U(1)-invariant special Lagrangian 3-folds. II. Existence of singular solutions

Dominic Joyce
Lincoln College, Oxford

1 Introduction

Special Lagrangian submanifolds (SL m-folds) are a distinguished class of real m-dimensional minimal submanifolds in $\mathbb{C}^m$, which are calibrated with respect to the m-form $\mathrm{Re}(\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_m)$. They can also be defined in Calabi–Yau manifolds, are important in String Theory, and are expected to play a rôle in the eventual explanation of Mirror Symmetry between Calabi–Yau 3-folds.

This is the second in a suite of three papers [6, 7] studying special Lagrangian 3-folds N in $\mathbb{C}^3$ invariant under the U(1)-action

$$\mathrm{e}^{i\theta} : (z_1, z_2, z_3) \mapsto (\mathrm{e}^{i\theta} z_1, \mathrm{e}^{-i\theta} z_2, z_3) \quad \text{for } \mathrm{e}^{i\theta} \in \mathrm{U}(1). \tag{1}$$

These three papers and [8] are surveyed in [9]. Locally we can write N as

$$\begin{aligned} N = \Big\{ &(z_1, z_2, z_3) \in \mathbb{C}^3 : \mathrm{Im}(z_3) = u\big(\mathrm{Re}(z_3), \mathrm{Im}(z_1 z_2)\big), \\ &\mathrm{Re}(z_1 z_2) = v\big(\mathrm{Re}(z_3), \mathrm{Im}(z_1 z_2)\big), \quad |z_1|^2 - |z_2|^2 = 2a \Big\}, \end{aligned} \tag{2}$$

where $a \in \mathbb{R}$ and $u, v : \mathbb{R}^2 \to \mathbb{R}$ are continuous functions. It was shown in [6] that when $a \neq 0$, N is an SL 3-fold in $\mathbb{C}^3$ if and only if u, v satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -2\big(v^2 + y^2 + a^2\big)^{1/2} \frac{\partial u}{\partial y}, \tag{3}$$

and then u, v are smooth and N is nonsingular.

The goal of this paper and its sequel [7] is to study what happens when $a = 0$. In this case, at points $(x, 0)$ with $v(x, 0) = 0$ the factor $-2(v^2 + y^2 + a^2)^{1/2}$ in (3) becomes zero, and then (3) is no longer elliptic. Because of this, when $a = 0$ the appropriate thing to do is to consider *weak solutions* of (3), which may have *singular points* $(x, 0)$ with $v(x, 0) = 0$. At such a point u, v may not be differentiable, and $\big(0, 0, x + iu(x, 0)\big)$ is a singular point of the SL 3-fold N in $\mathbb{C}^3$.

This paper will be concerned largely with technical analytic issues, to do with the existence, uniqueness and regularity of weak, singular solutions of (3) in the case $a = 0$. The sequel [7] will describe the singularities of solutions of (3) with $a = 0$, prove that under mild conditions the singularities are isolated and have a unique *multiplicity* and *type*, and show that for each $k \geqslant 1$ singularities with multiplicity k exist and occur in codimension k, in some sense.

In [7] we also construct large families of *special Lagrangian fibrations* on open subsets of $\mathbb{C}^3$. These fibrations are used in [8] as local models to study special Lagrangian fibrations of (almost) Calabi–Yau 3-folds, and to draw some conclusions about the *SYZ Conjecture* [10]. For a brief summary of the results of all four papers, see [9].

In [6] we showed that if S is a domain in $\mathbb{R}^2$ and $u, v \in C^1(S)$ satisfy (3), then v satisfies

$$\frac{\partial}{\partial x}\Big[\big(v^2+y^2+a^2\big)^{-1/2}\frac{\partial v}{\partial x}\Big]+2\,\frac{\partial^2 v}{\partial y^2}=0 \tag{4}$$

in S°, and also there exists $f \in C^2(S)$ with $\frac{\partial f}{\partial y}=u$ and $\frac{\partial f}{\partial x}=v$, unique up to addition of a constant, satisfying

$$\Big(\Big(\frac{\partial f}{\partial x}\Big)^2+y^2+a^2\Big)^{-1/2}\frac{\partial^2 f}{\partial x^2}+2\,\frac{\partial^2 f}{\partial y^2}=0. \tag{5}$$

Both (4) and (5) are *second-order quasilinear elliptic equations*. We then proved existence and uniqueness of solutions of the Dirichlet problems for (4) and (5) on (strictly convex) domains when $a \neq 0$.

The main results of this paper are Theorems 6.1, 6.2, 7.1 and 7.2 below, which prove existence and uniqueness of *singular* solutions to the Dirichlet problems for (4) and (5) when $a = 0$. They give detailed results on the *regularity* of the solutions — basically u, v are C^0, have weak derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ in L^p for p in various ranges, and u, v are real analytic away from their singular points. Also, f, u, v vary continuously with the boundary data. These singular solutions give many examples of *singular* $\mathrm{U}(1)$*-invariant SL* 3*-folds* in $\mathbb{C}^3$.

A fundamental question about compact SL 3-folds N in Calabi–Yau 3-folds M is: *how stable are they under large deformations*? Here we mean both deformations of N in a fixed M, and what happens to N as we deform M. The deformation theory of compact SL 3-folds under *small* deformations is already well understood, and is described in [5, §9]. But to extend this understanding to large deformations, one needs to take into account singular behaviour.

One possible moral of this paper is that *compact SL* 3*-folds are pretty stable under large deformations*. That is, we have shown existence and uniqueness for (possibly singular) U(1)-invariant SL 3-folds in $\mathbb{C}^3$ satisfying certain boundary conditions. This existence and uniqueness is *entirely unaffected* by singularities that develop in the SL 3-folds, which is quite surprising, as one might have expected that when singularities develop the existence and uniqueness properties would break down.

This is encouraging, as both the author's programme for constructing invariants of Calabi–Yau 3-folds in [4] by counting special Lagrangian homology 3-spheres, and proving some version of the SYZ Conjecture [10] in anything other than a fairly weak, limiting form, will require strong stability properties of compact SL 3-folds under large deformations; so these papers may be taken as a small piece of evidence that these two projects may eventually be successful.

In §6 and §7 we shall prove *existence and uniqueness of the Dirichlet problems* for (4) and (5) respectively when $a = 0$. That is, given a suitable domain S in $\mathbb{R}^2$ and boundary data $\phi \in C^{k+2,\alpha}(\partial S)$, we construct unique $v \in C^0(S)$ with $v|_{\partial S} = \phi$ satisfying (4) weakly, or unique $f \in C^1(S)$ with $f|_{\partial S} = \phi$ satisfying (5) with weak second derivatives.

The basic method is this. For each $a \in (0,1]$ we let v_a or f_a in $C^{k+2,\alpha}(S)$ be the unique solution of (4) or (5) with $v_a|_{\partial S} = \phi$ or $f_a|_{\partial S} = \phi$. Then we aim to prove that $v_a \to v$ in $C^0(S)$ or $f_a \to f$ in $C^1(S)$ as $a \to 0_+$ for some unique v, f which are (weak, singular) solutions of the Dirichlet problems for $a = 0$.

To show that these limits v, f exist, the main issue is to prove *a priori estimates* of v_a, f_a and their derivatives that are *uniform in a*. That is, we need bounds such as $\|v_a\|_{C^0} \leqslant C$ for all $a \in (0,1]$, with C independent of a. Getting such estimates is difficult, since equations (4) and (5) *really are* singular when $a = 0$, so many norms of v_a, f_a such as $\|\partial v_a\|_{C^0}$ can diverge to infinity as $a \to 0_+$, and uniform a priori estimates of these norms *do not exist*.

Although there are many results on a priori estimates for nonlinear elliptic equations in the literature, I could not find any that told me what I needed to know about (4) and (5), so I was forced to invent my own method. It gives a priori estimates for the first derivatives of bounded solutions of *nonlinear Cauchy-Riemann equations* such as (3). The underlying idea is geometrical, and comes from complex analysis.

We shall explain [6, Th. 6.9], which is the key tool. Suppose S, T are domains in $\mathbb{R}^2$, and $(u,v) : T \to S^\circ$, $(\hat u, \hat v) : S \to T$ satisfy (3) for some $a \neq 0$ with $(u,v)(x_0,y_0) = (u_0,v_0)$, $(\hat u,\hat v)(u_0,v_0) = (x_0,y_0)$ for $(u_0,v_0) \in S^\circ$, $(x_0,y_0) \in T^\circ$. Then the *graphs* $\Gamma, \hat\Gamma$ of (u,v) and $(\hat u,\hat v)$ are 2-submanifolds of $S \times T$ intersecting at (u_0,v_0,x_0,y_0).

As (3) is a *nonlinear Cauchy-Riemann equation*, it turns out there is an *almost complex structure* on $S \times T$, making $\Gamma, \hat\Gamma$ into *pseudo-holomorphic curves*. Therefore each intersection point in $\Gamma \cap \hat\Gamma$ has a positive integer *multiplicity*. Now by an argument like those used to count zeroes of holomorphic functions in complex analysis, by considering *winding numbers* along ∂S we find that the *total multiplicity* of $\Gamma \cap \hat\Gamma$ is 1. Thus, $(u_0,\ldots,y_0)$ has multiplicity 1, so the tangent spaces of $\Gamma, \hat\Gamma$ at $(u_0,\ldots,y_0)$ are *distinct*.

So suppose $(\hat u,\hat v) : S \to T$ satisfies (3) with $(\hat u,\hat v)(u_0,v_0) = (x_0,y_0)$. Then there *cannot exist* any solution $(u,v) : T \to S^\circ$ of (3) with $(u,v)(x_0,y_0) = (u_0,v_0)$ and first derivatives $\partial u, \partial v$ at (x_0,y_0) taking prescribed values, that is, those necessary to make $T_{(u_0,\ldots,y_0)}\Gamma, T_{(u_0,\ldots,y_0)}\hat\Gamma$ coincide. In this way we translate *existence results* for $(\hat u,\hat v) : S \to T$ satisfying (3) with prescribed values and derivatives at (u_0,v_0) into *nonexistence results* for $(u,v) : T \to S^\circ$ satisfying (3) with prescribed values and derivatives at (x_0,y_0).

Sections 4 and 5 use this to prove *a priori estimates* for $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ when (u,v) are bounded solutions of (3). In §4 we construct two families of solutions $(\hat u,\hat v)$ to (3), with $\hat v \approx \alpha + \beta y + \gamma x$ or $\hat v \approx \alpha + \beta y + \gamma xy$ for γ small.

Roughly speaking, these examples $(\hat{u},\hat{v}) : S \to T$ fill out all possible values and derivatives at (u_0, v_0) with $\partial\hat{v}$ small. Then [6, Th. 6.9] shows that for solutions $(u,v) : T \to S^\circ$ of (3), all values and derivatives with ∂v large are excluded. So we can give a priori bounds for $\partial u, \partial v$.

Here is another interesting analytic feature. Because of the involvement of y in (3)–(5), *x and y derivatives behave differently*. Roughly speaking, we find in §6 that if (u,v) are weak solutions of (3) when $a = 0$ then using the material of §5, we can show that $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}$ lie in L^p for $p \in [1, \frac{5}{2})$, and $\frac{\partial u}{\partial y}$ lies in L^q for $q \in [1,2)$, and $\frac{\partial v}{\partial x}$ lies in L^r for $r \in [1,\infty)$.

As $L^p_1 \hookrightarrow C^0$ for $p > 2$ by the Sobolev embedding theorem, v is continuous. But our L^q estimate of $\frac{\partial u}{\partial y}$ is too weak to show u is continuous in this way. Because of this, we prove a *nonstandard Sobolev embedding theorem*, Theorem 2.3. It allows us to use a stronger L^p norm of $\frac{\partial u}{\partial x}$ to compensate for the weaker L^q norm of $\frac{\partial u}{\partial y}$, and still prove u is continuous. Proving u,v are continuous is important geometrically: without continuity, N in (2) would not be *locally closed*, and one singular point of u,v would correspond to many in N.

Readers may wonder why we study both equations (4) and (5), rather than just one. The answer is that for different applications, each may be preferable. For instance, in §4 we construct our solutions with the Dirichlet problem for (5), but estimate them using elliptic regularity for (4). In [7] both Dirichlet problems are used, usually (5), and properties of (4) such as $v < v'$ on ∂S implies $v < v'$ on S crop up continually. In [8] the Dirichlet problem for (5) is best.

Acknowledgements. I would like to thank Mark Gross, Rafe Mazzeo and Rick Schoen for helpful conversations, and the referee for useful suggestions. I was supported by an EPSRC Advanced Fellowship whilst writing this paper.

2 Background material from analysis

In §2.1 we briefly summarize some background material we will need for later analytic results. Our principal reference is Gilbarg and Trudinger [2]. Section 2.2 proves a Sobolev embedding type result for functions u on subsets of $\mathbb{R}^2$, in terms of bounds for $\|\frac{\partial u}{\partial x}\|_{L^p}$ and $\|\frac{\partial u}{\partial y}\|_{L^q}$ with $p \neq q$.

2.1 Domains, function spaces and operators

First we define *domains* in $\mathbb{R}^2$, and *Banach spaces of functions* upon them.

Definition 2.1. A *domain* in $\mathbb{R}^2$ is a compact subset $S \subset \mathbb{R}^2$ which is topologically a disc with smooth (or sometimes piecewise smooth) *boundary* ∂S. The interior is $S^\circ = S \setminus \partial S$. A convex domain S is *strictly convex* if the curvature of ∂S is strictly positive.

For each $k \geqslant 0$, write $C^k(S)$ for the space of continuous functions $f : S \to \mathbb{R}$ with k continuous derivatives, with norm $\|f\|_{C^k} = \sum_{j=0}^k \sup_S |\partial^j f|$. For $\alpha \in$

$(0,1]$, the *Hölder space* $C^{k,\alpha}(S)$ is the subset of $f \in C^k(S)$ for which

$$[\partial^k f]_\alpha = \sup_{x\neq y\in S} \frac{\left|\partial^k f(x) - \partial^k f(y)\right|}{|x-y|^\alpha}$$

is finite, with norm $\|f\|_{C^{k,\alpha}} = \|f\|_{C^k} + [\partial^k f]_\alpha$. Set $C^\infty(S) = \bigcap_{k=0}^\infty C^k(S)$.

For $q \geqslant 1$, the *Lebesgue space* $L^q(S)$ is the set of locally integrable functions f on S for which the norm $\|f\|_{L^q} = \left(\int_S |f|^q \mathrm{d}\mathbf{x}\right)^{1/q}$ is finite. For $k \geqslant 0$, the *Sobolev space* $L^q_k(S)$ is the set of $f \in L^q(S)$ which are k times weakly differentiable with $|\nabla^j f| \in L^q(S)$ for $j \leqslant k$.

Next we discuss *differential operators* on domains.

Definition 2.2. Let S be a domain in $\mathbb{R}^2$. A *second-order linear differential operator* $P : C^2(S) \to C^0(S)$ may be written

$$\big(Pu\big)(x) = \sum_{i,j=1}^{2} a^{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{2} b^i(x)\frac{\partial u}{\partial x_i}(x) + c(x)u(x), \qquad (6)$$

for $u \in C^2(S)$. Here $a^{ij}, b^i, c \in C^0(S)$ are the *coefficients* of P, with $a^{ij} = a^{ji}$.

A *second-order quasilinear operator* $Q : C^2(S) \to C^0(S)$ may be written

$$\big(Qu\big)(x) = \sum_{i,j=1}^{2} a^{ij}(x,u,\partial u)\frac{\partial^2 u}{\partial x_i \partial x_j}(x) + b(x,u,\partial u), \qquad (7)$$

for $u \in C^2(S)$. Here $a^{ij}, b \in C^0\big(S \times \mathbb{R} \times (\mathbb{R}^2)^*\big)$ are the *coefficients* of Q, with $a^{ij} = a^{ji}$. We call P or Q *elliptic* if the symmetric 2×2 matrix (a^{ij}) is positive definite at every point. A second-order quasilinear operator $Q : C^2(S) \to C^0(S)$ is in *divergence form* if it is written as

$$\big(Qu\big)(x) = \sum_{j=1}^{2} \frac{\partial}{\partial x_j}\big(a^j(x,u,\partial u)\big) + b(x,u,\partial u), \qquad (8)$$

for functions $a^j \in C^1\big(S \times \mathbb{R} \times (\mathbb{R}^2)^*\big)$ and $b \in C^0\big(S \times \mathbb{R} \times (\mathbb{R}^n)^*\big)$.

Let Q be a quasilinear operator as in (7) or (8). We shall consider three different senses in which $Qu = 0$ can hold:

- We say $Qu = 0$ *holds* if $u \in C^2(S)$ and $Qu = 0$ in $C^0(S)$.
- We say $Qu = 0$ *holds with weak derivatives* if u is twice *weakly differentiable*, so that ∂u, $\partial^2 u$ exist almost everywhere, and $Qu = 0$ holds almost everywhere.
- For Q in *divergence form* (8), we say $Qu=0$ *holds weakly* if $u\in L^1_1(S)$ and

$$-\sum_{j=1}^{2} \int_S \frac{\partial \psi}{\partial x_j} \cdot a^j(x,u,\partial u)\mathrm{d}\mathbf{x} + \int_S \psi \cdot b(x,u,\partial u)\mathrm{d}\mathbf{x} = 0 \qquad (9)$$

for all $\psi \in C^1(S)$ supported in S°. We get (9) by multiplying (8) by ψ and integrating by parts. Note that (9) makes sense even if u is only once weakly differentiable.

Clearly, the first sense implies the second implies the third. But if Q is *elliptic*, under suitable assumptions on a^j, b one can show that a weak solution u is a solution, so all three senses coincide. See for instance [2, §8].

2.2 Continuity of functions with L^p, L^q derivatives

In §6 and §7 we will need the following result, to prove that u is continuous in weak solutions u, v of (3) when $a = 0$.

Theorem 2.3. *Let S be a domain in $\mathbb{R}^2$, and $p, q \in (1, \infty)$ with $p^{-1} + q^{-1} < 1$. Then there exist continuous functions $G, H : S \times S \to [0, \infty)$ depending only on S, p and q satisfying*

$$\begin{aligned} &G(w', x', w, x) = G(w, x, w', x'), \quad H(w', x', w, x) = H(w, x, w', x'), \\ &\textit{and} \quad G(w, x, w, x) = H(w, x, w, x) = 0 \quad \textit{for all } (w, x), (w', x') \in S, \end{aligned} \tag{10}$$

such that whenever $u \in C^1(S)$ then for all $(w, x), (w', x') \in S$ we have

$$\bigl|u(w, x) - u(w', x')\bigr| \leqslant \bigl\|\tfrac{\partial u}{\partial x}\bigr\|_{L^p} G(w, x, w', x') + \bigl\|\tfrac{\partial u}{\partial y}\bigr\|_{L^q} H(w, x, w', x'). \tag{11}$$

Before we prove the theorem, we explain its significance. The *Sobolev Embedding Theorem* [1, §2.3] implies that if S is a domain in $\mathbb{R}^2$ then $L^p_1(S) \hookrightarrow C^0(S)$ is a continuous inclusion when $p > 2$. That is, functions in a bounded subset of $L^p_1(S)$ are all uniformly continuous. Theorem 2.3 is essentially equivalent to this when $p = q$.

However, when $p \neq q$ and $p \leqslant 2$ or $q \leqslant 2$, our result is more general than this consequence of the Sobolev Embedding Theorem. The basic idea is that by taking a stronger norm of $\frac{\partial u}{\partial x}$ we can make do with a weaker norm of $\frac{\partial u}{\partial y}$, or vice versa, and still prove that u is continuous.

In fact one can prove a stronger result: the functions G, H of Theorem 2.3 actually satisfy $G(w, x, w', x') \leqslant C\bigl(|w - w'|^\gamma + |x - x'|^\delta\bigr)$ and $H(w, x, w', x') \leqslant C'\bigl(|w - w'|^\epsilon + |x - x'|^\zeta\bigr)$ for $\gamma, \delta > 0$ depending on p, β and $\epsilon, \zeta > 0$ depending on q, β and $C, C' > 0$. One can then use this to prove that u is *uniformly Hölder continuous*, rather than just uniformly continuous. But we will not need this.

We will prove the theorem explicitly in the case that S is a rectangle $R = [k, l] \times [m, n]$ in $\mathbb{R}^2$, and then explain briefly how to extend the proof to general S. We begin with a convolution formula for functions on R.

Proposition 2.4. *Let R be the closed rectangle $[k, l] \times [m, n]$ in $\mathbb{R}^2$, with $k < l$*

and $m < n$. *Fix* $\beta > 0$*, and define* $E : R\times R \to [0,1)$ *and* $F : R\times R \to (0,\infty]$ *by*

$$E(w,x,y,z)=\begin{cases}\max\bigl((y-w)^\beta(l-w)^{-\beta},(z-x)(n-x)^{-1}\bigr), & y\geqslant w,\ z\geqslant x,\\ \max\bigl((y-w)^\beta(l-w)^{-\beta},(x-z)(x-m)^{-1}\bigr), & y\geqslant w,\ z<x,\\ \max\bigl((w-y)^\beta(w-k)^{-\beta},(z-x)(n-x)^{-1}\bigr), & y<w,\ z\geqslant x,\\ \max\bigl((w-y)^\beta(w-k)^{-\beta},(x-z)(x-m)^{-1}\bigr), & y<w,\ z<x,\end{cases}$$

$$\text{and}\quad F(w,x,y,z) = -\frac{E(w,x,y,z)^{-1-1/\beta}-1}{(\beta+1)(l-k)(m-n)}. \tag{12}$$

Then for all $u \in C^1(R)$ *and* $(w,x) \in R$ *we have*

$$\begin{aligned}u(w,x) = {}&\frac{1}{(l-k)(n-m)}\int_m^n\int_k^l u(y,z)\mathrm{d}y\,\mathrm{d}z\\ &+\int_m^n\int_k^l \frac{\partial u}{\partial x}(y,z)(y-w)F(w,x,y,z)\mathrm{d}y\,\mathrm{d}z\\ &+\beta\int_m^n\int_k^l \frac{\partial u}{\partial y}(y,z)(z-x)F(w,x,y,z)\mathrm{d}y\,\mathrm{d}z.\end{aligned} \tag{13}$$

Proof. Let (w,x) and (r,s) lie in R. Then

$$\begin{aligned}u(w,x) &= u(r,s) - \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\Bigl(u\bigl(w+t^{1/\beta}(r-w),x+t(s-x)\bigr)\Bigr)\mathrm{d}t\\ &= u(r,s) - \int_0^1\Bigl(\beta^{-1}(r-w)t^{1/\beta-1}\frac{\partial u}{\partial x}\bigl(w+t^{1/\beta}(r-w),x+t(s-x)\bigr)\\ &\qquad + (s-x)\frac{\partial u}{\partial y}\bigl(w+t^{1/\beta}(r-w),x+t(s-x)\bigr)\Bigr)\mathrm{d}t\end{aligned}$$

Note that if $(w,x),(r,s) \in R$ then $\bigl(w+t^{1/\beta}(r-w),x+t(s-x)\bigr) \in R$ for all $t \in [0,1]$, as R is a rectangle. Integrate this equation over $(r,s) \in R$, regarding (w,x) as fixed. We get

$$\begin{aligned}(l-k)(n-m)u(w,x) &= \int_m^n\int_k^l u(r,s)\mathrm{d}r\,\mathrm{d}s\\ -\int_m^n\int_k^l\int_0^1&\Bigl(\beta^{-1}t^{1/\beta-1}(r-w)\frac{\partial u}{\partial x}\bigl(w+t^{1/\beta}(r-w),x+t(s-x)\bigr)\\ &+ (s-x)\frac{\partial u}{\partial y}\bigl(w+t^{1/\beta}(r-w),x+t(s-x)\bigr)\Bigr)\mathrm{d}t\,\mathrm{d}r\,\mathrm{d}s.\end{aligned}$$

Change variables from (r,s,t) to $(y,z,t) = \bigl(w+t^{1/\beta}(r-w),x+t(s-x),t\bigr)$ in the triple integral. Then $\mathrm{d}t\,\mathrm{d}r\,\mathrm{d}s = t^{-1-1/\beta}\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}z$, and $t^{1/\beta-1}(r-w) = t^{-1}(y-w)$, and $(s-x) = t^{-1}(z-x)$. Furthermore, if $(y,z) \in R$ and $t \in (0,1]$ then the condition for $(r,s) = \bigl(w+t^{-1/\beta}(y-w),x+t^{-1}(z-x)\bigr)$ to lie in R is

$E(w,x,y,z) < t \leqslant 1$, because this is how we defined E. Hence

$$(l-k)(n-m)u(w,x) = \int_m^n \int_k^l u(y,z) \mathrm{d}y \, \mathrm{d}z$$
$$- \int_m^n \int_k^l \Big((y-w)\frac{\partial u}{\partial x}(y,z) + \beta(z-x)\frac{\partial u}{\partial y}(y,z) \Big) \int_{E(w,x,y,z)}^1 \beta^{-1} t^{-2-1/\beta} \mathrm{d}t \, \mathrm{d}y \, \mathrm{d}z.$$

Equation (13) then follows by dividing by $(l-k)(n-m)$, doing the t integral explicitly, and substituting in (12). □

Now F is given entirely explicitly in (12), so the following is an exercise in Lebesgue integration, which we leave to the reader.

Proposition 2.5. *In the situation of Proposition 2.4, for $(w,x) \in R$ the function $(y,z) \mapsto (y-w)F(w,x,y,z)$ lies in $L^s(R)$ if $1 \leqslant s < 1 + \beta^{-1}$, and then the map $R \to L^s(R)$ taking (w,x) to the function $(y,z) \mapsto (y-w)F(w,x,y,z)$ is continuous. Similarly, the function $(y,z) \mapsto (z-x)F(w,x,y,z)$ lies in $L^t(R)$ if $1 \leqslant t < 1+\beta$, and then the corresponding map $R \to L^t(R)$ is continuous.*

We can now define the functions G, H in Theorem 2.3.

Definition 2.6. Let R be the closed rectangle $[k,l] \times [m,n]$ in $\mathbb{R}^2$, with $k < l$ and $m < n$. Suppose $p, q \in (1,\infty)$ with $p^{-1} + q^{-1} < 1$. This is equivalent to $1/(q-1) < p-1$. Choose β with $1/(q-1) < \beta < p-1$. Define $s = p/(p-1)$ and $t = q/(q-1)$. Then

$$\frac{1}{p} + \frac{1}{s} = 1, \quad \frac{1}{q} + \frac{1}{t} = 1, \quad 1 < s < 1 + \beta^{-1} \quad \text{and} \quad 1 < t < 1 + \beta.$$

Let F be as in Proposition 2.4, with this value of β. Define functions $G, H : R \times R \to [0,\infty)$ by

$$G(w,x,w',x') = \Big(\int_m^n \int_k^l \big| (y-w)F(w,x,y,z) - (y-w')F(w',x',y,z) \big|^s \mathrm{d}y \, \mathrm{d}z \Big)^{1/s},$$
$$H(w,x,w',x') = \beta \Big(\int_m^n \int_k^l \big| (z-x)F(w,x,y,z) - (z-x')F(w',x',y,z) \big|^t \mathrm{d}y \, \mathrm{d}z \Big)^{1/t}.$$

Then G is well-defined and continuous, as the functions

$$(y,z) \mapsto (y-w)F(w,x,y,z) \quad \text{and} \quad (y,z) \mapsto (y-w')F(w',x',y,z)$$

lie in $L^s(R)$ by Proposition 2.5 and depend continuously on (w,x) and (w',x'), and $G(w,x,w',x')$ is the L^s norm of their difference. Similarly, H is well-defined and continuous.

Proposition 2.7. *Theorem 2.3 holds when S is a rectangle $[k,l] \times [m,n]$.*

Proof. Let G, H be as in Definition 2.6. Then G, H are continuous, and (10) is immediate from the definition. Subtracting equation (13) with values (w, x) and (w', x') gives

$$\begin{aligned} u(w,x) - u(w,x) = \\ &\int_m^n \int_k^l \frac{\partial u}{\partial x}(y,z)\big((y-w)F(w,x,y,z) - (y-w')F(w',x',y,z)\big)\mathrm{d}y\,\mathrm{d}z \\ +\beta &\int_m^n \int_k^l \frac{\partial u}{\partial y}(y,z)\big((z-x)F(w,x,y,z) - (z-x')F(w',x',y,z)\big)\mathrm{d}y\,\mathrm{d}z. \end{aligned}$$

Equation (11) follows by using Hölder's inequality to estimate the first integral in terms of the L^p norm of $\frac{\partial u}{\partial x}$ and the L^s norm of the other factor, and the second integral in terms of the L^q norm of $\frac{\partial u}{\partial y}$ and the L^t norm of the other factor. □

To extend Theorem 2.3 to general domains S in $\mathbb{R}^2$, we mimic the proof of Proposition 2.4 to derive an analogue of (13) that holds on S, not just on a rectangle R. The main problem in doing this is that if $(w, x), (r, s) \in S$ then $\big(w + t^{1/\beta}(r - w), x + t(s - x)\big)$ may not lie in S for all $t \in [0, 1]$. So we choose a more general family of paths joining (w, x) to all points (r, s) in S. If these have the same qualitative behaviour near (w, x), scaling like $t^{1/\beta}$ in the x coordinate and t in the y coordinate, then the analogue of Proposition 2.5 should hold, and the rest of the proof follows with little change.

