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A tale of mathematical myth-making: E. T. Bell and the 'arithmetization of algebra'

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Throughout E. T. Bell's writings on mathematics, both those aimed at other mathematicians and those for a popular audience, we find him endeavouring to promote abstract algebra generally, and the postulational method in particular. Bell evidently felt that the adoption of the latter approach to algebra (a process that he termed the 'arithmetization of algebra') would lend the subject something akin to the level of rigour that analysis had achieved in the nineteenth century. However, despite promoting this point of view, it is not so much in evidence in Bell's own mathematical work. I offer an explanation for this apparent contradiction in terms of Bell's infamous *penchant* for mathematical 'myth-making'.

Introduction

Arguably one of the most significant developments in mathematics during the twentieth century was the advent of abstract algebra. The early decades of the century saw the culmination of a process that had been underway for almost a century: the shift from the traditional view of 'algebra' as the study of solutions of polynomial equations ('classical algebra'), to the structural 'algebra' of groups, rings, fields, etc. ('modern algebra' or 'abstract algebra'). Although the abstract algebraic viewpoint now dominates much of modern mathematics, its adoption was by no means immediate or uniform. A major boost to the subject was afforded by B. L. van der Waerden's seminal 1930 text *Moderne Algebra*, based upon the lectures of Emmy Noether in Göttingen and Emil Artin in Hamburg, but the process of persuading mathematicians that this was a useful point of view continued for several years.

A major proponent of abstract algebra in the USA was E. T. Bell, "a man of strong opinions about many things" (Goodstein and Babbitt 2013, p. 686), and one whose writings, for better or for worse, did much to shape the popular view of mathematics in the twentieth century. Born in Scotland in 1883, Bell appears to have travelled back and forth across the Atlantic a couple of times in his early years,[1] before entering Stanford University to study mathematics in 1902. Following a master's course at the University of Washington in Seattle, and some time spent as a high-school teacher, Bell obtained his doctorate, in number theory, from Columbia University in 1912. He returned to the University of Washington as an instructor, but, in 1926, moved to the institution with which he is now most associated: the California Institute of Technology. Bell remained a professor at Caltech until a year before his death in 1960.

Bell's main mathematical work was in number theory (indeed, he was awarded a prize by the American Mathematical Society for a particular pair of papers — see later), but he is probably best remembered for his extensive popular writings on mathematics. These writings abound with positive comments on the abstract point of view. For example, in an article entitled 'Fifty years of algebra in America, 1888–1938', he referred to

> the rapid growth from the age of relative algebraic innocence, when everything was special and detailed, to our present highly sophisticated abstraction ... (Bell 1938, p. 1)

Moreover, one of the remarkable things about Bell's advocacy of the abstract approach to algebra is the fact that he sang its praises not only in articles for mathematicians, but also in his extensive popular

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[1]The first two decades of Bell's life are shrouded in some mystery — a mystery that Bell appears to have cultivated deliberately (see Reid 2001). For a full biography of Bell, see Reid (1993).

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writings. Although a number theorist by background, Bell seems to have developed an early interest in abstract algebra; his entry into the subject has been described in the following terms:

> Algebra had changed greatly ... as a result of the work of Emmy Noether and her followers in Göttingen and of Emil Artin and Otto Schreier in Hamburg. What had been a very concrete subject had become very abstract. ... Bell realized the importance of what had happened and felt that he should educate himself further in the new developments. (Reid 1993, p. 262)

Bell's particular enthusiasm seems to have been for a branch of abstract algebra that has since fallen somewhat out of fashion, namely *postulational analysis*: the study of systems of postulates (or axioms) for their own sake, with the consideration of such issues as independence of postulates, and the search for a 'minimal' set of postulates for a given axiomatically-defined object. In his promotion of postulational analysis, Bell was in good company, for this was a subject that saw much study at the hands of many American (and some German) mathematicians in the first decades of the twentieth century. As Bell himself later remarked, in reference to a specific instance:

> The first decade of the twentieth century witnessed a somewhat feverish activity in the postulational analysis of groups, in which American algebraists produced numerous sets of postulates for groups, with full discussions of complete independence. By 1910, nobody could possibly misunderstand what a group is. (Bell 1945, p. 241)

Here, Bell was reproducing what had been a widely-held view at the beginning of the century (although it was largely out of favour by 1945): that in order to achieve a deep understanding of an algebraic structure, be it a group or a ring or any other such notion, it is necessary to conduct a thorough study of its defining postulates. In this way, abstract algebra might be put on a sound postulational basis. To Bell's mind, this would bring a degree of rigour to algebra that was analogous to that afforded to analysis in the nineteenth century by the so-called 'arithmetization of analysis', whereby, for example, intuitive notions of limits had been abandoned, and 'ε/δ'-style proofs had been made the norm (see, for example, Boyer 1968, Chapter XXV). Pursuing this analogy, Bell referred to the construction of a postulational basis for abstract algebra as the *arithmetization of algebra* (see, for example, Bell 1945, p. 255).

