

Reproducing the pressure–time signature of membrane filtration: The interplay between fouling, caking, and elasticity

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Abstract

We develop a mathematical model of direct-flow filtration operating at constant flux to understand the pressure–time signature. We combine fluid flow with membrane fouling and caking to explain the gradual increase in driving pressure that is often reported. We model the periodic backflashes used to clean such membranes and show that the elasticity of the membrane may explain the limited effectiveness of these backflashes. We also consider strategies for the operation of direct-flow filtration and show that tuning the flux and rate of backflushing, as well as the membrane material structure, may allow for improvements in membrane performance.

Keywords: Filtration, Fouling, Caking, Backflushing, Elasticity, Mathematical modelling

1. Introduction

1 In the constant-flux membrane filtration of liquid, the required driving pressure is known to
2 increase over time [1, 2, 3, 4]. This increase is attributed to the particles that are removed by the
3 filter, which cause *fouling* and *caking*, that is, the trapping of particles within and on the mem-
4 brane, respectively. One way of mitigating this increase in driving pressure, and hence to extend
5 the lifetime and efficiency of a membrane, is to perform a ‘backflush’ at regular intervals: the
6 direction of the fluid flux is briefly reversed, removing the *cake layer* and decreasing the driving
7 pressure required for the next period of (forwards) filtration. However, experiments show that
8 backflushing at regular intervals does not completely reduce the driving pressure back to its value
9 at the start of filtration [2, 4]. A schematic illustration of the pressure versus time trace, Figure 2,
10 shows that the driving pressure immediately following each successive backflush gradually in-
11 creases and, further, that the increase in driving pressure during each subsequent filtration cycle
12 grows. As a result, the filtration process becomes less efficient with time. The analogue of this
13 pressure–time *signature* when operating at constant pressure has also been reported. In this case,
14 the flux achieved at given pressure gradually declines over time with only partial recovery after
15 backflushing [5, 6]. Backflushing when a set pressure is recorded can improve the efficiency of
16 the system. However, the interval between backflashes decreases over time due to the successive
17 increases in the initial pressure after each backflush. Overall, the filtration time is significantly
18 reduced [3].

19 The limited effectiveness of regular backflashes described above is believed to be due to foul-
20 ing: particles enter the pores of the membrane, and are not readily removed by backflashes [7].
21 This is in contrast to caking – the build-up of particles on the surface of the membrane – which is
22 effectively removed by backflushing. In this paper, we hypothesize that the underlying physical
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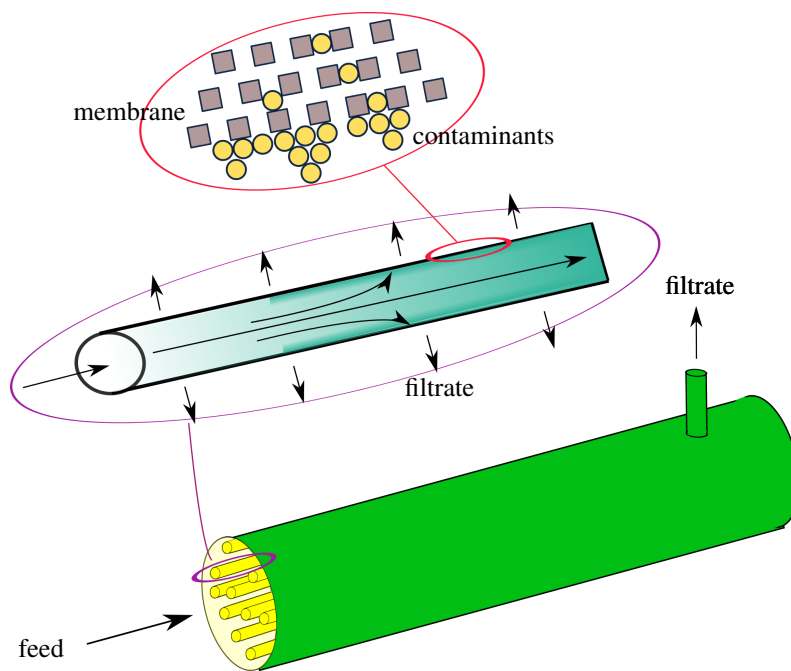


Figure 1: Sketch of a direct-flow membrane filtration device. A feed of contaminated water enters a tube with porous membrane walls. The membrane blocks the contaminants, allowing only clean water (filtrate) to pass through.

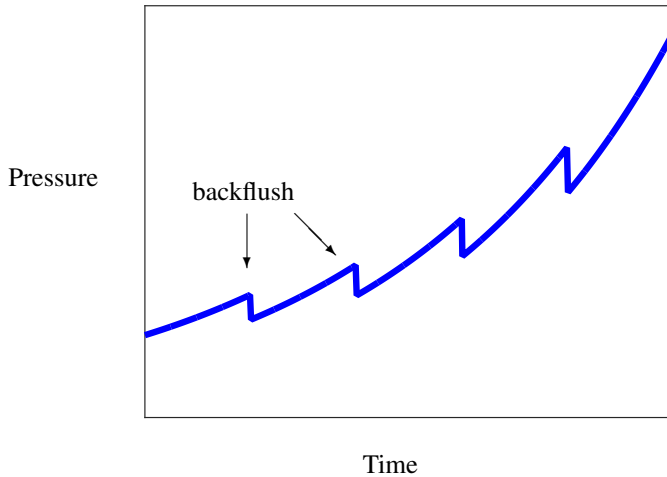


Figure 2: Sketch of the pressure versus time behaviour typically observed in a filtration device operating at constant flux. A cake layer of particles builds up on the membrane surface over time, increasing the pressure required to drive the constant flux. At regular intervals, backflushing is performed, removing the cake layer and reducing the filtration pressure. This causes the regular drops in pressure; note that, despite backflushing, the pressure never returns to its baseline value.

24 reason for this inability to remove fouling particles is the elastic deformation of the membrane
 25 during filtration and, in particular, the asymmetry in this deformation between forward flow (fil-
 26 tration) and backflushes. We model this deformation in detail, determining how it is coupled to
 27 the hydrodynamic pressures due to flow (which change because of caking and fouling). We
 28 also consider how deformation might be expected to modify the pore size, leading to increased
 29 fouling, and an accelerated decrease in the permeability of the membrane, as has been suggested
 30 qualitatively before [8]. All of these effects contribute to an increase in the driving pressure
 31 required for filtration.

32 Previous work has modelled fouling and caking extensively. In these models fouling is
 33 typically subcategorized as either concentration-polarization, complete, intermediate or inter-
 34 nal blocking [9, 10, 11, 12, 13]. Here, we are concerned primarily with the interaction of fluid
 35 flow, fouling, and caking with the elastic response of the membrane and cake layer, and how
 36 these factors affect the progression of fouling and caking. For simplicity, we consider a model
 37 in which fouling only occurs via internal pore blocking: we assume that a single particle plugs
 38 a single pore, and any particle that reaches but does not enter a membrane pore instead forms
 39 part of a cake. We consider the case of microfiltration (see Appendix A) in which the osmotic
 40 pressure is small compared to the fluid pressure [14]. We may therefore neglect osmotic effects
 41 in our model. However, it is crucial that internal fouling reduces the membrane permeability.

42 The idea of elastic deformation resulting in increased fouling has previously been consid-
 43 ered from a different perspective, the deformation of the contaminant itself, particularly under
 44 constant flux conditions [15]. For example, bacteria and macromolecules may deform to gain
 45 access to otherwise unavailable pores [16, 17]. Similarly, in ultrafiltration, flexible polymers
 46 are stretched by the flow to the point that their projected cross-sectional area is smaller than the

47 pore size, thus allowing the polymer to pass through the pore [18]. However, recent studies have
48 demonstrated that the membrane itself may deform under high pressures and, further, that this
49 deformation may lead to increased fouling as the pores increase in size [8].

50 The stress–strain relationship for hollow-fibre membranes has been characterized into three
51 types: elastic strain, transitional strain, and plastic strain. The elastic strain region is typically
52 valid for pressures of up to 10 bar [19], though there is considerable variation depending on the
53 material and structure of an individual membrane. In any case, elastic strain is a factor for all
54 types of filtration [20] while transitional and plastic strain may occur if the operating pressure is
55 allowed to reach the highest values.

56 The deformation of a cake layer of soft colloids or deformable particles has also been studied.
57 In particular, since the cake is observed to compress under pressure [21, 22] while the membrane
58 itself expands, there is an interesting coupling between membrane expansion and cake compression.
59 It has been demonstrated that the attachment of particles to the cake layer, via van der
60 Waals and electrostatic forces, is mostly irreversible [23]. We shall therefore assume that the
61 cake layer is stable (not easily broken up) during filtration, as opposed to a backflush. (A novel
62 route towards cake break-up is presented in [24].) Note, however, that growth of the cake layer
63 increases the shear stress on its surface (by a combination of increased fluid velocity and reduced
64 open area) and ultimately an equilibrium thickness is reached at which the shear stress is
65 enough to overcome the adhesive forces. Microfiltration experiments report that this threshold
66 cake thickness may be a significant fraction of the tube radius, up to 38% [25].

67 Finally, cake deposition and transmembrane pressure differences are often not uniform along
68 the membrane: they usually increase and decrease, respectively, monotonically with distance
69 along the membrane. However, in direct-flow systems (crossflow with a capped end) operation
70 can be performed with a transmembrane pressure close to uniform [26]. To study the fundamental
71 interaction of fouling and caking with elasticity in the simplest possible setting, we consider a
72 2D cross-section of a hollow-fibre membrane tube; we neglect axial variations.

73 In this paper, we present a model of the combination of caking, fouling, and elastic deformation
74 described above. We begin by showing that the timescale of poroelastic response is much
75 shorter than the rate of fouling. This motivates a quasi-static approximation to decouple the fouling
76 and caking from the induced elastic responses: the elastic effects may be assumed to occur
77 instantaneously, and appear as time-dependent parameters in a set of coupled ordinary differential
78 equations (ODEs) that we derive for the fouling and caking. We then consider different
79 operating strategies (particularly focussing on varying the rate of backflushing) that may be used
80 to reduce fouling and the effects of elastic deformation.

81 **2. Mathematical Modelling**

82 *2.1. Setup*

83 The axisymmetric 2D setup that we consider is depicted schematically in Figure 3. The wall
84 of the outer tube constitutes the porous membrane; we denote its elastic properties by \hat{E}_m and ν_m
85 (the Young’s modulus and Poisson ratio, respectively). We use hats to denote dimensional quantities.
86 Initially (*i.e.*, before deformation), the membrane has internal radius \hat{R} and thickness \hat{a}_m .
87 During filtration, a constant areal flux, \hat{Q} , of fluid flows radially from a source at the centre of
88 the tube towards and through the membrane with a velocity $\hat{V}(\hat{r})$. During the backflush phase,
89 the flux reverses sign so that the centre becomes a sink of strength \hat{Q} . We assume that the flow
90 is quasi-steady and occurs at low Reynolds number, with a constant viscosity $\hat{\mu}$; we take the
91 pressure outside the tube to be the (constant) pressure datum.

