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# Nonlinear Consensus on Networks: Equilibria, Effective Resistance, and Trees of Motifs*

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**Abstract.** We study a generic family of nonlinear dynamics on undirected networks generalizing linear consensus. We find a compact expression for its equilibrium points in terms of the topology of the network and classify their stability using the effective resistance of the underlying graph equipped with appropriate weights. Our general results are applied to some specific networks, namely trees, cycles, and complete graphs. When a network is formed by the union of two subnetworks joined in a single node, we show that the equilibrium points and stability in the whole network can be found by simply studying the smaller subnetworks instead. Applied recursively, this property opens the possibility of investigating dynamical behavior on families of networks made of trees of motifs.

**Key words.** consensus dynamics, network science, nonlinear dynamics, fixed points, effective resistance, trees

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## 1. Introduction.

A broad range of systems can be represented by dynamical systems on networks [38]. In this framework, each node is endowed with a time-varying state whose dynamics depend on its own state and the states of its neighbors. Important examples include models of collective behavior, where the decision process is distributed rather than centralized [13]. As reviewed in [50, 40], applications can be found in a variety of disciplines, such as ecology where it is used to model animal behavior and flocking [18, 44, 39]. In engineering, it is essential in designing decentralized control strategies for the movement of robots [32], solving the rendezvous problem [17], and coordinating decision making when multiple nosy sensors detect an event [2]. In the social sciences, it is used as a first model for opinion and language dynamics [16], and it can be applied in economics to coordinate a decentralized network of buyers [8]. The simplest algorithms and models for consensus are variations of the DeGroot model [19], usually called linear consensus dynamics, where the state of a node evolves towards the average value of its neighbors. These models are theoretically appealing thanks to their simplicity, as their dynamics is entirely determined by the spectral properties of the coupling matrix between the elements. However, linear models often arise as approximations for more complicated coupling functions and are not always sufficient to explain their complex behavior

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[18], which calls for the study of nonlinear consensus [43]. In many of these applications, nonlinearity has thus been introduced [3, 8], with recent developments made in the emerging field of opinion dynamics [12, 11]. The Kuramoto model of coupled oscillators [30, 4] is another prominent example of a nonlinear process, with applications in physics, chemistry, biology, and engineering [22, 37, 46].

The focus of this work is on studying the properties of a general class of nonlinear consensus dynamics taking place on a network. We derive exact results for arbitrary choices of undirected networks and a general class of nonlinear consensus dynamics, which contrasts with most other works that focus on mean-field approximations, specific types of networks, or specific classes of nonlinear dynamics. In particular, we manage to extend the theoretical analysis of [21], where the purely graph-theoretic notion of effective resistance was introduced for studying a particular nonlinear system. The role of effective resistances in nonlinear dynamical systems was also highlighted in [46], where resistances were shown to be a key determinant of the robustness of a nonlinear system to external perturbations. Another relevant line of research into generic nonlinear network systems can be found in [25, 24], where it is shown that certain properties (symmetries) of the underlying network restrict the possible dynamical features (equilibria, periodic states) of any possible system on the network. Finally, the work of Bronski and DeVille [15] presents complementary stability results based on the study of spanning trees and homology theories of signed graphs, which coincide with ours in simple cases, as we will show in this paper.

Our problem is formalized as follows. Let $n \in \mathbb{N}$ and $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a connected undirected network with $\mathcal{V} = \{1, \ldots, n\}$. If $\{i, j\} \in \mathcal{E}$, we say they are adjacent and write $i \sim j$. Note that we consider only networks without self-loops, so that $i \not\sim i$, as self-loops would not change the dynamics that we will study. Linear consensus dynamics on $\mathcal{G}$ is defined over the state space $\mathbf{x} \in \mathbb{R}^n$, where each node has a scalar state $x_i$, by

$$\dot{x}_i = -\sum_{j \sim i} (x_i - x_j) \qquad \text{for all} \qquad i \in \mathcal{V}.$$