What is rather curious, however, about Bell's trumpeting of the virtues of abstract algebra generally, and of postulational analysis in particular, is the fact that his own work in this direction was somewhat limited. Very few of Bell's mathematical papers truly embrace the abstract point of view. Some papers, for example, begin with an abstract set-up but then deal mostly with concrete examples. Although he clearly regarded the abstract point of view as an important one, and promoted the general programme of the 'arithmetization of algebra' wherever possible, Bell simply does not appear to have been comfortable conducting research in this area. Bell's treatment of other mathematical topics will help us to explain this apparent contradiction. As already noted, his popular writings did much to promote mathematics throughout several decades of the twentieth century, and, indeed, are often cited by mathematicians as the thing that drew them into mathematics in the first place (see, for example, Reid 1993, pp. 3–4). These same writings, however, are of course notoriously inaccurate where the biographies of mathematicians are concerned — most notably in Bell's romanticization of the life of Évariste Galois in *Men of mathematics*. Bell's wilful disregard of the facts in favour of simply telling a good story has been explained in terms of mathematical 'myth-making' (see, in particular, Rothman 1982): he had an impression of mathematics and mathematicians that he wanted to create in the popular consciousness, even if his representation was somewhat at variance with the historical facts. In this article, I argue that Bell's promotion of abstract algebra, despite his evident discomfort with it, stemmed from a similar desire to create an image of mathematics, this time on the more technical side of the subject. Along the way, I hope to convince the reader of the importance of taking a closer look at Bell's work: he may have been an irascible figure, who continues to divide opinion, but the scale of his impact on the popular twentieth-century view of mathematics surely means that he warrants further scholarly attention.

The article is structured as follows. We begin with a general introduction to the principles of postulational analysis, paying particular attention to Bell's opinions on the subject.[1] We then move to a consideration of

[1] I do not propose to give a comprehensive account of postulational analysis. Fuller surveys of this subject may be found in Corry (2000, §2), Corry (1996, 2nd ed., §3.5), Scanlan (1991) and Schlimm (2011). On the possible nineteenth-century origins of postulational analysis,

Bell's notion of 'algebraic arithmetic', which was closely connected with his ideas concerning the arithmetization of algebra. In essence, 'algebraic arithmetic' concerns the abstraction of certain ideas from number theory, and clearly reflects Bell's background in that area. His study of abstract problems rarely ventured outside 'algebraic arithmetic', though a further section of this article looks at some of the other abstract topics considered by Bell. At the end of the article, I make some final comments on Bell's enthusiasm for, but reluctance to participate fully in, abstract algebra, and examine the nature of his attempt at 'technical myth-making'.

Bell on postulational analysis

Postulational analysis was a subject that was taken up very enthusiastically by several American mathematicians in the early years of the twentieth century, being pursued almost as an obsession for several years before finally petering out. The initial boost for this subject in the USA seems to have come from David Hilbert's axiomatization of Euclidean geometry in his 1899 *Grundlagen der Geometrie* (see Corry 1996, 2nd ed., §3.3): seeing the axiomatic approach applied effectively to geometry prompted several American mathematicians to investigate its application in other areas. One of the most prominent proponents of postulational analysis was the Chicago-based mathematician E. H. Moore, who was later described by his student G. D. Birkhoff as "the great American protagonist, in his day, of the abstract point of view" (Birkhoff 1938, p. 284). Indeed, Moore was one of the United States' first algebraists (see, for example, Mac Lane 1989; more generally, see Parshall 1984). Inspired by Hilbert's work, Moore established a seminar in Chicago for the study first of *Grundlagen der Geometrie*, and then of axiomatizations for other systems. American mathematicians were soon investigating sets of postulates for a wide range of abstractly-defined objects: from groups and fields (Huntington 1901, Moore 1902), to linear associate algebras (Dickson 1903). In fact, Moore went on to base an entire monograph upon the postulational method (Moore and Barnard 1935) — see Siegmund-Schultze (1998).