We assume that the membrane consists of pores of equal size, \hat{a}_{pore} , that are uniformly distributed around the membrane. Due to the large number of pores typically found in a membrane, we describe the particle fouling and deposition via a continuum approach rather than modelling each pore individually. The pore size is allowed to vary during an experiment, $\hat{a}_{\text{pore}} = \hat{a}_{\text{pore}}(\hat{t})$, in response to the evolving deformation of the membrane (due to fouling and cake growth). The particles are assumed to have a distribution of sizes, $f(\hat{s})$, where $f(\hat{s})$ is non-zero only for $\hat{s} \in [\hat{a}_{\text{part}}^{\text{min}}, \hat{a}_{\text{part}}^{\text{max}}]$: the minimum particle size is $\hat{a}_{\text{part}}^{\text{min}}$, and the maximum particle size is $\hat{a}_{\text{part}}^{\text{max}}$. (Different functional forms of f may be considered, e.g. uniform, truncated Gaussian, *etc.* We take a uniform distribution to illustrate the phenomenology of the problem.) If $\hat{a}_{\text{pore}}(\hat{t}) < \hat{a}_{\text{part}}^{\text{min}}$ then particles are not able to foul the membrane and instead the cake layer grows; if $\hat{a}_{\text{pore}}(\hat{t}) > \hat{a}_{\text{part}}^{\text{max}}$ then all of the particles enter the membrane pores and contribute to fouling, but no cake layer grows. In general, particles in the size range $[\hat{a}_{\text{part}}^{\text{min}}, \hat{a}_{\text{pore}}(\hat{t})]$ contribute to fouling, while particles in the size range $[\hat{a}_{\text{pore}}(\hat{t}), \hat{a}_{\text{part}}^{\text{max}}]$ contribute to cake growth. The fraction of incoming particles that may enter the pore, f_p , is given by

$$f_p(\hat{t}) = \int_{\hat{a}_{\text{part}}^{\text{min}}}^{\hat{a}_{\text{pore}}(\hat{t})} f(\hat{s}) d\hat{s}, \quad (1)$$

with the remainder, the fraction $f_c = 1 - f_p$, contributing to cake growth. Finally, we shall assume that the membrane may be fouled even when a cake layer is present: particles in the cake may foul the membrane if the pores have become large enough for them to fit inside. For simplicity, and to highlight the fundamental aspect of this mechanism, we assume that backflushing removes the cake layer completely but does not alter the internal fouled state of the membrane. This could be easily modified to suit a different experimental observation.

Since filtration occurs at a constant flux, particles reach the membrane surface at a constant rate, λ , which is proportional to both the flow rate and particle concentration. The rules already given determine whether particles foul the membrane or form a cake layer. The growth of the cake layer adds resistance to the flow and exerts a mechanical stress on the membrane. We assume that this cake layer is also a poroelastic medium, with Young's modulus, \hat{E}_c , and Poisson ratio, ν_c . As particles continue to be advected, the thickness of the cake, \hat{d}_c , grows with time.

The domain $0 < \hat{r} < \hat{R} + \hat{d}_m$ divides into three regions: a fluid-filled space ($0 < \hat{r} < \hat{R} - \hat{d}_c$), the cake ($\hat{R} - \hat{d}_c < \hat{r} < \hat{R}$), and the membrane itself ($\hat{R} < \hat{r} < \hat{R} + \hat{d}_m$). Our aim is to solve for the time evolution of the stresses with the cake layer and membrane, and to include the effect of membrane deflection on the fouling of membrane pores.

We use the parameter values given in Appendix A, unless otherwise stated.

2.2. Poroelastic Timescale

The poroelastic timescale, \hat{T}_{pe} , reflects the material's control over the storage and release of elastic energy: the diffusion and dissipation of fluid pressure. The timescale \hat{T}_{pe} may be derived via a balance of the strain rate and elastic stress for the membrane [27],

$$\hat{T}_{\text{pe}} = \frac{\hat{\mu} \hat{R}^2}{\hat{E}_m \hat{k}_{m,0}}. \quad (2)$$

In water filtration, the viscosity is $\hat{\mu} = 10^{-3}$ Pa s, $\hat{R} = 10^{-3}$ m is a typical internal tube radius, $\hat{E}_m = 10^{10}$ Pa is a typical membrane Young's modulus (see Appendix A), and $\hat{k}_{m,0} = 10^{-16}$ m²

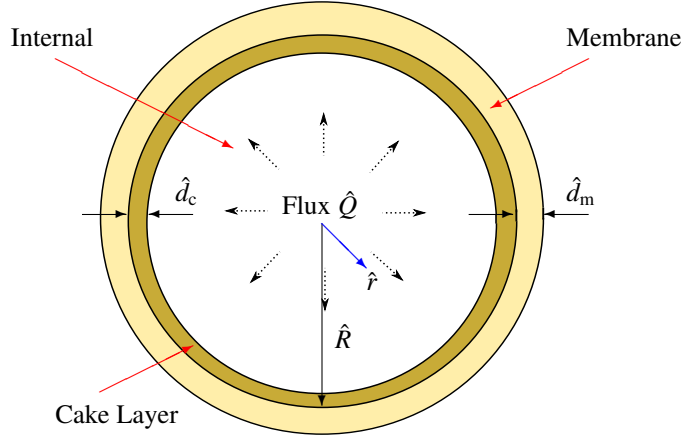


Figure 3: Cross-section through a cylindrical membrane. The filter has an undeformed internal radius \hat{R} . An areal fluid flux, \hat{Q} , flows radially from the centre of the tube with radial velocity \hat{V} . The membrane thickness is \hat{d}_m while the cake thickness, \hat{d}_c , evolves as additional material is filtered.

112 is a typical membrane permeability. This gives a value of $\hat{T}_{pe} \approx 1$ ms. The rates at which foul-
 113 ing and caking develop are on a timescale of the order of minutes (with backflushes performed
 114 approximately every 20 minutes). Hence, the poroelastic timescale is much shorter than the
 115 backflushing period, even for softer membranes. This justifies a quasi-static approximation, in
 116 which the elastic deformation is solved in steady state in response to the stresses for a particular
 117 cake size and amount of fouling.

118 2.3. Fouling and Caking

119 We present a simple model that accounts for the contribution of the incoming particles to
 120 both fouling and caking. We take the fraction of open, unfouled pores at a time \hat{t} to be given
 121 by $\mathcal{F}(\hat{t})$. A particle only enters a pore if its size is at most that of the pore. We assume that
 122 any particle smaller than a pore is equally likely to occupy the pore. This means that the first
 123 sufficiently small particle to reach an open pore enters it and blocks it. (This is a simplification
 124 since the particle only has to be smaller than the pore to fit in and may not block it completely;
 125 further particles could then be added to the pore, provided that they fit in the space remaining,
 126 but we do not consider such complications here.) In this model, only a single layer of particles
 127 foul the membrane internally; we do not account for depth in the fouling, *i.e.*, multiple particles
 128 fouling the same pore at different depths, but this could be incorporated using the ideas presented
 129 in [11].

If $\lambda \Delta \hat{t}$ particles, with size distribution $f(\delta)$, reach the membrane surface in an interval of
 time $\Delta \hat{t}$, the rate of fouling is given via the ODE,

$$\frac{d\mathcal{F}}{d\hat{t}} = -\frac{\lambda}{N_0} f_p(\hat{t}), \quad (3)$$

where $f_p(\hat{t})$ is given in Eq. (1), and N_0 is the number of pores initially (which we expect to be the
 number of pores that can fit into the circumference, scaled with the porosity of the membrane,

φ_m , *i.e.* $N_0 = \varphi_m \pi \hat{R} / \hat{a}_{\text{pore}}$). Eq. (3) is to be solved with the initial condition,

$$\mathcal{F}(0) = 1, \quad (4)$$

130 so that the membrane is initially unfouled.

The cake layer is an array of packed particles forming a porous medium with porosity φ_c , so that each additional monolayer of added particles leads to an increment $2\bar{a}_{\text{part}}/\varphi_c$ in the cake thickness, \hat{d}_c , where \bar{a}_{part} denotes the mean particle size. For a membrane surface with circumference $2\pi\hat{R}$, each layer of particles consists of $\varphi_c \pi \hat{R} / \bar{a}_{\text{part}}$ particles; that is, the perimeter of the surface divided by the diameter of a particle and scaled by the porosity of the cake layer. This of course is a first approximation; since the surface is curved, each layer has a smaller surface area, but such variations are insignificant if the cake thickness remains small compared to \hat{R} . The thickness of the cake layer, \hat{d}_c , therefore grows according to

$$\frac{d\hat{d}_c}{d\hat{t}} = \frac{2\lambda\bar{a}_{\text{part}}^2}{\varphi_c^2 \pi \hat{R}} (1 - f_p(\hat{t})). \quad (5)$$

We shall assume that, after each backflush, the cake layer is completely removed. If the backflushes are performed periodically at intervals of \hat{t}_{bf} , then at the beginning of the n^{th} filtration cycle, the initial condition on the cake layer is

$$\hat{d}_c(n\hat{t}_{\text{bf}}) = 0. \quad (6)$$

We non-dimensionalize the ODEs for fouling (3) and cake growth (5) by scaling the cake thickness with the undeformed membrane radius, \hat{R} , and the pore and particle sizes with the undeformed pore size, \hat{a}_{pore}^0 . We consider the problem over the fouling timescale. Specifically, we let

$$\hat{d}_c = \hat{R}d_c, \quad \hat{a}_{\text{pore}} = \hat{a}_{\text{pore}}^0 a_{\text{pore}}, \quad \hat{a}_{\text{part}} = \hat{a}_{\text{pore}}^0 a_{\text{part}}, \quad \hat{t} = \frac{\varphi_m \pi \hat{R}}{\hat{a}_{\text{pore}} \lambda} t. \quad (7)$$

The ODEs for fouling (3) and cake growth (5) may then be written as

$$\frac{d\mathcal{F}}{dt} = -f_p(t), \quad \mathcal{F}(0) = 1, \quad (8a)$$

$$\frac{dd_c}{dt} = \beta(1 - f_p(t)), \quad d_c(nt_{\text{bf}}) = 0, \quad (8b)$$

where f_p is given by Eq. (1) non-dimensionalized in the obvious way and

$$\beta = \frac{2\bar{a}_{\text{part}}}{\varphi_c \hat{R}}, \quad (9)$$

131 represents the ratio of the average cake-layer depth of a monolayer to the membrane thickness.
132 Typical parameters (provided in Appendix A) give the estimate $\beta \approx 0.02$.

