# Hall algebras and Green rings

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# Abstract

This thesis consists of two parts, both of which involve the study of algebraic structures constructed via the multiplication of modules.

In the first part we look at Hall algebras. We consider the Hall algebra of a cyclic quiver algebra with relations of length two and present a multiplication formula for the explicit calculation of products in this algebra. We then look at the case of a cyclic quiver with two vertices and describe the corresponding composition algebra as a quotient of the positive part of a quantised enveloping algebra of type $\widetilde{A}_1$.

We then look at quotients of Hall algebras of self-injective algebras. We give an abstract result describing the quotient of such a Hall algebra by the ideal generated by isomorphism classes of projective modules, and also a more explicit result describing quotients of Hall algebras of group algebras for cyclic 2-groups and some related polynomial algebras.

The second part of the thesis deals with Green rings. We compare the Green rings of a group algebra and the corresponding Jennings algebra for certain $p$-groups. It is shown that these two Green rings are isomorphic in the case of a cyclic $p$-group. In the case of the Klein four group it is shown that the two Green rings are not isomorphic, but that there exist quotients of these rings which are isomorphic. It is conjectured that the corresponding quotients will also be isomorphic in the case of a dihedral 2-group. The properties of these quotients are studied, with the aim of producing evidence to support this conjecture. The work on Green rings also includes some results on the realisation of quotients of Green rings as group rings over $\mathbb{Z}$.

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# Contents

# Introduction

The algebraic structures studied in this thesis, the Hall algebra and the Green ring, are both structures constructed by multiplying modules in some way. In the case of the Hall algebra, the multiplication is defined by counting filtrations of modules; this can be done with modules for any ring satisfying a mild finiteness condition. Green rings, by contrast, can only be defined with the modules for a bialgebra: the bialgebra co-multiplication is used to define the products of modules by taking tensor products. Green rings also have an addition given by taking direct sums of modules. As a consequence of this, Green rings of algebras of finite type are finite-dimensional over $\mathbb{Q}$. Hall algebras have as $\mathbb{Q}$-basis all isomorphism classes of modules for the relevant ring, and are thus never finite-dimensional over $\mathbb{Q}$.

Hall algebras were originally used to study filtrations of finite abelian $p$-groups, which can be done by considering the Hall algebra of the $p$-adic integers $\mathbb{Z}_{(p)}$. The study of Hall algebras of other rings, in particular of hereditary quiver algebras, was later seen to be of importance, largely because of the connection between Hall algebras and quantised enveloping algebras discovered by C. M. Ringel. In this thesis we shall look at Hall algebras of algebras, though we shall be concerned mainly with the case of algebras which are not hereditary.

In Chapter 2 we shall look at Hall algebras of cyclic quivers with and without relations. We begin by discussing a result of C. M. Ringel on the relationship between the Hall algebras of cyclic quiver algebras without relations and the quantised enveloping algebra of type $\widetilde{A}_{n-1}$. We then look at algebras of cyclic quivers with relations given by setting all paths of length two to be zero, and present a formula which allows us to calculate explicitly the product of any two modules in the corresponding Hall algebra (Theorem 2.2.7). We conclude this chapter by looking at the cyclic quiver with two vertices and relations of length two. It follows from Ringel's work that this is a homomorphic image of the positive part of the quantised enveloping algebra of type $\widetilde{A}_1$; we determine the kernel of this homomorphism (Theorem 2.3.5). Some further calculations involving the Hall algebras of a family of related algebras also given by a cyclic quiver with two vertices, with certain relations, can be found in Appendix A.

In Chapter 3 we look at quotients of Hall algebras of self-injective algebras. We begin with a result which describes the quotient of such a Hall algebra by the ideal generated by the isomorphism classes of projective modules (Theorem 3.1.1). We then consider the Hall algebra of a group algebra $kG$ and the ideal generated by the relatively $H$-projective modules for some subgroup $H \leqslant G$. We look at the example where $G$ is a cyclic 2-group and $H$ is a cyclic subgroup of index 2, and also at a related polynomial algebra. We describe a spanning set for our Hall algebra quotient in this case (Theorem 3.2.1) and calculate explicitly the product of any two elements of this spanning set.

The Green ring was introduced by J. A. Green as a tool for studying direct sums and tensor products of group algebra modules in cases where the characteristic of the field divides the order of the group. The definition of the Green ring extends naturally from the group algebra case to modules for any bialgebra. In Chapters 5 to 7 we compare the Green rings of finite $p$-groups $G$ and the corresponding Jennings algebras $U^{[p]}\mathrm{Jen}(G)$, in cases where $kG \simeq U^{[p]}\,\mathrm{Jen}(G)$ as algebras and the algebra is of finite or tame representation type. We also compare certain quotients of these Green rings, which we shall call Benson–Carlson quotients; these quotients were introduced by D. J. Benson and J. F. Carlson, who showed that they have no nilpotent elements.

Chapter 5 looks at the case where $G$ is a cyclic $p$-group, for any prime $p$. In this case the two Green rings considered—and hence also the two Benson–Carlson quotients—are isomorphic (Theorem 5.1.3). Chapter 6 looks at the case where $G$ is the Klein four group. In this case the two Benson–Carlson quotients are isomorphic (Theorem 6.2.1), but the two Green rings are not naturally isomorphic (Theorem 6.1.1).

In Chapter 7 we consider the case where $G = D_{4q}$ is a dihedral 2-group. It is conjectured that the two Benson–Carlson quotients will also be isomorphic in this case (Conjecture 7.0.2). The chapter is devoted to studying the Benson–Carlson quotients, with an emphasis on looking for evidence that may support Conjecture 7.0.2. The Benson–Carlson quotients are realised as group rings over $\mathbb{Z}$ (Theorem 7.1.2) and the problem of proving Conjecture 7.0.2 is reduced to the problem of proving that these two groups are isomorphic. The properties of the two groups are studied. In particular, we look at the subgroups of endo-trivial modules, which are indeed isomorphic. We then compare the properties of these subgroups which allowed us to deduce this with the properties of the whole groups, looking at properties of the map restricting modules to the Klein four subgroups of $D_{4q}$ and its relationship with the Auslander–Reiten quiver structure of $D_{4q}$. Following on from this, we provide an alternative method for reducing the problem of proving Conjecture 7.0.2 by looking at the Auslander–Reiten quiver for $D_{4q}$ and the relationship between this structure and taking tensor products of modules. Finally, we look at some example of products which are the same in both Benson–Carlson quotients.

In Chapter 8 we return to the result of Theorem 7.1.2 showing that the Benson–Carlson quotient of a dihedral 2-group can be a realised as a group ring over $\mathbb{Z}$, and show that analogous results hold for cyclic 2-groups (Theorem 8.1.1) and the Klein four group (Theorem 8.2.1).

A knowledge of the following topics of representation theory and related areas of algebra will be pre-supposed:

- Simple (irreducible), semisimple, indecomposable, projective and injective modules
- Direct sums and tensor products
- The Krull–Schmidt Theorem
- Radicals and socles of algebras and modules
- Self-injective algebras
- Filtrations and composition series
- Long and short exact sequences
- Opposite rings and dual modules
- Induction and restriction of modules

- Vertices of group algebra modules; relatively projective modules
- The Heller operator $\Omega$
- Graded algebras
- $\mathrm{Ext}^i(-,-)$
- Global dimension
- Quiver algebras and representations of quivers
- Finite, tame and wild representation types
- Finite-dimensional semisimple complex Lie algebras and their classification via Cartan matrices and Dynkin diagrams
- Universal enveloping algebras of Lie algebras
- $p$-restricted Lie algebras and universal enveloping algebras
- Basic Auslander–Reiten theory, including irreducible maps, the Auslander–Reiten translate $\tau$ (including the fact that $\tau = \Omega^2$ for group algebras), Auslander–Reiten quivers and stable Auslander–Reiten quivers.

All modules in this thesis will be finitely generated left modules, apart from dual modules, which must be right modules except in cases where the existence of an antipode allows us to define dual modules as left modules (see Section 4.1.1).

# Part I

# Hall algebras

# Chapter 1

# Definitions and background material

## 1.1 Hall algebras

**Definition 1.1.1** A ring $R$ is *finitary* if $\mathrm{Ext}^1_R(M_1, M_2)$ is finite for all finite $R$-modules $M_1$ and $M_2$.

The following lemma tells us that, if we wish to determine whether a ring is finitary, it is sufficient to check that the defining property holds for all finite simple modules.

**Lemma 1.1.2** *Let $R$ be a ring with $Ext^1_R(S_1, S_2)$ finite for all finite simple $R$-modules $S_1$, $S_2$. Then $R$ is finitary.*

For the proof of Lemma 1.1.2—and some further discussion of finitary rings—see [32, pp. 435–436].

**Definition 1.1.3** Let $R$ be a finitary ring. Take the set of isomorphism classes $[M]$ of all finite-dimensional $R$-modules as the basis for a free vector space over $\mathbb{Q}$. For modules $M$, $N_1$, $N_2$, let $F^M_{N_1,N_2}$ be the number of submodules $M_0$ of $M$ such that $M_0 \simeq N_2$ and $M/M_0 \simeq N_1$. Then define a multiplication on elements of our vector space by

$$[N_1] \circ [N_2] = \sum_{[M]} F^M_{N_1,N_2}[M] \tag{1.1}$$

and linear extension. This gives us an algebra over $\mathbb{Q}$, the *Hall algebra* $\mathcal{H}(R)$ of $R$.

Assuming that $R$ is finitary ensures that (1.1) will always be a finite sum; thus the Hall algebra multiplication is well-defined.

The following lemma collects some basic properties of the Hall algebra.

**Lemma 1.1.4** *1. If $[M]$ occurs with non-zero coefficient in some multiplication $[N_1] \circ [N_2]$ then there exists a short exact sequence*

$$0 \longrightarrow N_2 \longrightarrow M \longrightarrow N_1 \longrightarrow 0.$$

*2. $\mathcal{H}(R)$ is an associative algebra with a multiplicative identity.*

*3. $\mathcal{H}(R)$ is a $\mathbb{Z}^n$-graded algebra, where $n$ is the number of irreducible $A$-modules.*

*4. Suppose $R \simeq R^{\mathrm{op}}$. For $A$-modules $N_1$, $N_2$ we have*

$$[{N_2}^*] \circ [{N_1}^*] = \sum_{[M]} F^M_{N_1,N_2}[M^*].$$

**Proof:**

1. This is clear from the definition of $[N_1] \circ [N_2]$.

2. For $R$-modules $M$, $N_1$, $N_2$, $\ldots$, $N_t$, let $F^M_{N_1,\ldots,N_t}$ be the number of filtrations of $M$

$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_t = 0$$

such that $M_{i-1}/M_i \simeq N_i$ for $1 \leqslant i \leqslant t$. We have

$$[N_1] \circ ([N_2] \circ [N_3]) = \sum_{[M]} \sum_{[Y]} F^M_{N_1,Y} F^Y_{N_2,N_3} [M].$$

Now, given any $R$-module $Y$ with $F^Y_{N_2,N_3} \neq 0$ and any pair of filtrations

$$M \overset{N_1}{\supseteq} M_0 \overset{Y}{\supseteq} 0, \qquad\qquad Y \overset{N_2}{\supseteq} Y_0 \overset{N_3}{\supseteq} 0,$$

it is clear that the filtrations can be adjoined to give a filtration

$$M \overset{N_1}{\supseteq} M_0 \overset{N_2}{\supseteq} X_0 \overset{N_3}{\supseteq} 0$$

where the isomorphism $M_0 \simeq Y$ is used to find $X_0 \subseteq M_0$ corresponding to $Y_0 \subseteq Y$. All filtrations of $M$ with these factors can be found in this way. Hence

$$\sum_{[Y]} F^M_{N_1,Y} F^Y_{N_2,N_3} = F^M_{N_1,N_2,N_3},$$

so

$$[N_1] \circ ([N_2] \circ [N_3]) = \sum_{[M]} F^M_{N_1,N_2,N_3} [M].$$

A similar argument shows

$$([N_1] \circ [N_2]) \circ [N_3] = \sum_{[M]} F^M_{N_1,N_2,N_3} [M].$$

The zero module $[0_R]$ is the multiplicative identity for $\mathcal{H}(R)$.

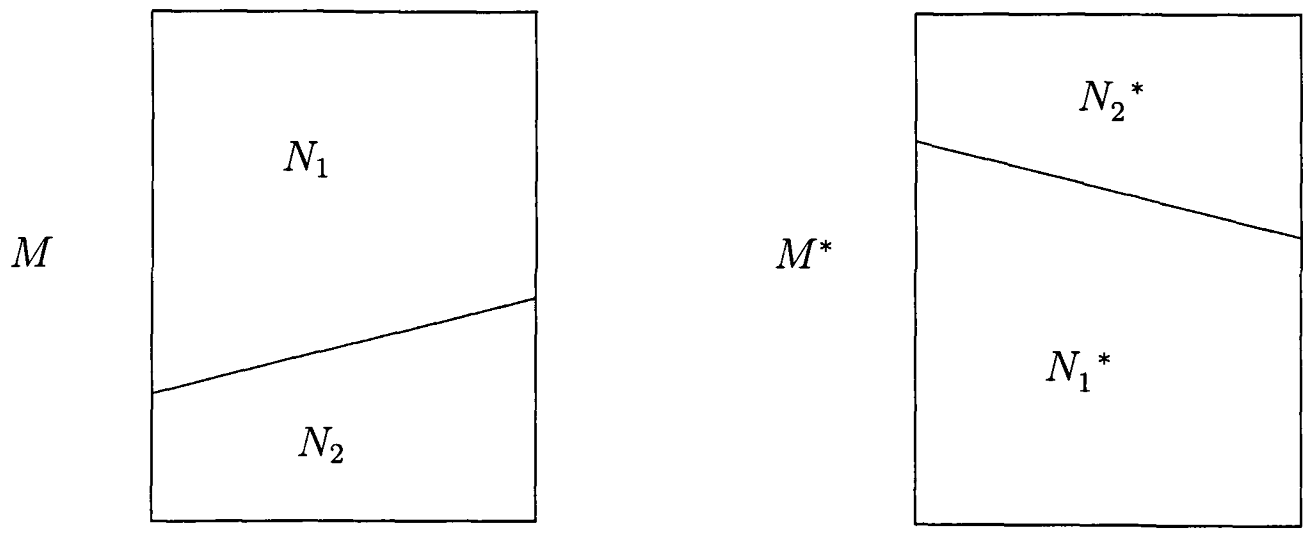
3. Let $S_1, \ldots S_n$ be the simple $R$-modules. Given an $R$-module $M$, let $\mathbf{dim}M = ([M : S_1], \ldots, [M : S_n])$ be the vector given by the composition multiplicities of the simple modules in $M$. Let $\mathcal{H}(R)_{\mathbf{d}}$ be the subspace of $\mathcal{H}(R)$ with basis the set of isomorphism classes $[M]$ with $\mathbf{dim}M = \mathbf{d}$. Then

$$\mathcal{H}(R) = \oplus_{\mathbf{d} \in \mathbb{Z}^n} \mathcal{H}(R)_{\mathbf{d}}$$

as a vector space. If $F^M_{N_1,N_2} \neq 0$ for some $M, N_1, N_2$ then it is clear by refining the filtration $M \supseteq N_2 \supseteq 0$ to a composition series for $M$ that $\mathbf{dim}M = \mathbf{dim}N_1 + \mathbf{dim}N_2$.

4. The dual $M^* = \mathrm{Hom}_R(M, k)$ of a left $R$-module $M$ is a left $R^{\mathrm{op}}$-module with $a\phi$ defined by $(a\phi)(m) = \phi(am)$ for $\phi \in M^*$, $a \in R$, $m \in M$. Since $R \simeq R^{\mathrm{op}}$, this allows us to think of $M^*$ as a left $R$-module.

   The coefficient of $[M]$ in the multiplication $[N_1] \circ [N_2]$ is given by the number of submodules $M_0$ of $M$ such that $M/M_0 \simeq N_1$ and $M_0 \simeq N_2$. But this is equal to the number of submodules $X$ of $M^*$ such that $M^*/X \simeq {N_2}^*$ and $X \simeq {N_1}^*$. To see this, just think of what the composition structure of $M$ and $M^*$ must look like in relation to these submodules and quotients: they can be represented as something like




   This number of submodules is by definition the coefficient of $[M^*]$ in the multiplication $[{N_2}^*] \circ [{N_1}^*]$.

□

**Definition 1.1.5** Given a Hall algebra $\mathcal{H}(R)$ for some ring $R$, the *composition algebra* $\mathcal{C}(R)$ is the subalgebra of $\mathcal{H}(R)$ generated by the isomorphism classes of simple $R$-modules.

The Hall algebra was originally studied by E. Steinitz [38] and later by Philip Hall [20], who both considered the Hall algebra $\mathcal{H}(\mathbb{Z}_{(p)})$ of the $p$-adic integers. The finite $\mathbb{Z}_{(p)}$-modules are just the finite abelian $p$-groups, so the algebra $\mathcal{H}(\mathbb{Z}_{(p)})$ is a tool for studying filtrations of such groups. Note that $\mathcal{H}(\mathbb{Z}_{(p)})$ is a commutative algebra, but that this is not true of Hall algebras in general. More information on this Hall algebra and its properties can be found in [27, Chapter II].

In this thesis we shall be concerned with Hall algebras $\mathcal{H}(A)$, where $A$ is a finitary algebra over a finite field $k$. The study of Hall algebras of this form was initiated in the 1990s by C. M. Ringel, with a series of papers in which he demonstrated some remarkable relationships between Hall algebras and quantised enveloping algebras (see Sections 1.3 and 1.4 below for a definition of these algebras).

## 1.2 Twisted Hall algebras

One tool used by Ringel in his study of the relationship between Hall algebras and quantised enveloping algebras was the introduction of a "twist" on the Hall and composition algebras.

**Definition 1.2.1** An algebra $A$ is *hereditary* if all submodules of projective $A$-modules are also projective.

Let $A$ be a finitary algebra over a finite field $k$. If $A$ is also hereditary, then the Hall algebra $\mathcal{H}(A)$ can be "twisted" to form the *twisted Hall algebra* $\mathcal{H}_*(A)$, also known as the *Ringel–Hall algebra.*

Let $A$ be a hereditary algebra. If there are $n$ simple $A$-modules, we can define a bilinear form from $\mathbb{N}_0{}^n \times \mathbb{N}_0{}^n$ to $\mathbb{N}_0$ by setting $\langle \underline{\dim}\, M, \underline{\dim}\, N \rangle = \sum_{i=0}^{\infty}(-1)^i \dim_k \operatorname{Ext}^i(M,N)$ for $A$-modules $M$ and $N$. It can be shown that this form depends only on the dimension vectors $\underline{\dim}\, M$ and $\underline{\dim}\, N$ (see [36, Section 2]). Since $A$ is hereditary, we have $\operatorname{Ext}^i(M,N) = 0$ for all $i \geqslant 2$. Hence $\langle \underline{\dim}\, M, \underline{\dim}\, N \rangle = \dim_k \operatorname{Hom}(M,N) - \dim_k \operatorname{Ext}^1(M,N)$, so the bilinear form is given by a finite sum.

We now introduce $v$ satisfying $v^2 = |k|$. If $|k|$ is a square in $\mathbb{Q}$ then take the positive square root. If not, then extend the field of coefficients for $\mathcal{H}(A)$ by taking $\mathcal{H}_*(A) = \mathcal{H}(A) \otimes_{\mathbb{Q}} \mathbb{Q}(v)$ as a vector space over $\mathbb{Q}(v)$.

**Definition 1.2.2** Let $A$ be a finitary, hereditary algebra over a finite field $k$. We construct the *twisted Hall algebra* $\mathcal{H}_*(A)$ by taking the $\mathbb{Q}(v)$-space $\mathcal{H}_*(A) = \mathcal{H}(A) \otimes_{\mathbb{Q}} \mathbb{Q}(v)$ and defining a multiplication by

$$[N_1] * [N_2] = v^{\langle \underline{\dim} N_1, \underline{\dim} N_2 \rangle} [N_1] \circ [N_2]$$

for $A$-modules $N_1$, $N_2$, and linear extension.

**Lemma 1.2.3** *The twisted Hall algebra $\mathcal{H}_*(A)$ is associative and $\mathbb{Z}^n$-graded.*

**Proof:** To see that $\mathcal{H}_*(A)$ is associative, we simply check that

$$[N_1] * ([N_2] * [N_3]) = v^{\sum_{i<j} \langle \underline{\dim} N_i, \underline{\dim} N_j \rangle} [N_1] \circ ([N_2] \circ [N_3])$$

and

$$[N_1] * ([N_2] * [N_3]) = v^{\sum_{i<j} \langle \underline{\dim} N_i, \underline{\dim} N_j \rangle} ([N_1] \circ [N_2]) \circ [N_3].$$

Associativity then follows from part 2 of Lemma 1.1.4.

The proof that $\mathcal{H}_*(A)$ is $\mathbb{Z}^n$-graded is exactly the same as in part 3 of Lemma 1.1.4. $\square$

**Definition 1.2.4** Given a twisted Hall algebra $\mathcal{H}_*(A)$, the *twisted composition algebra* $\mathcal{C}_*(A)$ is the subalgebra of $\mathcal{H}_*(A)$ generated by the simple $A$-modules.

## 1.3 Kac–Moody algebras

The construction of a quantised enveloping algebra begins with a generalised Cartan matrix corresponding to a Kac–Moody algebra. A Kac–Moody algebra can be thought of as a generalisation of a finite-dimensional semisimple complex Lie algebra. We begin by recalling some material relating to such Lie algebras. Let $L$ be a finite-dimensional semisimple Lie algebra with $n$ roots and Cartan matrix $A = (a_{ij})$, which must correspond to some Dynkin diagram of type A–G. Then $L$ is generated as a Lie algebra by elements $\{x_i, y_i, h_i : 1 \leqslant i \leqslant n\}$ which satisfy the following relations for all $i, j \in \{1, \ldots, n\}$:

$$\begin{array}{rl} \text{(i)} & [h_i, h_j] = 0 \\ \text{(ii)} & [x_i, y_j] = \delta_{ij} h_i \\ \text{(iii)} & [h_i, x_j] = a_{ij} x_j \\ \text{(iv)} & [h_i, y_j] = -a_{ij} y_j \\ \text{(v)} & (\mathrm{ad}(x_i))^{1-a_{ij}}(x_j) = 0 \\ \text{(vi)} & (\mathrm{ad}(y_i))^{1-a_{ij}}(y_j) = 0. \end{array} \tag{1.2}$$

Relations (i)–(iv) are the *Weyl relations*; relations (v) and (vi) are known as the *Serre relations.*

Furthermore, we have the following theorem, which guarantees the existence of such a Lie algebra for all possible Cartan matrices corresponding to Dynkin diagrams of types A–G:

**Theorem 1.3.1 (Serre)** *Let $A = (a_{ij})$ be a Cartan matrix corresponding to a Dynkin diagram of type A–G. Let $L$ be the complex Lie algebra with* *generators* $\{x_i, y_i, h_i : 1 \leqslant i \leqslant n\}$ *and relations (i)–(vi) as above. Then* $L$ *is a finite-dimensional semisimple Lie algebra.*

The Lie algebra $L$ can be written as a direct sum of subalgebras

$$L = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$$

where $\mathfrak{h} = \langle h_i : 1 \leqslant i \leqslant n\rangle_k$ is the Cartan subalgebra of $L$, and $\mathfrak{n}^- = \langle y_i : 1 \leqslant i \leqslant n\rangle_k$ and $\mathfrak{n}^+ = \langle x_i : 1 \leqslant i \leqslant n\rangle_k$ are the negative and positive parts of $L$.

For proofs of the results mentioned above and further information on these properties of Lie algebras, see [22, pp. 95–101].

Translating these properties into the language of the universal enveloping algebra, we see that $U(L)$ is generated as an associative algebra by elements $\{X_i, Y_i, H_i : 1 \leqslant i \leqslant n\}$ satisfying relations

$$\begin{array}{ll} \text{(i)} & H_iH_j = H_jH_i \\ \text{(ii)} & X_iY_j - Y_jX_i = \delta_{ij}H_i \\ \text{(iii)} & H_iX_j - X_jH_i = a_{ij}X_j \\ \text{(iv)} & H_iY_j - Y_jH_i = -a_{ij}Y_j \\ \text{(v)} & \sum_{r=0}^{1-a_{ij}}(-1)^r\binom{1-a_{ij}}{r}X_i^{1-a_{ij}-r}X_jX_i^r = 0 \\ \text{(vi)} & \sum_{r=0}^{1-a_{ij}}(-1)^r\binom{1-a_{ij}}{r}Y_i^{1-a_{ij}-r}Y_jY_i^r = 0 \end{array} \tag{1.3}$$

for all $i, j \in \{1, \ldots, n\}$. We also have that

$$U(L) = U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}^+)$$

and that $U(\mathfrak{n}^-) = \langle Y_i : 1 \leqslant i \leqslant n\rangle$, $U(\mathfrak{h}) = \langle H_i : 1 \leqslant i \leqslant n\rangle$ and $U(\mathfrak{n}^+) = \langle X_i : 1 \leqslant i \leqslant n\rangle$.

This idea that the Cartan matrix can be used as the starting point for defining a Lie algebra allows us to generalise the notion of a semisimple Lie algebra to include the possibility that the Lie algebra may be infinite-dimensional. In order to do this, we first need some idea of what would constitute a suitable matrix to use.

**Definition 1.3.2** A matrix $A = (a_{ij})$ is called a *generalised Cartan matrix* if it satisfies the following conditions:

(i) $a_{ii} = 2$ for all $i$.

(ii) $a_{ij} \in \mathbb{Z}_{\leqslant 0}$ for all $i \neq j$.

(iii) If $a_{ij} = 0$ then $a_{ji} = 0$.

**Definition 1.3.3** Let $A = (a_{ij})$ be a generalised Cartan matrix. The *Kac–Moody algebra* corresponding to $A$ is the complex Lie algebra $\widetilde{L}$ with generating set $\{x_i, y_i, h_i : 1 \leqslant i \leqslant n\}$ and relations (i)–(vi) as in (1.2).

These algebras were first introduced independently by Kac and Moody in the late 1960s. For the original definition of a Kac–Moody algebra, see [25, Chapter 1]; for a proof that this is equivalent to the definition given above, see [25, Theorem 9.11].

**Theorem 1.3.4** *Let $\widetilde{L}$ be a Kac–Moody algebra. Let $\mathfrak{n}^+$ be the subalgebra of $\widetilde{L}$ generated by $x_1, \ldots, x_n$ and let $\mathfrak{n}^-$ be the subalgebra of $\widetilde{L}$ generated by $y_1, \ldots, y_n$. Then*

$$\widetilde{L} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$$

*for some subalgebra $\mathfrak{h}$ of $\widetilde{L}$.*

**Remark:** In general, $\mathfrak{h} \neq \langle h_i : 1 \leqslant i \leqslant n \rangle$: in fact, we must have $\dim \mathfrak{h} = 2n - \operatorname{rank}(A)$ so these cannot be equal if $\det(A) = 0$.

This theorem is a direct consequence of Theorem 1.2 in [25].

The universal enveloping algebra of the Kac–Moody algebra $\widetilde{L}$ is the associative algebra $U(\widetilde{L})$ with generators $\{X_i, Y_i, H_i : 1 \leqslant i \leqslant n\}$ and relations (i)–(vi) as in (1.3). We have $U(\widetilde{L}) = U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}^+)$ for $\mathfrak{n}^-, \mathfrak{h}, \mathfrak{n}^+$

as in Theorem 1.3.4. Also, $U(\mathfrak{n}^-)$ is generated as an associative algebra by $\{Y_i : 1 \leqslant i \leqslant n\}$ and $U(\mathfrak{n}^+)$ by $\{X_i : 1 \leqslant i \leqslant n\}$. We shall be particularly interested in the positive part $U(\mathfrak{n}^+)$.

## 1.4 Quantised enveloping algebras

Let $A = (a_{ij})$ be a generalised Cartan matrix and let $\widetilde{L}$ be the corresponding Kac–Moody algebra. We will assume for the rest of this section that the matrix $A$ is *symmetrisable*; that is, that there exist $d_i \in \{1, 2, 3\}$ such that $(d_i a_{ij})$ is a symmetric matrix. Let $\widetilde{L}_{\mathbb{Q}}$ be the Lie algebra over $\mathbb{Q}$ with generators and relations as for $\widetilde{L}$, so that $\widetilde{L}_{\mathbb{Q}}$ satisfies $\widetilde{L}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C} \simeq \widetilde{L}$. We shall describe the quantised universal enveloping algebra $U_q(\widetilde{L}_{\mathbb{Q}})$, which is obtained by introducing an indeterminate $v$. The quantised enveloping algebra can then be specialised by taking $v$ to be a value in $\mathbb{Q}$, giving us a collection of associative $\mathbb{Q}$-algebras.

**Definition 1.4.1** Let $v \in \mathbb{Q}$ be an indeterminate and let $d \in \{1, 2, 3\}$. For $a \in \mathbb{Z}$, the *Gaussian integer* $[a]_d$ is defined by

$$[a]_d := \frac{v^{da} - v^{-da}}{v^d - v^{-d}}.$$

If $a > 0$ then $[a]_d = v^{d(a-1)} + v^{d(a-3)} + \cdots + v^{-d(a-3)} + v^{-d(a-1)}$. Note that if we put $v = 1$ then we recover the ordinary integer $a$.

**Definition 1.4.2** Let $a, r \in \mathbb{N}$ with $r \leqslant a$. Define the *Gaussian binomial coefficient*

$$\begin{bmatrix} a \\ r \end{bmatrix}_d := \frac{[a]_d[a-1]_d \cdots [a-r+1]_d}{[1]_d[2]_d \cdots [r]_d}.$$

Let

$$\begin{bmatrix} a \\ 0 \end{bmatrix}_d := 1.$$

**Definition 1.4.3** Let $A = (a_{ij})$ be a symmetrisable generalised Cartan matrix, with $d_i \in \{1, 2, 3\}$ such that $(d_i a_{ij})$ is symmetric. Let $v \in \mathbb{Q}$ be an indeterminate. The *quantised enveloping algebra* $U_q(\widetilde{L_{\mathbb{Q}}})$ is the associative $\mathbb{Q}(v)$-algebra with generators $\{E_i, F_i, K_i, K_i^{-1} : 1 \leqslant i \leqslant n\}$ and relations

$$\begin{array}{ll}
\text{(i)} & K_i K_j = K_j K_i \\
\text{(ii)} & E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{v^{d_i} - v^{-d_i}} \\
\text{(iii)} & K_i E_j K_i^{-1} = v^{d_i a_{ij}} E_j \\
\text{(iv)} & K_1 F_j K_i^{-1} = v^{-d_i a_{ij}} F_j \\
\text{(v)} & \sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{d_i} E_i^{1-a_{ij}-r} E_j E_i^r = 0 \\
\text{(vi)} & \sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{d_i} F_i^{1-a_{ij}-r} F_j F_i^r = 0 \\
\text{(vii)} & K_i K_i^{-1} = 1 = K_i^{-1} K_i
\end{array} \tag{1.4}$$

*for all* $i, j \in \{1, \ldots, n\}$.

This definition was introduced by V. G. Drinfeld and M. Jimbo in 1995; the defining relations are known as the *Drinfeld–Jimbo relations.* Quantised enveloping algebras were introduced primarily to model the solutions of the quantum Yang–Baxter equation, an equation arising in statistical mechanics, using deformations of Kac-Moody algebras. Early work on the construction of quantised enveloping algebras and their use in studying solutions of the Yang–Baxter equation can be found in [12] and [24].

It is clear that if we specialise by putting $v = 1$ in relations (v) and (vi) we recover the Serre relations (v) and (vi) of the relations (1.3) for the universal enveloping algebra of the Kac–Moody algebra, with $E_i$ corrseponding to $X_i$ and $F_i$ to $Y_i$ for each $i$. To see that the other relations can be thought of as deformations of the Weyl relations for the enveloping algebra, think of $K_i$ as corresponding to $v^{d_iH_i}$ (where we realise the enveloping algebra as a matrix algebra and use the formal definition for matrix exponentials $\exp(H_i)$ to construct these powers). This takes the additive structure of $U(\mathfrak{h})$ to a multiplicative structure in $U_q(\mathfrak{h})$, which is why we introduce a multiplicative inverse for $K_i$; this also explains the commutation relation (i). The relations (iii) and (iv) above can then be seen as multiplicative analogies of the Weyl relations (iii) and (iv) in (1.3), where we conjugate multiplicatively by $K_i$ instead of conjugating additively by $H_i$. Finally, to see the correspondence between relation (ii) above and relation (ii) in (1.3) for the enveloping algebra, think of

$$\delta_{ij}\frac{v^{d_iH_i} - v^{-d_iH_i}}{v^{d_i} - v^{-d_i}}$$

as a function in $v$. We cannot put $v = 1$, but we can use l'Hopital's Rule to find the limit as $v \longrightarrow 1$; treating it in this way we recover the Weyl relation (ii) of (1.3).

We shall be particularly interested in the positive part $U_q(\mathfrak{n}^+)$ of the quantised enveloping algebra, the subalgebra generated by $\{E_i : 1 \leqslant i \leqslant n\}$. This is just the algebra defined by generators $\{E_i : 1 \leqslant i \leqslant n\}$ and relations (v) as above. Note that if we restrict our attention to the positive part, the correspondence with the enveloping algebra is much more straightforward: we have

$$\frac{U_q(\mathfrak{n}^+)}{(v-1)} \simeq U(\mathfrak{n}^+)$$

as algebras over $\mathbb{Q}$.

## 1.5 The relationship between Hall algebras and quantised enveloping algebras

It is not possible to state the most general results relating Hall algebras and quantised enveloping algebras without introducing further technical constructions related to the composition algebra. Nevertheless, in this section we will give a brief indication of the nature of some of the work done on this relationship. The following theorem gives an example of such a relationship in perhaps the simplest case:

**Theorem 1.5.1 (Ringel)** *Let $A = (a_{ij})$ be the Cartan matrix corresponding to a semisimple complex Lie algebra of type A, D or E. Let $Q$ be the corresponding Dynkin quiver, with any orientation. Let $k$ be a finite field with $|k| = q$. Let $\mathcal{C}_*(kQ)$ be the twisted composition algebra of the quiver algebra $kQ$. Let $U_q(\mathfrak{n}^+)$ be the positive part of the quantised universal enveloping algebra given by $A$. Then*

$$\frac{U_q(\mathfrak{n}^+)}{(v^2 - q)} \simeq \mathcal{C}_*(kQ).$$

This theorem thus allows us to realise $U_q(\mathfrak{n}^+)/(v^2 - q)$ as a twisted composition algebra, with the consequence that properties of the composition algebra, such as module-theoretic properties, can be used to study the quantised enveloping algebra. We will see another similar result, with $U_q(\mathfrak{n}^+)$ given by a generalised Cartan matrix of type $\widetilde{A}_n$, in some more detail in Chapter 2.

Note that Theorem 1.5.1, as stated above, only allows us to realise the quantised enveloping algebra as a twisted composition algebra if the parameter $v$ in the quantised enveloping algebra is chosen to satisfy $v^2 = |k|$ for some finite field $k$. However, the theorem as orignially stated by Ringel does rather more than this: it gives an analogous isomorphism with $v$ retained as an arbitrary parameter, via the introduction of "Hall polynomials" giving the coefficients for Hall algebra multiplications, and a "generic twisted composition algebra" constructed using these polyonimals. See [36] for details of these constructions and the proof of Theorem 1.5.1.