As already indicated, Bell's writings are littered with positive references to the principles of postulational analysis. We take just a few examples, both from his popular works, and also from those of a more technical nature. We find, for instance, the following sentence in his *Mathematics: queen and servant of science*:

> Beginning in the 1890s and continuing well into the twentieth century, a beautiful art developed around postulate systems as things to be studied for their own sake. (Bell 1952, p. 25)

Indeed, the above quotation comes from a section (§2.2) of the book that is given over entirely to a description of the postulational method for the general reader; it features comments, for example, on the aesthetics of postulate systems, likening a set of *dependent* postulates to "an otherwise impressive cathedral [that] has been cheapened by too many gargoyles" (Bell 1952, p. 25). Chapter 3 of the same book contains a lengthy, but elementary, discussion of postulate systems, as does Chapter 9 of Bell's *The development of mathematics*, which leaves us in no doubt as to Bell's opinion of abstract mathematics generally:

> From the standpoint of mathematics as a whole, the methodology of deliberate generalization and abstraction, culminating in the twentieth century in a rapidly growing mathematics of structure, is doubtless the most significant contribution of all the successive attempts to extend the number concept. (Bell 1945, pp. 186–187)

Equally definite is Bell's comment in his article 'Fifty years of algebra in America' that

> [t]he critical analysis of postulate systems in the past third of a century has not only rectified weaknesses in earlier intuitive work, but has also suggested profitable new fields for exploration ... (Bell 1938, p. 16)

We note, however, that Bell's support of the postulational method was not all-encompassing: although he described the search for sets of independent postulates as being "at least as amusing a pursuit as solving crossword puzzles or playing solitaire" (Bell 1952, p. 25), he was critical of the ill-motivated 'axiomatic tinkering' that may, ultimately, have contributed to this area's fall from favour: he warned against "the decadent vice of playing with barren postulates" (Bell 1952, pp. 27–28). More generally, Goodstein and

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see Grattan-Guinness (2000) and Parshall (2011).

Babbitt (2013, p. 689, fn. 4) record Bell's distaste for certain mathematics that he regarded as "abstraction for its own sake, with no sense of problem". Curiously, Bell's advocacy of abstract mathematics did not extend to analysis, in connection with which he was apparently of the opinion that such abstraction "does nothing really new and finds no new results" (Taylor 1984, p. 610).

I conclude this section by recording an intriguing comment that appears at the beginning of Bell's chapter on Augustin-Louis Cauchy (Chapter 15) in *Men of mathematics*:

> In the first three decades of the nineteenth century mathematics quite suddenly became something quite noticeably different from what it had been in the heroic post-Newtonian age of the eighteenth. The change was in the direction of greater rigour in demonstration following an unprecedented generality and freedom of inventiveness. Something of the same sort is plainly visible again today ... (Bell 1937, vol. 1, p. 296)

Although Bell did not enlarge upon what was "plainly visible again today", it is not unreasonable to suggest that he meant the increasing abstraction of twentieth-century mathematics, and perhaps also the move towards postulational methods. The above quotation therefore contains an implicit comparison of the arithmetization of analysis on the one hand, and the arithmetization of algebra on the other.

Algebraic arithmetic

We have seen ample evidence for Bell's favour of postulational analysis. Moreover, he appears to have had a firm understanding of exactly what this approach entailed; if the quotations of the preceding section were not proof enough, then we could look to the particularly concise précis of the postulational method that he gave in his 1927 monograph *Algebraic arithmetic*:

> What is wanted ... is a workable set of postulates which will reveal at a glance those properties of the elements considered which are of mathematical importance and which it is desirable to abstract ... (Bell 1927c, p. 59)

Wherever Bell tackled problems in abstract algebra, these appear usually to have been connected in some way with the general notion of so-called *arithmetical theories*, for a pithy definition of which we may look to Constance Reid's *The search for E. T. Bell*:

> The expression *arithmetical theory* ... refers to an algebraic system in which there is unique factorization, as in the natural numbers 1, 2, 3, ... (Reid 1993, p. 159)

It is perhaps unsurprising that, as a number theorist with a burgeoning interest in abstract algebra, Bell should have been concerned with extending familiar ideas from arithmetic to more general situations:

> Throughout Bell's mathematical career he was always interested in discovering analogues of the Fundamental Theorem [of Arithmetic] in areas other than the integers. (Reid 1993, p. 159)

As with Bell's views on postulational analysis, arithmetical ideas occupy a prominent position in his popular writing, as well as in his technical works. In several places, for example, he discussed the ideal theories of Kummer, Dedekind, and others. See, for instance, Bell (1937, Chapter 27), Bell (1945, Chapter 10) and Bell (1952, Chapter 5). On the more technical front, Bell published an article entitled 'Successive generalizations in the theory of numbers' (Bell 1927b), to which we will come shortly.