133 2.4. Poroelasticity

134 In this section we consider the factors that determine the rate of fouling of the membrane.
135 The distribution of particles in a given size range is given by the quantity $f_p(t)$ in Eq. (8), but

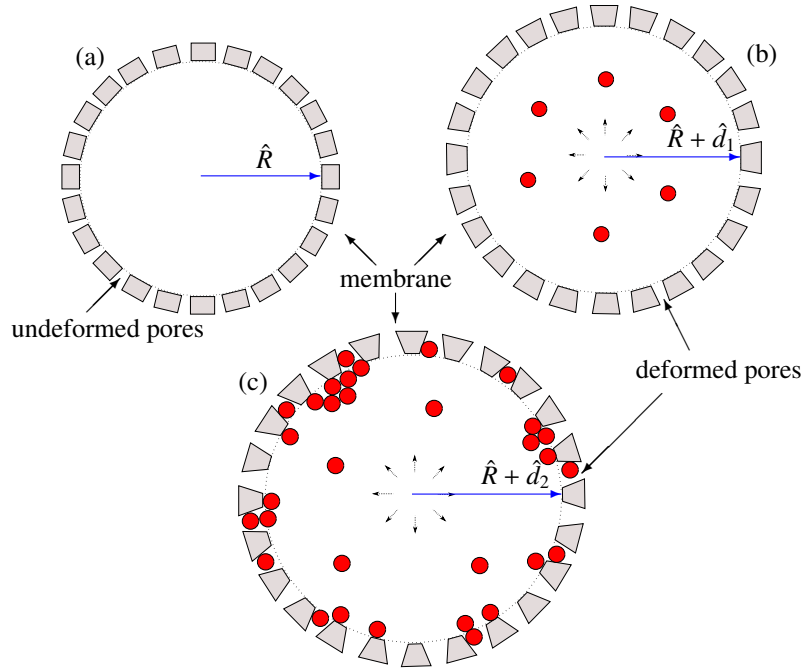


Figure 4: Schematic illustrating the expansion of a pore as fouling and caking occur. (a) The pores are initially uniform in size and undeformed, with radius \hat{R} . (b) A fluid flow through the membrane exerts a stress that deforms the membrane, inducing an increase in pore size. The tube radius increases by a small amount \hat{d}_1 . (c) Fouling of the pores and the build up of a cake layer results in higher pressures and stresses, and hence further deformation of the membrane, by $\hat{d}_2 > \hat{d}_1$, and to a larger pore size. The pressure required must increase to maintain a constant fluid flux through the membrane. The greater the fouling and thicker the cake layer, and hence the greater the deformation.

136 requires knowledge of the typical pore size at a particular operating pressure. We must therefore
 137 relate the fluid stress on the membrane to its deformation and thence to the change in the typical
 138 pore size. The deformation problem is made up of two parts: the deformation when no cake layer
 139 is present (at the beginning of the filtration process or after a backflush is performed) and when
 140 a cake layer is present. These two scenarios are fundamentally different.

141 Consider the schematic in Figure 4: in the absence of a cake layer (at the beginning of the
 142 experiment or immediately following a backflush) the membrane deforms elastically, by some
 143 distance $\hat{u}_m^0 = \hat{d}_1$, say, which is due solely to the fluid stress. When particles reach the membrane
 144 surface, if they are small enough they can enter and plug the membrane pores, reducing the
 145 permeability. As a result, a larger pressure must be applied to maintain a constant flux. Further-
 146 more, if a cake layer builds up on the surface of the membrane, an additional mechanical stress
 147 is exerted on the membrane. Altogether, this increases the deformation, to $\hat{u}_m = \hat{d}_2 > \hat{d}_1$ say. In
 148 addition to this, the cake layer may be deformed by the flow. As the pore radius increases, $f_p(t)$
 149 increases and so more irreversible fouling occurs. The fouling process worsens with each cycle.

150 2.4.1. Elastic deformation

151 Since the other aspects of the model are focused on deformations in the plane, we neglect
 152 axial elastic deformations and model the deformations of the membrane and cake via plane-strain

153 linear elasticity. (We discuss the circumstances under which this might be expected to hold at
 154 the end of the paper.) We use the Navier–Cauchy equation with Terzaghi’s principle to model
 155 the deformation of a porous material [27, 28]. The details for the poroelastic model are given
 156 in Appendix B. Here, we consider how the deformation affects the material properties of the
 157 membrane.

After deformation, the local dimensionless change in volume is given by the *dilation*

$$\Delta := \nabla \cdot \mathbf{u} = e_{rr} + e_{\theta\theta} = \frac{du}{dr} + \frac{u}{r}, \quad (10)$$

where e_{rr} and $e_{\theta\theta}$ are the strain components, and $u = u(r)$ is the dimensionless radial deformation of the porous medium, with the deformation and radial coordinate both scaled with \hat{R} [29]. Using the deformations given in Eq. (B.17), the local dilations (10) of the membrane and cake layer, $\Delta_m(r)$ and $\Delta_c(r)$ respectively, are given by

$$\Delta_m(r) = -\frac{\Gamma}{2\pi\Omega\mathcal{F}} \log r + 2A_m, \quad 1 < r < 1 + d_m, \quad (11a)$$

$$\Delta_c(r) = -\frac{\Gamma\gamma}{2\pi\Omega\omega\kappa} \log r + 2A_c, \quad 1 - d_c < r < 1, \quad (11b)$$

158 where A_m and A_c are (quasi-static) constants given in Eq. (B.18), which are determined as part
 159 of the solution to the problem, and Ω and Γ are dimensionless parameters defined in (B.12a) and
 160 (B.12b). The dilation decreases with radial position r (since the pressure decreases with r , the
 161 amount of compression/expansion changes too). An important observation for filtration appli-
 162 cations is that, for typical parameter values discussed in Appendix A, $\Delta_m > 0$ (the membrane
 163 expands) while $\Delta_c < 0$ (the cake is compressed).

164 2.4.2. Changes in pore size and porosity

While the mixture of void space and solid may change its volume locally, as discussed above, we assume that the solid portions of the porous media are both incompressible: any local dilation is accommodated solely by a local change in porosity. With this assumption, the deformed porosity, φ_d , can be written in terms of the undeformed (uniform) porosity, φ_0 , as

$$\varphi_d = 1 - \frac{1 - \varphi_0}{1 + \Delta} \approx \varphi_0 + \Delta(1 - \varphi_0), \quad (12)$$

165 for small deformations, $\Delta \ll 1$, consistent with our assumption of linear elasticity. Here Δ is
 166 given in Eq. (11) for the membrane and cake layer. However, we emphasize that, in general,
 167 $\Delta = \Delta(r)$, so that the initially uniform porosity becomes inhomogeneous, because of the inho-
 168 mogeneity of the deformations. It is also possible that the permeability becomes anisotropic as a
 169 result of this deformation, but we neglect such possibilities here.

We can now quantify the change in pore size. Suppose that, in the undeformed reference state, either porous media is composed of \hat{M} pores per unit area, each with equal size \hat{a}_{pore}^0 . The porosity, φ_0 , is the volume fraction of the pores, $\varphi_0 = \hat{M}\pi(\hat{a}_{\text{pore}}^0)^2$. After deformation, the pores have a new radius, \hat{a}_{pore} . However, the number of pores is unchanged following deformation and so the new pore size may be determined using the deformed porosity Eq. (12), with $\varphi_d = \hat{M}\pi\hat{a}_{\text{pore}}^2$, to give, in dimensionless terms,

$$a_{\text{pore}}(t) \approx 1 + \frac{1}{2}\Delta \frac{1 - \varphi_0}{\varphi_0}, \quad (13)$$

170 where we have assumed $\Delta \ll 1$. As expected, the pore size increases for $\Delta > 0$ and decreases
 171 for $\Delta < 0$ (recalling that $0 < \varphi_0 < 1$).

To understand fouling, the most important pore size is that at the membrane surface, since this determines whether a particle is accepted or rejected by the membrane. Using the displacement field within the membrane, (11a), to calculate $\Delta = \Delta_m(r = 1)$, we find that the pore size (13), $a_{\text{pore}}(t)$, takes the form

$$a_{\text{pore}}(t) = 1 + A_m(t) \frac{1 - \varphi_{m,0}}{\varphi_{m,0}}. \quad (14)$$

172 Here the constant, $A_m = A_m(t)$, is determined from the quasi-static membrane deformation solu-
 173 tion (B.18), and is now a time-dependent parameter that depends on the parameters of the system,
 174 including the dimensionless cake thickness, $d_c(t)$, and the degree of fouling, $\mathcal{F}(t)$.

175 At this stage, we have assumed that the deformation of the membrane is known. However, in
 176 reality this deformation depends on the fluid pressure, which in turn depends on how difficult it
 177 is to pump the imposed flux \hat{Q} through the membrane or, in other words, the permeability of the
 178 porous media. We therefore turn to determining this next.

179 2.4.3. Permeability of the porous media

A common model of porous media relates the permeability, \hat{k} , of a porous medium to its porosity, φ , and pore size, \hat{a}_{pore} . There are many choices for this constitutive relation, but a popular choice is the Kozeny–Carman relation [28, 30]. Since both φ and \hat{a}_{pore} depend on the dilation we may write

$$\hat{k}_d = \frac{\varphi_d^3}{(1 - \varphi_d)^2} \frac{\hat{a}_d^2}{36\tau} \approx \hat{k}_0 \left(1 + k_1 \Delta(r)\right), \quad (15)$$

where we have used Eqns. (12) and (13), and neglected $\mathcal{O}(\Delta^2)$ terms. In (15),

$$\hat{k}_0 = \frac{\varphi_0^3}{(1 - \varphi_0)^2} \frac{\hat{a}_0^2}{36\tau}, \quad (16a)$$

$$k_1 = \frac{2(2 - \varphi_0)}{\varphi_0}, \quad (16b)$$

180 where τ is the tortuosity of the pore space, which is assumed to be constant.

While Eq. (15) gives the permeability of those pores that are unblocked, we must recall that some portion, $1 - \mathcal{F}$, of the pores are fouled, and so the effective membrane permeability is time-dependent, $\hat{k}_{m,0} \mathcal{F}(t)$. We non-dimensionalize the permeabilities by scaling with the leading-order unfouled membrane permeability $\hat{k}_{m,0}$ so that

$$k_m = \mathcal{F} (1 + k_{m,1} \Delta_m), \quad (17a)$$

$$k_c = \kappa (1 + k_{c,1} \Delta_c), \quad (17b)$$

181 where Δ_m and Δ_c are the membrane and cake-layer dilations, given in eqn (11), κ is the ratio
 182 of the cake to membrane permeabilities (B.9), and $k_{m,1}$ and $k_{c,1}$ are the relevant k_1 (16b) for the
 183 membrane and cake, respectively.

