)(x-y) \eta_\delta(y) dy \right)^2 dx = 0. \quad (4.2.9)$$

Now putting the limits (4.2.7) and (4.2.9) into the original estimate (4.2.6), we have the limit:

$$\lim_{\delta \searrow 0} \int_{\mathbb{R}} (r_\delta^0)^{2m} |D_x v_\delta^0 - D_x \tilde{v}_0|^2 dx = 0.$$

Using the above and limit (4.2.5) into (4.2.4), we have

$$\lim_{\delta \searrow 0} \int_{\mathbb{R}} |(r_\delta^0)^m D_x v_\delta^0 - (\tilde{r}_0)^m D_x \tilde{v}_0|^2 dx = 0.$$

The convergence of other terms $(\tilde{r}_0)^m D_x \tilde{u}_0$ and $(\tilde{r}_0)^m D_x \tilde{e}_0$ are proven using exactly the same way. As a result, there exists a constant $\delta_0 \in (0, 1)$ such that

$$\sup_{\delta \in (0, \delta_0)} \int_{\mathbb{R}} (r_\delta^0)^{2m} \{ |D_x v_\delta^0|^2 + |D_x u_\delta^0|^2 + |D_x e_\delta^0|^2 \} (x) dx \leq 2C_0,$$

where $C_0 > 0$ is the original constant given in Theorem 4.1.1. \square

4.2.2 Truncation of initial data in the far-field region

We first solve an approximated problem of (1.3.6)₅ by replacing the boundary condition near the far field. To do so, the initial data must be modified with proper cut-off functions. Let $\phi \in C^\infty(\mathbb{R})$ be a test function such that $\phi(z) = 1$ if $z \in (-\infty, 0]$ and $\phi(z) = 0$ if $z \in [1, \infty)$. Given integer $k \in \mathbb{N}$, we let $\alpha = (n-1)/n = m/n$ and $\phi_k(x) := \phi(k^{-\alpha}(x-k))$ for $x \in [0, \infty)$. Then it follows that $\phi_k(x) \in C^\infty([0, \infty))$ and $\phi_k(x) = 1$ for $x \in [0, k]$ and $\phi_k(x) = 0$ for $x \in [k+k^\alpha, \infty)$. Moreover we also have the inequality:

$$|\phi_k^{(i)}(x)| \leq C_i k^{-i\alpha} \quad \text{for all } x \in [0, \infty) \quad \text{and for each } i \in \mathbb{N}, \quad (4.2.10)$$

where $C_i \equiv \sup_{y \in \mathbb{R}} |\phi^{(i)}(y)| > 0$ is a constant independent of $k \in \mathbb{N}$ and $a \in (0, 1)$. Now we defined the approximated initial data (v_k^0, u_k^0, e_k^0) as follows:

$$\begin{cases} v_k^0(x) := (v_0(x) - 1)\phi_k(x) + 1, \\ u_k^0(x) := u_0(x)\phi_k(x), \\ e_k^0(x) := (e_0(x) - 1)\phi_k(x) + 1, \end{cases} \quad \text{for } x \in [0, \infty), \quad (4.2.11)$$

and

$$r_k^0(x) = \left(a^n + n \int_0^x v_k^0(y) dy \right)^{\frac{1}{n}} \quad \text{for } x \in [0, \infty).$$

With this initial data, the k -th approximated Initial-Boundary Value Problem can be stated as: hello

$$\begin{cases} D_t v_k - D_x(r_k^m u_k) = 0 \\ D_t u_k + r_k^m D_x p(v_k, e_k) = \beta r_k^m D_x \left(\frac{D_x(r_k^m u_k)}{v_k} \right) \\ D_t e_k + p(v_k, e_k) D_x(r_k^m u_k) \\ = \beta \frac{|D_x(r_k^m u_k)|^2}{v_k} - 2m\mu D_x(r_k^{m-1} u_k^2) + \kappa D_x \left(\frac{r_k^{2m} D_x e_k}{v_k} \right) \\ r_k(x, t) := \left(a^n + n \int_0^x v_k(y, t) dy \right)^{\frac{1}{n}} \\ u_k(0, t) = u_k(k, t) = 0, \quad D_x e(0, t) = D_x e(k, t) = 0 \quad \text{for } t \in [0, T], \\ (v_k, u_k, e_k)(x, 0) = (v_k^0, u_k^0, e_k^0)(x) \quad \text{for } x \in [0, k]. \end{cases} \quad \text{for } (x, t) \in [0, k] \times [0, T], \quad (4.2.12)$$

From (4.2.11), one can easily see that the initial data (v_k^0, u_k^0, e_k^0) is compatible with the boundary condition, and under the assumption of Theorem, there exists a positive constant such that:

$$\min\{1, C_0^{-1}\} \leq v_k^0, e_k^0 \leq \max\{1, C_0\}. \quad (4.2.13)$$

Moreover the following proposition shows the convergence of approximated initial data in the suitable functional space.

Proposition 4.2.2. *Let $(v_0, u_0, e_0)(x)$ and $r_0(x) := \left(a^n + n \int_0^x v_0(y) dy \right)^{1/n}$ be the initial data under the assumption of Theorem 4.1.1 with given initial constant $C_0 > 0$, and let $(v_k^0, u_k^0, e_k^0, r_k^0)(x)$ be the truncated initial data constructed in (4.2.11). Then we have the following:*

- (i) *If $(v_0, u_0, e_0, r_0)(x)$ satisfies the regularities in Theorem 4.1.1(i), then there exists a constant $C_0 > 0$ independent of $k \in \mathbb{N}$ and $a \in (0, 1)$ such that*

$$\begin{cases} \sup_{k \in \mathbb{N}} \int_0^k \left\{ \frac{1}{2} |u_k^0|^2 + (\gamma - 1) \psi(v_k^0) + \psi(e_k^0) \right\} (x) dx \leq C_0, \\ \sup_{k \in \mathbb{N}} \int_0^k \left\{ |v_k^0 - 1|^2 + |u_k^0|^4 + |e_k^0 - 1|^2 \right\} (x) dx \leq C_0, \\ \sup_{k \in \mathbb{N}} \int_0^k (r_k^0)^{2m} \left\{ |D_x v_k^0|^2 + |D_x u_k^0|^2 + |D_x e_k^0|^2 \right\} (x) dx \leq C_0, \end{cases}$$

Moreover, the following convergences hold:

$$\begin{cases} (v_k^0, u_k^0, e_k^0, r_k^0)(x) \rightarrow (v_0, u_0, e_0, r_0)(x), \\ (D_x v_k^0, D_x u_k^0, D_x e_k^0)(x) \rightarrow (D_x v_0, D_x u_0, D_x e_0)(x), \end{cases} \quad \text{as } k \rightarrow \infty \text{ for a.e. } x \in [0, \infty).$$

and we also have that as $k \rightarrow \infty$:

$$\left\{ \begin{array}{l} (v_k^0 - 1, u_k^0, |u_k^0|^2, e_k^0 - 1) \rightarrow (v_0 - 1, u_0, |u_0|^2, e_0 - 1), \\ ((r_k^0)^m D_x v_k^0, (r_k^0)^m D_x u_k^0, (r_k^0)^m D_x e_k^0) \rightarrow (r_0^m D_x v_0, r_0^m D_x u_0, r_0^m D_x e_0), \\ ((r_k^0)^{2m} D_x^2 v_k^0, (r_k^0)^{2m} D_x^2 u_k^0, (r_k^0)^{2m} D_x^2 e_k^0) \rightarrow (r_0^{2m} D_x^2 v_0, r_0^{2m} D_x^2 u_0, r_0^{2m} D_x^2 e_0), \end{array} \right. \quad \text{in } L^2([0, \infty)).$$

(ii) If $(v_0, u_0, e_0, r_0)(x)$ satisfies the regularities in Theorem 4.1.1(ii), then in addition to the previous assertion, we also have:

$$\sup_{k \in \mathbb{N}} \int_0^k (r_k^0)^{4m} \{ |D_x^2 v_k^0|^2 + |D_x^2 u_k^0|^2 + |D_x^2 e_k^0|^2 \} (x) dx \leq C_0,$$

and the convergences

$$\left\{ \begin{array}{l} (D_x^2 v_k^0, D_x^2 u_k^0, D_x^2 e_k^0)(x) \rightarrow (D_x^2 v_0, D_x^2 u_0, D_x^2 e_0)(x) \text{ as } k \rightarrow \infty \text{ for a.e. } x \in [0, \infty), \\ ((r_k^0)^{2m} D_x^2 v_k^0, (r_k^0)^{2m} D_x^2 u_k^0, (r_k^0)^{2m} D_x^2 e_k^0) \rightarrow (r_0^{2m} D_x^2 v_0, r_0^{2m} D_x^2 u_0, r_0^{2m} D_x^2 e_0) \text{ in } L^2([0, \infty)). \end{array} \right.$$

Proof. The convergence results, initial entropy condition, point-wise bounds and L^2 estimates follows the same proof of Proposition 2.2.4 in Section 2.2.5. Hence we focuses on the convergence result and estimates for $(D_x v_k^0, D_x u_k^0, D_x e_k^0)$. Following a similar argument used for the proof for inequality (4.2.3), we can also obtain that

$$\left\{ \begin{array}{l} (a^n + nx \underline{C}_0^{-1})^{\frac{1}{n}} \leq r_k^0(x) \leq (a^n + nx \overline{C}_0)^{\frac{1}{n}} \\ (\overline{C}_0 \underline{C}_0)^{-\frac{1}{n}} \leq \frac{r_0(x)}{r_k^0(x)} \leq (\overline{C}_0 \underline{C}_0)^{\frac{1}{n}} \end{array} \right. \quad \text{for all } x \in [0, \infty), \quad (4.2.14)$$

where $\overline{C}_0 \equiv \max\{1, C_0\}$ and $\underline{C}_0 \equiv 1/\min\{1, C_0^{-1}\}$. Next, by construction (4.2.10) with $\alpha \equiv (n-1)/n = m/n$, and since $a \in (0, 1)$, $k \in \mathbb{N}$, we have the bound:

$$\sup_{x \in [0, \infty)} r_0^{2m}(x) |\phi'_k(x)|^2 \leq \sup_{x \in [k, k+k^\alpha]} \left(a^n + n \int_0^x v_0(y) dy \right)^{2m/n} C k^{-2\alpha} \leq C \{1 + 2nC_0\}^{2m/n} \equiv \tilde{C}_0, \quad (4.2.15)$$

where $C > 0$ and $C_0 > 0$ are constants independent of $a \in (0, 1)$ and $k \in \mathbb{N}$, hence so is $\tilde{C}_0 > 0$. This implies that

$$\int_0^k r_0^{2m} |\phi'_k|^2 |v_0 - 1|^2 dx = \sup_{y \in [0, \infty)} r_0^{2m}(y) |\phi'_k(y)|^2 \int_0^k |v_0 - 1|^2 dx \leq \tilde{C}_0 \int_0^\infty |v_0 - 1|^2 dx \leq \tilde{C}_0 C_0.$$

Using the above estimate, construction (4.2.11), and (4.2.14) we have obtained that

$$\int_0^k (r_k^0)^{2m} |D_x v_k^0|^2 \leq 2(\overline{C}_0 \underline{C}_0)^{2m/n} \int_0^k r_0^{2m} \{ |\phi'_k|^2 |v_0 - 1|^2 + \phi_k^2 |D_x v_0|^2 \} dx \leq 2(\overline{C}_0 \underline{C}_0)^{2m/n} \{ \tilde{C}_0 + 1 \} C_0.$$

where we emphasise that $\tilde{C}_0 \equiv C \{1 + 2nC_0\}^{2m/n}$ is independent of $a \in (0, 1)$ and $k \in \mathbb{N}$. The corresponding estimates for $\|(r_k^0)^m D_x u_0\|_{L^2(0, k)}$ and $\|(r_k^0)^m D_x e_0\|_{L^2(0, k)}$ follows by the exact same argument.

Next, we prove the estimate $\|(r_k^0)^{2m} D_x^2 v_0\|_{L^2(0, k)} \leq C_0$. From the construction (4.2.11), one can verify that $D_x^2 v_k^0 = \phi_k''(v_0 - 1) + 2\phi_k' D_x v_0 + \phi_k D_x^2 v_0$. So we have using (4.2.14) again to get

$$\int_0^k (r_k^0)^{4m} |D_x^2 v_k^0|^2 dx \leq 4(\overline{C}_0 \underline{C}_0)^{4m/n} \int_0^k r_0^{4m} \{ |\phi_k''|^2 |v_0 - 1|^2 + 4|\phi_k'|^2 |D_x v_0|^2 + |\phi_k|^2 |D_x^2 v_0|^2 \} dx$$

By construction (4.2.10) and (4.2.14) with $\alpha \equiv (n-1)/n = m/n$, we have the bound:

$$\sup_{x \in [0, \infty)} r_0^{4m}(x) |\phi_k''(x)|^2 \leq \sup_{x \in [k, k+k^\alpha]} (a^n + nC_0x)^{4m/n} Ck^{-4\alpha} \leq C\{1 + 2nC_0\}^{4m/n} \equiv \tilde{C}_0^2. \quad (4.2.16)$$

Using this, it follows that

$$\int_0^k r_0^{4m} |\phi_k''|^2 |v_0 - 1|^2 dx \leq \sup_{y \in [0, \infty)} r_0^{4m} |\phi_k''|^2(y) \int_0^k |v_0 - 1|^2 dx \leq \tilde{C}_0^2 \int_0^\infty |v_0 - 1|^2 dx \leq \tilde{C}_0^2 C_0.$$

For the second term, we use the bound (4.2.15) once again to obtain

$$\int_0^k r_0^{4m} |\phi_k'|^2 |D_x v_0|^2 dx \leq \sup_{y \in [0, \infty)} r_0^{2m} |\phi_k'|^2(y) \int_0^k r_0^{2m} |D_x v_0|^2 dx \leq \tilde{C}_0 \int_0^\infty r_0^{2m} |D_x v_0|^2 dx \leq \tilde{C}_0 C_0.$$

For the third term we have:

$$\int_0^k |\phi_k|^2 r_0^{4m} |D_x^2 v_0|^2 dx = \int_0^\infty |\phi_k|^2 r_0^{4m} |D_x^2 v_0|^2 dx \leq \int_0^\infty r_0^{4m} |D_x^2 v_0|^2 dx \leq C_0.$$

Next, we prove the almost everywhere convergence. From the construction, it follows that $\phi_k(x) \rightarrow 1$ as $k \rightarrow \infty$ for all $x \in [0, \infty)$, hence we have

$$(v_k^0, u_k^0, e_k^0)(x) = (v_0 \phi_k + 1 - \phi_k, \phi_k u_0, e_0 \phi_k + 1 - \phi_k)(x) \rightarrow (v_0, u_0, e_0)(x) \text{ as } k \rightarrow \infty \text{ for all } x \in [0, \infty).$$

Thus by Dominated convergence theorem, it follows that $\lim_{k \rightarrow \infty} r_k^0(x) = r_0(x)$ for each $x \in [0, \infty)$. Next, by the definition of extension, we have $D_x v_k^0 = \phi_k(x) D_x v_0(x) + (v_0(x) - 1) \phi_k'(x)$ for a.e. $x \in [0, \infty)$. Since $\phi_k(x) \rightarrow 1$ and $\phi_k'(x) \rightarrow 0$ as $k \rightarrow \infty$ for each $x \in [0, \infty)$, and $v_0(x) \leq C_0 < \infty$, it follows that $D_x v_k^0(x) \rightarrow D_x v_0(x)$ for $x \in [0, \infty)$ almost everywhere. The convergence for other two terms $D_x u_k^0$ and $D_x e_k^0$ can be shown in the exact same way. Similarly, we have

$$D_x^2 v_k^0 = \phi_k''(v_0 - 1) + 2\phi_k' D_x v_0 + \phi_k D_x^2 v_0.$$

Since we also have $\phi_k''(x) \rightarrow 0$ as $k \rightarrow \infty$ for each $x \in [0, \infty)$, it follows that $D_x^2 v_k^0(x) \rightarrow D_x^2 v_0(x)$ for $x \in [0, \infty)$ almost everywhere. The convergence for other two terms $D_x^2 u_k^0$ and $D_x^2 e_k^0$ can be shown in the exact same way.

Next, we show the strong convergence in L^2 . Since $r_k^0(x) \rightarrow r_0(x)$ and $D_x v_k^0 \rightarrow D_x v_0$ as was shown previously, it follows that $(r_k^0)^m D_x v_k^0(x) \rightarrow r_0^m D_x v_0(x)$ as $k \rightarrow \infty$ for a.e. $x \in [0, \infty)$. On the other hand, recalling from (4.2.14) and (4.2.15), for each $x \in [0, \infty)$, we have:

$$|(r_k^0)^m D_x v_k^0|(x) \leq (\overline{C}_0 \underline{C}_0)^{m/n} \left(\tilde{C}_0^{1/2} |v_0(x) - 1| + |r_0^m D_x v_0|(x) \right) \in L^2(0, \infty).$$

Thus, by Dominated Convergence theorem, the following limit holds:

$$\lim_{k \rightarrow \infty} \int_0^\infty |(r_k^0)^m D_x v_k^0 - r_0^m D_x v_0|^2(x) dx = 0.$$

The L^2 -strong convergence for $(r_k^0)^m D_x u_k^0$ and for $(r_k^0)^m D_x e_k^0$ can be obtained by the exact same way. Next, we have already proven in the previous part that $(r_k^0)^{2m} D_x^2 v_k^0(x) \rightarrow r_0^{2m} D_x^2 v_0(x)$ as $k \rightarrow \infty$ for a.e. $x \in [0, \infty)$. In addition by the inequalities (4.2.14), (4.2.15), and (4.2.16) it follows that for each $x \in [0, \infty)$, we also have the bound

$$|(r_k^0)^{2m} D_x^2 v_k^0|(x) \leq (\overline{C}_0 \underline{C}_0)^{2m/n} \{ \tilde{C}_0 |v_0(x) - 1| + 2\tilde{C}_0^{1/2} |r_0^m D_x v_0|(x) + |r_0^{2m} D_x^2 v_0|(x) \} \in L^2(0, \infty).$$

By Dominated Convergence theorem:

$$\lim_{k \rightarrow \infty} \int_0^\infty |(r_k^0)^{2m} D_x^2 v_k^0 - r_0^{2m} D_x^2 v_0|^2(x) dx = 0.$$

The L^2 -strong convergence for $(r_k^0)^{2m} D_x^2 u_k^0$ and for $(r_k^0)^{2m} D_x^2 e_k^0$ can be obtained by the exact same way. \square

In what follows, we will abbreviate the index k and denote $(v, u, e) \equiv (v_k, u_k, e_k)$ for simplicity.

4.2.3 Local-in-time well-posedness

With the approximate initial data we have constructed in the previous section, we obtain local-in-time strong solution to the approximate problem (4.2.12) using Theorem 3.1.1 in Chapter 3. To do so, we rewrite the initial as a new set of unknowns:

$$\tau_k^0(x) := v_k^0(x)r_k^0(x)^{-m}, \quad u_k^0(x) = u_k^0(x), \quad e_k^0(x, t) = e_k^0(x, t),$$

where the coefficient $r_k^0(x)$ is defined by:

$$r_k^0(x) := \left(a^n + n \int_0^x v_k^0(y) dy \right)^{\frac{1}{n}}.$$

With this new initial data, Theorem 3.1.1 in Chapter 3 implies that there exists a time $T_* > 0$, and a unique solution $(\tau_k, u_k, e_k, r_k)(x, t)$ to (3.1.1) in the domain $(x, t) \in [0, k] \times [0, T_*]$, such that the following are satisfied:

$$\left\{ \begin{array}{l} r_k(x, t) = a + \int_0^x \tau(y, t) dy = r_k^0(x) + \int_0^t u_k(x, s) ds, \quad \text{for all } (x, t) \in [0, k] \times [0, T_*], \\ C^{-1} \leq r_k^m \tau_k \leq C, \quad \text{for some constant } C = C(C_0, T, a, k) \quad \text{and } (x, t) \in [0, k] \times [0, T], \\ \tau_k \in C^0([0, T]; H^2(0, k)) \cap C^1([0, T]; H^1(0, k)), \\ (u_k, e_k - 1) \in C^0([0, T]; H^2(0, k)) \cap C^1([0, T]; L^2(0, k)) \cap L^2(0, T; H^3(0, k)), \\ (D_x e_k, u_k) \in C^0([0, T]; H_0^1). \end{array} \right.$$

We then redefine as:

$$v_k(x, t) := (r_k(x, t))^m \tau_k(x, t),$$

then one can show that $v_k \in C^0([0, T]; H^2(0, k)) \cap C^1([0, T]; H^1(0, k))$ and

$$\left\{ \begin{array}{l} r_k(x, t) = \left(a^n + n \int_0^x v_k(y, t) dy \right)^{\frac{1}{n}}, \\ D_t v_k = D_x (r_k^m u_k) \quad \text{and} \quad C \leq v_k(x, t) \leq C, \end{array} \right. \quad \text{for a.e. } (x, t) \in [0, k] \times [0, T_*].$$

Once the local-in-time solution of the approximated problem (4.2.12) is obtained, we show that a stronger estimate is satisfied by the solution in a domain before the maximal time of existence $T_* > 0$. This will then enable us to extend $T_* > 0$ via bootstrap argument. Moreover, we also ensure that the estimate is in fact independent of the approximation index $k > 0$. Hence one can take the limit $k \rightarrow \infty$ to attain solution to the original exterior problem (1.3.6) via compactness argument.

4.3 Higher Regularity Estimates

Since the local-in-time strong solution $(v_k, u_k, e_k)(x, t)$ obtained in the previous section satisfies the same set of equations and boundary conditions, (2.2.16) stated in Section 2.2.5 of Chapter 2, Theorem 2.3.1 also holds for $(v_k, u_k, e_k)(x, t)$ in the domain $(x, t) \in [0, k] \times [0, T]$ with $T > 0$ being the maximal time of existence instead, and the proof follows exactly the same procedure. Hence in the present section, we will abbreviate repeated demonstrations and Theorem 2.3.1 will be used for the derivation of different set of high regularity estimates which is stated below:

Theorem 4.3.1 (Higher Regularity Estimate). *Let $T > 0$ be the maximum time of existence, and (v, u, e) be the solution given in Theorem 3.1.1. Recall the term \mathcal{H}_2 from Theorem 4.1.1:*

$$\begin{aligned} \mathcal{H}_2[v, u, e, r](x, t) := & \{ |v - 1|^2 + r^{2m} |D_x v|^2 + r^{4m} |D_x^2 v|^2 + |u|^4 + r^{2m} |D_x u|^2 + r^{4m} |D_x^2 u|^2 \}(x, t) \\ & + \{ |e - 1|^2 + r^{2m} |D_x e|^2 + r^{4m} |D_x^2 e|^2 \}(x, t). \end{aligned}$$

Then there exists a constant $C_0 > 0$ depending only on the initial data such that

$$\left\{ \begin{array}{l} \sup_{0 \leq t \leq T} \int_0^k \left\{ \frac{1}{2} |u|^2 + (\gamma - 1) \psi(v) + \psi(e) \right\} (x, t) dx \leq C_0, \\ \int_0^T \int_0^k \left\{ \kappa \frac{r^{2m} |D_x e|^2}{v e^2} + \left(\lambda + \frac{2\mu}{n} \right) \frac{|D_x(r^m u)|^2}{v e} + 2m\mu \frac{v}{e} \left(\frac{D_x(r^m u)}{v \sqrt{n}} - \sqrt{n} \frac{u}{r} \right)^2 \right\} dx dt \leq C_0. \end{array} \right. \quad (4.3.1)$$

Moreover, there exists $C_1 = C_1(a, T, C_0) > 0$ independent of the approximating index $k \in \mathbb{N}$ such that:

$$\left\{ \begin{array}{l} C_1(a)^{-1} \leq v(x, t), \quad e(x, t) \leq C_1(a) \quad \text{and} \quad |u(x, t)| \leq C_1(a) \quad \text{for} \quad (x, t) \in [0, k] \times [0, T], \\ \sup_{0 \leq t \leq T} \int_0^k \left\{ \mathcal{H}_2[v, u, e, r] + |D_t v|^2 + |D_t u|^2 + |D_t e|^2 \right\} (x, t) dx \leq C_1(a), \\ \int_0^T \int_0^k \left\{ r^{2m} |D_t D_x v|^2 + r^{2m} |D_t D_x u|^2 + r^{2m} |D_t D_x e|^2 \right\} (x, t) dx dt \leq C_1(a), \\ \int_0^T \int_0^k \left\{ r^{6m} |D_x^3 u|^2 + r^{6m} |D_x^3 e|^2 \right\} (x, t) dx dt \leq C_1(a). \end{array} \right. \quad (4.3.2)$$

Moreover, for each $\varepsilon > 0$, there exists a constant $C_2(\varepsilon) = C_2(\varepsilon, T, C_0) > 0$ independent of $a > 0$ and $k \in \mathbb{N}$ such that

$$\left\{ \begin{array}{l} C_2(\varepsilon)^{-1} \leq v(x, t) \leq C_2(\varepsilon), \quad 0 \leq e(x, t) \leq C_2(\varepsilon), \quad |u(x, t)| \leq C_2(\varepsilon) \quad \text{for} \quad (x, t) \in [\varepsilon, k] \times [0, T], \\ \sup_{0 \leq t \leq T} \int_\varepsilon^k \left\{ \mathcal{H}_2[v, u, e, r] + |D_t v|^2 + |D_t u|^2 + |D_t e|^2 \right\} (x, t) dx \leq C_2(\varepsilon), \\ \int_0^T \int_\varepsilon^k \left\{ r^{2m} |D_t D_x v|^2 + r^{2m} |D_t D_x u|^2 + r^{2m} |D_t D_x e|^2 \right\} (x, t) dx dt \leq C_2(\varepsilon), \\ \int_0^T \int_\varepsilon^k \left\{ r^{6m} |D_x^3 u|^2 + r^{6m} |D_x^3 e|^2 \right\} (x, t) dx dt \leq C_2(\varepsilon). \end{array} \right. \quad (4.3.3)$$

In what follows, we mainly illustrate the statement (4.3.3), as (4.3.2) is easier to show. Moreover, it is important to note that we will be using the same cut-off functions $g_\varepsilon \in C^1$ introduced in (2.3.21)–(2.3.23), Section 2.3.3 of Chapter 2.

4.3.1 $L^\infty(0, T; H^1)$ -estimate of specific volume v

In this subsection, we prove the r^m -weighted L^2 bound of $D_x v$:

Lemma 4.3.1 (Exterior $L_T^\infty L^2$ estimate of $D_x v$). *Given $\varepsilon > 0$ and $T > 0$, we have the estimate*

$$\int_\varepsilon^k r^{2m} |D_x v|^2(x, t) dx \leq C(\varepsilon) \quad \forall t \in [0, T],$$

where $C(\varepsilon) = C(\varepsilon, T, C_0, n, \beta, \gamma, \kappa) > 0$ independent of $k > 0$ and $a > 0$.