Although Bell's mathematical work is now somewhat overshadowed by his popular writings (*Men of mathematics* in particular), he was, in his day, a celebrated number theorist, at least within the US mathematical community (see, for example, Goodstein and Babbitt 2013, p. 688), having been awarded the American Mathematical Society's Bôcher Memorial Prize in 1924 for a series of papers on so-called 'arithmetical paraphrases' (Bell 1921), which provided an overarching approach to certain problems in number theory (see Reid 1993, pp. 172–173 or Goodstein and Babbitt 2013, p. 695). This general methodology subsequently provided the basis for *Algebraic arithmetic*, the title of which is explained in the introduction in the following terms:

> Intermediate between the modern analytic theory of numbers and the classic arithmetic as developed by the school of Gauss, is an extensive region of the theory of numbers where the methods of algebra and analysis are freely used to yield relations between integers expressed wholly in finite terms and without reference, in the

> final propositions, to the operations or concepts of limiting processes. This part of the theory of numbers we shall call *algebraic arithmetic*. (Bell 1927c, p. 1)

Bell went on to help the reader distinguish between 'analytic arithmetic' and 'algebraic arithmetic' by presenting two examples of functions. Let $F(n)$ be the number of representations of a positive integer n as a sum of 4 integral squares, and let $s(n)$ be the sum of all the odd divisors of n. Then

$$F(n) = 8[2 + (-1)^n]s(n). \tag{1}$$

On the other hand, let $\pi(x)$ be the number of primes less than or equal to a positive integer x. Then

$$\lim_{x\to\infty}\left(\frac{\pi(x)}{x/\ln(x)}\right) = 1. \tag{2}$$

Bell noted that the formula presented in (1) fits the description of 'algebraic arithmetic' but that formula (2) "belongs to a totally different order of ideas" (Bell 1927c, p. 2).

It is the general approach of *Algebraic arithmetic* that is of interest to us here, rather than any particular result appearing therein. Indeed, the *approach* was also of paramount importance to Bell: "[t]he insistence will be upon general methods rather than specific instances" (Bell 1927c, p. 1). Perhaps somewhat unusually for the time (and maybe also a little surprisingly, given his number-theoretic background), Bell chose not to start with the positive rational numbers, a natural model for any subsequent arithmetical theory, but instead took an abstract field as his most basic notion.

Bell also appears to have derived some small amount of inspiration from logic. Early in *Algebraic arithmetic*, he defined *common arithmetic* to be the set Π of all propositions which follow from the postulates for an abstract field, including the postulates themselves. Moreover, Bell made heavy use of the notion of a 'doctrinal function', as introduced by his doctoral supervisor, C. J. Keyser (1913); a 'doctrinal function' is simply "a set of postulates together with the set of all logical consequences of those postulates" (Bell 1927c, p. 175). Thus, 'common arithmetic' is an example of a doctrinal function.

The postulational method certainly had a role to play in algebraic arithmetic:

> In the description of algebraic arithmetic we have used several terms whose significance is usually taken by consent as being obvious, but which will be clearly understood only when large tracts of extant arithmetic have been subjected to the postulational method. Among these is arithmetic itself. ... An abstract logical analysis, culminating in the relational formulation of existing arithmetical theories should disclose the essential characteristics common to all. (Bell 1927c, p. 3)

Bell observed also that such a general, postulational theory must preserve the central features of the ordinary arithmetic of the rational numbers, otherwise "the whole subject becomes entirely too simple and too structureless to be of interest" (Bell 1927c, p. 160). Some preliminary steps towards the construction of a general theory of this type ("a tentative first approximation to a postulation of arithmetic": Bell 1927c, p. 10) may be found in the book. In modern terminology, Bell took a commutative cancellative monoid G with identity 1. He then supposed that there is a subset $\Sigma \subseteq G$, implicitly assumed to be non-empty, none of whose elements can be written as the product of two non-units. Although Bell did not use the term, this is, notionally, the subset of 'irreducibles' of G. If G is non-trivial, and if every element of $G \setminus (\Sigma \cup \{1\})$ can be written as a product of elements of Σ in one way only (up to permutation of factors), then Bell termed G an *arithmetical semigroup*.