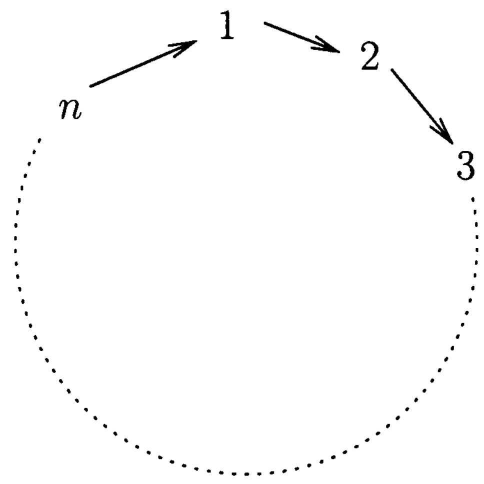
The early development of the relationship between composition algebras and quantised enveloping algebras can be traced in Ringel's papers [30], [31], [33] and [35]. The notes [36] provide a good introduction to Hall algebras and to this relationship, dealing with Dynkin quivers and quantised enveloping algebras of types A, D and E. The paper [33] entends these results to deal with the case of a quantised enveloping algebra given by a generalised symmetric Cartan matrix of type $\widetilde{A}$ , $\widetilde{D}$ or $\widetilde{E}$ ; in this case the construction of the generic composition algebra has to be modified to deal with the fact that for quiver algebras of infinite type the coefficients in the Hall algebra multiplications are no longer given by Hall polynomials.

The most general result on the relationship between Hall algebras and quantised enveloping algebras was proved by J. A. Green [17, Theorem 3]: this result provides a realisation of the positive part of a quantised enveloping algebra given by any symmetrisable generalised Cartan matrix. In this paper, Green defines a co-multiplication on $\mathcal{H}(A)$, making $\mathcal{H}(A)$ into a bialgebra (for $A$ any finitary, hereditary algebra over a finite field $k$). This bialgebra structure is then used to show that an appropriate generic composition algebra must be a member of a class of algebras also including the desired quantised enveloping algebra, using a definition of the quantised enveloping algebra introduced by Lusztig [26]; it is then deduced that the two algebras must in fact be isomorphic.

# Chapter 2

# The Hall algebra of a cyclic quiver algebra

Let $Q_n$ be the cyclic quiver




with $n$ vertices. Consider the quiver algebra $kQ_n$ for any finite field $k$. There are $n$ simple $kQ_n$-modules, corresponding to the $n$ vertices of $Q_n$, which we label $1, 2, \ldots, n$. These labels should be thought of as integers modulo $n$.

## 2.1 The corresponding quantised enveloping algebras

Suppose $n \geqslant 3$. Let $\mathfrak{g}$ be the Kac–Moody algebra of type $\widetilde{A}_{n-1}$, i.e. the Kac–Moody algebra with generalised Cartan matrix

$$\begin{pmatrix} 2 & -1 & & & -1 \\ -1 & 2 & -1 & & \\ & -1 & 2 & & \\ & & & \ddots & -1 \\ -1 & & & -1 & 2 \end{pmatrix}.$$

Let $U_q(\mathfrak{n}^+)$ be the positive part of the quantised enveloping algebra for $\mathfrak{g}$. Thus $U_q(\mathfrak{n}^+)$ is the $\mathbb{Q}(v)$-algebra with generators $E_1 \ldots, E_n$ and relations

$$E_i{}^2 E_j - (v + v^{-1}) E_i E_j E_i + E_j E_i{}^2 = 0 \text{ for } |i - j| = 1 \tag{2.1}$$

and

$$E_i E_j - E_j E_i = 0 \text{ for } |i - j| > 1. \tag{2.2}$$

Consider the composition algebra $\mathcal{C}(kQ_n)$. The structure of this algebra has been completely described by Ringel.

**Theorem 2.1.1 ([36, Theorem 3])** *The composition algebra $\mathcal{C}(kQ_n)$ has generators $[1], \ldots, [n]$ and is defined by the relations*

$$[i]^{\circ 2} \circ [i+1] - (q+1)[i] \circ [i+1] \circ [i] + q[i+1] \circ [i]^{\circ 2} = 0 \tag{2.3}$$

*and*

$$[i] \circ [i+1]^{\circ 2} - (q+1)[i+1] \circ [i] \circ [i+1] + q[i+1]^{\circ 2} \circ [i] = 0 \tag{2.4}$$

*for $1 \leqslant i \leqslant n$, and*

$$[i] \circ [j] - [j] \circ [i] = 0 \tag{2.5}$$

*for $|i - j| > 1$.*

It can be deduced from this theorem that certain specialisations of the quantised enveloping algebra of type $\widetilde{A}_{n-1}$ can be constructed as twisted composition algebras $\mathcal{C}_*(kQ_n)$.

**Corollary 2.1.2** *Let $\mathcal{C}_*(kQ_n)$ be the twisted composition algebra of the cyclic quiver algebra $kQ_n$. Let $U_q(\mathfrak{n}^+)$ be the positive part of the quantised enveloping algebra of type $\widetilde{A}_{n-1}$, with $v$ chosen to satisfy $v^2 = |k|$. Then $\mathcal{C}_*(kQ_n) \simeq U_q(\mathfrak{n}^+)$.*

**Proof:** The twisted composition algebra $\mathcal{C}_*(kQ_n)$ is generated by $[1], \dots [n]$. We check first that these satisfy the defining relations (2.1) and (2.2) for $U_q(\mathfrak{n}^+)$. The bilinear form calculations required are

$$\langle \underline{\dim}\ i, \underline{\dim}\ i \rangle = \dim_k \operatorname{Hom}(i,i) - \dim_k \operatorname{Ext}^1(i,i) = 1$$

and

$$\langle \underline{\dim}\ i, \underline{\dim}\ i+1 \rangle = \dim_k \operatorname{Hom}(i,i+1) - \dim_k \operatorname{Ext}^1(i,i+1) = -1$$

for $1 \leqslant i \leqslant n$; all other choices $i$ and $j$ have $\langle \underline{\dim}\ i, \underline{\dim}\ j \rangle = 0$. Thus relation (2.2) follows directly from relation (2.5) of Theorem 2.1.1. We also get

$$\begin{aligned}
&[i]^{*2} * [i+1] - (v+v^{-1})[i] * [i+1] * [i] + [i+1] * [i]^{*2} \\
&\quad = v^{-1} \left( [i]^{\circ 2} \circ [i+1] - (q+1)[i] \circ [i+1] \circ [i] + q[i+1] \circ [i]^{\circ 2} \right) = 0
\end{aligned}$$

by (2.3) and

$$\begin{aligned}
&[i+1]^{*2} * [i] - (v+v^{-1})[i+1] * [i] * [i+1] + [i] * [i+1]^{*2} \\
&\quad = \left( q[i+1]^{\circ 2} \circ [i] - (q+1)[i+1] \circ [i] \circ [i+1] + [i] \circ [i+1]^{\circ 2} \right) = 0
\end{aligned}$$

by (2.4).

This shows that $C_*(kQ_n)$ is a homomorphic image of $U_q(\mathfrak{n}^+)$. To see that they are in fact isomorphic, we compare dimensions of the $\underline{d}$-graded parts for all $\underline{d} \in {\mathbb{N}_0}^2$. For an algebra given by generators and relations as in Theorem 2.1.1 we can specialise by taking $q = 1$ (the algebra produced will not then be a composition algebra, since we cannot have $|k| = 1$, but it is legitimate if we use Ringel's theorem to think of $\mathcal{C}(kQ_n)$ purely in terms of generators and relations, forgetting the original context). We then have

$$\frac{\mathcal{C}(kQ_n)}{(q-1)} \simeq U(\mathfrak{n}^+).$$

Hence $\dim \mathcal{C}(kQ_n)_{\underline{d}} = \dim U(\mathfrak{n}^+)_{\underline{d}}$ for all $\underline{d}$. We then have $\dim C_*(kQ_n)_{\underline{d}} = \dim \mathcal{C}(kQ_n)_{\underline{d}}$ and $\dim U_q(\mathfrak{n}^+)_{\underline{d}} = \dim U(\mathfrak{n}^+)_{\underline{d}}$ for all $\underline{d}$, so there can be no further relations for $C_*(kQ_n)$. □

Now suppose $n = 2$. Let $\mathfrak{g}$ be the Kac–Moody algebra of type $\tilde{A}_1$, i.e. the Kac–Moody algebra with generalised Cartan matrix

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

Let $U_q(\mathfrak{n}^+)$ be the positive part of the quantised enveloping algebra for $\mathfrak{g}$. So $U_q(\mathfrak{n}^+)$ is the $\mathbb{Q}(v)$-algebra with generators $E_1, E_2$ and relations

$${E_1}^3E_2 - (v^2+1+v^{-2}){E_1}^2E_2E_1 + (v^2+1+v^{-2})E_1E_2{E_1}^2 - E_2{E_1}^3 = 0$$

and

$${E_2}^3E_1 - (v^2+1+v^{-2}){E_2}^2E_1E_2 + (v^2+1+v^{-2})E_2E_1{E_2}^2 - E_1{E_2}^3 = 0.$$

Since odd powers of $v$ do not occur in these relations, we can think of this as being a quantised algebra with indeterminate $v^2$ and can specialise by taking $v^2$ to be any value in $\mathbb{Q}$.

It has been shown by Ringel [34] that

$$\mathcal{C}(kQ_2) \simeq \frac{U_q(\mathfrak{n}^+)}{(v^2 - |k|)},$$

via the natural map $[1] \mapsto E_1$ and $[2] \mapsto E_2$. This tells us that a number of specialisations of $U_q(\mathfrak{n}^+)$ (all those where $v^2$ is taken to be a power of a prime) can be realised concretely as composition algebras for the quiver $Q_2$.

For $n \geqslant 2$, let $I$ be the ideal of $kQ_n$ generated by all paths of length 2. Put $A_n = kQ_n/I$. There are $n$ simple $A_n$-modules, corresponding to the $n$ vertices of $Q_n$, which we label $1, 2, \ldots, n$. There are also $n$ indecomposable projective $A_n$-modules, which have structure ${}^{1}_{2}, {}^{2}_{3}, \ldots, {}^{n}_{1}$. There are no further indecomposable $A_n$-modules. An arbitrary $A_n$-module will therefore have structure

$$M = \bigoplus_{i=1}^{n} \alpha_i . \begin{smallmatrix} i \\ i+1 \end{smallmatrix} \oplus \bigoplus_{i=1}^{n} \beta_i . i,$$

where both the labels $i$ for the simple $A_n$-modules and the indices of the corresponding coefficients (in this case $\alpha_i$ and $\beta_i$) should be thought of as integers modulo $n$ throughout this section.

It is clear that not all submodules of projective $A_n$-modules are projective, so $A_n$ is not hereditary for any $n \geqslant 2$. Furthermore, it can be seen that $A_n$ is of infinite global dimension for all $n \geqslant 2$. This means that the twisted Hall algebra $\mathcal{H}_*(A_n)$ is not well-defined. In the remaining part of this chapter we will study the Hall algebras $\mathcal{H}(A_n)$ for $n \geqslant 2$.

## 2.2 A multiplication formula

Consider the Hall algebra $\mathcal{H}(A_n)$. In this section we will present a formula for calculating explicitly the product $[M_1] \circ [M_3]$ of any two $A_n$-modules $M_1$ and $M_3$.

Let $q = |k|$. Let $G[m;n](q)$ be the number of $m$-dimensional subspaces of an $n$-dimensional vector space over $k$. This number is given by the formula

$$G[m;n](q) = \frac{(q^n - 1)(q^{n-1} - 1)\cdots(q^{n-m+1} - 1)}{(q^m - 1)(q^{m-1} - 1)\cdots(q - 1)}.$$

We also write $\operatorname{qexp}(n)$ for $q^n$.

As a starting point in our calculations, we will make use of the following result of Jin Yun Guo, which gives a method for calculating $[M_1] \circ [M_3]$ in the case where $M_3$ is a semisimple module.

**Proposition 2.2.1 ([19, Proposition 2.5])** *Let $M_1$ and $M_2$ be any two $A_n$-modules. Write these as*

$$M_1 = \bigoplus_{i=1}^{n} a_{i,2}^{(1)} . \begin{matrix} i \\ i+1 \end{matrix} \oplus \bigoplus_{i=1}^{n} a_{i,1}^{(1)} . i$$

*and*

$$M_2 = \bigoplus_{i=1}^{n} a_{i,2}^{(2)} . \begin{matrix} i \\ i+1 \end{matrix} \oplus \bigoplus_{i=1}^{n} a_{i,1}^{(2)} . i.$$

*Let $M_3$ be a semisimple $A_n$-module; write this as*

$$M_3 = \bigoplus_{i=1}^{n} a_{i,1}^{(3)} . i.$$

*The structures of these three modules are therefore determined by the three $n \times 2$ matrices*

$$\underline{a_1} = \begin{pmatrix} a_{1,1}^{(1)} & a_{1,2}^{(1)} \\ a_{2,1}^{(1)} & a_{2,2}^{(1)} \\ \vdots & \vdots \\ a_{n,1}^{(1)} & a_{n,2}^{(1)} \end{pmatrix} \qquad \underline{a_2} = \begin{pmatrix} a_{1,1}^{(2)} & a_{1,2}^{(2)} \\ a_{2,1}^{(2)} & a_{2,2}^{(2)} \\ \vdots & \vdots \\ a_{n,1}^{(2)} & a_{n,2}^{(2)} \end{pmatrix} \qquad \underline{a_3} = \begin{pmatrix} a_{1,1}^{(3)} & 0 \\ a_{2,1}^{(3)} & 0 \\ \vdots & \vdots \\ a_{n,1}^{(3)} & 0 \end{pmatrix}.$$

*For any $n \times 2$ matrix $\underline{c} = (c_{i,j})$, set as a convention $c_{i,j} = c_{i \,(\mathrm{mod}\ 2),j}$ for $i \notin \{1,2\}$ and $c_{i,j} = 0$ for $j \notin \{1,2\}$.*

*Let $\underline{b} = (b_{i,j})$ be the $n \times 2$ matrix determined by the following conditions:*

*Condition a:* $0 \leqslant b_{i,j} \leqslant a_{i,j}^{(2)}$ *for all* $i \in \{1, \ldots, n\}$ *and* $j \in \{1, 2\}$

*Condition b:* $b_{i,1} + b_{i-1,2} = a_{i,1}^{(3)}$ *for all* $i \in \{1, \ldots, n\}$

*Condition c:* $a_{i,j}^{(2)} - a_{i,j}^{(1)} = b_{i,j} - b_{i,j+1}$ *for all* $i \in \{1, \ldots, n\}$ *and* $j \in \{1, 2\}$

*if such a matrix* $\underline{b}$ *exists. If such a matrix exists then it is uniquely determined by the three matrices* $\underline{a_1}$, $\underline{a_2}$ *and* $\underline{a_3}$.

*The coefficient* $F_{M_1,M_3}^{M_2}$ *of* $[M_2]$ *in the Hall algebra multiplication* $[M_1] \circ [M_3]$ *is* 0 *if no* $\underline{b}$ *of this form exists, or, if such a matrix* $\underline{b}$ *exists, is given by*

$$F_{M_1,M_3}^{M_2} = \mathrm{qexp}\left(\sum_{i=1}^{n} b_{i,1}(a_{i-1,2}^{(2)} - b_{i-1,2})\right) \prod_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant 2}} G[a_{i,j}^{(2)}; b_{i,j}](q).$$

Starting from this result, we can move to the general case of multiplying any two $A_n$-modules. The method for deducing our general formula will depend on the fact that any $A_n$-module can be written as the direct sum of a projective module and a semisimple module; we therefore begin by giving some intermediate results on how to calculate multiplications involving only projective and semisimple $A_n$-modules.

**Lemma 2.2.2** *Let* $M_1 = \bigoplus_{i=1}^{n} \alpha_i \cdot {\begin{smallmatrix} i \\ i+1 \end{smallmatrix}}$ *and* $M_3 = \bigoplus_{i=1}^{n} \gamma_i \cdot {\begin{smallmatrix} i \\ i+1 \end{smallmatrix}}$ *be any two projective* $A_n$*-modules. Then*

$$\begin{aligned}[M_1] \circ [M_3] \;=\; & \left(\prod_{j=1}^{n} G[\alpha_j + \gamma_j; \alpha_j](q)\right) \mathrm{qexp}\left(\sum_{j=1}^{n} \alpha_j \gamma_{j+1}\right) \\ & \left[\bigoplus_{i=1}^{n} (\alpha_i + \gamma_i) \cdot {\begin{smallmatrix} i \\ i+1 \end{smallmatrix}}\right].\end{aligned}$$

**Proof:** Suppose $[M_2]$ appears with non-zero coefficient in $[M_1] \circ [M_3]$. Then there exists a short exact sequence

$$0 \longrightarrow M_3 \longrightarrow M_2 \longrightarrow M_1 \longrightarrow 0.$$

This must split, since $M_1$ is projective, so $M_2 \simeq M_1 \oplus M_3$. So

$$[M_1] \circ [M_3] = F^{M_2}_{M_1,M_3} \left[ \bigoplus_{i=1}^{n} (\alpha_i + \gamma_i) \cdot \begin{matrix} i \\ i+1 \end{matrix} \right]$$

for some $F^{M_2}_{M_1,M_3} \in \mathbb{Q}$.

We begin by showing that

$$\left[\alpha_i \cdot \begin{matrix} i \\ i+1 \end{matrix}\right] \circ \left[\gamma_i \cdot \begin{matrix} i \\ i+1 \end{matrix}\right] = G[\alpha_i + \gamma_i; \alpha_i](q) \left[(\alpha_i + \gamma_i) \cdot \begin{matrix} i \\ i+1 \end{matrix}\right] \tag{2.6}$$

for $1 \leqslant i \leqslant n$.

First we show that

$$\left[r \cdot \begin{matrix} i \\ i+1 \end{matrix}\right] \circ \left[\begin{matrix} i \\ i+1 \end{matrix}\right] = G[r+1; 1](q) \left[(r+1) \cdot \begin{matrix} i \\ i+1 \end{matrix}\right] \tag{2.7}$$

for all $r \geqslant 1$ and $1 \leqslant i \leqslant n$. Consider the product $\left[r\cdot\substack{i\\ i+1}\right] \circ [i] \circ [i+1]$. This equals $\left[r\cdot\substack{i\\ i+1}\right] \circ \left(\left[\substack{i\\ i+1}\right] + [i \oplus (i+1)]\right)$ (either by direct calculation, or via Proposition 2.2.1). But it is also equal to $\left[r\cdot\substack{i\\ i+1} \oplus i\right] \circ [i+1]$. We compare coefficients of $\left[(r+1)\cdot\substack{i\\ i+1}\right]$ in these two expressions for this multiplication. The product $\left[r\cdot\substack{i\\ i+1}\right] \circ [i \oplus (i+1)]$ has zero coefficient of $\left[(r+1)\cdot\substack{i\\ i+1}\right]$, since $r\cdot\substack{i\\ i+1}$ is projective; hence the coefficients of $\left[(r+1)\cdot\substack{i\\ i+1}\right]$ in the two products $\left[r\cdot\substack{i\\ i+1}\right] \circ \left(\left[\substack{i\\ i+1}\right] + [i \oplus (i+1)]\right)$ and $\left[r\cdot\substack{i\\ i+1}\right] \circ \left[\substack{i\\ i+1}\right]$ are equal. The coefficient of $\left[(r+1)\cdot\substack{i\\ i+1}\right]$ in $\left[r\cdot\substack{i\\ i+1} \oplus i\right] \circ [i+1]$ is $G[r+1;1](q)$ (again, this can be deduced from Proposition 2.2.1, though it should be clear that this coefficient is just the number of 1-dimensional subspaces of the $(r+1)$-dimensional vector space $\operatorname{soc}\left((r+1)\cdot\substack{i\\ i+1}\right)$ ). Equating these, we get that $G[r+1;1](q)$ is the coefficient of $\left[(r+1)\cdot\substack{i\\ i+1}\right]$ in $\left[r\cdot\substack{i\\ i+1}\right] \circ \left[\substack{i\\ i+1}\right]$. There are no other non-zero cofficients, since $r\cdot\substack{i\\ i+1}$ is projective. Hence $\left[r\cdot\substack{i\\ i+1}\right] \circ \left[\substack{i\\ i+1}\right] = G[r+1;1](q) \left[(r+1)\cdot\substack{i\\ i+1}\right]$.

Now consider the product of $\alpha_i + \gamma_i$ copies of $\left[\substack{i\\ i+1}\right]$. Bracketing this as $\alpha_i$ copies of $\left[\substack{i\\ i+1}\right]$ followed by $\gamma_i$ copies of $\left[\substack{i\\ i+1}\right]$ and calculating each of these products by repeated use of the formula (2.7), we get

$$\prod_{j=1}^{\alpha_i} G[j;1](q) \prod_{j=1}^{\gamma_i} G[j;1](q) \left[\alpha_i.{\,}^{\;\;i}_{i+1}\right] \circ \left[\gamma_i.{\,}^{\;\;i}_{i+1}\right].$$

Alternatively, if we calculate the whole expression directly using (2.7), we get

$$\prod_{j=1}^{\alpha_i+\gamma_i} G[j;1](q) \left[(\alpha_i+\gamma_i).{\,}^{\;\;i}_{i+1}\right].$$

Equating these, we have

$$\left[\alpha_i.{\,}^{\;\;i}_{i+1}\right] \circ \left[\gamma_i.{\,}^{\;\;i}_{i+1}\right] = \frac{\prod_{j=1}^{\alpha_i+\gamma_i} G[j;1](q)}{\prod_{j=1}^{\alpha_i} G[j;1](q) \prod_{j=1}^{\gamma_i} G[j;1](q)} \left[(\alpha_i+\gamma_i).{\,}^{\;\;i}_{i+1}\right]$$

which, when simplified, can be seen to be equal to

$$G[\alpha_i+\gamma_i;\alpha_i](q) \left[(\alpha_i+\gamma_i).{\,}^{\;\;i}_{i+1}\right]$$

as required. This completes the proof of (2.6).

Next, we show that

$$\left[r.{\,}^{\;\;i}_{i+1}\right] \circ \left[s.{\,}^{i+1}_{i+2}\right] = \operatorname{qexp}(rs) \left[r.{\,}^{\;\;i}_{i+1} \oplus s.{\,}^{i+1}_{i+2}\right] \tag{2.8}$$

for $1 \leqslant i \leqslant n$ and all $r, s \in \mathbb{N}_0$.

Since $r.{\,}^{\;\;i}_{i+1}$ is projective, we must have

$$\left[r.{\,}^{\;\;i}_{i+1}\right] \circ \left[s.{\,}^{i+1}_{i+2}\right] = F_{r,s} \left[r.{\,}^{\;\;i}_{i+1} \oplus s.{\,}^{i+1}_{i+2}\right]$$

for some $F_{r,s} \in \mathbb{Q}$. The coefficient $F_{r,s}$ is the number of submodules $M$ of $r.{\,}^{\;\;i}_{i+1} \oplus s.{\,}^{i+1}_{i+2}$ with $M \simeq s.{\,}^{i+1}_{i+2}$ and quotient $r.{\,}^{\;\;i}_{i+1}$. It is clear that all $s$ copies of the simple module $i+2$ in the socle of $r.{\,}^{\;\;i}_{i+1} \oplus s.{\,}^{i+1}_{i+2}$ must lie in $M$. So, we can take the quotient of both $r.{\,}^{\;\;i}_{i+1} \oplus s.{\,}^{i+1}_{i+2}$ and $M$ by the submodule $s.(i+2)$ and thus see that $F_{r,s}$ must be the number of submodules $N$ of

$r.{}^{\,i}_{i+1} \oplus s.(i+1)$ with $N \simeq s.(i+1)$ and quotient $r.{}^{\,i}_{i+1}$. The number of such submodules could be calculated directly; alternatively, note that, by definition, this says that $F_{r,s}$ is the coefficient defined by the multiplication $\left[r.{}^{\,i}_{i+1}\right] \circ [s.(i+1)] = F_{r,s}\left[r.{}^{\,i}_{i+1} \oplus s.(i+1)\right]$. So, it will be sufficient to calculate this product; since $s.(i+1)$ is a semisimple module this can be done using Guo's formula in Proposition 2.2.1. Let

$$\underline{a_1} = \begin{pmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & r \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \qquad \underline{a_2} = \begin{pmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & r \\ s & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \qquad \underline{a_3} = \begin{pmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ s & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}.$$

From Conditions a–c of Proposition 2.2.1, we get

$$\underline{b} = \begin{pmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ s & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}.$$

The formula at the end of Proposition 2.2.1 then gives $F_{r,s} = \operatorname{qexp}(rs)$, as claimed. This completes the proof of (2.8).

We now use this result to show that

$$\left[\alpha_1.{}^{1}_{2}\right] \circ \left[\alpha_2.{}^{2}_{3}\right] \circ \cdots \circ \left[\alpha_n.{}^{n}_{1}\right] = \operatorname{qexp}\left(\sum_{j=1}^{n-1} \alpha_j \alpha_{j+1}\right) \left[\bigoplus_{i=1}^{n} \alpha_i {}^{\;i}_{i+1}\right]. \quad (2.9)$$

Consider a product of the form

$$\left[\alpha_j.{}^{\;j}_{j+1}\right] \circ \left[\bigoplus_{i=j+1}^{n} \alpha_i.{}^{\;i}_{i+1}\right]$$

for some $1 \leqslant j \leqslant n-1$. This must be equal to

$$F_j \left[\bigoplus_{i=j}^{n} \alpha_i.{}^{\;i}_{i+1}\right]$$

for some $F_j \in \mathbb{Q}$, since both modules are projective. Any submodule $N$ of $\bigoplus_{i=j}^{n} \alpha_i \cdot {i \atop i+1}$ with $N \simeq \bigoplus_{i=j+1}^{n} \alpha_i \cdot {i \atop i+1}$ must include the whole of the submodule $\bigoplus_{i=j+2}^{n} \alpha_i \cdot {i \atop i+1}$. So, if we quotient out by this submodule, we see that $F_j$ is the coefficient in the multiplication

$$\left[\alpha_j \cdot {j \atop j+1}\right] \circ \left[\alpha_{j+1} \cdot {j+1 \atop j+2}\right] = F_j \left[\alpha_j \cdot {j \atop j+1} \oplus \alpha_{j+1} \cdot {j+1 \atop j+2}\right].$$

By (2.8), we know that this is $\operatorname{qexp}(\alpha_j \alpha_{j+1})$. Using this result and induction on $n-j$ then completes the proof of (2.9).

We can now use these ingredients to complete the proof of Lemma 2.2.2. We have

$$\begin{aligned}[][M_1] \circ [M_3] &= \operatorname{qexp}\left(-\textstyle\sum_{j=1}^{n-1}(\alpha_j\alpha_{j+1} + \gamma_j\gamma_{j+1})\right) \\ &\qquad \left[\alpha_1 \cdot {1 \atop 2}\right] \circ \left[\alpha_2 \cdot {2 \atop 3}\right] \circ \cdots \circ \left[\alpha_n \cdot {n \atop 1}\right] \\ &\qquad \circ \left[\gamma_1 \cdot {1 \atop 2}\right] \circ \left[\gamma_2 \cdot {2 \atop 3}\right] \circ \cdots \circ \left[\gamma_n \cdot {n \atop 1}\right]\end{aligned} \tag{2.10}$$

by (2.9). Consider the product of projectives

$$\left[\alpha_1 \cdot {1 \atop 2}\right] \circ \left[\alpha_2 \cdot {2 \atop 3}\right] \circ \cdots \circ \left[\alpha_n \cdot {n \atop 1}\right] \circ \left[\gamma_1 \cdot {1 \atop 2}\right] \circ \left[\gamma_2 \cdot {2 \atop 3}\right] \circ \cdots \circ \left[\gamma_n \cdot {n \atop 1}\right]. \tag{2.11}$$

By (2.8), this is

$$\operatorname{qexp}(\alpha_n\gamma_1) \left[\alpha_1 \cdot {1 \atop 2}\right] \circ \left[\alpha_2 \cdot {2 \atop 3}\right] \circ \cdots \circ \left[\alpha_n \cdot {n \atop 1} \oplus \gamma_1 \cdot {1 \atop 2}\right] \circ \left[\gamma_2 \cdot {2 \atop 3}\right] \circ \cdots \circ \left[\gamma_n \cdot {n \atop 1}\right].$$

Then, using a number of multiplications which all clearly have coefficient 1, this equals

$$\operatorname{qexp}(\alpha_n\gamma_1) \left[\alpha_1 \cdot {1 \atop 2}\right] \circ \left[\gamma_1 \cdot {1 \atop 2} \oplus \alpha_2 \cdot {2 \atop 3}\right] \circ \cdots \circ \left[\alpha_n \cdot {n \atop 1}\right] \circ \left[\gamma_2 \cdot {2 \atop 3}\right] \circ \cdots \circ \left[\gamma_n \cdot {n \atop 1}\right].$$

This is then equal to

$$\operatorname{qexp}(\alpha_n\gamma_1 - \alpha_2\gamma_1) \left[\alpha_1 \cdot {1 \atop 2}\right] \circ \left[\gamma_1 \cdot {1 \atop 2}\right] \circ \left[\alpha_2 \cdot {2 \atop 3}\right] \circ \cdots \circ \left[\alpha_n \cdot {n \atop 1}\right] \circ \left[\gamma_2 \cdot {2 \atop 3}\right] \circ \cdots \circ \left[\gamma_n \cdot {n \atop 1}\right]$$

by another application of (2.8). Similarly, we can gather all pairs involving the same indecomposable projective module together, to see that the product of projectives (2.11) is equal to

$$\operatorname{qexp}\left(\alpha_n\gamma_1 - \sum_{j=1}^{n-1}\gamma_j\alpha_{j+1}\right)$$
$$\left[\alpha_1\cdot{}^{1}_{2}\right] \circ \left[\gamma_1\cdot{}^{1}_{2}\right] \circ \left[\alpha_2\cdot{}^{2}_{3}\right] \circ \left[\gamma_2\cdot{}^{2}_{3}\right] \circ \cdots \circ \left[\alpha_n\cdot{}^{n}_{1}\right] \circ \left[\gamma_n\cdot{}^{n}_{1}\right].$$

Combining this with (2.10), we have

$$\begin{aligned}[M_1]\circ[M_3] &= \operatorname{qexp}\left(\alpha_n\gamma_1 - \sum_{j=1}^{n-1}(\gamma_j\alpha_{j+1} - \alpha_j\alpha_{j+1} - \gamma_j\alpha_{j+1})\right)\\ &\quad \left[\alpha_1\cdot{}^{1}_{2}\right] \circ \left[\gamma_1\cdot{}^{1}_{2}\right] \circ \left[\alpha_2\cdot{}^{2}_{3}\right] \circ \left[\gamma_2\cdot{}^{2}_{3}\right] \circ \cdots \circ \left[\alpha_n\cdot{}^{n}_{1}\right] \circ \left[\gamma_n\cdot{}^{n}_{1}\right].\end{aligned}$$

Using (2.6) to calculate each product $\left[\alpha_i\cdot{}^{\;i}_{i+1}\right] \circ \left[\gamma_i\cdot{}^{\;i}_{i+1}\right]$ in this expression, we then have

$$\begin{aligned}[M_1]\circ[M_3] &= \left(\prod_{j=1}^{n} G[\alpha_j+\gamma_j;\alpha_j](q)\right)\\ &\quad \operatorname{qexp}\left(\alpha_n\gamma_1 - \sum_{j=1}^{n-1}(\gamma_j\alpha_{j+1} - \alpha_j\alpha_{j+1} - \gamma_j\alpha_{j+1})\right)\\ &\quad \left[(\alpha_1+\gamma_1)\cdot{}^{1}_{2}\right] \circ \left[(\alpha_2+\gamma_2)\cdot{}^{2}_{3}\right] \circ \cdots \circ \left[(\alpha_n+\gamma_n)\cdot{}^{n}_{1}\right].\end{aligned}$$

Finally, by (2.9), we get

$$\begin{aligned}[M_1]\circ[M_3] &= \operatorname{qexp}\left(\alpha_n\gamma_1 + \sum_{j=1}^{n-1}(\alpha_j+\gamma_j)(\alpha_{j+1}+\gamma_{j+1})\right)\\ &\quad \operatorname{qexp}\left(-\sum_{j=1}^{n-1}(\gamma_j\alpha_{j+1} + \alpha_j\alpha_{j+1} + \gamma_j\gamma_{j+1})\right)\\ &\quad \left(\prod_{j=1}^{n} G[\alpha_j+\gamma_j;\alpha_j](q)\right)[M_1\oplus M_3].\end{aligned}$$

This simplifies to give the required result. □

**Lemma 2.2.3** *Let* $M_1 = \bigoplus_{i=1}^n \alpha_i \cdot \genfrac{}{}{0pt}{}{i}{i+1}$ *be a projective* $A_n$*-module and let* $M_3 = \bigoplus_{i=1}^n \delta_i . i$ *be a semisimple* $A_n$*-module. Then*

$$[M_1] \circ [M_3] = \mathrm{qexp}\left(\sum_{j=1}^{n} \delta_j \alpha_{j-1}\right) \left[\bigoplus_{i=1}^{n} \alpha_i \cdot \genfrac{}{}{0pt}{}{i}{i+1} \oplus \bigoplus_{i=1}^{n} \delta_i . i\right].$$

**Proof:** Since $M_1$ is projective, the only module appearing with non-zero coefficient in $[M_1] \circ [M_3]$ must be $[M_1 \oplus M_3]$. The coefficient can be calculated using Guo's method in Proposition 2.2.1. Let

$$\underline{a_1} = \begin{pmatrix} 0 & \alpha_1 \\ 0 & \alpha_2 \\ \vdots & \vdots \\ 0 & \alpha_n \end{pmatrix} \qquad \underline{a_2} = \begin{pmatrix} \delta_1 & \alpha_1 \\ \delta_2 & \alpha_2 \\ \vdots & \vdots \\ \delta_n & \alpha_n \end{pmatrix} \qquad \underline{a_3} = \begin{pmatrix} \delta_1 & 0 \\ \delta_2 & 0 \\ \vdots & \vdots \\ \delta_n & 0 \end{pmatrix}.$$

From Conditions a–c, it can be calculated that

$$\underline{b} = \begin{pmatrix} \delta_1 & 0 \\ \delta_2 & 0 \\ \vdots & \vdots \\ \delta_n & 0 \end{pmatrix}.$$

We then get $F^{M_2}_{M_1, M_3} = \mathrm{qexp}\left(\sum_{j=1}^n \delta_j \alpha_{j-1}\right)$ from the formula at the end of Proposition 2.2.1. $\square$

**Lemma 2.2.4** *Let* $M_1 = \bigoplus_{i=1}^n \alpha_i \cdot \genfrac{}{}{0pt}{}{i}{i+1}$ *be a projective* $A_n$*-module and let* $M_3 = \bigoplus_{i=1}^n \gamma_i \cdot \genfrac{}{}{0pt}{}{i}{i+1} \oplus \bigoplus_{i=1}^n \delta_i . i$ *be any* $A_n$*-module. Then*

$$\begin{aligned} [M_1] \circ [M_3] &= \left(\prod_{j=1}^n G[\alpha_j + \gamma_j; \alpha_j](q)\right) \mathrm{qexp}\left(\sum_{j=1}^n \alpha_j(\delta_{j+1} + \gamma_{j+1})\right) \\ &\quad \left[\bigoplus_{i=1}^n (\alpha_i + \gamma_i) \cdot \genfrac{}{}{0pt}{}{i}{i+1} \oplus \bigoplus_{i=1}^n \delta_i . i\right]. \end{aligned}$$

**Proof:** By Lemma 2.2.3

$$[M_3] = \mathrm{qexp}\left(-\sum_{j=1}^{n} \delta_j \gamma_{j-1}\right) \left[\bigoplus_{i=1}^{n} \gamma_i \cdot \begin{matrix} i \\ i+1 \end{matrix}\right] \circ \left[\bigoplus_{i=1}^{n} \delta_i . i\right].$$

Then, by Lemma 2.2.2

$$\begin{aligned}[M_1] \circ [M_3] &= \mathrm{qexp}\left(\textstyle\sum_{j=1}^{n} (\alpha_j \gamma_{j+1} - \delta_j \gamma_{j-1})\right) \left(\textstyle\prod_{j=1}^{n} G[\alpha_j + \gamma_j ; \alpha_j](q)\right) \\ &\quad \left[\textstyle\bigoplus_{i=1}^{n} (\alpha_i + \gamma_i) \cdot {}^{\,i}_{i+1}\right] \circ \left[\textstyle\bigoplus_{i=1}^{n} \delta_i . i\right].\end{aligned}$$

Then, using Lemma 2.2.3 again,

$$\begin{aligned}[M_1] \circ [M_3] &= \mathrm{qexp}\left(\textstyle\sum_{j=1}^{n} \delta_j (\alpha_{j-1} + \gamma_{j-1}) + \sum_{j=1}^{n} (\alpha_j \gamma_{j+1} - \delta_j \gamma_{j-1})\right) \\ &\quad \left(\textstyle\prod_{j=1}^{n} G[\alpha_j + \gamma_j ; \alpha_j](q)\right) [M_1 \oplus M_3],\end{aligned}$$

which simplifies to give the required formula. □

**Lemma 2.2.5** *Let* $M_1 = \bigoplus_{i=1}^{n} \alpha_i \cdot {}^{\,i}_{i+1}$ *be a projective* $A_n$*-module and let* $M_3 = \bigoplus_{i=1}^{n} \gamma_i \cdot {}^{\,i}_{i+1} \oplus \bigoplus_{i=1}^{n} \delta_i . i$ *be any* $A_n$*-module. Then*

$$\begin{aligned}[M_3] \circ [M_1] &= \left(\textstyle\prod_{j=1}^{n} G[\alpha_j + \gamma_j ; \alpha_j](q)\right) \mathrm{qexp}\left(\textstyle\sum_{j=1}^{n} \alpha_j (\delta_j \gamma_{j-1})\right) \\ &\quad \left[\textstyle\bigoplus_{i=1}^{n} (\alpha_i + \gamma_i) \cdot {}^{\,i}_{i+1} \oplus \bigoplus_{i=1}^{n} \delta_i . i\right].\end{aligned}$$

**Proof:** Since $M_1$ is projective $[M_3] \circ [M_1] = F[M_3 \oplus M_1]$ for some $F \in \mathbb{Q}$. It can be seen from the proof of Lemma 1.1.4, part 4 that $F$ is also the coefficient in the $\mathcal{H}(A_n^{\mathrm{op}})$ multiplication $[M_1^*] \circ [M_3^*] = F[M_1^* \oplus M_3^*]$. By an appropriate analogy of Lemma 2.2.4 for $A_n^{\mathrm{op}}$-modules, it can be seen that the coefficient will be

$$F = \left(\prod_{j=1}^{n} G[\alpha_j + \gamma_j ; \alpha_j](q)\right) \mathrm{qexp}\left(\sum_{j=1}^{n} \alpha_j (\delta_j \gamma_{j-1})\right)$$

as required. □

**Lemma 2.2.6** *Let* $M_1 = \bigoplus_{i=1}^n \beta_i.i$ *and* $M_3 = \bigoplus_{i=1}^n \delta_i.i$ *be two semisimple* $A_n$*-modules. Then*

$$\begin{aligned}[M_1] \circ [M_3] &= \sum_{\underline{\lambda}} \left( \prod_{j=1}^n G[\beta_i + \delta_i - \lambda_i - \lambda_{i-1}; \delta_i - \lambda_{i-1}](q) \right) \\ &\qquad [\textstyle\bigoplus_{i=1}^n \lambda_i. {i \atop i+1} \oplus \bigoplus_{i=1}^n (\beta_i + \delta_i - \lambda_i - \lambda_{i-1}).i]\end{aligned}$$

*where we sum over all* $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n)$ *such that* $0 \leqslant \lambda_i \leqslant \max\{\beta_i, \delta_{i+1}\}$ *for all* $i \in \{1, \ldots, n\}$.

