

Exploring hypergraphs with martingales

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Abstract

Recently, in [7] we adapted exploration and martingale arguments of Nachmias and Peres [17], in turn based on ideas of Martin-Löf [15], Karp [14] and Aldous [1], to prove asymptotic normality of the number L_1 of vertices in the largest component \mathcal{L}_1 of the random r -uniform hypergraph in the supercritical regime. In this paper we take these arguments further to prove two new results: strong tail bounds on the distribution of L_1 , and joint asymptotic normality of L_1 and the number M_1 of edges of \mathcal{L}_1 in the sparsely supercritical case. These results are used in [9], where we enumerate sparsely connected hypergraphs asymptotically.

1 Introduction and results

For $2 \leq r \leq n$ and $0 < p < 1$, let $H_{n,p}^r$ denote the random r -uniform hypergraph with vertex set $[n] = \{1, 2, \dots, n\}$ in which each of the $\binom{n}{r}$ possible hyperedges is present independently with probability p . One family of interesting questions concerning $H_{n,p}^r$ asks for analogues of the pioneering results of Erdős and Rényi [12] concerning the phase transition in the graph ($r = 2$) case of this model, as well as analogues of the many more detailed and precise results that followed. Throughout the paper we fix $r \geq 2$ and consider

$$p = p(n) = \lambda(r-2)!n^{-r+1}$$

with $\lambda = \lambda(n) = \Theta(1)$. The reason for this normalization is that, as shown by Schmidt-Pruzan and Shamir [20], with this choice $\lambda = 1$ is the critical point of the phase transition in $H_{n,p}^r$, above which a giant component emerges.

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For $r = 2$, a great deal is known; for $r \geq 3$, most past results concern the case $\lambda \neq 1$ constant, or (essentially equivalently), $\lambda = 1 \pm \Theta(1)$.¹ Here we are especially interested in what happens when $\lambda \rightarrow 1$, so much of the time we write $\lambda = 1 + \varepsilon$ or $\lambda = 1 - \varepsilon$, with $\varepsilon = \varepsilon(n) \rightarrow 0$. In [7], a result of Aldous [1] concerning critical random graphs ($r = 2$) is extended to $r \geq 3$; this implies in particular that the critical window of the phase transition in $H_{n,p}^r$ is when $\varepsilon^3 n = O(1)$, just as in the graph case. Here we study $H_{n,p}^r$ *outside* the critical window, i.e., when $\varepsilon^3 n \rightarrow \infty$.

If G is a (multi-)graph, then its *nullity* is

$$n(G) = c(G) + e(G) - |G|,$$

where $|G|$, $e(G)$ and $c(G)$ are the numbers of vertices, edges and components of G . In the hypergraph case, it is natural to define the nullity of H as the nullity of any multigraph obtained by replacing each hyperedge by a tree on the same set of vertices. In the r -uniform case, this reduces to the following definition:

$$n(H) = c(H) + (r - 1)e(H) - |H|.$$

For connected graphs and hypergraphs, one often studies instead the *excess* $n(G) - 1$ or $n(H) - 1$. However, while this definition is natural for connected graphs (where it reduces to $e(G) - |G|$), it seems less natural for hypergraphs, and we prefer to work with $n(H)$.

Let \mathcal{L}_1 be the component of $H_{n,p}^r$ containing the most vertices, chosen according to any rule if there is a tie. Let $L_1 = |\mathcal{L}_1|$ and $M_1 = e(\mathcal{L}_1)$ be the numbers of vertices and edges in \mathcal{L}_1 , and $N_1 = n(\mathcal{L}_1)$ its nullity, so

$$(r - 1)M_1 = L_1 + N_1 - 1.$$

Our main aim is to prove a bivariate central limit theorem (Theorem 1 below) for the random variable (L_1, N_1) (and hence for (L_1, M_1) and for (M_1, N_1)) throughout the sparsely supercritical regime, i.e., when $\lambda = 1 + \varepsilon$ with $\varepsilon^3 n \rightarrow \infty$ and $\varepsilon \rightarrow 0$. The corresponding result for $\varepsilon = \Theta(1)$ was proved recently by Behrisch, Coja-Oghlan and Kang [3], as part of a stronger result, a local limit theorem. Their methods are completely different from ours, and seem very unlikely to adapt to the case $\varepsilon \rightarrow 0$.

Our second aim is to prove, in Theorems 2 and 4 below, large-deviation bounds on L_1 in the supercritical and subcritical cases. As far as we are aware, even for $\varepsilon = \Theta(1)$ these results are new for hypergraphs, so here we do not assume that $\varepsilon \rightarrow 0$. As we show in a separate paper [9], it is possible to use ‘smoothing’ arguments to deduce from Theorem 1 its local limit analogue, and hence to give an asymptotic formula for the number of connected r -uniform

¹Given functions $f(n)$ and $g(n)$ with $g(n) > 0$ for $n \geq n_0$, we write $f(n) = O(g(n))$ if $\limsup_{n \rightarrow \infty} |f(n)|/g(n) < \infty$, i.e., there is a constant $C > 0$ such that $|f(n)| \leq Cg(n)$ for all $n \geq n_0$. We write $f(n) = \Theta(g(n))$ if there are *positive* constants $C > c > 0$ such that $cg(n) \leq f(n) \leq Cg(n)$ for all large enough n . Similarly, $f(n) = \Omega(g(n))$ if $\exists n_0, c > 0$ such that $f(n) \geq cg(n)$ for $n \geq n_0$.

hypergraphs with s vertices and m edges, for suitable $m = m(s)$. The tail bounds proved here are needed for these arguments as well as being (we hope) of interest in their own right.

To state our results precisely we need a number of definitions; we shall (mostly) follow the notation in [7]. For $\lambda > 1$ let ρ_λ be the unique positive solution to

$$1 - \rho_\lambda = e^{-\lambda\rho_\lambda}, \quad (1)$$

so ρ_λ is the survival probability of a Galton–Watson branching process whose offspring distribution is Poisson with mean λ , and define $\lambda_* < 1$, the parameter *dual* to λ , by

$$\lambda_* e^{-\lambda_*} = \lambda e^{-\lambda}.$$

It is easy to check that

$$\lambda_* = \lambda(1 - \rho_\lambda), \quad (2)$$

and that for any $A > 1$ there exist $C > c > 0$ such that $\lambda = 1 + \varepsilon \in (1, A]$ implies

$$1 - C\varepsilon \leq \lambda_* \leq 1 - c\varepsilon. \quad (3)$$

For $\lambda > 1$ and $r \geq 2$, define $\rho_{r,\lambda}$ by

$$1 - \rho_{r,\lambda} = (1 - \rho_\lambda)^{1/(r-1)}, \quad (4)$$

and set

$$\rho_{r,\lambda}^* = \frac{\lambda}{r} (1 - (1 - \rho_{r,\lambda})^r) - \rho_{r,\lambda}. \quad (5)$$

(The star here does not refer to duality; rather it is a notational convention adopted from [19].) If $\lambda = 1 + \varepsilon$ then, as $\varepsilon \rightarrow 0$, elementary but tedious calculations show that

$$1 - \lambda_* \sim \varepsilon, \quad \rho_{r,\lambda} \sim \frac{2\varepsilon}{r-1}, \quad \text{and} \quad \rho_{r,\lambda}^* \sim \frac{2}{3(r-1)^2} \varepsilon^3. \quad (6)$$

One way to see this is to use (1) to find (term-by-term) the first few terms in a series expansion for ρ_λ , and to substitute this expansion into (4) and then (5).²

In [7] we showed that throughout the supercritical regime, i.e., when $\varepsilon^3 n \rightarrow \infty$ and $\varepsilon = O(1)$, the random variable $L_1(H_{n,p}^r)$ is asymptotically normally distributed with mean $\rho_{r,\lambda} n$ and variance $\sigma_{r,\lambda}^2 n$, where a formula for $\sigma_{r,\lambda}$ is given in [7, Eq. (3)]. As noted there, when $\varepsilon \rightarrow 0$, $\sigma_{r,1+\varepsilon}^2 \sim 2\varepsilon^{-1}$. Hence, under this additional assumption, the main result of [7] says exactly that $L_1(H_{n,p}^r)$ is asymptotically normally distributed with mean $\rho_{r,\lambda} n$ and variance $2n/\varepsilon$. Our first result extends this univariate central limit theorem to a bivariate one.

²It turns out that with $\lambda = 1 + \varepsilon > 1$ we have $\rho_\lambda = 2\varepsilon - \frac{8}{3}\varepsilon^2 + O(\varepsilon^3)$. This gives

$$\rho_{r,\lambda} = \frac{2}{r-1} \varepsilon - \frac{2(r+2)}{3(r-1)^2} \varepsilon^2 + O(\varepsilon^3),$$

which is enough to establish (6).

Theorem 1. *Let $r \geq 2$ be fixed, and let $p = p(n) = (1 + \varepsilon)(r - 2)!n^{-r+1}$ where $\varepsilon = \varepsilon(n) \rightarrow 0$ and $\varepsilon^3 n \rightarrow \infty$. Let L_1 and N_1 be the order and nullity of the largest component \mathcal{L}_1 of $H_{n,p}^r$. Then*

$$\left(\frac{L_1 - \rho_{r,\lambda} n}{\sqrt{2n/\varepsilon}}, \frac{N_1 - \rho_{r,\lambda}^* n}{\sqrt{10/3(r-1)^{-1}\varepsilon^3 n}} \right) \xrightarrow{d} (Z_1, Z_2)$$

as $n \rightarrow \infty$, where $\rho_{r,\lambda}$ and $\rho_{r,\lambda}^*$ are defined as in (4) and (5) with $\lambda = 1 + \varepsilon$, \xrightarrow{d} denotes convergence in distribution, and (Z_1, Z_2) has a bivariate Gaussian distribution with mean 0, $\text{Var}[Z_1] = \text{Var}[Z_2] = 1$ and $\text{Cov}[Z_1, Z_2] = \sqrt{3/5}$.

The graph case of this result was proved by Pittel and Wormald [18] using very different methods, as part of a stronger result. As noted above, the corresponding result with $\varepsilon = \Theta(1)$ was proved recently by Behrisch, Coja-Oghlan and Kang [3]. Their formula for the quantity corresponding to $\rho_{r,\lambda}^*$ coincides with ours, though the different notation obscures this. (They write ρ for $1 - \rho$, and study M_1 rather than N_1 . Since $M_1 = (L_1 + N_1 - 1)/(r - 1)$, it is straightforward to translate.) We believe that our proof of Theorem 1 can be made to work replacing the assumption $\varepsilon \rightarrow 0$ by $\varepsilon = O(1)$, but the calculations would be more involved. Since the result for $\varepsilon = \Theta(1)$ is covered by that in [3], we assume that $\varepsilon \rightarrow 0$ to keep things simple.

We next turn to tail bounds on the distribution of $L_1(H_{n,p}^r)$ in the subcritical and supercritical cases. In reading these results, it is worth noting that in both cases, for deviations of order εn , i.e., of order the typical value of $L_1(H_{n,p}^r)$ in the supercritical case, we obtain a bound on the probability of order $\exp(-\Omega(\varepsilon^3 n))$. This formula, which we believe to be tight up to the constant, corresponds to the function $\exp(-\Omega(n))$ that one expects when $\lambda \neq 1$ is constant. We start with the subcritical case.

Theorem 2. *Let $r \geq 2$ be fixed and let $p = p(n) = (1 - \varepsilon)(r - 2)!n^{-r+1}$ where $\varepsilon^3 n \rightarrow \infty$ and $1 - \varepsilon$ is bounded away from 0. If $L = L(n)$ satisfies $\varepsilon^2 L \rightarrow \infty$ and $L = O(\varepsilon n)$, then there is a constant $C > 0$ such that*

$$\mathbb{P}(L_1(H_{n,p}^r) > L) \leq C \frac{\varepsilon n}{L} \exp(-\varepsilon^2 L/C) \quad (7)$$

for all large enough n .

