

THIN CYLINDRICAL CONDUCTIVITY INCLUSIONS IN A 3-DIMENSIONAL DOMAIN: POLARIZATION TENSOR AND UNIQUE DETERMINATION FROM BOUNDARY DATA

ELENA BERETTA, YVES CAPDEBOSCQ, FRÉDÉRIC DE GOURNAY, AND ELISA FRANCONI

ABSTRACT. We consider a 3-dimensional conductor containing an inclusion that can be represented as a cylinder with fixed axis and a small basis. As the size of the basis of the cylinder approaches zero, the voltage perturbation can be described by means of a polarization tensor. We give an explicit characterization of the polarization tensor of cylindrical inclusions in terms of the polarization tensor of its base, and we use this result to show that the axis of the inclusion can be uniquely determined by boundary values of the voltage perturbation. We also present a reconstruction algorithm and some numerical simulations.

1. INTRODUCTION

Let Ω be an open bounded smooth domain in \mathbb{R}^3 occupied by a conducting material, and let $\gamma_0 : \Omega \rightarrow \mathbb{R}^+$ represent the conductivity in Ω .

If we assign a current g on $\partial\Omega$ such that $\int_{\partial\Omega} g \, d\sigma = 0$, the voltage potential generated by this current is the solution u_0 to

$$(1) \quad \begin{cases} \operatorname{div}(\gamma_0 \nabla u_0) &= 0 \text{ in } \Omega, \\ \gamma_0 \frac{\partial u_0}{\partial n} &= g \text{ on } \partial\Omega, \\ \int_{\partial\Omega} u_0 \, d\sigma &= 0, \end{cases}$$

where last condition ensures the unique determination of the solution.

Let us suppose that Ω contains a small inclusion ω_ϵ , made of a different material with conductivity $\gamma_1 : \Omega \rightarrow \mathbb{R}^+$. The perturbed conductivity is given by

$$(2) \quad \gamma_\epsilon(x) = \begin{cases} \gamma_0(x) & x \in \Omega \setminus \omega_\epsilon, \\ \gamma_1(x) & x \in \omega_\epsilon, \end{cases}$$

If we apply the same current g on the boundary of the body containing the inclusion, the resulting potential is the solution u_ϵ to the boundary value problem

$$(3) \quad \begin{cases} \operatorname{div}(\gamma_\epsilon \nabla u_\epsilon) &= 0 \text{ in } \Omega, \\ \gamma_\epsilon \frac{\partial u_\epsilon}{\partial n} &= g \text{ on } \partial\Omega, \\ \int_{\partial\Omega} u_\epsilon \, d\sigma &= 0. \end{cases}$$

In recent years, a considerable amount of work has been dedicated to the case of small inclusions, that is, to subsets ω_ϵ whose Lebesgue measure tends to zero with ϵ . When this happens, the perturbation of the voltage potential is very small, in the sense that u_ϵ converges to u_0 in the $H^1(\Omega)$ norm. A number of asymptotic formulas have been proved for the asymptotic expansion of $(u_\epsilon - u_0)|_{\partial\Omega}$ with respect to ϵ for a variety of geometries. We recall here a general, geometry independent, result due to Capdeboscq & Vogelius [9]. Assume that

$$(4) \quad |\omega_\epsilon|^{-1} 1_{\omega_\epsilon}(\cdot) \text{ converges in the sense of measure to } \mu \text{ when } |\omega_\epsilon| \rightarrow 0.$$

Let N denote the Neumann function of the unperturbed domain: given $y \in \Omega$, let $N(\cdot, y)$ be the solution to

$$(5) \quad \begin{cases} \operatorname{div}_x (\gamma_0(x) \nabla_x N(x, y)) &= \delta_y(x) \text{ for } x \in \Omega, \\ \gamma_0(x) \frac{\partial N}{\partial n_x}(x, y) &= \frac{1}{|\partial\Omega|} \text{ for } x \in \partial\Omega, \\ \int_{\partial\Omega} N(x, y) d\sigma_x &= 0. \end{cases}$$

This function may be extended by continuity to $\partial\Omega$ and may also be defined as the solution to

$$\begin{cases} \operatorname{div}_x (\gamma_0(x) \nabla_x N(x, y)) &= 0 \text{ for } x \in \Omega, \\ \gamma_0(x) \frac{\partial N}{\partial n_x}(x, y) &= -\delta_y + \frac{1}{|\partial\Omega|} \text{ for } x \in \partial\Omega, \\ \int_{\partial\Omega} N(x, y) d\sigma_x &= 0. \end{cases}$$

The main result in [9] is the following:

Theorem 1.1. *Assume (4) holds. There exists a tensor $\{M_{ij}\}_{i,j=1}^3 \in L^2(\Omega, d\mu)$ such that, for $g \in H^{-1/2}(\partial\Omega)$ satisfying $\int_{\partial\Omega} g d\sigma = 0$, if we denote by u_ϵ and u_0 the solutions to boundary value problems (1) and (3) respectively, we have that, for $y \in \partial\Omega$,*

$$(u_\epsilon - u_0)(y) = |\omega_\epsilon| \sum_{i,j=1}^3 \int_{\Omega} (\gamma_1 - \gamma_0)(x) M_{ij}(x) \frac{\partial u_0}{\partial x_i}(x) \frac{\partial N}{\partial x_j}(x, y) d\mu(x) + o(|\omega_\epsilon|).$$

The $o(|\omega_\epsilon|)$ term is such that $|\omega_\epsilon|^{-1} \|o(|\omega_\epsilon|)\|_{L^\infty(\partial\Omega)}$ converges to zero as ϵ tends to zero, uniformly on $\{g \in H^{-1/2}(\partial\Omega) : \int_{\partial\Omega} g d\sigma = 0, \|g\|_{L^2(\partial\Omega)} \leq 1\}$.

The symmetric tensor M is the signature of the inclusion and it is called the polarization tensor. Later on, we will give more insight of how this tensor can be constructed. The concept of polarization tensor appears in various contexts. The term was coined by Polya, Schiffer & Szegő [22, 21]. Polarization tensors are well known in the theory of homogenization as the low volume fraction limit of the effective properties of the dilute two phase composites [16, 17, 19] (see [17] for an extensive list of references).

Explicit formulas for the polarization tensor are available in the case of diametrically small inclusions, i.e. those inclusions that can be written as $\omega_\epsilon = z + \epsilon B$ where z is a point in Ω and B is a bounded domain centered at the origin (see [4]). Even in the case of "thin" inclusion, that can be described as small neighborhood of hypersurfaces (a curve in \mathbb{R}^2 or a surface in \mathbb{R}^3), the polarization tensor has been explicitly characterized (see [6]).

In this work, we want to consider inclusions that can be represented as small neighborhood of a line segment in a 3-dimensional domain. For these cylindrically shaped inclusions we give an explicit description of the polarization tensor. This model has many possible applications, for example in non-destructive testing of materials and in geophysical prospection. We also want to look at this problem from the point of view of inverse problems. This approach was initiated by Friedman & Vogelius [12] who first used the polarization tensor for the detection of small inclusions. After that there have been many significant developments in this direction. For more information on this subject we refer to recent books [4, 5] and references therein.

In Section 2 we will set up general assumptions and recall the definition of polarization tensor. In Section 3 we will state and prove our main result for cylindrical inclusions. In Section 4 we observe that the asymptotic formula that we derive in Section 3 is useful for the reconstruction of the inclusion from boundary data. In particular we show that boundary data of the second order term of the expansion uniquely determine the axis of the cylinder. Section 5 contains a reconstruction algorithm and some numerical simulations.

2. GENERAL ASSUMPTIONS

In all that follows we will assume that our inclusions ω_ϵ are contained in a compact set $K_0 \subset \Omega$, with positive distance from $\partial\Omega$. The Borel measure μ defined by (4) will then be concentrated on K_0 .

We will assume that both γ_0 and γ_1 are smooth functions in Ω and that, for some positive constant c_0 , we have

$$c_0 < \gamma_i(x) < \frac{1}{c_0}, \quad \text{for } x \in \Omega, \quad i = 0, 1.$$

The polarization tensor M can be defined in several ways. In [9] it is defined by means of the following auxiliary problem.

For $j = 1, 2, 3$, let e_j denote the coordinate directions and let $v_\epsilon^j \in H^1(\Omega)$ be defined by

$$\begin{cases} \operatorname{div}(\gamma_\epsilon \nabla v_\epsilon^j) = \operatorname{div}(\gamma_0 e_j) & \text{in } \Omega, \\ \gamma_\epsilon \frac{\partial v_\epsilon^j}{\partial n} = \gamma_0 n_j & \text{on } \partial\Omega, \\ \int_{\partial\Omega} v_\epsilon^j d\sigma = 0, \end{cases}$$

where n_j is the j -th component of the unit normal direction n to $\partial\Omega$.

The tensor M is consistently defined in [9] as the following limit, defined possibly up to the extraction of a subsequence,

$$(6) \quad \int_{\Omega} M_{ij}(x) \phi(x) d\mu = \lim_{\epsilon \rightarrow 0} \frac{1}{|\omega_\epsilon|} \int_{\omega_\epsilon} \frac{\partial v_\epsilon^j}{\partial x_i}(x) \phi(x) dx,$$

for every smooth function ϕ .

We shall make use of the alternative equivalent definition ([10]).

Lemma 2.1. *Let M be the polarization tensor given by (6) and let ϕ be a positive smooth function on Ω , then, for every direction $\xi \in \mathbb{R}^3$,*

$$(7) \quad \begin{aligned} \int_{\Omega} (\gamma_1 - \gamma_0) M \xi \cdot \xi \phi d\mu &= \frac{1}{|\omega_\epsilon|} \int_{\omega_\epsilon} (\gamma_1 - \gamma_0) \frac{\gamma_0}{\gamma_1} |\xi|^2 \phi dx \\ &+ \frac{1}{|\omega_\epsilon|} \min_{w \in H^1(\Omega)} \int_{\Omega} \gamma_\epsilon \left| \nabla w + \mathbf{1}_{\omega_\epsilon} \frac{\gamma_1 - \gamma_0}{\gamma_1} \xi \right|^2 \phi dx + o(1). \end{aligned}$$

where $o(1)$ tends to zero with ϵ .