**Proof:** First, consider which modules may appear with non-zero coefficients in this product. A general $A_n$-module is of the form $\bigoplus_{i=1}^n \lambda_i. {i \atop i+1} \oplus \bigoplus_{i=1}^n \mu_i.i$. If this module appears with non-zero coefficient, then we must have $\lambda_i \leqslant \beta_i$ and $\lambda_i \leqslant \delta_{i+1}$, since each copy of ${i \atop i+1}$ must be produced by a copy of $i$ from the first factor and a copy of $i+1$ from the second factor. So, we have $\lambda_i \leqslant \max\{\beta_i, \delta_{i+1}\}$ for $1 \leqslant i \leqslant n$. Comparing gradings, we must have

$$\begin{pmatrix} \lambda_1 + \lambda_n + \mu_1 \\ \lambda_2 + \lambda_1 + \mu_2 \\ \vdots \\ \lambda_n + \lambda_{n-1} + \mu_n \end{pmatrix} = \begin{pmatrix} \beta_1 + \delta_1 \\ \beta_2 + \delta_2 \\ \vdots \\ \beta_n + \delta_n \end{pmatrix}$$

so $\mu_i = \beta_i + \delta_i - \lambda_i - \lambda_{i-1}$ for all $i \in \{1, \ldots, n\}$. Hence the product $[M_1] \circ [M_3]$ must be of the form

$$\sum_{\underline{\lambda}} F(\underline{\lambda}) \left[ \bigoplus_{i=1}^n \lambda_i. {i \atop i+1} \oplus \bigoplus_{i=1}^n (\beta_i + \delta_i - \lambda_i - \lambda_{i-1}).i \right].$$

where we sum over all $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n)$ such that $0 \leqslant \lambda_i \leqslant \max\{\beta_i, \delta_{i+1}\}$ for all $i \in \{1, \ldots, n\}$. It remains to calculate the coefficients $F(\underline{\lambda})$.

The coefficients $F(\underline{\lambda})$ can be calculated using Proposition 2.2.1. The matrices describing the structures of $M_1$, $M_2$ and $M_3$ are

$$\underline{a_1} = \begin{pmatrix} \beta_1 & 0 \\ \beta_2 & 0 \\ \vdots & \vdots \\ \beta_n & 0 \end{pmatrix} \qquad \underline{a_3} = \begin{pmatrix} \delta_1 & 0 \\ \delta_2 & 0 \\ \vdots & \vdots \\ \delta_n & 0 \end{pmatrix}.$$

$$\underline{a_2} = \begin{pmatrix} \beta_1 + \delta_1 - \lambda_1 - \lambda_n & \lambda_1 \\ \beta_2 + \delta_2 - \lambda_2 - \lambda_1 & \lambda_2 \\ \vdots & \vdots \\ \beta_n + \delta_n - \lambda_n - \lambda_{n-1} & \lambda_n \end{pmatrix}$$

Conditions a–c of Proposition 2.2.1 then determine the matrix $\underline{b}$:

$$\underline{b} = \begin{pmatrix} \delta_1 - \lambda_n & \lambda_1 \\ \delta_2 - \lambda_1 & \lambda_2 \\ \vdots & \vdots \\ \delta_n - \lambda_{n-1} & \lambda_n \end{pmatrix}.$$

Substituting this information into the formula in Proposition 2.2.1, we get the required expression

$$F(\underline{\lambda}) = \left( \prod_{j=1}^{n} G[\beta_i + \delta_i - \lambda_i - \lambda_{i-1}; \delta_i - \lambda_{i-1}](q) \right)$$

for the coefficients $F(\underline{\lambda})$. $\square$

**Theorem 2.2.7** *Let $M_1 = \bigoplus_{i=1}^{n} \alpha_i \cdot {}_{i+1}^{\;i} \oplus \bigoplus_{i=1}^{n} \beta_i . i$ and $M_3 = \bigoplus_{i=1}^{n} \gamma_i \cdot {}_{i+1}^{\;i} \oplus \bigoplus_{i=1}^{n} \delta_i . i$ be any two $A_n$-modules. Then*

$$[M_1] \circ [M_3] = \sum_{\underline{\lambda}} F(\underline{\lambda}) \left[ \bigoplus_{i=1}^{n} (\alpha_i + \gamma_i + \lambda_i) . \begin{matrix} i \\ i+1 \end{matrix} \oplus \bigoplus_{i=1}^{n} (\beta_i + \delta_i - \lambda_i - \lambda_{i-1}) . i \right]$$

*where*

$$\begin{aligned} F(\underline{\lambda}) \;\; = \;\; & \left( \textstyle\prod_{j=1}^{n} G[\beta_j + \delta_j - \lambda_j - \lambda_{j-1}; \delta_j - \lambda_{j-1}](q) \right) \\ & \left( \textstyle\prod_{j=1}^{n} G[\gamma_j + \alpha_j; \gamma_j](q) \right) \left( \textstyle\prod_{j=1}^{n} G[\alpha_j + \gamma_j + \lambda_j; \alpha_j + \gamma_j](q) \right) \\ & \operatorname{qexp} \left( \textstyle\sum_{j=1}^{n} \gamma_j (\beta_j + \alpha_{j-1} - \lambda_j) + \sum_{j=1}^{n} \alpha_j (\delta_{j+1} - \lambda_j) \right) \end{aligned}$$

*and we sum over all* $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n)$ *such that* $0 \leqslant \lambda_i \leqslant \max\{\beta_i, \delta_{i+1}\}$ *for* $1 \leqslant i \leqslant n$.

This theorem extends the result of Guo given in Proposition 2.2.1 and gives an explicit formula for calculating the product in $\mathcal{H}(A_n)$ of any two $A_n$-modules.

**Proof:** By Lemma 2.2.3,

$$[M_3] = \mathrm{qexp}\left(-\sum_{j=1}^{n} \delta_j \gamma_{j-1}\right) \left[\bigoplus_{i=1}^{n} \gamma_i \cdot {}_{\,i+1}^{\,i}\right] \circ \left[\bigoplus_{i=1}^{n} \delta_i . i\right].$$

Then, by Lemma 2.2.5,

$$\begin{aligned}[M_1] \circ [M_3] &= \left(\textstyle\prod_{j=1}^{n} G[\gamma_j + \alpha_j; \gamma_j](q)\right) \\ &\quad \mathrm{qexp}\left(\textstyle\sum_{j=1}^{n} \gamma_j(\beta_j + \alpha_{j-1}) - \sum_{j=1}^{n} \delta_j \gamma_{j-1}\right) \\ &\quad\quad [\textstyle\bigoplus_{i=1}^{n} (\alpha_i + \gamma_i) \cdot {}_{i+1}^{\,i} \oplus \bigoplus_{i=1}^{n} \beta_i . i] \circ [\bigoplus_{i=1}^{n} \delta_i . i].\end{aligned}$$

By Lemma 2.2.3 again,

$$\begin{aligned}[\textstyle\bigoplus_{i=1}^{n} (\alpha_i + \gamma_i) \cdot {}_{i+1}^{\,i} \oplus \bigoplus_{i=1}^{n} \beta_i . i] &= \mathrm{qexp}\left(-\textstyle\sum_{j=1}^{n} \beta_j(\alpha_{j-1} + \gamma_{j-1})\right) \\ &\quad [\textstyle\bigoplus_{i=1}^{n} (\alpha_i + \gamma_i) \cdot {}_{i+1}^{\,i}] \circ [\bigoplus_{i=1}^{n} \beta_i . i].\end{aligned}$$

By Lemma 2.2.6, $[\bigoplus_{i=1}^{n} \beta_i . i] \circ [\bigoplus_{i=1}^{n} \delta_i . i]$ is equal to

$$\begin{gathered}\textstyle\sum_{\underline{\lambda}} \left(\prod_{j=1}^{n} G[\beta_j + \delta_j - \lambda_j - \lambda_{j-1}; \delta_j - \lambda_{j-1}](q)\right) \\ [\textstyle\bigoplus_{i=1}^{n} \lambda_i \cdot {}_{i+1}^{\,i} \oplus \bigoplus_{i=1}^{n} (\beta_i + \delta_i - \lambda_i - \lambda_{i-1}) . i]\end{gathered}$$

where we sum over all $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n)$ such that $0 \leqslant \lambda_i \leqslant \max\{\beta_i, \delta_{i+1}\}$ for

$1 \leqslant i \leqslant n$. Then, by Lemma 2.2.4,

$$\begin{aligned}
&\left[\bigoplus_{i=1}^{n}(\alpha_i+\gamma_i)\cdot{}^{\,i}_{i+1}\right] \circ \left[\bigoplus_{i=1}^{n}\lambda_i\cdot{}^{\,i}_{i+1} \oplus \bigoplus_{i=1}^{n}(\beta_i+\delta_i-\lambda_i-\lambda_{i-1}).i\right] \\
&= \left(\prod_{j=1}^{n} G[\alpha_j+\gamma_j+\lambda_j;\alpha_j+\gamma_j](q)\right) \\
&\quad \operatorname{qexp}\left(\sum_{j=1}^{n}(\beta_j+\delta_j-\lambda_j-\lambda_{j-1})(\alpha_{j-1}+\gamma_{j-1})+\sum_{j=1}^{n}(\alpha_j+\gamma_j)\lambda_{j+1}\right) \\
&\quad\quad \left[\bigoplus_{i=1}^{n}(\alpha_i+\gamma_i+\lambda_i)\cdot{}^{\,i}_{i+1} \oplus \bigoplus_{i=1}^{n}(\beta_i+\delta_i-\lambda_i-\lambda_{i-1}).i\right].
\end{aligned}$$

Putting all this information together in a single expression and simplifying the power of $q$ gives the required formula for $[M_1] \circ [M_3]$. $\square$

## 2.3 The cyclic quiver with two vertices

We now focus on the case of a cyclic quiver with two vertices. Consider the algebra $A_2$ as defined above. There are four indecomposable $A_2$-modules: the two simple modules 1 and 2 corresponding to the two vertices of $Q_2$, and two projective indecomposable modules ${}^{1}_{2}$ and ${}^{2}_{1}$. As a special case of Theorem 2.2.7, we have the following formula for multiplication in $\mathcal{H}(A_2)$:

**Theorem 2.3.1** *Let $M_1 = \alpha.{}^{1}_{2}\oplus\beta.{}^{2}_{1}\oplus\lambda.1\oplus\mu.2$ and $M_3 = \gamma.1\oplus\delta.2\oplus\varepsilon.{}^{1}_{2}\oplus\eta.{}^{2}_{1}$ be any two $A$-modules. Then $[M_1] \circ [M_3]$ equals*

$$\sum_{i=0}^{\min\{\lambda,\delta\}}\ \sum_{j=0}^{\min\{\mu,\gamma\}} F_{i,j}\,[M_{i,j}]$$

*where $[M_{i,j}]$ denotes*

$$\left[(\alpha+\varepsilon+i).{}^{1}_{2}\oplus(\beta+\eta+j).{}^{2}_{1}\oplus(\lambda+\gamma-i-j).1\oplus(\mu+\delta-i-j).2\right]$$

*and*

$$\begin{aligned} F_{i,j} &= \operatorname{qexp}\left(\alpha(\delta-i)+\beta(\gamma-j)+\varepsilon(\lambda-i)+\eta(\mu-j)+\alpha\eta+\beta\varepsilon\right)\\ &\qquad G[\lambda+\gamma-i-j;\gamma-j](q)\, G[\mu+\delta-i-j;\delta-i](q)\\ &\qquad\quad G[\alpha+\varepsilon+i;\alpha+\varepsilon](q)\, G[\beta+\eta+j;\beta+\eta](q)\\ &\qquad\qquad G[\alpha+\varepsilon;\alpha](q)\, G[\beta+\eta;\beta](q). \end{aligned}$$

We also saw in the previous section that

$$\mathcal{C}(kQ_2) \simeq \frac{U_q(\mathfrak{n}^+)}{(v^2-q)},$$

where $U_q(\mathfrak{n}^+)$ is the positive part of the quantised Kac–Moody algebra of type $\widetilde{A}_1$. It is clear that any relation which holds in $\mathcal{C}(kQ_2)$ must also hold in $\mathcal{C}(A_2)$, since the $A_2$-modules form a subset of the $kQ_2$-modules and any relevant coefficients in the composition algebra multiplications will be the same in $\mathcal{C}(A_2)$ as in $\mathcal{C}(kQ_2)$. So, we must have $\mathcal{C}(A_2) \simeq \mathcal{C}(kQ_2)/I$, where $I$ is some ideal of $\mathcal{C}(kQ_2)$ generated by further relations which hold in $\mathcal{C}(A_2)$ but not in $\mathcal{C}(kQ_2)$. Equivalently, there exists a surjective algebra homomorphism

$$\phi : \frac{U_q(\mathfrak{n}^+)}{(v^2-|k|)} \twoheadrightarrow \mathcal{C}(A_2).$$

The aim of this section is to determine $\ker\phi$.

Consider the composition algebra $\mathcal{C}(A_2)$. A multiplication in $\mathcal{C}(A_2)$ is given by a word in the simple modules 1 and 2. Given such a multiplication $w$, we write $w^*$ for the word obtained by reversing $w$, and $w'$ for the word obtained by replacing all occurrences of 1 with 2 and vice versa. For example, $(1^2 2^3 1)^* = 1 2^3 1^2$ and $(1^2 2^3 1)' = 2^2 1^3 2$.

**Theorem 2.3.2** *Let $w$ be any multiplication in $\mathcal{C}(A_2)$ of grading $\binom{n}{n}$ for some $n$. Then $(w^*)' = w$ in $\mathcal{C}(A_2)$.*

**Proof:** Let $M$ be any $A_2$-module of grading $\binom{n}{n}$. We need to show that $[M]$ has the same coefficient in the two multiplications $w$ and $(w^*)'$.

The map $\phi : A_2 \longrightarrow A_2$, defined by $\phi(\alpha) = \beta$, $\phi(\beta) = \alpha$ and $\phi(e_i) = e_i$ for $i = 0, 1$, is an anti-automorphism of $A_2$. This tells us that $A_2 \simeq A_2^{\text{op}}$. An easy generalisation of Lemma 1.1.4 part 4 (and the fact that 1 and 2 are self-dual) then tells us that the coefficient of $[M]$ in $w$ is the same as the coefficient of $[M^*]$ in $w^*$. And the coefficient of $[M^*]$ in $w^*$ will clearly be the same as the coefficient of $[(M^*)']$ in $(w^*)'$, where $(M^*)'$ is the module obtained by replacing all occurrences of 1 by 2 and vice versa. Now note that, since $M$ has grading $\binom{n}{n}$, it must be of the form

$$M = a.\begin{smallmatrix}1\\2\end{smallmatrix} \oplus b.\begin{smallmatrix}2\\1\end{smallmatrix} \oplus (n-(a+b)).1 \oplus (n-(a+b)).2$$

for some $a, b, \in \mathbb{N}_0$. Then we have

$$M^* = a.\begin{smallmatrix}2\\1\end{smallmatrix} \oplus b.\begin{smallmatrix}1\\2\end{smallmatrix} \oplus (n-(a+b)).1 \oplus (n-(a+b)).2$$

and

$$(M^*)' = a.\begin{smallmatrix}1\\2\end{smallmatrix} \oplus b.\begin{smallmatrix}2\\1\end{smallmatrix} \oplus (n-(a+b)).2 \oplus (n-(a+b)).1 = M.$$

This completes the proof. □

Let $w$ be a word in $E_1$ and $E_2$ which gives a multiplication in $U(\mathfrak{n}^+)$ of grading $\binom{n}{n}$ for some $n$. Define $w^*$ and $w'$ by analogy with the definitions for multiplications in $\mathcal{C}(A_2)$. Let $I$ be the ideal of $U(\mathfrak{n}^+)/(v^2 - |k|)$ generated by all elements of the form $(w^*)' - w$ with $w$ of grading $\binom{n}{n}$. Theorem 2.3.2 then tells us that $I$ is contained in the kernel of the surjective algebra homomorphism $\phi$.

Let $U$ denote the quotient of $U_q(\mathfrak{n}^+)/(v^2-q)$ by our ideal $I$. Theorem 2.3.2 then tells us that $\mathcal{C}(A_2)$ is a homomorphic image of $U$. We will see below

(Theorem 2.3.5) that in fact $U \simeq \mathcal{C}(A_2)$. We will work towards a proof of this result by comparing dimensions of $U$ and $\mathcal{C}(A_2)$. The following result tells us about the dimensions of certain graded parts of $\mathcal{C}(A_2)$.

**Lemma 2.3.3** *The $\binom{n}{n-1}$-graded part of $\mathcal{C}(A_2)$ has dimension $\frac{1}{2}n\,(n+1)$ for all $n \in \mathbb{N}_0$.*

**Proof:** Since $\mathcal{C}(A_2) \leqslant \mathcal{H}(A_2)$, we know that dim $\mathcal{C}(A_2)_{\binom{n}{n-1}}$ is at most the number of isomorphism classes of $A_2$-modules of grading $\binom{n}{n-1}$. The number of such modules with $i$ projective indecomposable summands will be $i+1$, where $i$ can range between 0 and $n-1$. So, the total number of modules of this grading is $1+2+\cdots+n = \frac{1}{2}n\,(n+1)$. Hence $\dim \mathcal{C}(A_2)_{\binom{n}{n-1}} \leqslant \frac{1}{2}n\,(n+1)$. To see that we have equality, we need to show that $\mathcal{C}(A_2)_{\binom{n}{n-1}} = \mathcal{H}(A_2)_{\binom{n}{n-1}}$; that is, that $[M] \in \mathcal{C}(A_2)$ for all $A_2$-modules $M$ of grading $\binom{n}{n-1}$.

We prove the above claim by induction on $n$. the result clearly holds for $n = 0$. Suppose $[M] \in \mathcal{C}(A_2)$ for all $A_2$-modules of grading $\binom{n}{n-1}$, and consider modules of grading $\binom{n+1}{n}$.

Consider the products

$$[2] \circ [1] \circ \left[(n-2).\begin{smallmatrix}1\\2\end{smallmatrix} \oplus 1 \oplus 1 \oplus 2\right]$$

and

$$\left[(n-2).\begin{smallmatrix}1\\2\end{smallmatrix} \oplus 1 \oplus 1 \oplus 2\right] \circ [1] \circ [2].$$

By the inductive hypothesis, these both lie in $\mathcal{C}(A_2)$. Using the formula of Theorem 2.3.1, we have

$$\begin{aligned}
&[2] \circ [1] \circ \left[(n-2).\begin{smallmatrix}1\\2\end{smallmatrix} \oplus 1 \oplus 1 \oplus 2\right] - \left[(n-2).\begin{smallmatrix}1\\2\end{smallmatrix} \oplus 1 \oplus 1 \oplus 2\right] \circ [1] \circ [2] \\
&= -q(q^{n-2} + q^{n-3} + \cdots + q + 1)(q+1)\left[(n-1).\begin{smallmatrix}1\\2\end{smallmatrix} \oplus 1 \oplus 1 \oplus 2\right] \\
&\quad + q^{n-1}(q+1)\left[(n-2).\begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\end{smallmatrix} \oplus 1 \oplus 1 \oplus 2\right] \in \mathcal{C}(A_2).
\end{aligned}$$

Using Theorem 2.3.1 again, we also have

$$
\begin{aligned}
&(q^{n-2}+q^{n-3}+\cdots+q+1)[2]\circ[1]\circ\left[(n-1).{}^{1}_{2}\oplus 1\right]\\
&\quad -q^{n-2}\left[(n-2).{}^{1}_{2}\oplus{}^{2}_{1}\oplus 1\right]\circ[1]\circ[2]\\
&\qquad = q^{n-1}(q+1)(q^{n-2}+q^{n-3}+\cdots q+1)\left[(n-1).{}^{1}_{2}\oplus 1\oplus 1\oplus 2\right]\\
&\qquad\quad -q^{2n-3}(q+1)\left[(n-2).{}^{1}_{2}\oplus{}^{2}_{1}\oplus 1\oplus 1\oplus 2\right];
\end{aligned}
$$

by the inductive hypothesis this is also an element of $\mathcal{C}(A_2)$. So, we have two linear combinations of the same pair of modules which both lie in $\mathcal{C}(A_2)$; the matrix of coefficients corresponding to these combinations is

$$
X=\begin{pmatrix} q^{n-1}(q+1) & -q(q+1)(q^{n-2}+q^{n-3}+\cdots+q+1) \\ q^{2n-3}(q+1) & q^{n-1}(q+1)(q^{n-2}+q^{n-3}+\cdots+q+1) \end{pmatrix}.
$$

Since $q=|k|$ is neither 0 nor a root of 1, we have $\det X\neq 0$ and $X$ is invertible. Hence

$$
\left[(n-1).{}^{1}_{2}\oplus 1\oplus 1\oplus 2\right]\in\mathcal{C}(A_2)
$$

and

$$
\left[(n-2).{}^{1}_{2}\oplus{}^{2}_{1}\oplus 1\oplus 1\oplus 2\right]\in\mathcal{C}(A_2).
$$

Next, we see that $[M]\in\mathcal{C}(A_2)$ for all $M$ of grading $\binom{n+1}{n}$ with at most three simple direct summands. (Note that any module of this grading must have an odd number of simple direct summands, so this case covers all modules with either one or three such summands). We prove this by induction on the number of copies of ${}^{2}_{1}$ in $M$. The product $\left[(n-1).{}^{1}_{2}\oplus 1\right]\circ[1]\circ[2]$ is a linear combination of $\left[n.{}^{1}_{2}\oplus 1\right]$ and $\left[(n-1).{}^{1}_{2}\oplus 1\oplus 1\oplus 2\right]$. We have seen that $\left[(n-1).{}^{1}_{2}\oplus 1\oplus 1\oplus 2\right]\in\mathcal{C}(A_2)$; hence we also get $\left[n.{}^{1}_{2}\oplus 1\right]\in\mathcal{C}(A_2)$. These cover the case where there are no copies of ${}^{2}_{1}$ in $M$. Now suppose

$$
\left[(n-i).{}^{1}_{2}\oplus(i-1).{}^{2}_{1}\oplus 1\oplus 1\oplus 2\right]\in\mathcal{C}(A_2).
$$

We wish to show that

$$\left[(n-i).\begin{matrix}1\\2\end{matrix} \oplus i.\begin{matrix}2\\1\end{matrix} \oplus 1\right] \in \mathcal{C}(A_2)$$

and

$$\left[(n-i-1).\begin{matrix}1\\2\end{matrix} \oplus i.\begin{matrix}2\\1\end{matrix} \oplus 1 \oplus 1 \oplus 2\right] \in \mathcal{C}(A_2).$$

The product

$$[2] \circ [1] \circ \left[(n-i).\begin{matrix}1\\2\end{matrix} \oplus (i-1).\begin{matrix}2\\1\end{matrix} \oplus 1\right]$$

is a linear combination of

$$\left[(n-i).\begin{matrix}1\\2\end{matrix} \oplus (i-1).\begin{matrix}2\\1\end{matrix} \oplus 1 \oplus 1 \oplus 2\right]$$

and

$$\left[(n-i).\begin{matrix}1\\2\end{matrix} \oplus i.\begin{matrix}2\\1\end{matrix} \oplus 1\right].$$

Our inductive hypothesis states that the first of these modules lies in $\mathcal{C}(A_2)$; hence the second will also be in $\mathcal{C}(A_2)$. Now consider the product

$$\left[(n-i-1).\begin{matrix}1\\2\end{matrix} \oplus i.\begin{matrix}2\\1\end{matrix} \oplus 1\right] \circ [1] \circ [2].$$

This is a linear combination of

$$\left[(n-i-1).\begin{matrix}1\\2\end{matrix} \oplus i.\begin{matrix}2\\1\end{matrix} \oplus 1\right]$$

and

$$\left[(n-i-1).\begin{matrix}1\\2\end{matrix} \oplus i.\begin{matrix}2\\1\end{matrix} \oplus 1 \oplus 1 \oplus 2\right].$$

We have just seen that

$$\left[(n-i-1).\begin{matrix}1\\2\end{matrix} \oplus i.\begin{matrix}2\\1\end{matrix} \oplus 1\right] \in \mathcal{C}(A_2);$$

hence

$$\left[(n-i-1).\begin{matrix}1\\2\end{matrix} \oplus i.\begin{matrix}2\\1\end{matrix} \oplus 1 \oplus 1 \oplus 2\right] \in \mathcal{C}(A_2).$$

This completes the induction showing that $[M] \in \mathcal{C}(A_2)$ for all $M$ of grading $\binom{n+1}{n}$ with at most three simple direct summands.

Finally, we show that we have $[M] \in \mathcal{C}(A_2)$ for

$$M = r.\begin{matrix}1\\2\end{matrix} \oplus (n-r-i+1).\begin{matrix}2\\1\end{matrix} \oplus i.1 \oplus (i-1).2$$

any module of grading $\binom{n+1}{n}$. We prove this by induction on $i$: assume that all modules with fewer than $i$ copies of 1 in their direct sum decomposition lie in $\mathcal{C}(A_2)$. The cases $i = 1$ and $i = 2$ are done above. Consider

$$[1] \circ \left[r.\begin{matrix}1\\2\end{matrix} \oplus (n-r-i+1).\begin{matrix}2\\1\end{matrix} \oplus (i-2).1 \oplus (i-3).2\right] \circ [2],$$

which will be an element of $\mathcal{C}(A_2)$ by our original inductive hypothesis. This product will be a linear combination of

$$\left[(r+2).\begin{matrix}1\\2\end{matrix} \oplus (n-r-i+1).\begin{matrix}2\\1\end{matrix} \oplus (i-2).1 \oplus (i-3).2\right],$$

$$\left[(r+1).\begin{matrix}1\\2\end{matrix} \oplus (n-r-i+1).\begin{matrix}2\\1\end{matrix} \oplus (i-1).1 \oplus (i-2).2\right]$$

and

$$\left[r.\begin{matrix}1\\2\end{matrix} \oplus (n-r-i+1).\begin{matrix}2\\1\end{matrix} \oplus i.1 \oplus (i-1).2\right].$$

The inductive hypothesis tells us that the first two of these modules lie in $\mathcal{C}(A_2)$; hence we also have $[M] \in \mathcal{C}(A_2)$. This completes the proof that all modules of grading $\binom{n+1}{n}$ lie in $\mathcal{C}(A_2)$. □

We now look at the parts of $U$ with grading of this form and calculate the corresponding dimensions.

**Lemma 2.3.4** *The $\binom{n}{n-1}$-graded part of $U$ has dimension $\frac{1}{2}\,n\,(n+1)$ for all $n \in \mathbb{N}_0$.*

**Proof:** We have seen that $\mathcal{C}(A_2)$ is a homomorphic image of $U$ and that $\dim \mathcal{C}(A_2)_{\binom{n}{n-1}} = \frac{1}{2}n\,(n+1)$, which tells us that $\dim U_{\binom{n}{n-1}} \geqslant \frac{1}{2}n\,(n+1)$. So, it will be sufficient to exhibit $\frac{1}{2}n\,(n+1)$ elements of $U$ which span $U_{\binom{n}{n-1}}$. We claim that the set $S_n = \{(12)^a(21)^b1^c2^{c-1} : a+b+c = n \text{ and } a,b,c \in \mathbb{N}_o\}$ will have this property. The set $S_n$ has size $1+2+\cdots+n$, since for $a = n$ there is 1 possible value of $b$, for $a = n-1$ there are 2 possible values of $b$, and so on.

We begin by proving that $U_{\binom{n+1}{n}}$ is spanned by elements of the form $12w$, $21w$, $w12$ and $w21$ with $w$ any word in $U$ of grading $\binom{n}{n-1}$. The words in $U_{\binom{n+1}{n}}$ which are not of this form are those of the form $1^2v1^2$, $1^2v2^2$, $2^2v1^2$ and $2^2v2^2$ for some $v$. We have $1^2v1^2 = 1(1v1^2) = 1(2^2(v^*)'2)$ in $U$, which is of the form required. Consider next $1^2v2^2$. If $v$ begins with 2, then this is of the form $1^22u2^2 = 1(12u2^2) = 1^3(u^*)'12$, which is of the form required. If $v$ begins with 1, then the quantum Serre relations for $U$ can be applied repeatedly to show that $1^2v2^2$ lies in the span of three elements beginning $1^2$, $121$ and $21^2$. The first of these is dealt with by the previous case, and the other two are of the form required. A word of the form $2^2v1^2$ can be dealt with similarly, since $2^2v1^2 = (1^2v^*2^2)^*$. Finally, for any element of the form $2^2v2^2$, repeated use of the quantum Serre relations show that it must be in the span of elements of the form $2^21u2^2$, $21u2^2$ and $12u2^2$. So it is sufficient to show that $2^21u2^2$ lies in the span of the elements of our preferred form. Using either the quantum Serre relations again, or the relation $12^21 = 21^22$, we can show that it must lie in the span of some elements of the form $2^21^22y$ and $2^21^ry$ for $r \geqslant 3$. We have $2^21^22y = 2(21^22)y = 212^21y$, which is of the form required, and $2^21^ry \in \mathrm{sp}\{2121^{r-1}y, 21^221^{r-2}y, 21^321^{r-3}y\}$. This completes the proof that $U_{\binom{n+1}{n}}$ is spanned by products of the form $12w$, $21w$, $w12$ and

$w21$ for $w$ any word in $U$ of grading $\binom{n}{n-1}$.

We now prove that the set $S_n$ spans $U_{\binom{n}{n-1}}$ by induction on $n$. The base case $n = 1$ is clear. So, suppose $S_n$ spans $U_{\binom{n}{n-1}}$ and consider $U_{\binom{n+1}{n}}$. By the result of the previous paragraph, it is sufficient to show that all products of the form $12w$, $21w$, $w12$ and $w21$ with $w$ of grading $\binom{n}{n-1}$ lie in the span of $S_{n+1}$. Our inductive hypothesis tells us that $w$ lies in the span of $S_n$ in each case; we may assume without loss of generality that $w \in S_n$.