Remark 3. The formal statement is that for every $r \geq 2$ and every pair of functions $p(n)$ and $L(n)$ satisfying the given conditions, there exist $C > 0$ and n_0 such that (7) holds for all $n \geq n_0$. In other words, the constant C is allowed to depend on the choice of $r \geq 2$, and of the functions $p = p(n)$ and $L = L(n)$. This type of statement is convenient when it comes to the proof, since we can just take $p(n)$ and $L(n)$ as given, and not worry about how C depends on them. However, as usual in such contexts, uniformity over suitable sets of choices for $p(n)$ and $L(n)$ follows automatically. More precisely, given $r \geq 2$ and $A > 0$, Theorem 2 implies that there is a constant $C > 0$, depending only on r and A ,

such that (7) holds whenever $1 - \varepsilon \geq 1/A$, $L \leq A\varepsilon n$, and n , $\varepsilon^3 n$ and $\varepsilon^2 L$ are large enough.³

Theorem 2 gives a meaningful bound (a bound on the probability that is less than 1) only when L is at least some constant times $\log(\varepsilon^3 n)/\varepsilon^2$, which, as shown by Karoński and Łuczak [13], is the typical order of L_1 . For us, the most important case is that with $L = \Theta(\varepsilon n)$. We believe that, apart from the constant in the exponent, the bound given in Theorem 2 is best possible for essentially the entire range to which it applies.

In the supercritical case, we show that L_1 is concentrated around its mean, and that the number L_2 of vertices in the second-largest component is unlikely to be large.

Theorem 4. *Let $r \geq 2$ be fixed, let $p = p(n) = (1 + \varepsilon)(r - 2)!n^{-r+1}$ where $\varepsilon = O(1)$ and $\varepsilon^3 n \rightarrow \infty$, and define $\rho_{r,\lambda}$ as in (4) with $\lambda = 1 + \varepsilon$. If $\omega = \omega(n) \rightarrow \infty$ and $\omega = O(\sqrt{\varepsilon^3 n})$ then*

$$\mathbb{P}\left(|L_1(H_{n,p}^r) - \rho_{r,\lambda} n| \geq \omega \sqrt{n/\varepsilon}\right) = \exp(-\Omega(\omega^2)). \quad (8)$$

Moreover, if $L = L(n)$ satisfies $\varepsilon^2 L \rightarrow \infty$ and $L = O(\varepsilon n)$, then there exists $C > 0$ such that

$$\mathbb{P}(L_2(H_{n,p}^r) > L) \leq C \frac{\varepsilon n}{L} \exp(-\varepsilon^2 L/C) \quad (9)$$

for all large enough n .

Remark. Again, the constant C , and the implicit constant in the $\Omega(\cdot)$ notation in (8), may depend on the choice of the ‘input’ parameters $r \geq 2$, $(p(n))$, $(L(n))$ and $(\omega(n))$.

Since $\rho_{r,\lambda} n = \Theta(\varepsilon n)$, the bound (8) implies in particular that if $\delta = \delta(n) \leq 1/2$, say, and $\delta \sqrt{\varepsilon^3 n} \rightarrow \infty$, then there is a constant $c > 0$ such that

$$\mathbb{P}\left((1 - \delta)\rho_{r,\lambda} n \leq L_1(H_{n,p}^r) \leq (1 + \delta)\rho_{r,\lambda} n\right) \geq 1 - \exp(-c\delta^2 \varepsilon^3 n) \quad (10)$$

for n large enough. As in Remark 3 above, one can check that this constant depends only on r and the implicit constant in our assumption $\varepsilon = O(1)$.

For the largest component, much more precise results are known in the graph case, at least when $\varepsilon = \Theta(1)$: for $L_1(H_{n,p}^2)$, $p = c/n$, O’Connell [16] established a ‘large deviation principle’ tight up to a factor $1 + o(1)$ in the exponent in the

³Suppose not. Then for each $k = 1, 2, \dots$ we may find values n_k , ε_k and L_k with $1 - \varepsilon_k \geq 1/A$ and $L_k \leq A\varepsilon_k n_k$ such that (7) does not hold for these values with $C = k$, with, in addition, $\min\{n_k, \varepsilon_k^3 n_k, \varepsilon_k^2 L_k\} \geq k$. Passing to a subsequence we may assume that (n_k) is strictly increasing. But now we have partial functions $\varepsilon(n)$ and $L(n)$ (which we may complete to functions) satisfying the assumptions of Theorem 2. So there should be some C and k_0 such that (7) holds for this sequence, i.e., for all $(n_k, \varepsilon_k, L_k)$, $k \geq k_0$. Considering any $k > \max\{C, k_0\}$ now gives a contradiction.

error probability. Biskup, Chayes and Smith [5] proved a corresponding result for the number of vertices in ‘large’ components.

In the subcritical case, Karoński and Łuczak [13] proved very precise results about the limiting distribution of L_1 (essentially a local limit result, but conditional on the probability $1 - o(1)$ event that there are no complex components). Theorem 2 neither implies their result nor is implied by it: instead of considering ‘typical’ values of L_1 , we prove that the probability that L_1 is considerably larger than such typical values goes to zero rather quickly.

The rest of the paper is organized as follows. We shall prove Theorems 2, 4 and 1 in this order. First, in Section 2, we prove some simple lemmas that we shall need later. In Section 3, we recall the exploration argument from [7], and state some basic properties of corresponding random walk. In Section 4 we use this random walk to prove Theorem 2. Next, in section 5, we describe the approximation of the random walk by a martingale (as in [7]). We use this to prove Theorem 4 in Section 6 and our main result, Theorem 1, in Section 7.

2 Preliminaries

In this section we prove some probabilistic inequalities that will be needed later. Here (and indeed throughout the paper) we make no attempt to optimize the various constants that appear, or even to make them explicit.

Lemma 5. *Let $k > 0$. There is a constant $K = K(k)$ such that if $Y \sim \text{Bin}(n, p)$ with $np \leq \nu \leq k$, and X is a non-negative random variable with mean μ that is stochastically dominated by kY , then for $-1 \leq \theta \leq 1$ we have*

$$\mathbb{E}[(X - \mu)^2 e^{\theta(X - \mu)}] \leq K\nu.$$

Proof. For $0 \leq \alpha \leq k$, by the binomial theorem and the standard inequality $1 + x \leq e^x$ we have

$$\mathbb{E}[e^{\alpha Y}] = \sum_{k=0}^n \binom{n}{k} p^k (1-p)^{n-k} e^{\alpha k} = (1-p + pe^\alpha)^n \leq \exp(np(e^\alpha - 1)) \leq K_1, \quad (11)$$

where $K_1 = k(e^k - 1)$ is a constant depending only on k . Either by differentiating, or by using twice the observation that $Y \sim \text{Bin}(n, p)$ implies $\mathbb{E}[Y f(Y)] = np \mathbb{E}[f(Z + 1)]$ where $Z \sim \text{Bin}(n - 1, p)$, we deduce that

$$\mathbb{E}[Y e^{\alpha Y}] \leq np e^\alpha K_1 \leq \nu e^k K_1$$

and

$$\mathbb{E}[Y^2 e^{\alpha Y}] = \mathbb{E}[Y(Y - 1) e^{\alpha Y}] + \mathbb{E}[Y e^{\alpha Y}] \leq \nu^2 e^{2k} K_1 + \nu e^k K_1 \leq \nu K_2, \quad (12)$$

where $K_2 = (ke^{2k} + e^k)K_1$. For $0 \leq \theta \leq 1$, since $\exp(\theta x)$ and $x^2 \exp(\theta x)$ are increasing in $x \geq 0$, we have

$$\begin{aligned} \mathbb{E}[(X - \mu)^2 e^{\theta(X - \mu)}] &\leq \mathbb{E}[(X - \mu)^2 e^{\theta X}] \leq \mathbb{E}[X^2 e^{\theta X}] + \mu^2 \mathbb{E}[e^{\theta X}] \\ &\leq \mathbb{E}[k^2 Y^2 e^{\theta k Y}] + \mu^2 \mathbb{E}[e^{\theta k Y}] \leq k^2 \nu K_2 + \mu^2 K_1 \leq k^2 \nu K_2 + k \nu K_1 = \nu K_3, \end{aligned} \quad (13)$$

recalling (11) and (12), and noting that $\mu = \mathbb{E}[X] \leq \mathbb{E}[Y] = np \leq \nu \leq k$.

Since $\mu \leq k$ and $X \geq 0$, for $-1 \leq \theta < 0$ we have $e^{\theta(X-\mu)} \leq e^{-\theta\mu} \leq e^k$, so

$$\mathbb{E}[(X - \mu)^2 e^{\theta(X-\mu)}] \leq e^k \mathbb{E}[(X - \mu)^2] \leq e^k \nu K_3,$$

where in the last step we applied (13) with $\theta = 0$. This completes the proof of the lemma with $K = e^k K_3$, a constant depending only on k . \square

Our next lemma is a simple Hoeffding–Azuma-type martingale inequality that is doubtless a special case of (many) known results. Since the proof is very simple, it seems easiest just to give it.

Lemma 6. *Let $C > 0$ be a real number, and let $(M_t)_{t=0}^\ell$ be a martingale with respect to the filtration (\mathcal{F}_t) with $M_0 = 0$. Set $\Delta_t = M_t - M_{t-1}$, and suppose that for all $1 \leq t \leq \ell$ and all $\theta \in [-1, 1]$ we have*

$$\mathbb{E}[\Delta_t^2 e^{\theta \Delta_t} \mid \mathcal{F}_{t-1}] \leq C \text{ almost surely.} \quad (14)$$

Then

$$\mathbb{P}\left(\max_{0 \leq t \leq \ell} |M_t| \geq y\right) \leq 2 \exp(-y^2 / (2 \max\{y, C\ell\})). \quad (15)$$

Proof. By a standard stopping-time argument, to prove (15) it suffices to show that

$$\mathbb{P}(|M_\ell| \geq y) \leq 2 \exp(-y^2 / (2 \max\{y, C\ell\})). \quad (16)$$

Indeed, let $\tau = \inf\{t : |M_t| \geq y\} \leq \infty$ and consider the stopped martingale defined by $M'_t = M_{t \wedge \tau}$. (Thus $M'_t = M_t$ for all t if $\tau = \infty$.) This martingale also satisfies the assumptions of the lemma, and relation (16) for (M'_t) implies (15) for (M_t) .

If X is any random variable with $\mathbb{E}[X] = 0$ satisfying $\mathbb{E}[X^2 e^{\theta X}] \leq C$ for all $\theta \in [-1, 1]$ then, defining $f(\theta) = \mathbb{E}[e^{\theta X}] > 0$, we have $f(0) = 1$, $f'(0) = \mathbb{E}[X] = 0$ and, for $-1 \leq \theta \leq 1$,

$$f''(\theta) = \mathbb{E}[X^2 e^{\theta X}] \leq C.$$

It follows that for $-1 \leq \theta \leq 1$ we have

$$f(\theta) \leq 1 + C\theta^2/2 \leq \exp(C\theta^2/2).$$

For $1 \leq t \leq \ell$ let $\Delta_t = M_t - M_{t-1}$. Then $\mathbb{E}[\Delta_t \mid \mathcal{F}_{t-1}] = 0$ and, by assumption, for $-1 \leq \theta \leq 1$ we have $\mathbb{E}[\Delta_t^2 e^{\theta \Delta_t} \mid \mathcal{F}_{t-1}] \leq C$. It follows that

$$\mathbb{E}[e^{\theta \Delta_t} \mid \mathcal{F}_{t-1}] \leq \exp(C\theta^2/2).$$

A standard inductive argument now implies that $\mathbb{E}[e^{\theta M_\ell}] \leq \exp(C\theta^2 \ell / 2)$. Let $y \geq 0$. Then, by Markov's inequality, for $0 \leq \theta \leq 1$ we have

$$\mathbb{P}(M_\ell \geq y) \leq \mathbb{E}[e^{\theta M_\ell}] / e^{\theta y} \leq \exp(C\theta^2 \ell / 2 - \theta y).$$

For $y \leq C\ell$, taking $\theta = y/(C\ell) \in [0, 1]$ gives $\mathbb{P}(M_\ell \geq y) \leq \exp(-y^2/(2C\ell))$; for $y \geq C\ell$, taking $\theta = 1$ gives $\mathbb{P}(M_\ell \geq y) \leq \exp(C\ell/2 - y) \leq \exp(-y/2)$. We

may bound $\mathbb{P}(M_\ell \leq -y)$ similarly, using Markov's inequality to show that for $-1 \leq \theta \leq 0$ we have

$$\mathbb{P}(M_\ell \leq -y) \leq \mathbb{E}[e^{\theta M_\ell}] / e^{-\theta y} \leq \exp(C\theta^2 \ell / 2 + \theta y),$$

and then taking $\theta = -y/(C\ell)$ or $\theta = -1$. This completes the proof (16) and hence of the lemma. \square

3 The exploration process and its increments

Let us briefly recall some of the methods and results of [7], based on ‘exploring’ the component structure of $H_{n,p}^r$ step-by-step.⁴ Explorations of this type have been used on numerous occasions, including by Martin-Löf [15], Karp [14], Aldous [1] and Nachmias and Peres [17]. For hypergraphs, the form described here was used by Behrisch, Coja-Oghlan and Kang [2] and later by the present authors in [7]; in our opinion, the description and analysis in [7] is simpler than that in [2]. For further background, see [6].