3. POLARIZATION TENSOR FOR A CYLINDRICAL INCLUSION IN \mathbb{R}^3

In this work we will consider cylindrical inclusions having fixed height and small basis. For sake of simplicity we will fix our coordinate system in the center of the cylinder and the third coordinate direction (e_3) parallel to the axis of the cylinder. We will use the notation $x = (x_1, x_2, x_3)$. Consider an inclusion ω_ϵ given by

$$(8) \quad \omega_\epsilon = \omega_{2,\epsilon} \times (-l, l),$$

where $\omega_{2,\epsilon}$ is a bidimensional measurable set. An example of a possible inclusion is sketched in Figure 1. Note that the base of the cylinder can be quite general. We only require the minimal assumptions given in [9] for the two dimensional polarisation tensor associated to $\omega_{2,\epsilon}$ to be defined.

Namely, we assume that, for each ϵ , $\omega_{2,\epsilon} \subset D = D(0, r)$, the disk of radius r centered at the origin and that, for some $L > l$ the cylinder $K_0 = \overline{D} \times [-L, L]$ is contained in Ω . We will also assume that $\lim_{\epsilon \rightarrow 0} |\omega_{2,\epsilon}| = 0$ and

$$|\omega_{2,\epsilon}|^{-1} \mathbf{1}_{\omega_{2,\epsilon}}(\cdot) \text{ converges in the sense of measure to } \mu' \text{ when } \epsilon \rightarrow 0.$$

The Borel measure μ defined in (4) and μ' are related by

$$\int_{K_0} \psi d\mu = \frac{1}{2l} \int_{-l}^l \int_D \psi d\mu' dx_3, \quad \text{for each } \psi \in C(K_0).$$

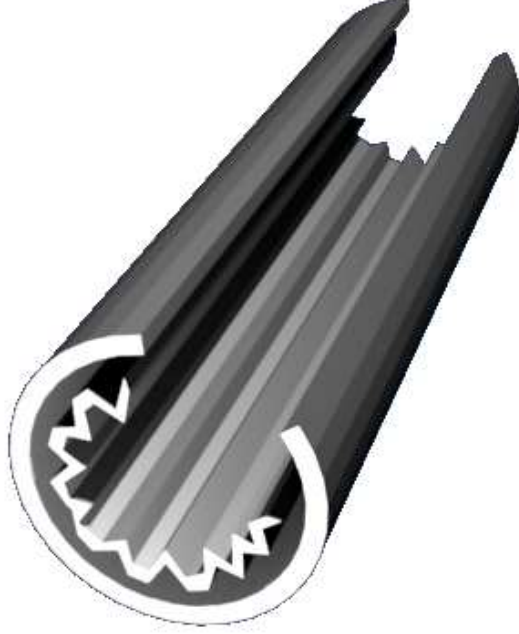


FIGURE 1. A cylindrical inclusion. The base of the cylinder $\omega_{2,\epsilon}$ has a small area.

Let γ_0 be the smooth conductivity of the material in Ω and let $\gamma_1 \neq \gamma_0$ be the smooth conductivity of the inclusion (for sake of generality we assume that both γ_0 and γ_1 are defined in the whole body Ω). The conductivity in the body containing the inclusion is given by (2).

Now let us slice our 3-dimensional body in sections that are parallel to the plane $\{x_3 = 0\}$. Let us denote by $\Omega_{x_3} = \{(x_1, x_2) \in \mathbb{R}^2 : (x_1, x_2, x_3) \in \Omega\}$ each of this slices. Let us define

$$\gamma_{2,\epsilon}(x) = (\gamma_1(x) - \gamma_0(x))\mathbf{1}_{\omega_{2,\epsilon}}(x_1, x_2) + \gamma_0(x).$$

For $x_3 \in (-l, l)$ this coefficient represents the conductivity of the slice Ω_{x_3} . For each x_3 , there is a 2×2 polarization tensor $m(x)$ that can be defined (as in Lemma 2.1) in the following way:

Lemma 3.1. *Let ϕ be a positive smooth function in Ω . Then, for every direction $\eta \in \mathbb{R}^2$,*

$$(9) \quad \int_{\Omega_{x_3}} (\gamma_1 - \gamma_0) \quad m\eta \cdot \eta \phi d\mu' = \frac{1}{|\omega_{2,\epsilon}|} \int_{\omega_{2,\epsilon}} (\gamma_1 - \gamma_0) \frac{\gamma_0}{\gamma_1} |\eta|^2 \phi dx_1 dx_2 \\ + \frac{1}{|\omega_{2,\epsilon}|} \min_{w \in H^1(\Omega_{x_3})} \int_{\Omega_{x_3}} \gamma_{2,\epsilon} \left| \nabla w + \mathbf{1}_{\omega_{2,\epsilon}} \frac{\gamma_1 - \gamma_0}{\gamma_1} \xi \right|^2 \phi dx_1 dx_2 + o(1).$$

where $o(1)$ tends to zero with ϵ .

Notice that, although the measure μ' is the same in each slice, the polarization tensor m may change because both conductivities γ_0 and γ_1 depend on x_3 .

Now, we state and prove the main results of this section.

Proposition 3.2. *If ω_ϵ is given by (8), the unit vector e_3 is an eigenvector for the polarization tensor $M(x)$, that is,*

$$(10) \quad M(x)e_3 \cdot e_3 = 1 \quad \text{for } \mu\text{-almost every } x \in \Omega.$$

Since the polarization tensor is symmetric (see [9]), this implies that there are other two eigenvectors in the orthogonal plane, the one spanned by e_1 and e_2 .

We prove that, in that plane, the polarization tensor coincide with the 2-dimensional tensor.

Proposition 3.3. *Let ω_ϵ be given by (8) and let m be the 2-dimensional polarization tensor defined by (9). Let η be any direction in \mathbb{R}^2 , and denote by η^* its extension $\eta^* = (\eta, 0)$. Then,*

$$M(x)\eta^* \cdot \eta^* = m(x)\eta \cdot \eta \quad \text{for } \mu\text{-almost every } x \in \Omega.$$

In order to prove Propositions (3.2) and (3.3) we are going to use definitions (7) and (9) of polarization tensors. Before doing that, we need to point out a variant of those formulas that is justified by [10, remark 1, p.185]. According to this remark, the minimum in formula (7) need not to be taken over the whole $H^1(\Omega)$, but it can be taken over $H_0^1(\Omega')$ for any convex set Ω' that contains the whole family of inclusions. In our case we can choose $\Omega' = K_0$. The same holds in formula (9) where the minimum can be taken over $H_0^1(D)$.

Let us fix a positive smooth function ϕ defined in Ω and, for $j = 1, 2, 3$, let us denote by Φ_ϵ^j the 3-dimensional minimizer in (7) corresponding to $\xi = e_j$. Each minimizer $\Phi_\epsilon^j \in H_0^1(K_0)$ is solution to

$$(11) \quad \operatorname{div}(\gamma_\epsilon \phi \nabla \Phi_\epsilon^j) = \operatorname{div}((\gamma_0 - \gamma_1) \mathbf{1}_{\omega_\epsilon} e_j \phi) \quad \text{in } K_0.$$

By Lemma 6.1 in the Appendix, the minimizer Φ_ϵ^j satisfies the estimates

$$(12) \quad \|\nabla \Phi_\epsilon^j\|_{L^2(K_0)} \leq C|\omega_\epsilon|^{1/2}, \quad \text{for } j = 1, 2, 3,$$

and

$$(13) \quad \|\Phi_\epsilon^j\|_{L^2(K_0)} \leq C|\omega_\epsilon|^{1/2+\alpha}, \quad \text{for } j = 1, 2, 3,$$

where the positive constants C and α depend on K_0 , c_0 and ϕ , but not on ϵ .

For the bidimensional tensor, we denote by ψ_ϵ^j , for $j = 1, 2$, the functions in $H_0^1(D)$ defined by

$$(14) \quad \operatorname{div}_{12}(\gamma_{2,\epsilon} \phi \nabla_{12} \psi_\epsilon^j) = \operatorname{div}_{12}((\gamma_0 - \gamma_1) \mathbf{1}_{\omega_{2,\epsilon}} e_j \phi) \quad \text{in } D,$$

where the notations div_{12} and ∇_{12} mean divergence and gradient with respect to the first two variables only. The third variable plays the role of a parameter. Due to Lemma 6.1 in the Appendix these functions satisfies the estimates

$$(15) \quad \|\nabla_{12} \psi_\epsilon^j\|_{L^2(D)} \leq C|\omega_{2,\epsilon}|^{1/2}, \quad \text{for } j = 1, 2,$$

and

$$(16) \quad \|\psi_\epsilon^j\|_{L^2(D)} \leq C|\omega_{2,\epsilon}|^{1/2+\alpha}, \quad \text{for } j = 1, 2,$$

where the positive constants C and α depend on K_0 , c_0 and ϕ , but not on ϵ .

Proof of Proposition 3.2. As it was noted in [10], it is easy to recover the optimal pointwise estimates of the polarization tensor M from (7). namely that

$$\min \left\{ 1, \frac{\gamma_0(x)}{\gamma_1(x)} \right\} |\xi|^2 \leq M_{ij}(x) \xi_i \xi_j \leq \max \left\{ 1, \frac{\gamma_0(x)}{\gamma_1(x)} \right\} |\xi|^2,$$

for every $\xi \in \mathbb{R}^3$ and for x μ -almost everywhere in Ω .

As a consequence, showing (10) will ensure that 1 is either the maximal or minimal eigenvalue of M , with eigenvector e_3 .

Let Φ_ϵ^3 be the minimizer corresponding to $\xi = e_3$.

Let us notice that Φ_ϵ^3 is a solution of (11) and, by De Giorgi-Nash estimates (see Theorem 8.24 in [13]), it is Hölder continuous and, for every $x \in K_1 \subset \subset K_0$

$$|\Phi_\epsilon^3(x)| \leq C \left(\|\Phi_\epsilon^3\|_{L^2(K_0)} + \|\mathbf{1}_{\omega_\epsilon} \psi\|_{L^4(K_0)} \right).$$

By (13) we deduce that

$$(17) \quad |\Phi_\epsilon^3(x)| \leq C \left(|\omega_\epsilon|^{1/2+\alpha} + |\omega_\epsilon|^{1/4} \right) \leq C|\omega_\epsilon|^{1/4}.$$

By definition (11) and integrating by parts,

$$\begin{aligned}
 \int_{K_0} \gamma_\epsilon |\nabla \Phi_\epsilon^3|^2 \phi \, dx &= \int_{\omega_\epsilon} (\gamma_0 - \gamma_1) \phi e_3 \cdot \nabla \Phi_\epsilon^3 \, dx, \\
 &= \int_{\omega_{2,\epsilon}} \int_{-l}^l (\gamma_0 - \gamma_1) \phi \frac{\partial}{\partial x_3} \Phi_\epsilon^3 \, dx_3 dx_1 \, dx_2, \\
 &= \left[\int_{\omega_{2,\epsilon}} (\gamma_0 - \gamma_1) \phi \Phi_\epsilon^3 \, dx_1 dx_2 \right]_{x_3=-l}^{x_3=l} \\
 &\quad - \int_{\omega_\epsilon} \Phi_\epsilon^3 \frac{\partial}{\partial x_3} ((\gamma_0 - \gamma_1) \phi) \, dx.
 \end{aligned} \tag{18}$$

By (17) (noticing that $\omega_{2,\epsilon} \times [-l, l] \subset \subset K_0$) and since γ_0, γ_1 and ϕ are smooth, we get

$$\left| \left[\int_{\omega_{2,\epsilon}} (\gamma_0 - \gamma_1) \phi \Phi_\epsilon^3 \, dx_1 dx_2 \right]_{x_3=-l}^{x_3=l} \right| \leq C |\omega_\epsilon|^{1/4} |\omega_{2,\epsilon}| \leq \frac{C}{l} |\omega_\epsilon|^{5/4}, \tag{19}$$

where C does not depend on ϵ and on l .