3 U(1)-invariant special Lagrangian 3-folds

For introductions to special Lagrangian geometry, see Harvey and Lawson [3, §III] and the author [5]. Here is the definition of special Lagrangian submanifolds in $\mathbb{C}^m$, taken from [3, §III].

Definition 3.1. Let $\mathbb{C}^m$ have complex coordinates $(z_1, \ldots, z_m)$, and define a metric g, a real 2-form ω and a complex m-form Ω on $\mathbb{C}^m$ by

$$\begin{gathered} g = |\mathrm{d}z_1|^2 + \cdots + |\mathrm{d}z_m|^2, \quad \omega = \frac{i}{2}(\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 + \cdots + \mathrm{d}z_m \wedge \mathrm{d}\bar{z}_m), \\ \text{and} \quad \Omega = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_m. \end{gathered} \tag{14}$$

Then $\operatorname{Re}\Omega$ and $\operatorname{Im}\Omega$ are real m-forms on $\mathbb{C}^m$. Let L be an oriented real submanifold of $\mathbb{C}^m$ of real dimension m. We say that L is a *special Lagrangian submanifold* of $\mathbb{C}^m$, or *SL m-fold* for short, if L is calibrated with respect to $\operatorname{Re}\Omega$. Equivalently [3, Cor. III.1.11], L is special Lagrangian (with some orientation) if $\omega|_L \equiv 0$ and $\operatorname{Im}\Omega|_L \equiv 0$.

We now recall a few fundamental results from [6] that will be used very often later. This paper is not designed to be read independently of [6], and many other results from [6] will be cited when they are needed. Readers are referred to [6] for proofs, discussion and motivation. The following result [6, Prop. 4.1] is the starting point for everything in [6, 7] and this paper.

Proposition 3.2. *Let S be a domain in $\mathbb{R}^2$ or $S = \mathbb{R}^2$, let $u, v : S \to \mathbb{R}$ be continuous, and $a \in \mathbb{R}$. Define*

$$N = \big\{ (z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 z_2 = v(x,y) + iy, \quad z_3 = x + iu(x,y), \\ |z_1|^2 - |z_2|^2 = 2a, \quad (x,y) \in S \big\}. \tag{15}$$

Then

(a) *If $a = 0$, then N is a (possibly singular) special Lagrangian 3-fold in $\mathbb{C}^3$, with boundary over ∂S, if u, v are differentiable and satisfy*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -2\big(v^2 + y^2\big)^{1/2} \frac{\partial u}{\partial y}, \tag{16}$$

except at points $(x, 0)$ in S with $v(x, 0) = 0$, where u, v need not be differentiable. The singular points of N are those of the form $(0, 0, z_3)$, where $z_3 = x + iu(x, 0)$ for $x \in \mathbb{R}$ with $v(x, 0) = 0$.

(b) *If $a \neq 0$, then N is a nonsingular SL 3-fold in $\mathbb{C}^3$, with boundary over ∂S, if and only if u, v are differentiable on all of S and satisfy*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -2\big(v^2 + y^2 + a^2\big)^{1/2} \frac{\partial u}{\partial y}. \tag{17}$$

In [6, Prop. 7.1] we show that solutions $u, v \in C^1(S)$ of (17) are derived from a *potential function* $f \in C^2(S)$ satisfying a *second-order quasilinear elliptic equation.*

Proposition 3.3. *Let S be a domain in $\mathbb{R}^2$ and $u, v \in C^1(S)$ satisfy (17) for $a \neq 0$. Then there exists $f \in C^2(S)$ with $\frac{\partial f}{\partial y} = u$, $\frac{\partial f}{\partial x} = v$ and*

$$P(f) = \Big(\Big(\frac{\partial f}{\partial x} \Big)^2 + y^2 + a^2 \Big)^{-1/2} \frac{\partial^2 f}{\partial x^2} + 2 \frac{\partial^2 f}{\partial y^2} = 0. \tag{18}$$

This f is unique up to addition of a constant, $f \mapsto f + c$. Conversely, all solutions of (18) yield solutions of (17).

Equation (18) may also be written in *divergence form* as

$$P(f) = \frac{\partial}{\partial x} \Big[A\Big(a, y, \frac{\partial f}{\partial x} \Big) \Big] + 2 \frac{\partial^2 f}{\partial y^2} = 0, \tag{19}$$

$$\text{where} \quad A(a, y, v) = \int_0^v \big(w^2 + y^2 + a^2 \big)^{-1/2} \mathrm{d}w, \tag{20}$$

so that $\frac{\partial A}{\partial v} = (v^2 + y^2 + a^2)^{-1/2}$. Note that A is undefined when $a = y = 0$.

In [6, Prop. 8.1] we show that if u, v satisfy (17) then v satisfies a *second-order quasilinear elliptic equation*, and conversely, any solution v of this equation extends to a solution u, v of (17).

Proposition 3.4. *Let S be a domain in $\mathbb{R}^2$ and $u, v \in C^2(S)$ satisfy* (17) *for $a \neq 0$. Then*

$$Q(v) = \frac{\partial}{\partial x}\Big[\big(v^2 + y^2 + a^2\big)^{-1/2}\frac{\partial v}{\partial x}\Big] + 2\,\frac{\partial^2 v}{\partial y^2} = 0. \tag{21}$$

Conversely, if $v \in C^2(S)$ satisfies (21) *then there exists $u \in C^2(S)$, unique up to addition of a constant $u \mapsto u + c$, such that u, v satisfy* (17).

Defining A as in (20), equation (21) may also be written

$$\frac{\partial^2}{\partial x^2}\big(A(a, y, v)\big) + 2\,\frac{\partial^2 v}{\partial y^2} = 0. \tag{22}$$

In [6, Th. 7.6] and [6, Th. 8.8] we prove existence and uniqueness of solutions to the Dirichlet problems for (18) and (21) in (strictly convex) domains in $\mathbb{R}^2$.

Theorem 3.5. *Let S be a strictly convex domain in $\mathbb{R}^2$, and let $a \neq 0$, $k \geqslant 0$ and $\alpha \in (0, 1)$. Then for each $\phi \in C^{k+2,\alpha}(\partial S)$ there exists a unique solution f of* (18) *in $C^{k+2,\alpha}(S)$ with $f|_{\partial S} = \phi$. This f is real analytic in S°, and satisfies $\|f\|_{C^1} \leqslant C\|\phi\|_{C^2}$, for some $C > 0$ depending only on S.*

Theorem 3.6. *Let S be a domain in $\mathbb{R}^2$. Then whenever $a \neq 0$, $k \geqslant 0$, $\alpha \in (0, 1)$ and $\phi \in C^{k+2,\alpha}(\partial S)$ there exists a unique solution $v \in C^{k+2,\alpha}(S)$ of* (21) *with $v|_{\partial S} = \phi$. Fix a basepoint $(x_0, y_0) \in S$. Then there exists a unique $u \in C^{k+2,\alpha}(S)$ with $u(x_0, y_0) = 0$ such that u, v satisfy* (17). *Furthermore, u, v are real analytic in S°.*

Combined with Propositions 3.2 and 3.3, these give existence and uniqueness results for U(1)-invariant special Lagrangian 3-folds in $\mathbb{C}^3$ satisfying certain boundary conditions.

4 Two families of model solutions of (17)

In this section we construct two families of solutions u, v and u', v' of (17) on the unit disc D in $\mathbb{R}^2$, and make detailed analytic estimates of u, v, u', v' and their derivatives. The reason for doing this is that in §5 we will use these examples and [6, Th. 6.9] to prove a priori estimates of the first derivatives of solutions u, v of (17) satisfying a bound $u^2 + v^2 < L^2$. These estimates are the key technical tool we will need to extend Theorems 3.5 and 3.6 to the case $a = 0$, and prove other important facts about singular solutions of (16).

The method we shall use is to start with the exact solutions $u = \beta x$, $v = \alpha + \beta y$ of (17), and add on a small perturbation. This perturbation is the sum of a known, exact solution of the linearization of (17) at $u = \beta x$, $v = \alpha + \beta y$, and an 'error term'. Most of the hard work below is in estimating this error term, and showing that in some circumstances it is small, so that the explicit approximate solution of (17) is close to an exact solution.

4.1 The main results

Here are the two main results of this section.

Theorem 4.1. *Let $K, L, M, N > 0$ be given. Then there exist $A, B, C > 0$ depending only on K, L, M, N such that the following is true.*

Suppose $a, x_0, y_0, u_0, v_0, p_0$ and q_0 are real numbers satisfying

$$\begin{aligned} &|p_0| \leqslant A \max\big((v_0^2+a^2)^{1/2}|y_0|^{1/2}, (v_0^2+a^2)^{3/4}\big), \qquad |y_0||q_0| \leqslant B(v_0^2+a^2)^{1/2}, \\ &|q_0| \leqslant C, \quad a \neq 0, \quad |a| \leqslant K, \quad x_0^2+y_0^2 \leqslant L^2, \qquad \text{and} \quad u_0^2+v_0^2 \leqslant M^2. \end{aligned} \tag{23}$$

Define $D_L = \big\{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leqslant L^2\big\}$. Then there exist $u, v \in C^\infty(D_L)$ satisfying (17)*, $(u-u_0)^2+(v-v_0)^2 < N^2$ on D_L, and*

$$u(x_0,y_0) = u_0, \ \ v(x_0,y_0) = v_0, \ \ \frac{\partial v}{\partial x}(x_0,y_0) = p_0 \ \text{ and } \ \frac{\partial v}{\partial y}(x_0,y_0) = q_0. \tag{24}$$

Theorem 4.2. *Let $K, L, M, N > 0$. Then there exist $A, B > 0$ depending only on K, L, M, N such that the following is true.*

Suppose $a, x_0, y_0, u_0, v_0, p_0$ and q_0 are real numbers satisfying

$$\begin{aligned} &|p_0| \leqslant A\,|y_0|^{5/2}, \qquad |q_0| \leqslant B, \qquad a \neq 0, \\ &|a| \leqslant K, \qquad x_0^2+y_0^2 \leqslant L^2, \qquad \text{and} \quad u_0^2+v_0^2 \leqslant M^2. \end{aligned} \tag{25}$$

Define $D_L = \big\{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leqslant L^2\big\}$. Then there exist $u', v' \in C^\infty(D_L)$ satisfying (17)*, $(u'-u_0)^2+(v'-v_0)^2 < N^2$ on D_L, and*

$$u'(x_0,y_0) = u_0, \ \ v'(x_0,y_0) = v_0, \ \ \frac{\partial v'}{\partial x}(x_0,y_0) = p_0 \ \text{ and } \ \frac{\partial v'}{\partial y}(x_0,y_0) = q_0. \tag{26}$$

We can interpret Theorems 4.1 and 4.2 like this: given a, x_0, y_0, u_0, v_0, L and N with $a \neq 0$ and $x_0^2+y_0^2 \leqslant L^2$, we wish to know what are the possible values of $p_0 = \frac{\partial v}{\partial x}(x_0,y_0)$ and $q_0 = \frac{\partial v}{\partial y}(x_0,y_0)$ for solutions u, v of (17) on D_L with $(u,v)(x_0,y_0) = (u_0,v_0)$ and $(u-u_0)^2+(v-v_0)^2 < N^2$. The theorems give ranges of values of p_0 and q_0 for which such solutions are guaranteed to exist, in terms of upper bounds K for $|a|$ and M for $\big|(u_0,v_0)\big|$. In §5 we will combine the theorems with [6, Th. 6.9] to derive a priori estimates of the first derivatives of solutions u, v of (17).

The only difference between Theorems 4.1 and 4.2 is the conditions on p_0 and q_0 in (23) and (25). Roughly speaking, when y_0 is small but $v_0^2+a^2$ is not small Theorem 4.1 is a stronger result than Theorem 4.2 — that is, the requirements on p_0 and q_0 in (25) are unnecessarily strict. And when v_0 and a are small but y_0 is not small, Theorem 4.2 is stronger — that is, the requirements on p_0 and q_0 in (23) are unnecessarily strict.

In §5 we will use Theorems 4.1 and 4.2 to prove *a priori estimates* for $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ when u, v satisfy (17) and a C^0 bound. Only by using *both*

Theorems 4.1 and 4.2 together will we be able to make these a priori estimates powerful enough for the applications in §6–§7, in particular, proving the *continuity* of singular solutions u, v of (16).

Theorems 4.1 and 4.2 are the only results of this section that will be used later, so readers not interested in the proofs should skip on to §5. A sketch of the proofs is given after Definition 4.4.

4.2 Definition of the solutions

We define the first of two families of solutions u, v of (17). In §4.7 we will use rescaled versions of these u, v to prove Theorem 4.1.

Definition 4.3. Let a, s, α, β and γ be real numbers satisfying

$$a \neq 0, \quad 0 < s \leqslant 1, \quad \alpha^2 + a^2 = s^2, \quad |\beta| \leqslant 1 \quad \text{and} \quad |\gamma| \leqslant \tfrac{1}{20}s. \tag{27}$$

Let D be the closed unit disc in $\mathbb{R}^2$. Define functions $F, G \in C^\infty(\mathbb{R})$ and $g \in C^\infty(D)$ by

$$F(y) = -\tfrac{1}{2}\int_0^y \bigl((\alpha + \beta w)^2 + w^2 + a^2\bigr)^{-1/2}\mathrm{d}w, \quad G(y) = \int_0^y F(w)\mathrm{d}w \tag{28}$$

$$\text{and} \quad g(x, y) = \alpha x + \beta xy + \gamma\bigl(\tfrac{1}{2}x^2 + G(y)\bigr). \tag{29}$$

Let $f \in C^\infty(D)$ be the unique solution of (18) on D with $f|_{\partial D} = g|_{\partial D}$. This exists by Theorem 3.5. Define $\phi = f - g$ and $\psi = \frac{\partial\phi}{\partial x}$. Then $\phi, \psi \in C^\infty(D)$ and $\phi|_{\partial D} = 0$. Define $u = \frac{\partial f}{\partial y}$ and $v = \frac{\partial f}{\partial x}$. Then u, v satisfy (17) and v satisfies (21). Also, as $f = g + \phi$, from (29) we get

$$u = \beta x + \gamma F(y) + \frac{\partial\phi}{\partial y} \quad \text{and} \quad v = \alpha + \beta y + \gamma x + \frac{\partial\phi}{\partial x} = \alpha + \beta y + \gamma x + \psi. \tag{30}$$

As $v = \alpha + \beta y + \gamma x + \psi$ satisfies (21), we get

$$\begin{aligned} &\bigl((\alpha + \beta y + \gamma x + \psi)^2 + y^2 + a^2\bigr)^{-1/2}\frac{\partial^2\psi}{\partial x^2} + 2\,\frac{\partial^2\psi}{\partial y^2} = \\ &\bigl((\alpha + \beta y + \gamma x + \psi)^2 + y^2 + a^2\bigr)^{-3/2}\bigl(\alpha + \beta y + \gamma x + \psi\bigr)\bigl(\tfrac{\partial\psi}{\partial x} + \gamma\bigr)^2. \end{aligned} \tag{31}$$

This is a second-order quasilinear elliptic equation on ψ. We will use elliptic regularity results for it to estimate the derivatives of ψ.

The function g constructed above is the sum of an exact solution $\alpha x + \beta xy$ of (18), and a multiple γ of an exact solution $\frac{1}{2}x^2 + G(y)$ of the *linearization* of (18) at $\alpha x + \beta xy$. Thus, when γ is small we expect g to be an approximate solution of (18), with error roughly of order γ^2.

Here is our second family of solutions u', v' of (17). In §4.7 we will use rescaled versions of these u', v' to prove Theorem 4.2.

Definition 4.4. Let a, s, α, β and γ be real numbers satisfying

$$a \neq 0, \quad 0 < s \leqslant 1, \quad \alpha^2 + a^2 = s^2, \quad |\beta| \leqslant \tfrac{1}{40} \quad \text{and} \quad |\gamma| \leqslant \tfrac{1}{40}. \tag{32}$$

Let D be the closed unit disc in $\mathbb{R}^2$. Define functions $F', G' \in C^\infty(\mathbb{R})$ and $g' \in C^\infty(D)$ by

$$F'(y) = -\tfrac{1}{2}\int_0^y w\big((\alpha+\beta w)^2 + w^2 + a^2\big)^{-1/2}\mathrm{d}w, \quad G'(y) = \int_0^y F'(w)\mathrm{d}w$$
$$\text{and} \quad g'(x,y) = \alpha x + \beta xy + \gamma\big(\tfrac{1}{2}x^2 y + G'(y)\big). \tag{33}$$

Let $f' \in C^\infty(D)$ be the unique solution of (18) on D with $f'|_{\partial D} = g'|_{\partial D}$. This exists by Theorem 3.5.

Define $\phi' = f' - g'$ and $\psi' = \frac{\partial \phi'}{\partial x}$. Then $\phi', \psi' \in C^\infty(D)$ and $\phi'|_{\partial D} = 0$. Define $u' = \frac{\partial f'}{\partial y}$ and $v' = \frac{\partial f'}{\partial x}$. Then u', v' satisfy (17) and v' satisfies (21). Also, as $f' = g' + \phi'$, from (33) we get

$$u' = \beta x + \gamma F'(y) + \frac{\partial \phi'}{\partial y} \quad \text{and} \quad v' = \alpha + \beta y + \gamma xy + \frac{\partial \phi'}{\partial x} = \alpha + \beta y + \gamma xy + \psi'.$$

Again, we start with an approximate solution $g'(x,y)$ of (18) which is the sum of the exact solution $\alpha x + \beta xy$ of (18) and an exact solution of the linearization of (18) at $\alpha x + \beta xy$, but this time it is a *different* solution of the linearization to that used in Definition 4.3.

Here is how the rest of the section is laid out. In §4.3 we derive *a priori C^0 estimates* for ϕ and ϕ'. We do this by noting that f, f' satisfy (18), and writing down explicit *super*- and *subsolutions* $f_\pm$, $f'_\pm$ of (18) with $f_- = f = f_+$ and $f'_- = f' = f'_+$ on ∂D, so that $f_- \leqslant f \leqslant f_+$ and $f'_- \leqslant f' \leqslant f'_+$ on D. Similarly, §4.4 derives a priori C^0 estimates for ψ, ψ', by noting that v, v' satisfy (21), and writing down explicit super- and subsolutions $v_\pm, v'_\pm$ of (21).

Section 4.5 extends these to a priori estimates of *higher derivatives* of ψ, ψ', using standard interior elliptic regularity results and a scaling argument. Section 4.6 shows that if a, v_0, p_0, q_0 satisfy some inequalities then in Definition 4.3 we can choose α, β, γ such that $v(x_0, y_0) = v_0$, $\frac{\partial v}{\partial x}(x_0, y_0) = p_0$ and $\frac{\partial v}{\partial y}(x_0, y_0) = q_0$, and similarly for u', v'. Finally, §4.7 completes the proofs.

4.3 A priori C^0 estimates for ϕ and ϕ'

Tedious manipulation of inequalities and completing the square proves:

Lemma 4.5. *In Definition 4.3 or 4.4, whenever $|w| \leqslant \frac{1}{10}s$ and $|y| \leqslant 1$ we have*

$$\tfrac{1}{4}(y^2 + s^2) \leqslant (\alpha + \beta y + w)^2 + y^2 + a^2 \leqslant 4(y^2 + s^2). \tag{34}$$

Here are explicit *super*- and *subsolutions* $f_\pm$ of (18) for f in D.

Proposition 4.6. *In the situation of Definition 4.3, define functions $f_\pm \in C^\infty(D)$ by $f_\pm = g \pm 8s^{-1}\gamma^2(1-x^2-y^2)$. Then $P(f_+) \leqslant 0$ and $P(f_-) \geqslant 0$, where P is defined in* (18).

Proof. Computation using (18) and (29) shows that $P(f_\pm) = \mp A + B$, where

$$A = 8s^{-1}\gamma^2\Big[2\big((\alpha+\beta y+\gamma x \mp 8s^{-1}\gamma^2 x)^2+y^2+a^2\big)^{-1/2}+4\Big] \quad \text{and}$$
$$B = \gamma\Big[\big((\alpha+\beta y+\gamma x\mp 8s^{-1}\gamma^2 x)^2+y^2+a^2\big)^{-1/2}-\big((\alpha+\beta y)^2+y^2+a^2\big)^{-1/2}\Big].$$

We show that $A \geqslant |B|$, which gives $P(f_+) \leqslant 0$ and $P(f_-) \geqslant 0$.

Define $w = \gamma x \mp 8s^{-1}\gamma^2 x$. Then as $|x| \leqslant 1$ and $|\gamma| \leqslant \frac{1}{20}s$ we see that $|w| \leqslant 2|\gamma|$. Applying the Mean Value Theorem to $h(z) = ((\alpha+\beta y+z)^2+y^2+a^2)^{-1/2}$ between $z=0$ and w implies that

$$B = -\gamma w(\alpha+\beta y+z)\big((\alpha+\beta y+z)^2+y^2+a^2\big)^{-3/2},$$

for some z between 0 and w. As $|\alpha+\beta y+z| \leqslant ((\alpha+\beta y+z)^2+y^2+a^2)^{1/2}$ and $|z| \leqslant |w| \leqslant 2|\gamma| \leqslant \frac{1}{10}s$, this gives

$$|B| \leqslant \frac{|\gamma||w|}{(\alpha+\beta y+z)^2+y^2+a^2} \leqslant \frac{2\gamma^2}{\frac{1}{4}(y^2+s^2)} \leqslant 8s^{-2}\gamma^2,$$

using the first inequality of Lemma 4.5 with z in place of w. Also, the second inequality of Lemma 4.5 yields

$$A \geqslant 8s^{-1}\gamma^2\big((y^2+s^2)^{-1/2}+4\big) \geqslant 8s^{-2}\gamma^2.$$

The last two equations give $A \geqslant |B|$, and the proof is complete. □

This shows that f_+ is a *supersolution* and f_- a *subsolution* of (18). As $1-x^2-y^2=0$ on ∂D, we have $f_\pm = g = f$ on ∂D. Therefore [6, Prop. 7.5] shows that $f_- \leqslant f \leqslant f_+$ on D. Subtracting g from each side gives:

Corollary 4.7. *In Definition 4.3, $|\phi| \leqslant 8s^{-1}\gamma^2(1-x^2-y^2)$ on D. Hence*

$$\|\phi\|_{C^0} \leqslant 8s^{-1}\gamma^2, \quad \textit{and} \quad \Big|\frac{\partial\phi}{\partial x}\Big|, \Big|\frac{\partial\phi}{\partial y}\Big| \leqslant 16s^{-1}\gamma^2 \quad \textit{on } \partial D. \tag{35}$$

The first inequality is the a priori C^0 estimate for ϕ that we want. The analogues of Proposition 4.6 and Corollary 4.7 for Definition 4.4 are:

Proposition 4.8. *Define functions $f'_\pm \in C^\infty(D)$ by $f'_\pm = g' \pm \gamma^2(1-x^2-y^2)$. Then $P(f'_+) \leqslant 0$ and $P(f'_-) \geqslant 0$, where P is defined in* (18).

Proof. Computation using (18) and (33) shows that $P(f'_\pm) = \mp A + B$, where

$$A = \gamma^2\Big[2\big((\alpha+\beta y+\gamma xy \mp \gamma^2 x)^2+y^2+a^2\big)^{-1/2}+4\Big] \quad \text{and}$$
$$B = \gamma y\Big[\big((\alpha+\beta y+\gamma xy\mp \gamma^2 x)^2+y^2+a^2\big)^{-1/2}-\big((\alpha+\beta y)^2+y^2+a^2\big)^{-1/2}\Big].$$

Calculations similar to the proof of Proposition 4.6 using (32) and (34) show that $A \geqslant |B|$, which gives $P(f'_+) \leqslant 0$ and $P(f'_-) \geqslant 0$. □

As for Corollary 4.7, we deduce an a priori C^0 estimate for ϕ':

Corollary 4.9. *In Definition 4.4, $|\phi'| \leqslant \gamma^2(1-x^2-y^2)$ on D. Hence*

$$\|\phi'\|_{C^0} \leqslant \gamma^2, \quad \text{and} \quad \left|\frac{\partial\phi'}{\partial x}\right|, \left|\frac{\partial\phi'}{\partial y}\right| \leqslant 2\gamma^2 \quad \text{on } \partial D.$$

4.4 A priori C^0 estimates for ψ and ψ'

For our next three results we work in the situation of Definition 4.3. As $\psi = \frac{\partial\phi}{\partial x}$, the second inequality of (35) shows that $|\psi| \leqslant 16s^{-1}\gamma^2$ on ∂D. We shall use this to construct *super*- and *subsolutions* $v_\pm$ for v, and hence derive an *a priori C^0 estimate* for ψ.

Proposition 4.10. *Define functions $v_\pm \in C^\infty(D)$ by*

$$v_\pm = \alpha + \beta y + \gamma x \pm s^{-1}\gamma^2(20-4x^2-4y^2). \tag{36}$$

Then $Q(v_+) \leqslant 0$ and $Q(v_-) \geqslant 0$, where Q is defined in (21).

Proof. From (21) we find that $Q(v_\pm) = \mp A - B$, where

$$A = 8s^{-1}\gamma^2\Big[\big((\alpha+\beta y+\gamma x \pm s^{-1}\gamma^2(20-4x^2-4y^2))^2+y^2+a^2\big)^{-1/2}+2\Big], \tag{37}$$

$$B = \frac{\big(\gamma \mp 8s^{-1}\gamma^2 x\big)^2\big(\alpha+\beta y+\gamma x \pm s^{-1}\gamma^2(20-4x^2-4y^2)\big)}{\big((\alpha+\beta y+\gamma x \pm s^{-1}\gamma^2(20-4x^2-4y^2))^2+y^2+a^2\big)^{3/2}}. \tag{38}$$

We shall show that $A \geqslant |B|$, which gives $Q(v_+) \leqslant 0$ and $Q(v_-) \geqslant 0$.

Define $w = \gamma x \pm s^{-1}\gamma^2(20-4x^2-4y^2)$. Then as $|x| \leqslant 1$, $x^2+y^2 \leqslant 1$ and $\gamma \leqslant \frac{1}{20}s$ we see that $|w| \leqslant \frac{1}{10}s$. Thus Lemma 4.5 implies that

$$(\alpha+\beta y+\gamma x \pm s^{-1}\gamma^2(20-4x^2-4y^2))^2+y^2+a^2 \geqslant \tfrac{1}{4}(y^2+s^2) \geqslant \tfrac{1}{4}s^2.$$

Raising this to the power $-\frac{1}{2}$ and using (38) and the inequalities

$$\big|\alpha+\beta y+\gamma x \pm s^{-1}\gamma^2(20-4x^2-4y^2)\big| \leqslant \big((\alpha+\beta y+\gamma x \pm s^{-1}\gamma^2(20-4x^2-4y^2))^2+y^2+a^2\big)^{1/2}$$

and $|\gamma \mp 8s^{-1}\gamma^2 x| \leqslant 2|\gamma|$ implies that

$$|B| \leqslant 4\gamma^2 \cdot 2s^{-1} \cdot \big((\alpha+\beta y+\gamma x \pm s^{-1}\gamma^2(20-4x^2-4y^2))^2+y^2+a^2\big)^{-1/2}.$$

Comparing this with (37) gives $A \geqslant |B|$, as we have to prove. □

Now $v = \alpha + \beta x + \gamma y + \frac{\partial\phi}{\partial x}$ and $|\frac{\partial\phi}{\partial x}| \leqslant 16s^{-1}\gamma^2$ on ∂D by Corollary 4.7. Thus (36) gives $v_- \leqslant v \leqslant v_+$ on ∂D. But as v satisfies (21), and $Q(v_+) \leqslant 0$ and $Q(v_-) \geqslant 0$ from above, [6, Prop. 8.5] implies that $v_- \leqslant v \leqslant v_+$. Subtracting $\alpha + \beta y + \gamma x$ from each side then yields

Corollary 4.11. *In Definition 4.3,* $|\frac{\partial \phi}{\partial x}| = |\psi| \leqslant s^{-1}\gamma^2(20-4x^2-4y^2)$ *on* D. *Hence* $\|\psi\|_{C^0} \leqslant 20s^{-1}\gamma^2$.

As $|\gamma| \leqslant \frac{1}{20}s$ this gives $|\psi| \leqslant |\gamma|$, so that $|\gamma x + \psi| \leqslant 2|\gamma| \leqslant \frac{1}{10}s$ as $|x| \leqslant 1$. Thus, applying Lemma 4.5 with $w = \gamma x + \psi$ shows that

Corollary 4.12. $\frac{1}{4}(y^2+s^2) \leqslant (\alpha+\beta y+\gamma x+\psi)^2+y^2+a^2 \leqslant 4(y^2+s^2)$ *on* D.

This gives an a priori bound on the factor in front of $\frac{\partial^2\psi}{\partial x^2}$ in (31), showing that (31) is *uniformly elliptic*. The analogues of Proposition 4.10 and Corollary 4.11 for Definition 4.4 are:

Proposition 4.13. *Define* $v'_\pm \in C^\infty(D)$ *by* $v'_\pm = \alpha+\beta y+\gamma xy \pm \gamma^2(\frac{9}{4}-\frac{1}{4}y^2)$. *Then* $Q(v'_+) \leqslant 0$ *and* $Q(v'_-) \geqslant 0$, *where* Q *is defined in* (21).