Compactly we may write $\dot{\mathbf{x}} = -L\mathbf{x}$, where $L$ is the Laplacian matrix of $\mathcal{G}$ (see (2.1)). As presented in [43], there are three obvious ways to extend this simple linear model by introducing a function $\hat{f} : \mathbb{R} \to \mathbb{R}$,

$$\dot{x}_i = \sum_{j \sim i} \left( \hat{f}(x_i) - \hat{f}(x_j) \right), \quad \dot{x}_i = \sum_{j \sim i} \hat{f}\left(x_i - x_j\right), \quad \dot{x}_i = \hat{f}\left( \sum_{j \sim i} (x_i - x_j) \right). \tag{1.1}$$

When $\hat{f} = -\text{id}$, all three cases recover the linear consensus model. If we denote by $f : \mathbb{R}^n \to \mathbb{R}^n$ the function that applies $\hat{f}$ to each component, we may write the first and third ODEs above more compactly as

$$\dot{\mathbf{x}} = Lf(\mathbf{x}) \qquad \text{and} \qquad \dot{\mathbf{x}} = f(L\mathbf{x}). \tag{1.2}$$

As shown in [43] for the case $\hat{f}(y) = \lambda y - y^3$, this matrix formulation facilitates the theoretical understanding of these systems. In this work we will focus on the second ODE in (1.1), which presents more challenges as it lacks an obvious matrix formulation. More precisely, let $\{f_e\}_{e\in\mathcal{E}}$ be a family of odd, continuously differentiable[1] functions, where we will denote $f_{\{i,j\}} = f_{ij} = f_{ji}$, and consider the ODE[2]

$$\dot{x}_i = \sum_{j\sim i} f_{ij}(x_i - x_j) \qquad \text{for all} \qquad i \in \mathcal{V}. \tag{\#}$$

An important point is that this equation allows us to model weighted networks by absorbing the corresponding weights in the functions $f_{ij}$. As $\mathcal{G}$ is undirected, we require $f_{ij}$ to be odd so that the system exhibits a conservation of quantity, which is desirable in many real scenarios. We may think, for instance, that a liquid flows between adjacent nodes. Moreover, this choice does not severely limit the applicability of our model, as in many applications a global reference frame is not available [36], and thus we need invariant dynamics under rotation and translation of the inertial frame. In [48] it is shown that this is satisfied if and only if the coupling functions $f_{ij}$ are quasi-linear, i.e., $f_{ij}(y) = k_{ij}(|y|)y$ for some functions $k_{ij}$, which are a collection of odd functions. It is also worth mentioning that the coupled phase oscillators model, of which the Kuramoto model is a subclass, is defined by

$$\dot{\theta}_i = w_i - \sum_{j=1}^{n} a_{\{i,j\}} \sin(\theta_i - \theta_j).$$

This model is not quite included in (#), due to the constants $w_i$. However, the differential of this system coincides with the differential of the case $w_i = 0$, which is included in (#). Thus, most of the presented results regarding linear stability of equilibria will be applicable to the coupled phase oscillator model.

In this paper, we have obtained four main results for the study of (#). First, we show that (#) can be expressed as a matrix product with a nonlinear coupling (3.3) which allows us to find a compact expression for the set of equilibrium points (3.6). This expression relates the equilibria with the topology of the underlying network through the cycle and cut space. Second, we find stability criteria for the equilibrium points using the concept of effective resistance, which in certain situations is tight (see Theorems 3.10 and 3.14). Third, when there are few edges contributing to instability, the Schur complement can be used to reduce the stability problem to a smaller network (see Theorem 3.8). Similarly, in subsection 3.5 we show that if a network consists of two subnetworks joined in a single common node, the task of finding equilibrium points and their stability in the whole network can be reduced to the same task for each of the subnetworks. This result motivates the study of particular classes of networks, i.e., building blocks, that will be combined to form more complex networks, which we refer to as trees of motifs (see subsection 3.5.1). In particular, we apply our general results to tree, cycle, and complete graphs and derive some additional properties that follow from their specific structures; see section 4. For the cycle graph, we show in Proposition 4.4 that a particular set of polynomials induces a continuum of equilibria in (#). This was observed for a particular case in [43] and [21], but no explanation for such behavior was given.

