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Spectral Conditions for Stability of One-Parameter Semigroups

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Let $\{S(t): t \geqslant 0\}$ be a C_0-semigroup on a Banach space Y with generator B and $\{T(t): t \geqslant 0\}$ be a bounded C_0-semigroup on a Banach space X with generator A. Suppose that $\sigma(B) \cap i\mathbf{R}$ is countable, $P\sigma(A^*) \cap i\mathbf{R}$ is empty and that there is a bounded linear operator $C: Y \to X$ with dense range which intertwines the two semigroups. Then $\|T(t)x\|_X \to 0$ as $t \to \infty$, for each x in X. This generalises results of W. Arendt and the author, Yu. I. Lyubich and Vũ Quôc Phóng, and Falun Huang.

1. INTRODUCTION AND BACKGROUND

In this paper, we will prove the following two abstract results concerning the asymptotic behaviour of C_0-semigroups of operators on Banach spaces.

THEOREM A. *Let $\mathscr{S} = \{S(t): t \geqslant 0\}$ be a C_0-semigroup on a Banach space Y with generator B and $\mathscr{T} = \{T(t): t \geqslant 0\}$ be a bounded C_0-semigroup on a Banach space X with generator A. Suppose that $\sigma(B) \cap i\mathbf{R}$ is countable, $P\sigma(A^*) \cap i\mathbf{R}$ is empty, and there is a bounded linear operator $C: Y \to X$ with dense range such that $CS(t) = T(t)C$ for all $t \geqslant 0$. Then $\|T(t)x\|_X \to 0$ as $t \to \infty$, for each x in X.*

THEOREM B. *Let $\mathscr{S}$ be a C_0-semigroup on a Banach space with generator B. Suppose that there is a bounded linear operator $C: Y \to X$ such that $\sup_{t \geqslant 0} \|CS(t)\|_{\mathscr{B}(Y, X)} < \infty$, $\sigma(B) \cap i\mathbf{R}$ is countable, and there exists y_0 in Y such that $\|CS(t)\, y_0\|_X$ does not converge to 0 as $t \to \infty$. Then there exist a non-zero g in Y^* and λ_0 in $i\mathbf{R}$ such that $|g(y)| \leqslant \limsup_{t \to \infty} \|CS(t)\, y\|_X$ for all y in Y and $S(t)^*\, g = e^{\lambda_0 t} g$ for all $t \geqslant 0$.*

Various special cases of Theorem A have previously been given, the earliest being the case when $X = Y$, $C = I$, and $\mathscr{T}$ is norm-continuous due

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to Sklyar and Shirman [15], and the case when $X = Y$, $C = I$ and $\sigma(A) \cap i\mathbf{R}$ is empty due to Falun Huang [10], [11]. More generally, the case when $X = Y$ and $C = I$ was obtained by Lyubich and Vũ Quôc Phóng [14] and independently by Arendt and the author [1], and is sometimes known as the ABLP Theorem (or, more accurately, the ABLV Theorem). Many variants of that result have subsequently been obtained—see [4] for a survey of work in this area up to 1992. More recently, Falun Huang [12] gave the case of Theorem A in which $P\sigma(B^*) \cap i\mathbf{R}$ is assumed to be empty and C to be an embedding.

Several proofs have already been given of a slightly weaker version of Theorem B. It was noted in [3, p. 803] and other proofs have been obtained by Greenfield [8] and Falun Huang [12].

Our arguments follow the same general route as [12], but we take the opportunity to make some simplifications and clarifications. While the basic approach is functional-analytic in the spirit of [12], complex analytic techniques are used in [12, Theorem 2.4] and, in simplifying that argument (Theorem 2), we exhibit some features in common with the methods of [1]. We also give some examples which complement one given in [12].