Proof. First, using continuity equation (4.2.12)₁ in the momentum equation (4.2.12)₂, we have:

$$\beta D_t \left(r^m D_x \log v - \frac{u}{\beta} \right) = r^m D_x P + m\beta r^{m-1} u D_x \log v. \quad (4.3.4)$$

Multiplying both sides of (4.3.4) by $r^m D_x \log v - u$ and integrating in $x \in [\varepsilon, k]$ we obtain:

$$\begin{aligned} & \beta \frac{d}{dt} \int_\varepsilon^k \left| r^m D_x \log v - \frac{u}{\beta} \right|^2 dx \\ & \leq 2 \int_\varepsilon^k \frac{r^{2m}}{v} |D_x e|^2 dx + C(\varepsilon) \int_\varepsilon^k |u|^2 dx + C(\varepsilon) \sup_{\varepsilon \leq y \leq k} (1 + e^2 + u^2) \int_\varepsilon^k |r^m D_x \log v|^2 dx, \end{aligned} \quad (4.3.5)$$

where the coefficients $C(\varepsilon) = C(\varepsilon, T, C_0)$ is independent of $a \in (0, 1)$ and $k \in \mathbb{N}$. Now integrating (4.3.5) in time, using Theorem 2.3.1(i), it follows that:

$$\beta \int_{\varepsilon}^k |r^m D_x \log v|^2 dx \leq C_0 + C_3 \int_0^t \sup_{\varepsilon \leq y \leq k} (1 + e^2 + u^2)(y, s) \int_{\varepsilon}^k |r^m D_x \log v|^2 dx ds.$$

Applying Grönwall Inequality, we obtain the following estimate:

$$\int_{\varepsilon}^k |r^m D_x \log v|^2(x, t) dx \leq \frac{C(\varepsilon)}{\beta} \exp\left(\frac{C_3}{\beta} \int_0^t \left\{1 + \sup_{\varepsilon \leq y \leq k} (e^2 + u^2)(y, s)\right\} ds\right).$$

Using Lemma 2.3.3–2.3.4, and Theorem 2.3.1(ii), we obtain the desired estimate. \square

4.3.2 $L^\infty(0, T; H^1)$ -estimate of (u, e)

Lemma 4.3.2 (Exterior $L_T^\infty H^1$ -estimate of u). *Given $\varepsilon > 0$ and $T > 0$, there exists a constant $C(\varepsilon) = C(\varepsilon, T, C_0, \beta, \gamma, \kappa) > 0$ independent of $k > 0$ and $a > 0$ such that:*

$$\int_0^T \int_{\varepsilon}^k \left\{ |D_t u|^2 + \frac{|D_t(r^m u)|^2}{r^{2m}} \right\} dx dt + \sup_{t \in [0, T]} \int_{\varepsilon}^k \left\{ r^{2m} |D_x u|^2 + |D_x(r^m u)|^2 + |D_t v|^2 \right\} (x, t) dx \leq C_0 + C(\varepsilon).$$

Proof. Multiplying momentum equation (4.2.12)₂ with $g_\varepsilon(x) D_t u$ and integrating by parts in $x \in [0, k]$, using boundary condition $D_t u(k, t) = 0$ for all $t \in [0, T]$, we have:

$$\begin{aligned} & \int_0^k g_\varepsilon |D_t u|^2 dx + \frac{\beta}{2} \frac{d}{dt} \int_0^k g_\varepsilon \frac{r^{2m}}{v} |D_x u|^2 dx \\ & \leq \frac{1}{4} \int_0^k g_\varepsilon |D_t u|^2 dx + C(\varepsilon) \left\{ 1 + \sup_{\varepsilon/2 \leq y \leq k} e^2(y, t) + \int_{\varepsilon/2}^k \frac{r^{2m}}{v} (|D_x e|^2 + |D_x u|^2) dx \right\} \\ & \quad + m\beta \sup_{\varepsilon/2 \leq y \leq k} \frac{|u|}{r}(y, t) \int_0^k g_\varepsilon \frac{r^{2m}}{v} |D_x u|^2 dx - \frac{\beta}{2} \int_0^k g_\varepsilon \frac{D_x(r^m u)}{v} \frac{r^{2m}}{v} |D_x u|^2 dx, \end{aligned} \quad (4.3.6)$$

where we have used (2.3.21)–(2.3.23), Theorem 2.3.1(i)–(ii), and Lemma 2.3.4–2.3.6, and Lemma 4.3.1 with the given constant ε replaced by $\varepsilon/2$. Next, with few lines of calculation, once can verify that

$$\frac{D_x(r^m u)}{v} \frac{r^{2m}}{v} |D_x u|^2 = \left| \frac{D_x(r^m u)}{v} \right|^2 D_x(r^m u) - 2m \frac{u}{r} \frac{r^{2m}}{v} |D_x u|^2 - 3m^2 r^{m-2} u^2 D_x u - m^3 \frac{v}{r^3} u^3.$$

Using this, we have

$$\begin{aligned} & - \int_0^k g_\varepsilon \frac{D_x(r^m u)}{v} \frac{r^{2m}}{v} |D_x u|^2 dx \\ & \leq C(\varepsilon) - \int_0^k g_\varepsilon \left| \frac{D_x(r^m u)}{v} \right|^2 D_x(r^m u) dx + \left\{ \frac{3m^2}{2} + 2m \sup_{\varepsilon/2 \leq y \leq k} \frac{|u|}{r}(y, t) \right\} \int_0^k g_\varepsilon \frac{r^{2m}}{v} |D_x u|^2 dx \end{aligned}$$

Substituting the above into (4.3.6) and integrating in time, we have

$$\begin{aligned} & \frac{3}{4} \int_0^t \int_0^k g_\varepsilon |D_t u|^2 dx ds + \frac{\beta}{2} \int_0^k g_\varepsilon \frac{r^{2m}}{v} |D_x u|^2(x, t) dx - \frac{\beta}{2} \int_0^k g_\varepsilon \frac{(r_k^0)^{2m}}{v_k^0} |D_x u_k^0|^2(x, t) dx \\ & \leq C(\varepsilon) - \frac{\beta}{2} \int_0^t \int_0^k g_\varepsilon \left| \frac{D_x(r^m u)}{v} \right|^2 D_x(r^m u) dx ds + C(\varepsilon) \int_0^t \sup_{\varepsilon/2 \leq y \leq k} |u| \int_0^k g_\varepsilon \frac{r^{2m}}{v} |D_x u|^2 dx ds, \end{aligned} \quad (4.3.7)$$

where we used Lemma 2.3.4. Next we recall effective viscosity flux: $F := \beta v^{-1} D_x(r^m u) - (\gamma - 1)v^{-1}e + (\gamma - 1)$ and $\frac{D_t u}{r^m} = D_x F$. Using this, we rewrite the following term:

$$I := -\frac{\beta}{2} \int_0^t \int_0^k g_\varepsilon \left| \frac{D_x(r^m u)}{v} \right|^2 D_x(r^m u) dx ds = \sum_{i=1}^3 I^{(i)}.$$

where we will indicate each term $I^{(i)}$ as we proceed to estimate it. First, for $I^{(1)}$, integrating by parts once, and using Lemma 2.3.4 and 2.3.7 we get

$$\begin{aligned} I^{(1)} &:= -\frac{1}{2\beta} \int_0^t \int_0^k g_\varepsilon F^2 D_x(r^m u) dx ds = \frac{1}{\beta} \int_0^t \int_0^k g_\varepsilon u F r^m D_x F dx ds + \frac{1}{2\beta} \int_0^t \int_0^k g'_\varepsilon F^2 r^m u dx ds \\ &\leq C(\varepsilon) + \frac{1}{4} \int_0^t \int_0^k g_\varepsilon |D_t u|^2 dx ds + C(\varepsilon) \int_0^t \sup_{y \in [\varepsilon/2, k]} |u|^2(y, s) \int_0^k g_\varepsilon \frac{|D_x(r^m u)|^2}{v} dx ds. \end{aligned}$$

Next, $I^{(2)}$ is estimated using Lemma 2.3.7 and Lemma 2.3.4–2.3.6:

$$\begin{aligned} I^{(2)} &:= -\frac{1}{\beta} \int_0^t \int_0^k g_\varepsilon F (P - (\gamma - 1)) D_x(r^m u) dx ds = -\frac{(\gamma - 1)}{\beta} \int_0^t \int_0^k g_\varepsilon F \frac{e - v}{v} D_x(r^m u) dx ds \\ &\leq C(\varepsilon) + C(\varepsilon) \int_0^t \sup_{y \in [\varepsilon/2, k]} |e(y, s)|^2 \int_0^k g_\varepsilon \frac{|D_x(r^m u)|^2}{v} dx ds \end{aligned}$$

Finally, we estimate $I^{(3)}$ using Lemma 2.3.7 and Lemma 2.3.4–2.3.6:

$$\begin{aligned} I^{(3)} &:= -\frac{1}{2\beta} \int_0^t \int_0^k g_\varepsilon (P - (\gamma - 1))^2 D_x(r^m u) = -\frac{(\gamma - 1)^2}{2\beta} \int_0^t \int_0^k g_\varepsilon \left\{ \frac{e - 1}{v} + \frac{1 - v}{v} \right\}^2 D_x(r^m u) \\ &\leq \frac{(\gamma - 1)^2}{2\beta} \int_0^t \int_0^k g_\varepsilon \left(\frac{(e - 1)^4}{v^3} + \frac{(v - 1)^4}{v^3} \right) dx ds + \frac{(\gamma - 1)^2}{2\beta} \int_0^t \int_0^k g_\varepsilon \frac{|D_x(r^m u)|^2}{v} dx ds \leq C(\varepsilon). \end{aligned}$$

In summary, we have the following estimate for I :

$$I \leq C(\varepsilon) + C(\varepsilon) \int_0^t \sup_{y \in [\varepsilon/2, k]} (e^2 + u^2)(y, s) \int_0^k g_\varepsilon \frac{|D_x(r^m u)|^2}{v} dx ds + \frac{1}{4} \int_0^t \int_0^k g_\varepsilon |D_t u|^2 dx ds.$$

Using the inequality: $v^{-1}|D_x(r^m u)|^2 \leq 2v^{-1}r^{2m}|D_x u|^2 + 2m^2r^{-2}vu^2$, it follows from (2.3.21)–(2.3.23), Theorem 2.3.1(i)–(ii), and Lemmas 2.3.3–2.3.4 that

$$I \leq C(\varepsilon) + C(\varepsilon) \int_0^t \sup_{y \in [\varepsilon/2, k]} (e^2 + u^2)(y, s) \int_0^k g_\varepsilon \frac{r^{2m}}{v} |D_x u|^2 dx ds + \frac{1}{4} \int_0^t \int_0^k g_\varepsilon |D_t u|^2 dx ds.$$

Substituting this estimate for I into (4.3.7), we get using (2.3.21)–(2.3.23) that

$$\begin{aligned} &\frac{1}{2} \int_0^t \int_0^k g_\varepsilon |D_t u|^2 dx ds + \frac{\beta}{2} \int_0^k g_\varepsilon \frac{r^{2m}}{v} |D_x u|^2(x, t) dx \\ &\leq C(\varepsilon) + C_0 + C(\varepsilon) \int_0^t \sup_{\varepsilon/2 \leq y \leq k} \{1 + u^2 + e^2\}(y, s) \int_0^k g_\varepsilon \frac{r^{2m}}{v} |D_x u|^2 dx ds \end{aligned}$$

Applying Grönwall Inequality, we finally conclude using Lemma 2.3.3–2.3.4 that

$$\frac{1}{2} \int_0^t \int_0^k g_\varepsilon |D_t u|^2 dx ds + \frac{\beta}{2} \int_0^k g_\varepsilon \frac{r^{2m}}{v} |D_x u|^2(x, t) dx \leq C_0 + C(\varepsilon).$$

Moreover, by Theorem 2.3.1(i) and (2.3.21)–(2.3.23), we also have

$$\begin{aligned} \int_0^T \int_0^k g_\varepsilon \frac{D_t(r^m u)}{r^m} dx dt &\leq \int_0^T \int_0^k g_\varepsilon \left\{ |D_t u|^2 + m \frac{|u|^4}{r^2} \right\} dx dt \leq C_0 + C(\varepsilon), \\ \int_0^k g_\varepsilon |D_x(r^m u)|^2 dx &\leq \int_0^k g_\varepsilon \left\{ r^{2m} |D_x u|^2 + m^2 \frac{v^2}{r^2} |u|^2 \right\} dx \leq C_0 + C(\varepsilon). \end{aligned}$$

By the continuity equation, $D_t v = D_x(r^m u)$, and Theorem 2.3.1(ii), we have in addition that

$$\int_0^k g_\varepsilon |D_t v|^2 dx = \int_0^k g_\varepsilon v \frac{|D_x(r^m u)|^2}{v} dx \leq C_0 + C(\varepsilon).$$

□

As an consequence of Lemma 4.3.2 and Sobolev Embedding Theorem, we obtain the following corollary:

Corollary 4.3.1. *Given $\varepsilon > 0$ and $T > 0$, there exists a constant $C(\varepsilon) = C(\varepsilon, T, C_0, n, \beta, \gamma, \kappa) > 0$ independent of $k > 0$ and $a > 0$ such that:*

$$|u(x, t)| \leq C(\varepsilon) \quad \forall (x, t) \in [\varepsilon, k] \times [0, T].$$

Proof. Using Sobolev Embedding theorem for 1D: $W^{1,1} \hookrightarrow C^0$, we have that

$$\begin{aligned} \sup_{y \in [\varepsilon, k]} u^2(y, t) &\leq C \int_\varepsilon^k u^2 dx + C \int_\varepsilon^k |u D_x u| dx = C \int_\varepsilon^k u^2 dx + C \int_\varepsilon^k |r^{-m} u D_x(r^m u) - m r^{-n} v u^2| dx \\ &\leq C_0 + C \sup_{y \in [\varepsilon, k]} \left(\frac{v}{r^{2m}} + m \frac{v}{r^n} \right)(y, t) \int_\varepsilon^k u^2(x, t) dx + C \int_\varepsilon^k \frac{|D_x(r^m u)|^2}{v}(x, t) dx \leq C(\varepsilon). \end{aligned}$$

Thus it follows that $\sup_{\varepsilon/2 \leq y \leq k} |u(y, t)| \leq C(\varepsilon)$ for all $t \in [0, T]$. □

Lemma 4.3.3 (Exterior $L_T^2 \dot{W}^{1,\infty}$ estimate of u). *Given $\varepsilon > 0$ and $T > 0$, there exists a constant $C(\varepsilon) = C(\varepsilon, T, C_0, n, \beta, \gamma, \kappa) > 0$ independent of $k \in \mathbb{N}$ and $a \in (0, 1)$ such that:*

$$\int_0^T \sup_{x \in [\varepsilon, k]} |D_x(r^m u)|^2(x, t) dt + \int_0^T \int_\varepsilon^k \{ r^{2m} |D_x^2(r^m u)|^2 + r^{4m} |D_x^2 u|^2 \}(x, t) dx dt \leq C_0 + C(\varepsilon).$$

Proof. Taking L^2 -norm in the region $(x, t) \in [\varepsilon, k] \times [0, T]$ on both sides of momentum equation (4.2.12)₂, and using (2.3.21)–(2.3.23), Lemmas 2.3.4–2.3.6, and 4.3.1, we obtain that:

$$\beta^2 \int_0^T \int_\varepsilon^k \frac{r^{2m}}{v^2} |D_x^2(r^m u)|^2 dx dt \leq C_0 + C(\varepsilon) + C(\varepsilon) \int_0^T \sup_{x \in [\varepsilon, k]} |D_x(r^m u)|^2(x, t) dt.$$

Now, from the Sobolev Embedding Theorem: $W^{1,1} \hookrightarrow C^0$, it follows that

$$\begin{aligned} C(\varepsilon) \int_0^T \sup_{x \in [\varepsilon, k]} |D_x(r^m u)|^2(x, t) dt &\leq C(\varepsilon) \int_0^T \int_\varepsilon^k |D_x(r^m u)|^2 + C(\varepsilon) \int_0^T \int_\varepsilon^k |D_x(r^m u) D_x^2(r^m u)| \\ &\leq C(\varepsilon) + \frac{\beta^2}{2} \int_0^T \int_\varepsilon^k t \frac{r^{2m}}{v^2} |D_x^2(r^m u)|^2 dx dt. \end{aligned}$$

Substituting this back into the previous equation, we obtain:

$$\frac{\beta^2}{2} \int_0^T \int_\varepsilon^k \frac{r^{2m}}{v^2} |D_x^2(r^m u)|^2 dx dt \leq C_0 + C(\varepsilon).$$

Finally using again the Sobolev Embedding Theorem, we have

$$\int_0^T \sup_{y \in [\varepsilon, k]} |D_x(r^m u)|^2(y, t) dt \leq C \int_0^T \left(\|D_x(r^m u)\|_{L^2(\varepsilon, k)}^2 + \left\| \frac{v^2}{r^{2m}} \right\|_{L^\infty(\varepsilon, k)} \left\| \frac{r^m}{v} D_x^2(r^m u) \right\|_{L^2(\varepsilon, k)}^2 \right) \leq C(\varepsilon).$$

Finally, we have $r^{2m} D_x^2 u = r^m D_x^2(r^m u) - m r^{m-1} u D_x v + m r^{-2} u v^2 - 2 m v r^{m-1} D_x u$. Hence, taking L^2 -norm, it follows from (2.3.21)–(2.3.23), Theorem 2.3.1(i)–(ii), and Lemma 2.3.3–2.3.6 that

$$\int_0^T \int_\varepsilon^k r^{4m} |D_x^2 u|^2 dx dt \leq C_0 + C(\varepsilon).$$

□

Corollary 4.3.2. *Given $\varepsilon > 0$ and $T > 0$, there exists a constant $C(\varepsilon) = C(\varepsilon, T, C_0, n, \beta, \gamma, \kappa) > 0$ independent of $k > 0$ and $a > 0$ such that:*

$$\int_0^T \int_\varepsilon^k r^{2m} |D_x D_t v|^2(x, t) dx dt \leq C_0 + C(\varepsilon).$$

Proof. By continuity equation $D_t v = D_x(r^m u)$ and above lemma, the corollary follows. □

Lemma 4.3.4 (Exterior $L_T^\infty \dot{H}^1$ estimate of e). *Given $\varepsilon > 0$ and $T > 0$, there exists a constant $C(\varepsilon) = C(\varepsilon, T, C_0) > 0$ independent of $k \in \mathbb{N}$ and $a \in (0, 1)$ such that:*

$$\sup_{0 \leq t \leq T} \int_\varepsilon^k r^{2m} |D_x e|^2(x, t) dx + \int_0^T \int_\varepsilon^k |D_t e|^2 dx dt \leq C_0 + C(\varepsilon).$$

Proof. Multiplying the energy equation (4.2.12) with $g_\varepsilon D_t e$ and integrating in $x \in [0, k]$, using the boundary condition $D_x e(k, t) = 0$ for all $t \in [0, T]$, (2.3.21)–(2.3.23), Theorem 2.3.1(ii) and Cauchy-Schwartz inequality, we have

$$\begin{aligned} & \frac{\kappa}{2} \frac{d}{dt} \int_0^k g_\varepsilon \frac{r^{2m}}{v} |D_x e|^2 dx + \int_0^k g_\varepsilon |D_t e|^2 dx \\ & \leq C(\varepsilon) \left(\sup_{\varepsilon/2 \leq y \leq k} |u(y, t)|^2 + \sup_{y \in [\varepsilon/2, k]} |D_x(r^m u)|^2 \right) + 16m^2 \mu^2 \int_0^k g_\varepsilon |D_x(r^{m-1} u^2)|^2 dx \\ & \quad + \left(\int_0^k g_\varepsilon \frac{r^{2m}}{v} |D_x e|^2(x, t) dx \right)^2 + C(\varepsilon) \int_{\varepsilon/2}^k \frac{r^{2m}}{v} |D_x e|^2(x, t) dx \\ & \quad + C(\varepsilon) \sup_{y \in [\varepsilon/2, k]} \{ |D_x(r^m u)(y, t)|^2 + e^2(y, t) \} \int_0^k g_\varepsilon |D_x(r^m u)|^2(x, t) dx. \end{aligned} \tag{4.3.8}$$

Moreover, using the identity: $D_x(r^{m-1} u^2) = 2r^{-1} u D_x(r^m u) - n v r^{-2} u^2$, it follows from (2.3.21)–(2.3.23), Theorem 2.3.1(i)–(ii), Lemma 4.3.2 and Corollary 4.3.1 that:

$$\int_0^k g_\varepsilon |D_x(r^{m-1} u^2)|^2 dx \leq \left\| \frac{v u}{r} \right\|_{L^\infty(\varepsilon/2, k)}^2 \int_0^k g_\varepsilon \frac{|D_x(r^m u)|^2}{v} dx + n^2 \left\| \frac{v u}{r^2} \right\|_{L^\infty(\varepsilon/2, k)}^2 \int_0^k g_\varepsilon |u|^2 dx \leq C(\varepsilon).$$

Substituting the above estimate in (4.3.8), integrating with respect to time in the interval $t \in [0, T]$ and using Theorem 2.3.1(i), Lemmas 2.3.4–2.3.6, and Lemmas 4.3.1–4.3.3 we obtain that

$$\begin{aligned} & \int_0^k g_\varepsilon \frac{r^{2m}}{v} |D_x e|^2 dx + \int_0^t \int_0^k g_\varepsilon |D_t e|^2 dx ds \\ & \leq C(\varepsilon) + \int_0^k g_\varepsilon \frac{(r_k^0)^{2m}}{v_k^0} |D_x e_k^0|^2 dx + C(\varepsilon) \int_0^t \left(\int_{\varepsilon/2}^k \frac{r^{2m}}{v} |D_x e|^2(x, s) dx \right) \int_0^k g_\varepsilon \frac{r^{2m}}{v} |D_x e|^2(x, s) dx ds. \end{aligned}$$

By Grönwall inequality, it then follows from Lemma 2.3.4 that

$$\int_0^k g_\varepsilon \frac{r^{2m}}{v} |D_x e|^2 dx + \int_0^t \int_0^k g_\varepsilon |D_t e|^2 dx ds \leq C_0 + C(\varepsilon).$$

□

As an consequence of Lemma 4.3.4 and Sobolev Embedding Theorem, we obtain the following corollary:

Corollary 4.3.3. *Given $\varepsilon > 0$ and $T > 0$, there exists a constant $C(\varepsilon) = C(\varepsilon, T, C_0, n, \beta, \gamma, \kappa) > 0$ independent of $k > 0$ and $a > 0$ such that:*

$$|e(x, t)| \leq C(\varepsilon) \quad \forall (x, t) \in [\varepsilon, k] \times [0, T].$$

Proof. Using Sobolev Embedding theorem for 1D: $H^1 \hookrightarrow C^0$, we have that

$$\begin{aligned} & \sup_{y \in [\varepsilon, k]} (e - 1)^2(y, t) \leq C \int_\varepsilon^k (e - 1)^2(x, t) dx + C \int_\varepsilon^k |e - 1| \cdot |D_x e|(x, t) dx \\ & \leq C \sup_{y \in [\varepsilon, k]} \left(1 + \frac{v}{r^{2m}}(y, t) \right) \int_\varepsilon^k (e - 1)^2(x, t) dx + \int_\varepsilon^k \frac{r^{2m}}{v} |D_x e|^2(x, t) dx \leq C(\varepsilon). \end{aligned}$$

From this, we have that $e(y, t) \leq 1 + \sqrt{C(\varepsilon)}$ for each $(y, t) \in [\varepsilon, k] \times [0, T]$. hence, taking supremum over $y \in [\varepsilon, k]$ on both sides, and taking the square root, we get the desired bound. □

4.3.3 $L^\infty(0, T; H^2)$ –estimate of (u, e)

Lemma 4.3.5 (Exterior $L_T^\infty L^2$ estimate of $D_t u$). *Given $\varepsilon > 0$ and $T > 0$, there exists a constant $C(\varepsilon) = C(\varepsilon, T, C_0, n, \beta, \gamma, \kappa) > 0$ independent of $k \in \mathbb{N}$ and $a \in (0, 1)$ such that:*

$$\int_0^T \int_\varepsilon^k \left\{ r^{2m} |D_x D_t u|^2 + |D_t D_x(r^m u)|^2 \right\} dx dt + \sup_{t \in [0, T]} \int_\varepsilon^k \left\{ |D_t u|^2 + \frac{|D_t(r^m u)|^2}{r^{2m}} \right\} (x, t) dx \leq C_0 + C(\varepsilon).$$

Proof. First, differentiating the momentum equation (4.2.12)₂ with respect to time, then multiplying both side of the resulting equation by $g_\varepsilon D_t(r^m u)$ and integrating in $x \in [0, k]$, by Cauchy-Schwartz inequality and the boundary condition $D_t u(k, t) = 0$ for all $t \in [0, T]$, it follows that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_0^k g_\varepsilon \frac{|D_t(r^m u)|^2}{r^{2m}} dx + \beta \int_0^k g_\varepsilon \frac{|D_t D_x(r^m u)|^2}{v} dx \\ & \leq \frac{\beta}{2} \int_0^k g_\varepsilon \frac{|D_t D_x(r^m u)|^2}{v} dx + C(\varepsilon) \left\{ \int_{\varepsilon/2}^k \left(\frac{|D_t(r^m u)|^2}{r^{2m}} + \left| D_t \left(\frac{e}{v} \right) \right|^2 \right) dx + |D_x(r^m u)|_{L^\infty(\varepsilon/2, k)}^2 \right\}, \end{aligned} \quad (4.3.9)$$

where in the last line we used (2.3.21)–(2.3.23), Theorem 2.3.1(i)–(ii), Lemma 4.3.2, and Corollary 4.3.1. Next, we write using continuity equation $D_t v = D_x(r^m u)$ that:

$$\int_{\varepsilon/2}^k \left| D_t \left(\frac{e}{v} \right) \right|^2 dx = \int_{\varepsilon/2}^k \left| \frac{D_t e}{v} - \frac{e}{v^2} D_x(r^m u) \right|^2 dx \leq C(\varepsilon) \left(\int_{\varepsilon/2}^k |D_t e|^2 dx + |e(\cdot, t)|_{L^\infty(\varepsilon/2, k)}^2 \right). \quad (4.3.10)$$

Now, substituting (4.3.10) into (4.3.9) and integrating the resulting inequality in $[0, t]$ for each $t \in (0, T]$, it follows from Theorem 2.3.1(i), Lemma 2.3.3, and Lemmas 4.3.2–4.3.4 that

$$\frac{1}{2} \int_0^k g_\varepsilon \frac{|D_t(r^m u)|^2}{r^{2m}}(\cdot, t) + \frac{\beta}{2} \int_0^t \int_0^k g_\varepsilon \frac{|D_t D_x(r^m u)|^2}{v} \leq \frac{1}{2} \limsup_{\tau \rightarrow 0^+} \int_0^k g_\varepsilon \frac{|D_t(r^m u)|^2}{r^{2m}}(\cdot, \tau) + C(\varepsilon). \quad (4.3.11)$$

Using momentum equation (4.2.12)₂, and the initial condition from Proposition 4.2.2, it follows that

$$\begin{aligned} & \limsup_{\tau \rightarrow 0^+} \int_0^k g_\varepsilon \frac{|D_t(r^m u)|^2}{r^{2m}}(x, \tau) dx \\ & \lesssim \int_0^k g_\varepsilon \left\{ \left| \frac{v_0 u_0}{r_0^2} \right|^2 + |r_0^{m-1} D_x u_0|^2 + \frac{r_0^{4m}}{v_0^4} |D_x v_0 D_x u_0|^2 + \frac{|r_0^{2m} D_x^2 u_0|^2}{v_0^2} + \frac{|r_0^m D_x e_0|^2}{v_0^2} \right\} dx \\ & \quad + \int_0^k g_\varepsilon \left\{ \frac{|e_0 - 1|^2}{v_0^4} |r_0^m D_x v_0|^2 + \frac{|r_0^m D_x v_0|^2}{v_0^4} + \frac{|u_0|^2}{r_0^4} \right\} dx \leq C_0 + C(\varepsilon). \end{aligned}$$

Substituting these estimates into (4.3.11), we have that

$$\frac{1}{2} \int_0^k g_\varepsilon \frac{|D_t(r^m u)|^2}{r^{2m}}(x, t) dx + \frac{\beta}{2} \int_0^t \int_0^k g_\varepsilon \frac{|D_t D_x(r^m u)|^2}{v} dx \leq C_0 + C(\varepsilon).$$

Finally, we also obtain that

$$\int_0^k g_\varepsilon |D_t u|^2(x, t) dx = \int_0^k g_\varepsilon \left| \frac{D_t(r^m u)}{r^m} - m \frac{u^2}{r} \right|^2(x, t) dx \leq C_0 + C(\varepsilon),$$

and with the identity: $r^m D_x D_t u = D_x D_t(r^m u) - m(m-1)vr^{-2}u^2 + 2mr^{m-1}u D_x u + mr^{-1}v D_t u$, it follows that

$$\int_0^T \int_0^k g_\varepsilon r^{2m} |D_x D_t u|^2 dx dt \leq C(\varepsilon).$$

□

In addition, we can also obtain a time-weighted version of above estimate due to the parabolic structure of the equation. It is important to note that this estimate does not require the initial condition:

$$\int_0^k (r_0^0)^{4m} |D_x^2 u_0^0|^2 dx \leq C_0.$$

Lemma 4.3.6 (Time Weighted Exterior $L_T^\infty L^2$ estimate of $D_t u$). *Given $\varepsilon > 0$ and $T > 0$, there exists a constant $C(\varepsilon) = C(\varepsilon, T, C_0, n, \beta, \gamma, \kappa) > 0$ independent of $k > 0$ and $a > 0$ such that:*

$$\begin{aligned} & \int_0^T \int_\varepsilon^k \sigma(t) \left\{ r^{2m} |D_x D_t u|^2 + |D_t D_x(r^m u)|^2 \right\}(x, t) dx dt + \sup_{t \in [0, T]} \int_\varepsilon^k \sigma(t) \left\{ |D_t u|^2 + \frac{|D_t(r^m u)|^2}{r^{2m}} \right\}(x, t) dx \\ & \leq C(\varepsilon), \quad \text{where } \sigma(t) = \min\{1, t\}. \end{aligned}$$

The proof for this lemma is exactly the same as previous Lemma 4.3.5 except that we make use of the time weighted estimates of Lemma 2.3.9 in Section 2.3.4. Thus we will abbreviate the detailed derivations.

Lemma 4.3.7 (Exterior $L_T^\infty H^2$ -Estimate of u). *Given $\varepsilon > 0$ and $T > 0$, there exists a constant $C(\varepsilon) = C(\varepsilon, T, C_0, n, \beta, \gamma, \kappa) > 0$ independent of $k \in \mathbb{N}$ and $a \in (0, 1)$ such that:*

$$\sup_{t \in [0, T]} \int_\varepsilon^k \left\{ r^{2m} |D_x^2(r^m u)|^2 + r^{4m} |D_x^2 u|^2 \right\}(x, t) dx \leq C_0 + C(\varepsilon).$$

Proof. From momentum equation (4.2.12)₂, we have that

$$\beta r^m D_x^2(r^m u) = v D_t u - (\gamma - 1) r^m \frac{e}{v} D_x v + (\gamma - 1) D_x e + \beta D_x(r^m u) r^m \frac{D_x v}{v}.$$

Taking $L^2(\varepsilon, k)$ -norm on both sides of the above equation. It follows from (2.3.21)–(2.3.23), Theorem 2.3.1(ii), Corollary 4.3.3, and Lemma 4.3.1, 4.3.3, 4.3.5 that:

$$\beta^2 \int_{\varepsilon}^k r^{2m} |D_x^2(r^m u)|^2(x, t) dx \leq C_0 + C(\varepsilon) + C(\varepsilon) \sup_{y \in [\varepsilon, k]} |D_x(r^m u)(y, t)|^2. \quad (4.3.12)$$

Now, using the Sobolev embedding theorem: $W^{1,1}(\varepsilon, k) \hookrightarrow C^0(\varepsilon, k)$, (2.3.21)–(2.3.23), and Lemma 4.3.2 we have

$$\sup_{y \in [\varepsilon, k]} |D_x(r^m u)(y, t)|^2 \leq C(\varepsilon) + \frac{\beta^2}{2C(\varepsilon)} \int_{\varepsilon}^k r^{2m} |D_x^2(r^m u)|^2 dx. \quad (4.3.13)$$

Substituting the above estimate into (4.3.12), it follows that:

$$\frac{\beta^2}{2} \int_k^{\varepsilon} r^{2m} |D_x^2(r^m u)|^2 dx \leq C_0 + C(\varepsilon). \quad (4.3.14)$$

Next, by computation, one obtains: $D_x^2(r^m u) = m r^{-1} u D_x v - r^{-m-2} v^2 u + 2m r^{-1} v D_x u + r^m D_x^2 u$. Multiplying both sides of this identity by r^m and then taking L^2 -norm in the domain $x \in [\varepsilon, k]$, it follows from Theorem 2.3.1(i)–(ii), Lemmas 4.3.1–4.3.2, Corollary 4.3.1 and estimate (4.3.14) that:

$$\int_{\varepsilon}^k r^{4m} |D_x^2 u|^2 dx \leq C_0 + C(\varepsilon).$$

This proves the lemma. \square

Using Lemma 4.3.6, we also obtain the following time weighted $L^\infty(0, T; H^2(0, k))$ estimates without requiring its corresponding initial condition.

Lemma 4.3.8 (Time Weighted Exterior $L_T^\infty H^2$ -Estimate of u). *Given $\varepsilon > 0$ and $T > 0$, there exists a constant $C(\varepsilon) = C(\varepsilon, T, C_0, n, \beta, \gamma, \kappa) > 0$ independent of $k \in \mathbb{N}$ and $a \in (0, 1)$ such that:*

$$\sup_{t \in [0, T]} \int_{\varepsilon}^k \sigma(t) \{r^{2m} |D_x^2(r^m u)|^2 + r^{4m} |D_x^2 u|^2\}(x, t) dx \leq C(\varepsilon).$$

The proof for this lemma follows exactly the same procedure as the previous Lemma 4.3.7, except that we use the time weighted estimates of Lemma 2.3.9 in Section 2.3.4. Thus we will not repeat the derivation here.

