

Global solution for massive Maxwell-Klein-Gordon equations

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Abstract

We derive the asymptotic properties of the mMKG system (Maxwell coupled with a massive Klein-Gordon scalar field), in the exterior of the domain of influence of a compact set. This complements the previous well known results, restricted to compactly supported initial conditions, based on the so called hyperboloidal method. That method takes advantage of the commutation properties of the Maxwell and Klein Gordon with the generators of the Poincaré group to resolve the difficulties caused by the fact that they have, separately, different asymptotic properties. Though the hyperboloidal method is very robust and applies well to other related systems it has the well known drawback that it requires compactly supported data. In this paper we remove this limitation based on a further extension of the vector-field method adapted to the exterior region. Our method applies, in particular, to nontrivial charges. The full problem could then be treated by patching together the new estimates in the exterior with the hyperboloidal ones in the interior. This purely physical space approach introduced here maintains the robust properties of the old method and can thus be applied to other situations such as the coupled Einstein Klein-Gordon equation.

1 Introduction

In this paper, we study the small data global solutions to the massive Maxwell-Klein-Gordon equations on \mathbb{R}^{3+1} . To define the equations, let $A = A_\mu dx^\mu$ be a 1-form. The covariant derivative associated to this 1-form is

$$D_\mu = \partial_\mu + \sqrt{-1}A_\mu,$$

which can be viewed as a $U(1)$ connection on the complex line bundle over \mathbb{R}^{3+1} with the standard flat metric $m_{\mu\nu}$. Then the curvature 2-form F is given by

$$F_{\mu\nu} = -\sqrt{-1}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu = (dA)_{\mu\nu}.$$

This is a closed 2-form or equivalently F satisfies the Bianchi identity

$$\partial_\gamma F_{\mu\nu} + \partial_\mu F_{\nu\gamma} + \partial_\nu F_{\gamma\mu} = 0. \quad (1)$$

Note also that,

$$(D_\mu D_\nu - D_\nu D_\mu)\phi = \sqrt{-1}F_{\mu\nu}\phi. \quad (2)$$

The massive Maxwell-Klein-Gordon equations (mMKG) is a system for the connection field A and the complex scalar field ϕ :

$$\begin{cases} \partial^\nu F_{\mu\nu} = \Im(\phi \cdot \overline{D_\mu \phi}) = J[\phi]_\mu; \\ D^\mu D_\mu \phi - \phi = \square_A \phi - \phi = 0, \end{cases} \quad (\text{mMKG})$$

in which the mass of the scalar field is normalized to be 1. It is well known that this system is gauge invariant in the sense that $(A - d\chi, e^{i\chi}\phi)$ solves the same equation (mMKG) for any potential function χ .

In this paper, we consider the Cauchy problem to (mMKG) with initial data (ϕ_0, ϕ_1, E, H) :

$$F_{0i}(0, x) = E_i(x), \quad {}^*F_{0i}(0, x) = H_i(x), \quad \phi(0, x) = \phi_0, \quad D_0\phi(0, x) = \phi_1,$$

where *F is the Hodge dual of the 2-form F . The (mMKG) equation system is an over-determined system which means that the data set has to satisfy the compatibility condition

$$\operatorname{div}(E) = \Im(\phi_0 \cdot \overline{\phi_1}), \quad \operatorname{div}(H) = 0$$

with the divergence taken on the initial hypersurface \mathbb{R}^3 . For solutions of (mMKG), the total energy as well as the total charge

$$q_0 = \frac{1}{4\pi} \int_{\mathbb{R}^3} \Im(\phi \cdot \overline{D_0\phi}) dx = \frac{1}{4\pi} \int_{\mathbb{R}^3} \operatorname{div}(E) dx \quad (3)$$

are conserved. The existence of nonzero charge has a long range effect on the asymptotic behaviour of the solution as the electric field $E_i = F_{0i}$ has a tail $q_0 r^{-3} x_i$ at any fixed time t .

The Maxwell-Klein-Gordon system (mMKG) has drawn extensive attention in the past. The pioneering works [3] and [4] of Eardley-Moncrief established a global existence result to the more general Yang-Mills-Higgs equations with sufficiently smooth initial data in \mathbb{R}^{3+1} . For the massless case, when the scalar field has the same radiation properties as the Maxwell field, Klainerman-Machedon [12] obtained a global existence and regularity result in the energy space by introducing the celebrated bilinear estimates for null forms [11]. These type of estimates, as well as further improvements made by various authors, have revolutionized our understanding of optimal well-posedness for classical field equations, such as Maxwell-Klein-Gordon, Yang-Mills, Wave Maps, Einstein Field Equations etc, whose ultimate aim is to provide an effective continuation argument¹ consistent with the scaling properties of the equations, see [9] for a general discussion of the topic. For such works, in the context of massless MKG, we refer to [13], [25], [16], [17], [20] and references therein.

The long time asymptotic behavior of smooth solutions to the massless MKG equations in \mathbb{R}^{3+1} is also relatively well understood. The vector field method introduced by Klainerman [8] is sufficiently robust to obtain the pointwise decay estimates for small solutions with non-trivial charges, see e.g. [19]. For the restricted case of compactly supported initial data, one can also use the conformal compactification method² [1], [21]. By combining the Eardley-Moncrief result with the conformal method one can also obtain similar results for large compactly supported data [2]. For large initial conditions, with nontrivial charges, the first result establishing quantitative energy flux decay of the solutions are due to S. Yang in [28]. Finally, by combining the modified vector field method of [28], [29] with the conformal compactification method, Yang-Yu in [30] and [31] have recently given a full description of the global asymptotic dynamics of massless MKG equations, with large initial data and nontrivial charges, in \mathbb{R}^{3+1} .

The decay properties of the massive Maxwell-Klein-Gordon (mMKG) equations are far less understood. In the case of an uncoupled, nonlinear scalar Klein Gordon equation [7], Klainerman found a variation of the vectorfield method, based on the standard hyperboloid foliation of the interior of a forward light cone, which allows one to derive³ global existence and asymptotic

¹This goal has been met successfully for semilinear fields such as Wave Maps, Maxwell-Klein-Gordon and Yang-Mills, but remains a major open problem for quasilinear ones such as the Einstein equations.

²The conformal compactification method does not in fact require compactly supported data but it imposes serious restriction on the rate of decay at infinity. It requires in particular that $E = O(|x|^{-4})$ and thus excludes non-trivial charges.

³A similar result based on Fourier methods and renormalization is due to J. Shatah [24]. Note that [24] does not require restrictions on the data.

behavior of solutions corresponding to small, compactly supported data. The hyperboloidal method can be easily extended to the mMKG equations provided that one maintains the restriction of the data, see⁴ [22]. The goal of this paper is to describe a new variation of the vectorfield method which allows us to dispense of the restriction to compactly supported data. We note that, in principle, the Fourier type methods recently introduced in [5] may also apply to the system considered here to derive comparable results. We believe however that the purely geometric, physical space, approach taken here has its own specific advantages which will prove to be of independent interest in various applications.

The main difference between the massive and massless cases is due to the different global asymptotic behavior of solutions to wave and Klein-Gordon equations. Recall that solutions of the standard wave equation in \mathbb{R}^{3+1} concentrate along outgoing null directions and decay faster in the interior while, to the contrary, solutions to the standard KG equation concentrate in the interior and decay faster along null directions. This difference can be neatly captured by the vectorfield method. Indeed while both \square and $\square - 1$ commute with the generators of the Lorentz group $\Omega_{\alpha\beta} = x_\alpha \partial_\beta - x_\beta \partial_\alpha$, it is only the former which has, in addition, good commutation properties with the scaling vectorfield $S = x^\alpha \partial_\alpha$ as well as the other conformal Killing vector fields of Minkowski space. The method pioneered by Klainerman in [7] takes advantage of the commutation properties of $\square - 1$ with $\Omega_{\alpha\beta}$ by considering higher order, invariant, energy estimates along the natural hyperboloidal foliation, asymptotic to a fixed outgoing null cone, spanned by these vector fields. The decay properties of solutions then follow by a simple global Sobolev inequality on the leaves of the foliation. The method has the advantage that it can be applied to coupled nonlinear wave and Klein-Gordon type equations and, indeed, it has been later widely used in various such applications, most notable the recent works of [18] and [26] on the coupled Einstein-Klein-Gordon equations, see also [22] and [6].

The obvious limitation of the hyperboloidal approach is that it is naturally restricted to solutions supported in a forward light cone, i.e. for compactly⁵ supported initial data. The main contribution of this paper is to provide the missing piece, i.e. to construct solutions to mMKG system in the complement of a fixed light cone. The full problem could then be treated by patching together the new estimates in the exterior with the hyperboloidal ones in the interior.

A first glimpse of what is needed to derive the decay of solutions to KG equations in the exterior of a light cone can be found in [10] where Klainerman gave a hierarchic multiplier-vectorfield approach to recover the fast decay along null directions. Unfortunately that approach does not directly apply to the Maxwell equation piece of the mMKG system, and we cannot rely on either commutation with the scaling vectorfield or the hyperboloidal foliation. The restriction to the exterior of an outgoing null cone provides however a small but crucial advantage in that we can derive energy-flux estimates with higher weights in $|u|$ (the outgoing optical function $u = \frac{1}{2}(t-r)$) in such regions, see e.g. [15], and that such estimates also apply to KG. Combining these two ingredients is not enough however, one needs to also derive subtle multiplier and commutation estimates and an appropriate version of the null condition to make everything work. We shall summarize the main steps in more detail in the next subsection after we introduce some notation and state the main result.

1.1 Main results

To state our main theorem, we first define some necessary notations. We use the standard polar local coordinate system (t, r, ω) of Minkowski space as well as the null coordinates $u = \frac{t-r}{2}$, $v = \frac{t+r}{2}$. We may use $u_+ = 1 + |u|$. Under the coordinates (u, v, ω) with $\omega \in \mathbb{S}^2$, all points with fixed (u, v)

⁴See also [23] for an integrated local energy estimate.

⁵As far as we know, the general case can only be treated by Fourier techniques, see [24] for nonlinear Klein-Gordon equations and see [5] for wave-KG coupled systems.

form a round sphere with radius r , which is denoted as $S(u, v)$. We now introduce a null frame $\{L, \underline{L}, e_1, e_2\}$ with

$$L = \partial_v = \partial_t + \partial_r, \quad \underline{L} = \partial_u = \partial_t - \partial_r$$

and $\{e_1, e_2\}$ an orthonormal basis of the sphere $S(u, v)$. We use \mathcal{D} to denote the covariant derivative associated to the connection field A on the sphere $S(u, v)$, which is defined by $\mathcal{D}_\nu = \Pi_\nu^\mu D_\mu$, with $\Pi_\nu^\mu = \delta_\nu^\mu + \frac{1}{2}(L^\mu \underline{L}_\nu + L_\nu \underline{L}^\mu)$. For any 2-form G , denote the null decomposition under the above null frame by

$$\alpha_B[G] = G_{Le_B}, \quad \underline{\alpha}_B[G] = G_{\underline{L}e_B}, \quad \rho[G] = \frac{1}{2}G_{\underline{L}L}, \quad \sigma[G] = G_{e_1e_2}, \quad B = 1, 2. \quad (4)$$

We interpret $\alpha, \underline{\alpha}$ as 1-forms tangent to the spheres $S(u, v)$. We refer to such quantities, more generally, as S -tangent tensors.

Without loss of generality we only prove estimates in the future, i.e., $t \geq 0$. Our analysis focuses on the exterior region $\{t + R \leq |x|\}$, for some fixed constant $R \geq 1$. We foliate this region by the standard double null foliation defined by the level surfaces of the optical functions u and v . Let \mathcal{H}_u denote the outgoing null hypersurface $\{t - r = 2u\} \cap \{t \geq 0, r \geq R\}$ and $\underline{\mathcal{H}}_v$ the incoming null hypersurface $\{t + r = 2v\} \cap \{t \geq 0, r \geq R\}$. We also use \mathcal{H}_u^v and $\underline{\mathcal{H}}_v^u$ to denote the truncated hypersurfaces

$$\begin{aligned} \mathcal{H}_u^v &:= \{(t, x) : t - |x| = 2u, \quad -2u \leq t + |x| \leq 2v\}, \quad \mathcal{H}_u := \mathcal{H}_u^\infty, \\ \underline{\mathcal{H}}_v^u &:= \{(t, x) : t + |x| = 2v, \quad -2v \leq t - |x| \leq 2u\}. \end{aligned}$$

On the initial hypersurface $\{t = 0\}$, define $\Sigma_0 = \mathbb{R}^3 \cap \{r \geq R\}$ and

$$\Sigma_0^{u_1, u_2} := \{(0, x) : -2u_1 \leq |x| \leq -2u_2\}, \quad \forall u_2 < u_1 \leq -\frac{1}{2}R, \quad \Sigma_0^u = \Sigma_0^{u, -\infty}.$$

In the exterior region, let \mathcal{D}_u^v be the domain bounded by $\mathcal{H}_u^v, \underline{\mathcal{H}}_v^u$ and the initial hypersurface:

$$\mathcal{D}_u^v := \{(t, x) : t - |x| \leq 2u, \quad t + |x| \leq 2v\}.$$

We fix the convention that the standard volume elements on $\Sigma_0, \mathcal{H}_u, \underline{\mathcal{H}}_v$ and \mathcal{D}_u^v are given by $r^2 dr d\omega, r^2 dv d\omega, r^2 du d\omega$ and $dx dt$ respectively. These will be dropped whenever there is no possible confusion, i.e. $\int_{\Sigma_0} f = \int_{\Sigma_0} f r^2 dr d\omega, \int_{\mathcal{H}_u} f = \int_{\mathcal{H}_u} f r^2 dv d\omega$ etc.

We denote by $E[f, G](\Sigma)$ the appropriate energy-flux of the 2-form G and complex scalar field f through the hypersurface Σ . For the hypersurfaces of interest to us,

$$\begin{aligned} E[f, G](\Sigma_0) &= \int_{\Sigma_0} (|G|^2 + |Df|^2 + |f|^2) dx, \quad |G|^2 = \rho^2 + |\sigma|^2 + \frac{1}{2}(|\alpha|^2 + |\underline{\alpha}|^2), \\ E[f, G](\mathcal{H}_u) &= \int_{\mathcal{H}_u} (|D_L f|^2 + |\mathcal{D}f|^2 + |f|^2 + \rho^2 + \sigma^2 + |\alpha|^2), \\ E[f, G](\underline{\mathcal{H}}_v) &= \int_{\underline{\mathcal{H}}_v} (|D_{\underline{L}} f|^2 + |\mathcal{D}f|^2 + |f|^2 + \rho^2 + \sigma^2 + |\underline{\alpha}|^2), \end{aligned} \quad (5)$$

where $\alpha, \underline{\alpha}, \rho, \sigma$ are the components of G defined in (4), $d\omega$ denotes the standard surface measure on the unit sphere \mathbb{S}^2 , and D is the covariant derivative associated to the connection A .

On the initial hypersurface, define the chargeless part of the electric field \tilde{E} :

$$\tilde{E}_i = E_i - q_0 r^{-2} \chi_{\{R \leq r\}} \omega_i, \quad \text{where } \omega_i = \frac{x_i}{r},$$

where q_0 is the total charge defined in (3). Similarly define the chargeless part of the Maxwell field

$$\tilde{F} = F - q_0 r^{-2} \chi_{\{t+R \leq r\}} dt \wedge dr.$$

Our assumption on the initial data is that for some positive constant $1 < \gamma_0 < 2$ the following weighted energy

$$\mathcal{E}_{k,\gamma_0} = \sum_{l \leq k} \int_{\Sigma_0} (1+r)^{\gamma_0+2l} (|\bar{D}\bar{D}^l \phi_0|^2 + |\bar{D}^l \phi_1|^2 + |\bar{D}^l \phi_0|^2 + |\bar{\nabla}^l \tilde{E}|^2 + |\bar{\nabla}^l H|^2) dx \quad (6)$$

is small for some positive integer $k \geq 2$, where \bar{D} is the projection of D to $\mathbb{R}^3 \times \{t=0\}$. We denote by ∇ the Levi-Civita connection in Minkowski space. $\bar{\nabla}$ is the projection of ∇ to $\mathbb{R}^3 \times \{t=0\}$. Note that the total charge is defined in terms of an integral on \mathbb{R}^3 , it is not bounded by the energy \mathcal{E}_{0,γ_0} which is only defined for $\mathbb{R}^3 \cap \{r \geq R\}$. Therefore it can be large even if \mathcal{E}_{0,γ_0} is small.

We are ready to state the main theorem of this paper.

Theorem 1 (Main theorem). *Consider the Cauchy problem for (mMKG) with the admissible initial data set (ϕ_0, ϕ_1, E, H) . There exists a positive constant ϵ_0 , depending only on $1 < \gamma_0 < 2$, $|q_0|$ and ϵ such that if $\mathcal{E}_{2,\gamma_0} < \epsilon_0$, the unique local solution (F, ϕ) of (mMKG) can be globally extended⁶ in time on the exterior region $\{(t, x) : t + R \leq |x|\}$, with a fixed constant $R \geq 1$.*

(1) *The global solution verifies the following pointwise estimates,*

$$\begin{aligned} r^2 |\not{D}\phi|^2 + u_+^2 |D_{\underline{L}}\phi|^2 + r^2 |D_L\phi|^2 &\leq C \mathcal{E}_{2,\gamma_0} r^{-\frac{5}{2}+\epsilon} u_+^{\frac{1}{2}-\gamma_0}, \quad |\phi|^2 \leq C \mathcal{E}_{2,\gamma_0} r^{-3} u_+^{-\gamma_0}; \\ |\tilde{\rho}|^2 + |\alpha|^2 + |\sigma|^2 &\leq C \mathcal{E}_{2,\gamma_0} r^{-2-\gamma_0} u_+^{-1}, \quad |\underline{\alpha}|^2 \leq C \mathcal{E}_{2,\gamma_0} r^{-2} u_+^{-\gamma_0-1}, \end{aligned}$$

where $\epsilon > 0$ is any positive constant. Here $\tilde{\rho} = \rho[\tilde{F}]$ and the other curvature components are for the full Maxwell field F .

(2) *The following generalized energy estimates (see the notation in (5)) hold true*

$$\begin{aligned} E[D_Z^k \phi, \mathcal{L}_Z^k \tilde{F}](\mathcal{H}_{u_1}^{-u_2}) + E[D_Z^k \phi, \mathcal{L}_Z^k \tilde{F}](\mathcal{H}_{-u_2}^{u_1}) &\leq C(u_1)_+^{-\gamma_0+2\zeta(Z^k)} \mathcal{E}_{2,\gamma_0}, \\ \int_{\mathcal{H}_{u_1}^{-u_2}} r |D_L D_Z^k \phi|^2 + \int_{\mathcal{H}_{-u_2}^{u_1}} r (|\not{D} D_Z^k \phi|^2 + |D_Z^k \phi|^2) &\leq C(u_1)_+^{1-\gamma_0+2\zeta(Z^k)} \mathcal{E}_{2,\gamma_0}, \\ \int_{\mathcal{H}_{u_1}^{-u_2}} r^{\gamma_0} |\alpha[\mathcal{L}_Z^k \tilde{F}]|^2 + \iint_{\mathcal{D}_{u_1}^{-u_2}} r^{\gamma_0-1} |(\alpha, \rho, \sigma)[\mathcal{L}_Z^k \tilde{F}]|^2 \\ + \int_{\mathcal{H}_{-u_2}^{u_1}} r^{\gamma_0} (|\rho[\mathcal{L}_Z^k \tilde{F}]|^2 + |\sigma[\mathcal{L}_Z^k \tilde{F}]|^2) &\leq C(u_1)_+^{2\zeta(Z^k)} \mathcal{E}_{2,\gamma_0} \end{aligned}$$

for all $u_2 < u_1 \leq -\frac{R}{2}$, $Z^k = Z_1 Z_2 \dots Z_k$ with $k \leq 2$ and $Z_i \in \Gamma$, where Γ is the set of generators of the Poincaré group.⁷ The definition of the signature function $\zeta(\cdot)$ can be found in Section 2.4.