[1]It would be enough for $f_e$ to be locally Lipschitz as long as they are continuously differentiable in a neighborhood of the equilibria of (#).

[2]Without loss of generality we could always consider the complete graph while choosing $f_{ij}$ to be null in some edges. We do not choose to do so, as in many cases we will require that all $f_{ij}$ coincide.

## 2. Mathematical preliminaries.

We give a brief recap of the main structures and results that we will need throughout the paper. A *network* or graph $\mathcal{G}$, which we will always consider undirected, is given by a pair of sets $(\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ is the set of *nodes*, which we will assume is of the form $\{1, \dots, n\}$, and $\mathcal{E}$ is a set of unordered, distinct pairs of nodes corresponding to the *edges*. We will sometimes abuse notation and write $i \in \mathcal{G}$ or $\{i, j\} \in \mathcal{G}$ instead of $i \in \mathcal{V}$ or $\{i, j\} \in \mathcal{E}$. Given a partition $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ of the nodes, we say that the set of edges from $\mathcal{V}_1$ to $\mathcal{V}_2$ forms a *cut-set* if it is nonempty.

Associated to a network we have two $\mathbb{R}$ vector spaces $C_0$, $C_1$ generated by the orthonormal bases $\mathcal{V}$, $\mathcal{E}$, respectively. They are related by the *boundary* and coboundary linear maps,

$$d : C_1 \longrightarrow C_0, \qquad d^\top : C_0 \longrightarrow C_1,$$

defined by $d(\{j, i\}) = i - j$ where $j < i$. Choosing an ordering in the bases, $d$ is simply the "signed" incidence matrix; see (3.2). The matrices $d$ and $d^\top$ are directly related with consensus dynamics, as the *Laplacian* may be defined by (see [10, Proposition 4.8])

$$L := dd^\top. \tag{2.1}$$

Additionally, the kernel and image of these linear maps have a strong topological interpretation [10]. Indeed, $\ker d^\top$, $\ker d$, and $\operatorname{im} d^\top = (\ker d)^\perp$ are generated by the connected components, the cycles, and the cut-sets, respectively. In particular, if the network is connected, $\ker d^\top = \langle \mathbf{1} \rangle$, and

$$d^\top|_{\langle \mathbf{1} \rangle^\perp} : \langle \mathbf{1} \rangle^\perp \longrightarrow \operatorname{im} d^\top = (\ker d)^\perp \tag{2.2}$$

is an isomorphism.

A *weighted network* is a network with a set of nonnegative weights $\{w_{\{i,j\}}\}_{\{i,j\} \in \mathcal{E}}$. Often we only consider the edges with positive weights; for instance, we say that a weighted network is connected if the set of edges with positive weights is connected in the usual sense. The Laplacian matrix of a weighted network with vector of weights $w = (w_{\{i,j\}})_{\{i,j\} \in \mathcal{E}}$ is defined as

$$L := d \operatorname{diag}(w)\, d^\top, \tag{2.3}$$

which is symmetric and positive semidefinite. Moreover, its null space is generated by the indicator vectors of the connected components of the underlying weighted network. As there is a one-to-one correspondence between weighted Laplacians and networks, we may directly refer to the connected components of the Laplacian.

Given an ODE, $\dot{\mathbf{x}} = g(\mathbf{x})$, we say that $\mathbf{x}^\star$ is an *equilibrium point* if $g(\mathbf{x}^\star) = 0$. We say that an equilibrium point is *attractive* if solutions that start close enough to it eventually converge to it. We say that an equilibrium point is *stable* if solutions that start close enough to it remain close for all positive times; otherwise, we say it is *unstable* (see supplementary material section SM1 for formal definitions). In practice it is easier to check whether an equilibrium is linearly stable, which implies that it is both stable and attractive. We say that an equilibrium point is *linearly stable* (resp., *unstable*) if all (resp., any) eigenvalues of the Jacobian matrix of $g$ at that point have negative (resp., positive) real part. Analogously we define linear stability of a matrix. Linear stability also has the advantage that it persists under small perturbations of the vector field [45], which is key when the system is an imperfect representation of a phenomenon, such as in scientific models.

