On global existence for variants of Teichmüller harmonic map flow



Craig Robertson

Keble College

University of Oxford

DPhil Mathematics

Trinity 2020

To my parents

Acknowledgements

I would like to thank Prof. Melanie Rupflin for her encouragement and guidance as my supervisor. I would also like to thank the Engineering and Physical Sciences Research Council [EP/L015811/1] for their continued support.

Contents

1 Introduction **6**

2 Existence for Teichmüller harmonic map flow from cylinders **12**

2.1 Outline of short-time existence proof 15

2.2 Admissible maps and metrics 16

2.3 Existence of minimisers to the variational problem 20

2.4 H^2-regularity under the integral three-point condition 21

2.5 H^2-estimates for minimisers of the variational problem 23

2.6 Convergence of the time-discretisation scheme 26

3 Global solutions to Teichmüller harmonic map flow from cylinders **31**

3.1 Long-time existence 31

3.2 Decomposing tangent vectors 32

3.3 Asymptotics 36

4 Finite-time degeneration for variants of Teichmüller harmonic map flow **39**

4.1 Statement of results 39

4.2 Proof of sufficient conditions for Teichmüller harmonic map flow 42

4.3 Proof of necessary and sufficient conditions for the rescaled flow 49

4.4 Construction of settings exhibiting finite-time degeneration 53

A Alternative approaches to existence for the flow from cylinders **66**

A.1 Problems with H^2-regularity under the standard three-point condition 66

A.2 Results for H^2-regular minimisers under the standard three-point condition . . . 69

Abstract

We resolve questions of existence and singularity formation for the Teichmüller harmonic map flow, a geometric flow introduced by Rupflin and Topping that deforms surfaces immersed in a Riemannian manifold into minimal surfaces.

We obtain global solutions of Teichmüller harmonic map flow in new settings from cylinders into nonpositively curved targets, and prove their asymptotic convergence to solutions of the Douglas–Plateau problem of finding minimal surfaces spanning given boundary curves in a Riemannian manifold. Due to the nonlinear nature of the formulation with non-local projection operators and unusual boundary conditions, even short-time existence presents a challenge, requiring consideration of a time-discretisation scheme and partial differential inequalities.

In the second part of the thesis, we present joint work with Rupflin establishing a mechanism by which solutions to Teichmüller harmonic map flow will degenerate in finite time into maps from surfaces of simpler topological type. We give sufficient criteria for degeneration for the flow both from closed surfaces and from cylinders in terms of the rate at which images of thin collars in the domain stretch out. Moreover, for the rescaled variant of the flow by Huxol, we show that such a rate is both necessary and sufficient for a degeneration. Furthermore, for both types of the flow from cylinders we construct settings in which the metric on the cylinder degenerates in finite time, settling an open question on whether such a finite-time degeneration is possible.

Chapter 1

Introduction

Geometric flows encapsulate the idea of deforming geometric objects towards critical points of some geometrically inspired quantity. In this thesis, we investigate questions which arise in flowing surfaces to critical points of the area functional, establishing existence for the corresponding flow from cylinders into non-Euclidean targets, and for more general targets showing that the topological type of the surface can change at finite time.

Since the seminal work of Eells and Sampson in [10] on the harmonic map flow, the idea of geometric flows has been successfully deployed in a large number of contexts, for example, in the work of Huisken in [16] on the mean curvature flow, and the one-dimensional analogue in the Gage–Hamilton–Grayson theorem for the curve shortening flow in [11]; the Ricci flow introduced by Hamilton in [15] and later studied by Perelman in [24]; but also the Willmore flow, the Yamabe flow, and many others.

Since by the nature of the motivating quantities we are led to deal with nonlinear PDEs and differential inequalities, even seemingly straightforward questions of existence and uniqueness often present interesting and complex challenges, let alone those of singularity formation and asymptotics.

My contributions to the field relate to the Teichmüller harmonic map flow, with a particular focus on the topics of existence and finite-time degeneration. This flow was introduced by Rupflin and Topping in [31] to address the following question for a closed surface M and closed Riemannian manifold (N, g_N):

> Given a map $u_0 \colon M \to (N, g_N)$, can we define a PDE that changes u_0 into a (possibly branched) minimal immersion?

Teichmüller harmonic map flow is a negative gradient flow of the Dirichlet energy,

$$E(u, g) = \tfrac{1}{2} \int_M |du|_g^2 \, dv_g, \tag{1.1}$$

where the energy is viewed as a function both of a map component $u \colon M \to N$ and a metric component g on the domain. We recall the main ideas of the construction from [31] below.

While minimal surfaces are those which are critical points of the area functional, we note that in isothermal coordinates x and y on M we may calculate

$$\text{Area}(u) = \int_M |u_x \wedge u_y| \, dx \, dy \leq \int_M |u_x| \, |u_y| \, dx \, dy \leq \int_M \tfrac{1}{2}|u_x|^2 + \tfrac{1}{2}|u_y|^2 \, dx \, dy = E(u, g), \tag{1.2}$$

with equality if and only if u is conformal as discussed below. We therefore have that the critical points of $u \mapsto E(u,g)$ which are conformal are exactly the minimal immersions, and furthermore this energy has significantly better analytic properties than the area functional, as we discuss below.

We recall that u is a harmonic map if $\nabla_u E = 0$ where $\nabla_u E(u,g)$ is described by the tension field τ_g by $-\nabla_u E = \tau_g(u)$ for $\tau_g(u) = \operatorname{tr}(\nabla du)$ defined using the Levi-Civita connection ∇ on u^*TN induced by g_N.

Note that by the Nash embedding theorem we may consider N isometrically embedded in some $\mathbb{R}^\nu$, so that we can view $u\colon M \to \mathbb{R}^\nu$. We could then equivalently compute τ_g extrinsically as $\tau_g(u) = \Delta_g u + A(u)(du,du)$, where Δ_g is the Laplace–Beltrami operator with respect to the standard Euclidean metric on $\mathbb{R}^\nu$, and A is the second fundamental form of the embedding $N \hookrightarrow \mathbb{R}^\nu$. However, throughout the thesis we will take the intrinsic viewpoint unless otherwise indicated.

To study the conformality of a map $u\colon M \to N$, we consider the Hopf differential which is defined to be $\Phi = 4(u^*g_N)^{(2,0)}$, i.e., $\Phi = \phi\, dz^2$ for $z = x + \mathrm{i}y$ a complex coordinate on M and $\phi = |\partial_x u|^2 - |\partial_y u|^2 - 2\mathrm{i}\langle \partial_x u, \partial_y u\rangle$. A map u is conformal if and only if the Hopf differential Φ vanishes, and we may compute $-\nabla_g E = \frac{1}{4}\operatorname{Re}\Phi$. Notice that Φ vanishes precisely if its real part vanishes; indeed, we may calculate

$$\begin{aligned}\operatorname{Re}[(\operatorname{Re}\phi + \mathrm{i}\operatorname{Im}\phi)\, dz^2] &= \operatorname{Re}[(\operatorname{Re}\phi + \mathrm{i}\operatorname{Im}\phi)(dx^2 - dy^2 + 2\mathrm{i}\, dx\, dy)] \\ &= \operatorname{Re}\phi(dx^2 - dy^2) - 2\operatorname{Im}\phi\, dx\, dy.\end{aligned} \tag{1.3}$$

We note that while harmonic maps are in general not conformal, the Hopf differential of a harmonic map will always be holomorphic. Furthermore, if the domain M is a sphere, or a disc with Plateau boundary conditions as below, all harmonic maps are necessarily conformal, and so our problem is solved using the classical harmonic map flow of Eells and Sampson in [10] (or for the disc, the corresponding flow of Courant in [6] we discuss in Chapter 2). For more general domains, harmonic maps are in general not conformal and hence not (branched) minimal immersions, forcing us to consider a flow of both the map and the domain metric.

Naively, one could then define a gradient flow by

$$\partial_t u = -\nabla_u E, \quad \partial_t g = -\nabla_g E \tag{1.4}$$

but this formulation ignores the inherent structure, so that unnecessary singularities could form in the metric component. The energy (u,g) is invariant under pullback by diffeomorphisms, i.e., $E(u,g) = E(u\circ f, f^*g)$ for $f \in C^\infty(M,M)$ homotopic to the identity, which motivates deforming only in a direction orthogonal to the action of the symmetries. We also have that the energy is conformally invariant with respect to the domain metric g, so it enough to evolve g in a set $\mathcal{M}_c$ of constant curvature metrics with curvature $1, 0, -1$, with suitable boundary conditions if M has boundary as discussed in Chapter 2.

We therefore seek solutions which at each time are equivalence classes of pairs $[(u,g)]$, where $u \in C^\infty(M,N)$, $g \in \mathcal{M}_c$, and $[(u_1,g_1)] = [(u_2,g_2)]$ if there is a smooth diffeomorphism f homotopic to the identity such that $u_2 = u_1 \circ f$ and $g_2 = f^*g_1$.

To define the gradient flow, the idea is to turn to Teichmüller theory as a mechanism to discuss paths through this solution space. For simplicity, here we describe only the case considered in [31] where M is a closed surface of positive genus, but we will introduce the equivalent flow for surfaces of cylindrical type in Chapter 2.

To discuss the evolution of the metric, we note that the diffeomorphisms of M which are homotopic to the identity form a group which acts naturally on $\mathcal{M}_c$, and that the tangent vectors to

the orbit of this group at $g \in \mathcal{M}_c$ are those of the form $L_X g$, where $X = \frac{d}{d\varepsilon}\big|_{\varepsilon=0} f_\varepsilon$ for a smooth family f_ε of such diffeomorphisms.

Following Tromba in [40], we can canonically decompose a tangent vector in $T_g\mathcal{M}_c$ into one of these Lie derivatives of the metric g and a trace- and divergence-free quadratic differential. These are characterised as the real parts of holomorphic quadratic differentials, so that

$$T_g\mathcal{M}_c = \operatorname{Re}(\mathcal{H}(g)) \oplus \{L_X g\}. \tag{1.5}$$

The quadratic differentials in $\operatorname{Re}(\mathcal{H}(g))$ are orthogonal to the action of the diffeomorphism group, and so we call these horizontal; by the reasoning above, it is most efficient to deform in the direction of these horizontal tangent vectors.

One can endow the space of equivalence classes of tangent vectors with an inner product in order to induce a formal metric structure; for a tangent vector V represented by $(\partial_t u, \operatorname{Re}\Theta)$ at $[(u, g)]$, we define

$$\|V\|^2 = \|\partial_t u\|_{L^2}^2 + \eta^{-2}\|\operatorname{Re}\Theta\|_{L^2}^2 \tag{1.6}$$

where $\eta > 0$ is a coupling constant which can be chosen freely, and we consider the inner product induced by this norm. As the differential of the energy in our solution space is

$$dE_{(u,g)}(\partial_t u, \partial_t g) = -\int_M \langle \tau_g(u), \partial_t u\rangle + \tfrac{1}{4}\langle \operatorname{Re}(\Phi(u,g)), \partial_t g\rangle, \tag{1.7}$$

where the tension τ_g and Hopf differential Φ are as defined above, for horizontal $\partial_t g = \operatorname{Re}\Theta$,

$$\begin{aligned} dE_{(u,g)}(\partial_t u, \partial_t g) &= -\int_M \langle \tau_g(u), \partial_t u\rangle + \tfrac{1}{4}\langle \operatorname{Re}(\Phi(u,g)), \operatorname{Re}\Theta\rangle \\ &= -\int_M \langle \tau_g(u), \partial_t u\rangle + \tfrac{1}{4}\langle P_g(\operatorname{Re}(\Phi(u,g))), \operatorname{Re}\Theta\rangle \\ &= -\left\langle \left(\tau_g(u), \tfrac{\eta^2}{4} P_g(\operatorname{Re}(\Phi(u,g)))\right), (\partial_t u, \operatorname{Re}\Theta)\right\rangle, \end{aligned} \tag{1.8}$$

where $P_g\colon T_g\mathcal{M}_c \to \operatorname{Re}(\mathcal{H}(g))$ is the L^2-orthogonal projection onto the horizontal part. The gradient flow is then characterised by the equations

$$\partial_t u = \tau_g(u), \qquad \partial_t g = \frac{\eta^2}{4} P_g(\operatorname{Re}(\Phi(u,g))). \tag{1.9}$$

If M is a torus, an equivalent flow introduced by Ding, Li and Liu in [7] (corresponding to the choice $\eta^2 = 2$) has been shown to have global weak solutions, with no finite-time degeneration of the metric component in the sense described below.

For higher genus surfaces, the results from the works [31, 27, 18, 34] of Rupflin and Topping (respectively Huxol, Rupflin and Topping) establish that for any initial data, the flow (1.9) admits a global weak solution which decomposes the initial map into a collection of minimal surfaces. Furthermore, this solution is smooth away from finitely many singular times, which are caused either by the bubbling off of harmonic spheres, or the degeneration of the metric component (or a combination of both). While we know that even for the classical harmonic map flow, this first type of singularity can indeed occur, the problem of whether the second type can actually happen in finite time for the Teichmüller harmonic map flow was left open. This is one of the questions we settle for both this problem and also the corresponding problem of evolving maps from cylinders, as we discuss in Chapter 4.

For maps from surfaces with boundary, the natural objective is now to flow to solutions of the Douglas–Plateau problem of finding minimal surfaces spanning a given set of curves in the

target. The corresponding gradient flow will result in a flow of metrics together with a flow of maps satisfying Plateau boundary conditions. While there is a well-established theory for solving PDEs with Dirichlet conditions, there is significantly less standard theory available for Plateau conditions, and we discuss some of the technicalities that arise in Chapter 2. There, we discuss in detail that for the disc, the flow is simplified since only the map, and not the metric, needs to be evolved as the moduli space of the disc is a point and the required three-point condition can be achieved through Möbius transformations. In this setting, work of Chang and Liu in [2, 3, 4] and later of Duzaar and Scheven in [9] establishes existence of global weak solutions, not just in the setting of flowing to minimal surfaces but also to surfaces of prescribed mean curvature (provided an isoperimetric condition is satisfied). In [28], Rupflin has shown long-time existence of weak solutions when the domain is a cylinder and the target is Euclidean.

In the first chapters of this thesis, we establish new existence and asymptotic convergence results for Teichmüller harmonic map flow from surfaces with boundary into non-Euclidean targets under Plateau boundary conditions, summarised in Theorems I–III below.

Difficulties arise already in the short-time existence problem as we discuss in Chapter 2, forcing us to consider a nonlinear partial differential inequality and a time-discretisation scheme to account for the unusual boundary conditions, making the analysis intricate. Furthermore, current methods such as [28] lead to boundary regularity issues due to the need to impose a three-point condition on the boundary, traditionally fixing the images of three points under the parametrisation of each boundary curve. Nevertheless, we can establish existence of solutions to Teichmüller harmonic map flow from the cylinder $C_0 = [-1,1] \times S^1$ with Plateau boundary conditions, using a modified version of the three-point condition as used in [4] which instead fixes three of the Fourier coefficients in the parametrisation of the boundary curve that we introduce in Section 2.2.

The main result of Chapter 2 is the following short-time existence result, stated in full there as Theorem 2.0.1.

Theorem I. Let N be a closed Riemannian manifold with nonpositive sectional curvature, and let $\Gamma_\pm$ be two disjoint closed C^3 Jordan curves which are contractible in N. Let (u_0, g_0) be initial data for which the restrictions of u_0 to the boundary of C_0 parametrise $\Gamma_\pm$ as described in Theorem 2.0.1. Then there is $T > 0$ so that a stationary weak solution (u, g) of Teichmüller harmonic map flow exists on $[0, T]$ which satisfies the associated energy inequality.

In Chapter 3, we extend this result to long-time existence for the flow from cylinders into non-Euclidean targets, and prove the desired asymptotic convergence in the case that the three-point condition does not degenerate. The main two results, Theorems 3.1.1 and 3.3.1, can be formulated as follows.

Theorem II. Let N be a closed Riemannian manifold with nonpositive sectional curvature, and let $\Gamma_\pm$ be two disjoint closed C^3 Jordan curves which are contractible in N. Let (u_0, g_0) be initial data for which the restrictions of u_0 to the boundary of C_0 parametrise $\Gamma_\pm$ as described in Theorem 3.1.1. Then there exists a stationary weak solution (u, g) of Teichmüller harmonic map flow which is defined for all times and satisfies the associated energy inequality.

Theorem III. Let N be a closed Riemannian manifold with nonpositive sectional curvature, and let $\Gamma_\pm$ be two disjoint closed C^3 Jordan curves which are contractible in N. Suppose that $(u, g)(t)$, $t \in [0, \infty)$, is a global weak solution of Teichmüller harmonic map flow afforded by Theorem II for which the three-point condition does not degenerate as $t \to \infty$ in the sense of Theorem 3.3.1. Then there is a sequence of times $t_i \to \infty$ such that the equivalence classes $[(u, g)(t_i)]$ converge to a critical point of Area in the sense described in Theorem 3.3.1.

We remark here that we have also considered the analagous flow under the standard three-point condition, and provide some additional partial results in Appendix A. One of the challenges to

overcome here is to exclude concentration of energy near these three special points and related regularity issues, and while we were unable to obtain a full proof, it is interesting to see that these are the only barriers to extending the argument from the Euclidean case.

The final chapter, mostly comprising the paper [26] published in J. Lond. Math. Soc. in joint work with my supervisor Melanie Rupflin, considers mechanisms which lead to a degeneration of the metric on the domain and hence to a change in its topology. We remark that in the works [31, 27, 34] of Rupflin and Topping, the problem of whether such a degeneration is possible in finite time was left open even for closed surfaces, except for the particular case of nonpositively curved targets in which such behaviour is excluded. In Chapter 4, we give criteria which force degeneration of the metric in finite time when the domain is a closed surface of genus $\gamma \geq 2$ or a cylinder, and exhibit initial data and targets which meet these criteria for both the Teichmüller harmonic map flow (1.9) and its rescaled variant (1.10) defined below, demonstrating such a degeneration can indeed occur. We summarise these results as Theorems IV–VI below.

In order to describe these results, we remark that it is known that the domain metric remains regular while the injectivity radius of the domain under this metric remains bounded away from zero, so the only possibility for a degeneration of the metric is that there is $T \in (0, \infty]$ such that as $t \uparrow T$, we have $\operatorname{inj}(M, g(t)) \to 0$. This means there will be a simple closed geodesic $\sigma(t)$ in M whose length tends to zero. For the hyperbolic surfaces we consider, this geodesic is surrounded by a so-called collar neighbourhood, which we denote by $\mathcal{C}(\sigma(t), g(t))$. Our first result in Chapter 4 is to establish a criterion on the length of the image of such a collar which forces a finite-time degeneration of the domain metric, stated in full in Chapter 4 as Theorem 4.1.1.

Theorem IV. Let M be either a closed oriented surface of genus $\gamma \geq 2$ or a cylinder, and let (N, g_N) be a complete Riemannian manifold with sectional curvature bounded from above. Let (u, g) be a weak solution of (1.9) which satisfies the energy inequality (4.2) and such that $(M, g(t))$ contains a long thin collar $\mathcal{C}(\sigma(t), g(t))$ whose image under $u(t)$ stretches out at a rate of at least $\operatorname{inj}(M, g(t))^{-(\frac{1}{4}+\delta)}$ for some $\delta > 0$, as described in Theorem 4.1.1 below. Then $\operatorname{inj}(M, g(t)) \to 0$ in finite time.

In addition to studying finite-time degeneration for the Teichmüller harmonic map flow of Rupflin and Topping, we also consider the corresponding situation for the rescaled flow

$$0 = \tau_g(u), \qquad \partial_t g = P_g(\operatorname{Re}(\Phi(u, g))), \tag{1.10}$$

introduced by Huxol in [17]. Here we obtain not only a sufficient but also a necessary condition for finite-time degeneration. Indeed, finite-time degeneration is precluded if the length of the collar is controlled by $\log(\operatorname{inj}(M, g(t))^{-1})^{\frac{1}{2}}$ as $t \uparrow T$, while any growth rate which is of order $\log(\operatorname{inj}(M, g(t))^{-1})^{\frac{1}{2}+\delta}$ for $\delta > 0$ will force the solution to degenerate in finite time. The following result is a summary of Theorem 4.1.2 below.

Theorem V. Let M be either a closed oriented surface of genus $\gamma \geq 2$ or a cylinder, and let (N, g_N) be a complete Riemannian manifold with sectional curvature bounded from above. Let (u, g) be a weak solution of (1.10) defined on an interval $[0, T)$ which is maximal in the sense that $T = \infty$ or $\inf_{[0,T)} \operatorname{inj}(M, g(t)) = 0$. Then the following hold true:

(i) Suppose that for every short geodesic $\sigma(t) \subset (M, g(t))$, the image of the corresponding collar under $u(t)$ stretches out at a rate no faster than $\log(\operatorname{inj}(M, g(t))^{-1})^{\frac{1}{2}}$ in the sense of Theorem 4.1.2. Then $T = \infty$, i.e., the flow cannot degenerate in finite time.

(ii) Conversely, suppose that $(M, g(t))$ contains a long thin collar $\mathcal{C}(\sigma(t), g(t))$ whose image under $u(t)$ stretches out at a rate of at least $\log(\operatorname{inj}(M, g(t))^{-1})^{\frac{1}{2}+\delta}$ for some $\delta > 0$, as described in Theorem 4.1.2 below. Then $\operatorname{inj}(M, g(t)) \to 0$ in finite time.

Furthermore, we construct settings in which the criteria are guaranteed to apply, thus establishing that finite-time degeneration does indeed occur for both variants of the flow from cylinders. Indeed, for the rescaled flow, we may choose a compact target, as set out in Theorem 4.1.3.

Theorem VI. There exists a complete Riemannian manifold (N, g_N) of bounded sectional curvature for which there are initial data $u_0 \colon C_0 \to N$ and g_0 such that the solution of Teichmüller harmonic map flow (1.9) degenerates in finite time. Furthermore, there exists a compact Riemannian manifold (N_c, g_{N_c}) for which there are corresponding initial data such that the solution of the rescaled flow (1.10) degenerates in finite time.

Chapter 2

Existence for Teichmüller harmonic map flow from cylinders

In this chapter, we will discuss short-time existence for solutions to Teichmüller harmonic map flow from cylinders into non-Euclidean targets.

This flow is designed to change combinations of maps and metrics into the corresponding solution of the Douglas–Plateau problem, evolving to a branched minimal immersion spanning two given curves in the target parametrised either over the cylinder or over two discs. Before we discuss the details of this construction, we first recall some of the important ideas from the analysis of the classical Plateau problem, named for Belgian physicist Joseph Plateau for his experiments with soap films in the 19^{th} century.

In the classical setting, the problem is of minimising the area functional among a given class of maps from a disc which span a prescribed boundary curve. Since it does not matter how this curve is parametrised, the natural boundary condition is not to specify the image of the boundary pointwise as for Dirichlet boundary conditions, but instead to allow for reparametrisations. The natural class of maps to consider is then those whose restriction to the boundary of the domain is again a weakly monotone parametrisation of the given curve in the target.

One main difficulty is that the direct method in the calculus of variations does not apply to the area functional on this space of maps (even under Dirichlet boundary conditions), in particular because the area functional is not coercive.

This problem of finding a minimal surface with a prescribed boundary curve was solved independently by Douglas and Radó in the 1930s for the case of a disctype surface spanning any simple closed curve in Euclidean space. In place of the area functional, Douglas introduced his A-functional, which had better analytic properties as noted in [12].

An alternative approach more commonly used today is due to Courant in [6], who employed the Dirichlet energy in place of the area functional. However, unlike the area functional, the Dirichlet energy is not invariant under arbitrary reparametrisations of the domain. In order to find a solution to the area minimisation problem, it is not enough to simply minimise the Dirichlet energy in the space of maps with Dirichlet boundary conditions; we must instead relax to Plateau boundary conditions. We denote by H^1_Γ the corresponding space of H^1-regular maps whose restrictions to the boundary of the disc parametrise a given curve Γ in the target.

However, a full such relaxation results in a space of maps which is not weakly closed, which

again causes issues with the direct method. Indeed, given any $X \in H^1_\Gamma$, we can choose a sequence of Möbius transformations of the disc such that the limit of the pullback of X by these transformations is a constant map, which then does not parametrise the boundary. Since the energy is conformally invariant, and the Möbius transformations are exactly the conformal diffeomorphisms of the disc to itself, sets of such maps with a given energy bound need not be weakly closed.

To address this problem, in [6] Courant imposed a three-point condition on the boundary ensuring that maps with bounded energy are uniformly continuous on the boundary, and consequently that the space of such maps is weakly closed. This allowed him to work in the space $H^1_{\Gamma,*}$ of maps which additionally fix three points on the boundary, which is weakly closed in H^1 as a result of the famous Courant–Lebesgue lemma, and hence for which the direct method in the calculus of variations applies.

This then settles the problem in case of disctype surfaces in a Euclidean target; a full account is given for example by Struwe in [38]. For general domains and targets, the corresponding problem is known as the Douglas–Plateau problem, and the situation is more complex. In particular, the two following simplifications that are present in the disc case no longer apply.

Firstly, there is only one metric on the disc up to conformal invariance. In other words, the moduli space of the disc (the space of conformal classes of metrics modulo diffeomorphisms of the disc) is trivial. Secondly, by pulling back by Möbius transformations, one can recover a member of the same conformal class of maps which additionally satisfies the three-point condition required to obtain a map in the weakly closed set $H^1_{\Gamma,*}$. Since the energy is conformally invariant, these special properties mean that minimising the Dirichlet energy over $H^1_{\Gamma,*}$ with respect to any fixed domain metric amounts to minimising the area functional.

This is no longer true for the Douglas–Plateau problem, and one has to take both of these considerations into account. We remark that the problem has been successfully resolved by Jost in [21], where he minimises the energy first within a conformal class, then subsequently over all conformal classes.

An alternative approach is to consider the energy as a function of both the map and of the metric simultaneously on the domain, and this is the approach taken by Chang and Liu in [2, 3, 4] who consider the resulting gradient flow to minimal discs. However, due to the simplifications for disctype surfaces mentioned in Chapter 1, a flow of the map component is sufficient.

The reason it is not sufficient to only evolve a map in the general case is that a minimal surface is one parametrised by a map which is not only harmonic but also (weakly) conformal. Unless the domain is particularly simple, such as a sphere or a disc, this is not guaranteed in general if we only evolve the map by harmonic map flow; indeed, if we consider maps from the cylinder $C_0 = [-1, 1] \times S^1$ into non-Euclidean targets now spanning two given boundary curves, we can expect that the obtained limit will not be conformal, and hence not a solution to the Douglas–Plateau problem.

We therefore need to additionally allow for the metric to evolve. For general hyperbolic surfaces with boundary, the space of conformal structures can become unwieldy, cf. [29]. In contrast, the moduli space of the cylinder is one-dimensional (representing the conformal moduli of an annulus), and we account for this by considering a one-parameter family of hyperbolic metrics on the cylinder. In addition, we will impose a three-point condition on each boundary curve to ensure that the direct method applies, and we will compensate by allowing pullback of the domain metric by select diffeomorphisms. All in all, we are now missing seven degrees of freedom, while the moduli space of the cylinder is only the one-dimensional space of rotations.

To overcome this, we follow the argument of Rupflin in [28] allowing the metric to flow in

these new directions, incorporating ideas from Teichmüller theory motivated by Tromba in [40]. However, [28] only addresses Euclidean targets, so that the equation for the map component is essentially a heat equation with an unusual boundary condition. If we wish to generalise this to curved targets, we have the additional challenge that the equation for the map is really a nonlinear PDE.

With current methods (e.g., [9, 38, 4, 28]), we could still establish regularity away from the three points on each boundary at which we impose our restrictions. However, issues arise in obtaining regularity in a neighbourhood of the fixed boundary points, as we discuss in Appendix A.

So instead of the standard three-point condition, we adapt the construction of Rupflin in [28] to a different three-point condition used by Struwe in [38] and Chang and Liu in [4], which we denote by (†) in place of (∗) below. In practice, this means we consider a gradient flow of the Dirichlet energy where we constrain the maps to the corresponding space $(H^1 \cap C)_{\Gamma,\dagger}(C_0, N)$ introduced at (2.11), and the metrics to a seven-dimensional manifold $\widetilde{\mathcal{M}}$ introduced at (2.12).

We remark that since the space of maps is now no longer a vector space, the flow equation for the map no longer takes the form of a PDE with standard boundary conditions, but is instead more natually written as a variational inequality. At the same time, the map will still be governed by the harmonic map heat flow in the interior, so that we can use existing theory in this region. For example, since we will focus on targets with nonpositive sectional curvature, the result of Eells and Sampson allows us to exclude the formation of bubbles in the interior; however, we still need to consider the behaviour at the boundary, where standard results cannot be applied.

Nevertheless, we are able to obtain short-time existence for Teichmüller harmonic map flow, using a time-discretisation scheme as in the work [4] of Chang and Liu and [9] of Duzaar and Scheven. We note that due to the non-local projection operator in the second component of the flow not present in either of these works, a careful analysis near the boundary is required in order to obtain convergence of the scheme.

We say (u, g) is a weak solution of Teichmüller harmonic map flow on $[0, T]$ if

$$u \in L^\infty([0,T], (H^1 \cap C)_{\Gamma,\dagger}(C_0, N)) \cap H^1([0,T] \times C_0) \quad \text{and} \quad g \in C^{0,1}([0,T], \widetilde{\mathcal{M}})$$

satisfy the partial differential inequality

$$\int_{[0,T]\times C_0} \langle du, dv \rangle_{g(t)} + \partial_t u \cdot v \, dv_{g(t)} dt \geq 0 \quad \text{for all} \quad v \in L^2([0,T], T_u^+(H^1 \cap C)_{\Gamma,\dagger}(C_0, N)) \tag{2.1}$$

and the evolution equation

$$\partial_t g = \frac{1}{4} P_g(\mathrm{Re}(\Phi(u, g))) \tag{2.2}$$

for almost every $t \in [0, T]$. Here $T_u^+(H^1 \cap C)_{\Gamma,\dagger}(C_0, N)$ is the solid tangent cone to the space of functions $(H^1 \cap C)_{\Gamma,\dagger}(C_0, N)$ which is introduced at (2.11) and $\widetilde{\mathcal{M}}$ is the finite-dimensional submanifold of hyperbolic metrics introduced at (2.12), while P_g is the projection onto the tangent space $\mathcal{V}(g)$ of $\widetilde{\mathcal{M}}$ at g introduced at (2.19) and $\Phi(u, g) = 4(u^* g_N)^{(2,0)}$ is the Hopf differential discussed in Chapter 1.

The solutions we obtain will furthermore satisfy an energy inequality of the form

$$E(u(t_1), g(t_1)) - E(u(t_2), g(t_2)) \geq \int_{t_1}^{t_2} \tfrac{1}{2} \|\partial_t u\|^2_{L^2(C_0,g)} + \|\partial_t g\|^2_{L^2(C_0,g)} \, dt \tag{2.3}$$

for almost every $t_1 < t_2$.

We also expect our solutions to be stationary, i.e., critical with respect to variations on the domain. To this end, we write $\Gamma(TC_0)$ for the space of smooth vectorfields on C_0 which are tangential to ∂C_0 on ∂C_0. We will obtain the stationarity condition that for almost every time $t \in [0,T]$,

$$\int_{C_0} \tfrac{1}{4} \operatorname{Re} \Phi(u(t), g(t)) \cdot L_Y g(t) + \tau_{g(t)} u(t) \cdot du(t)(Y)\, dv_{g(t)} = 0 \quad \text{for all } Y \in \Gamma(TC_0)_{u(t),\dagger}, \tag{2.4}$$

where $\Gamma(TC_0)_{u,\dagger}$ is the subspace of sections $Y = \frac{d}{d\varepsilon}\big|_{\varepsilon=0} f_\varepsilon \in \Gamma(TC_0)$ for which $f_\varepsilon \colon C_0 \to C_0$ is a family of diffeomorphisms with $f_0 = \mathrm{id}$ such that for every $\varepsilon > 0$, we have $u \circ f_\varepsilon \in (H^1 \cap C)_{\Gamma,\dagger}(C_0, N)$. We will obtain this stationarity property by choosing vectorfields derived from the associated variations $(u(t) \circ f_\varepsilon, g(t))$ in (2.1).

The main result of this chapter is the following short-time existence result.

Theorem 2.0.1 (Short-time existence)**.** Let N be a closed Riemannian manifold with nonpositive sectional curvature, and let $\Gamma_\pm$ be two disjoint closed C^3 Jordan curves which are contractible in N. Let $(u_0, g_0) \in (H^1 \cap C)_{\Gamma,\dagger}(C_0, N) \times \widetilde{\mathcal{M}}$ for the spaces defined in Section 2.2. Then there is $T > 0$ so that a weak solution (u, g) of Teichmüller harmonic map flow (2.1)–(2.2) exists on $[0, T]$ and satisfies both the energy inequality (2.3) and the stationarity condition (2.4).

In the following chapter, we extend this result to all times and additionally prove an asymptotic result for such global solutions. We remark that this result and its extensions in Chapter 3 apply more generally for targets which support no bubbles, i.e., for which there exist no nonconstant harmonic maps $\omega \colon S^2 \to N$. For now, we additionally assume a technical assumption on the contractibility of our boundary curves, but as explained below in Remark 2.4.4, we believe this is artificial and could be removed.

2.1 Outline of short-time existence proof

Before we embark on the technical details of the proof, we briefly explain how we will construct our solutions. We will make use of a time-discretisation scheme, as in the works of Haga, Hoshino and Kikuchi in [14] and Moser in [23].

Let $u_0 \in (H^1 \cap C)_{\Gamma,\dagger}(C_0, N)$ and $g_0 \in \widetilde{\mathcal{M}}$, and let $h > 0$ small. We will construct an approximate solution (u^h, g^h) to the flow as follows. Let $t_j = j \cdot h$, and set $u^h(t) = u_0$ on $[0, t_1]$. Choose g^h on $(t_j, t_{j+1}]$ as the solution of

$$\partial_t g^h = \frac{1}{4} P_g(\operatorname{Re}(\Phi(u^h(t_j), g^h))) \tag{2.5}$$

with initial data $g^h(t_j)$. Here and in the following, P_g is the projection onto the tangent space $\mathcal{V}(g)$ to the submanifold of hyperbolic metrics $\widetilde{\mathcal{M}}$ at g defined at (2.19). Then choose $u^h(t_{j+1})$ as the minimiser in $(H^1 \cap C)_{\Gamma,\dagger}(C_0, N)$ of

$$J^h_{u^h(t_j), g^h(t_{j+1})}(w) = E(w, g^h(t_{j+1})) + \frac{1}{2h} \|w - u^h(t_j)\|_{L^2(C_0, g^h(t_{j+1}))}. \tag{2.6}$$

We then set $u^h(t) = u(t_{j+1})$ on $(t_j, t_{j+1}]$, and iterate.

We show well-definedness of these approximate solutions below, and using suitable a priori estimates we will show that in the limit $h \to 0$, these approximate solutions converge to a weak solution of the flow on $[0, T]$ for some $T > 0$. More precisely, we will establish the following result.

Proposition 2.1.1. *Let N be a closed Riemannian manifold with nonpositive sectional curvature and $\Gamma_\pm$ contractible in N, and let $(u_0, g_0) \in (H^1 \cap C)_{\Gamma,\dagger}(C_0, N) \times \widetilde{\mathcal{M}}$ be initial data. Then there exist $T, h_0 > 0$ and a compact subset K of $\widetilde{\mathcal{M}}$ so the following hold true:*

(i) For every $h \in (0, h_0)$ the approximate solutions (u^h, g^h) exist on $[0, T]$, and $g^h(t) \in K$ for every $t \in [0, T]$.

(ii) For every $h_j \to 0$ there exists a subsequence such that (u^{h_j}, g^{h_j}) converges to a limit (u, g) in $C^0_t L^2_x \times C^{0,\alpha}_t C^k_x$ for any $\alpha < 1$ and any k, as well as in $H^1([0, T] \times C_0)$ for the map component, and this limit is a weak solution of Teichmüller harmonic map flow (2.1)–(2.2).

(iii) The limit (u, g) satisfies the energy inequality (2.3) and the stationarity condition (2.4).

We will prove the short-time existence in this chapter as follows. First, we introduce the spaces of maps and metrics used in the above results in Section 2.2. The main work will be analysing the corresponding variational problem in Section 2.3, as well as establishing the required regularity under the integral version of the three-point condition in Section 2.4. Finally, in Section 2.5, we prove H^2-estimates on solutions to the variational problem, which will be used in Section 2.6 to prove our proposition.

We are going to study the variational problem arising from minimising the functional

$$J^h_{f,g}(X) = \frac{1}{2}\int_{C_0} |dX|^2\, dv_g + \frac{1}{2h}\int_{C_0} |X - f|^2\, dv_g \tag{2.7}$$

with Plateau boundary conditions, for $f \in L^2(C_0, N)$ and $h \in (0, h_0)$ for $h_0 > 0$ sufficiently small chosen later. We prefer to use X below for a given map arising from this variational problem, and u for a curve of maps related to the solution of the flow.

2.2 Admissible maps and metrics

In this section we introduce the spaces $(H^1 \cap C)_{\Gamma,\dagger}(C_0, N)$ and $\widetilde{\mathcal{M}}$ of maps and metrics in which the solution to the flow will evolve. To this end, we first recall the spaces used in analysing the Plateau problem from the disc, used for example by Chang and Liu in [4].