The constant C in (1) and (2) depends only on γ_0 , $|q_0|$ and ϵ .

Remark 1. If $\gamma_0 \geq 2$, pointwise decay estimates as well as weighted energy estimates can be improved according to a slightly modified argument. Moreover if the initial data belong to higher order weighted Sobolev space ($k > 2$), we then can also derive the associated higher order weighted energy estimates. Note that the theorem can be adapted to large data, provided that R is sufficiently large. Indeed, if the total energy in \mathbb{R}^3 is bounded, the energy \mathcal{E}_{2,γ_0} defined on $\{r \geq R\}$ can be sufficiently small.

⁶We refer the reader to [3, 4] for the standard global existence without asymptotic behavior.

⁷Here $D_{Z^l}^l = D_{Z^l} := D_{Z_1} \dots D_{Z_l}$, $\mathcal{L}_Z^l = \mathcal{L}_{Z^l} := \mathcal{L}_{Z_1} \dots \mathcal{L}_{Z_l}$, and $D_{Z^0}, \mathcal{L}_{Z^0}$ are both the identity map.

Remark 2. *The estimates mentioned above can be used as boundary conditions on the outgoing null boundary of the causal future of a sufficiently large compact set where one can apply the hyperboloidal approach of [22] to derive global asymptotic results for general initial conditions.*

Remark 3. *Our approach can be applied to (mMKG) equations in Minkowski spacetimes of other dimensions. It can also be used to remove the restriction of the compactly supported Cauchy data for the 2-D problems treated by the hyperboloid foliation such as [27]. More importantly our approach can also be used to study quasilinear wave and Klein-Gordon systems, such as the model problem in [5] as well as the Einstein-Klein-Gordon equations⁸.*

Remark 4. *As mentioned earlier the exterior region provides considerable flexibility in allowing high weights in u_+ for the various flux integrals used here, as long as the initial data has sufficient decay in r . An obvious restriction however is given by the charge which limits the decay in r for the $\rho[F]$ component of the data. The way to overcome such restriction is to define the energy of the data for \tilde{F} , the chargeless part of F .*

We now briefly summarize our method. As expected, if we apply the standard multiplier approach to the linear Klein-Gordon equation with the multiplier rL , the energy identity generates the bulk term

$$B := \int_{\{r \geq R+t\}} |\phi|^2 dx dt$$

with an unfavorable sign. Note that the flux of the standard energy estimate provides a bound for $\int_{\mathcal{H}_u} |\phi|^2$ in terms of the initial energy. One can improve this, in the exterior region, to a bound of $u_+^\gamma \int_{\mathcal{H}_u} |\phi|^2$ in terms of the corresponding weighted energy norm of the data. Thus, by integrating $\int_{\mathcal{H}_u} |\phi|^2$ with respect to u (for $\gamma > 1$), we can control the bulk term B , as long as the initial weighted energy is bounded. This multiplier approach, combined with commutation with $\Omega_{\alpha\beta}$ and standard translations, $T_\alpha = \partial_\alpha$, is thus sufficient to derive the desired pointwise decay estimates for solutions of linear Klein-Gordon equations in the exterior and can also be adapted to control solutions to nonlinear KG equations.

Now consider the mMKG system. One can show, as before, that the total energy flux through the outgoing null hypersurface \mathcal{H}_u in the exterior region decays, sufficiently fast, with respect to u , i.e.

$$\int_{\mathcal{H}_u} (|D_L \phi|^2 + |\not{D}\phi|^2 + |\phi|^2) \lesssim u_+^{-\gamma_0} \mathcal{E}_{0,\gamma_0}, \quad \gamma_0 > 1,$$

provided that the initial data is bounded in the corresponding weighted energy space \mathcal{E}_{0,γ_0} . The key observation, once more, is that this energy flux decay is sufficient to bound the only unfavorable term

$$\int_{u \leq u_0} \left(\int_{\mathcal{H}_u} |\phi|^2 \right) du \lesssim u_0^{1-\gamma_0} \mathcal{E}_{0,\gamma_0},$$

generated when using the vector field rL as multiplier for the coupled system⁹.

Things become a lot more complicated when we try to derive the higher order derivative estimates due to the complexity of the quadratic error terms generated in the process. At the top level, when we commute with two vector fields X, Y the main error terms are due to the commutator¹⁰ $Com := [D_X D_Y, \square - 1]\phi$. We make use of the technique of double commutator in [29] to decompose Com into a combination of trilinear forms, such as $Q(F, D_X \phi, Y)$ and $Q(\mathcal{L}_X F, \phi, Y)$, where the definition of Q can be found in (27). To control Com we need to take into account that

⁸The last two authors of this paper are in the process of completing this goal.

⁹The bad term is due, of course, to the Klein-Gordon component of the system.

¹⁰See (63) for a precise form of Com .

Q verifies the null condition with respect to the fields¹¹ (ϕ, F) . Consider the case when $Y = \Omega_{0i}$ is a boost, as a typical example. In that case we can estimate, schematically, (see Lemma 7 for the complete inequalities),

$$|Q(F, D_X \phi, Y)| \lesssim r|F \cdot D_L D_X \phi| + \dots; \quad |Q(\mathcal{L}_X F, \phi, Y)| \lesssim r|\mathcal{L}_X F \cdot D_L \phi| + \dots. \quad (7)$$

To bound the standard energy flux for $E[D_X D_Y \phi](\mathcal{H}_u) + E[D_X D_Y \phi](\underline{\mathcal{H}}_v)$ (which can be found in the first inequality in Theorem 1 (2)), we need to control¹²

$$\int_{\{r>t+R\}} (|Q(F, D_X \phi, Y)| + |Q(\mathcal{L}_X F, \phi, Y)|) |D_0 D_X D_Y \phi| dx dt = I_1 + I_2. \quad (8)$$

Consider the first term I_1 in (8) in view of (7) and decomposing $2D_0 = D_L + D_{\underline{L}}$, we need to bound

$$I_1 \lesssim \int_{\{r>t+R\}} r|F \cdot D_L D_X \phi| (|D_{\underline{L}} D_X D_Y \phi| + |D_L D_X D_Y \phi|) dx dt.$$

For simplicity, we only discuss the treatment of the chargeless part of F , for which we have

$$I_1 \lesssim \|r^{-\frac{1}{2}-\epsilon}(|D_{\underline{L}} D_X D_Y \phi| + |D_L D_X D_Y \phi|)\|_{L^2(\{r>t+R\})} \|r^{\frac{3}{2}+\epsilon} \tilde{F} \cdot D_L D_X \phi\|_{L^2(\{r>t+R\})} + \dots$$

The first factor can be bounded by the energy fluxes on $\underline{\mathcal{H}}_v$ and \mathcal{H}_u , followed with direct integration. To bound the second factor we note that, since the radiative $\underline{\alpha}$ component of \tilde{F} decays only like r^{-1} in r , in view of its expected decay in Theorem 1 (1), we write,

$$\begin{aligned} \int_{\{r>t+R\}} r^{3+2\epsilon} |\tilde{F}|^2 |D_L D_X \phi|^2 &\lesssim \mathcal{E}_{2,\gamma_0} \int_{\{r>t+R\}} r^{-1+2\epsilon} u_+^{-\gamma_0-1} |r D_L D_X \phi|^2 \\ &\lesssim \mathcal{E}_{2,\gamma_0} \sup_{u \leq -\frac{1}{2}R} \|r D_L D_X \phi\|_{L^2(\mathcal{H}_u)}^2, \end{aligned}$$

which requires us to control

$$\|r D_L D_X \phi\|_{L^2(\mathcal{H}_u)}.$$

We can repeat the above estimate in the region $\{r \geq t - 2u_0\}$ with $u_0 \leq -\frac{1}{2}R$ so as to keep track of the negative power of u_0^+ in the bounds. To derive the improved bound for $\|r D_L D_X \phi\|_{L^2(\mathcal{H}_u)}$, we note the simple algebraic identity,

$$v|Lf| \lesssim \sum_i |\Omega_{0i} f| + \sum_{1 \leq i < j \leq 3} |\Omega_{ij} f| + |u| |\partial f|$$

for any smooth function f . By using the standard energy flux on \mathcal{H}_u and $r \leq 2v$ we have

$$\|r D_L D_X \phi\|_{L^2(\mathcal{H}_u)}^2 \lesssim \sum_{\mu, \nu} \|D_{\Omega_{\mu\nu}} D_X \phi\|_{L^2(\mathcal{H}_u)}^2 + u_+^2 \|D D_X \phi\|_{L^2(\mathcal{H}_u)}^2. \quad (9)$$

In view of the top-order standard energy flux bound, we expect that

$$\|D D_X \phi\|_{L^2(\mathcal{H}_u)}^2 \lesssim u_+^{-\gamma_0-2} \mathcal{E}_{2,\gamma_0}.$$

Thus, estimating the other terms on the right of (9) by $u_+^{-\gamma_0} \mathcal{E}_{2,\gamma_0}$, we obtain the desired estimate

$$\|r D_L D_X \phi\|_{L^2(\mathcal{H}_u)}^2 \lesssim u_+^{-\gamma_0} \mathcal{E}_{2,\gamma_0}.$$

¹¹This means roughly that the radiative components of F , i.e. $\underline{\alpha}[F]$, does not interact with $D_{\underline{L}} \phi$.

¹²For any one-form V , we fix the convention that $V_0 = V(\partial_t)$. Lifting it by Minkowski metric gives V^0 .

To treat the term I_2 in (8), we need the pointwise control for $D\phi$. It is worthwhile to point out that the weighted energy control (see the second inequality in Theorem 1 (2)) allows us to obtain a set of strong pointwise decay for the scalar field. However to treat the leading term generated by (7), we need a further improvement for the pointwise decay of $D_L\phi$. This is obtained by making use of the same algebraic identity as above combined with a pointwise estimate¹³ for $D_Z\phi$, with the help of the top order weighted energy for ϕ and global Sobolev inequalities.

To control the top order weighted estimate for the scalar field we proceed in the same manner except that we need to make use of the stronger decay in u_+ in the treatment of the error terms, see Proposition 7. To derive the weighted energy estimates¹⁴ for the Maxwell field we use again the multiplier method based on the vector fields $r^p L$. In this case however we can choose $1 \leq p \leq \gamma_0 < 2$. The error terms generated in this case are due to the inhomogeneous terms of the Maxwell equations which depend quadratically on the scalar field and its derivatives. This is, roughly, the reason why we can get stronger r -weighted estimates on the Maxwell field F than on the KG field ϕ . Our main theorem then follows by using a standard bootstrap argument.

At last, we remark that our result does not require small charges. This is achieved by carefully separating the terms related to the charge in the analysis. We direct the readers interested in this aspect of our result to the proof of Lemma 10 and Proposition 7.

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2 Preliminaries and energy identities

The proof of the main theorem is based on an energy method. We will mainly analyze the energy fluxes through various kinds of null hypersurfaces. To derive the pointwise bound of the solution, we need a global Sobolev embedding inequality adapted to the null hypersurfaces. In this section, we establish a preliminary Sobolev inequality and energy identities. In order to control higher order energy fluxes, we need to commute various differential operators with the operator $\square_A - 1$. Such commutators are treated in the last subsection.

Without loss of generality, we may assume the positive constant ϵ in the Main Theorem verifies $0 < \epsilon \leq \frac{1}{10}(\gamma_0 - 1)$. We make a convention in the sequel that the implicit constant in $A \lesssim B$ depends only on ϵ , γ_0 and $|q_0|$.

2.1 Sobolev inequalities

We may write the integral of a real scalar function f on any surface S as $\int_S f$, where the volume element on the surface S is omitted for simplicity. We have the following Sobolev inequality.

Lemma 1. (1) *For any smooth function f and constants verifying the relation $2\gamma = \gamma'_0 + 2\gamma_2$, we have, for all (u, v) , $-u < v \leq v_*$,*

$$\begin{aligned} \sup_{S(u,v)} |r^\gamma f|^4 &\lesssim \sum_{l \leq 1, 1 \leq i < j \leq 3} \int_{S(u, -u)} |r^\gamma \Omega_{ij}^l f|^4 r^{-2} + \sum_{k \leq 2, 1 \leq i < j \leq 3} \int_{\mathcal{H}_u^{-u, v_*}} r^{2\gamma_2} |\Omega_{ij}^k f|^2 r^{-2} \\ &\times \sum_{l \leq 1, 1 \leq i < j \leq 3} \int_{\mathcal{H}_u^{-u, v_*}} r^{\gamma'_0} |L \Omega_{ij}^l (r^\gamma f)|^2 r^{-2}. \end{aligned} \quad (10)$$

(2) *The same estimate holds true for any smooth complex scalar field ϕ with covariant derivative D associated to the connection field A .*

¹³Here Z is a generator of the Poincaré group.

¹⁴The standard energy estimates are, of course, much simpler.

(3) For any S -tangent tangent tensor field H , we have, for all (u, v) , $-u < v \leq v_*$,

$$\begin{aligned} \sup_{S(u,v)} |r^\gamma H|^4 &\lesssim \sum_{l \leq 1, 1 \leq i < j \leq 3} \int_{S(u, -u)} |r^\gamma \mathfrak{L}_{\Omega_{ij}}^l H|^4 r^{-2} + \sum_{k \leq 2, 1 \leq i < j \leq 3} \int_{\mathcal{H}_u^{-u, v_*}} r^{2\gamma_2} |\mathfrak{L}_{\Omega_{ij}}^k H|^2 r^{-2} \\ &\cdot \sum_{l \leq 1, 1 \leq i < j \leq 3} \int_{\mathcal{H}_u^{-u, v_*}} r^{\gamma'_0} |\nabla_L \mathfrak{L}_{\Omega_{ij}}^l (r^\gamma H)|^2 r^{-2}, \end{aligned} \quad (11)$$

where, for any vector field Z , $\mathfrak{L}_Z H$ denotes the projection of $\mathcal{L}_Z H$ to the sphere $S(u, v)$. Here $\nabla_L H$ denotes the projection of $\nabla_L H$ to $S(u, v)$.

Proof. The case for $\gamma = \gamma_2 = \frac{3}{2}$ was first established in [14]. The proof for this general version of this lemma is a minor modification of the original one. For reader's benefit, we give the proof here. The crucial idea is based on the following Poincaré inequality

$$\int_S |\Phi - \bar{\Phi}|^2 \leq C \left(\int_S |\nabla \Phi|^2 \right)$$

for any smooth function Φ and 2-sphere S , where the constant C depends only on the sphere and $\bar{\Phi}$ is the mean value of the function Φ on the surface. For the case when the sphere S is the standard unit sphere \mathbb{S}^2 , with C be the universal constant, we can apply the above inequality to $|f|^3$ to derive that

$$\int_{\mathbb{S}^2} |f|^6 d\omega \leq C \int_{\mathbb{S}^2} |\partial_\omega f|^2 d\omega \cdot \int_{\mathbb{S}^2} |f|^4 d\omega + C \left(\int_{\mathbb{S}^2} |f|^3 \right)^2 \leq C \int_{\mathbb{S}^2} (|f|^2 + |\partial_\omega f|^2) d\omega \cdot \int_{\mathbb{S}^2} |f|^4 d\omega.$$

For any constants $\gamma, \gamma_1, \gamma_2$ satisfying $3\gamma = \gamma_1 + 2\gamma_2$, we in particular have

$$\int_{\mathbb{S}^2} |r^\gamma f|^6 d\omega \leq C \int_{\mathbb{S}^2} (|r^{\gamma_1} f|^2 + |\partial_\omega (r^{\gamma_1} f)|^2) d\omega \cdot \int_{\mathbb{S}^2} |r^{\gamma_2} f|^4 d\omega.$$

Let $\gamma, \gamma'_0, \gamma_1$ and γ_2 verify the following relations

$$\gamma'_0 + 6\gamma_1 = 6\gamma, \quad 3\gamma_1 = 2\gamma + \gamma_2.$$

By integrating along the outgoing null hypersurface \mathcal{H}_u , we can obtain

$$\begin{aligned} \int_{S(u, v_1)} |r^\gamma f|^4 r^{-2} &\leq \int_{S(u, -u)} |r^\gamma f|^4 r^{-2} + 4 \int_{\mathcal{H}_u^{-u, v_1}} |L(r^\gamma f)| |r^\gamma f|^3 dv d\omega \\ &\leq \int_{S(u, -u)} |r^\gamma f|^4 r^{-2} + 4 \left(\int_{\mathcal{H}_u^{-u, v_1}} r^{\gamma'_0} |L(r^\gamma f)|^2 dv d\omega \right)^{\frac{1}{2}} \left(\int_{\mathcal{H}_u^{-u, v_1}} |r^{\gamma_1} f|^6 dv d\omega \right)^{\frac{1}{2}} \\ &\leq \int_{S(u, -u)} |r^\gamma f|^4 r^{-2} + 4C \left(\sup_v \int_{S(u, v)} |r^\gamma f|^4 r^{-2} \right)^{\frac{1}{2}} \\ &\quad \cdot \left(\int_{\mathcal{H}_u^{-u, v_1}} r^{\gamma'_0} |L(r^\gamma f)|^2 dv d\omega \right)^{\frac{1}{2}} \left(\int_{\mathcal{H}_u^{-u, v_1}} (|r^{\gamma_2} f|^2 + |\partial_\omega (r^{\gamma_2} f)|^2) dv d\omega \right)^{\frac{1}{2}}. \end{aligned}$$

Taking supremum on the left hand side with respect to $v_1 \in (-u, v_*]$ gives

$$\begin{aligned} \sup_{-u < v \leq v_*} \int_{S(u, v)} |r^\gamma f|^4 r^{-2} &\lesssim \int_{S(u, -u)} |r^\gamma f|^4 r^{-2} + \int_{\mathcal{H}_u^{-u, v_*}} r^{\gamma'_0} |L(r^\gamma f)|^2 dv d\omega \\ &\quad \cdot \int_{\mathcal{H}_u^{-u, v_*}} (|r^{\gamma_2} f|^2 + |\partial_\omega (r^{\gamma_2} f)|^2) dv d\omega. \end{aligned} \quad (12)$$

Finally, by using the standard Sobolev embedding on the sphere $S(u, v)$, we have

$$\sup_{S(u,v)} |\varphi| \lesssim \sum_{l \leq 1, 1 \leq i < j \leq 3} \left(\int_{S(u,v)} |\Omega_{ij}^l \varphi|^4 r^{-2} \right)^{\frac{1}{4}}.$$

The desired estimate (10) then follows from (12). (2) and (3) of Lemma 1 can be proved in the same way. \square

2.2 Energy identities

In this subsection, we derive fundamental energy identities for the massive MKG equations.