In addition, by using Cauchy-Schwarz inequality, and estimate (13) we obtain

$$\left| \int_{\omega_\epsilon} \Phi_\epsilon^3(x) \frac{\partial}{\partial x_3} ((\gamma_0 - \gamma_1) \phi) \, dx \right| \leq C |\omega_\epsilon|^{1+\alpha}. \tag{20}$$

By putting together (18), (19) and (20), we get

$$\int_{K_0} \gamma_\epsilon |\nabla \Phi_\epsilon^3|^2 \phi \, dx \leq C |\omega_\epsilon|^{1+\alpha'}, \quad \text{where } \alpha' = \min(1/4, \alpha). \tag{21}$$

Let us now write formula (7) for $\xi = e_3$:

$$\begin{aligned}
 \int_{\Omega} (\gamma_1 - \gamma_0) M e_3 \cdot e_3 \phi \, d\mu &= \frac{1}{|\omega_\epsilon|} \int_{\omega_\epsilon} (\gamma_1 - \gamma_0) \frac{\gamma_0}{\gamma_1} \phi \, dx \\
 &\quad + \frac{1}{|\omega_\epsilon|} \int_{K_0} \gamma_\epsilon \left| \nabla \Phi_\epsilon^3 + \mathbf{1}_{\omega_\epsilon} \frac{\gamma_1 - \gamma_0}{\gamma_1} e_3 \right|^2 \phi \, dx + o(1) \\
 &= \frac{1}{|\omega_\epsilon|} \int_{\omega_\epsilon} (\gamma_1 - \gamma_0) \phi \, dx + \frac{1}{|\omega_\epsilon|} \int_{K_0} \gamma_\epsilon |\nabla \Phi_\epsilon^3|^2 \phi \, dx \\
 &\quad + \frac{2}{|\omega_\epsilon|} \int_{K_0} \mathbf{1}_{\omega_\epsilon} \frac{\gamma_\epsilon (\gamma_1 - \gamma_0)}{\gamma_1} \phi e_3 \cdot \nabla \Phi_\epsilon^3 \, dx + o(1).
 \end{aligned} \tag{22}$$

By Cauchy-Schwarz inequality, and (21)

$$\left| \frac{2}{|\omega_\epsilon|} \int_{K_0} \mathbf{1}_{\omega_\epsilon} \frac{\gamma_\epsilon (\gamma_1 - \gamma_0)}{\gamma_1} \phi e_3 \cdot \nabla \Phi_\epsilon^3 \, dx \right| \leq C |\omega_\epsilon|^{\alpha'/2}. \tag{23}$$

By inserting (23) and (21) into (22) we get

$$\int_{\Omega} (\gamma_1 - \gamma_0) M e_3 \cdot e_3 \phi \, d\mu = \frac{1}{|\omega_\epsilon|} \int_{\omega_\epsilon} (\gamma_1 - \gamma_0) \phi \, dx + o(1),$$

and, by letting $\epsilon \rightarrow 0$, we get

$$\int_{\Omega} (\gamma_1 - \gamma_0) M e_3 \cdot e_3 \phi \, d\mu = \int_{\Omega} (\gamma_1 - \gamma_0) \phi \, d\mu,$$

which, in turn, implies (10). \square

Proof of Proposition 3.3. The idea of the proof consists in constructing an approximation of the correctors Φ_ϵ^1 and Φ_ϵ^2 by using the 2-dimensional correctors ψ_ϵ^1 and ψ_ϵ^2 defined by (14).

Let $f_\epsilon(x_3)$ be a function of x_3 only that we will specify better in Lemma 3.6. Let us define, for $j = 1, 2$,

$$(24) \quad \tilde{\Phi}_\epsilon^j(x) = \psi_\epsilon^j(x) f_\epsilon(x_3) \quad \text{for } x \in K_0.$$

Our proof will make use of the following technical results:

Lemma 3.4. *For $j = 1, 2$, the functions ψ_ϵ^j satisfies*

$$(25) \quad \left\| \frac{\partial}{\partial x_3} \psi_\epsilon^j \right\|_{L^2(K_0)} \leq C |\omega_{2,\epsilon}|^{\frac{1}{2} + \alpha},$$

for some positive C and α independent of ϵ and of l .

Lemma 3.5. *Let us denote by $\mathbf{1}_l(x_3) = \mathbf{1}_{(-l,l)}(x_3)$. Assume $f_\epsilon \in H^1(\mathbb{R})$ is chosen so that*

$$(26) \quad 0 \leq f_\epsilon \leq 1 \quad \text{and} \quad f_\epsilon(x_3) \mathbf{1}_l(x_3) = \mathbf{1}_l(x_3),$$

$$(27) \quad \|f'_\epsilon\|_{L^2(-L,L)} \leq C |\omega_{2,\epsilon}|^{-\frac{\alpha}{2}},$$

$$(28) \quad \|f_\epsilon(\cdot) (1 - \mathbf{1}_l(\cdot))\|_{L^2(-L,L)} \leq C |\omega_{2,\epsilon}|^{\frac{\alpha}{2}}.$$

Then, for $j = 1, 2$, the functions $\tilde{\Phi}_\epsilon^j$ and Φ_ϵ^j , given by (24) and (11), respectively, satisfy the inequality

$$\left\| \nabla (\tilde{\Phi}_\epsilon^j - \Phi_\epsilon^j) \right\|_{L^2(K_0)} \leq C |\omega_{2,\epsilon}|^{\frac{1+\alpha}{2}},$$

where C and α are independent of ϵ .

Lemma 3.6. *The function f_ϵ given by*

$$f_\epsilon(x_3) = \begin{cases} 0 & \text{if } x_3 < -l - 2|\omega_{2,\epsilon}|^\alpha \\ \frac{(x_3 + l + 2|\omega_{2,\epsilon}|^\alpha)^2}{2|\omega_{2,\epsilon}|^{2\alpha}} & \text{if } x_3 \in [-l - 2|\omega_{2,\epsilon}|^\alpha, -l - |\omega_{2,\epsilon}|^\alpha] \\ 1 - \frac{(x_3 + l)^2}{2|\omega_{2,\epsilon}|^{2\alpha}} & \text{if } x_3 \in [-l - |\omega_{2,\epsilon}|^\alpha, -l] \\ 1 & \text{if } x_3 \in [-l, 0], \end{cases}$$

and such that $f_\epsilon(-x_3) = f_\epsilon(x_3)$, satisfies assumptions (26), (27) and (28).

The proof of Lemma 3.6 is safely left to the reader. Lemma 3.4 and Lemma 3.5 are proven below. Let us first proceed with the proof of Proposition 3.3.

Let us take $\eta^* = (\eta, 0)$ where η is a unit vector in \mathbb{R}^2 . Let us consider formula (7) for $\xi = \eta^*$ and with the minimum taken over $H_0^1(K_0)$. We write the minimizer $w_\epsilon = \sum_{j=1}^2 \eta_j \Phi_\epsilon^j$ as $w_\epsilon = \sum_{j=1}^2 \eta_j \tilde{\Phi}_\epsilon^j + \sum_{j=1}^2 \eta_j (\Phi_\epsilon^j - \tilde{\Phi}_\epsilon^j)$ and obtain

$$(29) \quad \begin{aligned} \int_{\Omega} (\gamma_1 - \gamma_0) M \eta^* \cdot \eta^* \phi \, d\mu &= \int_{\Omega} (\gamma_1 - \gamma_0) \frac{\gamma_0}{\gamma_1} \phi \, d\mu \\ &+ \frac{1}{|\omega_\epsilon|} \int_{K_0} \gamma_\epsilon \left| \nabla \left(\sum_{j=1}^2 \eta_j \tilde{\Phi}_\epsilon^j \right) + \frac{\gamma_1 - \gamma_0}{\gamma_1} \mathbf{1}_{\omega_\epsilon} \eta^* \right|^2 \phi \, dx \\ &+ r_{1,\epsilon} + r_{2,\epsilon} + o(1), \end{aligned}$$

where we have set

$$r_{1,\epsilon} = \frac{1}{|\omega_\epsilon|} \int_{K_0} \gamma_\epsilon \left| \nabla \left(\sum_{j=1}^2 \eta_j (\Phi_\epsilon^j - \tilde{\Phi}_\epsilon^j) \right) \right|^2 \phi \, dx,$$

and

$$r_{2,\epsilon} = \frac{2}{|\omega_\epsilon|} \int_{K_0} \gamma_\epsilon \left(\nabla \left(\sum_{j=1}^2 \eta_j \tilde{\Phi}_\epsilon^j \right) + \frac{\gamma_1 - \gamma_0}{\gamma_1} \mathbf{1}_{\omega_\epsilon} \eta^* \right) \cdot \nabla \left(\sum_{j=1}^2 \eta_j (\Phi_\epsilon^j - \tilde{\Phi}_\epsilon^j) \right) \phi \, dx.$$

Let us first show that $r_{1,\epsilon}$ and $r_{2,\epsilon}$ are small. The term $r_{1,\epsilon}$ can be estimated by Lemma 3.5, so that

$$|r_{1,\epsilon}| \leq \frac{C}{|\omega_\epsilon|} \sum_{j=1}^2 \left\| \nabla \left(\Phi_\epsilon^j - \tilde{\Phi}_\epsilon^j \right) \right\|_{L^2(K_0)}^2 \leq \frac{C}{l} |\omega_{2,\epsilon}|^\alpha.$$