Lemma 4.3.9 (Exterior $L_T^\infty L^2$ -Estimate of e). *Given $\varepsilon > 0$ and $T > 0$, there exists a constant $C(\varepsilon) = C(\varepsilon, T, C_0, n, \beta, \gamma, \kappa) > 0$ independent of $k > 0$ and $a > 0$ such that:*

$$\sup_{t \in [0, T]} \int_{\varepsilon}^k |D_t e|^2(x, t) dx + \int_0^T \int_{\varepsilon}^k r^{2m} |D_x D_t e|^2(x, t) dx dt \leq C_0 + C(\varepsilon).$$

Proof. We first differentiate the internal energy equation (4.2.12)₃ with respect to time, then multiply both sides by $g_\varepsilon D_t e$ and integrating by parts, it follows from $D_t D_x e(k, t) = D_x e(k, t) = 0$ that

$$\frac{1}{2} \frac{d}{dt} \int_0^k g_\varepsilon |D_t e|^2 dx + \kappa \int_0^k g_\varepsilon \frac{r^{2m}}{v} |D_t D_x e|^2 dx = \sum_{i=1}^4 I_i. \quad (4.3.15)$$

First, we start by estimating I_1 : using Cauchy-Schwartz inequality (2.3.21)–(2.3.23), Theorem 2.3.1(ii), Corollary 4.3.1, and Lemma 4.3.4, we have

$$\begin{aligned}
I_1 &:= -\kappa \int_{\varepsilon/2}^{\varepsilon} g'_\varepsilon \frac{r^{2m}}{v} D_t D_x e D_t e dx - 2m\kappa \int_0^k g_\varepsilon \frac{r^{2m-1}}{v} u D_x e D_x D_t e dx \\
&\quad - 2m\kappa \int_{\varepsilon/2}^{\varepsilon} g'_\varepsilon \frac{r^{2m-1}}{v} u D_x e D_t e dx + \kappa \int_0^k g_\varepsilon \frac{r^{2m}}{v^2} D_x(r^m u) D_x e D_x D_t e dx \\
&\quad + \kappa \int_{\varepsilon/2}^{\varepsilon} g'_\varepsilon \frac{r^{2m}}{v^2} D_x(r^m u) D_x e D_t e dx \\
&\leq C(\varepsilon) + C(\varepsilon) \int_{\varepsilon/2}^{\varepsilon} |D_t e|^2 dx + \frac{\kappa}{2} \int_0^k g_\varepsilon \frac{r^{2m}}{v} |D_t D_x e|^2 dx + C(\varepsilon) \sup_{\varepsilon/2 \leq y \leq k} |D_x(r^m u)|^2(y, t).
\end{aligned}$$

Next, we estimate the term I_2 using Theorem 2.3.1(ii), Lemma 4.3.2:

$$\begin{aligned}
I_2 &:= 2\beta \int_0^k g_\varepsilon \frac{D_x(r^m u)}{v} D_x D_t(r^m u) D_t e dx - \beta \int_0^k g_\varepsilon \frac{(D_x(r^m u))^3}{v^2} D_t e dx \\
&\leq C(\varepsilon) \int_{\varepsilon/2}^{\varepsilon} |D_x D_t(r^m u)|^2 dx + \frac{3}{2}\beta \sup_{\varepsilon/2 \leq y \leq k} |D_x(r^m u)|^2(y, t) \int_0^k g_\varepsilon |D_t e|^2(x, t) dx \\
&\quad + C(\varepsilon) \sup_{\varepsilon/2 \leq y \leq k} |D_x(r^m u)|^2(y, t).
\end{aligned}$$

I_3 is estimated using Theorem 2.3.1(ii), Lemma 4.3.2 and Corollary 4.3.3:

$$\begin{aligned}
(\gamma - 1)^{-1} I_3 &:= - \int_0^k g_\varepsilon \frac{D_x(r^m u)}{v} |D_t e|^2 dx + \int_0^k g_\varepsilon \frac{|D_x(r^m u)|^2}{v^2} e D_t e - \int_0^k g_\varepsilon \frac{e}{v} D_t D_x(r^m u) D_t e dx \\
&\leq C(\varepsilon) + \sup_{\varepsilon/2 \leq y \leq k} |D_x(r^m u)|^2 \int_0^k g_\varepsilon |D_t e|^2 dx + C(\varepsilon) \int_{\varepsilon/2}^{\varepsilon} |D_t e|^2 dx + \int_{\varepsilon/2}^{\varepsilon} |D_t D_x(r^m u)|^2 dx.
\end{aligned}$$

Finally, to estimate I_4 , we use (2.3.21)–(2.3.23), Theorem 2.3.1(i)–(ii), Corollary 4.3.1, and Lemma 4.3.2, 4.3.5, to obtain

$$\begin{aligned}
I_4 &:= -2m\mu \int_0^k g_\varepsilon D_x D_t(r^{m-1} u^2) D_t e dx \\
&\leq C(\varepsilon) + \sup_{\varepsilon/2 \leq y \leq k} |D_x(r^m u)|^2(y, t) \int_0^k g_\varepsilon |D_t e|^2 dx + 2m\mu \int_{\varepsilon/2}^{\varepsilon} |D_x D_t(r^m u)|^2 dx + C(\varepsilon) \int_{\varepsilon/2}^{\varepsilon} |D_t e|^2 dx.
\end{aligned}$$

Substituting the estimates for I_1 – I_4 back into (4.3.15), it follows that

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int_0^k g_\varepsilon |D_t e|^2 dx + \frac{\kappa}{2} \int_0^k g_\varepsilon \frac{r^{2m}}{v} |D_t D_x e|^2 dx \\
&\leq C(\varepsilon) + C \sup_{\varepsilon/2 \leq y \leq k} |D_x(r^m u)|^2(y, t) \int_0^k g_\varepsilon |D_t e|^2 dx + C(\varepsilon) \int_{\varepsilon/2}^{\varepsilon} |D_x D_t(r^m u)|^2 dx + \\
&\quad + C(\varepsilon) \int_{\varepsilon/2}^{\varepsilon} |D_t e|^2 dx + \sup_{\varepsilon/2 \leq y \leq k} |D_x(r^m u)|^2(y, t).
\end{aligned}$$

Integrating in time and using Lemma 4.3.3, 4.3.4, and 4.3.5, we have

$$\begin{aligned}
&\frac{1}{2} \int_0^k g_\varepsilon |D_t e|^2(x, t) dx + \frac{\kappa}{2} \int_0^t \int_0^k g_\varepsilon \frac{r^{2m}}{v} |D_t D_x e|^2 dx ds \\
&\leq \frac{1}{2} \limsup_{\tau \rightarrow 0^+} \int_0^k g_\varepsilon |D_t e|^2(x, \tau) dx + C(\varepsilon) + C \int_0^t \sup_{\varepsilon/2 \leq y \leq k} |D_x(r^m u)|^2(y, s) \int_0^k g_\varepsilon |D_t e|^2(x, s) dx ds.
\end{aligned} \tag{4.3.16}$$

Now by taking $t \searrow 0$ in the internal energy equation (4.2.12)₃, and using the initial data estimate Proposition 4.2.2 we have that

$$\begin{aligned} & \limsup_{\tau \rightarrow 0^+} \int_0^k g_\varepsilon |D_t e|^2(x, \tau) dx \\ & \lesssim \int_0^k g_\varepsilon \left\{ \left| \frac{D_x(r_0^m u_0)}{v_0} \right|^2 (1 + |D_x(r_0^m u_0)|^2 + |e_0 - 1|^2) + \left| \frac{u_0}{r_0} \right|^2 |D_x(r_0^m u_0)|^2 + \frac{v_0^2}{r_0^4} u_0^4 \right\} dx \\ & \quad + \int_0^k g_\varepsilon \left\{ |r_0^{m-1} D_x e_0|^2 + \left| \frac{r_0^m D_x e_0}{v^2} \right|^2 |r_0^m D_x e_0|^2 + \left| \frac{r_0^{2m}}{v_0} D_x^2 e_0 \right|^2 \right\} dx \leq C_0 + C(\varepsilon). \end{aligned} \quad (4.3.17)$$

Substituting this into the main estimate (4.3.16), we have:

$$\begin{aligned} & \frac{1}{2} \int_0^k g_\varepsilon |D_t e|^2(x, t) dx + \frac{\kappa}{2} \int_0^t \int_0^k g_\varepsilon \frac{r^{2m}}{v} |D_t D_x e|^2 dx ds \\ & \leq C_0 + C(\varepsilon) + C \int_0^t \sup_{\varepsilon/2 \leq y \leq k} |D_x(r^m u)|^2(y, s) \int_0^k g_\varepsilon |D_t e|^2(x, s) dx ds. \end{aligned}$$

By Grönwall inequality, and using Lemma 4.3.3 once again, it follows that for all $t \in [0, T]$

$$\frac{1}{2} \int_0^k g_\varepsilon |D_t e|^2(x, t) dx + \frac{\kappa}{2} \int_0^t \int_0^k g_\varepsilon \frac{r^{2m}}{v} |D_t D_x e|^2(x, s) dx ds \leq C_0 + C(\varepsilon).$$

□

Similar to the case for $D_t u$, we can also obtain the time weighted $L^\infty(0, T; L^2)$ estimate of $D_t e$. We note that the following initial condition is not required for this estimate to hold:

$$\int_0^k (r_k^0)^{4m} |D_x^2 e_k^0|^2 dx \leq C_0.$$

Lemma 4.3.10 (Time Weighted Exterior $L_T^2 L^2$ -Estimate of e). *Given $\varepsilon > 0$ and $T > 0$, there exists a constant $C(\varepsilon) = C(\varepsilon, T, C_0, n, \beta, \gamma, \kappa) > 0$ independent of $k > 0$ and $a > 0$ such that:*

$$\sup_{t \in [0, T]} \int_\varepsilon^k \sigma(t) |D_t e|^2(x, t) dx + \int_0^T \int_\varepsilon^k \sigma(t) r^{2m} |D_x D_t e|^2(x, t) dx dt \leq C(\varepsilon),$$

where $\sigma(t) := \min\{1, t\}$.

The proof of this is the same as previous lemma, except that one uses the time weighted estimates of Lemma 2.3.9 in Section 2.3.4. We will not repeat the argument here.

Lemma 4.3.11 (Exterior $L_T^\infty H^2$ -Estimate of e). *Given $\varepsilon > 0$ and $T > 0$, there exists a constant $C(\varepsilon) = C(\varepsilon, T, C_0) > 0$ independent of $k > 0$ and $a > 0$ such that:*

$$\sup_{t \in [0, T]} \int_\varepsilon^k r^{4m} |D_x^2 e(x, t)|^2 dx \leq C(\varepsilon).$$

Proof. Writing the energy equation:

$$\begin{aligned} \kappa r^{2m} D_x^2 e &= -2m\kappa r^{m-1} v D_x e + \kappa \frac{r^{2m}}{v} D_x v D_x e + v D_t e - \beta |D_x(r^m u)|^2 + \\ & \quad + (\gamma - 1) e D_x(r^m u) + 2m\mu v D_x(r^{m-1} u^2). \end{aligned}$$

Taking $L^2(\varepsilon, k)$ norm on both sides, using (2.3.21)–(2.3.23), Theorem 2.3.1(i)–(ii), Lemmas 2.3.4, 4.3.1, 4.3.2, 4.3.4, 4.3.9, and Corollaries 4.3.1, 4.3.3, we have:

$$\kappa^2 \int_{\varepsilon}^k r^{4m} |D_x^2 e|^2 dx \leq C(\varepsilon) + C(\varepsilon) \sup_{y \in [\varepsilon, k]} |r^m D_x e(y, t)|^2 + \sup_{y \in [\varepsilon, k]} |D_x(r^m u)(y, t)|^2, \quad (4.3.18)$$

By the same argument used in the proof of Lemma 4.3.7, inequality (4.3.13), The third term is bounded as follows:

$$\sup_{y \in [\varepsilon, k]} |D_x(r^m u)(y, t)|^2 \leq C(\varepsilon) + \frac{\beta^2}{2} |r^{-m}|_{L^\infty(\varepsilon, k)}^2 \int_{\varepsilon}^k r^{2m} |D_x^2(r^m u)|^2 dx \leq C(\varepsilon).$$

Next, by the Sobolev Embedding theorem in 1D: $W^{1,1}(\varepsilon, k) \hookrightarrow C^0(\varepsilon, k)$, and Lemma 4.3.9 we have

$$\sup_{y \in [\varepsilon, k]} |r^m D_x e(y, t)|^2 \lesssim \int_{\varepsilon}^k r^{2m} |D_x e|^2 dx + \int_{\varepsilon}^k |r^m D_x e \cdot D_x(r^m D_x e)| dx \leq C(\varepsilon) + \frac{\kappa^2}{2C(\varepsilon)} \int_{\varepsilon}^k r^{4m} |D_x^2 e|^2 dx.$$

Substituting the above back into (4.3.18), we have

$$\kappa^2 \int_{\varepsilon}^k r^{4m} |D_x^2 e|^2 dx \leq C(\varepsilon) + \frac{\kappa^2}{2} \int_{\varepsilon}^k r^{4m} |D_x^2 e|^2 dx.$$

This proves the lemma. \square

Using Lemma 4.3.10, we also obtain the time weighted $L^\infty(0, T; H^2)$ estimate for e in the following Lemma:

Lemma 4.3.12 (Time Weighted Exterior $L_T^\infty H^2$ -Estimate of e). *Given $\varepsilon > 0$ and $T > 0$, there exists a constant $C(\varepsilon) = C(\varepsilon, T, C_0) > 0$ independent of $k > 0$ and $a > 0$ such that:*

$$\sup_{t \in [0, T]} \int_{\varepsilon}^k \sigma(t) r^{4m} |D_x^2 e(x, t)|^2 dx \leq C(\varepsilon), \quad \text{where } \sigma(t) := \min\{1, t\}.$$

The proof is same as that of Lemma 4.3.11 except for the time weight $\sigma(t) = \min\{1, t\}$. But this can be easily dealt with using the time weighted estimate of Lemma 2.3.9, hence we will not repeat the proof here.

4.3.4 $L^\infty(0, T; H^2)$ -estimate of v

Lemma 4.3.13 (Exterior $L_T^\infty L^2$ estimate of $D_x^2 v$). *Given $\varepsilon > 0$ and $T > 0$, we have the estimate*

$$\sup_{t \in [0, T]} \int_{\varepsilon}^k \{r^{4m} |D_x^2 v|^2 + r^{4m} |D_x^2 \log v|^2\}(x, t) dx \leq C_0 + C(\varepsilon).$$

where $C(\varepsilon) = C(\varepsilon, T, C_0, n, \beta, \gamma, \kappa) > 0$ independent of $k > 0$ and $a > 0$.

Proof. Taking spatial derivative D_x on both sides of the equation (4.3.4) from Lemma 4.3.1, and using the identity $D_t r = u$, and continuity equation $D_t v = D_x(r^m u)$, we have

$$\begin{aligned} \beta r^m D_t D_x^2 \log v &= D_t D_x u - m\beta \frac{D_x^2(r^m u)}{r} + m(\gamma - 1) \left\{ \frac{D_x e}{r} - \frac{e}{rv} D_x v \right\} \\ &\quad + (\gamma - 1) r^m \left\{ \frac{D_x^2 e}{v} - 2 \frac{D_x e D_x v}{v^2} + \frac{e}{v} |D_x \log v|^2 - \frac{e}{v} D_x^2 \log v \right\} \\ &\quad + m^2 \beta \frac{u}{r^2} D_x v + m\beta \frac{r^{m-1}}{v} D_x u D_x v. \end{aligned}$$

Multiplying both sides by $g_\varepsilon r^{3m} D_x^2 \log v$, and integrating in space, we obtain by Cauchy-Schwartz inequality, Theorem 2.3.1(ii), and (2.3.21)–(2.3.23) that

$$\frac{\beta}{2} \frac{d}{dt} \int_0^k g_\varepsilon r^{4m} |D_x^2 \log v|^2 \leq \left\{ \frac{5}{2} + 2m\beta \left| \frac{u}{r} \right|_{L^\infty(\varepsilon/2, k)} + (\gamma - 1) \left| \frac{e}{v} \right|_{L^\infty(\varepsilon/2, k)} \right\} \int_0^k g_\varepsilon r^{4m} |D_x^2 \log v|^2 + \sum_{i=1}^9 I_i,$$

where each term I_i will be properly indicated as we estimate it. By (2.3.21)–(2.3.23) and Lemmas 4.3.5–4.3.7, we have

$$I_1 + I_2 := \int_0^k g_\varepsilon \left\{ r^{2m} |D_t D_x u|^2 dx + m^2 \beta^2 \int_0^k r^{2m-2} |D_x^2 (r^m u)|^2 \right\} dx \leq C(\varepsilon).$$

Next, using Theorem 2.3.1(ii), (2.3.21)–(2.3.23), Lemmas 4.3.1, 4.3.4, and Corollary 4.3.3, it follows that

$$I_3 + I_4 := (\gamma - 1)^2 \int_0^k g_\varepsilon \left\{ m^2 r^{2m-2} |D_x e|^2 + m^2 \left| \frac{e}{rv} \right|_{L^\infty(\varepsilon/2, k)}^2 r^{2m} |D_x v|^2 \right\} dx \leq C(\varepsilon).$$

From Theorem 2.3.1(ii), and Lemma 4.3.11, we also have

$$I_5 := (\gamma - 1)^2 \int_0^k g_\varepsilon |v^{-2}|_{L^\infty(\varepsilon/2, k)} r^{4m} |D_x^2 e|^2 dx \leq C(\varepsilon).$$

From Sobolev embedding inequality, Theorem 2.3.1(ii) and Lemma 4.3.1, we have

$$I_6 := 4(\gamma - 1)^2 \int_0^k g_\varepsilon \left| \frac{r^m D_x e}{v^2} \right|_{L^\infty(\varepsilon/2, k)}^2 r^{2m} |D_x v|^2 dx \leq C(\varepsilon) |r^m D_x e|_{L^\infty(\varepsilon/2, k)}^2 \leq C(\varepsilon),$$

Next, using Sobolev embedding inequality, Theorem 2.3.1(ii), Lemmas 4.3.1, 4.3.13, Corollary 4.3.3, and (2.3.21)–(2.3.23), it follows that

$$I_7 := \frac{(\gamma - 1)^2}{2} \int_0^k g_\varepsilon \left| \frac{e}{v} \right|_{L^\infty(\varepsilon/2, k)}^2 r^{4m} |D_x \log v|^4 dx \leq C(\varepsilon) + C(\varepsilon) \int_0^k g_\varepsilon r^{4m} |D_x^2 \log v|^2 dx.$$

Next, by (2.3.21)–(2.3.23), Lemma 4.3.1, and Corollary 4.3.1, it follows that

$$I_8 := m^4 \beta^2 \int_0^k g_\varepsilon m^2 \left| \frac{u}{r^2} \right|_{L^\infty(\varepsilon/2, \infty)}^2 r^{2m} |D_x v|^2 dx \leq C(\varepsilon) \int_{\varepsilon/2}^k r^{2m} |D_x v|^2 dx \leq C(\varepsilon).$$

Using Sobolev embedding theorem, (2.3.21)–(2.3.23), Theorem 2.3.1(ii), Lemma 4.3.1 and 4.3.7, we have

$$I_9 := m^2 \beta^2 \int_0^k g_\varepsilon \left\{ \left| \frac{r^{m-1}}{v} D_x u \right|_{L^\infty(\varepsilon/2, \infty)}^2 r^{2m} |D_x v|^2 \right\} dx \leq C(\varepsilon) |r^m D_x u|_{L^\infty(\varepsilon/2, k)}^2 \leq C(\varepsilon),$$

Combining all estimates for terms $\{I_i\}_{i=1}^9$, it follows that

$$\frac{\beta}{2} \frac{d}{dt} \int_0^k g_\varepsilon r^{4m} |D_x^2 \log v|^2 dx \leq C(\varepsilon) + C(\varepsilon) \int_0^k g_\varepsilon r^{4m} |D_x^2 \log v|^2 dx,$$

where in addition, we also used (2.3.21)–(2.3.23), Theorem 2.3.1(ii), Corollary 4.3.1 and 4.3.3. Thus by Grönwall inequality, it follows that

$$\sup_{t \in [0, T]} \int_0^k g_\varepsilon r^{4m} |D_x^2 \log v|^2(x, t) dx \leq \int_0^k g_\varepsilon r^{4m} |D_x^2 \log v_0|^2(x) dx + C(\varepsilon).$$

By an application of Sobolev embedding theorem, we one can verify that

$$\int_0^k g_\varepsilon r_0^{4m} |D_x^2 \log v_0|^2(x) dx = \int_0^k g_\varepsilon r_0^{4m} \left| \frac{D_x^2 v_0}{v_0} - \frac{|D_x v_0|^2}{v_0^2} \right|^2(x) dx \leq C_0 + C(\varepsilon).$$

Hence, we conclude that

$$\sup_{t \in [0, T]} \int_0^k g_\varepsilon r^{4m} |D_x^2 \log v|^2(x, t) dx \leq C_0 + C(\varepsilon).$$

By Sobolev embedding theorem, and Theorem 2.3.1(ii), and Lemma 4.3.1, we also have that

$$\int_0^k g_\varepsilon r^{4m} |D_x^2 v|^2 dx = \int_0^k g_\varepsilon r^{4m} \left| v D_x^2 \log v + \frac{|D_x v|^2}{v} \right|^2 dx \leq C_0 + C(\varepsilon).$$

□

4.3.5 $L^2(0, T; H^3)$ -estimate of (u, e)

Lemma 4.3.14 (Exterior $L_T^2 H^3$ estimate of u). *Given $\varepsilon > 0$ and $T > 0$, there exists a constant $C(\varepsilon) = C(\varepsilon, T, C_0, n, \beta, \gamma, \kappa) > 0$ independent of $k > 0$ and $a > 0$ such that:*

$$\int_0^T \int_\varepsilon^k r^{6m} |D_x^3 u|^2 dx dt \leq C(\varepsilon).$$

Proof. We first taking spatial derivative D_x on both sides of momentum equation (4.2.12)₂ and then multiply by r^m . Next we take $L^2(\varepsilon, k)$ norm on both sides of the resulting equation and integrate in time, it follows that

$$\beta^2 C(\varepsilon)^{-2} \int_0^T \int_\varepsilon^k r^{6m} |D_x^3 u|^2 dx dt \leq \beta^2 \int_0^T \int_\varepsilon^k \frac{r^{6m}}{v^2} |D_x^3 u|^2 dx dt \leq \sum_{i=1}^{15} I_i,$$

where each term I_i will be properly indicated as we estimate it. By (2.3.21)–(2.3.23), Theorem 2.3.1(ii), and Lemmas 4.3.1, 4.3.4, 4.3.5, it follows that

$$\sum_{i=1}^3 I_i := \int_0^T \left\{ \|r^m D_t D_x u\|_{L^2(\varepsilon, k)}^2 + \|r^{m-1} D_x e\|_{L^2(\varepsilon, k)}^2 + \left\| \frac{e}{r^v} r^m D_x v \right\|_{L^2(\varepsilon, k)}^2 \right\} dt \leq C(\varepsilon).$$

Next, by Sobolev embedding theorem, Theorem 2.3.1(ii), Corollary 4.3.3 and Lemmas 4.3.1, 4.3.4, 4.3.11, 4.3.13, we have:

$$\begin{aligned} \sum_{i=4}^7 &:= \int_0^T \left\{ \left\| \frac{r^{2m}}{v} D_x^2 e \right\|_{L^2(\varepsilon, k)}^2 + \left\| \frac{r^{2m}}{v^2} D_x e D_x v \right\|_{L^2(\varepsilon, k)}^2 + \left\| \frac{e}{v^3} r^{2m} |D_x v|^2 \right\|_{L^2(\varepsilon, k)}^2 + \left\| \frac{e}{v^2} r^{2m} D_x^2 v \right\|_{L^2(\varepsilon, k)}^2 \right\} dt \\ &\leq C(\varepsilon) + \int_0^T \left\{ |r^m D_x e|_{L^\infty(\varepsilon, k)}^2 + |r^m D_x v|_{L^\infty(\varepsilon, k)}^2 \right\} dt \leq C(\varepsilon), \end{aligned}$$

Next, by (2.3.21)–(2.3.23), Theorem 2.3.1(i)–(ii), Corollary 4.3.1, 4.3.3, and Lemma 4.3.1, 4.3.2, 4.3.7, we obtain that

$$\begin{aligned} \sum_{i=8}^{11} I_i &:= \int_0^T \left\{ \|r^{m-2} v D_x u\|_{L^2(\varepsilon, k)}^2 + \|r^{2m-1} D_x^2 u\|_{L^2(\varepsilon, k)}^2 + \|r^{m-2} u D_x v\|_{L^2(\varepsilon, k)}^2 + \|r^{-3} v^2 u\|_{L^2(\varepsilon, k)}^2 \right\} dt \\ &\leq C(\varepsilon) \int_0^T \left\{ \|r^m D_x u\|_{L^2(\varepsilon, k)}^2 + \|r^{2m} D_x^2 u\|_{L^2(\varepsilon, k)}^2 + \|r^m D_x v\|_{L^2(\varepsilon, k)}^2 + \|u\|_{L^2(\varepsilon, k)}^2 \right\} dt \leq C(\varepsilon). \end{aligned}$$

Using Sobolev embedding theorem, (2.3.21)–(2.3.23), Theorem 2.3.1(ii), and Lemmas 4.3.2, 4.3.13, 4.3.7, we get:

$$\begin{aligned} \sum_{i=12}^{15} I_i &:= \int_0^T \left\{ \left\| \frac{r^{2m-1}}{v} D_x v D_x u \right\|_{L^2(\varepsilon, k)}^2 + \left\| \frac{r^{3m}}{v^3} |D_x v|^2 D_x u \right\|_{L^2(\varepsilon, k)}^2 \right\} dt \\ &\quad + \int_0^T \left\{ \left\| \frac{r^{3m}}{v^2} D_x u D_x^2 v \right\|_{L^2(\varepsilon, k)}^2 + \left\| \frac{r^{3m}}{v^2} D_x v D_x^2 u \right\|_{L^2(\varepsilon, k)}^2 \right\} dt \\ &\leq C(\varepsilon) \int_0^T \left\{ |r^m D_x v|_{L^\infty(\varepsilon, k)}^2 + |r^m D_x v|_{L^\infty(\varepsilon, k)}^4 + |r^m D_x u|_{L^\infty(\varepsilon, k)}^2 \right\} dt \leq C(\varepsilon). \end{aligned}$$

In summary we have that

$$\int_0^T \int_\varepsilon^k r^{6m} |D_x^3 u|^2 dx dt \leq \frac{C(\varepsilon)}{\beta^2} \sum_{i=1}^{15} I_i \leq C(\varepsilon). \quad \square$$

Lemma 4.3.15 (Exterior $L_T^2 H^3$ estimate of e). *Given $\varepsilon > 0$ and $T > 0$, there exists a constant $C(\varepsilon) = C(\varepsilon, T, C_0, n, \beta, \gamma, \kappa) > 0$ independent of $k > 0$ and $a > 0$ such that:*

$$\int_0^T \int_\varepsilon^k r^{6m} |D_x^3 e|^2 dx dt \leq C(\varepsilon).$$

Proof. We first take spatial derivative D_x on both sides of internal energy equation (4.2.12)₃ then substitute the momentum equation (4.2.12)₂. Next, taking $L^2 \equiv L^2(\varepsilon, k)$ norm on both sides of the resulting equations and integrating in time, then by (2.3.21)–(2.3.23), Theorem 2.3.1(ii), and Corollary 4.3.1, 4.3.3, it follows that

$$\kappa^2 C(\varepsilon)^{-2} \int_0^T \int_\varepsilon^k r^{6m} |D_x^3 e|^2 dx dt \leq \kappa^2 \int_0^T \int_\varepsilon^k \frac{r^{6m}}{v^2} |D_x^3 e|^2 dx dt \leq \sum_{i=1}^{12} I_i,$$

where each term I_i will be indicated as we estimate it. First, by Lemma 4.3.5, we have

$$I_1 := C(\varepsilon) \int_0^T \|r^m D_t D_x e\|_{L^2(\varepsilon, k)}^2 dt \leq C(\varepsilon).$$

Next, using Sobolev embedding theorem, (2.3.21)–(2.3.23), and Lemmas 4.3.2, 4.3.5, 4.3.7, it follows that

$$I_2 + I_3 := C(\varepsilon) \int_0^T \left\{ \|D_x(r^m u)\|_{L^\infty}^2 \|D_t u\|_{L^2}^2 + (\|D_x(r^m u)\|_{L^\infty}^2 + 1) \|r^m D_x^2(r^m u)\|_{L^2}^2 \right\} dt \leq C(\varepsilon).$$

By Theorem 2.3.1(i), and Lemmas 4.3.1, 4.3.2, 4.3.4, 4.3.11, it follows that

$$\sum_{i=4}^8 I_i := C(\varepsilon) \int_0^T \left\{ \|D_x(r^m u)\|_{L^2}^2 + \|u\|_{L^2}^2 + \|r^m D_x v\|_{L^2}^2 + \|r^m D_x e\|_{L^2}^2 + \|r^{2m} D_x^2 e\|_{L^2}^2 \right\} dt \leq C(\varepsilon).$$

Using Sobolev embedding theorem, and Lemma 4.3.4, 4.3.13, 4.3.11, we get:

$$\begin{aligned} \sum_{i=8}^{12} I_i &:= C(\varepsilon) \int_0^T \left\{ |r^m D_x v|_{L^\infty}^2 \|r^m D_x e\|_{L^2}^2 + |r^m D_x v|_{L^\infty}^4 \|r^m D_x e\|_{L^2}^2 \right\} dt \\ &\quad + C(\varepsilon) \int_0^T \left\{ |r^m D_x e|_{L^\infty}^2 \|r^{2m} D_x^2 v\|_{L^2}^2 + |r^m D_x v|_{L^\infty}^2 \|r^{2m} D_x^2 e\|_{L^2}^2 \right\} dt \leq C(\varepsilon). \end{aligned}$$

In conclusion, we obtained that

$$\int_0^T \int_\varepsilon^k r^{6m} |D_x^3 e|^2 dx dt \leq C(\varepsilon). \quad \square$$

4.4 Global-in-Time Existence to the Cauchy Problem: $k \rightarrow \infty$

4.4.1 Bootstrap argument

In this subsection, we argue that for each fixed $a \in (0, 1)$ and $k \in \mathbb{N}$, the maximal time of existence for local-in-time strong solution obtained in Theorem 3.1.1 of Chapter 3 can be extended to any choice of $T \in (0, \infty)$. To avoid repetition, we will only present the bootstrap argument for Theorem 4.1.1(i) as the case for Theorem 4.1.1(ii) is simpler.