For any 2-form \mathcal{G} , satisfying the Bianchi identity (1), any complex scalar field ϕ and connection field A , we define the associated energy momentum tensor

$$T[\phi, \mathcal{G}]_{\alpha\beta} = \mathcal{G}_{\alpha\mu} \mathcal{G}_{\beta}^{\mu} - \frac{1}{4} m_{\alpha\beta} \mathcal{G}_{\mu\nu} \mathcal{G}^{\mu\nu} + \Re(\overline{D_{\alpha}\phi} D_{\beta}\phi) - \frac{1}{2} m_{\alpha\beta} (\overline{D^{\mu}\phi} D_{\mu}\phi + |\phi|^2),$$

where $m_{\alpha\beta}$ is the Minkowski metric and D_{α} denotes the covariant derivative associated to the connection field A . Given a vector field X , we have the following identity

$$\partial^{\mu}(T[\phi, \mathcal{G}]_{\mu\nu} X^{\nu}) = \Re((\square_A - 1)\phi X^{\nu} \overline{D_{\nu}\phi}) + X^{\nu} F_{\nu\mu} J^{\mu}[\phi] + \partial^{\mu} \mathcal{G}_{\mu\beta} \mathcal{G}_{\nu}^{\beta} X^{\nu} + T[\phi, \mathcal{G}]^{\mu\nu} \pi_{\mu\nu}^X,$$

where $\pi_{\mu\nu}^X = \frac{1}{2} \mathcal{L}_X m_{\mu\nu}$ is the deformation tensor of the vector field X in Minkowski space and $J_{\mu}[\phi] = \Im(\phi \cdot \overline{D_{\mu}\phi})$. We also note that the term $F = F[A]$ appears from commuting covariant derivatives of ϕ as in (2). Throughout this paper, we raise and lower indices with respect to the Minkowski metric $m_{\mu\nu}$.

Take any smooth function χ . We have the following equality

$$\frac{1}{2} \partial^{\mu} (\chi \partial_{\mu} |\phi|^2 - \partial_{\mu} \chi |\phi|^2) = \chi (\overline{D_{\mu}\phi} D^{\mu}\phi + |\phi|^2) - \frac{1}{2} \square \chi \cdot |\phi|^2 + \chi \Re((\square_A - 1)\phi \cdot \overline{\phi}).$$

Let X, Y be smooth vector fields. We now define the vector field $\tilde{P}^{X,Y}[\phi, \mathcal{G}]$ with components

$$\tilde{P}_{\mu}^{X,Y}[\phi, \mathcal{G}] = T[\phi, \mathcal{G}]_{\mu\nu} X^{\nu} - \frac{1}{2} \partial_{\mu} \chi \cdot |\phi|^2 + \frac{1}{2} \chi \partial_{\mu} |\phi|^2 + Y_{\mu}, \quad (13)$$

where the vector field Y may depend on the scalar field ϕ . We then have the equality

$$\begin{aligned} \partial^{\mu} \tilde{P}_{\mu}^{X,Y}[\phi, \mathcal{G}] &= \Re((\square_A - 1)\phi (\overline{D_X\phi} + \chi \overline{\phi})) + \text{div}(Y) + X^{\nu} F_{\nu\mu} J[\phi]^{\mu} + \partial^{\mu} \mathcal{G}_{\mu\gamma} \mathcal{G}_{\nu}^{\gamma} X^{\nu} \\ &\quad + T[\phi, \mathcal{G}]^{\mu\nu} \pi_{\mu\nu}^X + \chi (\overline{D_{\mu}\phi} D^{\mu}\phi + |\phi|^2) - \frac{1}{2} \square \chi \cdot |\phi|^2. \end{aligned}$$

Here the operator \square is the wave operator in Minkowski space and the divergence of the vector field Y is also taken in the Minkowski space and $J[\phi]_{\mu} = \Im(\phi \cdot \overline{D_{\mu}\phi})$.

Now take any region \mathcal{D} in \mathbb{R}^{3+1} . Assume on this region the scalar field ϕ and the 2-form \mathcal{G} satisfies the following linear equations

$$\partial^{\nu} \mathcal{G}_{\mu\nu} = J_{\mu}, \quad \square_A \phi - \phi = h. \quad (14)$$

Here we note that the covariant operator \square_A is associated to the 2-form F . Then using the Stokes' formula, the above calculation leads to the following energy identity

$$\begin{aligned} &\iint_{\mathcal{D}} F_{X\mu} J[\phi]^{\mu} - \mathcal{G}_{X\gamma} J^{\gamma} + \Re(h \cdot (\overline{D_X\phi} + \chi \overline{\phi})) d\text{vol} \\ &+ \iint_{\mathcal{D}} \text{div}(Y) + T[\phi, \mathcal{G}]^{\mu\nu} \pi_{\mu\nu}^X + \chi (\overline{D_{\mu}\phi} D^{\mu}\phi + |\phi|^2) - \frac{1}{2} \square \chi \cdot |\phi|^2 d\text{vol} \\ &= \iint_{\mathcal{D}} \partial^{\mu} \tilde{P}_{\mu}^{X,Y}[\phi, \mathcal{G}] d\text{vol} = \int_{\partial \mathcal{D}} i_{\tilde{P}^{X,Y}[\phi, \mathcal{G}]} d\text{vol}, \end{aligned} \quad (15)$$

where $\partial\mathcal{D}$ denotes the boundary of the domain \mathcal{D} and $i_Z d\text{vol}$ denotes the contraction of the volume form $d\text{vol}$ with the vector field Z which gives the surface measure of the boundary. For example, for any basis $\{e_1, e_2, \dots, e_n\}$, we have $i_{e_1}(de_1 \wedge de_2 \wedge \dots \wedge de_n) = de_2 \wedge de_3 \wedge \dots \wedge de_n$.

In this paper, the domain \mathcal{D} will be a regular region bounded by a level set of t , an outgoing null hypersurfaces \mathcal{H}_u and an incoming null hypersurfaces \mathcal{H}_v . We now compute $i_{\tilde{P}^{X,Y}[\phi, \mathcal{G}]} d\text{vol}$ on these three kinds of hypersurfaces. Recall the volume form in Minkowski space

$$d\text{vol} = dx \wedge dt = -dt \wedge dx.$$

We thus can show that on Σ_0

$$\begin{aligned} i_{\tilde{P}^{X,Y}[\phi, \mathcal{G}]} d\text{vol} = & -(\Re(\overline{D^0\phi} D_X \phi) - \frac{1}{2} X^0 \overline{D^\gamma \phi} D_\gamma \phi - \frac{1}{2} X^0 |\phi|^2 - \frac{1}{2} \partial^0 \chi \cdot |\phi|^2 + \frac{1}{2} \chi \partial^0 |\phi|^2 + Y^0 \\ & + \mathcal{G}^{0\mu} \mathcal{G}_{\nu\mu} X^\nu - \frac{1}{4} X^0 \mathcal{G}_{\mu\nu} \mathcal{G}^{\mu\nu}) dx. \end{aligned} \quad (16)$$

On the outgoing null hypersurface \mathcal{H}_u , we can write the volume form

$$d\text{vol} = r^2 dr \wedge dt \wedge d\omega = 2r^2 dv \wedge du \wedge d\omega = -2r^2 du \wedge dv \wedge d\omega.$$

Notice that $\underline{L} = \partial_u$. We can compute on \mathcal{H}_u that

$$\begin{aligned} i_{\tilde{P}^{X,Y}[\phi, \mathcal{G}]} d\text{vol} = & -2(\Re(\overline{D^L\phi} D_X \phi) - \frac{1}{2} X^L \overline{D^\gamma \phi} D_\gamma \phi - \frac{1}{2} X^L |\phi|^2 - \frac{1}{2} \partial^L \chi |\phi|^2 + \frac{1}{2} \chi \partial^L |\phi|^2 + Y^L \\ & + \mathcal{G}^{L\mu} \mathcal{G}_{\nu\mu} X^\nu - \frac{1}{4} X^L \mathcal{G}_{\mu\nu} \mathcal{G}^{\mu\nu}) r^2 dv \wedge d\omega. \end{aligned} \quad (17)$$

Similarly, on \mathcal{H}_v we have

$$\begin{aligned} i_{\tilde{P}^{X,Y}[\phi, \mathcal{G}]} d\text{vol} = & 2(\Re(\overline{D^L\phi} D_X \phi) - \frac{1}{2} X^L \overline{D^\gamma \phi} D_\gamma \phi - \frac{1}{2} X^L |\phi|^2 - \frac{1}{2} \partial^L \chi |\phi|^2 + \frac{1}{2} \chi \partial^L |\phi|^2 + Y^L \\ & + \mathcal{G}^{L\mu} \mathcal{G}_{\nu\mu} X^\nu - \frac{1}{4} X^L \mathcal{G}_{\mu\nu} \mathcal{G}^{\mu\nu}) r^2 du \wedge d\omega. \end{aligned} \quad (18)$$

We now establish the r -weighted energy identities for solutions of the linear massive MKG equations (14) in the exterior region for the domain $\mathcal{D}_{u_1}^{v_1}$. In the energy identity (15), we choose the vector fields X, Y and the function χ as follows

$$X = r^p L, \quad Y = \frac{p}{2} r^{p-2} |\phi|^2 L, \quad \chi = r^{p-1}.$$

We then compute the non-vanishing components of the deformation tensor for the vector field X :

$$\pi_{\underline{L}\underline{L}} = 2pr^{p-1}, \quad \pi_{\underline{L}\underline{L}}^X = -pr^{p-1}, \quad \delta^{AB} \pi_{e_A e_B}^X = r^{p-1}, \quad A, B = 1, 2.$$

In the sequel, α, ρ, σ are the components of \mathcal{G} defined in (4). Therefore we can show that

$$\begin{aligned} \text{div}(Y) + T[\phi, \mathcal{G}]^{\mu\nu} \pi_{\mu\nu}^X + \chi(\overline{D^\mu \phi} D_\mu \phi + |\phi|^2) - \frac{1}{2} \square \chi |\phi|^2 \\ = \frac{p}{2} r^{-2} L(r^p |\phi|^2) + \frac{1}{2} r^{p-1} (p(|D_L \phi|^2 + |\alpha|^2) + (2-p)(|\mathcal{D}\phi|^2 + \rho^2 + \sigma^2)) \\ - \frac{1}{2} p(p-1) r^{p-3} |\phi|^2 - \frac{1}{2} p r^{p-1} |\phi|^2 \\ = \frac{1}{2} r^{p-1} (p(r^{-2} |D_L(r\phi)|^2 + |\alpha|^2) + (2-p)(|\mathcal{D}\phi|^2 + \rho^2 + \sigma^2)) - \frac{1}{2} p r^{p-1} |\phi|^2. \end{aligned}$$

Next we compute the boundary terms.

$$\begin{aligned}
\int_{\mathcal{H}_u} i_{\tilde{P}^{X,Y}[\phi,\mathcal{G}]} d\text{vol} &= \int_{\mathcal{H}_u} \{r^p(|D_L(r\phi)|^2 + r^2|\alpha|^2) - \frac{1}{2}L(r^{p+1}|\phi|^2)\}r^{-2}, \\
\int_{\underline{\mathcal{H}}_v} i_{\tilde{P}^{X,Y}[\phi,\mathcal{G}]} d\text{vol} &= - \int_{\underline{\mathcal{H}}_v} \{r^{p+2}(|\mathcal{D}\phi|^2 + |\rho|^2 + |\sigma|^2 + |\phi|^2) + \frac{1}{2}\underline{L}(r^{p+1}|\phi|^2)\}r^{-2}, \\
\int_{\Sigma_0^{u_1,u_2}} i_{\tilde{P}^{X,Y}[\phi,\mathcal{G}]} d\text{vol} &= \frac{1}{2} \int_{\Sigma_0^{u_1,u_2}} r^{p+2} (r^{-2}|D_L(r\phi)|^2 + |\mathcal{D}\phi|^2 + |\phi|^2 + |\alpha|^2 + |\rho|^2 + \sigma^2) \\
&\quad - \partial_r(r^{p+1}|\phi|^2) d\omega dr.
\end{aligned}$$

For the domain $\mathcal{D}_{u_1}^{-u_2}$ in the exterior region we have the identity

$$\int_{\mathcal{H}_{u_1}^{-u_2}} L(r^{p+1}|\phi|^2) dv d\omega - \int_{\underline{\mathcal{H}}_{u_1}^{-u_2}} \underline{L}(r^{p+1}|\phi|^2) dud\omega - \int_{\Sigma_0^{u_1,u_2}} \partial_r(r^{p+1}|\phi|^2) d\omega dr = 0.$$

The above calculations then lead to the first energy identity in the following lemma.

Lemma 2. *Assume that the triplet (\mathcal{G}, ϕ, A) verifies (14), i.e.,*

$$\partial^\nu \mathcal{G}_{\mu\nu} = J_\mu, \quad \square_A \phi - \phi = h.$$

Then the following identities hold true in the exterior region $\{r \geq R+t\}$ (with $F = dA$).

- (1) *For all $u_2 < u_1 \leq -\frac{R}{2}$ with $X = r^p L$, $0 < p < 2$, α, ρ, σ the null components of \mathcal{G} (as defined in (4)),*

$$\begin{aligned}
&\iint_{\mathcal{D}_{u_1}^{-u_2}} F_{X\mu} J[\phi]^\mu - \mathcal{G}_{X\gamma} J^\gamma + \Re(h \cdot (\overline{D_X \phi} + \chi \bar{\phi})) \\
&+ \frac{1}{2} \iint_{\mathcal{D}_{u_1}^{-u_2}} r^{p-1} \left(p(r^{-2}|D_L(r\phi)|^2 + |\alpha|^2) + (2-p)(|\mathcal{D}\phi|^2 + |\rho|^2 + \sigma^2) \right) \\
&+ \int_{\mathcal{H}_{u_1}^{-u_2}} r^p (r^{-2}|D_L(r\phi)|^2 + |\alpha|^2) + \int_{\underline{\mathcal{H}}_{u_1}^{-u_2}} r^p (|\mathcal{D}\phi|^2 + |\rho|^2 + \sigma^2 + |\phi|^2) \\
&= \frac{1}{2} \int_{\Sigma_0^{u_1,u_2}} r^p (r^{-2}|D_L(r\phi)|^2 + |\mathcal{D}\phi|^2 + |\phi|^2 + |\alpha|^2 + |\rho|^2 + \sigma^2) + \frac{1}{2} \iint_{\mathcal{D}_{u_1}^{-u_2}} p r^{p-1} |\phi|^2.
\end{aligned} \tag{19}$$

- (2) *For all $u_2 < u_1 \leq -\frac{R}{2}$ the following version of energy identity holds true*

$$\begin{aligned}
&2 \iint_{\mathcal{D}_{u_1}^{-u_2}} (F_{0\mu} J[\phi]^\mu - \mathcal{G}_{0\gamma} J^\gamma + \Re(h \cdot \overline{D_0 \phi})) \\
&+ E[\phi, \mathcal{G}](\mathcal{H}_{u_1}^{-u_2}) + E[\phi, \mathcal{G}](\underline{\mathcal{H}}_{u_1}^{-u_2}) = E[\phi, \mathcal{G}](\Sigma_0^{u_1,u_2}).
\end{aligned} \tag{20}$$

Indeed, to see (20), we note that ∂_t is killing, which gives $\pi_{\mu\nu}^{\partial_t} = 0$. (20) can be derived by applying in the region $\mathcal{D}_{u_1}^{-u_2}$ the energy identity (15) with $X = \partial_t$, $Y = 0$, $\chi = 0$.

Remark 5. *In application to the proof of our main theorem we will take either $\mathcal{G} = 0$ or $\phi = 0$ which allows us to control energy fluxes for Maxwell field and the scalar field separately.*

2.3 The decay of energy flux

In this subsection, we derive the decay of energy flux and the weighted energy flux in the exterior region. To obtain bounds for the energy flux, in this paper, we will constantly employ the following Gronwall's type inequality.

Lemma 3. *Let f_0 be a nonnegative non-increasing function, $a, b > 1$ and $R \geq 1$. Let f and g be two nonnegative continuous functions such that for any $\frac{1}{2}R \leq \tau \leq \iota$, there holds*

$$f(\tau, \iota) + g(\tau, \iota) \lesssim f_0(\tau) + \int_{\tau}^{\iota} f(\tau', \iota) \tau'^{-a} d\tau' + \int_{\tau}^{\iota} \iota'^{-b} g(\tau, \iota') d\iota'. \quad (21)$$

If $g(\tau, \iota)$ is non-increasing with respect to τ or $f(\tau, \iota)$ is non-decreasing with respect to ι , then

$$f(\tau, \iota) + g(\tau, \iota) \lesssim f_0(\tau), \quad \frac{1}{2}R \leq \tau \leq \iota. \quad (22)$$

Proof. We only consider the case that $g(\tau, \iota)$ is non-increasing with respect to τ , since the other case is similar. For any fixed $\iota_0 \geq \tau_0 \geq \frac{1}{2}R$, from (21) and the property of f_0 and g we can derive for $\tau_0 \leq \tau \leq \iota \leq \iota_0$ that

$$f(\tau, \iota) + g(\tau, \iota) \lesssim f_0(\tau_0) + \int_{\tau_0}^{\iota_0} g(\tau_0, \iota') \iota'^{-b} d\iota' + \int_{\tau}^{\iota} (f + g)(\tau', \iota) \tau'^{-a} d\tau',$$

where we also used g is nonnegative. Applying the standard Gronwall inequality to $f(\cdot, \iota) + g(\cdot, \iota)$, we can obtain

$$f(\tau, \iota) + g(\tau, \iota) \lesssim f_0(\tau_0) + \int_{\tau_0}^{\iota_0} g(\tau_0, \iota') \iota'^{-b} d\iota' \quad \text{for } \tau_0 \leq \tau \leq \iota \leq \iota_0,$$

which in particular implies that

$$f(\tau_0, \iota_0) + g(\tau_0, \iota_0) \lesssim f_0(\tau_0) + \int_{\tau_0}^{\iota_0} g(\tau_0, \iota') \iota'^{-b} d\iota', \quad \frac{1}{2}R \leq \tau_0 \leq \iota_0. \quad (23)$$

Applying the standard Gronwall inequality once again to the function $f(\tau_0, \cdot) + g(\tau_0, \cdot)$, we can derive $f(\tau_0, \iota_0) + g(\tau_0, \iota_0) \lesssim f_0(\tau_0)$ for $\frac{1}{2}R \leq \tau_0 \leq \iota_0$. Since τ_0 and ι_0 are arbitrary, we obtain the desired estimate (22). \square

Recall that the chargeless part of the Maxwell field

$$\tilde{F} = F - q_0 r^{-2} \chi_{\{t+R \leq r\}} dt \wedge dr.$$

For solution (F, ϕ) of the massive Maxwell Klein-Gordon equations, it is easy to check that in the exterior region we have

$$\square_A \phi - \phi = 0, \quad \partial^\nu \tilde{F}_{\mu\nu} = J[\phi]_\mu. \quad (24)$$

Proposition 1. *For the solution (F, ϕ) of the massive MKG equations, we have the following estimates on energy flux*

$$E[\phi, \tilde{F}](\mathcal{H}_{u_1}^{-u_2}) + E[\phi, \tilde{F}](\mathcal{H}_{-u_2}^{u_1}) \lesssim (u_1)_+^{-\gamma_0} \mathcal{E}_{0, \gamma_0} \quad (25)$$

as well as the estimates for the weighted energy flux

$$\int_{\mathcal{H}_{u_1}^{-u_2}} r(|D_L \phi|^2 + |\tilde{\alpha}|^2) + \int_{\mathcal{H}_{-u_2}^{u_1}} r(|\not{D}\phi|^2 + \tilde{\rho}^2 + \tilde{\sigma}^2 + |\phi|^2) \lesssim (u_1)_+^{1-\gamma_0} \mathcal{E}_{0, \gamma_0} \quad (26)$$

for all $u_2 \leq u_1 \leq -\frac{R}{2}$.