For $r_{2,\epsilon}$, using Cauchy-Schwarz inequality and Lemma 3.5, we obtain

$$\begin{aligned} |r_{2,\epsilon}| &\leq \frac{C}{|\omega_\epsilon|} |\omega_{2,\epsilon}|^{\frac{1+\alpha}{2}} \left\| \nabla \left(\sum_{j=1}^2 \eta_j \tilde{\Phi}_\epsilon^j \right) + \frac{\gamma_1 - \gamma_0}{\gamma_1} \mathbf{1}_{\omega_\epsilon} \eta^* \right\|_{L^2(K_0)} \\ &\leq \frac{C}{l} |\omega_{2,\epsilon}|^{\frac{\alpha-1}{2}} \left(\left\| \nabla \left(\sum_{j=1}^2 \eta_j \tilde{\Phi}_\epsilon^j \right) \right\|_{L^2(K_0)} + |\omega_\epsilon|^{\frac{1}{2}} \right). \end{aligned}$$

Let us note that

$$\|\nabla \tilde{\Phi}_\epsilon^j\|_{L^2(K_0)} \leq \|f_\epsilon \nabla_{12} \psi_\epsilon^j\|_{L^2(K_0)} + \|f'_\epsilon \psi_\epsilon^j + f_\epsilon \frac{\partial}{\partial x_3} \psi_\epsilon^j\|_{L^2(K_0)}.$$

Moreover, by (15) we get

$$\begin{aligned} \|f_\epsilon \nabla_{12} \psi_\epsilon^j\|_{L^2(K_0)} &= \left(\int_{-L}^L \int_D f_\epsilon^2 |\nabla_{12} \psi_\epsilon^j|^2 dx_1 dx_2 dx_3 \right)^{\frac{1}{2}} \\ &\leq 2L \|\nabla_{12} \psi_\epsilon^j\|_{L^2(D)} \leq C |\omega_{2,\epsilon}|^{\frac{1}{2}}, \end{aligned}$$

by (16) and (27), we obtain

$$\|\psi_\epsilon^j f'_\epsilon\|_{L^2(K_0)} \leq C |\omega_{2,\epsilon}|^{\frac{1}{2}+\alpha} |\omega_{2,\epsilon}|^{-\frac{\alpha}{2}},$$

and, by Lemma 3.4

$$\|f_\epsilon \frac{\partial}{\partial x_3} \psi_\epsilon^j\|_{L^2(K_0)} \leq C |\omega_{2,\epsilon}|^{\frac{1}{2}+\alpha}.$$

Hence, finally

$$|r_{2,\epsilon}| \leq \frac{C}{l} |\omega_{2,\epsilon}|^{\frac{\alpha-1}{2}} \left(|\omega_{2,\epsilon}|^{\frac{1}{2}} + |\omega_\epsilon|^{\frac{1}{2}} \right) \leq C |\omega_{2,\epsilon}|^{\frac{\alpha}{2}} \frac{1}{l}.$$

Now, we consider the second term of the right-hand-side in (29):

$$\begin{aligned} &\frac{1}{|\omega_\epsilon|} \int_{K_0} \gamma_\epsilon \left| \nabla \left(\sum_j \eta_j \tilde{\Phi}_\epsilon^j \right) + \frac{\gamma_1 - \gamma_0}{\gamma_1} \mathbf{1}_{\omega_\epsilon} \eta^* \right|^2 \phi dx \\ &= \frac{1}{|\omega_\epsilon|} \int_{-l}^l \int_D \gamma_\epsilon \left| \sum_j \eta_j \nabla_{12} \psi_\epsilon^j f_\epsilon + \frac{\gamma_1 - \gamma_0}{\gamma_1} \mathbf{1}_{\omega_{2,\epsilon}} \eta \right|^2 \phi dx \\ &+ \frac{1}{|\omega_\epsilon|} \int_{-L}^L (1 - \mathbf{1}_l(x_3)) \int_D \gamma_\epsilon \left| \sum_j \eta_j \nabla_{12} \psi_\epsilon^j f_\epsilon \right|^2 \phi dx \\ &+ \frac{1}{|\omega_\epsilon|} \int_{K_0} \gamma_\epsilon \left[\frac{\partial}{\partial x_3} \left(\sum_j \eta_j \psi_\epsilon^j \right) f_\epsilon + \sum_j \eta_j \psi_\epsilon^j f'_\epsilon \right]^2 \phi dx. \end{aligned}$$

Let us notice that, for $x_3 \in (-l, l)$, $f_\epsilon(x_3) = 1$ and, by (9),

$$\begin{aligned}
& \frac{1}{|\omega_\epsilon|} \int_{-l}^l \int_D \gamma_\epsilon \left| \sum_j \eta_j \nabla_{12} \psi_\epsilon^j f_\epsilon + \frac{\gamma_1 - \gamma_0}{\gamma_1} \mathbf{1}_{\omega_{2,\epsilon}} \eta \right|^2 \phi dx \\
&= \frac{1}{2l|\omega_{2,\epsilon}|} \int_{-l}^l \int_D \gamma_{2,\epsilon} \left| \sum_j \eta_j \nabla_{12} \psi_\epsilon^j + \frac{\gamma_1 - \gamma_0}{\gamma_1} \mathbf{1}_{\omega_{2,\epsilon}} \eta \right|^2 \phi dx \\
&= \frac{1}{2l} \int_{-l}^l \int_D (\gamma_1 - \gamma_0) m \eta \cdot \eta \phi d\mu' dx_3 - \frac{1}{2l|\omega_{2,\epsilon}|} \int_{-l}^l \int_{\omega_{2,\epsilon}} (\gamma_1 - \gamma_0) \frac{\gamma_0}{\gamma_1} \phi dx + o(1) \\
&= \int_\Omega (\gamma_1 - \gamma_0) m \eta \cdot \eta \phi d\mu - \int_\Omega (\gamma_1 - \gamma_0) \frac{\gamma_0}{\gamma_1} \phi d\mu + o(1).
\end{aligned}$$

Moreover, by (15) and (28),

$$\begin{aligned}
& \frac{1}{|\omega_\epsilon|} \int_{-L}^L (1 - \mathbf{1}_l(x_3)) \int_D \gamma_0 \left| \sum_j \eta_j \nabla_{12} \psi_\epsilon^j f_\epsilon \right|^2 \phi dx \\
&\leq \frac{C}{|\omega_\epsilon|} \left(\sum_j \|\nabla_{12} \psi_\epsilon^j\|_{L^2(D)}^2 \|(1 - \mathbf{1}_l) f_\epsilon\|_{L^2(-L, L)}^2 \right) \\
&\leq \frac{C}{l} |\omega_{2,\epsilon}|^\alpha
\end{aligned}$$

and, by Lemma 3.4, (16) and (27),

$$\begin{aligned}
& \frac{1}{|\omega_\epsilon|} \int_{K_0} \gamma_\epsilon \left[\frac{\partial}{\partial x_3} \left(\sum_j \eta_j \psi_\epsilon^j \right) f_\epsilon + \sum_j \eta_j \psi_\epsilon^j f'_\epsilon \right]^2 \phi dx \\
&\leq \frac{C}{|\omega_\epsilon|} \left(\left\| \frac{\partial}{\partial x_3} \psi_\epsilon^j \right\|_{L^2(K_0)}^2 + \|\psi_\epsilon^j\|_{L^2(K_0)}^2 \|f'_\epsilon\|_{L^2(-L, L)}^2 \right) \\
&\leq \frac{C}{|\omega_\epsilon|} (|\omega_{2,\epsilon}|^{1+2\alpha} + |\omega_{2,\epsilon}|^{1+2\alpha} |\omega_{2,\epsilon}|^{-\alpha}) = \frac{C}{l} |\omega_{2,\epsilon}|^\alpha.
\end{aligned}$$

As a result, identity (29) becomes

$$\int_\Omega (\gamma_1 - \gamma_0) M \eta^* \cdot \eta^* \phi d\mu = \int_\Omega (\gamma_1 - \gamma_0) m \eta \cdot \eta \phi d\mu + o(1),$$

that, passing to the limit for $\epsilon \rightarrow 0$, is our thesis. \square

Remark 3.7. *In the proof of Proposition 3.3, we underlined the dependence of the estimates upon l , the macroscopic length of the cylinder. Truly, we make no use of that information. This is merely a reminder of the fact that our approach cannot be used directly for arbitray shapes: when l tends to zero, these estimates become trivial.*

Proof of Lemma 3.4. Let us fix $j = 1$ or 2 . If we differentiate equation (14) with respect to x_3 , we obtain that for each x_3 , $\frac{\partial}{\partial x_3} \psi_\epsilon^j$ satisfies equation

$$\begin{aligned}
\operatorname{div}_{12} \left(\gamma_{2,\epsilon} \phi \nabla_{12} \left(\frac{\partial}{\partial x_3} \psi_\epsilon^j \right) \right) &= \operatorname{div}_{12} \left(\frac{\partial}{\partial x_3} ((\gamma_0 - \gamma_1) \phi) \mathbf{1}_{\omega_{2,\epsilon}} e_j \right) \\
&\quad - \operatorname{div}_{12} \left(\left(\frac{\partial}{\partial x_3} (\gamma_{2,\epsilon} \phi) \right) \nabla_{12} \psi_\epsilon^j \right),
\end{aligned}$$

hence we can write

$$\frac{\partial}{\partial x_3} \psi_\epsilon^j = a_{1,\epsilon}^j + a_{2,\epsilon}^j,$$

where for $i = 1, 2$, the function $a_{i,\epsilon}^j \in H_0^1(D)$ is the solution of

$$\operatorname{div}_{12} \left(\gamma_{2,\epsilon} \phi \nabla_{12} a_{i,\epsilon}^j \right) = f_{i,\epsilon}^j \text{ in } D,$$

and where

$$f_{1,\epsilon}^j = \operatorname{div}_{12} \left(\mathbf{1}_{\omega_{2,\epsilon}} e_j \frac{\partial}{\partial x_3} ((\gamma_0 - \gamma_1) \phi) \right), \quad f_{2,\epsilon}^j = -\operatorname{div}_{12} \left(\left(\frac{\partial}{\partial x_3} (\gamma_{2,\epsilon} \phi) \right) \nabla_{12} \psi_\epsilon^j \right).$$