The next part of Bell's description is rather vague — perhaps because he was trying to set down his ideas for a general intuitive procedure, rather than for a specific construction. Let F be what Bell referred to as "an instance of common arithmetic" — in modern terminology, an arbitrary abstract field. If it is possible to find a subring R of F from whose elements can be 'constructed', in some way or other, a set G of 'functions' which forms an arithmetical semigroup, then F is called a *restricted arithmetical theory*; if G is in fact a ring, then F is a *complete arithmetical theory*. In the above definition, Bell used the term 'function' somewhat loosely; in the case $F = \mathbb{Q}$, we have $R = \mathbb{Z} = G$ and the 'functions' derived from elements of R are simply the elements themselves, whereas if F is a general algebraic number field, R is the corresponding ring of algebraic integers and the relevant 'functions' are ideals. Note the use of the notion of an abstract ring, still quite a new concept at this time (see Corry 2000 or Corry 1996, 2nd ed., §4.5; see also Hollings 2014a).

Bell cited Dedekind's ideal theory as an example of a restricted arithmetical theory. However, he did not go very much further with this theory himself, commenting that a detailed postulational treatment was beyond the scope of the book. It is interesting to note that certain comments made by Bell suggest that he did indeed regard these methods as belonging to algebra, as well as to number theory: he quoted (and appeared to agree with) a proposal made by Eugene Cahen (1900) "to define arithmetic as that branch of algebra in which division is only exceptionally possible" (Bell 1927c, p. 62).

In the same year that *Algebraic arithmetic* was published, Bell also produced the above-mentioned article 'Successive generalizations in the theory of numbers' (Bell 1927b). Whereas the main purpose of the monograph was to unify various related notions, the article took the methodology of the monograph as an end in itself, presenting it almost as a latter-day Erlanger Programm:

> The foundations of geometry have had their share of attention. Few creative philosophers of this generation would deny that the critical insight gained from a postulational examination of geometry has clarified their outlook on time no less than on space. Is it too much to hope that a like scrutiny of modern arithmetic will also yield its rich reward in a clearer perception of thought itself? (Bell 1927b, p. 56)

Thus, Bell sought a deeper understanding of the foundations of arithmetic; for him, the way towards such an insight was through the development of an abstract arithmetical theory: an axiomatization of arithmetic (distinct from those attempted by Frege, Whitehead and Russell, *et al.*) which takes division as its basic operation and which emphasizes the role of prime factorization and related concepts. This would be just one manifestation (for Bell, the most important) of the more general programme for the 'arithmetization of algebra'.

Further arithmetization

Aside from his forming of general ideas in *Algebraic arithmetic*, Bell also developed some specific arithmetical theories. For example, an early paper (Bell 1915) built an arithmetical theory for certain numerical functions — concrete versions of some of the ideas of *Algebraic arithmetic* are already in evidence in this paper, and were subsequently developed on a more abstract level in Bell (1923) (see Reid 1993, pp. 190–191). A further paper ('Arithmetic of logic': Bell 1927a) constructed an arithmetical theory for Boolean algebras, whilst another (Bell 1931a) developed such a theory for certain collections of matrices.

Ideas of an even more abstract nature appear in a paper entitled 'Unique decomposition' (Bell 1930). Here, Bell catalogued different types of what he termed 'varieties': systems that may be obtained from an abstract field through the modification or suppression of one or more of the postulates. These had in fact already appeared in *Algebraic arithmetic* (Chapter I). Amongst the exotic terms adopted by Bell, we find, for example, 'ovum' (in modern terminology: a commutative semigroup) and 'ray' (a commutative monoid with zero, in which every non-zero element is a unit; alternatively, a group with zero adjoined). Fields and rings also appear, with their modern definitions.

Bell's listing of the axiomatic definitions of these various concepts, however, was hardly more than an exercise in taxonomy, for he did little with them in the abstract setting. Around halfway through the paper, he turned his attention suddenly to certain 'matric varieties': 'varieties' (in the sense of the first half of the paper) of real or complex one-rowed matrices, with the usual arithmetical operations extended to them component-wise. The development of an appropriate arithmetical theory for a particular special type of 'matric variety' takes up the rest of the paper, culminating in a version for these of the Fundamental Theorem of Arithmetic (Bell 1931b, Theorem 4.8). Thus, a paper on an ostensibly abstract subject concludes with the study of a very specific instance. Since the latter accounts for around half the paper, it is not merely an instance of the concrete examples that one often finds at the ends of papers on abstract subjects. The same is true in others of Bell's papers: Bell (1931a), for example, similarly begins in an entirely abstract style, before turning suddenly to 'varieties' of matrices. Bell (1933), on the other hand, is entirely abstract in its approach — but it is barely three pages long. Few of Bell's papers remain wholly abstract throughout.