Given a hypergraph H with vertex set $[n]$, we ‘explore’ H by revealing its edges in n steps as follows. In step $1 \leq t \leq n$ we pick a vertex v_t in a way that we shall specify in a moment, and reveal all edges incident with v_t but not with any of v_1, \dots, v_{t-1} . After t steps we have ‘explored’ the vertices v_1, \dots, v_t , and have revealed all edges incident with one or more of these vertices. An unexplored vertex is ‘active’ if it is incident with one or more revealed edges, and ‘unseen’ otherwise. We write \mathcal{A}_t for the set of active vertices after t steps, \mathcal{U}_t for the set of unseen vertices, and set $A_t = |\mathcal{A}_t|$. When choosing which vertex to explore next, we pick an active vertex if there is one (according to any rule), and an unseen vertex otherwise.

Let $0 = t_0 < t_1 < t_2 \cdots < t_\ell = n$ enumerate $\{t : A_t = 0\}$. Then, for $1 \leq i \leq \ell$, the set $V_i = \{v_{t_{i-1}+1}, \dots, v_{t_i}\}$ is the vertex set of a component of H . Indeed, for any t such that $A_t = 0$ there are no edges joining any v_i with $i \leq t$ to any v_j with $j > t$, so V_i is not joined to $[n] \setminus V_i$ in H , and if $A_t > 0$ then v_{t+1} is active at time t , and hence is in some edge containing some v_i , $i \leq t$; thus the subhypergraph of H induced by V_i is connected. Hence, for $1 \leq i \leq \ell$, t_i is the step at which we finish exploring the i th component of H .

Let

$$C_t = |\{i : 0 \leq i < t, A_i = 0\}|$$

be the number of components that we have started to explore within the first t steps, and define $X_t = A_t - C_t$. As we shall see in a moment, the increments of the process (X_t) are simpler to understand than those of (A_t) , so, as in [7], we shall primarily study (X_t) . We can read off the component sizes from the

⁴We aim for a presentation that is mostly self-contained: we shall need some specific results from [7] (see Lemmas 10, 13 and 20 and relation (60) below), but hope that, taking these on trust, it should be possible to follow the present paper without reading [7]. Having said this, there will be a few places where we shall give a little less detail than we might otherwise have done, since further detail is given in [7].

trajectory of (X_t) without too much trouble. Indeed, since $C_t = 1$ for $t = 1, 2, \dots, t_1$, we have $X_t = A_t - C_t \geq -1$ in this range with equality only at $t = t_1$. Similarly, X_t reaches a new ‘record low’ value $-i$ at time t_i :

$$t_i = \inf\{t : X_t = -i\}.$$

Let η_t be the number of vertices in $\mathcal{U}_{t-1} \setminus \{v_t\}$ that become active in step t , i.e., are contained in one or more hyperedges containing v_t and none of v_1, \dots, v_{t-1} . In step t , exactly η_t vertices become active. Moreover, either one vertex v_t that was previously active ceases to be active, or we start a new component and so $C_t = C_{t-1} + 1$. In either case, $X_t - X_{t-1} = \eta_t - 1$, so by induction

$$X_t = \sum_{i=1}^t (\eta_i - 1). \quad (17)$$

So far, we have not specified the hypergraph H that we are exploring. From now on, we take $H = H_{n,p}^r$. Let \mathcal{F}_t be the σ -algebra generated by all information revealed up to step t of the exploration process. This exploration process, the associated filtration (\mathcal{F}_t) , and the random sequences (X_t) , (η_t) and (to a lesser extent) (A_t) and (C_t) will be the tools that we use throughout the paper to study $H_{n,p}^r$.

We have not yet specified the function $p = p(n)$; we shall impose different assumptions in different sections. But throughout the paper, we take $r \geq 2$ constant, and assume that $p = p(n) = \Theta(n^{-r+1})$.

Lemma 7. *The distribution of η_t conditional on \mathcal{F}_{t-1} is stochastically dominated by $r - 1$ times a binomial random variable with mean*

$$\binom{n-t}{r-1} p \leq \binom{n}{r-1} p = O(1).$$

Proof. In step t we test exactly $\binom{n-t}{r-1}$ r -sets to see whether they are edges of H , namely all r -sets including v_t but none of v_1, \dots, v_{t-1} . None of these r -sets has been previously tested, so the random number E_t of edges that we find has a binomial distribution with mean $\binom{n-t}{r-1} p \leq \binom{n}{r-1} p = O(1)$. The number η_t of new active vertices is at most $(r-1)E_t$, with equality if and only if these edges intersect only at v_t , and contain no previously active vertices other than v_t . \square

In the rest of the paper we shall work with the Doob decomposition of the sequence (X_t) . Set

$$\begin{aligned} D_t &= \mathbb{E}[\eta_t - 1 \mid \mathcal{F}_{t-1}] \quad \text{and} \\ \Delta_t &= \eta_t - 1 - D_t = \eta_t - \mathbb{E}[\eta_t \mid \mathcal{F}_{t-1}], \end{aligned} \quad (18)$$

so by definition $\mathbb{E}[\Delta_t \mid \mathcal{F}_{t-1}] = 0$ and, from (17),

$$X_t = \sum_{i=1}^t (D_i + \Delta_i).$$

Then (Δ_t) is by definition a martingale difference sequence with respect to the filtration (\mathcal{F}_t) . We note two simple properties of the distribution of Δ_t which will be useful later.

Lemma 8. *Suppose that $p = p(n) = \lambda(n)(r-2)!n^{-r+1}$ with $\lambda(n) = \Theta(1)$. Then there is a constant C such that for all n and all $1 \leq t \leq n$ we have*

$$\text{Var}[\Delta_t \mid \mathcal{F}_{t-1}] \leq C$$

with probability 1. Furthermore, if $t = t(n) = o(n)$ and $a = a(n) = o(n)$ then

$$\text{Var}[\Delta_t \mid \mathcal{F}_{t-1}] \sim \lambda(r-1) \text{ whenever } A_{t-1} \leq a. \quad (19)$$

Proof. Condition on \mathcal{F}_{t-1} . By Lemma 7, the conditional distribution X of η_t is stochastically dominated by $(r-1)Y$ where $Y \sim \text{Bin}(\binom{n}{r-1}, p)$. Hence, writing $N = \binom{n}{r-1}$, we have

$$\begin{aligned} \text{Var}[\Delta_t \mid \mathcal{F}_{t-1}] &= \text{Var}[\eta_t \mid \mathcal{F}_{t-1}] \leq \mathbb{E}[\eta_t^2 \mid \mathcal{F}_{t-1}] \\ &= \mathbb{E}[X^2] \leq (r-1)^2 \mathbb{E}[Y^2] = (r-1)^2 (N(N-1)p^2 + Np) = O(1), \end{aligned}$$

proving the first statement.

For the second, when $t = o(n)$ and we have $A_{t-1} = o(n)$ active vertices, it is easy to see that X and $(r-1)Y$ are equal with probability $1 - o(1)$. (The probability that any of the $(r-1)Y$ vertices are ‘duplicates’ or lie in A_{t-1} is $o(1)$.) This, together with stochastic domination, implies that $\text{Var}[X] \sim (r-1)^2 \text{Var}[Y]$. But $\text{Var}[Y]$ is just $Np(1-p) \sim Np \sim \lambda/(r-1)$. \square

Note that if $a(n) = o(n)$ then, by considering worst-case values, one can check that the estimate (19) holds uniformly over all $0 \leq t \leq a(n)$ and all points in the sample space at which $A_{t-1} \leq a(n)$.

4 The subcritical tail bound

In this section we prove the easiest of our main results, Theorem 2; for this we use simpler methods than those in [7].

Proof of Theorem 2. Let $r \geq 2$ be fixed and let $p = p(n) = (1-\varepsilon)(r-2)!n^{-r+1}$ where $\varepsilon^3 n \rightarrow \infty$ and $1-\varepsilon$ is bounded away from 0. Fix a function $L = L(n)$ satisfying $\varepsilon^2 L \rightarrow \infty$ and $L = O(\varepsilon n)$. Our aim is to show that for n large enough we have

$$\mathbb{P}(L_1(H_{n,p}^r) > L) \leq C \frac{\varepsilon n}{L} \exp(-c\varepsilon^2 L),$$

for some constants $c, C > 0$ that may depend on all the choices made so far, just not (of course) on n .

We explore the random hypergraph $H_{n,p}^r$ as in Section 3, defining the filtration (\mathcal{F}_t) and random sequences (X_t) , (η_t) , (A_t) and (C_t) as in that section. Recall also the definition (18) of (D_t) and (Δ_t) . By Lemma 7,

$$\mathbb{E}[\eta_t \mid \mathcal{F}_{t-1}] \leq (r-1) \binom{n-t}{r-1} p \leq \frac{(n-t)^{r-1}}{(r-2)!} p = (1-\varepsilon)(1-t/n)^{r-1}.$$

Let

$$a_t = \varepsilon + (1-\varepsilon)t/n. \quad (20)$$

Then, crudely,

$$D_t = \mathbb{E}[\eta_t - 1 \mid \mathcal{F}_{t-1}] \leq (1-\varepsilon)(1-t/n)^{r-1} - 1 \leq (1-\varepsilon)(1-t/n) - 1 = -a_t. \quad (21)$$

Note that D_t is a random variable, but this deterministic bound holds with probability 1. Let

$$M_t = \sum_{i=1}^t \Delta_i = X_t - \sum_{i=1}^t D_i,$$

so (M_t) is a martingale with respect to (\mathcal{F}_t) . Since the sequence (a_t) is increasing, from (21) we see that for $t_1 < t_2$ we have

$$\begin{aligned} M_{t_2} - M_{t_1} &= X_{t_2} - X_{t_1} - \sum_{t=t_1+1}^{t_2} D_t \geq X_{t_2} - X_{t_1} + \sum_{t=t_1+1}^{t_2} a_t \\ &\geq X_{t_2} - X_{t_1} + (t_2 - t_1)a_{t_1}. \end{aligned} \quad (22)$$

Suppose that $L_1(H_{n,p}^r) > L$. Then there is some t (one less than the time at which we first start exploring a component with more than L vertices) such that $A_t = 0$, $A_{t+L} \geq 1$, and $C_{t+L} = C_{t+1} = C_t + 1$. Thus $X_{t+L} \geq X_t$. For $j \geq 0$ let \mathcal{E}_j denote the event that there is a t in the interval $jL \leq t < (j+1)L$ with $X_{t+L} \geq X_t$. What we have just noted tells us that

$$\mathbb{P}(L_1(H_{n,p}^r) > L) \leq \sum_{j=0}^{\infty} \mathbb{P}(\mathcal{E}_j),$$

so to complete the proof it suffices to bound the sum above.