Proof. From (21) we find that $Q(v'_\pm) = \mp A - B$, where $A = \gamma^2$ and

$$B = \frac{\gamma^2 y^2\bigl(\alpha+\beta y+\gamma xy \pm \gamma^2(\frac{9}{4}-\frac{1}{4}y^2)\bigr)}{\bigl((\alpha+\beta y+\gamma xy \pm \gamma^2(\frac{9}{4}-\frac{1}{4}y^2))^2+y^2+a^2\bigr)^{3/2}}.$$

As $|y|, |\alpha+\beta y+\gamma xy \pm \gamma^2(\frac{9}{4}-\frac{1}{4}y^2)| \leqslant ((\alpha+\beta y+\gamma xy \pm \gamma^2(\frac{9}{4}-\frac{1}{4}y^2))^2+y^2+a^2)^{1/2}$ we see that $|B| \leqslant \gamma^2 = A$, and so $Q(v'_+) \leqslant 0$ and $Q(v'_-) \geqslant 0$. □

Corollary 4.14. *In Definition 4.4,* $\|\frac{\partial \phi'}{\partial x}\|_{C^0} = \|\psi'\|_{C^0} \leqslant \frac{9}{4}\gamma^2$.

4.5 Estimates for higher derivatives of ψ and ψ'

For the next part of the proof we make use of an *overall symmetry* of equations (15)–(22), arising geometrically from *dilations* on $\mathbb{C}^3$. In the situation of Proposition 3.2, suppose $a \neq 0$ and u, v satisfy (17), so that N defined in (15) is an SL 3-fold. Let $r > 0$, and define $\tilde{N} = r^{-1}N$. Since $\tilde{N}$ is a dilation of N it is also an SL 3-fold, and also U(1)-invariant.

Now $\tilde{N}$ is of the form (15) with u, v and a replaced by

$$\tilde{u}(x,y) = r^{-1}u(rx, r^2y), \quad \tilde{v}(x,y) = r^{-2}v(rx, r^2y) \quad \text{and} \quad \tilde{a} = r^{-2}a.$$

Thus $\tilde{u}, \tilde{v}$ and $\tilde{a}$ also satisfy (17), and $\tilde{v}, \tilde{a}$ satisfy (21). If f is a potential for u, v as in Proposition 3.3, so that f, a satisfy (18), then $\tilde{f}(x,y) = r^{-3}f(rx, r^2y)$ is a potential for $\tilde{u}, \tilde{v}$, and $\tilde{f}, \tilde{a}$ satisfy (18).

The fact that x scales by r and y by r^2 is one reason why *x and y derivatives behave differently in this problem*, so that for instance in §6.5 we obtain L^p estimates for $\frac{\partial u}{\partial x}$ and L^q estimates for $\frac{\partial u}{\partial y}$ with p, q in different ranges.

Here is how we use the rescaling idea in this section. The function ψ of §4.2 transforms like v, so its rescaled version is $\tilde{\psi}(x,y) = r^{-2}\psi(rx, r^2y)$, which satisfies a rescaled version of (31). It turns out that on a ball away from the x axis, this rescaled equation is *uniformly elliptic uniformly in* $r \in (0,1]$. Thus we can use elliptic regularity results to estimate $\tilde{\psi}$ and its derivatives away from the x-axis independently of $r \in (0,1]$. Transforming back gives estimates of ψ and its derivatives near the x-axis, in terms of powers of y.

We shall work in the situation of Definition 4.3 until Theorem 4.21.

Definition 4.15. In Definition 4.3, let $\sqrt{s} \leqslant r \leqslant 1$. Write

$$\tilde{D} = \{(x,y) \in \mathbb{R}^2 : r^2x^2 + r^4y^2 \leqslant 1\},$$

and define $\tilde{\psi} \in C^\infty(\tilde{D})$ by $\tilde{\psi}(x,y) = r^{-2}\psi(rx, r^2y)$.

From Corollaries 4.11 and 4.12 and equation (31) we deduce:

Proposition 4.16. *In the situation above, $\tilde{\psi}$ satisfies $\|\tilde{\psi}\|_{C^0} \leqslant 20r^{-2}s^{-1}\gamma^2$,*

$$\begin{aligned}&\Big(\big(r^{-2}\alpha + \beta y + r^{-1}\gamma x + \tilde{\psi}\big)^2 + y^2 + r^{-4}a^2\Big)^{-1/2}\frac{\partial^2\tilde{\psi}}{\partial x^2} + 2\,\frac{\partial^2\tilde{\psi}}{\partial y^2} = \\ &\frac{\big(r^{-2}\alpha + \beta y + r^{-1}\gamma x + \tilde{\psi}\big)\big(\frac{\partial\tilde{\psi}}{\partial x} + r^{-1}\gamma\big)^2}{\big((r^{-2}\alpha + \beta y + r^{-1}\gamma x + \tilde{\psi})^2 + y^2 + r^{-4}a^2\big)^{3/2}}, \quad \textit{and}\end{aligned} \tag{39}$$

$$\tfrac{1}{4}\big(y^2 + r^{-4}s^2\big) \leqslant \big(r^{-2}\alpha + \beta y + r^{-1}\gamma x + \tilde{\psi}\big)^2 + y^2 + r^{-4}a^2 \leqslant 4\big(y^2 + r^{-4}s^2\big). \tag{40}$$

Now (39) involves constants $r^{-2}\alpha$, β, $r^{-1}\gamma$ and $r^{-2}a$ which are all uniformly bounded independently of r, as

$$|r^{-2}\alpha| \leqslant 1, \quad |\beta| \leqslant 1, \quad |r^{-1}\gamma| \leqslant \frac{1}{20} \quad \text{and} \quad |r^{-2}a| \leqslant 1.$$

Because of this we will be able to treat (39) as a quasilinear elliptic equation satisfying elliptic regularity bounds that are *independent* of $a, \alpha, \beta, \gamma, r$ and s, no matter how small r and s are.

In the next few results we shall construct a priori bounds for the derivatives of $\tilde{\psi}$ in the interior of $\tilde{D}$. We do this by using interior elliptic regularity results to bound Hölder norms of $\tilde{\phi}$ and $\tilde{\psi}$ on a series of small balls E_n in $\tilde{D}$.

Definition 4.17. Let $\overline{B}_R(x,y)$ be the closed ball of radius R about (x,y) in $\mathbb{R}^2$. Suppose $x' \in \mathbb{R}$ and $y' = \pm 1$ are such that $\overline{B}_{1/2}(x',y') \subset \tilde{D}$. Define a decreasing series of balls $E_2 \supset E_3 \supset \cdots$ in $\tilde{D}$ by $E_n = \overline{B}_{1/n}(x',y')$. Fix $\epsilon \in (0,1)$.

Proposition 4.18. *There exists $A_n > 0$ independent of $a, \alpha, \beta, \gamma, r, s$ such that $\big\|\tilde{\psi}|_{E_{n+3}}\big\|_{C^{n,\epsilon}} \leqslant A_n r^{-2}s^{-1}\gamma^2$ for each $n = 0, 1, 2, 3, \ldots$.*

Proof. Let us rewrite equation (39) in the form

$$\begin{aligned}\frac{\partial}{\partial x}\Big[\big((r^{-2}\alpha + \beta y + r^{-1}\gamma x + \tilde{\psi})^2 + y^2 + r^{-4}a^2\big)^{-1/2}\frac{\partial\tilde{\psi}}{\partial x}\Big] + \frac{\partial}{\partial y}\Big[2\,\frac{\partial\tilde{\psi}}{\partial y}\Big] \\ - \frac{r^{-1}\gamma\big(r^{-2}\alpha + \beta y + r^{-1}\gamma x + \tilde{\psi}\big)}{\big((r^{-2}\alpha + \beta y + r^{-1}\gamma x + \tilde{\psi})^2 + y^2 + r^{-4}a^2\big)^{3/2}}\,\frac{\partial\tilde{\psi}}{\partial x} = \\ \frac{r^{-2}\gamma^2}{\big((r^{-2}\alpha + \beta y + r^{-1}\gamma x + \tilde{\psi})^2 + y^2 + r^{-4}a^2\big)^{3/2}}.\end{aligned}$$

Regard this as a linear elliptic equation $Pu = f$ on $\tilde\psi$, where

$$\big(Pu\big)(x) = \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\Big[a^{ij}(x)\frac{\partial u}{\partial x_j}\Big] + \sum_{i=1}^{2} b^i(x)\frac{\partial u}{\partial x_i} + c(x)u(x),$$

so that $u = \tilde\psi$, $f = r^{-2}\gamma^2\big((r^{-2}\alpha + \beta y + r^{-1}\gamma x + \tilde\psi)^2 + y^2 + r^{-4}a^2\big)^{-3/2}$,

$$\begin{gathered} a^{11} = \big(\big(r^{-2}\alpha + \beta y + r^{-1}\gamma x + \tilde\psi\big)^2 + y^2 + r^{-4}a^2\big)^{-1/2}, \\ a^{12} = a^{21} = 0, \quad a^{22} = 2, \end{gathered} \tag{41}$$

$$b^1 = -\frac{r^{-1}\gamma\big(r^{-2}\alpha + \beta y + r^{-1}\gamma x + \tilde\psi\big)}{\big((r^{-2}\alpha+\beta y+r^{-1}\gamma x+\tilde\psi)^2+y^2+r^{-4}a^2\big)^{3/2}}, \text{ and } b^2 = c = 0. \tag{42}$$

To prove the proposition we shall apply three interior elliptic regularity results of Gilbarg and Trudinger [2] for second order linear elliptic operators with principal part in divergence form, to prove the cases $n = 0$, $n = 1$ and $n \geqslant 2$ respectively.

From [2, Th. 8.24, p. 202], if $\lambda, \Lambda_0, \nu_0 > 0$ and $\epsilon \in (0,1)$ with

$$\textstyle\sum_{i,j=1}^{2} a^{ij}\xi_i\xi_j \geqslant \lambda|\xi|^2 \quad \text{in } E_2 \text{ for all } \xi = (\xi_1,\xi_2) \in \mathbb{R}^2, \tag{43}$$

$$\big\|a^{ij}|_{E_2}\big\|_{C^0} \leqslant \Lambda_0, \quad \big\|b^i|_{E_2}\big\|_{C^0} \leqslant \nu_0 \quad \text{for all } i,j, \text{ and} \quad \big\|c|_{E_2}\big\|_{C^0} \leqslant \nu_0, \tag{44}$$

then there exists $C > 0$ depending only on $E_2, E_3, \lambda, \Lambda_0, \nu_0$ and ϵ such that if $u \in C^2(E_2)$ and $Pu = f$, then $\|u|_{E_3}\|_{C^{0,\epsilon}} \leqslant C\big(\|u\|_{C^0} + \|f\|_{C^0}\big)$.

As $\frac12 \leqslant |y| \leqslant \frac32$ for $(x,y) \in E_2$ and $0 \leqslant r^{-4}s^2 \leqslant 1$, by (40) we have

$$\tfrac{1}{16} \leqslant \big(r^{-2}\alpha+\beta y+r^{-1}\gamma x+\tilde\psi\big)^2+y^2+r^{-4}a^2 \leqslant 13 \quad \text{on } E_2. \tag{45}$$

Using (41)–(42) and (45) we can find $\lambda, \Lambda_0, \nu_0$ independent of $a, \alpha, \beta, \gamma, r, s$ such that (43)–(44) hold.

Thus there exists $C_0 > 0$ independent of $a, \alpha, \beta, \gamma, r, s$ such that

$$\begin{aligned} \big\|\tilde\psi|_{E_3}\big\|_{C^{0,\epsilon}} &\leqslant C_0\big(\big\|\tilde\psi|_{E_2}\big\|_{C^0} + \big\|f|_{E_2}\big\|_{C^0}\big) \\ &\leqslant C_0(20r^{-2}s^{-1}\gamma^2 + 64r^{-2}\gamma^2) \leqslant A_0 r^{-2}s^{-2}\gamma^2, \end{aligned}$$

setting $A_0 = 84C_0$, using Corollary 4.11, $0 < s \leqslant 1$ and $\|f|_{E_2}\|_{C^0} \leqslant 64r^{-2}\gamma^2$, which follows from (45) and the definition of f. This proves the case $n = 0$.

Combining the case $n = 0$ with (41) and $\|a^{ij}|_{E_2}\|_{C^0} \leqslant 4$ we can find $\Lambda_1 > 0$ depending on A_0 such that $\big\|a^{ij}|_{E_3}\big\|_{C^{0,\epsilon}} \leqslant \Lambda_1$. Applying [2, Th. 8.32, p. 210] shows that there exists $C_1 > 0$ depending only on $E_3, E_4, \lambda, \Lambda_1, \nu_0$ and ϵ such that if $u \in C^2(E_3)$ and $Pu = f$, then $\|u|_{E_4}\|_{C^{1,\epsilon}} \leqslant C_1\big(\|u\|_{C^0} + \|f\|_{C^0}\big)$. The argument above then shows that the case $n = 1$ holds with $A_1 = 84C_1$.

The case $n \geqslant 2$ is proved using the *Schauder interior estimates* [2, Th.s 6.2 & 6.17], by a technique known as *bootstrapping*. Suppose by induction that the proposition holds for $n = k - 1$, for $k \geqslant 2$. Then

$$\big\|a^{ij}|_{E_{k+2}}\big\|_{C^{k-1,\epsilon}} \leqslant \Lambda_k, \quad \big\|b^i|_{E_{k+2}}\big\|_{C^{k-2,\epsilon}} \leqslant \nu_k \quad \text{and} \quad \big\|c|_{E_{k+2}}\big\|_{C^{k-2,\epsilon}} \leqslant \nu_k,$$

for some Λ_k, ν_k depending on k, A_{k-1}, and all i, j. Therefore [2, Th.s 6.2 & 6.17] gives $C_k > 0$ depending on $k, E_{k+2}, E_{k+3}, \lambda, \Lambda_k, \nu_k$ such that if $u \in C^2(E_{k+2})$, $f \in C^{k-2,\epsilon}(E_{k+2})$ and $Pu = f$, then $\|u|_{E_{k+3}}\|_{C^{k,\epsilon}} \leqslant C_k\bigl(\|u\|_{C^0}+\|f\|_{C^{k-2,\epsilon}}\bigr)$.

Using (45), the definition of f and the case $n = k-1$ we can find $C'_k > 0$ depending only on k, ϵ and A_{k-1} such that $\bigl\|f|_{E_{k+2}}\bigr\|_{C^{k-2,\epsilon}} \leqslant C'_k r^{-2}\gamma^2$. The case $n = k$ of the proposition then follows with $A_k = C_k(20 + C'_k)$. Hence by induction, the proof is complete. □

As $\bigl|\partial^n\tilde\psi(x', y')\bigr| \leqslant \|\tilde\psi|_{E_{n+3}}\|_{C^{n,\epsilon}}$, we deduce:

Corollary 4.19. *There exist constants A_n for $n \geqslant 0$ independent of $a, \alpha, \beta, \gamma, r$ and s, such that if $x' \in \mathbb{R}$ and $y' = \pm 1$ with $\overline{B}_{1/2}(x', y') \subset \tilde D$, then*

$$\Bigl|\frac{\partial^{j+k}\tilde\psi}{\partial x^j \partial y^k}(x', y')\Bigr| \leqslant A_{j+k} r^{-2} s^{-1}\gamma^2 \quad \textit{for all } j, k \geqslant 0. \tag{46}$$

Here the A_n are given in Proposition 4.18. Now the only reason for setting $y' = \pm 1$ above was to be able to prove (45), using (40) and the inequality $\frac12 \leqslant |y| \leqslant \frac32$ on E_2. In the special case $r = \sqrt{s}$, the terms $r^{-4}s^2$ in (40) are 1, and we then only need $|y| \leqslant \frac32$ on E_2 to prove (45). Thus, when $r = \sqrt{s}$ the proofs above are valid for $|y'| \leqslant 1$ rather than just $|y'| = 1$, and we get:

Corollary 4.20. *Let $r = \sqrt{s}$ in Definition 4.15. If $x' \in \mathbb{R}$ and $|y'| \leqslant 1$ with $\overline{B}_{1/2}(x', y') \subset \tilde D$, then*

$$\Bigl|\frac{\partial^{j+k}\tilde\psi}{\partial x^j \partial y^k}(x', y')\Bigr| \leqslant A_{j+k} s^{-2}\gamma^2 \quad \textit{for all } j, k \geqslant 0, \tag{47}$$

where the A_n are as in Corollary 4.19.

We shall use the last two corollaries to estimate the derivatives of ψ. As by definition $\tilde\psi(x, y) = r^{-2}\psi(rx, r^2y)$, we see that for $(x, y) \in D$,

$$\frac{\partial^{j+k}\psi}{\partial x^j \partial y^k}(x, y) = r^{2-j-2k}\,\frac{\partial^{j+k}\tilde\psi}{\partial x^j \partial y^k}(x', y'), \quad \text{where } (x', y') = (r^{-1}x, r^{-2}y).$$

Put $r = |y|^{1/2}$ if $|y| \geqslant s$ and $r = \sqrt{s}$ if $|y| < s$. It turns out that $x^2 + y^2 \leqslant \frac14$ is sufficient to ensure that $\overline{B}_{1/2}(x', y') \subset \tilde D$. Thus, Corollaries 4.19 and 4.20 give

Theorem 4.21. *Suppose $(x, y) \in D$ with $x^2 + y^2 \leqslant \frac14$. Then for all $j, k \geqslant 0$,*

$$\Bigl|\frac{\partial^{j+k}\psi}{\partial x^j \partial y^k}(x, y)\Bigr| \leqslant \begin{cases} A_{j+k}|y|^{-j/2-k}s^{-1}\gamma^2, & |y| \geqslant s,\\ A_{j+k}s^{-1-j/2-k}\gamma^2, & |y| < s,\end{cases} \tag{48}$$

where $A_0, A_1, \ldots$ are positive constants independent of $x, y, a, \alpha, \beta, \gamma$ and s.

This is a measure of how closely the solution v of (21) defined in Definition 4.3 approximates $\alpha+\beta y+\gamma x$ in the interior of D. With some more work, the interior estimates (48) can be extended to the whole of D, using elliptic regularity results for regions with boundary, but we will not need this.

Extending the material above to the functions of Definition 4.4 is straightforward, so we just indicate the main differences. In Definition 4.15 we replace the assumption $\sqrt{s}\leqslant r\leqslant 1$ by $\max\bigl(|\gamma|,\sqrt{s}\,\bigr)\leqslant r\leqslant 1$. We then follow through to Corollary 4.19 with little essential change.

To prove an analogue of Corollary 4.20 we need an additional assumption, that $|\gamma|\leqslant\frac{1}{10}s$. This is because for Definition 4.3, when $r=\sqrt{s}$ the first inequality of (40) gives $\frac{1}{4}\leqslant(r^{-2}\alpha+\beta y+r^{-1}\gamma x+\tilde\psi)^2+y^2+r^{-4}a^2$, and we use this to bound the coefficient of $\frac{\partial^2\tilde\psi}{\partial x^2}$ in (39), and so to show that (39) is uniformly elliptic. However, for Definition 4.4 we need an extra condition like $|\gamma|\leqslant\frac{1}{10}s$ to show that the analogue of (39) is uniformly elliptic when y is small.

The analogue of Theorem 4.21 we get is:

Theorem 4.22. *Let $(x,y)\in D$ with $x^2+y^2\leqslant\frac{1}{4}$, $|y|\geqslant\gamma^2$ and $|y|\geqslant s$. Then*

$$\Bigl|\frac{\partial^{j+k}\psi'}{\partial x^j\partial y^k}(x,y)\Bigr|\leqslant A_{j+k}|y|^{-j/2-k}\gamma^2\quad\textit{for all } j,k\geqslant 0.$$

If $|\gamma|\leqslant\frac{1}{10}s$, then whenever $(x,y)\in D$ with $x^2+y^2\leqslant\frac{1}{4}$ and $|y|\leqslant s$, we have

$$\Bigl|\frac{\partial^{j+k}\psi'}{\partial x^j\partial y^k}(x,y)\Bigr|\leqslant A_{j+k}s^{-j/2-k}\gamma^2\quad\textit{for all } j,k\geqslant 0.$$

Here $A_0,A_1,\ldots$ are positive constants independent of x,y,a,α,β and γ.

4.6 Solutions of (21) with $v,\frac{\partial v}{\partial x},\frac{\partial v}{\partial y}$ prescribed at (x_0,y_0)

To prove a priori bounds for derivatives of solutions of (17) in §5, we will need to find examples of solutions u,v of (17) in D such that $u,v,\frac{\partial v}{\partial x},\frac{\partial v}{\partial y}$ take prescribed values u_0,v_0,p_0,q_0 at a given point (x_0,y_0) in D. As we are free to add a constant to u, it is enough to consider only v, regarded as a solution of (21), and ensure that $v,\frac{\partial v}{\partial x},\frac{\partial v}{\partial y}$ take prescribed values.

Theorem 4.23. *Let a,x_0,y_0,v_0,p_0 and q_0 be real numbers. Define*

$$\begin{gathered}\alpha_0=v_0-x_0p_0-y_0q_0,\quad \beta_0=q_0,\quad \gamma_0=p_0,\quad s_0=\sqrt{\alpha_0^2+a^2}\\ \textit{and}\quad J=\min\bigl(\tfrac{1}{40},\tfrac12 A_0^{-1},\tfrac{1}{16}A_1^{-1}\bigr),\end{gathered}\tag{49}$$

where $A_0,A_1>0$ are as in Theorem 4.21. Suppose

$$a\neq 0,\ x_0^2+y_0^2\leqslant\tfrac14,\ s_0^2\leqslant\tfrac12,\ |\beta_0|\leqslant\tfrac45\ \textit{and}\ |\gamma_0|\leqslant J\max(s_0|y_0|^{1/2},s_0^{3/2}).\tag{50}$$

Then there exist $\alpha,\beta,\gamma\in\mathbb{R}$ satisfying (27) *and*

$$|\alpha-\alpha_0|\leqslant\tfrac14 s_0,\quad |\beta-\beta_0|\leqslant\tfrac14 s_0\quad\textit{and}\quad |\gamma-\gamma_0|\leqslant|\gamma_0|,\tag{51}$$

such that the solution v of (21) *constructed in Definition 4.3 using a, α, β, γ satisfies*

$$v(x_0, y_0) = v_0, \quad \frac{\partial v}{\partial x}(x_0, y_0) = p_0 \quad \textit{and} \quad \frac{\partial v}{\partial y}(x_0, y_0) = q_0. \tag{52}$$

Proof. Let α, β, γ satisfy (51), and define $s = \sqrt{\alpha^2 + a^2}$. Using $|\alpha - \alpha_0| \leqslant \frac{1}{4}s_0$ one can easily show that

$$\tfrac{1}{2}s_0^2 \leqslant s^2 \leqslant 2s_0^2. \tag{53}$$

As $s_0^2 \leqslant \frac{1}{2}$ by (50) this gives $0 < s \leqslant 1$. The other inequalities $|\beta| \leqslant 1$ and $|\gamma| \leqslant \frac{1}{20}s$ in (27) also follow from (50). Thus (27) holds, and Definition 4.3 gives $v, \psi \in C^\infty(D)$ with $v = \alpha + \beta y + \gamma x + \psi$, such that v satisfies (21) and $\psi = 0$ on ∂D.

We shall show that there exist α, β, γ satisfying (51) for which v satisfies (52). Using $v = \alpha + \beta y + \gamma x + \psi$ and (49) we find that (52) is equivalent to

$$F_1(\alpha, \beta, \gamma) = (\alpha - \alpha_0) + y_0(\beta - \beta_0) + x_0(\gamma - \gamma_0) + \psi(x_0, y_0) = 0, \tag{54}$$

$$F_2(\alpha, \beta, \gamma) = (\beta - \beta_0) + \frac{\partial \psi}{\partial y}(x_0, y_0) = 0, \tag{55}$$

$$F_3(\alpha, \beta, \gamma) = (\gamma - \gamma_0) + \frac{\partial \psi}{\partial x}(x_0, y_0) = 0. \tag{56}$$

Define $\alpha_\pm = \alpha_0 \pm \frac{1}{4}s_0$, $\beta_\pm = \beta_0 \pm \frac{1}{4}s_0$ and $\gamma_\pm = \gamma_0 \pm |\gamma_0|$. Then (51) is equivalent to $\alpha_- \leqslant \alpha \leqslant \alpha_+$, $\beta_- \leqslant \beta \leqslant \beta_+$ and $\gamma_- \leqslant \gamma \leqslant \gamma_+$. Thus, (54)–(56) define functions $F_1, F_2, F_3 : [\alpha_-, \alpha_+] \times [\beta_-, \beta_+] \times [\gamma_-, \gamma_+] \to \mathbb{R}$. Using [6, Th. 7.7] one can show that the F_j are *continuous* functions.

Proposition 4.24. *Suppose $\gamma_0 \neq 0$. For all α, β, γ satisfying* (51) *we have*

$$\begin{gathered} F_1(\alpha_-, \beta, \gamma) < 0 < F_1(\alpha_+, \beta, \gamma), \quad F_2(\alpha, \beta_-, \gamma) < 0 < F_2(\alpha, \beta_+, \gamma), \\ \textit{and} \quad F_3(\alpha, \beta, \gamma_-) < 0 < F_3(\alpha, \beta, \gamma_+). \end{gathered} \tag{57}$$

Proof. To prove the first pair of inequalities, we shall show that for $\alpha = \alpha_\pm, \beta, \gamma$ satisfying (51), we have

$$|\alpha_\pm - \alpha_0| > |y_0||\beta - \beta_0| + |x_0||\gamma - \gamma_0| + \bigl|\psi(x_0, y_0)\bigr|. \tag{58}$$

Thus by (54), $F_1(\alpha_\pm, \beta, \gamma)$ has the same sign as $\alpha_\pm - \alpha_0$, and the first part of (57) follows. First suppose $|y_0| \geqslant s_0$. Then from (50)–(51) we have

$$|\alpha_\pm - \alpha_0| = \tfrac{1}{4}s_0, \quad |\beta - \beta_0| \leqslant \tfrac{1}{4}s_0 \quad \text{and} \quad |\gamma - \gamma_0| \leqslant |\gamma_0| \leqslant Js_0|y_0|^{1/2},$$

and Theorem 4.21 gives

$$\bigl|\psi(x_0, y_0)\bigr| \leqslant A_0 s^{-1}\gamma^2 \leqslant A_0(\tfrac{1}{2}s_0)^{-1}\bigl(2Js_0|y_0|^{1/2}\bigr) = 8A_0J^2s_0|y_0|,$$

using $\frac{1}{2}s_0 \leqslant s$ and $|\gamma| \leqslant 2Js_0|y_0|^{1/2}$. Thus (58) holds if

$$\tfrac{1}{4}s_0 > \tfrac{1}{4}s_0|y_0| + Js_0|x_0||y_0|^{1/2} + 8A_0J^2s_0|y_0|,$$

which follows from $s_0 > 0$, $|x_0| \leqslant \frac{1}{2}$, $|y_0| \leqslant \frac{1}{2}$, $J \leqslant \frac{1}{40}$ and $J \leqslant \frac{1}{2}A_0^{-1}$. The other four inequalities are proved in a similar way. □

The reason for supposing $\gamma_0 \neq 0$ is to get strict inequalities in the third part of (57). We can now finish the proof of Theorem 4.23. If $\gamma_0 = 0$ then $\alpha = \alpha_0$, $\beta = \beta_0$ and $\gamma = \gamma_0 = 0$ satisfy the conditions of the theorem, as then $v = \alpha + \beta y + \gamma x = \alpha_0 + \beta_0 y$ is an *exact* solution of (21), and $\psi \equiv 0$, so (54)–(56) hold. So suppose $\gamma_0 \neq 0$. Write $B = [\alpha_-, \alpha_+] \times [\beta_-, \beta_+] \times [\gamma_-, \gamma_+]$, and consider the map $\mathbf{F} = (F_1, F_2, F_3) : B \to \mathbb{R}^3$. By Proposition 4.24, $\mathbf{F}$ maps ∂B to $\mathbb{R}^3 \setminus \{0\}$. Furthermore, both ∂B and $\mathbb{R}^3 \setminus \{0\}$ are homotopic to $\mathcal{S}^2$, and one can show from the proposition that $\mathbf{F}_* : H_2(\partial B, \mathbb{Z}) \to H_2(\mathbb{R}^3 \setminus \{0\}, \mathbb{Z})$ is the identity map $\mathbb{Z} \to \mathbb{Z}$.

Suppose $\mathbf{F}$ maps $B \to \mathbb{R}^3 \setminus \{0\}$. Then $F(\partial B)$ is homologous to zero in $\mathbb{R}^3 \setminus \{0\}$, as it bounds $F(B)$. But $F_*([\partial B])$ generates $H_2(\mathbb{R}^3 \setminus \{0\}, \mathbb{Z})$ and so is nonzero, a contradiction. Thus $\mathbf{F}$ cannot map $B \to \mathbb{R}^3 \setminus \{0\}$, and there exists $(\alpha, \beta, \gamma) \in B^\circ$ with $\mathbf{F}(\alpha, \beta, \gamma) = 0$. From above this is equivalent to (52), and so α, β, γ satisfy the conditions in Theorem 4.23, completing the proof. □

Here is the analogue of Theorem 4.23 for Definition 4.4, proved similarly.