In the paper, we shall be considering C_0-semigroups $\mathcal{S} = \{S(t) : t \geqslant 0\}$ and $\mathcal{T} = \{T(t) : t \geqslant 0\}$ on Banach spaces Y and X respectively, with generators B and A respectively. We shall also consider an *intertwining operator* C for $(\mathcal{S}, \mathcal{T})$, that is a bounded linear operator $C : Y \to X$ such that $CS(t) = T(t)\,C$ for all $t \geqslant 0$. We can assume without loss of generality that C has dense range in X. Sometimes, we shall assume that C is injective, in which case we may suppress the operator C and regard Y as a Banach space continuously embedded in X and $\mathcal{S}$ as the restriction of $\mathcal{T}$ to Y.

The spectrum, resolvent set and point spectrum of an operator A are denoted by $\sigma(A)$, $\rho(A)$ and $P\sigma(A)$, respectively.

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2. The Results

We begin with a simple lemma.

Lemma 1. *Let $\mathcal{S}$ and $\mathcal{T}$ be C_0-semigroups on Banach spaces Y and X with generators B and A respectively, and let $C : Y \to X$ be an intertwining operator.*

(1) *If $y \in D(B)$, then $Cy \in D(A)$ and $ACy = CBy$.*

(2) *If $\lambda \in \rho(A) \cap \rho(B)$, then $C(\lambda I_Y - B)^{-1} = (\lambda I_X - A)^{-1}\,C$.*

(3) *If C has dense range, then $\{Cy : y \in D(B)\}$ is a core for A.*

Proof. (1) For y in $D(B)$,

$$t^{-1}(T(t)\,Cy - Cy) = C(t^{-1}(S(t)\,y - y)) \to CBy \qquad \text{as} \quad t \to 0+,$$

so $Cy \in D(A)$ and $ACy = CBy$.

(2) For y in Y and λ in $\rho(A) \cap \rho(B)$,

$$\begin{aligned}(\lambda I_X - A)\,C(\lambda I_Y - B)^{-1}\,y &= \lambda C(\lambda I_Y - B)^{-1}\,y - CB(\lambda I_Y - B)^{-1}\,y \\ &= C(\lambda I_Y - B)(\lambda I_Y - B)^{-1}\,y \\ &= Cy.\end{aligned}$$

Applying $(\lambda I_X - A)^{-1}$ shows that $C(\lambda I_Y - B)^{-1} = (\lambda I_X - A)^{-1}\,C$.

(3) Choose $\lambda \in \rho(A) \cap \rho(B)$. For x in $D(A)$, there is a sequence (y_n) in Y such that $\|Cy_n - (\lambda I_X - A)\,x\|_X \to 0$. Then $\|C(\lambda I_Y - B)^{-1}\,y_n - x\|_X \to 0$ and $\|AC((\lambda I_Y - B)^{-1}\,y_n - Ax\|_X \to 0$. ▮

The following is a slight generalisation of [12, Theorem 2.4]. The proof given there could be used here without significant change, but we prefer to give a similar, but slightly simpler, construction which shares with [1] the use of circular contours and factors such as $(1 + \lambda^2/r^2)$.

THEOREM 2. *Let $\mathcal{S}$ be a C_0-semigroup on Y with generator B and $\mathcal{T}$ be a C_0-group of isometries on X with generator A. Suppose that there is an intertwining operator $C\colon Y \to X$ with dense range in X. Then $\sigma(A) \subseteq \sigma(B) \cap i\mathbf{R}$.*

Proof. Since A generates a group of isometries, $\sigma(A) \subseteq i\mathbf{R}$ and $\|(\lambda I_X - A)^{-1}\|_{\mathcal{B}(X)} \leqslant (|\mathrm{Re}\,\lambda|)^{-1}$ $(\mathrm{Re}\,\lambda \neq 0)$.

Let $\lambda_0 \in \rho(B) \cap i\mathbf{R}$. We need to show that $\lambda_0 \in \rho(A)$. Replacing $S(t)$ by $e^{-\lambda_0 t}S(t)$ etc., we may assume that $\lambda_0 = 0$. Let $r > 0$ be such that $\bar{U} \subseteq \rho(B)$, where $U = \{\lambda \in \mathbf{C}\colon |\lambda| < r\}$.