Let $D \subset \mathbb{C}$ the unit disc, and N a closed Riemannian manifold. Let $\gamma\colon S^1 \to N$ be a fixed regular C^3 parametrisation of a given simple C^3 closed curve $\Gamma \subset N$. For $\alpha\colon S^1 \to \Gamma$ another (continuous) weakly monotone parametrisation of Γ with the same orientation, we consider the weakly monotone function $\overline{(\gamma^{-1} \circ \alpha)}\colon \mathbb{R} \to \mathbb{R}$ (see Figure 2.1) characterised by

$$\begin{aligned} e^{i\overline{(\gamma^{-1}\circ\alpha)}(\phi)} &= (\gamma^{-1} \circ \alpha)(e^{i\phi}) \qquad \text{for } \phi \in [0, 2\pi) \\ \overline{(\gamma^{-1} \circ \alpha)}(0) &\in [0, 2\pi) \\ \overline{(\gamma^{-1} \circ \alpha)}(t + 2\pi) &= \overline{(\gamma^{-1} \circ \alpha)}(t) + 2\pi \qquad \text{for } t \in \mathbb{R}. \end{aligned} \tag{2.8}$$

We will later consider maps $X\colon D \to N$ whose trace is a weakly monotone parametrisation of Γ and for which $x = \overline{(\gamma^{-1} \circ X|_{\partial D})}$ satisfies either the three-point condition introduced by Courant in [6] fixing three boundary points, that $\xi = x - \mathrm{id}$ satisfies

$$\xi(\frac{2\pi k}{3}) = 0,\ k = 0, 1, 2, \tag{$*$}$$

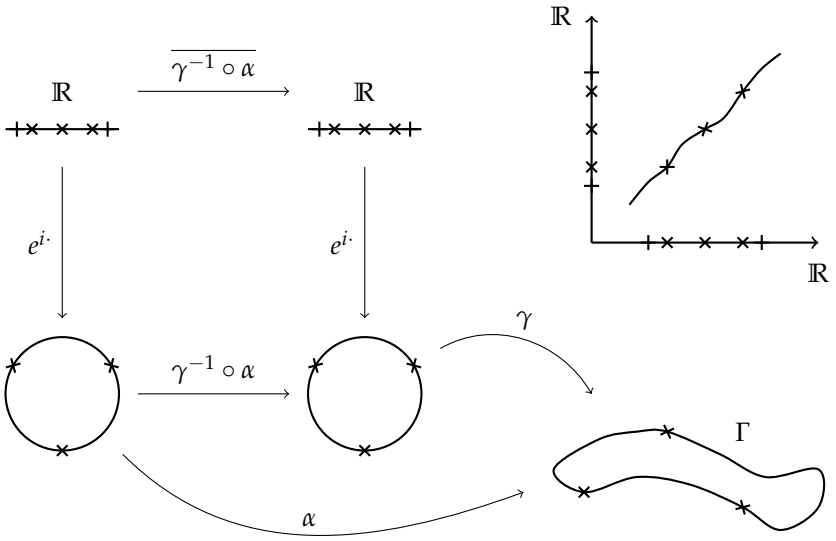


Figure 2.1: Motivation for the space M^*

or a modified version of the three-point condition used by Struwe in [38] and by Chang and Liu in [4], which asks that $\xi = x - \mathrm{id}$ satisfies

$$\int_0^{2\pi} \xi(\phi)\, d\phi = \int_0^{2\pi} \xi(\phi)\sin(\phi)\, d\phi = \int_0^{2\pi} \xi(\phi)\cos(\phi)\, d\phi = 0. \tag{$\dagger$}$$

We therefore define

$$\begin{aligned} M^* &= \{x \in C(\mathbb{R},\mathbb{R}) \text{ weakly monotone: } \xi = x - \mathrm{id} \text{ is } 2\pi\text{-periodic and satisfies } (*)\} \\ M^\dagger &= \{x \in C(\mathbb{R},\mathbb{R}) \text{ weakly monotone: } \xi = x - \mathrm{id} \text{ is } 2\pi\text{-periodic and satisfies } (\dagger)\} \end{aligned} \tag{2.9}$$

and as in [4], introduce the spaces

$$\begin{aligned} (H^1\cap C)_{\Gamma,*}(D,N) &= \{X \in H^1\cap C(D,N) : X(\partial D) = \Gamma, \overline{(\gamma^{-1}\circ X|_{\partial D})} \in M^*\} \\ (H^1\cap C)_{\Gamma,\dagger}(D,N) &= \{X \in H^1\cap C(D,N) : X(\partial D) = \Gamma, \overline{(\gamma^{-1}\circ X|_{\partial D})} \in M^\dagger\}. \end{aligned} \tag{2.10}$$

For our problem, we consider the analogous spaces of maps from the cylinder $C_0 = [-1,1]\times S^1$. Given two C^3 Jordan curves $\Gamma_\pm$ and corresponding regular C^3 parametrisations $\gamma_\pm : S^1 \to \Gamma_\pm$, we define

$$\begin{aligned} (H^1\cap C)_{\Gamma,*}(C_0,N) &= \{X \in H^1\cap C(C_0,N) : X(\partial C_\pm) = \Gamma_\pm, \overline{(\gamma_\pm^{-1}\circ X|_{\partial C_\pm})} \in M^*\} \\ (H^1\cap C)_{\Gamma,\dagger}(C_0,N) &= \{X \in H^1\cap C(C_0,N) : X(\partial C_\pm) = \Gamma_\pm, \overline{(\gamma_\pm^{-1}\circ X|_{\partial C_\pm})} \in M^\dagger\}, \end{aligned} \tag{2.11}$$

where we write $\partial C_\pm = \{\pm 1\}\times S^1$.

While the map component of the flow will evolve in this space of maps, the metric component will be constrained to the finite-dimensional manifold of metrics

$$\widetilde{\mathcal{M}} := \{h_{b,\phi}^* G_\ell : (b,\phi) \in \Omega^2 = D^2\times\mathbb{R}^2,\ \ell \in (0,\infty)\} \tag{2.12}$$

introduced in [28] which we define below, where the diffeomorphisms $h_{b,\phi}$ are chosen to compensate the imposition of a three-point condition. More precisely, we recall from [28] a curve of metrics on $C_0 = [-1,1]\times S^1$ is horizontal if $\partial_t g$ is given by the real part of a holomorphic

quadratic differential which is real on the boundary. These horizontal curves are described in terms of hyperbolic metrics G_ℓ which are given by

$$G_\ell = f_\ell^*\left(\rho_\ell^2(ds^2+d\theta^2)\right),\ f_\ell\colon C_0 \to [-X(\ell),X(\ell)]\times S^1 \text{ for } X(\ell)=\tfrac{2\pi}{\ell}\left(\tfrac{\pi}{2}-\arctan(\tfrac{\ell}{d})\right), \quad (2.13)$$

where f_ℓ are the smooth diffeomorphisms defined in Lemma 2.4 of [28], and $\rho_\ell(s)=\frac{\ell}{2\pi}\sec(\frac{\ell}{2\pi}s)$ are the associated conformal factors. Here $d>0$ is a fixed number which corresponds to the choice of uniformisation one uses to represent conformal classes by hyperbolic metrics with boundary curves of constant geodesic curvature, cf. [29], and can be characterised by

$$d = k_{G_\ell}|_{\partial C_\pm}\cdot L_{G_\ell}(\{\pm 1\}\times S^1) \quad \text{or equivalently by} \quad d^2 = L_{G_\ell}(\{\pm1\}\times S^1)^2-\ell^2. \quad (2.14)$$

Horizontal curves of metrics are then given by one-parameter families of metrics of the form $G_{\ell(t)}$, in general pulled back by a fixed diffeomorphism.

While for Teichmüller harmonic map flow from closed surfaces one evolves the metric orthogonally to all diffeomorphisms and hence by a horizontal curve, for the flow from cylinders one needs to impose a three-point condition for the map component on both boundary curves. This forces us to additionally evolve the metric component of the flow in the direction of a six-parameter family $h_{b,\phi}$ of diffeomorphisms, whose restrictions to the boundary $h_{b,\phi}|_{\{\pm1\}\times S^1}$ are given by Möbius transformations. To describe these, for $(b,\phi)\in\Omega = D\times\mathbb{R}$ we write

$$M_{b,\phi}(w) = e^{i\phi}\frac{w+b}{1+\bar b\cdot w} \quad (2.15)$$

for the corresponding Möbius transformation of the disc, and remark that if we take polar coordinates $w=re^{\mathrm{i}\theta}$ on D, we know that a given Möbius transformation $M\colon D\to D$ is determined by its action on the boundary, i.e., by the map $\hat M\colon\mathbb{R}\to\mathbb{R}$ characterised by $\hat M(0)\in[0,2\pi)$ and

$$M(w) = e^{\mathrm{i}\hat M(\theta)} \text{ for all } w=e^{\mathrm{i}\theta}\in\partial D. \quad (2.16)$$

We then choose our diffeomorphisms $h_{b,\phi}\colon C_0\to C_0$ for $(b,\phi)\in\Omega^2=D^2\times\mathbb{R}^2$ to be

$$h_{b,\phi}(s,\theta) = \begin{cases} \left(s,\lambda_1(|s|)\cdot\hat M_{b^+,\phi^+}(\theta)+\left(1-\lambda_1(|s|)\right)\cdot\left(\theta+\lambda_2(|s|)\cdot\phi^+\right)\right), & s\geq 0\\ \left(s,\lambda_1(|s|)\cdot\hat M_{b^-,\phi^-}(\theta)+\left(1-\lambda_1(|s|)\right)\cdot\left(\theta+\lambda_2(|s|)\cdot\phi^-\right)\right), & s<0, \end{cases} \quad (2.17)$$

where $\lambda_{1,2}\in C^\infty([0,1])$ are smooth cutoffs chosen to ensure independence of the vectorfields corresponding to $b^\pm$ and to $\phi^\pm$, and to avoid complications around the central geodesic. For concreteness, we may choose $\lambda_{1,2}$ to be 0 on $[0,\frac34]$ and $[0,\frac12]$ respectively, and 1 on $[\frac78,1]$ and $[\frac58,1]$ respectively.

As we can assume without loss of generality that $g(0)$ is of the form $h^*_{b_0,\phi_0}G_{\ell_0}$, the metric component of the flow is hence given by a curve $g(t)=h^*_{b(t),\phi(t)}G_{\ell(t)}$ that evolves by (1.9), with P_g denoting the $L^2(C_0,g)$-orthogonal projection onto the tangent space $\mathcal{V}(g)$ to the space of admissible metrics $\widetilde{\mathcal{M}}$. To analyse this space, let $\Gamma(TC_0)$ be the space of smooth vectorfields on C_0 which are tangential to ∂C_0 at ∂C_0, and for $(b_\infty,\phi_\infty)\in\Omega^2$, define

$$\mathcal{X}(b_\infty,\phi_\infty) = \left\{Y\in\Gamma(TC_0) : Y = \tfrac{d}{d\varepsilon}\big|_{\varepsilon=0}(h^{-1}_{b_\infty,\phi_\infty}\circ h_{b(\varepsilon),\phi(\varepsilon)}),\ (b(0),\phi(0))=(b_\infty,\phi_\infty)\right\}. \quad (2.18)$$

This consists of the vectorfields we will encounter in Lie derivatives of the pulled-back metrics below. Indeed, we can calculate that if $h_{b(\varepsilon),\phi(\varepsilon)}$ is a variation of h_{b_∞,ϕ_∞} and $g=h^*_{b_\infty,\phi_\infty}G_\ell$, then

$$\tfrac{d}{d\varepsilon}\big|_{\varepsilon=0}\left(h^*_{b(\varepsilon),\phi(\varepsilon)}G_\ell\right) = \tfrac{d}{d\varepsilon}\big|_{\varepsilon=0}\left((h^{-1}_{b_\infty,\phi_\infty}\circ h_{b(\varepsilon),\phi(\varepsilon)})^*(h^*_{b_\infty,\phi_\infty}G_\ell)\right) = L_{\frac{d}{d\varepsilon}\big|_{\varepsilon=0}\left(h^{-1}_{b_\infty,\phi_\infty}\circ h_{b(\varepsilon),\phi(\varepsilon)}\right)}g,$$

so that the tangent space to the space of metrics to $\widetilde{\mathcal{M}}$ at g decomposes L^2-orthogonally as

$$\mathcal{V}(g) = T_{h^*_{b_\infty,\phi_\infty} G_\ell}\widetilde{\mathcal{M}} = \operatorname{Re}\mathcal{H}(g) \oplus L_{\mathcal{X}(b_\infty,\phi_\infty)}g. \tag{2.19}$$

where $\mathcal{H}(g)$ is the space of holomorphic quadratic differentials on the cylinder which are real on the boundary.

We note that the above choice of diffeomorphisms will work under both the standard and integral versions of the three-point conditions, since both are designed to counteract the effect of Möbius transformations in neighbourhoods of the boundary curves.

We remark that on the boundary, all of the above vectorfields will agree by the group property of Möbius transformation. However, the diffeomorphisms $h_{b,\phi}$ themselves do not have this group property in the interior, and so the spaces $\mathcal{X}(b_\infty, \phi_\infty)$ contain different vectorfields for different $(b_\infty, \phi_\infty) \in \Omega^2$ in general.

As discussed above, in order to apply the direct method, it is important that the three-point condition provides us with weak closedness of our space of maps. Indeed, we must show that the traces of the maps will be equicontinuous. For the standard three-point condition, this is a result of the Courant–Lebesgue lemma, which we provide here in the version given by Jost as Lemma 1.1.1 in [21].

Lemma 2.2.1 (Courant–Lebesgue, [21])**.** *Let $X \in H^1(C_0, \mathbb{R}^\nu)$. Then for all $\delta \in (0,1)$ and $w \in C_0$, there exists $r \in (\delta, \sqrt{\delta})$ for which $X|_{\partial B_r(w)\cap C_0}$ is absolutely continuous. Furthermore, for every $w_1, w_2 \in \partial B_r(w) \cap C_0$, we have*

$$|X(w_1) - X(w_2)|^2 \leq \frac{8\pi}{\log \delta^{-1}} \int_{C_0} |dX|^2. \tag{2.20}$$

A standard argument found for example as Lemma 4.3 in [38, §A.4] can be used to show that maps satisfying the standard three-point condition with bounded energy must have equicontinuous traces. However, an alternative argument (also given in [38]) is needed in the case of our different three-point condition. We now extend this to our current situation.

Lemma 2.2.2. *For any compact subset K of $\widetilde{\mathcal{M}}$ and any $E_0 < \infty$, the family of maps $\{tr(X) : X \in (H^1 \cap C)_{\Gamma,\dagger}(C_0, N),\ E(X,g) \leq E_0$ for some $g \in K\}$ is equicontinuous.*

Proof. After pulling back with the diffeomorphisms (2.17), we can obtain maps u from the standard cylinder $[-Y, Y] \times S^1$. Since K is compact, the metrics will be uniformly equivalent, so it is enough to prove equicontinuity with respect to the Euclidean metric g_{E} on $[-Y, Y] \times S^1$.

We can then follow the argument of [38, pp.42–47], used there to prove the corresponding Proposition 2.8 for the disc, using Helly's selection theorem to obtain a weaker compactness result and ruling out the degeneration to a constant map by using the three-point condition.

Alternatively, if $\Gamma^\pm$ are contractible then we can apply the result for the disc directly, as discussed at Remark 2.4.4 below. □

Remark 2.2.3. We remark that while the proof is stated only for compact subsets of $\widetilde{\mathcal{M}}$, in the case that $\ell \to 0$ but still $|b| < b_0 < 1$, such results will still be valid on compact neighbourhoods of the boundary. This will be necessary to obtain our asymptotic convergence for the case in which the metric degenerates at infinite time.

In particular, if for some sequence X_j contained in one of the spaces defined in (2.10) respectively (2.11), there is $E_0 < \infty$ such that $E(X_j) \leq E_0$, then the traces $X_j|_{\partial D}$ respectively $X_j|_{\partial C_\pm}$ will be equicontinuous.

2.3 Existence of minimisers to the variational problem

In this section we show existence of a minimiser to our functional (2.7) in the associated space of maps with Plateau boundary conditions.

In order to obtain this, we first recall a result of Chang and Liu in [4] on the regularity of minimisers of this functional on subdomains $\Omega \subset C_0$ with Dirichlet boundary conditions. Since we will want to apply the result for domains obtained as subsets of the cylinder under a hyperbolic metric, we want to allow the domains to be equipped with metrics $g = \rho_g^2 g_{\mathrm{E}}$ which are conformal to the standard metric on the disc.

For such a subdomain and conformal metric g on Ω we write

$$\begin{aligned} J^h_{f,g}(X,\Omega) &= \frac{1}{2}\int_\Omega |dX|^2\, dv_g + \frac{1}{2h}\int_\Omega |X-f|^2\, dv_g \\ &= \frac{1}{2}\int_\Omega |dX|^2\, dv_{g_E} + \frac{1}{2h}\int_\Omega |X-f|^2\, \rho_g^2\, dv_{g_E}, \end{aligned} \tag{2.21}$$

where $\rho_g \in C^\infty(\Omega,(0,\infty))$ is the conformal factor associated to the change of metric. We remark that the proof given in [4] for $J^h_{f,g_{\mathrm{E}}}$ is valid for such $J^h_{f,g}$ on replacing the use of Lemma 2.2 in the proof of Lemma 2.4 of [4] with an analogue (similar to the extension given in Lemma 2.3) allowing for C^∞ weights in the second term.

Lemma 2.3.1 (Lemma 2.4, [4])**.** *Let N be a closed Riemannian manifold, and for $p \in N$ and $R \in (0, \frac{1}{3}\operatorname{inj}_N(p))$, let $B_R(p)$ be the corresponding closed geodesic ball in N. Let $\Omega \subset \mathbb{R}^2$ a bounded domain, and suppose $\psi \in C(\partial\Omega, B_R(p))$ admits an extension $\bar\psi \in H^1 \cap C(\Omega, B_R(p))$. Let $f \in L^2(C_0, N)$ and $h > 0$, and let $g = \rho_g^2 g_{\mathrm{E}}$ with $\Lambda^{-1} \le \rho_g^2 \le \Lambda$ for some $\Lambda > 0$. Suppose U_0 minimises the functional $J^h_{f,g}(X,\Omega)$ defined in (2.21) in the class*

$$\{X \in H^1(\Omega, N) : X(\Omega) \subset B_R(p),\ X|_{\partial\Omega} = \psi\}. \tag{2.22}$$

Then U_0 is continuous, with modulus of continuity depending only on N, Λ, h, the modulus of continuity of ψ, and upper bounds on $E(\bar\psi)$ and $\|f\|_{L^2}$.

We remark that since the minimisers considered in Lemma 2.3.1 are for Dirichlet boundary conditions on a closed subset, their existence is clear by the direct method. We use this result to prove the existence of a continuous minimiser of (2.7) with Plateau boundary conditions, where we use a method from [4] and [21], with the differences that our maps are now from the cylinder, and our metrics are now to be chosen from $\widetilde{\mathcal{M}}$. We stress that the main content of the result, which the bulk of the proof is devoted to, is that the minimiser can be chosen to be continuous.

Proposition 2.3.2. *Let N be a closed Riemannian manifold, and let $\Gamma_\pm$ be two disjoint closed C^3 Jordan curves which are contractible in N. Let $h > 0$, and let $g \in \widetilde{\mathcal{M}}$. Then for every $f \in L^2(C_0, N)$, the functional $J^h_{f,g}$ defined in (2.7) attains a minimiser X_0 in $(H^1 \cap C)_{\Gamma,\dagger}(C_0, N)$.*

Proof. We follow an argument from [4] and [21]. As in the proof of Lemma 2.2.2, we work on cylinders $C_Y = [-Y, Y] \times S^1$, and by pulling back by a fixed diffeomorphism $h_{b,\phi}$ as defined in (2.12), we may assume that our metric is given by $\tilde g = G_\ell = \rho_{\tilde g}^2 g_{\mathrm{E}}$ on C_Y.

Let $f \in L^2(C_Y, N)$. Given a minimising sequence $X_j \in (H^1 \cap C)_{\Gamma,\dagger}(C_Y, N)$ of $J^h_{f,\tilde g}$, we note that since $\|X_j\|_{H^1}$ is bounded, by Eberlein–Šmulian there is a subsequence converging weakly in H^1

to some $X_0 \in H^1(C_Y, N)$. If we can show that we can choose such a sequence $\widetilde{X_j}$ to be equicontinuous, then by Arzelà–Ascoli we can choose a subsequence which converges uniformly. This uniform convergence will not only imply that $X_0 \in H^1 \cap C(C_Y, N)$ but also that $X_0(\partial C_\pm)$ is a weakly monotone parametrisation of $\Gamma_\pm$. Since the modified three-point condition is preserved under uniform convergence, we conclude that $X_0 \in (H^1 \cap C)_{\Gamma,\dagger}(C_Y, N)$ is the required minimiser.

In order to obtain such a minimising sequence, we follow Jost in [21] to transform the problem to a setting in which we can apply Lemma 2.3.1. Let $R \in (0, \frac{1}{3}\operatorname{inj}_N)$ be fixed, and let $\tilde{R} > 0$ such that for all points $p \in N$ we have $N \cap \{q \in \mathbb{R}^\nu : |q-p|_{\mathbb{R}^\nu} < \tilde{R}\} \subset B_R(p)$. Let $E_0 = \sup E(X_j)$, which is finite since the X_j are minimising for $J^h_{f,\tilde{g}}$. By Courant–Lebesgue (Lemma 2.2.1), we can choose $\delta \in (0, \min\{1, Y\})$ such that for any $X \in H^1 \cap C(C_Y, N)$ with $E(X) \leq E_0$ and any $w \in C_Y$, there is $r = r(X, w) \in (\delta, \sqrt{\delta})$ such that $\operatorname{osc}_{\mathbb{R}^\nu}(X; \partial B_r(w)) < \tilde{R}$.

Fix now a finite cover $\{B_\delta(w_i)\}_{i=1}^m$ of C_Y. We will obtain the modified maps $\widetilde{X_j}$ in m steps, setting $X_{0,j} = X_j$ and constructing $X_{i+1,j}$ from $X_{i,j}$ by replacing it on a subdomain $\Omega_{i,j}$ using Lemma 2.3.1 which will ensure that the $(X_{i,j})_{j=1}^\infty$ are equicontinuous on the union $\bigcup_{k=1}^i B_\delta(w_k)$.

For $i \geq 0$, given $X_{i,j}$ we construct $X_{i+1,j}$ as follows. Let $r_{i,j} = r(X_{i,j}, w_i)$ and $\Omega_{i,j} = B_{r_{i,j}}(w_i) \cap C_Y$, so that $\operatorname{osc}_{\mathbb{R}^\nu}(X_{i,j}; \partial\Omega_{i,j}) < \tilde{R}$. We may then choose $p_{i,j} \in N$ such that $X_{i,j}(\partial\Omega_{i,j}) \subset B_R(p_{i,j})$.

Choose a minimiser U_j of $J^h_{f,\tilde{g}}(-, \Omega_{i,j})$ on $\Omega_{i,j}$ satisfying the Dirichlet condition $U_j = X_{i,j}$ on $\partial\Omega_{i,j}$ with $U_j(\Omega_{i,j}) \subset B_{2R}(p_{i,j})$. Note that our choice of R ensures that the ball $B_R(p_{i,j})$ is contractible in N, so there exists an $H^1 \cap C$ extension $\widetilde{\phi}_{i,j}$ of $X_{i,j}|_{\partial\Omega_{i,j}}$ into $\Omega_{i,j}$.

By Lemma 2.3.1, the U_j are equicontinuous, and satisfy $J^h_{f,\tilde{g}}(U_j, \Omega_{i,j}) \leq J^h_{f,\tilde{g}}(X_{i,j}, \Omega_{i,j})$ on $\Omega_{i,j}$. Define a new map $X_{i+1,j} \in (H^1 \cap C)_{\Gamma,\dagger}(C_Y, N)$ by $X_{i+1,j} = \chi_{\Omega_{i,j}} U_j + (1 - \chi_{\Omega_{i,j}}) X_{i,j}$. Then $X_{i+1,j}$ are equicontinuous on $\Omega_{i,j}$, and $J^h_{f,\tilde{g}}(X_{i+1,j}) \leq J^h_{f,\tilde{g}}(X_{i,j})$.

We note that we still have $X_{i+1,j} \in H^1 \cap C(C_Y, N)$, and its restriction to the boundary still parametrises $\Gamma_\pm$ in a weakly monotone way and satisfies the integral version of the three-point condition, since the construction does not alter the map on the boundary. In this way, for each j we obtain a map $X_{m,j}$ satisfying $J^h_{f,\tilde{g}}(X_{m,j}) \leq J^h_{f,\tilde{g}}(X_j)$. Setting $\widetilde{X_j} = X_{m,j}$, we find the desired minimising sequence. □

2.4 H^2-regularity under the integral three-point condition

In this section, we obtain H^2-regularity of the minimisers of J^h_f defined in (2.7) shown to exist in the previous section.

Proposition 2.4.1. *Let N be a closed Riemannian manifold and let $\Gamma_\pm$ be contractible in N, let $h > 0$, and let $g \in \widetilde{\mathfrak{M}}$. Suppose X is a minimiser of $J^h_{f,g}$ defined in (2.7) in the class $(H^1 \cap C)_{\Gamma,\dagger}(C_0, N)$ for some $W \in L^2(C_0, \mathbb{R}^\nu)$. Then $X \in H^2(C_0, N)$.*

We first note that the Euler–Lagrange inequality for our variational problem is given by

Proposition 2.4.2. *Suppose X is a minimiser of J^h_f defined in (2.7) for some $f \in L^2(C_0, N)$ in the class*

$(H^1 \cap C)_{\Gamma,\dagger}(C_0, N)$. *Then X satisfies the Euler–Lagrange inequality*

$$\int_{C_0} \nabla X \cdot \nabla Z + W \cdot Z \geq 0 \quad \textit{for all } Z \in T^+_X(H^1 \cap C)_{\Gamma,\dagger}(C_0, N) \tag{2.23}$$

for the map $W = h^{-1}(X - f)$, where

$$T^+_X(H^1 \cap C)_{\Gamma,\dagger}(C_0, N) = \begin{Bmatrix} Z \in (H^1 \cap C)(C_0, X^*TN) : Z|_{\partial C_\pm} \in (H^{\frac{1}{2}} \cap C)(\partial C_\pm, [X|_{\partial C_\pm}]^* T\Gamma_\pm) \\ \textit{and } Z|_{\partial C_\pm} = \gamma'_\pm(\gamma_\pm^{-1} \circ X|_{\partial C_\pm}) \cdot \xi \textit{ for some } \xi \in (\overline{\gamma_\pm^{-1} \circ X|_{\partial C_\pm}})^* TM^\dagger \end{Bmatrix}.$$

We see that since the class $T^+_X(H^1 \cap C)_{\Gamma,\dagger}(C_0, N)$ of admissible variations does not form a vector space, only a convex cone, we cannot obtain a system of PDEs with prescribed boundary values, and only a partial differential inequality for the map component. We remark that elements $Z \in T^+_X(H^1 \cap C)_{\Gamma,\dagger}(C_0, N)$ can equivalently be characterised in terms of one-sided variations $X_\varepsilon \in (H^1 \cap C)_{\Gamma,\dagger}(C_0, N)$ for $\varepsilon \in (0, \varepsilon_0)$ with $X_\varepsilon \to X$ as $\varepsilon \to 0$ and $Z = \frac{d}{d\varepsilon}\big|_{\varepsilon=0} X_\varepsilon$.

Let $X \in (H^1 \cap C)_{\Gamma,\dagger}(C_0, N)$ be a critical point of (2.7), and hence a solution of (2.23). Notice that given any $Z \in T^+_X(H^1 \cap C)_{\Gamma,\dagger}(C_0, N)$ with $\operatorname{supp}(Z) \subset (-1, 1) \times S^1$, we also have $-Z \in T^+_X(H^1 \cap C)_{\Gamma,\dagger}(C_0, N)$, as the boundary conditions are trivially satisfied for variations supported in the interior. Thus X weakly solves $\tau_g X = W$ in the interior, and so $X \in H^2(\Omega, N)$ for any compact subset $\Omega \subset (-1, 1) \times S^1$ by the classical theory for harmonic maps; see, for example, [14].

It remains to show that there is some $\eta > 0$ such that $X \in H^2(\pm[1 - \eta, 1] \times S^1, N)$. By symmetry, it is enough to consider only subcylinders $C_\eta = [1 - \eta, 1] \times S^1$. Furthermore, given any tubular neighbourhood around Γ_+ in N, we can choose η small enough that the image of C_η will be contained in this neighbourhood since our maps are continuous. We reduce this problem to the related problem on the disc, which was solved by Chang and Liu in [4].

Lemma 2.4.3 (Theorem 3.1, [4]). *Suppose $Y \in (H^1 \cap C)_{\Gamma,\dagger}(D, N)$ satisfies*

$$\int_D \nabla Y \cdot \nabla Z + \tilde{W} \cdot Z \geq 0 \quad \textit{for all } Z \in T^+_Y(H^1 \cap C)_{\Gamma_+,\dagger}(D, N), \tag{2.24}$$

for some $\tilde{W} \in L^2(D, \mathbb{R}^\nu)$, where

$$T^+_Y(H^1 \cap C)_{\Gamma_+,\dagger}(D, N) = \begin{Bmatrix} Z \in (H^1 \cap C)(D, Y^*TN) : Z|_{\partial D} \in (H^{\frac{1}{2}} \cap C)(\partial D, [Y|_{\partial D}]^* T\Gamma_+) \\ \textit{and } Z|_{\partial D} = \gamma'_+(\gamma_+^{-1} \circ Y|_{\partial D}) \cdot \xi \textit{ for some } \xi \in (\overline{\gamma_+^{-1} \circ Y|_{\partial D}})^* TM^\dagger \end{Bmatrix}.$$

Then $Y \in H^2(D, N)$.

We remark that while the result is stated for minimisers in [4], the proofs in fact only require criticality. Adapting these proofs directly for cylinders would not only allow a cleaner statement, but also allow us to drop the assumption that the boundary curves are contractible, as we remark below.

Remark 2.4.4. The assumption that the boundary curves are contractible is used only to ensure the existence of a map from the disc where we initially have a map from the cylinder. Reworking the results of Chang and Liu directly for the cylinder would allow us to drop this assumption.

Meanwhile, we show that Lemma 2.4.3 allows us to obtain our regularity result Proposition 2.4.1.

For $r \in (0, 1)$, we denote by A_r the annulus $D \setminus D^\circ_r$. Given $\eta \in (0, 1/2)$ as above, let $\beta = 1 - e^{-\eta}$, so that $\Phi \colon A_{1-\beta} \to C_\eta$ with $\Phi(re^{i\theta}) = (1 + \log r, \theta)$ is a conformal diffeomorphism.

Since X is continuous, using our assumption that Γ_+ is contractible in N in conjunction with Remark 2.4.4, for every $r \in [1-\beta, 1]$ the map $(X \circ \Phi)|_{\partial D_r}$ is homotopic to the map $(X \circ \Phi)|_{\partial D}$ whose image Γ_+ is contractible in N. We can hence extend $X \circ \Phi$ to a continuous map $\tilde{Y} \colon D \to N$ such that $\tilde{Y} = X \circ \Phi$ on $A_{1-3\beta/4}$.

For the next step, we finally use that we can consider N to be isometrically embedded into some $\mathbb{R}^\nu$ by Nash's embedding theorem. Fixing such an embedding, let $\delta_N > 0$ be small enough so that the nearest point projection Π_N into N is well-defined and smooth on the tubular neighbourhood $\{p \in \mathbb{R}^\nu : \operatorname{dist}(p, N) < \delta_N\}$. As $\tilde{Y}$ is continuous, we can choose $\varepsilon > 0$ such that for every $w \in D$, we have $\tilde{Y}(B_\varepsilon(w)) \subset B_{\delta_N}(\tilde{Y}(w)) \subset \mathbb{R}^\nu$, and hence $(\tilde{Y} * \phi_\varepsilon)(w) \in B_{\delta_N}(\tilde{Y}(w)) \subset \mathbb{R}^\nu$ for a standard mollifier ϕ_ε.

Let $\lambda_1 \in C^\infty(D, [0,1])$ be such that $\lambda_1 = 1$ on $D_{1-\beta}$ and $\lambda_1 = 0$ on $A_{1-\beta/2}$. We define a map $Y \colon D \to N$ by

$$Y = \Pi_N(\lambda_1 \cdot \tilde{Y} + (1-\lambda_1) \cdot (\tilde{Y} * \phi_\varepsilon)). \tag{2.25}$$

We will be able to apply Lemma 2.4.3 to this Y, since Y will satisfy an Euler–Lagrange inequality (2.24) for the choice $\tilde{W} = \Delta Y \cdot \rho_{(\Phi^* g)}$. Here we note that $\tilde{W}$ agrees with $(W \circ \Phi) \cdot |\det(d\Phi)|$ on $A_{1-\beta/2}$.

Lemma 2.4.5. *Let N be a closed Riemannian manifold and let $\Gamma_\pm$ be contractible in N. Suppose $X \in (H^1 \cap C)_{\Gamma,\dagger}(C_0, N)$ satisfies the Euler–Lagrange inequality (2.23) for some $W \in L^2(C_0, N)$. Then the map Y constructed in (2.25) has $Y \in (H^1 \cap C)_{\Gamma_+,\dagger}(D, N)$, and satisfies (2.24) where $\tilde{W} \in L^2(D, \mathbb{R}^\nu)$ is given by $\tilde{W} = \Delta Y \cdot \rho_{(\Phi^* g)}$.*

Proof. First note that $\tilde{W}$ is indeed in $L^2(D, \mathbb{R}^\nu)$ as the tension of an H^2 function with a smooth weight on all of D; Y is H^2 in the interior, and $\tilde{W}$ agrees with $(W \circ \Phi) \cdot |\det(d\Phi)|$ on $A_{1-\beta/2}$ for $\Phi \in C^1$. We note that we can decompose any $Z \in T^+_Y(H^1 \cap C)_{\Gamma_+,\dagger}(D, N)$ as $Z = Z_1 + Z_2$ with $\operatorname{supp}(Z_1) \subset D_{1-\beta/4}$ and $\operatorname{supp}(Z_2) \subset A_{1-\beta/2}$, for example by writing $Z = \lambda_2 \cdot Z + (1-\lambda_2) \cdot Z$ with $\lambda_2 \in C^\infty(D, [0,1])$ such that $\lambda_2 = 1$ on $D_{1-\beta/2}$ and $\lambda_2 = 0$ on $A_{1-\beta/4}$. It hence suffices to show that (2.24) is satisfied for both variations Z_1 with $\operatorname{supp}(Z_1) \subset D_{1-\beta/4}$ and Z_2 with $\operatorname{supp}(Z_2) \subset A_{1-\beta/2}$.

In the first case, Y is a weak solution of $\tau_g Y = \tilde{W}$ by construction. In the second, if $Z_2 \in T^+_Y(H^1 \cap C)_{\Gamma_+,\dagger}(D, N)$ with $\operatorname{supp}(Z_2) \subset A_{1-\beta/2}$, we see that $Z_2 \circ \Phi^{-1} \in T^+_X(H^1 \cap C)_{\Gamma,\dagger}(C_0, N)$ on comparing definitions. Thus by Proposition 2.4.2,

$$\begin{aligned}\int_D \nabla Y \cdot \nabla Z_2 + \tilde{W} \cdot Z_2 \, dv_{g_E} &= \int_{A_{1-\beta/2}} \nabla(X \circ \Phi) \cdot \nabla Z_2 + (W \circ \Phi) \cdot |\det(d\Phi)| \cdot Z_1 \, dv_{(\Phi^* g)} \\ &= \int_{C_0} \nabla X \cdot \nabla(Z_2 \circ \Phi^{-1}) + W \cdot (Z_2 \circ \Phi^{-1}) \, dv_g \geq 0. \qquad \square\end{aligned}$$

2.5 H^2-estimates for minimisers of the variational problem

As in [4], we obtain H^2-estimates for the minimiser of our variational problem. We begin by estimating first on balls where there is no concentration of energy.

Proposition 2.5.1 (H^2-estimate on cylinder)**.** *Let N be a closed Riemannian manifold and let $\Gamma_\pm$ be contractible in N. Let K be a compact subset of $\widetilde{\mathfrak{M}}$, and let $E_0 < \infty$. There exist $\varepsilon_1, r_0 > 0$ such that for all $r \in (0, r_0)$, we have the following: Let $X \in (H^1 \cap C)_{\Gamma,\dagger}(C_0, N) \cap H^2(C_0, N)$ with $E(X) \leq E_0$, let*

$g \in K$, and let $W \in L^2(C_0, \mathbb{R}^\nu)$ and $w \in C_0$. Suppose X satisfies (2.23) for this W and for all variations $Z \in T_X^+(H^1 \cap C)_{\Gamma,\dagger}(C_0, N)$ with $\mathrm{supp}(Z) \subset B_r(w)$. Suppose further that $E(X, B_r(w)) < \varepsilon_1$. Then

$$\int_{B^g_{r/2}(w)} |\nabla_g^2 X|_g^2 + |dX|_g^4 \, dv_g \leq \frac{C}{r^2} \int_{B^g_r(w)} |dX|_g^2 \, dv_g + C \int_{B^g_r(w)} |W|^2 \, dv_g, \tag{2.26}$$

for a constant C depending only on N, Γ, K, E_0, ε and r.

In order to prove this, we adapt the corresponding estimate on the disc given by Chang and Liu in [4].