184 We have considered the undeformed membranes as a porous medium with uniformly dis-
 185 tributed, identical pores. However, the elastic deformation is dependent on the radial position, r ,

186 as shown by the expressions for the dilation Δ_m , (11a). As a result, the porosity (12), pore
 187 size (13), and permeability (17a) all depend on r . In this sense, the membrane has been trans-
 188 formed into an asymmetric membrane in which the pores are larger at the inner surface (where
 189 rejection is meant to occur) than they are further into the membrane.

190 The effect of fouling on the the membrane permeability is two-fold. Firstly, from (17a) we
 191 see that permeability decreases with fouling, as would be expected. However, since fouling is
 192 also expected to increase the driving pressure and hence the elastic deformation, we find that
 193 there are higher order effects of fouling on permeability via the dilation Δ_m , which depends on
 194 pressure. In principle, this dilation can increase or decrease the permeability (depending on the
 195 sign of Δ_m).

196 2.4.4. Coupling to the fluid flow

197 The permeability of the membrane has a direct impact on the pressure gradient across the
 198 medium required to drive the (fixed) flux, as expressed by Darcy's law (B.2). As a result, the
 199 elastic expansion of the membrane has two contrasting effects on the flow: firstly, there may be
 200 an increase in fouling (caused by an increase in pore size as discussed in the previous subsection).
 201 Secondly, however, the elasticity-induced increase in pore size, and the concomitant increase in
 202 permeability, may reduce the pressure required to drive filtration.

Modifying the standard Darcy equation (B.2) to account for the variable permeability of (17a)
 we find that the dimensionless Darcy velocity through the membrane, scaled with \hat{Q}/\hat{R} , is

$$V = -\left(1 + \Delta_m(r) k_{m,1}\right) \mathcal{F} \frac{dp}{dr}, \quad (18)$$

where $k_{m,1}$ (16b) is a constant specified for the membrane in (17a), $\Delta_m = \Delta_m(r)$ is given by (11a)
 and we have scaled pressure via $\hat{p} = (\hat{Q}\hat{\mu}/\hat{k}_m)p$. Substituting the velocity profile, $V = 1/(2\pi r)$
 from (B.8a) and rearranging we find that

$$\frac{dp}{dr} = -\frac{1}{2\pi\mathcal{F}(1 + \Delta_m(r) k_{m,1})r} \approx -\frac{1}{2\pi\mathcal{F}r} + \frac{A_m k_{m,1}}{\pi\mathcal{F}r} - \frac{\Gamma k_{m,1} \log r}{4\pi^2\Omega\mathcal{F}^2r}, \quad (19)$$

203 for $1 < r < 1 + d_m$, with A_m, Γ, Ω , and $k_{m,1}$ all constants that are defined in Appendix B; further-
 204 more, we have exploited the small-strain approximation $\Delta_m(r) \ll 1$.

Integrating (19) subject to the ambient pressure condition, $p(r = 1 + d_m) = 0$, we find that
 the pressure field in the membrane (including the elastic correction), p_m , is

$$p_m = \frac{1}{2\pi\mathcal{F}} \log\left(\frac{1 + d_m}{r}\right) + \frac{k_{m,1}}{\pi\mathcal{F}} \log\left(\frac{1 + d_m}{r}\right) \left[\frac{\Gamma}{8\pi\Omega\mathcal{F}} \log[r(1 + d_m)] - A_m \right], \quad (20)$$

205 for $1 < r < 1 + d_m$. Here, the first term reproduces the pressure in the membrane without elastic
 206 deformation (B.8b). Examining the second term on the RHS of (20), we note that pore blocking
 207 increases the pressure change across the membrane (last term) whereas elasticity decreases this
 208 (second term). This reflects that the pressure needed to drive the flow in the flexible case is less
 209 than that in the rigid, undeformed case, because of the expansion of the pores.

We may repeat the above steps to determine the effect of the (compression-induced) perme-
 ability of the cake. We find now that

$$\frac{dp}{dr} = -\frac{1}{2\pi\kappa(1 + \Delta_c(r) k_{c,1})r} \approx -\frac{1}{2\pi\kappa r} + \frac{A_c k_{c,1}}{\pi\kappa r} - \frac{\Gamma\gamma k_{c,1} \log r}{4\pi^2\Omega\omega\kappa^2 r}, \quad (21)$$

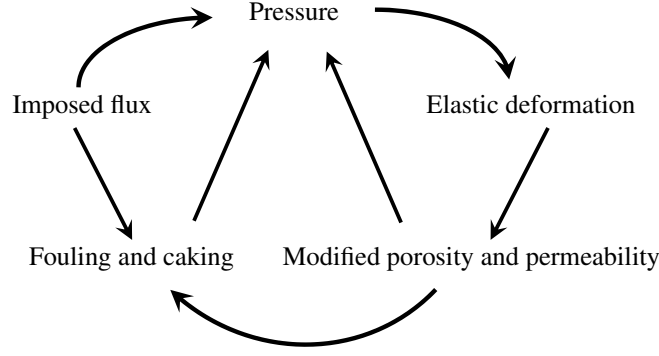


Figure 5: Flow chart illustrating the fouling process and our solution methodology.

210 with A_c , Γ , Ω , γ , ω , κ and $k_{c,1}$ all constants defined in Appendix B; again $\Delta_c(r) \ll 1$ for consis-
 211 tency with our small deformation theory.

Again (21) may be integrated (subject to the condition that the pressure be continuous at the membrane–cake interface) to give an elastically corrected pressure field within the cake layer, say p_c , given by

$$p_c = \frac{1}{2\pi\mathcal{F}} \log(1 + d_m) - \frac{1}{2\pi\kappa} \log r + \frac{k_{m,1}}{\pi\mathcal{F}} \log(1 + d_m) \left[\frac{\Gamma}{8\pi\Omega\mathcal{F}} \log(1 + d_m) - A_m \right] - \frac{k_{c,1}}{\pi\kappa} \log r \left[\frac{\Gamma\gamma}{8\pi\Omega\omega\kappa} \log r - A_c \right], \quad (22)$$

212 valid for $1 - d_c < r < 1$. Here, the first term corresponds to the leading-order pressure required,
 213 accounting for the cake layer, in the absence of deformation, as given by (B.8b). Examining the
 214 right-hand side of (22), we see that any blocking of the membrane is felt in the pressure across
 215 the cake layer as expected. However, here elasticity increases the pressure change across the
 216 cake layer as it is compressed. In figure 5 we illustrate the steps taken in the form of a flow chart.

The quantity that is of most interest from a practical point of view is the driving pressure, \hat{P}_{drive} ; this is the constant pressure within the internal void region and is given by evaluating the pressure field (22) at the inner cake boundary ($r = 1 - d_c$). We find that

$$\begin{aligned} \hat{P}_{\text{drive}} = & \frac{1}{2\pi\mathcal{F}} \log(1 + d_m) - \frac{1}{2\pi\kappa} \log(1 - d_c) \\ & + \frac{k_{m,1}}{\pi\mathcal{F}} \log(1 + d_m) \left[\frac{\Gamma}{8\pi\Omega\mathcal{F}} \log(1 + d_m) - A_m \right] \\ & - \frac{k_{c,1}}{\pi\kappa} \log(1 - d_c) \left[\frac{\Gamma\gamma}{8\pi\Omega\omega\kappa} \log(1 - d_c) - A_c \right]. \end{aligned} \quad (23)$$

217 Before moving on to discuss our results for the (quasi-static) evolution of filtration and foul-
 218 ing, it is worthwhile studying the form for the driving pressure (23). Fouling, characterized by

219 \mathcal{F} , and caking, characterized by the cake thickness d_c , appear explicitly in (23) and that both
 220 act to increase the driving pressure (since $\mathcal{F} < 1$ and $\delta_c > 0$), as should be expected. We also
 221 note that the first two terms of (23) represent the leading-order pressure without any deformation
 222 while the third and fourth terms represent the first-order corrections to the pressure due to the
 223 expansion of the membrane (which reduces the driving pressure) and compression of the cake
 224 (which increases the driving pressure), respectively.

225 The approximations (20), (22) and (23) hold for small deformations, $\Delta_m(r) \ll 1$ and
 226 $\Delta_c(r) \ll 1$. As deformations get larger, the driving pressure \hat{P}_{drive} must be determined by di-
 227 rect numerical integration of (19) and (21).

228 Finally, we note that the effect of elastic deformation on the porosity, pore size, and per-
 229 meability enter as first-order changes to the fluid flow and membrane fouling. Other physical
 230 parameters, such as the Young's modulus \hat{E} and Poisson ratio ν may well be affected by elastic
 231 deformation. However, since these changes in parameters would alter the fluid flow and mem-
 232 brane fouling at still higher order, we neglect such variations here.

233 3. Results

234 The driving pressure given in (23) illustrates the detrimental effect fouling can have on the ef-
 235 ficiency of the system. As fouling (*i.e.*, decreasing \mathcal{F}) reduces the permeability of the membrane,
 236 the driving pressure must increase to maintain a constant flux. Eventually the driving pressure
 237 becomes so high that the membrane is no longer practical.

238 In this section we present results for how the fouling and cake growth develop as functions
 239 of time. This is achieved by solving the set of ODEs (8) numerically. However, a key indicator
 240 of how fouling and cake growth would affect filtration is through the evolution of the driving
 241 pressure given by (23).

242 A key concern is on the effect of regular backflushing and so we envisage performing a
 243 backflush at regular intervals of time t_{bf} ; we assume that each backflush fully removes the cake
 244 layer, which is expressed via the condition (8b). The system (8) is solved numerically, with
 245 a particle size distribution, f , and the membrane surface pore size, $a_{\text{pore}}(t)$, given by Eq. (14).
 246 The numerical integration is performed using the MATLAB routine `ode45`. At each time step,
 247 Δt , the deformed pore size, $a_{\text{pore}}(t)$ (14), is updated for the current values of $\mathcal{F}(t)$ and $d_c(t)$
 248 that determine its elastic deformation. As t passes through nt_{bf} , d_c is reset to 0 (representing a
 249 backflush), but $\mathcal{F}(t)$ remains unchanged. This is outlined in Algorithm 1.