We shall show that $a_{i,\epsilon}^j$ can be bounded as required. Concerning $a_{1,\epsilon}^j$, we can rely on Lemma 6.1 to obtain

$$\|a_{1,\epsilon}^j\|_{L^2(D)} \leq C |\omega_{2,\epsilon}|^{1/2+\alpha}.$$

Let us now turn to $a_{2,\epsilon}^j$. Notice that we can write

$$\begin{aligned} \operatorname{div}_{12} \left(\gamma_{2,\epsilon} \phi \nabla_{12} a_{2,\epsilon}^j \right) &= -\operatorname{div}_{12} \left(\frac{\partial}{\partial x_3} (\gamma_{2,\epsilon} \phi) \nabla_{12} \psi_\epsilon^j \right) \\ &= -\operatorname{div}_{12} \left((\gamma_{2,\epsilon} \phi) \frac{\partial}{\partial x_3} (\log(\gamma_{2,\epsilon} \phi)) \nabla_{12} \psi_\epsilon^j \right) \\ &= -\operatorname{div}_{12} \left((\gamma_{2,\epsilon} \phi) \nabla_{12} \left(\left(\frac{\partial}{\partial x_3} (\log(\gamma_{2,\epsilon} \phi)) \right) \psi_\epsilon^j \right) \right) \\ &\quad + \operatorname{div}_{12} \left((\gamma_{2,\epsilon} \phi) \psi_\epsilon^j \nabla_{12} \left(\frac{\partial}{\partial x_3} (\log(\gamma_{2,\epsilon} \phi)) \right) \right). \end{aligned}$$

And this means that

$$(30) \quad a_{2,\epsilon}^j = -\psi_\epsilon^j \frac{\partial}{\partial x_3} (\log(\gamma_{2,\epsilon} \phi)) + b_\epsilon^j,$$

with

$$\operatorname{div}_{12} (\gamma_{2,\epsilon} \phi \nabla_{12} b_\epsilon^j) = \operatorname{div}_{12} \left((\gamma_{2,\epsilon} \phi) \psi_\epsilon^j \nabla_{12} \left(\frac{\partial}{\partial x_3} (\log(\gamma_{2,\epsilon} \phi)) \right) \right).$$

By standard energy estimates, and by (16) we can conclude that

$$\|\nabla_{12} b_\epsilon^j\|_{L^2(D)} \leq C |\omega_{2,\epsilon}|^{\frac{1}{2}+\alpha}.$$

By Poincaré estimates for b_ϵ^j , by (30) and (16) again, we can conclude that

$$\|a_{2,\epsilon}^j\|_{L^2(D)} \leq C |\omega_{2,\epsilon}|^{\frac{1}{2}+\alpha}.$$

□

Proof of Lemma 3.5. By (24) and (14), for $j = 1, 2$, given any function $\Psi \in H_0^1(K_0)$, we have

$$\begin{aligned} \int_{K_0} \gamma_\epsilon \phi \nabla \tilde{\Phi}_\epsilon^j \nabla \Psi \, dx &= \int_{K_0} \gamma_\epsilon \phi \nabla (f_\epsilon \psi_\epsilon^j) \nabla \Psi \, dx \\ &= \int_{K_0} \gamma_\epsilon \phi f_\epsilon \nabla_{12} \psi_\epsilon^j \nabla_{12} \Psi \, dx + \int_{K_0} \gamma_\epsilon \phi \left(f'_\epsilon \psi_\epsilon^j + f_\epsilon \frac{\partial}{\partial x_3} \psi_\epsilon^j \right) \frac{\partial}{\partial x_3} \Psi \, dx \\ &= \int_{\omega_\epsilon} (\gamma_0 - \gamma_1) \phi \frac{\partial}{\partial x_j} \Psi \, dx + \int_{K_0} (1 - \mathbf{1}_l) \gamma_\epsilon \phi f_\epsilon \nabla_{12} \psi_\epsilon^j \nabla_{12} \Psi \, dx \\ &\quad + \int_{K_0} \gamma_\epsilon \phi \left(f'_\epsilon \psi_\epsilon^j + f_\epsilon \frac{\partial}{\partial x_3} \psi_\epsilon^j \right) \frac{\partial}{\partial x_3} \Psi \, dx \end{aligned}$$

while, by (11),

$$\int_{K_0} \gamma_\epsilon \phi \nabla \Phi_\epsilon^j \nabla \Psi \, dx = \int_{\omega_\epsilon} (\gamma_0 - \gamma_1) \phi \frac{\partial}{\partial x_j} \Psi \, dx$$

and, hence,

$$\begin{aligned} & \int_{K_0} \gamma_\epsilon \phi \left(\nabla \Phi_\epsilon^j - \nabla \tilde{\Phi}_\epsilon^j \right) \nabla \Psi \, dx = \\ & \int_{K_0} (1 - \mathbf{1}_l) \gamma_\epsilon \phi f_\epsilon \nabla_{12} \psi_\epsilon^j \nabla_{12} \Psi \, dx + \int_{K_0} \gamma_\epsilon \phi \left(f'_\epsilon \psi_\epsilon^j + f_\epsilon \frac{\partial}{\partial x_3} \psi_\epsilon^j \right) \frac{\partial}{\partial x_3} \Psi \, dx \end{aligned}$$

from which it follows that

$$\begin{aligned} & \|\nabla \Phi_\epsilon^j - \nabla \tilde{\Phi}_\epsilon^j\|_{L^2(K_0)} \\ & \leq \|\nabla_{12} \psi_\epsilon^j\|_{L^2(D)} \|(1 - \mathbf{1}_l) f_\epsilon\|_{L^2(-L, L)} + \|f'_\epsilon \psi_\epsilon^j\|_{L^2(K_0)} + \|f_\epsilon \frac{\partial}{\partial x_3} \psi_\epsilon^j\|_{L^2(K_0)} \\ & \leq C \left(|\omega_{2,\epsilon}|^{\frac{1}{2}} |\omega_{2,\epsilon}|^{\frac{\alpha}{2}} + |\omega_{2,\epsilon}|^{\frac{1}{2}+\alpha} |\omega_{2,\epsilon}|^{-\frac{\alpha}{2}} + |\omega_{2,\epsilon}|^{\frac{1}{2}+\alpha} \right) \leq C |\omega_{2,\epsilon}|^{\frac{1+\alpha}{2}} \end{aligned}$$

□

4. RECONSTRUCTION OF THE AXIS OF THE CYLINDER FROM BOUNDARY DATA OF THE CORRECTION TERM

Let ω_ϵ be a cylinder whose axis is a segment $\sigma \subset \subset \Omega$ and whose basis can be written as $\omega_{2,\epsilon} = \epsilon \omega_2$, where ω_2 is a bidimensional domain of measure $|\omega_2| = 1$.

Denote by m the polarization tensor for $\epsilon \omega_2$ as defined in Lemma 3.1. From Propositions 3.2 and 3.3, it follows that, for $y \in \partial\Omega$,

$$\begin{aligned} (u_\epsilon - u_0)(y) &= \epsilon^2 \int_\sigma (\gamma_1 - \gamma_0)(x) \left[\frac{\partial u_0}{\partial \tau}(x) \frac{\partial N}{\partial \tau}(x, y) \right. \\ & \quad \left. + m(x) \widetilde{\nabla u_0}(x) \cdot \widetilde{\nabla N}(x, y) \right] d\sigma_x + o(\epsilon^2), \end{aligned}$$

where τ is the tangent direction to σ and, for any vector $v \in \mathbb{R}^3$ we denote by $\tilde{v} = v - (v \cdot \tau)\tau$ the non tangential part of v .

Let us denote by u_σ the function

$$(31) \quad u_\sigma(y) = \int_\sigma (\gamma_1 - \gamma_0)(x) \left[\frac{\partial u_0}{\partial \tau}(x) \frac{\partial N}{\partial \tau}(x, y) + m(x) \widetilde{\nabla u_0}(x) \cdot \widetilde{\nabla N}(x, y) \right] d\sigma_x,$$

defined for $y \in \Omega \setminus \sigma$.

In this section we want to address the following problem: do the boundary values of the correction term u_σ uniquely determine the segment σ ?

In order to answer to this question let us focus on some properties of this correction term.

First of all we observe that u_σ is solution to

$$(32) \quad \operatorname{div}(\gamma_0 \nabla u_\sigma(x)) = 0 \quad \text{for } x \in \Omega \setminus \bar{\sigma}.$$

Moreover

$$(33) \quad \gamma_0 \frac{\partial u_\sigma}{\partial n} = 0 \quad \text{on } \partial\Omega,$$

because of the boundary condition on the Neumann function and of the fact that σ is far from the boundary.

If we integrate by parts in equation (31), and denote by P and Q the endpoints of segment σ (such that $\tau = (Q - P)/|Q - P|$) we have

$$\begin{aligned} (34) \quad u_\sigma(y) &= N(Q, y) \left((\gamma_1 - \gamma_0) \frac{\partial u_0}{\partial \tau}(Q) \right) - N(P, y) \left((\gamma_1 - \gamma_0) \frac{\partial u_0}{\partial \tau}(P) \right) \\ &- \int_\sigma N(x, y) \frac{\partial}{\partial \tau} \left((\gamma_1 - \gamma_0) \frac{\partial u_0}{\partial \tau} \right) d\sigma_x \\ &+ \int_\sigma (\gamma_1 - \gamma_0) m(x) \widetilde{\nabla u_0}(x) \cdot \widetilde{\nabla N}(x, y) d\sigma_x. \end{aligned}$$

Although the formulation of the correction term u_σ may look similar to the one that was established for inclusions that are small neighborhood of a curve in the plane (see [1]), we point out that the correction term given by (34) is singular at every point of the segment σ . Moreover, as it will be clear from the proof of the following proposition, at the endpoints of the segment σ , the correction term does not have worse singularities than at the other point of the segment. This behavior is different from the case studied in [1] where the correction term presents stronger singularities at the endpoints of the segment than at any other point.

Proposition 4.1. *Let γ_0 and γ_1 be smooth positive functions. Let Σ be an open subset of $\partial\Omega$ and let σ and σ' be two segments strictly contained in Ω . Let u_0 be a smooth solution to $\operatorname{div}(\gamma_0 \nabla u_0) = 0$ in Ω such that $\nabla u_0 \neq 0$ in Ω , and let u_σ and $u_{\sigma'}$ be defined by (31) for segments σ and σ' respectively.*

If

$$(35) \quad u_\sigma = u_{\sigma'} \quad \text{on} \quad \Sigma,$$

then

$$\sigma = \sigma'.$$

Proof. For sake of simplicity, let us carry out the proof in the case of a constant conductivity γ_0 . The general case is briefly discussed at the end.