Lemma 2.5.2 (Theorem 4.2, [4])**.** *Let $E_0 < \infty$. There exists $\varepsilon_1 > 0$ such that for all $r \in (0,1)$, we have the following: Let $Y \in (H^1 \cap C)_{\Gamma,\dagger}(D, N) \cap H^2(D, N)$ with $E(Y) \leq E_0$, and let $\tilde{W} \in L^2(D, \mathbb{R}^\nu)$ and $w \in D$. Suppose Y satisfies (2.24) for this $\tilde{W}$ and variations $Z \in T_Y^+(H^1 \cap C)_{\Gamma,\dagger}(D, N)$ with $\mathrm{supp}(Z) \subset B_r(w)$. Suppose further that $E(Y, B_r(w)) < \varepsilon_1$. Then*

$$\int_{B_{r/2}(w)} |\nabla^2 Y|^2 + \int_{B_{r/2}(w)} |\nabla Y|^4 \leq \frac{C}{r^2} \int_{B_r(w)} |\nabla Y|^2 + C \int_{B_r(w)} |\tilde{W}|^2, \tag{2.27}$$

for a constant C depending only on N, Γ, E_0, ε and r.

Proof of Proposition 2.5.1. Let $E_0 < \infty$. We first note that it suffices to show the result for $B_{cr}(w)$ for some fixed constant $c > 0$ in place of $B_{r/2}(w)$, since we can then cover $B_{r/2}(w)$ with a fixed number of balls of radius cr and recover the estimate (with a modified constant).

If $B_r(w) \cap \partial C_\pm = \emptyset$, then the estimate (2.26) follows by standard elliptic theory since the $g \in K$ are uniformly equivalent and since X satisfies $\tau_g X = W$ in the interior.

So suppose $B_r(w) \cap \partial C_\pm \neq \emptyset$, so that $B_r(w)$ is contained in a $2r$-neighbourhood of one of the boundary curves, say ∂C_+.

Repeating the construction of the previous section at (2.25), again using that $\Gamma_\pm$ are contractible with reference to Remark 2.4.4, we define a map $Y \colon D \to N$ which has $Y = X \circ \Phi$ on $B_r(w)$ and satisfies the assumptions of Lemma 2.5.2. Since $\Phi \in C^3$ is fixed, the result for X follows. □

Notice that these estimates are valid at any point on the boundary, in contrast to the situation if we were to impose a standard three-point condition fixing three special points as discussed in Appendix A.

Our next step is to show that energy cannot concentrate in regions around boundary points, using these estimates and for the first time using that the target supports no bubbles. To do this, we adapt an argument of Rupflin in [28] to our nonlinear setting.

Lemma 2.5.3 (No concentration)**.** *Suppose N is as in Proposition 2.5.1 and is additionally such that there are no nontrivial harmonic maps $\omega \colon S^2 \to N$, and let K be a compact subset of $\widetilde{\mathcal{M}}$. For any $\Lambda, E_0 < \infty$ and $\varepsilon > 0$, there is $r > 0$ such that the following holds:*

Let $X \in (H^1 \cap C)_{\Gamma,\dagger}(C_0, N) \cap H^2(C_0, N)$ with $E(X, g) \leq E_0$ and which satisfies (2.23) with $g \in K$ and $W \in L^2(C_0, g)$ with $\|W\|_{L^2} \leq \Lambda$. Then we have for any $w \in C_0$,

$$E(X, B_r^g(w)) \leq \varepsilon. \tag{2.28}$$

Proof. We argue by contradiction, following [28]. We first remark that since K is a compact subset of $\widetilde{\mathcal{M}}$, it will be enough to consider $g_i = h^*_{b_i,\phi_i} G_{\ell_i}$ for $|b_i^\pm| \leq b_0 < 1$ and $\ell_i > \ell_0$.

Suppose that for some $\Lambda, E_0 < \infty$, $b_0 < 1$ and $\ell_0, \varepsilon > 0$ there is a sequence (X_i, g_i, W_i) as above and a sequence $r_i \to 0$ with

$$\sup_{w \in C_0} E(X_i, B^{g_i}_{r_i}(w)) = \varepsilon. \tag{2.29}$$

We may assume that $\varepsilon < \varepsilon_1$ from Proposition 2.5.1. We choose $w_i \in C_0$ so that

$$E(X_i, B^{g_i}_{r_i}(w_i)) = \varepsilon = \max_{w \in C_0} E(u, B^{g_i}_{r_i}(w)). \tag{2.30}$$

We perform a standard blow-up argument in two cases; first where $\mathrm{dist}_{g_i}(w_i, \partial C_0)/r_i \to \infty$ for some subsequence, and then where $\mathrm{dist}_{g_i}(w_i, \partial C_0)/r_i \leq C$ for some $C < \infty$. In the first case, we rescale to maps $V_i(x) = X_i(\exp^{g_i}_{w_i}(r_i x))$ defined on larger and larger balls in $\mathbb{R}^2$. We note that the resulting metrics can be written as $(\exp^{g_i}_{w_i}(r_i \cdot))^* g_i = r_i^2 \widetilde{g}_i$ for metrics $\widetilde{g}_i$ which converge locally to g_{E}.

On compact subsets of $(\mathbb{R}^2, g_{\mathrm{E}})$, Proposition 2.5.1 gives us that the H^2-norm of the V_i are bounded uniformly, and since also $\|\tau_{g_i} X_i\|_{L^2} = \|W_i\|_{L^2}$ is bounded, we have $\tau_{\widetilde{g}_i} X_i \to 0$ locally in L^2. Thus, on passing to a subsequence we have that V_i converges strongly in H^1_{loc} and weakly in H^2_{loc} to a limit $V_\infty \colon \mathbb{R}^2 \to N$ which is harmonic with bounded energy. By the removable singularity result of Sacks–Uhlenbeck in [36], we may extend this to a harmonic map from S^2. Since our target was assumed to support no bubbles, this map must be constant, but at the same time

$$E(V_\infty, B_1(0)) = \lim_{i \to \infty} E(V_i, B_1(0), \widetilde{g}_i) = \lim_{i \to \infty} E(X_i, B^{g_i}_{r_i}(w_i)) \geq \varepsilon, \tag{2.31}$$

a contradiction.

In the second case, consider the nearest point projections $\widetilde{w}_i$ of the points w_i onto the boundary of C_0. Then on rescaling around these points $\widetilde{w}_i$ in local isothermal coordinates, we obtain maps defined on larger and larger subsets of the half-plane $\mathbb{H}$ which satisfy uniform H^2 bounds on compact sets and have energy at least ε in the ball $B_{C+1}(0) \cap \mathbb{H}$.

We again obtain a harmonic limit with bounded energy, now defined on the half-plane, and since the maps $X_i|_{\partial C_0}$ are equicontinuous from Lemma 2.2.2, this limit must be constant on $\partial \mathbb{H}$. We therefore have a contradiction in all cases. □

As an aside, we remark that in the case that N supports bubbles, then the contradiction argument above fails, as we cannot conclude that harmonic maps with bounded energy from $\mathbb{R}^2$ must be constants. However, for the construction resulting in maps from $\mathbb{H}$ which are constant on $\partial \mathbb{H}$, we have the following result of Lemaire in [22].

Lemma 2.5.4 (Theorem 3.2, [22]). *If M is a contractible compact Riemann surface with boundary, and N is a connected Riemannian manifold, then any $v \colon M \to N$ which is harmonic and constant on ∂M is constant in M.*

We therefore have a restriction on the types of bubbles that might form; in the case that N is allowed to support bubbles, we need only consider bubbles arising from the interior of M with $\mathrm{dist}_{g_i}(p_i, \partial C_0) \cdot r_i^{-1} \to \infty$, i.e., those which do not approach the boundary.

To summarise the results of this section, we give the following estimate on the full cylinder as a corollary.

Corollary 2.5.5. *In the setting of Lemma 2.5.3, we have*

$$\|X\|^2_{H^2} \leq C(\Lambda, E_0, \varepsilon, W, N, \Gamma, K). \tag{2.32}$$

2.6 Convergence of the time-discretisation scheme

We are now in a position to prove Proposition 2.1.1. Here, we follow the argument used by Rupflin in §3.4 of [28].

To obtain existence for the approximate solution on a uniform time interval $[0, T]$, we note that since we can always carry out the second step of obtaining a minimiser, it remains to show there is $T > 0$ so that for every $h \in (0, h_0)$, we always have existence for the metric equation (2.5) in the first step.

As the metric component is characterised by a system of ODEs, we hence need to make sure that the parameters b, ϕ in the diffeomorphisms described above at (2.17) remain in a compact subset of their domain, and the parameter ℓ remains bounded away from zero.

We may derive the following energy estimate for approximate solutions

$$E((u^h, g^h)(t_k)) \leq E((u^h, g^h)(t_{\tilde{k}})) - \int_{t_{\tilde{k}}}^{t_k} \left(\tfrac{1}{2}\|D_t^h u^h\|^2_{L^2(C_0, \tilde{g}^h)} + \|\partial_t g^h\|^2_{L^2(C_0, g^h)}\right) dt \tag{2.33}$$

for $\tilde{k} < k$, where $\tilde{g}^h|_{(t_k, t_{k+1})} = g^h(t_{k+1})$ is a piecewise constant curve of metrics and $D_t^h u^h = h^{-1}(u^h(t+h) - u^h(t))$ is a temporal difference quotient. Indeed, as the metric evolves by the projection of $-\nabla_g E$ onto $T_g\widetilde{\mathcal{M}}$, we have

$$E(u^h(t_k), g^h(t_k)) - E(u^h(t_k), g^h(t_{k+1})) = \int_{t_k}^{t_{k+1}} \|\partial_t g^h\|^2_{L^2(C_0, g^h(t_{k+1}))}\, dt, \tag{2.34}$$

and since $u^h(t_{k+1})$ is chosen as the minimiser of $J^h_{u^h(t_k), g^h(t_{k+1})}$, at t_{k+1} itself we see directly that

$$E(u^h(t_{k+1}), g^h(t_{k+1})) + \frac{1}{2h}\|u^h(t_{k+1}) - u^h(t_k)\|^2_{L^2(C_0, g^h(t_{k+1}))} \leq E(u^h(t_k), g^h(t_{k+1})). \tag{2.35}$$

Over a range of times indexed between $\tilde{k} < k$ as above, the estimate (2.33) results.

We first note that given $g \in \widetilde{\mathcal{M}}$, we can choose $\varepsilon > 0$ such that every curve g starting at g_0 and with $L_{L^2}(g) < \varepsilon$ remains in a compact region of $\widetilde{\mathcal{M}}$. If g^h is defined on a given time interval $[0, t]$, we can estimate the length of the curve of metrics by

$$L_{L^2}(g^h|_{[0,t]}) = \int_0^t \|\partial_t g^h\|_{L^2} \leq t^{\frac{1}{2}} \left(\int_0^t \|\partial_t g^h\|_{L^2}\right)^{\frac{1}{2}} dt \leq t^{\frac{1}{2}} E_0^{\frac{1}{2}}. \tag{2.36}$$

Choosing T small enough so that $T^{\frac{1}{2}} E_0^{\frac{1}{2}} < \varepsilon$, we see that any approximate solution (u^h, g^h) with initial data (u_0, g_0) exists on $[0, T]$ and g^h remains in a compact region of $\widetilde{\mathcal{M}}$.

For a given sequence $h_j \to 0$, we consider the corresponding approximate solutions $(u_j, g_j) = (u^{h_j}, g^{h_j})$ on such an interval $[0, T]$. Notice that for $t \in [0, T]$, $u_j(t)$ satisfies

$$\int_{C_0} \langle du_j, dv\rangle_{\tilde{g}_j(t)} + D_t^{h_j} u_j \cdot v\, dv_{\tilde{g}_j(t)} \geq 0 \quad \text{for all } v \in T^+_{u_j}(H^1 \cap C)_{\Gamma,\dagger}(C_0, N)) \tag{2.37}$$

for $\tilde{g}_j(t)$ as above, so that the results of the previous sections can be applied.

We first claim that u_j (subsequentially) converges to some limit u in $L^\infty_t L^2_x$, i.e.,

$$\sup_{t\in[0,T]} \|u_j(t) - u(t)\|_{L^2} \to 0 \quad \text{as } j \to \infty, \tag{2.38}$$

where the convergence can be understood with respect to any of the metrics $g_j(t)$, since they are uniformly equivalent as elements of the finite-dimensional smooth manifold $\widetilde{\mathcal{M}}$.

To see this, define a piecewise linear map $t \mapsto \tilde{u}_j(t)$ where $\tilde{u}_j$ agrees with u_j at every $t_k^{h_j}$. Then using the energy inequality (2.33) we can estimate

$$\begin{aligned}\|\tilde{u}_j(t_2) - \tilde{u}_j(t_1)\|_{L^2} \leq \int_{t_1}^{t_2} \|D_t^{h_j} u_j\|_{L^2}\, dt &\leq (t_2 - t_1)^{\frac{1}{2}} \left(\int_{t_1}^{t_2} \|D_t^{h_j} u^h\|_{L^2}^2\, dt \right)^{\frac{1}{2}} \\ &\leq (2E_0)^{\frac{1}{2}} (t_2 - t_1)^{\frac{1}{2}}.\end{aligned} \tag{2.39}$$

This gives uniform $C_t^{0,\frac{1}{2}} L_x^2$ bounds for $\tilde{u}_j$, and so on passing to a subsequence, we can obtain convergence in $C_t^0 L_x^2$ to a limit u. If we further estimate

$$\sup_t \|u_j(t) - \tilde{u}_j(t)\|_{L^2} \leq \sup_k \int_{t_k^{h_j}}^{t_{k+1}^{h_j}} \|D_t^{h_j} u_j\|_{L^2}\, dt \leq h_j^{\frac{1}{2}} (2E_0)^{\frac{1}{2}} \to 0 \quad \text{as } j \to \infty, \tag{2.40}$$

we obtain the convergence claimed in (2.38).

Furthermore, we see that the limiting map is in fact in $L^\infty([0,T], H^1(C_0, N))$ due to uniform energy bounds, and that $du_j \rightharpoonup du$ weakly in $L^2([0,T] \times C_0)$.

By Lemma 2.2.2, we know the traces of u_j on ∂C_0 are equicontinuous, so that for each $t \in [0,T]$,

$$u_j(t)|_{\partial C_0} \to u(t)|_{\partial C_0} \quad \text{uniformly on } \partial C_0. \tag{2.41}$$

We also have from the energy inequality uniform $L^2([0, T - h_j] \times C_0)$ bounds on $D_t^{h_j} u_j$, so that u is weakly differentiable in time on $[0,T] \times C_0$ with $D_t^{h_j} u_j \rightharpoonup \partial_t u$ in $L^2_{\text{loc}}([0,T] \times C_0)$.

While it is also clear that a limiting curve of metrics will still be admissible, we need to show that the velocity of the limiting curve of metrics g is actually given by the projections of the Hopf differentials of the limiting curve of maps u. Following [28], we will use the H^2-estimates proved above to obtain strong H^1-convergence of the map component at times we have bounds on our inhomogeneity, and therefore obtain strong L^1-convergence of the Hopf differentials.

We first define for $\Lambda < \infty$ the set of times

$$A_j^\Lambda = \{t \in [0,T] : \|D_t^{h_j} u_j\|_{L^2} \leq \Lambda\} \tag{2.42}$$

and notice that its measure satisfies $\mathcal{L}^1(A_j^\Lambda) \geq T - \frac{CE_0}{\Lambda^2}$ due to the energy inequality (2.33). If we define

$$A^\Lambda = \limsup A_j^\Lambda = \bigcap_{n=1}^{\infty} \bigcup_{j \geq n} A_j^\Lambda, \tag{2.43}$$

we therefore have too that $\mathcal{L}^1(A^\Lambda) \geq T - \frac{CE_0}{\Lambda^2}$.

We note that for suitable subsequences h_j, the sets A_j^Λ are those on which our results above are applicable, since each map $u_j(t)$ satisfies (2.23) for $W = D_t^{h_j} u_j(t)$.

To show that g solves the desired equation (2.2), we will show that it agrees with the solution $\hat{g} \in C_t^{0,1} \widetilde{\mathcal{M}}$ of the initial value problem

$$\partial_t \hat{g} = \frac{1}{4} P_{\hat{g}}(\text{Re}(\Phi(u, \hat{g}))), \quad \hat{g}(0) = g_0. \tag{2.44}$$

This curve $\hat{g}$ is guaranteed to exist on some interval $[0, \tilde{T}]$ since it is characterised as the unique solution of a system of ordinary differential equations with locally Lipschitz-continuous right-hand side, since on this interval the metrics are constrained to a compact subset of $\widetilde{\mathcal{M}}$. Possibly by reducing T, we will then obtain an approximate solution which converges to a solution of (2.2).

Fix $\varepsilon > 0$ and $\Lambda < \infty$. Then by Corollary 2.5.5, we have uniform H^2 bounds on $u_j(t)$ and $u_k(t)$ for every $t \in A_j^\Lambda \cap A_k^\Lambda$, so combined with the convergence of u_j in $C_t^0 L_x^2$, by Ehrling's lemma we can conclude there is $j_0 < \infty$ such that for every $j, k > j_0$ and every $t \in A_j^\Lambda \cap A_k^\Lambda$, we have $\|u_j(t) - u_k(t)\|_{H^1(C_0)} < \varepsilon$. This means that for all $t \in A_j^\Lambda \cap A^\Lambda$, we can find a sequence $k_n \to \infty$ so that the same bound holds for $\|u_j(t) - u(t)\|_{H^1(C_0)}$. As in Lemma 3.3 of [28], the difference in the Hopf differentials can thus be estimated as

$$\begin{aligned}\|\operatorname{Re}_{g_j} \Phi(u_j, g_j)(t) - \operatorname{Re}_{\hat{g}} \Phi(u, \hat{g})(t)\|_{L^1} &\leq CE_0 \|(g_j - \hat{g})(t)\|_{C^0} + CE_0^{\frac{1}{2}} \|(u_j - u)(t)\|_{H^1} \\ &\leq C\|(g_j - \hat{g})(t)\|_{C^0} + C\varepsilon,\end{aligned}$$

for any $t \in A_j^\Lambda \cap A^\Lambda$ and $j \geq j_0(\varepsilon, \Lambda)$, so for such t we have

$$\|\partial_t (g_j - \hat{g})(t)\|_{C^k} \leq C\|(g_j - \hat{g})(t)\|_{C^k} + C\varepsilon. \tag{2.45}$$

On the set $[0, T] \setminus (A_j^\Lambda \cap A^\Lambda)$, we can still bound $\|\partial_t g_j(t)\|_{C^k}$ and $\|\hat{g}(t)\|_{C^k}$ by CE_0 as our metrics are constrained to a compact subset of our finite-dimensional manifold, so that altogether we have

$$\|\partial_t (g_j - \hat{g})(t)\|_{C^k} \leq C\|(g_j - \hat{g})(t)\|_{C^k} + C\varepsilon + CE_0 \mathbb{1}_{[0,T]\setminus(A_j^\Lambda \cap A^\Lambda)}. \tag{2.46}$$

By Grönwall, we then have that for any $t \in [0, T]$ and $j \geq j_0$ that

$$\|(g_j - \hat{g})(t)\|_{C^k} \leq e^{CT} \int_0^t |C\varepsilon + CE_0 \mathbb{1}_{[0,T]\setminus(A_j^\Lambda \cap A^\Lambda)}| \, dt \leq C\varepsilon + \frac{C}{\Lambda^2}. \tag{2.47}$$

Taking $\Lambda \to \infty$ and $\varepsilon \to 0$, we see that $g_j \to \hat{g}$ uniformly, and so $g = \hat{g}$ indeed solves (2.2).

We now show that the pair (u, g) satisfies the first requirement (2.1) of the flow. To this end, we provide the following lemma on approximating test functions.

Lemma 2.6.1 (Continuous dependence of tangent cones). *Let $N \hookrightarrow \mathbb{R}^\nu$ compact, and let $u \in H^1_{\Gamma,\dagger}(C_0, N)$ and $u^i \in H^1_{\Gamma,\dagger}(C_0, N)$ any sequence converging strongly to u in $H^1(C_0)$. Then for any $v \in T^+_u(H^1 \cap C)_{\Gamma,\dagger}(C_0, N)$, there are $v^i \in T^+_{u^i}(H^1 \cap C)_{\Gamma,\dagger}(C_0, N)$ such that $v^i \to v$ strongly in $H^1(C_0)$.*

Proof. We have that $u|_{\partial C_\pm}, u^i|_{\partial C_\pm} \in H^{\frac{1}{2}} \cap C(\partial C_\pm, \Gamma_\pm)$ and $v|_{\partial C_\pm} \in T^+_{u|_{\partial C_\pm}} H^{\frac{1}{2}} \cap C(\partial C_\pm, \Gamma_\pm)$ and that $u^i|_{\partial C_\pm} \to u|_{\partial C_\pm}$ in $C^0 \cap H^{\frac{1}{2}}(\partial C_\pm)$ by Lemma 2.2.2.

Write $u|_{\partial C_\pm} = \gamma_\pm \circ \varphi_\pm$, $u^i|_{\partial C_\pm} = \gamma_\pm \circ \varphi^i_\pm$, and $v|_{\partial C_\pm} = \lambda_\pm \cdot \gamma'_\pm(\varphi_\pm) \cdot (\psi_\pm - \varphi_\pm)$, and define maps $\tilde{v}^i_\pm := \lambda_\pm \cdot \gamma'_\pm(\varphi^i_\pm) \cdot (\psi_\pm - \varphi^i_\pm)$, so $\tilde{v}^i_\pm \in H^{\frac{1}{2}} \cap C(\partial C_\pm, \Gamma_\pm)$ with $\tilde{v}^i_\pm \to v|_{\partial C_\pm}$ in $H^{\frac{1}{2}} \cap C(\partial C_\pm)$.

We use a tubular neighbourhood argument to obtain our sequence of maps v^i, and write Π_N for the nearest point projection from the tubular neighbourhood $N + B^\nu_{\delta_N}$ onto N, where δ_N is chosen so that the projection is well-defined.

For i sufficiently large, there are $\psi^i_\pm \in H^1(S^1 \times [0,1], B^\nu_{\delta_N} \subset \mathbb{R}^\nu)$ with $\|\psi^i_\pm\|_{H^1(S^1\times[0,1])} \to 0$ and $\mathrm{tr}_{S^1\times\partial[0,1]}(\psi^i_\pm) = (\tilde{v}^i_\pm - v|_{\partial C_\pm}, 0)$. To see this, we can take the harmonic extension of $\tilde{v}^i_\pm - v|_{\partial C_\pm}$ with a suitable cutoff.

We may then define $v^i \colon C_0 \to N$ by

$$v^i(s,\theta) = \Pi_N\big(v(s,\theta) + \psi^i_\pm(1 \mp s, \theta) \cdot \mathbb{1}_{\pm[0,1]}(s)\big). \tag{2.48}$$

By construction, $v^i \in T^+_{u^i}(H^1 \cap C)_{\Gamma,\dagger}(C_0, N)$, and we show $v^i \to v$ strongly in $H^1(C_0)$. Indeed, we may bound

$$\|v^i - v\|_{H^1(C_0)} \le \|\Pi_N\|_{C^1}\|\psi^i_\pm\|_{H^1(S^1\times[0,1])} \to 0. \qquad \square$$

We now return to the proof that the map component of our limit satisfies (2.1). Following the argument of [28] and [9], first notice that for almost every $t \in [0,T]$, there is $\Lambda < \infty$ such that $t \in A^\Lambda$. For a given such t, choose a subsequence such that $\|D^{h_j}_t u_j(t)\|_{L^2} \to \liminf_{j\to\infty} \|D^{h_j}_t u_j\|_{L^2} \le \Lambda$. Then by Corollary 2.5.5 we see that $u_j(t)$ converges to $u(t)$ weakly in $H^2(C_0)$ and strongly in $H^1(C_0)$.

Combining the uniform H^2- and $W^{1,4}$-estimates for u_j with the uniform convergence of the metrics above, we find $\tau_{g_j(t)}u_j(t) \rightharpoonup \tau_{g(t)}u(t)$ weakly in $L^2(C_0)$. Our bound on $\tau_{g_j(t)}u_j(t) = D^{h_j}_t u_j$ above then translates to

$$\|\tau_{g(t)}u(t)\|_{L^2} \le \lim_{j\to\infty} \|\tau_{g_j(t)}u_j(t)\|_{L^2} \le \Lambda. \tag{2.49}$$

Passing to the limit in the Euler–Lagrange equation using Lemma 2.6.1, we conclude that $u(t)$ solves (2.23) for a function $f(t) = \tau_{g(t)}u(t)$ whose $L^2(C_0, g)$-norm is again bounded by

$$\liminf_{j\to\infty} \|D^{h_j}_t u_j\|_{L^2(C_0,g_j)} \le \Lambda. \tag{2.50}$$

Repeating for almost every $t \in [0,T]$, we obtain a function $f \colon [0,T] \times C_0 \to N$ with

$$\begin{aligned} \|f\|^2_{L^2([0,T]\times C_0)} &\le \int_0^T \liminf_{j\to\infty} \|D^{h_j}_t u_j(t)\|^2_{L^2(C_0)}\, dt \\ &\le \liminf_{j\to\infty} \|D^{h_j}_t u_j\|^2_{L^2([0,T]\times C_0)} \le 2E_0, \end{aligned} \tag{2.51}$$

where we again use (2.33). In order to show that this function f is indeed the time derivative of u, we construct new sequences of functions

$$\begin{aligned} u^\Lambda_j(t) &= (1 - \mathbb{1}_{A^\Lambda_j})(t) \cdot u(t) + \mathbb{1}_{A^\Lambda_j}(t) \cdot u_j(t), \\ f^\Lambda_j(t) &= (1 - \mathbb{1}_{A^\Lambda_j})(t) \cdot f(t) + \mathbb{1}_{A^\Lambda_j}(t) \cdot D^{h_j}_t u_j(t). \end{aligned} \tag{2.52}$$

These have the property that $u^\Lambda_j(t)$ again satisfies (2.23) for the function $f^\Lambda_j(t)$, but in addition $\|f^\Lambda_j(t)\|_{L^2} \le \Lambda$ for every j and every $t \in A^\Lambda_j \cap A^\Lambda$ (i.e., for those t where we have $u^\Lambda_j(t) \ne u(t)$).

We will first show that for any sequence $\Lambda_j \to \infty$, we have $f^{\Lambda_j}_j(t) \rightharpoonup \partial_t u$ weakly in $L^2([0,T] \times C_0)$ as $j \to \infty$. Indeed, it suffices to show that $f^{\Lambda_j}_j(t) - D^{h_j}_t u_j \rightharpoonup 0$ since we already have

$D_t^{h_j} u_j \rightharpoonup \partial_t u$, and we see that for any $\phi \in L^2([0,T] \times C_0)$,

$$\begin{aligned}
&\int_{[0,T]\times C_0} \left(f_j^{\Lambda_j}(t) - D_t^{h_j} u_j\right) \cdot \phi \, dv_g \, dt \\
&\leq \left(\|f_j^{\Lambda_j}(t)\|_{L^2([0,T]\times C_0)} + \|D_t^{h_j} u_j\|_{L^2([0,T]\times C_0)} \right) \|\phi\|_{L^2(B_j^{\Lambda_j} \times C_0)} \\
&\leq C E_0^{\frac{1}{2}} \|\phi\|_{L^2(B_j^{\Lambda_j} \times C_0)}
\end{aligned} \tag{2.53}$$

where we write $B_j^{\Lambda_j}$ for the complement $[0,T] \setminus A_j^{\Lambda_j}$. The claimed convergence follows since the measure $\mathcal{L}^1(B_j^{\Lambda_j}) \to 0$ as $\Lambda_j \to \infty$.

We now show how this convergence will allow us to pass to the limit to obtain (2.1). By Lemma 2.6.1, we can approximate a given variation $v \in L^2([0,T], T_u^+(H^1 \cap C)_{\Gamma,\dagger}(C_0, N))$ by $v_j \in L^2([0,T], T^+_{u_j^{\Lambda_j}}(H^1 \cap C)_{\Gamma,\dagger}(C_0, N))$ in the sense that $\|dv_j - dv\|_{L^2([0,T]\times C_0)} \to 0$. In order to verify (2.1), we therefore need convergence of the terms in the appropriate version of (2.37), i.e., in

$$\int_0^T \int_{C_0} \langle du_j^{\Lambda_j}, dv_j \rangle_g + f_j^{\Lambda_j} \cdot v_j \, dv_g \geq 0. \tag{2.54}$$

In order to pass to the limit in the first term of (2.54), we show strong convergence of $du_j^{\Lambda_j}$ to du in $L^2([0,T] \times C_0)$. For fixed $\Lambda < \infty$, given $t \in A_j^\Lambda$ we have that $u_j^\Lambda(t)$ is bounded uniformly in $H^2(C_0)$. Furthermore, we know $u_j^\Lambda \rightharpoonup u$ strongly in $L^2([0,T] \times C_0)$, so by Ehrling's lemma, we can find $\varepsilon_j \to 0$ such that there are $C_j < \infty$ with $C_j \|u_j^\Lambda - u\|_{L^2([0,T]\times C_0)} \to 0$ and

$$\|u_j^\Lambda - u\|_{L^2([0,T];H^1(C_0))} \leq \varepsilon_j \|u_j^\Lambda - u\|_{L^2([0,T];H^2(C_0))} + C_j \|u_j^\Lambda - u\|_{L^2([0,T]\times C_0)}. \tag{2.55}$$

We can therefore conclude for each fixed $\Lambda < \infty$ that

$$\|du_j^\Lambda - du\|_{L^2([0,T]\times C_0)} \to 0. \tag{2.56}$$

But since the symmetric difference of the set of times where $du_j^{\Lambda_j}$ and du_j^Λ do not agree satisfies $\mathcal{L}^1(A_j^{\Lambda_j} \triangle A_j^\Lambda) \leq C(\Lambda^{-2} + \Lambda_j^{-2})$, we conclude that $\|du_j^{\Lambda_j} - du\|_{L^2([0,T]\times C_0)} \to 0$.

We now show that the energy inequality will also hold in the limit. We recall from above that for almost every $t_1 \in [0,T]$ (specifically those $t_1 \in \bigcup_{\Lambda>0} A^\Lambda$), we have strong convergence of $u_j(t_1)$ in $H^1(C_0)$. We therefore have convergence of the energies $E(u_j(t_1), g_j(t_1)) \to E(u(t_1), g(t_1))$. Furthermore, since $u_j(t)$ converges to $u(t)$ at least weakly in $H^1(C_0)$, we can pass to the limit in (2.33) to obtain

$$E(u(t_1), g(t_1)) - E(u(t_2), g(t_2)) \geq \int_{t_1}^{t_2} \tfrac{1}{2} \|\partial_t u\|^2_{L^2(C_0,g)} + \|\partial_t g\|^2_{L^2(C_0,g)} \, dt \tag{2.57}$$

for every $t_2 \in [t_1, T]$, as required. We conclude by remarking that the stationarity condition (2.4) is then a simple consequence of (2.1), resulting from variations $(u(t) \circ f_\varepsilon, g(t))$ where $u \circ f_\varepsilon \in (H^1 \cap C)_{\Gamma,\dagger}(C_0, N)$.

Chapter 3

Global solutions to Teichmüller harmonic map flow from cylinders

In this chapter, we show long-time existence for solutions of Teichmüller harmonic map flow from cylinders into non-Euclidean targets, and analyse the asymptotics of such solutions. This asymptotic convergence for the global solutions we obtain demonstrates that the flow indeed deforms given initial data into a solution of the Douglas–Plateau problem in this newly considered setting.

3.1 Long-time existence

In this section, we obtain the existence of global solutions to the flow considered in Chapter 2.

Theorem 3.1.1 (Long-time existence)**.** Let N be a complete Riemannian manifold with nonpositive sectional curvature, and let $\Gamma_\pm$ be two disjoint closed C^3 Jordan curves which are contractible in N. Let $(u_0, g_0) \in (H^1 \cap C)_{\Gamma,\dagger}(C_0, N) \times \widetilde{\mathcal{M}}$ for the spaces defined in Section 2.2. Then there exists a stationary weak solution (u, g) of Teichmüller harmonic map flow which is defined for all times and satisfies the energy inequality (2.3).

A careful reading of the short-time existence proof in Section 2.6 tells us we can always extend a solution further in time unless there is a finite-time singularity in the metric component. Furthermore, the only way in which the flow can become singular in finite time is if the metrics leave a compact subset of the space of metrics

$$\widetilde{\mathcal{M}} := \{h^*_{b,\phi} G_\ell : (b, \phi) \in \Omega^2 = D^2 \times \mathbb{R}^2,\ \ell \in (0, \infty)\}$$

introduced at (2.12). We cite the following result of Rupflin from [28] in order to rule out that a finite-time singularity can be caused by the behaviour of the diffeomorphisms $(h_{b,\phi})$ by which we pullback the metrics. We note that it applies as stated, since it is a result for curves of metrics in $\widetilde{\mathcal{M}}$.

Lemma 3.1.2 (Lemma 4.1, [28])**.** *Let $(h_{b,\phi})$ be the diffeomorphisms defined in (2.17), and suppose $g(t) = h^*_{b(t),\phi(t)} G_{\ell(t)}$, $t \in [0, T)$ is a Lipschitz continuous curve of metrics such that the diffeomorphisms $h_{b,\phi}$ become singular as $t \to T$, in the sense that (at least) one of the values $|b^\pm| \to 1$ or $|\phi^\pm| \to \infty$ as*

$t \to T$. *Then*

$$\int_0^T \|\partial_t g\|_{L^2(C_0,g)} = \infty. \tag{3.1}$$

By the energy inequality (2.3), this means that the flow from finite-energy initial data cannot have the parameters b or ϕ approaching the boundary of their domain in finite time, and thus that any degeneration must be caused by the parameter ℓ, i.e., that either $\ell \to 0$ or $\ell \to \infty$. Furthermore, we know from equation (A.7) of [28] that we can bound the evolution of ℓ for large values of ℓ along any such curve by

$$\left|\frac{d\ell}{dt}\right| \leq C\ell \|\partial_t g\|_{L^2}, \tag{3.2}$$

and hence that

$$|\log(\ell(t)) + 1| \leq |\log(\ell(0)) + 1| + C\int_0^t \|\partial_t g\|_{L^2} \leq C\left(1 + t^{\frac{1}{2}}\right), \tag{3.3}$$

so that $\ell \to \infty$ is impossible in finite time. Indeed, one can furthermore use the assumption that the boundary curves $\Gamma^{\pm}$ are disjoint to get an a priori upper bound on ℓ as equation (3.10) of [28], which even excludes the possibility that $\ell \to \infty$ at infinite time.

For negatively curved targets, or more generally targets that support no bubbles, we thus have that the only possible reason for a finite-time singularity is if the metric degenerates in the sense that $\mathrm{inj}(C_0, g(t)) \to 0$ as t approaches a finite time. While this can indeed happen if $t \to \infty$, e.g., for two maps spanning two circles at sufficient distance in Euclidean space, we will show that such a degeneration $\mathrm{inj}(C_0, g(t)) \to 0$ cannot occur in finite time. Following the work of [32], we can bound a weighted energy and show this ensures that the injectivity radius of the manifold is bounded away from zero. This result is established in [32] for negatively curved targets for surfaces without boundary, and in [28] for Euclidean targets in the case of a standard three-point condition; we need to show this main result still applies in our situation. More precisely, we provide the following lemma based on Lemma 2.4 of [32], cf. Lemma 4.4 of [28] for the corresponding adaptation. This original adaptation works identically here, since the diffeomorphisms are the same and the original result is for nonpositively curved targets. We note that the assumption there that $\|u_0\|_{L^\infty} < M$ is not required since N is closed.

Lemma 3.1.3. *Let N be a closed Riemannian manifold with nonpositive sectional curvature. Then for any $\ell_1 > 0$ and $E_0 < \infty$, there are $C < \infty$ and $\varepsilon_0 > 0$ such that the following holds.*

Let $(u_0, g_0) \in (H^1 \cap C)_{\Gamma,\dagger}(C_0, N) \times \widetilde{\mathcal{M}}$ be initial data with $E(u_0, g_0) < E_0$ and $\mathrm{inj}(C_0, g_0) \geq \ell_1$, and let (u, g) be a stationary weak solution of Teichmüller harmonic map flow (2.1)–(2.2) on a maximal interval $[0, T)$ afforded by Theorem 2.0.1. Then for $t \in [0, T)$ we have uniform bounds

$$I(t) := \int_{C_0} e(u(t), g(t))\, \rho^{-2}(t)\, dv_{g(t)} \leq C, \tag{3.4}$$

where $e(u, g) = \frac{1}{2}|du|_g^2$ is the energy density and $\rho(t)(x, \theta) = \rho_{\ell(t)}(s_{\ell(t)}(x))$ the conformal factor, and

$$\mathrm{inj}(C_0, g(t)) \geq \varepsilon_0. \tag{3.5}$$

3.2 Decomposing tangent vectors

This section is dedicated to exploring the structure of our spaces of vectorfields, which will allow us to use the three-point condition to obtain the desired asymptotic result for the flow in the

final section of the chapter. We recall the spaces $\Gamma(TC_0)_{u,\dagger}$ introduced at (2.4) of vectorfields derived from variations respecting the three-point condition, and the space $\mathcal{X}(b_\infty, \phi_\infty)$ introduced at (2.18) of vectorfields which feature in Lie derivatives obtained from variations in our space of metrics $\widetilde{\mathcal{M}}$.