For the given initial data (v_0, u_0, e_0) in Theorem 4.1.1(i) with the initial constant $C_0 > 0$, we first mollify it with parameter $\delta \in (0, 1)$ as in Section 4.2.1 to obtain a smooth function $(v_\delta^0, u_\delta^0, e_\delta^0) \in C^\infty([0, \infty))$. Then the truncation procedure described in Section 4.2.1 is applied to get the modified initial data $(v_{\delta,k}^0, u_{\delta,k}^0, e_{\delta,k}^0) \in C^\infty([0, k])$, where $k \in \mathbb{N}$ is the truncation parameters. By Proposition 4.2.1 and 4.2.2, it follows that there exists an constant $\delta_0 \in (0, 1)$ such that

$$\begin{aligned} u_{\delta,k}^0(0) = D_x e_{\delta,k}^0(0) = 0 \quad \text{and} \quad u_{\delta,k}^0(k) = D_x e_{\delta,k}^0(k) = 0 \\ \sup_{\substack{k \in \mathbb{N} \\ \delta \in (0, \delta_0)}} \int_0^k \{(\gamma - 1)\psi(v_{\delta,k}^0) + \frac{1}{2}|u_{\delta,k}^0|^2 + \psi(e_{\delta,k}^0)\}(x)dx \leq C_0, \\ \sup_{\substack{k \in \mathbb{N} \\ \delta \in (0, \delta)}} \int_0^k \{|v_{\delta,k}^0 - 1|^2 + |u_{\delta,k}^0|^4 + |e_{\delta,k}^0 - 1|^2\}(x)dx \leq C_0, \\ \sup_{\substack{k \in \mathbb{N} \\ \delta \in (0, \delta_0)}} \int_0^k (r_{\delta,k}^0)^{2m} \{|D_x v_{\delta,k}^0|^2 + |D_x u_{\delta,k}^0|^2 + |D_x e_{\delta,k}^0|^2\}(x)dx \leq C_0, \end{aligned}$$

and there exists a constant $C_{\delta,k} = C_{\delta,k}(C_0, \delta, k) > 0$ depending on the parameters $\delta \in (0, 1)$, $k \in \mathbb{N}$ and the initial constant $C_0 > 0$ such that

$$\int_0^k (r_{\delta,k}^0)^{4m} \{|D_x^2 v_{\delta,k}^0|^2 + |D_x^2 u_{\delta,k}^0|^2 + |D_x^2 e_{\delta,k}^0|^2\}(x)dx \leq C_{\delta,k} < \infty, \quad \text{for each } (\delta, k) \in (0, 1) \times \mathbb{N}.$$

Hence for each fixed $(\delta, k) \in (0, 1) \times \mathbb{N}$, one can apply the local-in-time existence result, Theorem 3.1.1 in Chapter 3, it follows that there exists a time $T_{\delta,k} > 0$ possibly depending on the parameters δ, k such that a strong solution $(v_{\delta,k}, u_{\delta,k}, e_{\delta,k}, r_{\delta,k})(x, t)$ to the k -th approximate Lagrangian equations (4.2.12) exists in the domain $(x, t) \in [0, k] \times [0, T_{\delta,k})$, and $(v_{\delta,k}, u_{\delta,k}, e_{\delta,k})(x, t)$ exists in the space:

$$\begin{aligned} E[u_{\delta,k}, e_{\delta,k}](T) := \sup_{0 \leq t \leq T} \{ \|(u_{\delta,k}, e_{\delta,k})(\cdot, t)\|_{H^2(0,k)}^2 + \|(D_t u_{\delta,k}, D_t e_{\delta,k})(\cdot, t)\|_{L^2(0,k)}^2 \} + \\ + \int_0^T \{ \|(D_t u_{\delta,k}, D_t e_{\delta,k})(\cdot, t)\|_{H^1(0,k)}^2 + \|(u_{\delta,k}, e_{\delta,k})(\cdot, t)\|_{H^3(0,k)}^2 \} dt < \infty, \end{aligned}$$

for each $T \in [0, T_{\delta,k})$. Note that this norm provide enough information to completely determine solution $(v_{\delta,k}, u_{\delta,k}, e_{\delta,k})(x, t)$ since the specific volume $v_{\delta,k}$ can be expressed as

$$v_{\delta,k}(x, t) = (r_{\delta,k}(x, t))^m \left\{ v_{\delta,k}^0(x) (r_{\delta,k}^0(x))^{-m} + \int_0^t D_x u_{\delta,k}(x, s) ds \right\},$$

where

$$r_{\delta,k}(x, t) = r_{\delta,k}^0(x) + \int_0^t u_{\delta,k}(x, s) ds \quad \text{and} \quad r_{\delta,k}^0(x) := \left(a^n + n \int_0^x v_{\delta,k}^0(y) dy \right)^{\frac{1}{n}}.$$

Now assume $T_{\delta,k} < \infty$. Since it is the maximal time of existence for the strong solution, it follows that

$$\liminf_{t \nearrow T_{\delta,k}} E(u_{\delta,k}, e_{\delta,k})(t) = \infty,$$

However by Theorem 4.3.1, there exists a constant $C_1(\delta, k) = C_1(\delta, k, a, C_0, T_{\delta, k}) > 0$ such that

$$C_1(\delta, k)^{-1} \leq v_{\delta, k}(x, t) \leq C_1(\delta, k) \quad \text{for } (x, t) \in [0, k] \times [0, T_{\delta, k}],$$

and

$$\begin{aligned} & \sup_{t \in [0, T_{\delta, k}]} \left\{ \|(u_{\delta, k}, e_{\delta, k} - 1)(\cdot, t)\|_{L^2(0, k)}^2 + \|(r_{\delta, k}^m D_x u_{\delta, k}, r_{\delta, k}^m D_x e_{\delta, k})(\cdot, t)\|_{L^2(0, k)}^2 \right\} \\ & + \sup_{t \in [0, T_{\delta, k}]} \left\{ \|(D_t u_{\delta, k}, D_t e_{\delta, k})(\cdot, t)\|_{L^2(0, k)}^2 + \|(r_{\delta, k}^{2m} D_x^2 u_{\delta, k}, r_{\delta, k}^{2m} D_x^2 e_{\delta, k})(\cdot, t)\|_{L^2(0, k)}^2 \right\} \\ & + \int_0^{T_{\delta, k}} \int_0^k \left\{ |(r_{\delta, k}^{3m} D_x^3 u_{\delta, k}, r_{\delta, k}^{3m} D_x^3 e_{\delta, k})|^2 + |(r_{\delta, k}^m D_t D_x u_{\delta, k}, r_{\delta, k}^m D_t D_x e_{\delta, k})| \right\} (x, t) dx dt \leq C_1(\delta, k) < \infty. \end{aligned}$$

For fixed $\delta \in (0, 1)$ and $k \in \mathbb{N}$, we have that for all $(x, t) \in [0, k] \times [0, T_{\delta, k}]$

$$\left(a^n + nk(C_1(\delta, k))^{-1} \right)^{\frac{1}{n}} \leq r_{\delta, k}(x, t) = \left(a^n + n \int_0^x v_{\delta, k}(y, t) dy \right)^{\frac{1}{n}} \leq (a^n + nkC_1(\delta, k))^{1/n}$$

Thus there exists a constant $\tilde{C}_1(\delta, k) = \tilde{C}_1(\delta, k, a, C_0, T_{\delta, k}) \in (0, \infty)$ such that

$$E[u_{\delta, k}, e_{\delta, k}](T_{\delta, k}) \leq \tilde{C}_1(\delta, k) < \infty.$$

This is a contradiction to (4.4.1). Hence we conclude that $T_{\delta, k} = \infty$.

4.4.2 Strong Convergence

Let $a \in (0, 1)$, $k \in \mathbb{N}$ and $\delta \in (0, 1)$ be fixed, then we have shown in the previous section that for every $T > 0$ a strong solution $(v_k, u_k, e_k)(x, t)$ or $(v_{\delta, k}, u_{\delta, k}, e_{\delta, k})(x, t)$ in the domain $(x, t) \in [0, k] \times [0, T]$ exists. In this section, we take limit as $k \rightarrow \infty$ to obtain a strong solution for the system (1.3.6) in $(x, t) \in [0, \infty) \times [0, T]$. For the proof of Theorem 4.1.1(ii), we denote $(v_k, u_k, e_k)(x, t)$ as the solution obtained in the previous Section 4.4.1 in the domain $(x, t) \in [0, k] \times [0, T]$, and

$$r_k(x, t) = \left(a^n + n \int_0^x v_k(y, t) dy \right)^{1/n} = r_k^0(x) + \int_0^t u_k(x, s) ds,$$

with the function $r_k^0(x)$ defined in Section 4.2.2 by

$$r_k^0(x) = \left(a^n + n \int_0^x v_k^0(y) dy \right)^{1/n}, \quad \text{where } v_k^0(y) = v_0(y)\phi_k(y) + (1 - \phi_k(y)).$$

On the other hand, for the proof of Theorem 4.1.1(i), we construct the following sequence of solutions: for each $T > 0$, let $(v_{\delta, k}, u_{\delta, k}, e_{\delta, k})(x, t)$ be the solution obtained in Section 4.4.1. Then setting $\delta_k \equiv 1/k$ for each $k \in \mathbb{N}$, and we denote:

$$(v_k, u_k, e_k)(x, t) := (v_{\delta_k, k}, u_{\delta_k, k}, e_{\delta_k, k})(x, t) \quad \text{for } (x, t) \in [0, k] \times [0, T],$$

and

$$r_k(x, t) := \left(a^n + n \int_0^x v_{\delta_k, k}(y, t) dy \right)^{1/n} = r_{\delta_k, k}^0(x) + \int_0^t u_{\delta_k, k}(x, s) ds,$$

with the function $r_{\delta, k}^0(x)$ defined by

$$r_{\delta, k}^0(x) = \left(a^n + n \int_0^x v_{\delta, k}^0(y) dy \right)^{1/n}.$$

In both cases, these sequences of solutions are then extended to the whole domain $(x, t) \in (0, \infty) \times [0, T]$ via the following operation: Same as in Section 4.2.2, we let $\varphi_k(x) \in C^\infty(0, \infty)$ be such that $\varphi(x) = 1$ for $x \in [0, k/2]$ and $\varphi_k(x) = 0$ for $x \in [k, \infty)$, and $\sup_{x \in [0, k]} |\varphi_k^{(i)}(x)| \leq Ck^{-i}$ for all $i \in \mathbb{N}$. We define

$$\begin{aligned} \tilde{v}_k(x, t) &:= (v_k(x, t) - 1)\varphi_k(x) + 1, & \tilde{u}_k(x, t) &:= u_k(x, t)\varphi_k(x), \\ \tilde{e}_k(x, t) &:= (e_k(x, t) - 1)\varphi_k(x) + 1, & \tilde{r}_k(x, t) &:= \left(a^n + n \int_0^x \tilde{v}_k(y, t) dy\right)^{1/n}. \end{aligned} \quad (4.4.1)$$

The following estimates are true for initial data assumed in both (i) and (ii) of Theorem 4.1.1.

Lemma 4.4.1. *Let $(\tilde{v}_k, \tilde{u}_k, \tilde{e}_k)(x, t)$ be the extended function constructed in (4.4.1), then there exists a constant $C(a) = C(a, T, C_0)$ independent of $k \in \mathbb{N}$ such that the following estimates hold:*

$$\begin{aligned} \sup_{t \in [0, T]} \int_0^\infty \tilde{r}_k^{2m} \{ |D_x \tilde{v}_k|^2 + |D_x \tilde{u}_k|^2 + |\tilde{e}_k|^2 \} (x, t) dx &\leq C(a), \\ \int_0^T \int_0^\infty \{ |D_t \tilde{v}_k|^2 + |D_t \tilde{u}_k|^2 + |D_t \tilde{e}_k|^2 \} (x, t) dx dt &\leq C(a), \end{aligned}$$

Proof. We only show the case for $\tilde{r}_k^{2m} |D_x \tilde{v}_k|^2$ as the other cases follows exactly the same. First there exists a constant $C(a) = C(a, T, C_0) > 0$ independent of $k \in \mathbb{N}$ such that for all $(x, t) \in [0, k] \times [0, T]$

$$C(a)^{-1} \leq \frac{\tilde{r}_k(x, t)}{r_k(x, t)} \leq C(a) \quad \text{and} \quad (\tilde{r}_k(x, t))^m |\varphi'(x)| \leq C(a) k^{-2/3}.$$

Proof for the first assertion is same as (4.2.3), the only difference here is that we instead use the upper and lower bound: $\min\{1, C(a)^{-1}\} \leq v_k(x, t), \tilde{v}_k(x, t) \leq \max\{1, C(a)\}$. The second assertion can be verified using the fact that $m \equiv n - 1 < n$, construction of the cut-off function φ_k , and definition of \tilde{r}_k . Hence, using this inequality, Lemma 2.3.4 and 4.3.1, it follows that

$$\int_0^\infty \tilde{r}_k^{2m} |D_x \tilde{v}_k|^2 dx \leq 2 \int_0^\infty \left\{ \left(\frac{\tilde{r}_k}{r_k}\right)^{2m} \varphi_k^2 r_k^{2m} |D_x v_k|^2 + \tilde{r}_k^{2m} |\varphi_k'|^2 |v_k - 1|^2 \right\} dx \leq C(a).$$

□

Proposition 4.4.1. *For all $(x_1, x_2, t) \in [0, \infty)^2 \times [0, T]$,*

$$\begin{cases} \sup_{k \in \mathbb{N}} |\tilde{v}_k(x_1, t) - \tilde{v}_k(x_2, t)| \leq C(a) |x_1 - x_2|^{1/2}, \\ \sup_{k \in \mathbb{N}} |\tilde{u}_k(x_1, t) - \tilde{u}_k(x_2, t)| \leq C(a) |x_1 - x_2|^{1/2}, \\ \sup_{k \in \mathbb{N}} |\tilde{e}_k(x_1, t) - \tilde{e}_k(x_2, t)| \leq C(a) |x_1 - x_2|^{1/2}, \end{cases}$$

and for all $t_1 < t_2 \in [0, T]$ and $x \in [0, \infty)$,

$$\begin{cases} \sup_{k \in \mathbb{N}} |\tilde{v}_k(x, t_1) - \tilde{v}_k(x, t_2)| \leq C(a) |t_1 - t_2|^{1/4}, \\ \sup_{k \in \mathbb{N}} |\tilde{u}_k(x, t_1) - \tilde{u}_k(x, t_2)| \leq C(a) |t_1 - t_2|^{1/4}, \\ \sup_{k \in \mathbb{N}} |\tilde{e}_k(x, t_1) - \tilde{e}_k(x, t_2)| \leq C(a) |t_1 - t_2|^{1/4}. \end{cases}$$

Proof. We only present the proof for \tilde{v}_k as the cases for \tilde{u}_k, \tilde{e}_k follows in the exact same way. By Fundamental Theorem of Calculus, and estimates from Lemma 4.4.1, we have for all $x_1, x_2 \in [0, \infty)$ and $t \in [0, T]$:

$$|\tilde{v}_k(x_2, t) - \tilde{v}_k(x_1, t)| \leq |x_2 - x_1|^{1/2} a^{-m} \left(\int_0^k \tilde{r}_k^{2m} |D_x \tilde{v}_k|^2(x, t) dx \right)^{1/2} \leq C(a) |x_2 - x_1|^{1/2}.$$

Next, we show the Hölder continuity in time. First we fix a triplet $(x, t_1, t_2) \in [0, \infty) \times [0, T]^2$ such that $0 \leq t_1 < t_2$, and set $h := \sqrt{t_2 - t_1}$, then by Mean Value theorem

$$\frac{1}{h} \int_x^{x+h} |\tilde{v}_k(y, t_1) - \tilde{v}_k(y, t_2)| dy = |\tilde{v}_k(z, t_1) - \tilde{v}_k(z, t_2)| \quad \text{for some } z \in (x, x+h).$$

Next using triangular inequality, and the Fundamental Theorem of Calculus, we have for $k/2 \geq L'$:

$$\begin{aligned} & |\tilde{v}_k(x, t_1) - \tilde{v}_k(x, t_2)| - |\tilde{v}_k(z, t_1) - \tilde{v}_k(z, t_2)| \leq \int_x^z (|D_x \tilde{v}_k(y, t_1)| + |D_x \tilde{v}_k(y, t_2)|) dy \\ & \leq |x - z|^{1/2} a^{-m} \sum_{i=1}^2 \left(\int_x^z r_k^{2m} |D_x \tilde{v}_k(\cdot, t_i)|^2 \right)^{1/2} \leq h^{1/2} a^{-m} \sum_{i=1}^2 \left(\int_0^k r_k^{2m} |D_x \tilde{v}_k(\cdot, t_i)|^2 \right)^{1/2} \leq C(a) h^{1/2}. \end{aligned}$$

Thus combining with the previous assertion, and using the continuity equation $D_t v_k = D_x(r_k^m u_k)$, the following inequality holds for $k/2 \geq L'$

$$\begin{aligned} & |\tilde{v}_k(x, t_1) - \tilde{v}_k(x, t_2)| \leq |\tilde{v}_k(z, t_1) - \tilde{v}_k(z, t_2)| + C(a) h^{1/2} \leq C(a) h^{1/2} + \frac{1}{h} \int_x^{x+h} \int_{t_1}^{t_2} |D_t \tilde{v}_k| \\ & \leq C(a) h^{1/2} + h^{-1/2} |t_2 - t_1|^{1/2} \left(\int_0^T \int_0^k |D_t \tilde{v}_k|^2(y, t) dy dt \right)^{1/2} \leq C(a) |t_2 - t_1|^{1/4}. \end{aligned}$$

In summary, we have the inequality:

$$\begin{aligned} & \sup_{k \in \mathbb{N}} |\tilde{v}_k(x_2, t) - \tilde{v}_k(x_1, t)| \leq C(a) |x_2 - x_1|^{1/2} \quad \text{for all } t \in [0, T] \quad \text{and } x_1, x_2 \in [0, \infty), \\ & \sup_{k \in \mathbb{N}} |\tilde{v}_k(x, t_2) - \tilde{v}_k(x, t_1)| \leq C(a) |t_2 - t_1|^{1/4} \quad \text{for all } x \in [0, \infty) \quad \text{and } t_1 < t_2 \in [0, T]. \end{aligned} \tag{4.4.2}$$

□

Using these continuity estimates, the following convergence result is obtained. We again emphasises here that this lemma holds for the initial data assumed in both (i) and (ii) of Theorem 4.1.1.

Lemma 4.4.2. *Let $\{(\tilde{v}_k, \tilde{u}_k, \tilde{e}_k)\}_{k \in \mathbb{N}}$ be the collection of extended approximated solutions constructed in (4.4.1). Then there exists a continuous function $(v, u, e)(x, t)$ in $[0, \infty) \times (0, T]$, and a subsequence which is still denoted as $(\tilde{v}_k, \tilde{u}_k, \tilde{e}_k)_{k \in \mathbb{N}}$ such that for each compact subset $K \subset \subset [0, \infty) \times [0, T]$:*

$$\lim_{k \rightarrow \infty} \sup_{(x, t) \in K} |(\tilde{v}_k - v, \tilde{u}_k - u, \tilde{e}_k - e)(x, t)| = 0.$$

In addition, for all $(x_1, x_2, t) \in [0, \infty)^2 \times [0, T]$,

$$\begin{cases} |v(x_1, t) - v(x_2, t)| \leq C(a) |x_1 - x_2|^{1/2}, \\ |u(x_1, t) - u(x_2, t)| \leq C(a) |x_1 - x_2|^{1/2}, \\ |e(x_1, t) - e(x_2, t)| \leq C(a) |x_1 - x_2|^{1/2}, \end{cases}$$

and for all $t_1 < t_2 \in [0, T]$ and $x \in [0, \infty)$,

$$\begin{cases} |v(x, t_1) - v(x, t_2)| \leq C(a)|t_1 - t_2|^{1/4}, \\ |u(x, t_1) - u(x, t_2)| \leq C(a)|t_1 - t_2|^{1/4}, \\ |e(x, t_1) - e(x, t_2)| \leq C(a)|t_1 - t_2|^{1/4}. \end{cases}$$

Proof. We only present the proof for $v(x, t)$ as the cases for $u(x, t)$, $e(x, t)$ follows in a similar manner. First fix an integer $L \in \mathbb{N}$, and we set the compact subset $K_L := [0, L] \times [0, T]$. From Proposition 4.4.1, we have

$$\begin{aligned} \sup_{k \in \mathbb{N}} |\tilde{v}_k(x_2, t) - \tilde{v}_k(x_1, t)| &\leq C(a)|x_2 - x_1|^{1/2} \quad \text{for all } t \in [0, T] \text{ and } x_1, x_2 \in [0, L], \\ \sup_{k \in \mathbb{N}} |\tilde{v}_k(x, t_2) - \tilde{v}_k(x, t_1)| &\leq C(a)|t_2 - t_1|^{1/4} \quad \text{for all } x \in [0, L] \text{ and } t_1 < t_2 \in [0, T]. \end{aligned}$$

Moreover, $\sup_{k \in \mathbb{N}} \sup_{(x, t) \in K_L} \tilde{v}_k(x, t) \leq C(a)$. Since $K_L \subseteq K_{L+1}$, $\text{Int}(K_L) \neq \emptyset$, and $[0, \infty) \times [0, T] = \cup_{L \in \mathbb{N}} K_L$. Then by Proposition A.0.1 from Appendix A, there exists a continuous function $v(x, t) : [0, \infty) \times [0, T]$ and a subsequence still denoted as $\{\tilde{v}_k\}_{k \in \mathbb{N}}$ such that

$$\lim_{k \rightarrow \infty} \sup_{(x, t) \in K} |\tilde{v}_k(x, t) - v(x, t)| = 0 \quad \text{for each compact subset } K \subset\subset [0, \infty) \times [0, T].$$

Using the above uniform convergence with Proposition 4.4.1, it follows that

$$\begin{aligned} |v(x_2, t) - v(x_1, t)| &\leq C(a)|x_2 - x_1|^{1/2} \quad \text{for all } t \in [0, T] \text{ and } x_1, x_2 \in [0, \infty), \\ |v(x, t_2) - v(x, t_1)| &\leq C(a)|t_2 - t_1|^{1/4} \quad \text{for all } x \in [0, \infty) \text{ and } t_1 < t_2 \in [0, T]. \end{aligned}$$

Finally we remark that for the proof of $u(x, t)$ and $e(x, t)$, the argument follows exactly the same. \square

From the construction (4.4.1) and the uniform bounds from Theorem 4.3.1, it follows that there exists a constant $C(a) = C(a, T, C_0) > 0$ independent of $k \in \mathbb{N}$ such that for a.e. $(x, t) \in [0, \infty) \times [0, T]$,

$$\begin{cases} \min\{1, C(a)^{-1}\} \leq \sup_{k \in \mathbb{N}} \tilde{v}_k(x, t) \leq \max\{1, C(a)\}, \\ \sup_{k \in \mathbb{N}} |\tilde{u}_k(x, t)| \leq C(a), \quad \min\{1, C(a)^{-1}\} \leq \sup_{k \in \mathbb{N}} \tilde{e}_k(x, t) \leq \max\{1, C(a)\}, \end{cases}$$

and for each $\varepsilon \in (0, 1]$ there exists constants $C(\varepsilon) = C(\varepsilon, T, C_0) > 0$ independent of $a \in (0, 1)$ and $k \in \mathbb{N}$ such that for a.e. $(x, t) \in [\varepsilon, \infty) \times [0, T]$,

$$\begin{cases} \min\{1, C(\varepsilon)^{-1}\} \leq \sup_{k \in \mathbb{N}} \tilde{v}_k(x, t) \leq \max\{1, C(\varepsilon)\}, \\ \sup_{k \in \mathbb{N}} |\tilde{u}_k(x, t)| \leq C(\varepsilon), \quad \sup_{k \in \mathbb{N}} \tilde{e}_k(x, t) \leq \max\{1, C(\varepsilon)\}. \end{cases}$$

Combining this result with Lemma 4.4.2, we also obtain the following:

Corollary 4.4.1. *Let (v, u, e) be the function obtained in Lemma 4.4.2, then there exists constants $C(a) = C(a, T, C_0) > 0$ independent of $k \in \mathbb{N}$, and $C(\varepsilon) = C(\varepsilon, T, C_0) > 0$ for each $\varepsilon \in (0, 1]$ which are independent of both $a \in (0, 1)$ and $k \in \mathbb{N}$ such that for all ,*

$$\begin{cases} C(a)^{-1} \leq v(x, t) \leq C(a), \quad |u(x, t)| \leq C(a), \quad C(a)^{-1} \leq e(x, t) \leq C(a) & \text{for } (x, t) \in [0, \infty) \times [0, T], \\ C(\varepsilon)^{-1} \leq v(x, t) \leq C(\varepsilon), \quad |u(x, t)| \leq C(\varepsilon), \quad 0 \leq e(x, t) \leq C(\varepsilon) & \text{for } (x, t) \in [\varepsilon, \infty) \times [0, T]. \end{cases}$$

4.4.3 Construction of particle path

Next, we construct particle path in the limit $k \rightarrow \infty$. We remark that all the results obtained in this section is applicable to the initial data assumed in both (i) and (ii) of Theorem 4.1.1.

Lemma 4.4.3. *Let (v, u, e) and $(\tilde{v}_k, \tilde{u}_k, \tilde{e}_k)_{k \in \mathbb{N}}$ be the function and subsequence obtained in Lemma 4.4.2. If we define*

$$r(x, t) := \left(a^n + n \int_0^x v(y, t) dy \right)^{\frac{1}{n}},$$

then there exists a further subsequence such that

$$\tilde{r}_k(x, t) \rightarrow r(x, t) \quad \text{for all } (x, t) \in [0, \infty) \times [0, T],$$

Moreover, we also have for all $(x, t) \in [0, \infty) \times [0, T]$:

$$r(x, t) = r_0(x) + \int_0^t u(x, s) ds, \quad \text{where } r_0(x) := \left(a^n + n \int_0^x v_0(y) dy \right)^{\frac{1}{n}},$$

and the following bound holds for $r(x, t)$

$$a^n + nx\psi_-^{-1}\left(\frac{C_0}{x}\right) \leq (r(x, t))^n \leq a^n + nx\psi_+^{-1}\left(\frac{C_0}{x}\right) \quad \forall (x, t) \in [0, \infty) \times [0, T],$$

where $C_0 > 0$ is the constant depending on $(\gamma - 1) > 0$ and the initial data, and $\psi_-^{-1}(\cdot), \psi_+^{-1}(\cdot)$ are respectively the left and right branch inverse of the convex function $\psi(\zeta) = \zeta - \log \zeta - 1$.

Proof. Using Lemma 4.4.2 and the bound $\sup_{k \in \mathbb{N}} \tilde{v}_k(x, t) \leq C(a)$ for all $(x, t) \in [0, \infty) \times [0, T]$, it follows from Dominated Convergence theorem that

$$\lim_{k \rightarrow \infty} \tilde{r}_k(x, t) = \lim_{k \rightarrow \infty} \left(a^n + n \int_0^x \tilde{v}_k(y, t) dy \right)^{1/n} = \left(a^n + n \int_0^x v(y, t) dy \right)^{1/n} = r(x, t).$$

This proves the first assertion. Next, from the construction Theorem 3.1.1, it follows that

$$\left(a^n + n \int_0^x v_k(y, t) dy \right)^{\frac{1}{n}} = r_k^0(x) + \int_0^t u_k(x, s) ds \quad \text{for } (x, t) \in [0, k] \times [0, T],$$

where recalling from Section 4.2.2:

$$r_k^0(x) = \left(a^n + n \int_0^x v_k^0(y) dy \right)^{\frac{1}{n}}, \quad \text{and } v_k^0(y) = v_0(y)\phi_k(y) + (1 - \phi_k(y)).$$

It then follows from a similar argument as before:

$$\lim_{k \rightarrow \infty} r_k^0(x) = r_0(x) = \left(a^n + n \int_0^x v_0(y) dy \right)^{\frac{1}{n}} \quad \text{for each } x \in [0, \infty).$$

Now for the same fixed point $(x, t) \in [0, \infty) \times [0, T]$, we use the bound: $\sup_{k \in \mathbb{N}} |u_k(x, t)| \leq C(a) \in L^1(0, T)$, and Lemma 4.4.2 to apply Dominated Convergence theorem, and it follows that:

$$r(x, t) = \lim_{k \rightarrow \infty} \tilde{r}_k(x, t) = \lim_{k \rightarrow \infty} r_k^0(x) + \lim_{k \rightarrow \infty} \int_0^t \tilde{u}_k(x, s) ds = r_0(x) + \int_0^t u(x, s) ds.$$

Lastly, the point-wise bound for $r(x, t)$ can be obtained by combining the uniform convergence: Lemma 4.4.2 and point-wise bound for $\tilde{r}_k(x, t)$: Lemma 2.3.1. \square

4.4.4 Weak derivatives and weak convergence

Next, we show the existence of weak derivatives of $(v, u, e)(x, t)$ obtained in the previous section. First recalling from Theorem 4.1.1, the terms $\mathcal{H}_1[v, u, e, r](x, t)$ and $\mathcal{H}_2[v, u, e, r](x, t)$ were defined as:

$$\begin{aligned} \mathcal{H}_1[v, u, e, r](x, t) = & \{|v - 1|^2 + r^{2m}|D_x v|^2 + |u|^4 + r^{2m}|D_x u|^2 + \sigma r^{4m}|D_x^2 u|^2\}(x, t) \\ & + \{|e - 1|^2 + r^{2m}|D_x e|^2 + \sigma r^{4m}|D_x^2 e|^2\}(x, t), \end{aligned}$$

where $\sigma = \sigma(t)$ is the weight defined by $\sigma(t) = \min\{1, t\}$ for $t \in [0, \infty)$ and

$$\begin{aligned} \mathcal{H}_2[v, u, e, r](x, t) := & \{|v - 1|^2 + r^{2m}|D_x v|^2 + r^{4m}|D_x^2 v|^2 + |u|^4 + r^{2m}|D_x u|^2 + r^{4m}|D_x^2 u|^2\}(x, t) \\ & + \{|e - 1|^2 + r^{2m}|D_x e|^2 + r^{4m}|D_x^2 e|^2\}(x, t). \end{aligned}$$

Weak convergence results for Theorem 4.1.1(i)

Lemma 4.4.4. *Let $(v, u, e)(x, t)$, $(x, t) \mapsto r(x, t)$, and $(\tilde{v}_k, \tilde{u}_k, \tilde{e}_k)_{k \in \mathbb{N}}$ be the functions and sequence obtained in Lemma 4.4.2-4.4.3. Then weak derivatives: (Dv, Du, De) for $D \in \{D_t, D_x\}$, and $(D_x D_t v, D_x D_t u, D_x D_t e)$ exists in $L^2_{loc}((0, \infty) \times (0, T])$, and there exists a further subsequence still denoted as $(\tilde{v}_k, \tilde{u}_k, \tilde{e}_k)_{k \in \mathbb{N}}$ such that as $k \rightarrow \infty$, we have the weak star convergence:*

$$\left\{ \begin{array}{l} (\tilde{v}_k - 1, \tilde{u}_k^2, \tilde{e}_k - 1) \overset{*}{\rightharpoonup} (v - 1, u^2, e - 1), \\ (D_x \tilde{v}_k, D_x \tilde{u}_k, D_x \tilde{e}_k) \overset{*}{\rightharpoonup} (D_x v, D_x u, D_x e), \\ (D_t \tilde{v}_k, t^{1/2} D_t \tilde{u}_k, t^{1/2} D_t \tilde{e}_k) \overset{*}{\rightharpoonup} (D_t v, t^{1/2} D_t u, t^{1/2} D_t e), \\ (t^{1/2} D_x^2 \tilde{u}_k, t^{1/2} D_x^2 \tilde{e}_k) \overset{*}{\rightharpoonup} (t^{1/2} D_x^2 u, t^{1/2} D_x^2 e), \end{array} \right. \quad \text{in } L^\infty(0, T; L^2([0, \infty))),$$

and the following weak convergence in $L^2(0, T; L^2([0, \infty)))$: holds

$$\left\{ \begin{array}{l} (D_x \tilde{u}_k, D_x \tilde{e}_k) \rightharpoonup (D_x u, D_x e), \\ (D_t \tilde{v}_k, D_t \tilde{u}_k, D_t \tilde{e}_k) \rightharpoonup (D_t v, D_t u, D_t e), \\ (D_x D_t \tilde{v}_k, t^{1/2} D_x D_t \tilde{u}_k, t^{1/2} D_x D_t \tilde{e}_k) \rightharpoonup (D_x D_t v, t^{1/2} D_x D_t u, t^{1/2} D_x D_t e). \end{array} \right.$$

Moreover, there exists a constant $C(a) = C(a, T, C_0) > 0$ such that:

$$\left\{ \begin{array}{l} \sup_{0 \leq t \leq T} \int_0^\infty \{\mathcal{H}_1[v, u, e, r] + |D_t v|^2 + \sigma |D_t u|^2 + \sigma |D_t e|^2\}(x, t) dx \leq C(a), \\ \int_0^T \int_0^\infty r^{2m} \{|D_t D_x v|^2 + \sigma(t) |D_t D_x u|^2 + \sigma(t) |D_t D_x e|^2\}(x, t) dx dt \leq C(a). \end{array} \right.$$

Proof. For the weak and weak star convergences, we only show the proof for $D_t D_x \tilde{e}_k$ as the other cases can be shown using exactly the same method. Assume we have already shown that the weak derivatives $D_t e$ and $D_x e$ exists, our aim is then to show $D_x D_t e$ exists. First, using the extension (4.4.1), Theorem 4.3.1 and the fact that $r_k(x, t) \geq a$, we have

$$\int_0^T \int_0^\infty t |D_t D_x \tilde{e}_k|^2 dx \leq a^{-2m} \int_0^T \int_0^k t r_k^{2m} |\varphi_k|^2 |D_t D_x e_k|^2 dx dt + \int_0^T \int_0^k t |\varphi_k'|^2 |D_t e_k|^2 dx dt \leq C(a).$$