Let $w = u_\sigma - u_{\sigma'}$. By (32), function w is solution to

$$\operatorname{div}(\gamma_0 \nabla w) = 0 \quad \text{in} \quad \Omega \setminus (\bar{\sigma} \cup \bar{\sigma'}).$$

Moreover, by (33) and (35), w has zero Cauchy data on Σ , hence, by unique continuation property

$$w \equiv 0 \quad \text{on} \quad \Omega \setminus (\bar{\sigma} \cup \bar{\sigma'}).$$

We argue by contradiction and assume that $\sigma \neq \sigma'$. This means that there is an endpoint, say P , that belongs to σ but not to σ' . Of course, this means that there is a segment γ with endpoint P that belong to $\sigma \setminus \sigma'$. We fix at P the origin of our coordinate system and we set e_3 as the tangent direction τ .

Let v be a direction different from τ . Consider a line $s(t) = vt$ approaching the origin as t goes to zero. There is a positive number t_0 such that $s(t) \in \Omega \setminus (\bar{\sigma} \cup \bar{\sigma'})$ for $0 < t < t_0$, hence

$$u_\sigma(s(t)) = w(s(t)) + u_{\sigma'}(s(t)) = u_{\sigma'}(s(t))$$

is bounded for $t \in (0, t_0)$, since $d(s(t), \sigma') > 0$. We want to show that this is a contradiction to the fact that $\nabla u_0 \neq 0$.

The Neumann function N can be written as

$$(36) \quad N(x, y) = \Gamma(|x - y|) + h(x, y),$$

where $\Gamma(|x - y|) = \frac{1}{4\pi\gamma_0|x-y|}$ and h is a harmonic function in Ω . By inserting expression (36) into (34) we have that, for $t \in (0, t_0)$,

$$(37) \quad \begin{aligned} u_\sigma(s(t)) &= -(\gamma_1(s(t)) - \gamma_0) \frac{\partial u_0}{\partial \tau}(s(t)) \Gamma(|s(t)|) \\ &- \int_\sigma \frac{\partial}{\partial \tau} \left((\gamma_1 - \gamma_0) \frac{\partial u_0}{\partial \tau} \right) \Gamma(|x - s(t)|) d\sigma_x \\ &+ \int_\sigma (\gamma_1 - \gamma_0) m(x) \widetilde{\nabla u_0}(x) \cdot \widetilde{\nabla \Gamma}(x, s(t)) d\sigma_x + \tilde{h}(t), \end{aligned}$$

where $\tilde{h}(t)$ is a bounded function.

Let B_0 be a ball centered at the origin with radius R , such that $0 < R < |\sigma|/2$ and $B_0 \subset K_0$. Let us estimate the right term of (37).

Let us define $v_0(x) = (\gamma_1(x) - \gamma_0) \partial u_0(x) / \partial \tau$. By the regularity assumptions on $u_0 \in C^2(\Omega)$ and γ_1 , the function v_0 and its derivatives are bounded on K_0 .

We first consider the first term in (37), which we rewrite in the following form

$$(38) \quad -(\gamma_1(s(t)) - \gamma_0) \frac{\partial u_0}{\partial \tau}(s(t)) \Gamma(|s(t)|) = -v_0(0) \Gamma(|s(t)|) - (v_0(s(t)) - v_0(0)) \Gamma(|s(t)|).$$

Note that the last right-hand side of (38) is bounded for $t \in (0, t_0)$ due to the regularity of v_0 .

We now write

$$\begin{aligned}
 & \int_{\sigma} \frac{\partial v_0}{\partial \tau}(x) \Gamma(|x - s(t)|) d\sigma_x = \int_{\sigma \setminus B_0} \frac{\partial v_0}{\partial \tau}(x) \Gamma(|x - s(t)|) d\sigma_x \\
 & + \int_{\sigma \cap B_0} \left(\frac{\partial v_0}{\partial \tau}(x) - \frac{\partial v_0}{\partial \tau}(0) \right) \Gamma(|x - s(t)|) d\sigma_x + \frac{\partial v_0}{\partial \tau}(0) \int_{\sigma \cap B_0} \Gamma(|x - s(t)|) d\sigma_x \\
 (39) \quad & := I_1 + I_2 + I_3
 \end{aligned}$$

For $x \in \sigma \setminus B_0$ and $t < R/2$, it is true that $|x - s(t)| \geq R - t \geq R/2$ and therefore

$$|I_1| \leq \frac{C}{R}.$$

On the other hand, because of the regularity of v_0 , we can estimate

$$|I_2| \leq C \int_0^R \frac{x_3}{[t^2(v_1^2 + v_2^2) + (x_3 - tv_3)^2]^{1/2}} dx_3 \leq C \quad \text{for } t \in [0, t_0]$$

The last integral in (39) can be explicitly calculated and estimated by

$$|I_3| \leq C \left| \ln \left(\frac{R}{t} \right) \right|.$$

Now, let us turn to the last term in (37). Arguing as before, we divide it into three parts, but this time we define $V_0(x) := (\gamma_1 - \gamma_0(x))m(x)\nabla u_0(x)$.

$$\begin{aligned}
 & \int_{\sigma} V_0(x) \cdot \widetilde{\nabla \Gamma}(x, s(t)) d\sigma_x = \int_{\sigma \setminus B_0} V_0(x) \cdot \widetilde{\nabla \Gamma}(x, s(t)) d\sigma_x \\
 & + \int_{\sigma \cap B_0} (V_0(x) - V_0(0)) \cdot \widetilde{\nabla \Gamma}(x, s(t)) d\sigma_x + V_0(0) \cdot \int_{\sigma \cap B_0} \widetilde{\nabla \Gamma}(x, s(t)) d\sigma_x \\
 & := J_1 + J_2 + J_3
 \end{aligned}$$

For $x \in \sigma \setminus B_0$ and $t < R/2$ we have that $|x - s(t)| \geq R - t \geq R/2$ and, hence,

$$|J_1| \leq \frac{C}{R^2}.$$

By regularity of V_0 , we can estimate

$$|J_2| \leq C \int_0^R \frac{x_3 t}{[t^2(v_1^2 + v_2^2) + (x_3 - tv_3)^2]^{3/2}} dx_3 \leq C \quad \text{for } t \in [0, t_0].$$

Now, we evaluate term J_3 that contains a singularity of leading order:

$$\begin{aligned}
 J_3 &= V_0(0) \cdot \int_0^R \frac{(-v_1 t, -v_2 t)}{[t^2(v_1^2 + v_2^2) + (x_3 - tv_3)^2]^{3/2}} dx_3 \\
 &= -V_0(0) \cdot \frac{(v_1, v_2)}{(v_1^2 + v_2^2)} \left(\frac{z}{\sqrt{1+z^2}} \Big|_{\frac{-v_3}{\sqrt{v_1^2+v_2^2}}}^{\frac{R-v_3 t}{\sqrt{v_1^2+v_2^2} t}} \right) \\
 &= -V_0(0) \cdot \frac{(v_1, v_2)}{(v_1^2 + v_2^2)t} v_3 + O \left(t \ln \left(\frac{1}{t} \right) \right).
 \end{aligned}$$

Collecting all this estimates we conclude that, for sufficiently small t ,

$$(40) \quad u_{\sigma}(s(t)) = (\gamma_1(0) - \gamma_0) \left(-\frac{\partial u_0}{\partial \tau}(0) \frac{C_1}{t} + m(0) \widetilde{\nabla u_0}(0) \cdot \frac{(v_1, v_2)}{\sqrt{v_1^2 + v_2^2} t} v_3 \right) + O \left(\ln \frac{1}{t} \right).$$

The leading order of remainder term is the contribution of the I_3 integral. Since the function $u_{\sigma}(s(t))$ has to be bounded for $t \in (0, t_0)$ and for any direction $v \neq e_3$, we can choose $v_3 = 0$ in (40) and conclude that

$$(41) \quad \frac{\partial u_0}{\partial \tau}(0) = \frac{\partial u_0}{\partial x_3}(0) = 0.$$

Now let us choose a direction v such that $v_3 \neq 0$, and obtain

$$m(0)\widetilde{\nabla u_0}(0) \cdot (v_1, v_2) = 0 \quad \text{for any } (v_1, v_2) \in \mathbb{R}^2,$$

which, in turn, implies that

$$m(0)\widetilde{\nabla u_0}(0) = 0.$$

Now, we notice that $\nabla u_0(0) = \left(\frac{\partial u_0}{\partial x_1}(0), \frac{\partial u_0}{\partial x_2}(0) \right)$ and that $m(0)$ is a symmetric and positive definite tensor, from which, together with (41), it follows that

$$\nabla u_0(0) = 0$$

which contradicts our assumptions.

Let us now consider the case of a smooth coefficient γ_0 . The Neumann function defined by (5) has the same singularities as the function Γ (see [18, ch.1 , sec.8]) and the same estimates can be carried out. \square

5. A RECONSTRUCTION ALGORITHM

In this section, we investigate how the results presented in the previous section can be translated in a numerical algorithm which would locate a cylindrical inclusion buried inside a three-dimensional domain. Several algorithms have been developed for that purpose in two dimensions, see e.g. [1, 2], and tested numerically [20]. They can be adapted to three dimensions, and we detail here a variant for our case.

5.1. Description of the algorithm. To fix ideas, the domain Ω is the ball B_R of radius R centered in the origin. As a particular case of the one considered in the previous section, we assume the inclusion ω_ϵ to be a cylinder whose axis is the segment σ and whose basis is $\epsilon\omega_2$, where ω_2 is a bidimensional domain of fixed area. Let L denote the length of σ , and τ its tangent direction. The conductivity is supposed equal to one in $\Omega \setminus \omega_\epsilon$, and equal to γ_1 in ω_ϵ .

Let u_ϵ^i , $i = 1, 2, 3$ be the solution of (3) corresponding to the boundary currents

$$\varphi^i = \frac{\partial x^i}{\partial n}.$$

The unperturbed solution is $u_0^i = x^i$ in $\overline{B_R}$.

In a first step, we find the direction τ . We compute the power perturbation, namely we calculate for all $i, j \in \{1, 2, 3\}^2$

$$\begin{aligned} \delta W_{ij} &= \int_{\partial B_R} (u_\epsilon^i - u_0^i) \phi^j ds \\ (42) \quad &= |\omega_\epsilon| \int_{\Omega} (1 - \gamma_1) M \nabla x^i \cdot \nabla x^j d\mu + o(|\omega_\epsilon|), \\ &= |\omega_\epsilon| (1 - \gamma_1) M e^i \cdot e^j + o(|\omega_\epsilon|). \end{aligned}$$

The matrix (δW_{ij}) is symmetric up to numerical errors, so diagonalizing its symmetric part, we obtain its eigendirections and eigenvalues. The eigenvalues have a constant sign, that of $(1 - \gamma_1)$. If $\gamma_1 > 1$ (resp. $\gamma_1 < 1$), the eigenvalue corresponding to the direction τ is the largest (resp. the smallest) in absolute value. We note by ν_1 and ν_2 the other two eigendirections, forming with τ an orthonormal basis.