Theorem 4.25. *Let a, x_0, y_0, v_0, p_0 and q_0 be real numbers with $y_0 \neq 0$. Define*

$$\alpha_0 = v_0 - 2x_0p_0 - y_0q_0, \quad \beta_0 = q_0 - \frac{x_0p_0}{y_0}, \quad \gamma_0 = \frac{p_0}{y_0}$$
$$\text{and} \quad J = \min\left(\tfrac{1}{80}, \tfrac{1}{4}A_0^{-1/2}, \tfrac{1}{4}A_1^{-1}\right),$$

where $A_0, A_1 > 0$ are as in Theorem 4.22. Suppose

$$a \neq 0, \;\; x_0^2 + y_0^2 \leqslant \tfrac{1}{4}, \;\; |\alpha_0| \leqslant \tfrac{1}{2}, \;\; |\beta_0| \leqslant \tfrac{1}{80} \;\; \text{and} \;\; |\gamma_0| \leqslant J\,|y_0|^{3/2}.$$

Then there exist $\alpha, \beta, \gamma \in \mathbb{R}$ satisfying (32) *and*

$$|\alpha - \alpha_0| \leqslant \tfrac{1}{2}, \quad |\beta - \beta_0| \leqslant \tfrac{1}{80} \quad \text{and} \quad |\gamma - \gamma_0| \leqslant |\gamma_0|,$$

such that the solution v' of (21) *constructed in Definition 4.4 using a, α, β, γ satisfies*

$$v'(x_0, y_0) = v_0, \quad \frac{\partial v'}{\partial x}(x_0, y_0) = p_0 \quad \text{and} \quad \frac{\partial v'}{\partial y}(x_0, y_0) = q_0.$$

4.7 Proof of Theorems 4.1 and 4.2

We now prove Theorem 4.1, by the rescaling method of §4.5. Define

$$r=\min\bigl(1,(3L)^{-1},(3K)^{-1/2}\bigr),\quad \hat a=r^2a,\quad \hat v_0=r^2v_0,$$
$$\hat x_0=rx_0,\quad \hat y_0=r^2y_0,\quad \hat p_0=rp_0\quad\text{and}\quad \hat q_0=q_0.$$

Let J be as in (49), and let $A,B,C>0$ be chosen such that

$$A\leqslant \tfrac12 Jr^2,\quad A\leqslant\frac{1}{8L^{3/2}},\quad A\leqslant\frac{1}{8L(M^2+K^2)^{1/4}},\quad B\leqslant\tfrac18\quad\text{and}\quad C\leqslant\tfrac45.$$

Define $\hat\alpha_0,\hat\beta_0,\hat\gamma_0$ and $\hat s_0$ as in (49), using $\hat a,\hat v_0,\hat p_0,\hat q_0,\hat x_0$ and $\hat y_0$. We will show that (23) implies that (50) holds for $\hat a,\hat x_0,\hat y_0,\hat s_0,\hat\beta_0$ and $\hat\gamma_0$.

As $r,a\neq 0$ we have $\hat a\neq 0$, and $r\leqslant(3L)^{-1}$, $r\leqslant 1$ and $x_0^2+y_0^2\leqslant L^2$ imply that $\hat x_0^2+\hat y_0^2\leqslant\frac14$. So the first two inequalities of (50) hold. Now $\hat\alpha_0=r^2(v_0-x_0p_0-y_0q_0)$. Using $|a|\leqslant K$, $|x_0|\leqslant L$, $|y_0|\leqslant L$, $|v_0|\leqslant M$ and the first inequality of (23), we find that

$$|x_0p_0|\leqslant A\max\bigl(L^{3/2},L(K^2+M^2)^{1/4}\bigr)(v_0^2+a^2)^{1/2}\leqslant\tfrac18(v_0^2+a^2)^{1/2},$$

as $A\leqslant 1/8L^{3/2}$ and $A\leqslant 1/8L(M^2+K^2)^{1/4}$.

Also $|y_0q_0|\leqslant\frac18(v_0^2+a^2)^{1/2}$ from the second inequality of (23) and $B\leqslant\frac18$. Thus $|x_0p_0+y_0q_0|\leqslant\frac14(v_0^2+a^2)^{1/2}$, and following the proof of (53) we find that

$$\tfrac12 r^4(v_0^2+a^2)\leqslant\hat s_0^2=r^4(v_0-x_0p_0-y_0q_0)^2+r^4a^2\leqslant 2r^4(v_0^2+a^2).\tag{59}$$

Now $r\leqslant 1$, $r\leqslant(3L)^{-1}$ and $|v_0|\leqslant L$ imply that $2r^4v_0^2\leqslant\frac29$, and $r\leqslant(3K)^{-1/2}$, $|a|\leqslant K$ yield $2r^4a^2\leqslant\frac29$. Thus $\hat s_0^2\leqslant\frac49<\frac12$, the third inequality of (50). The fourth inequality $|\hat\beta_0|\leqslant\frac45$ follows from $\hat\beta_0=q_0$, $|q_0|\leqslant C$ and $C\leqslant\frac45$.

The final inequality $|\hat\gamma_0|\leqslant J\max(\hat s_0|\hat y_0|^{1/2},\hat s_0^{3/2})$ follows using (59), the first inequality of (23), and $A\leqslant\frac12 Jr^2$. Thus (50) holds for $\hat a,\hat x_0,\hat y_0,\hat v_0,\hat p_0$ and $\hat q_0$. Therefore, by Theorem 4.23, there exist $\hat\alpha,\hat\beta,\hat\gamma\in\mathbb{R}$ satisfying (51) such that if $\hat f,\hat\phi,\hat\psi,\hat u,\hat v\in C^\infty(D)$ are constructed in Definition 4.3 using $\hat a,\hat\alpha,\hat\beta,\hat\gamma$ then

$$\hat v(\hat x_0,\hat y_0)=\hat v_0,\quad\frac{\partial\hat v}{\partial x}(\hat x_0,\hat y_0)=\hat p_0\quad\text{and}\quad\frac{\partial\hat v}{\partial y}(\hat x_0,\hat y_0)=\hat q_0.\tag{60}$$

Define $u,v\in C^\infty(D_L)$ by

$$u(x,y)=r^{-1}\hat u(rx,r^2y)-r^{-1}\hat u(\hat x_0,\hat y_0)+u_0\quad\text{and}\quad v(x,y)=r^{-2}\hat v(rx,r^2y).$$

As $r\leqslant 1$ and $r\leqslant(3L)^{-1}$ it follows that if $(x,y)\in D_L$ then $(rx,r^2y)\in D$, so u,v are well-defined. Also, u,v and a satisfy (17) as $\hat u,\hat v$ and $\hat a$ do, and $u(x_0,y_0)=u_0$ follows from the definition of u, so (60) implies (24).

It remains only to show that $(u-u_0)^2+(v-v_0)^2<N^2$ on D_L. This is implied by $|u-u_0|\leqslant\frac12 N$ and $|v-v_0|\leqslant\frac12 N$ on D_L, which in turn follows from

$$\bigl|\hat u-\hat u(\hat x_0,\hat y_0)\bigr|\leqslant\tfrac12 rN\quad\text{and}\quad\bigl|\hat v-\hat v_0\bigr|\leqslant\tfrac12 r^2N\tag{61}$$

on D. But by [6, Cor. 4.4] the maxima and minima of $\hat{u}$ and $\hat{v}$ are achieved on ∂D. Thus it is enough for (61) to hold on ∂D, and further the maxima of $\hat{u} - \hat{u}(x_0, y_0), \hat{v} - \hat{v}_0$ on ∂D are nonnegative, and the minima nonpositive.

To prove this we use (30) to write $\hat{u}, \hat{v}$ in terms of $x, y, \frac{\partial \hat{\phi}}{\partial y}$ and $\frac{\partial \hat{\phi}}{\partial x}$, and then apply (35) to show that $\bigl|\frac{\partial \hat{\phi}}{\partial x}\bigr|, \bigl|\frac{\partial \hat{\phi}}{\partial y}\bigr| \leqslant 16\hat{s}^{-1}\hat{\gamma}^2$ on ∂D. Using (49), (51) and (23) we can derive upper bounds for $\hat{u} - \hat{u}(x_0, y_0)$ and $\hat{v} - \hat{v}_0$ on ∂D. If they are less than $\frac{1}{2}rN$ and $\frac{1}{2}r^2N$ respectively then (61) holds on ∂D, and we are finished. This will be true provided $A, B, C > 0$ are chosen small enough to satisfy certain inequalities involving N. We leave the details to the reader.

This completes the proof of Theorem 4.1. The proof of Theorem 4.2 is similar, using Theorem 4.25 rather than Theorem 4.23.

5 A priori estimates for $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$

We can now use the results of §4 and [6, Th. 6.9] to derive a priori interior and global estimates for derivatives of solutions u, v of (17) satisfying a C^0 bound.

5.1 Interior estimates for $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$

Here are *a priori interior estimates* for $\partial u, \partial v$ when u, v satisfy (17).

Theorem 5.1. *Let $K, L > 0$ be given, and S, T be domains in $\mathbb{R}^2$ with $T \subset S^\circ$. Then there exists $R > 0$ depending only on K, L, S and T such that the following is true.*

Suppose that $a \in \mathbb{R}$ with $a \neq 0$ and $|a| \leqslant K$, and that $u, v \in C^1(S)$ satisfy (17) and $u^2 + v^2 < L^2$. Then whenever $(x_0, y_0) \in T$, we have

$$\Bigl|\frac{\partial u}{\partial x}(x_0, y_0)\Bigr| \leqslant \sqrt{2}\, R\bigl(v(x_0, y_0)^2 + y_0^2 + a^2\bigr)^{-1}, \tag{62}$$

$$\Bigl|\frac{\partial u}{\partial y}(x_0, y_0)\Bigr| \leqslant R\bigl(v(x_0, y_0)^2 + y_0^2 + a^2\bigr)^{-5/4}, \tag{63}$$

$$\Bigl|\frac{\partial v}{\partial x}(x_0, y_0)\Bigr| \leqslant 2R\bigl(v(x_0, y_0)^2 + y_0^2 + a^2\bigr)^{-3/4}, \tag{64}$$

$$\text{and} \quad \Bigl|\frac{\partial v}{\partial y}(x_0, y_0)\Bigr| \leqslant \sqrt{2}\, R\bigl(v(x_0, y_0)^2 + y_0^2 + a^2\bigr)^{-1}. \tag{65}$$

This gives good estimates of $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ except when $v, y, a \approx 0$. But the equations (17) are singular exactly when $v = y = a = 0$. So, we have good estimates of the derivatives of u, v except when we are close to a singular point.

The proof of Theorem 5.1 uses [6, Th. 6.9], which was explained in §1, and which readers are advised to consult at this point. Applying this result with $(\hat{u}, \hat{v})$ equal to one of the families (u, v), (u', v') constructed in §4 gives nonexistence results for (u, v) with $(u, v)(x_0, y_0) = (u_0, v_0)$ and prescribed $\partial v(x_0, y_0)$.

In this way we exclude all values of $\partial v(x_0,y_0)$ except those allowed by (64)–(65), which are equivalent to (62)–(63) by (17).

Theorem 5.1 follows from the next two theorems, which combine [6, Th. 6.9] with Theorems 4.1 and 4.2 respectively. Note that we need *both* Theorems 4.1 and 4.2 to prove Theorem 5.1.

Theorem 5.2. *Let $K,L>0$ be given, and S,T be domains in $\mathbb{R}^2$ with $T\subset S^\circ$. Then there exists $R'>0$ depending only on K,L,S and T such that the following is true.*

Suppose that $a\in\mathbb{R}$ with $a\neq 0$ and $|a|\leqslant K$, and that $u,v\in C^1(S)$ satisfy (17) and $u^2+v^2<L^2$. Then whenever $(x_0,y_0)\in T$, we have

$$\left|\frac{\partial u}{\partial x}(x_0,y_0)\right|\leqslant \sqrt{2}\,R'\bigl(y_0^2+a^2\bigr)^{-1/2}, \tag{66}$$

$$\left|\frac{\partial u}{\partial y}(x_0,y_0)\right|\leqslant R'\bigl(v(x_0,y_0)^2+y_0^2+a^2\bigr)^{-1/4}\bigl(y_0^2+a^2\bigr)^{-1/2}, \tag{67}$$

$$\left|\frac{\partial v}{\partial x}(x_0,y_0)\right|\leqslant 2R'\bigl(v(x_0,y_0)^2+y_0^2+a^2\bigr)^{1/4}\bigl(y_0^2+a^2\bigr)^{-1/2}, \tag{68}$$

$$\text{and}\quad \left|\frac{\partial v}{\partial y}(x_0,y_0)\right|\leqslant \sqrt{2}\,R'\bigl(y_0^2+a^2\bigr)^{-1/2}. \tag{69}$$

Proof. As u,v satisfy (17), equations (66) and (67) follow from (68) and (69). So it is enough to prove (68) and (69). Define

$$M=\sup_{(x,y)\in T}(x^2+y^2)^{1/2}\quad\text{and}\quad N=\sup\bigl\{\epsilon>0:B_\epsilon(x,y)\subset S\ \ \forall(x,y)\in T\bigr\}.$$

Then $M,N>0$ are well-defined, as T is compact and $T\subset S^\circ$.

Let $A,B,C>0$ be as in Theorem 4.1, using these K,L,M,N, and define

$$R'=\max\bigl(2^{1/4}A^{-1},2^{-1/2}LB^{-1},2^{-1/2}(M^2+K^2)^{1/2}C^{-1}\bigr). \tag{70}$$

Then R' depends only on K,L,S and T, as M,N and A,B,C do. Define

$$\begin{aligned} &u_0=u(x_0,y_0), && v_0=v(x_0,y_0), && p_0=\tfrac{\partial v}{\partial x}(x_0,y_0), && q_0=\tfrac{\partial v}{\partial y}(x_0,y_0),\\ &\hat x_0=u_0, && \hat y_0=v_0, && \hat u_0=x_0, && \hat v_0=y_0,\end{aligned} \tag{71}$$

$$\hat p_0=-\frac{p_0}{\frac12(v_0^2+y_0^2+a^2)^{-1/2}p_0^2+q_0^2}\quad\text{and}\quad \hat q_0=\frac{q_0}{\frac12(v_0^2+y_0^2+a^2)^{-1/2}p_0^2+q_0^2}. \tag{72}$$

A straightforward calculation now shows:

Lemma 5.3. *Suppose either (68) or (69) does not hold. Then $a,\hat x_0,\hat y_0,\hat u_0,$ $\hat v_0,\hat p_0$ and $\hat q_0$ satisfy equation (23), replacing x_0 by $\hat x_0$, and so on.*

We can now finish the proof of Theorem 5.2. Suppose that either (68) or (69) does not hold. Then by Lemma 5.3, the hypotheses of Theorem 4.1 hold, with x_0,y_0,u_0,v_0,p_0,q_0 replaced by $\hat x_0,\hat y_0,\hat u_0,\hat v_0,\hat p_0,\hat q_0$. Hence Theorem 4.1 gives $\hat u,\hat v\in C^\infty(D_L)$ satisfying (17), $(\hat u-\hat u_0)^2+(\hat v-\hat v_0)^2<N^2$ and (24).

But $(\hat{u}_0,\hat{v}_0)=(x_0,y_0)$ which lies in T, and so the open ball $B_N(x_0,y_0)$ of radius N about (x_0,y_0) lies in S, by definition of N. Therefore $(\hat{u},\hat{v})$ maps $D_L\rightarrow S$. Applying [6, Th. 6.9] then shows that there do not exist $u,v\in C^1(S)$ satisfying (17), $u^2+v^2<L^2$ and (24), contradicting the definitions of u_0,v_0,p_0 and q_0. Therefore both (68) and (69) hold, and the theorem is complete. □

In the same way, combining Theorem 4.2 and [6, Th. 6.9] we prove:

Theorem 5.4. *Let $K,L>0$ be given, and S,T be domains in $\mathbb{R}^2$ with $T\subset S^\circ$. Then there exists $R''>0$ depending only on K,L,S and T such that the following is true.*

Suppose that $a\in\mathbb{R}$ with $a\neq 0$ and $|a|\leqslant K$, and that $u,v\in C^1(S)$ satisfy (17) and $u^2+v^2<L^2$. Then whenever $(x_0,y_0)\in T$, we have

$$\Bigl|\frac{\partial u}{\partial x}(x_0,y_0)\Bigr|\leqslant\sqrt{2}\,R''\bigl(v(x_0,y_0)^2+y_0^2+a^2\bigr)^{1/4}\bigl|v(x_0,y_0)\bigr|^{-5/2},\qquad(73)$$

$$\Bigl|\frac{\partial u}{\partial y}(x_0,y_0)\Bigr|\leqslant R''\bigl|v(x_0,y_0)\bigr|^{-5/2},\qquad(74)$$

$$\Bigl|\frac{\partial v}{\partial x}(x_0,y_0)\Bigr|\leqslant 2R''\bigl(v(x_0,y_0)^2+y_0^2+a^2\bigr)^{1/2}\bigl|v(x_0,y_0)\bigr|^{-5/2},\qquad(75)$$

$$\textit{and}\quad\Bigl|\frac{\partial v}{\partial y}(x_0,y_0)\Bigr|\leqslant\sqrt{2}\,R''\bigl(v(x_0,y_0)^2+y_0^2+a^2\bigr)^{1/4}\bigl|v(x_0,y_0)\bigr|^{-5/2}.\qquad(76)$$

We now prove Theorem 5.1. Let R',R'' be as in Theorems 5.2, 5.4, and put

$$R=\max\bigl(2^{1/2}(L^2+\sup_T|y|^2+K^2)^{1/2}R',2^{5/4}R''\bigr).$$

Let $(x_0,y_0)\in T$, and divide into the two cases (a) $v(x_0,y_0)^2\leqslant y_0^2+a^2$ and (b) $v(x_0,y_0)^2>y_0^2+a^2$. In case (a) we have

$$\begin{aligned}R'(y_0^2+a^2)^{-1/2}&\leqslant 2^{1/2}R'\bigl(v(x_0,y_0)^2+y_0^2+a^2\bigr)^{-1/2}\\&\leqslant 2^{1/2}(L^2+\sup_T|y|^2+K^2)^{1/2}R'\bigl(v(x_0,y_0)^2+y_0^2+a^2\bigr)^{-1}\\&\leqslant R\bigl(v(x_0,y_0)^2+y_0^2+a^2\bigr)^{-1},\end{aligned}$$

and in case (b) we have

$$\begin{aligned}R''\bigl(v(x_0,y_0)^2+y_0^2+a^2\bigr)^{1/4}\bigl|v(x_0,y_0)\bigr|^{-5/2}&\leqslant 2^{5/4}R''\bigl(v(x_0,y_0)^2+y_0^2+a^2\bigr)^{-1}\\&\leqslant R\bigl(v(x_0,y_0)^2+y_0^2+a^2\bigr)^{-1}.\end{aligned}$$

Equations (62)–(65) then follow from (66)–(69) in case (a), and (73)–(76) in (b).

5.2 Global estimates for $\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$

In analysis, interior estimates can usually be be extended to global estimates on the whole domain S by imposing suitable boundary conditions on ∂S. We shall now extend the results of §5.1 to all of S, provided $\partial u,\partial v$ satisfy certain inequalities on ∂S. Here is our main result.

Theorem 5.5. *Let $J, K, L > 0$ be given, and S be a domain in $\mathbb{R}^2$ such that $T_{(x,0)}\partial S$ is parallel to the y-axis for each $(x, 0)$ in ∂S. Then there exists $H > 0$ depending only on J, K, L and S such that the following is true.*

Suppose that $a \in \mathbb{R}$ with $a \neq 0$ and $|a| \leqslant K$, and that $u, v \in C^1(S)$ satisfy (17) and $u^2 + v^2 < L^2$ on S, and $\bigl|\frac{\partial v}{\partial x}\bigr| \leqslant J$, $\bigl|\frac{\partial v}{\partial y}\bigr| \leqslant J(y^2+a^2)^{-1/2}$ and $\bigl|\frac{\partial v}{\partial y}\bigr| \leqslant J(v^2+y^2+a^2)^{1/4}|v|^{-5/2}$ on ∂S. Then for all $(x_0, y_0) \in S$, we have

$$\Bigl|\frac{\partial u}{\partial x}(x_0,y_0)\Bigr| = \Bigl|\frac{\partial v}{\partial y}(x_0,y_0)\Bigr| \leqslant H\bigl(v(x_0,y_0)^2 + y_0^2 + a^2\bigr)^{-1}, \qquad (77)$$

$$\Bigl|\frac{\partial v}{\partial x}(x_0,y_0)\Bigr| \leqslant J \quad \textit{and} \quad \Bigl|\frac{\partial u}{\partial y}(x_0,y_0)\Bigr| \leqslant \tfrac{1}{2}J\bigl(v(x_0,y_0)^2 + y_0^2 + a^2\bigr)^{-1/2}. \qquad (78)$$

This is a global estimate for derivatives of solutions u, v of (17) on a domain S satisfying certain bounds on ∂S, similar to the interior estimates in Theorem 5.1. Here (77) is essentially the same as (62) and (65), but (78) is stronger than (63) and (64). This is because $\frac{\partial v}{\partial x}$ satisfies a *maximum principle* on S [6, Prop. 8.12], so $\bigl|\frac{\partial v}{\partial x}\bigr| \leqslant J$ on ∂S implies $\bigl|\frac{\partial v}{\partial x}\bigr| \leqslant J$ on S.

The bounds needed in Theorem 5.5 for ∂v on ∂S are quite strong, in that we assume a bound on *all* of ∂v, but in applications such as the Dirichlet problem for v, we initially only know a bound on *half* of ∂v. In §6 we will implicitly show how to extend this to bound all of ∂v, so that Theorem 5.5 applies.

The rest of the section proves Theorem 5.5. Consider the following situation.

Definition 5.6. Let S be a domain in $\mathbb{R}^2$, such that for every point of the form $(x, 0)$ in ∂S, the tangent line $T_{(x,0)}\partial S$ is parallel to the y-axis. Let $J, K, L > 0$ be given, and suppose that $a \in \mathbb{R}$ with $a \neq 0$ and $|a| \leqslant K$, and that $u, v \in C^1(S)$ satisfy (17) and $u^2 + v^2 < L^2$ on S, and $\bigl|\frac{\partial v}{\partial x}\bigr| \leqslant J$, $\bigl|\frac{\partial v}{\partial y}\bigr| \leqslant J(y^2+a^2)^{-1/2}$ and $\bigl|\frac{\partial v}{\partial y}\bigr| \leqslant J(v^2+y^2+a^2)^{1/4}|v|^{-5/2}$ on ∂S.

We shall show that under these assumptions, $\partial u, \partial v$ satisfy estimates similar to (62)–(65) on the whole of S. From [6, Prop. 8.12] and (17) we have:

Corollary 5.7. *We have $\bigl|\frac{\partial v}{\partial x}\bigr| \leqslant J$ and $\bigl|\frac{\partial u}{\partial y}\bigr| \leqslant \frac{1}{2}J(v^2 + y^2 + a^2)^{-1/2}$ on S.*

Thus the problem is to estimate $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$. We begin by bounding $\frac{\partial v}{\partial y}$ away from the x-axis.

Proposition 5.8. *Let $\epsilon > 0$ be small, and set $S_\epsilon = \bigl\{(x,y) \in S : |y| > \epsilon\bigr\}$. Then there exists $G > 0$ depending only on S, J, K, L, ϵ such that $\bigl\|\frac{\partial v}{\partial y}|_{S_\epsilon}\bigr\|_{C^0} \leqslant G$.*

Proof. The proposition will follow from an *interior regularity result* for *quasilinear elliptic equations* on *domains in $\mathbb{R}^2$ with boundary portions*. For *linear* operators of a certain form this is done in Gilbarg and Trudinger [2, p. 302-4], and will be discussed in the proof of Proposition 7.4. For *quasilinear* operators we can deduce what we need from the proof of [2, Th. 15.2].

This says that if Q is a quasilinear elliptic operator of the form (7), then if a^{ij} and b and their first derivatives satisfy certain complicated estimates on a domain S, and $v \in C^2(S)$ with $Qv = 0$, then $\|\partial v\|_{C^0} \leqslant G$ for some $G > 0$ depending on S, $\|v\|_{C^0}$, $\|\partial v|_{\partial S}\|_{C^0}$, and quantities in the estimates on a^{ij} and b.

We may extend this result to an interior regularity result for *domains with boundary portions*, using the ideas of [2, p. 302-4]. That is, we suppose that the estimates hold in $S_{\epsilon/2}$, but deduce the a priori bound on S_ϵ. The important point here is that $S_{\epsilon/2}$ is *noncompact*, and its boundary $\partial S_{\epsilon/2}$ is part of ∂S. The closure $\overline{S}_{\epsilon/2}$ has an *extra boundary portion*, two line segments with $y = \pm\epsilon/2$. But we do *not* need a bound for ∂v on these line segments.

The price of this is that we can only bound ∂v away from $y = \pm\epsilon/2$, which is why we end up with an a priori bound for $\|\partial v|_{S_\epsilon}\|_{C^0}$ rather than $\|\partial v|_{S_{\epsilon/2}}\|_{C^0}$. Since $|\frac{\partial v}{\partial x}| \leqslant J$, $|a| \leqslant K$, $|v| < L$, and $\frac{1}{2}\epsilon \leqslant y \leqslant \sup_S |y|$ on $S_{\epsilon/2}$, it is not difficult to show that the necessary estimates on a^{ij} and b hold at v with constants depending only on S, J, K, L and ϵ. It then follows that $\left\|\frac{\partial v}{\partial y}|_{S_\epsilon}\right\|_{C^0} \leqslant G$ for some $G > 0$ depending only on S, J, K, L and ϵ. □

It remains to bound $\frac{\partial v}{\partial y}$ near the x-axis. We do this by extending Theorems 5.2 and 5.4 from interior domains T to all points (x_0, y_0) in S near the x-axis. Here is the extension of Theorem 5.2.

Proposition 5.9. *There exist constants $\epsilon > 0$ depending only on S, and $R' > 0$ depending only on S, J, K, L and ϵ, such that if $(x_0, y_0) \in S$ with $|y_0| < \epsilon$ then equations* (66)–(69) *hold.*

Proof. Let small $\epsilon > 0$ and large $R' > 0$ be chosen, to satisfy conditions we will give later. Since u, v are C^1 and (66)–(69) are closed conditions, it is enough to prove the proposition for $(x_0, y_0) \in S^\circ$ rather than S. So suppose for a contradiction that $(x_0, y_0) \in S^\circ$ with $|y_0| < \epsilon$, and that (66)–(69) do not all hold. As (68)–(69) imply (66)–(67) and (68) follows from Corollary 5.7 when $R' \gg J$, this means that (69) does not hold.

We follow the proof of Theorem 5.2. Set $M = \sup_{(x,y)\in S}(x^2+y^2)^{1/2}$ and $N = \epsilon$. Let $A, B, C > 0$ be as in Theorem 4.1, using these K, L, M, N, and choose R' greater than or equal to the r.h.s. of (70). Define $u_0, v_0, p_0, q_0, \hat{x}_0, \hat{y}_0, \hat{u}_0, \hat{v}_0, \hat{p}_0$ and $\hat{q}_0$ as in (71)–(72). Then Lemma 5.3 shows that $a, \hat{x}_0, \hat{y}_0, \hat{u}_0$, $\hat{v}_0, \hat{p}_0$ and $\hat{q}_0$ satisfy (23). Hence Theorem 4.1 gives $\hat{u}, \hat{v} \in C^\infty(D_L)$ satisfying (17), $(\hat{u} - \hat{u}_0)^2 + (\hat{v} - \hat{v}_0)^2 < \epsilon^2$ and (24).

Now $|p_0| \leqslant J$ by Corollary (5.7) and $|q_0| > \sqrt{2}R'(y_0^2 + a^2)^{-1/2}$ as (69) does not hold. Hence $p_0 = O(1)$ and q_0 is large. Equation (72) then gives $\hat{q}_0 \approx q_0^{-1}$ and $\hat{p}_0 = O(\hat{q}_0^2)$. The material of §4 then implies that $\hat{u}, \hat{v}$ and their first derivatives approximate the affine maps

$$\hat{u}(x, y) \approx \hat{u}_0 + \hat{q}_0(x - \hat{x}_0), \qquad \hat{v}(x, y) \approx \hat{v}_0 + \hat{q}_0(y - \hat{y}_0). \tag{79}$$

Define $U = (\hat{u}, \hat{v})(D_L)$. Then $U \subset B_\epsilon(x_0, y_0)$, the open ball of radius ϵ about (x_0, y_0), as $\hat{u}_0 = x_0$, $\hat{v}_0 = y_0$ and $(\hat{u} - \hat{u}_0)^2 + (\hat{v} - \hat{v}_0)^2 < \epsilon^2$. Furthermore, (79) implies that U is approximately a closed disc of radius $|\hat{q}_0|L$, and that the map $(\hat{u}, \hat{v}) : D_L \to U$ is invertible with differentiable inverse.