Thus there exists a further subsequence such that

$$t^{1/2} D_t D_x \tilde{e}_k \rightharpoonup t^{1/2} D_t D_x e \quad \text{as } k \rightarrow \infty \quad \text{weakly in } L^2(0, T; L^2([0, \infty))), \quad (4.4.3)$$

Next, fixing an integer $L \in \mathbb{N}$, it follows from $\tilde{v}_k(x, t), v(x, t) \leq C(a)$ that

$$\tilde{r}_k(x, t), r(x, t) \leq (a^n + nC(a)L)^{1/n} \leq C(a, L) < \infty. \quad (4.4.4)$$

If $\xi \in L^2(0, T; L^2([0, L]))$, then $r^m \xi \in L^2(0, T; L^2([0, L]))$ and we have for all $k/2 \geq L$

$$\begin{aligned} & \left| \int_0^T \int_0^L \tilde{r}_k^m t^{1/2} D_x D_t \tilde{e}_k \xi dx dt - \int_0^T \int_0^L r^m t^{1/2} D_x D_t e \xi dx dt \right| \\ & \leq C(a) \left(\int_0^T \int_0^L \left| 1 - \frac{r^m}{\tilde{r}_k^m} \right|^2 \xi^2 dx dt \right)^{1/2} + \left| \int_0^T \int_0^L r^m \xi (t^{1/2} D_x D_t \tilde{e}_k - t^{1/2} D_x D_t e) dx dt \right|. \end{aligned}$$

Due to the bounds (4.4.4) and point-wise convergence $\tilde{r}_k(x, t) \rightarrow r(x, t)$ from Lemma 4.4.3, one can use Dominated Convergence theorem to obtain:

$$\lim_{k \rightarrow \infty} \left(\int_0^T \int_0^L \left| \frac{\tilde{r}_k^m - r^m}{\tilde{r}_k^m} \right|^2 \xi^2 dx dt \right)^{1/2} = 0.$$

Using the weak convergence (4.4.3), we also have

$$\lim_{k \rightarrow \infty} \left| \int_0^T \int_0^L r^m \xi (t^{1/2} D_x D_t \tilde{e}_k - t^{1/2} D_x D_t e) dx dt \right| = 0.$$

Hence we have just proved that $t^{1/2} \tilde{r}_k^m D_x D_t \tilde{e}_k \rightharpoonup t^{1/2} r^m D_x D_t e$ weakly in $L^2(0, T; L^2([0, L]))$ as $k \rightarrow \infty$ for each $L \in \mathbb{N}$. By the weak lower semi-continuity of the Sobolev norms, we have

$$\int_0^T \int_0^L tr^{2m} |D_x D_t e|^2(x, t) dx dt \leq \liminf_{k \rightarrow \infty} \int_0^T \int_0^L t \tilde{r}_k^{2m} |D_x D_t \tilde{e}_k|^2(x, t) dx dt \leq C(a).$$

Since $C(a) > 0$ is independent of $L \in \mathbb{N}$, applying Monotone Convergence theorem, it follows that

$$\int_0^T \int_0^\infty tr^{2m} |D_x D_t e|^2(x, t) dx dt = \lim_{L \rightarrow \infty} \int_0^T \int_0^L tr^{2m} |D_x D_t e|^2(x, t) dx dt \leq C(a). \quad \square$$

Corollary 4.4.2. *Let $(v, u, e)(x, t)$ and $(x, t) \mapsto r(x, t)$ be the functions obtained in Lemma 4.4.2-4.4.3. Then there exists a constant $C(a) = C(a, T, C_0) > 0$ independent of the approximating index $k \in \mathbb{N}$ such that:*

$$\begin{aligned} & \int_0^T \left\{ \|r^m D_x u(\cdot, s)\|_{L^\infty}^2 + \|D_x(r^m u)(\cdot, s)\|_{L^\infty}^2 + \|r^m D_x e(\cdot, s)\|_{L^\infty}^2 \right\} ds \leq C(a), \\ & \int_0^T \sigma(t) \left\{ \|D_t u(\cdot, s)\|_{L^\infty}^2 + \|D_t e(\cdot, s)\|_{L^\infty}^2 \right\} ds \leq C(a). \end{aligned}$$

Proof. This follows from an application of Sobolev embedding theorem $H^1(0, \infty) \hookrightarrow C^0(0, \infty)$ and previous Lemma 4.4.4. \square

Lemma 4.4.5. *Let $(v, u, e)(x, t)$ and $(x, t) \mapsto r(x, t)$ be the functions obtained in Lemma 4.4.2-4.4.3. Then, for each $\varepsilon \in (0, 1]$ there exists a constant $C(\varepsilon) = C(\varepsilon, T, C_0) > 0$ independent of $a \in (0, 1)$ and $k \in \mathbb{N}$ such that:*

$$\begin{cases} \sup_{0 \leq t \leq T} \int_\varepsilon^\infty \left\{ \mathcal{H}_1[v, u, e, r] + |D_t v|^2 + \sigma |D_t u|^2 + \sigma |D_t e|^2 \right\}(x, t) dx \leq C(\varepsilon), \\ \int_0^T \int_\varepsilon^\infty r^{2m} \left\{ |D_t D_x v|^2 + \sigma(t) |D_t D_x u|^2 + \sigma(t) |D_t D_x e|^2 \right\}(x, t) dx dt \leq C(\varepsilon). \end{cases}$$

Proof. The proof is the same as Lemma 4.4.4, except we consider in the domain $[\varepsilon, \infty) \times [0, T]$ instead of $[0, \infty) \times [0, T]$. \square

Weak Convergence for Theorem 4.1.1(ii)

Lemma 4.4.6. *Let $(v, u, e)(x, t)$, $(x, t) \mapsto r(x, t)$, and $(\tilde{v}_k, \tilde{u}_k, \tilde{e}_k)_{k \in \mathbb{N}}$ be the functions and sequence obtained in Lemma 4.4.2-4.4.3. Then weak derivatives: (Dv, Du, De) for $D \in \{D_t, D_x\}$, and $(D_x D_t v, D_x D_t u, D_x D_t e)$ exists in $L^2_{loc}((0, \infty) \times (0, T])$, and there exists a further subsequence still denoted as $(\tilde{v}_k, \tilde{u}_k, \tilde{e}_k)_{k \in \mathbb{N}}$ such that as $k \rightarrow \infty$, we have the weak star convergence:*

$$\left\{ \begin{array}{l} (\tilde{v}_k - 1, \tilde{u}_k^2, \tilde{e}_k - 1) \overset{*}{\rightharpoonup} (v - 1, u^2, e - 1), \\ (D_x \tilde{v}_k, D_x \tilde{u}_k, D_x \tilde{e}_k) \overset{*}{\rightharpoonup} (D_x v, D_x u, D_x e), \\ (D_t \tilde{v}_k, D_t \tilde{u}_k, D_t \tilde{e}_k) \overset{*}{\rightharpoonup} (D_t v, D_t u, D_t e), \\ (D_x^2 \tilde{u}_k, D_x^2 \tilde{e}_k) \overset{*}{\rightharpoonup} (D_x^2 u, D_x^2 e), \end{array} \right. \quad \text{in } L^\infty(0, T; L^2[0, \infty)),$$

and the following weak convergence:

$$\left\{ \begin{array}{l} (D_x \tilde{u}_k, D_x \tilde{e}_k) \rightharpoonup (D_x u, D_x e), \\ (D_x D_t \tilde{v}_k, D_x D_t \tilde{u}_k, D_x D_t \tilde{e}_k) \rightharpoonup (D_x D_t v, D_x D_t u, D_x D_t e), \\ (D_x^3 \tilde{u}_k, D_x^3 \tilde{e}_k) \rightharpoonup (D_x^3 u, D_x^3 e). \end{array} \right. \quad \text{in } L^2(0, T; L^2[0, \infty)),$$

Moreover, there exists a constant $C(a) = C(a, T, C_0) > 0$ such that:

$$\left\{ \begin{array}{l} \sup_{0 \leq t \leq T} \int_0^\infty \{ \mathcal{H}_2[v, u, e, r] + |D_t v|^2 + |D_t u|^2 + |D_t e|^2 \}(x, t) dx \leq C(a), \\ \int_0^T \int_0^\infty r^{2m} \{ |D_t D_x v|^2 + |D_t D_x u|^2 + |D_t D_x e|^2 \}(x, t) dx dt \leq C(a), \\ \int_0^T \int_0^\infty r^{6m} \{ |D_x^3 u|^2 + |D_x^3 e|^2 \}(x, t) dx dt \leq C(a). \end{array} \right.$$

The proof of this convergence result is exactly the same as Lemma 4.4.4, except it is simpler since there is no time weight $\sigma(t)$. Thus we will not repeat the argument here.

Corollary 4.4.3. *Let $(v, u, e)(x, t)$ and $r(x, t)$ be the functions obtained in Lemma 4.4.2-4.4.3. Then there exists a constant $C(a) = C(a, T, C_0) > 0$ independent of the approximating index $k \in \mathbb{N}$ such that:*

$$\int_0^T \{ \|r^m D_x u(\cdot, s)\|_{L^\infty(0, \infty)}^2 + \|D_x(r^m u)(\cdot, s)\|_{L^\infty(0, \infty)}^2 + \|r^m D_x e(\cdot, s)\|_{L^\infty(0, \infty)}^2 \} ds \leq C(a),$$

$$\int_0^T \{ \|D_t u(\cdot, s)\|_{L^\infty(0, \infty)}^2 + \|D_t e(\cdot, s)\|_{L^\infty(0, \infty)}^2 \} ds \leq C(a),$$

Proof. This follows from an application of Sobolev embedding theorem $H^1(0, \infty) \hookrightarrow C^0(0, \infty)$ and previous Lemma 4.4.6. \square

Lemma 4.4.7. *Let $(v, u, e)(x, t)$ and $(x, t) \mapsto r(x, t)$ be the functions obtained in Lemma 4.4.2-4.4.3. Then, for each $\varepsilon \in (0, 1]$ there exists a constant $C(\varepsilon) = C(\varepsilon, T, C_0) > 0$ independent of $a \in (0, 1)$ and $k \in \mathbb{N}$ such that:*

$$\left\{ \begin{array}{l} \sup_{0 \leq t \leq T} \int_\varepsilon^\infty \{ \mathcal{H}_2[v, u, e, r] + |D_t v|^2 + |D_t u|^2 + |D_t e|^2 \}(x, t) dx \leq C(\varepsilon), \\ \int_0^T \int_\varepsilon^\infty r^{2m} \{ |D_t D_x v|^2 + |D_t D_x u|^2 + |D_t D_x e|^2 \}(x, t) dx dt \leq C(\varepsilon), \\ \int_0^T \int_\varepsilon^\infty r^{6m} \{ |D_x^3 u|^2 + |D_x^3 e|^2 \}(x, t) dx dt \leq C(\varepsilon). \end{array} \right.$$

Proof. The proof is the same as Lemma 4.4.6, except we consider in the domain $[\varepsilon, \infty) \times [0, T]$ instead of $[0, \infty) \times [0, T]$. \square

4.4.5 Entropy estimates

In this section we recover the entropy estimate for which the limit function $(v, u, e)(x, t)$ obtained in the previous Section 4.4.4 satisfies. Note that the arguments presented in this section can be applied to the initial data assumed in both (i) and (ii) of Theorem 4.1.1.

Lemma 4.4.8. *Let $(v, u, e)(x, t)$ be the function obtained in Lemma 4.4.2, then there exists a constant $C_0 > 0$ depending only on the initial data such that the following estimate holds:*

$$\begin{aligned} & \sup_{t \in [0, T]} \int_0^\infty \left\{ \frac{1}{2} |u|^2 + (\gamma - 1) \psi(v) + \psi(e) \right\} (x, t) dx \leq C_0, \\ & \int_0^T \int_0^\infty \left\{ \kappa \frac{r^{2m} |D_x e|^2}{v e^2} + \left(\lambda + \frac{2\mu}{n} \right) \frac{|D_x(r^m u)|^2}{v e} + 2m\mu \frac{v}{e} \left| \frac{D_x(r^m u)}{v \sqrt{n}} - \sqrt{n} \frac{u}{r} \right|^2 \right\} dx dt \leq C_0, \end{aligned}$$

where $\psi(\zeta) = \zeta - 1 - \log \zeta$ is a convex function.

Proof. First we set an integer $L \in \mathbb{N}$. Since $C(a)^{-1} \leq \tilde{e}_k(x, t) \leq C(a)$, and $\psi(\zeta) \rightarrow \infty$ only when $\zeta \rightarrow 0^+$ or $\zeta \rightarrow \infty$. It follows that for all $t \in (0, T]$ there exists a constant $C(a) > 0$, such that $\sup_{k \in \mathbb{N}} \psi(\tilde{e}_k) \leq C(a) \in L^2(0, L)$. Moreover, due to the point-wise convergence $\tilde{e}_k \rightarrow e$ from Lemma 4.4.2, it follows that $\psi(\tilde{e}_k(x, t)) \rightarrow \psi(e)(x, t)$ for each $(x, t) \in [0, \infty) \times [0, T]$, hence we can apply Dominated Convergence theorem to obtain:

$$\int_0^L \psi(e)(x, t) dx = \lim_{k \rightarrow \infty} \int_0^L \psi(\tilde{e}_k)(x, t) dx \leq \limsup_{k \rightarrow \infty} \int_0^k \psi(e_k)(x, t) dx \leq C_0,$$

where we have used the entropy estimate, Theorem 2.3.1(i). Since $C_0 > 0$ is independent of $L \in \mathbb{N}$, and $\psi(e) \geq 0$ we use Monotone Convergence theorem to obtain that

$$\int_0^\infty \psi(e)(x, t) dx = \lim_{L \rightarrow \infty} \int_0^L \psi(e)(x, t) dx \leq C_0, \quad \text{for all } t \in (0, T].$$

The proof for $\psi(v)$ and $|u|^2$ follows in the exact same way. Next, we consider the dissipation terms. Fixing an integer $L \in \mathbb{N}$, and Let $\phi \in L^2([0, T]; L^2(0, L))$, using the Lemma 4.4.2–4.4.4 and Dominated Convergence theorem, it follows that as $k \rightarrow \infty$,

$$\begin{aligned} & \left| \int_0^T \int_0^L \frac{\tilde{r}_k^m D_x \tilde{e}_k}{\tilde{v}_k^{1/2} \tilde{e}_k} \phi dx dt - \int_0^T \int_0^\infty \frac{r^m D_x e}{v^{1/2} e} \phi dx dt \right| \\ & \leq C(a) \left(\int_0^T \int_0^L \left| \frac{\tilde{r}_k^m}{\tilde{r}_k^m \tilde{v}_k^{1/2} \tilde{e}_k} - \frac{r^m}{\tilde{r}_k^m v^{1/2} e} \right|^2 \phi^2 dx dt \right)^{1/2} + \left| \int_0^T \int_0^L \frac{r^m}{v^{1/2} e} (D_x \tilde{e}_k - D_x e) \phi dx dt \right| \rightarrow 0. \end{aligned}$$

Thus by weak lower semi-continuity of L^2 -norm, we have

$$\int_0^T \int_0^L \frac{r^{2m} |D_x e|^2}{v e^2} dx dt \leq \liminf_{k \rightarrow \infty} \int_0^T \int_0^L \frac{\tilde{r}_k^{2m} |D_x \tilde{e}_k|^2}{\tilde{v}_k \tilde{e}_k^2} dx dt \leq \liminf_{k \rightarrow \infty} \int_0^T \int_0^k \frac{\tilde{r}_k^{2m} |D_x \tilde{e}_k|^2}{v_k e_k^2} dx dt \leq C_0.$$

Since $C_0 > 0$ is independent of $L \in \mathbb{N}$, we can apply Monotone Convergence theorem to obtain:

$$\int_0^T \int_0^\infty \frac{r^{2m} |D_x e|^2}{v e^2} dx dt = \lim_{L \rightarrow \infty} \int_0^T \int_0^L \frac{r^{2m} |D_x e|^2}{v e^2} dx dt \leq C_0.$$

The other two terms in the dissipation estimates can be obtained in the exact same manner. \square

4.4.6 Initial and boundary conditions

Next we show that $(v, u, e)(x, t)$ obtained above indeed satisfies the initial and boundary condition.

Lemma 4.4.9. *Let $(v, u, e)(x, t)$ be the function obtained in Lemma 4.4.2, then $\lim_{t \rightarrow 0^+}(v, u, e)(\cdot, t)$ exists in $L^2(0, \infty)$ and*

$$\lim_{t \rightarrow 0^+} (v, u, e)(x, t) = (v_0, u_0, e_0)(x) \quad \text{for } x \in [0, \infty).$$

Moreover, the boundary conditions are satisfied as follows

- (i) For Theorem 4.1.1(i), the functions $t \mapsto u(0, t)$ is well defined in $C^0((0, T])$ and $t \mapsto D_x e(0, t)$ is well defined in $L^2(0, T)$ such that:

$$\begin{aligned} u(0, t) &= 0 && \text{for all } t \in (0, T], \\ D_x e(0, t) &= 0 && \text{for a.e. } t \in (0, T], \end{aligned}$$

and the following boundary condition in the far field limit also holds:

$$\begin{aligned} \lim_{x \rightarrow \infty} u(x, t) &= 0 && \text{for all } t \in (0, T], \\ \lim_{x \rightarrow \infty} D_x e(x, t) &= 0 && \text{for a.e. } t \in (0, T], \end{aligned}$$

- (ii) For Theorem 4.1.1(ii), the functions $t \mapsto (u(0, t), D_x e(0, t))$ are well defined continuous functions in $C^0([0, T])$ and

$$u(0, t) = D_x e(0, t) = 0 \quad \text{for all } t \in (0, T].$$

Finally, the following boundary condition in the far field limit also holds:

$$\lim_{x \rightarrow \infty} u(x, t) = \lim_{x \rightarrow \infty} D_x e(x, t) = 0 \quad \text{for all } t \in (0, T].$$

Proof. First from Lemma 4.4.4 or 4.4.6, and the fact that $r(x, t) \geq a$ for all $(x, t) \in [0, \infty) \times [0, T]$, the following regularities holds:

$$\begin{aligned} (D_t v, D_t u, D_t e) &\in L^2(0, T; L^2(0, \infty)) \subset L^2(0, T; H^{-1}(0, \infty)), \\ \text{and } (v, u, e) &\in L^2(0, T; H^1(0, \infty)). \end{aligned}$$

It follows that $(v, u, e) \in C^0([0, T]; L^2(0, \infty))$, and we conclude that $(v, u, e)(\cdot, 0) = \lim_{t \rightarrow 0^+}(v, u, e)(\cdot, t)$ exists in $L^2(0, \infty)$. Next, let $\phi(x) \in C_c^1(0, L)$ for some fixed integer $L \in \mathbb{N}$ then:

$$\begin{aligned} (u(\cdot, 0) - u_0(\cdot), \phi)_{L^2} &= (u(\cdot, 0) - \tilde{u}_k(\cdot, 0), \phi)_{L^2} + (\tilde{u}_k(\cdot, 0) - u_0(\cdot), \phi)_{L^2} \\ &= \frac{1}{T} \int_0^T (D_t u(\cdot, t) - D_t \tilde{u}_k(\cdot, t), \phi)_{L^2} (t - T) dt + \frac{1}{T} \int_0^T (u(\cdot, t) - \tilde{u}_k(\cdot, t), \phi)_{L^2} dt \\ &\quad + (\tilde{u}_k(\cdot, 0) - u_0(\cdot))_{L^2} \rightarrow 0 \quad \text{as } k \rightarrow \infty, \end{aligned}$$

where in the last line, we have used the weak convergence in $L^2(0, T; L^2(0, \infty))$ from Lemma 4.4.4 or 4.4.6, and Proposition 4.2.2 of Section 4.2. By Fundamental Lemma of Calculus of Variation, it follows the initial condition: $u(x, 0) = u_0(x)$ for almost every $x \in [0, \infty)$. In the exact same way, we also have $e(x, 0) = e_0(x)$ and $v(x, 0) = v_0(0)$ for almost every $x \in [0, \infty)$.

Next, we show the boundary conditions for Theorem 4.1.1(i). From Lemma 4.4.4, we have

$$D_t u \in L^2(0, T; L^2(0, \infty)), \quad \sigma^{1/2} u \in L^2(0, T; H^2(0, \infty)), \quad \sigma^{1/2} D_x e \in L^2(0, T; H^1(0, \infty)),$$

where $\sigma = \sigma(t) = \min\{1, t\}$. Thus by the interpolation theorem, it implies that $u \in C^0([\tau, T]; H^1(0, \infty))$ for each $\tau \in (0, T)$. By Sobolev embedding theorem in 1D, $H^1(0, \infty) \hookrightarrow C^0([0, \infty))$, we have that $u \in C^0([\tau, T] \times [0, \infty))$, and $D_x e \in L^2(\tau, T; C^0([0, \infty)))$ for each $\tau \in (0, T)$. Hence for each $\tau \in (0, T)$, $u(0, t) = \lim_{x \searrow 0} u(x, t)$ exists in $C^0([\tau, T])$ and $D_x e(0, t) = \lim_{x \searrow 0} D_x e(x, t)$ exists in $L^2(\tau, T)$. Next, let $\chi(t) \in C_c^1([0, T])$ be a test function, then there exists $\tau \in (0, T)$ such that $\text{supp}(\chi) \subseteq [\tau, T]$. Since the approximation solution satisfies the boundary condition $\tilde{u}_k(0, t) = 0$ for all $t \in [0, T]$, we have:

$$\begin{aligned} & \int_0^T u(0, t)\chi(t)dt = \int_\tau^T (u(0, t) - \tilde{u}_k(0, t))\chi(t)dt \\ &= \int_\tau^T \int_0^1 \chi(t)(D_x u(x, t) - D_x \tilde{u}_k(x, t))(x-1)dxdt + \int_\tau^T \int_0^1 \chi(t)(u(x, t) - \tilde{u}_k(x, t))dxdt \\ &\rightarrow 0 \text{ as } k \rightarrow \infty. \end{aligned}$$

where we used the weak convergence in Lemma 4.4.4, and point-wise convergence in Lemma 4.4.2. Thus, it follows that $u(0, t) = 0$ for all $t \in (0, T]$. By the exact same argument, one can also show $D_x e(0, t) = 0$ for almost every $t \in (0, T)$. Next, we show the far field behaviour. By Fundamental Theorem of Calculus, and using the previously proven boundary condition $D_x e(0, t) = 0$ it follows that for almost every $(x, t) \in [0, \infty) \times (0, T]$

$$\begin{aligned} |r^{2m} D_x e(x, t)| &= \left| \int_0^x D_x(r^{2m} D_x e)(y, t)dy \right| = \left| \int_0^x \left\{ 2m \frac{v}{r} r^m D_x e + r^{2m} D_x^2 e \right\} dy \right| \\ &\leq x^{1/2} \left(4m^2 \int_0^x \frac{v^2}{r^2} r^{2m} |D_x e|^2(\cdot, t) \right)^{1/2} + t^{-1/2} x^{1/2} \left(\int_0^x r^{4m} t |D_x^2 e|^2(\cdot, t) \right)^{1/2} \leq C(a) x^{1/2} (1 + t^{-1/2}), \end{aligned}$$

where we have used the estimates obtained in Lemma 4.4.4. Next by the point-wise bound $C(a)^{-1} \leq v(x, t) \leq C(a)$ we have $(a^n + nC(a)^{-1}x)^{1/n} \leq r(x, t) \leq (a^n + nC(a)x)^{1/n}$. Combining this with the previous estimate, it follows that

$$|D_x e(x, t)| \leq C(a)(1 + t^{-1/2}) \left(a^n x^{-\frac{n}{4m}} + nC(a)^{-1} x^{\frac{4m-n}{4m}} \right)^{-m/n}.$$

Since $4m = 4(n-1) > n$ for $n = 2, 3$, we conclude the limit: $\lim_{x \rightarrow \infty} |D_x e(x, t)| = 0$ for a.e. $t \in (0, T]$. Using similar argument one can also show that $\lim_{x \rightarrow \infty} |u(x, t)| = 0$ for all $t \in (0, T]$.

For the proof of Theorem 4.1.1(ii), the argument is essentially the same, except that the solution (v, u, e) satisfies additional regularities. Thus we will skip this part of the proof to avoid repetition. \square

4.4.7 Strong solution

In this section, we show that the limit function $(v, u, e)(x, t)$ satisfies the equations (1.3.6) point-wise almost everywhere. We will only present the proof for Theorem 4.1.1(i), since the case for Theorem 4.1.1(ii) can be argued using exactly the same way.

Lemma 4.4.10. *Let $(v, u, e)(x, t)$ be the function obtained in Lemma 4.4.2, then for $(x, t) \in (0, \infty) \times (0, T)$ almost everywhere, the following equations are satisfied:*

$$\begin{cases} D_t v = D_x(r^m u), \\ D_t u = \beta r^m D_x \left(\frac{D_x(r^m u)}{v} \right) - (\gamma - 1) r^m D_x \left(\frac{e}{v} \right), \\ D_t e = \beta \frac{D_x(r^m u)}{v} (\beta D_x(r^m u) - (\gamma - 1) \frac{e}{v}) - 2m\mu D_x(r^{m-1} u^2) + \kappa D_x \left(\frac{r^{2m}}{v} D_x e \right). \end{cases}$$

Proof. We start by fixing an integer $L \in \mathbb{L}$ and letting $\phi \in C_c^\infty([0, T] \times [0, L])$. Then for large enough $k \in \mathbb{N}$ such that $k \geq 2L$ we have from the weak convergence Lemma 4.4.4 and strong convergences Lemma 4.4.2:

$$\int_0^T \int_0^L D_t v \phi dx dt = \lim_{k \rightarrow \infty} \int_0^T \int_0^L D_t \tilde{v}_k \phi dx dt = - \lim_{k \rightarrow \infty} \int_0^T \int_0^L \tilde{r}_k^m \tilde{u}_k D_x \phi dx dt = \int_0^T \int_0^L D_x(r^m u) \phi dx dt.$$

Since the integer $L > 0$ is arbitrary, it follows that $D_t v(x, t) = D_x(r^m u)(x, t)$ for $(x, t) \in [0, \infty) \times [0, T]$ almost everywhere. A similar argument can be used to show the point-wise almost everywhere equality of the momentum equation (1.3.6)₂, the only difference is the non-linear elliptic term, which will be shown as follows:

$$\begin{aligned} & \int_0^T \int_0^L \frac{1}{\beta} \left(D_t u + r^m (\gamma - 1) D_x \left(\frac{e}{v} \right) \right) \phi dx dt = \lim_{k \rightarrow \infty} \int_0^T \int_0^L \frac{1}{\beta} \left(D_t \tilde{u}_k + \tilde{r}_k^m (\gamma - 1) D_x \left(\frac{\tilde{e}_k}{\tilde{v}_k} \right) \right) \phi dx dt \\ &= - \lim_{k \rightarrow \infty} \int_0^T \int_0^L m \tilde{r}_k^{-1} D_t \tilde{v}_k \phi dx dt - \lim_{k \rightarrow \infty} \int_0^T \int_0^L \tilde{r}_k^m \tilde{v}_k^{-1} D_t \tilde{v}_k D_x \phi dx dt \\ &= - \int_0^T \int_0^L m r^{-1} D_t v \phi dx dt - \int_0^T \int_0^L r^m v^{-1} D_t v D_x \phi dx dt = \int_0^T \int_0^L r^m D_x \left(\frac{D_x(r^m u)}{v} \right) \phi dx dt, \end{aligned}$$

where in the last line we used $D_x r = r^{-m} v$ from Lemma 4.4.3 and $D_t v = D_x(r^m u)$ proven above. All terms in the energy equation can be shown using the same argument as above, except for the following non-linear term in the equations:

$$\begin{aligned} & \lim_{k \rightarrow \infty} \int_0^T \int_0^L \frac{|D_x(\tilde{r}_k^m \tilde{u}_k)|^2}{\tilde{v}_k} \phi dx dt = \lim_{k \rightarrow \infty} \int_0^T \int_0^L \frac{D_t \tilde{v}_k D_x(\tilde{r}_k^m \tilde{u}_k)}{\tilde{v}_k} \phi dx dt \\ &= - \lim_{k \rightarrow \infty} \int_0^T \int_0^L \left\{ \tilde{r}_k^m \tilde{u}_k \frac{\phi}{v} D_t D_x \tilde{v}_k + \tilde{r}_k^m \tilde{u}_k D_t \tilde{v}_k D_x \left(\frac{\phi}{v} \right) \right\} dx dt \\ &= - \int_0^T \int_0^L \left\{ r^m u \frac{\phi}{v} D_t D_x v + r^m u D_t v D_x \left(\frac{\phi}{v} \right) \right\} dx dt = \int_0^T \int_0^L \frac{|D_x(r^m u)|^2}{v} \phi dx dt, \end{aligned}$$

where in the last line we again used $D_x r = r^{-m} v$ from Lemma 4.4.3 and $D_t v = D_x(r^m u)$ proven above. \square

4.4.8 Uniqueness

In this section, we show that the solution $(v, u, e)(x, t)$ obtained in Sections 4.4.2–4.4.7 is unique. We only present the argument for Theorem 4.1.1(i), since the case for Theorem 4.1.1(ii) is exactly the same. The uniqueness result is summarised in the following lemma:

Lemma 4.4.11. *Let $(v, u, e)(x, t)$ and $(\tilde{v}, \tilde{u}, \tilde{e})(x, t)$ be two solutions for Lemma 4.4.10, and*

$$\begin{aligned} r(x, t) &:= \left(a^n + n \int_0^x v(y, t) dy \right)^{1/n} = r_0(x) + \int_0^t u(x, s) ds, \\ \tilde{r}(x, t) &:= \left(a^n + n \int_0^x \tilde{v}(y, t) dy \right)^{1/n} = r_0(x) + \int_0^t \tilde{u}(x, s) ds. \end{aligned}$$

Then $(v, u, e)(x, t) = (\tilde{v}, \tilde{u}, \tilde{e})(x, t)$ for $(x, t) \in [0, \infty) \times [0, T]$ almost everywhere.