We begin with the following lemma, designed to replace Lemma 4.2 of [28] in the analogous argument, which does not depend on specific properties of the target manifold. We remark that in a departure from the result in [28], due to our version of the three-point condition, the decomposition below must also depend on the map component.

Lemma 3.2.1. *Let N be a closed Riemannian manifold, and let $u \in (H^1 \cap C)_{\Gamma,\dagger}(C_0, N)$. Then for any $(b_\infty, \phi_\infty) \in \Omega^2 = D^2 \times \mathbb{R}^2$, there is a decomposition*

$$\Gamma(TC_0) = \Gamma(TC_0)_{u,\dagger} \oplus \mathcal{X}(b_\infty, \phi_\infty), \tag{3.6}$$

where $\Gamma(TC_0)_{u,\dagger}$ is the space appearing in (2.4).

Before we begin the proof, we recall some properties of tangent vectorfields relating to Möbius transformations. We will write $\mathcal{G}$ for the group of Möbius transformations of the disc, and recall that we can represent a given Möbius transformation M by its action on the boundary, namely

$$M(e^{\mathrm{i}\theta}) = e^{\mathrm{i}\hat{M}(\theta)}, \tag{3.7}$$

so we may write $\hat{M}_{b,\phi} \in \mathcal{G} \subset C^1(S^1, S^1)$ for the particular transformation given by $(b, \phi) \in \Omega = D \times \mathbb{R}$.

We may view $T_{\mathrm{id}}\mathcal{G} \subset C^1(S^1, S^1)$ as follows. We can calculate, for instance, that

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\left(\frac{w+\varepsilon}{1+\varepsilon w}\right) = \frac{1+\varepsilon w-(w+\varepsilon)w}{(1+\varepsilon w)^2}\bigg|_{\varepsilon=0} = 1-w^2, \tag{3.8}$$

for a family $M_{\varepsilon,0}$ representing variations in the (real) parameter $\operatorname{Re} b$. Considering variations in the other parameters, we see that as maps on the whole disc, the tangent space is the span of the maps $\mathrm{i}w$, $1-w^2$ and $\mathrm{i}(1+w^2)$. Indeed, we may explicitly calculate that for $(b(\varepsilon), \phi(\varepsilon)) \in \Omega$ with $(b(0), \phi(0)) = (0,0)$ we have

$$\tfrac{d}{d\varepsilon}\big|_{\varepsilon=0} M_{b(\varepsilon),\phi(\varepsilon)} = \operatorname{Re} b'(0) \cdot \tfrac{d}{d\varepsilon}\big|_{\varepsilon=0} M_{\varepsilon,0} + \operatorname{Im} b'(0) \cdot \tfrac{d}{d\varepsilon}\big|_{\varepsilon=0} M_{\mathrm{i}\varepsilon,0} + \phi'(0) \cdot \tfrac{d}{d\varepsilon}\big|_{\varepsilon=0} M_{0,\varepsilon}. \tag{3.9}$$

We may then calculate that on the boundary

$$1 - e^{2\mathrm{i}\theta} = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\left(\frac{e^{\mathrm{i}\theta}+\varepsilon}{1+\varepsilon e^{\mathrm{i}\theta}}\right) = \tfrac{d}{d\varepsilon}\big|_{\varepsilon=0}(e^{\mathrm{i}\hat{M}_{\varepsilon,0}(\theta)}) = \mathrm{i}e^{\mathrm{i}\theta}\tfrac{d}{d\varepsilon}\big|_{\varepsilon=0}\hat{M}_{\varepsilon,0}(\theta), \tag{3.10}$$

so that $\tfrac{d}{d\varepsilon}\big|_{\varepsilon=0}\hat{M}_{\varepsilon,0}(\theta) = \mathrm{i}(e^{\mathrm{i}\theta} - e^{-\mathrm{i}\theta}) = -2\sin\theta$, and in a similar way

$$T_{\mathrm{id}}\mathcal{G} = \operatorname{span}\{1, \sin\theta, \cos\theta\} \subset C^1(S^1; S^1). \tag{3.11}$$

Explicitly, for $(b(\varepsilon), \phi(\varepsilon)) \in \Omega$ with $(b(0), \phi(0)) = (0,0)$ as above,

$$\tfrac{d}{d\varepsilon}\big|_{\varepsilon=0}\hat{M}_{b(\varepsilon),\phi(\varepsilon)} = \phi'(0) - 2\operatorname{Re} b'(0)\sin\theta - 2\operatorname{Im} b'(0)\cos\theta, \tag{3.12}$$

and in particular

$$\tfrac{d}{d\varepsilon}\big|_{\varepsilon=0}\hat{M}^{-1}_{b(\varepsilon),\phi(\varepsilon)}(\theta) = \tfrac{d}{d\varepsilon}\big|_{\varepsilon=0}\hat{M}_{-b(\varepsilon)e^{i\phi(\varepsilon)},-\phi(\varepsilon)}(\theta) = -\tfrac{d}{d\varepsilon}\big|_{\varepsilon=0}\hat{M}_{b(\varepsilon),\phi(\varepsilon)}(\theta). \tag{3.13}$$

Notice that while $\mathcal{X}(b_\infty, \phi_\infty) \neq \mathcal{X}(0,0)$ in general, they are identical in a neighbourhood of ∂C_0. Indeed, on $\{\operatorname{dist}(\partial C_0, (s,\theta)) < \frac{1}{8}\}$, for $(b(\varepsilon), \phi(\varepsilon)) \in \Omega$ with $(b(0), \phi(0)) = (b_\infty, \phi_\infty)$ we have

$$h^{-1}_{b_\infty,\phi_\infty} \circ h_{b(\varepsilon),\phi(\varepsilon)}(s,\theta) = \hat{M}^{-1}_{b_\infty,\phi_\infty} \circ \hat{M}_{b(\varepsilon),\phi(\varepsilon)}(\theta) = \hat{M}_{\tilde{b}(\varepsilon),\tilde{\phi}(\varepsilon)}(\theta) \tag{3.14}$$

where $(\tilde{b}(0), \tilde{\phi}(0)) = (0,0)$. Thus every variation in $\mathcal{X}(b_\infty, \phi_\infty)$ restricts on either boundary curve to a variation in $T_{\mathrm{id}}\mathcal{G}$.

Proof of Lemma 3.2.1. The following is motivated by an argument of Struwe in [38, p.48f.]. Let N and u be as in the lemma, and let $Y \in \Gamma(TC_0)$ and $(b_\infty, \phi_\infty) \in \Omega^2$. We seek $Y_2 \in \mathcal{X}(b_\infty, \phi_\infty)$ such that $Y - Y_2 \in \Gamma(TC_0)_{u,\dagger}$.

On restriction to one of the boundary circles, say ∂C_+, in light of (3.13), such a vectorfield $Y - Y_2$ results from a variation $M^{-1}_{b_\varepsilon,\phi_\varepsilon} + \varepsilon Y|_{\partial C_+}$, where $(b_0, \phi_0) = (0,0)$.

We therefore let $\tilde{u}^\pm = \gamma_\pm^{-1} \circ u|_{\partial C_\pm} \colon S^1 \to S^1$, and define $F_u^\pm \in C^1(\mathbb{R} \times \Omega; \mathbb{R}^3)$ by

$$(\varepsilon, b, \phi) \mapsto \int_0^{2\pi} \tilde{u}^\pm \circ (\hat{M}^{-1}_{b,\phi} + \varepsilon Y|_{\partial C_\pm})(\theta) \begin{pmatrix} 1 \\ \sin\theta \\ \cos\theta \end{pmatrix} d\theta. \tag{3.15}$$

This is well-defined in a neighbourhood of $(0,0,0)$ since for $\eta \in C^1(S^1; S^1)$ with $\|d\eta\|_{L^\infty} < 1$, the map $\mathrm{id} + \eta$ is also a diffeomorphism of S^1, where we identify S^1 with $\mathbb{R}/2\pi$ where necessary.

Since $\Gamma(TC_0)_{u,\dagger}$ consists of those vectorfields respecting the three-point condition (†), we know that any vectorfields $Y - Y_2$ obtained in the above construction from variations $h_{b_\varepsilon,\phi_\varepsilon}$ for which $F_u^\pm(\varepsilon, b_\varepsilon^\pm, \phi_\varepsilon^\pm) = F_u^\pm(0,0,0) = (2\pi^2, -2\pi, 0)$ will belong to $\Gamma(TC_0)_{u,\dagger}$.

We aim to use the implicit function theorem to show that there is $\varepsilon_0 > 0$ so that there is a one-parameter family $(b_\varepsilon, \phi_\varepsilon)_\varepsilon \subset \Omega$ with $(b_0, \phi_0) = (0,0)$ such that $F_u^+(\varepsilon, b_\varepsilon, \phi_\varepsilon) = F_u^+(0,0,0)$. Moreover, we could then conclude that the mapping $\varepsilon \mapsto (b_\varepsilon, \phi_\varepsilon)$ is unique and C^1 in these neighbourhoods.

In order to apply the implicit function theorem, it suffices to show that $D_{(b,\phi)}F_u^+(0,0,0) \colon T_{(0,0)}\Omega \to \mathbb{R}^3$ is surjective, which we show below. Once we have this, by the group property of $\mathcal{G}$ we may define a new set of parameters $\tilde{b}_\varepsilon^+, \tilde{\phi}_\varepsilon^+$ such that

$$\hat{M}_{\tilde{b}_\varepsilon^+, \tilde{\phi}_\varepsilon^+} = \hat{M}_{b_\infty^+, \phi_\infty^+} \circ \hat{M}_{b_\varepsilon, \phi_\varepsilon}. \tag{3.16}$$

With the analogous construction on ∂C_-, we may define a family of diffeomorphisms $h_{\tilde{b}_\varepsilon, \tilde{\phi}_\varepsilon}$ from these families of parameters $(\tilde{b}_\varepsilon^\pm, \tilde{\phi}_\varepsilon^\pm)$. Then we have $\frac{d}{d\varepsilon}\big|_{\varepsilon=0}(h^{-1}_{b_\infty,\phi_\infty} \circ h_{\tilde{b}_\varepsilon,\tilde{\phi}_\varepsilon}) \in \mathcal{X}(b_\infty, \phi_\infty)$ by definition, and by construction, $Y - \frac{d}{d\varepsilon}\big|_{\varepsilon=0}(h^{-1}_{b_\infty,\phi_\infty} \circ h_{\tilde{b}_\varepsilon,\tilde{\phi}_\varepsilon})$ lies in the space $\Gamma(TC_0)_{u,\dagger}$, as required to establish the lemma.

It remains to show the claim. Direct calculations show that at $(v_b, v_\phi) \in T_{(0,0)}\Omega \cong \mathbb{R}^3$,

$$\begin{aligned}
D_{(b,\phi)}F_u^+(0,0,0)(v_b, v_\phi) &= \frac{d}{d\tau}\bigg|_{\tau=0} F_u^+(0, \tau v_b, \tau v_\phi) \\
&= \frac{d}{d\tau}\bigg|_{\tau=0} \int_0^{2\pi} \tilde{u}^+(\theta) \begin{pmatrix} 1 \\ \sin \hat{M}_{\tau v_b, \tau v_\phi}(\theta) \\ \cos \hat{M}_{\tau v_b, \tau v_\phi}(\theta) \end{pmatrix} \frac{d}{d\theta} \hat{M}_{\tau v_b, \tau v_\phi}(\theta)\, d\theta \\
&= \int_0^{2\pi} \tilde{u}^+(\theta) \begin{pmatrix} -2\cos\theta & 2\sin\theta & 0 \\ -2\sin 2\theta & -2\cos 2\theta & \cos\theta \\ -2\cos 2\theta & 2\sin 2\theta & -\sin\theta \end{pmatrix} \begin{pmatrix} \operatorname{Re} v_b \\ \operatorname{Im} v_b \\ v_\phi \end{pmatrix} d\theta \\
&= \begin{pmatrix} 0 & -4\pi & 0 \\ -2B & -2A & 0 \\ -2A & 2B & 2\pi \end{pmatrix} \begin{pmatrix} \operatorname{Re} v_b \\ \operatorname{Im} v_b \\ v_\phi \end{pmatrix},
\end{aligned} \tag{3.17}$$

where $A = \int_0^{2\pi} \tilde{u}^+(\theta)\cos 2\theta\, d\theta$ and $B = \int_0^{2\pi} \tilde{u}^+(\theta)\sin 2\theta\, d\theta$. It is then enough to show that for $\tilde{u}^+ \in M^\dagger$, we cannot have $B = 0$, since the determinant of this transformation is $16\pi^2 B$. However, since $\tilde{u}^+$ is weakly monotone we may calculate

$$\begin{aligned} B &= \int_0^{2\pi} \tilde{u}^+(\theta)\sin 2\theta\, d\theta = \int_0^{\frac{\pi}{2}} [\tilde{u}^+(\theta) - \tilde{u}^+(\theta+\tfrac{\pi}{2}) + \tilde{u}^+(\theta+\pi) - \tilde{u}^+(\theta+\tfrac{3\pi}{2})]\sin 2\theta\, d\theta \\ &= \int_0^{\frac{\pi}{2}} [\tilde{u}^+(\theta) - \tilde{u}^+(\theta+\tfrac{\pi}{2})]\sin 2\theta\, d\theta + \int_0^{\frac{\pi}{2}} [\tilde{u}^+(\theta+\pi) - \tilde{u}^+(\theta+\tfrac{3\pi}{2})]\sin 2\theta\, d\theta \leq 0, \end{aligned}$$

with equality only if $\tilde{u}^+$ is constant, a contradiction. □

In order to pass to the limit in the stationarity condition later, we prove the following continuity result. For given $(b,\phi) \in \Omega^2$, we write

$$\Pi_u^\dagger \colon \Gamma(TC_0) \to \Gamma(TC_0)_{u,\dagger} \quad \text{and} \quad \Pi_u^{\mathfrak{X}} \colon \Gamma(TC_0) \to \mathfrak{X}(b,\phi) \tag{3.18}$$

for the projections associated to $u \in H^1_{\Gamma,\dagger}(C_0,N)$ and $(b,\phi) \in \Omega^2$ which are well-defined in light of Lemma 3.2.1, so that $Y = \Pi_u^\dagger(Y) + \Pi_u^{\mathfrak{X}}(Y)$.

Lemma 3.2.2. *Let N be a closed Riemannian manifold, let $u_\infty \in (H^1 \cap C)_{\Gamma,\dagger}(C_0,N)$ and let $u_i \in (H^1 \cap C)_{\Gamma,\dagger}(C_0,N)$ any sequence converging strongly to u_∞ in $H^1(C_0)$. Then for any $Y \in \Gamma(TC_0)$ and any $(b,\phi) \in \Omega^2$, we have*

$$\Pi_{u_i}^\dagger(Y) \to \Pi_{u_\infty}^\dagger(Y) \quad \text{in } C^1(C_0). \tag{3.19}$$

Proof. Let Y, u_i, u_∞ and (b,ϕ) as in the lemma, and let $E_0 = 2E(u_\infty, g_E)$. We claim that there are $\delta_0 > 0$ and $r_0 > 0$ so that for every $u \in (H^1 \cap C)_{\Gamma,\dagger}(C_0,N)$ with $\|u - u_\infty\|_{H^1} < r_0$, we have

$$|\det D_{(b,\phi)} F_u^+(0,0,0)| \geq \delta_0. \tag{3.20}$$

As in the proof of Lemma 3.2.1, we see that

$$\begin{aligned} |\det D_{(b,\phi)} F_u^+(0,0,0)| &= 16\pi^2 \left| \int_0^{2\pi} \tilde{u}^+(\theta)\sin 2\theta\, d\theta \right| \\ &= \left| \det D_{(b,\phi)} F_{u_\infty}^+(0,0,0) + 16\pi^2 \int_0^{2\pi} (\tilde{u}^+(\theta) - \widetilde{u_\infty}^+(\theta))\sin 2\theta\, d\theta \right| \\ &\leq |\det D_{(b,\phi)} F_{u_\infty}^+(0,0,0)| - 64\pi^2 \|\tilde{u}^+ - \widetilde{u_\infty}^+\|_{C^0(\partial C_+)}. \end{aligned} \tag{3.21}$$

Since $E(u_i, g_E) \leq E_0$ for sufficiently large i, by Lemma 2.2.2, we have equicontinuity of the traces, so that there is $r_0 > 0$ such that if $\|u - u_\infty\|_{H^1} < r_0$, then

$$64\pi^2 \|\tilde{u}^+ - \widetilde{u_\infty}^+\|_{C^0} \leq \tfrac{1}{2} |\det D_{(b,\phi)} F_{u_\infty}^+(0,0,0)|, \tag{3.22}$$

since this right-hand side is positive. Thus the claim is satisfied for $\delta_0 = \frac{1}{2}|\det D_{(b,\phi)} F_{u_\infty}^+(0,0,0)|$.

Since we have uniform C^k bounds for F_u^+ on $B_{r_0}(u_\infty)$, namely that

$$\|F_u^+\|_{C^k} \leq \|F_{u_\infty}^+\|_{C^k} + C\|\tilde{u}^+ - \widetilde{u_\infty}^+\|_{C^0}, \tag{3.23}$$

the interval $(-\varepsilon_0, \varepsilon_0)$ provided by the implicit function theorem can therefore be taken uniform in $u \in B_{r_0}(u_\infty) \subset (H^1 \cap C)_{\Gamma,\dagger}(C_0,N)$, and as in the proof of Lemma 3.2.1 the assignment

$$B_{r_0}(u_\infty) \to C^1([-\varepsilon_0,\varepsilon_0];\Omega);\ u \mapsto (b_\varepsilon^+, \phi_\varepsilon^+)(u,Y) \tag{3.24}$$

is C^1. Since the construction of $\Pi_u^\dagger(Y)$ from the parameters $(b_\varepsilon^+, \phi_\varepsilon^+)(u,Y)$ is also smooth, we conclude that $\Pi_{u_i}^\dagger(Y) \to \Pi_{u_\infty}^\dagger(Y)$ in C^1. □

3.3 Asymptotics

We are now in a position to obtain an asymptotic result for solutions to the flow, to show that under suitable assumptions they indeed converge to solutions of the Plateau problem at infinite time, whether that be a minimal immersion parametrised over the cylinder or instead over two discs. The key technical assumption is that the three-point condition on the boundary does not degenerate in this limit.

We note that the work of Rupflin and Schrecker in [30] provides some results in the case that the three-point condition does degenerate, obtaining convergence to a solution of the Plateau problem, but needing to allow for harmonic bubbles to form at the boundary of the domain. The argument is quite involved, requiring bounds on the rate at which the three-point condition degenerates, which can be engineered by adapting the coupling of the flow equations, and furthermore requiring the boundary curves satisfy a separation condition that minimal discs spanning one boundary curve do not intersect the other curve.

However, in the absence of such a degeneration, we obtain the following result, cf. Theorem 2.7 of [28], though we note that the technical results of the previous section are necessary in order for the argument to carry over to our non-Euclidean setting.

Theorem 3.3.1 (Asymptotics)**.** Let N be a closed Riemannian manifold with nonpositive sectional curvature. Suppose that $(u,g)(t)$, $t\in[0,\infty)$, is a stationary weak solution of Teichmüller harmonic map flow which satisfies the energy inequality, and for which the three-point condition does not degenerate in the sense that $\limsup_{t\to\infty} 1-\max_{\pm}|b^{\pm}(t)|>0$. Further suppose that for almost every time, $u(t)\in H^2(C_0)$. Then there is a sequence of times $t_i\to\infty$ such that the equivalence classes $[(u,g)(t_i)]$ converge to a critical point of Area in one of the following ways, where $f_i=h_{0,2\pi n_i}$ with $n_i^{\pm}=\lfloor\phi^{\pm}(t_i)/2\pi\rfloor$.

(i) (Non-degenerate metric) If $\mathrm{inj}(C_0,g(t_i))\not\to 0$ as $i\to\infty$, then $f_i^*(u(t_i),g(t_i))$ converges to a limit (u_∞,g_∞) with $g_\infty\in\widetilde{\mathcal{M}}$ and $u_\infty\in(H^1\cap C)_{\Gamma,\dagger}(C_0,N)$ is a (possibly branched) minimal immersion. Here the convergence is smooth convergence of the metric component on all of C_0, and uniform convergence of the map component as well as strong H^1- and weak H^2-convergence on the whole cylinder C_0.

(ii) (Degenerate metric) If $\mathrm{inj}(C_0,g(t_i))\to 0$, then $f_i^*(u(t_i),g(t_i))$ converges locally on $C_0\setminus(\{0\}\times S^1)$ to a limit (u_∞,g_∞) such that:

- each of the cylinders $(C^{\pm},g_\infty)$, $C^{\pm}=\{0<\pm s\leq 1\}\times S^1$ is isometric to the hyperbolic cusp
$$\big([0,\infty)\times S^1,\rho_0(s)^2(ds^2+d\theta^2)\big),\quad \rho_0(s)=(2\pi\eta+s)^{-1}. \tag{3.25}$$
- the two maps $u_\infty|_{C^{\pm}}$ can be extended across the punctures to give two (possibly branched) minimal immersions $\tilde{u}_\infty^{\pm}\in(H^1\cap C)_{\Gamma_\pm,\dagger}(\bar{D},N)$.

Here the convergence is smooth local convergence on $C^{\pm}$ for the metrics, and locally uniform, strong H^1_{loc}- and weak H^2_{loc}-convergence on $C^{\pm}$ for the maps.

We remark that the regularity required of solutions here is the same as that of the solutions obtained in Theorem 3.1.1, so that the convergence is established for the global solutions we have shown to exist.

Proof of Theorem 3.3.1. We follow the argument presented by Rupflin in §5 of [28]. Let (u,g) be a global stationary weak solution of Teichmüller harmonic map flow with the required regularity satisfying the energy inequality and the non-degeneracy property on the three-point condition.

We first note that by the energy inequality at any choice $(t_1, t_2) = (0, t^*)$ for $t^* \in [0, \infty)$ (which we remark from the discussion at (2.57) are times at which the estimate holds), we have

$$\begin{aligned} E(u_0, g_0) \geq E(u, g)(0) - E(u, g)(t^*) &\geq c \int_0^{t^*} \|\partial_t u\|_{L^2(C_0, g(t))} + \|\partial_t g\|_{L^2(C_0, g(t))}\, dt \\ &= c \int_0^{t^*} \|\tau_{g(t)} u(t)\|_{L^2(C_0, g(t))} + \tfrac{1}{4} \|P^{\mathcal{V}}_{g(t)}(\operatorname{Re}\Phi(u, g)(t))\|_{L^2(C_0, g(t))}\, dt. \end{aligned} \tag{3.26}$$

By the non-degeneracy of the three-point condition, we may choose a sequence $\widetilde{t_i} \to \infty$ with

$$1 - |b^{\pm}(\widetilde{t_i})| \geq \frac{1}{2} \limsup_{t \to \infty} \left(1 - \max_{\pm} |b^{\pm}(t)|\right) > 0. \tag{3.27}$$

Then we know that there is $\eta > 0$ uniform in i such that for every $t^* \in [\widetilde{t_i} - \eta, \widetilde{t_i} + \eta]$, we have

$$1 - |b^{\pm}(t^*)| \geq \frac{1}{4} \limsup_{t \to \infty} \left(1 - \max_{\pm} |b^{\pm}(t)|\right). \tag{3.28}$$

We can then choose $t_i \in [\widetilde{t_i} - \eta, \widetilde{t_i} + \eta]$ for which both the stationarity condition is satisfied and by the estimate (3.26) for which

$$\|\tau_{g(t_i)} u(t_i)\|_{L^2(C_0, g(t_i))} \to 0 \tag{3.29}$$

$$\|P^{\mathcal{V}}_{g_i}(\operatorname{Re}\Phi(u, g)(t_i))\|_{L^2(C_0, g(t_i))} \to 0. \tag{3.30}$$

Furthermore, by (3.28) we also have $\liminf_{i \to \infty}(1 - |b^{\pm}(t_i)|) > 0$.

Following [28], we pull back by the diffeomorphisms f_i defined in the theorem, which ensures that the metrics are given by $g_i = h^*_{b_i, \tilde{\phi}_i} G_{\ell(t_i)}$ for parameters which remain in a compact subset of Ω^2, but still that the resulting (u_i, g_i) satisfy (3.29), (3.30), the differential inequality

$$\int_{C_0} \langle du_i, dv \rangle_{g_i} + \tau_{g_i} u_i \cdot v\, dv_{g_i} \geq 0 \quad \text{for all } v \in T^+_{u_i}(H^1 \cap C)_{\Gamma,\dagger}(C_0, N)) \tag{3.31}$$

and the stationarity equation

$$\int_{C_0} \operatorname{Re}\Phi(u_i, g_i) \cdot L_Y g_i + \tau_{g_i} u_i \cdot du_i(Y)\, dv_{g_i} = 0 \quad \text{for all } Y \in \Gamma(TC_0)_{u_i,\dagger}. \tag{3.32}$$

We first consider the non-degenerate case in which $\ell_i = \ell(t_i)$ does not tend to zero in the limit $i \to \infty$, in which we show convergence to a (branched) minimal immersion parametrised over a cylinder.

As in [28], we find that the metrics g_i converge smoothly to $g_\infty = h^*_{b_\infty, \phi_\infty} G_{\ell_\infty}$ along a subsequence for which $b_i^{\pm}, \tilde{\phi}_i^{\pm}, \ell_i$ converge to their respective limits. As Corollary 2.5.5 ensures that the maps u_i are bounded uniformly in H^2, we can pass to a further subsequence such that we furthermore have that the maps u_i converge weakly in $H^2(C_0)$ and strongly in $W^{1,p}(C_0)$ for any $p < \infty$ to a limit $u_\infty \in H^2(C_0)$. Then (3.29) gives that the Hopf differentials $\Phi(u_i, g_i)$ converge to a holomorphic limit $\Phi(u_\infty, g_\infty)$ strongly in $L^1(C_0)$.

Using the estimates on projections of quadratic differentials given in Lemma 3.3 of [28], we may still find that

$$P^{\mathcal{V}}_{g_\infty}(\operatorname{Re}\Phi(u_\infty, g_\infty)) = \lim_{i \to \infty} P^{\mathcal{V}}_{g_i}(\operatorname{Re}\Phi(u_i, g_i)) = 0 \tag{3.33}$$

where the convergence is in any C^k, so in particular

$$\int_{C_0} \operatorname{Re}\Phi(u_\infty, g_\infty) \cdot L_Y g_\infty \, dv_{g_\infty} = 0 \tag{3.34}$$

for the vectorfields $Y \in \mathfrak{X}(b_\infty, \phi_\infty)$ generating the diffeomorphisms $h_{b,\phi}$, compare (2.19).

However, we must invoke Lemma 3.2.2 in order to pass to the limit in the stationarity condition (3.32) to conclude that (3.34) holds for any $Y \in \Gamma(TC_0)_{u_\infty,\dagger}$, and hence by Lemma 3.2.1 for any $Y \in \Gamma(TC_0)$.

We note that since our convergence at the boundary is improved on that of [28] with our version of the three-point condition, we can apply Stokes' theorem directly to (3.34) to deduce that Φ_∞ must be real on the boundary. Indeed, we remark that $\frac{1}{2}L_Y g$ is the formal adjoint of the divergence operator, and that by tangentiality we can write $Y|_{\partial C_\pm} = \lambda^\pm(\theta)\frac{d}{d\theta}$ for some scalar functions $\lambda^\pm$. Thus by Stokes', we may then conclude from (3.34) that

$$\begin{aligned} &\int_{\partial C_+} \operatorname{Re}\Phi(u_\infty, g_\infty) \left(\lambda^+(\theta)\frac{d}{d\theta}\right) \rho^{-2}(\theta)\, d\theta \\ &- \int_{\partial C_-} \operatorname{Re}\Phi(u_\infty, g_\infty) \left(\lambda^-(\theta)\frac{d}{d\theta}\right) \rho^{-2}(\theta)\, d\theta = 0. \end{aligned} \tag{3.35}$$

for every $\lambda^\pm \in C^\infty(S^1)$. We note that here $\Phi_\infty = \phi_\infty\, dz^2$ for the complex coordinate z obtained on pulling back the Euclidean coordinate $x + i\theta$ along the diffeomorphism f_{ℓ_∞} of (2.13). Furthermore, since we know Φ_∞ is holomorphic, it must be of the form $\Phi_\infty = c\, dz^2$ for some $c \in \mathbb{R}$.

However, we may also conclude from (3.33) that the projection of Φ_∞ onto $\operatorname{Re}\mathcal{H}(g_\infty) \subset \mathcal{V}(g_\infty)$ must vanish, so that $\Phi_\infty \equiv 0$ and u_∞ is weakly conformal, and since it spans Γ it is nonconstant. By the corresponding result of Gulliver, Osserman and Royden in [13], u_∞ is then a possibly branched minimal immersion.

We now turn to the degenerate case in which $\ell_i \to 0$, in which we show convergence to two minimal discs.

As in [28], we can obtain the desired subconvergence indicated in the theorem, now on the subcylinders $C^\pm = \pm(0,1] \times S^1$. The same argument as above (noting that Lemma 3.2.2 applies now in the sense of Remark 2.2.3) allows us to conclude that the stationarity condition holds in the limit

$$\int_{C^\pm} \operatorname{Re}\Phi(u_\infty, g_\infty) \cdot L_Y g_\infty \, dv_{g_\infty} = 0 \tag{3.36}$$

at least for vectorfields $Y \in \Gamma(TC_0)_{u_\infty,\dagger}$ which are supported in a single subcylinder $C^\pm$.

This is sufficient to conclude as in [28] that $u_\infty|_{C^\pm}$ can be extended to weakly conformal harmonic maps from D, and hence that the resulting maps $u_\infty^\pm$ are minimal immersions. □

Chapter 4

Finite-time degeneration for variants of Teichmüller harmonic map flow

This chapter is devoted to demonstrating finite-time degeneration of solutions of Teichmüller harmonic map flow and its rescaled variant, a result published in J. Lond. Math. Soc. in joint work with my supervisor, Melanie Rupflin. This chapter consists mainly of this paper, available at [26], supplemented with some more detailed calculations, sketches, and proofs.

We remark that the results in this chapter are not based on the preceding existence theory; existence is obtained in specific settings where Plateau boundary conditions do not present issues, and indeed the existence of minimal spheres in the target is used to advantage.

4.1 Statement of results

We consider the question of whether solutions of variants of Teichmüller harmonic map flow from surfaces M to general targets can degenerate in finite time. For the original flow from closed surfaces of genus at least 2, as well as the flow from cylinders, we prove that such a finite-time degeneration must occur in situations where the image of thin collars is 'stretching out' at a rate of at least $\mathrm{inj}(M,g)^{-(\frac{1}{4}+\delta)}$, and we construct targets in which the flow from cylinders must indeed degenerate in finite time. For the rescaled Teichmüller harmonic map flow, the condition that the image stretches out is not only sufficient but also necessary and we prove the following sharp result: Solutions of the rescaled flow cannot degenerate in finite time if the image stretches out at a rate of no more than $|\log(\mathrm{inj}(M,g))|^{\frac{1}{2}}$, but must degenerate in finite time if it stretches out at a rate of at least $|\log(\mathrm{inj}(M,g))|^{\frac{1}{2}+\delta}$ for some $\delta > 0$.

We remark that these specific rates arise naturally from our estimates (cf. Lemmas 4.2.3 and 4.3.1 respectively) which ensure that known differential inequalities governing $\mathrm{inj}(M,g)^{-1}$ imply divergence (or otherwise) of this quantity in finite time.

To begin, we recall that the solution obtained in the joint works [31, 27, 18, 34] of Rupflin and Topping (respectively Huxol, Rupflin and Topping) is smooth away from finitely many singular times and we distinguish between two different types of possible finite-time singularities. If the injectivity radius remains bounded away from zero as t approaches a singular time $T < \infty$, then the metric component of the flow remains regular across T while the singularity of

the map component is caused by the bubbling off of a finite collection of harmonic spheres, a phenomenon that is well known already from the theory of classical harmonic map flow, cf. [37]. Below we will be concerned with the second type of possible finite-time singularities in which the metric degenerates due to the injectivity radius of (M, g) tending to zero as t approaches a finite time T.

The second type of finite-time singularities is excluded if the domain is a torus, in which case an equivalent flow was already introduced and studied by Ding, Li and Liu in [7], since the Teichmüller space of the torus (equipped with the Weil–Petersson metric) is complete.

For higher genus surfaces, the delicate analysis in [32] guarantees that if the target (N, g_N) has non-positive curvature, or more generally if the target supports no bubbles, then solutions of (1.9) cannot degenerate in finite time. This naturally raises the question of whether solutions of the flow can degenerate in finite time at all and if so, what mechanism can lead to such a finite-time degeneration.

We remark that a careful analysis of the flow at such potential finite-time degenerations has been carried out in the joint works [33, 34] of Rupflin and Topping and recall the following properties of solutions (u, g) of the flow: If $\mathrm{inj}(M, g(t)) \to 0$ as $t \uparrow T$ for some $T < \infty$, then there are a finite number of geodesics $\sigma_j(t)$ in $(M, g(t))$, with $\sigma_j(t)$ homotopic to $\sigma_j(t')$, whose length $\ell_j(t) = L_{g(t)}(\sigma_j(t)) \to 0$ as $t \uparrow T$. These geodesics are surrounded by long thin collar neighbourhoods $\mathcal{C}(\sigma_j(t))$ which are isometric to cylinders $\left([-X(\ell_j(t)), X(\ell_j(t))] \times S^1, \rho^2(s)(ds^2 + d\theta^2)\right)$, compare Lemma 4.2.1 below. On the thick part of these collars, and indeed on the thick part of the whole surface, the metric converges as described in [33, 34] and so does the map component, up to the usual bubbling off of harmonic spheres. The results of [34] also ensure that away from finitely many regions where further bubbles may form, the map component u maps the degenerating part of the collar very close to curves in the sense that the oscillation of u over circles $\{s\} \times S^1$ tends to zero, see [34] for details. As already seen in [32] we furthermore know that the behaviour of the map component u away from the degenerating part of a collar has hardly any influence on the evolution of the length of the central geodesic of such a thin collar.

This raises the question of what behaviour of the map component on the degenerating parts of a collar can be responsible for a degeneration of the metric. As we shall see, the driving force for a finite-time degeneration of the flow is not merely the formation of a bubble on a very thin collar in the domain but rather a very stretched-out image in the target of such a thin collar. It is hence the behaviour of the map component on the long cylinders connecting different bubble regions on the degenerating part of the collar (or connecting them to the thick part of the surface) that can force the flow to degenerate. To be more precise, we shall see that if the length of one of the curves in the target close to which these cylinders are mapped is large compared to $\ell^{-\frac{1}{4}}$, ℓ the length of the central geodesic of the corresponding collar in the domain, then finite-time degeneration must occur. Such a phenomenon can, for example, be caused by the formation of a winding bubble on the collar as we shall discuss in detail in Section 4.4, compare also [39] where the existence of winding singularities of classical harmonic map flow was established.

As a measure of how long these curves in the image are we shall consider

$$\mathscr{L}(u, \mathcal{C}(\sigma)) := \int_{-X(\ell)}^{X(\ell)} \fint_{S^1} |u_s| \, d\theta \, ds, \tag{4.1}$$

(s, θ) collar coordinates on the collar $\mathcal{C}(\sigma)$ around a simple closed geodesic σ, cf. Lemma 4.2.1.

Our first main result establishes that degeneration in finite time must occur if the length $\mathscr{L}(u, \mathcal{C}(\sigma))$ is of order $\ell^{-(\frac{1}{4}+\delta)}$ for some $\delta > 0$. We stress that we do not ask that $\mathscr{L}(u, \mathcal{C}(\sigma))$ becomes unbounded, which would already amount to imposing a priori that the solution degenerates in finite time, but just ask for a suitable relation between $\mathscr{L}(u, \mathcal{C}(\sigma))$ and the length of the central

geodesic of the corresponding collar, and of course that ℓ is below some fixed threshold. For suitable targets, such as the warped products that we construct in Section 4.4, these properties can be proven to hold true for all maps with suitable symmetries and energy bounds, which will allow us to obtain examples of finite-time degeneration in Section 4.4.

Our main results, which we state in Theorems 4.1.1 and 4.1.2 below, apply not only for solutions of the flow (1.9), respectively (1.10), on closed surfaces of genus at least 2, but also for solutions on cylinders. We recall the relevant properties of the flow (1.9) from cylinders introduced in [28], which is designed to flow to minimal surfaces with prescribed boundary curves, in Section 2.2 and note that in this case one considers (1.9) together with Plateau boundary conditions. As our analysis will be carried out in the interior of the domain, the specific nature of the boundary conditions is however irrelevant in our proofs, and all we will require of solutions of (1.9) is that they satisfy an energy inequality of the form

$$E(u(t_1),g(t_1)) - E(u(t_2),g(t_2)) \geq c \int_{t_1}^{t_2} \|\partial_t u\|^2_{L^2(M,g)} + \|\partial_t g\|^2_{L^2(M,g)}\, dt \text{ for some } c > 0 \tag{4.2}$$

for almost every $t_1 < t_2$, as expected for a gradient flow, and, in case N is non-compact, that for almost every t the image of $u(t)$ is bounded. We note that these properties are satisfied in particular for the solutions obtained in [27, 28, 34], and note that for the (unique) weak solutions of the flow on closed domains which have non-increasing energy we indeed have

$$\frac{dE}{dt} = -\int_M |\tau_g(u)|^2 + \left(\frac{\eta}{4}\right)^2 |P_g(\mathrm{Re}(\Phi(u,g)))|^2\, dv_g, \tag{4.3}$$

away from finitely many singular times.