Given a real symmetric matrix $M$, we define its *inertia*, and denote it by $\mathrm{In}(M)$, as the tuple whose entries represent the number of positive, negative, and zero eigenvalues (including multiplicities). This quantity will be useful as it determines the linear stability of the matrix. It is well known that a symmetric matrix can be expressed as $M = \sum_{i=1}^{k} \lambda_i \mathbf{v}_i \mathbf{v}_i^\top$, where $\lambda_i$ are the nonzero eigenvalues and $\mathbf{v}_i$ are the corresponding eigenvectors forming an orthonormal set. We define the *pseudoinverse* $M^\dagger$ of $M$ as

$$M^\dagger := \sum_{i=1}^{k} \lambda_i^{-1} \mathbf{v}_i \mathbf{v}_i^\top,$$

so $\ker M = \ker M^\dagger$, and if $V \perp \ker M$, then $M^\dagger|_V = (M|_V)^{-1}$. It is worth mentioning that we do not need to know the eigenvalues of a matrix to find its pseudoinverse, as it can be computed through column/row operations and matrix mutiplications [41].

We denote by $\mathbf{0}_k$ (resp., $\mathbf{1}_k$) the column vectors with all entries 0 (resp., 1) and $\mathbf{0}_{k,k'} = \mathbf{0}_k \mathbf{0}_{k'}^\top$. When the dimension is clear we will suppress the subindex. We denote by $\mathbf{e}_1, \dots, \mathbf{e}_k$ the standard basis of $\mathbb{R}^k$ and by $I_k$ the identity matrix.

## 3. Development of the model.

**3.1. First results.** From now on we study the system (#) defined on an undirected, connected network $\mathcal{G}$. The following two observations were already mentioned in [21].

The mean state $\overline{\mathbf{x}} = \frac{1}{n}\sum_{i=1}^{n} x_i$ is a conserved quantity of the system (#). To see this, simply note that the functions $f_{ij}$ are odd, and thus,

$$\dot{\overline{\mathbf{x}}} = \langle \nabla \overline{\mathbf{x}}, \dot{\mathbf{x}} \rangle = \frac{1}{n}\langle \mathbf{1}, \dot{\mathbf{x}} \rangle = 0,$$

i.e., $\overline{\mathbf{x}}$ is a first integral of (#). Thus, the planes defined by $\overline{\mathbf{x}} = k$ for some constant $k$, which are precisely the affine perpendicular planes to $\mathbf{1}$, are invariant. Moreover, as only differences of elements $x_i$ appear in (#), the vector field in $\mathbf{x}$ and $\mathbf{x} + \lambda \mathbf{1}$ is the same for all $\lambda \in \mathbb{R}$. So we conclude that the dynamics in each plane is exactly the same, and we can simply restrict our study to one of these planes. From now on, we will study (#) in the vector space $\langle \mathbf{1} \rangle^\perp$ or, equivalently, the invariant plane given by $\overline{\mathbf{x}} = 0$, unless stated otherwise. In [21] a similar approach was taken by limiting the study to the state space $\mathbb{R}^n / \langle \mathbf{1} \rangle$.

The fact that $\overline{\mathbf{x}}$ is a conserved quantity is also desirable, as it shows that our model preserves some of the key properties of linear consensus dynamics. In the linear case, $\overline{\mathbf{x}}$ is not only conserved but also all entries of a solution tend to this value.