Concluding remarks

Following his ‘Unique decomposition’ of 1930, Bell seems to have had little more to say on the subject of the arithmetization of algebra, except for in those places in his popular writings that I have already indicated (Bell 1940 is another rare example). We have seen that he went to a great deal of effort to promote this approach to algebra, and yet his own contributions to its subsequent development were minimal.

Bell’s pursuit of abstract algebraic problems appears to have had a good start. Regarding the 1923 paper that was mentioned at the beginning of the preceding section, Reid (1993, p. 191) recorded some comments made by the number theorist Lincoln Durst:

> Here we find Bell at the threshold of what came to be called ‘modern’ or ‘abstract’ mathematics in the nineteen twenties and thirties. At the time he wrote this, the terminology had not settled down to the standard collection of words we use now, but his treatment is very much in the abstract or modern manner. He just doesn’t have the modern terms at his disposal yet.

Moreover, Goodstein and Babbitt (2013, p. 689) quote a 1926 letter from Bell to his former student and sometime-colleague H. P. Robertson, in which he noted that he had “recently opened up a whole new field in ‘General Arith[metic]’ where there are hundreds of things to be done ...”. These were surely the words of a man on the brink of an entirely new programme of research. However, by 1930, we see Bell begin ‘Unique decomposition’ with abstract considerations, only to veer, partway through, back into the specific. Bell was certainly not alone in his inconsistent application of the abstract approach: Wussing (1969, English transl., p. 238) records some other examples from the early development of abstract group theory.

The explanation for Bell’s veering away from the abstract appears in fact to be quite straightforward: like many mathematicians faced by the new conception of algebra, Bell found that he simply was not comfortable with the new ideas, despite his apparent enthusiasm for them (motivated perhaps by simple aesthetics — suggested by the quotations earlier) and his great efforts to get to grips with them. Reid (1993, p. 262) quotes R. P. Dilworth, a Caltech colleague of Bell:

> E. T. was not at home in abstraction. ... He had worked all his life with the integers. ... He was just never at home in the new stuff.

Reid (1993, p. 262) also records that Bell’s Caltech course on modern algebra, based on van der Waerden’s book, was somewhat confused and poorly organized (see also the comments of Goodstein and Babbitt 2013, p. 693). Although Bell “felt he *ought* to be interested in this abstract side of algebra and to follow it” (Reid 1993, p. 295), he was simply much more comfortable with the number-theoretic ideas and techniques that he had been developing since the beginning of his career. Indeed, it appears that he elected not to pursue a fledgling interest in the rather different subject of relativity for this same reason, admitting to a colleague in 1922 that he would

> hate to scrap the detailed knowledge of the theory of numbers which has taken so long to acquire. (Goodstein and Babbitt 2013, p. 689)

Moreover, by the late 1920s, Bell had research students who were beginning to consider abstract problems for their theses (see, for example, Ward 1928, or, a little later, Clifford 1933, Worth 1933, and Poole 1935),[1] and so, having made an effort to promote the ‘arithmetization of algebra’, Bell was probably happy to leave this abstract mathematics to others.

I have suggested that Bell’s promotion of the ‘arithmetization of algebra’, most particularly in his popular writings, is another example of his desire to shape the collective image of mathematics. However,

[1]On the theses of Ward and Clifford, see, respectively, Sections 4.3 and 4.5 of Hollings (2014b); on Poole’s dissertation, see the scattered comments in Sections 4.2, 6.5, 6.6 and 8.4. On Ward’s thesis, see also Goodstein and Babbitt (2013, p. 692). Of the dissertations cited here, it is Ward’s that is the closest in spirit to Bell’s ideas on algebraic arithmetic: it seeks an axiomatization of basic arithmetic, with a particular focus on factorization properties. Worth’s dissertation enumerates various different types of ‘subvarieties’ (in Bell’s terminology) of an abstract field.

it is interesting to note that the view that he presented here was arguably no more accurate than some of his portrayals of mathematicians in *Men of mathematics*, for, by the time of his writing, the strictly postulational point of view was no longer in fashion — but Bell's apparently unshakeable belief in the importance of this area led him to advance it nevertheless. Thus, Bell's general readers were being fed an image of mathematics that would not necessarily have been recognized in all its details by practicing mathematicians.

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