Theorem 4.1.1. *Let M be either a closed oriented surface of genus $\gamma \geq 2$ or a cylinder, let (N, g_N) be a complete Riemannian manifold with sectional curvature bounded from above by some $\bar{\kappa} \in \mathbb{R}$, and let $E_0 < \infty$, $c_0 > 0$, and $\delta \in (0,1)$ be any fixed numbers. Then there exists a number $\bar{\ell} > 0$ depending only on E_0, c_0, δ, the genus of M (respectively the fixed constant d in the definition (2.13) of the flow on cylinders) and $\bar{\kappa}$ so that the following holds true:*
Let (u,g) be a weak solution of (1.9) with $E(u,g) \leq E_0$ which satisfies the energy inequality (4.2) and which is defined on an interval $[0,T)$ that is maximal in the sense that $T = \infty$ or $\inf_{t\in[0,T)} \mathrm{inj}(M, g(t)) = 0$. Suppose that there exist a homotopically non-trivial simple closed curve σ_0 in M, a number $\varepsilon_1 > 0$, and a time $T_1 < T$ so that for every $t \in [T_1, T)$

(i) the length $\ell(t)$ of the simple closed geodesic $\sigma(t)$ homotopic to σ_0 satisfies $\ell(t) \leq \bar{\ell}$, and
(ii) for those $t \in [T_1, T)$ for which $\|\tau_g(u)(t)\|_{L^2(M,g(t))} \leq \varepsilon_1$, we furthermore have that

$$\mathscr{L}(u(t), \mathcal{C}(\sigma(t)) \geq c_0\, \ell(t)^{-\frac{1}{4}(1+\delta)}, \qquad \text{where } \mathscr{L} \text{ is defined in (4.1).}$$

Then $T < \infty$, i.e. the solution (u,g) of the flow (1.9) must degenerate in finite time.

Here we call (u,g) a weak solution of (1.9) if g is a Lipschitz curve of hyperbolic metrics that satisfies the second equation in (1.9) for almost every t and $u \in H^1([0,T) \times M, N)$ (respectively $u \in H^1_{loc}$ if $T = \infty$) is a weak solution of the first equation in (1.9), where we may assume without loss of generality that N is isometrically embedded in some Euclidean space by Nash's embedding theorem.

While it would be interesting to know if the rate $\ell^{-(\frac{1}{4}+\delta)}$ can be improved, a much more important question is whether the stretching out of collars is not only a sufficient but also a necessary condition for finite-time degeneration of the metric, or conversely if there are solutions of (1.9) that degenerate in finite time but for which the quantity $\mathscr{L}$ introduced in (4.1) remains bounded.

For weak solutions of the rescaled flow

$$0 = \tau_g(u), \qquad \partial_t g = P_g(\mathrm{Re}(\Phi(u,g))), \tag{4.4}$$

which was introduced by Huxol in [17], we shall prove that the flow degenerates if and only if the image is stretching out, and indeed that finite-time degeneration is excluded if $\mathscr{L}(u, \mathcal{C}(\sigma))$ is controlled by $\log(\ell^{-1})^{\frac{1}{2}}$, while any growth rate of $\mathscr{L}(u, \mathcal{C}(\sigma))$ of order $\log(\ell^{-1})^{\frac{1}{2}+\delta}$, $\delta > 0$, will force the solution to degenerate in finite time. Here we say that (u, g) is a weak solution of (4.4) if the maps $u(t)$ are harmonic with bounded energy and g is a Lipschitz curve of hyperbolic metrics for which (4.4) is satisfied for almost every time.

Theorem 4.1.2. *Let M be either a closed oriented surface of genus $\gamma \geq 2$ or a cylinder and let (N, g_N) be a complete Riemannian manifold with sectional curvature bounded from above.*
Let (u, g) be a weak solution of (4.4) defined on an interval $[0, T)$ which is maximal in the sense that $T = \infty$ or $\inf_{[0,T)} \mathrm{inj}(M, g(t)) = 0$. Then the following hold true:

(i) Suppose that there exist $C_1 < \infty$ and $\bar{\ell} \in (0, 1)$ so that for every $t \in [0, T)$ and every simple closed geodesic $\sigma(t) \subset (M, g(t))$ with length $\ell(t) = L_{g(t)}(\sigma(t)) \leq \bar{\ell}$ we have

$$\mathscr{L}(u(t), \sigma(t)) \leq C_1 \log(\ell(t)^{-1})^{\frac{1}{2}}.$$

Then $T = \infty$, i.e. the flow cannot degenerate in finite time.

(ii) Conversely for any numbers $\delta, c_0 > 0$ there exists a constant $\bar{\ell} \in (0, 1)$ (with the same dependencies as in Theorem 4.1.1), so that the following holds true:
Suppose that there exists a homotopically non-trivial simple closed curve $\sigma_0 \subset M$ and a number $T_1 < T$ so that on $[T_1, T)$ we have $\ell(t) = L_{g(t)}(\sigma(t)) \leq \bar{\ell}$ and

$$\mathscr{L}(u(t), \mathcal{C}(\sigma(t))) \geq c_0 \log(\ell(t)^{-1})^{\frac{1}{2}(1+\delta)},$$

where $\sigma(t)$ is the unique geodesic in $(M, g(t))$ homotopic to σ_0. Then $T < \infty$, that is, the flow must degenerate in finite time.

In Section 4.4, we will construct settings in which our main results apply, thus establishing that finite-time degeneration does indeed occur for both variants of the flow from cylinders.

Theorem 4.1.3. *Let (N, g_N) be a complete Riemannian manifold of bounded sectional curvature as constructed in Section 4.4 and let $u_0 \colon [-1, 1] \times S^1 \to N$ and g_0 be any pair of initial map and metric as described in Lemma 4.4.3. Then the solution of (1.9) degenerates in finite time.*
Furthermore, choosing (N_c, g_{N_c}) to be a compact Riemannian manifold as constructed in the last part of Section 4.4 we have that any weak solution of the rescaled flow (4.4) with map component as described in Remark 4.4.7 must degenerate in finite time.

4.2 Proof of sufficient conditions for Teichmüller harmonic map flow

In this section we prove our main result for solutions of the flow (1.9). We initially carry out this proof for smooth solutions and explain at the end of the section what minor modifications are needed to prove the result also for weak solutions.

We first recall the classical collar lemma

Lemma 4.2.1 (Keen–Randol [25]). *Let (M, g) be a closed oriented hyperbolic surface and let σ be a simple closed geodesic of length ℓ. Then there is a neighbourhood $\mathcal{C}(\sigma)$ around σ, a so-called collar, which is isometric to the cylinder $[-X(\ell), X(\ell)] \times S^1$ equipped with the metric $\rho^2(s)(ds^2 + d\theta^2)$ where*

$$\rho(s) = \rho_\ell(s) = \frac{\ell}{2\pi}\cos\Big(\frac{\ell s}{2\pi}\Big)^{-1} \text{ and } X(\ell) = \frac{2\pi}{\ell}\left(\frac{\pi}{2} - \arctan\left(\sinh\Big(\frac{\ell}{2}\Big)\right)\right).$$

Throughout the paper we will use the following estimates on hyperbolic cylinders with $\ell \in (0, \operatorname{arsinh}(1))$ as considered in (4.14) and (4.15) that hold true for constants that depend at most on the numbers $\bar{c}_{1,2}$ from (4.15). We have a uniform upper bound on $\rho(s)$, say $\rho(s) \le C_3$, and thus also on $|\partial_s \log(\rho))| \le \rho$ allowing us in particular to bound

$$\rho(s) \le e^{C_3|s-s'|}\rho(s') \text{ for any } s, s' \in [-X(\ell), X(\ell)]. \tag{4.5}$$

For points for which ρ is small, we use instead that $|\partial_s \log(\rho))| \le \rho$ also implies that

$$e^{-1}\cdot\rho(s_0) \le \rho(s) \le e\cdot\rho(s_0) \text{ for all } s, s_0 \in [-X(\ell), X(\ell)] \text{ with } |s - s_0| \le e^{-1}\rho^{-1}(s_0). \tag{4.6}$$

Indeed, if for any s_0 the maximal number $h > 0$ for which the above estimate holds true on the interval $I = [s_0, s_0 + h]$ were less than $\min(e^{-1}\rho^{-1}(s_0), X - s_0)$, then

$$1 = |\log(\rho(s_0 + h)) - \log(\rho(s_0))| \le h \sup_I \rho \le he\rho(s_0) < 1 \tag{4.7}$$

would lead to a contradiction.

We shall also use that for any $0 \le 2\ell \le \varepsilon < \operatorname{inj}_g(X(\ell))$ the ε-thin part of such a hyperbolic collar is described in collar coordinates by $(-X_\varepsilon(\ell), X_\varepsilon(\ell)) \times S^1$, where

$$X_\varepsilon := \tfrac{2\pi}{\ell}\big[\tfrac{\pi}{2} - \arcsin\big(\sinh(\tfrac{\ell}{2})\cdot(\sinh\varepsilon)^{-1}\big)\big] \text{ and hence so that } \frac{C}{\varepsilon} \ge X(\ell) - X_\varepsilon(\ell) \ge \frac{c}{\varepsilon} \tag{4.8}$$

for some $C, c > 0$. A short calculation furthermore shows that

$$\int_0^{X(\ell)} \rho^{\frac{1}{2}} \le 2\sqrt{2}\pi\,\ell^{-\frac{1}{2}} \text{ and } \operatorname{Area}([0,s]\times S^1) = 2\pi\int_0^s \rho^2 \le 2\pi\rho(s). \tag{4.9}$$

To see these, first note that $X(\ell) < \frac{2\pi}{\ell}\frac{\pi}{2}$ for $\ell > 0$, so that $s \in [0, X(\ell)]$ implies that $\frac{\ell s}{2\pi} \in [0, \frac{\pi}{2})$. Using that $\sin x \ge \frac{x}{2}$ for $x \in [0, \frac{\pi}{2})$, we see that

$$\begin{aligned}\int_0^{X(\ell)} \rho^{\frac{1}{2}}(s)\,ds &= \int_0^{X(\ell)} \Big(\frac{2\pi}{\ell}\Big)^{-\frac{1}{2}}\cos\Big(\frac{\ell s}{2\pi}\Big)^{-\frac{1}{2}} ds = \Big(\frac{2\pi}{\ell}\Big)^{-\frac{1}{2}}\int_0^{\frac{\ell X(\ell)}{2\pi}} (\cos x)^{-\frac{1}{2}}\cdot\frac{2\pi}{\ell}\,dx \\ &\le \Big(\frac{2\pi}{\ell}\Big)^{\frac{1}{2}}\int_0^{\frac{\pi}{2}} (\cos x)^{-\frac{1}{2}}\,dx = \Big(\frac{2\pi}{\ell}\Big)^{\frac{1}{2}}\int_0^{\frac{\pi}{2}} (\sin x)^{-\frac{1}{2}}\,dx \\ &\le \Big(\frac{2\pi}{\ell}\Big)^{\frac{1}{2}}\int_0^{\frac{\pi}{2}} \Big(\frac{x}{2}\Big)^{-\frac{1}{2}} dx = 2\sqrt{2}\pi\ell^{-\frac{1}{2}}.\end{aligned} \tag{4.10}$$

Since $\sin \le 1$, we have $\tan \le \frac{1}{\cos}$ on $[0, \frac{\pi}{2})$, and so

$$\begin{aligned}\operatorname{Area}([0,s]\times S^1) &= 2\pi\int_0^s \rho^2(s')\,ds' = 2\pi\int_0^s \Big(\frac{2\pi}{\ell}\Big)^{-2}\cos\Big(\frac{\ell s'}{2\pi}\Big)^{-2} ds' \\ &= 2\pi\Big(\frac{2\pi}{\ell}\Big)^{-1}\int_0^{\frac{\ell s}{2\pi}} (\cos x)^{-2}\,dx = 2\pi\Big(\frac{2\pi}{\ell}\Big)^{-1}\tan\Big(\frac{\ell s}{2\pi}\Big) \\ &\le 2\pi\Big(\frac{2\pi}{\ell}\Big)^{-1}\cos\Big(\frac{\ell s}{2\pi}\Big)^{-1} = 2\pi\rho(s).\end{aligned} \tag{4.11}$$

We recall that the evolution of the length of a simple closed geodesic under Teichmüller harmonic map flow is essentially determined by a weighted integral of the Hopf differential

$$\Phi(u,g) = \left(|u_s|^2 - |u_\theta|^2 - 2\mathrm{i}\langle u_s, u_\theta\rangle\right)(ds + \mathrm{i}d\theta)^2 \tag{4.12}$$

of u over the corresponding collar neighbourhood $\mathcal{C}(\sigma)$. To be more precise, let (u,g) be a solution of (1.9) defined on either a closed surface of genus at least 2 or on a cylinder and let σ_0 be a simple closed curve in M which is homotopically non-trivial. If M is a closed surface of genus at least 2, then Lemma 2.3 of [32] asserts that the evolution of $\ell(t) = L_{g(t)}(\sigma(t))$, $\sigma(t)$ the unique geodesic homotopic to σ_0, under (1.9) is so that

$$\left|\frac{d}{dt}\log(\ell^{-1}) - \frac{\eta^2}{16\pi^3}\cdot\ell\int_{-X(\ell)}^{X(\ell)}\int_{S^1}\left(|u_s|^2 - |u_\theta|^2\right)\rho^{-2}\,d\theta\,ds\right| \leq C\ell\,\eta^2 E_0, \tag{4.13}$$

$(s,\theta)\in[-X(\ell),X(\ell)]\times S^1$ collar coordinates on $\mathcal{C}(\sigma)\subset M$, in which the metric is given by $g=\rho^2(ds^2+d\theta^2)$, compare Lemma 4.2.1. Here C depends only on the genus of the surface.

As remarked in Lemma 4.4 of [28] and its proof, the same estimate applies also if M is a cylinder, now for a constant C that only depends on the fixed parameter $d>0$ from the definition (2.13) of the metric component of the flow on cylinders, compare Section 2.2. For cylinders, the collar $\mathcal{C}(\sigma)$ agrees with the whole cylinder and is isometric to $\left([-X(\ell),X(\ell)]\times S^1,\rho^2(ds^2+d\theta^2)\right)$ where ρ is still given by (4.14) but now $X(\ell)=\frac{2\pi}{\ell}(\frac{\pi}{2}-\arctan(\frac{\ell}{d}))$.

To treat both situations at the same time, in the following we consider maps $u\colon \mathcal{C}(\ell)\to(N,g_N)$ that are defined on hyperbolic cylinders of the form

$$\left(\mathcal{C}(\ell),g\right)\cong\left([-X(\ell),X(\ell)]\times S^1,\rho^2(ds^2+d\theta^2)\right),\quad \rho(s)=\tfrac{\ell}{2\pi}\cos(\tfrac{\ell}{2\pi}s)^{-1} \tag{4.14}$$

where we ask that $X(\ell)>0$ is so that for some $\bar{c}_{1,2}>0$

$$\tfrac{2\pi}{\ell}\left(\tfrac{\pi}{2}-\bar{c}_1\ell\right)\geq X(\ell)\geq\tfrac{2\pi}{\ell}\left(\tfrac{\pi}{2}-\bar{c}_2\ell\right)\quad\text{if}\quad \ell\in(0,\operatorname{arsinh}(1)). \tag{4.15}$$

We note that for collars in closed hyperbolic surfaces, (4.15) holds true for universal constants $\bar{c}_{1,2}>0$, while for cylinders the constants $\bar{c}_{1,2}$ depend only on the fixed constant $d>0$ used in the definition of the metrics (2.13), compare Section 2.2 and [29].

As a first step towards the proof of Theorem 4.1.1, we show that for any map from such a hyperbolic cylinder, the weighted integral of the Hopf differential appearing in (4.13) is bounded from below by the following estimate.

Lemma 4.2.2. *Let (N,g_N) be a complete Riemannian manifold with sectional curvature bounded from above by some $\bar{\kappa}\in\mathbb{R}$. Then for any smooth map $u\colon\mathcal{C}=\mathcal{C}(\ell)\to(N,g_N)$ from a hyperbolic cylinder $(\mathcal{C},g)$ as in (4.14) and (4.15) with $\ell\in(0,\operatorname{arsinh}(1))$, and for any subset $J\subset[-X(\ell),X(\ell)]$, we have*

$$\ell\int_{J\times S^1}\left(|u_s|^2-|u_\theta|^2\right)\rho^{-2}\,d\theta\,ds\geq -C\cdot(1+\|\tau_g(u)\|^2_{L^2(\mathcal{C},g)})\cdot(1+\log(\ell^{-1})) \tag{4.16}$$

where C depends only on $\bar{\kappa}$, an upper bound E_0 on the energy of u and $\bar{c}_{1,2}>0$ from (4.15).

Combining Lemma 4.2.2 with (4.13) hence tells us that for solutions of (1.9) we may always bound

$$\frac{d}{dt}\log(\ell^{-1})\geq -C\cdot(1+\|\tau_g(u)\|^2_{L^2(\mathcal{C},g)})\cdot[1+\log(\ell^{-1})]-C\ell. \tag{4.17}$$

We shall apply this very weak lower bound at those times t where we do not have good control on the tension, and hence, in the setting of Theorem 4.1.1, do not impose a lower bound on $\mathscr{L}(u,\mathcal{C})$. Conversely, for maps u for which the assumption (ii) of Theorem 4.1.1 is satisfied, we obtain a far stronger bound.

Lemma 4.2.3. *Suppose that in the setting of Lemma 4.2.2, we have additionally that*

$$\mathscr{L}(u, \mathcal{C}(\ell)) := \int_{-X(\ell)}^{X(\ell)} \fint_{S^1} |u_s| \, d\theta \, ds \geq c_0 \, \ell^{-\frac{1}{4}(1+\delta)}, \tag{4.18}$$

for some $c_0 > 0$ and some $\delta \in (0,1)$. Then

$$\ell \int_{\mathcal{C}} \left(|u_s|^2 - |u_\theta|^2 \right) \rho^{-2} \, d\theta \, ds \geq c_1 \ell^{-\delta} - C \left(1 + \|\tau_g(u)\|^2_{L^2(\mathcal{C},g)} \right) \left(1 + \log(\ell^{-1}) \right) \tag{4.19}$$

for some $c_1 > 0$ and $C \in \mathbb{R}$ that depend only on c_0, δ and as always $\bar{\kappa}$, E_0 and $\bar{c}_{1,2} > 0$.

We can now combine Lemma 4.2.3 with (4.13) to conclude that for solutions (u, g) of (1.9) and times t with $\|\tau_{g(t)}(u(t))\|_{L^2(M,g(t))} \leq 1$ we have

$$\frac{d}{dt} \log(\ell^{-1}) \geq c_1 \, \ell^{-\delta} - C \cdot [1 + \log(\ell^{-1})] - C\ell, \tag{4.20}$$

for collars $\mathcal{C}(\sigma(t))$ around geodesics of length $\ell(t) \in (0, \operatorname{arsinh}(1))$ on which (4.18) holds. Given any numbers $c_0 > 0$ and $\delta \in (0,1)$, we can and will choose $\bar{\ell} \in (0,1)$ sufficiently small so that for $\ell \in (0, \bar{\ell})$, the first term in the above expression dominates, say so that $\frac{c_1}{2} \ell^{-\delta} \geq C[1 + \log(\ell^{-1})] + C\ell$. Hence (4.20) implies that if (u, g) is a solution of (1.9), then

$$\frac{d}{dt} \log(\ell^{-1}) \geq \frac{c_1}{2} \, \ell^{-\delta} \text{ whenever (4.18) holds, } \ell \in (0, \bar{\ell}) \text{ and } \|\tau_g(u)\|_{L^2(M,g)} \leq 1. \tag{4.21}$$

We shall give the proof of these two lemmas later and first explain how they can be used to prove Theorem 4.1.1.

Proof of Theorem 4.1.1. Let (u, g) and $\sigma(t)$ be as in Theorem 4.1.1 and let $\ell(t) = L_{g(t)}(\sigma(t))$. We let $D := \left\{ t \in [T_1, T) : \|\tau_g(u)(t)\|_{L^2(M,g(t))} \leq \varepsilon_1 \right\}$ be the set of times where the second assumption of the theorem ensures that (4.18) holds on $\mathcal{C}(\sigma(t))$. As we may assume without loss of generality that $\varepsilon_1 \leq 1$ and that $\bar{\ell} \in (0,1)$ is chosen as above we may hence bound $\frac{d}{dt} \log(\ell^{-1})$ from below according to (4.21) on D. Conversely, for times $t \notin D$ we bound $\frac{d}{dt} \log(\ell^{-1})$ using (4.17). Combined we obtain that $f(t) := 1 + \log(\ell(t)^{-1})$ satisfies a differential inequality of the form

$$f'(t) \geq \frac{c_1}{2} \ell(t)^{-\delta} \mathbb{1}_D - g(t) f(t) = c_2 \, e^{\delta f(t)} \mathbb{1}_D - g(t) f(t) \text{ on } [T_1, T). \tag{4.22}$$

Here $g = C \cdot \mathbb{1}_{D^c} \cdot (1 + \|\tau_g(u)\|_{L^2(\mathcal{C},g)})$, which we note is integrable over $[T_1, T)$, even if $T = \infty$, since the energy inequality (4.2) assures that the measure of this complementary set satisfies $\mathcal{L}^1(D^c) \leq C E_0 \, \varepsilon_1^{-2} < \infty$.

Hence $f(t) \leq F(t) := \exp\left(\int_{T_1}^t g(t') \, dt' \right) \cdot f(t) \leq e^{\|g\|_{L^1}} f(t)$ on $[T_1, T)$, which allows us to conclude from (4.22) that

$$F'(t) \geq c_2 \, e^{\delta f(t)} \mathbb{1}_D \geq c_2 \, e^{\tilde{\delta} F(t)} \mathbb{1}_D,$$

where $\tilde{\delta} = \delta e^{-\|g\|_{L^1}} > 0$. Integration over $[T_1, t)$ then yields

$$0 < e^{-\tilde{\delta} F(t)} \leq e^{-\tilde{\delta} F(T_1)} - c_2 \tilde{\delta} \cdot \mathcal{L}^1(D \cap [T_1, t)) \leq 1 - c_2 \tilde{\delta} \cdot (t - T_1 - C E_0 \, \varepsilon_1^{-2})$$

which leads to a contradiction for t sufficiently large. Hence, we cannot have that $T = \infty$, so the solution has to degenerate in finite time. □

To prove the above Lemmas 4.2.2 and 4.2.3, we shall use the following standard estimates on the angular energy and on the H^2-norm of maps from Euclidean cylinders on regions with low energy, compare, for example, [8, 18, 37]. Here and in the following we write for short $\mathscr{C}_\Lambda(s) := [s-\Lambda, s+\Lambda] \times S^1$ and $\mathscr{C}_\Lambda = \mathscr{C}_\Lambda(0)$ and will equip these cylinders with the Euclidean metric $g_E = ds^2 + d\theta^2$ unless specified otherwise.

Lemma 4.2.4 (compare, for example, [8, 18, 37]). *Let (N, g_N) be a complete Riemannian manifold with sectional curvature bounded from above by some $\bar{\kappa} \in \mathbb{R}$. Then there exist numbers $\varepsilon_0 = \varepsilon_0(\bar{\kappa}) > 0$ and $C = C(\bar{\kappa}) \in \mathbb{R}$ so that the following holds true for H^2 maps $u \colon \mathscr{C}_X \to (N, g_N)$ away from*

$$\mathcal{A} := \{s : |s| \leq X - 1 \text{ and } E(u; \mathscr{C}_1(s)) \geq \varepsilon_0\} \cup [-X, -X+1] \cup [X-1, X].$$

Let $\varphi_0 \in C_c^\infty((-1,1), \mathbb{R}^+)$ be a fixed cut-off function with $\varphi_0 \equiv 1$ on $[-\frac{1}{2}, \frac{1}{2}]$ and $|\varphi_0'| + |\varphi_0''| \leq C$. Then for any $s_0 \notin \mathcal{A}$, we have that

$$\int \varphi_0(\cdot - s_0)^2 \, |\nabla^2 u|^2 \, d\theta \, ds \leq C E(u; \mathscr{C}_1(s_0)) + C \int \varphi_0(\cdot - s_0)^2 \, |\tau_{g_E}(u)|^2 \, d\theta \, ds \tag{4.23}$$

where $\nabla^2 u$ is the (intrinsic) Hessian of the map $u \colon (\mathscr{C}_X, g_E) \to (N, g_N)$, i.e. $\nabla^2 u = \nabla du$ is computed using the connection on $T^*\mathscr{C}_X \otimes u^*TN$ induced by the Levi-Civita connections.
Furthermore, the angular energy $\vartheta(s) = \int_{\{s\} \times S^1} |u_\theta|^2 \, d\theta$ satisfies

$$\vartheta(s) \leq C E_0 \exp(-\mathrm{dist}_{g_E}(s, \mathcal{A})) + C \int_{\mathscr{C}_X} |\tau_{g_E}(u)|^2 \, e^{-|s-q|} \, dq \, d\theta, \tag{4.24}$$

for every s with $\mathrm{dist}_{g_E}(s, \mathcal{A}) \geq 1$, E_0 an upper bound on the energy of u.

We recall that the tension with respect to the hyperbolic metric $\rho^2(ds^2 + d\theta^2)$ is related to the Euclidean tension by $\tau_g(u) = \rho^{-2} \, \tau_{g_E}(u)$. We also have $\sup_{|q| \leq X(\ell)} e^{-|s-q|/2} \rho(q) \leq C\rho(s)$ for a constant C that depends only on an upper bound on ρ on the collar, and hence only on $\bar{c}_1$ from (4.15) and an upper bound on ℓ. We hence immediately obtain from (4.24) that for maps from hyperbolic collars as considered in Lemmas 4.2.2 and 4.2.3

$$\vartheta(s) \leq C \exp(-\mathrm{dist}_{g_E}(s, \mathcal{A})) + C\rho(s)^2 \int_{\mathcal{C}} |\tau_g(u)|^2 \, e^{-|s-q|/2} \, dv_g \text{ for } s \notin \mathcal{A}, \tag{4.25}$$

$\mathcal{A}$ as in the above Lemma 4.2.4, compare also [32].

In the proofs of both Lemmas 4.2.2 and 4.2.3 we will encounter weighted integrals of this angular energy and as such it will be useful to split our collar $\mathcal{C} = \mathcal{C}(\ell)$ into $\mathcal{B} \times S^1$, for

$$\mathcal{B} := \left\{ s \in [-X(\ell), X(\ell)] : \mathrm{dist}_{g_E}(s, \mathcal{A}) \geq 4 \log(\rho(s)^{-1}) + 2 \right\} \tag{4.26}$$

where (4.25) yields strong bounds on angular energies, and $\mathcal{B}^c \times S^1$, whose measure is very small compared to $X(\ell)$ if ℓ is small, compare Lemma 4.2.5 below.

We first note that given any map u from a collar as in Lemma 4.2.2 or Lemma 4.2.3 and any subset $K \subset \mathcal{B}$ we can use (4.25) to estimate

$$\begin{aligned} \ell \int_{K \times S^1} |u_\theta|^2 \rho^{-2} \, d\theta \, ds &\leq \ell \int_{\mathcal{B}} \vartheta(s) \, \rho^{-2}(s) \, ds \\ &\leq C\ell \int_{\mathcal{B}} \left(\rho^2(s) + \int_{\mathcal{C}} |\tau_g(u)|^2 \, e^{-|s-q|/2} \, dv_g \right) ds \\ &\leq C\ell (1 + \|\tau_g(u)\|^2_{L^2(\mathcal{C}, g)}), \end{aligned} \tag{4.27}$$

where we use in the last step that $\int \rho^2 \leq \mathrm{Area}(\mathcal{C}, g)$ and that the area of the collars we consider is bounded above in terms of only $\bar{c}_1$. In the setting of Lemma 4.2.2 we will use this bound to simply estimate for any given subset $J \subset [-X(\ell), X(\ell)]$ that

$$\ell \int_{(J\cap\mathcal{B})\times S^1} \left(|u_s|^2 - |u_\theta|^2\right) \rho^{-2}\, d\theta\, ds \geq -C\ell(1 + \|\tau_g(u)\|^2_{L^2(\mathcal{C})}), \tag{4.28}$$

while in the setting of Lemma 4.2.3 we will combine the above bound on the angular energy with a much stronger lower bound on $\int_{\mathcal{B}\times S^1} |u_s|^2\rho^{-2}$ that we will later derive from the main assumption (4.18) of that lemma.

In both cases, we have to consider the high-energy regions $\mathcal{B}^c$ separately from $\mathcal{B}$, and it will be useful to note that $\mathcal{B}^c$ has the following simple properties.

Lemma 4.2.5. *Let u be a map from a hyperbolic collar $\mathcal{C}$ as in Lemma 4.2.2 with energy bounded by E_0, and let $\mathcal{B}$ be defined by (4.26). Then the number of connected components of $\mathcal{B}^c$ is bounded above by $\frac{E_0}{\varepsilon_0} + 2$, and the length of each connected component I_j of $\mathcal{B}^c$ is bounded by*

$$|I_j| \leq C \cdot \left(\sup_{I_j} \log(\rho^{-1}) + 1\right), \tag{4.29}$$

for $C = 8\left(\frac{E_0}{\varepsilon_0} + 2\right)$. Furthermore, for points contained in the same connected component, we have that the conformal factor is of comparable size, namely

$$\sup_{I_j} \rho \leq C \inf_{I_j} \rho, \text{ and hence in fact } |I_j| \leq C \cdot (\inf_{I_j} \log(\rho^{-1}) + 1) \tag{4.30}$$

for a constant C that depends only on ε_0 and E_0, and the constant $\bar{c}_1$ from (4.15).

Proof of Lemma 4.2.5. The first claim immediately follows from the definition of $\mathcal{B}$ as each connected component I_j of $\mathcal{B}^c$ must either contain $\pm X(\ell)$ or an interval of length 2 with energy at least ε_0. Similarly, for each connected component I_j of $\mathcal{B}^c$, we can choose a maximal family of disjoint intervals $I_j^k \subset I_j$ of length 2 with energy at least ε_0 and note that the total number of such intervals is bounded by $\frac{E_0}{\varepsilon_0}$. As each point in I_j will have distance at most $4\sup_{I_j} \log(\rho^{-1}) + 3$ from one of these intervals or from one of the endpoints $\pm X(\ell)$, we hence obtain that (4.29) holds. To prove (4.30), we first recall that since $|\partial_s \log(\rho))| \leq \rho$ we have $e^{-1}\rho(s_0) \leq \rho(s) \leq e\rho(s_0)$ for any points $s, s_0 \in [-X(\ell), X(\ell)]$ with $|s - s_0| \leq e^{-1}\rho(s_0)^{-1}$, compare (4.6). If $\sup_{I_j} \rho^{-1}$ is large enough so that $|I_j| \leq C(\log(\sup_{I_j} \rho^{-1}) + 1)$ is smaller then $e^{-1}\sup_{I_j} \rho^{-1}$, we hence obtain that (4.30) holds true for $C = e$. Conversely, for any other connected component I_j of $\mathcal{B}$ we have a uniform upper bound on $\sup_{I_j} \rho^{-1}$ (depending only on ε_0 and E_0) and hence a uniform upper bound on $|I_j|$, so (4.30) follows from (4.5). □

We are now in a position to complete the proofs of Lemmas 4.2.2 and 4.2.3.

Proof of Lemma 4.2.2. We first note that if $\ell \geq \ell_1$, for some number $\ell_1 > 0$ determined below, then (4.16) is trivially satisfied as $\rho^{-2} \leq 4\pi^2\ell_1^{-2}$. So, assume that $\ell \in (0, \ell_1)$. In light of (4.28), it is enough to show that for any given subset $J \subset [-X(\ell), X(\ell)]$ and each j

$$\ell \int_{(J\cap I_j)\times S^1} \left(|u_s|^2 - |u_\theta|^2\right) \rho^{-2}\, d\theta\, ds \geq -C \cdot (1 + \|\tau_g(u)\|^2_{L^2(\mathcal{C},g)}) \cdot (1 + \log(\ell^{-1}))$$

where I_j are the connected components of the set $\mathcal{B}^c$ whose properties were discussed in Lemma 4.2.5 and whose measure satisfies in particular $\mathcal{L}^1(\mathcal{B}^c) \leq C(1 + \log(\ell^{-1})) < 2X(\ell)$, provided ℓ_1

is chosen suitably small. So, $\mathcal{B}$ must be non-empty and hence each connected component I_j of $\mathcal{B}^c$ has at least one endpoint s_j which is an element of $\mathcal{B}$. Writing for short

$$\psi(s) := \int_{\{s\}\times S^1} |u_s|^2 - |u_\theta|^2 \, d\theta, \tag{4.31}$$

we can therefore bound for each $s \in I_j$

$$\begin{aligned}\psi(s) &\geq \psi(s_j) - \int_{I_j} |\partial_s \psi| \, ds \geq -\vartheta(s_j) - \int_{I_j} |\partial_s \psi| \, ds \\ &\geq -C\rho^4(s_j) - C\rho^2(s_j)\|\tau_g(u)\|_{L^2}^2 - \int_{I_j} |\partial_s \psi| \, ds,\end{aligned}$$

where we applied (4.25) in the last step and where norms are computed over the corresponding collar. Using (4.30), $|I_j| \leq C(\log(\ell^{-1}) + 1)$ and that $\ell\rho^{-1} \leq 2\pi$, we may hence bound

$$\begin{aligned}\ell \int_{J\cap I_j} \psi(s)\rho^{-2}(s) \, ds &\geq -C\ell|I_j| \cdot \rho^2(s_j) - C\ell|I_j| \cdot \|\tau_g(u)\|_{L^2}^2 - |I_j| \int_{I_j} \rho^{-1}|\partial_s \psi| \\ &\geq -C(1 + \log(\ell^{-1})) \cdot \left[1 + \|\tau_g(u)\|_{L^2}^2 + \int_{I_j} \rho^{-1}|\partial_s \psi|\right].\end{aligned} \tag{4.32}$$

We now complete the proof of Lemma 4.2.2 by proving that $\int \rho^{-1}|\partial_s \psi| \leq C(1 + \|\tau_g(u)\|_{L^2}^2)$. To this end, we note that ψ can be viewed as an integral of the real part of the function

$$\phi = |u_s|^2 - |u_\theta|^2 - 2\mathrm{i}\langle u_s, u_\theta\rangle \tag{4.33}$$

which represents the Hopf differential $\Phi = \phi \, dz^2$ of u in collar coordinates. As the antiholomorphic derivative of the Hopf differential is controlled in terms of the (Euclidean) tension, in particular $|\partial_s \operatorname{Re}(\phi) - \partial_\theta \operatorname{Im}(\phi)| = |\operatorname{Re}(\bar{\partial}\phi)| \leq 2\left|\tau_{g_{\mathrm{E}}}(u)\right| |du|_{g_{\mathrm{E}}}$, we have that

$$\begin{aligned}\int_{-X(\ell)}^{X(\ell)} \rho^{-1}|\partial_s \psi| \, ds &\leq 2\int_{\mathcal{C}} \rho^{-1} \left|\tau_{g_{\mathrm{E}}}(u)\right| |du|_{g_{\mathrm{E}}} \, d\theta \, ds \\ &\leq CE_0^{\frac{1}{2}} \|\tau_g(u)\|_{L^2(\mathcal{C},g)} \leq C(1 + \|\tau_g(u)\|_{L^2(\mathcal{C},g)}^2)\end{aligned}$$

where we used $\tau_g(u) = \rho^{-2}\,\tau_{g_{\mathrm{E}}}(u)$ in the penultimate step. Together with (4.28) and (4.32), this completes the proof of the lemma. □

Proof of Lemma 4.2.3. We first remark that it suffices to consider the case that $\ell \in (0, \ell_2)$ for a constant $\ell_2 > 0$ chosen below, as for larger values of ℓ the claim already follows from Lemma 4.2.2. To prove the lower bound (4.19) on the weighted integral $\int \rho^{-2}\psi \, ds$,

$$\psi := \int_{\{s\}\times S^1} |u_s|^2 - |u_\theta|^2 \, d\theta \tag{4.34}$$

as above, we divide $[-X(\ell), X(\ell)] = K \cup K^c$ into the subset $K \subset \mathcal{B}$ defined below, and its complement K^c on which we shall simply apply Lemma 4.2.2 to bound

$$\ell \int_{K^c} \rho^{-2}\psi \, ds \geq -C \cdot (1 + \|\tau_g(u)\|_{L^2(\mathcal{C},g)}^2) \cdot (1 + \log(\ell^{-1})). \tag{4.35}$$

More precisely, we define $K = \{s \in \mathcal{B} : \operatorname{inj}_g(s, \cdot) < \varepsilon(\ell)\}$, for $\varepsilon(\ell) := \Lambda \cdot \ell^{\frac{1}{2}(1+\delta)}$ and $\Lambda > 0$ determined below, which we know is non-empty for $\ell \in (0, \ell_2)$ since $\delta < 1$, provided ℓ_2 is

sufficiently small. We will prove below that this choice of $\varepsilon(\ell)$ ensures that the (Lebesgue) measure of K^c is small enough so that the assumption (4.18) implies that also

$$\frac{1}{2\pi}\int_{K\times S^1}|u_s| \geq \frac{c_0}{2}\ell^{-\frac{1}{4}(1+\delta)}. \tag{4.36}$$

At the same time the hyperbolic area of $K\times S^1$ is also small, namely

$$\int_{K\times S^1}\rho^2 = \mathrm{Area}(K\times S^1, g) \leq \mathrm{Area}(\varepsilon(\ell)\text{-thin}(\mathcal{C}), g) \leq C\varepsilon(\ell) \leq C\ell^{\frac{1}{2}(1+\delta)},$$

compare e.g. [35, (A.2)]. Combined, these two estimates imply that

$$\ell\int_{K\times S^1}|u_s|^2\rho^{-2}\,d\theta\,ds \geq \ell\Big(\int_{K\times S^1}\rho^2\,d\theta\,ds\Big)^{-1}\cdot\Big(\int_{K\times S^1}|u_s|\,d\theta\Big)^2 \geq c_1\ell^{-\delta}$$

for some $c_1 > 0$. We note that combining this estimate with (4.35) and the estimate on the angular energy on K already obtained in (4.27) immediately yields the claim of the lemma.