Algorithm 1

Require: Solving the ODEs (8) for fouling and caking

Input: backflush time, t_{bf} ; number of backflushes, m ; particle-size distribution, f

1: **Initialization** $\mathcal{F}(0) = 1$, $d_c(0) = 0$

2: **for** $n = 1 : m$ **do**

3: Solve ODEs (8) for $\mathcal{F}(t)$ and $d_c(t)$ from $t = (n - 1)t_{\text{bf}}$ to $t = nt_{\text{bf}}$

4: At each time step calculate the pore size, $a_{\text{pore}}(t, r)$, from Eq. (14)

5: Determine the driving pressure, $\hat{P}_{\text{drive}}(t)$, from Eq. (23)

6: Reset the ICs at $t = nt_{\text{bf}}$ with an unchanged membrane fouling \mathcal{F} and remove the cake by
 setting $d_c = 0$

7: **end**

Output: $\mathcal{F}(t)$, $d_c(t)$, $a_{\text{pore}}(t, r)$, $\hat{P}_{\text{drive}}(t)$

250 *3.1. Development of fouling and caking*

251 To illustrate the range of behaviours exhibited by our model, we first assume that the particle
252 distribution, f , is uniform with a minimum particle size $\hat{a}_{\text{part}}^{\text{min}} = 4 \mu\text{m}$ (microfiltration) and maxi-
253 mum particle size $\hat{a}_{\text{part}}^{\text{max}} = 5 \mu\text{m}$. The undeformed pore size is taken to be $4.1 \mu\text{m}$ so that initially
254 10% of the particles are able to foul the membrane.

255 In figure 6 we show how the various properties of the system evolve with time, depending on
256 the number of backflushes (zero, one or two). For later comparison, the dashed curves in figure 6
257 show how fouling and cake growth proceed when elastic effects are neglected. In the undeformed
258 case, the pore size, a_{pore} , remains constant and the rate of fouling and cake buildup are now
259 decoupled. The right-hand side of each ODE is constant and, hence, the system has solutions
260 that are linear with time. The fouling follows a linear progression that remains unchanged with
261 the number of backflushes (Figure 6(b)), whereas the cake layer grows linearly until a backflush
262 removes it and cake growth starts over again (Figure 6(c)).

263 Adding in the various effects of elasticity discussed above gives rise to the solid curves shown
264 in figure 6. The key difference here is that initially the rate of fouling is relatively weak (since
265 only 10% of particles are small enough to foul the membrane, as in the rigid case). However,
266 as the cake grows, so does the pressure, which in turn opens the pores (Figure 6(a)), allowing
267 more particles to foul the membrane. The effect of this can be seen in the proportion of the
268 membrane that is fouled, which decreases significantly faster in the elastic case than in the rigid
269 case (Figure 6(b)). The increased fouling also slows the rate of cake growth (Figure 6(c)) since
270 particles enter the system at the same rate and more are entering the pore spaces. Note also that
271 these effects grow more noticeable with time: the rate of fouling increases with time, as has been
272 widely acknowledged in the membrane industry [6].

273 The number of backflushes also alters the elastic response of the system. Elastic deformation
274 leads to additional fouling and so the pore size after each subsequent backflush remains larger
275 than at the beginning of the last cycle. Nevertheless, it is primarily the size of the cake layer that
276 causes the pores to enlarge: each time the cake layer is removed by backflushing, the pore size
277 drops (Figure 6(a)). As a result, more regular backflushes reduce the number of particles that can
278 foul the membrane and reduces the rate of fouling (Figure 6(b)), as expected.

279 As the filtration process generally operates at constant flux, its efficiency is measured by the
280 driving pressure, \hat{P}_{drive} , which responds quasi-statically to changes in \mathcal{F} and d_c . In Figure 6(d)
281 we show the evolution of the driving pressure for the system. This figure resembles the schematic
282 of the driving pressure in a typical membrane filtration operation (Figure 2) used to motivate this
283 study. Varying the parameters in our model would change the quantitative nature of the graph
284 while leaving the qualitative shape unaffected; this would allow us, in principle, to compare with
285 experimental data.

286 Finally, if the pore size is initially smaller than all the particles, then no fouling occurs until
287 the fluid and cake stresses deform the membrane sufficiently that particles can foul the membrane.
288 The system then behaves as described above. We will discuss this, in the context of a strategy to
289 prevent fouling, in §3.3.

290 *3.2. Competing mechanisms*

291 The preliminary numerical analysis of the preceding section shows that the effects of elastic-
292 ity are not straightforward: the elasticity of the membrane increases the pore size, and hence the
293 permeability of the membrane, decreasing the driving pressure. However, this increase in pore
294 size also means that a greater proportion of particles are able to enter the pores and foul the mem-
295 brane irreversibly, which ultimately increases the pressure. There is thus a competition between

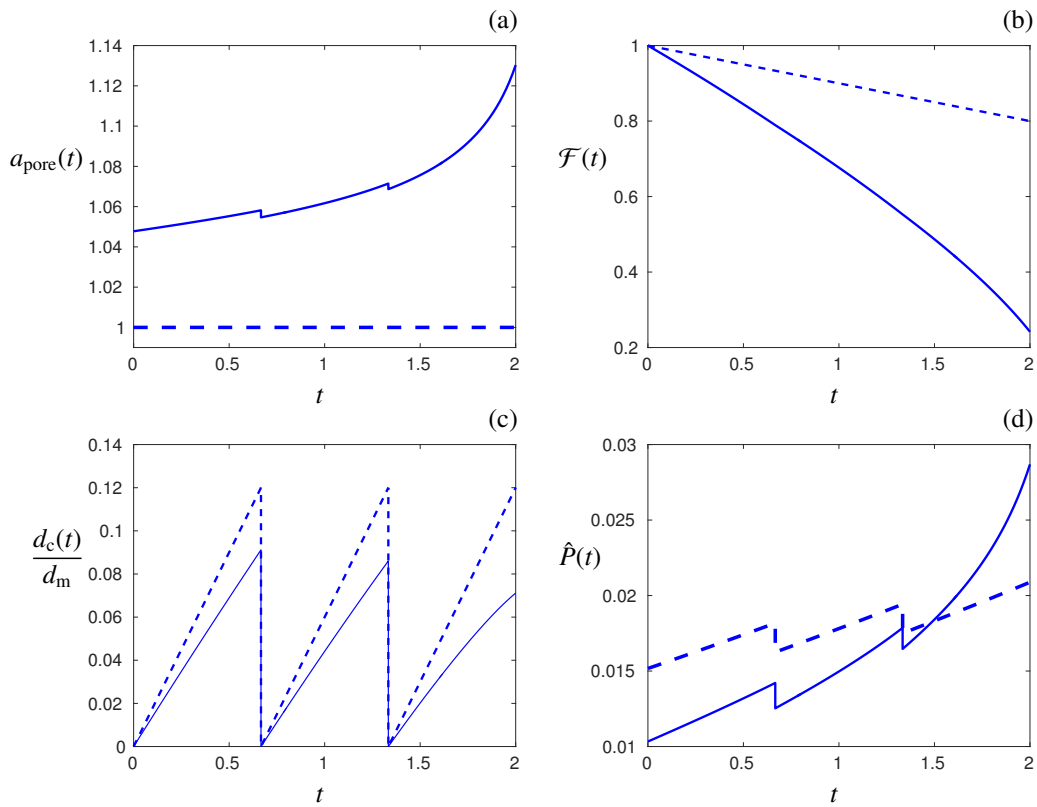


Figure 6: The evolution of (a) pore size, (b) proportion of unfouled pores, (c) cake thickness (as a fraction of membrane thickness) and (d) driving pressure, \hat{P} (23), with time. Results are shown for a deformed membrane (solid curves) and the rigid case (dotted curves). Here, two backflushes are performed during operation and the dimensionless membrane thickness is $d_m = 0.1$.

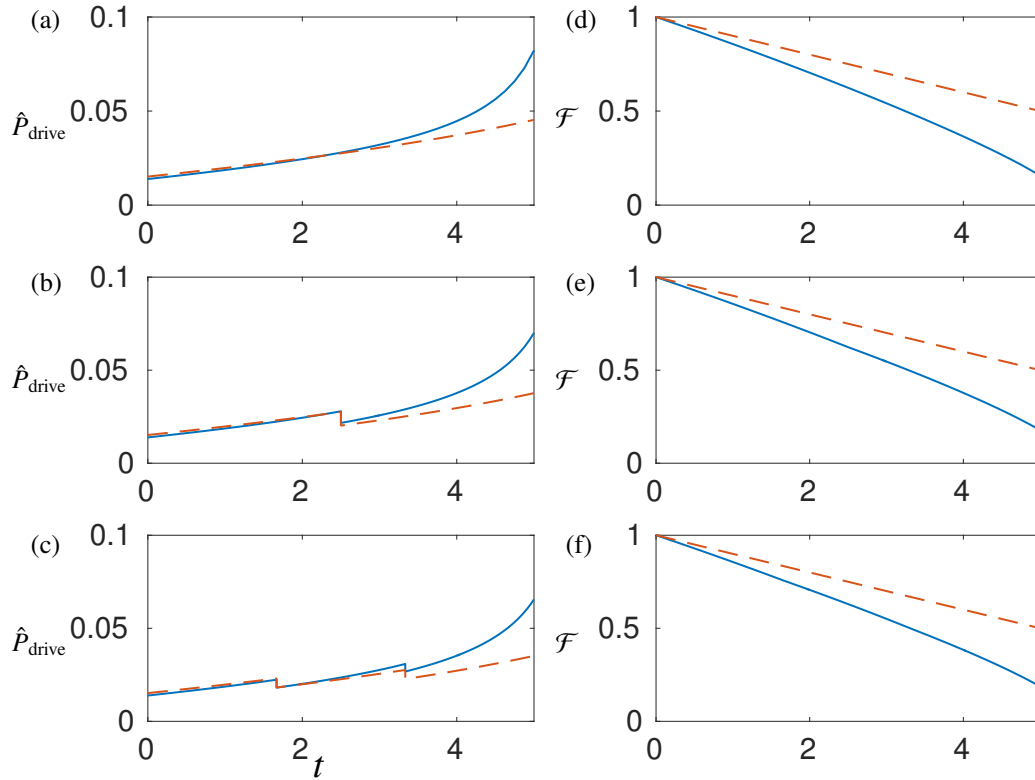


Figure 7: Evolution of driving pressure $\hat{P}_{\text{drive}}(t)$ (23) and fouling $\mathcal{F}(t)$ (8a) over a time period $[0,3]$ with (a,d) no backflushes, (b,e) one backflush, and (c,f) two backflushes. We compare the pressure with (solid) and without (dashed) elastic effects. Initially, elastic effects reduce the driving pressure, but increased fouling means that after some time the pressure exceeds that of the rigid case.

296 increased permeability and increased fouling as a result of elastic deformation. An analogous,
 297 albeit reversed, competition occurs in the cake layer: elastic deformation squeezes the cake layer,
 298 reducing its permeability but, because of the fouling of the membrane caused by deformation,
 299 the rate of cake growth is slowed (see Figure 6(c)).

300 To see the effect of these competing mechanisms, in figure 7 we illustrate the decrease in
 301 driving pressure (left panel) and fouling (right panel) over time as we increase backflushing
 302 rates, with (solid) and without (dashed) elastic effects. Initially, elastic effects reduce the driving
 303 pressure by opening the pores to allow fluid to pass more easily. However, eventually this is
 304 detrimental to the system as increased fouling clogs up the pores. Then, the pressure becomes
 305 significantly greater with elastic effects than without.