In a second step, we find the median plane of the inclusion. By a linear combination, we compute the solution corresponding to the boundary data $\varphi^\tau = \tau \cdot n$, which is

$$u_\epsilon^\tau = \sum_{i=1}^3 u_\epsilon^i \cdot \tau^i.$$

We have

$$\begin{aligned}\delta W_\tau &= \int_{\partial B_R} (u_\epsilon^\tau - x \cdot \tau) \phi^\tau ds \\ &= |\omega_\epsilon| (1 - \gamma_1) M\tau \cdot \tau + o(|\omega_\epsilon|).\end{aligned}$$

Consider now the harmonic test function

$$\psi = \frac{1}{2} \left((x \cdot \tau)^2 - (x \cdot \nu_1)^2 \right).$$

Integrating against $u_\epsilon^\tau - x \cdot \tau$, we obtain

$$\begin{aligned}D_\psi &= \int_{\partial B_R} (u_\epsilon^\tau - x \cdot \tau) \frac{\partial \psi}{\partial n} ds \\ &= |\omega_\epsilon| \int_{\Omega} (1 - \gamma_1) M\tau \cdot \tau (x \cdot \tau) d\mu + o(|\omega_\epsilon|), \\ &= \frac{1}{2} |\omega_\epsilon| (1 - \gamma_1) M\tau \cdot \tau ((A \cdot \tau) + (B \cdot \tau)) + o(|\omega_\epsilon|).\end{aligned}$$

Where A and B are the endpoints of segment σ . We therefore can deduce the position of median plane of σ , $x_\tau = X_m \cdot \tau$, where X_m is the midpoint of σ .

$$x_\tau = \frac{D_\psi}{\delta W_\tau} = \frac{1}{2} ((A \cdot \tau) + (B \cdot \tau)) + o(1).$$

In a third step, we find the other coordinates of the midpoint X_m . We now use harmonic test functions of the form $\Phi_a = |x - a|^{-1}$, where a varies in the plate

$$P_1 = \{a = R_1\tau + \alpha\nu_1 + \beta\nu_2, \text{ with } -R \leq \alpha, \beta \leq R\},$$

where $R_1 > R$, the radius of the ball B_R . Projected on this plate, the inclusion ω_ϵ is a small disc centered in (α_0, β_0) . Integrating the boundary data $u_\epsilon^\tau - x \cdot \tau$ against $\nabla \Phi_a \cdot n$ we obtain

$$\begin{aligned}D^1(\alpha, \beta) &= \left| \int_{\partial B_R} (u_\epsilon^\tau - x \cdot \tau) \frac{\partial \Phi_a}{\partial n} ds \right| \\ &= |\omega_\epsilon| \left| \int_{\Omega} (1 - \gamma_1) M\tau \cdot \nabla \Phi_a d\mu \right| + o(|\omega_\epsilon|), \\ &= |\omega_\epsilon| (1 - \gamma_1) M\tau \cdot \tau \left| \frac{1}{|A - a|} - \frac{1}{|B - a|} \right| + o(|\omega_\epsilon|).\end{aligned}$$

Note that $D^1(\alpha, \cdot)$ attains its maximum at (α, β_0) , and $D^1(\cdot, \beta)$ attains its maximum at (α_0, β) . We can therefore find these coordinates by a simple dichotomy, alternating direction at each step. Fixing α , for $\beta \in [\beta_{\min}, \beta_{\max}]$ we restrict the values in an interval of size at most $\frac{1}{2}(\beta_{\max} - \beta_{\min})$ by computing values of $D^1(\alpha, \beta)$ at $\beta_{\min} + \frac{i}{4}(\beta_{\max} - \beta_{\min})$, $i = 1, 2, 3, 4$. We then repeat the procedure with the minimal β found fixed. After n iterations, the size of the box $[\alpha_{\min}, \alpha_{\max}] \times [\beta_{\min}, \beta_{\max}]$ is smaller than $(2R)^2 \times 2^{-n}$. At the maximal point, we have

$$(43) \quad D^1(\alpha_0, \beta_0) = |\omega_\epsilon| |\gamma_1 - 1| M\tau \cdot \tau \left(\frac{1}{R_1 - x_\tau - \frac{1}{2}L} - \frac{1}{R_1 - x_\tau + \frac{1}{2}L} \right)$$

and the coordinates of the midpoint of σ are $X_m = x_\tau\tau + \alpha_0\nu_1 + \beta_0\nu_2$.

Finally, to find the length of the inclusion we repeat this procedure on the plate

$$P_2 = \{a = -R_2\tau + \alpha\nu_1 + \beta\nu_2, \text{ with } -R \leq \alpha, \beta \leq R\},$$

where $R_2 > R$, and $(-R_2 + R_1) \neq 2x_\tau$, leading to

$$(44) \quad D^2(\alpha_0, \beta_0) = |\omega_\epsilon| |\gamma_1 - 1| M\tau \cdot \tau \left(\frac{1}{R_2 + x_\tau - \frac{1}{2}L} - \frac{1}{R_2 + x_\tau + \frac{1}{2}L} \right)$$

and finally compute the length L of σ , from formulae (43-44),

$$L = 2 \left(\frac{D^2(\alpha_0, \beta_0)(x_\tau + R_2)^2 - D^1(\alpha_0, \beta_0)(R_1 - x_\tau)^2}{D^2(\alpha_0, \beta_0) - D^1(\alpha_0, \beta_0)} \right)^{1/2}.$$

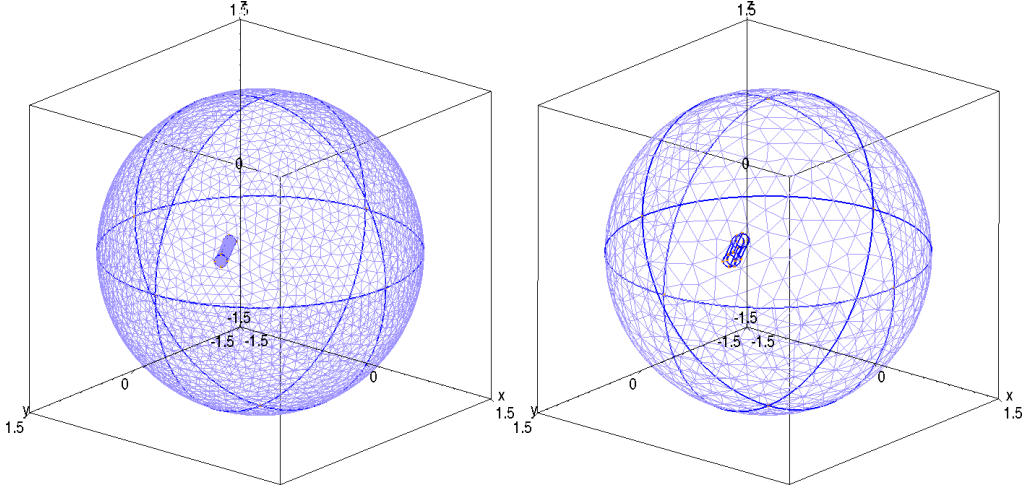


FIGURE 2. Simulation for a cylinder of revolution. On the left, the mesh used for the simulation (the volume elements are not represented). On the right, the reconstructed inclusion together with the original one.

Note that we have not used so far the properties of the polarization tensor. In the direction τ , we know that $M\tau \cdot \tau = 1$, hence, asymptotically, $\delta W\tau \cdot \tau = 1$ and the eigenvalue λ_τ corresponding to the direction τ is asymptotically $\lambda_\tau = |\omega_\epsilon|(1 - \gamma_1)$. If we represent the cross section of the inclusion by an equivalent ellipse, the other two eigenvalues λ_1 and λ_2 are given by the two-dimensional formula (see e.g. [17, 8]) $|\omega_\epsilon|(1 - \gamma_1)\frac{2(r+1)}{r+\gamma_1}$ and $|\omega_\epsilon|(1 - \gamma_1)\frac{2(r+1)}{r\gamma_1+1}$, where r is the ratio of the major axis a over the minor axis b of the ellipse. We therefore extract from (42) that

$$\frac{\lambda_1}{\lambda_\tau} = \frac{(r+1)}{r+\gamma_1} + o(1) \text{ and } \frac{\lambda_2}{\lambda_\tau} = \frac{(r+1)}{1+r\gamma_1} + o(1),$$

which gives in turn $r = a/b$ and γ_1 . We can then estimate a and b , using $\lambda_\tau = \pi abL(\gamma_1 - 1) + o(abL)$.

5.2. Numerical Simulation for a disk-base cylinder. Numerical simulations are performed using Gmsh and Getfem++ [14, 23]. The domain is a ball of radius 1.5 centered in 0.

The inclusion is a rod with a circular cross-section, of radius $\epsilon = \frac{1}{10\sqrt{\pi}}$, and of length $L = 1$, with a direction

$$\tau = (\sin(\pi/3)\cos(\pi/4), \sin(\pi/3)\sin(\pi/4), \cos(\pi/3)).$$

The conductivity in the inclusion is $\gamma_1 = 3$.

The midpoint is $X_m := (x_m, y_m, z_m) \approx (0.006, 0.506, 0.150)$.

The three dimensional mesh has 12625 nodes, and the underlying two dimensional mesh is represented in Figure 2. We compute of the polarization matrix (and we symmetrize it to reduce numerical errors) $\frac{1}{2}(\delta W_{ij} + \delta W_{ji})$ and we find that its eigenvalues are $\lambda_0 = -1.83777 \times 10^{-2}$, $\lambda_1 = -1.04347 \times 10^{-2}$, and $\lambda_2 = -1.04434 \times 10^{-2}$. Thus, $\gamma_1 > 1$ and the first eigenvalue corresponds to the direction of the cylinder. Its eigenvector is $\tilde{\tau} = (-0.612494, -0.612253, -0.499998)$, and $|\tilde{\tau} + \tau| \approx 2 \times 10^{-4}$.

The computed median plane abscissa is $\tilde{x}_\tau = -0.387759$, and $|\tilde{x}_\tau + X_m \cdot \tau| = 1 \times 10^{-3}$.