Let $(u',v'):U\rightarrow D_L$ be this inverse map. Then [6, Prop. 6.8] implies that u',v' satisfy (17) in U. Moreover, equations (24), (71) and (72) imply that $u=u'$, $v=v'$, $\partial u=\partial u'$ and $\partial v=\partial v'$ at (x_0,y_0). Thus $(u',v')-(u,v)$ has a zero of *multiplicity* at least 2 at (x_0,y_0), in the sense of [6, Def. 6.3].

Suppose now that $U\subset S$. Then (u',v') and (u,v) satisfy (17) in U. Now (u',v') takes ∂U to ∂D_L, the circle of radius L, and winds round ∂D_L once in the positive sense. Since $u^2+v^2<L^2$ it follows that the winding number of $(u',v')-(u,v)$ about 0 along ∂U is the same as that of (u',v'), which is 1. So by [6, Th. 6.7] there is 1 zero of $(u',v')-(u,v)$ in U°, counted with multiplicity. But this is a contradiction, as $(u',v')-(u,v)$ has a zero of multiplicity at least 2 at $(x_0,y_0)\in U^\circ$. This proves the proposition when $U\subset S$.

It remains to deal with the case that $U\not\subset S$. Then U must intersect ∂S. Since $|y_0|<\epsilon$ and $U\subset B_\epsilon(x_0,y_0)$ it follows that $|y|<2\epsilon$ on U. So if ϵ is small enough, (x_0,y_0) and U must be close to a point of the form $(x,0)$ in ∂S. By Definition 5.6 the tangent line $T_{(x,0)}\partial S$ is parallel to the y-axis. Thus, by making ϵ small we can assume that U intersects a portion of ∂S close to $(x,0)$, and with tangent spaces nearly parallel to the y-axis.

For $s\in\mathbb{R}$, define $U_s=\bigl\{(x+s,y):(x,y)\in U\bigr\}$ and $u'_s,v'_s:U_s\rightarrow\mathbb{R}$ by $(u'_s,v'_s)(x,y)=(u',v')(x-s,y)$. Then $(u'_s,v'_s):U_s\rightarrow D_L$ satisfy (17). Suppose for simplicity that $(1,0)$ points inward to S at $(x,0)$. Then $U_0=U$, and as s increases from zero, U_s moves inwards into S, until U_t lies wholly in S for some $t=O(\epsilon)$.

The argument above shows that the number of zeroes of $(u'_t,v'_t)-(u,v)$ in U_t, counted with multiplicity, is 1. We shall show that as s increases from 0 to t, the number of zeroes of $(u'_s,v'_s)-(u,v)$ in $U_s\cap S$, counted with multiplicity, can only increase. Hence, the number of zeroes of $(u',v')-(u,v)$ in $U\cap S$, counted with multiplicity, is no more than 1, and as above we have a contradiction with [6, Th. 6.7] because (x_0,y_0) is a zero of multiplicity at least 2, so the proof will be finished.

The only way in which the number of zeroes of $(u'_s,v'_s)-(u,v)$ in $U_s\cap S$ with multiplicity can change as s changes, is if a zero crosses the boundary $\partial(U_s\cap S)$. This consists of two portions, $\partial U_s\cap S$ and $U_s\cap\partial S$. But on $\partial U_s\cap S$ we have $(u'_s)^2+(v'_s)^2=L^2$ and $u^2+v^2<L^2$, so no zeroes can cross this part of the boundary. Thus, zeroes can only enter or leave $U_s\cap S$ across $U_s^\circ\cap\partial S$.

The next lemma computes $\frac{\mathrm{d}}{\mathrm{d}s}$ of a zero $\bigl(x(s),y(s)\bigr)$ of $(u'_s,v'_s)-(u,v)$, in terms of $\partial v'_s$ and ∂v. The proof is an elementary calculation using (17), and is left to the reader.

Lemma 5.10. *Suppose $\bigl(x(s),y(s)\bigr)$ is a zero of $(u'_s,v'_s)-(u,v)$ in $U_s\cap S$. Then*

$$\begin{gathered}\Bigl(\bigl(\tfrac{\partial}{\partial y}(v'_s-v)\bigr)^2+\tfrac{1}{2}(v^2+y^2+a^2)^{-1/2}\bigl(\tfrac{\partial}{\partial x}(v'_s-v)\bigr)^2\Bigr)\cdot\frac{\mathrm{d}}{\mathrm{d}s}\begin{pmatrix}x(s)\\ y(s)\end{pmatrix}=\\ \begin{pmatrix}\frac{\partial}{\partial y}v'_s\cdot\frac{\partial}{\partial y}(v'_s-v)+\frac{1}{2}(v^2+y^2+a^2)^{-1/2}\frac{\partial}{\partial x}v'_s\frac{\partial}{\partial x}(v'_s-v)\\ \frac{\partial}{\partial x}v'_s\cdot\frac{\partial}{\partial y}(v'_s-v)-\frac{\partial}{\partial y}v'_s\cdot\frac{\partial}{\partial x}(v'_s-v)\end{pmatrix}.\end{gathered}\tag{80}$$

When $\big(x(s), y(s)\big)$ actually crosses ∂S we have $\big|\frac{\partial v}{\partial y}\big| \leqslant J(y^2+a^2)^{-1/2}$, by assumption. Also, from (79) we can show that $\frac{\partial}{\partial y} v'_s \approx q_0$ and $\frac{\partial}{\partial x} v'_s = O(1)$. Therefore, provided $R' \gg J$, careful calculation shows that dominant terms on both sides of (80) are $(\frac{\partial}{\partial y} v'_s)^2$, and so $\frac{\mathrm{d}}{\mathrm{d}s}\big(x(s), y(s)\big) \approx (1,0)$. But from above, $T_{(x,y)}\partial S$ is nearly parallel to the y-axis, and $(1,0)$ points inwards to S. Hence, as s increases from 0 to t, zeroes of $(u'_s, v'_s) - (u,v)$ can only move into $U'_s \cap S$, not out. This completes the proof of Proposition 5.9. □

By using the solutions of Theorem 4.2 instead of Theorem 4.1, using a similar proof we obtain a global analogue of Theorem 5.4. Combining these two results using the method of Theorem 5.1 then yields Theorem 5.5.

6 The Dirichlet problem for v when $a = 0$

Theorem 3.6 shows that the Dirichlet problem for equation (21) is uniquely solvable in arbitrary domains in $\mathbb{R}^2$ for $a \neq 0$. In this section we will show that the Dirichlet problem for (21) also has a unique *weak* solution when $a = 0$, for strictly convex domains S invariant under $(x,y) \mapsto (x,-y)$.

6.1 The main results

The following theorem is an analogue of Theorem 3.6 for the case $a = 0$, and the first main result of the paper.

Theorem 6.1. *Let S be a strictly convex domain in $\mathbb{R}^2$ invariant under the involution $(x,y) \mapsto (x,-y)$, let $k \geqslant 0$ and $\alpha \in (0,1)$. Suppose $\phi \in C^{k+2,\alpha}(\partial S)$ with $\phi(x,0) \neq 0$ for points $(x,0)$ in ∂S. Then there exists a unique weak solution v of* (21) *in $C^0(S)$ with $a = 0$ and $v|_{\partial S} = \phi$.*

Fix a basepoint $(x_0, y_0) \in S$. Then there exists a unique $u \in C^0(S)$ with $u(x_0,y_0) = 0$ such that u, v are weakly differentiable in S and satisfy (16) *with weak derivatives. The weak derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ satisfy $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \in L^p(S)$ for $p \in [1, \frac{5}{2})$, and $\frac{\partial u}{\partial y} \in L^q(S)$ for $q \in [1,2)$, and $\frac{\partial v}{\partial x}$ is bounded on S. Also u, v are $C^{k+2,\alpha}$ in S and real analytic in S° except at singular points $(x,0)$ with $v(x,0) = 0$.*

Combined with Proposition 3.2 the theorem can be used to construct large numbers of U(1)-*invariant singular special Lagrangian* 3-*folds* in $\mathbb{C}^3$. This is the principal motivation for the paper. The singularities of these special Lagrangian 3-folds will be studied in [7]. The restriction to boundary data ϕ with $\phi(x,0) \neq 0$ for points $(x,0)$ in ∂S is to avoid singular points on the boundary ∂S.

Our second main result extends [6, Th. 8.9] to include the case $a = 0$.

Theorem 6.2. *Let S be a strictly convex domain in $\mathbb{R}^2$ invariant under the involution $(x,y) \mapsto (x,-y)$, let $k \geqslant 0$, $\alpha \in (0,1)$, and $(x_0,y_0) \in S$. Define X to be the set of $\phi \in C^{k+2,\alpha}(\partial S)$ with $\phi(x,0) = 0$ for some $(x,0) \in \partial S$. Then the map $C^{k+2,\alpha}(\partial S) \times \mathbb{R} \setminus X \times \{0\} \to C^0(S)^2$ taking $(\phi, a) \mapsto (u,v)$ is continuous,* *where (u,v) is the unique solution of* (17) *(with weak derivatives when $a=0$) with $v|_{\partial S}=\phi$ and $u(x_0,y_0)=0$, constructed in Theorem 3.6 when $a\neq 0$, and in Theorem 6.1 when $a=0$. This map is also continuous in stronger topologies on (u,v) than the C^0 topology.*

The proofs of Theorems 6.1 and 6.2 will take up the rest of the section. Here is how they are laid out. Let S,ϕ be as in Theorem 6.1. In §6.2, for each $a\in(0,1]$ we define $v_a\in C^{k+2,\alpha}(S)$ to be the unique solution of (21) in S with $v_a|_{\partial S}=\phi$, and u_a,f_a such that u_a,v_a satisfy (17) and $\frac{\partial f_a}{\partial y}=u_a$, $\frac{\partial f_a}{\partial x}=v_a$. The idea is to show that $u_a,v_a,f_a\rightarrow u_0,v_0,f_0$ as $a\rightarrow 0_+$, for unique, suitably differentiable $u_0,v_0,f_0\in C^0(S)$. Then u_0,v_0 are the weak solutions u,v in Theorem 6.1.

To show that these limits u_0,v_0,f_0 exist, the main issue is to prove *a priori estimates* of u_a,v_a,f_a that are *uniform in a*. That is, we need bounds such as $\|u_a\|_{C^0}\leqslant C$ for all $a\in(0,1]$, with C independent of a. Given strong enough uniform a priori estimates, the existence of *some* weak limits u_0,v_0,f_0 becomes essentially trivial, using compact embeddings of Banach spaces.

Getting such uniform a priori estimates is difficult, since equations (17) and (21) *really are* singular when $a=0$, so many norms of u_a,v_a such as $\|\partial u_a\|_{C^0}$, $\|\partial v_a\|_{C^0}$ can diverge to infinity as $a\rightarrow 0_+$, and uniform a priori estimates of these norms *do not exist*. The part of Theorem 6.1 that gave the author most trouble was finding estimates strong enough to prove that u is *continuous*.

This is important geometrically, as if u,v are not continuous then the SL 3-fold N in (15) is not locally closed, and one singular point of u,v will correspond to many singular points of $\overline{N}$ rather than one. To show u is continuous we use the *nonstandard Sobolev embedding result* Theorem 2.3, which allows us to trade off a stronger L^p estimate of $\frac{\partial u}{\partial x}$ against a weaker L^q estimate of $\frac{\partial u}{\partial y}$.

The a priori estimates that we need are built up step by step in §6.3–§6.5. In §6.3 we construct super- and subsolutions for v_a near points $(x,0)$ in ∂S. This gives a positive lower bound for $|v_a|$ near $(x,0)$, which proves that (17) and (21) are *uniformly elliptic* in a close to ∂S. Then §6.4 proves uniform C^0 estimates for u_a,v_a,f_a and some derivatives on S or ∂S.

In §6.5 we use the results of §5 to prove uniform L^p estimates for $\frac{\partial u_a}{\partial x}$, $\frac{\partial u_a}{\partial y}$, $\frac{\partial v_a}{\partial x}$ and $\frac{\partial v_a}{\partial y}$, and deduce uniform continuity of the u_a,v_a from Theorem 2.3. Section 6.6 proves the existence of limits u_0,v_0,f_0 in C^0, and that they satisfy (16), (18) and (21) in the appropriate weak senses. Section 6.7 proves uniqueness of the solutions, and completes the proofs.

The requirement that S be a strictly convex domain in $\mathbb{R}^2$ invariant under $(x,y)\mapsto(x,-y)$ is unnecessarily strong. All the proofs above actually use is that S should be a domain in $\mathbb{R}^2$, and that for every point $(x,0)$ in ∂S, the tangent to ∂S at $(x,0)$ should be parallel to the y-axis, and S should be strictly convex near $(x,0)$.

The author believes that Theorem 6.1 actually holds for arbitrary domains S in $\mathbb{R}^2$. To extend the proof to such S would need suitable super- and subsolutions for v_a near $(x,0)$ in ∂S, generalizing those of §6.3. Perhaps one can use the above Theorem 6.1 to construct such super- and subsolutions.

6.2 A family of solutions of (21)

We shall consider the following situation.

Definition 6.3. Let S be a strictly convex domain in $\mathbb{R}^2$ which is invariant under the involution $(x,y) \mapsto (x,-y)$. Then there exist unique $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$ and $(x_i, 0) \in \partial S$ for $i = 1,2$. Let $k \geqslant 0$ and $\alpha \in (0,1)$, and suppose $\phi \in C^{k+2,\alpha}(\partial S)$ with $\phi(x_i, 0) \neq 0$ for $i = 1,2$. For each $a \in (0,1]$, let $v_a \in C^{k+2,\alpha}(S)$ be the unique solution of (21) in S with this value of a and $v_a|_{\partial S} = \phi$, which exists and is unique by Theorem 3.6.

For each $a \in (0,1]$, let $u_a \in C^{k+2,\alpha}(S)$ be the unique function such that u_a, v_a and a satisfy (17), and $u_a(x_1, 0) = 0$. This exists by Proposition 3.4. Let $f_a \in C^{k+3,\alpha}(S)$ be the unique solution of (18) satisfying $\frac{\partial f_a}{\partial y} = u_a$, $\frac{\partial f_a}{\partial x} = v_a$ and $f_a(x_1, 0) = 0$. This exists by Proposition 3.3.

We will show that v_a converges in $C^0(S)$ to $v_0 \in C^0(S)$ as $a \to 0_+$, and that v_0 is the unique weak solution of the Dirichlet problem for (21) on S when $a = 0$. The reason for supposing that $\phi(x_i, 0) \neq 0$ is to avoid having singular points on the boundary ∂S.

We begin by defining a family of solutions $v_{a,\gamma}$ of (21) which we will use in §6.3 as super- and subsolutions to bound the v_a near the $(x_i, 0)$ for $i = 1,2$.

Definition 6.4. Let $R > 0$, and define D_R to be the closed disc of radius R about $(0,0)$ in $\mathbb{R}^2$. For each $a \in (0,1]$ and $\gamma > 0$, define $v_{a,\gamma} \in C^\infty(D_R)$ to be the unique solution of (21) in D_R with this value of a and $v_a|_{\partial S} = \gamma x$, which exists and is unique by Theorem 3.6. Considering how $v_{a,\gamma}$ transforms under the involutions $(x,y) \mapsto (-x,y)$ and $(x,y) \mapsto (x,-y)$, by uniqueness we see that $v_{a,\gamma}$ satisfies the identities $v_{a,\gamma}(-x,y) = -v_{a,\gamma}(x,y)$, $v_{a,\gamma}(x,-y) = v_{a,\gamma}(x,y)$ and $v_{a,\gamma}(0,y) = 0$.

Provided γ is large enough, these $v_{a,\gamma}$ satisfy certain inequalities on D_R.

Proposition 6.5. *There exists $C > 0$ depending only on R such that whenever $a \in (0,1]$ and $\gamma \geqslant C$, the function $v_{a,\gamma}$ of Definition 6.4 satisfies $v_{a,\gamma}(x,y) > 0$ when $x > 0$, $v_{a,\gamma}(x,y) < 0$ when $x < 0$ and $v_{a,\gamma}(0,y) = 0$, and*

$$\left|v_{a,\gamma}\right| \leqslant \gamma|x| \quad \textit{and} \quad v_{a,\gamma}^2 + y^2 \geqslant x^4 \quad \textit{on } D_R. \tag{81}$$

Proof. In [6, Th. 5.1] the author defined an explicit family of solutions $\hat{u}_a, \hat{v}_a$ of (17) on $\mathbb{R}^2$ for all $a \geqslant 0$, with the properties that $\hat{v}_a(x,y) > 0$ when $x > 0$, $\hat{v}_a(x,y) < 0$ when $x < 0$ and $\hat{v}_a(0,y) = 0$ for all y, and

$$\hat{v}_a^2 + y^2 \equiv (x^2 + \hat{u}_a^2)(x^2 + \hat{u}_a^2 + 2a). \tag{82}$$

Choose $C > 0$ such that $|\hat{v}_a(x,y)| \leqslant C|x|$ whenever $(x,y) \in \partial D_R$, for all $a \in [0,1]$. This is possible because the $\hat{v}_a$ are smooth on ∂D_R and depend smoothly on a, and $\hat{v}_a(0,y) \equiv 0$.

Define S_R to be the semicircle $\{(x,y) \in D_R : x \geqslant 0\}$. Let $\gamma \geqslant C$. Then for each $a \in (0,1]$ we have $\hat{v}_a \leqslant Cx \leqslant \gamma x = v_{a,\gamma}$ on ∂S_R. Since $\hat{v}_a$ and $v_{a,\gamma}$ satisfy (21) in S_R, we see from [6, Prop. 8.5] on S_R with $v = \hat{v}_a$ and $v' = v_{a,\gamma}$ that $\hat{v}_a \leqslant v_{a,\gamma}$ in S_R. But $\hat{v}_a(x,y) > 0$ for $x > 0$, and thus $v_{a,\gamma}(x,y) > 0$ when $x > 0$.

Now $v_{a,\gamma}^2 \geqslant \hat{v}_a^2$ in S_R, as $v_{a,\gamma} \geqslant \hat{v}_a \geqslant 0$ there. But it follows from (82) that $\hat{v}_a^2 + y^2 \geqslant x^4$ in $\mathbb{R}^2$. Hence $v_{a,\gamma}^2 + y^2 \geqslant x^4$ in S_R. Also $v_{a,\gamma} \leqslant \gamma x$ on ∂S_R, and

$$Q(\gamma x) = -(\gamma^2 x^2 + y^2 + a^2)^{-3/2} \gamma^3 x \leqslant 0 \quad \text{in } S_R.$$

Thus, applying [6, Prop. 8.5] on S_R with $v = v_{a,\gamma}$ and $v' = \gamma x$ gives $v_{a,\gamma} \leqslant \gamma x$ on S_R, so that $|v_{a,\gamma}| \leqslant \gamma |x|$ on S_R. This proves all the assertions of the proposition on S_R, that is, when $x \geqslant 0$. The case $x < 0$ follows immediately using the identity $v_{a,\gamma}(-x,y) = -v_{a,\gamma}(x,y)$. $\square$

6.3 Super- and subsolutions for v_a near $(x_j, 0)$

Next we find a uniform positive lower bound for $v_a^2 + y^2 + a^2$ near ∂S for all $a \in (0,1]$. This will ensure that (21) is uniformly elliptic at the v_a near ∂S. The difficulty is to estimate the v_a near points $(x,0)$ in ∂S, that is, near $(x_1, 0)$ and $(x_2, 0)$. We will do this by using the solutions $v_{a,\gamma}$ of Proposition 6.5 as super- and subsolutions.

Proposition 6.6. *In the situation above, there exists $\delta > 0$ such that whenever $(x,y) \in S$ with $x \leqslant x_1 + \delta$ then $v_a(x,y)^2 + y^2 \geqslant (x - x_1 - \delta)^4$ for all $a \in (0,1]$, and whenever $(x,y) \in S$ with $x \geqslant x_2 - \delta$ then $v_a(x,y)^2 + y^2 \geqslant (x - x_2 + \delta)^4$ for all $a \in (0,1]$.*

Proof. We first prove the estimate near $(x_1, 0)$. Suppose $\phi(x_1, 0) > 0$. Choose large $\gamma > 0$ and small $\delta, R > 0$ such that the following conditions hold:

(a) $\gamma \geqslant C$, where C is given in Proposition 6.5;

(b) for all $(x,y) \in S$ with $(x - x_1 - \delta)^2 + y^2 = R^2$, we have $\gamma(x_1 - \delta - x) \leqslant \inf_{\partial S} \phi$;

(c) for all $(x,y) \in \partial S$ with $(x - x_1 - \delta)^2 + y^2 \leqslant R^2$ and $x \geqslant x_1 + \delta$, we have $\phi(x,y) \geqslant 0$; and

(d) for all $(x,y) \in \partial S$ with $(x - x_1 - \delta)^2 + y^2 \leqslant R^2$ and $x \leqslant x_1 + \delta$, we have $\phi(x,y) \geqslant \gamma(x_1 + \delta - x)$.

For small enough δ and R, part (c) holds automatically and parts (b) and (d) are approximately equivalent to $\gamma\delta - \gamma R^2/2\kappa \leqslant \inf_{\partial S} \phi$ and $\gamma\delta \leqslant \phi(x_1, 0)$, where $\kappa > 0$ is the radius of curvature of ∂S at $(x_1, 0)$. It is then easy to see that if γ, δ, R satisfy $1 \ll R^{-2} \ll \gamma \ll \delta^{-1}$ then all the conditions hold.

Define $T = \{(x,y) \in S : (x - x_1 - \delta)^2 + y^2 \leqslant R^2\}$, and for each $a \in (0,1]$ define $v'_a \in C^\infty(T)$ by $v'_a(x,y) = v_{a,\gamma}(x_1 + \delta - x, y)$, where $v_{a,\gamma}$ is given in Proposition 6.5. Then v'_a and a satisfy (21) in T, as $v_{a,\gamma}$ and a do. Now T is a domain with piecewise-smooth boundary, which consists of two portions, the first an arc of the circle of radius R about $(x_1+\delta,y)$, and the second a part of ∂S.

We claim that $v_a \geqslant v'_a$ on ∂T. On the first portion of ∂T, the circle arc, we have $v'_a(x,y) = \gamma(x_1-\delta-x)$ by definition of $v_{a,\gamma}$, and thus the claim follows from part (b) above. On the second portion of ∂T, the part of ∂S, the claim follows from parts (c) and (d) above and the facts that $v'_a(x,y) \leqslant 0$ for $x \geqslant x_1+\delta$ and $v'_a(x,y) \leqslant \gamma(x_1+\delta-x)$ for $x \geqslant x_1+\delta$, which in turn follow from the statements in Proposition 6.5 that $v_{a,\gamma}(x,y) \leqslant 0$ when $x \leqslant 0$ and $v_{a,\gamma}(x,y) \leqslant \gamma x$ when $x \geqslant 0$.

Thus v_a and v'_a both satisfy (21) in T, and $v_a \geqslant v'_a$ on ∂T. So by [6, Prop. 8.5] we have $v_a \geqslant v'_a$ on T. If $(x,y) \in S$ with $x \leqslant x_1+\delta$ then $(x,y) \in T$, so that $v_a(x,y) \geqslant v'_a(x,y)$, and also $v'_a(x,y) \geqslant 0$ and $v'_a(x,y)^2+y^2 \geqslant (x-x_1-\delta)^4$ by Proposition 6.5. Combining these gives $v_a(x,y)^2+y^2 \geqslant (x-x_1-\delta)^4$.

This proves the estimate near $(x_1,0)$ in the case $\phi(x_1,0) > 0$. For the case $\phi(x_1,0) < 0$ we instead define $v'_a(x,y) = -v_{a,\gamma}(x_1+\delta-x,y)$, and use it as a supersolution rather than a subsolution. For the estimate near $(x_2,0)$ we use the a similar argument, with $v'_a(x,y) = \pm v_{a,\gamma}(x-x_2+\delta,y)$. This completes the proof. □

The proposition implies that if $(x,y) \in S$ is close to $(x_1,0)$ or $(x_2,0)$ then $v_a(x,y)^2+y^2$ is uniformly bounded below by a positive constant. But if $(x,y) \in S$ is close to ∂S then either (x,y) is close to $(x_1,0)$ or $(x_2,0)$, or else $|y|$ is bounded below by a positive constant, and hence $v_a(x,y)^2+y^2$ is uniformly bounded below by a positive constant. Thus we may prove:

Corollary 6.7. *There exist $\epsilon, J > 0$ such that whenever $(x,y) \in S$ lies within distance 2ϵ of ∂S, then $v_a(x,y)^2+y^2+a^2 \geqslant J^2$ for all $a \in (0,1]$.*

6.4 Estimates on u_a, v_a and f_a on ∂S

Corollary 6.7 implies that (21) is *uniformly elliptic* at v_a near ∂S for all $a \in (0,1]$. We shall use this to prove estimates on u_a, v_a and f_a near ∂S. We begin by bounding the ∂v_a on ∂S.

Proposition 6.8. *There exists $K > 0$ with $\|\partial v_a|_{\partial S}\|_{C^0} \leqslant K$ for all $a \in (0,1]$.*

Proof. Gilbarg and Trudinger [2, Th. 14.1, p. 337] show that if $v \in C^2(S)$ satisfies a quasilinear equation $Qv = 0$ of the form (7) on a domain S and $v|_{\partial S} = \phi \in C^2(\partial S)$, then $\|\partial v|_{\partial S}\|_{C^0} \leqslant K$ for some $K > 0$ depending only on S, upper bounds for $\|v\|_{C^0}$ and $\|\phi\|_{C^2}$, and certain constants to do with Q, which ensure that Q is uniformly elliptic and b not too large.

Examining the proof shows that it is enough for the conditions to hold within distance 2ϵ of ∂S. We apply this to the solutions v_a of (21) for all $a \in (0,1]$. Corollary 6.7 implies that (21) is uniformly elliptic at the v_a within distance 2ϵ of ∂S for all $a \in (0,1]$, and the other conditions of [2, Th. 14.1] easily follow. Thus the theorem gives $K > 0$ such that $\|\partial v_a|_{\partial S}\|_{C^0} \leqslant K$ for all $a \in (0,1]$. □

This implies uniform C^0 bounds for u_a, v_a and C^1 bounds for f_a.

Corollary 6.9. *There exist constants* $K_1, \ldots, K_5 > 0$ *such that*

$$\begin{gathered}\|u_a\|_{C^0} \leqslant K_1,\ \ \|\partial u_a|_{\partial S}\|_{C^0} \leqslant K_2,\ \ \|v_a\|_{C^0} \leqslant K_3,\ \ \|\partial v_a|_{\partial S}\|_{C^0} \leqslant K_4, \\ \|\tfrac{\partial v_a}{\partial x}\|_{C^0} \leqslant K_4, \quad \textit{and} \quad \|f_a\|_{C^1} \leqslant K_5 \quad \textit{for all } a \in (0,1].\end{gathered} \tag{83}$$

Proof. As $|v_a|$ is maximum on ∂S by [6, Cor. 4.4] and $v_a|_{\partial S} = \phi$ we have $\|v_a\|_{C^0} = K_3 = \|\phi\|_{C^0}$ for all a. Proposition 6.8 gives $\|\partial v_a|_{\partial S}\|_{C^0} \leqslant K_4$ for all a, with $K_4 = K$. Thus $\bigl|\frac{\partial v_a}{\partial x}|_{\partial S}\bigr| \leqslant K_4$. But the maximum of $\bigl|\frac{\partial v_a}{\partial x}\bigr|$ is achieved on ∂S by [6, Prop. 8.12]. Hence $\|\frac{\partial v_a}{\partial x}\|_{C^0} \leqslant K_4$. As $(v_a^2+y^2+a^2)^{-1/2} \leqslant J^{-1}$ on ∂S by Corollary 6.7, we see from (17) and $\|\partial v_a|_{\partial S}\|_{C^0} \leqslant K_4$ that $\|\partial u_a|_{\partial S}\|_{C^0} \leqslant K_2$, with $K_2 = \max(\frac{1}{2}J^{-1}, 1)K_4$.

Now $u_a(x_1, 0) = 0$ by definition and $\|\partial u_a|_{\partial S}\|_{C^0} \leqslant K_2$, so $\|u_a|_{\partial S}\|_{C^0} \leqslant \frac{1}{2}K_2\, l(\partial S)$, where $l(\partial S)$ is the length of ∂S. But u_a is maximum on ∂S by [6, Cor. 4.4], so $\|u_a\|_{C^0} \leqslant K_1$, where $K_1 = \frac{1}{2}K_2\, l(\partial S)$. Similarly, $\|\partial f_a\|_{C^0} \leqslant \|u_a\|_{C^0} + \|v_a\|_{C^0} \leqslant K_1 + K_3$, and $f_a(x_1, 0) = 0$, so $\|f_a\|_{C^0} \leqslant \|\partial f_a\|_{C^0} \cdot \mathrm{diam}(S)$. Hence $\|f_a\|_{C^1} \leqslant K_5$, with $K_5 = (K_1 + K_3)(1 + \mathrm{diam}(S))$. □

Corollary 6.9 bounded $\frac{\partial v_a}{\partial x}$ uniformly in S for all $a \in (0,1]$. We now use Theorem 5.5 to prove a slightly weaker result for $\frac{\partial v_a}{\partial y}$.