Proof. Define $(V, U, E)(x, t) \equiv (v - \tilde{v}, u - \tilde{u}, e - \tilde{e})(x, t)$, then it satisfies the following system for $(x, t) \in$

$[0, \infty) \times [0, T]$ almost everywhere:

$$\left\{ \begin{array}{l} D_t V = D_x(r^m U) + D_x\{(r^m - \tilde{r}^m)\tilde{u}\}, \\ D_t U = \beta r^m D_x \left\{ \frac{D_x(r^m U)}{v} + \frac{D_x\{(r^m - \tilde{r}^m)\tilde{u}\}}{v} - \frac{V D_x(\tilde{r}^m \tilde{u})}{v\tilde{v}} \right\} \\ \quad + (r^m - \tilde{r}^m) \frac{D_t \tilde{u}}{\tilde{r}^m} - (\gamma - 1) r^m D_x \left\{ \frac{E}{v} - \frac{\tilde{e}V}{v\tilde{v}} \right\}, \\ D_t E = \kappa D_x \left\{ \frac{r^{2m}}{v} D_x E + \frac{r^{2m} - \tilde{r}^{2m}}{v} D_x \tilde{e} - \frac{\tilde{r}^{2m} V}{v\tilde{v}} D_x \tilde{e} \right\} - \beta \frac{|D_x(r^m u)|^2 V}{v\tilde{v}} \\ \quad + \frac{\beta}{\tilde{v}} \left\{ D_x\{(r^m - \tilde{r}^m)u\} + D_x(\tilde{r}^m U) \right\} \cdot \left\{ D_x(r^m u) + D_x(\tilde{r}^m \tilde{u}) \right\} \\ \quad - (\gamma - 1) \frac{D_x(r^m u)}{v} \left\{ E - \frac{\tilde{e}V}{\tilde{v}} \right\} - (\gamma - 1) \frac{\tilde{e}}{\tilde{v}} \left\{ D_x\{(r^m - \tilde{r}^m)u\} + D_x(\tilde{r}^m U) \right\} \\ \quad - 2\mu m D_x \left\{ (r^{m-1} - \tilde{r}^{m-1})u^2 + \tilde{r}^{m-1}(u + \tilde{u})U \right\}. \end{array} \right. \quad (4.4.5)$$

Multiplying $r^{-2m}V$ on both sides of (4.4.5)₁, integrating in $(x, t) \in [0, \infty) \times [0, T]$, and using Cauchy-Schwartz inequality, we have

$$\begin{aligned} \frac{1}{2} \int_0^\infty r^{-2m} |V|^2(x, t) dx &\leq \frac{\beta}{8} \int_0^t \int_0^\infty \frac{|D_x U|^2}{v} dx + C(a) \int_0^t s \int_0^s \left\| \frac{U}{r^m}(\cdot, \tau) \right\|_{L^2}^2 d\tau ds + \\ &\quad + C(a) \int_0^t s \sup_{y \in [0, \infty)} |\tilde{r}^m D_x \tilde{u}|^2(y, s) \int_0^s \left\| \frac{U}{r^m}(\cdot, \tau) \right\|_{L^2}^2 d\tau + C(a) \int_0^t \int_0^\infty \frac{|V|^2}{r^{2m}} dx. \end{aligned}$$

Taking supremum over $s \in [0, t]$, it follows that

$$\begin{aligned} \sup_{0 \leq s \leq t} \|r^{-m} V(\cdot, s)\|_{L^2}^2 &- \frac{\beta}{8} \int_0^t \left\| \frac{D_x U}{v^{1/2}}(\cdot, s) \right\|_{L^2}^2 ds \\ &\leq C(a) \int_0^t \left\{ s^2 + s^2 \|\tilde{r}^m D_x \tilde{u}(\cdot, s)\|_{L^\infty}^2 \right\} \sup_{0 \leq \tau \leq s} \|r^{-m} U(\cdot, \tau)\|_{L^2}^2 ds + C(a) \int_0^t \sup_{0 \leq \tau \leq s} \|r^{-m} V(\cdot, \tau)\|_{L^2}^2 ds. \end{aligned} \quad (4.4.6)$$

Next, multiplying $r^{-2m}U$ to (4.4.5)₂, integrating in $(x, t) \in [0, \infty) \times [0, T]$, we have

$$\begin{aligned} \sup_{0 \leq s \leq t} \|r^{-m} U(\cdot, s)\|_{L^2}^2 &+ \frac{\beta}{2} \int_0^t \left\| \frac{D_x U}{v^{1/2}}(\cdot, s) \right\|_{L^2}^2 ds \\ &\leq \mathcal{B}_\infty - \mathcal{B}_0 + C(a) \int_0^t \sup_{0 \leq \tau \leq s} \|(r^{-m} V, r^{-m} U, r^{-m} E)(\cdot, \tau)\|_{L^2}^2 ds + \\ &\quad + C(a) \int_0^t \|D_x(\tilde{r}^m \tilde{u})(\cdot, s)\|_{L^\infty}^2 \sup_{0 \leq \tau \leq s} \|r^{-m} V(\cdot, \tau)\|_{L^2}^2 ds + \\ &\quad + C(a) \int_0^t \left\{ s \|\tilde{r}^m D_x \tilde{u}(\cdot, s)\|_{L^\infty}^2 + s^2 \|D_t \tilde{u}(\cdot, s)\|_{L^\infty}^2 \right\} \sup_{0 \leq \tau \leq s} \|r^{-m} U(\cdot, \tau)\|_{L^2}^2 ds \end{aligned} \quad (4.4.7)$$

where \mathcal{B}_z for $z = 0, \infty$ are the boundary limit giving by:

$$\mathcal{B}_z := \beta \lim_{x \rightarrow z} \int_0^t \frac{U}{r^m} \left\{ \frac{D_x(r^m U)}{v} + \frac{D_x\{(r^m - \tilde{r}^m)\tilde{u}\}}{v} - \frac{V D_x(\tilde{r}^m \tilde{u})}{v\tilde{v}} - (\gamma - 1) \left(\frac{E}{v} - \frac{\tilde{e}V}{v\tilde{v}} \right) \right\} (x, s) ds.$$

It follows from the boundary condition Lemma 4.4.9 that we have $\mathcal{B}_0 = 0$. To deal with \mathcal{B}_∞ , we first note that from Corollary 4.4.2:

$$\int_0^t \|D_x(r^m u)(\cdot, s)\|_{L^\infty} ds \leq \int_0^t \left\{ 1 + \|D_x(r^m u)(\cdot, s)\|_{L^\infty}^2 \right\} ds \leq C(a) < \infty$$

Thus $D_x(r^m u)(x, t) < \infty$ for $(x, t) \in [0, \infty) \times [0, T]$ almost everywhere. Moreover, from the boundary condition, Lemma 4.4.9, we have $\lim_{x \rightarrow \infty} u(x, t) = 0$ for all $t \in [0, T]$. Thus it follows that

$$\lim_{x \rightarrow \infty} \frac{u}{r^m} \frac{D_x(r^m u)}{v}(x, s) = 0 \quad \text{for } (x, s) \in [0, \infty) \times [0, T] \text{ almost everywhere,}$$

$$\text{and } \left| \frac{u}{r^m} \frac{D_x(r^m u)}{v}(x, s) \right| \leq C(a) \|D_x(r^m u)(\cdot, s)\|_{L^\infty} \in L^1(0, T).$$

Thus by Dominated Convergence theorem, it follows that

$$\lim_{x \rightarrow \infty} \int_0^t \frac{u}{r^m} \frac{D_x(r^m u)}{v}(x, s) ds = 0.$$

In similar manner, one can also prove the above convergence with other terms in \mathcal{B}_∞ and it can be shown that $\mathcal{B}_\infty = 0$.

Next, multiplying $r^{-2m}E$ to the equation (4.4.5)₃, integrating in $(x, t) \in [0, \infty) \times [0, T]$, it follows that:

$$\begin{aligned} & \sup_{0 \leq s \leq t} \|r^{-m}E(\cdot, s)\|_{L^2}^2 + \frac{\kappa}{2} \int_0^t \int_0^\infty \frac{|D_x E|^2}{v} dx ds + \frac{\beta}{4} \int_0^t \int_0^\infty \frac{|D_x U|^2}{v} dx ds \\ & \leq \tilde{\mathcal{B}}_\infty - \tilde{\mathcal{B}}_0 + C(a) \int_0^t \left\{ 1 + \|D_x(r^m u)\|_{L^\infty}^2 + \|\tilde{r}^m D_x \tilde{e}\|_{L^\infty}^2 \right\}(s) \sup_{0 \leq \tau \leq s} \|r^{-m}V(\cdot, \tau)\|_{L^2}^2 ds \\ & \quad + C(a) \int_0^t \left\{ 1 + \|D_x(r^m u)\|_{L^\infty}^2 + \|D_x(\tilde{r}^m \tilde{u})\|_{L^\infty}^2 \right\}(s) \sup_{0 \leq \tau \leq s} \|r^{-m}E(\cdot, \tau)\|_{L^2}^2 ds \\ & \quad + C(a) \int_0^t \left\{ s + s\|\tilde{r}^m D_x \tilde{e}\|_{L^\infty}^2 + s\|r^m D_x u\|_{L^\infty}^2 \right\}(s) \sup_{0 \leq \tau \leq s} \|r^{-m}U(\cdot, \tau)\|_{L^2}^2 d\tau. \end{aligned} \quad (4.4.8)$$

where the boundary terms $\tilde{\mathcal{B}}_z$ are given by:

$$\begin{aligned} \tilde{\mathcal{B}}_z & := \kappa \lim_{x \rightarrow z} \int_0^t \frac{E}{r^{2m}} \left\{ \frac{r^{2m}}{v} D_x E + \frac{r^{2m} - \tilde{r}^{2m}}{v} D_x \tilde{e} - \frac{\tilde{r}^{2m} D_x \tilde{e}}{v \tilde{v}} V \right\}(x, s) ds \\ & \quad - 2m\mu \lim_{x \rightarrow z} \int_0^t \frac{E}{r^{2m}} \left\{ (r^{m-1} - \tilde{r}^{m-1})u^2 + \tilde{r}^{m-1}(u + \tilde{u})U \right\}(x, s) ds \end{aligned}$$

By a similar process used for the previous terms $\mathcal{B}_0 = \mathcal{B}_\infty = 0$, but using instead the boundary conditions $D_x e(0, t) = D_x \tilde{e}(0, t) = 0$ and $\lim_{x \rightarrow \infty} D_x e(x, t) = \lim_{x \rightarrow \infty} D_x \tilde{e}(x, t) = 0$, it can be shown with Lemma 4.4.9 and Corollary 4.4.2 that $\tilde{\mathcal{B}}_\infty = \tilde{\mathcal{B}}_0 = 0$.

Combining (4.4.6), (4.4.7), and (4.4.8), we obtain the estimate:

$$\begin{aligned} & \sup_{0 \leq s \leq t} \left\| \left(\frac{V}{r^m}, \frac{U}{r^m}, \frac{E}{r^m} \right) (\cdot, s) \right\|_{L^2}^2 + \frac{1}{8} \int_0^t \int_0^\infty \left\{ \beta \frac{|D_x U|^2}{v} + \kappa \frac{|D_x E|^2}{v} \right\} dx ds \\ & \leq \int_0^t h(s) \sup_{0 \leq \tau \leq s} \left\| \left(\frac{V}{r^m}, \frac{U}{r^m}, \frac{E}{r^m} \right) (\cdot, \tau) \right\|_{L^2}^2 ds, \end{aligned} \quad (4.4.9)$$

where

$$h(t) := C(a) \left\{ 1 + \|r^m D_x u(\cdot, t)\|_{L^\infty}^2 + \|\tilde{r}^m D_x \tilde{u}(\cdot, t)\|_{L^\infty}^2 + \|\tilde{r}^m D_x \tilde{e}(\cdot, t)\|_{L^\infty}^2 + t \|D_t \tilde{u}(\cdot, t)\|_{L^\infty}^2 \right\}.$$

In light of Corollary 4.4.2, we have that $\int_0^T h(t) dt \leq C(a)$. Applying Grönwall inequality to (4.4.9) and from the above we have

$$\sup_{0 \leq t \leq T} \left\| \left(\frac{V}{r^m}, \frac{U}{r^m}, \frac{E}{r^m} \right) (\cdot, t) \right\|_{L^2}^2 = 0.$$

Since $r(x, t) \geq a > 0$, it follows that $V = U = E = 0$ in $L^\infty(0, T; L^2(0, \infty))$. Therefore we conclude that $v(x, t) = \tilde{v}(x, t)$, $u(x, t) = \tilde{u}(x, t)$, and $e(x, t) = \tilde{e}(x, t)$ for $(x, t) \in [0, \infty) \times [0, T]$ almost everywhere. \square

4.5 Limit as $a \searrow 0$

We wish to construct a solution in the Lagrangian domain $(x, t) \in [0, \infty) \times [0, T]$ without any dependence in $a \in (0, 1)$, using Theorem 4.1.1(i). In order to do this, the initial data $(v_0, u_0, e_0)(x)$ given in Theorem 4.1.2 is first truncated near the origin with parameter $a \in (0, 1)$, such that the resulting function $(v_a^0, u_a^0, e_a^0)(x)$ satisfies the assumption of Theorem 4.1.1(i). From this, a collection of strong solutions $\{(v_a, u_a, e_a)(x, t)\}_{a \in (0, 1)}$ is obtained such that one can apply compactness argument in the limit $a \searrow 0$.

4.5.1 Truncation of initial data near the origin

In this section, we provide the procedure for truncating the initial data given in Theorem 4.1.2 with parameter $a \in (0, 1)$. First, we denote $\eta_a := a^n > 0$. Consider the smooth cut-off function $\phi(x) \in C^\infty([0, \infty))$ given by $\phi(x) = 0$ for $x \in [0, 1/2]$ and $\phi(x) = 1$ for $x \in [1, \infty)$. We set $\phi_a(x) := \phi(x/\eta_a)$. Then

$$|D_x \phi_a(x)| \leq \eta_a^{-1} |\phi'(\frac{x}{\eta_a})| \leq C a^{-n} \text{ for all } x \in [0, \infty), \text{ where } C \equiv \sup_{y \in [0, \infty)} |\phi'(y)|. \quad (4.5.1)$$

Next, let $(v_0, u_0, e_0)(x)$ be the initial data given in Theorem 4.1.2. We modify this solution as follows:

$$\begin{cases} v_a^0(x) := (v_0(x) - 1)\phi_a(x) + 1, \\ u_a^0(x) := u_0(x)\phi_a(x), \\ e_a^0(x) := (e_0(x) - 1)\phi_a(x) + 1, \\ r_a^0(x) := \left(a^n + n \int_0^x v_a^0(y) dy\right)^{\frac{1}{n}}, \end{cases} \text{ for } x \in [0, \infty) \quad (4.5.2)$$

The next proposition shows that for each $a \in (0, 1)$, the approximate solution (v_a^0, u_a^0, e_a^0) constructed above preserves the r_a^0 weighted $H^1(0, \infty)$ norm, and it also converges to the original initial data (v_0, u_0, e_0) in the L^2 sense.

Proposition 4.5.1. *Let $(v_a^0, u_a^0, e_a^0, r_a^0)(x)$ be the functions defined in (4.5.2), then as $a \searrow 0$ we have the convergence:*

$$\begin{cases} (v_a^0, u_a^0, e_a^0, r_a^0)(x) \rightarrow (v_0, u_0, e_0, r_0)(x) & \text{for each } x \in (0, \infty), \\ (v_a^0 - 1, |u_a^0|^2, e_a^0 - 1) \rightarrow (v_0 - 1, |u_0|^2, e_0 - 1) & \text{strongly in } L^2(0, \infty). \end{cases}$$

Moreover, there exists $\tilde{C}(C_0, n) > 0$ independent of $a \in (0, 1)$ such that

$$\begin{cases} u_a^0(0) = D_x e_a^0(0) = 0, \\ \min\{1, C_0^{-1}\} \leq v_a^0(x), e_a^0(x) \leq \max\{1, C_0\} \text{ for } x \in [0, \infty), \\ \sup_{a \in (0, 1)} |u_a^0(x)| \leq C_0 \text{ for } x \in [0, \infty), \\ \sup_{a \in (0, 1)} \int_0^\infty \{|v_a^0 - 1|^2 + |u_a^0|^4 + |e_a^0 - 1|^2\}(x) dx \leq C_0, \\ \sup_{a \in (0, 1)} \int_0^\infty (r_a^0)^{2m} \{|D_x v_a^0|^2 + |D_x u_a^0|^2 + |D_x e_a^0|^2\}(x) dx \leq \tilde{C}(C_0, n). \end{cases}$$

Proof. With the construction given in (4.5.2), one can immediately see that the following holds

$$\left\{ \begin{array}{l} u_a^0(0) = D_x e_a^0(0) = 0, \\ \min\{1, C_0^{-1}\} \leq v_a^0(x), e_a^0(x) \leq \max\{1, C_0\} \text{ for } x \in [0, \infty), \\ \sup_{a \in (0,1)} |u_a^0(x)| \leq C_0 \text{ for } x \in [0, \infty), \\ \sup_{a \in (0,1)} \int_0^\infty \{|v_a^0 - 1|^2 + |u_a^0|^4 + |e_a^0 - 1|^2\}(x) dx \leq C_0. \end{array} \right.$$

Thus we mainly focus on the convergence results and the r_a^0 weighted estimates. By construction, we have $\phi_a(x) \rightarrow 1$ as $a \searrow 0$ for all $x \in (0, \infty)$. This immediately implies that $(v_a^0, u_a^0, e_a^0)(x) \rightarrow (v_0, u_0, e_0)(x)$ as $a \searrow 0$ for all $x \in (0, \infty)$. Next, since $\sup_{a \in (0,1)} v_a^0(x) \leq \max\{1, C_0\}$, we have from Dominated Convergence theorem that

$$\lim_{a \rightarrow 0^+} r_a^0(x) = \lim_{a \searrow 0} \left(a^n + n \int_0^x v_a^0(y) dy \right)^{1/n} = \left(n \int_0^x v_0(y) dy \right)^{1/n} = r_0(x) \text{ for } x \in [0, \infty).$$

Furthermore, since for all $x \in [0, \infty)$ we have:

$$\begin{aligned} |(v_a^0(x) - 1) - (v_0(x) - 1)|^2 &= |(\phi_a(x) - 1)(v_0(x) - 1)|^2, \\ (|u_a^0(x)|^2 - |u_0(x)|^2)^2 &= |(\phi_a(x) + 1)(\phi_a(x) - 1)| \cdot |u_0(x)|^2, \\ |(e_a^0(x) - 1) - (e_0(x) - 1)|^2 &= |(\phi_a(x) - 1)(e_0(x) - 1)|^2. \end{aligned}$$

Noting once again that $\phi_a(x) \rightarrow 1$ for all $x \in (0, \infty)$ as $a \searrow 0$, and $(v_0 - 1, |u_0|^2, e_0 - 1) \in L^2(0, \infty)$, we have by Dominated Convergence theorem that

$$(v_a^0 - 1, |u_a^0|^2, e_a^0 - 1) \rightarrow (v_0 - 1, |u_0|^2, e_0 - 1) \text{ as } a \searrow 0 \text{ strongly in } L^2(0, \infty).$$

Next we show the r_a^0 weighted L^2 integrability of $(D_x v_a^0, D_x u_a^0, D_x e_a^0)$. From the construction, we have that

$$D_x v_a^0(x) = D_x((v_0(x) - 1)\phi_a(x) + 1) = (v_0(x) - 1)D_x \phi_a(x) + \phi_a(x)D_x v_0(x).$$

Using the bound $C_0^{-1} \leq v_0(x) \leq C_0$ and $\min\{1, C_0^{-1}\} \leq v_a^0(x) \leq \max\{1, C_0\}$, we have that for all $x \in [\eta_a/2, \infty)$, with $\eta_a \equiv a^n$

$$\frac{r_a^0(x)}{r_0(x)} \leq \left(\frac{a^n + n \max\{1, C_0\}x}{nC_0^{-1}x} \right)^{1/n} \leq C_0^{1/n} \left(\frac{2}{n} + \max\{1, C_0\} \right)^{1/n}.$$

Moreover we also have that for $x \in \text{supp}(D_x \phi_a) \subseteq [\eta_a/2, \eta_a]$:

$$(r_a^0(x))^{2m} \leq (a^n + n \max\{1, C_0\}\eta_a)^{2m/n} = (1 + C_0 \max\{1, C_0\})^{2m/n} a^{2m}.$$

Combining with the estimate for $D_x \phi_a$, it follows that from the construction (4.5.1) that

$$\begin{aligned} \int_0^\infty (r_a^0(x))^{2m} |D_x \phi_a(x)|^2 dx &\leq (1 + C_0 \max\{1, C_0\})^{2m/n} a^{2m} C a^{-2n} \int_{\eta_a/2}^{\eta_a} dx \\ &= C(1 + C_0 \max\{1, C_0\})^{2m/n} a^{-2} \frac{\eta_a}{2} = \frac{C}{2} (1 + C_0 \max\{1, C_0\})^{2m/n} a^{n-2} \leq \frac{C}{2} (1 + C_0 \max\{1, C_0\})^{2m/n}, \end{aligned}$$

since $a \in (0, 1)$, and $n - 2 > 0$ for $n = 2, 3$. Using these inequality, we have that

$$\begin{aligned} \int_0^\infty (r_a^0)^{2m} |D_x u_a^0|^2(x) dx &\leq 2 \sup_{x \in [0, \infty)} |u_0(x)|^2 \int_0^\infty |r_a^0(x)|^{2m} |D_x \phi_a(x)|^2 dx + 2C_0 \int_0^\infty r_0^{2m} |D_x u_0|^2(x) dx \\ &\leq CC_0^2 (1 + C_0 \max\{1, C_0\})^{2m/n} + 2C_0 := \tilde{C}(C_0, n). \end{aligned}$$

With the same argument, using $v_0(x) \leq C_0$ and $e_0(x) \leq C_0$ for $x \in [0, \infty)$, we can also obtain:

$$\int_0^\infty (r_a^0)^{2m} \{|D_x v_a^0|^2 + |D_x e_a^0|^2\}(x) dx \leq \tilde{C}(C_0, n).$$

□

For each $a \in (0, 1)$, we denote $(v_a^0, u_a^0, e_a^0)(x)$ and $r_a^0(x)$ be the approximate initial data constructed above. Then Theorem 4.1.1(i) combined with Proposition 4.5.1 guarantees the existence of a unique global-in-time strong solution $(v_a, u_a, e_a)(x, t)$ and $r_a(x, t)$ in the domain $(x, t) \in [a, \infty) \times [0, T]$, where we also have

$$r_a(x, t) = \left(a^n + n \int_0^x v_a(y, t) dy \right)^{1/n} = r_a^0(x) + \int_0^t u_a(x, s) ds \quad \text{with} \quad r_a^0(x) := \left(a^n + n \int_0^x v_a^0(y) dy \right)^{1/n}.$$

4.5.2 Strong convergence

In light of Lemma 4.4.5, we have the following estimates

Lemma 4.5.1. *Let $(v_a, u_a, e_a)(x, t)$ and $(x, t) \mapsto r_a(x, t)$ be the functions obtained in Theorem 4.1.1(i). Then, for each $\varepsilon \in (0, 1]$ there exists a constant $C(\varepsilon) = C(\varepsilon, T, C_0) > 0$ independent of $a \in (0, 1)$ such that:*

$$\left\{ \begin{array}{l} \sup_{0 \leq t \leq T} \int_\varepsilon^\infty \{ \mathcal{H}_1[v_a, u_a, e_a, r_a] + \mathcal{H}_1[v_a, u_a, e_a, 1] \}(x, t) dx \leq C(\varepsilon), \\ \sup_{t \in [0, T]} \int_\varepsilon^\infty \{ |D_t v_a|^2 + \sigma(t) |D_t u_a|^2 + \sigma(t) |D_t e_a|^2 \} dx \leq C(\varepsilon), \\ \int_0^T \int_\varepsilon^\infty r_a^{2m} \{ |D_t D_x v_a|^2 + \sigma(t) |D_t D_x u_a|^2 + \sigma(t) |D_t D_x e_a|^2 \} (x, t) dx dt \leq C(\varepsilon), \\ \int_0^T \int_\varepsilon^\infty \{ |D_t D_x v_a|^2 + \sigma(t) |D_t D_x u_a|^2 + \sigma(t) |D_t D_x e_a|^2 \} (x, t) dx dt \leq C(\varepsilon), \end{array} \right.$$

where the term \mathcal{H}_1 is defined as

$$\begin{aligned} \mathcal{H}_1[v, u, e, \zeta](x, t) := & \{ |v - 1|^2 + \zeta^{2m} |D_x v|^2 + |u|^4 + \zeta^{2m} |D_x u|^2 + \sigma \zeta^{4m} |D_x^2 u|^2 \}(x, t) \\ & + \{ |e - 1|^2 + \zeta^{2m} |D_x e|^2 + \sigma \zeta^{4m} |D_x^2 e|^2 \}(x, t), \end{aligned}$$

with $\sigma = \sigma(t) \equiv \min\{1, t\}$.

Proof. Let $\psi_-(\cdot)$ be the left inverse of the convex function $\psi(\zeta) = \zeta - 1 - \log \zeta$, one can verify that for each constant $C_0 > 0$, the function $x \mapsto x \psi_-^{-1}(C_0/x)$ is a monotone increasing map from $[0, \infty)$ unto $[0, \infty)$. Using this and Lemma 4.4.3, it follows that $C(\varepsilon)^{-1} = n \varepsilon \psi_-(C_0/\varepsilon) \leq r_a(x, t)$ for all $(x, t) \in [\varepsilon, \infty) \times [0, T]$. Combining this with Lemma 4.4.5, we have

$$\begin{aligned} & \int_0^T \int_\varepsilon^\infty \{ |D_t D_x v_a|^2 + t |D_t D_x u_a|^2 + t |D_t D_x e_a|^2 \} (x, t) dx dt \\ & \leq C(\varepsilon)^{2m} \int_0^T \int_\varepsilon^\infty r_a^{2m} \{ |D_t D_x v_a|^2 + t |D_t D_x u_a|^2 + t |D_t D_x e_a|^2 \} (x, t) dx dt \leq C(\varepsilon). \end{aligned}$$

the estimates for \mathcal{H}_1 can be obtained in the similar way. This concludes the proof of this lemma. □

Proposition 4.5.2. For each fixed $\varepsilon \in (0, 1]$, there exists a constant $C(\varepsilon) = C(\varepsilon, T, C_0) > 0$ independent of $a \in (0, 1)$ such that for all $(x_1, x_2, t) \in [\varepsilon, \infty)^2 \times (0, T]$,

$$\left\{ \begin{array}{l} \sup_{a \in (0,1)} |v_a(x_1, t) - v_a(x_2, t)| \leq C(\varepsilon)|x_1 - x_2|^{1/2}, \\ \sup_{a \in (0,1)} |u_a(x_1, t) - u_a(x_2, t)| \leq C(\varepsilon)|x_1 - x_2|^{1/2}, \\ \sup_{a \in (0,1)} |e_a(x_1, t) - e_a(x_2, t)| \leq C(\varepsilon)|x_1 - x_2|^{1/2}, \end{array} \right.$$

and for all $t_1 < t_2 \in (0, T]$ and $x \in [\varepsilon, \infty)$,

$$\left\{ \begin{array}{l} \sup_{a \in (0,1)} |v_a(x, t_1) - v_a(x, t_2)| \leq C(\varepsilon)|t_1 - t_2|^{1/4}, \\ \sup_{a \in (0,1)} |u_a(x, t_1) - u_a(x, t_2)| \leq C(\varepsilon)|t_1 - t_2|^{1/4}, \\ \sup_{a \in (0,1)} |e_a(x, t_1) - e_a(x, t_2)| \leq C(\varepsilon)|t_1 - t_2|^{1/4}. \end{array} \right.$$

Proof. We only present the proof for v_a as the cases for u_a, e_a follows in the exact same way. By Fundamental Theorem of Calculus, and estimates from Lemma 4.5.1, we have for all $x_1, x_2 \in [\varepsilon, \infty)$ and $a \in (0, 1)$:

$$|v_a(x_2, t) - v_a(x_1, t)| \leq |x_2 - x_1|^{1/2} \left(\int_{\varepsilon}^{\infty} |D_x v_a|^2(x, t) dx \right)^{1/2} \leq C(\varepsilon)|x_2 - x_1|^{1/2}.$$

Next, we show the Hölder continuity in time. First we fix a triplet $(x, t_1, t_2) \in [\varepsilon, \infty) \times (0, T]^2$ such that $0 < t_1 < t_2$. It follows from Mean Value theorem that

$$\frac{1}{h} \int_x^{x+h} |v_a(y, t_1) - v_a(y, t_2)| dy = |v_a(z, t_1) - v_a(z, t_2)| \quad \text{for some } z \in (x, x+h).$$

Next using triangular inequality, and the Fundamental Theorem of Calculus, we have:

$$|v_a(x, t_1) - v_a(x, t_2)| - |v_a(z, t_1) - v_a(z, t_2)| \leq h^{1/2} \sum_{i=1}^2 \left(\int_{\varepsilon}^{\infty} |D_x v_a(y, t_i)|^2 dy \right)^{1/2} \leq C(\varepsilon)h^{1/2}.$$

Thus combining with the previous assertion, the following inequality holds:

$$\begin{aligned} |v_a(x, t_1) - v_a(x, t_2)| &\leq C(\varepsilon)h^{1/2} + \frac{1}{h} \int_x^{x+h} \int_{t_1}^{t_2} |D_t v_a(\zeta, t)| dt d\zeta \\ &\leq C(\varepsilon)h^{1/2} + h^{-1/2}|t_2 - t_1|^{1/2} \left(\int_0^T \int_{\varepsilon}^{\infty} |D_t v_a(y, t)| dy dt \right)^{1/2} \leq C(\varepsilon)|t_2 - t_1|^{1/4}. \end{aligned}$$

In summary, we have the inequality:

$$\begin{aligned} \sup_{a \in (0,1)} |v_a(x_2, t) - v_a(x_1, t)| &\leq C(\varepsilon)|x_2 - x_1|^{1/2} \quad \text{for all } t \in [0, T] \quad \text{and } x_1, x_2 \in [\varepsilon, \infty), \\ \sup_{a \in (0,1)} |v_a(x, t_2) - v_a(x, t_1)| &\leq C(\varepsilon)|t_2 - t_1|^{1/4} \quad \text{for all } x \in [\varepsilon, \infty) \quad \text{and } t_1 < t_2 \in [0, T]. \end{aligned}$$

□

Lemma 4.5.2. *Let $\{(v_a, u_a, e_a)\}_{a \in (0,1)}$ be the collection of solutions obtained in Theorem 4.1.1(i). Then there exists a locally continuous function $(v, u, e) : (0, \infty) \times [0, T] \rightarrow (0, \infty) \times \mathbb{R} \times (0, \infty)$, and a subsequence $\{a_j\}_{j \in \mathbb{N}}$ with $a_j \rightarrow 0$ as $j \rightarrow \infty$ such that for each compact subset $K \subset\subset [0, \infty) \times [0, T]$:*

$$\lim_{j \rightarrow \infty} \sup_{(x,t) \in K} |(v_{a_j} - v, u_{a_j} - u, e_{a_j} - e)(x, t)| = 0$$

Proof. We only present the proof for $v(x, t)$ as the cases for $u(x, t)$, $e(x, t)$ follows in a similar manner. First fix an integer $L \in \mathbb{N}$, and we set the compact subset $K_L := [L^{-1}, L] \times [0, T]$. From Proposition 4.5.2, we have

$$\begin{aligned} \sup_{a \in (0,1)} |v_a(x_2, t) - v_a(x_1, t)| &\leq C(L^{-1})|x_2 - x_1|^{1/2} \quad \text{for all } t \in [0, T] \text{ and } x_1, x_2 \in [L^{-1}, L], \\ \sup_{a \in (0,1)} |v_a(x, t_2) - v_a(x, t_1)| &\leq C(L^{-1})|t_2 - t_1|^{1/4} \quad \text{for all } x \in [L^{-1}, L] \text{ and } t_1 < t_2 \in [0, T]. \end{aligned}$$

Moreover, for each $L \in \mathbb{N}$, from Theorem 4.1.1, $\sup_{a \in (0,1)} \sup_{(x,t) \in K_L} v_a(x, t) \leq C(L^{-1})$ for some constant $C(L^{-1})$ independent of $a \in (0, 1)$. Since $K_L \subset K_{L+1}$, $\text{Int}(K_L) \neq \emptyset$, and $(0, \infty) \times [0, T] = \cup_{L \in \mathbb{N}} K_L$, by Proposition A.0.1 from Appendix A, there exists a continuous function $v(x, t) : (0, \infty) \times [0, T] \rightarrow (0, \infty)$ and a further subsequence, still denoted as $a_j \searrow 0$ such that

$$\lim_{j \rightarrow \infty} \sup_{(x,t) \in K} |v_{a_j}(x, t) - v(x, t)| = 0 \quad \text{for all compact subset } K \subset\subset (0, \infty) \times [0, T].$$

Using the above uniform convergence with Proposition 4.5.2, it follows that

$$\begin{aligned} |v(x_2, t) - v(x_1, t)| &\leq C(\varepsilon)|x_2 - x_1|^{1/2} \quad \text{for all } t \in [0, T] \text{ and } x_1, x_2 \in [\varepsilon, \infty), \\ |v(x, t_2) - v(x, t_1)| &\leq C(\varepsilon)|t_2 - t_1|^{1/4} \quad \text{for all } x \in [\varepsilon, \infty) \text{ and } t_1 < t_2 \in [0, T]. \end{aligned}$$

Finally we remark that for the proof of $u(x, t)$ and $e(x, t)$, the argument follows exactly the same. \square

From Theorem 4.1.1, it follows that for each $\varepsilon \in (0, 1]$ there exists constants $C(\varepsilon) = C(\varepsilon, T, C_0) > 0$ independent of $a \in (0, 1)$ such that $C(\varepsilon) \leq \sup_{a \in (0,1)} v_a(x, t) \leq C(\varepsilon)$, $\sup_{a \in (0,1)} |u_a(x, t)| \leq C(\varepsilon)$, and $\sup_{a \in (0,1)} e_a(x, t) \leq C(\varepsilon)$ for all $(x, t) \in [\varepsilon, \infty) \times [0, T]$. Combining this result with Lemma 4.5.2, we also obtain the following:

Corollary 4.5.1. *Let (v, u, e) be the function obtained in Lemma 4.5.2, then for each $\varepsilon \in (0, 1]$, there exists a constant $C(\varepsilon) = C(\varepsilon, T, C_0) > 0$ which are independent of both $a \in (0, 1)$ such that for all $(x, t) \in [\varepsilon, \infty) \times [0, T]$*

$$C(\varepsilon)^{-1} \leq v(x, t) \leq C(\varepsilon) \quad |u(x, t)| \leq C(\varepsilon) \quad e(x, t) \leq C(\varepsilon),$$

4.5.3 Construction of particle path function

Lemma 4.5.3. *Let $(v, u, e)(x, t)$ and $(v_{a_j}, u_{a_j}, e_{a_j})_{j \in \mathbb{N}}$ be the function and subsequence obtained in Lemma 4.5.2. If we define $r(x, t) := r_0(x) + \int_0^t u(x, s) ds$ with $r_0(x) = (n \int_0^x v_0(y) dy)^{1/n}$, then*

$$\lim_{j \rightarrow \infty} r_{a_j}(x, t) = r(x, t) \quad \text{for all } (x, t) \in (0, \infty) \times [0, T].$$

In addition, $r(x, t)$ satisfies the following bound:

$$nx\psi_-^{-1}\left(\frac{C_0}{x}\right) \leq r^n(x, t) \leq nx\psi_+^{-1}\left(\frac{C_0}{x}\right) \quad \forall (x, t) \in (0, \infty) \times [0, T],$$

where $C_0 > 0$ is a constant depending only on the initial data, and $\psi_-^{-1}(\cdot), \psi_+^{-1}(\cdot)$ are respectively the left and right branch inverse of the convex function $\psi(\zeta) = \zeta - \log \zeta - 1$.