Let $x \in X$. There is a sequence (y_n) in Y such that $\|Cy_n - x\|_X \to 0$. Let

$$f_n(\lambda) = \left(1 + \frac{\lambda^2}{r^2}\right) C(\lambda I_Y - B)^{-1}\,y_n \qquad (\lambda \in \bar{U}).$$

Then f_n is continuous on $\bar{U}$ and analytic in U. For λ in $\bar{U} \backslash i\mathbf{R}$, Lemma 1 shows that

$$f_n(\lambda) = \left(1 + \frac{\lambda^2}{r^2}\right) (\lambda I_X - A)^{-1}\,Cy_n.$$

For λ in $\partial U \backslash i\mathbf{R}$, $\lambda = re^{i\theta}$ where $\cos\theta \neq 0$, and

$$\begin{aligned}\|f_n(\lambda)\|_X &\leqslant \left|1+\frac{\lambda^2}{r^2}\right| \,\|(\lambda I_X - A)^{-1}\|_{\mathscr{B}(X)} \,\|Cy_n\|_X \\ &\leqslant 2\,|\cos\theta|\, \frac{1}{|r\cos\theta|}\,\|Cy_n\|_X \\ &= \frac{2\,\|Cy_n\|_X}{r}.\end{aligned}$$

By the Maximum Modulus Theorem,

$$\|f_n(\lambda)\|_X \leqslant \frac{2\,\|Cy_n\|_X}{r}$$

for all λ in U. Thus, for λ in $U \backslash i\mathbf{R}$,

$$\|(\lambda I_X - A)^{-1}\, x\|_X = \lim_{n\to\infty} \left|\frac{r^2}{r^2+\lambda^2}\right| \,\|f_n(\lambda)\|_X \leqslant \frac{2r\,\|x\|_X}{|r^2+\lambda^2|}.$$

This shows that

$$\|(\lambda I_X - A)^{-1}\|_{\mathscr{B}(X)} \leqslant \frac{2r}{|r^2+\lambda^2|} \leqslant \frac{8}{3r} \quad \left(|\lambda| \leqslant \frac{r}{2},\ \lambda \notin i\mathbf{R}\right).$$

It follows that $\lambda_0 \in \rho(A)$. ∎

Remark. In [12, Theorem 2.4], it is claimed that if, in addition to the assumptions of Theorem 2, C is injective and $\sigma(B) \cap i\mathbf{R}$ is countable, then $\sigma(B) = \sigma(A)$. However, there appears to be a gap in the argument, because it is not clear that approximate eigenvalues of B belong to $\sigma(A)$. We do not know whether the claimed result is true, but we will see in Proposition 3 that it holds if $\mathscr{S}$ is also assumed to be bounded.

There is a similar gap in the proof of [11, Theorem 3.1], but in that case the result is known to be correct. Indeed, the argument in [14] shows how to fill the gap (and also generalises the result). A short proof of [11, Theorem 3.1] can be obtained by applying a method of Korevaar [13](see [4, p. 40]).

PROPOSITION 3. *Let $\mathscr{S}$ be a bounded C_0-semigroup on Y with generator B and $\mathscr{T}$ be a bounded C_0-group on X with generator A. Suppose that $\sigma(B) \cap i\mathbf{R}$ is countable and there is an injective intertwining operator $C\colon Y \to X$ with dense range in X. Then $\mathscr{S}$ extends to a bounded C_0-group on Y and $\sigma(B) = \sigma(A)$.*