Proof. By definition of \tilde{F} , we can obtain that

$$F_{0\mu}J[\phi]^\mu - \tilde{F}_{0\gamma}J^\gamma + \Re(h \cdot \overline{D_0\phi}) = (F_{0\mu} - \tilde{F}_{0\mu})J[\phi]^\mu = q_0 r^{-2} \Im(\phi \cdot \overline{D_{\partial_r}\phi}).$$

Apply the energy estimate (20) to (\tilde{F}, ϕ) . By using Cauchy-schwarz inequality, we can derive that

$$\begin{aligned} & E[\phi, \tilde{F}](\mathcal{H}_{u_1}^{-u_2}) + E[\phi, \tilde{F}](\underline{\mathcal{H}}_{-u_2}^{u_1}) \\ & \lesssim E[\phi, \tilde{F}](\Sigma_0^{u_1, u_2}) + \iint_{\mathcal{D}_{u_1}^{-u_2}} r^{-2} |D\phi| |\phi| \\ & \lesssim (u_1)_+^{-\gamma_0} \mathcal{E}_{0, \gamma_0} + \iint_{\mathcal{D}_{u_1}^{-u_2}} r^{-2} (|D_L\phi|^2 + |\not{D}\phi|^2 + |D_{\underline{L}}\phi|^2 + |\phi|^2) \\ & \lesssim (u_1)_+^{-\gamma_0} \mathcal{E}_{0, \gamma_0} + \int_{u_1}^{u_2} u_+^{-2} E[\phi, \tilde{F}](\mathcal{H}_u^{-u_2}) du + \int_{-u_1}^{-u_2} v_+^{-2} E[\phi, \tilde{F}](\underline{\mathcal{H}}_v^{u_1}) dv, \end{aligned}$$

where we used that in the exterior region $\{t + R \leq r\}$, we always have $v \leq r \leq 2v$. Applying Lemma 3 with

$$\tau = -u_1, \iota = -u_2, E[\phi, \tilde{F}](\mathcal{H}_u^{-u_2}) = f(-u, -u_2), E[\phi, \tilde{F}](\underline{\mathcal{H}}_v^{u_1}) = g(-u_1, v),$$

we derive the energy flux decay estimate (25).

To see (26), we apply the r -weighted energy identity (19) to (\tilde{F}, ϕ) with $p = 1$. First we note that

$$F_{X\mu}J[\phi]^\mu - \tilde{F}_{X\gamma}J^\gamma + \Re(h \cdot \overline{D_X\phi + \chi\phi}) = r(F_{L\underline{L}} - \tilde{F}_{L\underline{L}})J^{\underline{L}}[\phi] = -\frac{1}{2}q_0 r^{-1}J_L[\phi]$$

as h vanishes in this case and other components of $F - \tilde{F}$ vanish as well. This implies

$$\begin{aligned} & \frac{1}{2} \iint_{\mathcal{D}_{u_1}^{-u_2}} (r^{-2} |D_L(r\phi)|^2 + |\tilde{\alpha}|^2 + |\not{D}\phi|^2 + \tilde{\rho}^2 + \tilde{\sigma}^2) \\ & + \int_{\mathcal{H}_{u_1}^{-u_2}} r (r^{-2} |D_L(r\phi)|^2 + |\tilde{\alpha}|^2) + \int_{\underline{\mathcal{H}}_{-u_2}^{u_1}} r (|\not{D}\phi|^2 + \tilde{\rho}^2 + \tilde{\sigma}^2 + |\phi|^2) \\ & \lesssim (u_1)_+^{1-\gamma_0} \mathcal{E}_{0, \gamma_0} + \iint_{\mathcal{D}_{u_1}^{-u_2}} (|\phi|^2 + r^{-1} |D_L\phi| |\phi|) \\ & \lesssim (u_1)_+^{1-\gamma_0} \mathcal{E}_{0, \gamma_0} + \int_{u_1}^{u_2} E[\phi, \tilde{F}](\mathcal{H}_u^{-u_2}) du \lesssim (u_1)_+^{1-\gamma_0} \mathcal{E}_{0, \gamma_0}, \end{aligned}$$

where we employed the first estimate in (25) to derive the last inequality. Thus the proof of the weighted energy flux decay estimate (26) is completed. \square

2.4 Commutators

In Proposition 1, we established estimates for the energy flux and weighted energy flux of the lowest order. For higher order estimates we make use of the commuting vector field approach. The Killing vector fields that will be used as commutators are the generators of the Poincaré group

$$\Gamma = \{\partial, \Omega_{\mu\nu} = x^\mu \partial_\nu - x^\nu \partial_\mu\},$$

where $x^0 = -t$. It is convenient to define the following signature of ∂ and Ω , capturing their different weights in x :

$$\zeta(\partial_\mu) = -1, \quad \zeta(\Omega_{\mu\nu}) = 0.$$

For $Z^k = \Pi_{i=1}^k Z_i$ with $Z_i \in \Gamma$, we have $\zeta(Z^k) = \sum_{i=1}^k \zeta(Z_i)$. In particular, $\zeta(Z^0) = 0$. For convenience we denote by $\Gamma^k = \{Z^k = Z_1 \cdots Z_k, Z_i \in \Gamma\}$ and Γ^0 is $\{Z^0\}$.

To derive pointwise decay estimate we need energy decay estimates for higher order derivatives of solutions with respect to any vector-field $Z \in \Gamma$. For the scalar field ϕ , it is natural to take the covariant derivative $D_Z \phi = Z^\mu D_\mu \phi = Z^\mu (\partial_\mu \phi + \sqrt{-1} A_\mu \phi)$, associated to the connection field A , while for the Maxwell field F and 1-form J we take the Lie derivative

$$\begin{aligned} (\mathcal{L}_Z F)_{\mu\nu} &= Z(F_{\mu\nu}) - F(\mathcal{L}_Z \partial_\mu, \partial_\nu) - F(\partial_\mu, \mathcal{L}_Z \partial_\nu), \\ (\mathcal{L}_Z J)_\mu &= Z(J_\mu) - J(\mathcal{L}_Z \partial_\mu). \end{aligned}$$

We record the following useful commutator identities:

Lemma 4. *For any killing vector field Z , we have*

$$\begin{aligned} [\square_A - 1, D_Z] \phi &= 2iZ^\nu F_{\mu\nu} D^\mu \phi + i\partial^\mu (Z^\nu F_{\mu\nu}) \phi, \\ \partial^\mu (\mathcal{L}_Z G)_{\mu\nu} &= (\mathcal{L}_Z \delta G)_\nu \end{aligned}$$

for any complex scalar field ϕ and any closed 2-form G . Here δG is the 1-form defined by $(\delta G)_\nu = \partial^\mu G_{\mu\nu}$.

Proof. The proof is standard, see for example [29]. □

According to the above lemma, the Maxwell field commutes with the Lie derivatives along Killing fields. To treat the commutator error terms generated by $\square_A - 1$, it is helpful to define a trilinear form for any two form \mathcal{G} , any complex scalar field f and killing vector field $Z \in \Gamma$,

$$Q(\mathcal{G}, f, Z) = 2iZ^\nu \mathcal{G}_{\mu\nu} D^\mu f + i\partial^\mu (Z^\nu \mathcal{G}_{\mu\nu}) f. \quad (27)$$

We have the following double commutator identity (see [29, Lemma 9]).

Lemma 5. *For all $X, Y \in \Gamma$, we have*

$$[D_Y, [\square_A - 1, D_X]] \phi = Q(\mathcal{L}_Y F, \phi, X) + Q(F, \phi, [Y, X]) - 2F_{Y\mu} F^\mu_X \phi. \quad (28)$$

This lemma gives the structure for the error terms of second order derivatives for the scalar fields, which will be used together with the following formula.

Lemma 6. *Given two vector fields $X, Y \in \Gamma$, we have*

$$(\square_A - 1) D_X D_Y \phi = Q(F, D_Y \phi, X) + Q(F, D_X \phi, Y) + [D_X, [\square_A - 1, D_Y]] \phi + D_X D_Y ((\square_A - 1) \phi).$$

Proof. In fact we can compute that

$$\begin{aligned} &(\square_A - 1) D_X D_Y \phi \\ &= [\square_A - 1, D_X] D_Y \phi + D_X ((\square_A - 1) D_Y \phi) \\ &= Q(F, D_Y \phi, X) + D_X ([\square_A - 1, D_Y] \phi + D_Y ((\square_A - 1) \phi)) \\ &= Q(F, D_Y \phi, X) + [D_X, [\square_A - 1, D_Y]] \phi + [\square_A - 1, D_Y] D_X \phi + D_X D_Y ((\square_A - 1) \phi) \\ &= Q(F, D_Y \phi, X) + Q(F, D_X \phi, Y) + [D_X, [\square_A - 1, D_Y]] \phi + D_X D_Y ((\square_A - 1) \phi). \end{aligned}$$

□

Therefore, in view of (28), if ϕ solves the equation $\square_A \phi = \phi$, the righthand side of the above identities are linear combination of the trilinear forms of Q except the cubic term $F_{Y\mu} F_X^\mu \phi$.

The next lemma shows that $Q(\mathcal{G}, \phi, Z)$ indeed verifies a null type structure, that is the “bad” component $\underline{\alpha}[\tilde{F}]$ does not interact with $D_{\underline{L}}\phi$.

Lemma 7. *For any 2-form $\mathcal{G} = (\alpha, \rho, \sigma, \underline{\alpha})$ and a complex scalar field ϕ , in the exterior region $\{t + R \leq r\}$ we have*

$$|Q(\mathcal{G}, \phi, Z)| \lesssim r^{\zeta(Z)+1}(|\alpha||D\phi| + |\underline{\mathcal{G}}||D_L\phi| + |\sigma||\not{D}\phi|) + u_+^{\zeta(Z)+1}(|\rho||D_{\underline{L}}\phi| + |\underline{\alpha}||\not{D}\phi|) \\ + \left(u_+^{\zeta(Z)+1}|J_{\underline{L}}| + r^{\zeta(Z)+1}(|J_L| + |\not{J}|) + r^{\zeta(Z)}|\mathcal{G}|\right)|\phi| \quad (29)$$

for all $Z \in \Gamma$, where $\underline{\mathcal{G}}$ denotes all the components of \mathcal{G} except $\alpha[\mathcal{G}]$. Here $(J_L, J_{\underline{L}}, \not{J})$, $\not{J} = (J_{e_1}, J_{e_2})$ denote the null components of the 1-form $J = -\delta\mathcal{G}$.

Proof. Recall the definition of Q in (27), we have in view of (mMKG) that

$$Q(\mathcal{G}, \phi, Z) = 2iZ^\nu \mathcal{G}_{\mu\nu} D^\mu \phi + i(\partial^\mu Z^\nu \mathcal{G}_{\mu\nu} - J_Z)\phi.$$

When $Z = \partial_\mu$, we can estimate that

$$|Q(\mathcal{G}, \phi, \partial_\mu)| \lesssim (|\alpha| + |\underline{\alpha}| + |\sigma|)|\not{D}\phi| + (|\rho| + |\underline{\alpha}|)|D_L\phi| + |D_{\underline{L}}\phi|(|\rho| + |\alpha|) + |J||\phi|.$$

As in this case $\zeta(\partial_\mu) = -1$, (29) holds indeed for $Z = \partial_\mu$.

When $Z = \Omega_{ij}$, we can easily show that

$$|Q(\mathcal{G}, \phi, \Omega_{ij})| \lesssim r(|\alpha||D_{\underline{L}}\phi| + |\underline{\alpha}||D_L\phi| + |\sigma||\not{D}\phi| + |\not{J}||\phi|) + |\mathcal{G}||\phi|.$$

For $Z = \Omega_{0i}$, we can write that

$$\Omega_{0i} = t\partial_i + x_i\partial_t = \omega_i(t\partial_r + r\partial_t) + t(\partial_i - \omega_i\partial_r) = \omega_i(vL - u\underline{L}) + t(\partial_i - \omega_i\partial_r).$$

Therefore we can bound that

$$|Q(\mathcal{G}, \phi, \Omega_{0j})| \lesssim r(|\alpha||D\phi| + (|\rho| + |\underline{\alpha}|)|D_L\phi| + |\sigma||\not{D}\phi|) + u_+(|\rho||D_{\underline{L}}\phi| + |\underline{\alpha}||\not{D}\phi|) \\ + (r|\not{J}| + r|J_L| + u_+|J_{\underline{L}}| + |\mathcal{G}|)|\phi|.$$

Since $\zeta(\Omega) = 0$, we conclude that the estimate (29) hold for all $Z \in \Gamma$. \square

Our next lemma shows that the quadratic part of the cubic terms $F_{Y\mu} F_X^\mu \phi$ also posses a necessary null structure. The following result will be used in Section 3.5.

Lemma 8. *For all $X, Y \in \Gamma$ and $r \geq R$, we have the following estimate*

$$|F_{Y\mu} F_X^\mu| \lesssim u_+^{\zeta(XY)+2}|\underline{\alpha}|^2 + u_+^{\zeta(XY)}r^2(|\sigma|^2 + |\alpha|^2 + |\rho|^2 + |\alpha||\underline{\alpha}|). \quad (30)$$

Proof. If both $X, Y \in \{\partial_\mu\}$, we can simply bound that

$$|F_X^\mu F_{Y\mu}| \lesssim |F|^2.$$

For this case $\zeta(XY) = -2$, in particular estimate (30) holds.

If both $X, Y \in \{\Omega_{\mu\nu}\}$, we can write the Lorentz boost as in the previous lemma

$$\Omega_{0j} = \omega_j(vL - u\underline{L}) + t(\partial_j - \omega_j\partial_r).$$

Then we can bound that

$$\begin{aligned} |F_X^\mu F_{Y\mu}| &\lesssim r(|\rho| + |\underline{\alpha}|)(u_+|\rho| + r|\alpha|) + (u_+|\underline{\alpha}| + r|\alpha| + r|\sigma|)^2 \\ &\lesssim u_+^2|\underline{\alpha}|^2 + r^2(|\alpha|^2 + |\sigma|^2 + |\rho|^2 + |\alpha||\underline{\alpha}|). \end{aligned}$$

Since $\zeta(XY) = 0$ in this case, we conclude that estimate (30) is proved.

If, without loss of generality, $X = \partial_\mu$, $Y = \Omega_{\nu\gamma}$, we then can show that

$$|F_X^\mu F_{Y\mu}| \lesssim r|\alpha||F| + r|\rho|(|\rho| + |\underline{\alpha}|) + r|\sigma|(|\underline{\alpha}| + |\sigma|) + u_+|\underline{\alpha}|(|\alpha| + |\underline{\alpha}| + |\sigma|).$$

Considering that $\zeta(XY) = -1$ for this situation, by using Cauchy-Schwarz's inequality, we have shown that estimate (30) holds for all $X, Y \in \Gamma$. \square

3 Proof of main theorem

In this section, we prove Theorem 1 by a bootstrap argument.

3.1 Bootstrap argument

For some small positive constant Δ_0 , verifying $1 > \Delta_0 \geq \mathcal{E}_{2,\gamma_0}$, to be determined later, we make a set of bootstrap assumptions on the the solution (F, ϕ) to the massive MKG equations in the exterior region. The bootstrap assumptions mainly consist of the higher order energy flux decay as well as the r -weighted energy flux decay of the solution. For the scalar field, the highest order r -weighted energy estimates can at most have weights r while for the chargeless part of the Maxwell field, the weights are chosen to be r^{γ_0} for some $\gamma_0 > 1$.

Let $v_* > \frac{R}{2}$ be a fixed constant. We suppose the following estimates hold

$$E[D_Z^k \phi, \mathcal{L}_Z^k \tilde{F}](\mathcal{H}_{u_1}^{-u_2}) + E[D_Z^k \phi, \mathcal{L}_Z^k \tilde{F}](\mathcal{H}_{-u_2}^{u_1}) \leq 2(u_1)_+^{-\gamma_0+2\zeta(Z^k)} \Delta_0, \quad (31)$$

$$\int_{\mathcal{H}_{u_1}^{-u_2}} r |D_L D_Z^k \phi|^2 + \int_{\mathcal{H}_{-u_2}^{u_1}} r (|\mathcal{D} D_Z^k \phi|^2 + |D_Z^k \phi|^2) \leq 2(u_1)_+^{1-\gamma_0+2\zeta(Z^k)} \Delta_0, \quad (32)$$

$$\begin{aligned} \int_{\mathcal{H}_{u_1}^{-u_2}} r^{\gamma_0} |\alpha[\mathcal{L}_Z^k \tilde{F}]|^2 + \iint_{\mathcal{D}_{u_1}^{-u_2}} r^{\gamma_0-1} |(\alpha, \rho, \sigma)[\mathcal{L}_Z^k \tilde{F}]|^2 \\ + \int_{\mathcal{H}_{-u_2}^{u_1}} r^{\gamma_0} (|\rho[\mathcal{L}_Z^k \tilde{F}]|^2 + |\sigma[\mathcal{L}_Z^k \tilde{F}]|^2) \leq 2(u_1)_+^{2\zeta(Z^k)} \Delta_0 \end{aligned} \quad (33)$$

for all $-v_* \leq u_2 < u_1 \leq -\frac{R}{2}$, $Z^k \in \Gamma^k$, $k \leq 2$.

Here we remark that $k = 0$, due to $\zeta(Z^0) = 0$, (31) and (32) can be improved in view of Proposition 1. The improvement of all other estimates relies on the bootstrap assumptions. At the end of this section, we will be able to show the inequalities (31)-(33) hold with $2\Delta_0$ replaced by $C(\mathcal{E}_{2,\gamma_0} + \Delta_0^2)$. We then will choose Δ_0 to achieve the improvements of the bootstrap assumptions. We note that the improved estimates are independent of the choice of v_* , which allows us to achieve Theorem 1 (2) for any $v_* > R/2$ by the principle of continuation.

The strategy of the proof is that with these assumptions, we first prove the pointwise decay estimates for the scalar field ϕ as well as the chargeless part of the Maxwell field \tilde{F} . We then obtain energy estimates and the r -weighted energy estimates, which improves the above bootstrap assumptions due to the smallness of the initial data. Combined with the standard local existence result, we conclude that the solution exists globally in time in the exterior region. As a consequence, we also obtain the decay estimates for the solutions.

We will rely on Lemma 1 and a similar Sobolev inequality to derive the pointwise decay. For this purpose, we need to bound the initial value of the solution (F, ϕ) and its derivatives in terms of the given data, which is treated below.

Lemma 9. Let $2 \leq p \leq 4$ and $u \leq -\frac{1}{2}R$. For the admissible data (F, ϕ) of (mMKG) we have the following.