Proposition 6.10. *There exists* $K_6 > 0$ *such that*

$$\bigl|\tfrac{\partial u_a}{\partial x}\bigr| = \bigl|\tfrac{\partial v_a}{\partial y}\bigr| \leqslant K_6\bigl(v_a^2 + y^2 + a^2\bigr)^{-1} \quad \textit{in } S \textit{ for all } a \in (0,1]. \tag{84}$$

Proof. The bound $\|\partial v_a|_{\partial S}\|_{C^0} \leqslant K$ for all $a \in (0,1]$ implies that there exists $J > 0$ such that the boundary conditions on ∂v in Theorem 5.5 hold for a and v_a for all $a \in (0,1]$. Set $K = 1$ and $L = K_1 + K_3 + 1$, so that $u_a^2 + v_a^2 \leqslant K_1^2 + K_3^2 < L^2$ in S by (83). Applying Theorem 5.5 with these J, K, L and S gives $H > 0$ such that (77) holds in S with $v = v_a$ for all $a \in (0,1]$. Setting $K_6 = H$ completes the proof. □

6.5 Estimates for $\frac{\partial u_a}{\partial x}, \frac{\partial u_a}{\partial y}, \frac{\partial v_a}{\partial x}, \frac{\partial v_a}{\partial y}$ in L^p and u_a, v_a in C^0

Here is an exact expression for a weighted L^2 norm of ∂v_a.

Proposition 6.11. *Let* $\alpha \in [0, \frac{1}{2})$, *and define* $J(a, v) = -\int_0^v (w^2 + a^2)^{-\alpha}\mathrm{d}w$. *Then for each* $a \in (0,1]$ *we have*

$$\int_S (v_a^2+a^2)^{-\alpha}\Bigl[\tfrac{1}{2}\bigl(v_a^2+y^2+a^2\bigr)^{-1/2}\Bigl(\frac{\partial v_a}{\partial x}\Bigr)^2+\Bigl(\frac{\partial v_a}{\partial y}\Bigr)^2\Bigr]\mathrm{d}x\,\mathrm{d}y = \int_{\partial S} J(a, v_a)\mathrm{d}u_a. \tag{85}$$

Proof. Stokes' Theorem gives

$$\begin{aligned}\int_{\partial S} J(a,v_a)\mathrm{d}u_a &= -\int_S \mathrm{d}u_a \wedge \mathrm{d}\big(J(a,v_a)\big) = \int_S (v_a^2+a^2)^{-\alpha}\mathrm{d}u_a\wedge\mathrm{d}v_a \\ &= \int_S (v_a^2+a^2)^{-\alpha}\Big[\frac{\partial u_a}{\partial x}\frac{\partial v_a}{\partial y} - \frac{\partial u_a}{\partial y}\frac{\partial v_a}{\partial x}\Big]\mathrm{d}x\wedge\mathrm{d}y \\ &= \int_S (v_a^2+a^2)^{-\alpha}\Big[\tfrac{1}{2}\big(v_a^2+y^2+a^2\big)^{-1/2}\Big(\frac{\partial v_a}{\partial x}\Big)^2 + \Big(\frac{\partial v_a}{\partial y}\Big)^2\Big]\mathrm{d}x\,\mathrm{d}y,\end{aligned}$$

using (17) to rewrite $\frac{\partial u_a}{\partial x}, \frac{\partial u_a}{\partial y}$ in terms of $\frac{\partial v_a}{\partial y}, \frac{\partial v_a}{\partial x}$. □

We use this to derive uniform L^p estimates on $\frac{\partial u_a}{\partial x}, \frac{\partial u_a}{\partial y}, \frac{\partial v_a}{\partial x}$ and $\frac{\partial v_a}{\partial y}$.

Proposition 6.12. *Let $p \in [2, \frac{5}{2})$, $q \in [1,2)$ and $r \in [1,\infty)$. Then there exist $C_p, C'_q, C''_r > 0$ such that $\|\frac{\partial u_a}{\partial x}\|_{L^p} = \|\frac{\partial v_a}{\partial y}\|_{L^p} \leqslant C_p$, $\|\frac{\partial u_a}{\partial y}\|_{L^q} \leqslant C'_q$ and $\|\frac{\partial v_a}{\partial x}\|_{L^r} \leqslant C''_r$ for all $a \in (0,1]$.*

Proof. Let $p \in [2,\frac{5}{2})$ and define $\alpha = p-2$, so that $\alpha \in [0,\frac{1}{2})$. Then (84) gives

$$\big|\tfrac{\partial v_a}{\partial y}\big|^p = \big|\tfrac{\partial v_a}{\partial y}\big|^\alpha \cdot \big|\tfrac{\partial v_a}{\partial y}\big|^2 \leqslant K_6^\alpha (v_a^2+y^2+a^2)^{-\alpha}\big|\tfrac{\partial v_a}{\partial y}\big|^2 \leqslant K_6^\alpha(v_a^2+a^2)^{-\alpha}\big|\tfrac{\partial v_a}{\partial y}\big|^2.$$

Integrating this over S gives

$$\big\|\tfrac{\partial v_a}{\partial y}\big\|_{L^p}^p \leqslant K_6^\alpha \textstyle\int_S (v_a^2+a^2)^{-\alpha}\big|\tfrac{\partial v_a}{\partial y}\big|^2\mathrm{d}x\,\mathrm{d}y \leqslant K_6^\alpha \int_{\partial S} J(a,\phi)\mathrm{d}u_a,$$

using equation (85) and $v_a|_{\partial S} = \phi$.

Now from $J(a,v) = -\int_0^v (w^2+a^2)^{-\alpha}\mathrm{d}w$ we see that $J(a,0) = 0$ and $\big|\frac{\partial J}{\partial v}(a,v)\big| \leqslant |v|^{-2\alpha}$, so integrating yields $\big|J(a,v)\big| \leqslant (1-2\alpha)^{-1}|v|^{1-2\alpha}$. Putting $v=\phi$ in this and estimating $\mathrm{d}u_a$ using (83) then gives

$$\int_{\partial S} J(a,\phi)\mathrm{d}u_a \leqslant (1-2\alpha)^{-1}\|\phi\|_{C^0}^{1-2\alpha} K_2\, l(\partial S), \tag{86}$$

where $l(\partial S)$ is the length of ∂S. Combining the last two equations and taking p^{th} roots gives $\|\frac{\partial v_a}{\partial y}\|_{L^p} \leqslant C_p$ with $C_p = \big(K_6^\alpha(1-2\alpha)^{-1}\|\phi\|_{C^0}^{1-2\alpha}K_2\, l(\partial S)\big)^{1/p}$.

To prove the second inequality, let $q\in[1,2)$ and put $\alpha = \frac{3}{4} - \frac{1}{2q}$, a different value of α. Then $\alpha \in [\frac{1}{4},\frac{1}{2})$, and using (7) we see that

$$(v_a^2+y^2+a^2)^{1/2-\alpha}\big|\tfrac{\partial u_a}{\partial y}\big|^2 \leqslant \tfrac{1}{4}(v_a^2+a^2)^{-\alpha}(v_a^2+y^2+a^2)^{-1/2}\big|\tfrac{\partial v_a}{\partial x}\big|^2.$$

Integrating this over S and using (85) and (86) shows that

$$\begin{aligned}\int_S (v_a^2+y^2+a^2)^{1/2-\alpha}\big|\tfrac{\partial u_a}{\partial y}\big|^2\mathrm{d}x\,\mathrm{d}y &\leqslant \tfrac{1}{4}\int_{\partial S} J(a,\phi)\mathrm{d}u_a \\ &\leqslant \tfrac{1}{4}(1-2\alpha)^{-1}\|\phi\|_{C^0}^{1-2\alpha}K_2\, l(\partial S).\end{aligned} \tag{87}$$

Also $|y|^{-1/2} \in L^1(S)$, and therefore

$$\int_S (v_a^2 + y^2 + a^2)^{-1/4} \mathrm{d}x\,\mathrm{d}y \leqslant \int_S |y|^{-1/2} \mathrm{d}x\,\mathrm{d}y = \big\| |y|^{-1/2} \big\|_{L^1}. \tag{88}$$

If $f, g \in L^1(S)$ and $\beta \in (0,1)$ then Hölder's inequality gives

$$\int_S |f|^\beta |g|^{1-\beta} \mathrm{d}x\,\mathrm{d}y \leqslant \Big(\int_S |f| \mathrm{d}x\,\mathrm{d}y \Big)^\beta \cdot \Big(\int_S |g| \mathrm{d}x\,\mathrm{d}y \Big)^{1-\beta}.$$

Applying this to equations (87) and (88) with $\beta = q/2$ yields

$$\big\| \tfrac{\partial u_a}{\partial y} \big\|_{L^q}^q \leqslant \big(\tfrac{1}{4}(1-2\alpha)^{-1} \|\phi\|_{C^0}^{1-2\alpha} K_2\, l(\partial S) \big)^{q/2} \big\| |y|^{-1/2} \big\|_{L^1}^{1-q/2},$$

as the powers of $(v_a^2 + y^2 + a^2)$ cancel. Defining C_q' to be the q^{th} root of the r.h.s. then gives the second inequality. Finally, (83) implies that $\| \frac{\partial v_a}{\partial x} \|_{L^r} \leqslant K_4 \operatorname{vol}(S)^{1/r} = C_r''$ for all $r \in [1, \infty)$ and $a \in (0,1]$. □

We use the L^p bounds for $\frac{\partial u_a}{\partial x}, \frac{\partial u_a}{\partial y}, \frac{\partial v_a}{\partial x}$ and $\frac{\partial v_a}{\partial y}$ to show that the u_a, v_a are *uniformly continuous* uniformly in $a \in (0,1]$. This will imply that the limits u_0, v_0 of u_a, v_a as $a \to 0_+$ are continuous.

Proposition 6.13. *In the situation of Definition 6.3, there exist continuous functions $M, N : S \times S \to [0, \infty)$ satisfying $M(x', y', x, y) = M(x, y, x', y')$, $N(x', y', x, y) = N(x, y, x', y')$ and $M(x, y, x, y) = N(x, y, x, y) = 0$ for all $(x, y), (x', y') \in S$, such that for all $(x, y), (x', y') \in S$ and $a \in (0,1]$ we have*

$$\begin{aligned} &\big| u_a(x,y) - u_a(x',y') \big| \leqslant M(x,y,x',y') \\ \textit{and} \quad &\big| v_a(x,y) - v_a(x',y') \big| \leqslant N(x,y,x',y'). \end{aligned} \tag{89}$$

Proof. Choose $p \in [2, \frac{5}{2})$ and $q \in [1, 2)$ with $p^{-1} + q^{-1} < 1$. Then Theorem 2.3 gives continuous functions $G, H : S \times S \to [0, \infty)$. Combining Proposition 6.12 and Theorem 2.3, we see that for all $(x, y), (x', y') \in S$ and $a \in (0,1]$, we have

$$\big| u_a(x,y) - u_a(x',y') \big| \leqslant C_p G(x,y,x',y') + C_q' H(x,y,x',y') = M(x,y,x',y').$$

Similarly, choosing $\hat{p} \in [1, \infty)$ and $\hat{q} \in [2, \frac{5}{2})$ with $\hat{p}^{-1} + \hat{q}^{-1} < 1$, for all $(x, y), (x', y') \in S$ and $a \in (0,1]$ Proposition 6.12 and Theorem 2.3 give

$$\big| v_a(x,y) - v_a(x',y') \big| \leqslant C_{\hat{p}}'' \hat{G}(x,y,x',y') + C_{\hat{q}} \hat{H}(x,y,x',y') = N(x,y,x',y'),$$

where $\hat{G}, \hat{H}$ are the continuous functions $S \times S \to [0, \infty)$ given by Theorem 2.3, starting from $\hat{p}, \hat{q}$ rather than p, q. The conditions $M(x', y', x, y) = M(x, y, x', y')$, $N(x', y', x, y) = N(x, y, x', y')$ and $M(x, y, x, y) = N(x, y, x, y) = 0$ follow from the corresponding conditions on G, H in Theorem 2.3. □

6.6 Existence of weak solutions when $a = 0$

We can now construct a weak solution v_0 of the Dirichlet problem for (21) when $a = 0$.

Proposition 6.14. *Let $(a_n)_{n=1}^{\infty}$ be a sequence in $(0,1]$ with $a_n \to 0$ as $n \to \infty$. Then there exists a subsequence $(a_{n_i})_{i=1}^{\infty}$ and $f_0 \in C^1(S)$ such that $f_{a_{n_i}} \to f_0$ in $C^1(S)$ as $i \to \infty$. Let $u_0 = \frac{\partial f_0}{\partial y}$ and $v_0 = \frac{\partial f_0}{\partial x}$. Then $u_{a_{n_i}} \to u_0$ and $v_{a_{n_i}} \to v_0$ in $C^0(S)$ as $i \to \infty$, and $v_0|_{\partial S} = \phi$.*

Proof. By Ascoli's Theorem [1, Th. 3.15], the inclusion $C^1(S) \hookrightarrow C^0(S)$ is *compact*. But Corollary 6.9 gives $\|f_{a_n}\|_{C^1} \leqslant K_5$ for all n. Thus the f_{a_n} all lie in a compact subset of $C^0(S)$, and there exists a subsequence $(a_{n_i})_{i=1}^{\infty}$ such that $f_{a_{n_i}} \to f_0$ in $C^0(S)$ for some $f_0 \in C^0(S)$ as $i \to \infty$.

Define $A = \{u_a : a \in (0,1]\}$, and regard A as a subset of $C^0(S)$. Then A is *bounded*, as $\|u_a\|_{C^0} \leqslant K_1$ by (83), and *equicontinuous*, by Proposition 6.13. Hence by Ascoli's Theorem [1, Th. 3.15] A is precompact in $C^0(S)$. So the sequence $(u_{a_{n_i}})_{i=1}^{\infty}$ in A must have a convergent subsequence $(u_{a_{\hat{n}_i}})_{i=1}^{\infty}$ in $C^0(S)$, converging to some $u_0 \in C^0(S)$. We shall show that $u_0 = \frac{\partial f_0}{\partial y}$ in S.

Suppose $y_1 < y_2$ and $x \in \mathbb{R}$ with $(x, y) \in S$ whenever $y \in [y_1, y_2]$. Then

$$f_{a_{\hat{n}_i}}(x, y_2) - f_{a_{\hat{n}_i}}(x, y_1) = \int_{y_1}^{y_2} u_{a_{\hat{n}_i}}(x, y)\,\mathrm{d}y.$$

Let $i \to \infty$ in this equation. As $(a_{\hat{n}_i})_{i=1}^{\infty}$ is a subsequence of $(a_{n_i})_{i=1}^{\infty}$ and $f_{a_{n_i}} \to f_0$ in $C^0(S)$ as $i \to \infty$, the left hand side converges to $f_0(x, y_2) - f_0(x, y_1)$. Thus, as $u_{a_{\hat{n}_i}}$ converges uniformly to u_0 in S, we have

$$f_0(x, y_2) - f_0(x, y_2) = \int_{y_1}^{y_2} u_0(x, y)\,\mathrm{d}y.$$

Differentiating this equation with respect to y_2 and using continuity of u_0 shows that $\frac{\partial f_0}{\partial y}$ exists in S and equals u_0.

Thus the limit u_0 is unique, and so the whole sequence $(u_{a_{n_i}})_{i=1}^{\infty}$ converges in $C^0(S)$ to u_0, rather than just some subsequence $(u_{a_{\hat{n}_i}})_{i=1}^{\infty}$, as otherwise we could choose a subsequence converging to a different limit. Using the same argument we prove that $v_0 = \frac{\partial f_0}{\partial x}$ exists in S and lies in $C^0(S)$, and $v_{a_{n_i}} \to v_0$ in $C^0(S)$. As $v_{a_{n_i}}|_{\partial S} = \phi$ it follows that $v_0|_{\partial S} = \phi$.

Since $f_0 \in C^0(S)$ and $\frac{\partial f_0}{\partial x}, \frac{\partial f_0}{\partial y}$ exist and lie in $C^0(S)$ we have $f_0 \in C^1(S)$, and as $f_{a_{n_i}} \to f_0$, $\frac{\partial}{\partial x} f_{a_{n_i}} \to \frac{\partial}{\partial x} f_0$ and $\frac{\partial}{\partial y} f_{a_{n_i}} \to \frac{\partial}{\partial y} f_0$ in $C^0(S)$ as $i \to \infty$, we have $f_{a_{n_i}} \to f_0$ in $C^1(S)$ as $i \to \infty$. This completes the proof. □

In the next result, when we say that u_0, v_0 satisfy (16) and f_0 satisfies (19) *with weak derivatives*, we mean that the corresponding derivatives exist weakly and satisfy the equations.

Proposition 6.15. *The function v_0 satisfies* (21) *weakly with $a = 0$. The derivatives $\frac{\partial u_0}{\partial x}$, $\frac{\partial u_0}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$ exist weakly and satisfy* (16)*, with $\frac{\partial u_0}{\partial x} = \frac{\partial v_0}{\partial y} \in L^p(S)$ for $p \in [1, \frac{5}{2})$, and $\frac{\partial u_0}{\partial y} \in L^q(S)$ for $q \in [1, 2)$, and $\frac{\partial v_0}{\partial x}$ bounded on S. The function f_0 satisfies* (19) *with weak derivatives and $a = 0$.*

Proof. We shall show that v_0 satisfies (22) weakly, which is equivalent to (21). Let $A(a, y, v)$ be as in (20). Then by (17), for all $a \in (0, 1]$ we have

$$\frac{\partial}{\partial x}\big(A(a, y, v_a)\big) = (v_a^2 + y^2 + a^2)^{-1/2}\frac{\partial v_a}{\partial x} = 2\frac{\partial u_a}{\partial y}.$$

Thus Proposition 6.12 gives $\|\frac{\partial}{\partial x}A(a, y, v_a)\|_{L^q} \leqslant 2C'_q$ and $\|\frac{\partial v_a}{\partial y}\|_{L^p} \leqslant C_p$ for $p \in [2, \frac{5}{2})$, $q \in (1, 2)$ and all $a \in (0, 1]$.

The inclusions $L^p(S) \hookrightarrow L^1(S)$ and $L^q(S) \hookrightarrow L^1(S)$ are *compact* by Aubin [1, Th. 2.33]. Thus the functions $\frac{\partial}{\partial x}A(a, y, v_a)$ for all $a \in (0, 1]$ and $\frac{\partial v_a}{\partial y}$ for all $a \in (0, 1]$ lie in compact subsets of $L^1(S)$. So we may choose subsequences of the sequences $(\frac{\partial}{\partial x}A(a_{n_i}, y, v_{a_{n_i}}))_{i=1}^\infty$ and $(\frac{\partial v_{a_{n_i}}}{\partial y})_{i=1}^\infty$ which converge in $L^1(S)$.

By an argument similar to that in the proof of Proposition 6.14, we can show that these limits are the weak derivatives $\frac{\partial}{\partial x}A(0, y, v_0)$ and $\frac{\partial}{\partial y}v_0$. So the limits are unique, and the whole sequence converges to them, not just a subsequence. That is, $\frac{\partial}{\partial x}A(0, y, v_0)$ and $\frac{\partial}{\partial y}v_0$ both exist weakly in $L^1(S)$, and $\frac{\partial}{\partial x}A(a_{n_i}, y, v_{a_{n_i}}) \to \frac{\partial}{\partial x}A(0, y, v_0)$ and $\frac{\partial}{\partial y}v_{a_{n_i}} \to \frac{\partial}{\partial y}v_0$ in $L^1(S)$ as $i \to \infty$.

Let $\psi \in C_0^1(S)$. As $v_{a_{n_i}}$ and a_{n_i} satisfy (22), multiplying by ψ and integrating by parts gives

$$-\int_S \frac{\partial \psi}{\partial x} \cdot \frac{\partial}{\partial x}A(a_{n_i}, y, v_{a_{n_i}})\,\mathrm{d}x\,\mathrm{d}y - 2\int_S \frac{\partial \psi}{\partial y} \cdot \frac{\partial v_{a_{n_i}}}{\partial y}\,\mathrm{d}x\,\mathrm{d}y = 0,$$

Letting $i \to \infty$ in this equation shows that

$$-\int_S \frac{\partial \psi}{\partial x} \cdot \frac{\partial}{\partial x}A(0, y, v_0)\,\mathrm{d}x\,\mathrm{d}y - 2\int_S \frac{\partial \psi}{\partial y} \cdot \frac{\partial v_0}{\partial y}\,\mathrm{d}x\,\mathrm{d}y = 0,$$

since $\frac{\partial}{\partial x}A(a_{n_i}, y, v_{a_{n_i}}) \to \frac{\partial}{\partial x}A(0, y, v_0)$ and $\frac{\partial}{\partial y}v_{a_{n_i}} \to \frac{\partial}{\partial y}v_0$ in $L^1(S)$ as $i \to \infty$, and $\frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y} \in C^0(S)$. Thus v_0 satisfies (22) weakly with $a = 0$.

The proof above shows that $\frac{\partial v_0}{\partial y}$ exists weakly and is the limit in $L^1(S)$ of $\frac{\partial}{\partial y}v_{a_{n_i}}$ as $i \to \infty$. Since $\frac{\partial}{\partial y}v_{a_{n_i}} = \frac{\partial}{\partial x}u_{a_{n_i}}$, it easily follows that $\frac{\partial u_0}{\partial x}$ exists weakly and is the limit in $L^1(S)$ of $\frac{\partial}{\partial x}u_{a_{n_i}}$ as $i \to \infty$. Also, as $\frac{\partial}{\partial x}A(a_{n_i}, y, v_{a_{n_i}}) = -2\frac{\partial}{\partial y}u_{a_{n_i}}$, a similar argument shows that $\frac{\partial u_0}{\partial y}$ exists weakly and is the limit in $L^1(S)$ of $\frac{\partial}{\partial y}u_{a_{n_i}}$ as $i \to \infty$.

Since $v_{a_{n_i}}^2 + y^2 + a_{n_i}^2$ is bounded above we can use this and (17) to deduce that $\frac{\partial v_0}{\partial x}$ exists weakly and is the limit in $L^1(S)$ of $\frac{\partial}{\partial x}v_{a_{n_i}}$ as $i \to \infty$. Thus, the first derivatives of u_0, v_0 and the second derivatives of f_0 exist weakly, and are the limits in $L^1(S)$ of the corresponding derivatives of $u_{a_{n_i}}, v_{a_{n_i}}, f_{a_{n_i}}$ as $i \to \infty$. The estimates in Proposition 6.12 and Corollary 6.9 then imply that

$\frac{\partial u_0}{\partial x} = \frac{\partial v_0}{\partial y} \in L^p(S)$ for $p \in [1, \frac{5}{2})$, $\frac{\partial u_0}{\partial y} \in L^q(S)$ for $q \in [1, 2)$, and $\frac{\partial v_0}{\partial x}$ is bounded on S. Taking the limit in $L^1(S)$ as $i \to \infty$ of equations (17) and (19) then completes the proof. □

Away from singular points $(x, 0)$ with $v_0(x, 0) = 0$ we can prove much stronger regularity of u_0, v_0. Near a nonsingular point (x, y), equation (18) with $a = 0$ is strictly elliptic. We use results of [2, §6] to show that f_0 is $C^{2,\alpha}$ near (x, y) and satisfies (18) with $a = 0$, and then [6, Th. 3.6] gives:

Proposition 6.16. *Except at singular points $(x, 0)$ with $v_0(x, 0) = 0$, the functions u_0, v_0 are $C^{k+2,\alpha}$ in S and real analytic in S°.*

6.7 Uniqueness of weak solutions when $a = 0$

Weak solutions of the Dirichlet problem for (21) are unique.

Proposition 6.17. *Let $v, v' \in C^0(S) \cap L^1_1(S)$ be weak solutions of (21) on S with $a = 0$ and $v|_{\partial S} = v'|_{\partial S} = \phi$. Then $v = v'$.*

Proof. Following Proposition 3.4 but using weak solutions we find that there exist $u, u' \in L^1_1(S)$ such that u, v and u', v' satisfy (16) with weak derivatives. Using the ideas of §6.4 we can show that u, u', v, v' are $C^{k+2,\alpha}$ near ∂S, so u, u', v, v' are bounded. From Proposition 3.3 we see that there exist $f, f' \in C^{0,1}(S) \cap L^2_1(S)$ with $\frac{\partial f}{\partial y} = u$, $\frac{\partial f}{\partial x} = v$, $\frac{\partial f'}{\partial y} = u'$ and $\frac{\partial f'}{\partial x} = v'$ weakly, that satisfy (19) with weak derivatives, and f, f' are $C^{k+3,\alpha}$ near ∂S.

Let $\gamma \in \mathbb{R}$. Then as in [6, Prop. 7.5] we find that $f - f' + \gamma y$ weakly satisfies an equation $L(f - f' + \gamma y) = 0$, where L is a linear elliptic operator of the form (6) with $c \equiv 0$. Thus by the maximum principle for elliptic operators [2, Th. 3.1], which holds for weak solutions by [2, p. 45-6], the maximum and minimum of $f - f' + \gamma y$ occur on ∂S. Furthermore, one can use [2, Lem. 3.4] to show that either the normal derivatives of $f - f' + \gamma y$ at the maximum and minimum are nonzero, or else $f - f' + \gamma y$ is constant in S.

Let the maximum of $f - f' + \gamma y$ occur at $(x, y) \in \partial S$. Then

$$\frac{\partial}{\partial x}\bigl(f - f' + \gamma y\bigr)\big|_{(x,y)} = v(x, y) - v'(x, y) = \phi(x, y) - \phi(x, y) = 0.$$

But the derivative of $f - f' + \gamma y$ at (x, y) tangent to ∂S is also zero. Thus, if $\frac{\partial}{\partial x}$ is not tangent to ∂S at (x, y) then the normal derivative of $f - f' + \gamma y$ at (x, y) is zero, and so $f - f' + \gamma y$ is constant.

Therefore either $f - f' + \gamma y$ is constant, or else the maximum (and similarly the minimum) of $f - f' + \gamma y$ occur at points (x, y) in ∂S where $\frac{\partial}{\partial x}$ is tangent to ∂S. But as S is strongly convex there are only two points $\mathbf{x}_1, \mathbf{x}_2$ in ∂S with $\frac{\partial}{\partial x}$ tangent to ∂S. Choose $\gamma \in \mathbb{R}$ uniquely so that $\frac{\partial}{\partial y}\bigl(f - f' + \gamma y\bigr) = 0$ at $\mathbf{x}_1$. Then either $f - f' + \gamma y$ is constant, or else the maximum and minimum of $f - f' + \gamma y$ both occur at $\mathbf{x}_2$, again implying that $f - f' + \gamma y$ is constant. Taking $\frac{\partial}{\partial x}$ gives $v = v'$. □

This implies that the limit v_0 chosen in Proposition 6.14 is *unique*. Thus, the entire sequence $(v_{a_n})_{n=1}^{\infty}$ converges to v_0 in $C^0(S)$ rather than just the subsequence $(v_{a_{n_i}})_{i=1}^{\infty}$, since otherwise we could have chosen a different limit v_0. As this is true for an arbitrary sequence $(a_n)_{n=1}^{\infty}$ in $(0,1]$, this shows that $v_a \to v_0$ in $C^0(S)$ as $a \to 0_+$. Similarly, the limits u_0, f_0 are also unique, as the freedoms to add constants are fixed by $u_0(x_1, 0) = f_0(x_1, 0) = 0$. So we deduce:

Corollary 6.18. *As $a \to 0_+$ in $(0,1]$ we have $u_a \to u_0$, $v_a \to v_0$ in $C^0(S)$ and $f_a \to f_0$ in $C^1(S)$.*

Theorem 6.1 now follows from Propositions 6.14–6.17. For Theorem 6.2, continuity in the a variable at $a = 0$ for fixed ϕ follows from Corollary 6.18, since $u_a \to u_0$ and $v_a \to v_0$ in $C^0(S)$ as $a \to 0_+$. To prove continuity in the ϕ variable we need to show that small $C^{k+2,\alpha}$ changes in ϕ result in small C^0 changes in u, v, and this can be seen by examining the proofs above.

7 The Dirichlet problem for f when $a = 0$

Theorem 3.5 shows that the Dirichlet problem for equation (18) is uniquely solvable in strictly convex domains for $a \neq 0$. In this section we will use the material of §5 and §6 to show that the Dirichlet problem also has a unique solution when $a = 0$, but with weak second derivatives.

7.1 The main results

Here is an analogue of Theorem 3.5 in the case $a = 0$.