Proof. By changing to equivalent norms, we may assume that $\mathcal{S}$ is a contraction semigroup and $\mathcal{T}$ is a group of isometries. Suppose that $y \in Y$ and $\|S(t)\, y\|_Y \to 0$ as $t \to \infty$. Then $\|Cy\|_X = \|T(t)\, Cy\|_X = \|CS(t)\, y\|_X \to 0$, so $Cy = 0$. Since C is injective, it follows that $y = 0$. By a result of Greenfield [9], $\mathcal{S}$ extends to a C_0-group of isometries on Y. Hence $\sigma(B) = \sigma(B) \cap i\mathbf{R}$ which is a countable subset of $i\mathbf{R}$, so the isolated points are dense in $\sigma(B)$. Each isolated point is an eigenvalue of B and hence of A by Lemma 1 (1). It follows that $\sigma(B) \subseteq \sigma(A)$, and Theorem 2 shows that $\sigma(B) = \sigma(A)$. ∎

PROPOSITION 4. *Let $\mathcal{S}$ be a C_0-semigroup on Y with generator B, and $\mathcal{T}$ be a C_0-semigroup of isometries on X with generator A. Suppose that $\sigma(B) \cap i\mathbf{R} \neq i\mathbf{R}$ and that there is an intertwining operator $C\colon Y \to X$ with dense range. Then $\mathcal{T}$ extends to a C_0-group of isometries on X.*

Proof. Suppose that $\mathcal{T}$ does not extend. By [14, Lemma, p. 38] or [11, Theorem 2.4], for each λ with $\operatorname{Re} \lambda < 0$, $\lambda I_X - A$ does not have dense range in X. By Lemma 1, C maps the range of $\lambda I_Y - B$ into the range of $\lambda I_X - A$. Since C has dense range, it follows that $\lambda I_Y - B$ does not have dense range in Y, so $\lambda \in \sigma(B)$. Since $\sigma(B)$ is closed, it follows that $\sigma(B) \cap i\mathbf{R} = i\mathbf{R}$, which is a contradiction. ∎

THEOREM 5. *Let $\mathcal{S}$ be a C_0-semigroup on Y with generator B and $\mathcal{T}$ be a bounded C_0-semigroup on X with generator A. Suppose that $\sigma(B) \cap i\mathbf{R}$ is countable, $P\sigma(A^*) \cap i\mathbf{R}$ is empty, and that there is an intertwining operator $C\colon Y \to X$ with dense range. Then $\|T(t)x\|_X \to 0$ as $t \to \infty$, for each x in X.*

Proof. Define a seminorm l on X by

$$l(x) = \limsup_{t \to \infty} \|T(t)\, x\|_X.$$

Let $X_0 = \{x \in X : l(x) = 0\}$, and let X_1 be the completion of X/X_0 with respect to the norm induced by l. Let $Q\colon X \to X_1$ be the natural map and $C_1 = QC$. Since $l(T(t)x) = l(x)$, $\mathcal{T}$ induces a C_0-semigroup $\mathcal{T}_1$ of isometries on X_1, and C_1 is an intertwining map for $(\mathcal{S}, \mathcal{T}_1)$ with dense range. Let A_1 be the generator of $\mathcal{T}_1$. By Proposition 4, $\mathcal{T}_1$ extends to a C_0-group on X. By Theorem 2, $\sigma(A_1) \subseteq \sigma(B) \cap i\mathbf{R}$, so $\sigma(A_1)$ is a countable subset of $i\mathbf{R}$.

The remainder of the proof is now fairly standard (see [14, p. 40], [12, Theorem 3.1], [7, Proposition 4.1]). Suppose that $X_1 \neq \{0\}$. Then $\sigma(A_1)$ is non-empty, and therefore has an isolated point λ_0 in $i\mathbf{R}$. Moreover, λ_0 is an eigenvalue of A_1 and A_1^*, so there is a non-zero f in X_1^* such that $T_1(t)^* f = e^{\lambda_0 t} f\ (t \geqslant 0)$. Let $g = Q^* f \in X^*$. Then $g \neq 0$ and $T(t)^*\, g = e^{\lambda_0 t} g$, so $\lambda_0 \in P\sigma(A^*) \cap i\mathbf{R}$, which is a contradiction. Hence, $X_1 = \{0\}$, so $X_0 = X$, and therefore $\|T(t)x\|_X \to 0$ for each x in X. ∎