It hence remains to prove that (4.36) holds true for a suitably chosen Λ. To this end we first recall that $\varepsilon\text{-thin}(\mathcal{C}) = \{p\in\mathcal{C} : \mathrm{inj}_g(p) < \varepsilon\}$ is given in collar coordinates by a cylinder $(-X_\varepsilon(\ell), X_\varepsilon(\ell))\times S^1$, for a number $X_\varepsilon(\ell)$ which is such that $X(\ell) - X_\varepsilon(\ell) \leq C\varepsilon^{-1}$, compare (4.8). Combined with the properties of $\mathcal{B}^c$ obtained in Lemma 4.2.5 this yields

$$\mathcal{L}^1(K^c) \leq \sum_j |I_j| + C\varepsilon^{-1} \leq C(\log(\ell^{-1})+1) + C\Lambda^{-1}\ell^{-\frac{1}{2}(1+\delta)}.$$

For Λ sufficiently large and $\ell_2 > 0$ sufficiently small, we hence obtain that for $\ell\in(0,\ell_2)$

$$\frac{1}{2\pi}\int_{K^c\times S^1}|u_s|\,d\theta\,ds \leq C(\mathcal{L}^1(K^c))^{\frac{1}{2}} \leq C\big[(\log(\ell^{-1}))^{\frac{1}{2}} + 1 + \Lambda^{-\frac{1}{2}}\ell^{-\frac{1}{4}(1+\delta)}\big] \leq \frac{c_0}{2}\ell^{-\frac{1}{4}(1+\delta)}.$$

Combined with (4.18) this implies the claim (4.36), which completes the proof of the lemma. □

Remark 4.2.6. The above proof of Theorem 4.1.1 applies with only minor modifications if (u,g) is a weak solution of (1.9): For almost every t the maps $u(t)$ are of class H^2 in the interior of M so Lemmas 4.2.4 and 4.2.5 may be applied. Lemmas 4.2.2 and 4.2.3 also apply without change at these times as their proofs are based only on Lemmas 4.2.4 and 4.2.5. Since g is Lipschitz, and hence differentiable almost everywhere, we can thus estimate $\frac{d\ell}{dt}$ precisely as in the above proof for almost every time, which yields the result.

4.3 Proof of necessary and sufficient conditions for the rescaled flow

We now turn to the proof of our second main result which gives a sharp criterion for finite-time degeneration for solutions of the rescaled flow. For this we exploit that at each time t the map component $u(t)$ of a solution of (4.4) is a harmonic map. We note that harmonic maps on degenerating surfaces have been studied by Chen–Tian [5] and Zhu [42]. We recall in particular that the Hopf differential of harmonic maps is holomorphic which implies that the quantity $\psi(s) = \int_{\{s\}\times S^1}|u_s|^2 - |u_\theta|^2\,d\theta$ considered above is constant on each collar around a simple closed geodesic $\sigma(t)$ in $(M, g(t))$. Knowing that a map is harmonic furthermore allows us to prove the following stronger relationship between ψ and $\mathscr{L}(u, \mathcal{C}(\sigma))$, which will be the main tool in the proof of Theorem 4.1.2.

Lemma 4.3.1. *For any numbers $\bar\kappa, C_1 \in \mathbb{R}$, respectively, $\bar\kappa \in \mathbb{R}$ and $\delta, c_0 > 0$, there exist numbers $\bar\ell, C_2 > 0$, respectively, $\bar\ell, c_1 > 0$ so that the following holds true for any complete Riemannian manifold (N, g_N) whose sectional curvature is bounded from above by $\bar\kappa$ and any hyperbolic collar $\mathcal{C}(\ell)$ as in (4.14), (4.15) for which $\ell \in (0, \bar\ell)$.*

Suppose that $u\colon \mathcal{C}(\ell) \to (N, g_N)$ is a harmonic map for which the quantity $\mathscr{L}$ introduced in (4.1) is bounded above by $\mathscr{L}(u, \mathcal{C}(\ell)) \leq C_1(\log(\ell^{-1}))^{\frac{1}{2}}$. Then $\psi = \int_{\{s\}\times S^1} |u_s|^2 - |u_\theta|^2 \, d\theta$, which is constant on the collar, is bounded from above by

$$\psi \leq C_2 \ell^2 [\log(\ell^{-1}) + 1]. \tag{4.37}$$

Conversely, for harmonic maps $u\colon \mathcal{C}(\ell) \to (N, g_N)$ with $\mathscr{L}(u, \mathcal{C}(\ell)) \geq c_0(\log(\ell^{-1}))^{\frac{1}{2}+\delta}$ we have

$$\psi \geq c_1 \ell^2 \log(\ell^{-1}(t))^{1+\delta}. \tag{4.38}$$

For the proof of this lemma we use the following variation of the H^2 estimate (4.23) from Lemma 4.2.4, in which we can replace the full energy on the right-hand side by only the angular energy. For the sake of completeness, we include a short proof at the end of this section. We recall the notation $\mathscr{C}_\Lambda(s) := [s - \Lambda, s + \Lambda] \times S^1$.

Lemma 4.3.2. *In the setting of Lemma 4.2.4, we have that for any $s_0 \notin \mathcal{A}$*

$$\int_{s_0-\frac{1}{2}}^{s_0+\frac{1}{2}} \int_{S^1} |\nabla^2 u|^2 d\theta \, ds \leq C \int_{s_0-1}^{s_0+1} \vartheta(s) \, ds + C \, \|\tau_{g_E}(u)\|^2_{L^2(\mathscr{C}_2(s_0), g_E)}, \tag{4.39}$$

where C depends only on an upper bound on the sectional curvature of (N, g_N).

Given a harmonic map u from a hyperbolic collar as in Lemma 4.3.1 we can combine the above lemma with the angular energy estimate of Lemma 4.2.4 to conclude that for $\ell \in (0, 1)$

$$\vartheta(s) + \int_{s-\frac{1}{2}}^{s+\frac{1}{2}} \int_{S^1} |\nabla^2 u|^2 d\theta \, ds \leq C\ell^4 \text{ on } \widetilde{\mathcal{B}} := \{s : \operatorname{dist}_{g_E}(s, \mathcal{A}) \geq 4\log(\ell^{-1}) + 2\}. \tag{4.40}$$

We also note that the measure of the complement of $\widetilde{\mathcal{B}} \subset [-X(\ell), X(\ell)]$ can be bounded as explained in the first part of the proof of Lemma 4.2.5, resulting in

$$\mathcal{L}^1(\widetilde{\mathcal{B}}^c) \leq C(\log(\ell^{-1}) + 1) \leq X(\ell), \tag{4.41}$$

where the last inequality holds true as $\ell \in (0, \bar\ell)$ for $\bar\ell \in (0, 1)$ chosen sufficiently small.

Proof of Lemma 4.3.1. Let u be as in the lemma. In both cases we compare ψ to $\alpha(s) := \fint_{\{s\}\times S^1} |u_s| \, d\theta$, and already note that by Cauchy–Schwarz

$$\alpha(s)^2 \leq \fint_{\{s\}\times S^1} |u_s|^2 \, d\theta \leq \psi + \vartheta(s) \text{ and that } \mathscr{L}(u, \mathcal{C}(\sigma)) = \int_{-X(\ell)}^{X(\ell)} \alpha \, ds. \tag{4.42}$$

We now estimate

$$\begin{aligned}
\psi^{\frac{1}{2}} \cdot |\sqrt{2\pi}\alpha(s) - \psi^{\frac{1}{2}}| &= \frac{\psi^{\frac{1}{2}}}{\sqrt{2\pi}\alpha(s) + \psi^{\frac{1}{2}}} \cdot |2\pi\alpha(s)^2 - \psi| \\
&\leq \int_{\{s\}\times S^1} |\alpha(s)^2 - |u_s|^2 + |u_\theta|^2| \, d\theta \\
&\leq C\bigg(\int_{\{s\}\times S^1} \big|\alpha(s) - |u_s|\big|^2 \, d\theta \bigg)^{\frac{1}{2}} \bigg(\int_{\{s\}\times S^1} |u_s|^2 \, d\theta \bigg)^{\frac{1}{2}} + \vartheta(s) \\
&\leq C\bigg(\int_{\{s\}\times S^1} |\nabla^2_{s,\theta} u|^2 \, d\theta \bigg)^{\frac{1}{2}} \cdot \big[\psi + \vartheta(s)\big]^{\frac{1}{2}} + \vartheta(s),
\end{aligned}$$

where we used (4.42) in both the penultimate and the last step. We thus obtain from (4.40) that

$$\psi^{\frac{1}{2}} \cdot \left|\psi^{\frac{1}{2}} - \sqrt{2\pi}\int_{s-\frac{1}{2}}^{s+\frac{1}{2}} \alpha\right| \leq C\ell^2 \cdot [\psi^{\frac{1}{2}} + \ell^2] \text{ for every } s \in \widetilde{\mathcal{B}}. \tag{4.43}$$

Suppose now that $\mathscr{L}(u, \mathcal{C}(\sigma)) \leq C_1 \log(\ell^{-1})^{\frac{1}{2}}$. As (4.41) implies that $\mathcal{L}^1(\widetilde{\mathcal{B}}) \geq X(\ell) \geq c\ell^{-1} > 0$, we can now combine (4.43) and (4.42) to conclude that in this case,

$$\psi = \fint_{\widetilde{\mathcal{B}}} \psi \, ds \leq C\psi^{\frac{1}{2}}\left[X(\ell)^{-1}\int_{-X(\ell)}^{X(\ell)} \alpha(s)\, ds + \ell^2\right] + C\ell^4 \leq \frac{1}{2}\psi + C\ell^2 \log(\ell^{-1}) + C\ell^4 \tag{4.44}$$

which gives the claimed bound (4.37) on ψ.

It hence remains to prove that if instead $\mathscr{L}(u, \mathcal{C}(\sigma)) \geq c_0 \log(\ell^{-1})^{\frac{1}{2}(1+\delta)}$, then we obtain the lower bound (4.38) on ψ. To this end, we let $\widetilde{\mathcal{B}}_-$ be the set of all $s \in \widetilde{\mathcal{B}}$ for which the interval $[s - \frac{1}{2}, s + \frac{1}{2}]$ is fully contained in $\widetilde{\mathcal{B}}$. We note that since the number of connected components of $\widetilde{\mathcal{B}}$ is bounded uniformly in terms of E_0 and ε_0, we have that $\mathcal{L}^1(\widetilde{\mathcal{B}} \setminus \widetilde{\mathcal{B}}_-) \leq C$ and hence $\mathcal{L}^1((\widetilde{\mathcal{B}}_-)^c) \leq C(1 + \log(\ell^{-1}))$, compare (4.41). Thus,

$$\int_{(\widetilde{\mathcal{B}}_-)^c} \alpha \leq C\mathcal{L}^1((\widetilde{\mathcal{B}}_-)^c)^{\frac{1}{2}} E_0^{\frac{1}{2}} \leq C(\log(\ell^{-1})^{\frac{1}{2}} + 1) \leq \frac{c_0}{2}\log(\ell^{-1})^{\frac{1}{2}(1+\delta)},$$

where the last estimate holds as we may assume that $\bar{\ell}$ was chosen sufficiently small (depending on $c_0, \delta > 0$). Hence, we must have that

$$\int_{\widetilde{\mathcal{B}}_-} \alpha \geq \frac{c_0}{2}\log(\ell^{-1})^{\frac{1}{2}(1+\delta)}. \tag{4.45}$$

We first claim that this implies that $\psi \geq c\ell^4 > 0$. Indeed if we had $\psi \leq c\ell^4$ we would have that $\alpha(s) \leq \vartheta(s)^{\frac{1}{2}} + c\ell^2 \leq C\ell^2$ for all $s \in \widetilde{\mathcal{B}}$ and hence $\int_{\widetilde{\mathcal{B}}_-} \alpha \leq \int_{\widetilde{\mathcal{B}}} \alpha \leq C\ell$ contradicting (4.45) if $\bar{\ell} > 0$ is sufficiently small.

The right-hand side of (4.43) is hence bounded by $C\ell^2\psi^{\frac{1}{2}}$ for every $s \in \widetilde{\mathcal{B}}$, and thus,

$$\psi^{\frac{1}{2}} \geq \sqrt{2\pi}\mathcal{L}(\widetilde{\mathcal{B}})^{-1}\int_{\widetilde{\mathcal{B}}}\int_{s-\frac{1}{2}}^{s+\frac{1}{2}} \alpha - C\ell^2 \geq c\ell\int_{\widetilde{\mathcal{B}}_-} \alpha - C\ell^2 \geq \sqrt{2c_1}\ell\log(\ell^{-1})^{\frac{1}{2}(1+\delta)} - C\ell^2$$

for some $c_1 > 0$, yielding (4.38) for $\bar{\ell}$ sufficiently small. □

We are now in a position to prove our second main theorem.

Proof of Theorem 4.1.2. Suppose first that (u, g) is a smooth solution of the rescaled flow (4.4) and let $\sigma_0 \subset M$ be a homotopically non-trivial simple closed curve. We denote by $\sigma(t)$ the geodesic in $(M, g(t))$ homotopic to σ_0, and by $\psi(t)$ the corresponding quantity considered in Lemma 4.3.1. We note that ψ is related to the principal part of the Fourier expansion $\Phi(u, g) = \sum b_j e^{jz}\, dz^2$, $z = s + \mathrm{i}\theta$ on $\mathcal{C}(\sigma(t))$ by $\psi = 2\pi\,\mathrm{Re}(b_0)$. We also recall that if g is either a horizontal curve of hyperbolic metrics on a closed surface, that is, such that $\partial_t g = \mathrm{Re}(\Psi(t))$ for holomorphic quadratic differentials Ψ, or a curve of metrics on the cylinder as described in Section 2.2, then $\ell(t) = L_{g(t)}(\sigma(t))$ evolves by $\frac{d\ell}{dt} = -\frac{2\pi^2}{\ell}\,\mathrm{Re}(b_0(\Psi))$, see, for example, [41]. In the present situation hence

$$\frac{d}{dt}\log(\ell^{-1}) = \pi\ell^{-2} \cdot \psi, \tag{4.46}$$

which allows us to derive the theorem from Lemma 4.3.1 as follows:

Suppose first that (u,g) is as in part (i) of the theorem and let $\bar\ell > 0$ and C_2 be as in Lemma 4.3.1. This lemma yields that at any time $t \in [0,T)$ for which $\ell(t) = L_{g(t)}(\sigma(t)) \in (0,\bar\ell)$ we have $\psi(t) \le C_2 \ell(t)^2 [\log(\ell^{-1}(t)) + 1]$ on $\mathcal{C}(\sigma(t))$, and hence also

$$\frac{d}{dt}(\log(\ell^{-1}(t)) + 1) \le C_2 \pi (\log(\ell^{-1}(t)) + 1),$$

which excludes the possibility that $\ell(t) \to 0$ in finite time. As this argument applies for every non-trivial simple closed curve in M, we hence find that the injectivity radius cannot go to zero in finite time, that is, that a finite-time degeneration of the metric is excluded.

Conversely, if (u,g) and $\sigma(t)$ are as in part (ii) of the theorem for $\bar\ell$ chosen as in Lemma 4.3.1, then this lemma yields that for every $t \in [T_1, T)$, we have $\psi(t) \ge c_1 \ell^2 \log(\ell^{-1}(t))^{1+\delta}$ on $\mathcal{C}(\sigma(t))$ for a number $c_1 > 0$ that is independent of time. Inserting this lower bound on ψ into (4.46) then immediately gives that

$$\frac{d}{dt} \log(\ell^{-1}(t)) \ge c_1 \pi \log(\ell^{-1})^{1+\delta} \text{ for } t \in [T_1, T).$$

As $\delta > 0$, the solution of the flow must hence degenerate in finite time.

If (u,g) is only a weak solution, the above proof still applies because the maps are still harmonic and hence smooth in the interior, so we can directly apply Lemma 4.3.2 at every time, and obtain the same estimates as above at every time at which the curve of metrics is differentiable. □

We finally provide a proof of Lemma 4.3.2 which follows by well-known arguments that have been used in particular in [8, 37] to establish H^2 bounds such as (4.23) in low-energy regions.

Proof of Lemma 4.3.2. Given any $s_0 \notin \mathcal{A}$ we let $\varphi = \varphi_0(\cdot - s_0)$, where φ_0 is a cut-off function as in Lemma 4.2.4. As $\nabla^2_{s,s} u = \tau(u) - \nabla^2_{\theta,\theta} u$, where we write for short $\tau = \tau_{g_E}$, we have

$$\begin{aligned} I := \int \varphi^2 |\nabla^2 u|^2 &= \int \varphi^2 \Big(|\tau(u) - \nabla^2_{\theta,\theta} u|^2 + |\nabla^2_{\theta,\theta} u|^2 + 2 |\nabla^2_{s,\theta} u|^2 \Big) \\ &\le \|\varphi \tau(u)\|^2_{L^2} + 2 \|\varphi \tau(u)\|_{L^2} \cdot \Big(\int \varphi^2 |\nabla^2_{\theta,\theta} u|^2 \Big)^{\frac12} + 2 \int \varphi^2 |\nabla^2_{\theta,\theta} u|^2 \\ &\quad + 4 \int \varphi |\varphi'| |u_\theta| \, |\nabla^2_{s,\theta} u| + 2\bar\kappa \int \varphi^2 |u_\theta|^2 \, |u_s|^2 + 2 \int \varphi^2 \nabla^2_{\theta,\theta} u \cdot \nabla^2_{s,s} u, \end{aligned} \tag{4.47}$$

where $\bar\kappa \ge 0$ is an upper bound on the sectional curvature of the target. To justify the curvature term, working in normal coordinates $\frac{\partial}{\partial y}$ at $u(p)$, $p \in \mathscr{C}_1(s_0)$, we can calculate

$$\begin{aligned} \nabla_s \nabla_s \nabla_\theta u &= \nabla_s \nabla_s \Big(\frac{\partial u^\alpha}{\partial \theta} \frac{\partial}{\partial y^\alpha} \circ u \Big) = \nabla_s \Big(\frac{\partial^2 u^\alpha}{\partial s \, \partial \theta} \frac{\partial}{\partial y^\alpha} \circ u + \frac{\partial u^\alpha}{\partial \theta} \frac{\partial u^\beta}{\partial s} (\nabla_{\frac{\partial}{\partial y^\beta}} \frac{\partial}{\partial y^\alpha}) \circ u \Big) \\ &= \frac{\partial^3 u^\alpha}{\partial s^2 \, \partial \theta} \frac{\partial}{\partial y^\alpha} \circ u + \Big(\frac{\partial^2 u^\alpha}{\partial s \, \partial \theta} \frac{\partial u^\beta}{\partial s} + \frac{\partial u^\alpha}{\partial \theta} \frac{\partial^2 u^\beta}{\partial s^2} \Big) (\nabla_{\frac{\partial}{\partial y^\beta}} \frac{\partial}{\partial y^\alpha}) \circ u \\ &\quad + \frac{\partial^2 u^\alpha}{\partial s \, \partial \theta} \frac{\partial u^\gamma}{\partial s} (\nabla_{\frac{\partial}{\partial y^\gamma}} \frac{\partial}{\partial y^\alpha}) \circ u + \frac{\partial u^\alpha}{\partial \theta} \frac{\partial u^\beta}{\partial s} \frac{\partial u^\gamma}{\partial s} (\nabla_{\frac{\partial}{\partial y^\gamma}} \nabla_{\frac{\partial}{\partial y^\beta}} \frac{\partial}{\partial y^\alpha}) \circ u \\ &= \frac{\partial^3 u^\alpha}{\partial s^2 \, \partial \theta} \frac{\partial}{\partial y^\alpha} \circ u + \frac{\partial u^\alpha}{\partial \theta} \frac{\partial u^\beta}{\partial s} \frac{\partial u^\gamma}{\partial s} (\nabla_{\frac{\partial}{\partial y^\gamma}} \nabla_{\frac{\partial}{\partial y^\beta}} \frac{\partial}{\partial y^\alpha}) \circ u. \end{aligned}$$

Similarly,

$$\nabla_\theta \nabla_s \nabla_s u = \frac{\partial^3 u^\alpha}{\partial s^2 \, \partial \theta} \frac{\partial}{\partial y^\alpha} \circ u + \frac{\partial u^\alpha}{\partial \theta} \frac{\partial u^\beta}{\partial s} \frac{\partial u^\gamma}{\partial s} (\nabla_{\frac{\partial}{\partial y^\gamma}} \nabla_{\frac{\partial}{\partial y^\beta}} \frac{\partial}{\partial y^\alpha}) \circ u,$$

and so writing $R(\cdot,\cdot)$ for the Riemann curvature tensor we find

$$\begin{aligned}
-\nabla_s\nabla_s\nabla_\theta u + \nabla_\theta\nabla_s\nabla_s u &= \frac{\partial u^\alpha}{\partial\theta}\frac{\partial u^\beta}{\partial s}\frac{\partial u^\gamma}{\partial s}\big(-\nabla_{\frac{\partial}{\partial y^\beta}}\nabla_{\frac{\partial}{\partial y^\gamma}}\frac{\partial}{\partial y^\alpha} + \nabla_{\frac{\partial}{\partial y^\alpha}}\nabla_{\frac{\partial}{\partial y^\beta}}\frac{\partial}{\partial y^\gamma}\big)\circ u \\
&= \frac{\partial u^\alpha}{\partial\theta}\frac{\partial u^\beta}{\partial s}\frac{\partial u^\gamma}{\partial s}\big(-\nabla_{\frac{\partial}{\partial y^\beta}}\nabla_{\frac{\partial}{\partial y^\alpha}}\frac{\partial}{\partial y^\gamma} + \nabla_{\frac{\partial}{\partial y^\alpha}}\nabla_{\frac{\partial}{\partial y^\beta}}\frac{\partial}{\partial y^\gamma}\big)\circ u \\
&= \frac{\partial u^\alpha}{\partial\theta}\frac{\partial u^\beta}{\partial s}\frac{\partial u^\gamma}{\partial s}\big[R(\frac{\partial}{\partial y^\alpha},\frac{\partial}{\partial y^\beta})\frac{\partial}{\partial y^\gamma}\big]\circ u \\
&= R(u_\theta,u_s)u_s.
\end{aligned}$$

Thus we can bound

$$\begin{aligned}
2\int\varphi^2\nabla_\theta u\cdot(-\nabla_s\nabla_s\nabla_\theta u + \nabla_\theta\nabla_s\nabla_s u) &= 2\int\varphi^2 u_\theta\cdot R(u_\theta,u_s)u_s \\
&= 2\int\varphi^2\,K([u_\theta\wedge u_s])\,|u_\theta\wedge u_s|^2 \\
&\le 2\bar\kappa\int\varphi^2|u_\theta|^2\,|u_s|^2,
\end{aligned}$$

where $[u_\theta\wedge u_s]$ denotes the tangent plane spanned by the section. As the last term of (4.47) is equal to $-2\int\varphi^2|\nabla^2_{\theta,\theta}u|^2 + 2\int\varphi^2\tau(u)\nabla^2_{\theta,\theta}u$, we hence obtain that

$$\begin{aligned}
I &\le \|\varphi\tau(u)\|^2_{L^2} + CI^{\frac12}\cdot\big[\|\varphi\tau(u)\|_{L^2} + \hat\vartheta(s_0)^{\frac12}\big] + II \\
&\le C\|\varphi\tau(u)\|^2_{L^2} + C\hat\vartheta(s_0) + \frac14 I + II
\end{aligned} \tag{4.48}$$

where we write for short $\hat\vartheta(s_0) = \int_{s_0-1}^{s_0+1}\vartheta(s)ds$ and set $II := 2\bar\kappa\int\varphi^2|u_\theta|^2|u_s|^2$. Using that $W^{1,1}_0([-1,1]\times S^1)$ embeds continuously into $L^2([-1,1]\times S^1)$, we may now estimate

$$\begin{aligned}
II &\le C\big\||\varphi|u_s|^2\big\|_{L^2}\cdot\big\||\varphi|u_\theta|^2\big\|_{L^2} \le C\big\|\nabla(\varphi|u_s|^2)\big\|_{L^1}\cdot\big\|\nabla(\varphi|u_\theta|^2)\big\|_{L^1} \\
&\le C\big[E(u;\mathscr{C}_1(s_0)) + E(u;\mathscr{C}_1(s_0))^{\frac12}\|\varphi\nabla^2 u\|_{L^2}\big]\cdot\big[\hat\vartheta(s_0) + \hat\vartheta(s_0)^{\frac12}\|\varphi\nabla^2u\|_{L^2}\big] \\
&\le C\varepsilon_0\hat\vartheta(s_0) + CI^{\frac12}\cdot\big[\varepsilon_0\hat\vartheta(s_0)^{\frac12} + \varepsilon_0^{\frac12}\hat\vartheta(s_0) + C\varepsilon_0^{\frac12}\hat\vartheta(s_0)^{\frac12}I^{\frac12}\big] \\
&\le \frac14 I + C\hat\vartheta(s_0) + C\hat\vartheta(s_0)I.
\end{aligned}$$

We can now use the H^2 estimate (4.23) from Lemma 4.2.4 to control the last term in this estimate by $C\hat\vartheta(s_0)I \le C\hat\vartheta(s_0)(\varepsilon_0 + \|\varphi\tau(u)\|^2_{L^2}) \le C\hat\vartheta(s_0) + C\|\varphi\tau(u)\|^2_{L^2}$ and insert the resulting bound on II into (4.48) to obtain the claim of the lemma. □

4.4 Construction of settings exhibiting finite-time degeneration

In this section, we construct settings in which Theorem 4.1.1 and Theorem 4.1.2 assure that solutions of Teichmüller harmonic map flow (1.9), respectively, of the rescaled flow (4.4), from the cylinder $\mathscr{C} = [-1,1]\times S^1$ degenerate in finite time. This smooth target will be obtained as a warped product $N = \mathbb{R}\times_f\widetilde N$ of $\mathbb{R}$ with

$$(\widetilde N, g_{\widetilde N}) := (\mathbb{R}^3,\rho^2_{\widetilde N}g_{\mathrm E}),\ \text{where}\ \rho^2_{\widetilde N} = C_{\widetilde N}\exp\left(1-\frac{1}{1-|x|^2}\right)\mathbb{1}_{\{|x|<1\}} + 1, \tag{4.49}$$

for $C_{\widetilde{N}} > 0$ a (large) constant that we choose below. Before we analyse solutions $u = (v, w) \colon \mathscr{C} \times [0,T] \to \mathbb{R} \times_f \widetilde{N}$ of the flow, or even just discuss the choice of the warping function f, we discuss some key properties of maps $w \colon \mathscr{C} \to \widetilde{N}$. To this end, we first note that for large $C_{\widetilde{N}}$, the metric $g_{\widetilde{N}}$, which is Euclidean outside the Euclidean unit ball but highly concentrated in $\{|x|_{\mathrm{E}} < 1\}$, is so that the function $r \mapsto r^2 \rho_{\widetilde{N}}^2(r)$ has exactly two local extrema $0 < r_{\max} < r_{\min} < 1$ on $[0,1]$, where $r_{\max} \downarrow \frac{\sqrt{5}-1}{2}$ and $r_{\min} \uparrow 1$ as $C_{\widetilde{N}} \to \infty$, as indicated in Figures 4.1 and 4.2.

We will later choose the w component of our initial map $u_0 = (v_0, w_0)$ of the flow so that its image is disjoint from the region $\{|x|_{\mathrm{E}} < 1\}$ and first prove the following lower bound on the area of maps $w \colon \mathscr{C} \to (\widetilde{N}, g_{\widetilde{N}})$ which are obtained by continuously deforming such a map.

Lemma 4.4.1. *For any $E_0 < \infty$ and $z_0 > 1$ and for sufficiently large $C_{\widetilde{N}}$ (depending on E_0 and z_0). we have that the following holds true for the metric $g_{\widetilde{N}}$ defined in (4.49).*
Let $(w_t)_{t\in[0,T)}$ be any continuous family of smooth, rotationally symmetric maps $w_t \colon \mathscr{C} := [-1,1] \times S^1 \to \mathbb{R}^3$, $w_t(x,\theta) = \big(r_t(x)e^{\mathrm{i}\theta}, z_t(x)\big)$, with fixed boundary values $r_t(\pm 1) = r_0 \in (0, z_0]$ and $z_t(\pm 1) = \pm z_0$. Suppose that $\mathrm{Area}_{(\widetilde{N}, g_{\widetilde{N}})}(w_t) \leq E_0$ for every $t \in [0,T)$ and that the image of the initial map w_0 is disjoint from the Euclidean open unit ball. Then for every $t \in [0,T)$

$$\mathrm{Area}_{(\widetilde{N}, g_{\widetilde{N}})}(w_t) \geq 2\pi. \tag{4.50}$$

Furthermore, denoting by $\psi(p)$ the angle between p and the positive z-axis and by $|p|$ the Euclidean distance to the origin, we have that for every $t \in [0,T)$

$$w_t(\mathscr{C}) \cap \{p \in \mathbb{R}^3 : |p| = r_{\max} \text{ and } \psi(p) \in [\tfrac{\pi}{4}, \tfrac{3\pi}{4}]\} = \emptyset. \tag{4.51}$$

Proof. We will first explain how (4.51) implies (4.50) (for $C_{\widetilde{N}}$ sufficiently large), and provide the proof of (4.51) at the end. We note that (4.51) allows us to conclude that for each $t \in [0,T)$, there must be an interval $[x_1^t, x_2^t] \subset [-1,1]$ such that

$$|w_t| > r_{\max} \text{ and } \psi(w_t) \in (\tfrac{\pi}{4}, \tfrac{3\pi}{4}) \text{ on } (x_1^t, x_2^t) \times S^1 \text{ with } \psi(w_t)(\{x_1^t, x_2^t\}) = \{\tfrac{\pi}{4}, \tfrac{3\pi}{4}\}.$$

To see this, we note that the number of such intervals for the initial w_0 must be odd as by assumption $|w_0| \geq 1 > r_{\max}$ on all of $\mathscr{C}$ and as the points $w_t(\{\pm 1\} \times S^1)$ of the fixed boundary circles have ψ coordinate in $(0, \frac{\pi}{4}]$, respectively, $[\frac{3\pi}{4}, \pi)$ since $r_0 \leq z_0$. Combined with (4.51), this property of the boundary data ensures also that along a family of maps (w_t) as in the lemma such intervals can only be lost or gained in pairs, so their number remains odd, and so non-zero.

The claim (4.50) hence follows if we prove a lower bound of 2π for the area of any given surface S_γ that is obtained by rotating a curve $\gamma = |\gamma| e^{\mathrm{i}\psi} \in \mathbb{R}^2$, parametrised over some $[a,b]$, around the z-axis, for which $\psi(a) = \frac{\pi}{4}$, $\psi(b) = \frac{3\pi}{4}$ and $|\gamma| \geq r_{\max}$ as well as $\psi \in (\frac{\pi}{4}, \frac{3\pi}{4})$ on (a,b). Given such a γ, say parametrised by arclength and hence with $|\gamma| \cdot |\psi'| \leq 1$, we can estimate

$$\begin{aligned}\mathrm{Area}_{(\widetilde{N}, g_{\widetilde{N}})}(S_\gamma) &= 2\pi \int_a^b \rho_{\widetilde{N}}^2 \circ \gamma \cdot |\gamma| \sin(\psi) \geq 2\pi \min_{[a,b]} \left(|\gamma|^2 \rho_{\widetilde{N}}^2 \circ \gamma\right) \int_a^b \sin(\psi) |\psi'| \\ &\geq 4\pi r_{\min}^2 \, \rho_{\widetilde{N}}(r_{\min})^2 \cos(\frac{\pi}{4}) \geq 2\sqrt{2}\pi r_{\min}^2 \geq 2\pi\end{aligned}$$

for $C_{\widetilde{N}}$ sufficiently large, where we used that the minimum of $r \mapsto r^2 \rho_{\widetilde{N}}^2(r)$ on $[r_{\max}, \infty)$ is achieved at $r_{\min}$ and that $r_{\min} \uparrow 1$ as $C_{\widetilde{N}} \to \infty$.

Finally, to prove the claim (4.51), we show that for $C_{\widetilde{N}}$ chosen sufficiently large, the area of any rotationally symmetric surface with the given boundary conditions that violates (4.51) must have area larger than E_0. So let $\gamma = |\gamma| e^{\mathrm{i}\psi}$ be any smooth curve in $\mathbb{R}^2$, say parametrised by

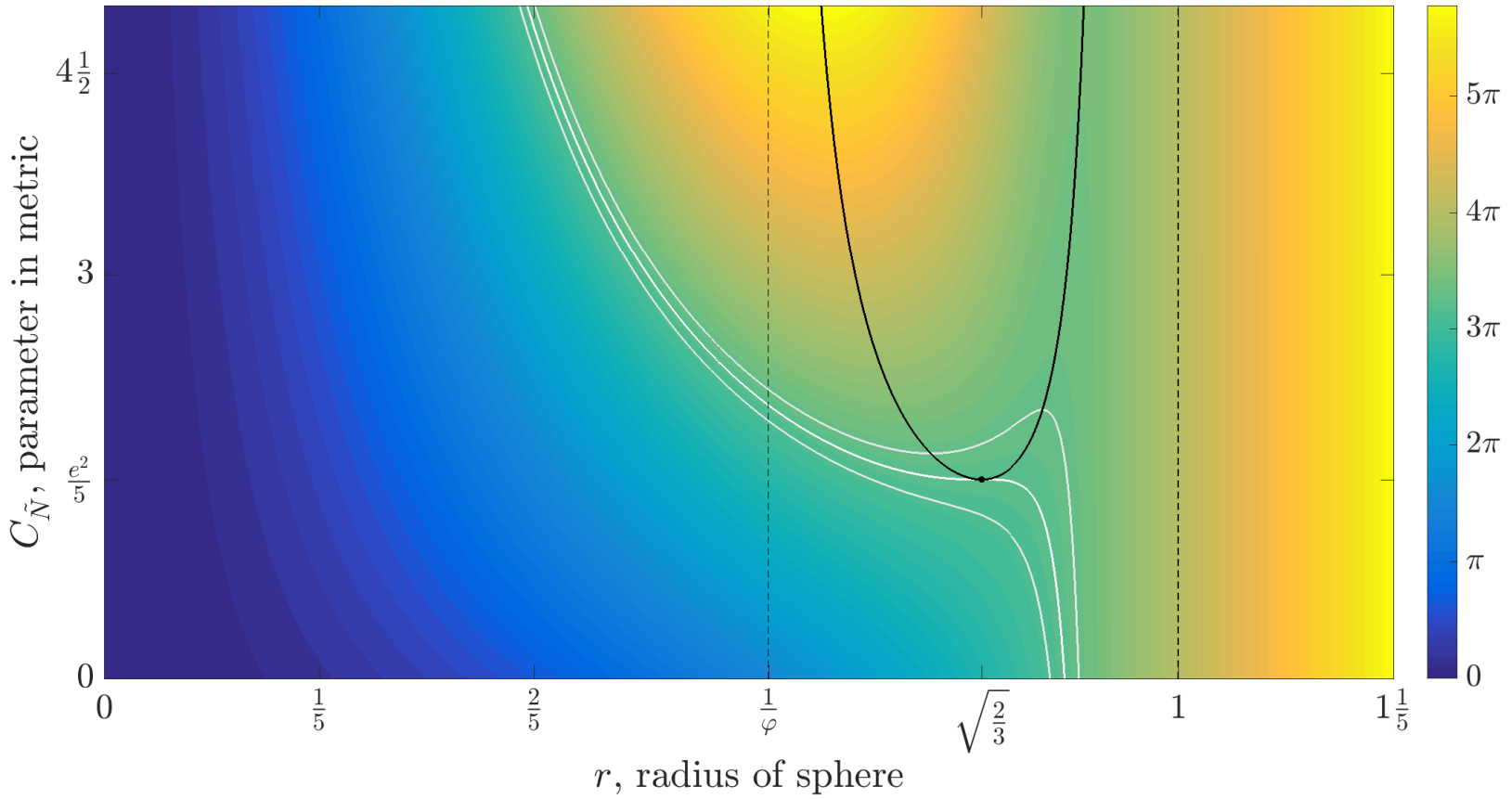


Figure 4.1: The area of spheres under the metric $g_{\widetilde{N}}$; the black line depicts the critical points, with a maximum followed by a minimum occurring for $C_{\widetilde{N}} > \frac{e^2}{5}$. Level sets in a neighbourhood of the area at $C_{\widetilde{N}} = \frac{e^2}{5}, r = \sqrt{\frac{2}{3}}$ are shown in white.