Theorem 7.1. *Let S be a strictly convex domain in $\mathbb{R}^2$ invariant under the involution $(x, y) \mapsto (x, -y)$, let $k \geqslant 0$ and $\alpha \in (0,1)$. Then for each $\phi \in C^{k+3,\alpha}(\partial S)$ there exists a unique weak solution f of* (19) *in $C^1(S)$ with $a = 0$ and $f|_{\partial S} = \phi$. Furthermore f is twice weakly differentiable and satisfies* (18) *with weak derivatives.*

Let $u = \frac{\partial f}{\partial y}$ and $v = \frac{\partial f}{\partial x}$. Then $u, v \in C^0(S)$ are weakly differentiable and satisfy (16) *with weak derivatives, and v satisfies* (21) *weakly with $a = 0$. The weak derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ satisfy $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \in L^p(S)$ for $p \in [1,2]$, and $\frac{\partial u}{\partial y} \in L^q(S)$ for $q \in [1,2)$, and $\frac{\partial v}{\partial x}$ is bounded on S. Also u, v are $C^{k+2,\alpha}$ in S and real analytic in S° except at singular points $(x, 0)$ with $v(x, 0) = 0$.*

Combined with Proposition 3.2 the theorem can be used to construct large numbers of U(1)-*invariant singular special Lagrangian* 3-*folds* in $\mathbb{C}^3$. This is the principal motivation for the paper. The singularities of these special Lagrangian 3-folds will be studied in [7].

Our second theorem extends [6, Th. 7.7] to include the case $a = 0$.

Theorem 7.2. *Let S be a strictly convex domain in $\mathbb{R}^2$ invariant under the involution $(x, y) \mapsto (x, -y)$, let $k \geqslant 0$ and $\alpha \in (0,1)$. Then the map $C^{k+3,\alpha}(\partial S) \times \mathbb{R} \to C^1(S)$ taking $(\phi, a) \mapsto f$ is continuous, where f is the unique solution of*

(18) *(with weak derivatives) with* $f|_{\partial S} = \phi$ *constructed in Theorem 3.5 when* $a \neq 0$*, and in Theorem 7.1 when* $a = 0$*. This map is also continuous in stronger topologies on* f *than the* C^1 *topology.*

The proofs of these theorems take up the rest of the paper, and are similar to those of Theorems 6.1 and 6.2. Here is how they are laid out. Let S, ϕ be as in Theorem 7.1. In §7.2, for each $a \in (0, 1]$ we define $f_a \in C^{k+3,\alpha}(S)$ to be the unique solution of (18) in S with $f_a|_{\partial S} = \phi$, and $u_a = \frac{\partial f_a}{\partial y}$, $v_a = \frac{\partial f_a}{\partial x}$. As in §6, we aim to show that $u_a, v_a, f_a \to u_0, v_0, f_0$ as $a \to 0_+$, where u_0, v_0, f_0 are u, v, f in Theorem 7.1, and the main issue is to prove *a priori estimates* of u_a, v_a, f_a that are *uniform in* a.

However, there are some important differences with §6. On the positive side, Theorem 3.5 immediately gives uniform bounds for $\|f_a\|_{C^1}$, $\|u_a\|_{C^0}$ and $\|v_a\|_{C^0}$. But as we have no analogue of the $\phi(x, 0) \neq 0$ assumption in Theorem 6.1, we must allow v_a to be zero at or near points $(x, 0)$ in ∂S. Because of this, equation (18) at f_a need *not* be uniformly elliptic in a close to ∂S, so the methods of §6.3–§6.4 for estimating u_a, v_a, f_a near ∂S do not work.

Instead, we do something different. The second derivatives of ϕ give us a bound on half of $\partial u_a, \partial v_a$ on ∂S. Using this, and supposing ∂S positively curved at points $(x, 0)$, in §7.3 we use a boundary version of the method of §5 to estimate the other half of $\partial u_a, \partial v_a$ on ∂S near points $(x, 0)$.

Section 7.4 then gives partial analogues of the a priori estimates of §6.4–§6.5. As our bounds on $\frac{\partial v_a}{\partial y}$ on ∂S are not strong enough to apply the global estimates Theorem 5.5 uniformly in a, we instead have to work in interior domains $T \subset S^\circ$ for our L^p estimates of $\frac{\partial u_a}{\partial x}, \frac{\partial v_a}{\partial y}$ when $p \in (2, \frac{5}{2})$.

In §7.5 we establish uniform continuity of the u_a, v_a, as in §6.5. But because we only have interior L^p estimates of $\frac{\partial u_a}{\partial x}$, we must do some extra work to show u_a is uniformly continuous near points $(x, 0)$ in ∂S. Finally, §7.6 proves existence of the limit solutions f_0, u_0, v_0, which is just as in §6.6, and uniqueness, completing the proofs.

The hypotheses of Theorems 7.1 and 7.2 can be relaxed slightly, without changing the proofs: rather than requiring S invariant under $(x, y) \mapsto (x, -y)$ we can ask only that each point $(x, 0)$ in ∂S has tangent $T_{(x,0)}\partial S$ parallel to the y-axis, and rather than requiring $\phi \in C^{k+3,\alpha}(\partial S)$ we can ask only that $\phi \in C^3(\partial S)$.

Note that these two theorems do not have the awkward restriction that $\phi(x, 0) \neq 0$ for points $(x, 0) \in \partial S$ in Theorems 6.1 and 6.2. For this reason we find them more convenient for applications such as constructing special Lagrangian fibrations on subsets of $\mathbb{C}^3$, and we will generally use them in preference to Theorems 6.1 and 6.2 in the sequel [7].

7.2 A family of solutions f_a to (18)

Consider the following situation:

Definition 7.3. Let S be a strictly convex domain in $\mathbb{R}^2$ which is invariant under the involution $(x, y) \mapsto (x, -y)$. Then there exist unique $x_1, x_2 \in \mathbb{R}$ with

$x_1 < x_2$ and $(x_i, 0) \in \partial S$ for $i = 1, 2$. Let $k \geqslant 0$ and $\alpha \in (0,1)$, and suppose $\phi \in C^{k+3,\alpha}(\partial S)$. For each $a \in (0,1]$, let $f_a \in C^{k+3,\alpha}(S)$ be the unique solution of (18) in S with this value of a and $f_a|_{\partial S} = \phi$, which by Theorem 3.5 exists and satisfies $\|f_a\|_{C^1} \leqslant C\|\phi\|_{C^2}$ for all $a \in (0,1]$, where $C > 0$ depends only on S, and in particular is independent of a. Set

$$X = C\|\phi\|_{C^2} \quad \text{and} \quad Y = \sup_{(x,y)\in S} |y|. \tag{90}$$

Define $u_a, v_a \in C^{k+2,\alpha}(S)$ by $u_a = \frac{\partial f_a}{\partial y}$ and $v_a = \frac{\partial f_a}{\partial x}$. Then

$$\|f_a\|_{C^1}, \|u_a\|_{C^0}, \|v_a\|_{C^0} \leqslant X \quad \text{for all } a \in (0,1], \tag{91}$$

and u_a, v_a and a satisfy (17), and v_a and a satisfy (18).

We will show that f_a converges in $C^1(S)$ to $f_0 \in C^1(S)$ as $a \to 0_+$, and that f_0 is the unique weak solution of the Dirichlet problem for (18) on S when $a = 0$. The main difficulty in doing this is to establish uniform continuity of u_a and v_a for all $a \in (0,1]$, as we did for the v Dirichlet problem in §6.5. Once we have done this, we can follow the proofs of §6 with few changes.

First we bound the f_a in C^2 away from the x-axis.

Proposition 7.4. *Let $\epsilon > 0$ be small, and set $S_\epsilon = \{(x,y) \in S : |y| > \epsilon\}$. Then there exists $G > 0$ such that $\|f_a|_{S_\epsilon}\|_{C^2} \leqslant G$ for all $a \in (0,1]$.*

Proof. Let $S_{\epsilon/2}$ and $S_{\epsilon/4}$ be defined in the obvious way. We prove the proposition in two steps. Regard P as in (18) as a *linear* operator, with coefficients a^{ij} depending on v_a. Firstly we use estimates on $S_{\epsilon/4}$ to bound $\|f_a|_{S_{\epsilon/2}}\|_{C^{1,\gamma}}$ uniformly in $a \in (0,1]$ for some $\gamma \in (0,1)$. This gives a uniform bound Λ' for $\|a^{ij}|_{S_{\epsilon/2}}\|_{C^{0,\gamma}}$. Secondly, we bound $\|f_a|_{S_\epsilon}\|_{C^{2,\gamma}}$ uniformly in a.

The results we need are *interior regularity results* for linear elliptic operators on *noncompact regions* in $\mathbb{R}^2$ with *boundary portions*. In the first step we use Gilbarg and Trudinger [2, Th. 12.4, p. 302], extended to the boundary case as in [2, p. 303-4]. This deals with equations of the form $Pu = f$, where P is a linear elliptic operator of the form (6) with $a^{ij} \in C^0(S)$ and $b^i = c = 0$. Note that (18) is of this form.

Now $\frac{1}{16}\epsilon^2 \leqslant v_a^2 + y^2 + a^2 \leqslant X^2 + Y^2 + 1$ on $S_{\epsilon/4}$. Raising this to the power $-\frac{1}{2}$ we see from (18) that $\sum_{i,j=1}^2 a^{ij}\xi_i\xi_j \geqslant \lambda(\xi_1^2 + \xi_2^2)$ for all $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ and $\|a^{ij}\|_{C^0} \leqslant \Lambda$ hold in $S_{\epsilon/4}$, with $\lambda = \min\bigl((X^2 + Y^2 + 1)^{-1/2}, 2\bigr)$ and $\Lambda = \max(4\epsilon^{-1}, 2)$. Therefore [2, p. 302-4] gives $\gamma \in (0,1)$ depending only on Λ/λ, and $G' > 0$ such that $\|f_a|_{S_{\epsilon/2}}\|_{C^{1,\gamma}} \leqslant G'$ for all $a \in (0,1]$.

This gives a uniform bound Λ' for $\|a^{ij}|_{S_{\epsilon/2}}\|_{C^{0,\gamma}}$. Following [2, Lem. 6.18], which gives a priori $C^{2,\gamma}$ interior estimates for solutions of linear elliptic equations with $C^{0,\gamma}$ coefficients in domains with boundary portions, we find that there exists $G > 0$ depending on $\lambda, \Lambda, \gamma, G', S_{\epsilon/2}$ and S_ϵ such that $\|f_a|_{S_\epsilon}\|_{C^{2,\gamma}} \leqslant G$ for all $a \in (0,1]$. Hence $\|f_a|_{S_\epsilon}\|_{C^2} \leqslant G$ for all a, as we want. □

Thus, $\partial^2 f_a$ is uniformly bounded on ∂S except arbitrarily close to the points $(x_i, 0)$ for $i = 1, 2$. So we shall study u_a, v_a and f_a near these points. Fix $i = 1$ or 2. As S is invariant under $(x, y) \mapsto (x, -y)$, the tangent to ∂S at $(x_i, 0)$ is parallel to the y-axis. Thus we may use y as a parameter on ∂S near $(x_i, 0)$, and write $(x, y) \in \partial S$ near $(x_i, 0)$ as $\bigl(x(y), y\bigr)$. Regard ϕ as a function of y near $(x_i, 0)$. Then differentiating the equation $f_a\bigl(x(y), y\bigr) = \phi(y)$ once and twice w.r.t. y gives

$$\dot\phi = u_a(x, y) + v_a(x, y)\dot x \qquad \text{and} \tag{92}$$

$$\begin{aligned} \ddot\phi &= \frac{\partial u_a}{\partial x}\,\dot x + \frac{\partial u_a}{\partial y} + \frac{\partial v_a}{\partial x}\,\dot x^2 + \frac{\partial v_a}{\partial y}\,\dot x + v_a\,\ddot x \\ &= \frac{\partial u_a}{\partial y}\bigl(1 - 2(v_a^2 + y^2 + a^2)^{1/2}\dot x^2\bigr) + 2\frac{\partial u_a}{\partial x}\,\dot x + v_a\,\ddot x, \end{aligned} \tag{93}$$

writing '$\dot{}$' for $\frac{\mathrm{d}}{\mathrm{d}y}$, and using (17) in the final line.

Now when $y = 0$ we have $\dot x = 0$ and $\ddot x = \kappa_i$, where κ_i is the curvature of ∂S at $(x_i, 0)$, measured in the direction of increasing x. Thus $\dot x \approx \kappa_i y$ to leading order in y near $(x_i, 0)$. As S is strictly convex, it follows that $\kappa_1 > 0$ and $\kappa_2 < 0$. We can use (93) to prove:

Proposition 7.5. *There exist $\epsilon, H > 0$ and smooth functions $F_{i,a} : [-\epsilon, \epsilon] \to \mathbb{R}$ for $i = 1, 2$ and $a \in (0, 1]$ with $\bigl|F_{i,a}(y) - 2\kappa_i y\bigr| \leqslant |\kappa_i y|$, such that for all $(x, y) \in \partial S$ close to $(x_i, 0)$ and with $|y| \leqslant \epsilon$, we have*

$$\Bigl|\frac{\partial u_a}{\partial y}(x, y) + F_{i,a}(y)\frac{\partial u_a}{\partial x}(x, y)\Bigr| < H \quad \textit{for all } a \in (0, 1]. \tag{94}$$

Here $F_{i,a}(y) = 2\dot x\bigl(1 - 2(v_a^2 + y^2 + a^2)^{1/2}\dot x^2\bigr)^{-1}$, so it does depend on a and v_a. But the approximations above give $F_{i,a}(y) \approx 2\kappa_i y$ for small y, and one can show that for $\epsilon > 0$ depending only on S and upper bounds $1, X$ for $|a|, |v_a|$, if $|y| \leqslant \epsilon$ then $\bigl|F_{i,a}(y) - 2\kappa_i y\bigr| \leqslant |\kappa_i y|$.

7.3 An a priori bound for $\frac{\partial u_a}{\partial x}, \frac{\partial v_a}{\partial y}$ on ∂S

Proposition 7.4 bounds $\partial u_a, \partial v_a$ on ∂S away from the x-axis, and Proposition 7.5 in effect bounds half of ∂u_a, and hence ∂v_a, on ∂S near the a-axis. The following theorem in effect bounds the other half of ∂u_a and ∂v_a near the x-axis. The proof, which is unfortunately rather long and complicated, adapts the method of §5.2.

Theorem 7.6. *There exist $\delta, J > 0$ such that for all $(x_0, y_0) \in \partial S$ close to $(x_i, 0)$ for $i = 1$ or 2 and with $|y_0| \leqslant \delta$, we have*

$$\Bigl|\frac{\partial u_a}{\partial x}(x_0, y_0)\Bigr| = \Bigl|\frac{\partial v_a}{\partial y}(x_0, y_0)\Bigr| \leqslant J\bigl(y_0^2 + a^2\bigr)^{-1/2} \quad \textit{for all } a \in (0, 1]. \tag{95}$$

Proof. Choose small $\delta > 0$ and large $J > 0$, to satisfy conditions we will give during the proof. Suppose, for a contradiction, that $(x_0, y_0) \in \partial S$ is close to $(x_i, 0)$ for $i = 1$ or 2 with $|y_0| \leqslant \delta$, and that (95) does not hold for some given $a \in (0, 1]$. Let ϵ be as in Proposition 7.5, and suppose $\delta \leqslant \epsilon$. Then Proposition 7.5 implies that (94) holds at (x_0, y_0). Define

$$u_0 = u_a(x_0, y_0),\ \ v_0 = v_a(x_0, y_0),\ \ p_0 = \tfrac{\partial v_a}{\partial x}(x_0, y_0) \quad \text{and} \quad q_0 = \tfrac{\partial v_a}{\partial y}(x_0, y_0).$$

Then as (95) does not hold we have

$$|q_0| > J(y_0^2 + a^2)^{-1/2}, \tag{96}$$

and from (17) and Proposition 7.5 we deduce that

$$\begin{gathered} |p_0| < 2(v_0^2 + y_0^2 + a^2)^{1/2}\bigl|F_{i,a}(y_0)\bigr| \cdot |q_0| + 2(v_0^2 + y_0^2 + a^2)^{1/2} H, \\ \text{where} \quad \bigl|F_{i,a}(y_0) - 2\kappa_i y_0\bigr| \leqslant |\kappa_i y_0|. \end{gathered} \tag{97}$$

These imply that q_0 is large, and that p_0 is small compared to q_0.

Set $L = 10X$, and define $\hat{x}_0, \hat{y}_0, \hat{u}_0, \hat{v}_0, \hat{p}_0, \hat{q}_0$ as in (72). Then as in the proof of Proposition 5.9, we can show that if $\delta > 0$ is small enough and $J > 0$ large enough there exist $\hat{u}, \hat{v} \in C^\infty(D_L)$ satisfying (17) and (24), and with $(\hat{u} - \hat{u}_0, \hat{v} - \hat{v}_0)$ bounded by a small constant. Furthermore, as q_0 is large and p_0 small compared to q_0, we find that $\hat{u}, \hat{v}$ approximate the affine maps (79) up to their first derivatives.

Define $U = (\hat{u}, \hat{v})(D_L)$. Then as in the proof of Proposition 5.9, U is approximately a closed disc of radius $|\hat{q}_0|L$, and $(\hat{u}, \hat{v}) : D_L \to U$ is invertible with differentiable inverse $(u', v') : U \to D_L$. Moreover, u', v' satisfy (17) in U, and by construction we have $u_a = u'$, $v_a = v'$, $\partial u_a = \partial u'$ and $\partial v_a = \partial v'$ at (x_0, y_0). Thus $(u', v') - (u_a, v_a)$ has a zero of *multiplicity* at least 2 at (x_0, y_0), in the sense of [6, Def. 6.3].

To complete the proof, we follow a similar strategy to Proposition 5.9. Roughly speaking, we shall show that the winding number of $(u', v') - (u_a, v_a)$ about 0 along the boundary of $U \cap S$ is at most 1. But this contradicts [6, Th. 6.7], as the number of zeroes of $(u', v') - (u_a, v_a)$ in $U \cap S$ counted with multiplicity should be at most 1, but we already know there is a zero of multiplicity at least 2 at (x_0, y_0). This then proves the theorem.

There are two problems with this strategy. The first is that (x_0, y_0) lies on the boundary of $U \cap S$ rather than the interior, and so the winding number of $(u', v') - (u_a, v_a)$ about 0 along $\partial(U \cap S)$ is not defined, and [6, Th. 6.7] does not apply. To deal with this we perturb (x_0, y_0) a very little way into the interior of $U \cap S$, and construct a slightly different (u', v') to intersect (u_a, v_a) with multiplicity 2 at the new point (x_0, y_0) instead. Since u_a, v_a are C^1 and (96) and (97) are open conditions, we can still assume that (96) and (97) hold at the new (x_0, y_0).

The second problem is how to prove that the winding number of $(u', v') - (u_a, v_a)$ about 0 along $\partial(U \cap S)$ is at most 1, given that we do not know much about the behaviour of (u_a, v_a). We shall use the method of [6, Th. 7.10], which bounds the number of zeroes of $(u_1, v_1) - (u_2, v_2)$ in S in terms of the stationary points of the difference $f_1 - f_2$ of their potentials. Here is the crucial step in the proof.

Proposition 7.7. *Let $f' \in C^\infty(U)$ be a potential for u', v', as in Proposition 3.3. Then $f' - f_a$ has at most two stationary points on the curve $U \cap \partial S$.*

Proof. From above, $\hat{u}, \hat{v}$ approximate the affine maps in (79) up to their first derivatives. Inverting this, we find that u', v' also approximate the affine maps

$$u'(x, y) \approx u_0 + q_0(x - x_0), \qquad v'(x, y) \approx v_0 + q_0(y - y_0) \tag{98}$$

up to their first derivatives. Hence the potential f' approximates the quadratic

$$f'(x, y) \approx u_0(y - y_0) + v_0(x - x_0) + q_0(x - x_0)(y - y_0) + c \tag{99}$$

up to its second derivatives, for some $c \in \mathbb{R}$. We are being vague about what we mean by 'approximates' here. An exact statement can be derived by using Theorem 4.21 to estimate $(\hat{u}, \hat{v})$ and its derivatives, and then inverting.

As in §7.2, we may parametrize ∂S near $(x_i, 0)$ as $\bigl(x(y), y\bigr)$ with $x(y) \approx x_i + \frac{1}{2}\kappa_i y^2$ for small y, where $\kappa_i \neq 0$ is the curvature of ∂S at $(x_i, 0)$. It follows that the restriction of f' to ∂S is approximately

$$f'\bigl(x(y), y\bigr) \approx \tfrac{1}{2}\kappa_i q_0 y^3 + \tfrac{1}{2}\kappa_i(v_0 - q_0 y_0)y^2 + (u_0 + q_0 x_i - q_0 x_0)y + c', \tag{100}$$

where $c' = c - u_0 y_0 + (q_0 + v_0)(x_i - x_0)$. By making J large enough we can suppose that $|q_0| \gg 2|\kappa_i|^{-1}$ by (96). Thus $f'|_{U \cap \partial S}$ approximates a cubic polynomial in y with *large third derivative*.

Now $f_a|_{\partial S} = \phi \in C^{k+3,\alpha}(\partial S)$. So by choosing J large compared to $\|\phi\|_{C^3}$ we can ensure that $\bigl|\frac{\mathrm{d}^3}{\mathrm{d}y^3} f_a|_{U \cap \partial S}\bigr| \ll 3\bigl|\kappa_i q_0\bigr|$. Therefore, provided the approximations are valid, $\frac{\mathrm{d}^3}{\mathrm{d}y^3}(f' - f_a)|_{U \cap \partial S}$ has the same sign as $\kappa_i q_0$ on $U \cap \partial S$, and it easily follows that $f' - f_a$ has at most two stationary points on $U \cap \partial S$.

To make this into a rigorous argument, we need to consider the approximations above very carefully using the estimates of §4.5, and make sure that for small ϵ and large J, the effect on the second and third derivatives of f' of the 'error terms' we have neglected are always significantly smaller than the 'leading terms' given in (99) and (100). We will not do this in detail, as it is long and dull, but here is a sketch of the important steps.

Firstly, U is approximately a closed disc of radius $|\hat{q}_0|L$ with centre $(x_0, y_0) - \hat{q}_0(u_0, v_0)$. Since $u_0^2 + v_0^2 \leqslant X^2$ and $L = 10X$, it follows that for $(x, y) \in U$ we have $|y - y_0| \leqslant \frac{11}{10}|\hat{q}_0|L$, approximately. As $\hat{q}_0 \approx q_0^{-1}$, we see from (96) that if $(x, y) \in U$ then $|y - y_0| \leqslant \frac{11}{10} J^{-1} L (y_0^2 + a^2)^{1/2}$, approximately. So by choosing $J \gg L$ we easily show that

$$\tfrac{1}{4}(y_0^2 + a^2) \leqslant y^2 + a^2 \leqslant 4(y_0^2 + a^2) \quad \text{on } U.$$

As $(\hat{u}, \hat{v})$ maps $D_L \to U$, this implies that

$$\tfrac{1}{4}(y_0^2 + a^2) \leqslant \hat{v}^2 + a^2 \leqslant 4(y_0^2 + a^2) \quad \text{on } D_L.$$

Now $(\hat{u},\hat{v})$ is constructed using the solutions of Theorem 4.1, and these are estimated in terms of a parameter s, which is roughly a lower bound for $(v^2+y^2+a^2)^{1/2}$. The equation above implies that $(\hat{v}^2+y^2+a^2)^{1/2} \geqslant \frac{1}{2}(y_0^2+a^2)^{1/2}$ on D_L. This means that for the solutions $(\hat{u},\hat{v})$ the parameter s in §4 is approximately $(y_0^2+a^2)^{1/2}$, up to a bounded factor. Therefore we can use Theorem 4.21 to estimate the derivatives of $\hat{v}$ in terms of powers of $(y_0^2+a^2)^{1/2}$.

Secondly, some detail on the estimates we need. For the argument above to work, we need the approximation (99) to hold up to third derivatives with errors small compared to q_0. That is, we need

$$\frac{\partial u'}{\partial x}=\frac{\partial v'}{\partial y}=q_0+o(q_0),\ \ \frac{\partial u'}{\partial y}=o(q_0),\ \ \frac{\partial v'}{\partial x}=o(q_0),\ \ \partial^2 u'=o(q_0),\ \ \partial^2 v'=o(q_0),$$

using the $o(\dots)$ order notation in the obvious way. Since $(\hat{u},\hat{v})$ is the inverse map of (u',v'), calculation shows these are equivalent to:

$$\begin{gathered}\frac{\partial\hat{u}}{\partial x}=\frac{\partial\hat{v}}{\partial y}=q_0^{-1}+o(q_0^{-1}),\quad \frac{\partial\hat{u}}{\partial y}=o(q_0^{-1}),\\ \frac{\partial\hat{v}}{\partial x}=o(q_0^{-1}),\quad \partial^2\hat{u}=o(q_0^{-2})\quad\text{and}\quad \partial^2\hat{v}=o(q_0^{-2}).\end{gathered}\tag{101}$$

Now $(\hat{u},\hat{v})$ were constructed in Theorem 4.1 by rescaling the solutions u,v studied in §4.2–§4.6. For simplicity let us ignore this rescaling process, and identify $\hat{u},\hat{v}$ with the solutions u,v of §4.2–§4.6. Then from (30) we have $\hat{v}=\alpha+\beta y+\gamma x+\psi$, where $\alpha,\beta,\gamma\in\mathbb{R}$ with $\beta\approx q_0^{-1}$ and $\gamma\approx p_0q_0^{-2}$. Hence we can use Theorem 4.21 to estimate the derivatives of $\hat{v}$, in terms of powers of $\gamma\approx p_0q_0^{-2}$ and $s\approx(y_0^2+a^2)^{1/2}$. Using (17) we can then deduce estimates for the derivatives of $\hat{u}$.

Thirdly, we divide into the two cases (a) $|v_0|\leqslant(y_0^2+a^2)^{3/8}$, and (b) $|v_0|>(y_0^2+a^2)^{3/8}$. In case (a), we can use Theorem 4.21 to prove (101), because (97) implies that $p_0=O\big((y_0^2+a^2)^{3/8}|y_0||q_0|\big)$, and this gives an estimate for γ, which turns out to be exactly what we need.

However, in case (b), the most direct approach is insufficient to prove (101) on all of $U\cap\partial S$, so we have to do something different. We divide $U\cap\partial S$ into three connected regions, (i) $|y-y_0|\leqslant\frac{1}{2}|v_0q_0|$; (ii) $y>y_0+\frac{1}{2}|v_0q_0|$; and (iii) $y<y_0-\frac{1}{2}|v_0q_0|$. In region (i) we see from (98) that $|v'|\geqslant\frac{1}{2}|v_0|$, approximately. So when we invert (u',v') to get $(\hat{u},\hat{v})$, the curve in D_L corresponding to region (i) will satisfy $|y|\geqslant\frac{1}{2}|v_0|$, approximately. Thus when we apply Theorem 4.21 to estimate the 'errors', as above, the top line of (48) is strong enough to prove that $\frac{\mathrm{d}^3}{\mathrm{d}y^3}(f'-f_a)|_{U\cap\partial S}$ has the same sign as $\kappa_i q_0$ on region (i).

In regions (ii) and (iii), we instead use similar arguments to show that $\frac{\mathrm{d}^2}{\mathrm{d}y^2}(f'-f_a)|_{U\cap\partial S}$ has the same sign as $\kappa_i q_0$ on region (ii), and the opposite sign on region (iii). Thus we see that $\frac{\mathrm{d}^2}{\mathrm{d}y^2}(f'-f_a)|_{U\cap\partial S}$ has exactly one zero on $U\cap\partial S$, and so $\frac{\mathrm{d}}{\mathrm{d}y}(f'-f_a)|_{U\cap\partial S}$ has at most two zeroes on $U\cap\partial S$. This completes the proof of Proposition 7.7. □

Divide the boundary $\partial(U\cap S)$ into two curves $\gamma_1=\partial U\cap S$ and $\gamma_2=U\cap\partial S$. As U is approximately a closed disc of small radius $|\hat{q}_0|L$, and ∂S is approximately parallel to the y-axis near U, it easily follows that γ_1 and γ_2 are homeomorphic to $[0,1]$. As we have moved (x_0, y_0) a little way into the interior of $U \cap S$ it follows that $(u', v') \neq (u_a, v_a)$ on $\partial(U \cap S)$. Define θ_j to be the angle that $(u', v') - (u, v)$ winds around zero along the curve γ_j for $j = 1, 2$, where γ_j has the orientation induced from ∂U or ∂S. Then $\theta_1 + \theta_2 = 2\pi k$, where k is the winding number of $(u', v') - (u, v)$ about 0 along ∂U. Theorem 7.6 will follow from the next two lemmas.

Lemma 7.8. *In the situation above,* $\theta_1 < 2\pi$.

Proof. From above, U is approximately a closed disc of radius $|\hat{q}_0|L$ with centre $(x_0, y_0) - \hat{q}_0(u_0, v_0)$, and ∂S is near U approximately parallel to the y-axis, and passes very near (x_0, y_0). Since $u_0^2 + v_0^2 \leqslant X^2$ and $L = 10X$, it follows that ∂S is to first approximation a straight line which passes within distance $\frac{1}{10}|\hat{q}_0|L$ of the centre of the approximate closed disc U. Thus γ_1 occupies between $\pi - 2\sin^{-1}\frac{1}{10}$ and $\pi + 2\sin^{-1}\frac{1}{10}$ radians, approximately, of the approximate circle ∂U.