If \mathcal{E}_j holds, then by definition there is a $t \in [jL, (j+1)L]$ such that $X_{t+L} \geq X_t$. Then, by (22), we have

$$M_{t+L} - M_t \geq La_t \geq La_{jL}. \quad (23)$$

Consider the martingale (M'_k) defined by $M'_k = M_{jL+k} - M_{jL}$, $k = 0, \dots, 2L$. If (23) holds then $M'_{t+L-jL} - M'_{t-jL} \geq La_{jL}$, so by the triangle inequality $\max\{|M'_{t+L-jL}|, |M'_{t-jL}|\} \geq La_{jL}/2$. Since $0 \leq t - jL \leq L$, we find that if \mathcal{E}_j holds, then

$$\max_{0 \leq k \leq 2L} |M'_k| \geq La_{jL}/2.$$

By Lemmas 5 and 7, the martingale differences $\Delta_t = M'_t - M'_{t-1}$, $1 \leq t \leq 2L$, satisfy the hypothesis (14) of Lemma 6 for some constant $C > 0$.⁵ We may of course assume that $C \geq 1/4$. Then, by Lemma 6, applied with $\ell = 2L$ and $y = La_{jL}/2 \leq L/2 \leq 2CL = C\ell$, we have

$$\mathbb{P}(\mathcal{E}_j) \leq 2 \exp\left(-\frac{y^2}{2C\ell}\right) = 2 \exp\left(-\frac{L^2 a_{jL}^2/4}{4CL}\right) = 2 \exp(-ca_{jL}^2 L)$$

where $c = 1/(16C)$ is a positive constant. From (20),

$$a_{jL} = \varepsilon + (1 - \varepsilon)jL/n \geq \max\{\varepsilon, (1 - \varepsilon)jL/n\}.$$

Recalling that $1 - \varepsilon$ is bounded away from zero by assumption, and considering $j < \varepsilon n/L + 1$ and $j \geq \varepsilon n/L + 1$ separately, it follows that

$$\sum_j \mathbb{P}(\mathcal{E}_j) \leq 2 \left\lceil \frac{\varepsilon n}{L} \right\rceil \exp(-c\varepsilon^2 L) + 2 \sum_{j \geq \varepsilon n/L + 1} \exp(-c' j^2 L^3/n^2), \quad (24)$$

for some constant $c' > 0$. Clearly

$$\begin{aligned} \sum_{j \geq \varepsilon n/L + 1} \exp(-c' j^2 L^3/n^2) &\leq \sum_{j \geq \varepsilon n/L + 1} \exp(-c' j \varepsilon L^2/n) \\ &\leq \exp(-c' \varepsilon^2 L) \sum_{j \geq 1} \exp(-c' j \varepsilon L^2/n), \end{aligned}$$

and if $x > 0$ then

$$0 < \sum_{j \geq 1} e^{-jx} = \frac{e^{-x}}{1 - e^{-x}} = \frac{1}{e^x - 1} < \frac{1}{x}.$$

It follows that

$$\sum_{j \geq \varepsilon n/L + 1} \exp(-c' j^2 L^3/n^2) = O\left(\frac{n}{\varepsilon L^2}\right) \exp(-c' \varepsilon^2 L).$$

Finally, by assumption $\varepsilon^2 L \rightarrow \infty$, so $n/(\varepsilon L^2) = o(\varepsilon n/L)$ and, from (24),

$$\sum_j \mathbb{P}(\mathcal{E}_j) = O\left(\frac{\varepsilon n}{L}\right) \exp(-\min\{c, c'\} \varepsilon^2 L),$$

completing the proof of Theorem 2. \square

⁵In principle, as we have phrased the argument, C may depend on the choice of $r \geq 2$ and also on the choice of the function $p(n)$. Since we assume $p(n) \leq (r-2)!n^{-r+1}$, it is not hard to see that C depends only on r .

5 Martingale approximation

In preparation for the proof of Theorem 4, we recall and extend some results from [7], approximating the random sequence $(X_t)_{t=0}^n$ by the sum of a certain deterministic sequence and a martingale.

For the rest of the paper we make the following assumption.

Standard Assumption 9. The integer $r \geq 2$ is fixed, $\varepsilon = \varepsilon(n)$ is a function satisfying $\varepsilon > 0$, $\varepsilon = O(1)$ and $\varepsilon^3 n \rightarrow \infty$. Furthermore, $\lambda = \lambda(n) = 1 + \varepsilon$ and $p = p(n) = \lambda(r-2)!n^{-r+1}$.

As discussed in Remark 3, all new constants introduced may depend on the choice of r and of the function $\varepsilon(n)$.

We start with some definitions, following the notation in [7]. Firstly, for $1 \leq t \leq n$, set

$$\alpha_t = p \binom{n-t-1}{r-2}.$$

Note that for all t we have $0 \leq \alpha_t \leq p \binom{n}{r-2} = O(1/n)$, so in particular $\max_t \alpha_t < 1/2$, say, if n is large enough. Let

$$\beta_t = \prod_{i=1}^t (1 - \alpha_i). \quad (25)$$

Then

$$\beta_t = \exp(-O(t/n)) \quad (26)$$

uniformly in $0 \leq t \leq n$. In particular, there is a constant $\beta > 0$ such that for n large enough,

$$\beta \leq \beta_t \leq 1$$

for all $0 \leq t \leq n$. Set

$$x_t = x_{n,t} = n - t - n\beta_t.$$

We showed in [7] that this deterministic sequence is a good approximation to the expected trajectory of the random process $(X_t)_{0 \leq t \leq n}$, and that (x_t) is in turn well approximated by a certain (convex) continuous function. We now give the details of these approximations.

Given an integer $r \geq 2$ and a positive real number λ , define the function $g = g_{r,\lambda}$ on $[0, 1]$ by

$$g(\tau) = g_{r,\lambda}(\tau) = 1 - \tau - \exp\left(-\frac{\lambda}{r-1}(1 - (1-\tau)^{r-1})\right). \quad (27)$$

Since λ depends on n , we have a different function g_n for each n . As usual, we suppress the dependence on n in the notation.

Lemma 10. *Suppose that our Standard Assumption 9, holds. Define a function $g = g_n$ as in (27). Then*

$$x_t = ng(t/n) + O(1) \quad (28)$$

uniformly in $0 \leq t \leq n$. Also,

$$g(0) = 0, \quad g'(0) = \lambda - 1, \quad g''(\tau) \leq 0, \quad \text{and} \quad \sup_{\tau \in [0,1]} |g''(\tau)| = O(1), \quad (29)$$

and, writing ρ for $\rho_{r,\lambda}$,

$$g(\rho) = 0 \quad \text{and} \quad g'(\rho) = -(1 - \lambda_*) = -\Theta(\varepsilon). \quad (30)$$

Proof. The proof is just elementary calculation. The calculations giving (28) and (29) are described in [7] (see equations (15) and (16) there), so we omit them. The final statement (30) follows easily from (4), simple calculations and, for the final equality, (3) (recalling that λ is bounded by assumption). \square

Corollary 11. *Suppose that our Standard Assumption 9 holds. Then there are constants $0 < c_2, c_3 < 1$ (which may depend as usual on the choice of the function $\varepsilon(n)$, but not on n) such that for all n and all $0 \leq \tau \leq c_2\varepsilon$ we have*

$$g(\tau) \geq c_3\varepsilon\tau, \quad g(\rho - \tau) \geq c_3\varepsilon\tau, \quad \text{and} \quad g(\rho + \tau) \leq -c_3\varepsilon\tau, \quad (31)$$

where $g = g_n$ is defined in (27).

Proof. Immediate from (29) and (30). \square

We resume our analysis of the exploration process, filtration (\mathcal{F}_t) , and random sequences (X_t) and (η_t) introduced in Section 3, next considering the martingale approximation to (X_t) . Define $\beta_t = \beta_{n,t}$ as in (25), and $\Delta_t = \eta_t - \mathbb{E}[\eta_t | \mathcal{F}_{t-1}]$ as in (18). Set

$$S_t = \sum_{i=1}^t \beta_i^{-1} \Delta_i \quad \text{and} \quad \tilde{X}_t = x_t + \beta_t S_t. \quad (32)$$

Then (S_t) is a martingale with respect to (\mathcal{F}_t) , since β_i is deterministic and Δ_i is \mathcal{F}_i -measurable with $\mathbb{E}[\Delta_i | \mathcal{F}_{i-1}] = 0$. It follows that (S_t) is unlikely to be very large.

Lemma 12. *Suppose that $r \geq 2$ is fixed and $p = p(n) = \Theta(n^{-r+1})$. For any $1 \leq t = t(n) \leq n$ and $y = y(n) = O(t)$ we have*

$$\mathbb{P}(\max_{i \leq t} |S_i| \geq y) \leq 2 \exp(-\Omega(y^2/t)).$$

Proof. Note that

$$S_i - S_{i-1} = \beta_i^{-1} \Delta_i = \beta_i^{-1} \eta_i - \mathbb{E}[\beta_i^{-1} \eta_i | \mathcal{F}_{i-1}],$$

and that $\beta_i \geq \beta > 0$. The result thus follows from Lemma 7, Lemma 5 (applied to the conditional distribution of $\beta_i^{-1} \eta_i$ given \mathcal{F}_{i-1}), and Lemma 6. \square

To close this section we quote Lemma 3 from [7]. This result shows that $\tilde{X}_t = x_t + \beta_t S_t$ is a very good approximation to X_t . Recall from Section 3 that C_t is the number of components that we have started to explore by time t .

Lemma 13. *Suppose that $r \geq 2$ is fixed and $p = p(n) = \Theta(n^{-r+1})$. Then there is a constant $c_1 > 0$ such that for all $n \geq 1$ we have*

$$|X_t - \tilde{X}_t| \leq c_1 t C_t / n \quad (33)$$

for $0 \leq t \leq n$. \square

6 Large deviations in the supercritical case

In this section we shall prove Theorem 4. First, we give a definition and two lemmas; these will be used in the next section also. Throughout this section we assume our Standard Assumption 9, that $p = p(n) = \lambda(n)(r-2)!n^{-r+1}$, where $\lambda(n) = 1 + \varepsilon(n)$ with $\varepsilon > 0$, $\varepsilon = O(1)$ bounded, and $\varepsilon^3 n \rightarrow \infty$ as $n \rightarrow \infty$. We explore the random hypergraph $H_{n,p}^r$ as in Section 3, and consider the filtration (\mathcal{F}_t) and random sequences (X_t) , (A_t) and (C_t) associated to this exploration. We shall also consider the deterministic sequence (x_t) , function g , and martingale (S_t) defined in Section 5.

Definition 14. Given a deterministic ‘cut-off’ $t_0 = t_0(n)$, let

$$\begin{aligned} Z &= -\inf\{X_t : t \leq t_0\}, \\ T_0 &= \inf\{t : X_t = -Z\} \quad \text{and} \\ T_1 &= \inf\{t : X_t = -Z - 1\}. \end{aligned}$$

Thus Z is the number of components completely explored by time t_0 , T_0 is the time at which we finish exploring the last such component, and T_1 is the time at which we finish exploring the next component. Note that $Z + 1 = C_{t_0+1}$, and that by definition $T_0 \leq t_0 < T_1$.

We continue following the strategy of [7], itself based on that of [6], modifying the calculations to obtain the tighter error bounds claimed in Theorem 4. The next lemma shows that we are unlikely to see too many components near the start of the process.

Lemma 15. *Suppose that our Standard Assumption 9 holds. Let $t_0 = t_0(n)$ satisfy $1 \leq t_0 \leq \min\{n/(2c_1), c_2 \varepsilon n\}$, where c_1 is the constant in Lemma 13 and c_2 is that in Corollary 11. Then for any $y = y(n)$ satisfying $y \rightarrow \infty$ and $y = O(t_0)$ we have*

$$\mathbb{P}(C_{t_0} \geq y) \leq 2 \exp(-\Omega(y^2/t_0)).$$

Proof. Define Z and T_0 as in Definition 14, and (S_t) as in (32). Let \mathcal{A} be the event

$$\mathcal{A} = \{|S_t| \leq y/4 \text{ for all } 0 \leq t \leq t_0\}.$$

By Lemma 12 we have $\mathbb{P}(\mathcal{A}^c) \leq 2 \exp(-\Omega(y^2/t_0))$.