306 Driving pressures overall are reduced as the number of backflushes that are performed in a
 307 given interval of time increases. Figure 8 shows how the amount of fouling and driving pressure
 308 change after a fixed time interval of filtration, $t_f = 5$, but with different numbers of backflushes
 309 performed in that interval. The results of figure 8 are therefore plotted as a percentage reduction
 310 in fouling (a) and driving pressure (b) compared with the case of no backflushing. Note that
 311 for increasing frequency of backflushing, the degree of fouling and driving pressure at the end

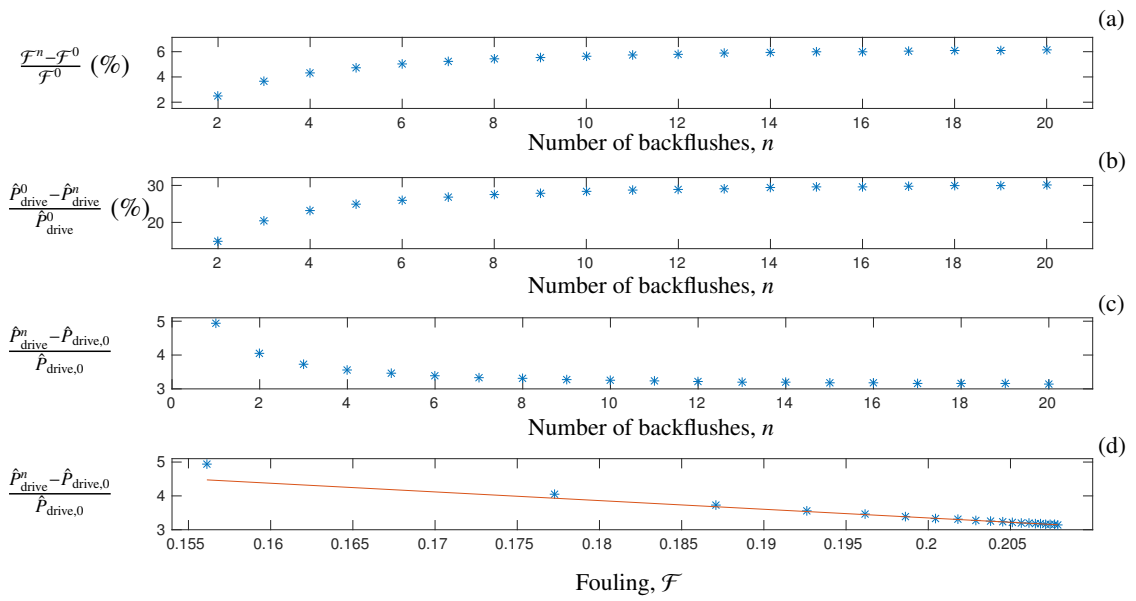


Figure 8: The effects of backflushing on pressure and fouling for a number of backflushes n performed in a fixed time interval. (a) The percentage reduction in fouling (8a). (b) The percentage decrease in relative driving pressure (23) at the final time \hat{t}_f . As the number of backflushes performed in a fixed time interval increases, both the fouling and driving pressure are reduced compared with the case of no backflushing. (c) The pressure change between start and end times (initially, $\hat{P}_{drive,0} = \hat{P}_{drive}(t=0)$). (d) The pressure change between start and end times as a function of the final amount of fouling for different numbers of backflushes (cf. (c)).

312 of the numerical experiment decrease, but that ultimately the effect plateaus after approximately
 313 10 backflashes in the cycle (figure 8). This plateau in the percentage reduction in fouling and
 314 driving pressures reaches 6% and 30%, respectively. The pressure increase during each run is
 315 reduced significantly by backflushing, from a factor of approximately 5 to 3 (Figure 8(c)), with a
 316 direct correspondence to a reduction in fouling. The backflushing effect on pressure and fouling
 317 shows an approximately linear relationship between the two (d).

318 For long enough operating times, pressure always increases regardless of the material pa-
 319 rameters, and eventually with fouling can become excessively large (Figure 6(d)). However, we
 320 can consider operating conditions to prevent this: not only to remain in a region where elasticity
 321 reduces the pressure, but also one in which the pores do not expand enough to allow any fouling.

322 3.3. A strategy for fouling prevention

323 Pore fouling tends to be irreversible, and has more severe consequences than cake buildup
 324 (which can be mitigated by backflushing). However, internal fouling cannot occur if the pore
 325 size is maintained at a value below that of the minimum particle size. Our previous results have
 326 shown that a significant contributor to the opening of pores that can be important in fouling is the
 327 mechanical stress of the cake layer acting on the membrane. It is natural then to wonder whether
 328 one can perform backflashes often enough that this mechanical pressure does not increase to the
 329 point at which the pores are opened enough to allow fouling to occur. In this section, we show
 330 that it is, at least in principle, possible to operate filtration in such a regime.

Suppose the smallest particle is a fraction η larger than the undeformed (and uniform) pore
 size. Fouling will only be possible at times $t > t_{\text{crit}}$ where t_{crit} is defined by the equation

$$\eta = a_{\text{pore}}(t_{\text{crit}}) - 1 = A_{\text{m}}(t_{\text{crit}}) \left(\frac{1}{\varphi_0} - 1 \right). \quad (24)$$

331 Here we have used (14) and A_{m} defined in (B.18), which depends on the cake thickness d_{c}
 332 (figure 9(a)). In the absence of fouling, the fraction of active membrane pores is $\mathcal{F} = 1$ for all
 333 times. The cake-growth ODE (8b) can then be solved independently. The time, t_{crit} , at which the
 334 value of d_{c} is large enough to extend the pores by a fraction η determines the minimum backflush
 335 frequency, $f_{\text{bf}}^* = 1/t_{\text{crit}}$.

336 A proof of concept of this strategy is shown in Figure 9(b); again we note that the pro-
 337 cedure must be tuned for other material and operating parameters. In dimensional terms we
 338 take pores of initial diameter $3.957 \mu\text{m}$, with the particle size distribution being uniform on
 339 $4 \mu\text{m} \leq a_0 \leq 5 \mu\text{m}$. This corresponds to $\eta = 0.00525$. The cake thickness required to achieve
 340 fouling is then $d_{\text{c}} = 0.1096 d_{\text{m}}$, via (24). Prior to this point, the solution to (8b) is linear in time;
 341 in particular, we find that in half a time unit the cake grows by an amount $\Delta d_{\text{c}} = 0.1 d_{\text{m}}$. Hence,
 342 operating at a backflush frequency of 2 per unit time should be just enough to prevent fouling.
 343 We demonstrate this by backflushing at two different frequencies: once and twice per unit time
 344 over 350 time units (we consider such a long time interval to demonstrate the detrimental effect
 345 of fouling once it begins to occur). The inset of Figure 9(b) illustrates the onset of fouling in
 346 each cycle once the cake layer has had enough time to grow.

347 4. Discussion & Conclusions

In this paper, we have developed a model for the evolution of fouling and cake development
 in membrane filtration. Our model incorporates the effect of a small elastic (plane-strain) de-
 formation of the membrane and cake layer; the two phenomena of fouling and caking are coupled by

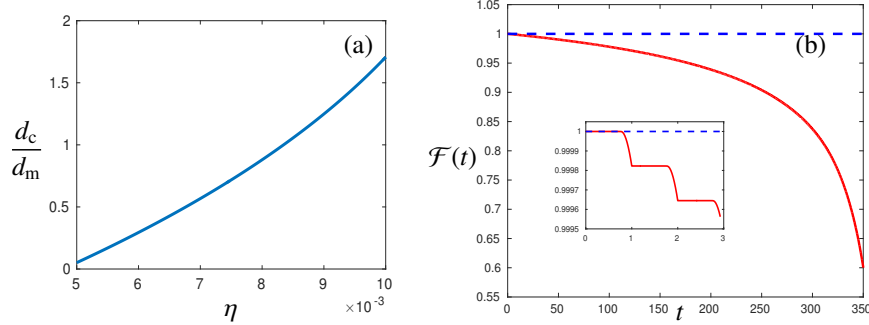


Figure 9: A proof of concept for a strategy to prevent fouling. We consider a uniform distribution of particles in the range $4 \mu\text{m} \leq a_0 \leq 5 \mu\text{m}$, and a membrane pore size $3.957 \mu\text{m}$. Once the membrane pores expand beyond $4 \mu\text{m}$, fouling begins to occur. (a) The critical cake thickness, d_c , at which the pore size increases enough to allow some fouling as a function of the relative difference in radii, η , between the smallest particle and the undeformed pore size. Provided that the critical pore size remains below this value, then (according to our model) fouling will not occur. (b) The time evolution of the fraction of open (unfouled) pores, \mathcal{F} , with a dimensionless backflush interval $t_{\text{bf}} = 1$ (solid) or $t_{\text{bf}} = 1/2$ (dashed). For sufficiently high frequencies of backflushing, the pores do not expand sufficiently to permit fouling (dashed blue curve). The inset shows a close-up of three backflushing cycles in which the elastic effects eventually cause fouling (only with the lower backflushing frequency).

a quasi-static elastic response of the two porous media. Various simplifications and approximations were made in the development of our model. For example, we neglected axial deformations caused by deformation and, in addition, any change in axial stress due to tangential stresses acting on the membrane wall. Our assumption of plane strain, $e_{zz} = 0$, introduces an axial stress $\sigma_{zz} = \nu(\sigma_{rr} + \sigma_{\theta\theta}) \sim \mu Q \hat{d}_m / (\hat{k}_{m,0} \hat{R})$. The effect of viscous shear stresses, $\mu \partial u / \partial r$, acting on the wall is expected to lead to a change in the thickness-integrated tension along the length of the membrane $\Delta T = T(L) - T(0) \sim \mu Q L^2 / \hat{R}^4$. For plane strain to be a realistic assumption, we require $\Delta T \ll \hat{d}_m \sigma_{zz} \sim \mu Q \hat{d}_m^2 / (\hat{k}_{m,0} \hat{R})$, which in turn requires

$$\frac{\hat{k}_{m,0}}{\hat{d}_m^2} \ll \frac{\hat{R}^2}{L^2} \ll 1, \quad (25)$$

348 with the second inequality required to ensure a long, slender geometry. Given the typical values
 349 for microfiltration given in Appendix A, we find that this requirement is readily satisfied in
 350 practice.

351 Despite these simplifications, our model captures the advantages and disadvantages of elastic
 352 effects in membrane filtration (Figure 7). Initially, pores are expanded, which increases the flux
 353 of filtered water. However, with that comes increased deposition and irreversible fouling. In the
 354 long term, this increases fouling and operating pressures.