THEOREM 6. *Let $\mathscr{S}$ be a C_0-semigroup on a Banach space Y with generator B. Suppose that there is a bounded linear operator $C\colon Y \to X$ such that $\sup_{t \geqslant 0} \|CS(t)\|_{\mathscr{B}(Y, X)} < \infty$, $\sigma(B) \cap i\mathbf{R}$ is countable, and there exists y_0 in Y such that $\|CS(t)\, y_0\|_X$ does not converge to 0 as $t \to \infty$. Then there exist a non-zero g in Y^* and λ_0 in $i\mathbf{R}$ such that $|g(y)| \leqslant \limsup_{t \to \infty} \|CS(t)\, y\|_X$ for all y in Y and $S(t)^*\, g = e^{\lambda_0 t} g$ for all $t \geqslant 0$.*

Proof. Define a seminorm p on Y by

$$p(y) = \limsup_{t \to \infty} \|CS(t)\, y\|_X.$$

Let $Y_0 = \{y \in Y : p(y) = 0\}$ and let Y_1 be the completion of Y/Y_0 in the norm induced by p. Then $\mathscr{S}$ induces a C_0-semigroup $\mathscr{S}_1$ of isometries on Y_1, with generator B_1 (say). The natural map $Q\colon Y \to Y_1$ intertwines $\mathscr{S}$ and $\mathscr{S}_1$ and has dense range. By Proposition 4, $\mathscr{S}_1$ extends to a C_0-group on Y_1. By Theorem 2, $\sigma(B_1) \subseteq \sigma(B) \cap i\mathbf{R}$, so $\sigma(B_1)$ is countable. Since $Y_1 \neq \{0\}$, it follows as in Theorem 5 that $\sigma(B_1)$ has an isolated point λ_0 in $i\mathbf{R}$ and that there exists f in Y_1^* with $\|f\|_{Y_1^*} = 1$ such that $S_1(t)^*\, f = e^{\lambda_0 t} f$ $(t \geqslant 0)$. Then we may take $g = Q^* f$. ∎

Remarks. (1) We have chosen to present independent proofs of Theorems 5 and 6, but Theorem 5 is really the special case of Theorem 6 when there is a constant c such that $\|CS(t)\, y\|_X \leqslant c\, \|Cy\|_X$ for all $t \geqslant 0$ and all y in Y.

(2) As mentioned in the introduction, Falun Huang [12, Theorem 3.1] gave a version of Theorem 5 in which the assumption that $P\sigma(A^*) \cap i\mathbf{R}$ is non-empty was replaced by the stronger assumption that $P\sigma(B^*) \cap i\mathbf{R}$ is non-empty and it was also assumed that C is injective. There appears to be a gap in the argument in [12, p. 188] where it is asserted that $\sigma(B_1)$ is contained in $\sigma(B)$ when B_1 is the generator of the C_0-semigroup induced by $\mathscr{S}$ on Y/Y_0, where $Y_0 = \{y \in Y : \|T(t)\, y\|_X \to 0\}$ (taking $C = I$ now). Such a spectral inclusion holds if Y_0 is invariant under the resolvents $(\lambda I_Y - B)^{-1}$ for all λ in $\rho(B)$. In the present context, invariance holds for λ in $\rho(B) \cap \rho(A)$, but it is not clear for λ in $\rho(B) \cap \sigma(A)$.

(3) Simple two-dimensional examples show that in Theorem 5 it is not always possible to arrange that $|g(y)| \leqslant \|Cy\|_X$ for all y in Y.