Since $t_0 \leq n/(2c_1)$, Lemma 13 implies that for $t \leq t_0$ we have $|X_t - \tilde{X}_t| \leq C_t/2$. Since $X_{T_0} = -Z$, $C_{T_0} = Z$ and $T_0 \leq t_0$, it follows that $\tilde{X}_{T_0} \leq -Z/2$.

Since $T_0 \leq t_0 \leq c_2 \varepsilon n$, from (31) we have $g(T_0/n) \geq 0$. By (28) it follows that $x_{T_0} \geq -O(1)$, so from (32) we have $\beta_{T_0} S_{T_0} = \tilde{X}_{T_0} - x_{T_0} \leq -Z/2 + O(1)$. Hence $S_{T_0} \leq -Z/2 + O(1)$. By the definition of \mathcal{A} , it follows that whenever \mathcal{A} holds, then

$$C_{t_0} \leq Z + 1 \leq 2|S_{T_0}| + O(1) \leq y/2 + O(1) < y$$

for n large enough. \square

By our Standard Assumption 9, we have $\lambda = \lambda(n) = O(1)$ and $\lambda > 1$. Hence, by (3), there is a constant c_0 such that

$$\lambda_* \leq 1 - c_0 \varepsilon. \quad (34)$$

There exist a constant c and an integer n_0 such that for all $n \geq n_0$ we have

$$c\varepsilon n \leq \min\{c_0 \varepsilon n / (4(r-1)\lambda), c_2 \varepsilon n, n/(2c_1), \rho_{r,\lambda} n/4\}, \quad (35)$$

where c_1 is as in Lemma 13 and c_2 as in Corollary 11. Indeed, $\rho_{r,\lambda} = \Theta(\varepsilon)$ from (6), and λ and ε are $O(1)$ by assumption, so all terms on the right are $\Omega(\varepsilon n)$. From now on, we shall always assume $n \geq n_0$. In addition to the function $\varepsilon(n)$, we fix a function $\omega(n)$ satisfying

$$\omega = \omega(n) \rightarrow \infty \quad \text{and} \quad \omega \leq c\sqrt{\varepsilon^3 n} \quad (36)$$

with c as in (35). Any new constants introduced may depend on the choice of $\omega(n)$ as well as that of r and $\varepsilon(n)$.

We shall work with the ‘initial cut-off’

$$t_0 = \omega \sqrt{n/\varepsilon}, \quad (37)$$

ignoring the rounding to integers, which causes no complications. Since $n \geq n_0$, from (35) we have

$$t_0 \leq \min\{c_0 \varepsilon n / (4(r-1)\lambda), c_2 \varepsilon n, n/(2c_1), \rho_{r,\lambda} n/4\}. \quad (38)$$

Recalling (6), set

$$t_1 = \rho_{r,\lambda} n = \Theta(\varepsilon n). \quad (39)$$

Note for later that, from (30), $g(\rho_{r,\lambda}) = 0$, so (28) implies that

$$x_{t_1} = O(1). \quad (40)$$

The convex function $g(\tau)$ is positive on $(0, \rho)$ and passes through zero at $\tau = \rho$. Hence, roughly speaking, we expect that near $t = t_1 = \rho n$ the random trajectory (X_t) will be close to 0, and that around this point it will reach a new record low value. We shall show that with high probability this happens within t_0 steps of t_1 .

Lemma 16. *Suppose that our Standard Assumption 9 holds and that $\omega(n)$ satisfies (36). Defining t_0 and t_1 as above and T_1 as in Definition 14, we have*

$$\mathbb{P}(t_1 - t_0 \leq T_1 \leq t_1 + t_0) = 1 - \exp(-\Omega(\omega^2)).$$

Proof. As above, let c_1 and c_3 be the constants in Lemma 13 and Corollary 11, and define Z , T_0 and T_1 as in Definition 14. Let \mathcal{B}_1 be the event

$$\mathcal{B}_1 = \{C_{t_0} \leq c_3 \varepsilon t_0 / (4 \max\{c_1, 1\})\}.$$

By Lemma 15,

$$\mathbb{P}(\mathcal{B}_1^c) \leq 2 \exp(-\Omega(\varepsilon^2 t_0)) = 2 \exp(-\Omega(\omega \sqrt{\varepsilon^3 n})) = \exp(-\Omega(\omega^2)).$$

Let \mathcal{B}_2 be the event

$$\mathcal{B}_2 = \{|S_t| \leq c_3 \varepsilon t_0 / 5 \text{ for all } 0 \leq t \leq t_1 + t_0\}.$$

Since $t_1 = \Theta(\varepsilon n)$ and $c_3 \varepsilon t_0 / 5 = O(\varepsilon^2 n) = O(\varepsilon n)$, by Lemma 12 we have

$$\mathbb{P}(\mathcal{B}_2^c) \leq 2 \exp\left(-\Omega(\varepsilon^2 t_0^2 / (\varepsilon n))\right) = \exp(-\Omega(\omega^2)),$$

since $t_0 = \omega \sqrt{n/\varepsilon}$.

To complete the proof of the lemma we shall establish the deterministic claim that, for n large enough,

$$\mathcal{B}_1 \cap \mathcal{B}_2 \implies t_1 - t_0 \leq T_1 \leq t_1 + t_0. \quad (41)$$

To see this, suppose that \mathcal{B}_1 and \mathcal{B}_2 hold. For $t \leq \min\{T_1, t_1 + t_0\}$ relations (28) and (33) and the definition (32) give

$$\begin{aligned} |ng(t/n) - X_t| &\leq |ng(t/n) - x_t| + |\tilde{X}_t - x_t| + |X_t - \tilde{X}_t| \\ &\leq O(1) + \beta_t |S_t| + c_1 C_{t_0} \\ &\leq O(1) + c_3 \varepsilon t_0 / 5 + c_3 \varepsilon t_0 / 4, \end{aligned}$$

using $\beta_t \leq 1$ and the assumption that $\mathcal{B}_1 \cap \mathcal{B}_2$ holds in the last step. Hence

$$|ng(t/n) - X_t| \leq c_3 \varepsilon t_0 / 2 \quad (42)$$

for n large enough.

Suppose for a contradiction that $T_1 < t_1 - t_0$. Then (42) applies for $t = T_1 \in [t_0, t_1 - t_0]$. From (29) the function g is concave. Hence, from (31), we have $ng(t/n) \geq c_3 \varepsilon t_0$ for $t \in [t_0, t_1 - t_0]$, so

$$0 > -(Z + 1) = X_{T_1} \geq c_3 \varepsilon t_0 - c_3 \varepsilon t_0 / 2 > 0,$$

a contradiction. Thus $T_1 \geq t_1 - t_0$.

Suppose instead that $T_1 > t_1 + t_0$. Then (42) applies with $t = t_1 + t_0$. Hence, by the definition of $T_1 = \inf\{t : X_t = -Z - 1\}$, the last bound in (31) and (42),

$$-Z \leq X_{t_1+t_0} \leq -c_3 \varepsilon t_0 + c_3 \varepsilon t_0 / 2 = -c_3 \varepsilon t_0 / 2.$$

Thus $C_{t_0} \geq Z \geq c_3 \varepsilon t_0 / 2$, contradicting the assumption that \mathcal{B}_1 holds. This completes the proof of (41) and thus of the lemma. \square

We are now ready to prove Theorem 4.

Proof of Theorem 4. The conditions of Theorem 4 include our Standard Assumption 9, which we thus assume. The conditions also state that $\omega = \omega(n)$ satisfies $\omega \rightarrow \infty$ and $\omega = O(\sqrt{\varepsilon^3 n})$. To apply the lemmas above we need the additional condition (36), i.e., $\omega \leq c\sqrt{\varepsilon^3 n}$ with c as in (35). We may impose this without problems since, in proving (8), we may reduce ω by a constant factor, changing the implicit constant to compensate. As in (37) and (39), we set $t_0 = \omega\sqrt{n/\varepsilon}$ and $t_1 = \rho n$. Our first aim is to show that

$$\mathbb{P}\left(|L_1(H_{n,p}^r) - t_1| > 2t_0\right) = \exp(-\Omega(\omega^2)),$$

which (changing ω by an irrelevant factor of 3, say) is exactly (8).

Let \mathcal{C} be the component that we explore from time $T_0 + 1$ to time T_1 . We have $T_0 \leq t_0$ by definition, while from Lemma 16, with probability $1 - \exp(-\Omega(\omega^2))$ we have $|T_1 - t_1| \leq t_0$. Thus \mathcal{C} has between $t_1 - 2t_0$ and $t_1 + t_0$ vertices. Moreover, since $t_0 \leq t_1/4$ by (38), any component explored before \mathcal{C} has at most $t_0 < |\mathcal{C}|$ vertices. To complete the proof of (8) it remains to show that with very high probability no component explored after time T_1 has more than $|\mathcal{C}|$ vertices.

Stopping the exploration at time T_1 , the unexplored part of $H_{n,p}^r$ has exactly the distribution of $H_{n-T_1,p}^r$. We shall apply Theorem 2 to this hypergraph; to obtain the result we need we must show that its ‘branching factor’

$$\Lambda = (n - T_1)^{r-1} p / (r - 2)! = \lambda(1 - T_1/n)^{r-1}$$

is $1 - \Omega(\varepsilon)$. Since $T_1 \geq t_1 - 2t_0 = \rho n - 2t_0$, we have

$$\begin{aligned} \Lambda &\leq \lambda(1 - \rho + 2t_0/n)^{r-1} \\ &\leq \lambda(1 - \rho)^{r-1} + 2(r-1)\lambda t_0/n \leq \lambda(1 - \rho)^{r-1} + c_0\varepsilon/2, \end{aligned}$$

using the first condition in (38) in the last step. By (2) and (4) we have

$$\lambda(1 - \rho)^{r-1} = \lambda(1 - \rho_{r,\lambda})^{r-1} = \lambda(1 - \rho_\lambda) = \lambda_*,$$

so, recalling (34),

$$\Lambda \leq 1 - c_0\varepsilon + c_0\varepsilon/2 = 1 - \Omega(\varepsilon).$$

Hence, by Theorem 2 (applied with $n - T_1 = \Theta(n)$ in place of n and $1 - \Lambda = \Omega(\varepsilon)$ in place of ε , and with $L = |\mathcal{C}| = \Theta(\varepsilon n)$), with probability $1 - \exp(-\Omega(\varepsilon^3 n)) = 1 - \exp(-\Omega(\omega^2))$, the hypergraph $H_{n-T_1,p}^r$ has no component with at least as many vertices as \mathcal{C} . It follows that

$$\mathbb{P}(\mathcal{C} \text{ is the unique largest component of } H_{n,p}^r) = 1 - \exp(-\Omega(\omega^2)), \quad (43)$$

completing the proof of (8).

The bound (9) follows easily from (8), Theorem 2 and a standard duality argument; let us outline this briefly. Condition not only on the number L_1

of vertices in the largest component \mathcal{L}_1 of $H = H_{n,p}^r$, but also on the vertex set of this component. The conditional distribution of $H^- = H - \mathcal{L}_1$ is then that of $H_{n-L_1,p}^r$ conditioned on a monotone decreasing event (that there is no component with more than L_1 vertices, plus an extra condition to deal with the possibility of ties; see, e.g., [9, Section 8]). Taking $\omega = c\sqrt{\varepsilon^3 n}$ with c as in (35), so $t_0 = c\varepsilon n$, as above we have $|L_1 - t_1| \leq 2t_0$ with probability $1 - \exp(-\Omega(\omega^2)) = 1 - \exp(-\Omega(\varepsilon^3 n))$. It follows as above that the ‘branching factor’ of $H_{n-L_1,p}^r$ is $1 - \Omega(\varepsilon)$ (in fact $1 - \Theta(\varepsilon)$, but we only need an upper bound). Since conditioning on a decreasing event can only decrease the probability of having a component of more than a given size, we may apply Theorem 2 to see that $\mathbb{P}(L_1(H^-) \geq L) \leq C\varepsilon n L^{-1} \exp(-\varepsilon^2 L/C)$. By assumption $L = O(\varepsilon n)$, so increasing C if necessary we may absorb the additional $\exp(-\Omega(\varepsilon^3 n))$ error probability into the expression in (9). \square

7 Bivariate central limit theorem

7.1 Martingale CLTs

In this section we shall prove Theorem 1. For this we need a martingale central limit theorem. Although the result we need is well known, there are many possible variants, and it is not so easy to find a form convenient for combinatorial applications in the literature; the following is (up to a trivial change noted below) Corollary 1 of Brown and Eagleson [10]; we thank Svante Janson for supplying this reference.