We use $R_1 = R_2 = 1.6$, and we find coordinates midpoint of the segment to be $X_{m,1} = (0.04, 0.44, 0.19)$ using the minimization on P_1 , or $X_{m,2} = (0.03, 0.47, 0.16)$ using the minimization on P_2 . Since P_2 is closer to the inclusion (because $x_\tau < 0$), we should expect this approximation to be the best one. They are however of the same order, $|X_{m,2} - X_m| \approx 5 \times 10^{-2}$ and $|X_{m,1} - X_m| \approx 8 \times 10^{-2}$.

The data $\max D^1 = 8.82 \times 10^{-3}$ and $\max D^2 = 2.67 \times 10^{-2}$ give an estimated length $\tilde{L} = 0.984$.

Finally, since $\lambda_1 = \lambda_2$, it is clear that the equivalent ellipse is a disk: this leads to an estimation of $\tilde{\gamma}_1 \approx 2.5$, and of the radius of the disk of $\tilde{\epsilon} \approx \frac{11}{100\sqrt{\pi}}$.

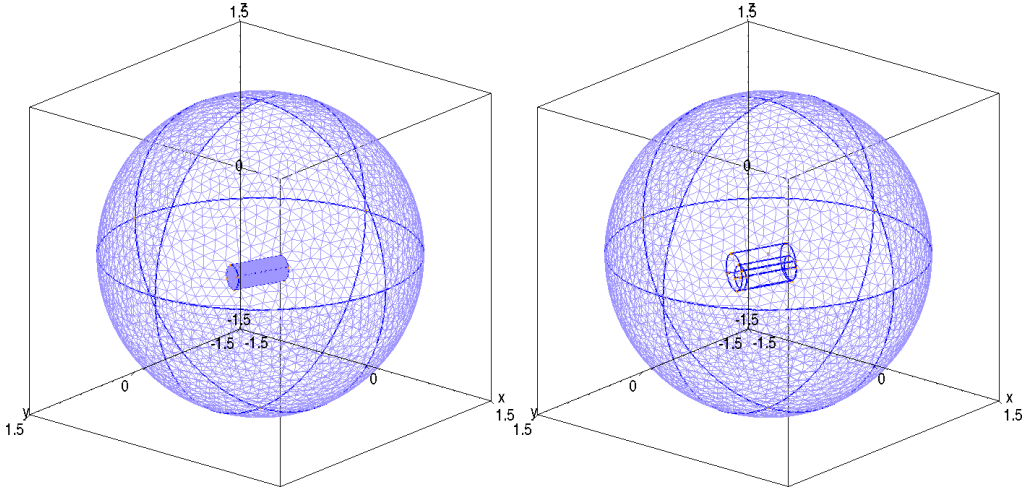


FIGURE 3. Simulation for a cylinder with elliptic cross-section. On the left, the mesh used for the simulation (the volume elements are not represented). On the right, the reconstructed inclusion together with the original one.

5.3. Numerical simulation with a ellipse-base cylinder. In this second simulation, the inclusion is a rod with an elliptic cross-section, of major axis length $a = \frac{2}{10\sqrt{\pi}}$, of minor axis length $b = \frac{1}{10\sqrt{\pi}}$ and of length $L = \frac{6}{5}$. The main direction of the cylinder is τ , the direction of the major axis is ν_1 and the direction of the minor axis is ν_2 given by

$$\begin{aligned}\tau &= \left(\cos\left(\frac{\pi}{7}\right) \sin\left(\frac{3\pi}{8}\right), \sin\left(\frac{\pi}{7}\right) \sin\left(\frac{3\pi}{8}\right), \cos\left(\frac{3\pi}{8}\right) \right), \\ \nu_1 &= \left(-\cos\left(\frac{\pi}{7}\right) \cos\left(\frac{3\pi}{8}\right), -\sin\left(\frac{\pi}{7}\right) \cos\left(\frac{3\pi}{8}\right), \sin\left(\frac{3\pi}{8}\right) \right), \\ \nu_2 &= \left(-\sin\left(\frac{\pi}{7}\right), \cos\left(\frac{\pi}{7}\right), 0 \right).\end{aligned}$$

The conductivity in the inclusion is $\gamma = \frac{1}{2}$.

The midpoint of the inclusion is $X_m = (-0.001, 0.041, -0.170)$. The three dimensional mesh has 15306 nodes, and the underlying two dimensional mesh is represented in Figure 3. We compute the polarization matrix (symmetrized to reduce numerical errors) $\frac{1}{2}(\delta W_{ij} + \delta W_{ji})$ and we find that its eigenvalues are $\lambda_0 = 1.72031 \times 10^{-2}$, $\lambda_1 = 1.22152 \times 10^{-2}$, and $\lambda_2 = 1.39706 \times 10^{-2}$, corresponding to the eigenvectors (in column)

$$\begin{bmatrix} 0.434006 & 0.832474 & -0.344421 \\ -0.900910 & 0.400773 & -0.166563 \\ -6.2 \times 10^{-4} & 0.382582 & 0.923922 \end{bmatrix}$$

Thus, $\gamma_1 > 1$ and the second eigenvalue corresponds to the direction of the cylinder, the first one to the minor axis, and the third to the major axis. The error in the main direction is 2×10^{-4} and in the cross section, it is 8×10^{-4} . The computed median plane abscissa is $\tilde{x}_\tau = -0.049244$, and $|\tilde{x}_\tau - X_m \cdot \tau| = 2 \times 10^{-4}$.

We use again $R_1 = R_2 = 1.6$, and we find the minimum to be $X_{m,1} = (-0.026, 0.053, -0.128)$ using the minimization on P_1 , or $X_{m,2} = (-0.041, 0.069, -0.111)$ using the minimization on P_2 . They are again of the same order, $|X_{m,1} - X_m| \approx 5 \times 10^{-2}$ and $|X_{m,2} - X_m| \approx 8 \times 10^{-2}$.

The data $D^1 = 1.14 \times 10^{-2}$ and $D^2 = 9.86 \times 10^{-3}$ gives an estimated length $L = 1.31$.

Finally, turning to the polarization tensor, from the ratios λ_0/λ_1 and λ_2/λ_1 , we obtain $\frac{b}{a} = 0.43$ and $\gamma_1 = 0.58$, which gives in turn, using $\lambda_1 = \pi a^2 L \frac{b}{a} (1 - \gamma_1)$, $a \approx \frac{31}{100\sqrt{\pi}}$ and $b \approx \frac{13}{100\sqrt{\pi}}$.

5.4. Discussion. From the numerical simulations, we observe that we can recover the location of the inclusions quite accurately. The error in the estimation of the center of the inclusion, when

compared to the size of the domain is less than 2%. The error on the length of the inclusion is larger in the case of the ellipse (9%) as the median plane is located close the middle of the two projection planes.

The estimation of the contrast is less precise, the error is about 16% for the disk and for the ellipse, and this in turn introduces compounded errors in the estimation of the length of the major and minor axis. This could be due to the coarseness of the numerical mesh. But we suspect that it is also linked to the relatively slow convergence of the first eigenvalue of $\frac{1}{|\omega_\epsilon|(1-\gamma_1)}\delta W$ towards 1 – which is an upper asymptotic bound not reached except for infinitely thin inclusions.

6. CONCLUDING REMARKS

In this paper we showed that the polarization tensor of cylindrical inclusions can be deduced from the polarization tensor of cross section orthogonal to the axis of the cylinder. When conductivity in the background and in the cylinder vary smoothly, the polarization tensor in every cross section is only a function of the contrast γ_1/γ_0 in that cross section, and can be obtained by a 2-dimensional calculation. Note that our arguments do not depend on the dimension, and does not require the base to be of small diameter. For example, iterating this result between dimension 1 and dimension d , we would recover the polarization tensor of a flat thin plate, already obtained in [6, 10], from that of a small segment in dimension 1.

The case of a base of small diameter is new, and we show that it allows uniquely determine the axis of the cylinder from one boundary measurement. We believe that a similar form of the polarization tensor holds for small neighborhoods of a general smooth curve. In this case the singularities of the correction term along the curve should be sufficient to be able to determine the curve itself from the knowledge of boundary data. This will be subject of a forthcoming paper.

In the last section of the paper, we verified that these results could be used to find a cylindrical inclusion with synthetic data. We showed that it is possible to reconstruct quite accurately the position of a cylinder with circular or elliptical cross section.

APPENDIX

Lemma 6.1. [10, 3] *Let $a \in L^\infty(\Omega)$ such that $c_1 < a < c_1^{-1}$ for some positive constant c_1 . Suppose that $\phi_\epsilon \in H_0^1(\Omega)$ is such that*

$$\operatorname{div}(a\nabla\phi_\epsilon) = \operatorname{div}(F_\epsilon) \text{ in } \Omega,$$

where either

$$\begin{aligned} F_\epsilon &= 1_{\omega_\epsilon}(x)F_0(x) \text{ with } \|F_0\|_{L^\infty(\omega_\epsilon)^d} \leq F_C, \\ \text{or } F_\epsilon &= 1_{\omega_\epsilon}(x)F_\epsilon(x) \text{ with } \|F_\epsilon\|_{L^2(\Omega)^d} \leq F_C|\omega_\epsilon|^{1/2}. \end{aligned}$$

where F_C is a constant independent of ϵ . Then,

$$\|\nabla\phi_\epsilon\|_{L^2(\Omega)^d} \leq \frac{1}{\sqrt{c_1}}|\omega_\epsilon|^{1/2}F_C,$$

Furthermore, there exists $\alpha > 0$ and $C > 0$, independent on ϵ , such that

$$\|\phi_\epsilon\|_{L^2(\Omega)} \leq C|\omega_\epsilon|^{\frac{1}{2}+\alpha}F_C.$$

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DIPARTIMENTO DI MATEMATICA “G. CASTELNUOVO” UNIVERSITÀ DI ROMA “LA SAPIENZA”,
PIAZZALE ALDO MORO 5, 00185 ROMA, ITALY
E-mail address: BERETTA@MAT.UNIROMA1.IT

MATHEMATICAL INSTITUTE, UNIVERSITY OF OXFORD, OXFORD OX1 3LB, UK
E-mail address: CAPDEBOSCQ@MATHS.OX.AC.UK

LABORATOIRE DE MATHÉMATIQUES DE VERSAILLES, UVSQ, 78035 VERSAILLES, FRANCE
E-mail address: FREDERIC.DE-GOURNAY@MATH.UVSQ.FR

DIPARTIMENTO DI MATEMATICA “U. DINI”, VIALE MORGAGNI 67A, 50134 FIRENZE, ITALY
E-mail address: FRANCINI@MATH.UNIFI.IT