(1) For all $Z^l \in \Gamma^l$ and $l \leq 2$,

$$\int_{\mathbb{S}^2} |D_Z^l \phi|^p(u, -u, \omega) d\omega \lesssim u_+^{-2+\frac{p}{2}(-\gamma_0+2\zeta(Z^l))} \mathcal{E}_{l,\gamma_0}^{\frac{p}{2}}. \quad (34)$$

In particular, for all $Z^l \in \Gamma^l$ and $l \leq 1$,

$$\int_{\mathbb{S}^2} |D_Z^l \phi|^p(u, -u, \omega) d\omega \lesssim u_+^{2-2p+\frac{p}{2}(-\gamma_0+2\zeta(Z^l))} \mathcal{E}_{l+1,\gamma_0}^{\frac{p}{2}}. \quad (35)$$

(2) For all $Z = \Omega_{ij}$ and $l \leq 1$,

$$\int_{\mathbb{S}^2} |\mathcal{L}_Z^l \tilde{F}|^p(u, -u, \omega) d\omega \lesssim u_+^{2-2p-\frac{p}{2}\gamma_0} \mathcal{E}_{l+1,\gamma_0}^{\frac{p}{2}}. \quad (36)$$

Proof. Let us consider (34) first. Note that we have the embedding

$$\left(\int_{\Sigma_0^u} |D_Z^l \phi|^6 dx \right)^{\frac{1}{3}} \lesssim \int_{\Sigma_0^u} (|D_{\partial r} D_Z^l \phi|^2 + |\not{D} D_Z^l \phi|^2 + |D_Z^l \phi|^2) dx \lesssim \mathcal{E}_{l,\gamma_0}. \quad (37)$$

Since \mathcal{E}_{l,γ_0} is finite, there holds

$$\liminf_{u_+ \rightarrow \infty} \int_{\mathbb{S}^2} |D_Z^l \phi|^6(u, -u, \omega) d\omega = 0, \forall l \leq 2.$$

By interpolation, for all $l \leq 2$ and $2 \leq p \leq 6$, we can obtain

$$\liminf_{u_+ \rightarrow \infty} \int_{\mathbb{S}^2} |D_Z^l \phi|^p(u, -u, \omega) d\omega = 0.$$

Note that for $l \leq 2$, by integrating from the spatial infinity, we have

$$\int_{\mathbb{S}^2} |D_Z^l \phi|^2(u, -u, \omega) d\omega \lesssim \int_{-2u}^{\infty} \int_{\mathbb{S}^2} |D_{\partial r} D_Z^l \phi| |D_Z^l \phi| dr d\omega \lesssim u_+^{-\gamma_0-2+2\zeta(Z^l)} \mathcal{E}_{l,\gamma_0} \quad (38)$$

and if $l \leq 1$, we can derive

$$\int_{\mathbb{S}^2} |D_Z^l \phi|^2(u, -u, \omega) d\omega \lesssim u_+^{-3} \|r D_{\partial r} D_Z^l \phi\|_{L^2(\Sigma_0^u)} \|D_Z^l \phi\|_{L^2(\Sigma_0^u)} \lesssim u_+^{-\gamma_0-3+2\zeta(Z^l)} \mathcal{E}_{l+1,\gamma_0}. \quad (39)$$

Next, we consider the L^4 estimates. Similar to the proof of (12) in Lemma 1, we can obtain for a complex scalar field f that

$$\int_{S(u,-u)} |f|^4 r^{-2} \lesssim \lim_{u_+ \rightarrow \infty} \int_{S(u,-u)} |f|^4 r^{-2} + \int_{\Sigma_u^0} |D_{\partial r} f|^2 r^{-2} dx \cdot \int_{\Sigma_0^u} (|f|^2 + r^2 |\not{D} f|^2) r^{-2} dx.$$

Applying the above inequality to $f = D_Z^l \phi$, with $l \leq 2$, gives

$$\int_{\mathbb{S}^2} |D_Z^l \phi|^4(u, -u, \omega) \lesssim u_+^{-2} \sum_{k \leq 1} \|\bar{D}^k D_Z^l \phi\|_{L^2(\Sigma_0^u)}^4 \lesssim u_+^{-2-2\gamma_0+4\zeta(Z^l)} \mathcal{E}_{l,\gamma_0}^2$$

where \bar{D} is the projection of D to $\mathbb{R}^3 \times \{t = 0\}$.

When $l \leq 1$, we have the improved estimate

$$\begin{aligned} \int_{\mathbb{S}^2} |D_Z^l \phi|^4(u, -u, \omega) &\lesssim u_+^{-6} \|r D_{\partial r} D_Z^l f\|_{L^2(\Sigma_0^u)}^2 \sum_{k \leq 1, 1 \leq i < j \leq 3} \|D_{\Omega_{ij}}^k D_Z^l \phi\|_{L^2(\Sigma_0^u)}^2 \\ &\lesssim u_+^{-6-2\gamma_0+4\zeta(Z^l)} \mathcal{E}_{l+1, \gamma_0}^2. \end{aligned}$$

where we used the fact that $r|\not{D}f| \lesssim \sum_{1 \leq i < j \leq 3} |D_{\Omega_{ij}} f|$ for any complex scalar field. By interpolating the above two estimates with (38) and (39) respectively, the estimates in (1) of this lemma can be proved.

To see (36), we first note that it is direct to check $|\mathcal{L}_Z^l \tilde{F}| \approx |\mathcal{L}_Z^l \tilde{E}_i| + |\mathcal{L}_Z^l \tilde{H}_i|$. Then it suffices to show (36) holds for $\mathcal{L}_Z^l \tilde{E}_i$ and $\mathcal{L}_Z^l \tilde{H}_i$ with $Z = \Omega_{ij}$ and $l \leq 1$. We then can repeat the proof for (35) to obtain the estimate in (36) to $\mathcal{L}_Z^l \tilde{E}$ and $\mathcal{L}_Z^l \tilde{H}$ separately. Thus we can complete the proof of (36). \square

3.2 Decay estimates for the scalar field

We first derive the decay estimates for the scalar field with the help of the bootstrap assumptions.

Proposition 2. *Under the bootstrap assumptions (31) and (32), in $\{r \geq t + R, v \leq v_*\}$, we have the decay estimates for the scalar fields*

$$r^2 |\not{D}\phi|^2 + u_+^2 |D_{\underline{L}}\phi|^2 + r^2 |D_L\phi|^2 + u_+^{-2\zeta(Z)} |D_Z\phi|^2 \lesssim \Delta_0 r^{-\frac{5}{2}+\epsilon} u_+^{\frac{1}{2}-\gamma_0}, \quad (40)$$

$$|\phi|^2 \lesssim \Delta_0 r^{-3} u_+^{-\gamma_0}, \quad (41)$$

where $Z \in \Gamma$.

Proof. Let's first consider the decay estimates for $D_Z\phi$ for all $Z \in \Gamma$. We will make a slight modification on Lemma 1. We can show that, by integrating from the initial slice,

$$\begin{aligned} & \left| \int_{\mathbb{S}^2} r^{\frac{5}{2}} |D_Z^l \phi|^p(u, v, \omega) d\omega - \int_{\mathbb{S}^2} r^{\frac{5}{2}} |D_Z^l \phi|^p(u, -u, \omega) d\omega \right| \\ & \lesssim \int_{-u}^v \int_{\mathbb{S}^2} r^{\frac{5}{2}} |D_L D_Z^l \phi| |D_Z^l \phi|^{p-1} dv' d\omega + \int_{-u}^v \int_{\mathbb{S}^2} r^{\frac{3}{2}} |D_Z^l \phi|^p dv' d\omega \\ & \lesssim (\|r^{\frac{1}{2}} D_L D_Z^l \phi\|_{L^2(\mathcal{H}_u^{-u_2})} + \|r^{-\frac{1}{2}} D_Z^l \phi\|_{L^2(\mathcal{H}_u^{-u_2})}) \| |D_Z^l \phi|^{p-1} \|_{L^2(\mathcal{H}_u^{-u_2})}. \end{aligned}$$

Here $Z^l \in \Gamma^l$ with $l \leq 2$.

When $p = 2$, we can bound the term of $\| |D_Z^l \phi|^{p-1} \|_{L^2(\mathcal{H}_u^{-u_2})}$ by the energy flux through \mathcal{H}_u . While for $p = 4$, similar to (37) we can bound the integral by using Sobolev embedding on the outgoing null hypersurface \mathcal{H}_u as follows,

$$\left(\int_{-u}^{-u_2} \int_{\mathbb{S}^2} r^2 |D_Z^l \phi|^6 dv d\omega \right)^{\frac{1}{3}} \lesssim \int_{\mathcal{H}_u^{-u_2}} \{ |D_L D_Z^l \phi|^2 + |\not{D} D_Z^l \phi|^2 + |D_Z^l \phi|^2 \}.$$

Also by using (34), we can obtain for $l \leq 2$,

$$\begin{aligned} \int_{\mathbb{S}^2} r^{\frac{5}{2}} |D_Z^l \phi|^2(u, v, \omega) d\omega &\lesssim u_+^{\frac{1}{2}} u_+^{-\gamma_0+2\zeta(Z^l)} \Delta_0, \\ \int_{\mathbb{S}^2} r^{\frac{5}{2}} |D_Z^l \phi|^4(u, v, \omega) d\omega &\lesssim u_+^{\frac{1}{2}} (u_+^{-\gamma_0+2\zeta(Z^l)} \Delta_0)^2. \end{aligned}$$

Then interpolation implies that for $2 \leq p \leq 4$, $l \leq 2$,

$$\begin{aligned} \int_{\mathbb{S}^2} |D_Z^l \phi|^p(u, v, \omega) d\omega &\lesssim \|D_Z^l \phi(u, v, \cdot)\|_{L_\omega^2}^{4-p} \|D_Z^l \phi(u, v, \cdot)\|_{L_\omega^4}^{2p-4} \\ &\lesssim \Delta_0^{\frac{p}{2}} r^{-\frac{5}{2}} u_+^{\frac{1}{2} + \frac{p}{2}(-\gamma_0 + 2\zeta(Z^l))}. \end{aligned}$$

Take $Z^l = \Omega_{ij}Z$ and $p = 2 + \epsilon$. By using Sobolev embedding on the unit sphere we obtain the decay estimates for $D_Z \phi$,

$$|D_Z \phi|^2 \lesssim \Delta_0 r^{-\frac{5}{2} + \epsilon} u_+^{\frac{1}{2} + 2\zeta(Z) - \gamma_0}.$$

Now by taking $Z = \Omega_{ij}$, we can derive the decay estimate for $|\not{D}\phi|$ which could be bounded above by the sum of $r^{-1}|D_{\Omega_{ij}}\phi|$. For the other components such as $D_L \phi$ and $D_{\underline{L}} \phi$, first we can take $Z = \partial_\mu$ to conclude that

$$|D\phi|^2 \lesssim \Delta_0 r^{-\frac{5}{2} + \epsilon} u_+^{\frac{1}{2} - \gamma_0 - 2}.$$

In particular, we have

$$u_+^2 |D_{\underline{L}} \phi|^2 \lesssim \Delta_0 r^{-\frac{5}{2} + \epsilon} u_+^{\frac{1}{2} - \gamma_0}.$$

For $D_L \phi$, we use the Lorentz boost $Z = \Omega_{0j}$. Notice that

$$\Omega_{0j} = \omega_j(vL - u\underline{L}) + t(\partial_j - \omega_j \partial_r).$$

In the exterior region $\{t + R \leq r\}$, we have

$$|D_{t(\partial_j - \omega_j \partial_r)} \phi|^2 \lesssim \sum_{l,k} |D_{\Omega_{lk}} \phi|^2.$$

Therefore we can show that

$$r^2 |D_L \phi|^2 \lesssim |D_{vL} \phi|^2 \lesssim \sum_j |D_{\Omega_{0j}} \phi|^2 + |D_{u\underline{L}} \phi|^2 + \sum_{l,k} |D_{\Omega_{lk}} \phi|^2 \lesssim \Delta_0 r^{-\frac{5}{2} + \epsilon} u_+^{\frac{1}{2} - \gamma_0}. \quad (42)$$

Therefore the decay estimate (40) holds.

To show the improved decay estimate (41) for ϕ , we need the associated improved energy flux decay. Recall the bootstrap assumption (31) implies that

$$\int_{\mathcal{H}_u^{-u_2}} |D_Z^l \phi|^2 \lesssim E[D_Z^l \phi](\mathcal{H}_u^{-u_2}) \lesssim \Delta_0 u_+^{-\gamma_0 + 2\zeta(Z^l)}, \quad \forall l \leq 2.$$

Therefore similar to (42), we can show that

$$\begin{aligned} \int_{\mathcal{H}_u^{-u_2}} r^2 |D_L D_Z^l \phi|^2 &\lesssim \int_{\mathcal{H}_u^{-u_2}} (|D_{\Omega_{0j}} D_Z^l \phi|^2 + |D_{\Omega_{ij}} D_Z^l \phi|^2 + u_+^2 |D D_Z^l \phi|^2) \\ &\lesssim \Delta_0 u_+^{-\gamma_0 + 2\zeta(Z^l)}, \quad \forall l \leq 1. \end{aligned}$$

Now we substitute the above two estimates with $Z^l = \Omega_{ij}^l$, $l \leq 2$, $\forall 1 \leq i < j \leq 3$ to the following inequality, which is derived by applying Lemma 1 to $f = \phi$, $\gamma = \frac{3}{2}$, $\gamma'_0 = \gamma_2 = 1$,

$$\begin{aligned} \sup_{S(u,v)} |r^{\frac{3}{2}} \phi|^4 &\lesssim \sum_{l \leq 1, 1 \leq i < j \leq 3} \int_{S(u, -u)} |r D_{\Omega_{ij}}^l \phi|^4 + \sum_{k \leq 2, 1 \leq i < j \leq 3} \int_{\mathcal{H}_u^{-u, v_*}} |D_{\Omega_{ij}}^k f|^2 \\ &\times \sum_{l \leq 1, 1 \leq i < j \leq 3} \int_{\mathcal{H}_u^{-u, v_*}} r^{-1} |D_L D_{\Omega_{ij}}^l (r^{\frac{3}{2}} \phi)|^2 \lesssim \Delta_0^2 u_+^{-2\gamma_0}, \end{aligned}$$

where the term on the sphere $S(u, -u)$ has been treated by (35). Thus we have completed the proof of (41). \square

3.3 Decay estimates for the Maxwell field

In this subsection, we derive the pointwise decay estimates for the Maxwell field under the bootstrap assumptions.

Proposition 3. *In the exterior region $\{t + R \leq r, v \leq v_*\}$, under the bootstrap assumptions (31) and (33), we have*

$$|\tilde{\rho}|^2 + |\alpha|^2 + |\sigma|^2 \lesssim \Delta_0 r^{-2-\gamma_0} u_+^{-1}, \quad (43)$$

$$|\underline{\alpha}|^2 \lesssim \Delta_0 r^{-2} u_+^{-\gamma_0-1}, \quad (44)$$

where $\tilde{\rho} = \rho[\tilde{F}]$ ¹⁵ and clearly $|\rho|^2 \lesssim (\Delta_0 + q_0^2) r^{-2} u_+^{-2}$.

Assuming Proposition 3, in view of $F = \tilde{F} + q_0 r^{-2} dt \wedge dr$, we can have the rough estimate

$$|F_{X\mu}|^2 \lesssim (\Delta_0 + q_0^2) u_+^{-2} r^{2\zeta(X)}, \quad \forall X \in \Gamma. \quad (45)$$

Proof. To prove these pointwise estimates, we apply Lemma 1 with the help of the (weighted) energy flux decay through the outgoing null hypersurface \mathcal{H}_u or the incoming null hypersurfaces \mathcal{H}_v . We first need some preliminary results. Let us show that for any 2-form G ,

$$|\underline{L}(\rho[G])| \lesssim |\rho[\mathcal{L}_{\partial} G]|, \quad |\underline{L}(\sigma[G])| \lesssim |\sigma[\mathcal{L}_{\partial} G]|, \quad (46)$$

$$|\nabla_L \alpha_A[G]| \lesssim |\alpha[\mathcal{L}_{\partial} G]|, \quad |\nabla_L \underline{\alpha}_A[G]| \lesssim |\underline{\alpha}[\mathcal{L}_{\partial} G]|, \quad (47)$$

$$\alpha[\mathcal{L}_{\Omega_{ij}}^l G] = \mathcal{L}_{\Omega_{ij}}^l \alpha[G], \quad \underline{\alpha}[\mathcal{L}_{\Omega_{ij}}^l G] = \mathcal{L}_{\Omega_{ij}}^l \underline{\alpha}[G], \quad (48)$$

$$\rho[\mathcal{L}_{\Omega_{ij}}^l G] = \Omega_{ij}^l \rho[G], \quad \sigma[\mathcal{L}_{\Omega_{ij}}^l G] = \Omega_{ij}^l \sigma[G]. \quad (49)$$

Indeed, we recall from [19, Page 58-59] that

$$\partial_r \rho = \omega^i \rho[\mathcal{L}_{\partial_i} G], \quad \partial_r \sigma = \omega^i \sigma[\mathcal{L}_{\partial_i} G], \quad \partial_r \alpha_A[G] = \omega^i \alpha_A[\mathcal{L}_{\partial_i} G]. \quad (50)$$

It is easy to check that

$$\partial_t \rho[G] = \rho[\mathcal{L}_{\partial_t} G], \quad \partial_t \sigma[G] = \sigma[\mathcal{L}_{\partial_t} G], \quad \partial_t \alpha_A[G] = \alpha_A[\mathcal{L}_{\partial_t} G]. \quad (51)$$

Hence we can obtain (46) by combining the above identities. (48) and (49) can be proved in view of [19, Page 58-59 (5.20), (5.21) and (5.28)].

To see (47), we first note

$$\nabla_L \alpha_A[G] = L \alpha_A[G] - \alpha[G] \nabla_L e_A = \partial_t \alpha_A[G] + \partial_r \alpha_A[G],$$

where we used the fact that $\nabla_L e_A = 0$. Hence, in view of (50) and (51), we can derive the first inequality in (47). The second inequality in (47) can be proved in the same way.

With the help of (48) and (49), the estimate (36) holds for the components of $\mathcal{L}_{\Omega_{ij}}^l \alpha[\tilde{F}]$, $\mathcal{L}_{\Omega_{ij}}^l \underline{\alpha}[\tilde{F}]$, $\Omega_{ij}^l \rho[\tilde{F}]$ and $\Omega_{ij}^l \sigma[\tilde{F}]$, which gives the bounds for the initial values needed in the sequel.