Lemma 17. *For each n , let $(M_{n,t})_{t=0}^{k(n)}$ be a martingale with respect to a filtration $(\mathcal{F}_{n,t})$, with $M_{n,0} = 0$ for all n . Writing $\Delta_{n,t} = M_{n,t} - M_{n,t-1}$, let*

$$V_n = \sum_{t=1}^{k(n)} \text{Var}[\Delta_{n,t} \mid \mathcal{F}_{t-1}]$$

be the sum of the conditional variances of the increments. Suppose that

$$V_n \xrightarrow{\mathbb{P}} \sigma^2 \tag{44}$$

as $n \rightarrow \infty$, where $\xrightarrow{\mathbb{P}}$ denotes convergence in probability. Suppose also that for any constant $\delta > 0$ we have

$$\sum_{t=1}^{k(n)} \mathbb{E}[\Delta_{n,t}^2 1_{|\Delta_{n,t}| \geq \delta} \mid \mathcal{F}_{t-1}] \xrightarrow{\mathbb{P}} 0. \tag{45}$$

Then $M_{n,k(n)} \xrightarrow{d} N(0, \sigma^2)$.

The only difference between the statement above and Corollary 1 in [10] is that there $k(n) = n$, i.e., the array is triangular. As noted in [10], this loses no

generality, since n plays no role in the result above except as an index. (Thus we may pad rows with zeros and/or add zero rows to transform a general array into a triangular one.) Condition (45) is the ‘Lindeberg’ condition, in a conditional form.

Lemma 17 extends without problems to higher dimensions, i.e., to simultaneous convergence of several martingales; we shall need the following two-dimensional version.

Lemma 18. *For each $n \geq 1$ and $j \in \{1, 2\}$ let $(M_{j,n,t})_{t=0}^{k(n)}$ be a martingale with respect to a filtration $(\mathcal{F}_{n,t})$, with $M_{j,n,0} = 0$. Writing $\Delta_{j,n,t} = M_{j,n,t} - M_{j,n,t-1}$, suppose that the Lindeberg condition (45) holds for $j = 1$ and for $j = 2$, and that*

$$V_{j,n} = \sum_{t=1}^{k(n)} \text{Var}[\Delta_{j,n,t} \mid \mathcal{F}_{t-1}] \xrightarrow{\text{P}} \sigma_j^2 \quad (46)$$

for $j = 1, 2$ and

$$V_{1,2,n} = \sum_{t=1}^{k(n)} \text{Cov}[\Delta_{1,n,t}, \Delta_{2,n,t} \mid \mathcal{F}_{t-1}] \xrightarrow{\text{P}} \sigma_{1,2}. \quad (47)$$

Then $(M_{1,n,k(n)}, M_{2,n,k(n)})$ converges in distribution to a bivariate normal distribution (N_1, N_2) with $N_j \sim N(0, \sigma_j^2)$ and $\text{Cov}[N_1, N_2] = \sigma_{1,2}$.

Proof. By the Cramér–Wold Theorem [11] (see e.g., Billingsley [4, Theorem 29.4]), a sequence of random vectors converges in distribution to a given random vector if and only if all the one-dimensional projections converge in distribution. Thus it suffices to show that for any constants α and β , $\alpha M_{1,n,k(n)} + \beta M_{2,n,k(n)}$ converges in distribution to a Gaussian with mean 0 and the appropriate variance, namely $\alpha^2 \sigma_1^2 + 2\alpha\beta \sigma_{1,2} + \beta^2 \sigma_2^2$. This follows by applying Lemma 17 to $M_{n,t} = \alpha M_{1,n,t} + \beta M_{2,n,t}$. Indeed, using the formula $\text{Var}[\alpha X + \beta Y] = \alpha^2 \text{Var}[X] + 2\alpha\beta \text{Cov}[X, Y] + \beta^2 \text{Var}[Y]$, which applies just as well to conditional variances, the variance condition (44) follows from the assumptions on $V_{1,n}$, $V_{2,n}$ and $V_{1,2,n}$. In establishing the Lindeberg condition we may assume without loss of generality that $\alpha = \beta = 1$. It is easy to see that

$$(X + Y)^2 1_{|X+Y| \geq 2\delta} \leq 4X^2 1_{|X| \geq \delta} + 4Y^2 1_{|Y| \geq \delta}.$$

(Indeed, the first or second term on its own is an upper bound according to whether $|X| \geq |Y|$ or $|X| < |Y|$.) Hence the Lindeberg condition for $(M_{1,n,t} + M_{2,n,t})$ follows from the same condition for $(M_{1,n,t})$ and $(M_{2,n,t})$. \square

7.2 Application to $H_{n,p}^r$

Let N_t denote the nullity of the hypergraph formed by all edges exposed within the first t steps of the exploration described in Section 3. Since nullity is additive over components, the component \mathcal{C} explored between time T_0 and T_1 has nullity $N_{T_1} - N_{T_0}$. We now study the joint distribution of this quantity and $T_1 - T_0$.

In this section we assume the following stronger form of our Standard Assumption 9.

Strong Assumption 19. The integer $r \geq 2$ is fixed, $\varepsilon = \varepsilon(n)$ is a function satisfying $\varepsilon > 0$, $\varepsilon \rightarrow 0$ and $\varepsilon^3 n \rightarrow \infty$. Furthermore, $\lambda = \lambda(n) = 1 + \varepsilon$ and $p = p(n) = \lambda(r-2)!n^{-r+1}$.

As usual, we consider the exploration process defined in Section 3, and the associated random sequences (X_t) , (A_t) and (η_t) . Recall that \mathcal{A}_t denotes the set of active vertices at time t , and $A_t = |\mathcal{A}_t|$. Let E_t be the set of edges revealed during step t . Then, whether or not we start exploring a new component in step t , we have

$$N_t - N_{t-1} = (r-1)|E_t| - \left| \bigcup_{e \in E_t} (e \setminus \{v_t\}) \setminus \mathcal{A}_{t-1} \right|.$$

Indeed, we have added $|E_t|$ edges to the ‘revealed graph’, and the vertices in the union above, which were previously isolated, have now been connected to v_t . (If $\mathcal{A}_{t-1} \neq \emptyset$, then the vertices in \mathcal{A}_{t-1} were already in the same component as v_t .)

Let ξ_t be the number of vertices in $\mathcal{A}_{t-1} \setminus \{v_t\}$ included in one or more edges in E_t , and set

$$\zeta_t = \sum_{e, f \in E_t} |(e \cap f) \setminus \{v_t\}|,$$

where the sum is over all unordered pairs of distinct edges in E_t . Then

$$\xi_t \leq N_t - N_{t-1} \leq \xi_t + \zeta_t. \quad (48)$$

Considering the number of triples (e, w, f) where e and f are edges tested at step t and $w \in (e \cap f) \setminus \{v_t\}$, by linearity of expectation we have

$$\mathbb{E}[\zeta_t] \leq \binom{n-t}{r-1} (r-1) \binom{n-t-1}{r-2} p^2 = O(n^{-1}). \quad (49)$$

As we shall see later, this implies that we can essentially ignore ζ_t , and consider only the ξ_t .

Let $A'_t = |\mathcal{A}_t \setminus \{v_{t+1}\}|$ be the number of active vertices after t steps other than v_{t+1} . Thus $A'_t = A_t - 1$ if $A_t \neq 0$ and $A'_t = 0$ if $A_t = 0$. In particular, $A'_t = A_t + O(1)$. Let $\pi_t = \pi_{1,t}$ be the probability that a given vertex u not among v_1, \dots, v_t is contained in $\bigcup_{e \in E_t} e$. (This quantity is denoted π_1 in [7].) Since there are $c_t = \binom{n-t-1}{r-2}$ edges tested at step t that contain u , we have

$$\begin{aligned} \pi_t &= 1 - (1-p)^{c_t} = pc_t + O(p^2 c_t^2) = pc_t + O(1/n^2) \\ &= \lambda(1-t/n)^{r-2}/n + O(1/n^2), \end{aligned} \quad (50)$$

recalling that $p = \lambda(r-2)!n^{-r+1}$, with $\lambda = \lambda(n) = 1 + \varepsilon$. In particular, for $t = O(\varepsilon n)$ we have

$$\pi_t = (1 + O(\varepsilon))/n. \quad (51)$$

From the definition of ξ_{t+1} and the linearity of expectation,

$$\mathbb{E}[\xi_{t+1} \mid \mathcal{F}_t] = A'_t \pi_{t+1} = A_t \pi_t + O(1/n). \quad (52)$$

Let $\pi_{2,t}$ be the probability that two given (distinct) vertices $u, w \in [n] \setminus \{v_1, \dots, v_t\}$ are contained in $\bigcup_{e \in E_t} e$. Considering the cases where u, w are in the same $e \in E_t$ and in distinct $e, f \in E_t$ it is easy to see that

$$\pi_{2,t} \leq p \binom{n-t-2}{r-3} + \pi_{1,t}^2 = O(1/n^2).$$

It follows that

$$\mathbb{E}[\xi_{t+1}(\xi_{t+1} - 1) \mid \mathcal{F}_t] = A'_t(A'_t - 1)\pi_{2,t+1} = O((A_t/n)^2). \quad (53)$$

Similarly,

$$\mathbb{E}[\xi_{t+1}\eta_{t+1} \mid \mathcal{F}_t] = A'_t(n-t-1-A'_t)\pi_{2,t+1} \leq nA'_t\pi_{2,t+1} = O(A_t/n). \quad (54)$$

These bounds are enough to extend the argument we used in [7] to prove a univariate central limit theorem for $L_1(H_{n,p}^r)$, to prove Theorem 1. Roughly speaking, we shall use the estimates above to decompose (N_t) into two parts. The first part is a martingale that is essentially independent of (X_t) , and the second depends on (X_t) in a simple way. Then we can apply Lemma 18 to prove the result. As usual in this type of argument, we must calculate the expectation terms very accurately, but it suffices to estimate the variance terms within a factor of $1 + o(1)$.

For the rest of the paper we consider $\varepsilon = \varepsilon(n)$ satisfying our Strong Assumption 19, and a function $\omega = \omega(n)$ satisfying

$$\omega \rightarrow \infty \quad \text{with} \quad \omega = o((\varepsilon^3 n)^{1/6}).$$

Define $t_1 = \rho_{r,\lambda} n$ as before (in (39)), recalling that $t_1 = O(\varepsilon n)$. As before, set

$$t_0 = \omega \sqrt{n/\varepsilon}.$$

In addition, define (S_t) as in (32), and Z, T_0 and T_1 as in Definition 14. We shall work with these quantities for the rest of the paper.

As usual, we say that an event $\mathcal{E} = \mathcal{E}(n)$ holds *whp* (with high probability), if $\mathbb{P}(\mathcal{E}(n)) \rightarrow 1$ as $n \rightarrow \infty$.

Lemma 20. *Let*

$$\begin{aligned} \mathcal{E}_1 &= \{Z \leq \omega^{-1} \sqrt{\varepsilon n} \text{ and } T_0 \leq \omega^{-1} \sqrt{n/\varepsilon}\}, \\ \mathcal{E}_2 &= \left\{ \max_{t \leq t_1 + t_0} |S_t| \leq \omega \sqrt{\varepsilon n} \right\}, \text{ and} \\ \mathcal{E}_3 &= \{t_1 - t_0 \leq T_1 \leq t_1 + t_0\}, \end{aligned}$$

and set $\mathcal{E} = \mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3$. Then \mathcal{E} holds whp.