355 Despite the inevitable long-term consequences of elasticity, we can take advantage of the
 356 short-term reduction in pressure by considering our operating strategy. According to our model
 357 the backflush frequency may be tuned to ensure that the cake never grows thick enough to deform
 358 the membrane to the point where it fouls irreversibly (Figure 9). In this way, it should be possible
 359 to tune the operating conditions (particularly the operating flux and backflush frequency)
 360 as well as the membrane characteristics (permeability, pore size, material, *etc.*) to avoid fouling
 361 altogether.

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366 **Nomenclature**

Subscripts

m	Membrane
c	Cake
bf	Backflush
0	Undeformed
1	Elastic perturbation

Superscript

$\hat{}$	Dimensional
$-$	Average

Symbols

E	Young's modulus
k	Permeability
κ	Ratio of permeabilities
ϕ	Porosity
ν	Poisson ratio
μ	Viscosity
Q	Flux
λ	Particle deposition rate
N_0	Initial number of open pores
β	Cake layer to membrane thickness
d	Layer thickness
\mathcal{F}	Fouling
p	Pressure
σ	Stress
e	Strain
t	Time
P	Driving pressure
r	Radial coordinate
u	Deformation
A, B	Deformation parameters
$\Gamma, \gamma, \Omega, \omega$	Deformation dimensionless numbers
Δ	Dilation
a_{pore}	Pore radius
a_{part}	Particle radius

367 Appendix A. Parameter estimates

368 In the results of §3, we fix the parameter values of the system, unless otherwise stated.
369 Our main purpose of that section is to understand whether the proposed mechanism (elasticity-
370 induced pore expansion) works, so we do not compare a wide range of all the parameter values.

371 We take a uniform distribution of particles with sizes 4–5 μm , representative of microfil-
372 tration. To have maximum permeability while still rejecting particles, the undeformed pore
373 size, \hat{a}_0 , must be chosen to be as large as possible without allowing for fouling; we thus take
374 $\hat{a}_0 = 3.957 \mu\text{m}$. Specifically, we choose this \hat{a}_0 so that a small deformation may lead to fouling.
375 Alternatively, for illustrative purposes we take a pore size of $\hat{a}_0 = 4.1 \mu\text{m}$ so that 10% of particles
376 can enter the membrane.

377 We take typical material parameter values for the undeformed state, which will also apply
378 for the linearly elastic deformed state. We consider a tube with an internal radius $\hat{R} = 10^{-3} \text{ m}$
379 and membrane thickness $\hat{d}_m = 10^{-4} \text{ m}$ [1] (*i.e.*, a dimensionless membrane thickness $d_m = 10^{-1}$).
380 The filtration velocity is of the order of millimetres to centimetres per second, say $0.5 \times 10^{-2} \text{ m/s}$,
381 similar to those in a range of filtration experiments [31, 32]. This corresponds to an areal flux
382 through the annulus of $\hat{Q} = \pi \times 10^{-5} \text{ m}^2/\text{s}$,

383 The membrane properties are taken to be those of hollow-fibre membranes. The (unde-
384 formed) membrane permeability for microfiltration is of the order $\hat{k}_{m,0} = 10^{-16} \text{ m}^2$ [32]. Given
385 the pore size and permeability, the porosity may be determined by relations such as Kozeny–
386 Carman [28, 30], giving $\varphi_{m,0} \approx 0.35$. Typical values for the Young’s modulus of membranes vary
387 to a large degree, for example, of order 10^8 Pa for ultrafiltration hollow fibre membranes [33] to
388 order 10^{11} Pa for microfiltration membranes with a silicon-nitride layer [34]. Here we take the
389 Young’s modulus of the membrane, $\hat{E}_m = 10^{10} \text{ Pa}$. We choose a Poisson ratio, $\nu_m = 0.3$, although
390 a large range including negative values (*e.g.*, honeycomb and carbon nanotube structures) are re-
391 ported [34, 35, 36].

392 To the authors’ knowledge, parameter values for the cake layer are limited. As such, the
393 cake-layer properties are chosen for convenience, guided by some intuition. A value for cake per-
394 meability is difficult to determine as the cake layer is a collection of loose particles packed at the
395 membrane surface, whose packing is non-trivial and beyond the scope of this work [15, 37]. Nev-
396 ertheless, it is reasonable to consider a large permeability of the cake, $\hat{k}_{c,0}$ when caking is begin-
397 ning, with smaller values later on when the cake is more compact. Since the cake layer represents
398 a collection of particles of similar size to the membrane pore size, we assume that it has a similar
399 (undeformed) permeability to the membrane, $\hat{k}_{c,0} = 10^{-16} \text{ m}^2$. We take the porosity, $\varphi_{c,0} = 0.4$,
400 to be slightly larger than the membrane as it is not a unified solid material. Analogously, we take
401 the Young’s modulus to be an order of magnitude lower than the membrane, $\hat{E}_c = 2 \times 10^9 \text{ Pa}$.
402 Finally, we take the Poisson ratio to be lower than the membrane, $\nu_c = 0.2$ (which has been mea-
403 sured for a closely packed colloidal crystal, and has been used in the modelling of a colloidal
404 gel [38]).

405 Appendix B. Poroelastic Model

406 In the context of poroelastic deformations, the fluid–structure interactions must be modelled
407 in the same reference frame. However, it is usual to model the fluid flow in an Eulerian frame, and
408 the solid mechanics in a Lagrangian frame. Assuming linear elasticity, the difference between
409 these reference frames becomes negligible [39]. Further to this, in the context of linear elasticity,
410 \hat{R} is both the undeformed and deformed membrane radius.

411 *Appendix B.1. Governing Equations*

412 A membrane tube or channel is usually long and thin, resulting in a small reduced Reynolds
413 number [40]. Assuming that the flow is quasi-steady, we use the steady thin-layer Stokes equa-
414 tions (or lubrication equations) for the flow inside the tube.

The fluid flow within the central void of the 2D annular cross-section of the filter (Figure 3) is given by the axisymmetric Stokes equations

$$\frac{1}{\hat{r}} \frac{\partial}{\partial \hat{r}} (\hat{r} \hat{V}) = 0, \quad (\text{B.1a})$$

$$\frac{\hat{\mu}}{\hat{r}} \frac{\partial}{\partial \hat{r}} \left(\hat{r} \frac{\partial \hat{V}}{\partial \hat{r}} \right) = \frac{\partial \hat{p}}{\partial \hat{r}}, \quad (\text{B.1b})$$

where \hat{V} is the radial velocity, \hat{p} is the radial pressure, and $\hat{\mu}$ is the fluid viscosity. The flow in the porous cake layer and membrane is given by Darcy's equation in polar coordinates

$$\hat{V} = -\frac{\hat{k}}{\hat{\mu}} \frac{\partial \hat{p}}{\partial \hat{r}}, \quad (\text{B.2a})$$

415 where \hat{k} is the permeability of the porous medium. The pressure, \hat{p} , and velocity, \hat{V} , are contin-
416 uous at the cake-layer–membrane boundary. We note that the fluid velocity in the porous media
417 is the Darcy velocity and depends on the porosity of each medium. It is this velocity that is
418 assumed to be continuous at each interface. We consider the case of a constant flux \hat{Q} passing
419 through the system. The reference pressure outside the membrane is taken to be zero.

420 For the setup of a 2D annulus with constant influx (Figure 3), the pressure is constant in the
421 internal void, consistent with thin-layer Stokes flow [41]. However, a pressure difference, the
422 transmembrane pressure difference (TMP), exists across the membrane driving the flow. The
423 stress from the fluid pressure deforms the membrane and cake elastically.

The steady-state linear elastic deformation of a solid material is governed by the steady Navier–Cauchy equation with no body forces [29],

$$\hat{\nabla} \cdot \hat{\sigma}^t = 0, \quad (\text{B.3})$$

where $\hat{\nabla}$ is the dimensional gradient operator and $\hat{\sigma}^t$ is the total stress in the material. For a poroelastic material, the total stress is often given according to Terzaghi's principle [27, 28] by

$$\hat{\sigma}_{ij}^t = \hat{\sigma}_{ij} - \hat{p} \delta_{ij}, \quad (\text{B.4})$$

424 where $\hat{\sigma}$ is the elastic stress of the material (due to strain), and $-\hat{p}\mathbf{I}$ is the contribution due to the
425 fluid pore pressure (where \mathbf{I} denotes the identity matrix). Hence, the excess of stress over pore
426 pressure drives the deformation.

The linearly elastic deformations are modelled using plane strain, whereby the elastic stress components may be written in terms of the deformation, \hat{u} , as

$$\hat{\sigma}_{rr} = \frac{\hat{E}}{(1+\nu)(1-2\nu)} \left((1-\nu) \frac{d\hat{u}}{d\hat{r}} + \nu \frac{\hat{u}}{\hat{r}} \right), \quad (\text{B.5a})$$

$$\hat{\sigma}_{\theta\theta} = \frac{\hat{E}}{(1+\nu)(1-2\nu)} \left(\nu \frac{d\hat{u}}{d\hat{r}} + (1-\nu) \frac{\hat{u}}{\hat{r}} \right), \quad (\text{B.5b})$$

427 where \hat{E} is the Young's modulus and ν is the Poisson ratio [29]. We use subscripts m and c to
 428 denote values in the membrane or cake respectively. For the full membrane tube, plane strain
 429 would model the case of constant TMP, *i.e.*, constant permeate flux along the membrane, as
 430 found in direct-flow filtration [26].

431 Appendix B.2. Non-dimensionalization

We non-dimensionalize using the following scalings

$$\begin{aligned} \hat{r} &= \hat{R}r, & \hat{u}_m^0 &= \hat{R}u_m^0, & \hat{u}_m &= \hat{R}u_m, & \hat{u}_c &= \hat{R}u_c, & \hat{V} &= \frac{\hat{Q}}{\hat{R}}V, & \hat{d}_c &= \hat{R}d_c, \\ \hat{d}_m &= \hat{R}d_m, & \hat{p} &= \frac{\hat{Q}\hat{\mu}}{\hat{k}_{m,0}}p, & \hat{P}_1 &= \frac{\hat{Q}\hat{\mu}}{\hat{k}_{m,0}}P_1, & \hat{P}_2 &= \frac{\hat{Q}\hat{\mu}}{\hat{k}_{m,0}}P_2, & \hat{\sigma}_{rr} &= \hat{P}_1\sigma_{rr}, & \hat{\sigma}_{\theta\theta} &= \hat{P}_1\sigma_{\theta\theta}. \end{aligned} \quad (\text{B.6})$$

432 Here, \hat{u}_m^0 is the membrane deformation in the absence of any cake layer, \hat{u}_m and \hat{u}_c are the
 433 deformations of the membrane and the cake during operation, respectively, \hat{d}_m and \hat{d}_c are the
 434 thickness of the membrane and cake layer respectively, \hat{P}_1 is the initial pressure on the inner
 435 membrane surface, and P_2 is the initial pressure on the cake inner surface.