Let's first consider the decay estimate for α . Apply Lemma 1(3) with $H = \alpha$, $\gamma'_0 = 0$ and $\gamma = \gamma_2 = 1 + \frac{1}{2}\gamma_0$. By using (48), we can derive

$$\int_{\mathcal{H}_u^{-u_2}} |\mathcal{L}_{\Omega_{ij}}^k \tilde{\alpha}|^2 r^{\gamma_0} = \int_{\mathcal{H}_u^{-u_2}} |\alpha[\mathcal{L}_{\Omega_{ij}}^k \tilde{F}]|^2 r^{\gamma_0} \lesssim \Delta_0, \quad \forall k \leq 2$$

¹⁵To distinguish from the component of F , we add the tilde to the Greek letter when it is the component of \tilde{F} . It is clear that $\alpha = \underline{\tilde{\alpha}}$, $\underline{\tilde{\alpha}} = \underline{\alpha}$, $\tilde{\sigma} = \sigma$.

and

$$\int_{\mathcal{H}_u^{-u_2}} |\nabla_L \mathcal{L}_{\Omega_{ij}}^l (r^{1+\frac{1}{2}\gamma_0} \tilde{\alpha})|^2 dv d\omega \lesssim \int_{\mathcal{H}_u^{-u_2}} (r^{\gamma_0} |\alpha[\mathcal{L}_{\partial} \mathcal{L}_{\Omega_{ij}}^l \tilde{F}]|^2 + r^{\gamma_0-2} |\alpha[\mathcal{L}_{\Omega_{ij}}^l \tilde{F}]|^2) \lesssim \Delta_0 u_+^{-2}.$$

The integral on $S(u, -u)$ is controlled in view of (36). Then by using (11) we obtain that

$$r^{2+\gamma_0} |\alpha|^2 \lesssim \Delta_0 u_+^{-1}.$$

For the other components, we make use of the energy flux through the incoming null hypersurface. However for $\tilde{\rho}$ and σ , the extra decay in r relies on the r -weighted energy flux through the incoming null hypersurface while for $\underline{\alpha}$ we can only use the energy flux. Let $-\frac{R}{2} \leq -u < v \leq v_*$. For $\tilde{\rho}$ and σ , from the bootstrap assumption (33), we first conclude that

$$\int_{\underline{\mathcal{H}}_v^u} r^{\gamma_0} (|\rho[\mathcal{L}_{\partial} \mathcal{L}_{\Omega_{ij}}^k \tilde{F}]|^2 + |\sigma[\mathcal{L}_{\partial} \mathcal{L}_{\Omega_{ij}}^k \tilde{F}]|^2) \lesssim \Delta_0 u_+^{-2}, \quad \forall k \leq 1.$$

By using (46) and (49), we have

$$|\underline{L} \mathcal{L}_{\Omega_{ij}}^k \rho[\tilde{F}]| \lesssim |\rho[\mathcal{L}_{\partial} \mathcal{L}_{\Omega_{ij}}^k \tilde{F}]|, \quad |\underline{L} \mathcal{L}_{\Omega_{ij}}^k \sigma[\tilde{F}]| \lesssim |\sigma[\mathcal{L}_{\partial} \mathcal{L}_{\Omega_{ij}}^k \tilde{F}]|.$$

We therefore can derive that

$$\int_{\underline{\mathcal{H}}_v^u} r^{\gamma_0} (|\underline{L}(\mathcal{L}_{\Omega_{ij}}^k \tilde{\rho})|^2 + |\underline{L}(\mathcal{L}_{\Omega_{ij}}^k \tilde{\sigma})|^2) \lesssim \Delta_0 u_+^{-2}, \quad \forall k \leq 1.$$

By using (49) and the bootstrap assumption (33), we have

$$\int_{\underline{\mathcal{H}}_v^u} r^{\gamma_0} |\Omega_{ij}^k(\tilde{\rho}, \tilde{\sigma})|^2 = \int_{\underline{\mathcal{H}}_v^u} r^{\gamma_0} |(\rho, \sigma)[\mathcal{L}_{\Omega_{ij}}^k \tilde{F}]|^2 \lesssim \Delta_0, \quad \forall k \leq 2.$$

Combining the above two estimates, we can derive

$$\begin{aligned} \int_{\underline{\mathcal{H}}_v^u} |\underline{L}(\mathcal{L}_{\Omega_{ij}}^k r^{1+\frac{1}{2}\gamma_0}(\tilde{\rho}, \tilde{\sigma}))|^2 du d\omega &\lesssim \int_{\underline{\mathcal{H}}_v^u} r^{\gamma_0} |\underline{L} \mathcal{L}_{\Omega_{ij}}^k(\tilde{\rho}, \tilde{\sigma})|^2 + r^{\gamma_0-2} |\mathcal{L}_{\Omega_{ij}}^k(\tilde{\rho}, \tilde{\sigma})|^2 \\ &\lesssim \Delta_0 u_+^{-2}, \quad \forall k \leq 1. \end{aligned}$$

We now apply Lemma 1 (1) to $f = (\tilde{\rho}, \tilde{\sigma})$, $\gamma = \gamma_2 = 1 + \frac{1}{2}\gamma_0$ on the incoming null hypersurface $\underline{\mathcal{H}}_v^u$. In this case, we replace L derivative by \underline{L} derivative. The integral on the sphere $S(-v, v)$ can be bounded in view of (36). As we assumed that $\mathcal{E}_{2,\gamma_0} \leq \Delta_0$, we then derive from Lemma 1 that

$$r^{2+\gamma_0} (|\tilde{\rho}|^2 + |\tilde{\sigma}|^2) \lesssim \Delta_0 u_+^{-1}.$$

Finally for $\tilde{\alpha}$, we make use of the energy flux through the incoming null hypersurface. In Lemma 1 (3), let $\gamma = 1$, $\gamma_2 = 1$, $H = \tilde{\alpha}$ on the incoming null hypersurface $\underline{\mathcal{H}}_v^u$. In view of (48), we can bound that

$$\int_{\underline{\mathcal{H}}_v^u} |\mathcal{L}_{\Omega_{ij}}^k \tilde{\alpha}|^2 \lesssim E[\mathcal{L}_{\Omega_{ij}}^k \tilde{F}](\underline{\mathcal{H}}_v^u) \lesssim \Delta_0 u_+^{-\gamma_0}, \quad \forall k \leq 2.$$

In view of (47) and (48), we can also show that for $l \leq 1$,

$$\begin{aligned} \int_{\underline{\mathcal{H}}_v^u} |\nabla_L \mathcal{L}_{\Omega_{ij}}^l (r \tilde{\alpha})|^2 du' d\omega &\lesssim \int_{\underline{\mathcal{H}}_v^u} |\nabla_L \mathcal{L}_{\Omega_{ij}}^l \tilde{\alpha}|^2 + r^{-2} |\mathcal{L}_{\Omega_{ij}}^l \tilde{\alpha}|^2 \\ &\lesssim \int_{\underline{\mathcal{H}}_v^u} |\alpha[\mathcal{L}_{\partial} \mathcal{L}_{\Omega_{ij}}^l \tilde{F}]|^2 + r^{-2} |\alpha[\mathcal{L}_{\Omega_{ij}}^l \tilde{F}]|^2 \\ &\lesssim \Delta_0 u_+^{-2-\gamma_0}. \end{aligned}$$

The integral on $S(-v, v)$ is controlled by (36). Thus by using Lemma 1, we derive that

$$r^2|\underline{\alpha}|^2 \lesssim \Delta_0 u_+^{-1-\gamma_0}.$$

□

As a corollary of Proposition 3, we derive below a result which will be crucial in the last subsection.

Corollary 1. *Let $\mathcal{I}_Y[F] = r(|F_{LY}| + |F_{AY}|) + u_+|F_{LY}|$ for $Y \in \Gamma$. In the exterior region $\{t + R \leq r, v \leq v_*\}$, there holds*

$$\mathcal{I}_Y^2[F] \lesssim (q_0^2 + \Delta_0 + \Delta_0 r^{2-\gamma_0} u_+^{-1}) u_+^{2\zeta(Y)}. \quad (52)$$

Proof. We first can check that if $Y = \partial$, $\mathcal{I}_Y[F] \lesssim r|F|$, and if $Y = \Omega_{ij}, \Omega_{0i}$

$$\mathcal{I}_Y[F] \lesssim r^2(|\alpha| + |\sigma|) + r u_+ (|\underline{\alpha}| + |\rho|).$$

Thus in view of Proposition 3, if $\zeta(Y) = 0$, $Y \in \Gamma$,

$$\mathcal{I}_Y[F] \lesssim \Delta_0^{\frac{1}{2}} r^{1-\frac{1}{2}\gamma_0} u_+^{-\frac{1}{2}} + \frac{u_+}{r} q_0,$$

and if $\zeta(Y) = -1$, $Y \in \Gamma$,

$$\mathcal{I}_Y[F] \lesssim r^{-1} q_0 + \Delta_0^{\frac{1}{2}} u_+^{-\frac{1+\gamma_0}{2}} \lesssim (\Delta_0^{\frac{1}{2}} + q_0) u_+^{-1}.$$

The result then follows by combining the above estimates. □

3.4 Energy decay estimates for the Maxwell field

In this subsection, we obtain energy and weighted energy estimates for the Maxwell field. We prove the following result.

Proposition 4. *Under the bootstrap assumptions (31)-(33), the energy fluxes for the Maxwell field verify the following decay estimates*

$$E[\mathcal{L}_Z^k \tilde{F}](\mathcal{H}_{u_1}^{-u_2}) + E[\mathcal{L}_Z^k \tilde{F}](\underline{\mathcal{H}}_{-u_2}^{u_1}) \lesssim (\mathcal{E}_{k,\gamma_0} + \Delta_0^2) (u_1)_+^{-\gamma_0+2\zeta(Z^k)} \quad (53)$$

as well as the r -weighted energy decay estimates

$$\begin{aligned} & \iint_{\mathcal{D}_{u_1}^{-u_2}} r^{\gamma_0-1} (|\alpha[\mathcal{L}_Z^k \tilde{F}]|^2 + |\rho[\mathcal{L}_Z^k \tilde{F}]|^2 + |\sigma[\mathcal{L}_Z^k \tilde{F}]|^2) + \int_{\mathcal{H}_{u_1}^{-u_2}} r^{\gamma_0} |\alpha[\mathcal{L}_Z^k \tilde{F}]|^2 \\ & + \int_{\underline{\mathcal{H}}_{-u_2}^{u_1}} r^{\gamma_0} (|\rho[\mathcal{L}_Z^k \tilde{F}]|^2 + |\sigma[\mathcal{L}_Z^k \tilde{F}]|^2) \lesssim (\mathcal{E}_{k,\gamma_0} + \Delta_0^2) (u_1)_+^{2\zeta(Z^k)} \end{aligned} \quad (54)$$

for all $-v_* \leq u_2 < u_1 \leq -\frac{R}{2}$, $k \leq 2$, $Z^k \in \Gamma^k$.

By choosing \mathcal{E}_{2,γ_0} and Δ_0 suitably small, we can then improve the bootstrap assumption (33) as well as the bootstrap assumption (31) for the Maxwell field. The proof for the above proposition is based on the following estimates on $J[\phi] = \Im(\phi \cdot \overline{D}\phi)$.

Proposition 5. *Under the bootstrap assumptions (31)-(33), we can bound the term $J[\phi]$ as follows:*

$$\iint_{\mathcal{D}_{u_1}^{-u_2}} r^{\gamma_0} u_+^{1+\epsilon} |\mathcal{L}_Z^k \beta|^2 + r^{\gamma_0+1} |\mathcal{L}_Z^k J_L|^2 + u_+^{1+\gamma_0+\epsilon} |\mathcal{L}_Z^k J_{\underline{L}}|^2 \lesssim \Delta_0^2 (u_1)_+^{2\zeta(Z^k)} \quad (55)$$

for all $-v_* \leq u_2 < u_1 \leq -\frac{R}{2}$, $k \leq 2$, $Z^k \in \Gamma^k$, where $\mathcal{L}_Z^k \beta$, $\mathcal{L}_Z^k J_L$ and $\mathcal{L}_Z^k J_{\underline{L}}$ represent the angular, L and \underline{L} components of the one form $\mathcal{L}_Z^k J[\phi]$ respectively.

The rest of this subsection is devoted to proving the above two propositions. We first show that Proposition 5 implies Proposition 4. To obtain the energy estimates (53), we apply the energy identity (20) to $\phi = 0$ and $\mathcal{G} = \mathcal{L}_Z^k \tilde{F}$. By the assumption on the initial data, we derive that

$$E[\mathcal{L}_Z^k \tilde{F}](\mathcal{H}_{u_1}^{-u_2}) + E[\mathcal{L}_Z^k \tilde{F}](\underline{\mathcal{H}}_{-u_2}^{u_1}) \lesssim \mathcal{E}_{k,\gamma_0}(u_1)_+^{-\gamma_0+2\zeta(Z^k)} + \iint_{\mathcal{D}_{u_1}^{-u_2}} |(\mathcal{L}_Z^k \tilde{F})_{0\gamma}(\mathcal{L}_Z^k J[\phi])^\gamma|.$$

The error term can be bounded with the help of Cauchy-schwarz inequality,

$$\begin{aligned} |(\mathcal{L}_Z^k \tilde{F})_{0\gamma}(\mathcal{L}_Z^k J[\phi])^\gamma| &\lesssim |(\mathcal{L}_Z^k \tilde{F})_{L\gamma}(\mathcal{L}_Z^k J[\phi])^\gamma| + |(\mathcal{L}_Z^k \tilde{F})_{\underline{L}\gamma}(\mathcal{L}_Z^k J[\phi])^\gamma| \\ &\lesssim u_+^{-1-\epsilon}(|\alpha[\mathcal{L}_Z^k \tilde{F}]|^2 + |\rho[\mathcal{L}_Z^k \tilde{F}]|^2) + u_+^{1+\epsilon}|\mathcal{L}_Z^k J[\phi]_{\underline{L}}|^2 + r^{-1-\epsilon}|\alpha[\mathcal{L}_Z^k \tilde{F}]|^2 + r^{1+\epsilon}|\mathcal{L}_Z^k \tilde{F}[\phi]|^2. \end{aligned}$$

We then employ (55) to treat the nonlinear term involving $J[\phi]$, and apply Lemma 3 to derive the estimate (53).

Next for the r -weighted energy estimates (54), we apply the identity (19) to $\phi = 0$, $\mathcal{G} = \mathcal{L}_Z^k \tilde{F}$ and $p = \gamma_0 < 2$ with the help of the second identity in Lemma 4 and (24). Then the right hand side can be bounded by the initial data and all the other terms possess positive signs except the nonlinear term $(\mathcal{L}_Z^k \tilde{F})_{X\gamma}(\mathcal{L}_Z^k J[\phi])^\gamma$. We therefore can derive that

$$\begin{aligned} &\iint_{\mathcal{D}_{u_1}^{-u_2}} r^{\gamma_0-1}(|\alpha[\mathcal{L}_Z^k \tilde{F}]|^2 + |\rho[\mathcal{L}_Z^k \tilde{F}]|^2 + |\sigma[\mathcal{L}_Z^k \tilde{F}]|^2) \\ &+ \int_{\mathcal{H}_{u_1}^{-u_2}} r^{\gamma_0}|\alpha[\mathcal{L}_Z^k \tilde{F}]|^2 + \int_{\underline{\mathcal{H}}_{-u_2}^{u_1}} r^{\gamma_0}(|\rho[\mathcal{L}_Z^k \tilde{F}]|^2 + |\sigma[\mathcal{L}_Z^k \tilde{F}]|^2) \\ &\lesssim \mathcal{E}_{k,\gamma_0}(u_1)_+^{2\zeta(Z^k)} + \iint_{\mathcal{D}_{u_1}^{-u_2}} |\mathcal{L}_Z^k \tilde{F}_{X\gamma}(\mathcal{L}_Z^k J[\phi])^\gamma|. \end{aligned}$$

For the last integral, with $J = J[\phi]$, we first can estimate that

$$\begin{aligned} |(\mathcal{L}_Z^k \tilde{F})_{X\gamma}(\mathcal{L}_Z^k J)^\gamma| &\lesssim r^{\gamma_0}(|\rho[\mathcal{L}_Z^k \tilde{F}]||\mathcal{L}_Z^k J_L| + |\alpha[\mathcal{L}_Z^k \tilde{F}]||\mathcal{L}_Z^k \tilde{J}|) \\ &\lesssim r^{\gamma_0}u_+^{-1-\epsilon}|\alpha[\mathcal{L}_Z^k \tilde{F}]|^2 + r^{\gamma_0}u_+^{1+\epsilon}|\mathcal{L}_Z^k \tilde{J}|^2 + \epsilon_1 r^{\gamma_0-1}|\rho[\mathcal{L}_Z^k \tilde{F}]|^2 + \epsilon_1^{-1}r^{\gamma_0+1}|\mathcal{L}_Z^k J_L|^2 \end{aligned}$$

for all $\epsilon_1 > 0$. The integral of the first term can be absorbed by using Gronwall's inequality and the third term can be absorbed for sufficiently small ϵ_1 which depends only on γ_0 and the implicit universal constant. Once we have chosen ϵ_1 , the rest two terms involving the terms of J can be bounded by using Proposition 5. Next, we prove Proposition 5.

Proof for Proposition 5. We first consider the case when $k = 0$. In this case, we will show

$$\iint_{\mathcal{D}_{u_1}^{-u_2}} r^{\gamma_0}u_+^{1+\epsilon}|\tilde{J}|^2 + r^{\gamma_0+1}|J_L|^2 + u_+^{1+\gamma_0+\epsilon}|J_{\underline{L}}|^2 \lesssim \Delta_0^2(u_1)_+^{-1-\gamma_0+\epsilon}. \quad (56)$$

Note that in view of (41) and (25), we can derive

$$\begin{aligned} &\iint_{\mathcal{D}_{u_1}^{-u_2}} |\phi|^2(r^{\gamma_0}u_+^{1+\epsilon}|\mathcal{D}\phi|^2 + r^{\gamma_0+1}|D_L\phi|^2 + u_+^{1+\gamma_0+\epsilon}|D_{\underline{L}}\phi|^2) \\ &\lesssim \Delta_0(u_1)_+^{-2+\epsilon} \iint_{\mathcal{D}_{u_1}^{-u_2}} |\mathcal{D}\phi|^2 + |D_L\phi|^2 + r^{-1-\epsilon}u_+^{1+\epsilon}|D_{\underline{L}}\phi|^2 \\ &\lesssim \Delta_0\mathcal{E}_{0,\gamma_0}(u_1)_+^{-1-\gamma_0+\epsilon}. \end{aligned}$$

In view of the definition of $J[\phi]$, this gives (56).