Proof. Under our Standard Assumption 9, which of course is implied by our Strong Assumption 19, we proved in [7] that \mathcal{E}_1 holds whp – see the paragraph after (20) on page 448 of [7].

For \mathcal{E}_2 apply Lemma 12, noting that $t_1 + t_0 = \Theta(\varepsilon n)$, and that $\omega\sqrt{\varepsilon n} = O(\varepsilon n)$, since $\omega = o((\varepsilon^3 n)^{1/6}) = O(\sqrt{\varepsilon^3 n}) = O(\sqrt{\varepsilon n})$, with room to spare.

Finally, \mathcal{E}_3 holds whp by Lemma 16. \square

For the rest of the paper the events \mathcal{E}_i and \mathcal{E} are as above. In our next lemma we establish some consequences of the event \mathcal{E} holding. Let

$$I = [t_1 - t_0, t_1 + t_0].$$

Lemma 21. *If \mathcal{E} holds then*

- (i) $C_{t_1+t_0} = O(\omega\sqrt{\varepsilon n})$,
- (ii) $\max_{t \in I} A_t, \max_{t \leq t_0} A_t = O(\omega\sqrt{\varepsilon n})$, and
- (iii) $\max_{t \leq t_0+t_1} A_t = O(\varepsilon^2 n)$.

Proof. Suppose $\mathcal{E} = \mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3$ holds. Since $C_{T_1} = Z + 1$ and by assumption $T_1 \in I$ we have, very crudely, that

$$C_{t_1+t_0} \leq C_{T_1} + (t_1 + t_0 - T_1) \leq Z + 1 + 2t_0 \leq 3\omega\sqrt{n/\varepsilon}.$$

Since $t_1 + t_0 = O(\varepsilon n)$, it follows from Lemma 13 that $|X_t - \tilde{X}_t| = O(\omega\sqrt{\varepsilon n})$, uniformly in $t \leq t_1 + t_0$. Recalling (29) and (30), for $t \leq t_0$ or $t \in I$ we have $g(t/n) = O(\varepsilon t_0/n)$ and hence $x_t = ng(t/n) + O(1) = O(\varepsilon t_0) = O(\omega\sqrt{\varepsilon n})$. Since \mathcal{E}_2 holds it follows that

$$|X_t| \leq |x_t| + |S_t| + |X_t - \tilde{X}_t| = O(\omega\sqrt{\varepsilon n}), \quad (55)$$

uniformly in $t \in [0, t_0] \cup I$. Let $T = \max\{t \in I : A_t = 0\}$ be the last time that we finish exploring a component within the interval $t \in I$; this makes sense since $T_1 \in I$. Then $A_T = 0$ so $C_T = -X_T$. Hence $C_{t_1+t_0} \leq C_T + 1 = O(\omega\sqrt{\varepsilon n})$, proving (i).

For $t \leq t_1 + t_0$ we have $A_t \leq |X_t| + C_t \leq |X_t| + C_{t_1+t_0}$. Hence (ii) follows from (i) and (55). Recalling from (6) that $\rho = \Theta(\varepsilon)$, from (29) it is easy to check that $\sup_{\tau \leq \rho} g(\tau) = O(\varepsilon^2)$. The argument for (iii) is very similar to that for (ii), using this estimate to show that $x_t = O(\varepsilon^2 n)$ for $t \leq t_1 + t_0$, in place of the tighter bound $O(\omega\sqrt{\varepsilon n})$ we used in case (ii). \square

In the rest of the paper we use the following standard notation for probabilistic asymptotics: given random variables (Z_n) and a function $f(n) > 0$, we write $Z_n = o_p(f(n))$ if $Z_n/f(n)$ converges to 0 in probability as $n \rightarrow \infty$. We (briefly) write $Z_n = O_p(1)$ to mean that Z_n is bounded in probability.

Lemma 22. *Let \mathcal{C} be the component explored between times T_0 and T_1 . Then*

$$n(\mathcal{C}) = N_{T_1} - N_{T_0} = \sum_{t=1}^{t_1} \xi_t + o_p(\sqrt{\varepsilon^3 n}).$$

Proof. That $n(\mathcal{C}) = N_{T_1} - N_{T_0}$ is immediate from the additivity of nullity over components. From (48) we have

$$\left| N_{T_1} - N_{T_0} - \sum_{t=T_0+1}^{T_1} \xi_t \right| \leq \sum_{t=T_0+1}^{T_1} \zeta_t \leq \sum_{t=1}^n \zeta_t = O_p(1) = o_p(\sqrt{\varepsilon n}),$$

where for the second-last step we used the expectation bound (49) and Markov's inequality.

Since $\xi_t \geq 0$ and $T_0 \leq t_0$ hold by definition, whenever \mathcal{E}_3 holds we have

$$\left| \sum_{t=T_0+1}^{T_1} \xi_t - \sum_{t=1}^{t_1} \xi_t \right| \leq \sum_{t=1}^{t_0} \xi_t + \sum_{t=t_1-t_0+1}^{t_1+t_0} \xi_t = B,$$

say. By Lemma 21(ii) and (52), since $\max_t \pi_t = O(1/n)$, we have

$$\mathbb{E}[1_{\mathcal{E}} B] \leq 3t_0 O(\omega \sqrt{\varepsilon n})/n = O(\omega^2) = o(\sqrt{\varepsilon^3 n}).$$

Since \mathcal{E} holds whp, it follows that $B = o_p(\sqrt{\varepsilon^3 n})$. \square

Following (a modified form of) the strategy in [19], we now consider the Doob decomposition of the sequence $(\sum_{i=1}^t \xi_i)$. More precisely, writing (as before) \mathcal{F}_t for the σ -algebra generated by all information revealed up to step t of the exploration process, set

$$D_t^* = \mathbb{E}[\xi_t \mid \mathcal{F}_{t-1}] \quad \text{and} \quad \Delta_t^* = \xi_t - D_t^*. \quad (56)$$

Lemma 23. *Define $\rho^* = \rho_{r,\lambda}^*$ as in (5). Then*

$$\sum_{t=1}^{t_1} D_t^* = \rho^* n + \sum_{t=1}^{t_1} \gamma_t \Delta_t^* + o_p(\sqrt{\varepsilon^3 n}),$$

where the γ_t are deterministic and satisfy

$$\gamma_t = \frac{t_1 - t}{n} + O(\varepsilon^2), \quad (57)$$

uniformly in $1 \leq t \leq t_1$.

Proof. From Lemma 13 and the definitions $X_t = A_t - C_t$ and $\tilde{X}_t = x_t + \beta_t S_t$, we have

$$A_t = X_t + C_t = \tilde{X}_t + (X_t - \tilde{X}_t) + C_t = \tilde{X}_t + O(C_t) = x_t + \beta_t S_t + O(C_t).$$

Hence, recalling (52),

$$D_{t+1}^* = \mathbb{E}[\xi_{t+1} \mid \mathcal{F}_t] = x_t \pi_t + \beta_t S_t \pi_t + O(C_t/n) + O(1/n),$$

so

$$\sum_{t=1}^{t_1} D_t^* = \sum_{t=0}^{t_1-1} x_t \pi_t + \sum_{t=0}^{t_1-1} \beta_t S_t \pi_t + O(E), \quad (58)$$

where $E = \sum_{t=0}^{t_1-1} C_t/n$. We shall estimate the terms on the right-hand side of (58) in reverse order.

Whenever the event $\mathcal{E} = \mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3$ defined in Lemma 20 holds, for $t \leq t_1 - t_0$ we have $C_t \leq Z + 1 = O(\omega^{-1}\sqrt{\varepsilon n})$. Also, by Lemma 21(i), for $t_1 - t_0 < t \leq t_1$ we have $C_t \leq C_{t_0+t_1} = O(\omega\sqrt{\varepsilon n})$. Since \mathcal{E} holds whp, it follows that whp

$$E = O\left((t_1 - t_0)\omega^{-1}\sqrt{\varepsilon n}/n + t_0\omega\sqrt{\varepsilon n}/n\right) = O\left(\omega^{-1}\sqrt{\varepsilon^3 n} + \omega^2\right) = o\left(\sqrt{\varepsilon^3 n}\right).$$

Thus $E = o_p(\sqrt{\varepsilon^3 n})$.

Turning to the middle term in the right-hand side of (58), let

$$\gamma_i = \sum_{t=i}^{t_1-1} \frac{\beta_t \pi_t}{\beta_i}.$$

From the definition (32) of S_t , we have

$$\sum_{t=0}^{t_1-1} \beta_t S_t \pi_t = \sum_{t=1}^{t_1-1} \sum_{i=1}^t \frac{\beta_t \pi_t}{\beta_i} \Delta_i = \sum_{i=1}^{t_1-1} \gamma_i \Delta_i.$$

Recalling (26) and (51),

$$\gamma_i = \sum_{t=i}^{t_1-1} \frac{1 + O(\varepsilon)}{n} = \frac{t_1 - i}{n} + O(\varepsilon^2).$$

Finally, turning to the main term in (58), we shall make use of the function $g = g(\tau)$ defined in (27), and the related function

$$h(\tau) = g(\tau)\lambda(1 - \tau)^{r-2}.$$

From the definition (27) of g and relations (28) and (50) we have

$$x_t \pi_t = h(t/n) + O(1/n). \tag{59}$$

An elementary calculation shows that

$$\begin{aligned} \int_0^\rho h(\tau) d\tau &= \left[\exp\left(-\frac{\lambda}{r-1}(1 - (1-\tau)^{r-1})\right) - \frac{\lambda(1-\tau)^r}{r} \right]_{\tau=0}^\rho \\ &= \left[1 - \tau - g(\tau) - \frac{\lambda(1-\tau)^r}{r} \right]_{\tau=0}^\rho, \end{aligned}$$

substituting in the definition (27) of g for the second step. Recalling from (29) and (30) that $g(\rho) = 0 = g(0)$, it follows that

$$\int_0^\rho h(\tau) d\tau = -\rho + \frac{\lambda}{r}(1 - (1-\rho)^r) = \rho^*,$$

where $\rho^* = \rho_{r,\lambda}^*$ is defined in (5). It is easy to check that h' is uniformly bounded on $[0, 1]$; it thus follows easily from (59) that

$$\sum_{t=0}^{t_1-1} x_t \pi_t = n \int_0^\rho h(\tau) d\tau + O(t_1/n) = \rho^* n + o(1).$$

Combining the estimates just proved, Lemma 23 follows from (58). \square

We note the following simple corollary for later.

Corollary 24. *We have*

$$\sum_{t=1}^{t_1} D_t^* = (1 + o_p(1)) \rho^* n.$$

Proof. Recall that Δ_t is \mathcal{F}_t -measurable with $\mathbb{E}[\Delta_t \mid \mathcal{F}_{t-1}] = 0$. Hence,

$$\text{Var} \left[\sum_{t=1}^{t_1} \gamma_t \Delta_t \right] = \sum_{t=1}^{t_1} \gamma_t^2 \text{Var}[\Delta_t] = O(\varepsilon^3 n),$$

since there are $t_1 = O(\varepsilon n)$ terms, each γ_t is $O(\varepsilon)$ from (57), and, from Lemma 8, $\text{Var}[\Delta_t] = O(1)$. The result thus follows from Lemma 23 and the observation that $\sqrt{\varepsilon^3 n} = o(\rho^* n)$, recalling (6) and that $\varepsilon^3 n \rightarrow \infty$. \square

After this preparation, we are ready to complete the proof of Theorem 1.

Proof of Theorem 1. Suppose that our Strong Assumption 19 holds, and that $\omega(n)$ satisfies (36). Define t_0 and t_1 as in (37) and (39), and Z , T_0 and T_1 as in Definition 14.