For $w \in S_n$, it is immediately clear that $12w \in S_{n+1}$. For an arbitrary $w = (12)^a(21)^b1^c2^{c-1} \in S_n$, we have $21w = (21)(12)^a(21)^b1^c2^{c-1} = (12)^a(21)^{b+1}1^c2^{c-1} \in S_{n+1}$, since the relation $12^21 = 21^22$ of $U$ tells us that $12$ and $21$ commute. For $w = (12)^a(21)^b1^c2^{c-1}$, we see that $w12$ begins $12$ or $21$ (and is thus covered by the previous two cases) unless $a = b = 0$. If $a = b = 0$ then $w12 = 1^c2^{c-1}12 = 1(1^{c-1}2^{c-1}12) = 1^221^{c-1}1^{c-1}$ in $U$. Repeated use of the quantum Serre relations shows that $1^221^{c-1}2^{c-1}$ lies in the span of $1^{c+1}2^c$, $121^c2^{c-1}$ and $21^{c+1}2^{c-1}$. Then $1^{c+1}2^c \in S_{n+1}$, and $121^c2^{c-1}$ and $21^{c+1}2^{c-1}$ are both covered by the previous two cases. Finally, for $w = (12)^a(21)^b1^c2^{c-1}$, the word $w21$ also begins with $12$ or $21$ unless $a = b = 0$. If $a = b = 0$ then $w21 = 1^c2^c1 = 1(1^{c-1}2^c1) = 121^c2^{c-1}$, which also begins with $12$ and is thus covered by the previous cases. Hence all elements of $U_{\binom{n+1}{n}}$ lie in the span of $S_{n+1}$, completing the inductive step. $\square$

These results on dimensions then allow us to prove the following result which describes the structure of $\mathcal{C}(A_2)$.

**Theorem 2.3.5** *Let $U$ be the quotient of $U_q(\mathfrak{n}^+)/(v^2 - q)$ by the ideal $I$ generated by all elements of the form $(w^*)' - w$ with $w$ of grading $\binom{n}{n}$ for some $n$. Then $U \simeq \mathcal{C}(A_2)$.*

**Proof:** We need to show that there are no relations in $\mathcal{C}(A_2)$ which do not also hold in $U$. Suppose, for a contradiction, that $\sum_w a_w w = 0$, for some $a_w \in \mathbb{Q}$, is such a relation of grading $\binom{m}{n}$. We may assume without loss of generality that $n \geqslant m$, since the algebra $\mathcal{C}(A_2)$ is symmetric in 1 and 2. Then $[1]^{\circ(n+1-m)} \sum_w a_w w = 0$ is also a relation which holds in $\mathcal{C}(A_2)$. This relation has grading $\binom{n+1}{n}$. By Lemmas 2.3.3 and 2.3.4, we have

$$\dim \mathcal{C}(A_2)_{\binom{n+1}{n}} = \dim U_{\binom{n+1}{n}}.$$

Since $\mathcal{C}(A_2)$ is a homomorphic image of $U$, this equality of dimensions tells us that there cannot be any relations of grading $\binom{n+1}{n}$ which hold in $\mathcal{C}(A_2)$ but not in $U$. So, we must have $[1]^{\circ(n+1-m)} \sum_w a_w w = 0$ in $U$. However, pre-multiplication by 1 is an injective map from $U$ to itself, so we must then have $\sum_w a_w w = 0$ in $U$, giving the required contradiction. $\square$

# Chapter 3

# Describing some quotients of Hall algebras

In this chapter we shall look at some quotients of Hall algebras of self-injective algebras. We begin by giving a general result which describes the quotient of a Hall algebra by the ideal generated by isomorphism classes of projective modules. We follow this by looking for a more explicit description of quotients of Hall algebras of certain group algebras $kG$, where we take the quotient by the ideal generated by all relatively $H$-projective modules for a specified subgroup $H \leqslant G$.

## 3.1 The ideal generated by projective modules

**Theorem 3.1.1** *Let $\Lambda$ be a self-injective finitary algebra over a finite field $k$. Let $A = \Lambda/\mathrm{soc}\Lambda$. Then $\mathcal{H}(A) \simeq \mathcal{H}(\Lambda)/I$, where $I$ is the ideal of $\mathcal{H}(\Lambda)$ generated by the isomorphism classes of projective $\Lambda$-modules.*

**Proof:** First, note that the set of isomorphism classes of indecomposable $A$-modules can be identified with the set of isomorphism classes of non-projective indecomposable $\Lambda$-modules. Next we claim that $I$ is given by the span over $\mathbb{Q}$ of all isomorphism classes of $\Lambda$-modules which are not projective-free. Assuming this charcterisation of $I$, we see that a linear combination of terms in $\mathcal{H}(\Lambda)$ will be zero in $\mathcal{H}(\Lambda)/I$ if and only if all the projective-free terms sum to zero. Since projective-free $\Lambda$-modules correspond precisely to $A$-modules, this implies that a linear combination of terms in $\mathcal{H}(\Lambda)$ will be zero in $\mathcal{H}(\Lambda)/I$ if and only if the corresponding linear combination of terms in $\mathcal{H}(A)$ is zero; that is, $\mathcal{H}(\Lambda)/I$ and $\mathcal{H}(A)$ satisfy precisely the same relations. Hence $\mathcal{H}(A) \simeq \mathcal{H}(\Lambda)/I$ via the obvious isomorphism. So, it suffices to prove the characterisation of $I$ claimed above.

Let $M$ be a non-zero projective-free $\Lambda$-module. Suppose some projective $\Lambda$-module $P$ occurs in a filtration of $M$; say we have

$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_r = 0$$

with $M_i/M_{i+1} \simeq P$ for some $i$. The short exact sequence

$$0 \to M_{i+1} \to M_i \to P \to 0$$

must split, since $P$ is projective. So, we have $M_i = M_{i+1} \oplus P$. This tells us that $P$ is a submodule of $M$. Now we look at the short exact sequence

$$0 \to P \to M \to M/P \to 0.$$

Since $\Lambda$ is a self-injective algebra, all projective modules are also injective and so this sequence must also split; this tells us that $P$ is a direct summand of $M$. This cannot occur, however, since we have assumed that $M$ is projective-free. Hence $[M]$ must have zero coefficient in any multiplication in $\mathcal{H}(\Lambda)$ involving a projective $\Lambda$-module, which tells us that $[M] \notin I$.

Now suppose $M$ is a $\Lambda$-module which is not projective-free. We have a decomposition $M = P \oplus N$ with $P$ projective and non-zero and $N$ projective-free. Consider the multiplication $[P] \circ [N]$ in $\mathcal{H}(\Lambda)$, which gives an element of $I$. If $[X]$ has a non-zero coefficient in this product, then there exists a filtration

$$X \supseteq N \supseteq 0$$

with $X/N \simeq P$ and hence a short exact sequence

$$0 \to N \to X \to P \to 0.$$

This must split, since $P$ is projective. Hence the only module with non-zero coefficient in the multiplication $[P] \circ [N]$ is $[P \oplus N] = [M]$; it is clear that the relevant coefficient $\alpha \in \mathbb{Q}$ must indeed be non-zero. So, we have $[M] = (1/\alpha)[P] \circ [N] \in I$. This completes the proof. □

## 3.2 The ideal generated by relatively projective modules

It would be good to be able to extend this result by giving a general description of the quotient $\mathcal{H}(\Lambda)/I$, where $\Lambda = kG$ for some finite group $G$ and $I$ is the ideal of $\mathcal{H}(\Lambda)$ generated by all relatively $H$-projective $kG$-modules for some fixed subgroup $H \leqslant G$. However, this seems to be a rather too ambitious target; instead we shall look at some concrete examples where, for particular choices of $G$ and $H$, we are able to describe the quotient $\mathcal{H}(kG)/I$.

We shall look at the example $G = C_{2^n}$ and $H = C_{2^{n-1}}$ for some $n \geqslant 2$, with $k$ a field of characteristic 2. Let $G = \langle g \rangle$. The indecomposable $kG$-modules are $M_1, \ldots, M_{2^n}$, where $g$ acts on $M_i$ as a Jordan block matrix of size $i$ with eigenvalue 1. The relatively $H$-projective modules will be all those $kG$-modules such that all indecomposable summands are of even length. Thus we can see that this is an approachable example because a large proportion of $kG$-modules are relatively $H$-projective, which will ensure that $\mathcal{H}(kG)/I$ is "not too big", perhaps making it easier to decribe.

Notice that $kG \simeq k[X]/\langle X^{2^n} \rangle$ by putting $X = 1 - g$. So, describing our quotient will be equivalent to describing $\mathcal{H}(\Lambda)/I$, where $\Lambda = k[X]/\langle X^{2^n} \rangle$ and $I$ is the ideal generated by all modules with all summands of even length. We shall generalise the problem and look instead at $\Lambda = k[X]/\langle X^{2r} \rangle$ for any $r \geqslant 1$.

Let $\Lambda = k[X]/\langle X^{2r} \rangle$ for some $r \geqslant 1$, with $k$ any finite field. This algebra has $2r$ indecomposable modules $M_1, \ldots, M_{2r}$, where the action of $X$ on $M_i$ is given by a Jordan block matrix of size $i$ with eigenvalue 0. Notice that $\Lambda$-modules correspond to partitions with no part of length greater than $2r$. If $\lambda = (\lambda_1, \ldots, \lambda_t)$ is such a partition, we shall write $M_\lambda$ for the $\Lambda$-module $M_{\lambda_1} \oplus \cdots \oplus M_{\lambda_t}$.

Consider the Hall algebra $\mathcal{H}(\Lambda)$. Note that this is a graded algebra, with grading given by module dimension (since there is only one simple $\Lambda$-module and hence only one possible dimension vector for each dimension). We clearly have $\Lambda \simeq \Lambda^{\mathrm{op}}$ since $\Lambda$ is commutative, so all duals of left $\Lambda$-modules can also be thought of as left $\Lambda$-modules. Note also that each indecomposable—and hence each $\Lambda$-module—is self-dual. So, for all $M_\lambda$, $M_\mu$ we have $[M_\mu] \circ [M_\lambda] = [{M_\mu}^*] \circ [{M_\lambda}^*]$. Then by Lemma 1.1.4 part 4, this is

$$\sum_{[M_\nu]} F_{M_\lambda, M_\mu}^{M_\nu} [{M_\nu}^*] = \sum_{[M_\nu]} F_{M_\lambda, M_\mu}^{M_\nu} [M_\nu]$$

which is by definition $[M_\lambda] \circ [M_\mu]$. This tells us that $\mathcal{H}(\Lambda)$ is commutative.

Let $I$ be the ideal of $\mathcal{H}(\Lambda)$ generated by those $[M_\lambda]$ such that all parts of $\lambda$ are even. We shall consider the quotient algebra $\mathcal{H}(\Lambda)/I$.

**Theorem 3.2.1** *The algebra $\mathcal{H}(\Lambda)/I$ is spanned as a vector space by the set $\{[nM_1] + I : n \geqslant 1\} \cup \{[M_{2s+1}] + I : 1 \leqslant s \leqslant r-1\}$.*

Before giving the proof of this theorem, we shall collect some results which will be used in the proof.

## 3.3 The Hall algebra $\mathcal{H}(k[X]/\langle X^t \rangle)$

Let $\Lambda = k[X]/\langle X^t \rangle$ for some $t \in \mathbb{N}$, where $k$ can again be any finite field. Let $M_1 \ldots, M_t$ be the indecomposable $\Lambda$-modules, and let $\Lambda$-modules be labelled by partitions with no part of length greater than $t$, as in the case $t = 2r$ above. The following lemmas will present results concerning the structure of $\mathcal{H}(\Lambda)$ which we will use in the case $t = 2r$ to prove Proposition 3.2.1.

**Lemma 3.3.1** *Let $\lambda$,$\mu$ partition $n$, and suppose $\lambda = \lambda' \cup \lambda''$. Suppose $[M_\mu]$ occurs with non-zero coefficient in the multiplication $[M_{\lambda'}] \circ [M_{\lambda''}]$. Then $\mu \geqslant \lambda$ in the lexicographic ordering of partitions.*

**Proof:** We shall use induction on $n$. As a base case, take $n = 3$. If $\lambda$ can be written as a non-trivial union $\lambda = \lambda' \cup \lambda''$ then we must have $\lambda = (2, 1)$ or $\lambda = (1, 1, 1)$. Suppose $\mu < \lambda$. The only such possibility is $\lambda = (2, 1)$ and $\mu = (1, 1, 1)$. Let $\lambda' = (2)$, $\lambda'' = (1)$. The module $M_\mu$ is semisimple and hence $M_2$ (which is not semisimple) cannot occur as a quotient. Hence $[M_\mu]$ cannot occur with non-zero coefficient in $[M_{\lambda'}] \circ [M_{\lambda''}]$, and the result holds for $n = 2$.

Now suppose the result holds for partitions of length less than $n$. We suppose that $\mu \leqslant \lambda$ and also that $[M_\mu]$ occurs with non-zero coefficient in $[M_{\lambda'}] \circ [M_{\lambda''}]$, and will show that in this case we must have $\mu = \lambda$.

Let $\lambda_1$ be the largest part of $\lambda$. We can assume without loss of generality that $\lambda_1$ occurs in $\lambda''$ since $\mathcal{H}(\Lambda)$ is commutative. Since $[M_\mu]$ occurs in $[M_{\lambda'}] \circ [M_{\lambda''}]$, we have the following filtration of $M_\mu$:

$$M_\mu \supseteq M_{\lambda''} \supseteq 0$$

where the first quotient is $M_{\lambda'}$. So, $M_{\lambda_1}$ is a submodule of $M_\mu$ and we have a short exact sequence

$$0 \longrightarrow M_{\lambda_1} \longrightarrow M_\mu \longrightarrow M_\mu / M_{\lambda_1} \longrightarrow 0.$$

Since $\mu \leqslant \lambda$, this sequence has no summand of length greater than $\lambda_1$ and can thus be thought of as a sequence of $k[X]/\langle X^{\lambda_1} \rangle$-modules. The module $M_{\lambda_1}$ is then injective, so the sequence must split, telling us that $M_{\lambda_1}$ is a direct summand of $M_\mu$ and that $M_\mu/M_{\lambda_1}$ is the direct sum complement of $M_{\lambda_1}$ in $M_\mu$. So, we have $\mu = (\lambda_1, \mu_2, \ldots, \mu_t)$ and

$$\frac{M_\mu}{M_{\lambda''}} \simeq \frac{M_{\mu \setminus \lambda_1}}{M_{\lambda'' \setminus \lambda_1}}.$$

So, there exists a filtration

$$M_{\mu \setminus \lambda_1} \supseteq M_{\lambda'' \setminus \lambda_1} \supseteq 0$$

of $M_{\mu \setminus \lambda_1}$ where the first quotient is $M_{\lambda'}$; equivalently $[M_{\mu \setminus \lambda_1}]$ appears with non-zero coefficient in $[M_{\lambda'}] \circ [M_{\lambda'' \setminus \lambda_1}]$. Then, by the inductive hypothesis $\mu \setminus \lambda_1 \geqslant \lambda \setminus \lambda_1$, and hence $\mu \geqslant \lambda$. But we assumed initially that $\mu \leqslant \lambda$. Hence $\lambda = \mu$ as required. □

**Lemma 3.3.2** *Suppose $[M_\nu]$ appears with non-zero coefficient in the multiplication $[M_\lambda] \circ [M_\mu]$. Then we must have (#parts of $\nu$) $\leqslant$ (#parts of $\lambda$) + (#parts of $\mu$).*

**Proof:** If $[M_\nu]$ occurs with non-zero coefficient in $[M_\lambda] \circ [M_\nu]$ then there must exist a short exact sequence

$$0 \longrightarrow M_\mu \longrightarrow M_\nu \longrightarrow M_\lambda \longrightarrow 0.$$

Applying $\mathrm{Hom}_\Lambda(M_1, -)$ to this, we get the long exact sequence

$$0 \longrightarrow \mathrm{Hom}(M_1, M_\mu) \longrightarrow \mathrm{Hom}(M_1, M_\nu) \longrightarrow \mathrm{Hom}(M_1, M_\lambda)$$

$$\xrightarrow{\phi} \mathrm{Ext}^1(M_1, M_\mu) \longrightarrow \cdots .$$

Let $|S(M)|$ denote the number of indecomposable summands of a module $M$. Exactness of this sequence tells us that $\dim(\mathrm{im}\phi) = \dim \mathrm{Hom}(M_1, M_\mu) + \dim \mathrm{Hom}(M_1, M_\lambda) - \dim \mathrm{Hom}(M_1, M_\nu) = |S(M_\mu)| + |S(M_\lambda)| - |S(M_\nu)| =$ (#parts of $\mu$) + (#parts of $\lambda$) − (#parts of $\nu$). But this dimension cannot be negative; hence (#parts of $\lambda$) + (#parts of $\mu$) − (#parts of $\nu$) $\geqslant 0$ as required. □

**Lemma 3.3.3** *Let $M_\lambda$ be a fixed $\Lambda$-module and suppose $[M_i \oplus \cdots \oplus M_i]$ occurs with non-zero coefficient in the multiplication $[M_\lambda] \circ [M_\mu]$, with $1 \leqslant i \leqslant t$. Suppose also that neither $\lambda$ nor $\mu$ has any part of length greater than $i$. Then there is only one possible candidate for the module $M_\mu$; this can be specified.*

**Proof:** There must exist a short exact sequence

$$0 \longrightarrow M_\mu \longrightarrow M_i \oplus \cdots \oplus M_i \xrightarrow{\pi} M_\lambda \longrightarrow 0. \tag{3.1}$$

This can be viewed as a sequence of $k[X]/\langle X^i \rangle$-modules, since no summands of length greater than $i$ appear. Viewed in this way, $M_i \oplus \cdots \oplus M_i$ must then be projective, and the map $\pi$ must factor through a projective cover for $M_\lambda$. Thus we must have $M_\mu = \ker \pi = \Omega(M_\lambda) \oplus M_i \oplus \cdots \oplus M_i$, where the number of copies of $M_i$ depends on the number of copies of $M_i$ in the middle of the short exact sequence (3.1) and will be determined by the fact that the dimensions of the modules in (3.1) must add up correctly. This tells us that there is only one possible choice for the module $M_\mu$, since the Heller operator $\Omega$ is well-defined. $\square$

We are now equipped to prove the theorem.

## 3.4 Proof of Theorem 3.2.1

Let $\Lambda$ be as in Theorem 3.2.1. We begin by showing that if $\dim M_\lambda$ is even and/or greater than $2r$ and $M_\lambda$ is not semisimple (i.e. $\lambda \neq (1, 1, \ldots, 1)$) then $[M_\lambda] \in I$. We shall prove this by induction using an appropriate ordering of the partitions corresponding to $\Lambda$-modules. The base case will be $\lambda = (2)$; we have $[M_{(2)}] \in I$ by definition. Now suppose we have $[M_\mu] \in I$ for all $\mu$ satisfying all of

(i) $|\mu|$ even and/or $|\mu| > 2r$

(ii) $\mu \neq (1, 1, \ldots, 1)$

(iii) $|\mu| < |\lambda|$, or

$|\mu| = |\lambda|$ and $\mu > \lambda$ in the lexicographic ordering.

Suppose $\lambda$ can be written $\lambda = \lambda' \cup \lambda''$, where neither $\lambda'$ nor $\lambda''$ is the empty partition, with $|\lambda'|$ even and $\lambda' \neq (1, 1, \ldots, 1)$. Then by hypothesis $[M_{\lambda'}] \in I$. Consider the multiplication $[M_{\lambda'}] \circ [M_{\lambda''}]$. Lemma 3.3.1 tells us that the modules which will appear in this multiplication are $[M_\lambda]$ and modules of the form $[M_\mu]$ with $|\mu| = |\lambda|$ and $\mu > \lambda$ in the lexicographic ordering. By assumption, all such modules lie in $I$; hence $[M_\lambda]$ is a linear combination of elements of $I$ and must also lie in $I$.

Now suppose $\lambda$ cannot be written in this form. Since we assume that $|\lambda|$ is even or $|\lambda| > 2r$, we cannot have $\lambda$ a single-part partition. The only possibility is therefore that $\lambda$ has just two parts, both of which are odd. We divide such possibilities into two cases.

Case 1: $\lambda = (s,t)$ with $s \neq t$. Look at the multiplication $[M_{s-1}] \circ [M_{t+1}]$. Lemmas 3.3.1 and 3.3.2 tell us that the modules which will appear in this multiplication are $[M_{(s,t)}]$, $[M_{(s-1,t+1)}]$ and modules which are assumed by our inductive hypothesis to lie in $I$. But $[M_{(s-1,t_1)}]$ has both summands of even length and therefore lies in $I$ by definition. Hence $[M_{(s,t)}]$ is the only new module appearing. It can be shown by explicit construction of the appropriate submodule and quotient that it does indeed appear with non-zero coefficient. Hence it can be expressed as a linear combination of elements of $I$.

Case 2: $\lambda = (s,s)$. To deal with this case we introduce a slight modification to the ordering used in our induction and show first that $[M_{(s,s-1,1)}] \in I$. To see this, look at the multiplication $[M_{(s-1,s-1)}] \circ [M_2]$. Lemmas 3.3.1 and 3.3.2 tell us that the modules which could possibly appear in this multiplication will be modules assumed to lie in $I$, along with $[M_{(s,s-1,1)}]$, $[M_{(s,s)}]$ and $[M_{(s-1,s-1,2)}]$. The last of these has all summands even and therefore lies in $I$ by definition. Lemma 3.3.3 tells us that if we think of our modules as $k[X]/\langle X^s \rangle$-modules, if $[M_{(s,s)}]$ does indeed appear then $M_2 = \Omega(M_{(s-1,s-1)})$. But it is clear that $\Omega(M_{(s-1,s-1)}) = M_{(1,1)}$. Hence $[M_{(s,s)}]$ cannot occur, and so we have only $[M_{s,s-1,1)}]$ and modules known to be in $I$; hence $[M_{s,s-1,1)}] \in I$. Now if we look at $[M_{(s-1)}] \circ [M_{(s,1)}]$, Lemma 3.3.1 tells us we will get some non-zero multiple of $[M_{(s,s)}]$ along with $[M_{(s,s-1,1)}]$ and other modules already known to lie in $I$, telling us that $[M_{(s,s)}] \in I$.

This completes the inductive proof that those modules of the form given lie in $I$.

Finally, we show that if $\dim M_\lambda = n$ is odd and at most $2r-1$ and $\lambda \neq (1,1,\ldots,1)$, $\lambda \neq (n)$ then $\{[M_\lambda]+I, [M_{(n)}]+I\}$ is a linearly dependent set. Suppose that this result holds for all $\mu$ with $\mu \vdash n$ and $\mu > \lambda$ in the lexicographic ordering. The conditions on $\lambda$ tell us that it must be possible to partition $\lambda$ as $\lambda = \lambda' \cup \lambda''$ with $|\lambda'|$ even. Then if we look at the multiplication $[M_{\lambda'}] \circ [M_{\lambda''}]$, Lemma 3.3.1 tells us that this will involve $[M_\lambda]$ and modules assumed to be linearly dependent on $[M_{(n)}]$ modulo $I$. Hence $[M_\lambda]$ is also linearly dependent on $[M_{(n)}]$ modulo $I$, as claimed. □

## 3.5 An explicit description of the quotient in Theorem 3.2.1

Let $\Lambda$ be the algebra of Theorem 3.2.1. We have seen that $\mathcal{H}(\Lambda)/I$ is spanned as a vector space by elements corresponding to the semisimple $\Lambda$-modules and a finite number of indecomposable $\Lambda$-modules of odd dimension. This spanning set, although infinite, is much smaller than the basis for $\mathcal{H}(\Lambda)$; also, the corresponding modules are a nicely defined subset of the set of $\Lambda$-modules.

Let $[M_\lambda]+I$, $[M_\mu]+I$ be elements of the spanning set for $\mathcal{H}(\Lambda)/I$ described above. Consider $([M_\lambda]+I) \circ ([M_\mu]+I) = [M_\lambda] \circ [M_\mu] + I$. Since the longest parts of $\lambda$ and $\mu$ must both be odd, a single module of odd dimension cannot occur with non-zero coefficient in $[M_\lambda] \circ [M_\mu]$: this tells us that $[M_{2s+1}]$ can never occur. Theorem 3.2.1 then tells us that $[M_\lambda] \circ [M_\mu] + I = \alpha[nM_1] + I$

for some $\alpha \in \mathbb{Q}$ and the appropriate dimension $n$.

Case (i): Suppose $M_\lambda$ and $M_\mu$ are not both semisimple, so we have either $M_\lambda = M_{2s+1}$ or $M_\mu = M_{2t+1}$ for some $s, t \in \{1, \ldots, r-1\}$ (or both). Then $[nM_1]$ cannot occur in the multiplication $[M_\lambda] \circ [M_\mu]$. This tells us that we must have $[M_\lambda] \circ [M_\mu] \in I$, i.e. $[M_\lambda] \circ [M_\mu] = 0$ modulo $I$.

Case (ii): Suppose $M_\lambda = nM_1$ and $M_\mu = mM_1$. Then $[(n+m)M_1]$ will occur in the multiplication $[nM_1] \circ [mM_1]$. Since $(n+m)M_1$ is the trivial $\Lambda$-module of dimension $n+m$, all subspaces of $(n+m)M_1$ will be $\Lambda$-submodules. Any submodule of dimension $m$ will be isomorphic to $[mM_1]$, and the corresponding quotient will be isomorphic to $[nM_1]$. So, the coefficient of $[(n+m)M_1]$ in the multiplication $[nM_1] \circ [mM_1]$ will be equal to the number of $m$-dimensional subspaces of an $(n+m)$-dimensional vector space,

$$G[m+n; m](|k|) = \frac{(|k|^{m+n}-1)(|k|^{m+n-1}-1)\cdots(|k|^{n+1}-1)}{(|k|^m-1)(|k|^{m-1}-1)\cdots(|k|-1)}.$$

All other terms appearing in the multiplication $[nM_1] \circ [mM_1]$ will be multiples of elements which are known by Theorem 3.2.1 to lie in $I$. So, we will have $[nM_1] \circ [mM_1] = G[m+n; m](|k|)[(n+m)M_1]$ modulo $I$.

# Part II

# Green rings

# Chapter 4

# Definitions and background material

## 4.1 Hopf algebras

Let $A$ be an associative algebra over $k$. The tensor product $A \otimes A$ over $k$ is also an associative algebra, with multiplication given by $(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) := a_1 a_2 \otimes b_1 b_2$. Note that $A \otimes k$ is also an associative algebra, with a natural isomorphism $\iota_1 : A \otimes k \longrightarrow A$ given by $\iota_1(a \otimes 1) = a$ for all $a \in A$. Similarly, we have a natural isomorphism $\iota_2 : k \otimes A \longrightarrow A$.

**Definition 4.1.1** Let $A$ be an associative algebra over $k$. This is a *bialgebra* if it is equipped with a *co-multiplication* $\Delta : A \longrightarrow A \otimes A$ and a *co-unit* $\varepsilon : A \longrightarrow k$, which are both linear and which satisfy the following properties:

(i) co-unit property: $\iota_2 \circ (\varepsilon \otimes \mathrm{id}_A) \circ \Delta = \mathrm{id}_A = \iota_1 \circ (\mathrm{id}_A \otimes \varepsilon) \circ \Delta$.

(ii) co-associativity: $(\Delta \otimes \mathrm{id}_A) \circ \Delta = (\mathrm{id}_A \otimes \Delta) \circ \Delta$.

(iii) $\Delta$ and $\varepsilon$ are both algebra homomorphisms.

The triple $(A, \Delta, \varepsilon)$ is called a *co-algebra*.

Sweedler notation: The image of an element $a \in A$ under the co-multiplication $\Delta$ will, in general, be of the form $\sum_i a_{i1} \otimes a_{i2}$ for some $a_{i1}, a_{i2} \in A$. We will use the short-hand notation introduced by Sweedler [39] and write $\Delta(a) = a^{(1)} \otimes a^{(2)}$.

**Example 4.1.2** Let $kG$ be a group algebra. Take $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$ for all elements $g \in G$ and extend linearly to $kG$. This makes $kG$ into a bialgebra.

**Example 4.1.3** Let $U(L)$ be the universal enveloping algebra for some Lie algebra $L$. Take $\Delta(1) = 1 \otimes 1$ and $\varepsilon(1) = 1$, and $\Delta(X) = X \otimes 1 + 1 \otimes X$ and $\varepsilon(X) = 0$ for all elements $X \in L$. Extend to $U(L)$ via the condition that $\Delta$ and $\varepsilon$ are algebra homomorphisms.

Let $\tau : A \otimes A \to A \otimes A$ be the *transposition map* given by $\tau(a_1 \otimes a_2) = a_2 \otimes a_1$.

**Definition 4.1.4** A bialgebra $A$ is *co-commutative* if $\tau \circ \Delta(a) = \Delta(a)$ for all $a \in A$.

**Example 4.1.5** A group algebra $kG$, with $\Delta$ defined as in Example 4.1.2, is co-commutative. A universal enveloping algebra $U(L)$, with $\Delta$ defined as in Example 4.1.3, is also co-commutative.

**Definition 4.1.6** A *Hopf algebra* is a bialgebra $A$ equipped with a linear map $S : A \to A$ satisfying the relation

$$a^{(1)} S(a^{(2)}) = \varepsilon(a) 1_A = S(a^{(1)}) a^{(2)}$$

for all $a \in A$. The map $S$ is called the *antipode*.

If the underlying bialgebra $A$ is co-commutative then the antipode $S$ will be unique: see [39, Section 4.0] for an alternative definition of the antipode which implies uniqueness.

**Example 4.1.7** The bialgebra $kG$, with antipode $S(g) = g^{-1}$ for all $g \in G$, is a Hopf algebra.

**Example 4.1.8** The bialgebra $U(L)$, with antipode $S(X) = -X$ for all $X \in L$, is a Hopf algebra.

Note that in both the examples above we have $S^2 = \mathrm{id}_A$. This is in fact the case for all co-commutative Hopf alegbras (see [39, Proposition 4.0.1]). This property will be important to us later.

### 4.1.1 Hopf algebra modules

Let $A$ be a bialgebra over $k$. Let $M, N$ be $A$-modules. The tensor product $M \otimes N$ over $k$ can be made into an $A$-module using the co-multiplication $\Delta$ on $A$: given $a \in A$ and $m \otimes n \in M \otimes N$, set $a(m \otimes n) = \Delta(a)(m \otimes n)$.

**Example 4.1.9** Let $kG$ be a group algebra and let $M, N$ be $kG$-modules. Putting $g(m \otimes n) = (g \otimes g)(m \otimes n) = gm \otimes gn$ for $g \in G$ makes $M \otimes N$ into a $kG$-module in the usual way.

Let $A$ be a bialgebra over $k$. The field $k$ is given an $A$-module structure by setting $a.r = \varepsilon(a).r$ for all $a \in A$ and $r \in k$. This module is called the *trivial module.*

**Example 4.1.10** Let $U(L)$ be a universal enveloping algebra. The trivial module $k$ is given by $X.r = 0$ for all $X \in L$ and $r \in k$.

Let $A$ be a Hopf algebra over $k$. Let $M$ and $N$ be $A$-modules. We define the action of $A$ on $\mathrm{Hom}_k(M, N)$, using the antipode $S$, by setting

$$(a(\phi))(m) = a^{(1)}(\phi(S(a^{(2)})(m)))$$

for all $a \in A$, $m \in M$ and $\phi \in \mathrm{Hom}_k(M, N)$.

**Example 4.1.11** Let $kG$ be a group algebra and let $M$ and $N$ be $kG$-modules. Then the action of $kG$ on $\mathrm{Hom}_k(M, N)$ is given by $(g\phi)(m) = g(\phi(g^{-1}m))$ for $g \in G$.

**Example 4.1.12** Let $U(L)$ be a universal enveloping algebra and let $M$ and $N$ be $U(L)$-modules. The action of $U(L)$ on $\mathrm{Hom}_k(M, N)$ is given by $(X\phi)(m) = X(\phi(m)) - \phi(Xm)$ for $X \in L$.

This action allows us to define the dual module $M^* = \mathrm{Hom}_k(M, k)$. Note that the existence of an antipode $S$ has allowed us to view the dual module $M^*$ as a *left* $A$-module.

In the work that follows we will make use of various isomorphisms relating tensor products and homomorphism groups as Hopf algebra modules. A good account of the isomorphisms required can be found in [18, Section 1]. All the results of this account are generalisations of familiar isomorphisms of group algebra modules.

## 4.2 Green rings

**Definition 4.2.1** Let $\Lambda$ be a bialgebra. Let $A(\Lambda)$ be the ring generated by all isomorphism classes of $\Lambda$-modules, with the only relations being $[M]+[N] = [M \oplus N]$ and with multiplication defined by $[M] \circ [N] = [M \otimes N]$. This is the *Green ring* (also sometimes called the *representation ring*) of $\Lambda$.

It follows from the basic properties of direct sums and tensor products that the addition defined on $A(\Lambda)$ is both associative and commutative. Multiplication is also associative. If the bialgebra $\Lambda$ is co-commutative then the multiplication in the Green ring will also be commutative: we have $M \otimes N \simeq N \otimes M$ as $\Lambda$-modules via the map $\phi(m \otimes n) = n \otimes m$, since $a\phi(m \otimes n) = a(n \otimes m) = (a^{(1)} \otimes a^{(2)})(n \otimes m) = (a^{(2)} \otimes a^{(1)})(n \otimes m) = a^{(2)}n \otimes a^{(1)}m = \phi(a^{(1)}m \otimes a^{(2)}n) = \phi(a(m \otimes n))$ for all $a \in \Lambda$.

The Krull–Schmidt Theorem tells us that each element of $A(\Lambda)$ can be written uniquely as a sum of isomorphism classes of indecomposable $\Lambda$-modules. So, the additive group of $A(\Lambda)$ is a free abelian group with generators the isomorphism classes of indecomposable $\Lambda$-modules. It is clear that this additive structure of $A(\Lambda)$ depends only on the structure of $\Lambda$ as an algebra and not on the co-algebra structure.

### 4.2.1 Green rings of group algebras and universal enveloping algebras

We wish to look at the Green rings of group algebras and universal enveloping algebras, and to compare the Green rings given by these two co-multiplications. One aspect of our investigation will involve looking at the ideal of the Green ring defined by the following proposition:

**Proposition 4.2.2** *Let $k$ be an algebraically closed field of characteristic $p$. Let $H$ be either a group algebra or a universal enveloping algebra over $k$. Let $A(H;p)$ be the span in $A(H)$ of all $H$-modules $M$ such that all direct summands of $M$ have dimension divisible by $p$. Then $A(H;p)$ is an ideal of $A(H)$.*

In the case where $H$ is a group algebra, this proposition is a version of [7, Lemma 2.5].

To see that Proposition 4.2.2 also holds when $H$ is a universal enveloping algebra, we will look at a result of Benson & Carlson about tensor products of group algebra modules; in the following section we will see that the corresponding result for universal enveloping algebras also holds. We begin by checking some basic properties of enveloping algebras. We look first at the following, which corresponds to the result for group algebras given as Proposition 3.1.7 in [6]. The question of whether the analogous result also holds for an arbitrary Hopf algebra appears to be an open problem.

**Proposition 4.2.3** *Let $U(L)$ be a universal enveloping algebra over $k$. Let $M$, $N$ be $U(L)$-modules. Then* $\mathrm{Hom}_{U(L)}(k, \mathrm{Hom}_k(M, N)) \simeq \mathrm{Hom}_{U(L)}(M, N)$.