Next, we consider the cases when $k = 1, 2$. By the definition of Lie derivatives, for any $X \in \Gamma$, we can show that

$$\begin{aligned}\mathcal{L}_X J_\mu[\phi] &= X \Im(\phi \cdot \overline{D_\mu \phi}) - \Im(\phi \cdot \overline{D_{[X, \partial_\mu]} \phi}) \\ &= \Im(D_X \phi \cdot \overline{D_\mu \phi}) + \Im(\phi \cdot \overline{D_\mu D_X \phi}) + \Im(\phi \cdot \overline{i F_{X\mu} \phi}) \\ &= \Im(D_X \phi \cdot \overline{D_\mu \phi}) + \Im(\phi \cdot \overline{D_\mu D_X \phi}) - F_{X\mu} |\phi|^2.\end{aligned}\tag{57}$$

In particular we can bound that

$$|\mathcal{L}_X J_\mu[\phi]| \lesssim |D_X \phi| |D_\mu \phi| + |\phi| |D_\mu D_X \phi| + |F_{X\mu}| |\phi|^2.$$

For the second order derivative, we can show that

$$\begin{aligned}\mathcal{L}_X \mathcal{L}_Y J_\mu[\phi] &= X(\mathcal{L}_Y J_\mu[\phi]) - (\mathcal{L}_Y J[\phi])([X, \partial_\mu]) \\ &= X(\Im(D_Y \phi \cdot \overline{D_\mu \phi}) + \Im(\phi \cdot \overline{D_\mu D_Y \phi}) - F_{Y\mu} |\phi|^2) \\ &\quad - \Im(D_Y \phi \cdot \overline{D_{[X, \partial_\mu]} \phi}) - \Im(\phi \cdot \overline{D_{[X, \partial_\mu]} D_Y \phi}) + F_{Y[X, \partial_\mu]} |\phi|^2 \\ &= \Im(D_X D_Y \phi \cdot \overline{D_\mu \phi}) + \Im(D_Y \phi \cdot \overline{(D_\mu D_X \phi + i F_{X\mu} \phi)}) \\ &\quad + \Im(D_X \phi \cdot \overline{D_\mu D_Y \phi}) + \Im(\phi \cdot \overline{(D_\mu D_X D_Y \phi + i F_{X\mu} D_Y \phi)}) \\ &\quad - (\mathcal{L}_X F)_{Y\mu} |\phi|^2 - F_{[X, Y]\mu} |\phi|^2 - F_{Y\mu} D_X |\phi|^2.\end{aligned}$$

Therefore we can estimate that

$$\begin{aligned}|\mathcal{L}_X \mathcal{L}_Y J_\mu[\phi]| &\lesssim |D_X D_Y \phi| |D_\mu \phi| + |D_Y \phi| |D_\mu D_X \phi| + |F_{X\mu}| |\phi| |D_Y \phi| + |D_X \phi| |D_\mu D_Y \phi| \\ &\quad + |\phi| |D_\mu D_X D_Y \phi| + |(\mathcal{L}_X F)_{Y\mu}| |\phi|^2 + |F_{[X, Y]\mu}| |\phi|^2 + |F_{Y\mu}| |\phi| |D_X \phi|.\end{aligned}$$

Let's only consider the second order derivatives as the one derivative case should be easier and the corresponding estimate can follow in a similar way. For the first term on the right hand side of the above inequality, we bound $|D_\mu \phi|$ by the pointwise estimates obtained in Proposition 2 and the other term $|D_X D_Y \phi|$ by the bound of energy flux along the outgoing null hypersurface in (31). Indeed from Proposition 2, we first can bound that

$$r^{\gamma_0} u_+^{1+\epsilon} |\not{D}\phi|^2 + r^{\gamma_0+1} |D_L \phi|^2 + u_+^{1+\gamma_0+\epsilon} |D_{\underline{L}} \phi|^2 \lesssim \Delta_0 u_+^{-3+2\epsilon}.$$

Therefore we can show that

$$\iint_{\mathcal{D}_{u_1}^{-u_2}} |D_X D_Y \phi|^2 (r^{\gamma_0} u_+^{1+\epsilon} |\not{D}\phi|^2 + r^{\gamma_0+1} |D_L \phi|^2 + u_+^{1+\gamma_0+\epsilon} |D_{\underline{L}} \phi|^2) \lesssim \Delta_0^2 (u_1)_+^{2\zeta(XY)-2-\gamma_0+2\epsilon}.$$

Next for $|D_X \phi| |D_\mu D_Y \phi|$ or $|D_Y \phi| |D_\mu D_X \phi|$, it suffices to consider one of them due to symmetry. From Proposition 2, we have

$$|D_X \phi|^2 \lesssim \Delta_0 r^{-\frac{5}{2}+\epsilon} u_+^{\frac{1}{2}-\gamma_0+2\zeta(X)}.$$

Therefore we can bound that

$$\begin{aligned}&\iint_{\mathcal{D}_{u_1}^{-u_2}} |D_X \phi|^2 (r^{\gamma_0} u_+^{1+\epsilon} |\not{D} D_Y \phi|^2 + r^{\gamma_0+1} |D_L D_Y \phi|^2 + u_+^{1+\gamma_0+\epsilon} |D_{\underline{L}} D_Y \phi|^2) \\ &\lesssim \Delta_0 (u_1)_+^{2\zeta(X)} \iint_{\mathcal{D}_{u_1}^{-u_2}} u_+^{-1+2\epsilon} |\not{D} D_Y \phi|^2 + r u_+^{-2+\epsilon} |D_L D_Y \phi|^2 + r^{-1-\epsilon} u_+^{3\epsilon} |D_{\underline{L}} D_Y \phi|^2 \\ &\lesssim \Delta_0^2 (u_1)_+^{2\zeta(XY)-\gamma_0+2\epsilon}.\end{aligned}$$

Next we estimate $|\phi||D_\mu D_X D_Y \phi|$. For this term, we can bound ϕ by the L^∞ norm in (41) and the other one by the bound in (31) for the energy fluxes through incoming and outgoing null hypersurfaces. More precisely we have

$$\begin{aligned} & \iint_{\mathcal{D}_{u_1}^{-u_2}} |\phi|^2 (r^{\gamma_0} u_+^{1+\epsilon} |\not{D} D_X D_Y \phi|^2 + r^{\gamma_0+1} |D_L D_X D_Y \phi|^2 + u_+^{1+\gamma_0+\epsilon} |D_{\underline{L}} D_X D_Y \phi|^2) \\ & \lesssim \Delta_0 (u_1)_+^{-2+\epsilon} \iint_{\mathcal{D}_{u_1}^{-u_2}} |\not{D} D_X D_Y \phi|^2 + |D_L D_X D_Y \phi|^2 + r^{-1-\epsilon} u_+^{1+\epsilon} |D_{\underline{L}} D_X D_Y \phi|^2 \\ & \lesssim \Delta_0^2 (u_1)_+^{2\zeta(XY)-1-\gamma_0+\epsilon}. \end{aligned}$$

The remaining ones are cubic nonlinear terms. By symmetry, it suffices to consider $|F_{X\mu}||\phi||D_Y \phi|$ and $(|(\mathcal{L}_X F)_{Y\mu}| + |F_{[X,Y]\mu}|)|\phi|^2$. In view of (45), we can roughly bound such terms as follows,

$$|F_{X\mu}|^2 \lesssim (\Delta_0 + q_0^2) u_+^{-2+2\zeta(X)}, \quad |F_{[X,Y]\mu}|^2 \lesssim (\Delta_0 + q_0^2) u_+^{2\zeta(XY)-2}.$$

Therefore we have

$$\begin{aligned} & \iint_{\mathcal{D}_{u_1}^{-u_2}} r^{\gamma_0+1+\epsilon} (|F_{X\mu}|^2 |\phi|^2 |D_Y \phi|^2 + |F_{[X,Y]\mu}|^2 |\phi|^4) \\ & \lesssim \Delta_0 (\Delta_0 + q_0^2) (u_1)_+^{-4+\epsilon+2\zeta(X)} \iint_{\mathcal{D}_{u_1}^{-u_2}} |D_Y \phi|^2 + u_+^{2\zeta(Y)} |\phi|^2 \\ & \lesssim \Delta_0^2 (\Delta_0 + q_0^2) (u_1)_+^{2\zeta(XY)-3-\gamma_0+\epsilon}. \end{aligned}$$

For the last one $|(\mathcal{L}_X F)_{Y\mu}||\phi|^2$, we need the following facts

$$|\mathcal{L}_X F - \mathcal{L}_X \tilde{F}| \lesssim r^{\zeta(X)-2} |q_0|, \quad |\mathcal{G}_{X\nu}| \lesssim r^{\zeta(X)+1} |\mathcal{G}_{\mu\nu}| \text{ where } X \in \Gamma. \quad (58)$$

Here \mathcal{G} is a two form. We can bound ϕ by the pointwise bound in (41) and $(\mathcal{L}_X \tilde{F})_{Y\mu}$ by the bound of energy flux in (31). We thus can derive that

$$\begin{aligned} & \iint_{\mathcal{D}_{u_1}^{-u_2}} |\phi|^4 (r^{\gamma_0} u_+^{1+\epsilon} |(\mathcal{L}_X F)_{Ye_A}|^2 + r^{\gamma_0+1} |(\mathcal{L}_X F)_{YL}|^2 + u_+^{1+\gamma_0+\epsilon} |(\mathcal{L}_X F)_{Y\underline{L}}|^2) \\ & \lesssim \Delta_0^2 \iint_{\mathcal{D}_{u_1}^{-u_2}} r^{-5+\gamma_0+\epsilon} u_+^{-2\gamma_0} |(\mathcal{L}_X F)_{Y\mu}|^2 \\ & \lesssim \Delta_0^2 \iint_{\mathcal{D}_{u_1}^{-u_2}} r^{-3+\gamma_0+\epsilon} u_+^{-2\gamma_0+2\zeta(Y)} |\mathcal{L}_X F|^2 \\ & \lesssim \Delta_0^2 (\Delta_0 + q_0^2) (u_1)_+^{2\zeta(XY)-2-2\gamma_0+\epsilon}. \end{aligned}$$

For deriving the last inequality, we used (58) to decompose $\mathcal{L}_X F$. With the help of the fact that $-3 + \gamma_0 + \epsilon < -1$, which can be seen by our assumption of γ_0 , we then estimated the component $\underline{\alpha}[\mathcal{L}_X \tilde{F}]$ by using the energy flux through the incoming null hypersurfaces and other components of $\mathcal{L}_X \tilde{F}$ by using energy flux on the outgoing null hypersurfaces. Combining all the above estimates, we can derive estimate (55). This finished the proof for Proposition 5. \square

3.5 Decay of Energy estimates for the scalar field

We derive the energy decay estimates for the scalar field in this section under the bootstrap assumptions (31)-(33). We show that

Proposition 6. *Under the bootstrap assumptions (31)-(33), we have the energy flux decay for the scalar field*

$$E[D_Z^k \phi](\mathcal{H}_{u_1}^{-u_2}) + E[D_Z^k \phi](\underline{\mathcal{H}}_{-u_2}^{u_1}) \lesssim (\mathcal{E}_{k,\gamma_0} + \Delta_0^2)(u_1)_+^{-\gamma_0+2\zeta(Z^k)} \quad (59)$$

and the r -weighted energy flux decay estimate

$$\int_{\mathcal{H}_{u_1}^{-u_2}} r |D_L D_Z^k \phi|^2 + \int_{\underline{\mathcal{H}}_{-u_2}^{u_1}} r (|\mathcal{D} D_Z^k \phi|^2 + |D_Z^k \phi|^2) \lesssim (\mathcal{E}_{k,\gamma_0} + \Delta_0^2)(u_1)_+^{1-\gamma_0+2\zeta(Z^k)} \quad (60)$$

for all $-v_* \leq u_2 < u_1 \leq -\frac{R}{2}$, $k \leq 2$, $Z^k \in \Gamma^k$.

To show the above proposition, we need the following estimates on the commutators.

Lemma 10. *For $1 \leq k \leq 2$, let $Z^k = Z_1 \cdots Z_k \in \Gamma^k$. Under the bootstrap assumptions (31)-(33), when $k = 2$, we can bound*

$$\begin{aligned} \iint_{\mathcal{D}_{u_1}^{-u_2}} r^{1+\epsilon} u_+^{1+\epsilon} |(\square_A - 1) D_Z^k \phi|^2 &\lesssim (\mathcal{E}_{0,\gamma_0} + \Delta_0^2)(u_1)_+^{2\zeta(Z^k)+1-\gamma_0} + q_0^2 \iint_{\mathcal{D}_{u_1}^{-u_2}} u_+^{2\zeta(Z^k)+2\epsilon} |D\phi|^2 \\ &\quad + q_0^2 \iint_{\mathcal{D}_{u_1}^{-u_2}} u_+^{2(\zeta(Z^k)+2\epsilon)} \sum_{Z \in \{Z_1, Z_2\}} u_+^{-2\zeta(Z)} |D D_Z \phi|^2, \end{aligned} \quad (61)$$

where $-v_* \leq u_2 < u_1 \leq -\frac{1}{2}R$ in the above estimate. When $k = 1$, the same estimate holds with the terms in the second line vanished.

We can actually prove the improved estimate

Proposition 7. *Under the bootstrap assumptions (31)-(33), we have for $Z^k \in \Gamma^k$ with $k \leq 2$ ¹⁶*

$$\iint_{\mathcal{D}_{u_1}^{-u_2}} r^{1+\epsilon} u_+^{1+\epsilon} |(\square_A - 1) D_Z^k \phi|^2 \lesssim (\mathcal{E}_{k,\gamma_0} + \Delta_0^2)(u_1)_+^{2\zeta(Z^k)+1-\gamma_0}, \quad (62)$$

where $-v_* \leq u_2 < u_1 \leq -\frac{1}{2}R$.

Proposition 7 has to be proved together with (59) in Proposition 6 by using Lemma 10. We now prove Lemma 10 first, then use it to prove Proposition 7 and Proposition 6.

Proof of Lemma 10. We start with considering $(\square_A - 1) D_Z^k \phi$. When $k = 1$, in view of Lemma 4 and the definition in (27), we have

$$(\square_A - 1) D_Z \phi = Q(F, \phi, Z).$$

When $k = 2$, let $Z^2 = XY$ with $X, Y \in \Gamma$. Lemma 5 and Lemma 6 imply that

$$\begin{aligned} (\square_A - 1) D_X D_Y \phi &= Q(F, D_X \phi, Y) + Q(F, D_Y \phi, X) + Q(F, \phi, \mathcal{L}_X Y) \\ &\quad + Q(\mathcal{L}_X F, \phi, Y) - 2F_{X\mu} F_Y^\mu \phi. \end{aligned} \quad (63)$$

Note that the commutators consist of two parts: one has the same structure as $Q(F, \phi, Z)$, which is quadratic in F and $D\phi$ and will be referred to as the quadratic part; the other one is a set of

¹⁶Note that when $k = 0$, as ϕ verifies the massive MKG equation, we have $\square_A \phi - \phi = 0$. Thus (62) automatically holds.

the cubic terms. Let's first estimate the cubic terms for which we can use Lemma 8. Indeed we can show that

$$\begin{aligned}
& \iint_{\mathcal{D}_{u_1}^{-u_2}} r^{1+\epsilon} u_+^{1+\epsilon} |F_{X\mu} F_Y^\mu|^2 |\phi|^2 \\
& \lesssim \iint_{\mathcal{D}_{u_1}^{-u_2}} r^{1+\epsilon} u_+^{1+\epsilon+2\zeta(XY)} (u_+^2 |\underline{\alpha}|^2 + r^2 (|\sigma|^2 + |\alpha|^2 + |\rho|^2 + |\alpha||\underline{\alpha}|))^2 |\phi|^2 \\
& \lesssim \iint_{\mathcal{D}_{u_1}^{-u_2}} (\Delta_0 + q_0^2)^2 r^{-\gamma_0+1+\epsilon} u_+^{-3+\gamma_0+\epsilon+2\zeta(XY)} |\phi|^2 \\
& \lesssim (\Delta_0 + q_0^2)^2 \mathcal{E}_{0,\gamma_0}(u_1)_+^{2\zeta(XY)-1+2\epsilon-\gamma_0}, \tag{64}
\end{aligned}$$

where we also employed Proposition 3 and (25).

Next we will control the main part of the commutator, which takes the quadratic form $Q(F, f, X)$. We first apply Lemma 7 to scalar field $D_Z^l \phi$ with $\mathcal{G} = F$ for all $l \leq k-1$. Denote $\phi_{,l} = D_Z^l \phi$ and $Z^1 \in \Gamma$. We can employ Proposition 2, Proposition 3, (45) and the fact that $\Delta_0 \leq 1$ to bound that

$$\begin{aligned}
|Q(F, \phi_{,l}, X)|^2 & \lesssim r^{2\zeta(X)+2} (|\alpha|^2 |D\phi_{,l}|^2 + |\underline{F}|^2 |D_L \phi_{,l}|^2 + |\sigma|^2 |\mathcal{D}\phi_{,l}|^2) \\
& \quad + u_+^{2\zeta(X)+2} (|\rho|^2 |D_L \phi_{,l}|^2 + |\underline{\alpha}|^2 |\mathcal{D}\phi_{,l}|^2) \\
& \quad + \left(u_+^{2\zeta(X)+2} |J_L[\phi_{,l}]|^2 + r^{2\zeta(X)+2} (|J_L[\phi_{,l}]|^2 + |\mathcal{J}[\phi_{,l}]|^2) + r^{2\zeta(X)} |F|^2 \right) |\phi_{,l}|^2 \\
& \lesssim u_+^{2\zeta(X)} r^{-1-\epsilon} ((q_0^2 + \Delta_0) u_+^{-1+\epsilon} |D\phi_{,l}|^2 + \Delta_0 (r^{1+\epsilon} u_+^{-2} |D_L \phi_{,l}|^2 + u_+^{-3+\epsilon} |\phi_{,l}|^2)) \\
& \quad + q_0^2 r^{2\zeta(X)-2} |D_L \phi_{,l}|^2
\end{aligned}$$

where \underline{F} represents all the components of F except $\alpha[F]$.

Now from the bootstrap assumption (31) and the fact that $l \leq k-1$, we can show that

$$\int_{\mathcal{H}_u^{-u_2}} |DD_Z^l \phi|^2 \lesssim \Delta_0 u_+^{-\gamma_0+2\zeta(Z^l)+2\zeta(\partial)} = \Delta_0 u_+^{-\gamma_0-2+2\zeta(Z^l)}.$$

By making use of the Lorentz boost, we have the improved energy flux decay

$$\begin{aligned}
\int_{\mathcal{H}_u^{-u_2}} r^2 |D_L D_Z^l \phi|^2 & \lesssim \int_{\mathcal{H}_u^{-u_2}} (|D_{\Omega_{0j}} D_Z^l \phi|^2 + |D_{\Omega_{ij}} D_Z^l \phi|^2 + u_+^2 |DD_Z^l \phi|^2) \\
& \lesssim \Delta_0 u_+^{-\gamma_0+2\zeta(\Omega Z^l)} = \Delta_0 u_+^{-\gamma_0+2\zeta(Z^l)}.
\end{aligned}$$

Therefore by using the above estimate and (31) we can obtain

$$\begin{aligned}
\iint_{\mathcal{D}_{u_1}^{-u_2}} r^{1+\epsilon} u_+^{1+\epsilon} |Q(F, D_Z^l \phi, X)|^2 & \lesssim \Delta_0^2 (u_1)_+^{2\zeta(XZ^l)-1-\gamma_0+2\epsilon} \\
& \quad + q_0^2 \iint_{\mathcal{D}_{u_1}^{-u_2}} u_+^{2\zeta(XZ^l)+2\epsilon-2\zeta(Z^l)} |DD_Z^l \phi|^2 \tag{65}
\end{aligned}$$

for $l \leq k-1$ and $k \leq 2$. This in particular implies the $k=1$ case in Lemma 10 if we take $l=0$.