436 The problem is scaled with the material parameters of the membrane as this is a well-defined
 437 material, unlike the cake. We assume that the Young's modulus, \hat{E} , and Poisson ratio, ν , of both
 438 the membrane and cake layer, and the permeability of the cake layer, remain constant to leading
 439 order (deformation enters only at higher order).

As the membrane is deformed, both the pore size and porosity change. However, as the
 porosity is reduced due to membrane fouling, characterized by the density of open pores, \mathcal{F} , we
 assume that the dimensionless membrane permeability, k_m , decreases uniformly. Both deforma-
 tion and fouling contribute to the change in membrane structure, and hence, permeability. We
 write the dimensionless permeability, k_m , as

$$k_m = (k_{m,0} + k_{m,1})\mathcal{F}, \quad (\text{B.7})$$

440 where $k_{m,0}$ is the leading order undeformed permeability and $k_{m,1}$ is the effect of a small elastic
 441 deformation on the permeability.

442 Appendix B.3. Fluid Flow

The leading-order (undeformed) solution to the 2D symmetric radial fluid-flow problem can
 be written down immediately as

$$V = \frac{1}{2\pi r}, \quad (\text{B.8a})$$

$$p = \begin{cases} \frac{1}{2\pi\mathcal{F}} \log(1 + d_m) - \frac{1}{2\pi\kappa} \log(1 - d_c), & r < 1 - d_c, \\ -\frac{1}{2\pi\kappa} \log r + \frac{1}{2\pi\mathcal{F}} \log(1 + d_m), & 1 - d_c < r < 1, \\ -\frac{1}{2\pi\mathcal{F}} \log r + \frac{1}{2\pi\mathcal{F}} \log(1 + d_m), & 1 < r < 1 + d_m, \\ 0, & r > 1 + d_m, \end{cases} \quad (\text{B.8b})$$

where

$$\kappa = \frac{\hat{k}_{c,0}}{\hat{k}_{m,0}}, \quad (\text{B.9})$$

443 is the ratio of undeformed dimensional permeabilities between the cake and membrane.

The radial pressure, \hat{P} , inside the void that drives the flow is constant in space, as expected for flow in a thin channel, and so equals the pressure at the internal-cake interface. However, it is time-dependent via \mathcal{F} and d_c as described by (8). The dimensionless reference pressures, \hat{P}_1 at the membrane–cake-layer interface ($r = 1$), and \hat{P}_2 at the internal-cake interface ($r = 1 - d_c$), are given by

$$\hat{P}_1 = \frac{1}{2\pi\mathcal{F}} \log(1 + d_m), \quad (\text{B.10a})$$

$$\hat{P}_2 = \frac{1}{2\pi\mathcal{F}} \log(1 + d_m) - \frac{1}{2\pi\kappa} \log(1 - d_c). \quad (\text{B.10b})$$

444 Appendix B.4. Deformation

When the filtration process begins (originally or after a backflush) there is no cake layer present. The membrane deformation, u_m^0 , is given by the dimensionless version of the Navier–Cauchy equation (B.3), with boundary conditions given by the (plane strain) stresses (B.5) on the inner ($r = 1$) and outer ($r = 1 + d_c$) walls of the membrane; that is, the fluid pressure at the boundaries,

$$\frac{d}{dr} \left(\frac{1}{r} \frac{d}{dr} (ru_m^0) \right) = \frac{\Gamma}{\Omega} \frac{dp}{dr} = -\frac{\Gamma}{\Omega} \frac{1}{2\pi\mathcal{F}r} \quad (1 < r < 1 + d_m), \quad (\text{B.11a})$$

$$\Omega \left((1 - \nu_m) \frac{du_m^0}{dr} + \nu_m \frac{u_m^0}{r} \right) \Big|_{r=1} = -1, \quad (\text{B.11b})$$

$$\Omega \left((1 - \nu_m) \frac{du_m^0}{dr} + \nu_m \frac{u_m^0}{r} \right) \Big|_{r=1+d_m} = 0, \quad (\text{B.11c})$$

with p given in (B.8b), and

$$\Omega = \frac{\hat{E}_m}{\hat{P}_1(1 + \nu_m)(1 - 2\nu_m)}, \quad (\text{B.12a})$$

$$\Gamma = \frac{2\pi}{(1 - \nu_m) \log(1 + d_m)}, \quad (\text{B.12b})$$

445 are dimensionless parameters that apply (for reference) to the unfouled ($\mathcal{F} = 1$) membrane; Ω is
 446 a parameter related to the physics of the system, representing a stiffness since $\hat{E}_m/\hat{P}_1 \propto \hat{E}_m/\hat{Q}$; Γ
 447 is a purely geometrical parameter.

The solution to (B.11) for u_m^0 reads

$$u_m^0 = -\frac{\Gamma}{8\pi\Omega\mathcal{F}} (-r + 2r \log r) + A_m^0 r + \frac{B_m^0}{r}, \quad (\text{B.13})$$

for $1 < r < 1 + d_m$, where

$$A_m^0 = \frac{8\pi\mathcal{F} + \Gamma \left\{ [(1 + d_m)^2 - 1] (1 - 2\nu_m) + 2(1 + d_m)^2 \log(1 + d_m) \right\}}{8\pi\mathcal{F}\Omega [(1 + d_m)^2 - 1]}, \quad (\text{B.14a})$$

$$B_m^0 = \frac{(1 + d_m)^2 [4\pi\mathcal{F} + \Gamma \log(1 + d_m)]}{4\pi\mathcal{F}\Omega (1 - 2\nu_m) [(1 + d_m)^2 - 1]}. \quad (\text{B.14b})$$

When a cake layer is present, the deformation of this layer, u_c , and membrane deformation, u_m , are given by a set of Navier–Cauchy equations (B.3). As the cake layer is retained by the membrane, continuity of stress holds at the cake–membrane interface. Since the fluid stress is continuous here, the elastic stress must also be continuous. Furthermore, we must have continuity of the deformation at the cake–membrane interface. However, before the cake layer was deposited, the membrane was deformed by u_m^0 . We therefore impose continuity of additional deformation once the cake is deposited (*i.e.*, $u_c = u_m - u_m^0$ at the cake–membrane interface). The remaining boundary conditions are again due to the pressure on the inner- and outer-most surfaces. These conditions are written as

$$\frac{d}{dr} \left(\frac{1}{r} \frac{d}{dr} (ru_m) \right) = \frac{\Gamma}{\Omega} \frac{dp}{dr} = -\frac{\Gamma}{\Omega} \frac{1}{2\pi\mathcal{F}r} \quad (1 < r < 1 + d_m), \quad (\text{B.15a})$$

$$\frac{d}{dr} \left(\frac{1}{r} \frac{d}{dr} (ru_c) \right) = \frac{\Gamma}{\Omega} \frac{\gamma}{\omega} \frac{dp}{dr} = -\frac{\Gamma}{\Omega} \frac{\gamma}{\omega} \frac{1}{2\pi\kappa r} \quad (1 - d_c < r < 1), \quad (\text{B.15b})$$

$$\Omega \omega \left((1 - \nu_c) \frac{du_c}{dr} + \nu_c \frac{u_c}{r} \right) \Big|_{r=1-d_c} = -\frac{\hat{P}_2}{\hat{P}_1} = -\left(1 - \frac{\mathcal{F}}{\kappa} \frac{\log(1 - d_c)}{\log(1 + d_m)} \right), \quad (\text{B.15c})$$

$$\Omega \left((1 - \nu_m) \frac{du_m}{dr} + \nu_m \frac{u_m}{r} \right) \Big|_{r=1+d_m} = 0, \quad (\text{B.15d})$$

$$\omega \left((1 - \nu_c) \frac{du_c}{dr} + \nu_c \frac{u_c}{r} \right) \Big|_{r=1} = \left((1 - \nu_m) \frac{du_m}{dr} + \nu_m \frac{u_m}{r} \right) \Big|_{r=1}, \quad (\text{B.15e})$$

$$u_c(r = 1) = u_m(r = 1) - u_m^0(r = 1), \quad (\text{B.15f})$$

where p is given in (B.8b), and the dimensionless parameters

$$\omega = \frac{\hat{E}_c}{(1 + \nu_c)(1 - 2\nu_c)} \Big/ \frac{\hat{E}_m}{(1 + \nu_m)(1 - 2\nu_m)}, \quad (\text{B.16a})$$

$$\gamma = \frac{1 - \nu_m}{1 - \nu_c}, \quad (\text{B.16b})$$

⁴⁴⁸ reflect the material parameter ratios between the cake layer and membrane.

The solution to (B.15) for u_m and u_c reads

$$u_m = -\frac{\Gamma}{8\pi\Omega\mathcal{F}} (-r + 2r \log r) + A_m r + \frac{B_m}{r}, \quad 1 < r < 1 + d_m, \quad (\text{B.17a})$$

$$u_c = -\frac{\Gamma\gamma}{8\pi\Omega\omega\kappa} (-r + 2r \log r) + A_c r + \frac{B_c}{r}, \quad 1 - d_c < r < 1, \quad (\text{B.17b})$$

where A_m , B_m , A_c , and B_c are constants determined by substituting the general solutions (B.17) into the boundary conditions (B.15c–f). These constants may be determined explicitly, but for

brevity we show their expressions in matrix form,

$$\begin{pmatrix} A_m \\ B_m \\ A_c \\ B_c \end{pmatrix} = \begin{pmatrix} (1+d_m)^2 & -1+2\nu_m & 0 & 0 \\ 0 & 0 & (1-d_c)^2 & -1+2\nu_c \\ 1 & -1+2\nu_m & -\omega & \omega(1-2\nu_c) \\ -1 & -1 & 1 & 1 \end{pmatrix}^{-1} \Psi, \quad (\text{B.18})$$

where

$$\Psi = \begin{pmatrix} \frac{\Gamma(1+d_m)^2}{8\pi\mathcal{F}\Omega} [1-2\nu_m+2\log(1+d_m)] \\ \frac{(1-d_c)^2}{8\pi\kappa\Omega\omega} \left[-8\pi\kappa + \Gamma\gamma(1-2\nu_c) + 2\frac{\log(1-d_c)}{\log(1+d_m)} (4\pi\mathcal{F} + \Gamma\gamma\log(1+d_m)) \right] \\ \frac{\Gamma}{8\pi\kappa\Omega} \left[\gamma(-1+2\nu_c) + \frac{\kappa(1-2\nu_m)}{\mathcal{F}} \right] \\ -u_m^0(1) + \frac{\gamma}{8\pi\Omega} \left[\frac{1}{\mathcal{F}} - \frac{\gamma}{\kappa\omega} \right] \end{pmatrix}. \quad (\text{B.19})$$

449 When the cake thickness is zero, A_m and B_m reduce to (B.14), obtained in the absence of a cake.
 450 We note that the solution incorporates the deformation with no cake layer, u_m^0 (B.13), after the
 451 last backflush or at the beginning of the experiment.

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