**Proof:** Let $\phi : \mathrm{Hom}_{U(L)}(k, \mathrm{Hom}_k(M, N)) \to \mathrm{Hom}_k(M, N)$ be given by $\phi(\alpha)(m) = \alpha(1)(m)$ for $\alpha \in \mathrm{Hom}_{U(L)}(k, \mathrm{Hom}_k(M, N))$ and $m \in M$. This is an embedding of $\mathrm{Hom}_{U(L)}(k, \mathrm{Hom}_k(M, N))$ into $\mathrm{Hom}_k(M, N)$. Now, for all $X \in L$ and for all $\alpha \in \mathrm{Hom}_{U(L)}(k, \mathrm{Hom}_k(M, N))$, we have $X.\alpha(1) = \alpha(X.1) = 0$. Hence $(X.\alpha(1))(m) = 0$ for all $m \in M$. Then, by definition of the action of $U(L)$ on $\mathrm{Hom}_k(M, N)$, we get $X(\alpha(1)(m)) - \alpha(1)(Xm) = 0$. That is, $X(\phi(\alpha)(m)) = \phi(\alpha)(Xm)$. So, we see that the image of $\phi$ consists of precisely those $\alpha \in \mathrm{Hom}_k(M, N)$ that are in $\mathrm{Hom}_{U(L)}(M, N)$. Hence $\phi$ is the required isomorphism. □

This result then allows us to check the following:

**Lemma 4.2.4** *Let $U(L)$ be a universal enveloping algebra over $k$. Let $M$, $N$ be $U(L)$-modules. Then* $\mathrm{Hom}_{U(L)}(k, M \otimes N) \simeq \mathrm{Hom}_{U(L)}(N^*, M)$.

**Proof:** We have $\mathrm{Hom}_{U(L)}(N^*, M) \simeq \mathrm{Hom}_{U(L)}(k, \mathrm{Hom}_k(N^*, M))$, by Proposition 4.2.3 . Then, by Proposition 1.2(a) of [18], this is isomorphic to $\mathrm{Hom}_{U(L)}(k, N^{**} \otimes M)$. Since the antipode for $U(L)$ squares to the identity on $U(L)$, we have $\mathrm{Hom}_{U(L)}(k, N^{**} \otimes M) \simeq \mathrm{Hom}_{U(L)}(k, N \otimes M)$, by Proposition 1.4 of [18]. Since the co-multiplication on $U(L)$ is co-commutative, this gives the required result. □

We are now able to give the required result of Benson & Carlson.

**Theorem 4.2.5** *Let $k$ be an algebraically closed field of characteristic $p$. Let $H$ be a Hopf algebra over $k$ such that $\mathrm{Hom}_H(k, U \otimes V) \simeq \mathrm{Hom}_H(V^*, U)$ for all $H$-modules $U$, $V$. Let $M$ and $N$ be indecomposable $H$-modules. Then $M \otimes N$ has the trivial module $k$ as a direct summand if and only if the following two conditions are satisfied:*

*(i) $M \simeq N^*$*

*(ii) $p \not| \dim N$.*

*Moreover, if $k$ is a component of $N^* \otimes N$ then it has multiplicity one.*

This is a modified version of Theorem 2.1 of [7]. In this paper by Benson and Carlson, the result is stated in the case where $H = kG$ for some finite group $G$. However, examination of the proof given in [7] shows that their proof will immediately give us the theorem in the form stated above. In particular, Lemma 4.2.4 tells us that this theorem holds for restricted universal enveloping algebras, as well as for group algebras.

Theorem 4.2.5 will prove to be useful for various purposes in the work that follows. Most immediately, it gives us an easy proof of Proposition 4.2.2.

**Proof of Proposition 4.2.2:** Let $M$ be an indecomposable $H$-module of dimension divisible by $p$. Let $N$ be any $H$-module. We must show that all direct summands of $M \otimes N$ have dimension divisible by $p$. Suppose, for a contradiction, that $M \otimes N \simeq U \oplus R$ for some $U$ and $R$ with $U$ indecomposable and $p \not| \dim U$. Consider $M \otimes N \otimes U^* \simeq (U \otimes U^*) \oplus (R \otimes U^*)$. By Theorem 4.2.5 applied to $U$, this has $k$ as a direct summand. But Theorem 4.2.5 tells us that $M \otimes (N \otimes U^*)$ has no trivial direct summand, giving the required contradiction. $\square$

As a consequence of Proposition 4.2.2, we will be able to study quotients of the form $A(kG)/A(kG;p)$ and $A(U(L))/A(U(L);p)$. We shall call these the *Benson–Carlson quotients*, as they were first introduced and studied by Benson and Carlson in the case of a group algebra.

## 4.3 The Jennings Algebra

Let $G$ be a finite $p$-group and let $k$ be a field of characteristic $p$. In this section, we describe the construction of a restricted universal enveloping algebra, the Jennings algebra $U^{[p]}\mathrm{Jen}(G)$, whose co-multiplication and Green ring we will compare with the group algebra $kG$. The construction behind the Jennings algebra was first given by S. A. Jennings in his PhD thesis and his paper [23]; an account of the construction can also be found in [6, Section 3.14].

Let $\Gamma_r(G)$ be the central series of $G$ with properties

(i) $\Gamma_r(G) \lhd G$

(ii) If $g \in \Gamma_r(G)$ then $g^p \in \Gamma_{pr}(G)$.

(iii) $[\Gamma_r(G), \Gamma_s(G)] \subseteq \Gamma_{r+s}(G)$

for all $r, s$, and such that each $\Gamma_r(G)$ is minimal with respect to the above properties. It can be shown that this sequence terminates in $\{1\}$. This is the *Jennings series* of $G$.

It is clear from the defining properties above that each of the quotients $\Gamma_r(G)/\Gamma_{r+1}(G)$ is an elementary abelian $p$-group. So, each quotient can be regarded additively as a vector space over $\mathbb{F}_p$. Extending the field of coefficients to $k$ and taking the direct sum of all these, we have the graded vector space

$$\mathrm{Jen}(G) = \bigoplus_{r \geqslant 1} \left( k \otimes_{\mathbb{F}_p} \left( \frac{\Gamma_r(G)}{\Gamma_{r+1}(G)} \right) \right).$$

The vector space $\mathrm{Jen}(G)$ can be given the structure of a $p$-restricted Lie algebra, with the Lie bracket given by commutators $[g, h]$ as in the group $G$ and the $p$-mapping given by taking $p$th powers of group elements, and linear extension. It can be checked that these maps are well-defined (that is, that they do not depend on our choice of coset representatives) and that they satisfy the axioms required for a Lie bracket and a $p$-mapping. The details of these proofs can be found in [15, Chapter 5.6]. We can then take the restricted universal enveloping algebra $U^{[p]}\mathrm{Jen}(G)$ of $\mathrm{Jen}(G)$. This is the *Jennings algebra* of $G$.

The group algebra $kG$ and the Jennings algebra $U^{[p]}\mathrm{Jen}(G)$ are both finite-dimensional associative Hopf algebras over $k$. In certain cases it turns out that we have $kG \simeq U^{[p]}\mathrm{Jen}(G)$ as algebras. However, the fact that the co-multiplication is defined differently for group algebras and universal enveloping algebras means that we may have $kG$ and $U^{[p]}\mathrm{Jen}(G)$ which are isomorphic as algebras but not as co-algebras. In such cases we can inverstigate the relationship between the Green rings $A(kG)$ and $A(U^{[p]}\mathrm{Jen}(G))$. In order to restrict ourselves to cases where we have a description of the modules which generate the Green rings, we will look only at those $p$-groups which are of finite or tame representation type.

**Theorem 4.3.1** *Let $G$ be a finite $p$-group and let $k$ be an infinite field of characteristic $p$.*

1. *The group algebra $kG$ is of finite representation type if and only if $G$ is cyclic.*
2. *The group algebra $kG$ is of tame representation type if and only if $p = 2$ and $G$ is the Klein four group or is a dihedral, semidihedral or generalised quaternion group.*
3. *The group algebra $kG$ is of wild representation type in all other cases.*

For the proof of this theorem, see [8] and [29].

We now identify which $p$-groups of finite or tame representation type have $kG \simeq U^{[p]}\mathrm{Jen}(G)$ as algebras.

**Theorem 4.3.2** *Let $G$ be a finite $p$-group and let $k$ be an infinite field of characteristic $p$.*

1. *We have $kG$ of finite representation type and $kG \simeq U^{[p]}\mathrm{Jen}(G)$ as algebras if and only if $G$ is cyclic.*
2. *We have $kG$ of tame representation type and $kG \simeq U^{[p]}\mathrm{Jen}(G)$ as algebras if and only if $p = 2$ and $G$ is either the Klein four group or a dihedral $2$-group.*

**Proof:** We need to show that $kC_{p^n} \simeq U^{[p]}\mathrm{Jen}(C_{p^n})$ for all $p$ and $n \geqslant 1$, that $kV_4 \simeq U^{[2]}\mathrm{Jen}(V_4)$ and that $kD_{2^n} \simeq U^{[2]}\mathrm{Jen}(D_{2^n})$ for the dihedral group $D_{2^n}$

of order $2^n$ with $n \geqslant 3$. We also need to show that $kSD_{2^n} \not\simeq U^{[2]}\mathrm{Jen}(SD_{2^n})$ and $kQ_{2^n} \not\simeq U^{[2]}\mathrm{Jen}(Q_{2^n})$, where

$$SD_{2^n} = \langle x, y : x^{2^{n-1}} = y^2 = 1, y^{-1}xy = x^{2^{n-2}-1} \rangle$$

with $n \geqslant 4$ is the semidihedral group of order $2^n$ and

$$Q_{2^n} = \langle x, y : x^{2^{n-1}} = 1, x^{2^{n-2}} = y^2, y^{-1}xy = x^{-1} \rangle$$

with $n \geqslant 3$ is the generalised quaternion group of order $2^n$.

First consider $C_{p^n} = \langle x \rangle$. The Jennings algebra can easily be constructed and shown to satisfy

$$U^{[p]}\mathrm{Jen}(C_{p^n}) \simeq \frac{k[X]}{\langle X^{p^n} \rangle}.$$

Setting $X \mapsto 1 - g$, it is clear that this is isomorphic to $kC_{p^n}$. Similarly, we have

$$U^{[2]}\mathrm{Jen}(V_4) \simeq \frac{k[X, Y]}{\langle X^2, Y^2 \rangle} \simeq kV_4.$$

The case $G = D_{2^n}$ is another easy calculation and can be found as [11, Lemma 27]; this result is used in this thesis in calculating the cohomology of the dihedral 2-groups.

To see that $kDS_{2^n} \not\simeq U^{[2]}\mathrm{Jen}(SD_{2^n})$ and $kQ_{2^n} \not\simeq U^{[2]}\mathrm{Jen}(Q_{2^n})$, we use the result $U^{[2]}\mathrm{Jen}(SD_{2^n}) \simeq U^{[2]}\mathrm{Jen}(Q_{2^n}) \simeq U^{[2]}\mathrm{Jen}(D_{2^n}) \simeq kD_{2^n}$ [11, Corollary 6]. It can easily be checked that $kSD_{2^n} \not\simeq kD_{2^n}$ and $kQ_{2^n} \not\simeq kD_{2^n}$ (see for example [13, Section III.13]). $\square$

We can therefore summarise our intended line of investigation as follows:

(i) Compare the Green rings $A(kC_{p^n})$ and $A(U^{[p]}\mathrm{Jen}(C_{p^n}))$.

(ii) Compare the Green rings $A(kV_4)$ and $A(U^{[2]}\mathrm{Jen}(V_4))$.

(iii) Compare the Green rings $A(kD_{2^n})$ and $A(U^{[2]}\mathrm{Jen}(D_{2^n}))$.

## 4.4 Modules and tensor products for finite $p$-groups with $kG \simeq U^{[p]}\mathrm{Jen}(G)$ as algebras

Before beginning our comparison of the Green rings of group algebras and Jennings algebras we will survey what is known about the indecomposable $kG$-modules and the Green rings $A(kG)$ for the $p$-groups in our list.

### 4.4.1 Cyclic $p$-groups

Let $G = C_{p^n}$. Putting $X \mapsto 1 - g$ we have

$$kG \simeq \frac{k[X]}{\langle X^{p^n} \rangle}.$$

The indecomposable $kG$-modules are $M_1, \ldots, M_{p^n}$, where $X$ acts on $M_i$ as a Jordan block matrix of size $i$ with eigenvalue 0.

The Green ring $A(C_{p^n})$ is also well understood. In particular we have the following theorem of J. A. Green.

**Theorem 4.4.1** **([16, Theorem 2])** *Let $k$ be a field of characteristic $p$, and let $G$ be a cyclic group of order a power of $p$. Then $A(kG)$ is semisimple.*

This theorem is the main result of [16]; this paper is the origin of the study of Green rings as algebraic objects, explaining why they have since been given this name.

### 4.4.2 The Klein four group

Let $k$ be a field of characteristic 2 and let $V_4 = \langle g, h \rangle$ be the Klein four group. Putting $X = 1 + g$ and $Y = 1 + h$ we see that

$$kV_4 \simeq \frac{k[X, Y]}{\langle X^2, Y^2 \rangle}.$$

The indecomposable $kV_4$-modules were classified by Kronecker; a formulation of the classification similar to that used below was given by Basev [2]. The actions of $X$ and $Y$ on the families of indecomposable modules are as follows:

$$A_n: \quad X \mapsto \left(\begin{array}{c|c} \mathbf{0} & \mathbf{A_X} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right) \quad Y \mapsto \left(\begin{array}{c|c} \mathbf{0} & \mathbf{A_Y} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right)$$

where $\mathbf{A_X}$ and $\mathbf{A_Y}$ are the $(n+1) \times n$ matrices

$$\mathbf{A_X} = \begin{pmatrix} 1 & 0 & & 0 \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{pmatrix} \quad \mathbf{A_Y} = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \end{pmatrix}$$

for all $n \in \mathbb{N}_0$.

$$B_n: \quad X \mapsto \left(\begin{array}{c|c} \mathbf{0} & \mathbf{B_X} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right) \quad Y \mapsto \left(\begin{array}{c|c} \mathbf{0} & \mathbf{B_Y} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right)$$

where $\mathbf{B_X}$ and $\mathbf{B_Y}$ are the $n \times (n+1)$ matrices

$$\mathbf{B_X} = \begin{pmatrix} 1 & & 0 \\ 0 & \ddots & \\ & \ddots & 1 \\ 0 & & 0 \end{pmatrix} \quad \mathbf{B_Y} = \begin{pmatrix} 0 & & 0 \\ 1 & \ddots & \\ & \ddots & 0 \\ 0 & & 1 \end{pmatrix}$$

for all $n \in \mathbb{N}$.

$$C_n(\pi): \quad X \mapsto \left(\begin{array}{c|c} \mathbf{0} & \mathbf{I_{mn}} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right) \quad Y \mapsto \left(\begin{array}{c|c} \mathbf{0} & \mathbf{C_Y} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right)$$

for all $n \in \mathbb{N}$ and all irreducible polynomials $\pi = T^m + u_{m-1}T^{m-1} + \cdots + u_1T + u_0$ over $k$, where $\mathbf{C_Y}$ is an $mn \times mn$ matrix consisting of $m \times m$ blocks

$$\mathbf{C_Y} = \begin{pmatrix} \mathbf{M} & & & \mathbf{0} \\ \mathbf{N} & \ddots & & \\ & \ddots & \ddots & \\ \mathbf{0} & & \mathbf{N} & \mathbf{M} \end{pmatrix}$$

with

$$\mathbf{M} = \begin{pmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ u_0 & u_1 & \cdots & u_{m-2} & u_{m-1} \end{pmatrix} \quad \mathbf{N} = \begin{pmatrix} & & \\ & & \\ 1 & & \end{pmatrix}.$$

$$C_n(\infty):\ X \mapsto \left(\begin{array}{c|c} \mathbf{0} & \mathbf{C_X} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right) \ Y \mapsto \left(\begin{array}{c|c} \mathbf{0} & \mathbf{I_n} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right)$$

where $\mathbf{C_X}$ is the $n \times n$ matrix

$$\mathbf{C_X} = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}$$

for all $n \in \mathbb{N}$.

$$D:\ X \mapsto \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \ Y \mapsto \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

The module $D$ is both the only projective indecomposable module and the only injective indecomposable module for our algebra.

This notation for the indecomposable modules is that used in [10].

The multiplications of any pair of modules in the Green ring $A(kV_4)$ have been calculated by Conlon [10]; the results are recorded in the tables on page 74.

### 4.4.3 Dihedral 2-groups

Let $k$ be a field of characteristic 2. Let

$$kD_{4q} = \langle x, y : x^{2q} = y^2 = 1, y^{-1}xy = x^{-1} \rangle,$$

with $q \geqslant 2$ and $q$ a power of 2, be the dihedral group of order $4q$. If we put $X \mapsto 1 + y$ and $Y \mapsto 1 + xy$, we have $kD_{4q}$ isomorphic to

$$\Lambda = \frac{k\langle X, Y\rangle}{\langle X^2, Y^2, (XY)^q - (YX)^q \rangle}.$$

The indecomposable $kD_{4q}$-modules were classified by Ringel [28]. There is a single projective indecomposable module, which is just ${}_\Lambda\Lambda$. The non-projective indecomposable modules can be described as belonging to two

| $n \leqslant n'$ | $A_n$ | $B_n$ | $D$ |
|---|---|---|---|
| $A_{n'}$ | $nn'D + A_{n+n'}$ | $n(n'+1)D + A_{n'-n}$ | $(2n'+1)D$ |
| $B_{n'}$ | $n(n'+1)D + B_{n'-n}$ | $nn'D + B_{n+n'}$ | $(2n'+1)D$ |
| $C_{n'}(\pi')$ | $nn'm'D + C_{n'}(\pi')$ | $nn'm'D + C_{n'}(\pi')$ | $2n'm'D$ |
| $C_{n'}(\infty)$ | $nn'D + C_{n'}(\infty)$ | $nn'D + C_{n'}(\infty)$ | $2n'D$ |
| $D$ | $(2n+1)D$ | $(2n+1)D$ | $4D$ |

| $n \leqslant n'$ | $C_n(\pi)$ | $C_n(\infty)$ |
|---|---|---|
| $A_{n'}$ | $nn'mD + C_n(\pi)$ | $nn'D + C_n(\infty)$ |
| $B_{n'}$ | $nn'mD + C_n(\pi)$ | $nn'D + C_n(\infty)$ |
| $C_{n'}(\pi')$ | If $\pi \neq \pi'$: $nmn'm'D$<br>If $\pi = \pi'$: $nm(n'm-1)D + 2C_n(\pi)$<br>Except special case: for $\pi \neq T, T+1$<br>$[C_1(\pi)]^2 = m(m-1)D + C_2(\pi)$ | $nn'm'D$ |
| $C_{n'}(\infty)$ | $nn'mD$ | $n(n'-1)D + 2C_n(\infty)$ |
| $D$ | $2nmD$ | $2nD$ |

Figure 4.1: Multiplication tables for the Green rings $A(kV_4)$

classes, the *string modules* and the *band modules*. (This terminology was not used by Ringel, but comes from the fact that $\Lambda$ is a special biserial algebra as defined in [37]: the non-projective indecomposable $\Lambda$-modules are thus string modules and band modules as defined for string algebras. For an explanation of the classification of the indecomposable modules for a special biserial algebra, see [13, Chapter II].)

**The string modules**

Let $W$ be the set of words in the letters $X$, $Y$, $X^{-1}$ and $Y^{-1}$ such that

(i) $X$ and $X^{-1}$ are always followed by $Y$ or $Y^{-1}$, and vice versa.

(ii) no word in $W$ contains any of $(XY)^q$, $(YX)^q$, $(X^{-1}Y^{-1})^q$, $(Y^{-1}X^{-1})^q$.

Also include in $W$ the word 1 of length 0. For a word $w = a_1a_2\cdots a_n$ in $W$, define $w^{-1} := a_n^{-1}\cdots a_2^{-1}a_1^{-1}$, where we set $(X^{-1})^{-1} = X$ and $(Y^{-1})^{-1} = Y$ and $1^{-1} = 1$. Then we define an equivalence relation $\sim$ on $W$ by setting $w \sim w^{-1}$ for all $w \in W$.

The indecomposable string modules for $\Lambda$ are in one-one correspondence with equivalence classes of $W$ under $\sim$. Given a word $w = a_1a_2\cdots a_n$, we have a natural basis $v_1, v_2, \ldots, v_{n+1}$ for the corresponding module $M(w)$. The actions of $X$ and $Y$ with respect to this basis are defined by the diagram

$$v_1 \text{ --- } v_2 \text{ --- } v_3 \cdots\cdots v_n \text{ --- } v_{n+1}$$

where the $i$th edge is an arrow pointing to the left and labelled with $a_i$ if $a_i = X$ or $a_i = Y$ and is an arrow pointing to the right and labelled with $a_i^{-1}$ if $a_i = X^{-1}$ or $a_i = Y^{-1}$. This diagram indicates how $X$ and $Y$ map the basis vectors onto each other; where there is no arrow, a basis vector is mapped to zero.

**Example 4.4.2** Take $w = XY^{-1}X$. Then $M(w)$ is a 4-dimensional $\Lambda$-module with action of $X$ and $Y$ defined by the diagram

$$v_1 \xleftarrow{X} v_2 \xrightarrow{Y} v_3 \xleftarrow{X} v_4.$$

So

$$X \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Y \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

on $M(w)$.

### Band modules

Let $W$ be the set of words as defined above. For $w \in W$, define $w^r$ by juxtaposing $r$ copies of $w$. Note that this will not always be a word in $W$. Let $W_2$ be the set of words $w$ in $W$ such that (i) all powers of $w$ are also in $W$, and (ii) $w$ is not itself a power of any shorter word. Given a word $w = a_1a_2\cdots a_n$ in $W_2$, say that a *rotation* of $w$ is any word of the form $a_{i+1}\cdots a_na_1a_2\cdots a_i$ for some $i$. Define an equivalence relation $\sim_2$ on $W_2$ by setting $w \sim_2 w^{-1}$ and $w \sim_2 u$ for any rotation $u$ of $w$.

The indecomposable band modules for $\Lambda$ are constructed as follows. Let $w = a_1a_2\cdots a_n$ be a representative of an equivalence class in $W_2$. We can assume (by rotating and/or inverting) that $a_1$ is $X$ or $Y$. Draw a circular diagram as in Figure 4.2, where the edge between $i$ and $i+1$ (modulo $n$) is an arrow pointing anticlockwise and labelled with $a_i$ if $a_i = X$ or $a_i = Y$ and is an arrow pointing clockwise and labelled with $a_i^{-1}$ if $a_i = X^{-1}$ or $a_i = Y^{-1}$. Let $V$ be a finite-dimensional vector space over $k$ and let $\phi : V \to V$ be an indecomposable linear transformation. The band module $M(w, V, \phi)$ has basis given by attaching a copy of $V$ to each vertex of the circular diagram of

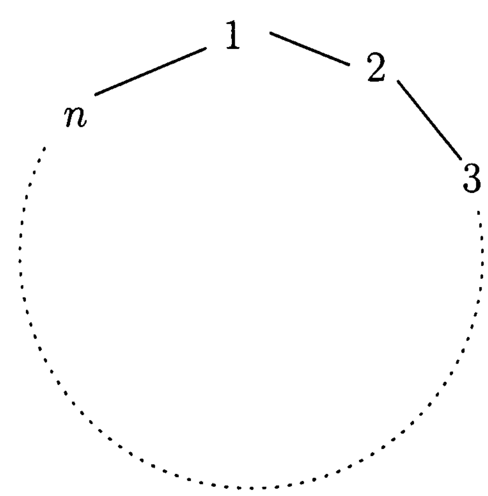



Figure 4.2:

Figure 4.2, letting $a_1$ act as $\phi$ and letting $a_i$ act as $\mathrm{Id}_V$ for all $i \geqslant 2$. The band modules for $\Lambda$ are in one-one correspondence with triples $(w, V, \phi)$ where $w$ is a representative of an equivalence class in $W_2$ and $V$ is a finite-dimensional vector space over $k$ (up to isomorphism) and $\phi : V \to V$ is an indecomposable linear transformation (up to equivalence of transformations).

**Example 4.4.3** Let $w = XY^{-1}$. Then the action of $X$ and $Y$ on $M(w, V, \phi)$ is given by

$$\begin{tikzcd}[column sep=huge] \overset{V}{\bullet} & \overset{V}{\bullet} \arrow[l, shift right, "X = \phi"'] \arrow[l, shift left, "Y = \mathrm{Id}_V"] \end{tikzcd}$$

and we have

$$X \mapsto \left(\begin{array}{c|c} \mathbf{0} & \phi \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right) \quad \text{and} \quad Y \mapsto \left(\begin{array}{c|c} \mathbf{0} & \mathbf{I_V} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right)$$

on $M(w, V, \phi)$.

Note that if $w = a_1 a_2 \cdots a_n$ is in $W_2$ then $n$ must be even, in order for powers of $w$ to be in $W$. We therefore have $\dim M(w, V, \phi) = n. \dim V$, and hence all band modules are even-dimensional.

**Remark:** The indecomposable $kV_4$-modules presented in Section 4.4.2 can also be described as string and band modules in an analogous way, as $kV_4$ is also a special biserial algebra.

### Almost split sequences

In the work that follows we will make use of information about the components of the Auslander–Reiten quiver for $kD_{4q}$ which contain string modules. Here we use the results and notation of [3, Appendix] (see also [6, Section 4.17]).

Define two functions $R_q : W \to W$ and $L_q : W \to W$ as follows. If $w \in W$ ends in $XY^{-1}(X^{-1}Y^{-1})^{q-1}$ or $YX^{-1}(Y^{-1}X^{-1})^{q-1}$ then $wR_q$ is obtained from $w$ by cancelling that part; otherwise define $wR_q = wX^{-1}(YX)^{q-1}Y$ or $wR_q = wY^{-1}(XY)^{q-1}X$, whichever is a word. Similarly, if $w \in W$ begins with $(XY)^{q-1}XY^{-1}$ or $(YX)^{q-1}YX^{-1}$ then $wL_q$ is obtained by cancelling this part; otherwise $wL_q = X^{-1}(Y^{-1}X^{-1})^{q-1}Yw$ or $wL_q = Y^{-1}(X^{-1}Y^{-1})^{q-1}Xw$, whichever is a word. Note that $R_q$ and $L_q$ depend on the actual word $w \in W$ and not just on the equivalence class under $\sim$. It is clear that $R_q$ and $L_q$ commute. We then have $\Omega^2(M(w)) \simeq M(wR_qL_q)$ (this construction for taking even powers of $\Omega$ was described in Ringel's paper [28]). The almost split sequence terminating in $M(w)$ is

$$0 \longrightarrow M(wR_qL_q) \longrightarrow M(wL_q) \oplus M(wR_q) \longrightarrow M(w) \longrightarrow 0,$$

unless $w$ or $w^{-1}$ is $(XY)^{q-1}XY^{-1}(X^{-1}Y^{-1})^{q-1}$, in which case it is

$$0 \longrightarrow M(wR_qL_q) \longrightarrow M(wL_q) \oplus M(wR_q) \oplus {}_\Lambda\Lambda \longrightarrow M(w) \longrightarrow 0.$$

The Auslander–Reiten quiver components for $kD_{4q}$ involving string modules are: two 1-tubes

$$M(u_1) \rightleftarrows M(u_1R_q) \rightleftarrows M(u_1R_q^2) \rightleftarrows \cdots\cdots$$

and

$$M(u_2) \rightleftarrows M(u_2R_q) \rightleftarrows M(u_2R_q^2) \rightleftarrows \cdots\cdots$$

where $u_1 = (XY)^{q-1}X$ and $u_2 = (YX)^{q-1}Y$, and an infinite set of components of type $\mathbb{Z}A_\infty^\infty$, of the form shown in Figure 4.3, one of which has the projective module $\Lambda$ attached (Figure 4.4). Note that the two 1-tubes described above involve only modules of even dimension, so all odd-dimensional $kD_{4q}$-modules lie in some Auslander–Reiten quiver component of type $\mathbb{Z}A_\infty^\infty$.

The components of the Auslander–Reiten quiver containing band modules will not be used in the work that follows, though it is known that all band modules lie in 1-tubes (see again [3, Appendix] and [6, Section 4.17]).

### The Green ring

In contrast with the case of the cyclic $p$-groups and the Klein four group, relatively little is known about the structure of the Green ring $A(kD_{4q})$. In particular, there is no description of how tensor products of $kD_{4q}$-modules decompose, as there is with Conlon's table for $A(kV_4)$. Research into the structure of $A(kD_{4q})$ has focussed on the question of whether it is semisimple, or equivalently whether $A(kD_{4q})$ has any nilpotent elements. It was shown by Zemanek [40] that, unlike $A(kC_{p^n})$, the Green ring $A(kD_{4q})$ will have nilpotent elements. Methods for constructing such elements can be found in [41] and in the survey articles [4] and [5]. The existence of nilpotent elements of order greater than 2 was shown in [21]; it is not known whether there are any nilpotent elements of order greater than 3.

$$
\begin{tikzcd}[column sep=small]
 & & M(wR_q) \arrow[dr] & & M(wL_q^{-1}) \cdots \\
\cdots M(wR_qL_q) \arrow[urr, start anchor=east] \arrow[drr, start anchor=east] & & & M(w) \arrow[ur] \arrow[dr] & \\
 & & M(wL_q) \arrow[ur] \arrow[dr] & & M(wR_q^{-1}) \cdots \\
\cdots M(wL_q^2) \arrow[urr, start anchor=east] \arrow[drr, start anchor=east] & & & M(wR_q^{-1}L_q) \arrow[ur] \arrow[dr] & \\
 & & M(wR_q^{-1}L_q^2) \arrow[ur] & & M(wR_q^{-2}L_q) \cdots \\
 & & \vdots & & \vdots
\end{tikzcd}
$$

Figure 4.3: Auslander–Reiten quiver components of string modules

$$
\begin{tikzcd}
\vdots & & \vdots & \\
\cdots M(u_1^{-1}u_2R_q) \arrow[dr] & & M(u_1u_2^{-1}R_q) \arrow[dr] & \\
 & M(u_1u_2^{-1}) \arrow[ur] \arrow[dr] \arrow[r, dashed] & \Lambda \arrow[r, dashed] & M(u_1u_2^{-1}) \cdots \\
\cdots M(u_1^{-1}u_2L_q) \arrow[ur] & & M(u_1u_2^{-1}L_q) \arrow[ur] & \\
\vdots & & \vdots &
\end{tikzcd}
$$

Figure 4.4: The Auslander–Reiten quiver component containing $\Lambda$

# Chapter 5

# Cyclic $p$-groups

## 5.1 Comparing Green rings

Let $G = C_{p^n} = \langle g \rangle$ be a cyclic $p$-group and let $k$ be a field of characteristic $p$. We have seen that

$$kG \simeq \frac{k[X]}{\langle X^{p^n} \rangle} \simeq U^{[p]}\mathrm{Jen}(G).$$

Co-multiplication for the group algebra is given by $\Delta(X) = \Delta(1) - \Delta(g) = 1 \otimes 1 - g \otimes g = 1 \otimes 1 - (1-X) \otimes (1-X) = 1 \otimes 1 - (1 \otimes 1 - X \otimes 1 - 1 \otimes X + X \otimes X) = X \otimes 1 + 1 \otimes X - X \otimes X$. Co-multiplication for the enveloping algebra is given by $\Delta(X) = 1 \otimes X + X \otimes 1$.

So, we see that $kG$ and $U^{[p]}\mathrm{Jen}(G)$ are isomorphic as algebras, but that they have different co-multiplications and hence may not be isomorphic as co-algebras. We consider the question of whether they are indeed isomorphic as bialgebras.

**Definition 5.1.1** Let $A$ be a bialgebra. An element $a \in A$ is said to be *group-like* if it satisfies $\Delta(a) = a \otimes a$.

**Lemma 5.1.2** *The group algebra $kG$ and the restricted enveloping algebra $U^{[p]}\mathrm{Jen}(G)$ are not isomorphic as bialgebras.*

**Proof:** It is clear that $kG$ has a basis of group-like elements. So, if we had $U^{[p]}\mathrm{Jen}(G)) \simeq kG$ as bialgebra, then $U^{[p]}\mathrm{Jen}(G)$ would also have such a basis.

A general element of $U^{[p]}\mathrm{Jen}(G)$ is of the form $a = \sum_{r=0}^{p^n-1} a_r X^r$. Since $\Delta : A \to A \otimes A$ is an algebra homomorphism,

$$\Delta(a) = \sum_{r=0}^{p^n-1} \sum_{t=0}^{r} a_r \binom{r}{t} X^{r-t} \otimes X^t.$$

We need to compare this with

$$a \otimes a = \sum_{r=0}^{p^n-1} \sum_{t=0}^{p^n-1} a_r a_t (X^r \otimes X^t + X^t \otimes X^r).$$

If $a$ is group-like then we must have $a_r a_t = \binom{r+t}{t} a_{r+t}$ for all $r, t$ with $r + t \leqslant p^n - 1$ and $a_r a_t = 0$ for all $r, t$ with $r + t \geqslant p^n$. We have $a_0 = 0$ or $a_0 = 1$, since $a_0^2 = a_0$. Fix $a_1$. The relations $a_1 a_i = (i+1) a_{i+1}$ for $1 < i < p-1$ will then give $a_i = \frac{a_1^i}{i!}$. However, we must also have $a_1 a_{p-1} = \binom{p}{p-1} a_p = 0$, i.e. $a_1 \frac{a_1^{p-1}}{(p-1)!} = 0$. Hence we must have $a_1 = 0$ and thus $a_i = 0$ for $1 < i < p$. Similarly, for $1 < i < p-1$, the relations $a_p a_{ip} = \binom{p(i+1)}{p} a_{p(i+1)}$ will give

$$a_{ip} = \frac{1}{\binom{2p}{p}\binom{3p}{p}\cdots\binom{ip}{p}} a_p^i.$$

We must also have $a_p a_{p(p-1)} = \binom{p^2}{p(p-1)} a_p^2 = 0$, i.e.

$$a_p \frac{1}{\binom{2p}{p}\binom{3p}{p}\cdots\binom{p(p-1)}{p}} a_p^{p-1} = 0.$$

Hence we have $a_p = 0$ and thus $a_{ip} = 0$ for $1 < i < p$. Repeated use of similar arguments shows that, for $1 < i \leqslant p-1$, we have $a_{ip^t} = 0$ for $0 \leqslant t < n$. Now look an arbitrary coefficient $a_j$. We can write $j = sp + i$ with $0 \leqslant s < n$ and $0 \leqslant i \leqslant p-1$. In the case $i = 0$ we have seen above that $a_j = 0$. If $i \neq 0$, consider $a_{sp} a_i = \binom{sp+i}{i} a_j$. We know that $a_i = 0$; the binomial coefficient $\binom{sp+i}{i}$ is non-zero. Hence $a_j = 0$. So, the only group-like elements of $U^{[p]}\mathrm{Jen}(G)$ are 0 and 1. In particular, $U^{[p]}\mathrm{Jen}(G)$ cannot have a basis of group-like elements, so $U^{[p]}\mathrm{Jen}(G)$ and $kG$ are not isomorphic as bialgebras. □

Let the Green rings of the two algebras above be denoted $A(kG)$ and $A(U^{[p]}\mathrm{Jen}(G))$. We will investigate the relationship between these two Green rings; since $kG$ and $U^{[p]}\mathrm{Jen}(G)$ are only isomorphic as algebras but not as bialgebras, we might expect two different Green rings.