We will estimate the term $Q(\mathcal{L}_Y F, \phi, X)$ by applying Lemma 7 to $\mathcal{L}_Y F$ and the scalar field ϕ . To treat the term of $\delta(\mathcal{L}_Y F)$, we can use the second identity in Lemma 4 and (mMKG). Also by

using Proposition 2 and (58), we then can bound that

$$\begin{aligned}
|Q(\mathcal{L}_Y F, \phi, X)|^2 &\lesssim r^{2\zeta(X)+2}(|\alpha[\mathcal{L}_Y F]|^2 |D\phi|^2 + |\underline{\mathcal{L}}_Y F|^2 |D_L \phi|^2 + |\sigma[\mathcal{L}_Y F]|^2 |\mathcal{D}\phi|^2) \\
&\quad + u_+^{2\zeta(X)+2}(|\rho[\mathcal{L}_Y F]|^2 |D_{\underline{L}} \phi|^2 + |\underline{\alpha}[\mathcal{L}_Y F]|^2 |\mathcal{D}\phi|^2) \\
&\quad + \left(u_+^{2\zeta(X)+2} |\mathcal{L}_Y J_{\underline{L}}|^2 + r^{2\zeta(X)+2} (|\mathcal{L}_Y J_L|^2 + |\mathcal{L}_Y \mathcal{J}|^2) + r^{2\zeta(X)} |\mathcal{L}_Y F|^2 \right) |\phi|^2 \\
&\lesssim \Delta_0 r^{2\zeta(X)-\frac{1}{2}+\epsilon} u_+^{\frac{1}{2}-\gamma_0} (u_+^{-2} |\alpha[\mathcal{L}_Y \tilde{F}]|^2 + r^{-2} |\underline{\mathcal{L}}_Y \tilde{F}|^2 + r^{-2} |\sigma[\mathcal{L}_Y \tilde{F}]|^2) \\
&\quad + \Delta_0 u_+^{2\zeta(X)+\frac{5}{2}-\gamma_0} r^{-\frac{5}{2}+\epsilon} (u_+^{-2} |\rho[\mathcal{L}_Y \tilde{F}]|^2 + r^{-2} |\underline{\alpha}[\mathcal{L}_Y \tilde{F}]|^2) \\
&\quad + \left(u_+^{2\zeta(X)+2} |\mathcal{L}_Y J_{\underline{L}}|^2 + r^{2\zeta(X)+2} (|\mathcal{L}_Y J_L|^2 + |\mathcal{L}_Y \mathcal{J}|^2) \right) |\phi|^2 \\
&\quad + u_+^{2\zeta(XY)} r^{-2} q_0^2 (|D\phi|^2 + r^{-2} |\phi|^2),
\end{aligned} \tag{66}$$

where the terms in (66) can be incorporated into those above the line.

Recall from (57) that

$$|\mathcal{L}_Y J_\mu[\phi]| \lesssim |D_Y \phi| |D_\mu \phi| + |\phi| |D_\mu D_Y \phi| + |F_{Y\mu}| |\phi|^2.$$

By using Proposition 2, we can therefore bound that

$$\begin{aligned}
&\left(u_+^{2\zeta(X)+2} |\mathcal{L}_Y J_{\underline{L}}|^2 + r^{2\zeta(X)+2} (|\mathcal{L}_Y J_L|^2 + |\mathcal{L}_Y \mathcal{J}|^2) \right) |\phi|^2 \\
&\lesssim \Delta_0^2 r^{\epsilon-\frac{7}{2}} u_+^{2\zeta(X)-2\gamma_0-\frac{3}{2}} (|D_Y \phi|^2 + u_+^{\frac{3}{2}} r^{-\frac{1}{2}} |DD_Y \phi|^2) + \Delta_0 u_+^{2\zeta(X)} \mathcal{I}_Y^2[F] u_+^{-\gamma_0} r^{-3} |\phi|^2,
\end{aligned}$$

where the definition and the treatment of $\mathcal{I}_Y[F]$ can be found in (52). The above estimate together with the bootstrap assumptions implies that

$$\begin{aligned}
&\iint_{\mathcal{D}_{u_1}^{-u_2}} r^{1+\epsilon} u_+^{1+\epsilon} |Q(\mathcal{L}_Y F, \phi, X)|^2 \\
&\lesssim \Delta_0 \iint_{\mathcal{D}_{u_1}^{-u_2}} u_+^{\frac{3}{2}+\epsilon-\gamma_0} r^{2\zeta(X)+\frac{1}{2}+2\epsilon} (u_+^{-2} |\alpha[\mathcal{L}_Y \tilde{F}]|^2 + r^{-2} |\underline{\mathcal{L}}_Y \tilde{F}|^2 + r^{-2} |\sigma[\mathcal{L}_Y \tilde{F}]|^2) \\
&\quad + \Delta_0 \iint_{\mathcal{D}_{u_1}^{-u_2}} r^{2\epsilon-\frac{5}{2}} u_+^{2\zeta(X)-2\gamma_0-\frac{1}{2}+\epsilon} (|D_Y \phi|^2 + u_+^{\frac{3}{2}} r^{-\frac{1}{2}} |DD_Y \phi|^2) + u_+^{-\gamma_0-1+2\epsilon+2\zeta(XY)} |\phi|^2 \\
&\quad + q_0^2 \iint_{\mathcal{D}_{u_1}^{-u_2}} u_+^{2\zeta(XY)+2\epsilon} (|D\phi|^2 + r^{-2} |\phi|^2) \\
&\lesssim (\Delta_0^2 + \mathcal{E}_{0,\gamma_0}) (u_1)_+^{2\zeta(XY)+1-2\gamma_0+3\epsilon} + q_0^2 \iint_{\mathcal{D}_{u_1}^{-u_2}} u_+^{2\zeta(XY)+2\epsilon} |D\phi|^2,
\end{aligned}$$

where we used $\gamma_0 > 1 + 3\epsilon$ to bound the term of $\underline{\mathcal{L}}_Y \tilde{F}$. We also used Proposition 1 for deriving the last inequality.

Thus we proved

$$\iint_{\mathcal{D}_{u_1}^{-u_2}} r^{1+\epsilon} u_+^{1+\epsilon} |Q(\mathcal{L}_X F, \phi, Y)|^2 \lesssim (\Delta_0^2 + \mathcal{E}_{0,\gamma_0}) (u_1)_+^{2\zeta(Z^k)+1-2\gamma_0+3\epsilon} + q_0^2 \iint_{\mathcal{D}_{u_1}^{-u_2}} u_+^{2\zeta(Z^k)+2\epsilon} |D\phi|^2.$$

Now we treat the first three terms on the right of (63) by applying (65) to $Q(F, D_Z^{l_1} \phi, Z^{l_2})$, with some $0 \leq l_1 \leq l_2 \leq 1$. Note that $[X, Y] \in \Gamma$ since $X, Y \in \Gamma$. With $(Z^{l_1}, Z^{l_2}) = (X, Y), (Y, X), (Z^0, [X, Y])$, we derive from (65) that

$$\begin{aligned}
&\iint_{\mathcal{D}_{u_1}^{-u_2}} r^{1+\epsilon} u_+^{1+\epsilon} |Q(F, D_Z^{l_1} \phi, Z^{l_2})|^2 \lesssim \Delta_0^2 (u_1)_+^{2\zeta(Z^2)-1-\gamma_0+2\epsilon} + q_0^2 \iint_{\mathcal{D}_{u_1}^{-u_2}} u_+^{2\zeta(Z^2)+2\epsilon} |D\phi|^2 \\
&\quad + q_0^2 \iint_{\mathcal{D}_{u_1}^{-u_2}} u_+^{2\zeta(Z^2)+2\epsilon} \sum_{Z \in \{X, Y\}} u_+^{-2\zeta(Z)} |DD_Z \phi|^2,
\end{aligned}$$

where we used the fact that the sum of signatures of each (Z^{l_1}, Z^{l_2}) satisfies $\zeta(Z^{l_1}) + \zeta(Z^{l_2}) = \zeta(XY)$, which is $\zeta(Z^2)$. Combining the above two estimates with (64) implies (61). We thus finished the proof for Lemma 10. \square

Now we are ready to prove Proposition 6 and Proposition 7.

Proof of Proposition 6 and Proposition 7. It suffices to consider the cases that $k = 1, 2$ for Proposition 6, since in Proposition 1, we have completed the case $k = 0$.

We apply the energy identity (20) to $(\phi, \mathcal{G}) = (D_Z^k \phi, 0)$, which implies

$$\begin{aligned} & E[D_Z^k \phi](\mathcal{H}_{u_1}^{-u_2}) + E[D_Z^k \phi](\underline{\mathcal{H}}_{-u_2}^{u_1}) \\ & \lesssim \mathcal{E}_{k, \gamma_0} (u_1)_+^{-\gamma_0 + 2\zeta(Z^k)} + \iint_{\mathcal{D}_{u_1}^{-u_2}} (|F_{0\mu}| |J[D_Z^k \phi]^\mu| + |(\square_A - 1) D_Z^k \phi| |D_0 D_Z^k \phi|). \end{aligned}$$

We first bound the nonlinear terms in the integral on the right hand side as

$$\begin{aligned} & |F_{0\mu}| |J[D_Z^k \phi]^\mu| + |(\square_A - 1) D_Z^k \phi| |D_0 D_Z^k \phi| \\ & \lesssim r^{-1-\epsilon} |DD_Z^k \phi|^2 + r^{1+\epsilon} |(\square_A - 1) D_Z^k \phi|^2 + r^{1+\epsilon} |F_{0\mu}|^2 |D_Z^k \phi|^2. \end{aligned}$$

The integral of the first term can be controlled by the fluxes on $\mathcal{H}_{u_1}^{-u_2}$ and $\underline{\mathcal{H}}_{-u_2}^{u_1}$. For the last term, due to $\zeta(\partial_t) = -1$, we apply (45) to obtain $|F_{0\mu}|^2 \lesssim (\Delta_0 + q_0^2) r^{-2} u_+^{-2}$. We can bound that

$$\iint_{\mathcal{D}_{u_1}^{-u_2}} r^{1+\epsilon} |F_{0\mu}|^2 |D_Z^k \phi|^2 \lesssim \iint_{\mathcal{D}_{u_1}^{-u_2}} (\Delta_0 + q_0^2) u_+^{-3+\epsilon} |D_Z^k \phi|^2.$$

Thus we can obtain

$$\begin{aligned} & E[D_Z^k \phi](\mathcal{H}_{u_1}^{-u_2}) + E[D_Z^k \phi](\underline{\mathcal{H}}_{-u_2}^{u_1}) \lesssim (\mathcal{E}_{k, \gamma_0} + \Delta_0^2) (u_1)_+^{-\gamma_0 + 2\zeta(Z^k)} \\ & \quad + \int_{u_2}^{u_1} E[D_Z^k \phi](\mathcal{H}_u^{-u_2}) u_+^{-1-\epsilon} du + \int_{-u_1}^{-u_2} v^{-1-\epsilon} E[D_Z^k \phi](\underline{\mathcal{H}}_v^{u_1}) dv \\ & \quad + \iint_{\mathcal{D}_{u_1}^{-u_2}} r^{1+\epsilon} |(\square_A - 1) D_Z^k \phi|^2. \end{aligned} \tag{67}$$

The last term will be estimated by using Lemma 10. We consider (59) for the case $k = 1$ and $Z^1 = \partial$. In view of (67), we have

$$\begin{aligned} & E[D_Z \phi](\mathcal{H}_{u_1}^{-u_2}) + E[D_Z \phi](\underline{\mathcal{H}}_{-u_2}^{u_1}) \lesssim (\mathcal{E}_{1, \gamma_0} + \Delta_0^2) (u_1)_+^{-\gamma_0 - 2} \\ & \quad + \int_{u_2}^{u_1} E[D \phi](\mathcal{H}_u^{-u_2}) (u_+^{-2+2\epsilon} + u_+^{-1-\epsilon}) du + \int_{-u_1}^{-u_2} v^{-1-\epsilon} E[D_Z \phi](\underline{\mathcal{H}}_v^{u_1}) dv. \end{aligned}$$

By taking all $Z \in \{\partial\}$ and using Lemma 3, we can obtain

$$E[D \phi](\mathcal{H}_{u_1}^{-u_2}) + E[D \phi](\underline{\mathcal{H}}_{-u_2}^{u_1}) \lesssim (\mathcal{E}_{1, \gamma_0} + \Delta_0^2) (u_1)_+^{-\gamma_0 - 2}. \tag{68}$$

We can substitute (68) to (61) to control the estimate of $D\phi$. This also completed the case $k = 1$ of Proposition 7.

Let us consider the case $k = 2$ and $Z^2 = \partial Z$ with $Z \in \Gamma$. Due to $\zeta(\partial Z) = -1 + \zeta(Z)$, by using (61) and (67) we can derive

$$\begin{aligned} & E[DD_Z \phi](\mathcal{H}_{u_1}^{-u_2}) + E[DD_Z \phi](\underline{\mathcal{H}}_{-u_2}^{u_1}) \lesssim (\mathcal{E}_{2, \gamma_0} + \Delta_0^2) (u_1)_+^{-\gamma_0 + 2\zeta(\partial Z)} \\ & \quad + \int_{u_2}^{u_1} u_+^{2\zeta(\partial Z) + 2\epsilon} (u_+^2 |DD_{\partial} \phi|^2 + u_+^{-2\zeta(Z)} |DD_Z \phi|^2) du \\ & \quad + \int_{u_2}^{u_1} u_+^{-1-\epsilon} E[DD_Z \phi](\mathcal{H}_u^{-u_2}) du + \int_{-u_1}^{-u_2} v^{-1-\epsilon} E[DD_Z \phi](\underline{\mathcal{H}}_v^{u_1}) dv. \end{aligned} \tag{69}$$

By summing over $Z \in \{\partial\}$, applying Lemma 3 gives (59) for all $Z^2 = Z_1 Z_2$ with $Z_1, Z_2 \in \{\partial\}$. We then let $Z \in \{\Omega_{\mu\nu}\}$ in (69). By substituting the estimate (59) for $Z^2 = \partial\partial$ into (69), also by using Lemma 3, we obtain the following energy flux estimate

$$E[DD_Z\phi](\mathcal{H}_{u_1}^{-u_2}) + E[DD_Z\phi](\underline{\mathcal{H}}_{-u_2}^{u_1}) \lesssim (\mathcal{E}_{2,\gamma_0} + \Delta_0^2)(u_1)_+^{-\gamma_0+2\zeta(\partial Z)}, \forall Z = \Omega_{ij}, \Omega_{0j}.$$

Thus we proved (59) for $Z^2 = \partial Z$, $Z \in \Gamma$. Substituting the result back to Lemma 10 yields

$$\begin{aligned} \iint_{\mathcal{D}_{u_1}^{-u_2}} r^{1+\epsilon} u_+^{1+\epsilon} |(\square_A - 1)D_Z^k \phi|^2 &\lesssim (\mathcal{E}_{2,\gamma_0} + \Delta_0^2)(u_1)_+^{2\zeta(Z^k)+1-\gamma_0} \\ &+ q_0^2(\mathcal{E}_{2,\gamma_0} + \Delta_0^2) \int_{u_2}^{u_1} u_+^{2\zeta(Z^k)+2\epsilon-\gamma_0-2} du \end{aligned}$$

for any $Z^k \in \Gamma^k, k = 2$. By direct integration, we can obtain Proposition 7.

(59) with the general $Z^k \in \Gamma^k$ can be proved by applying (67) with the help of (62) and Lemma 3.

To prove (60), we apply the energy identity (19) with $(\phi, \mathcal{G}) = (D_Z^k \phi, 0)$ to derive

$$\begin{aligned} \int_{\mathcal{H}_{u_1}^{-u_2}} r^{-1} |D_L(rD_Z^k \phi)|^2 + \int_{\underline{\mathcal{H}}_{-u_2}^{u_1}} r(|\not{D}D_Z^k \phi|^2 + |D_Z^k \phi|^2) \\ \lesssim \mathcal{E}_{k,\gamma_0}(u_1)_+^{1-\gamma_0+2\zeta(Z^k)} + \iint_{\mathcal{D}_{u_1}^{-u_2}} |D_Z^k \phi|^2 \\ + \iint_{\mathcal{D}_{u_1}^{-u_2}} (r|F_{L\mu}| |J[D_Z^k \phi]^\mu| + |(\square_A - 1)D_Z^k \phi| |D_L(rD_Z^k \phi)|). \end{aligned} \quad (70)$$

The second term is estimated by using (59) as follows

$$\iint_{\mathcal{D}_{u_1}^{-u_2}} |D_Z^k \phi|^2 \lesssim (\mathcal{E}_{k,\gamma_0} + \Delta_0^2)(u_1)_+^{1-\gamma_0+2\zeta(Z^k)}, \quad k \leq 2.$$

Next we treat the nonlinear terms on the right hand side of (70). Note that $r^2 J[D_Z^k \phi] = J[rD_Z^k \phi]$, and recall from Proposition 3 that

$$|\rho|^2 \lesssim (\Delta_0 + q_0^2)r^{-2}u_+^{-2}, \quad |\alpha|^2 \lesssim \Delta_0 r^{-2-\gamma_0}u_+^{-1}.$$

We then can bound that

$$\begin{aligned} r|F_{L\mu}| |J[D_Z^k \phi]^\mu| + |(\square_A - 1)D_Z^k \phi| |D_L(rD_Z^k \phi)| \\ \lesssim r^{-1}u_+^{-1-\epsilon} |D_L(rD_Z^k \phi)|^2 + ru_+^{1+\epsilon} (|(\square_A - 1)D_Z^k \phi|^2 + |\rho|^2 |D_Z^k \phi|^2) \\ + r^{-2-\epsilon} |\not{D}(rD_Z^k \phi)|^2 + r^{2+\epsilon} |\alpha|^2 |D_Z^k \phi|^2 \\ \lesssim r^{-1}u_+^{-1-\epsilon} |D_L(rD_Z^k \phi)|^2 + r^{-2-\epsilon} |\not{D}(rD_Z^k \phi)|^2 + ru_+^{1+\epsilon} |(\square_A - 1)D_Z^k \phi|^2 + (\Delta_0 + q_0^2)u_+^{-2+\epsilon} |D_Z^k \phi|^2. \end{aligned}$$

The integral of the the first two terms can be bounded by the weighted fluxes on $\mathcal{H}_{u_1}^{-u_2}$ and $\underline{\mathcal{H}}_{-u_2}^{u_1}$. The integral of the last term can be bounded by using (59) as follows,

$$\iint_{\mathcal{D}_{u_1}^{-u_2}} (\Delta_0 + q_0^2)u_+^{-2+\epsilon} |D_Z^k \phi|^2 \lesssim (\mathcal{E}_{k,\gamma_0} + \Delta_0^2)(u_1)_+^{2\zeta(Z^k)-1+\epsilon-\gamma_0}.$$

The commutator can be treated by using (62). Combining all these estimates, by using Lemma 3, we have

$$\int_{\mathcal{H}_{u_1}^{-u_2}} r^{-1} |D_L(rD_Z^k \phi)|^2 + \int_{\underline{\mathcal{H}}_{-u_2}^{u_1}} r(|\not{D}D_Z^k \phi|^2 + |D_Z^k \phi|^2) \lesssim (\mathcal{E}_{k,\gamma_0} + \Delta_0^2)(u_1)_+^{1-\gamma_0+2\zeta(Z^k)}.$$

Note that $\int_{\mathcal{H}_{u_1}^{-u_2}} r^{-1} |D_Z^k \phi|^2$ can be treated by (59). Thus we can obtain the r -weighted energy flux decay (60) for the scalar field. □

3.6 Improving the bootstrap assumptions

Let C be the implicit constants in Proposition 4 and Proposition 6. Without loss of generality we may assume $C \geq 2$. This ensures that $\Delta_0 \geq \mathcal{E}_{2,\gamma_0}$. By our convention, the constant C depends only on ϵ , γ_0 and $|q_0|$. Let \mathcal{E}_{2,γ_0} , Δ_0 verify the following conditions:

$$C\mathcal{E}_{2,\gamma_0} = \frac{1}{2}\Delta_0, \quad C\Delta_0 \leq \frac{1}{2},$$

that is,

$$\mathcal{E}_{2,\gamma_0} \leq \frac{1}{4C^2}.$$

Then we have

$$C(\mathcal{E}_{2,\gamma_0} + \Delta_0^2) \leq \Delta_0.$$

We thus have improved the bootstrap assumptions (31)-(33).

Thus in view of Proposition 2 and Proposition 3, we can obtain the pointwise estimates in Theorem 1 part (1). In view of Proposition 4 and Proposition 6, we can obtain the set of estimates on energy fluxes in Theorem 1 part (2).

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