(4) It is routine to adapt the statements and proofs of this section to the case when $\mathscr{S}$ and $\mathscr{T}$ are discrete semigroups of the form $\{S^n : n \geqslant 0\}$ and $\{T^n : n \geqslant 0\}$, for some bounded operators S and T (see [1, Section 5], [12, Section 4]).

4. Some Examples

When one is given a C_0-semigroup explicitly, it is usually easier to verify directly whether $T(t)$ converges to 0 in the strong operator topology than it is to check that $\sigma(A) \cap i\mathbf{R}$ is countable. Consequently, practical applications of the ABLP Theorem and its variants usually arise in examples where the generator A is more accessible than the semigroup (see [2, Proposition 3.2], for example), or where the semigroup is constructed in some indirect way, and in applications arising from further abstract theory (see [1, Remark 3.3], [5, Theorem 4.1], [12, Theorem 3.4]).

Falun Huang [12, Example 3.5] has given an example where Theorem 5 is applicable on X, but the ABLP Theorem is not. In his example, the semigroup $\mathscr{S}$ is bounded on Y and $\sigma(B) \cap i\mathbf{R}$ is empty. In this section, we give two more examples where Theorem 5 is applicable and which complement the example in [12]. In Example 7 (an extension of [1, Example 2.5]), $\mathscr{S}$ is not bounded on Y and $\sigma(B) \cap i\mathbf{R}$ is empty. In Example 8, $\mathscr{S}$ is bounded on Y, $\sigma(B) \cap i\mathbf{R} = \{0\}$, $0 \in P\sigma(B^*)$ but $0 \notin P\sigma(A^*)$. In Example 8 and in Falun Huang's example, strong convergence on X is evident; in Example 7, it is much less so, although it can be established by direct calculation.

Example 7. Let $Y = c_0$, and

$$(S(t)\,y)_{2n-1} = \exp(-t/n + int)(y_{2n-1} + ty_{2n}),$$
$$(S(t)\,y)_{2n} = \exp(-t/n + int)\; y_{2n}.$$

Then $\mathscr{S}$ is an unbounded semigroup on Y and $\sigma(B) \cap i\mathbf{R}$ is empty. Define

$$\|y\|_X = \sup_{n \geqslant 1} \sup_{t \geqslant 0} 2^{-n} e^{-t/n} \max(|y_{2n-1} + ty_{2n}|, |y_{2n}|).$$

Then $\|\cdot\|_X$ is a norm on Y with respect to which $S(t)$ is contractive. Let X be the completion of $(Y, \|\cdot\|_X)$, and $T(t)$ be the continuous extension of $S(t)$ to X. Then the conditions of Theorem 5 are all satisfied.

Example 8. Let $X = L^1(0, \infty)$, $Y = l^1$ and

$$(T(t)\,f)(s) = f(s+t) \qquad (f \in X;\ s, t \geqslant 0)$$
$$(S(t)\,y)_n = e^{-t} \sum_{r=1}^{n} \frac{t^{n-r}}{(n-r)!}\, y_r \qquad (y \in Y).$$

Then $\mathscr{T}$ is a C_0-semigroup on X which converges strongly to 0 as $t \to \infty$, $\sigma(A) = \{\lambda \in \mathbf{C} : \operatorname{Re} \lambda \leqslant 0\}$ and $P\sigma(A^*) \cap i\mathbf{R}$ is empty. Moreover, $\mathscr{S}$ is a bounded norm-continuous C_0-semigroup on Y, whose generator B is given by:

$$(By)_n = \begin{cases} y_{n-1} - y_n & (n \geqslant 2) \\ -y_1 & (n = 1). \end{cases}$$

In particular, $\sigma(B) = \{\lambda \in \mathbf{C} : |\lambda + 1| \leqslant 1\}$, $\sigma(B) \cap i\mathbf{R} = \{0\}$ and 0 is an eigenvalue of B^* with eigenvector $(1, 1, 1, ...)$ in $l^\infty = Y^*$.