**Theorem 5.1.3** *The Green rings $A(kG)$ and $A(U^{[p]}\mathrm{Jen}(G))$ are isomorphic, via the natural isomorphism $[M_i] \mapsto [M_i]$.*

**Proof:** We need to check that the decompositions of $M_i \otimes M_j$ as a $kG$-module and as a $U^{[p]}\mathrm{Jen}(G)$-module are the same, where we may assume without loss of generality that $i \geq j$. Since $kG$ and $U^{[p]}\mathrm{Jen}(G)$ are both generated by the single generator $X$, it is sufficient to check that the action of $X$ on $M_i \otimes M_j$ considered as a $kG$-module is equivalent to the action of $X$ on $M_i \otimes M_j$ considered as a $U^{[p]}\mathrm{Jen}(G)$-module. Let $M_i = \langle u_1, \ldots, u_i \rangle$, $Xu_s = u_{s+1}$ and let $M_j = \langle v_1, \ldots, v_j \rangle$, $Xv_r = v_{r+1}$.

First consider $M_i \otimes M_j$ as a $kG$-module. As a basis for $M_i \otimes M_j$, take $E = \{e_{rs} = u_s \otimes v_r : 1 \leqslant s \leqslant i, 1 \leqslant r \leqslant j\}$. Set $u_s \otimes v_r = 0$ if $s > i$ or $r > j$. Then $X(e_{rs}) = Xu_s \otimes v_r + u_s \otimes Xv_r - Xu_s \otimes Xv_r = u_{s+1} \otimes v_r + u_s \otimes v_{r+1} - u_{s+1} \otimes v_{r+1} = e_{r,s+1} + e_{r+1,s} - e_{r+1,s+1}$.

Now consider $M_i \otimes M_j$ as a $U^{[p]}\mathrm{Jen}(G)$-module. As a basis, we take $B = \{b_{rs} : 1 \leqslant r \leqslant i, 1 \leqslant s \leqslant j\}$, where $b_{1s} = u_s \otimes v_1$ and

$$b_{rs} = \sum_{k=0}^{i-s} \binom{k+r-2}{r-2} u_{s+k} \otimes v_r$$

for $r \geqslant 2$. Then $X(b_{1s}) = u_{s+1} \otimes v_1 + u_s \otimes v_2 = u_{s+1} \otimes v_1 + \sum_{k=0}^{i-s} u_{s+k} \otimes v_2 - \sum_{k=1}^{i-s} u_{s+k} \otimes v_2 = b_{1,s+1} + b_{2,s} - b_{2,s+1}$. For $r \geqslant 2$, we have

$$\begin{aligned} X(b_{rs}) &= \sum_{k=0}^{i-s} \binom{k+r-2}{r-2} u_{s+k+1} \otimes v_r + \sum_{k=0}^{i-s} \binom{k+r-2}{r-2} u_{s+k} \otimes v_{r+1} \\ &= b_{r,s+1} + \sum_{k=0}^{i-s} \binom{k+r-2}{r-2} u_{s+k} \otimes v_{r+1} \end{aligned} \tag{5.1}$$

by the convention that $u_t \otimes v_r = 0$ if $t > i$.

Now look at $b_{r+1,s} - b_{r+1,s+1}$. This is

$$\sum_{k=0}^{i-s} \binom{k+r-1}{r-1} u_{s+k} \otimes v_{r+1} - \sum_{k=0}^{i-s-1} \binom{k+r-1}{r-1} u_{s+k+1} \otimes v_{r+1}.$$

The coefficient of $u_{s+k} \otimes v_{r+1}$ in this expression is

$$\binom{k+r-1}{r-1} - \binom{k+r-2}{r-1}.$$

It can easily be seen that this is equal to

$$\binom{k+r-2}{r-2}$$

which is the coefficient of $u_{s+k} \otimes v_{r+1}$ in (5.1). Hence $X(b_{rs}) = b_{r+1,s} + b_{r,s+1} - b_{r+1,s+1}$.

So, we see that the action of $X$ on $M_i \otimes M_j$ as a $kG$-module with respect to the basis $E$, and the action of $X$ on $M_i \otimes M_j$ as a $U^{[p]}\mathrm{Jen}(G)$-module with respect to the basis $B$, are given by precisely the same matrices. □

# Chapter 6

# The Klein four group

## 6.1 Comparing Green rings

Let $V_4 = \langle g, h \rangle$ be the Klein four group and let $k$ be a field of characteristic 2. We have seen that

$$kV_4 \simeq \frac{k[X,Y]}{\langle X^2, Y^2 \rangle} \simeq U^{[2]}\mathrm{Jen}(V_4).$$

The co-multiplication on $kV_4$ is given by $\Delta(X) = X \otimes 1 + 1 \otimes X + X \otimes X$ and $\Delta(Y) = Y \otimes 1 + 1 \otimes Y + Y \otimes Y$. The co-multiplication on $U^{[2]}\mathrm{Jen}(V_4)$ is given by $\Delta(X) = X \otimes 1 + 1 \otimes X$ and $\Delta(Y) = Y \otimes 1 + 1 \otimes Y$.

As above, we will compare the Green rings $A(kV_4)$ and $A(U^{[2]}(L))$.

Suppose first that $k = \overline{k}$ and look at some multiplications in the Green ring $A(U^{[2]}\mathrm{Jen}(V_4))$. Consider the tensor product $C_1(T+\alpha) \otimes C_1(T+\alpha)$ with $\alpha \in k$ and $\alpha \neq 0, 1$. Let $\{u_1, u_2\}$ and $\{v_1, v_2\}$ be bases for the two copies of $C_1(T+\alpha)$ such that $X$ acts as

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and $Y$ acts as

$$\begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}.$$

Let $B = \{u_1 \otimes v_1, u_1 \otimes v_2, u_1 \otimes v_2 + u_2 \otimes v_1, u_2 \otimes v_2\}$; this is a basis for $C_1(T+\alpha) \otimes C_1(T+\alpha)$. With respect to this basis, $X$ acts as

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $Y$ acts as

$$\begin{pmatrix} 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So we see that $[C_1(T+\alpha)] \circ [C_1(T+\alpha)] = 2[C_1(T+\alpha)]$ in the Green ring $A(U^{[2]}\mathrm{Jen}(V_4))$. This contrasts with the Green ring $A(kV_4)$, where we have $[C_1(T+\alpha)] \circ [C_1(T+\alpha)] = [C_2(T+\alpha)]$. So, in the case where $k = \overline{k}$, we certainly do not have a natural isomorphism between the Green rings where each $U^{[2]}\mathrm{Jen}(V_4)$-module is mapped to the corresponding $kV_4$-module, as we had in Theorem 5.1.3.

In the case $k = \overline{k}$ all other tensor products of indecomposable $U^{[2]}\mathrm{Jen}(V_4)$-modules can also be calculated; it turns out that all such products other than the one calculated above are the same in $A(U^{[2]}\mathrm{Jen}(V_4))$ as in $A(kV_4)$.

Now consider the general case, where $k$ can be any field of characteristic 2. Conlon's paper calculating all products of indecomposable $kV_4$-modules [10] uses a technique which allows us to deduce the decomposition of tensor products of $kV_4$-modules in the case where $k \neq \overline{k}$ from the decomposition of $\overline{k}V_4$-modules. We can apply the same technique in the case of $U^{[2]}\mathrm{Jen}(V_4)$-modules; this enables us to give the complete multiplication tables for tensor products of $U^{[2]}\mathrm{Jen}(V_4)$-modules:

| $n \leqslant n'$ | $A_n$ | $B_n$ | $D$ |
|---|---|---|---|
| $A_{n'}$ | $nn'D + A_{n+n'}$ | $n(n'+1)D + A_{n'-n}$ | $(2n'+1)D$ |
| $B_{n'}$ | $n(n'+1)D + B_{n'-n}$ | $nn'D + B_{n+n'}$ | $(2n'+1)D$ |
| $C_{n'}(\pi')$ | $nn'm'D + C_{n'}(\pi')$ | $nn'm'D + C_{n'}(\pi')$ | $2n'm'D$ |
| $C_{n'}(\infty)$ | $nn'D + C_{n'}(\infty)$ | $nn'D + C_{n'}(\infty)$ | $2n'D$ |
| $D$ | $(2n+1)D$ | $(2n+1)D$ | $4D$ |

| $n \leqslant n'$ | $C_n(\pi)$ | $C_n(\infty)$ |
|---|---|---|
| $A_{n'}$ | $nn'mD + C_n(\pi)$ | $nn'D + C_n(\infty)$ |
| $B_{n'}$ | $nn'mD + C_n(\pi)$ | $nn'D + C_n(\infty)$ |
| $C_{n'}(\pi')$ | If $\pi \neq \pi'$: $nmn'm'D$<br>If $\pi = \pi'$: $nm(n'm-1)D + 2C_n(\pi)$ | $nn'm'D$ |
| $C_{n'}(\infty)$ | $nn'mD$ | $n(n'-1)D + 2C_n(\infty)$ |
| $D$ | $2nmD$ | $2nD$ |

It is clear from the tables above that, for any field of characteristic 2, we do not have $A(kV_4) \simeq A(U^{[2]}\mathrm{Jen}(V_4))$ via a natural isomorphism fixing all $kV_4$-modules, as we had in Theorem 5.1.3.

**Theorem 6.1.1** *Let $k$ be a field of characteristic 2. As algebras we have $kV_4 \simeq U^{[2]}\mathrm{Jen}(V_4)$, but there is no natural isomorphism between $A(kV_4)$ and $A(U^{[2]}\mathrm{Jen}(V_4))$.*

**Proof:** Consider the rings $A(kV_4)_{\mathbb{Q}} := A(kV_4) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $A(U^{[2]}\mathrm{Jen}(V_4))_{\mathbb{Q}} := A(U^{[2]}\mathrm{Jen}(V_4)) \otimes_{\mathbb{Z}} \mathbb{Q}$; that is, extend the ring of coefficients for the two Green rings to $\mathbb{Q}$. We will show that there is no natural isomorphism between $A(kV_4)_{\mathbb{Q}}$ and $A(U^{[2]}\mathrm{Jen}(V_4))_{\mathbb{Q}}$, by showing that $A(kV_4)_{\mathbb{Q}}$ contains fewer idempotents than $A(U^{[2]}\mathrm{Jen}(V_4))_{\mathbb{Q}}$.

Comparing the tables above with those in Section 4.4.2, note that the only products in $A(kV_4)_{\mathbb{Q}}$ which differ from the corresponding products in $A(U^{[2]}\mathrm{Jen}(V_4))_{\mathbb{Q}}$ are the squares $[C_1(\pi)]^2$ for $\pi \neq T, T+1$. So, any idempotent in $A(kV_4)_{\mathbb{Q}}$ which does not involve elements of the type $C_1(\pi)$ with $\pi \neq T, T+1$ must also be an idempotent in $A(U^{[2]}\mathrm{Jen}(V_4))_{\mathbb{Q}}$.

Suppose $e = e^2$ for some $e \in A(kV_4)_{\mathbb{Q}}$ which involves a summand of type $C_1(\pi)$ with $\pi \neq T, T+1$. It can be seen from the structure of the modules $A_n$ and $B_n$ that $A_n = \Omega^n(k)$ and $B_n = \Omega^{-n}(k)$. So, the element $e$ can be written in the form

$$e = \sum_{\pi\in\Pi}\sum_{n\in I}\alpha_{n,\pi}[C_n(\pi)] + \sum_{n\in J}\beta_n[C_n(\infty)] + \sum_{n\in K}\gamma_n[\Omega^n(k)] + \delta[D]$$

with $\alpha_{n,\pi}, \beta_n, \gamma_n, \delta \in k$, where $I, J \subseteq \mathbb{N}$ and $K \subseteq \mathbb{Z}$ are finite indexing sets and $\Pi$ is a finite set of indecomposable polynomials over $k$. We have assumed that $\alpha_{1,\pi'} \neq 0$ for some $\pi' \in \Pi\backslash\{T, T+1\}$. Suppose we square the element $e$ in this form, using Conlon's multiplication tables and then compare coefficients of each indecomposable module. The first thing to note is that we must have $\gamma_n = 0$ for all $n \in K \setminus \{0\}$. For, our multiplication tables tell us that $\gamma_n[\Omega^n(k)]$ can appear in $e^2$ only from terms of the form $\gamma_r\gamma_s[\Omega^r(k)] \circ [\Omega^s(k)]$. If $r_0 = \max\{r : r \in K\} > 0$ then $\gamma_{r_0}[\Omega^{r_0}]^2 = \gamma_{r_0}^2[\Omega^{2r_0}(k)]$, but $2r_0 \notin K$ and hence we must have $\gamma_{r_0} = 0$. Similarly, if $r_1 = \min\{r : r \in K\} < 0$ then $2r_1 \notin K$ and we must have $\gamma_{r_1} = 0$. We can now replace the indexing set $K$ by $K \setminus \{r_0\}$ or $K \setminus \{r_1\}$ as appropriate, and repeat the argument. Iterating this process, we deduce that $K = \{0\}$ or $K = \emptyset$. Comparing coefficients of $\Omega^0(k)$ in $e$ and $e^2$, we have $\gamma_0^2 = \gamma_0$, so $\gamma_0 = 1$ or $\gamma_0 = 0$. If $\gamma_0 = 1$, then we can replace $e$ by $1 - e$, which is also an idempotent; this is then covered by the case $\gamma_0 = 0$.

Assume $\gamma_0 = 0$. let $n' = \max\{n \in I : \alpha_{n',\pi} \neq 0\}$, as before. If $n' > 2$ then the coefficient of $[C_{n'}(\pi')]$ in $e^2$ is $2(\alpha_{n',\pi'})^2$, which by assumption equals $\alpha_{n',\pi'}$, so we have $\alpha_{n',\pi'} = 1/2$. Now consider the coefficients of $[C_{n''}(\pi')]$, where $n'' = \max\{n \in I \setminus \{n'\} : \alpha_{n'',\pi'} \neq 0\}$. If $n'' > 2$ then the coefficient of $[C_{n''}(\pi')]$ in $e^2$ is $\alpha_{n',\pi'}\alpha_{n'',\pi} + 2(\alpha_{n'',\pi})^2 = 2\alpha_{n'',\pi} + 2(\alpha_{n'',\pi})^2$. By assumption this equals $\alpha_{n'',\pi}$, giving $\alpha_{n'',\pi} = -1/2$. Repeating this process tells us that the non-zero $\alpha_{n,\pi}$ are alternately $1/2$ and $-1/2$. Now look at the coefficient of $[C_1(\pi')]$ in $e^2$. This will be $\pm 2\alpha_{1,\pi'} + 4\alpha_{2,\pi'}\alpha_{1,\pi'} = \alpha_{1,\pi'}$. Since $\alpha_{1,\pi'} \neq 0$, this gives $\alpha_{2,\pi'} = (1 \mp 2)/4$. Finally, consider the coefficient of $[C_2(\pi')]$ in $e^2$. This will be $\pm 2\alpha_{2,\pi'} + 2(\alpha_{2,\pi'})^2 + (\alpha_{1,\pi'})^2$ and should equal $\alpha_{1,\pi'}$. However, the two possibilities for $\alpha_{2,\pi'}$ give us $(\alpha_{1,\pi'})^2 = 9/8$ or $(\alpha_{1,\pi'})^2 = -7/8$, neither of which have roots in $\mathbb{Q}$. So, there cannot be such an idempotent in $A(kV_4)_{\mathbb{Q}}$. Hence $A(kV_4)_{\mathbb{Q}}$ contains no idempotents which do not correspond directly to idempotents in $A(U^{[2]}\mathrm{Jen}(V_4))_{\mathbb{Q}}$.

Finally, we see that $A(U^{[2]}\mathrm{Jen}(V_4))_{\mathbb{Q}}$ *does* contain idempotents involving summands of the type $C_1(\pi)$ with $\pi \neq T, T+1$. For example, for any such $\pi$ of degree $m$, we have $([C_1(\pi)] - (m/2)[D])^2 = [C_1(\pi)]^2 - m[C_1(\pi)][D] + (m^2/4)[D]^2 = m(m-1)[D] + 2[C_1(\pi)] - 2m[D] + m^2[D] = 2[C_1(\pi)] - m[D]$, so we see that $(1/2)[C_1(\pi)] - (m/4)[D]$ is an idempotent in $A(U^{[2]}\mathrm{Jen}(V_4))_{\mathbb{Q}}$. Hence $A(U^{[2]}\mathrm{Jen}(V_4))_{\mathbb{Q}}$ contains more idempotents than $A(kV_4)_{\mathbb{Q}}$. □

## 6.2 Comparing Benson–Carlson quotients

Having seen that $A(kV_4) \not\cong A(U^{[2]}\mathrm{Jen}(V_4))$, we now consider the case where $k$ is algebraically closed and look at the Benson-Carlson quotients

$$\frac{A(kV_4)}{A(kV_4; 2)}$$

and

$$\frac{A(U^{[2]}\mathrm{Jen}(V_4))}{A(U^{[2]}\mathrm{Jen}(V_4);2)}.$$

**Theorem 6.2.1** *Let $k$ be an algebraically closed field of characteristic 2. Then*

$$\frac{A(kV_4)}{A(kV_4;2)} \simeq \frac{A(U^{[2]}\mathrm{Jen}(V_4))}{A(U^{[2]}\mathrm{Jen}(V_4);2)}$$

*via the natural isomorphism $[M] \mapsto [M]$ for all cosets of odd-dimensional $kV_4$-modules $M$.*

**Proof:** We need to check that, for all odd-dimensional $kV_4$-modules $M$ and $N$, we have $M \otimes_1 N \equiv M \otimes_2 N$ modulo even-dimensional summands, where $\otimes_1$ denotes the tensor product for the group algebra $kV_4$ and $\otimes_2$ denotes the tensor product for the universal enveloping algebra $U^{[2]}\mathrm{Jen}(V_4)$. The odd-dimensional $kV_4$-modules are the modules $A_n$ and $B_n$ in Conlon's notation. Our multiplication tables in Section 4.4.2 and Section 6.1 tell us immediately that all products involving only modules of the form $A_n$ and $B_n$ are the same in $A(kV4)$ and $A(U^{[2]}\mathrm{Jen}(V_4))$. Alternatively, it can be seen from the structure of these modules that $\{A_n : n \in \mathbb{N}_0\} \cup \{B_n : n \in \mathbb{N}\} = \{\Omega^n(k) : n \in \mathbb{Z}\}$. Taking two such modules $\Omega^m(k)$ and $\Omega^n(k)$, we have $\Omega^m(k) \otimes_i \Omega^n(k) \equiv \Omega^{m+n}(k)$ modulo projectives, for $i = 1$ and $i = 2$. So, they certainly have the same product modulo even-dimensional summands. □

# Chapter 7

# Dihedral 2-groups: comparing Benson–Carlson quotients

Let $k$ be a field of characteristic 2, and let $D_{4q}$ with $q \geqslant 2$ be the dihedral group of order $4q$. We have seen that

$$kD_{4q} \simeq \frac{k\langle X, Y\rangle}{\langle X^2, Y^2, (XY)^q - (YX)^q\rangle} \simeq U^{[2]}\mathrm{Jen}(D_{4q}).$$

The co-multiplication on $kD_{4q}$ is given by $\Delta(X) = X \otimes 1 + 1 \otimes X + X \otimes X$ and $\Delta(Y) = Y \otimes 1 + 1 \otimes Y + Y \otimes Y$. The co-multiplication on $U^{[2]}\mathrm{Jen}(D_{4q})$ is given by $\Delta(X) = X \otimes 1 + 1 \otimes X$ and $\Delta(Y) = Y \otimes 1 + 1 \otimes Y$.

Suppose $k$ is algebraically closed. We wish to compare the Benson–Carlson quotients

$$\frac{A(kD_{4q})}{A(kD_{4q}; 2)}$$

and

$$\frac{A(U^{[2]}\mathrm{Jen}(D_{4q}))}{A(U^{[2]}\mathrm{Jen}(D_{4q}); 2).}$$

**Conjecture 7.0.2** *Let $k$ be an algebraically closed field of characteristic 2. Then*

$$\frac{A(kD_{4q})}{A(kD_{4q}; 2)} \simeq \frac{A(U^{[2]}\mathrm{Jen}(D_{4q}))}{A(U^{[2]}\mathrm{Jen}(D_{4q}); 2)}$$

*via the natural isomorphism* $[M] \mapsto [M]$ *for all cosets of odd-dimensional* $kD_{4q}$*-modules* $M$.

Note that if this conjecture is correct, then we would have

$$\frac{A(kG)}{A(kG;p)} \simeq \frac{A(U^{[p]}\mathrm{Jen}(G))}{A(U^{[p]}\mathrm{Jen}(G);p)}$$

for all $p$-groups $G$ of finite or tame representation type with $kG \simeq U^{[p]}\mathrm{Jen}(G)$ as algebras. If correct, this conjecture would provide an alternative way of looking at multiplications in the Green ring $A(kD_{4q})$ and could help to improve our understanding of such products.

## 7.1 Properties of the Benson–Carlson quotients

Here we study some properties of the two Benson–Carlson quotients, with particular emphasis on looking for evidence which would seem to support Conjecture 7.0.2 or which might go some way towards proving the conjecture.

Throughout this chapter, we will write $\Lambda$ to denote the algebra $kD_{4q} \simeq U^{[2]}\mathrm{Jen}(D_{4q})$ in contexts where we are concerned only with the algebra structure and not with the two different co-multiplications. When we do need to distinguish between the two co-multiplications, we will write $\otimes_1$ for tensor products of $kD_{4q}$-modules and $\otimes_2$ for tensor products of $U^{[2]}\mathrm{Jen}(D_{4q})$-modules, if the appropriate co-multiplication to use may not otherwise be clear from the context.

### 7.1.1 The group formed by the odd-dimensional indecomposable modules

**Lemma 7.1.1** *Let* $M$, $N$ *be two indecomposable* $\Lambda$*-modules.*

1. *The product $M \otimes_1 N$ of $kD_{4q}$-modules decomposes as a direct sum of indecomposables as one odd-dimensional string module plus band modules and projective summands.*

2. *The product $M \otimes_2 N$ of $U^{[2]}\mathrm{Jen}(D_{4q})$-modules decomposes as a direct sum of indecomposables as one odd-dimensional string module plus band modules and projective summands.*

**Proof:** The structure of the following proof is such that it will prove both parts of the above lemma, without needing to distinguish between the two co-multiplications for the two different Hopf algebras.

The classification of indecomposable $\Lambda$-modules in [28] tells us that $M$ and $N$ must be string modules corresponding to strings in $X$ and $Y$ of even length. Let $\Lambda_X$ be the subalgebra of $\Lambda$ generated by $X$. There are precisely two indecomposable $\Lambda_X$-modules, the trivial module $k$ where $X$ acts as 0 and the projective module

$$\begin{array}{c}\cdot\\ \downarrow X\\ \cdot\end{array} \qquad \text{with} \qquad X \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

From the structure of string modules, it is clear that $M|_{\Lambda_X} = k \oplus$ projectives and $N|_{\Lambda_X} = k \oplus$ projectives. Hence $(M \otimes N)|_{\Lambda_X} = k \oplus$ projectives, since taking the tensor product of modules commutes with restriction to $\Lambda_X$.

Similarly, let $\Lambda_Y$ be the subalgebra of $\Lambda$ generated by $Y$. By symmetry, the two indecomposable $\Lambda_Y$-modules have the same structure as the two indecomposable $\Lambda_X$-modules. As in the case of $\Lambda_X$, we then have $(M \otimes N)|_{\Lambda_Y} = k \oplus$ projectives.

Now think about the direct sum decomposition of $M \otimes N$ as a $\Lambda$-module. It is clear that every odd-dimensional summand (which must be a string module) will contribute a copy of $k$ when restricted to $\Lambda_X$. Furthermore, any even-dimensional string module given by a string which begins and ends with $Y^{\pm 1}$ will contribute two copies of $k$ when restricted to $\Lambda_X$. Hence $M \otimes N$ has at most one odd-dimensional indecomposable summand and $M \otimes N$ cannot have any summands which are even-dimensional string modules given by strings beginning and ending with $Y^{\pm 1}$.

Similarly, if we consider restrictions to $\Lambda_Y$, we see that $M \otimes N$ cannot have any summands which are even-dimensional string modules given by strings beginning and ending with $X^{\pm 1}$. Any string module is given by a string $u_1 u_2 \cdots u_n$, where $u_i = X^{\pm 1}$ implies $u_{i+1} = Y^{\pm 1}$ and $u_i = Y^{\pm 1}$ implies $u_{i+1} = X^{\pm 1}$, so for any string module of even dimension (that is, where the string has odd length) we must have either $u_1 = X^{\pm 1}$ and $u_n = X^{\pm 1}$, or $u_1 = Y^{\pm 1}$ and $u_n = Y^{\pm 1}$. So, we have in fact seen that $M \otimes N$ cannot have any summands which are even-dimensional string modules.

We have seen above that $M \otimes N$ must have at most one odd-dimensional string summand. However, these are the only odd-dimensional $\Lambda$-modules; the fact that $\dim(M \otimes N)$ is odd tells us that there must be at least one such summand. Hence $M \otimes N$ decomposes exactly as claimed. □

**Theorem 7.1.2** 1. *The cosets of odd-dimensional indecomposable $kD_{4q}$-modules form an abelian group under the tensor product multiplication in*

$$\frac{A(kD_{4q})}{A(kD_{4q};2)}.$$

2. *The cosets of odd-dimensional indecomposable $U^{[2]}\mathrm{Jen}(D_{4q})$-modules form an abelian group under the tensor product multiplication in*

$$\frac{A(U^{[2]}\mathrm{Jen}(D_{4q}))}{A(U^{[2]}\mathrm{Jen}(D_{4q});2)}.$$

**Proof:** As with the previous lemma, we can give a proof which works for both parts of the above theorem.

Let $M$ and $N$ be odd-dimensional indecomposable $\Lambda$-modules, and let $[M]$ and $[N]$ be the corresponding cosets in $A(\Lambda)/A(\Lambda; 2)$. By Lemma 7.1.1 we have $[M] \otimes [N] = [U]$ for some odd-dimensional indecomposable $\Lambda$-module $U$. It is clear that our multiplication will be associative and commutative, since this follows from the corresponding properties of $A(kD_{4q})$ and $A(U^{[2]}\mathrm{Jen}(D_{4q}))$. It is also clear that the coset of the trivial module $k$ (on which both $X$ and $Y$ act as 0) will be a multiplicative identity. Given an odd-dimensional indecomposable module $M$, Theorem 4.2.5 tells us that $M \otimes M^*$ has a summand isomorphic to $k$. By Lemma 7.1.1, $M \otimes M^*$ has precisely one odd-dimensional summand. Hence this summand must be $k$ and we have $[M] \otimes [M^*] = [k]$ in both $A(kD_{4q})/A(kD_{4q}; 2)$ and $A(U^{[2]}\mathrm{Jen}(D_{4q}))/A(U^{[2]}\mathrm{Jen}(D_{4q}); 2)$, telling us that $[M^*]$ is the inverse for $[M]$ in both cases. $\square$

We now introduce some notation for these groups: write $\Gamma(kD_{4q})$ for the group given by part 1 of Theorem 7.1.2, and write $\Gamma(U^{[2]}\mathrm{Jen}(D_{4q}))$ for the group given by part 2 of the theorem.

Note that the two Benson–Carlson quotients we are considering are in fact the group rings over $\mathbb{Z}$ of the two groups given by Theorem 7.1.2. That is,

$$\frac{A(kD_{4q})}{A(kD_{4q}; 2)} \simeq \mathbb{Z}\Gamma(kD_{4q})$$

and

$$\frac{A(U^{[2]}\mathrm{Jen}(D_{4q}))}{A(U^{[2]}\mathrm{Jen}(D_{4q}); 2)} \simeq \mathbb{Z}\Gamma(U^{[2]}\mathrm{Jen}(D_{4q})).$$

So, to prove Conjecture 7.0.2, it would be sufficient to prove the following:

**Conjecture 7.1.3** *Let $k$ be an algebraically closed field of characteristic 2. Then*

$$\Gamma(kD_{4q}) \simeq \Gamma(U^{[2]}\mathrm{Jen}(D_{4q}))$$

*via the natural isomorphism $[M] \mapsto [M]$ for all cosets of odd-dimensional indecomposable $kD_{4q}$-modules $M$.*

We now study the properties of the groups $\Gamma(kD_{4q})$ and $\Gamma(U^{[2]}\mathrm{Jen}(D_{4q}))$. For $[M]$ an element of $\Gamma(kD_{4q})$ or $\Gamma(U^{[2]}\mathrm{Jen}(D_{4q}))$, let $\overline{M}$ denote the unique odd-dimensional indecomposable summand of $M$.

**Theorem 7.1.4** *The groups $\Gamma(kD_{4q})$ and $\Gamma(U^{[2]}\mathrm{Jen}(D_{4q}))$ are both torsion-free.*

**Proof:** Let $N$ be an indecomposable $\Lambda$-module, and suppose $[N^{\otimes r}] = [k]$ in $A(kD_{4q})/A(kD_{4q};2)$ or $A(U^{[2]}\mathrm{Jen}(D_{4q}))/A(U^{[2]}\mathrm{Jen}(D_{4q});2)$, for some $r \geqslant 1$. Then in particular, $k$ is a direct summand of $\overline{N^{\otimes(r-1)}} \otimes N$. Then, by Theorem 4.2.5, we must have $\overline{N^{\otimes(r-1)}} \simeq N^*$. We will show that this is not possible unless $N \simeq k$.

Let $N$ be an odd-dimensional indecomposable $\Lambda$-module with $N \not\simeq k$. The classification of indecomposable $\Lambda$-modules [28] tells us that $N$ must be a string module given by some string $w = u_1 u_2 \cdots u_{2n}$ of even length. If $u_1 = X^{\pm 1}$ then $u_{2n} = Y^{\pm 1}$ (and vice versa). Assume without loss of generality that $u_1 = X^{\varepsilon}$ with $\varepsilon \in \{\pm 1\}$. We claim that $\overline{N^{\otimes t}}$ is also given by a word beginning with $X^{\varepsilon}$, for all $t \geqslant 1$. This would be sufficient to show that $\overline{N^{\otimes(r-1)}} \not\simeq N^*$, since the dual module $N^*$ is given by the string $u_1^{-1} u_2^{-1} \cdots u_{2n}^{-1}$. If the claim is correct then this is different from the string corresponding to $\overline{N^{\otimes t}}$. The only string equivalent to $u_1^{-1} u_2^{-1} \cdots u_{2n}^{-1}$ under Ringel's equivalence relation on strings is $u_{2n}u_{2n-1}\cdots u_1$. This is also different from the string corresponding to $\overline{N^{\otimes t}}$, since $u_{2n} = Y^{\pm 1}$. So, the strings corresponding to $N^*$ and $\overline{N^{\otimes t}}$ are not equivalent, which tells us that $\overline{N^{\otimes t}} \not\cong N^*$, as required.

So, it remains to show that if $N$ is given by a string beginning with $X^\varepsilon$ then $\overline{N^{\otimes t}}$ is also given by a string beginning with $X^\varepsilon$ for all $t \geqslant 1$. We first take the case $\varepsilon = 1$. We will give a proof by induction on $t$, so suppose $N$ and $\overline{N^{\otimes(t-1)}}$ are both given by words beginning with $X$. Pictorially, the string module structures of $N$ and $\overline{N^{\otimes(t-1)}}$ are as in Figure 7.1.

$$N = \begin{tikzcd} & b_2 \arrow[dl, "X"'] & \cdots\cdots\cdots \\ b_1 & & \end{tikzcd} \quad \text{and} \quad \overline{N^{\otimes(t-1)}} = \begin{tikzcd} & c_2 \arrow[dl, "X"'] & \cdots\cdots\cdots \\ c_1 & & \end{tikzcd}$$

Figure 7.1: Module structures of $N$ and $N^{\otimes(t-1)}$

Consider $N \otimes \overline{N^{\otimes(t-1)}}$. In this product (with either co-multiplication) we have $b_1 \otimes c_1 = X(b_2 \otimes c_2)$, so $b_1 \otimes c_1 \in \operatorname{im} X$. Also $b_1 \otimes c_1 \in \ker X \cap \ker Y$ and $b_1 \otimes c_1 \notin \operatorname{im} Y$. Suppose, for a contradiction, that $\overline{N^{\otimes t}}$ is given by a word beginning with $X^{-1}$. The calculation above tells us that there exists some element $a \in N^{\otimes t}$ such that $a \in \operatorname{im} X \cap \ker X \cap \ker Y$ and $a \notin \operatorname{im} Y$. Lemma 7.1.1 tells us that $N^{\otimes t}$ decomposes as $\overline{N^{\otimes t}}$ (a string module given by a string of even length beginning with $X^{-1}$), plus band modules, plus projective summands. Consider the natural basis for $N^{\otimes t}$ giving the decomposition in this form. The element $a$ must be a linear combination $\sum_{i \in I} X m_i$ with $\{m_i : i \in I\}$ some subset of the natural basis. By inspection of the pictures for: (i) a string module given by a string of even length beginning with $X^{-1}$; (ii) a band module; (iii) a projective module, we see that $Xm_i$ adjoins an edge labelled with $Y$ for all $i \in I$. So, this part of the module must have structure either as in Figure 7.2, or as in Figure 7.3 for some $n_i$. But, if the first

$$\begin{tikzcd}
m_i \arrow[dr, "X"] & & \\
& Xm_i \arrow[dr, "Y"] & \\
& & YXm_i
\end{tikzcd}$$

Figure 7.2:

$$\begin{tikzcd}
m_i \arrow[dr, "X"] & & n_i \arrow[dl, "Y"'] \\
& Xm_i &
\end{tikzcd}$$

Figure 7.3:

picture ever arises, we have $Xm_i \notin \ker Y$ and $a \notin \ker Y$. So, in all cases, we must have the structure given by the second picture. But then $Xm_i \in \mathrm{im}Y$ and $a \in \mathrm{im}Y$. This shows that $\mathrm{im}X \cap \ker Y \subseteq \mathrm{im}Y$, giving the contradiction we require. Hence $\overline{N^{\otimes t}}$ cannot be given by a string beginning with $X^{-1}$ and must (up to equivalence of strings) be given by a string beginning with $X$.

Now take the case $\varepsilon = -1$, so $N$ is given by a string beginning with $X^{-1}$. Then $N^*$ is given by a string beginning with $X$. So, by the case above, $\overline{(N^*)^{\otimes t}} \simeq (\overline{N^{\otimes t}})^*$ is given by a word beginning with $X$. Hence $\overline{N^{\otimes t}}$ is given by a word beginning with $X^{-1}$. This completes the proof. $\square$

**Theorem 7.1.5** *The groups $\Gamma(kD_{4q})$ and $\Gamma(U^{[2]}\mathrm{Jen}(D_{4q}))$ are not finitely generated.*

**Proof:** We begin by proving that $\Gamma(kD_{4q})$ is not finitely generated. Suppose, for a contradiction, that $\Gamma(kD_{4q})$ is finitely generated by elements

$[M_1], [M_2], \ldots, [M_n]$. Then for all odd-dimensional indecomposable $kD_{4q}$-modules $U$, we must have $[U] = [M_1^{\otimes r_1} \otimes M_2^{\otimes r_2} \otimes \cdots \otimes M_n^{\otimes r_n}]$ in $\Gamma(kD_{4q})$.