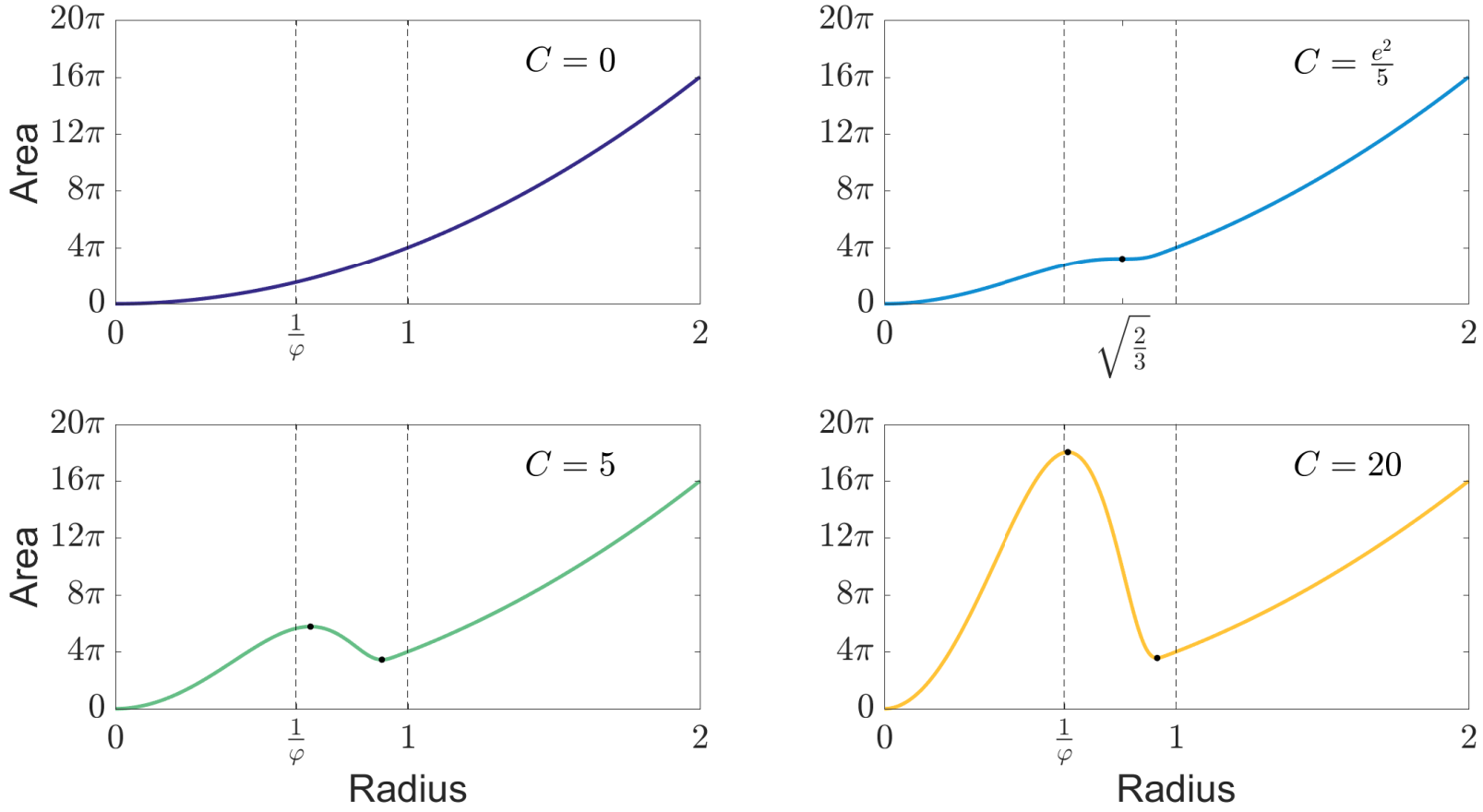


Figure 4.2: The area of spheres under the metric $g_{\widetilde{N}}$, shown at specific values of $C_{\widetilde{N}}$.

arclength over some $[a,b]$, with $\gamma(a) = (-z_0, r_0)$ and $\gamma(b) = (z_0, r_0)$ and suppose that there exists $t_0 \in (a,b)$ so that $\gamma(t_0) = r_{\max} e^{i\psi_1}$ for some $\psi_1 \in [\frac{\pi}{4}, \frac{3\pi}{4}]$.

Let now $\varepsilon := \min(z_0 - 1, \frac{1}{10}) > 0$ which ensures in particular that $[t_0 - \varepsilon, t_0 + \varepsilon] \subset [a,b]$. Since $r_{\max} \downarrow \frac{\sqrt{5}-1}{2} \approx 0.62$ as $C_{\widetilde{N}} \to \infty$, we may assume that $C_{\widetilde{N}}$ is large enough to ensure that $\frac{1}{2} + \frac{1}{10} \leq r_{\max} \leq \frac{3}{4} - \frac{1}{10}$ so that the above choice of ε furthermore implies that

$$\tfrac{1}{2} \leq r_{\max} - \varepsilon \leq |\gamma| \leq r_{\max} + \varepsilon \leq \tfrac{3}{4} \text{ and } \psi \in [\tfrac{\pi}{4} - 2\varepsilon, \tfrac{3\pi}{4} + 2\varepsilon] \subset [\tfrac{\pi}{6}, \tfrac{5\pi}{6}] \text{ on } [t_0 - \varepsilon, t_0 + \varepsilon].$$

We hence obtain

$$\begin{aligned} \operatorname{Area}_{(\widetilde{N}, g_{\widetilde{N}})}(S_\gamma) &\geq \operatorname{Area}_{(\widetilde{N}, g_{\widetilde{N}})}(S_{\gamma|_{[t_0-\varepsilon, t_0+\varepsilon]}}) = 2\pi \int_{t_0-\varepsilon}^{t_0+\varepsilon} \rho_{\widetilde{N}}^2 \circ \gamma \cdot |\gamma| \sin(\psi) \\ &\geq 4\pi\varepsilon C_{\widetilde{N}} \exp\left(1 - (1 - (\tfrac{3}{4})^2)^{-1}\right) \cdot \tfrac{1}{4} \end{aligned}$$

which yields that $\operatorname{Area}_{(\widetilde{N}, g_{\widetilde{N}})}(S_\gamma) > E_0$ provided $C_{\widetilde{N}} = C_{\widetilde{N}}(E_0, z_0)$ is chosen sufficiently large. This completes the proof of the claim (4.51) and thus of the lemma. □

We now construct our target as a warped product

$$N = \mathbb{R} \times_f \widetilde{N} \text{ with metric } g_N = dv^2 + f(v) \cdot g_{\widetilde{N}}, \tag{4.52}$$

where the warping function is of the form $f = f_0(\cdot - \bar{v})$, $\bar{v} > 0$ chosen later, for f_0 as assumed below. This ensures that we obtain the correct decay while still avoiding later dependencies when showing an upper bound on the injectivity radius.

Assumption 4.4.2. We let $f_0 \colon \mathbb{R} \to [1, \infty)$ be a non-increasing function that satisfies $f_0 \equiv 8$ on $(-\infty, 0]$ and $f_0 > 7$ on $(-\infty, 1)$ for which $-f_0'$ is decreasing on $[1, \infty)$ with $0 < -f_0' \leq \frac{1}{8}$ and that has one of the following two types of asymptotic behaviours:

(i) For some $\delta \in (0,1)$ and $c_3, \Lambda > 0$, we have that

$$f_0(v) = 1 + c_3 v^{-(\frac{2}{1+\delta} - 1)} \text{ on } [\Lambda, \infty).$$

(ii) For some $\alpha, c_3, \Lambda > 0$, we have that

$$f_0(v) = 1 + c_3 e^{-\alpha(v - \Lambda)} \text{ on } [\Lambda, \infty).$$

We observe that in both cases the resulting target (N, g_N) is complete and has bounded curvature, where by construction the curvature bound $\bar{\kappa}$ is independent of the choice of $\bar{v}$ and will use functions with asymptotics (i) to prove the first part of Theorem 4.1.3, and functions with asymptotics (ii) in the proof of the second part of the theorem.

In the following, we consider maps $u = (v, w) \colon \mathscr{C} = [-1,1] \times S^1 \to N$ with symmetries

$$\begin{gathered} v = v(x) = v(-x), \quad w(x,\theta) = \big(r(x) e^{i\theta}, z(x)\big) \\ \text{with } r(x) = r(-x) \text{ and } z(-x) = -z(x) \end{gathered} \tag{4.53}$$

and fixed boundary data of

$$v(\pm 1) = 0 \text{ and } r(\pm 1) = \frac{1}{4}, \quad z(\pm 1) = \pm z_0 > 1. \tag{4.54}$$

In general, to evolve a given initial map u_0 from the cylinder towards a minimal surface with prescribed boundary curves, one needs to consider (1.9) together with Plateau boundary condition. However, if our initial map $u_0 = (v_0, w_0)$ is of the above form and if we choose the initial

metric $g_0 = G_{\bar\ell}$ on the cylinder as described in (2.13), then due to the symmetries this problem is reduced to solving (1.9) with *Dirichlet* boundary conditions

$$u(\pm 1, \theta) = u_0(\pm 1, \theta) \tag{4.55}$$

and the evolution of the metric reduces to an ODE for the length $\ell(t)$ of the central geodesic with $g(t) = G_{\ell(t)}$ as described in (2.13). Thus, standard parabolic theory yields the existence of a (unique) solution (u, g) of the flow (1.9) along which the energy decays according to (4.3). We note that this solution remains smooth for all times unless $\operatorname{inj}(\mathscr{C}, g(t)) \to 0$ as $t \uparrow T < \infty$, as a singularity of just the map component, which would need to be caused by the bubbling off of *finitely* many harmonic spheres, is excluded by the symmetries.

We will thus be able to conclude that the flow degenerates in finite time, provided we can establish that the assumptions (i) and (ii) of Theorem 4.1.1 hold. To this end, we fix the constant $C_{\widetilde N}$ in the definition of $(\widetilde N, g_{\widetilde N})$ so that Lemma 4.4.1 applies for maps of area no more than 10π and choose the initial data as follows.

Lemma 4.4.3. *Let (N, g_N) be as in (4.52), where $\bar v > 0$ is any fixed number. Then there is a smooth map $u_0 = (v_0, w_0) \colon \mathscr{C} \to N$ with the symmetries (4.53) and boundary data (4.54) that satisfies $|w_0| \geq 1$ on all of $\mathscr{C}$, and a metric $g_0 = G_{\bar\ell}$ as defined in (2.13) such that $E(u_0, g_0) \leq 10\pi$.*

Proof. For $\varepsilon > 0$ and $\ell > 0$ determined below, we construct such a map

$$u_0 = (v_0, w_0) \colon (\mathscr{C}, G_\ell) \cong \big([-X(\ell), X(\ell)] \times S^1, \rho_\ell^2(ds^2 + d\theta^2)\big) \to (N, g_N)$$

as follows: We first let $\Lambda_1(\varepsilon) > 1$ be the unique number for which $S^2 \setminus B_\varepsilon(P^\pm)$, $P^\pm := (0, 0, \pm 1)$, is conformal to the cylinder $[-\Lambda_1(\varepsilon) + 1, \Lambda_1(\varepsilon) - 1] \times S^1$ and choose w_0 on this central part of the collar to be such a conformal parametrisation, which we can assume to have the symmetries as in (4.53), and which is chosen so that $\{\pm(\Lambda_1(\varepsilon) - 1)\} \times S^1$ is mapped to the corresponding boundary curve $S^2 \cap \partial B_\varepsilon(P^\pm) \subset \mathbb{R}^3$.

We then let $\Lambda_2(\varepsilon)$ be the unique number so that we can choose w_0 on $[X(\ell) - (\Lambda_2(\varepsilon) - 1), X(\ell)] \times S^1$ to be a conformal parametrisation of the annulus $\overline{D}_{\frac14} \setminus D_\varepsilon \times \{z_0\}$, again with the required symmetry and boundary conditions (4.53) and (4.54). Here we of course assume that $\ell = \ell(\varepsilon) > 0$ is small enough so that $\Lambda_1(\varepsilon) + \Lambda_2(\varepsilon) - 2 \leq X(\ell)$, but furthermore ask that ℓ is sufficiently small so that choosing $w_0|_{[\Lambda_1(\varepsilon), X(\ell) - \Lambda_2(\varepsilon)] \times S^1}(s, \theta) := (0, 0, 1 + \frac{s - \Lambda_1(\varepsilon)}{X(\ell) - \Lambda_1(\varepsilon) - \Lambda_2(\varepsilon)}(z_0 - 1))$ as a linear parametrisation of the line connecting P^+ and $(0, 0, z_0)$ gives energy

$$E(w|_{[\Lambda_1(\varepsilon), X(\ell) - \Lambda_2(\varepsilon)] \times S^1}) = \pi(z_0 - 1)^2 (X(\ell) - \Lambda_1(\varepsilon) - \Lambda_2(\varepsilon))^{-1} \leq \varepsilon.$$

We finally complete w_0 to a smooth map with the desired symmetries which can be done so that on the intermediate regions $[\Lambda_1(\varepsilon) - 1, \Lambda_1(\varepsilon)] \times S^1$ and $[X(\ell) - \Lambda_2(\varepsilon), X(\ell) - (\Lambda_2(\varepsilon) - 1)] \times S^1$, we have $|\nabla w_0| \leq C\varepsilon$ for some universal $C > 0$, as shown in Figure 4.3.

To construct a suitable v component for the initial data, we first fix $v^* = v^*(\bar v)$ so that $f(v^*) = 2$. We then ask that $v_0 \colon [-X(\ell), X(\ell)] \times S^1 \to \mathbb{R}$ is a smooth map with $v_0(s, \theta) = v_0(s) = v_0(-s)$ so that $v_0(\pm X(\ell)) = 0$, $v_0 \equiv v^*$ on the central part $|s| < \Lambda_1(\varepsilon)$, and $|v_0'| \leq 2\frac{v^*}{X(\ell) - \Lambda_1(\varepsilon)}$. For $\ell = \ell(\varepsilon, \bar v) > 0$ small enough, we hence obtain that also $E(v_0) = 2\pi \frac{4(v_0^*)^2}{X(\ell) - \Lambda_1(\varepsilon)} \leq \varepsilon$. In total we hence obtain an energy of no more than

$$E(u_0, g_0) \leq 2 \max_{\mathbb{R}} f \cdot \operatorname{Area}(D_{\frac14} \setminus D_\varepsilon) + f(v_0^*) \operatorname{Area}(S^2) + C\varepsilon \leq 9\pi + C\varepsilon \leq 10\pi$$

provided we initially chose $\varepsilon > 0$ sufficiently small. □

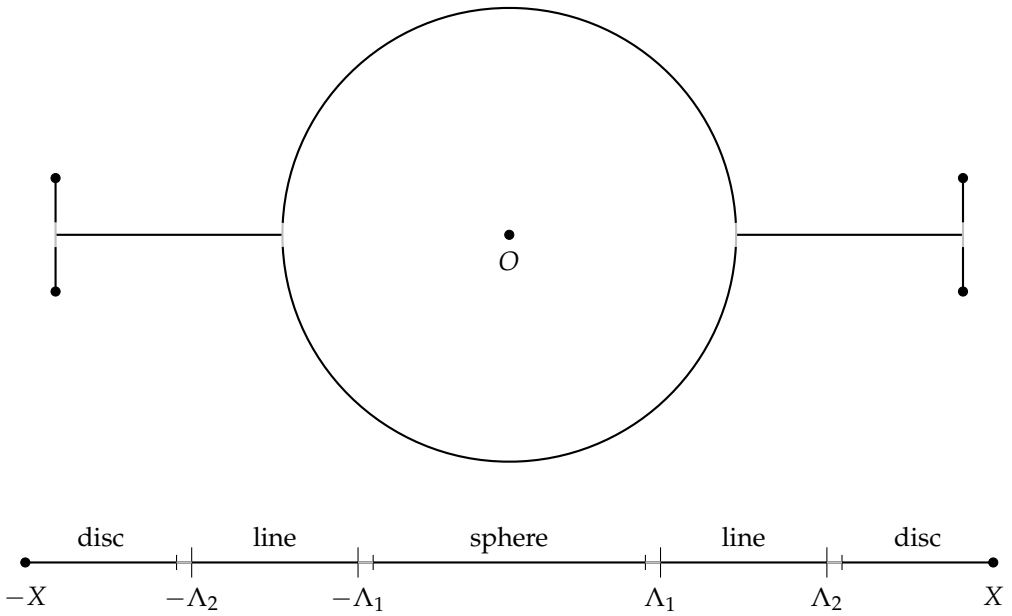


Figure 4.3: A cross section of the image in $\tilde{N}$ of the initial data constructed in Lemma 4.4.3, shown with the corresponding divisions of the domain.

We note that this choice of initial data assures that along the flow the energy of each map $u(t)\colon (\mathscr{C}, g(t)) \to (N, g_N)$ remains below 10π. As the warping function f satisfies $f \geq 1$ everywhere, we thus obtain in particular an upper bound of 10π on the energy, and thus on the area, of the component $w(t)\colon (\mathscr{C}, g(t)) \to (\widetilde{N}, g_{\widetilde{N}})$, allowing us to apply Lemma 4.4.1 (since we chose $C_{\widetilde{N}}$ accordingly). We hence know that along the flow the component $w(t)$ remains disjoint from the set described in (4.51) and hence in particular that points on the central curve $\{s = 0\}$ are mapped to points which have Euclidean distance at least $r \geq r_{\max}$ from the z-axis.

While Lemma 4.4.1 already yields a lower bound of 2π on the area, and thus on the energy, of the whole second component w, we now prove that already the restriction of w to a small central region of the domain will need to have at least energy 2π.

Lemma 4.4.4. *Let $w\colon \mathscr{C} = [-1,1] \times S^1 \to (\widetilde{N}, g_{\widetilde{N}})$ be any map with symmetries and boundary conditions as in (4.53) and (4.54) which satisfies (4.51) and which is so that $|w| \geq r_{\max}$ on $\{0\} \times S^1$. Let now G_ℓ be any metric on $\mathscr{C}$ as described in (2.13) and let $(s,\theta) \in [-X(\ell), X(\ell)] \times S^1$ be the corresponding collar coordinates, where we assume that ℓ is small enough so that $X(\ell) \geq 8$. Then we have a lower bound on the energy of the restriction of w to the central part of*

$$E(w|_{\{(s,\theta):|s|\leq 8\}}) = \frac{1}{2}\int_{-8}^{8}\int_{S^1} (|w_s|^2 + |w_\theta|^2)\,\rho_{\widetilde{N}}^2\, d\theta\, ds \geq 2\pi. \tag{4.56}$$

Proof. If the image of $w|_{\{(s,\theta):|s|\leq 8\}}$ intersects the cone $\{p : \psi(p) = \frac{\pi}{4} \text{ or } \psi(p) = \frac{3\pi}{4}\}$ then we know from the proof of Lemma 4.4.1 that the area of the image of this map in $(\widetilde{N}, g_{\widetilde{N}})$ is at least 2π and hence so is its energy. Otherwise, the image of $w|_{\{(s,\theta):|s|\leq 8\}}$ is contained in $\{p : |p| \geq r_{\max} \text{ and } \psi(p) \in (\frac{\pi}{4}, \frac{3\pi}{4})\}$ which assures in particular that the radial component of $w(s,\theta) = (r(s)e^{i\theta}, z(s))$ is at least $r(s) \geq r_{\max} \cdot \sin(\frac{\pi}{4})$. As $\rho_{\widetilde{N}} \geq 1$, we hence obtain that

$$E(w|_{\{(s,\theta):|s|\leq 8\}}) \geq \frac{1}{2}\int_{-8}^{8}\int_{S^1} |w_\theta|^2\, d\theta\, ds \geq 16\pi \cdot \frac{1}{2} r_{\max}^2 > 2\pi, \text{ as } r_{\max} > \frac{1}{2}. \qquad \square$$

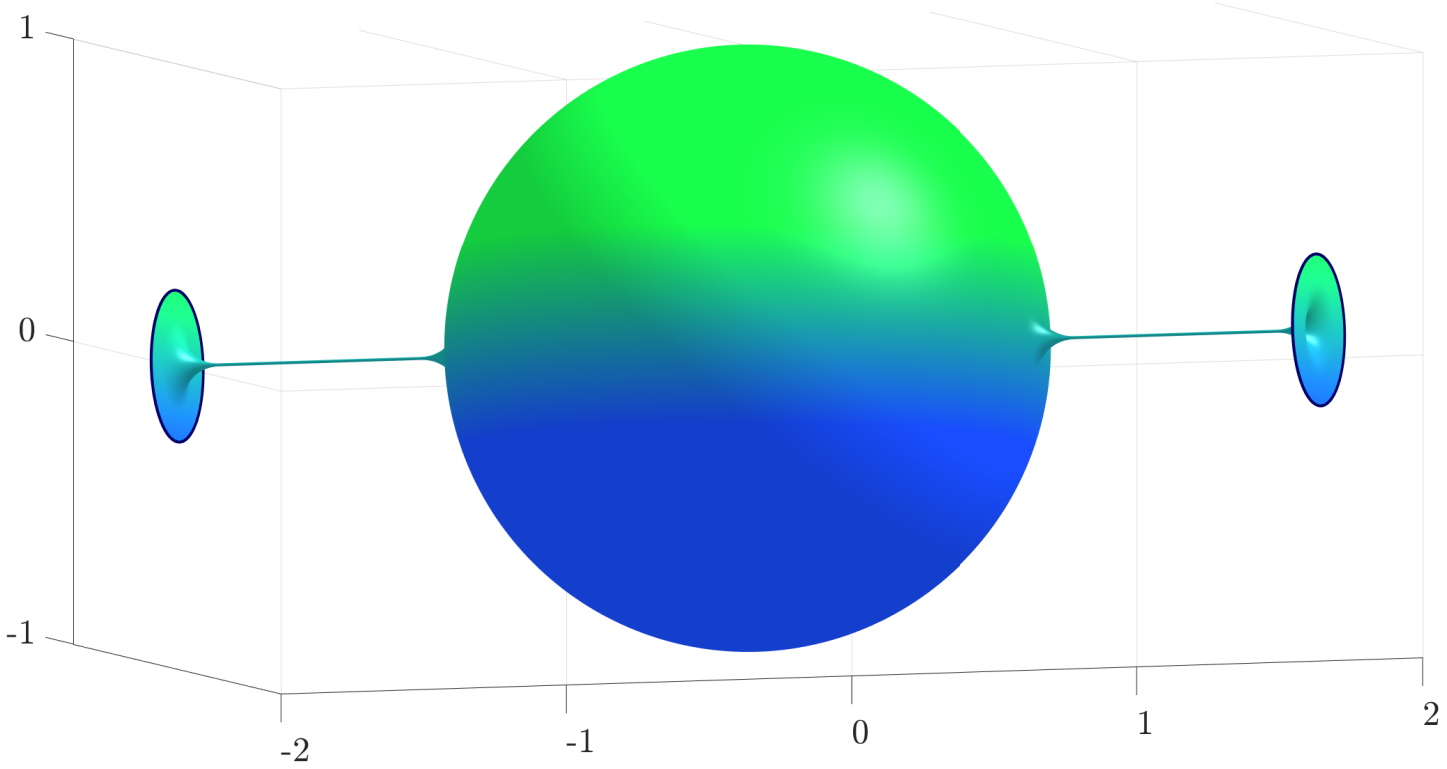


Figure 4.4: An illustration of an image of the flow in $\tilde{N}$.

To decrease the energy of the map $u = (v, w)$ into the warped product $\mathbb{R} \times_f (\widetilde{N}, g_{\widetilde{N}})$, the flow thus wants to stretch out the v component so that the energy of w on this central part is weighted with a small warping factor $f(v)$. This stretched-out image of the v component will in turn drive the degeneration of the metric as proven in Theorem 4.1.1.

We will obtain the required lower bound on $\mathscr{L}(u(t), \mathcal{C}(\sigma(t)))$ in Lemma 4.4.6, where we will use in particular that the v component must map the central region to the part of the line where $-f'$ is decreasing.

Lemma 4.4.5. *Let (N, g_N) be defined as in (4.52) where $f = f_0(\cdot - \bar{v})$ for some $\bar{v} \geq 7$ and some smooth function f satisfying Assumption 4.4.2. Let (u_0, g_0) be initial data as obtained in Lemma 4.4.3 and let $(u = (v, w), g = G_\ell)$ be the corresponding solution of (1.9). Then on the whole existence interval, we have $\ell \leq \ell_2(\bar{v}) := \frac{5\pi^2}{\bar{v}^2}$ and furthermore*

$$v(s) \geq \bar{v} + 1 \text{ for every } |s| \leq 8 \tag{4.57}$$

where (s, θ) are the collar coordinates of $(\mathscr{C}, g)$.

Proof. Lemma 4.4.1 assures that at any time the energy of $w \colon (\mathscr{C}, g) \to (\widetilde{N}, g_{\widetilde{N}})$ is at least 2π and hence $10\pi \geq E(u, g) \geq \min_{\mathscr{C}} f(v) \cdot 2\pi = 2\pi f(\max v)$. This ensures that $f(\max v) \leq 5$ and hence $\max v \geq \bar{v} + 1$. As $v(\pm X(\ell)) = 0$, we hence get

$$2\bar{v} \leq 2 \max v \leq \int_{-X(\ell)}^{X(\ell)} |v'| \, ds \leq (\pi^{-1} E(u, g))^{\frac{1}{2}} \cdot (2X(\ell))^{\frac{1}{2}} \leq (20X(\ell))^{\frac{1}{2}},$$

and thus $\frac{\pi^2}{\ell} \geq X(\ell) \geq \frac{1}{5}\bar{v}^2$, which implies the first claim of the lemma. This estimate ensures in particular that $X(\ell) \geq 8$ for $\bar{v} \geq 7$. By Lemma 4.4.4, thus already the restriction of w to the central region $\{(s, \theta) : |s| \leq 8\}$ must have energy at least 2π and hence $\min_{[-8,8]} f \circ v \leq 5$. As $|f'| \leq \frac{1}{8}$ and $v(x) = v(-x)$ we conclude that $\operatorname{osc}_{[-8,8]} f \circ v \leq \frac{1}{8} \int_0^8 |v'| \, ds \leq \frac{1}{8} \cdot \left(\frac{1}{2\pi} E(u, g)\right)^{\frac{1}{2}} \cdot \sqrt{8} \leq \sqrt{\frac{5}{8}}$. Therefore, $\max_{[-8,8]} f(v(s)) \leq 6$ and hence $v(s) \geq \bar{v} + 1$ for every $|s| \leq 8$ as claimed. □

We are now in a position to prove that the image of maps as considered above stretches out as required in the setting of our main results.

Lemma 4.4.6. *For any $c_3, \Lambda > 0$ and $\delta \in (0, 1)$, respectively $c_3, \Lambda, \alpha > 0$, there exist $\ell_1, c_0 > 0$ such that the following holds true.*
Let (N, g_N) be defined as in (4.52) for a warping function $f = f_0(\cdot - \bar{v})$, $\bar{v} \geq 7$, where f_0 satisfies Assumption 4.4.2 either with decay (i) (for c_3, Λ, δ) or decay (ii) (for c_3, Λ, α). Let $u \colon (\mathscr{C}, g = G_\ell) \to (N, g_N)$ be a map with tension $\|\tau_g(u)\|_{L^2(\mathscr{C}, g)} \leq 1$ and energy $E(u, g) \leq 10\pi$ for which (4.56) and (4.57) hold and suppose that $\ell \leq \ell_1$. Then in the case that f_0 has polynomial decay

$$v_{\max} := \max_{\mathcal{C}(\ell)} v \geq c_0 \, \ell^{-\frac{1}{4}(1+\delta)}, \tag{4.58}$$

while in the case that f_0 has exponential decay

$$v_{\max} := \max_{\mathcal{C}(\ell)} v \geq c_0 \log(\ell^{-1}). \tag{4.59}$$

Proof. The v component of the tension field is given in collar coordinates (s, θ) by

$$\tau_g(u)^v = \rho^{-2} \partial_{ss} v - f'(v) \, e_g(w) \tag{4.60}$$

where $e_g(w) = \frac{1}{2}|dw|_g^2$ denotes the energy density of $w\colon (\mathscr{C}, g) \to (\widetilde{N}, g_{\widetilde{N}})$. As the symmetry of v ensures that $\partial_s v(0) = 0$, we can integrate (4.60) over $([0,s] \times S^1, \rho^2(ds^2 + d\theta^2))$ to obtain that

$$\begin{aligned} -2\pi\partial_s v(s) &= -\int_0^s \int_{S^1} \partial_{ss} v \rho^{-2}\, dv_g = -\int_0^s \int_{S^1} f'(v)\, e_g(w) + \tau_g(u)^v\, dv_g \\ &\geq -f'(v_{\max}) E(w|_{[0,\min(s,8)]\times S^1}) - \mathrm{Area}([0,s]\times S^1, g)^{\frac{1}{2}}, \end{aligned}$$

where we use that $\|\tau_g(u)\|_{L^2(\mathscr{C},g)} \leq 1$ and that $v([-8,8])$ is contained in $[\bar{v}+1, \infty)$ where $-f'$ is decreasing. Combined with (4.56) and (4.9), we hence conclude that

$$-\partial_s v(s) \geq -\tfrac{1}{2} f'(v_{\max}) - \rho^{\frac{1}{2}}(s)(2\pi)^{-\frac{1}{2}} \text{ for } s \geq 8$$

and $-\partial_s v(s) \geq -\rho^{\frac{1}{2}}(s)(2\pi)^{-\frac{1}{2}}$ for $|s| \leq 8$. Using (4.9) as well as that $X(\ell) \geq \frac{\pi^2}{\ell} - C$, we get

$$\begin{aligned} v_{\max} &\geq -\tfrac{1}{2}(X(\ell)-8) f'(v_{\max}) - (2\pi)^{-\frac{1}{2}} \int_0^{X(\ell)} \rho^{\frac{1}{2}}(s)\, ds \geq -4\ell^{-1} f'(v_{\max}) - 4\ell^{-\frac{1}{2}} \\ &= -4\ell^{-1} f_0'(v_{\max} - \bar{v}) - 4\ell^{-\frac{1}{2}} \geq 4(-f_0'(v_{\max})\ell^{-1} - \ell^{-\frac{1}{2}}), \end{aligned} \tag{4.61}$$

where we use that $\ell \in (0, \ell_1)$ for a sufficiently small ℓ_1 and that $-f_0'$ is decreasing on $[1,\infty)$.

We first note that for $\ell_1 \in (0,1)$ chosen sufficiently small, we must have that $v_{\max} \geq \Lambda$. Indeed, as $v_{\max} \geq \bar{v} + 1 \geq 1$ we would otherwise have that $-f_0'(v_{\max}) \geq -f_0'(\Lambda) = c > 0$, where $c = c_3(\frac{2}{1+\delta} - 1)\Lambda^{-\frac{2}{1+\delta}}$, respectively, $c = c_3\alpha$. Hence (4.61) would imply that $\Lambda > v_{\max} \geq 4c\ell^{-1} - 4\ell^{-\frac{1}{2}}$ which leads to a contradiction as $\ell \leq \ell_1$ for sufficiently small ℓ_1.

Suppose first that f_0 as in part (i) of Assumption 4.4.2. If $-\frac{1}{2}\ell^{-1} f_0'(v_{\max}) \geq \ell^{-\frac{1}{2}}$, then (4.61) gives

$$v_{\max} \geq -2\ell^{-1} f_0'(v_{\max}) = 2c_3(\tfrac{2}{1+\delta} - 1) v_{\max}^{-\frac{2}{1+\delta}} \ell^{-1},$$

which implies that $v_{\max} \geq c_0 \ell^{-\frac{1}{4}(1+\delta)}$ for some $c_0 = c_0(\delta, c_3) > 0$ as $\frac{4}{1+\delta} \geq 1 + \frac{2}{1+\delta}$. This establishes the claim (4.58) in this case. Conversely, if $-\frac{1}{2}\ell^{-1} f_0'(v_{\max}) \leq \ell^{-\frac{1}{2}}$, then we get

$$2\ell^{\frac{1}{2}} \geq -f_0'(v_{\max}) = c_3(\tfrac{2}{1+\delta} - 1) v_{\max}^{-\frac{2}{1+\delta}}$$

which again implies that (4.58) holds true for some $c_0 = c_0(\delta, c_3) > 0$.

Suppose now that f_0 decays instead exponentially as in part (ii) of the assumption. We hence know that $-f_0'(v_{\max}) = \alpha c_3 e^{-\alpha(v_{\max} - \Lambda)}$. If $-\frac{1}{2}\ell^{-1} f_0'(v_{\max}) \geq \ell^{-\frac{1}{2}}$, then (4.61) yields

$$v_{\max} \geq 2\alpha c_3 \ell^{-1} e^{-\alpha(v_{\max} - \Lambda)} = c e^{-\alpha v_{\max}} \ell^{-1}, \text{ for some } c > 0.$$

This implies in particular that $e^{2\alpha v_{\max}} \geq \tilde{c}\ell^{-1}$ for some $\tilde{c} > 0$, which, for sufficiently small ℓ_1, yields the claim. Otherwise $-f_0'(v_{\max}) \leq 2\ell^{\frac{1}{2}}$ which again yields that $v_{\max} \geq c \log(\ell^{-1})$ for some $c > 0$ that depends only on α, Λ, c_3, provided ℓ_1 is sufficiently small. □

We can now prove the first part of Theorem 4.1.3 as follows. Let $\delta \in (0,1)$ be any fixed number and let $f = f_0(\cdot - \bar{v})$ for f_0 as in Assumption 4.4.2 with polynomial decay (i) (for some $c_3, \Lambda > 0$). We first observe that the sectional curvature of (N, g_N) is bounded by a constant $\bar{\kappa}$ which is independent of $\bar{v}$ and recall that also the numbers $\ell_1 > 0$ and $c_0 > 0$ obtained in Lemma 4.4.6 do not depend on $\bar{v}$. We furthermore fix $E_0 = 10\pi$ and $\varepsilon_0 = 1$ and let now $\bar{\ell} > 0$ be the number obtained in Theorem 4.1.1 corresponding to this choice of $\bar{\kappa}$, E_0, δ and ε_0, where we stress that $\bar{\ell}$ is independent of $\bar{v}$ which we have yet to choose.

Theorem 4.1.1 hence assures that any solution of (1.9) that satisfies the assumptions (i) (for this $\bar{\ell}$) and (ii) of the theorem must degenerate in finite time. We now finally choose the parameter $\bar{v}$ by which we shift the warping function so that $\ell_2(\bar{v}) \leq \bar{\ell}$ holds true for the number $\ell_2(\bar{v}) = \frac{5\pi^2}{\bar{v}^2}$ obtained in Lemma 4.4.5. Choosing our initial data (u_0, g_0) as in Lemma 4.4.3, we obtain from Lemma 4.4.5 that the resulting solution (u, g) satisfies the first assumption of Theorem 4.1.1 for every time in the maximal existence interval $[0, T)$. This lemma also assures that (4.57) is satisfied for every $t \in [0, T)$, while Lemma 4.4.4 implies that (4.56) holds true. We can thus apply the first estimate (4.58) from Lemma 4.4.6 to conclude that also the second assumption of the theorem is satisfied. Hence Theorem 4.1.1 implies that the solution of Teichmüller harmonic map flow for any such initial data must degenerate in finite time as claimed in the first part of Theorem 4.1.3.

We finally construct a compact target for which solutions of (4.4) degenerate in finite time. For this we will modify the construction of Topping [39] so that the resulting warped product

$$(N_c, g_{N_c}) = (T^2, \gamma) \times_F (\widehat{N}, g_{\widehat{N}})$$

contains a totally geodesic submanifold Σ which is isometric to $[0, \infty) \times_f (\widehat{N}, g_{\widehat{N}})$, for a coupling function $f = f_0(\cdot - \bar{v})$, $\bar{v} \geq 7$, where we now ask that f_0 is as in Assumption 4.4.2 with exponential decay (ii) for $\alpha = 2\pi$, see below for the precise choice of (N_c, g_{N_c}). Here $(\widehat{N}, g_{\widehat{N}})$ is the 3-torus obtained by identifying opposite faces of a suitably large cuboid in $(\widetilde{N}, g_{\widetilde{N}})$ around the origin. We note that alternatively one could also construct such a warped product with a sphere as done in [39] to obtain a target for which Theorem 4.1.3 holds.

To obtain the existence of a suitable solution of the flow (4.4) we first note the following remark.

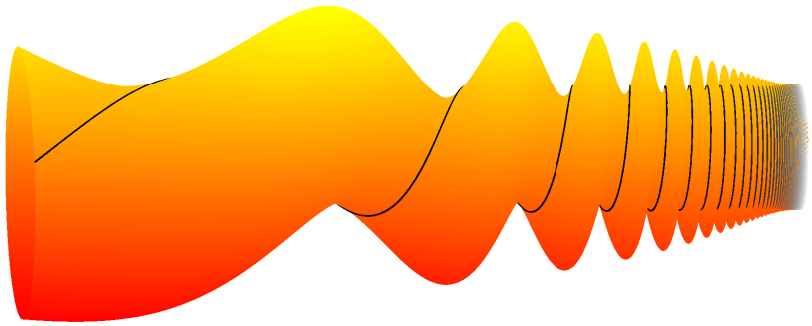
Remark 4.4.7. For any $\ell \in (0, \infty)$ there exists a harmonic map u_ℓ from $([-1, 1] \times S^1, G_\ell)$ to Σ which satisfies the assumptions of Lemma 4.4.6. Indeed such a map can, for example, be obtained as a limit as $t \to \infty$ of a solution of the classical harmonic map flow for an initial map u_0 chosen as in Lemma 4.4.3.