Let

$$w_n(s) = \frac{s^{n-1} e^{-s}}{(n-1)!} \qquad (s \geqslant 0, n \geqslant 1),$$

so w_n is a unit vector in X. Let $(\alpha_n)_{n \geqslant 1}$ be a sequence in l^1 such that $\alpha_n \neq 0$ and $\beta_k := \alpha_k^{-1} \sum_{n=k+1}^\infty |\alpha_n| \to 0$ as $k \to \infty$. For example, we could take $\alpha_n = 1/n!$. Let

$$f_k = \sum_{n=1}^\infty \alpha_{k+n-1} w_n \in X \qquad (k \geqslant 1).$$

Let Z be the closed linear span of $\{f_k : k \geqslant 1\}$ in X. Since

$$w_1 + \frac{1}{\alpha_k} \sum_{n=k+1}^\infty \alpha_n w_{n-k+1} \in Z, \tag{$*$}$$

the distance from w_1 to Z is at most β_k, for any $k \geqslant 1$. Since $\beta_k \to 0$, it follows that $w_1 \in Z$. Now, $(*)$ implies that

$$w_2 + \frac{1}{\alpha_{k+1}} \sum_{n=k+2}^\infty \alpha_n w_{n-k+1} \in Z,$$

so the distance from w_2 to Z is at most β_{k+1}. Thus $w_2 \in Z$. Iterating this argument shows that $w_n \in Z$ for all $n \geqslant 1$. Since $\{w_n : n \geqslant 1\}$ forms a total set in X, it follows that $Z = X$.

Since $\sup_{k \geqslant 1} \|f_k\|_X < \infty$, there is a bounded linear map $C: Y \to X$ such that $Ce_k = f_k$, where $\{e_k : k \geqslant 1\}$ is the standard basis of Y. Since $Z = X$, C has dense range in X. Moreover,

$$\begin{aligned}(T(t)\, Ce_k)(s) &= (T(t)\, f_k)(s) \\ &= \sum_{n=1}^\infty \alpha_{k+n-1} w_n(s+t) \\ &= \sum_{n=1}^\infty \sum_{r=0}^{n-1} \alpha_{n+k-1} \frac{s^r t^{n-1-r}}{r!\,(n-1-r)!} e^{-(s+t)}\end{aligned}$$

$$= \sum_{r=0}^{\infty} \sum_{n=r+1}^{\infty} \alpha_{n+k-1} \frac{s^r t^{n-1-r}}{r!\,(n-1-r)!} e^{-(s+t)}$$

$$= \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \alpha_{k+r+m} \frac{s^r}{r!} \frac{t^m}{m!} e^{-(s+t)}$$

$$= \sum_{m=0}^{\infty} \frac{t^m}{m!} f_{m+k}(s)\, e^{-t}$$

$$= (CS(t)\, e_k)(s).$$

Thus C is an intertwining operator for $\mathscr{S}$ and $\mathscr{T}$, so all the conditions of Theorem 5 are satisfied.

In fact, C is injective, so that Y is continuously and densely embedded in X and all the conditions of [12, Theorem 3.1] are satisfied except that $P\sigma(B^*) \cap i\mathbf{R}$ is non-empty. To see this, suppose that $y = (y_k)_{k \geqslant 1} \in Y$ and $Cy = 0$. Then

$$0 = \sum_{k=1}^{\infty} y_k f_k = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} y_k \alpha_{n+k-1} w_n.$$

It follows that $\sum_{k=1}^{\infty} y_k \alpha_{n+k-1} = 0$ for all $n \geqslant 1$. In particular,

$$y_1 + \frac{1}{\alpha_n} \sum_{k=2}^{\infty} \alpha_{n+k-1}\, y_k = 0,$$

so $|y_1| \leqslant \beta_n \sup_{k \geqslant 2} |y_k|$. Letting $n \to \infty$, it follows that $y_1 = 0$. Iterating this argument shows that $y = 0$.

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