Recall that $kD_{4q} = k\langle X, Y\rangle / \langle X^2, Y^2, (XY)^q = (YX)^q\rangle$. Let

$$A = (XY)^q + (XY)^{q-1}X + Y(XY)^{q-1} + \sum_{i=0}^{(\log_2 q)-1} ((XY)^{q-2^i} + (YX)^{q-2^i})$$

and $B = X$. Then

$$AB = (YX)^q + \sum_{i=0}^{(\log_2 q)-1} (XY)^{q-2^i} X$$

and

$$BA = (XY)^q + \sum_{i=0}^{(\log_2 q)-1} X(YX)^{q-2^i} = AB$$

and $A^2 = B^2 = 0$. So, $A$ and $B$ generate a subalgebra $\Lambda'$ isomorphic to $kV_4$. Note that $A = 1 + z$, where $z$ is the central involution in the group $D_{4q}$, so $\Lambda'$ is indeed the group algebra of a subgroup of $D_{4q}$ isomorphic to $V_4$. Consider the restrictions $M_i|_{\Lambda'}$. We have

$$M_i|_{\Lambda'} = C \oplus \bigoplus_{s \in \mathbb{Z}} \beta_{i,s} \Omega^s(k) \oplus \gamma_i D$$

where $C$ is some direct sum of non-projective even-dimensional indecomposable $kV_4$-modudules, with $\beta_{i,s}, \gamma_i \in \mathbb{N}_0$ and only finitely many of the coefficients $\beta_{i,s}$ non-zero.

Now, given an odd-dimensional indecomposable $kD_{4q}$-module $U$, we have supposed $U \simeq \overline{M_1^{\otimes r_1} \otimes \cdots M_n^{\otimes r_n}}$. Equivalently, $U$ is a direct summand of $M_1^{\otimes r_1} \otimes \cdots \otimes M_n^{\otimes r_n}$. Then $U|_{\Lambda'}$ is a direct summand of $(M_1^{\otimes r_1} \otimes \cdots \otimes M_n^{\otimes r_n})|_{\Lambda'}$, which is isomorphic to $(M_1|_{\Lambda'})^{\otimes r_1} \otimes \cdots \otimes (M_n|_{\Lambda'})^{\otimes r_n}$. Now, by inspection of our multiplication tables for $kV_4$-modules in Section 4.4.2 , we see that a $kV_4$-module of type $C_t(0)$ for $t \in \mathbb{Z}$ can only occur as a summand of

$(M_1|_{\Lambda'})^{\otimes r_1} \otimes \cdots \otimes (M_n|_{\Lambda'})^{\otimes r_n}$ if $C_t(0)$ occurs with non-zero coefficient in the restriction $M_i|_{\Lambda'}$ for some $i$. Only finitely many $C_t(0)$ occur in these restrictions. So, we see that only finitely many $C_t(0)$ can occur with non-zero coefficient in restrictions $U|_{\Lambda'}$ of odd-dimensional indecomposable $kD_{4q}$-modules $U$.

To give the required contradiction, we now present a family of $kD_{4q}$-modules $U_1, U_2, \ldots$ such that the restrictions $U_1|_{\Lambda'}, U_2|_{\Lambda'}, \ldots$ involve infinitely many different $C_t(0)$. Let $U_r$ be the odd-dimensional string module given by the string $XY(X^{-1}(YX)^{q-1}Y)^r X^{-1}Y$, for $r \geqslant 1$. Looking at the pictorial representation of these modules and thinking about the actions of $A$ and $B$, it can be seen that $U_r|_{\Lambda'} \simeq C_{q+2+(r-1)2^{q-1}}(0)$. This gives the required infinite list of $C_t(0)$ appearing in restrictions to $\Lambda'$. This tells us that $\Gamma(kD_{4q})$ cannot be finitely generated.

We now move to the case of $\Gamma(U^{[2]}\mathrm{Jen}(D_{4q}))$. With the co-multiplication of $U^{[2]}\mathrm{Jen}(D_{4q})$, the subalgebra generated by $A$ and $B$ above is isomorphic to the restricted enveloping algebra $U^{[2]}\mathrm{Jen}(kV_4)$, which is isomorphic as an algebra to $kV_4$ but has a different co-multiplication. However, tensor products of string modules *are* always the same for $kV_4$-modules and for $U^{[2]}\mathrm{Jen}(kV_4)$-modules. So, we can argue exactly as in the case of $\Gamma(kD_{4q})$ to see that $\Gamma(U^{[2]}\mathrm{Jen}(D_{4q}))$ is not finitely generated. □

### 7.1.2 The subgroup of endo-trivial modules

Let $H$ be a Hopf algebra over $k$, and let $M$ be an $H$-module. Recall that $M$ is said to be *endo-trivial* if $\mathrm{End}_k(M) \simeq k \oplus \text{projectives}$. This is equivalent to $M \otimes M^* \simeq k \oplus \text{projectives}$. It is clear that if the co-multiplication on $H$ is co-commutative (so that the Green ring $A(H)$ is then commutative) then the set of endo-trivial $H$-modules will form a group, which we denote $T(H)$, under tensor product multiplication modulo projectives, with identity $k$. Thus we see that the group $T(D_{4q})$ of endo-trivial $kD_{4q}$-modules forms a subgroup of our group $\Gamma(kD_{4q})$. Similarly, $T(U^{[2]}\mathrm{Jen}(D_{4q})) \leqslant \Gamma(U^{[2]}\mathrm{Jen}(D_{4q}))$. In this section we will study multiplications in $T(kD_{4q})$ and $T(U^{[2]}\mathrm{Jen}(D_{4q}))$, and we will see that $T(kD_{4q}) \simeq T(U^{[2]}\mathrm{Jen}(D_{4q}))$ via the natural map on $\Gamma(kD_{4q})$ fixing all modules.

The endo-trivial $kD_{4q}$-modules have been classified by Carlson and Thévenaz in [9]; it follows from Theorem 5.4 of [9] that the projective-free endo-trivial modules are precisely $\{\Omega^a(L^{\otimes b}) : a, b \in \mathbb{Z}\}$, where $L$ is the unique indecomposable module such that the sequence

$$0 \longrightarrow L \longrightarrow k \uparrow_{\langle y\rangle}^{D_{4q}} \longrightarrow k \longrightarrow 0 \tag{7.1}$$

is exact, and $L^{\otimes b}$ is understood to mean $(L^*)^{\otimes(-b)}$ if $b < 0$. For example, in the case $q = 2$ we have $L = M(XY)$. Theorem 5.4 of [9] also describes the group $T(kD_{4q})$: we have an isomorphism $\mathrm{Res} : T(kD_{4q}) \to T(kH) \oplus T(kH') \simeq \mathbb{Z} \oplus \mathbb{Z}$ where $\mathrm{Res}(M) = (\Omega^0_{V_4}(M|_{kH}), \Omega^0_{V_4}(M|_{kH'}))$, with $H = \langle x^q, y\rangle$ and $H' = \langle x^q, xy\rangle$ the two subgroups of $D_{4q}$ isomorphic to the Klein four-group. Explicitly, the map Res is given by $\mathrm{Res}(\Omega^a_{D_{4q}}(L^{\otimes b})) = ((\Omega^{a+b}_{V_4}(k), \Omega^{a-b}_{V_4}(k))$ modulo projectives. Exactly the same method shows that $\mathrm{Res} : T(U^{[2]}\mathrm{Jen}(D_{4q})) \to T(k\langle A, X\rangle) \oplus T(k\langle A, Y\rangle) \simeq \mathbb{Z} \oplus \mathbb{Z}$ is an isomorphism, where

$$A = (XY)^q + (XY)^{q-1}X + Y(XY)^{q-1} + \sum_{i=0}^{(\log_2 q)-1} ((XY)^{q-2^i} + (YX)^{q-2^i})$$

and $k\langle A, X\rangle$ and $k\langle A, Y\rangle$ have co-multiplication as in $U^{[2]}\mathrm{Jen}(V_4)$. From this it is clear immediately that $T(kD_{4q}) \simeq T(U^{[2]}\mathrm{Jen}(D_{4q}))$ and that we must also have $T(U^{[2]}\mathrm{Jen}(D_{4q})) = \{\Omega^a(L^{\otimes b}) : a, b \in \mathbb{Z}\}$.

To see that multiplication in $T(D_{4q})$ is exactly the same as multiplication in $T(U^{[2]}\mathrm{Jen}(D_{4q}))$, note that since Res is injective it will be sufficient to show that $\mathrm{Res}(M \otimes_1 N) \simeq \mathrm{Res}(M \otimes_2 N)$ for all $M$ and $M$ endo-trivial. Since Res is a homomorphism of abelian groups we have $\mathrm{Res}(M \otimes_1 N) = \mathrm{Res}(M) \otimes \mathrm{Res}(N)$ under the tensor product multiplication in $kV_4$, and also $\mathrm{Res}(M \otimes_2 N) = \mathrm{Res}(M) \otimes \mathrm{Res}(N)$ under the tensor product multiplication in $U^{[2]}\mathrm{Jen}(V_4)$. But since $\mathrm{Res}(M)$ and $\mathrm{Res}(N)$ are both modules of the form $\Omega^r_{V_4}(k)$, we have seen in Section 4.4.2 that the product $\mathrm{Res}(M) \otimes \mathrm{Res}(N)$ is the same under both tensor product multiplications. This gives us $\mathrm{Res}(M \otimes_1 N) \simeq \mathrm{Res}(M \otimes_2 N)$ and hence $M \otimes_1 N \simeq M \otimes_2 N$ as required.

### 7.1.3 The restriction map applied to all indecomposable odd-dimensional modules

It is clear that the map $\mathrm{Res}_1 : T(kD_{4q}) \to T(H) \oplus T(H')$ and the map $\mathrm{Res}_2 : T(U^{[2]}\mathrm{Jen}(D_{4q})) \to T(k\langle A, X\rangle) \oplus T(k\langle A, Y\rangle)$ of the previous section can be extended naturally to maps $\mathrm{Res}_1 : \Gamma(kD_{4q}) \to A(kH) \times A(kH')$ and $\mathrm{Res}_2 : \Gamma(U^{[2]}\mathrm{Jen}(D_{4q})) \to A(k\langle A, X\rangle) \times A(k\langle A, Y\rangle)$, by again taking $\mathrm{Res}(M) = (M|_{kH}, M|_{kH'})$ modulo projectives in both cases. Note that $\mathrm{Res}_i(M \otimes_i N) = (M|_{kH} \otimes_i N|_{kH}, M|_{kH'} \otimes_i N|_{kH'})$ for $i = 1, 2$, so Res preserves the tensor product multiplication. The facts that enable us to deduce that $M \otimes_1 N \simeq M \otimes_2 N$ when $M, N \in T(kD_{4q})$ are (i) the injectivity of Res on $T(kD_{4q})$ and $T(U^{[2]}\mathrm{Jen}(D_{4q}))$, and (ii) the fact that $(U \otimes_1 V, U' \otimes_1 V') = (U \otimes_2 V, U' \otimes_2 V')$ for all $(U, U'), (V, V') \in \mathrm{im}(\mathrm{Res})$. So, in investigating the properties of Res when applied to the whole group $\Gamma$ of odd-dimensional modules, we will pay particular attention to whether these two properties hold in the general case.

Recall that the image of Res on $T(kD_{4q})$ is $\{(\Omega^r_{V_4}(k), \Omega^s_{V_4}(k)) : r, s \in Z\}$ which is precisely the set of pairs of endo-trivial $kV_4$-modules. By Lemma 2.9 of [9], any module $M$ with $(M|_{kH}, M|_{kH'}) \in \{(\Omega^r_{V_4}(k), \Omega^s_{V_4}(k)) : r, s \in Z\}$ modulo projectives is endo-trivial. So, the set of endo-trivial $kD_{4q}$-modules is precisely the set of modules with restrictions of this form, and we must expect the appearance of summands corresponding to band modules for $kV_4$ in the elements of $\mathrm{Res}(\Gamma(kD_{4q}))$.

The endo-trivial $kD_{4q}$-modules can be located in the Auslander–Reiten quiver via the following theorem of Karin Erdmann [14]:

**Theorem 7.1.6** *Let $L$ be the endo-trivial $kD_{4q}$-module defined by the short exact sequence*

$$0 \longrightarrow L \longrightarrow k \uparrow_{\langle y \rangle}^{D_{4q}} \longrightarrow k \longrightarrow 0$$

*of Section 7.1.2. There exist irreducible maps $L^{\otimes r} \to \Omega^{-1}(L^{\otimes(r+1)})$ and $L^{\otimes r} \to \Omega^{-1}(L^{\otimes(r-1)})$ for all $r \in \mathbb{Z}$.*

This tells us that the endo-trivial $kD_{4q}$-modules are precisely the modules lying in the stable Auslander–Reiten quiver components of $k$ and $L$, which have structure as shown on page 104.

Replacing each label $M$ of these quiver components with the label $(r, s)$ where $\mathrm{Res}(M) = (\Omega^r_{V_4}(k), \Omega^s_{V_4}(k))$, we get the pictures on page 105.

The fact that all endo-trivial modules are located in these components—and the patterns observable in these pictures—suggest that a sensible approach to studying the properties of Res on $\Gamma(kD_{4q})$ could be to look at how the images of modules under this map relate to the position of modules in the stable Auslander–Reiten quiver.

$$
\begin{tikzcd}[column sep=small]
\cdots & \Omega^2(L^{\otimes(-2)}) \arrow[dr] & & \Omega^0(L^{\otimes(-2)}) \arrow[dr] & & \\
& & \Omega(L^{\otimes(-1)}) \arrow[ur] \arrow[dr] & & \Omega^{-1}(L^{\otimes(-1)}) & \cdots \\
\cdots & \Omega^2(k) \arrow[ur] \arrow[dr] & & k \arrow[ur] \arrow[dr] & & \\
& & \Omega(L) \arrow[ur] \arrow[dr] & & \Omega^{-1}(L) & \cdots \\
\cdots & \Omega^2(L^{\otimes 2}) \arrow[ur] & & \Omega^0(L^{\otimes 2}) \arrow[ur] & & \\
& \vdots & & \vdots & &
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
& \vdots & & \vdots & & \\
\cdots & \Omega^2(L^{\otimes(-1)}) \arrow[dr] & & \Omega^0(L^{\otimes(-1)}) \arrow[dr] & & \\
& & \Omega(k) \arrow[ur] \arrow[dr] & & \Omega^{-1}(k) & \cdots \\
\cdots & \Omega^2(L) \arrow[ur] \arrow[dr] & & L \arrow[ur] \arrow[dr] & & \\
& & \Omega(L^{\otimes 2}) \arrow[ur] \arrow[dr] & & \Omega^{-1}(L^{\otimes 2}) & \cdots \\
\cdots & \Omega^2(L^{\otimes 3}) \arrow[ur] & & \Omega^0(L^{\otimes 3}) \arrow[ur] & & \\
& \vdots & & \vdots & &
\end{tikzcd}
$$

Figure 7.4: The stable Auslander–Reiten quiver components of $k$ and $L$

$$
\begin{tikzcd}
{} & (0,4) \arrow[l, dotted, no head] \arrow[dr] & & (-2,2) \arrow[dr] & & (-4,0) \arrow[r, dotted, no head] & {} \\
& & (0,2) \arrow[ur] \arrow[dr] & & (-2,0) \arrow[ur] \arrow[dr] & & \\
{} & (2,2) \arrow[l, dotted, no head] \arrow[ur] \arrow[dr] & & (0,0) \arrow[ur] \arrow[dr] & & (-2,-2) \arrow[r, dotted, no head] & {} \\
& & (2,0) \arrow[ur] \arrow[dr] & & (0,-2) \arrow[ur] \arrow[dr] & & \\
{} & (4,0) \arrow[l, dotted, no head] \arrow[ur] \arrow[d, dotted, no head] & & (2,-2) \arrow[ur] \arrow[d, dotted, no head] & & (0,-4) \arrow[r, dotted, no head] \arrow[d, dotted, no head] & {} \\
& {} & & {} & & {} &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& {} & & {} & & {} & \\
{} & (1,3) \arrow[u, dotted, no head] \arrow[l, dotted, no head] \arrow[dr] & & (-1,1) \arrow[u, dotted, no head] \arrow[dr] & & (-3,-1) \arrow[u, dotted, no head] \arrow[r, dotted, no head] & {} \\
& & (1,1) \arrow[ur] \arrow[dr] & & (-1,-1) \arrow[ur] \arrow[dr] & & \\
{} & (3,1) \arrow[l, dotted, no head] \arrow[ur] \arrow[dr] & & (1,-1) \arrow[ur] \arrow[dr] & & (-1,-3) \arrow[r, dotted, no head] & {} \\
& & (3,-1) \arrow[ur] \arrow[dr] & & (1,-3) \arrow[ur] \arrow[dr] & & \\
{} & (5,-1) \arrow[l, dotted, no head] \arrow[ur] \arrow[d, dotted, no head] & & (3,-3) \arrow[ur] \arrow[d, dotted, no head] & & (1,-5) \arrow[r, dotted, no head] \arrow[d, dotted, no head] & {} \\
& {} & & {} & & {} &
\end{tikzcd}
$$

Figure 7.5: Restrictions of the stable Auslander–Reiten quiver components of $k$ and $L$

## The restriction map and the Auslander–Reiten quiver

Let M be an odd-dimensional indecomposable $\Lambda$-module. We know that $M$ lies in a stable Auslander–Reiten quiver component of structure $\mathbb{Z}A_\infty^\infty$; we look at how the restrictions to $H$ and $H'$ of all modules in this component are related.

First we note that $M|_H$ and $M|_{H'}$ must both be of the form

$$C \oplus \Omega^n(k) \oplus \text{projectives}$$

for some $n \in Z$ and some $C$ a direct sum of non-projective even-dimensional indecomposable $kV_4$-modules (possibly zero). For, we have seen in the proof of Lemma 7.1.1 that $M|_{\langle X \rangle} = k \oplus \text{projectives}$. We have

$$\Omega^n(k)|_{\langle X \rangle} = k \oplus \text{projectives}$$

and $M|_{\langle X \rangle} = (M|_H)|_{\langle X \rangle}$. Hence there cannot be more than one summand of the form $\Omega^n(k)$ in $M|_H$. In fact, there must be precisely one, since $M$ is odd-dimensional and hence must have at least one odd-dimensional indecomposable summand. An analogous argument looking at $M|_{\langle Y \rangle}$ will provide the same result about the structure of $M|_{H'}$.

We now use the following result to relate the structure of restrictions of other modules in Auslander–Reiten quiver component of $M$ to the structure of $M|_H$ and $M|_{H'}$:

**Proposition 7.1.7 ([6], Proposition 4.12.10)** *Let $G$ be a group with subgroup $G'$. An Auslander–Reiten sequence*

$$0 \longrightarrow \Omega^2(M) \longrightarrow E \longrightarrow M \longrightarrow 0$$

*of $kG$-modules splits on restriction to $G'$ if and only if $G'$ does not contain a vertex of $M$.*

Applying this in our situation, every odd-dimensional $kD_{4q}$-module has vertex $D_{4q}$. So, every Auslander–Reiten sequence of odd-dimensional $kD_{4q}$-modules must split on restriction to $H$ and on restriction to $H'$.

Suppose for simplicity that $M$ does not lie in the Auslander–Reiten quiver component of the projective indecomposable module (this can be treated separately with minor modifications to the arguments below). Then the Auslander–Reiten sequence terminating in $M$ is of the form

$$0 \longrightarrow \Omega^2(M) \longrightarrow U \oplus V \longrightarrow M \longrightarrow 0$$

with $U$ and $V$ odd-dimensional and indecomposable. Then, by Proposition 7.1.7, we have

$$(U \oplus V)|_H \simeq M|_H \oplus \Omega^2(M)|_H \simeq M|_H \oplus \Omega^2(M|_H) \oplus \text{projectives},$$

since $\Omega^2(M)|_H \simeq \Omega^2(M|_H) \oplus$ projectives.

We consider first the non-projective even-dimensional summand $C$ of $M|_H$. It is clear, since the string module $M$ does not involve any parameters, that $C = \sum_r \alpha_r C_r(0)$ for some $\alpha_r \in k$. These modules all satisfy $\Omega(C_r(0)) = C_r(0)$, from which we see that $\Omega^2(M)|_H$ has exactly the same non-projective even-dimensional indecomposable summands as $M|_H$. Then, by Proposition 7.1.7, any such summand appearing in $U|_H$ must appear in $M|_H$. Applying the same argument to $U|_H$ and $\Omega^{-2}(U)|_H$, we see that any indecomposable non-projective even-dimensional summand $C_r(0)$ appearing in $M|_H$ must occur in $U|_H$. So, we see that exactly the same such summands must occur. Periodicity of these modules and iteration of the above argument then tells us that this must be the case for any module $N$ in the Auslander–Reiten quiver component of $M$; it remains only to see whether the multiplicities may differ.

Let $C_r(0)$ be some indecomposable even-dimensional module which occurs in $M|_H$. Let $n_r(M|_H)$ be the multiplicity of $C_r(0)$ as a summand of $M|_H$, and similarly for other $kD_{4q}$-modules. Then, by Proposition 7.1.7, $n_r(U|_H) + n_r(V|_H) = n_r(M|_H) + n_r(\Omega^2(M)|_H) = 2n_r(M|_H)$, since we have $n_r(\Omega^2(M)|_H) = n_r(M|_H)$ as $C_r(0)$ is periodic of period 1. For exactly the same reason we have $n_r(\Omega^{-2}(U)|_H) = n_r(U|_H)$ and hence $n_r(\Omega^{-2}(U)|_H) + n_r(V|_H) = 2n_r(M|_H)$. This applies at every point in the Auslander–Reiten quiver component, which tells us that $n_r(-|_H) : \Gamma(kD_{4q}) \to \mathbb{N}_0$ gives a subadditive function on each copy of $A_\infty^\infty$ in the component. It is easy to see that every subadditive function on $A_\infty^\infty$ is constant (for example, see the proof of Proposition 4.5.7 (ii) in [6]). The constant must be the same in all cases, since $n_r(\Omega^t(M)|_H) = n_r(M|_H)$ for all $t \in \mathbb{Z}$. Hence each module in this Auslander–Reiten quiver component has exactly the same non-projective even-dimensional summand when restricted to $H$.

Now consider the odd-dimensional summand $\Omega^n(k)$ of $M|_H$. By Proposition 7.1.7, $(U \oplus V)|_H \simeq \Omega^n(k) \oplus \Omega^{n+2}(k) \oplus$ (even-dimensional summands). So, either

$$U|_H = \Omega^n(k) \oplus \text{(even-dimensional summands)}$$

and

$$V|_H = \Omega^{n+2}(k) \oplus \text{(even-dimensional summands)},$$

or vice versa (since both restrictions must have exactly one odd-dimensional indecomposable summand). Suppose $U|_H$ has odd-dimensional summand $\Omega^n(k)$. Look at the Auslander–Reiten sequence

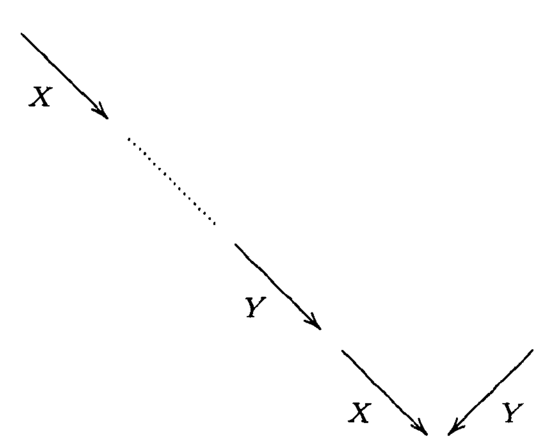
$$0 \longrightarrow \Omega^2(U) \longrightarrow \Omega^2(M) \oplus M' \longrightarrow U \longrightarrow 0.$$

By applying Proposition 7.1.7 again, we see that $M'|_H$ has odd-dimensional summand $\Omega^n(k)$. Repeating this process we see that the exponent $n$ will be constant along a diagonal through $M$ and all other diagonals parallel to this. Conversely, if $U|_H$ has odd-dimensional summand $\Omega^{n+2}(k)$, then we get exponents constant along the other diagonal through $M$ and all diagonals parallel to this.

Shown pictorially, we have seen that, modulo projectives, the restrictions to $H$ of modules in the Auslander–Reiten quiver component of $M$ (given by their positions in the Auslander–Reiten quiver) must be either as in the picture on page 110, or the component obtained by taking this picture and reflecting it in any horizontal line.

The same arguments will apply to restrictions to $H'$. So, either restrictions to $H$ and restrictions to $H'$ are the same for all modules in this component, or we get constant exponents on the two different diagonals for the two different subgroups $H$ and $H'$.

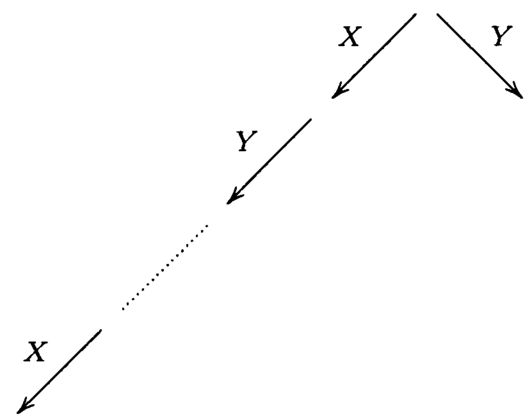
It can be seen, however, that we do not have $U|_H \simeq U|_{H'}$ for all modules $U$ in this Auslander–Reiten quiver component. For, suppose $U|_H \simeq U|_{H'}$. If $U$ is given by a word beginning with $X$, consider the module $V$ formed by adding a "hook" of the form




to $U$ (this corresponds to Benson's operation $L_q$). When restricted to $H$, we get $U|_H \oplus rD$ for some $r$. However, when restricted to $H'$, the extra part

$$
\begin{tikzcd}[column sep=small]
\cdots \sum \alpha_r C_r(0) \oplus \Omega^{n}(k) \arrow[dr] & & \sum \alpha_r C_r(0) \oplus \Omega^{n-2}(k) \arrow[dr] & \\
& \sum \alpha_r C_r(0) \oplus \Omega^{n}(k) \arrow[ur] \arrow[dr] & & \sum \alpha_r C_r(0) \oplus \Omega^{n-2}(k) \\
\cdots \sum \alpha_r C_r(0) \oplus \Omega^{n+2}(k) \arrow[ur] \arrow[dr] & & \sum \alpha_r C_r(0) \oplus \Omega^{n}(k) \arrow[ur] \arrow[dr] & \\
& \sum \alpha_r C_r(0) \oplus \Omega^{n+2}(k) \arrow[ur] \arrow[dr] & & \sum \alpha_r C_r(0) \oplus \Omega^{n}(k) \cdots \\
\cdots \sum \alpha_r C_r(0) \oplus \Omega^{n+4}(k) \arrow[ur] & & \sum \alpha_r C_r(0) \oplus \Omega^{n+2}(k) \arrow[ur] &
\end{tikzcd}
$$

will be "connected" to $U|_H$. Hence $V|_H \not\simeq V|_{H'}$. If $U$ is given by a word beginning with $X^{-1}$, consider the module $V$ formed by adding a "hook" of the form




(this corresponds to $L_q^{-1}$). Then we again have $V|_H \simeq U|_H \oplus rD \not\simeq V|_{H'}$. This covers all possibilities for $U$ (up to equivalence of words under $\sim$).

So, we can see that if we replace each vertex $M$ in the Auslander–Reiten quiver with a label $(M|_H, M|_{H'})$ modulo projectives, we will get a pattern for all components of odd-dimensional $kD_{4q}$-modules as shown on page 112, where $C_1 = \sum_r \alpha_r C_r(0)$ for some $\alpha_r \in k$ and $C_2 = \sum_t \beta_t C_t(0)$ for some $\beta_t \in k$. The projective summands of each restriction can then be calculated by considering dimensions.

### Properties of the restriction map

We have noted above that, for $M$ an odd-dimensional $\Lambda$-module,

$$M|_H = \sum_r \alpha_r C_r(0) \oplus \Omega^s(k) \oplus mD.$$

Let $N$ be another such module. Consulting our tables for multiplication in $A(kV_4)$ and $A(U^{[2]}\mathrm{Jen}(V_4))$ in Section 4.4.2 and Section 6.1, we see that any tensor products involoving only modules of the form $C_r(0)$ for $r \in \mathbb{N}$ and $\Omega^s(k)$ for $s \in \mathbb{Z}$ and $D$ will be the same in both Green rings. So, we have $M|_H \otimes_1 N|_H \simeq M|_H \otimes_2 N|_H$. Similarly, we also have $M|_{H'} \otimes_1 N|_{H'} \simeq$

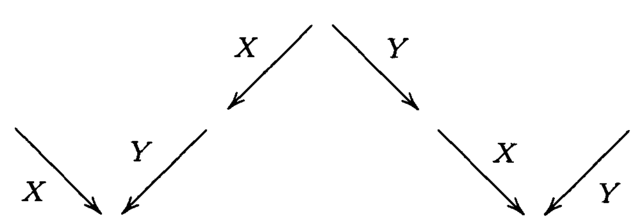
$$
\begin{tikzcd}[column sep=small]
 & & \vdots & & \vdots \\
 & & (C_1 \oplus \Omega^n(k), C_2 \oplus \Omega^{m+2}(k)) \arrow[dr] & & (C_1 \oplus \Omega^{n-2}(k), C_2 \oplus \Omega^m(k)) \\
\cdots\cdots & (C_1 \oplus \Omega^{n+2}(k), C_2 \oplus \Omega^{m+2}(k)) \arrow[ur] \arrow[dr] & & (C_1 \oplus \Omega^n(k), C_2 \oplus \Omega^m(k)) \arrow[ur] \arrow[dr] & \\
 & & (C_1 \oplus \Omega^{n+2}(k), C_2 \oplus \Omega^m(k)) \arrow[ur] \arrow[dr] & & (C_1 \oplus \Omega^n(k), C_2 \oplus \Omega^{m-2}(k)) \\
\cdots\cdots & (C_1 \oplus \Omega^{n+4}(k), C_2 \oplus \Omega^m(k)) \arrow[ur] & & (C_1 \oplus \Omega^{n+2}(k), C_2 \oplus \Omega^{m-2}(k)) \arrow[ur] &
\end{tikzcd}
$$

$M|_{H'} \otimes_2 N|_{H'}$. In other words, $(U \otimes_1 V, U' \otimes_1 V') = (U \otimes_2 V, U' \otimes_2 V')$ for all $(U, U'), (V, V') \in \mathrm{im}(\mathrm{Res})$, the second of the two properties of the restriction map on endo-trivial modules which we were particularly interested in.

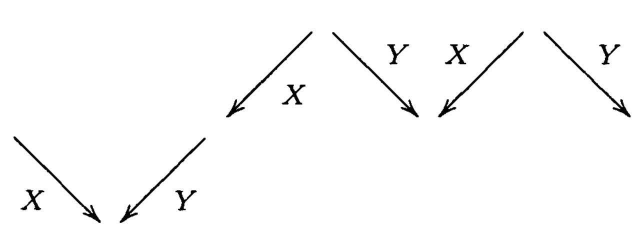
We now address the question of whether Res may be an injective map on $\Gamma(\Lambda)$.

**Lemma 7.1.8** *The map* $\mathrm{Res} : \Gamma(\Lambda) \to A(kH) \times A(kH')$ *is not injective.*

**Proof:** This can be seen easily by providing two modules with the same image under Res: for example,




and




are both $\Lambda$-modules for any $q \geqslant 2$, and both have image

$$(C_2(0) \oplus C_1(0) \oplus k, C_2(0) \oplus C_1(0) \oplus k)$$

under Res if $q = 2$, or image

$$(3C_1(0) \oplus k, 3C_1(0) \oplus k)$$

under Res if $q > 2$. $\square$

It follows from this result that we cannot argue as we did for $T(kD_{4q})$ and $T(U^{[2]}\mathrm{Jen}(D_{4q}))$ to show that tensor product multiplications are the same in

$\Gamma(kD_{4q})$ and $\Gamma(U^{[2]}\mathrm{Jen}(D_{4q}))$. We now look at how common such pairs of modules with the same image under Res are: this can be thought of as an investigation of how far the map Res is from being injective. To do this we consider the number of modules of dimension $2n+1$ and the number of possible images under Res of such modules, and compare how these two numbers increase as functions of $n$. Since we are interested primarily in limiting behaviour, let us assume that $q$ is arbitrarily large. That is, assume that $q > 2n$, which will ensure that the indecomposable $kD_{4q}$-modules of dimension $2n+1$ are the same as the indecomposable $kD_\infty$-modules of this dimension.

In Ringel's paper giving the construction of the indecomposable modules for dihedral 2-groups [28], it is shown that there are $2^{2n} = 4^n$ indecomposable $kD_{4q}$-modules of dimension $2n+1$.

Let $M$ be an indecomposable $kD_{4q}$-module of dimension $2n+1$. We have $M|_H = \sum_r \alpha_r C_r(0) \oplus \Omega^s(k) \oplus mD$ for some $r, m \in \mathbb{N}$, $\alpha_r \in k$, $s \in \mathbb{Z}$. Now $\dim C_r(0) = 2r$. Hence, if $r'$ is the largest integer with $\alpha_{r'} \neq 0$, we have that

$$(\underbrace{r', \ldots, r'}_{\alpha_{r'}}, \underbrace{r'-1, \ldots, r'-1}_{\alpha_{r'-1}}, \ldots, \underbrace{1, \ldots, 1}_{\alpha_1})$$

must partition some number less than or equal to $n$. So, the number of possible non-projective even-dimensional summands $C = \sum_r \alpha_r C_r(0)$ of $M|_H$ must be at most $\sum_{t=1}^{n} p(t)$ where $p(t)$ us the number of partitions of $t$. Looking now at the odd-dimensional summand $\Omega^s(k)$, we have $\dim \Omega^s(k) = 2|s| + 1$, So there are at most $2n$ possible $\Omega^s(k)$ which could appear in our restriction. If $C$ and $\Omega^s(k)$ have been chosen, then the number of projective summands $m$ of $M|_H$ is then determined by $\dim M = \dim M|_H$. So, the number of possible restrictions to $H$ of $kD_{4q}$-modules of dimension $2n+1$ is certainly at most $2n.\sum_{t=1}^{n} p(t)$.

Similarly, the number of possible restrictions to $H'$ of $kD_{4q}$-modules of dimension $2n+1$ is at most $2n.\sum_{t=1}^{n} p(t)$. So, the number of possible images under Res of such modules is at most

$$\left(2n.\sum_{t=1}^{n} p(t)\right)^2 \leqslant \left(2n^2 p(n)\right)^2 = 4n^4 p(n)^2 \sim \frac{n^2}{12} e^{2\pi\sqrt{2n/3}}$$

by Stirling's Formula for the number of partitions of $n$. It is clear even from this crude estimate that the number of indecomposable $kD_{4q}$-modules of dimension $2n+1$ grows much faster as $n$ increases than the number of possible images of such modules under the map Res. So, we see that in the limiting case where $q$ and $n$ are large, the map Res is increasingly far from being injective. This tells us that the example given in Lemma 7.1.8 is far from being an anomaly, and shows that we have no hope of adopting the methods used in section 7.1.2 to show that multiplications in $\Gamma(kD_{4q})$ and $\Gamma(U^{[2]}\mathrm{Jen}(D_{4q}))$ are always the same.