Proof. Let $(x, t) \in (0, \infty) \times [0, T]$, then there exists $\varepsilon > 0$ such that $x \geq \varepsilon$. From Lemma 4.5.2, it follows that $u_a(x, s) \rightarrow u(x, s)$ for all $s \in [0, T]$. Moreover, from Theorem 4.1.1, we also have the bound $u_a(x, s) \leq C(\varepsilon) \in L^1([0, T])$. Hence using Dominated Convergence theorem, it follows that

$$\lim_{j \rightarrow \infty} r_{a_j}(x, t) = \lim_{j \rightarrow \infty} r_a^0(x) + \lim_{j \rightarrow \infty} \int_0^t u_{a_j}(x, s) ds = \lim_{j \rightarrow \infty} r_a^0(x) + \int_0^t u(x, s) ds,$$

where we used Proposition 4.5.1 with $\lim_{a \rightarrow 0^+} r_a^0(x) = r_0(x)$. This proves the first assertion. Next, recall from Lemma 4.4.3, we have

$$a^n + nx\psi_-^{-1}\left(\frac{C_0}{x}\right) \leq (r_a(x, t))^n \leq a^n + nx\psi_+^{-1}\left(\frac{C_0}{x}\right) \quad \text{for all } (x, t) \in [0, \infty) \times [0, T],$$

Thus, combining with the above convergence, we have

$$nx\psi_-^{-1}\left(\frac{C_0}{x}\right) \leq (r(x, t))^n \leq nx\psi_+^{-1}\left(\frac{C_0}{x}\right) \quad \text{for all } (x, t) \in (0, \infty) \times [0, T].$$

□

4.5.4 Weak derivatives and weak convergence

Lemma 4.5.4. *Let $(v, u, e)(x, t)$, $(x, t) \mapsto r(x, t)$, and $(v_{a_j}, u_{a_j}, e_{a_j})_{j \in \mathbb{N}}$ be the functions and sequence obtained in Lemma 4.5.2-4.5.3. Then the weak derivatives:*

$$(Dv, Du, De) \text{ for } D \in \{D_t, D_x\}, \text{ and } (D_x D_t v, D_x D_t u, D_x D_t e)$$

exists in $L^2_{loc}((0, \infty) \times (0, T])$, and there exists a further subsequence still denoted as $(v_{a_j}, u_{a_j}, e_{a_j})_{j \in \mathbb{N}}$ such that For each fixed $\varepsilon \in (0, 1]$ we have the following weak star convergence as $j \rightarrow \infty$

$$\left\{ \begin{array}{l} (D_x v_{a_j}, D_x u_{a_j}, D_x e_{a_j}) \overset{*}{\rightharpoonup} (D_x v, D_x u, D_x e), \\ (D_t v_{a_j}, \sigma^{1/2} D_t u_{a_j}, \sigma^{1/2} D_t e_{a_j}) \overset{*}{\rightharpoonup} (D_t v, \sigma^{1/2} D_t u, \sigma^{1/2} D_t e), \text{ in } L^\infty(0, T; L^2(0, \infty)), \\ (\sigma^{1/2} D_x^2 u_{a_j}, \sigma^{1/2} D_x^2 e_{a_j}) \overset{*}{\rightharpoonup} (\sigma^{1/2} D_x^2 u, \sigma^{1/2} D_x^2 e), \end{array} \right.$$

where $\sigma = \min\{1, t\}$, and the following weak convergence holds in $L^2(0, T; L^2(0, \infty))$:

$$\left\{ \begin{array}{l} (D_x u_{a_j}, D_x e_{a_j}) \rightharpoonup (D_x u, D_x e), \\ (D_t v_{a_j}, D_t u_{a_j}, D_t e_{a_j}) \rightharpoonup (D_t v, D_t u, D_t e), \\ (D_x D_t v_{a_j}, \sigma^{1/2} D_x D_t u_{a_j}, \sigma^{1/2} D_x D_t e_{a_j}) \rightharpoonup (D_x D_t v, \sigma^{1/2} D_x D_t u, \sigma^{1/2} D_x D_t e), \end{array} \right.$$

Moreover, for each $\varepsilon \in (0, 1]$, there exists a constant $C(\varepsilon) = C(\varepsilon, T, C_0) > 0$ such that:

$$\left\{ \begin{array}{l} \sup_{0 \leq t \leq T} \int_\varepsilon^\infty \{\mathcal{H}_1[v, u, e, r] + |D_t v|^2 + \sigma |D_t u|^2 + \sigma |D_t e|^2\}(x, t) dx \leq C(\varepsilon), \\ \int_0^T \int_\varepsilon^\infty \{r^{2m} |D_t D_x v|^2 + \sigma r^{2m} |D_t D_x u|^2 + \sigma r^{2m} |D_t D_x e|^2\}(x, t) dx dt \leq C(\varepsilon). \end{array} \right.$$

Proof. For the weak and weak star convergences, we only show the proof for $D_t D_x e_{a_j}$ as the other cases can be shown using exactly the same method. Assume we have already shown that the weak derivatives $D_t e$ and $D_x e$ exists, our aim is then to show $D_x D_t e$ exists. First, fixing a $\varepsilon_N = N^{-1} \in (0, 1]$ for each $N \in \mathbb{N}$, from Lemma 4.5.1, we have

$$\int_0^T \int_{\varepsilon_N}^\infty \sigma(t) |D_t D_x e_{a_j}|^2 dx \leq C(\varepsilon) < \infty.$$

Thus there exists a further subsequence $a_j^{(N)} \rightarrow 0$ as j such that

$$\sigma^{1/2} D_t D_x e_{a_j^{(N)}} \rightharpoonup f^{(N)} \text{ as } j \rightarrow \infty \text{ weakly in } L^2([0, T]; L^2(\varepsilon_N, \infty)),$$

for some $f^{(N)} \in L^2([0, T] \times [\varepsilon_N, \infty))$. Now we inductively construct a chain of sequence $\{a_j^{(N+1)}\}_{j \in \mathbb{N}} \subseteq \{a_j^{(N)}\}_{j \in \mathbb{N}}$ for each $N \in \mathbb{N}$, such that $a_j^{(N)} \rightarrow 0^+$ as $j \rightarrow \infty$, and a countable collection of functions $\{f^{(N)}\}_{N \in \mathbb{N}}$. If we redefine $\{a_j\}_{j \in \mathbb{N}}$ to be the diagonal sequence: $a_j := a_j^{(j)}$ then it follows that

$$\sigma^{1/2} D_t D_x e_{a_j} \rightharpoonup f^{(N)} \text{ as } j \rightarrow \infty \text{ weakly in } L^2(0, T; L^2([\varepsilon_N, \infty))) \text{ for all } N \in \mathbb{N}.$$

Applying the Fundamental Lemma of Calculus of Variation, one can verify that if $M \geq N$ then $f^{(M)}(x, t) = f^{(N)}(x, t)$ for $(x, t) \in [\varepsilon_N, \infty) \times (0, T]$ almost everywhere. Now we redefine $f^{(N)}(x, t)$ to the whole domain: $(x, t) \in (0, \infty) \times (0, T]$ for each $N \in \mathbb{N}$ as follows:

$$f^{(N)}(x, t) := \begin{cases} f^{(N)}(x, t) & \text{if } (x, t) \in [\varepsilon_N, \infty) \times (0, T], \\ 0 & \text{if } (x, t) \in (0, \infty) \times (0, T] \setminus [\varepsilon_N, \infty) \times (0, T], \end{cases}$$

and using this we define

$$g_M(x, t) := f^{(1)}(x, t) + \sum_{i=1}^M (f^{(M+1)}(x, t) - f^{(M)}(x, t)), \text{ for a.e. } (x, t) \in (0, \infty) \times (0, T].$$

For a given $(x, t) \in (0, \infty) \times (0, T]$, there exists $N \in \mathbb{N}$ such that $\varepsilon_N \leq x$, and $x < \varepsilon_{N-1}$ thus with this point it follows from the previous uniqueness result that $g_M(x, t) = f^{(N)}(x, t)$ for all $M \geq N$. Thus the function:

$$g(x, t) = \lim_{M \rightarrow \infty} g_M(x, t) \text{ is well-defined for almost every } (x, t) \in (0, \infty) \times (0, T].$$

And by construction, we have that $g \in L^2_{\text{loc}}((0, \infty) \times (0, T])$ and

$$\sigma^{1/2} D_t D_x e_{a_j} \rightharpoonup g \text{ as } j \rightarrow \infty \text{ weakly in } L^2(0, T; L^2(\varepsilon, \infty)) \text{ for all } \varepsilon \in (0, 1]. \quad (4.5.3)$$

Next, taking the test function $\phi \in C_c^\infty((0, \infty) \times (0, T])$, then $t^{-1/2} \phi \in C_c^\infty((0, \infty) \times (0, T])$ and we get:

$$\int_0^T \int_0^\infty t^{-1/2} \phi t^{1/2} D_x D_t e_{a_j} dx dt = - \int_0^T \int_0^\infty D_t e_{a_j} D_x \phi dx dt = \int_0^T \int_0^\infty e_{a_j} D_t D_x \phi dx dt.$$

Taking limit $j \rightarrow \infty$ on both sides, using (4.5.3) and Lemma 4.5.2, we have

$$\int_0^T \int_0^\infty \phi t^{-1/2} g dx dt = \int_0^T \int_0^\infty e D_t D_x \phi dx dt$$

It follows that the weak derivative exists and $t^{1/2} D_t D_x e(x, t) = g(x, t)$ for a.e. $(x, t) \in (0, \infty) \times (0, T]$. Next, fixing an integer $L \in \mathbb{N}$, it follows from Lemma 4.4.3 and 4.5.3 that

$$\begin{aligned} r_{a_j}(x, t), r(x, t) &\leq \left(1 + nx\psi_+^{-1}\left(\frac{C_0}{x}\right)\right)^{1/n} \leq \left(1 + nL\psi_+^{-1}\left(\frac{C_0}{L}\right)\right)^{1/n} \leq C(L) < \infty, \\ r_{a_j}(x, t), r(x, t) &\geq \left(nx\psi_+^{-1}\left(\frac{C_0}{x}\right)\right)^{1/n} \geq \left(n\varepsilon\psi_+^{-1}\left(\frac{C_0}{\varepsilon}\right)\right)^{1/n} \geq C(\varepsilon)^{-1} > 0. \end{aligned} \quad (4.5.4)$$

If $\xi \in L^2(0, T; L^2(\varepsilon, L))$, then $r^m \xi \in L^2(0, T; L^2(0, L))$ and we have

$$\begin{aligned} & \left| \int_0^T \int_\varepsilon^L r_{a_j}^m \sigma^{1/2} D_x D_t e_{a_j} \xi dx dt - \int_0^T \int_\varepsilon^L r^m \sigma^{1/2} D_x D_t e \xi dx dt \right| \\ & \leq C(\varepsilon) \left(\int_0^T \int_\varepsilon^L |r_{a_j}^m - r^m|^2 |\xi|^2 dx dt \right)^{1/2} + \left| \int_0^T \int_\varepsilon^L r^m \xi (\sigma^{1/2} D_x D_t e_{a_j} - \sigma^{1/2} D_x D_t e) dx dt \right|. \end{aligned}$$

Due to the bounds (4.5.4) and point-wise convergence $r_{a_j}(x, t) \rightarrow r(x, t)$ from Lemma 4.5.3, one can use Dominated Convergence theorem to obtain:

$$\lim_{j \rightarrow \infty} \left(\int_0^T \int_\varepsilon^L |r_{a_j}^m - r^m|^2 |\xi|^2 dx dt \right)^{1/2} = 0.$$

Using the weak convergence (4.5.3), we also have

$$\lim_{j \rightarrow \infty} \left| \int_0^T \int_\varepsilon^L r^m \xi (\sigma^{1/2} D_x D_t e_{a_j} - \sigma^{1/2} D_x D_t e) dx dt \right| = 0.$$

Hence we have just proved that $\sigma^{1/2} r_{a_j}^m D_x D_t e_{a_j} \rightharpoonup \sigma^{1/2} r^m D_x D_t e$ as $j \rightarrow \infty$ weakly in $L^2(0, T; L^2(\varepsilon, L))$ for each $L \in \mathbb{N}$. By the weak lower semi-continuity of the Sobolev norms, we have

$$\int_0^T \int_\varepsilon^L tr^{2m} |D_x D_t e|^2(x, t) dx dt \leq \liminf_{j \rightarrow \infty} \int_0^T \int_\varepsilon^L tr_{a_j}^{2m} |D_x D_t e_{a_j}|^2(x, t) dx dt \leq C(\varepsilon).$$

Since $C(\varepsilon) > 0$ is independent of $L \in \mathbb{N}$, applying Monotone Convergence theorem, it follows that

$$\int_0^T \int_\varepsilon^\infty tr^{2m} |D_x D_t e|^2(x, t) dx dt = \lim_{L \rightarrow \infty} \int_0^T \int_\varepsilon^L tr^{2m} |D_x D_t e|^2(x, t) dx dt \leq C(\varepsilon).$$

□

4.5.5 Entropy estimates

Lemma 4.5.5. *Let $(v, u, e)(x, t)$ be the function obtained in Lemma 4.5.2, then there exists a constant $C_0 > 0$ depending only on the initial data such that the following estimate holds:*

$$\sup_{t \in [0, T]} \int_0^\infty \left\{ \frac{1}{2} |u|^2 + (\gamma - 1) \psi(v) \right\} (x, t) dx \leq C_0,$$

where $\psi(\zeta) = \zeta - 1 - \log \zeta$ is a convex function.

Proof. First we set an integer $L \in \mathbb{N}$ and a quantity $\varepsilon \in (0, 1]$. Since $C(\varepsilon)^{-1} \leq \sup_{j \in \mathbb{N}} v_{a_j}(x, t) \leq C(\varepsilon)$ for $(x, t) \in [\varepsilon, \infty) \times [0, T]$, and $\psi(\zeta) \rightarrow \infty$ only when $\zeta \rightarrow 0^+$ or $\zeta \rightarrow \infty$. It follows that for all $t \in (0, T]$, $\sup_{j \in \mathbb{N}} \psi(v_{a_j}) \leq C(\varepsilon) \in L^1(\varepsilon, L)$. Moreover, due to the point-wise convergence $v_{a_j} \rightarrow v$ from Lemma 4.5.2, it follows that $\psi(v_{a_j})(x, t) \rightarrow \psi(v)(x, t)$ for each $(x, t) \in [\varepsilon, L] \times [0, T]$, hence we can apply Dominated Convergence theorem to obtain:

$$\int_\varepsilon^L \psi(v)(x, t) dx = \lim_{j \rightarrow \infty} \int_\varepsilon^L \psi(v_{a_j})(x, t) dx \leq \limsup_{j \rightarrow \infty} \int_0^\infty \psi(v_{a_j})(x, t) dx \leq C_0,$$

where we have used the entropy estimate from Theorem 4.1.1. Since $C_0 > 0$ is independent of $L \in \mathbb{N}$ and $\varepsilon \in (0, 1]$, taking $\varepsilon \equiv L^{-1}$ and the fact that $\psi(v) \geq 0$ we use Monotone Convergence theorem to obtain that

$$\int_0^\infty \psi(v)(x, t) dx = \lim_{L \rightarrow \infty} \int_{L^{-1}}^L \psi(v)(x, t) dx \leq C_0, \quad \text{for all } t \in (0, T].$$

The proof for $|u|^2$ follows in the exact same way. □

4.5.6 Weak forms

First, we show that the weak derivative of $r(x, t)$ exists.

Proposition 4.5.3. *Let $(v, u, e)(x, t)$ be the function obtained in Lemma 4.5.2, then $D_x r(x, t)$ exists and*

$$D_x r(x, t) = r^{-m}(x, t)v(x, t) \text{ for a.e. } (x, t) \in (0, \infty) \times [0, T],$$

Moreover, $x \mapsto r(x, t)$ is monotone increasing for each $t \in [0, T]$ and

$$\underline{r}(t) := \lim_{x \rightarrow 0^+} r(x, t) \text{ exists in } \underline{r}(t) \in [0, \infty) \text{ for each } t \in [0, T].$$

As a consequence, $r(x, t)$ also takes the form:

$$r(x, t) = \left((\underline{r}(t))^n + n \int_0^x v(y, t) dy \right)^{1/n} \text{ for } (x, t) \in [0, \infty) \times [0, T].$$

Proof. By the construction from Lemma 4.5.3, it follows that $D_x (r_{a_j}(x, t))^n = n v_{a_j}(x, t)$ for all $(x, t) \in [0, \infty) \times [0, T]$. Let $\phi \in C_c^\infty((0, \infty) \times [0, T])$, multiplying ϕ to the above equation, and integrating by parts, it follows that:

$$- \int_0^T \int_0^\infty (r_{a_j}(x, t))^n D_x \phi(x, t) dx dt = n \int_0^T \int_0^\infty v_{a_j}(x, t) \phi(x, t) dx dt.$$

Next, since $\phi \in C_c^\infty((0, \infty) \times [0, T])$, there exists $\varepsilon > 0$ such that $\text{supp}(\phi) \subset [\varepsilon, \infty) \times [0, T]$. It then follows from Lemma 4.5.2, we have the convergence: $(v_{a_j}, r_{a_j})(x, t) \rightarrow (v, r)(x, t)$ as $j \rightarrow \infty$ for all $(x, t) \in \text{supp}(\phi)$. Moreover, from Corollary 4.5.1 and Lemma 4.5.3, there exists a constant $C(\phi) > 0$ depending on the support of ϕ such that $C(\phi)^{-1} \leq \sup_{j \in \mathbb{N}} v_{a_j}(x, t)$, $\sup_{j \in \mathbb{N}} r_{a_j}(x, t) \leq C(\phi)$ for all $(x, t) \in \text{supp}(\phi)$. Hence applying Dominated Convergence theorem, we have

$$- \int_0^T \int_0^\infty (r(x, t))^n D_x \phi(x, t) dx dt = n \int_0^T \int_0^\infty v(x, t) \phi(x, t) dx dt.$$

This shows that the spatial weak derivative of $r^n(x, t) := (r(x, t))^n$ exists and $D_x r^n(x, t) = n \cdot v(x, t)$ for $(x, t) \in (0, \infty) \times [0, T]$ almost everywhere. Next using Lemma 4.5.3: $0 < n x \psi^{-1}(C_0/x) \leq r^n(x, t)$ for all $(x, t) \in (0, \infty) \times [0, T]$. Thus by Chain Rule for weak derivative, we have $D_x r(x, t) = (r(x, t))^{-m} v(x, t)$ for $(x, t) \in (0, \infty) \times [0, T]$ almost everywhere. Next, for each $0 < x_1 < x_2$, by choosing a sequence of smooth test functions which approximates the indicator set $\mathbb{1}_{x \in [x_1, x_2]}$, one can show that

$$(r(x_2, t))^n - (r(x_1, t))^n = n \int_{x_1}^{x_2} v(y, t) dy \text{ for all } (x_1, x_2, t) \in (0, \infty)^2 \times [0, T]$$

Since $v(x, t) \geq C(\varepsilon) > 0$ for $(x, t) \in [\varepsilon, \infty) \times [0, T]$, it follows from the above equation that if $x_2 > x_1 > 0$ then $r(x_2, t) > r(x_1, t)$ for all $t \in [0, T]$. Using the lower bound $0 < (n x \psi^{-1}(C_0/x))^{1/n} \leq r^n(x, t)$ for all $(x, t) \in (0, \infty) \times [0, T]$ and the fact that $x \mapsto r(x, t)$ is monotone increasing, we have

$$\underline{r}(t) := \lim_{x \rightarrow 0^+} r(x, t) \text{ exists in } \underline{r}(t) \in [0, \infty) \text{ for each } t \in [0, T].$$

Now since Lemma 4.5.5 and Jensen's inequality:

$$\psi\left(\frac{1}{x} \int_0^x v(y, t) dy\right) \leq \frac{1}{x} \int_0^x \psi(v)(y, t) dy \leq \frac{C_0}{x}$$

Thus, for each $x \in (0, \infty)$ and $t \in [0, T]$: $0 < x\psi_-^{-1}(C_0/x) < \|v(\cdot, t)\|_{L^1(0,x)} \leq x\psi_+^{-1}(C_0/x) < \infty$. Integrating over $D_x(r(x, t))^n = nv(x, t)$, using the limit $r(x, t) \rightarrow \underline{r}(t)$ for each $t \in [0, T]$, and rearranging the terms it follows that

$$r(x, t) = \left((\underline{r}(t))^n + n \int_0^x v(y, t) dy \right)^{1/n}.$$

□

For each $j \in \mathbb{N}$, the solution $(v_{a_j}, u_{a_j}, e_{a_j})(x, t)$ obtained in Theorem 4.1.1(i) satisfies the equations almost everywhere. In particular its corresponding weak forms also holds. Applying Lemma 4.5.4 to this weak form for each test function $\phi \in C_c^\infty((0, \infty) \times [0, T])$ bounded away from $x = 0$, we obtain the following result:

Lemma 4.5.6. *Let $(v, u, e)(x, t)$ be the function obtained in Lemma 4.5.2, then the following weak forms are satisfied: For all test function $\varphi \in C^\infty((0, \infty) \times [0, T])$ such that there exists $N_\varphi \in \mathbb{N}$ with $\text{supp}(\varphi(\cdot, t)) \subseteq [N_\varphi^{-1}, N_\varphi]$ for all $t \in [0, T]$ then it follows that*

(i) *Continuity equation:*

$$\int_0^\infty v(x, t)\varphi(x, t)dx - \int_0^\infty v_0(x)\varphi(x, 0)dx - \int_0^t \int_0^\infty (vD_t\varphi - r^m u D_x\varphi)(x, s)dx ds = 0.$$

(ii) *Momentum equation:*

$$\begin{aligned} & \int_0^\infty u(x, t)\varphi(x, t)dx - \int_0^\infty u_0(x)\varphi(x, 0)dx - \int_0^t \int_0^\infty u(x, s)D_t\varphi(x, s)dx ds \\ &= - \int_0^t \int_0^\infty \left\{ \beta \frac{r^m}{v} D_x u + m \frac{u}{r} - (\gamma - 1) \frac{e}{v} \right\} (r^m D_x \varphi + m \frac{v}{r} \varphi) dx ds. \end{aligned}$$

(iii) *Equation for energy density:*

$$\begin{aligned} & \int_0^\infty (e + \frac{|u|^2}{2})\varphi(x, t)dx - \int_0^\infty (e_0 + \frac{|u_0|^2}{2})\varphi(x, 0)dx - \int_0^t \int_0^\infty (e + \frac{|u|^2}{2})D_t\varphi dx ds \\ &= \int_0^t \int_0^\infty r^m u \left\{ (\gamma - 1) \frac{e}{v} - m\lambda \frac{u}{r} - (2\mu + \lambda) \frac{r^m}{v} D_x u \right\} D_x \varphi dx ds - \int_0^t \int_0^\infty \kappa \frac{r^{2m}}{v} D_x e D_x \varphi dx ds. \end{aligned}$$

Since the weak derivatives $(D, D^2)(v, u, e, r)(x, t)$ for $D \in \{D_t, D_x\}$ exists in $L^2([\varepsilon, \infty) \times [0, T])$ for all $\varepsilon > 0$, by Fundamental Lemma of Calculus of Variation and the above weak forms, one can obtain the following statement:

Lemma 4.5.7. *Let $(v, u, e)(x, t)$ be the solution obtained in Lemma 4.5.2. Then the following equations are satisfied for $(x, t) \in (0, \infty) \times (0, T)$ almost everywhere:*

$$\left\{ \begin{array}{l} D_t v = D_x(r^m u), \\ D_t u = \beta r^m D_x \left(\frac{D_x(r^m u)}{v} \right) - (\gamma - 1) r^m D_x \left(\frac{e}{v} \right), \\ D_t e = \frac{D_x(r^m u)}{v} \{ \beta D_x(r^m u) - (\gamma - 1)e \} - 2m\mu D_x(r^{m-1}u^2) + \kappa D_x \left(\frac{r^{2m}}{v} D_x e \right). \end{array} \right.$$

Appendix A

Some variants of Arzelà-Ascoli theorem.

This appendix will be devoted to the proof of two propositions on Arzelà-Ascoli theorem, which are used in Sections 2.4.3–2.4.4 for the limit process as $k \rightarrow \infty$, and in Sections 2.5.2–2.5.3 for the limit process as $a \searrow 0$.

Proposition A.0.1. *Let $D \subseteq \mathbb{R}^2$ be a connected domain with Lipschitz boundary, such that $\text{Int}(D) \neq \emptyset$, where $\text{Int}(\cdot)$ denotes the set of interior points. Assume $\{f_i : D \rightarrow \mathbb{R}\}_{i \in \mathbb{N}}$ is a collection of functions, and $\{(D_\alpha, C_\alpha)\}_{\alpha \in \mathbb{N}}$ is a collection of compact subsets $D_\alpha \subset\subset D$ and constants $C_\alpha \in (0, \infty)$ such that for each $\alpha \in \mathbb{N}$,*

$$\begin{aligned} D_\alpha &\subseteq D_{\alpha+1}, \quad \text{Int}(D_\alpha) \neq \emptyset, \quad D = \cup_{\alpha \in \mathbb{N}} D_\alpha, \\ \sup_{i \in \mathbb{N}} \sup_{x \in D_\alpha} f_i(x) &\leq C_\alpha, \quad \sup_{i \in \mathbb{N}} |f_i(x) - f_i(y)| \leq C_\alpha |x - y| \quad \text{for all } x, y \in D_\alpha. \end{aligned} \quad (\text{A.0.1})$$

Then there exists a continuous function $f : D \rightarrow \mathbb{R}$ and a sub-sequence $\{i_j\}_{j \in \mathbb{N}}$ with $i_j \rightarrow \infty$ as $j \rightarrow \infty$ such that

$$\begin{aligned} \lim_{j \rightarrow \infty} \sup_{x \in K} |f_{i_j}(x) - f(x)| &= 0 \quad \text{for all compact subset } K \subset\subset D, \\ |f(x) - f(y)| &\leq C_\alpha |x - y| \quad \text{for all } x, y \in D_\alpha, \text{ for each } \alpha \in \mathbb{N}. \end{aligned}$$

Proof. We split the proof into 3 steps: the desired sub-sequence, the limit function and uniform convergence and continuity.