Let \mathcal{C} be the component of $H_{n,p}^r$ explored between times T_0 and T_1 . By (43), whp \mathcal{C} is the unique component \mathcal{L}_1 of $H_{n,p}^r$ with the most vertices. We need one final result from [7], namely Eq. (21) there, which says that

$$T_1 = t_1 + \tilde{X}_{t_1}/(1 - \lambda_*) + o_p(\sqrt{n/\varepsilon}). \quad (60)$$

(The quantity σ_0 appearing in [7] is simply $\sqrt{\varepsilon n}$.) Now $t_1 = \rho n$ by definition. From Lemma 20 (considering \mathcal{E}_1) we have $T_0 \leq \omega^{-1} \sqrt{n/\varepsilon}$ whp, and thus $T_0 = o_p(\sqrt{n/\varepsilon})$. Hence

$$\begin{aligned} |\mathcal{C}| &= T_1 - T_0 = \rho n + \tilde{X}_{t_1}/(1 - \lambda_*) + o_p(\sqrt{n/\varepsilon}) \\ &= \rho n + \beta_{t_1} S_{t_1}/(1 - \lambda_*) + o_p(\sqrt{n/\varepsilon}), \end{aligned}$$

since $\tilde{X}_{t_1} = x_{t_1} + \beta_{t_1} S_{t_1} = \beta_{t_1} S_{t_1} + O(1)$ by (32) and (40). From (6), we have $1 - \lambda_* \sim \varepsilon$, while from (26) we have $\beta_{t_1} \sim 1$. Recalling Lemma 22, to complete the proof of Theorem 1 it thus suffices to show that the pair

$$\left(S_{t_1}, \sum_{t=1}^{t_1} \xi_t - \rho^* n \right)$$

is asymptotically bivariate normal with zero mean, variance $2\varepsilon n$ for the first coordinate, $\frac{10}{3(r-1)^2}\varepsilon^3 n$ for the second, and covariance $\frac{2}{r-1}\varepsilon^2 n$.

From (56) and Lemma 23,

$$\sum_{t=1}^{t_1} \xi_t = \sum_{t=1}^{t_1} D_t^* + \sum_{t=1}^{t_1} \Delta_t^* = \rho^* n + \sum_{t=1}^{t_1} (\gamma_t \Delta_t + \Delta_t^*) + o_p(\sqrt{\varepsilon^3 n}), \quad (61)$$

where the γ_t are deterministic and satisfy (57). Set

$$\hat{\Delta}_t = \gamma_t \Delta_t + \Delta_t^* \quad \text{and} \quad \hat{S}_i = \sum_{t=1}^i \hat{\Delta}_t. \quad (62)$$

Then (61) implies that

$$\hat{S}_{t_1} = \sum_{t=1}^{t_1} \xi_t - \rho^* n + o_p(\sqrt{\varepsilon^3 n}).$$

Thus to prove Theorem 1 it suffices to show that (S_{t_1}, \hat{S}_{t_1}) is asymptotically bivariate normal with mean zero and variance as above. More precisely, it suffices to show that

$$((\varepsilon n)^{-1/2} S_{t_1}, (\varepsilon^3 n)^{-1/2} \hat{S}_{t_1}) \xrightarrow{d} (X, Y) \quad (63)$$

where (X, Y) is bivariate normal with

$$\text{Var}[X] = \sigma_1^2 = 2, \quad \text{Var}[Y] = \sigma_2^2 = \frac{10}{3(r-1)^2} \quad \text{and} \quad \text{Cov}[X, Y] = \sigma_{1,2} = \frac{2}{r-1}. \quad (64)$$

For this we shall use Lemma 18.

First, by the definitions (18) and (56), $\mathbb{E}[\Delta_t \mid \mathcal{F}_{t-1}] = \mathbb{E}[\Delta_t^* \mid \mathcal{F}_{t-1}] = 0$, so $\mathbb{E}[\hat{\Delta}_t \mid \mathcal{F}_{t-1}] = 0$, and $(S_t, \hat{S}_t)_{t=0}^{t_1}$ is a martingale. The remaining assumptions of Lemma 18 are captured in the following claim.

Claim 25. *As $n \rightarrow \infty$ we have*

$$\sum_{t=1}^{t_1} \text{Var}[\beta_t^{-1} \Delta_t \mid \mathcal{F}_{t-1}] = (2 + o_p(1))\varepsilon n, \quad (65)$$

$$\sum_{t=1}^{t_1} \text{Var}[\hat{\Delta}_t \mid \mathcal{F}_{t-1}] = (1 + o_p(1)) \frac{10}{3(r-1)^2} \varepsilon^3 n, \quad (66)$$

$$\sum_{t=1}^{t_1} \text{Cov}[\hat{\Delta}_t, \beta_t^{-1} \Delta_t \mid \mathcal{F}_{t-1}] = (1 + o_p(1)) \frac{2}{(r-1)} \varepsilon^2 n. \quad (67)$$

Moreover, indicating the dependence on n explicitly for a change, the rescaled martingales

$$M_{1,n,t} = (\varepsilon(n)n)^{-1/2} S_{n,t} \quad \text{and} \quad M_{2,n,t} = (\varepsilon(n)^3 n)^{-1/2} \hat{S}_{n,t}, \quad (68)$$

defined for $0 \leq t \leq t_1(n)$, satisfy the Lindeberg condition (45).

Assuming the claim for the moment then, rescaling as in (68), the bounds (65)–(67) give exactly the variance conditions (46) and (47) of Lemma 18, with σ_1^2 , σ_2^2 and $\sigma_{1,2}$ as in (64). Thus Lemma 18 implies (63) which, as noted above, implies Theorem 1. It remains only to prove the claim. The Lindeberg condition asserts, roughly speaking, that it is unlikely that any single step in either martingale contributes significantly to the total variance of the martingale over t_1 steps. As in almost all combinatorial settings, this condition holds with plenty of room to spare. Indeed, the (unrescaled) martingales have step sizes of order 1, with strong tail bounds (inherited from the binomial distribution), and their final variances are much larger than 1, so the Lindeberg condition holds with plenty of room to spare. We give a full proof in the appendix to [8].

It remains to establish (65)–(67). This concerns only steps $1, \dots, t_1$ of our random exploration process, so from now on we only consider $0 \leq t \leq t_1$. Since we are aiming for convergence in probability, and the event \mathcal{E} defined in Lemma 20 holds whp, much of the time we assume that \mathcal{E} holds.

By Lemma 21(iii), when \mathcal{E} holds we have

$$\max_{t \leq t_1} A_t = O(\varepsilon^2 n). \quad (69)$$

Since $t \leq t_1$, the bound (26) implies that $\beta_t \sim 1$. Thus, when \mathcal{E} holds,

$$\text{Var}[\beta_t^{-1} \Delta_t \mid \mathcal{F}_{t-1}] \sim \text{Var}[\Delta_t \mid \mathcal{F}_{t-1}] \sim r - 1, \quad (70)$$

where the final estimate follows from Lemma 8 and the bound (69) above, recalling that $\varepsilon = o(1)$. (It also follows from [7, Eq. (7)], for example.) Hence, on \mathcal{E} ,

$$\sum_{t=1}^{t_1} \text{Var}[\beta_t^{-1} \Delta_t \mid \mathcal{F}_{t-1}] \sim (r - 1)t_1 \sim 2\varepsilon n.$$

Since \mathcal{E} holds whp, this implies (65). Next, recalling that $D_t^* = \mathbb{E}[\xi_t \mid \mathcal{F}_{t-1}]$, we have

$$\begin{aligned} \text{Var}[\Delta_t^* \mid \mathcal{F}_{t-1}] &= \text{Var}[\xi_t - D_t^* \mid \mathcal{F}_{t-1}] \\ &= \text{Var}[\xi_t \mid \mathcal{F}_{t-1}] \\ &= \mathbb{E}[\xi_t \mid \mathcal{F}_{t-1}] + \mathbb{E}[\xi_t(\xi_t - 1) \mid \mathcal{F}_{t-1}] - \mathbb{E}[\xi_t \mid \mathcal{F}_{t-1}]^2 \\ &= \mathbb{E}[\xi_t \mid \mathcal{F}_{t-1}] + O(A_{t-1}^2/n^2 + 1/n^2), \end{aligned}$$

by (51)–(53). Hence, by (69), when \mathcal{E} holds we have

$$\text{Var}[\Delta_t^* \mid \mathcal{F}_{t-1}] = D_t^* + O(\varepsilon^4).$$

Now, on \mathcal{E} ,

$$\begin{aligned} \text{Cov}[\Delta_t^*, \beta_t^{-1} \Delta_t \mid \mathcal{F}_{t-1}] &\sim \text{Cov}[\Delta_t^*, \Delta_t \mid \mathcal{F}_{t-1}] \\ &= \text{Cov}[\xi_t, \eta_t \mid \mathcal{F}_{t-1}] \\ &= \mathbb{E}[\xi_t \eta_t \mid \mathcal{F}_{t-1}] - \mathbb{E}[\xi_t \mid \mathcal{F}_{t-1}] \mathbb{E}[\eta_t \mid \mathcal{F}_{t-1}] \\ &= O(A_t/n + 1/n) = O(\varepsilon^2) = o(\varepsilon), \end{aligned} \quad (71)$$

using (54), (52), the bound $\mathbb{E}[\eta_t \mid \mathcal{F}_{t-1}] = O(1)$ (which follows from Lemma 7), and (69). Hence

$$\begin{aligned}\text{Var}[\hat{\Delta}_t \mid \mathcal{F}_{t-1}] &= \text{Var}[\Delta_t^* \mid \mathcal{F}_{t-1}] + \gamma_t^2 \text{Var}[\Delta_t \mid \mathcal{F}_{t-1}] + \gamma_t \text{Cov}[\Delta_t^*, \Delta_t \mid \mathcal{F}_{t-1}] \\ &= D_t^* + (r-1)(t_1 - t)^2/n^2 + o(\varepsilon^2),\end{aligned}$$

recalling (70) and (57). Thus,

$$\begin{aligned}\sum_{t=1}^{t_1} \text{Var}[\hat{\Delta}_t \mid \mathcal{F}_{t-1}] &= \sum_{t=1}^{t_1} D_t^* + (r-1) \frac{t_1^3}{3n^2} + o(\varepsilon^3 n) \\ &= \rho^* n + \frac{8\varepsilon^3}{3(r-1)^2} n + o_p(\varepsilon^3 n) \\ &= (1 + o_p(1)) \frac{10}{3(r-1)^2} \varepsilon^3 n,\end{aligned}$$

by Corollary 24 and (6). This proves (66). Finally, since $\beta_t \sim 1$ and $\hat{\Delta}_t = \gamma_t \Delta_t + \Delta_t^*$, when \mathcal{E} holds we have

$$\begin{aligned}\text{Cov}[\hat{\Delta}_t, \beta_t^{-1} \Delta_t \mid \mathcal{F}_{t-1}] &\sim \text{Cov}[\hat{\Delta}_t, \Delta_t \mid \mathcal{F}_{t-1}] \\ &= \gamma_t \text{Var}[\Delta_t \mid \mathcal{F}_{t-1}] + \text{Cov}[\Delta_t^*, \Delta_t \mid \mathcal{F}_{t-1}] \\ &= \frac{t_1 - t}{n} (r-1) + o(\varepsilon),\end{aligned}$$

from (57), (70) and (71). Hence

$$\sum_{t=1}^{t_1} \text{Cov}[\hat{\Delta}_t, \beta_t^{-1} \Delta_t \mid \mathcal{F}_{t-1}] = (r-1) \frac{t_1^2}{2n} + o_p(\varepsilon^2 n) = (1 + o_p(1)) \frac{2}{(r-1)} \varepsilon^2 n,$$

establishing (67). This completes the proof of Claim 25 and hence of Theorem 1. \square

As we have already remarked, in a follow-up paper [9] we prove a local limit version of Theorem 1, using Theorems 1 and 4 as tools in the proof. This local limit theorem is then used to prove an asymptotic formula for the number of connected r -uniform hypergraphs with a given number of vertices and edges, in the case where the nullity is small compared to the number of edges.

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