### 7.1.4 Tensor products and Auslander–Reiten theory

In this section we shall look at how tensor products of odd-dimensional $kD_{4q}$-modules or $U^{[2]}\mathrm{Jen}(D_{4q})$-modules are related to the Auslander–Reiten quiver; this will allow us to make a substantial reduction to the problem of determining whether $\overline{M \otimes_1 N} \simeq \overline{M \otimes_2 N}$ for all odd-dimensional indecomposable $kD_{4q}$-modules $M$ and $N$.

The key tool in this section will be a theorem of Green, Marcos and Solberg [18]. First we require the following definition explaining the terminology used in [18].

**Definition 7.1.9** Let $H$ be a finite-dimensional Hopf algebra with $S^2 = \mathrm{id}_H$

over an algebraically closed field $k$. A *splitting trace module* is a finitely generated indecomposable $H$-module such that $\mathrm{char}(k)$ does not divide $\dim M$.

Note that this is not the original definition of a splitting trace module: the original definition and an explanation of why the terminology chosen is appropriate can be found in Proposition 3.1 and Corollary 3.2 of [18]. The definition taken above comes from the result of Proposition 3.3 of [18]; from this definition it is clear that all odd-dimensional $\Lambda$-modules are splitting trace modules.

We can now state the result we need on tensor products and Auslander–Reiten sequences:

**Theorem 7.1.10 ([18, Theorem 8])** *Let $H$ be a finite-dimensional Hopf algebra with $S^2 = \mathrm{id}_H$, and let $M$ be a splitting trace module. If*

$$0 \longrightarrow \tau(k) \longrightarrow E \longrightarrow k \longrightarrow 0 \tag{7.2}$$

*is an almost split sequence, then*

$$0 \longrightarrow M \otimes \tau(k) \longrightarrow M \otimes E \longrightarrow M \longrightarrow 0$$

*is an almost split sequence modulo injectives.*

This result was generalised from the case where $H$ is a group algebra; two formulations of the original result can be found in [1, Theorem 3.6] and [7, Proposition 2.15].

We can now use Theorem 7.1.10 to prove the following lemma describing a particular tensor product. First recall Benson's description of the Auslander–Reiten sequences of odd-dimensional $\Lambda$-modules outlined in Section 4.4.3.

**Lemma 7.1.11** *Let $M$ be an odd-dimensional $\Lambda$-module. We may assume that $M = M(w)$ where $w$ is a word beginning $X^{\varepsilon}$ for some $\varepsilon \in \{\pm 1\}$. Then the tensor product $M(w) \otimes_i \Omega(L) \equiv M(wR_q)$ modulo projectives for both $i = 1$ and $i = 2$.*

**Proof:** The following proof holds for both $i = 1$ and $i = 2$; we therefore drop the subscript $i$.

The almost split sequence terminating in $k$ is

$$0 \longrightarrow \Omega(k) \longrightarrow \Omega(L) \oplus \Omega(L^*) \longrightarrow k \longrightarrow 0.$$

So, by Theorem 7.1.10,

$$0 \longrightarrow M(w) \otimes \Omega^2(k) \longrightarrow M(w) \otimes \Omega(L) \oplus M(w) \otimes \Omega(L^*) \longrightarrow M(w) \longrightarrow 0$$

is almost split modulo projectives. But, by Benson's construction, the almost split sequence terminating in $M(w)$ is

$$0 \longrightarrow M(wR_qL_q) \longrightarrow M(wR_q) \oplus M(wL_q) \longrightarrow M(w) \longrightarrow 0.$$

Since the almost split sequence terminating in a particular module is unique up to isomorphism, we must therefore have either $M(w) \otimes \Omega(L) \equiv M(wR_q)$ or $M(w) \otimes \Omega(L) \equiv M(wL_q)$ modulo projectives.

We now look at restrictions to $H = k\langle A, X\rangle \simeq kV_4$. Suppose $M|_H = \sum_r \alpha_r C_r(0) \oplus \Omega^t(k) \oplus nD$. Then $(M(w) \otimes \Omega(L))|_H = M(w)|_H \otimes \Omega(L)|_H = (\sum \alpha_r C_r(0) \oplus \Omega^t(k) \oplus nD) \otimes (\Omega^2(k) \oplus mD) = \sum \alpha_r C_r(0) \oplus \Omega^{t+2}(k) \oplus$projectives by our tables of multiplications for $kV_4$-modules and $U^{[2]}\mathrm{Jen}(V_4)$-modules. So, it is enough to show that $M(wL_q)|_H \not\simeq \sum \alpha_r C_r(0) \oplus \Omega^{t+2}(k) \oplus$projectives.

From the work in Section 7.1.3 on restrictions to Klein-four subgroups and Auslander–Reiten quiver structure, we know that either

$$M(wL_q)|_H \simeq \sum \alpha_r C_r(0) \oplus \Omega^{t+2}(k) \oplus \text{projectives}$$

or

$$M(wL_q)|_H \simeq \sum \alpha_r C_r(0) \oplus \Omega^t(k) \oplus \text{projectives},$$

since there exist irreducible maps $M(wL_q) \to M(w)$. Suppose, for a contradiction, that $M(wL_q)|_H \simeq \sum \alpha_r C_r(0) \oplus \Omega^{t+2}(k) \oplus$ projectives. Note that $\dim M(wL_q) = \dim M(w) \pm 2q$ (according to whether $M(wL_q)$ is obtained by adding or removing a hook from $M(w)$). Also, $\dim \Omega^{t+2}(k) = \dim \Omega^t(k) \pm 4$ (according to whether $|t+2|$ is greater or less than $|t|$). So, by comparing dimensions, our assumption tells us that the number of projective summands of $M(w)|_H$ and $M(wL_q)|_H$ must differ either by $\pm(q-2)/2$ or by $\pm(q+2)/2$. We will show that this is not the case.

First consider the case where $wL_q = X^{-1}(Y^{-1}X^{-1})^{q-1}Yw$. Let $v_1, v_3, \ldots$ $v_{q-1}$ be the first $q/2$ odd numbered basis elements of the corresponding natural basis for the string module $M(wL_q)$. Then for $i \in \{1, 3, \ldots, q-1\}$ we have $AXv_i \neq 0$, so $v_i$ must be the top of a projective $kH$-module, which must then be a direct summand of $M(wL_q)|_H$. But any element $v$ which is the top of a projective summand of $M(w)|_H$ will clearly also be the top of a projective summand of $M(wL_q)|_H$. So, in this case $M(wL_q)|_H$ has $q/2$ more projective summands than $M(w)|_H$.

Now consider the case where $w$ begins with $(XY)^{q-1}XY^{-1}$ and $wL_q$ is obtained from $w$ by cancelling this part. Then the natural basis elements $v_{q+2}, v_{q+4}, \ldots, v_{2q}$ of $M(w)$ will all be the tops of projective summands of $M(w)|_H$; all projective summands of $M(wL_q)|_H$ will also correspond to projective summands of $M(w)|_H$. So in this case $M(w)|_H$ has $q/2$ more projective summands than $M(wL_q)|_H$. □

We now use this to show how products of elements in the groups $\Gamma(kD_{4q})$

and $\Gamma(U^{[2]}\mathrm{Jen}(D_{4q}))$ are related to the structure of the Auslander–Reiten quiver.

**Theorem 7.1.12** *Let $M = M(u)$ and $N$ be odd-dimensional indecomposable $\Lambda$-modules. Let $r \in \{1, 2\}$ and suppose $\overline{M(u) \otimes_r N} = M(w)$. Assume that $u$ and $w$ both begin $X^\varepsilon$ for some $\varepsilon \in \{\pm 1\}$. Then for all $i, j \in \mathbb{Z}$ we have $\overline{M(uR_q^iL_q^j) \otimes_r N} \simeq M(wR_q^iL_q^j)$.*

**Remark:** Recall that the elements $M(uR_q^iL_q^j)$ for $i, j \in \mathbb{Z}$ are the vertices of the Auslander–Reiten quiver component containing $M$, with their position in the quiver relative to $M$ determined by the indices $i$ and $j$. So, this result tells us that the component on $\overline{M \otimes N}$ can be obtained by taking the component of $M$ and replacing all vertices with their image under the map $U \mapsto \overline{U \otimes N}$. So, tensoring with $N$ induces a graph isomorphism from the component of $M(u)$ to the component of $M(w)$. Alternatively, Theorem 7.1.12 says that, if we know $\overline{M \otimes N}$ then $\overline{U \otimes N}$ is determined for all $U$ in the same Auslander–Reiten quiver component as $U$, since $\overline{U \otimes N}$ is determined by its position relative to $\overline{M \otimes N}$ in the corresponding component.

**Proof of Theorem 7.1.12:** As for Lemma 7.1.11, we drop the subscript $r$ and provide a proof that holds for both $r = 1$ and $r = 2$.

First we look at modules in the same column of the Auslander–Reiten quiver as $M(u)$; that is, we show that

$$\overline{M(uR_q^iL_q^{-i}) \otimes N} = M(wR_q^iL_q^{-i}) \text{ for all } i \in \mathbb{Z}. \tag{7.3}$$

We prove this for $i \geqslant 0$ by induction on $i$; the case $i \leqslant 0$ is analogous using induction on $-i$. The base case $i = 0$ holds by definition. Assume $\overline{M(uR_q^{i-1}L_q^{1-i}) \otimes N} = M(wR_q^{i-1}L_q^{1-i})$. Now

$$\overline{M(uR_q^iL_q^{-i}) \otimes N} = \overline{\Omega^{-2}(k) \otimes M(uR_q^{i+1}L_q^{i-1}) \otimes N}$$

by Ringel's construction for taking even powers of $\Omega$ [28]. Then, by two applications of Lemma 7.1.11, this is

$$\overline{\Omega^{-2}(k) \otimes \Omega^2(L^{\otimes 2}) \otimes M(uR_q^{i-1}L_q^{1-i}) \otimes N},$$

which equals

$$\overline{\Omega^{-2}(k) \otimes \Omega^2(L^{\otimes 2}) \otimes M(wR_q^{i-1}L_q^{1-i})}$$

by the inductive hypothesis. Then, applying Lemma 7.1.11 again, this is $\overline{\Omega^{-2}(k) \otimes M(wR_q^{i+1}L_q^{1-i})}$. Finally, Ringel's construction tells us that this is $M(wR_q^iL_q^{-i})$ as required.

Now we look at modules in the preceeding column of the Auslander–Reiten quiver; that is, the modules $E$ such that there exist irreducible homomorphisms $E \to M$, and all other modules in the same column. So, we show that

$$\overline{M(uR_q^{i+1}L_q^{-i}) \otimes N} = M(wR_q^{i+1}L_q^{-i}) \text{ for all } i \in \mathbb{Z}. \tag{7.4}$$

As above, we can prove this by induction on $i$ and $-i$. Here we give the case $i \geqslant 0$. For the base case $i = 0$, we have $\overline{M(uR_q) \otimes N} = \overline{\Omega(L) \otimes M(u) \otimes N}$ by Lemma 7.1.11. By definition this is $\overline{\Omega(L) \otimes M(w)}$, which equals $M(wR_q)$ by Lemma 7.1.11. Now suppose $\overline{M(uR_q^iL_q^{1-i}) \otimes N} = M(wR_q^iL_q^{1-i})$. By Ringel's construction, $\overline{M(uR_q^{i+1}L_q^{-i}) \otimes N} = \overline{\Omega^{-2}(k) \otimes M(uR_q^{i+2}L_q^{1-i}) \otimes N}$. Two applications of Lemma 7.1.11 show that this is

$$\overline{\Omega^{-2}(k) \otimes \Omega^2(L^{\otimes 2}) \otimes M(uR_q^iL_q^{i-1}) \otimes N}.$$

By the inductive hypothesis this is $\overline{\Omega^{-2}(k) \otimes \Omega^2(L^{\otimes 2}) \otimes M(wR_q^iL_q^{1-i})}$. Then using Lemma 7.1.11 and Ringel's construction again, this is $M(wR_q^{i+1}L_q^{-i})$ as required.

We can now use (7.3) and (7.4) to look at the general case $M(uR_q^i L_q^j)$. This will be done in two cases, according to whether $i+j$ is even or odd. First suppose $i+j$ is even. Then we can re-write $\overline{M(uR_q^i L_q^j) \otimes N}$ as

$$\overline{M(uR_q^{(i-j)/2} L_q^{(j-i)/2} R_q^{(i+j)/2} L_q^{(i+j)/2}) \otimes N},$$

which is

$$\overline{\Omega^{(i+j)}(k) \otimes M(uR_q^{(i-j)/2} L_q^{(j-i)/2}) \otimes N}$$

by Ringel's construction. Then by (7.3) this is

$$\overline{\Omega^{(i+j)}(k) \otimes M(wR_q^{(i-j)/2} L_q^{(j-i)/2})}$$

and applying Ringel's construction again this is $M(wR_q^i L_q^j \otimes N)$ as required.

Similarly, if $i+j$ is odd, we write $\overline{M(uR_q^i L_q^j)}$ as

$$\overline{M(uR_q^{(i-j-1)/2+1} L_q^{-(i-j-1)/2+1} R_q^{(i+j-1)/2} L_q^{(i+j-1)/2}) \otimes N},$$

which is

$$\overline{\Omega^{(i+j-1)}(k) \otimes M(uR_q^{(i-j-1)/2+1} L_q^{-(i-j-1)/2}) \otimes N}$$

by Ringel's construction. Then by (7.4) this is

$$\overline{\Omega^{(i+j-1)}(k) \otimes M(wR_q^{(i-j-1)/2+1} L_q^{-(i-j-1)/2})}$$

and by Ringel's construction this is $M(wR_q^i L_q^j)$. Hence

$$\overline{M(uR_q^i L_q^j) \otimes N} = M(wR_q^i L_q^j)$$

in all cases. $\square$

Relating this to our problem of comparing the two groups $\Gamma(kD_{4q})$ and $\Gamma(U^{[2]}\mathrm{Jen}(D_{4q}))$, what this tells us is that if $\overline{M \otimes_1 N} \simeq \overline{M \otimes_2 N}$ then we have

$\overline{U \otimes_1 N} \simeq \overline{U \otimes_2 N}$ for all $U$ in the Auslander–Reiten quiver component of $M$. So, in order to show that all multiplications in $\Gamma(kD_{4q})$ and $\Gamma(U^{[2]}\mathrm{Jen}(D_{4q}))$ are the same, it would be sufficient to restrict our attention to multiplications involving only a subset of the odd-dimensional $kD_{4q}$-modules, containing one appropriately chosen module from each Auslander–Reiten quiver component. For example, we could take a module $M$ of minimal dimension from each component.

We conclude this section with some remarks on the class of modules described above. First, note that this class does not form a subgroup of $\Gamma(kD_{4q})$ of $\Gamma(U^{[2]}\mathrm{Jen}(D_{4q}))$. For example, the module $L$ is of the form described above, but $L^{\otimes 2}$ (in either group) lies in the same component as the trivial module $k$ and thus is not of this form. Second, the subgroup generated by all modules of this form is an infinitely generated subgroup of $\Gamma(kD_{4q})$ and of $\Gamma(U^{[2]}\mathrm{Jen}(D_{4q}))$. To see this, look again at the proof of Theorem 7.1.5: the modules used in this proof to show that $\Gamma(kD_{4q})$ and $\Gamma(U^{[2]}\mathrm{Jen}(D_{4q}))$ are not finitely generated are all of minimal dimension in the relevant components of the Auslander–Reiten quiver (since applying $R_q$ or $L_q$ will increase the dimension in all cases). Hence we would need an infinite set of generators for any subgroup of $\Gamma$ containing all these modules.

### 7.1.5 The action of $T(\Lambda)$ on $\Gamma(\Lambda)$

The group $T(\Lambda)$ of endo-trivial $\Lambda$-modules, whose multiplication we can describe fully, is a subgroup of the group $\Gamma(\Lambda)$ of odd-dimensional $\Lambda$-modules. Another way to look at their relationship is to note that $T(\Lambda)$ acts via tensor product multiplication on $\Gamma(\Lambda)$. Theorem 7.1.12 of the previous section allows us to describe the orbits of this action.

**Theorem 7.1.13** *Let $T(\Lambda)$ act on $\Gamma(\Lambda)$ by tensor product multiplication modulo even-dimensional modules. The $T(\Lambda)$-orbit of a module $M \in \Gamma(\Lambda)$ under this action is determined by the AR quiver structure as $\{N : N$ lies in the AR quiver component of $M\} \cup \{N : N$ lies in the AR quiver component of $\Omega(M)\}$.*

**Proof:** The modules in $T(\Lambda)$ are precisely the modules lying in the AR quiver components of $k$ and $\Omega(k)$. The orbit of $M$ is the set of products of $M$ with all such modules. But, by Theorem 7.1.12, taking all such products modulo even-dimensional summands gives us precisely the AR quiver components of $k \otimes M = M$ and $\Omega(k) \otimes M = \Omega(M)$. Hence the modules lying in these two components must be precisely those in the $T(\Lambda)$-orbit of $M$. $\square$

### 7.1.6 Some products which can be seen to be the same in both Green ring quotients

We conclude this chapter by discussing some classes of tensor products where it can be seen that $\overline{M \otimes_1 N} \simeq \overline{M \otimes_2 N}$, providing some further justification for the hypothesis that the natural map

$$\frac{A(kD_{4q})}{A(kD_{4q};2)} \to \frac{A(U^{[2]}\mathrm{Jen}(D_{4q}))}{A(U^{[2]}\mathrm{Jen}(D_{4q});2)}$$

given by fixing all modules will be an isomorphism of rings.

#### Multiplying by an endo-trivial module

We have already seen in Section 7.1.2 that if $M$ and $N$ are both endo-trivial modules then $\overline{M \otimes_1 N} \simeq \overline{M \otimes_2 N}$. However, the results of the previous section allow us to go further.

**Theorem 7.1.14** *Let $M$ be an indecomposable endo-trivial $\Lambda$-module and let $N$ be any odd-dimensional indecomposable $\Lambda$-module. Then $\overline{M \otimes_1 N} \simeq \overline{M \otimes_2 N}$.*

**Proof:** From the classification of the endo-trivial modules by Carlson and Theéevenaz [9], we know that $M$ must lie in either the Auslander–Reiten quiver component of the trivial module $k$, or the Auslander–Reiten quiver component of $\Omega(k)$. So, by Theorem 7.1.12 it is enough to check that $\overline{k \otimes_1 N} \simeq \overline{k \otimes_2 N}$ and $\overline{\Omega(k) \otimes_1 N} \simeq \overline{\Omega(k) \otimes_2 N}$. But this is clear, since $k \otimes_i N \simeq N$ and $\overline{\Omega(k) \otimes_i N} \simeq \Omega(N)$ for $i = 1, 2$. □

**Multiplying by $M(X(YX)^{q/2-1}(Y^{-1}X^{-1})^{q/2-1}Y^{-1})$**

Let $M = M(X(YX)^{q/2-1}(Y^{-1}X^{-1})^{q/2-1}Y^{-1})$ throughout this section. We will describe a method for constructing $M \otimes_i N$ for $i \in \{1, 2\}$ and $N$ any odd-dimensional indecomposable $kD_{4q}$-module. This method will not depend on $i$, showing that $\overline{M \otimes_1 N} \simeq \overline{M \otimes_2 N}$.

Note first that $M$ is characterised by the short exact sequence

$$0 \longrightarrow k \longrightarrow k \uparrow_{k\langle A\rangle}^{kD_{4q}} \longrightarrow M \longrightarrow 0.$$

Taking the tensor product with $N$, we have a short exact sequence

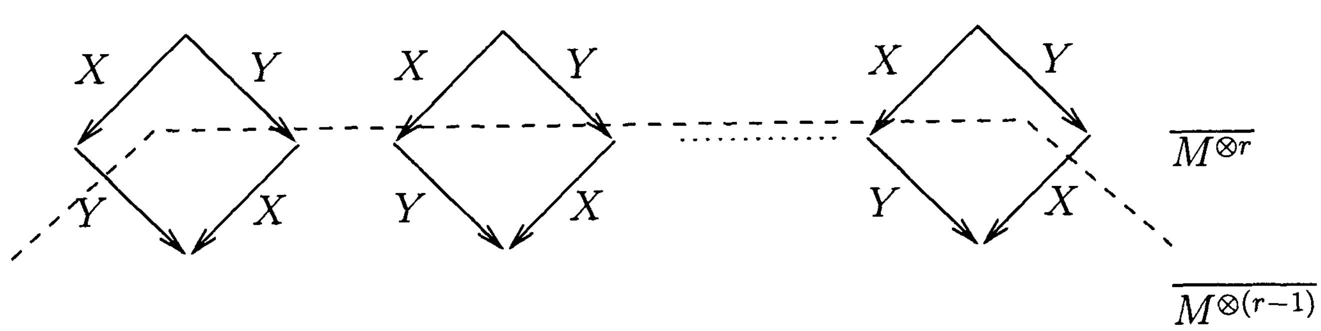
$$0 \longrightarrow N \longrightarrow k \uparrow_{k\langle A\rangle}^{kD_{4q}} \otimes_i N \longrightarrow M \otimes_i N \longrightarrow 0$$

which will uniquely characterise $M \otimes_i N$. So, it will be sufficient to check that $k \uparrow_{k\langle A\rangle}^{kD_{4q}} \otimes_i N$ does not depend on $i$. But, by the Nakayama relations for induction and restriction (see, for example, [6, Proposition 2.8.3 and Proposition 3.3.3]), we have $k \uparrow_{k\langle A\rangle}^{kD_{4q}} \otimes_i N \simeq (k \otimes_i N|_{k\langle A\rangle}) \uparrow^{kD_{4q}} = (N|_{k\langle A\rangle}) \uparrow^{kD_{4q}}$ which indeed does not depend on $i$.

As an example of this method of calculation, look at the case $q = 2$. Then $M = M(XY^{-1})$ and the induced module—which we shall call $B$— has structure

$$
k \uparrow_{k\langle A\rangle}^{kD_8} = \begin{tikzcd}[column sep=small] & \bullet \arrow[dl, "X"'] \arrow[dr, "Y"] & \\ \bullet \arrow[dr, "Y"'] & & \bullet \arrow[dl, "X"] \\ & \bullet & \end{tikzcd}
$$

We now use the method described above to see that $\overline{M^{\otimes r}} = M((XY^{-1})^r)$ for all $r \geqslant 1$. For $r = 1$ this is trivially true. Suppose the result holds for $r-1$. By the method above, we should look at $(\overline{M^{\otimes(r-1)}}|_{k\langle A\rangle}) \uparrow^{kD_8}$. But $\overline{M^{\otimes(r-1)}}|_{k\langle A\rangle}$ is just $2r+1$ copies of the trivial module, so $(\overline{M^{\otimes(r-1)}}|_{k\langle A\rangle}) \uparrow^{kD_8} = (2r+1)B$. To find $\overline{M^{\otimes r}}$ we then take the quotient of this by $\overline{M^{\otimes(r-1)}}$ and select the odd-dimensional summand:




It is clear from this diagram of modules that this is $M((XY^{-1})^r)$ as claimed.

## 7.2 Summary of the approach of Chapter 7

In this chapter we have looked for evidence which might support Conjecture 7.0.2, stating that the two Benson–Carlson quotients we are comparing are naturally isomorphic. In particular, we have presented two alternative approaches to reducing the problem of proving this conjecture: in Section 7.1.1

we saw that it would be sufficient to prove that the two groups which allowed us to realise the Benson–Carlson quotients as group rings are isomorphic; in Section 7.1.4 we saw that we can use Auslander–Reiten theory to reduce the problem to proving that all tensor products of modules in a certain class are the same for both algebras. The remainder of the chapter consists of work done following these two approaches and details how much progress has been made on the problem.

# Chapter 8

# Realising the Benson–Carlson quotients as group rings

In the previous chapter we saw, as a consequence of Theorem 7.1.2, that the Benson–Carlson quotients of a dihedral 2-group and the corresponding Jennings algebra can be realised as group rings over $\mathbb{Z}$. That is, we have

$$\frac{A(kD_{4q})}{A(kD_{4q};2)} \simeq \mathbb{Z}\Gamma(kD_{4q})$$

and

$$\frac{A(U^{[2]}\mathrm{Jen}(D_{4q}))}{A(U^{[2]}\mathrm{Jen}(D_{4q});2)} \simeq \mathbb{Z}\Gamma(U^{[2]}\mathrm{Jen}(D_{4q}))$$

for some groups $\Gamma(kD_{4q})$ and $\Gamma(U^{[2]}\mathrm{Jen}(D_{4q}))$. In this chapter we will show that similar results hold for the 2-groups $C_{2^n}$ and $V_4$ studied in Chapters 5 and 6. Furthermore, in these cases—unlike the case of the dihedral 2-groups—we are able to identify the groups in question.

## 8.1 Cyclic 2-groups

Let $k$ be an algebraically closed field of characteristic 2. We will look at the Benson–Carlson quotient $A(kC_{2^n})/A(kC_{2^n};2)$; it can be seen that results analogous to those in this section will hold for the Benson–Carlson quotient of the corresponding Jennings algebra $U^{[2]}\mathrm{Jen}(C_{2^n})$.

**Theorem 8.1.1** *The Benson–Carlson quotient of $kC_{2^n}$ satisfies*

$$\frac{A(kC_{2^n})}{A(kC_{2^n};2)} \simeq \mathbb{Z}(C_2 \times \cdots \times C_2),$$

*where there are $n-1$ copies of $C_2$ in the direct product.*

**Proof:** We begin by showing that cosets of odd-dimensional indecomposable $kC_{2^n}$-modules form a group $\Gamma(kC_{2^n})$ under multiplication in the Benson–Carlson quotient. As in the case of the dihedral 2-groups, this shows that the Benson–Carlson quotient can be realised as a group ring over $\mathbb{Z}$.

First we show that the product of any two odd-dimensional indecomposable $kC_{2^n}$-modules has exactly one odd-dimensional indecomposable summand; this ensures that we have a well-defined multiplication for our group. We prove this by induction on $n$. It is immediately apparent that this holds for $n = 1$, since the only odd-dimensional indecomposable $kC_2$-module is the trivial module. Let $C_{2^n} = \langle g \rangle$, let $H = \langle g^2 \rangle \simeq C_{2^{n-1}}$ and suppose we have a well-defined multiplication for odd-dimensional indecomposable $kH$-modules.

The indecomposable $kC_{2^n}$-modules are $M_1, \ldots, M_{2^n}$, where $g$ acts on $M_i$ as a Jordan block matrix of size $i$ with eigenvalue 1. Let $M_{2i+1}$ and $M_{2j+1}$ be any two odd-dimensional indecomposable modules. It can be seen that $M_{2i+1}|_H \simeq M_i \oplus M_{i+1}$ as $kH$-modules for $0 \leqslant i \leqslant 2^{n-1}-1$. Hence

$$\begin{aligned}
&(M_{2i+1} \otimes M_{2j+1})|_H \simeq M_{2i+1}|_H \otimes M_{2j+1}|_H \\
&\quad \simeq (M_i \oplus M_{i+1}) \otimes (M_j \oplus M_{j+1}) \\
&\quad\quad \simeq (M_i \otimes M_j) \oplus (M_i \otimes M_{j+1}) \oplus (M_{i+1} \otimes M_j) \oplus (M_{i+1} \otimes M_{j+1}).
\end{aligned}$$

These four summands are (in some order)

(i) the product of two even-dimensional $kH$-modules

(ii) the product of an even-dimensional $kH$-module and an odd-dimensional $kH$-module

(iii) the product of an odd-dimensional $kH$-module and an even-dimensional $kH$-module

(iv) the product of two odd-dimensional $kH$-modules.

Proposition 4.2.2 tells us that all summands of the products (i), (ii) and (iii) must be even-dimensional. Our inductive hypothesis states that the fourth product must have precisely one odd-dimensional inedcomposable summand. So, we see that $(M_{2i+1} \otimes M_{2j+1})|_H$ has precisely one odd-dimensional indecomposable summand. Any odd-dimensional indecomposable summand of $M_{2i+1} \otimes M_{2j+1}$ will produce an odd-dimensional indecomposable summand upon restriction to $H$. Hence $M_{2i+1} \otimes M_{2j+1}$ has at most one odd-dimensional inedecomposable summand; since it is odd-dimensional it must have precisely one such summand, as required.

As in Theorem 7.1.2, it is clear that our multiplication will be associative and commutative, and that the trivial module $k$ will be a multiplicative identity. By Theorem 4.2.5, the trivial module $k$ will be a direct summand of $M \otimes M^*$ for any odd-dimensional indecomposable $kC_{2^n}$-modules $M$; the fact that our multiplication is well-defined then gives $M \otimes M^* \equiv k$ modulo $A(kC_{2^n}; 2)$. Hence $M^*$ is the inverse for $M$. This completes the proof that

$$\frac{A(kC_{2^n})}{A(kC_{2^n}; 2)} \simeq \mathbb{Z}\Gamma(kC_{2^n})$$

for some abelian group $\Gamma(kC_{2^n})$.

To identify $\Gamma(kC_{2^n})$ it is sufficient to note that $M^* \simeq M$ for all indecomposable $kC_{2^n}$-modules; this tells us that all elements of $\Gamma(kC_{2^n})$ are self-inverse. Hence $\Gamma(kC_{2^n})$ is an elementary abelian 2-group. The order of $\Gamma(kC_{2^n})$ is given by the number of odd-dimensional indecomposable $kC_{2^n}$-modules, which is $2^{n-1}$. Hence $\Gamma(kC_{2^n})$ is a product of $n-1$ copies of $C_2$. □

## 8.2 The Klein four group

Let $k$ be an algebraically closed field of characteristic 2.

**Theorem 8.2.1** *The Benson–Carlson quotient of $kV_4$ can be realised as a group over $\mathbb{Z}$ via the isomorphism*

$$\frac{A(kV_4)}{A(kV_4;2)} \simeq \mathbb{Z}C_\infty.$$

**Proof:** The odd-dimensional indecomposable $kV_4$-modules are $\{\Omega^r(k) : r \in \mathbb{Z}\}$. These multiply in $A(kV_4)/A(kV_4;2)$ as $\Omega^r(k) \otimes \Omega^s(k) \equiv \Omega^{r+s}(k)$ modulo $A(kV_4;2)$. It is clear that this multiplication makes these modules into a group $\Gamma(kV_4)$, which is isomorphic to the additive group of the integers, or equivalently (written multiplicatively) to the infinite cyclic group $C_\infty$. □

It is clear that an analogous result will hold for the Benson–Carlson quotient of the corresponding Jennings algebra $U^{[2]}\mathrm{Jen}(V_4)$.

# Appendix A

# Some composition algebra relations for a family of cyclic quiver algebras

Let $Q_2$ be the cyclic quiver with two vertices

$$\begin{tikzcd} \overset{1}{\bullet} \arrow[r, shift left, "\alpha"] & \overset{2}{\bullet} \arrow[l, shift left, "\beta"] \end{tikzcd}$$

and let $k$ be any finite field. We define a family of quiver algebras as follows:

$$A_{1,1} = \frac{kQ_2}{\langle \alpha, \beta \rangle}$$

$$A_{2,1} = \frac{kQ_2}{\langle \beta \rangle}$$

$$A_{2,2} = \frac{kQ_2}{\langle \alpha\beta, \beta\alpha \rangle}$$

$$A_{3,2} = \frac{kQ_2}{\langle \alpha\beta \rangle}$$

$$A_{3,3} = \frac{kQ_2}{\langle \alpha\beta\alpha, \beta\alpha\beta \rangle}$$

etc. Note that $A_{2,2}$ is the algebra defined in Section 2.3. The structures of the indecomposable projective summands of these algebras are:

$$A_{1,1} = 1 \oplus 2$$

$$A_{2,1} = \begin{array}{c} 1 \\ 2 \end{array} \oplus \begin{array}{c} \\ 2 \end{array}$$

$$A_{2,2} = \begin{array}{c} 1 \\ 2 \end{array} \oplus \begin{array}{c} 2 \\ 1 \end{array}$$

$$A_{3,2} = \begin{array}{c} 1 \\ 2 \\ 1 \end{array} \oplus \begin{array}{c} \\ 2 \\ 1 \end{array}$$

$$A_{3,3} = \begin{array}{c} 1 \\ 2 \\ 1 \end{array} \oplus \begin{array}{c} 2 \\ 1 \\ 2 \end{array}$$

and so on.

Consider the composition algebras $\mathcal{C}(A_{i,j})$. Note that if $i_1 \leqslant i_2$ and $j_1 \leqslant j_2$, then all $A_{i_1,j_1}$-modules are also $A_{i_2,j_2}$-modules. The relevant coefficients in any composition algebra multiplication will be the same. So, any relation which holds in $\mathcal{C}(A_{i_2,j_2})$ must hold in $\mathcal{C}(A_{i_1,j_1})$; equivalently, $\mathcal{C}(A_{i_1,j_1})$ is some quotient of $\mathcal{C}(A_{i_2,j_2})$.

Note that $A_{1,1}$ and $A_{2,1}$ are both hereditary algebras. Furthermore, their quivers are both finite unions of Dynkin quivers of type A. The composition algebras for such quiver algebras have been completely determined by Ringel in [36]. Theorem 2.3.5 gives us a complete description of $\mathcal{C}(A_{2,2})$.

The following table presents relations of degree up to $\binom{3}{3}$ for the composition algebras $\mathcal{C}(A_{1,1})$ to $\mathcal{C}(A_{3,3})$. The quantum Serre relations of Section 2.3, which we have seen must hold in all of these algebras, have been omitted. Relations which are consequences of those listed together with the quantum Serre relations have also been omitted.

| Algebra | Degree | Relations |
|---|---|---|
| $\mathcal{C}(A_{1,1})$ | $\binom{1}{1}$ | $12 = 21$ |
| $\mathcal{C}(A_{2,1})$ | $\binom{2}{1}$ | $1^2 2 + q21^2 = (q+1)121$ |
| | $\binom{1}{2}$ | $12^2 + q2^2 1 = (q+1)212$ |
| $\mathcal{C}(A_{2,2})$ | $\binom{2}{2}$ | $12^2 1 = 21^2 2$ |
| | $\binom{3}{3}$ | $12^2 121 = 2121^2 2$<br>$1212^2 1 = 21^2 212$<br>$1^2 2^2 12 = 121^2 2^2$<br>$1^2 2^3 1 = 2^2 1^3 2$<br>$12^3 1^2 = 2^2 1^3 2$<br>$2^2 1^2 21 = 212^2 1^2$ |
| $\mathcal{C}(A_{3,2})$ | $\binom{2}{3}$ | $12^2 12 + 2^2 1^2 2 = 21^2 2^2 + 212^2 1$ |
| | $\binom{3}{3}$ | $21^2 2^2 1 + 21^2 212 = 12^2 1^2 2 + 2121^2 2$<br>$(q+1+q^{-1})(1212^2 1 - 12^2 1^2 2) = 1^2 2^3 1 - 21^3 2^2$<br>$(q+1+q^{-1})(12^2 121 - 21^2 2^2 1) = 2^2 1^3 2 - 12^3 1^2$ |
| $\mathcal{C}(A_{3,3})$ | $\binom{3}{3}$ | $12^2 121 + 2^2 1^2 21 + 12^2 1^2 2 + 2121^2 2$<br>$= 21^2 2^2 1 + 21^2 2^2 1 + 212^2 1^2 + 21^2 212$<br>$1^2 2^3 1 + (q+1+q^{-1})(12^2 121 + 2^2 1^2 21 + 12^2 1^2 2)$<br>$= 21^3 2^2 + (q+1+q^{-1})(21^2 2^2 1 + 212^2 1^2 + 1212^2 1)$ |

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