Step 1: the desired sub-sequence. By Arzelà-Ascoli theorem, it follows that for each $\alpha \in \mathbb{N}$ there exists a continuous function $f^{(\alpha)}(x, t) : D_\alpha \rightarrow \mathbb{R}$ and a sub-sequence $\{i_j^{(\alpha)}\}_{j \in \mathbb{N}}$ such that $i_j^{(\alpha)} \rightarrow \infty$ as $j \rightarrow \infty$, and

$$\lim_{j \rightarrow \infty} \sup_{x \in D_\alpha} |f^{(\alpha)}(x) - f_{i_j^{(\alpha)}}(x)| = 0. \quad (\text{A.0.2})$$

Based on (A.0.2), one can inductively construct a chain of sub-sequences $\{i_j^{(\alpha+1)}\}_{j \in \mathbb{N}} \subseteq \{i_j^{(\alpha)}\}_{j \in \mathbb{N}}$ for each $\alpha \in \mathbb{N}$. If one takes the diagonal sub-sequences $i_j := i_j^{(j)}$ then for all $\alpha \in \mathbb{N}$,

$$\lim_{j \rightarrow \infty} \sup_{x \in D_\alpha} |f^{(\alpha)}(x) - f_{i_j}(x)| = 0. \quad (\text{A.0.3})$$

If $x \in D_\alpha$ for some integer $\alpha \in \mathbb{N}$ then for any other integer $q \geq \alpha + 1$, we have for all $x \in D_\alpha$

$$|f^{(\alpha)}(x) - f^{(q)}(x)| \leq |f^{(\alpha)}(x) - f_{i_j}(x)| + |f_{i_j}(x) - f^{(q)}(x)| \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

Thus if $q \geq \alpha$ are two integers then

$$f^{(q)}(x) = f^{(\alpha)}(x) \quad \text{for all } x \in D_\alpha. \quad (\text{A.0.4})$$

Step 2: the limit function. For each $\alpha \in \mathbb{N}$, $f^{(\alpha)}(x)$ is extended into D by

$$h^{(\alpha)}(x) := \begin{cases} f^{(\alpha)}(x) & \text{if } x \in D_\alpha, \\ 0 & \text{if } x \in D \setminus D_\alpha. \end{cases}$$

With this, one can define the function $g_N(x)$ in the domain $x \in D$ as,

$$g_N(x) := h^{(1)}(x) + \sum_{k=1}^{N-1} \{h^{(k+1)}(x) - h^{(k)}(x)\}, \quad \text{for each } N \in \mathbb{N} \text{ and } N \geq 2.$$

Then for a fixed point $y \in D$, there exists $\alpha \in \mathbb{N}$ such that $y \in D_\alpha$ and $y \notin D_{\alpha-1}$, with $D_0 := \emptyset$. By (A.0.4), and the assumption $D_q \subseteq D_{q+1}$ for each $q \in \mathbb{N}$, it follows that $h^{(q)}(y) = 0$ if $q \leq \alpha - 1$, and $h^{(q)}(y) = h^{(\alpha)}(y)$ if $q \geq \alpha$. Hence it follows that for all $N \geq \alpha + 1$,

$$g_N(y) = h^{(1)}(y) + \sum_{k=1}^{N-1} \{h^{(k+1)}(y) - h^{(k)}(y)\} = h^{(N)}(y) = h^{(\alpha)}(y) = f^{(\alpha)}(y).$$

Thus for this point $y \in D$, one has $\lim_{N \rightarrow \infty} g_N(y) = f^{(\alpha)}(y)$. Since this is true for any arbitrary point $x \in D$, one gets that the function $f(x) : D \rightarrow \mathbb{R}$ given by

$$f(x) := \lim_{N \rightarrow \infty} g_N(x),$$

is well-defined and in particular, $f(x) = f^{(\alpha)}(x)$ if $x \in D_\alpha$ for some $\alpha \in \mathbb{N}$. By the uniform convergence (A.0.3), it follows that for each $\alpha \in \mathbb{N}$,

$$\lim_{j \rightarrow \infty} \sup_{x \in D_\alpha} |f(x) - f_{i_j}(x)| = \lim_{j \rightarrow \infty} \sup_{x \in D_\alpha} |f^{(\alpha)}(x) - f_{i_j}(x)| = 0. \quad (\text{A.0.5})$$

Step 3: uniform convergence and continuity. Let $K \subset\subset D$ be any compact subset. Then $\{\text{Int}(D_\alpha)\}_{\alpha \in \mathbb{N}}$ is an open covering for K , hence there exists a finite sub-covering $\{D_{\alpha_k}\}_{k=1}^N$ of K , for some $N \in \mathbb{N}$. Notice $D_q \subseteq D_{q+1}$ for each $q \in \mathbb{N}$, therefore there exists $\alpha \in \mathbb{N}$ such that $K \subseteq D_\alpha$. Combining with (A.0.5), it follows that:

$$\lim_{j \rightarrow \infty} \sup_{x \in K} |f_{i_j}(x) - f(x)| = 0.$$

For the continuity of f , fixing a point $x \in D$, then there exists $\alpha \in \mathbb{N}$ such that $x \in D_\alpha$. Since $\text{Int}(D_\alpha) \neq \emptyset$, there exists a $\varepsilon_1 > 0$ such that $B_{\varepsilon_1}(x) \subset D_\alpha$. According to (A.0.1), one has

$$\sup_{i \in \mathbb{N}} |f_i(x) - f_i(y)| \leq C_\alpha |x - y| \quad \text{for all } y \in D_\alpha.$$

Applying the uniform convergence (A.0.5), it follows that $|f(x) - f(y)| \leq C_\alpha |x - y|$ for all $y \in D_\alpha$. Given $\varepsilon > 0$, if one sets $\delta := \min\{\varepsilon_1, \varepsilon/C_\alpha\} > 0$ then for $y \in B_\delta(x) \subseteq B_{\varepsilon_1}(x) \subset D_\alpha$,

$$|f(x) - f(y)| \leq C_\alpha |x - y| \leq \varepsilon.$$

The continuity of $x \mapsto f(x)$ has been proved. \square

Before stating the next proposition, we define: for each connected interval $I \subseteq [0, \infty)$, the functional space $\tilde{H}_0^1(I, r^m dr)$ is the closure of

$$\mathcal{D}_0(I) := \left\{ \phi \in C^\infty(I) : \exists N > a \text{ such that } [a, N] \subset I \text{ and } \phi(r) = 0 \text{ for } r \in I \cap [N, \infty) \right\}$$

via the $H^1(I, r^m dr)$ -norm. We also denote its dual space as $\tilde{H}^{-1}(I, r^m dr)$. Then $\tilde{H}^{-1}(I, r^m dr)$ endowed with the norm:

$$\|f\|_{\tilde{H}^{-1}(I, r^m dr)} := \sup \left\{ \langle f, g \rangle_{\tilde{H}^{-1}, \tilde{H}_0^1} : \|g\|_{\tilde{H}_0^1(I, r^m dr)} \leq 1 \right\} \text{ is a Banach space.}$$

Proposition A.0.2. *Let $I = [a, b]$ be an interval for some $a, b \in \mathbb{R}$, and let $D = [d, \infty)$ be an unbounded interval with some fixed $d \in (0, 1)$. Suppose that $\{f_i(r, t) : D \times I \rightarrow \mathbb{R}\}_{i \in \mathbb{N}}$ is a collection of functions satisfying $f_i(r, t) \in L^\infty(I; \tilde{H}^{-1}(D_L, r^m dr))$ for each $i, L \in \mathbb{N}$, where $D_L := D \cap [0, L] = [d, L]$. In addition, we suppose that there exists a constant $C > 0$ for which the following bounds are satisfied: for all $t_1, t_2 \in I$,*

$$\begin{aligned} \sup_{i \in \mathbb{N}} \|f_i(\cdot, t_2) - f_i(\cdot, t_1)\|_{\tilde{H}^{-1}(D_L, r^m dr)} &\leq C|t_2 - t_1| \quad \text{for all } L \in \mathbb{N}, \\ \sup_{i \in \mathbb{N}} \sup_{t \in I} \int_D |f_i - 1|^2(r, t) r^m dr &\leq C. \end{aligned} \tag{A.0.6}$$

Then there exists a sub-sequence $\{i_k\}_{k \in \mathbb{N}}$ and a function f such that for all $L \in \mathbb{N}$,

$$\begin{aligned} f - 1 &\in L^\infty(I; L^2(D, r^m dr)), \quad f \in C^0(I; \tilde{H}^{-1}(D_L, r^m dr)), \\ \lim_{k \rightarrow \infty} \sup_{t \in I} \|f_{i_k}(\cdot, t) - f(\cdot, t)\|_{\tilde{H}^{-1}(D_L, r^m dr)} &= 0, \\ \|f(\cdot, t_1) - f(\cdot, t_2)\|_{\tilde{H}^{-1}(D_L, r^m dr)} &\leq C|t_1 - t_2| \quad \text{for all } t_1, t_2 \in I. \end{aligned} \tag{A.0.7}$$

where the constant $C > 0$ is independent of $L \in \mathbb{N}$.

Proof. We split the proof in 4 steps: the desired sub-sequence, the desired Cauchy sequence, uniform convergence, and independence of domain size.

Step 1: the desired sub-sequence. Fixing a large integer $L \in \mathbb{N}$, it follows that for all $i \in \mathbb{N}$,

$$\begin{aligned} \int_{D_L} |f_i|^2 dr &\leq 2 \int_{D_L} |f_i - 1|^2 dr + 2(L - d) \\ &\leq 2d^{-m} \int_{D_L} |f_i - 1|^2 r^m dr + 2L \leq 2d^{-m}C + 2L < \infty. \end{aligned}$$

Hence, the following uniform estimate hold:

$$\sup_{i \in \mathbb{N}} \sup_{t \in I} \int_{D_L} |f_i|^2(r, t) r^m dr \leq 2d^{-m}C + 2L < \infty. \tag{A.0.8}$$

Let $\{t_j\}_{j \in \mathbb{N}} \subset I$ be a countable dense subset. Then from (A.0.8), one has that for each $j \in \mathbb{N}$, there exists a function $g_{j, L} \in L^2(D_L, dr)$ and a sub-sequence $\{i_k^{(j)}\}_{k \in \mathbb{N}}$ such that

$$f_{i_k^{(j)}}(\cdot, t_j) \rightharpoonup g_{j, L}(\cdot) \quad \text{weakly as } k \rightarrow \infty \quad \text{in } L^2(D_L, dr).$$

Thus one can inductively obtain a chain of sub-sequence $\{i_k^{(j+1)}\}_{k \in \mathbb{N}} \subseteq \{i_k^{(j)}\}_{k \in \mathbb{N}}$ for each $j \in \mathbb{N}$, and if one takes the diagonal sequence $i_k := i_k^{(k)}$, then for all $j \in \mathbb{N}$,

$$f_{i_k}(\cdot, t_j) \rightharpoonup g_{j, L}(\cdot) \quad \text{weakly as } k \rightarrow \infty \quad \text{in } L^2(D_L, dr). \tag{A.0.9}$$

Step 2: the desired Cauchy sequence. If one defines

$$M^i(r, t) := \int_{D \cap [0, r]} f_i(\zeta, t) d\zeta, \quad \text{for each } r \leq L,$$

then by (A.0.9), it follows that for each $(r, j) \in [0, L] \times \mathbb{N}$,

$$\lim_{k \rightarrow \infty} M^{i_k}(r, t_j) = \lim_{k \rightarrow \infty} \int_{D \cap [0, r]} f_{i_k}(\zeta, t_j) d\zeta \rightarrow \int_{D \cap [0, r]} g_{j, L}(\zeta) d\zeta =: M_{j, L}(r). \tag{A.0.10}$$

For all $\phi \in \tilde{H}_0^1(D_L, r^m dr)$, one has

$$\begin{aligned}
& \int_{D_L} (f_{i_k}(r, t_j) - g_j(r))\phi(r)r^m dr = \int_{D_L} \partial_r(M^{i_k}(r, t_j) - M_{j,L}(r))\phi(r)r^m dr \\
& = - \int_{D_L} (M^{i_k}(r, t_j) - M_{j,L}(r))(\partial_r\phi + m\frac{\phi}{r})r^m dr \\
& \leq \left(\int_{D_L} |M^{i_k}(r, t_j) - M_{j,L}(r)|^2 r^m dr \right)^{\frac{1}{2}} \left(\int_{D_L} \{|\partial_r\phi|^2 + m^2\frac{|\phi|^2}{d^2}\} r^m dr \right)^{\frac{1}{2}} \\
& \leq \frac{C}{d} \|\phi\|_{H^1(D_L, r^m dr)} \left(\int_{D_L} |M^{i_k}(r, t_j) - M_{j,L}(r)|^2 r^m dr \right)^{\frac{1}{2}}.
\end{aligned}$$

Therefore by (A.0.10) and Dominated Convergence theorem, we have that for $j \in \mathbb{N}$, as $k \rightarrow \infty$,

$$\|f_{i_k}(\cdot, t_j) - g_j(\cdot)\|_{\tilde{H}^{-1}(D_L, r^m dr)} \leq \frac{C}{d} \left(\int_{D_L} |M^{i_k}(r, t_j) - M_{j,L}(r)|^2 r^m dr \right)^{\frac{1}{2}} \rightarrow 0.$$

In particular, since $L \in \mathbb{N}$ can be arbitrarily chosen, the following statement holds true:

$$\{f_{i_k}(\cdot, t_j)\}_{k \in \mathbb{N}} \text{ is a Cauchy sequence in } \tilde{H}^{-1}(D_L, r^m dr) \text{ for each } (j, L) \in \mathbb{N}^2. \quad (\text{A.0.11})$$

Step 3: uniform convergence. Given $\delta > 0$, if one sets $I_j := (t_j - \delta/(3C), t_j + \delta/(3C))$ for each $j \in \mathbb{N}$, then $t_j \in I_j$ and

$$\sup_{k \in \mathbb{N}} \|f_{i_k}(\cdot, t) - f_{i_k}(\cdot, t_j)\|_{\tilde{H}^{-1}(D_L, r^m dr)} \leq C|t - t_j| \leq \frac{\delta}{3}, \text{ for all } t \in I_j. \quad (\text{A.0.12})$$

Since $\{t_j\}_{j \in \mathbb{N}} \subset I$ is dense, $\{I_j\}_{j \in \mathbb{N}}$ is an open covering of $I = [a, b]$. It follows that there exists a finite sub-covering $\{I_j\}_{j=1}^M$ for some $M \in \mathbb{N}$. Now, for each $j = 1, \dots, M$, it follows from (A.0.11) that there exists $N_j \in \mathbb{N}$ such that

$$\|f_{i_p}(\cdot, t_j) - f_{i_q}(\cdot, t_j)\|_{\tilde{H}^{-1}(D_L, r^m dr)} \leq \frac{\delta}{3} \text{ for all } p, q \geq N_j. \quad (\text{A.0.13})$$

Setting $N := \max\{N_j : j = 1, \dots, M\}$, then for all $t \in I$ there exists $j \in \{1, \dots, M\}$ such that $t \in I_j$, hence by (A.0.12)–(A.0.13), it follows that for all $p, q \geq N$,

$$\begin{aligned}
& \|f_{i_p}(\cdot, t) - f_{i_q}(\cdot, t)\|_{\tilde{H}^{-1}(D_L, r^m dr)} \\
& \leq \|f_{i_p}(\cdot, t) - f_{i_p}(\cdot, t_j)\|_{\tilde{H}^{-1}(D_L, r^m dr)} + \|f_{i_p}(\cdot, t_j) - f_{i_q}(\cdot, t_j)\|_{\tilde{H}^{-1}(D_L, r^m dr)} \\
& \quad + \|f_{i_q}(\cdot, t_j) - f_{i_q}(\cdot, t)\|_{\tilde{H}^{-1}(D_L, r^m dr)} \leq \delta/3 + \delta/3 + \delta/3 = \delta.
\end{aligned}$$

Since this is true for any $t \in I$, one has

$$\sup_{t \in I} \|f_{i_p}(\cdot, t) - f_{i_q}(\cdot, t)\|_{\tilde{H}^{-1}(D_L, r^m dr)} \leq \delta \text{ for all } p, q \geq N.$$

Therefore, $\{f_{i_k}\}_{k \in \mathbb{N}}$ is a Cauchy sequence in $L^\infty(I; \tilde{H}^{-1}(D_L, r^m dr))$, which is a Banach space. Thus there exists $g_L \in L^\infty(I; \tilde{H}^{-1}(D_L, r^m dr))$, such that

$$\lim_{k \rightarrow \infty} \sup_{t \in I} \|f_{i_k}(\cdot, t) - g_L(\cdot, t)\|_{\tilde{H}^{-1}(D_L, r^m dr)} = 0. \quad (\text{A.0.14})$$

It also follows from the Lipschitz continuity assumption of the proposition and (A.0.14) that

$$\|g_L(\cdot, t_1) - g_L(\cdot, t_2)\|_{\tilde{H}^{-1}(D_L, r^m dr)} \leq C|t_1 - t_2| \text{ for all } t_1, t_2 \in I,$$

hence $g_L \in C^0(I; \tilde{H}^{-1}(D_L, r^m dr))$.

Step 4: independence of domain size. We claim there is $f - 1 \in L^\infty(0, T; L^2(D, r^m dr))$ independent of $L \in \mathbb{N}$, such that $f = g_L$ in $L^\infty(I, \tilde{H}^{-1}(D_L, r^m dr))$. It follows from (A.0.6) that there exists a function $f - 1 \in L^\infty(I; L^2(D, r^m dr))$ and a sub-sequence which is still denoted as $\{f_i\}_{i \in \mathbb{N}}$ such that

$$f_i - 1 \xrightarrow{*} f - 1 \text{ as } i \rightarrow \infty \text{ in } L^\infty(I; L^2(D, r^m dr)). \quad (\text{A.0.15})$$

Let $\varphi \in L^1(I, \tilde{H}_0^1(D_L, r^m dr))$, and let $\{i_k\}_{k \in \mathbb{N}}$ be the sub-sequence in (A.0.14). Then using (A.0.14) and (A.0.15), it follows that as $k \rightarrow \infty$

$$\begin{aligned} & \left| \int_I \langle f - g_L, \varphi \rangle_{\tilde{H}^{-1}, \tilde{H}_0^1} dt \right| \leq \left| \int_I \langle f - f_{i_k}, \varphi \rangle_{\tilde{H}^{-1}, \tilde{H}_0^1} dt \right| + \left| \int_I \langle f_{i_k} - g_L, \varphi \rangle_{\tilde{H}^{-1}, \tilde{H}_0^1} dt \right| \\ & \leq \left| \int_I \int_{D_L} (f - f_{i_k})(r, t) \varphi(r, t) r^m dr dt \right| + \int_I \|f_{i_k} - g_L\|_{\tilde{H}^{-1}} \|\varphi\|_{\tilde{H}_0^1} dt \rightarrow 0, \end{aligned}$$

which implies that $f = g_L$ in $L^\infty(I, \tilde{H}^{-1}(D_L, r^m dr))$. Since f is independent of $L \in \mathbb{N}$, one concludes from (A.0.14) that (A.0.7) holds for all $L \in \mathbb{N}$. \square

Appendix B

Some results regarding convex functions

This appendix will be devoted to giving some properties on convex functions, which will be used in Sections 2.4.5, 2.5.1, and 2.5.3.

Proposition B.0.1. *Let $G(\cdot)$, $\psi(\cdot)$, $H(\cdot)$ be the convex functions defined in (2.5.1). If G_+^{-1} , ψ_+^{-1} , H_+^{-1} are their corresponding right branch inverses, then they are concave, strictly increasing continuous functions defined on the domain:*

$$G_+^{-1} : [0, \infty) \rightarrow [1, \infty), \quad \psi_+^{-1} : [0, \infty) \rightarrow [1, \infty), \quad H_+^{-1} : [-e^{-1}, \infty) \rightarrow [e^{-1}, \infty).$$

It follows that functions $(f_1, f_2, f_3)(y, z)$ given in (2.5.2) is well defined for $(y, z) \in (0, \infty)^2$, and for each fixed $z > 0$, $y \mapsto f_i(y; z)$ is positive monotone increasing for $i = 1, \dots, 3$. In addition, for each fixed $z > 0$ one has:

$$\lim_{y \searrow 0} f_i(y; z) = 0 \quad \text{for } i \in \{1, 2, 3\}.$$

Proof. First, $G(\zeta)$, $\psi(\zeta)$ are uniformly convex and it achieves minimum at $\zeta = 1$ with $G(1) = 0$. Hence, $G(\zeta)$ and $\psi(\zeta)$ are strictly increasing in $[1, \infty)$. Thus the inverse functions, $\psi_+^{-1}, G_+^{-1} : [0, \infty) \rightarrow [1, \infty)$ exists and satisfies $G(G_+^{-1}(\xi)) = \psi(\psi_+^{-1}(\xi)) = \xi$ for all $\xi \in [0, \infty)$. In addition, $H(\zeta)$ is uniformly convex and achieves minimum at $\zeta = e^{-1}$ with $H(e^{-1}) = -e^{-1}$. Hence, $H(\zeta)$ is strictly increasing in $[e^{-1}, \infty)$. Thus the inverse, $H_+^{-1} : [e^{-1}, \infty) \rightarrow [-e^{-1}, \infty)$ exists and satisfies $H(H_+^{-1}(\xi)) = \xi$ for all $\xi \in [-e^{-1}, \infty)$. Also since $H(1) = 0$, the restricted function $H_+^{-1} : [0, \infty) \rightarrow [1, \infty)$ is monotone increasing, with $H_+^{-1}(0) = 1$.

Next, by Inverse Function theorem, one has

$$\frac{dG_+^{-1}}{d\xi}(\xi) = \left(\frac{dG}{d\zeta}(G_+^{-1}(\xi)) \right)^{-1} = \frac{1}{\log(G_+^{-1}(\xi))}, \quad \text{for } \xi \in (0, \infty).$$

Since $\xi \mapsto G_+^{-1}(\xi)$ is monotone increasing bijective map, $G_+^{-1}(\xi) \rightarrow \infty$ as $\xi \rightarrow \infty$, hence

$$\lim_{\xi \rightarrow \infty} \frac{dG_+^{-1}}{d\xi}(\xi) = \lim_{\xi \rightarrow \infty} \frac{1}{\log(G_+^{-1}(\xi))} = 0,$$

which, along with L'Hôpital's rule, yields that

$$\lim_{y \searrow 0} f_1(y; z) := \lim_{y \searrow 0} y G_+^{-1}\left(\frac{z}{y}\right) \stackrel{\text{H}}{=} z \lim_{y \searrow 0} \frac{dG_+^{-1}}{d\xi}\left(\frac{z}{y}\right) = 0.$$

In addition,

$$\frac{\partial f_1}{\partial y}(y; z) = G_+^{-1}(zy^{-1}) - \frac{zy^{-1}}{\log(G_+^{-1}(zy^{-1}))} \quad \text{for } (y, z) \in (0, \infty)^2,$$

where $G_+^{-1}(zy^{-1}) \in (1, \infty)$ for each $(y, z) \in (0, \infty)^2$. It follows from $G(\zeta) = 1 - \zeta + \zeta \log \zeta$ and the identity $G_+^{-1}(G(\zeta)) = \zeta$ that

$$\frac{\partial f_1}{\partial y}(y; z) = \frac{-1 + G_+^{-1}(zy^{-1})}{\log(G_+^{-1}(zy^{-1}))} > 0 \text{ for all } (y, z) \in (0, \infty)^2,$$

which implies that for fixing $z > 0$, $f_1(y; z)$ is monotone increasing for $y \in (0, \infty)$.

Next, by Inverse Function theorem, it follows that:

$$\frac{d\psi_+^{-1}}{d\xi}(\xi) = \left(\frac{d\psi}{d\zeta}(\psi_+^{-1}(\xi)) \right)^{-1} = \left(1 - \frac{1}{\psi_+^{-1}(zy^{-1})} \right)^{-1} \text{ for } \xi \in (0, \infty).$$

Since $\psi_+^{-1}(\xi) \rightarrow \infty$ as $\xi \rightarrow \infty$ for each fixed $z > 0$, by L'Hôpital's rule, one has

$$\lim_{y \searrow 0} y\psi_+^{-1}\left(\frac{z}{y}\right) \stackrel{\text{H}}{=} z \lim_{y \searrow 0} \left(1 - \frac{1}{\psi_+^{-1}(zy^{-1})} \right)^{-1} = z, \text{ for } z > 0.$$

Thus it follows that

$$\lim_{y \searrow 0} f_2(y; z) = -z + \lim_{y \searrow 0} y\psi_+^{-1}\left(\frac{z}{y}\right) = -z + z = 0.$$

Moreover, fixing $z > 0$ and taking the derivative of $y \mapsto f_2(y; z)$, one obtains that

$$\frac{\partial f_2}{\partial y}(y; z) = \psi_+^{-1}\left(\frac{z}{y}\right) - \frac{z}{y^2} \frac{d\psi_+^{-1}}{d\xi}\left(\frac{z}{y}\right) = \frac{\psi_+^{-1}(zy^{-1}) \log(\psi_+^{-1}(zy^{-1}))}{\psi_+^{-1}(zy^{-1}) - 1} > 0 \text{ for } y \in (0, \infty),$$

thus it follows that $y \rightarrow f_2(y; z)$ is increasing for each fixed $z > 0$.

Finally, by Inverse Function theorem, it follows that

$$\frac{dH_+^{-1}}{d\xi}(\xi) = \left(\frac{dH}{d\zeta}(H_+^{-1}(\xi)) \right)^{-1} = \frac{1}{1 + \log(H_+^{-1}(\xi))}, \text{ for } \xi \in [0, \infty).$$

Since, $H_+^{-1}(\xi) \rightarrow \infty$ as $\xi \rightarrow \infty$, one has

$$\lim_{\xi \rightarrow \infty} \frac{dH_+^{-1}}{d\xi}(\xi) = \lim_{\xi \rightarrow \infty} \frac{1}{1 + \log(H_+^{-1}(\xi))} = 0,$$

which, along with the L'Hôpital's rule, implies that

$$\lim_{y \searrow 0} f_3(y; z) \stackrel{\text{H}}{=} \lim_{y \searrow 0} \frac{-zy^{-2}}{-y^{-2}} \frac{dH_+^{-1}}{d\xi}(zy^{-1}) = \lim_{y \searrow 0} \frac{dH_+^{-1}}{d\xi}(zy^{-1}) = 0.$$

Furthermore, for fixing $z > 0$, one also has that for all $y \in (0, \infty)$,

$$\frac{\partial f_3}{\partial y}(y; z) = H_+^{-1}(zy^{-1}) - \frac{z}{y} \frac{dH_+^{-1}}{d\xi}(zy^{-1}) = \frac{H_+^{-1}(zy^{-1})}{1 + \log H_+^{-1}(zy^{-1})} > 0.$$

Therefore $y \mapsto f_3(y; z)$ is strictly increasing and positive for each fixed $z > 0$. □

Before proving the next proposition, we state the so-called Mazur's Lemma:

Theorem B.0.1 (Mazur's Lemma). *Let $(B, \|\cdot\|_B)$ be a Banach space, and $S \subseteq B$ be a convex subset. Then S is closed in the strong topology if and only if it is closed in the weak topology.*

For further detail, we refer the readers to Theorem 3.7 in Chapter 3.3 of [1].

Proposition B.0.2. *Let $d > 0$, $T > 0$, and $G(\zeta) := 1 - \zeta + \zeta \log \zeta$. Assume there exists constants $\tilde{C}_1, \tilde{C}_2 > 0$, $\tilde{\delta} \geq 0$, and a sequence $\{\theta_i\}_{i \in \mathbb{N}} \subset L^2(0, T; L^2([d, \infty), r^m dr))$ satisfying:*

$$\begin{cases} \theta_i - 1 \rightharpoonup \theta - 1 \text{ weakly in } L^2(0, T; L^2([d, \infty), r^m dr)), \\ \sup_{i \in \mathbb{N}} \operatorname{ess\,sup}_{t \in [0, T]} \int_d^\infty G(\theta_i)(r, t) r^m dr \leq \tilde{C}_1, \\ \tilde{\delta} \leq \theta_i \leq \tilde{C}_2 \text{ for a.e. } (r, t) \in [d, \infty) \times [0, T] \text{ and for each } i \in \mathbb{N}. \end{cases} \quad (\text{B.0.1})$$

Then the limit function θ satisfies

$$\operatorname{ess\,sup}_{t \in [0, T]} \int_d^\infty G(\theta)(r, t) r^m dr \leq \tilde{C}_1.$$

Proof. Fix an integer $N \in \mathbb{N}$. Denote $B_N := L^2(0, T; L^2([d, N], r^m dr))$. Then B_N is a Banach space. Set S_N to be the subset:

$$\begin{aligned} S_N := \left\{ w \in B_N : \operatorname{ess\,sup}_{t \in [0, T]} \int_d^N G(w)(r, t) r^m dr \leq \tilde{C}_1, \right. \\ \left. \text{and } \tilde{\delta} \leq w(r, t) \leq \tilde{C}_2 \text{ for a.e. } (r, t) \in [d, N] \times [0, T] \right\}. \end{aligned}$$

First, we claim that S_N is a convex subset of B_N . Since $G(z) = 1 - z + z \log z$ is convex, it follows that for each $w_1, w_2 \in S_N$ and $0 < \eta < 1$,

$$\begin{aligned} & \operatorname{ess\,sup}_{t \in [0, T]} \int_d^N G(\eta w_1 + (1 - \eta)w_2)(r, t) r^m dr \\ & \leq \eta \operatorname{ess\,sup}_{t \in [0, T]} \int_d^N G(w_1)(r, t) r^m dr + (1 - \eta) \operatorname{ess\,sup}_{t \in [0, T]} \int_d^N G(w_2)(r, t) r^m dr \leq \tilde{C}_1, \\ & \tilde{\delta} \leq (\eta w_1 + (1 - \eta)w_2)(r, t) \leq \tilde{C}_2 \text{ for a.e. } (r, t) \in [d, N] \times [0, T], \end{aligned}$$

which implies that S_N is indeed a convex subset of B_N .

Next, we wish to show that, if $\{w_i\}_{i \in \mathbb{N}} \subset S_N$ is a sequence such that

$$\lim_{i \rightarrow \infty} \|w_i - w\|_{B_N} = \lim_{i \rightarrow \infty} \int_0^T \int_d^N |w_i - w|^2(r, t) r^m dr dt = 0,$$

for some $w \in B_N$, then $w \in S_N$ as well. Since $w_i \rightarrow w$ strongly in $L^2(0, T; L^2([d, N], r^m dr))$, there exists a sub-sequence, which will be still denoted as $\{w_i\}_{i \in \mathbb{N}}$ such that

$$w_i(r, t) \rightarrow w(r, t) \text{ for } (r, t) \in [d, N] \times [0, T] \text{ a.e.} \quad (\text{B.0.2})$$

Due to $\{w_i\}_i \subset S_N$, the convergence in (B.0.2) implies that

$$\tilde{\delta} \leq w(r, t) \leq \tilde{C}_2 \text{ for } (r, t) \in [d, N] \times [0, T] \text{ a.e.} \quad (\text{B.0.3})$$

Since $G(\cdot)$ is convex and achieves its minimum at $G(\zeta = 1) = 0$, and $G(\zeta = 0) = 1$, it follows that for all $i \in \mathbb{N}$ with $\tilde{\delta} \leq w_i \leq \tilde{C}_2$, $G(w_i(r, t)) \leq \max\{1, G(\tilde{\delta}), G(\tilde{C}_2)\}$. Moreover, since $G(\cdot)$ is continuous, it

follows from (B.0.2) that $G(w_i(r, t)) \rightarrow G(w(r, t))$ as $i \rightarrow \infty$ for a.e. $(r, t) \in [d, N] \times [0, T]$. By Dominated Convergence theorem, one obtains for each $0 \leq t_1 < t_2 \leq T$,

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \int_d^N G(w_i)(r, t) r^m dr dt = \lim_{i \rightarrow \infty} \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \int_d^N G(w_i)(r, t) r^m dr dt \leq \tilde{C}_1.$$

Since this is true for any interval $[t_1, t_2] \subseteq [0, T]$, then by Lebesgue Differentiation theorem,

$$\operatorname{ess\,sup}_{t \in [0, T]} \int_d^N G(w)(r, t) r^m dr \leq \tilde{C}_1. \quad (\text{B.0.4})$$

It follows from (B.0.3)-(B.0.4) that $w \in S_N$ as well, which along with Theorem B.0.1, implies that

$$S_N \text{ is closed in the weak topology of } B_N := L^2(0, T; L^2([d, N], r^m dr)). \quad (\text{B.0.5})$$

By (B.0.2), it follows that $\{\theta_i\}_{i \in \mathbb{N}} \subset S_N$ and $\theta_i \rightharpoonup \theta$ in $B_N = L^2(0, T; L^2([d, N], r^m dr))$. Assertion (B.0.5) then implies that $\theta \in S_N$, and in particular,

$$\operatorname{ess\,sup}_{t \in [0, T]} \int_d^N G(\theta)(r, t) r^m dr \leq \tilde{C}_1.$$

Now since this is true for all $N \in \mathbb{N}$, and $G(\cdot)$ is non-negative, it follows from Monotone Convergence theorem that:

$$\operatorname{ess\,sup}_{t \in [0, T]} \int_d^\infty G(\theta)(r, t) r^m dr = \lim_{N \rightarrow \infty} \operatorname{ess\,sup}_{t \in [0, T]} \int_d^N G(\theta)(r, t) r^m dr \leq \tilde{C}_1.$$

□

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