But $(u', v') : U \to D_L$ takes γ_1 to a portion of the circle ∂D_L, approximately preserving angles. Therefore $(u', v')(\gamma_1)$ sweeps out an arc of ∂D_L with angle approximately in the interval $[2{\cdot}94, 3{\cdot}34]$ radians. Arguing in more detail, using the estimates on $\hat{u}, \hat{v}$ in §4, one can show that if J is big enough then $(u', v')(\gamma_1)$ is no more than 5 radians of the circle ∂D_L.

Now on γ_1 we have $\bigl|(u', v')\bigr| = L$ and $\bigl|(u_a, v_a)\bigr| \leqslant X = \frac{1}{10}L$. As the angle swept out by (u', v') about 0 along γ_1 is no more than 5 radians, it follows that the angle swept out by $(u', v') - (u_a, v_a)$ about 0 along γ_1 is no more than $5 + 2\sin^{-1}\frac{1}{10} = 5 \cdot 20$ radians. Thus $\theta_1 \leqslant 5 \cdot 20 < 2\pi$. $\square$

Lemma 7.9. *In the situation above,* $\theta_2 < 2\pi$.

Proof. As above, we may parametrize ∂S near $(x_i, 0)$ as $\bigl(x(y), y\bigr)$ with $x(y) \approx x_i + \frac{1}{2}\kappa_i y^2$ for small y. So γ_2 is parametrized by $\bigl(x(y), y\bigr)$ for y in some small interval $[y_1, y_2]$. Therefore

$$\frac{\mathrm{d}}{\mathrm{d}y}\bigl(f' - f_a\bigr)|_{U\cap\partial S} = \bigl(u' - u_a, v' - v_a\bigr) \cdot \bigl(1, \dot{x}(y)\bigr) \quad \text{for } y \in [y_1, y_2],$$

where $\dot{x}(y) = \frac{\mathrm{d}}{\mathrm{d}y}x(y)$. Now Proposition 7.7 implies that $\frac{\mathrm{d}}{\mathrm{d}y}\bigl(f' - f_a\bigr)|_{U\cap\partial S}$ can change sign at most twice on γ_2. This shows that the angle between $(u' - u_a, v' - v_a)$ and $(1, \dot{x})$ can cross over $\pm\frac{\pi}{2}$ modulo 2π at most twice on γ_2. So we shall prove the lemma by comparing the angles that $(u' - u_a, v' - v_a)$ and $(1, \dot{x})$ rotate through about 0 along γ_2.

For simplicity, suppose that $i = 2$, so that γ_2 is oriented in the direction of increasing y and $\kappa_i = \kappa_2 < 0$, and suppose that $q_0 > 0$. The other possibilities of $i = 1$ or 2 and $q_0 > 0$ or $q_0 < 0$ follow in the same way. Then from the proof of Proposition 7.7, as $\kappa_i q_0 < 0$ we see that $\frac{\mathrm{d}}{\mathrm{d}y}\bigl(f' - f_a\bigr)|_{U\cap\partial S}$ is negative near $y = y_1$ and $y = y_2$, and from the proof of Lemma 7.8 we see that at $y = y_1$ we have $v' - v_a < 0$, and at $y = y_2$ we have $v' - v_a > 0$, and $u' - u_a$ is small compared to $v' - v_a$ at both $y = y_1$ and $y = y_2$.

If $\frac{\mathrm{d}}{\mathrm{d}y}\big(f' - f_a\big)|_{U\cap\partial S} < 0$ on all of γ_2, then $(u'-u_a, v'-v_a)\cdot(1,\dot{x}) < 0$ on γ_2, which approximately says that $u' - u_a < 0$ on γ_2, as $\dot{x}$ is small. A little thought shows that the angle θ_2 which $(u'-u_a, v'-v_a)$ rotates through along γ_2 is approximately $-\pi$.

If $\frac{\mathrm{d}}{\mathrm{d}y}\big(f' - f_a\big)|_{U\cap\partial S}$ changes sign twice on γ_2, then $(u'-u_a, v'-v_a)$ rotates through an extra angle of $2\pi, 0$ or -2π compared to $(1,\dot{x})$. Hence θ_2 is approximately $\pi, -\pi$ or -3π. By explaining what we mean by $u' - u_a$ being small compared to $v' - v_a$ at $y = y_1$ and $y = y_2$, we may easily show that $\theta_2 < 2\pi$. $\square$

We can now finish the proof of Theorem 7.6. From above $\theta_1 + \theta_2 = 2\pi k$, where k is the winding number of $(u', v') - (u, v)$ about 0 along $\partial(U \cap S)$. But $\theta_1, \theta_2 < 2\pi$ by the last two lemmas. Hence $k < 2$, and so the winding number of $(u', v') - (u_a, v_a)$ about 0 along $\partial(U \cap S)$ is at most 1. But $(u', v') - (u_a, v_a)$ has a zero of multiplicity at least 2 at (x_0, y_0) in $(U \cap S)^\circ$, so [6, Th. 6.7] gives a contradiction. $\square$

7.4 Estimates for $\frac{\partial u_a}{\partial x}, \frac{\partial u_a}{\partial y}, \frac{\partial v_a}{\partial x}, \frac{\partial v_a}{\partial y}$ on ∂S and in $L^p(S)$

Next we prove analogues of the estimates of §6.4–§6.5 in the situation of this section. Combining Theorem 7.6 and the results of §7.2, we deduce:

Corollary 7.10. *There exist constants $K_1, K_2 > 0$ such that*

$$\begin{aligned}&\Big|\frac{\partial u_a}{\partial x}\Big| = \Big|\frac{\partial v_a}{\partial y}\Big| \leqslant K_1(y^2+a^2)^{-1/2}, \quad \Big|\frac{\partial u_a}{\partial y}\Big| \leqslant K_2 \quad \textit{and}\\ &\Big|\frac{\partial v_a}{\partial x}\Big| \leqslant 2K_2(v_a^2+y^2+a^2)^{1/2} \quad \textit{on } \partial S\textit{, for all } a \in (0,1].\end{aligned} \tag{102}$$

Proof. Theorem 7.6 implies that $\big|\frac{\partial v_a}{\partial y}\big| \leqslant J(y^2+a^2)^{-1/2}$ for all $a \in (0,1]$ and $(x,y) \in \partial S$ with $|y| \leqslant \delta$, for some $\delta, J > 0$. Apply Proposition 7.4 with ϵ replaced by this δ. This gives $G > 0$ such that $\big|\frac{\partial v_a}{\partial y}\big| \leqslant G$ for all $a \in (0,1]$ and $(x,y) \in \partial S$ with $|y| > \delta$. Combining these shows that $\big|\frac{\partial v_a}{\partial y}\big| \leqslant K_1(y^2+a^2)^{-1/2}$ on ∂S for all $a \in (0,1]$, where $K_1 = \max\big(J, G(Y^2+1)^{1/2}\big)$. This proves the first inequality of (102).

Proposition 7.5 gives $\epsilon, H > 0$ and functions $F_{i,a}$ such that if $(x,y) \in \partial S$ is close to $(x_i, 0)$ and $|y| \leqslant \epsilon$ then by (94) we have

$$\begin{aligned}\Big|\frac{\partial u_a}{\partial y}(x,y)\Big| &\leqslant \big|F_{i,a}(y)\big|\Big|\frac{\partial u_a}{\partial x}(x,y)\Big| + H\\ &\leqslant 3|\kappa_i||y|K_1(y^2+a^2)^{-1/2} + H \leqslant 3|\kappa_i|K_1 + H,\end{aligned}$$

using the first inequality of (102) and $\big|F_{i,a}(y) - 2\kappa_i y\big| \leqslant |\kappa_i y|$, so that $\big|F_{i,a}(y)\big| \leqslant 3|\kappa_i||y|$. Applying Proposition 7.4 with this value of ϵ gives $G > 0$ such that $\big|\frac{\partial u_a}{\partial y}\big| \leqslant G$ for all $a \in (0,1]$ and $(x,y) \in \partial S$ with $|y| > \epsilon$. Combining these shows that $\big|\frac{\partial u_a}{\partial y}\big| \leqslant K_2$ on ∂S for all $a \in (0,1]$, where $K_2 = \max(3|\kappa_1|K_1 +$

$H, 3|\kappa_2|K_1 + H, G)$. This proves the second inequality of (102), and the third follows from (17). □

As $|v_a| \leqslant X$ by (91) we see from (102) that $\bigl|\frac{\partial v_a}{\partial x}\bigr| \leqslant K_3$ on ∂S for all $a \in (0,1]$, where $K_3 = 2K_2(X^2+Y^2+1)^{1/2}$. Thus from [6, Prop. 8.12] we deduce:

Corollary 7.11. *There exists $K_3 > 0$ such that $\bigl|\frac{\partial v_a}{\partial x}\bigr| \leqslant K_3$ in S for all $a \in (0,1]$.*

These two corollaries will serve as an analogue of Corollary 6.9. Unfortunately we do not have an analogue of Proposition 6.10 on all of S, as Corollary 7.10 is not strong enough to apply Theorem 5.5. However, by applying Theorem 5.1 with $K = 1$ and $L = X$ we deduce the following analogue of Proposition 6.10 in interior domains $T \subset S^\circ$.

Proposition 7.12. *Let $T \subset S^\circ$ be a subdomain. Then there exists $K_4 > 0$ with*

$$\bigl|\tfrac{\partial u_a}{\partial x}\bigr| = \bigl|\tfrac{\partial v_a}{\partial y}\bigr| \leqslant K_4\bigl(v_a^2 + y^2 + a^2\bigr)^{-1} \quad \text{in } T \text{ for all } a \in (0,1]. \tag{103}$$

The integrals of (85) are bounded uniformly in a.

Proposition 7.13. *Let $\alpha \in [0, \frac{1}{2})$, and define $J(a,v)$ as in Proposition 6.11. Then there exists $D_\alpha > 0$ depending only on α such that $\int_{\partial S} J(a, v_a)\mathrm{d}u_a \leqslant D_\alpha$ for all $a \in (0,1]$.*

Proof. We have $\bigl|J(a,v_a)\bigr| \leqslant (1-2\alpha)^{-1}|v_a|^{1-2\alpha} \leqslant (1-2\alpha)^{-1}X^{1-2\alpha}$ as in the proof of Proposition 6.12, and so

$$\begin{aligned}\int_{\partial S} J(a,v_a)\mathrm{d}u_a &\leqslant (1-2\alpha)^{-1}X^{1-2\alpha}\int_{\partial S}|\mathrm{d}u_a| \\ &= (1-2\alpha)^{-1}X^{1-2\alpha}\int_0^1 \Bigl|\frac{\partial u_a}{\partial x}\cdot\frac{\mathrm{d}x}{\mathrm{d}s} + \frac{\partial u_a}{\partial y}\cdot\frac{\mathrm{d}y}{\mathrm{d}s}\Bigr|\mathrm{d}s,\end{aligned}$$

where $s \mapsto \bigl(x(s), y(s)\bigr)$ is a parametrization $[0,1] \mapsto \partial S$ of ∂S.

Now $\bigl|\frac{\partial u_a}{\partial x}\bigr| \leqslant K_1(y^2+a^2)^{-1/2}$ and $\bigl|\frac{\partial u_a}{\partial y}\bigr| \leqslant K_2$ on ∂S by Corollary 7.10. Also, when y is small we see that $\bigl(x(s), y(s)\bigr)$ is close to $(x_i, 0)$ for $i = 1$ or 2, and then $\frac{\mathrm{d}x}{\mathrm{d}s} \approx \kappa_i y \frac{\mathrm{d}y}{\mathrm{d}s}$. Thus $\bigl|\frac{\mathrm{d}x}{\mathrm{d}s}\bigr| \leqslant C|y|\bigl|\frac{\mathrm{d}y}{\mathrm{d}s}\bigr|$ for $s \in [0,1]$ and some $C > 0$ depending only on S. Hence

$$\int_0^1 \Bigl|\frac{\partial u_a}{\partial x}\cdot\frac{\mathrm{d}x}{\mathrm{d}s} + \frac{\partial u_a}{\partial y}\cdot\frac{\mathrm{d}y}{\mathrm{d}s}\Bigr|\mathrm{d}s \leqslant \int_0^1 \bigl(K_1(y^2+a^2)^{-1/2}C|y| + K_2\bigr)\Bigl|\frac{\mathrm{d}y}{\mathrm{d}s}\Bigr|\mathrm{d}s.$$

As $(y^2+a^2)^{-1/2}|y| \leqslant 1$, combining the last two equations proves the proposition, with $D_\alpha = (1-2\alpha)^{-1}X^{1-2\alpha}(K_1C + K_2)\int_0^1 \bigl|\frac{\mathrm{d}y}{\mathrm{d}s}\bigr|\mathrm{d}s$. □

Following the proof of Proposition 6.12, we show:

Proposition 7.14. *Let $q\in[1,2)$ and $r\in[1,\infty)$. Then there exist $C_2, C_q', C_r''>0$ such that $\|\frac{\partial u_a}{\partial x}\|_{L^2}=\|\frac{\partial v_a}{\partial y}\|_{L^2}\leqslant C_2$, $\|\frac{\partial u_a}{\partial y}\|_{L^q}\leqslant C_q'$ and $\|\frac{\partial v_a}{\partial x}\|_{L^r}\leqslant C_r''$ for all $a\in(0,1]$. Let $T\subset S^\circ$ be a subdomain, and let $p\in(2,\frac{5}{2})$. Then there exists $C_p^T>0$ such that $\|\frac{\partial u_a}{\partial x}|_T\|_{L^p}=\|\frac{\partial v_a}{\partial y}|_T\|_{L^p}\leqslant C_p^T$ for all $a\in(0,1]$.*

The difference between this and Proposition 6.12 is because the proof of Proposition 6.12 for $p>2$ involved Proposition 6.10, but its analogue in this section is Proposition 7.12, which only holds in interior domains $T\subset S^\circ$. Hence we can only prove L^p estimates for $\frac{\partial u_a}{\partial x},\frac{\partial v_a}{\partial y}$ for $p>2$ on such subdomains T.

7.5 Uniform continuity of the u_a, v_a

Now we can prove the analogue of Proposition 6.13, which shows that the u_a, v_a are *uniformly continuous* for all $a\in(0,1]$. The proof for v_a is as in §6.5, but because we can only bound $\frac{\partial u_a}{\partial x}$ in L^p in interior domains $T\subset S^\circ$, we have to introduce some new ideas to show that u_a are uniformly continuous near $(x_i,0)$ for $i=1,2$.

Theorem 7.15. *In the situation of Definition 7.3, there exist continuous $M,N: S\times S\rightarrow[0,\infty)$ satisfying $M(x',y',x,y)=M(x,y,x',y')$, $N(x',y',x,y)=N(x,y,x',y')$ and $M(x,y,x,y)=N(x,y,x,y)=0$ for all $(x,y),(x',y')\in S$, such that for all $(x,y),(x',y')\in S$ and $a\in(0,1]$ we have*

$$\begin{aligned} \bigl|u_a(x,y)-u_a(x',y')\bigr| &\leqslant M(x,y,x',y') \\ \text{and}\quad \bigl|v_a(x,y)-v_a(x',y')\bigr| &\leqslant N(x,y,x',y'). \end{aligned} \tag{104}$$

Proof. Choose $p>2$ and set $q=2$. Then Theorem 2.3 gives continuous functions $G,H:S\times S\rightarrow[0,\infty)$. Combining Proposition 7.14 and Theorem 2.3, we see that for all $(x,y),(x',y')\in S$ and $a\in(0,1]$, we have

$$\bigl|v_a(x,y)-v_a(x',y')\bigr|\leqslant C_p''G(x,y,x',y')+C_2H(x,y,x',y')=N(x,y,x',y').$$

This defines $N:S\times S\rightarrow[0,\infty)$ satisfying all the conditions of the theorem.

The next three lemmas construct versions of M on subsets of S.

Lemma 7.16. *Let $T\subset S^\circ$ be a subdomain. Then there exists a continuous $M^T: T\times T\rightarrow[0,\infty)$ such that for all $(x,y),(x',y')\in T$ and $a\in(0,1]$ we have $M^T(x',y',x,y)=M^T(x,y,x',y')$, $M^T(x,y,x,y)=0$ and $\bigl|u_a(x,y)-u_a(x',y')\bigr|\leqslant M^T(x,y,x',y')$.*

Proof. Choose $p\in(2,\frac{5}{2})$ and $q\in[1,2)$ with $p^{-1}+q^{-1}<1$. Then Theorem 2.3 with T in place of S gives continuous functions $G,H:T\times T\rightarrow[0,\infty)$. Combining Proposition 7.14 and Theorem 2.3, we see that for all $(x,y),(x',y')\in T$ and $a\in(0,1]$, we have

$$\bigl|u_a(x,y)-u_a(x',y')\bigr|\leqslant C_p^TG(x,y,x',y')+C_q'H(x,y,x',y')=M^T(x,y,x',y').$$

This M^T satisfies the conditions of the lemma. □

Lemma 7.17. *Let $\epsilon > 0$ be small, and let G be as in Proposition 7.4. Then whenever $(x,y),(x',y') \in S$ and either $y,y' > \epsilon$ or $y,y' < -\epsilon$ we have*

$$\bigl|u_a(x,y) - u_a(x',y')\bigr| \leqslant G\bigl((x-x')^2 + (y-y')^2\bigr)^{1/2} \quad \textit{for all } a \in (0,1]. \quad (105)$$

Proof. Let S_ϵ be as in Proposition 7.4. Then S_ϵ has two connected components, with $y > \epsilon$ and $y < -\epsilon$, each of which is convex. The condition that either $y,y' > \epsilon$ or $y,y' < -\epsilon$ ensures that (x,y) and (x',y') lie in the same connected component of S_ϵ. Hence the straight line segment joining (x,y) and (x',y') lies in S_ϵ, and so $|\partial u_a| \leqslant G$ on this line segment by Proposition 7.4. Thus, $\bigl|u_a(x,y) - u_a(x',y')\bigr|$ is bounded by G times the length of the segment. □

Lemma 7.18. *There exists a continuous $M^{\partial S} : \partial S \times \partial S \to [0,\infty)$ such that for all $(x,y),(x',y') \in \partial S$ and $a \in (0,1]$ we have $M^{\partial S}(x',y',x,y) = M^{\partial S}(x,y,x',y')$, $M^{\partial S}(x,y,x,y) = 0$ and $\bigl|u_a(x,y) - u_a(x',y')\bigr| \leqslant M^{\partial S}(x,y,x',y')$.*

Proof. First suppose $(x,y),(x',y') \in \partial S$ are close to $(x_i,0)$ for $i=1$ or 2. Then as in §7.2 we can parametrize ∂S near $(x_i,0)$ as $\bigl(x(y),y\bigr)$, and (92) gives

$$u_a(x,y) = \dot\phi(y) - \dot x(y) v_a(x,y) \quad \text{and} \quad u_a(x',y') = \dot\phi(y') - \dot x(y') v_a(x',y'),$$

where '$\dot{}$' is short for $\frac{\mathrm{d}}{\mathrm{d}y}$. Hence

$$\begin{aligned} u_a(x,y) - u_a(x',y') = \dot\phi(y) - \dot\phi(y') &- \tfrac12\bigl(\dot x(y) - \dot x(y')\bigr)\bigl(v_a(x,y) + v_a(x',y')\bigr) \\ &- \tfrac12\bigl(\dot x(y) + \dot x(y')\bigr)\bigl(v_a(x,y) - v_a(x',y')\bigr). \end{aligned}$$

Since $|v_a| \leqslant X$ by (91), using the second inequality of (104) we see that

$$\begin{aligned} \bigl|u_a(x,y) - u_a(x',y')\bigr| \leqslant \bigl|\dot\phi(y) - \dot\phi(y')\bigr| &+ X\bigl|\dot x(y) - \dot x(y')\bigr| \\ &+ \tfrac12\bigl|\dot x(y) + \dot x(y')\bigr| N(x,y,x',y'). \end{aligned}$$

As $\dot\phi$ and $\dot x$ are well-defined and continuous near $(x_i,0)$ on ∂S, and N is continuous, the r.h.s. of this equation defines a continuous function $M^{\partial S}$ near $\bigl((x_i,0),(x_i,0)\bigr)$ in $\partial S \times \partial S$ which satisfies the conditions of the lemma. Also, Lemma 7.17 shows how to define $M^{\partial S}$ on the subsets of ∂S with $y > \epsilon$ and $y < -\epsilon$ for small $\epsilon > 0$. It is then not difficult to patch together these functions on subsets of $\partial S \times \partial S$ using the triangle inequality to construct a suitable function $M^{\partial S}$ on all of $\partial S \times \partial S$, and we leave this as an exercise. □

Let us review what we have proved so far. The last three lemmas show that the u_a are uniformly continuous for all $a \in (0,1]$ in interior domains $T \subset S^\circ$, and away from the x-axis in S, and on ∂S, respectively. Taken together, these imply that the u_a are uniformly continuous except possibly at $(x_i,0)$ for $i=1,2$, and that the restrictions of the u_a to ∂S are uniformly continuous at $(x_i,0)$. It remains only to show that the u_i are uniformly continuous at $(x_i,0)$.

By [6, Cor. 4.4], if u, v satisfy (17) on a domain S then the maximum of u is achieved on ∂S. It is proved by applying the maximum principle to a second-order linear elliptic equation satisfied by u in [6, eq. (27), Prop. 4.3]. Now this equation has no terms in u and $\frac{\partial u}{\partial x}$, and so if $\alpha \in \mathbb{R}$ then $u - \alpha x$ satisfies the same equation. Hence we may prove:

Lemma 7.19. *Let $T \subset S$ be a subdomain, $\alpha \in \mathbb{R}$ and $a \in (0,1]$. Then the maximum of $u_a - \alpha x$ on T is achieved on ∂T.*

Using this we shall show that the u_a are uniformly continuous at $(x_i, 0)$.

Lemma 7.20. *Let $i = 1,2$ and $\epsilon > 0$. Then there exists $\delta > 0$ such that if $(x,y) \in S$ and $|x - x_i| < \delta$ then $\bigl|u_a(x,y) - u_a(x_i,0)\bigr| < \epsilon$ for all $a \in (0,1]$.*

Proof. Let i, ϵ be as above. Then as S is strictly convex, and $T_{(x_i,0)}\partial S$ is parallel to the y-axis, and $M^{\partial S}$ is continuous with $M^{\partial S}(x_i,0,x_i,0) = 0$, there exists $\gamma > 0$ such that $M^{\partial S}(x,y,x_i,0) \leqslant \frac{\epsilon}{2}$ for all $(x,y) \in \partial S$ such that $|x - x_i| \leqslant \gamma$. Define $T = \bigl\{(x,y) \in S : |x - x_i| \leqslant \gamma\bigr\}$. Then T is a subdomain of S, with piecewise-smooth boundary consisting of a portion of ∂S, and a straight line segment on which $|x - x_i| = \gamma$.

Observe that on ∂T we have

$$u_a(x_i,0) - \frac{\epsilon}{2} - \frac{2X}{\gamma}|x - x_i| \leqslant u_a(x,y) \leqslant u_a(x_i,0) + \frac{\epsilon}{2} + \frac{2X}{\gamma}|x - x_i|. \quad (106)$$

This is because on the part of ∂T coming from ∂S we have $\bigl|u_a(x,y) - u_a(x_i,0)\bigr| \leqslant M^{\partial S}(x,y,x_i,0) \leqslant \frac{\epsilon}{2}$, by Lemma 7.18 and the definition of γ, and on the part of ∂T with $|x - x_i| = \gamma$ we have $\frac{2X}{\gamma}|x - x_i| = 2X$, and the result follows from $|u_a| \leqslant X$ by (91).

Now the l.h.s. and r.h.s. of (106) are both of the form $\alpha x + \beta$ on T for $\alpha, \beta \in \mathbb{R}$, since $|x - x_i| = x - x_i$ on T if $i = 1$, and $|x - x_i| = x_i - x$ on T if $i = 2$. Therefore Lemma 7.19 implies that as (106) holds on ∂T, it holds on T. Thus $\bigl|u_a(x,y) - u_a(x_i,0)\bigr| \leqslant \frac{\epsilon}{2} + \frac{2X}{\gamma}|x - x_i|$ on T. Choosing $\delta = \min(\gamma, \epsilon\gamma/4X)$, the lemma easily follows. □

We have now shown that the u_a for $a \in (0,1]$ are uniformly continuous everywhere in S. With some effort it can be shown that one can piece together the various functions constructed above to construct a continuous function $M : S \times S \to [0,\infty)$ satisfying the conditions of the theorem, and we leave this as an exercise for the reader. In fact it is rather easier to construct M which is *lower semicontinuous* rather than continuous, and lower semicontinuity is all we will need to show that limits of the u_a are continuous. This completes the proof of Theorem 7.15. □

7.6 Existence and uniqueness of weak solutions of (19)

We can now follow the proofs of §6.5 more-or-less unchanged in the situation of this section, to prove:

Proposition 7.21. *Propositions 6.14–6.16 hold for f_a, u_a, v_a in §7.2, except that in Proposition 6.15 we have $\frac{\partial u_0}{\partial x} = \frac{\partial v_0}{\partial y} \in L^p(S)$ only for $p \in [1,2]$.*

In particular, this gives *existence* of a solution f_0 of (19) with weak derivatives, and $f_0|_{\partial S} = \phi$. The reason for the difference with Proposition 6.15 is that in Proposition 7.14 we have a priori L^p estimates for $\frac{\partial u_a}{\partial x}, \frac{\partial v_a}{\partial y}$ when $p \in (2, \frac{5}{2})$ only in *interior subdomains* $T \subset S^\circ$. Here is an analogue of Proposition 6.17, which shows that weak solutions of the Dirichlet problem for (19) are *unique*.

Proposition 7.22. *Suppose $f, f' \in C^{0,1}(S)$ are weak solutions of (19) with $a = 0$ and $f|_{\partial S} = f'|_{\partial S}$. Then $f = f'$.*

Proof. Define $u, v, u', v' \in L^1(S)$ to be the weak derivatives $\frac{\partial f}{\partial y}, \frac{\partial f}{\partial x}, \frac{\partial f'}{\partial y}, \frac{\partial f'}{\partial x}$. Now as $f, f' \in C^{0,1}(S)$ and $f|_{\partial S} = f'|_{\partial S}$, it follows that $f - f'$ lies in the closure of $C^1_0(S)$ in $C^{0,1}(S)$. Hence there exists a sequence (ψ_n) in $C^1_0(S)$ converging to $f - f'$ in $C^{0,1}(S)$, where $C^1_0(S)$ is the subspace of $\psi \in C^1(S)$ supported in S°. As f' satisfies (19) weakly and $\psi_n \in C^1_0(S)$ we have

$$-\int_S \frac{\partial \psi_n}{\partial x} \cdot A(0, y, v')\, \mathrm{d}x\, \mathrm{d}y - 2\int_S \frac{\partial \psi_n}{\partial y} \cdot u'\, \mathrm{d}x\, \mathrm{d}y = 0. \tag{107}$$

But as $\psi_n \to f - f'$ in $C^{0,1}(S)$ it follows that $\frac{\partial \psi_n}{\partial x} \to v - v'$ and $\frac{\partial \psi_n}{\partial x} \to u - u'$ in $L^\infty(S)$ as $n \to \infty$. So letting $n \to \infty$ in (107) and using the Dominated Convergence Theorem shows that

$$-\int_S (v - v') \cdot A(0, y, v')\, \mathrm{d}x\, \mathrm{d}y - 2\int_S (u - u') \cdot u'\, \mathrm{d}x\, \mathrm{d}y = 0.$$

Applying the same argument with f instead of f' yields

$$-\int_S (v - v') \cdot A(0, y, v)\, \mathrm{d}x\, \mathrm{d}y - 2\int_S (u - u') \cdot u\, \mathrm{d}x\, \mathrm{d}y = 0.$$

Subtracting the last two equations gives

$$\int_S (v - v') \cdot \big(A(0, y, v) - A(0, y, v')\big)\, \mathrm{d}x\, \mathrm{d}y + 2\int_S (u - u')^2\, \mathrm{d}x\, \mathrm{d}y = 0.$$

Now $\frac{\partial A}{\partial v} = \big(v^2 + y^2 + a^2\big)^{-1/2}$ by (20). Thus if $y \neq 0$ the Mean Value Theorem shows that $A(0, y, v) - A(0, y, v') = (w^2 + y^2)^{-1/2}(v - v')$, for some w between v and v'. Therefore

$$\int_S (w^2 + y^2)^{-1/2}(v - v')^2 \mathrm{d}x\, \mathrm{d}y + 2\int_S (u - u')^2\, \mathrm{d}x\, \mathrm{d}y = 0.$$

As the integrands are nonnegative they are zero, so $u = u'$ and $v = v'$ in $L^1(S)$, and hence $f = f'$ in $C^{0,1}(S)$, as $f|_{\partial S} = f'|_{\partial S}$. □

Note that we have shown in Proposition 7.21 that a solution f_0 of the Dirichlet problem for (19) on S exists that is twice weakly differentiable and satisfies (19) with weak derivatives, but Proposition 7.22 shows that f_0 is unique in the possibly larger class of weak solutions of (19), which need only be once weakly differentiable. So this is a stronger result than we actually need.

Finally, Theorem 7.1 follows from Propositions 7.21 and 7.22, and Theorem 7.2 in the same way as Theorem 6.2.

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