This allows us to obtain a solution $(u_{\ell(t)}, G_\ell(t))$ of the rescaled flow (4.4) for which we can now argue exactly as in the proof of the first part of Theorem 4.1.3 to conclude that it must degenerate in finite time, except that we now apply Theorem 4.1.2 instead of Theorem 4.1.1 (and of course choose $\bar{\ell}$ accordingly) and that we use the second estimate (4.59) of Lemma 4.4.6 instead of the first estimate (4.58) of that lemma. As Σ is a totally geodesic submanifold of our compact target this hence yields an example of a solution to (4.4) into a compact target that degenerates in finite time as claimed in the second part of Theorem 4.1.3.

We finally explain how such a target can be chosen. As motivation, Figure 4.5 illustrates a portion of the torus component T^2 of our warped product manifold $T^2 \times_F \tilde{N}$. We will fix the image of the boundary circles at a point towards the left on the winding line, and the image of the central region (which is mapped near the minimal sphere in $\widetilde{N}$) along the line towards the right. The decay of the warping function F ensures that decreasing energy corresponds to moving the image in T^2 associated to the minimal sphere further along the line to the right, which causes the winding singularity to arise.

The torus component (T^2, γ) of (N_c, g_{N_c}) is exactly as in [39], and, as in [39], we consider the submanifold $\Sigma = \{z = \frac{1}{w}, w \in (0, 1]\} \times_F (\widetilde{N_c}, g_{\widetilde{N_c}})$, which we recall is totally geodesic if the coupling function F satisfies $\partial_x F|_{\{x=0\}} = 0$ with respect to the coordinates $x = \frac{1}{w} - z$, $y = z$ used in [39].

Indeed, writing ∇ for the Levi-Civita connection on the direct product $T^2 \times \tilde{N}$, and $\bar{\nabla}$ for that on the warped product $N = T^2 \times_F \tilde{N}$, we have from Bishop–O'Neill [1] that the relation between



$T^2 \times_F \widetilde{N}$

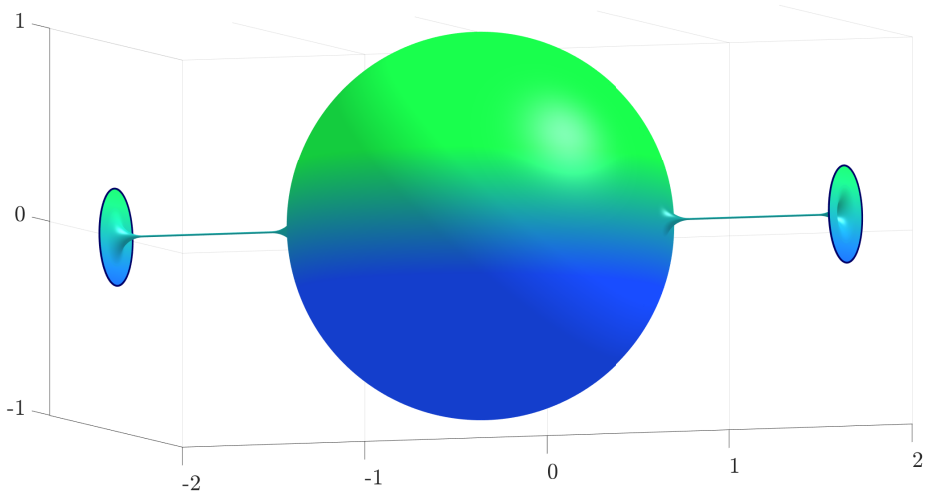


Figure 4.5: A representation of $T^2 \times_F \widetilde{N}$, with the torus component of the totally geodesic submanifold marked in black.

these connections for a vector field $X = (X_1, X_2)$ in $T_{(p_1,p_2)}N \cong T_{p_1}T^2 \oplus T_{p_2}\tilde{N}$ is

$$\bar{\nabla}_X X = \nabla_X X + \frac{X_1(F)}{F}(0, X_2) - \frac{1}{2} g_{\tilde{N}}(X_2, X_2) \cdot (\nabla F, 0). \tag{4.62}$$

Let $\gamma = (\gamma_1, \gamma_2)$ be a geodesic in Σ. Then γ_1 is a geodesic of T^2, so for the direct product spaces,

$$\nabla^{\Sigma}_{\gamma'}\gamma' = \nabla^{\{x=0\}}_{\gamma_1'}\gamma_1' + \nabla^{\tilde{N}}_{\gamma_2'}\gamma_2' = \nabla^{T^2}_{\gamma_1'}\gamma_1' + \nabla^{\tilde{N}}_{\gamma_2'}\gamma_2' = \nabla^{N}_{\gamma'}\gamma'. \tag{4.63}$$

We can therefore compute that in T_pN,

$$\begin{aligned} 0 = \bar{\nabla}^{\Sigma}_{\gamma'}\gamma' &= \nabla^{\Sigma}_{\gamma'}\gamma' + \frac{\gamma_1'(F)}{F}(0, 0, \gamma_2') - \frac{1}{2} g_{\tilde{N}}(\gamma_2', \gamma_2') \cdot (0, \partial_y F, 0) \\ &= \nabla^{N}_{\gamma'}\gamma' + \frac{\gamma_1'(F)}{F}(0, 0, \gamma_2') - \frac{1}{2} g_{\tilde{N}}(\gamma_2', \gamma_2') \cdot (0, \partial_y F, 0) \\ &= \bar{\nabla}^{N}_{\gamma'}\gamma' + \frac{1}{2} g_{\tilde{N}}(\gamma_2', \gamma_2') \cdot (\partial_x F, 0, 0). \end{aligned} \tag{4.64}$$

Thus if $\partial_x F = 0$ on Σ, then Σ is totally geodesic in N.

From [1] we have that the sectional curvature of a plane Π tangent to N at a point $p = (x, y, \tilde{p})$, given by an orthonormal basis $v_1 + w_1, v_2 + w_2$ in T_pN, is

$$\begin{aligned} K(\Pi) = {} & \kappa_{T^2}(x, y) \, \|v_1 \wedge v_2\|^2_{T^2} - F^{\frac{1}{2}}(x, y) \, g_{\tilde{N}}(w_2, w_2) \, \nabla^2(F^{\frac{1}{2}})(v_1, v_1) \\ & + 2F^{\frac{1}{2}}(x, y) \, g_{\tilde{N}}(w_1, w_2) \, \nabla^2(F^{\frac{1}{2}})(v_1, v_2) - F^{\frac{1}{2}}(x, y) \, g_{\tilde{N}}(w_1, w_1) \, \nabla^2(F^{\frac{1}{2}})(v_2, v_2) \\ & + F(x, y) \left(K_{g_{\tilde{N}}}(w_1, w_2) + \|\nabla(F^{\frac{1}{2}}(x, y))\| \right) \|w_1 \wedge w_2\|^2_{\tilde{N}}. \end{aligned} \tag{4.65}$$

We can calculate that the Gaussian curvature of T^2 with its metric is

$$\kappa_{T^2}(x, y) = -6(x + y - (\bar{v} - 1))^{-2} \leq 0. \tag{4.66}$$

We therefore see that for any Π,

$$K(\Pi) \leq C\|F^{\frac{1}{2}}\|_{L^\infty}\|g_{\tilde{N}}\|_{L^\infty}\|\nabla^2(F^{\frac{1}{2}})\|_{L^\infty} + \|F\|_{L^\infty}(\bar{\kappa}_{\tilde{N}} + \|\nabla(F^{\frac{1}{2}})\|_{L^\infty}) \leq C, \tag{4.67}$$

so that N has bounded sectional curvature independent of $\bar{v}$.

We choose F to be of the form

$$F(x, y) = h(x + y - \bar{v})e^{-2\pi(x+y-\bar{v}-\Lambda)}\left(\sin 2\pi\left(x - \tfrac{1}{8}\right) + \sqrt{2}\right) + k(x + y - \bar{v}) + 1,$$

for smooth functions $h, k \geq 0$, where we note that the case $h \equiv 1$ and $k \equiv 0$ would correspond to the coupling function used in [39]. For our construction, we need $\partial_x F|_{\{x=0\}} = 0$, that is,

$$k'(v) + e^{-2\pi(v-\Lambda)}\tfrac{\sqrt{2}}{2}h'(v) = 0, \tag{4.68}$$

as well as that $f_0(v) := F(0, v + \bar{v}) = h(v)e^{-2\pi(v-\Lambda)}\frac{\sqrt{2}}{2} + k(v) + 1$ satisfies Assumption 4.4.2 with decay (ii). Suitable functions h, k can, for example, be obtained as follows: We first select k so that for c_4 chosen below and some $\Lambda = \Lambda(c_4) > 0$, we have $k \equiv 7$ on $(-\infty, 0]$, $k \equiv 0$ on $[\Lambda, \infty)$ and so that the derivative of k satisfies $0 \leq -k' \leq e^{\pi}c_4$ on all of $\mathbb{R}$ with $-k' \equiv e^{\frac{\pi}{2}}c_4$ on $[\frac{1}{2}, \frac{3}{4}]$, $-k' \equiv \frac{1}{4}c_4$ on $[1, \Lambda - 1]$ as well as $-k'' \leq 0$ on $[1, \Lambda]$.

We then set $h(v) := \sqrt{2}\int_0^v e^{2\pi(t-\Lambda)}|k'(t)|dt$, which ensures not only that (4.68) is satisfied, but also that f_0 has the behaviour on $(-\infty, 0]$ and $[\Lambda, \infty)$ required in Assumption 4.4.2. As (4.68) furthermore implies that $-f_0'(v) = \sqrt{2}\pi e^{-2\pi(v-\Lambda)}h(v) \geq 0$ and as a short calculation shows that for c_4 sufficiently small also $-f_0'(v) \leq \frac{1}{8}$ for all $v \in \mathbb{R}$, it remains to check that $-f_0'' \leq 0$ on $[1, \infty)$ which is equivalent to $h' \leq 2\pi h$. This can be easily seen as the choice of k ensures that $2\pi h(1) \geq h'(1)$ and that $2\pi h' - h'' \geq 0$ on $[1, \infty)$.

Appendix A

Alternative approaches to existence for the flow from cylinders

This appendix will be devoted to the flow analogous to that considered in Chapter 2 under the standard three-point condition, and the framework for the corresponding existence results along with the partial results we have obtained. In Appendix A.1, we consider regularity issues and proposed solutions in these circumstances, and in Appendix A.2 we outline the remainder of the argument and the issues involved in preventing concentration of energy not present for the three-point condition used in Chapter 2.

While this material does not constitute a full result in this direction, nonetheless we summarise the situation, since for asymptotic convergence in the situation where a three-point condition degenerates, one may be forced to instead consider such a standard three-point condition in place of the one used above. Indeed, the original motivation for choosing a condition fixing three points was to simplify the eventual asymptotic argument.

A.1 Problems with H^2-regularity under the standard three-point condition

In this section, we briefly report on the hurdles that we encountered in attempting to obtain H^2-regularity of a minimiser of (2.7) normalised with the standard three-point condition; in the next section, we show how the rest of the argument would proceed.

More precisely, we sought to prove the following. We write $C_0^* = C_0 \setminus \{(\pm 1, 2\pi k/3)\}$; as in [28], we do not expect regularity at the points fixed by the three-point condition.

Conjecture A.1.1. Let N be a closed Riemannian manifold, let $h > 0$, and let $g \in \widetilde{\mathcal{M}}$. Suppose X is a minimiser of $J^h_{f,g}$ defined in (2.7) in the class $(H^1 \cap C)_{\Gamma,*}(C_0, N)$ for some $W \in L^2(C_0, \mathbb{R}^\nu)$. Then $X \in H^2_{\mathrm{loc}}(C_0^*, N)$.

We remark that in the Euclidean case, this regularity was a simple consequence of the work of Duzaar and Scheven in [9], since in that situation one can view the equation as a special case (of mean curvature $H = 0$) of the equation $\partial_t u - \Delta u = -2H(u)\, u_s \wedge u_\theta$. However, this is not possible in our nonlinear setting.

We would like to be able to test against functions like $Z = \frac{1}{2} h^{-2}\big(\chi(X, X_+) + \chi(X, X_-)\big)$ used by Chang and Liu in [4] as described below which work in place of $\partial_{\theta\theta} X$ which is not admissible. However, while this satisfies the integral version of the three-point condition since it is formed from rotations of the original map, it fails to satisfy the standard three-point condition; the special points are mapped rather than to the value originally specified instead to averages of the nearby function values.

Since we only hope to obtain regularity away from the points fixed by the condition, we could imagine using a cutoff function supported away from these points. However, a test function constructed in such a way would no longer be admissible, since there is no way to guarantee that the restriction to the boundary will be monotone; counterexamples exist for instance where the original trace looks locally like $\max(\cdot, 0)$ in the region where the cutoff function is small but increasing, and can be constructed to work at all scales.

Since we could not use a standard cutoff argument, we explored other avenues, which we report below but which so far have also failed to allow us to close the argument.

The first was a method of odd reflection. Following Ji and Wang in [20], by modifying the metric in a neighbourhood of Γ, we can assume Γ is a closed geodesic of N. For $\delta < \frac{1}{2} \operatorname{inj}_N$, let $B_\delta = \{(u, v) \in N \times N : d_N(u, v) < \delta\}$, and define

$$\chi(u, v) = \exp_u^{-1} v \quad \text{and} \quad \psi(u, v) = \exp_u(-\exp_u^{-1} v). \tag{A.1}$$

(For $N = \mathbb{R}^n$, these would simply be $\chi_{\mathbb{R}^n}(u, v) = v - u$ and $\psi_{\mathbb{R}^n}(u, v) = 2u - v$.)

Extend $X|_{[-1,1]\times[0,\frac{2\pi}{3}]}$ by odd reflection in N to $\tilde{X} \in H^1([-1, 1] \times [-h, \frac{2\pi}{3} + h], N)$ as

$$\tilde{X}(z, \theta) = \begin{cases} \psi\big(X(z, 0), X(z, -\theta)\big) & \text{for } \theta \in (-h, 0) \\ X(z, \theta) & \text{for } \theta \in \big(0, \frac{2\pi}{3}\big) \\ \psi\big(X(z, \frac{2\pi}{3}), X(z, \frac{4\pi}{3} - \theta)\big) & \text{for } \theta \in \big(\frac{2\pi}{3}, \frac{2\pi}{3} + h\big)\,. \end{cases} \tag{A.2}$$

This allows us to define restricted angular translates $X_\pm \colon C_0 \to N$ by

$$X_\pm(z, \theta) = \begin{cases} \tilde{X}(z, \theta \pm h) & \text{for } \theta \in \big(0, \frac{2\pi}{3}\big) \\ X(z, \theta) & \text{for } \theta \in \big(\frac{2\pi}{3}, 2\pi\big)\,. \end{cases} \tag{A.3}$$

We would seek to emulate Lemma 3.1 of Chang and Liu in [4] for these new maps $X_\pm$, showing that in the setting of Conjecture A.1.1, there is $h_0 > 0$ such that for all $h \in (0, h_0)$,

$$\int_{C_0} |\nabla \partial_{\pm h} X|^2 \, d\theta \, dz \le C \left(\int_{C_0} |\nabla X|^2 \left(|\partial_h X|^2 + |\partial_{-h} X|^2 \right) d\theta \, dz + \|W\|_{L^2}^2 \right). \tag{A.4}$$

where $\partial_{\pm h} X = h^{-1}(X_\pm - X)$ with $X_\pm$ as defined in (A.3).

In order to prove this, we would want to choose a test function

$$Z = \tfrac{1}{2} h^{-2}\big(\chi(X, X_+) + \chi(X, X_-)\big), \tag{A.5}$$

where χ is as in (A.1) and $X_\pm$ are as in (A.3). The $X_\pm$ are individually discontinuous at $[-1, 1] \times \{0, \frac{2\pi}{3}\}$, but Z is still continuous on C_0.

To see whether $X_\pm \in H^1(C_0, N)$, notice the target is correct by construction, and while we can calculate (for instance for X_-, similarly X_+)

$$\int_{C_0} |\partial_\theta X_-|^2 \le \|\psi\|_{C^1}^2 \int_{-1}^{1} \int_0^h |\partial_\theta X|^2 + \int_{-1}^{1} \int_0^{\frac{2\pi}{3} - h} |\partial_\theta X|^2 + \int_{-1}^{1} \int_{\frac{2\pi}{3}}^{2\pi} |\partial_\theta X|^2 < \infty, \tag{A.6}$$

the $\partial_z X_\pm$ component can become unbounded.

Another attempt was to follow the argument used in [19] by Imbusch and Struwe in order to construct test functions that preserve our three point condition.

We define a space of diffeomorphisms $\mathcal{F} = (f_{b,\phi})$ similar to the $(h_{b,\phi})$ introduced at (2.17) by

$$f_{b^\pm,\phi^\pm}(s,\theta) = \left(s, \hat{M}_{b(s,\theta),\phi(s,\theta)}(\theta)\right) \tag{A.7}$$

where

$$b(s,\theta) = \begin{cases} \lambda_1(|s|)b^+ + (1-\lambda_1(|s|))\,\frac{1}{2}(b^+ + b^-), & s \geq 0 \\ \lambda_1(|s|)b^- + (1-\lambda_1(|s|))\,\frac{1}{2}(b^+ + b^-), & s < 0, \end{cases} \tag{A.8}$$

$$\phi(s,\theta) = \begin{cases} \lambda_2(|s|)\phi^+ + (1-\lambda_2(|s|))\,\frac{1}{2}(\phi^+ + \phi^-), & s \geq 0 \\ \lambda_2(|s|)\phi^+ + (1-\lambda_2(|s|))\,\frac{1}{2}(\phi^+ + \phi^-), & s < 0, \end{cases} \tag{A.9}$$

where $\lambda_{1,2}$ and $\hat{M}$ are as defined at (2.17). Instead of agreeing with the identity around a central geodesic, they instead remain conformal when $b^\pm = 0$ and $\phi^+ = \phi^-$, and the space is closed under inverses.

For such a diffeomorphism f_h with $|\, f_h - \mathrm{id}\,| \leq h$, write $\partial_h^+ X = h^{-1}(X \circ f_h - X)$ and $\partial_h^- X = h^{-1}(X \circ f_h^{-1} - X)$. Note that these are a generalisation of the difference quotients introduced by Chang and Liu in [4], recovered by the choice of $f_h(s,\theta) = (s, \theta + h)$. We also introduce difference quotients with respect to the space of diffeomorphisms

$$\partial_h^{\mathcal{F}} X = \sup\left\{ h^{-1}(X \circ f_h - X) : f_h \in \mathcal{F} \text{ with } |\, f_h - \mathrm{id}\,| < h \right\}. \tag{A.10}$$

The following result can be obtained from the similar property for $\mathcal{F}$ replaced with the group of Möbius transforms of the disc as Proposition 3.2(ii) of Imbusch and Struwe in [19].

Lemma A.1.2. *Suppose $X \in \mathcal{M}^*$, and $f_h \in \mathcal{F}$ with $|\, f_h - \mathrm{id}\,| \leq h$. Then there is $h_0 > 0$ such that for all $h \in (0, h_0)$ and $\varepsilon \in (0, \frac{1}{2}h^2)$, there is $f \in \mathcal{F}$ with $(X + \varepsilon \partial_h^- \partial_h^+ X) \circ f \in \mathcal{M}^*$ and $|\, f - \mathrm{id}\,| \leq Ch$ as $h \to 0$.*

We then hoped to prove an estimate corresponding to that used to prove Proposition 2.2 of Imbusch and Struwe in [19], that in the setting of Conjecture A.1.1, there are $h_0, \delta_0 > 0$ such that for all $h \in (0, h_0)$, all $\delta \in (0, \delta_0)$, and each pair $(f_h, f) \in \mathcal{F} \times \mathcal{F}$ with $|\, f_h - \mathrm{id}\,| \leq h$ given above,

$$\begin{aligned} \int_{C_0} \nabla X \cdot \nabla(\partial_h^- \partial_h^+ X \circ f) \leq \delta \Bigg(\int_{C_0} |\nabla(\partial_h^+ X \circ f)|^2 + |\nabla(\partial_h^- X \circ f)|^2 + \left| \frac{\nabla(X \circ f - X)}{h} \right|^2 \Bigg) \\ + C \int_{C_0} |\nabla X|^2 \left(|\partial_h^+ X|^2 + |\partial_h^- X|^2 + \left| \frac{X \circ f - X}{h} \right|^2 \right) + C(1 + \|W\|_{L^2}^2). \end{aligned}$$

We note that χ defined in (A.1) has the properties that for $(u, v) \in C^1(C_0 \times C_0, B_\delta)$,

$$\begin{aligned} |\chi(u,v) - (v-u)| &\leq C|v-u|^2, \\ |\nabla\chi(u,v) - \nabla(v-u)| &\leq C|\nabla u|\,|v-u|^2 + C|\nabla(v-u)|\,|v-u|. \end{aligned} \tag{A.11}$$

For $\varepsilon \in (0, \frac{1}{2}h^2)$ as above, define a test function by

$$Z = (1 - \frac{2\varepsilon}{h^2})\,\chi(X, X \circ f) + \frac{\varepsilon}{h^2}\left(\chi(X, X \circ \widetilde{f_+}) + \chi(X, X \circ \widetilde{f_-}) \right), \tag{A.12}$$

where $\widetilde{f_+} = f_h \circ f$ and $\widetilde{f_-} = f_h^{-1} \circ f$, for f corresponding to X and f_h given by Lemma A.1.2. This is well-defined for h_0 sufficiently small. However, with our inhomogeneity due to the eventual time derivative, we end up with terms like $\int_{C_0} W \cdot (X \circ f - X)$ that can't be bounded using the symmetry arguments of Imbusch and Struwe in [19], and therefore cannot close the argument.

Our final suggestion to deal with this issue would be to adapt some of the ideas of Duzaar and Scheven in [9], replacing arguments involving second order difference quotients using an intermediate step involving $W^{1,4}$-regularity and Calderón–Zygmund estimates. While this approach is plausibly viable, with our nonlinear setting and absence of the structure present in their flow equations, we opted to instead pursue the approach using the alternative three-point condition where the regularity could be established.

In any case, even with such a regularity result, we would still need to exclude concentration of energy at the points fixed by the three-point condition. As discussed in the next section, there are significant issues arising from the nonlinearity of our setting.

A.2 Results for H^2-regular minimisers under the standard three-point condition

In this section, we outline the argument for short-time existence now under the standard three-point condition. We detail all necessary results required assuming the regularity result discussed in Section A.1. We note that the key difficulty in [28] was to exclude concentration of energy at the three special points on each boundary curve. While the other parts are straightforward extensions of material in [28], we discuss this aspect at Lemma A.2.4.

Without the regularity result discussed in the previous section, we instead need as an assumption that the critical points of the variational problem we consider are H^2 away from the three points. In this case, we can use the arguments in [28] and [9] to obtain the following estimate, cf. Proposition 3.6 of [28] and the corresponding Theorem 8.3 of [9]. We write $C^* = C_0 \setminus \{P^\pm\}$, and for $d > 0$ we write $C^*(d) = C_0 \setminus \bigcup B_d^g(P^\pm)$, where $P^\pm = 2\pi k/3$ are the points fixed under the standard three-point condition.

Lemma A.2.1 (H^2-estimate, up to boundary away from $P^\pm$)**.** *Let K be a compact subset of $\widetilde{\mathcal{M}}$, and let $E_0 < \infty$. There exist $\varepsilon_1, r_0 > 0$ such that for all $r \in (0, r_0)$, we have the following: Let $u \in (H^1 \cap C)_{\Gamma,*}(C_0, N) \cap H^2_{loc}(C_0^*, N)$ with $E(u, g) \leq E_0$, let $g \in K$, and let $f \in L^2(C_0, \mathbb{R}^\nu)$ and $w \in C_0$. Suppose u satisfies (2.23) for this f and for all variations with $\operatorname{supp}(v) \subset B_r^g(w)$ with $P^\pm \notin B_r^g(w)$. Suppose further that $E(u, B_r^g(w)) < \varepsilon_1$. Then we have*

$$\int_{B^g_{r/2}(w)} |\nabla_g^2 u|_g^2 + |du|_g^4 \, dv_g \leq \frac{C}{r^2} \int_{B_r^g(w)} |du|_g^2 \, dv_g + C \int_{B_r^g(w)} |f|^2 \, dv_g. \tag{A.13}$$

We first show we have no concentration of energy in a neighbourhood of general boundary points in the case that N supports no bubbles. The proof of Lemma 3.8 in [28] works more generally for targets supporting no bubbles, and with Lemma A.2.1 in place of Proposition 3.3 in [28], the result follows.

Lemma A.2.2 (No concentration, I)**.** *Suppose N additionally supports no bubbles, and let K be a compact subset of $\widetilde{\mathcal{M}}$. For any $\Lambda, E_0 < \infty$ and $d, \varepsilon > 0$, there is $r > 0$ such that the following holds:*

Let $u \in (H^1 \cap C)_{\Gamma,*}(C_0, N) \cap H^2_{loc}(C_0^*, N)$ with $E(u, g) \leq E_0$ and which satisfies (2.23) with $g \in K$

and with $f \in L^2(C_0, \mathbb{R}^\nu)$ *with* $\|f\|_{L^2} \leq \Lambda$. *Then we have for any* $w \in C^*(d)$,

$$E(u, B_r^g(w)) \leq \varepsilon. \tag{A.14}$$

If N supports bubbles, then this contradiction argument fails, as we cannot conclude that harmonic maps with bounded energy from $\mathbb{R}^2$ must be constants. However, for the construction resulting in maps from $\mathbb{H}$ which are constant on $\partial\mathbb{H}$, we can use Lemma 2.5.4 to find a restriction on the types of bubbles that can form.

Lemma A.2.3 (Bubble formation)**.** *Let* K *be a compact subset of* $\widetilde{\mathcal{M}}$, *let* $\Lambda, E_0 < \infty$ *and let* $d, \varepsilon > 0$. *Suppose* $u_i \in H^1_{\Gamma,*}(C_0, N) \cap H^2_{loc}(C_0^*, N)$ *with* $E(u_i, g_i) \leq E_0$ *satisfying* (2.23) *for* $g \in K$ *and variations* v *with* $supp(v) \subset C^*$. *Suppose further that there are sequences* $r_i \to 0$, $w_i \in C^*(d/2)$ *and* $\lambda_i \uparrow \infty$ *such that*

$$E(u_i, B_{r_i}^{g_i}(w_i)) = \varepsilon = \max_{B^{g_i}_{\lambda_i r_i}(w_i)} E(u_i, B_{r_i}^{g_i}(\cdot)). \tag{A.15}$$

Then $\operatorname{dist}(w_i, \partial C_0) \cdot r_i^{-1} \to \infty$ *for some subsequence.*

In the case that N is allowed to support bubbles, we remark that we therefore only need consider bubbles arising from the interior of M with $\operatorname{dist}_{g_i}(p_i, \partial C_0) \cdot r_i^{-1} \to \infty$; i.e., those which do not approach the boundary.

It remains to consider the concentration of energy at the points $P^\pm$. Topology can now be an issue since the method used in the Euclidean case involves interpolating between any two functions into the target. As a first step, we could restrict our attention to targets N which have no topology and are Euclidean off some compact set, but even in this situation, we will encounter significant issues not seen in the Euclidean setting.

In the Euclidean setting, Rupflin in [28] is able to obtain the following result preventing concentration of energy at the points $P^\pm$.

Lemma A.2.4 (Lemma 3.11, [28])**.** *Suppose* $(N, g_N) = (\mathbb{R}^n, g_E)$, *and let* K *be a compact subset of* $\widetilde{\mathcal{M}}$. *For any* $\Lambda, M, E_0 < \infty$, *and* $d_0, \varepsilon > 0$, *there is* $r > 0$ *such that the following holds:*

Let $u \in H^1_{\Gamma,*}(C_0, N) \cap H^2_{loc}(C_0^*, N)$ *with* $E(u, g) \leq E_0$, *which has* $\|u\|_{L^\infty} \leq M$ *and which satisfies* (2.23) *for* $f \in L^2$ *with* $\|f\|_{L^2} \leq \Lambda$ *and* $g \in K$. *Then for any* $x_0 = (x, \theta) \in C_0$ *with* $|x| \geq d_0$, *we have*

$$E(u, B_r^g(x_0)) < \varepsilon. \tag{A.16}$$

The proof of this result relies on the linear nature of the Euler–Lagrange inequality in such a setting, and furthermore involves interpolating between variations used to test the inequality, which is not possible in a general Riemannian target.

While we were able to circumvent the interpolation issue in the curved setting by considering the maps χ and ψ defined in (A.1), the Euler–Lagrange inequality now takes the form

$$\int_{C_0} \langle du_i, dv_i \rangle + \int_{C_0} \Gamma(u_i)(du_i, du_i) \cdot v_i + \int_{C_0} f_i \cdot v_i \geq 0 \tag{A.17}$$

for $v_i \in T^+_{u_i} H^1_{\Gamma,*}(C_0, N)$. This leads to a term which is quadratic in u with no sign, which we are unable to control. Since near points where the energy concentrates we cannot expect $W^{1,4}$ convergence of the map component, new insights would be required in order to adapt the argument used in the Euclidean case, and indeed it is unclear whether such a result must hold in general.

If we were able to overcome this second obstacle, we would proceed as in [28], passing through an analogue of the H^1-control for solutions obtained in Corollary 3.13 of [28] to conclude the following, based on the argument used in Section 2.6.

Conjecture A.2.5. Let N be a complete Riemannian manifold with nonpositive sectional curvature, and let $(u_0, g_0) \in (H^1 \cap C)_{\Gamma,*}(C_0, N) \times \widetilde{\mathcal{M}}$ be initial data. Then there exist $T, h_0 > 0$ and a compact subset K of $\widetilde{M}$ so the following hold true:

(i) For every $h \in (0, h_0)$ the approximate solutions (u^h, g^h) exist on $[0, T]$, and $g^h(t) \in K$ for every $t \in [0, T]$.

(ii) For every $h_j \to 0$ there exists a subsequence such that (u^{h_j}, g^{h_j}) converges to a limit (u, g) in $C^0_t L^2_x \times C^{0,\alpha}_t C^k_x$ for any $\alpha < 1$ and any k, and this limit is a weak solution of Teichmüller harmonic map flow (2.1)–(2.2) for variations in $L^2([0, T], T^+_u(H^1 \cap C)_{\Gamma,*}(C_0, N))$.

(iii) The limit (u, g) satisfies the energy inequality (2.3) and corresponding stationarity condition (2.5) of [28].

We remark that a potential concentration of energy is of concern to us only because our flow has a metric component. Such a result precluding concentration of energy is not required in the stationary case or for disctype surfaces, and indeed is not obtained for example in the works of Duzaar and Scheven or Chang and Liu. These issues encouraged us to proceed with the approach of Chapters 2 and 3 using the alternative three-point condition. Indeed, such issues lead us to question whether the standard three-point condition is suitable for such nonlinear problems.

Bibliography

[1] R. L. Bishop and B. O'Neill, *Manifolds of negative curvature*, Trans. Amer. Math. Soc. **145** (1969) 1–49.

[2] K. Chang and J. Liu, *Heat flow for the minimal surface with Plateau boundary condition*, Acta Math. Sin. (Engl. Ser.) **19** (2003) 1–28.

[3] K. Chang and J. Liu, *Another approach to the heat flow for Plateau problem*, J. Differ. Equations **189** (2003) 46–70.

[4] K. Chang and J. Liu, *An evolution of minimal surfaces with Plateau condition*, Calc. Var. PDE **19** (2004) 117–163.

[5] J. Chen and G. Tian, *Compactification of moduli space of harmonic mappings*, Comment. Math. Helv. **74** (1999) 201–237.

[6] R. Courant, *The existence of minimal surfaces of given topological structure under prescribed boundary conditions*, Acta Math. **72** (1940) 51–98.

[7] W. Ding, J. Li and Q. Liu, *Evolution of minimal torus in Riemannian manifolds*, Invent. Math. **165** (2006) 225–242.

[8] W. Ding and G. Tian, *Energy identity for a class of approximate harmonic maps from surfaces*, Comm. Anal. Geom. **3** (1995) 543–554.

[9] F. Duzaar and C. Scheven, *The evolution of H-surfaces with a Plateau boundary condition*, Ann. Inst. H. Poincaré Anal. Non Linéaire **32** (2015) 109–157.

[10] J. Eells and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86** (1964) 109–160.

[11] M. E. Gage, *An isoperimetric inequality with applications to curve shortening*, Duke Math. J. **50** (1983) 1225–1229, *doi:10.1215/S0012-7094-83-05052-4*.

[12] J. Gray and M. Micallef, *The work of Jesse Douglas on minimal surfaces*, Bull. Amer. Math. Soc. **45** (2008) 293–302, *doi.org/10.1090/S0273-0979-08-01192-0*.

[13] R. D. Gulliver, R. Osserman and H. L. Royden, *A theory of branched immersions of surfaces*, Amer. J. Math. **95** (1973) 750–812.

[14] J. Haga, K. Hoshino and N. Kikuchi, *Construction of harmonic map flows through the method of discrete Morse flows*, Comput. Vis. Sci. **7** (2004) 53–59, *doi.org/10.1007/s00791-004-0130-7*.

[15] R. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differ. Geom. **17** (1982) 255-306.

[16] G. Huisken, *Flow by mean curvature of convex surfaces into spheres*, J. Differ. Geom. **20** (1984) 237–266.

[17] T. Huxol, *Limiting behaviour of the Teichmüller harmonic map flow*, Preprint, 2017, *arXiv:1711.08844 [math.DG]*.

[18] T. Huxol, M. Rupflin and P. M. Topping, *Refined asymptotics of the Teichmüller harmonic map flow into general targets*, Calc. Var. PDE **55** (2016) 85, *doi:10.1007/s00526-016-1019-2*.

[19] C. Imbusch and M. Struwe, *Variational problems for minimal surfaces*, Progr. Nonlinear Differential Equations Appl. **35** (1999) 477–498.

[20] M. Ji and G. Y. Wang, *Minimal surfaces in Riemannian manifolds*, Mem. Amer. Math. Soc. **495** (1993).

[21] J. Jost, *Two-dimensional geometric variational problems*, Pure & Applied Mathematics (Wiley-Interscience, Chichester, 1991).

[22] L. Lemaire, *Harmonic applications of Riemannian surfaces*, J. Differ. Geom. **13** (1978) 51–78.

[23] R. Moser, *Weak solutions of a biharmonic map heat flow*, Adv. Calc. Var. **2** (2009) 73–92, *doi.org/10.1515/ACV.2009.004*.

[24] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, 2002, *arXiv:math.DG/0211159*.

[25] B. Randol, *Cylinders in Riemann surfaces*, Comment. Math. Helv. **54** (1979) 1–5.

[26] C. Robertson and M. Rupflin, *Finite-time degeneration for variants of Teichmüller harmonic map flow*, J. Lond. Math. Soc. **102** (2020) 535–556, *doi:10.1112/jlms.12327*.

[27] M. Rupflin, *Flowing maps to minimal surfaces: Existence and uniqueness of solutions*, Ann. Inst. H. Poincaré Anal. Non Linéaire **31** (2014) 349–368.

[28] M. Rupflin, *Teichmüller harmonic map flow from cylinders*, Math. Ann. **368** (2017) 1227–1276.

[29] M. Rupflin, *Hyperbolic metrics on surfaces with boundary*, J. Geom. Anal. (2020), *doi.org/10.1007/s12220-020-00386-2*.

[30] M. Rupflin and M. R. I. Schrecker, *Analysis of boundary bubbles for almost minimal cylinders*, Calc. Var. PDE **57** (2018) 121. *doi.org/10.1007/s00526-018-1397-8*.

[31] M. Rupflin and P. M. Topping, *Flowing maps to minimal surfaces*, Amer. J. Math. **138** (2016) 1095–1115.

[32] M. Rupflin and P. M. Topping, *Teichmüller harmonic map flow into nonpositively curved targets*, J. Differ. Geom. **108** (2018) 135–184.

[33] M. Rupflin and P. M. Topping, *Horizontal curves of hyperbolic metrics*, Calc. Var. PDE **57** (2018) 106, *doi.org/10.1007/s00526-018-1386-y*.

[34] M. Rupflin and P. M. Topping, *Global weak solutions of the Teichmüller harmonic map flow into general targets*, Anal. PDE **12** (2018) 815–842.

[35] M. Rupflin, P. M. Topping and M. Zhu, *Asymptotics of the Teichmüller harmonic map flow*, Adv. Math. **244** (2013) 874–893.

[36] J. Sacks and K. Uhlenbeck, *The existence of minimal immersions of 2-spheres*, Ann. of Math. **113** (1981) 1–24.

[37] M. Struwe, *On the evolution of harmonic mappings of Riemannian surfaces*, Comment. Math. Helv. **60** (1985) 558–581.

[38] M. Struwe, *Plateau's Problem and the Calculus of Variations. (MN-35)*, (Princeton University Press, 1989).

[39] P. M. Topping, *Winding behaviour of finite-time singularities of the harmonic map heat flow*, Math. Z. **247** (2004) 279–302.

[40] A. Tromba, *Teichmüller theory in Riemannian geometry*, Lectures in Mathematics. ETH Zürich (Birkhäuser, 1992).

[41] S. Wolpert, *Behavior of geodesic-length functions on Teichmüller space*, J. Differ. Geom. **79** (2008) 277–334.

[42] M. Zhu, *Harmonic maps from degenerating Riemann surfaces*, Math. Z. **264** (2010) 63.