Another important property preserved from linear consensus is that the dynamics given by (#) are gradient dynamics in $\mathbb{R}^n$. Indeed, if we let $F_{ij}$ be an antiderivative of $f_{ij}$ for all $\{i,j\} \in \mathcal{E}$ and define

$$V(\mathbf{x}) := - \sum_{\{i,j\} \in \mathcal{E}} F_{ij}(x_i - x_j), \tag{3.1}$$

which is well defined as all $F_{ij}$ are even, we get

$$[-\nabla V(\mathbf{x})]_i = \sum_{j\sim i} f_{ij}(x_i - x_j) = \dot{x}_i,$$

so $\dot{\mathbf{x}} = -\nabla V(\mathbf{x})$. Thus, $V$ is a potential of the system (#), and $V(\mathbf{x}(t))$ is a decreasing function in time (except in equilibrium points); see supplementary equation (SM1.2). In particular, by LaSalle's invariance principle [45, Theorem 6.15] all forward (resp., backward) bounded orbits "tend to" (resp., "come from") equilibrium points. In Proposition SM1.3 we show that if $f_{ij}$ satisfy certain conditions, then all forward orbits are bounded and thus converge to equilibria.

**3.2. Matrix reformulation.** Define $f : C_1 \longrightarrow C_1$ by $f((y_e)_{e\in\mathcal{E}}) = (f_e(y_e))_{e\in\mathcal{E}}$. Then, $[f(d^\top(\mathbf{x}))]_{\{j,k\}} = f_{kj}(x_k - x_j)$ where $\{j,k\} \in \mathcal{E}$ with $j < k$, and

$$[d]_{i,\{j,k\}} = \begin{cases} 1 & \text{if } k = i, \\ -1 & \text{if } j = i, \\ 0 & \text{else.} \end{cases} \tag{3.2}$$

So we get

$$\begin{aligned} [d(f(d^\top(\mathbf{x})))]_i &= \sum_{\{j,k\}\in\mathcal{E}} [d]_{i,\{j,k\}} \cdot [f(d^\top(\mathbf{x}))]_{\{j,k\}} \\ &= \sum_{\substack{\{j,i\}\in\mathcal{E} \\ j<i}} f_{ij}(x_i - x_j) + \sum_{\substack{\{i,k\}\in\mathcal{E} \\ i<k}} -f_{ki}(x_k - x_i) = \sum_{j\sim i} f_{ij}(x_i - x_j) = \dot{x}_i, \end{aligned}$$

where in the second-to-last equality we use the facts that $f_{ki} = f_{ik}$ and that they are odd functions. Hence we can rewrite (#) as

$$\dot{\mathbf{x}} = (d \circ f \circ d^\top)(\mathbf{x}). \tag{3.3}$$

This is very reminiscent of the compact forms in (1.2). Indeed, if we recall (2.1), the three ODEs in (1.1) can be expressed as

$$\dot{\mathbf{x}} = (d \circ d^\top \circ f)(\mathbf{x}), \qquad \dot{\mathbf{x}} = (d \circ f \circ d^\top)(\mathbf{x}), \qquad \dot{\mathbf{x}} = (f \circ d \circ d^\top)(\mathbf{x}), \tag{3.4}$$

where in the first and last cases, $f$ is given by not necessarily odd functions in nodes, i.e., $f((x_v)_{v\in\mathcal{V}}) = (f_v(x_v))_{v\in\mathcal{V}}$ (in (1.1) we chose $f_v = \hat{f}$). In [43] it is shown that the matrix form of the first and third ODEs in (3.4) is very useful for studying the structure of the corresponding systems. We will show that the same is true for (3.3).

Equation (3.3) also allows us to represent system (#) as a dynamical system on the edges. That is, in the coordinates $\mathbf{y} = d^\top\mathbf{x}$, system (#) has the form

$$\dot{\mathbf{y}} = (d^\top \circ d \circ f)(\mathbf{y}) \tag{3.5}$$

and is defined on the cut space $(\ker d)^\perp$; see (2.2). Note that the value in an edge $\{j,i\} \in \mathcal{E}$ is given by $x_i - x_j$, where $j < i$.

**3.3. Equilibrium points.** In this section, we find expressions that relate the set of equilibrium points of (#) with the topology of the underlying network. First, using (3.5) and $\ker(d^\top d) = \ker d$ makes it clear that the equilibrium points in the *edge space*, i.e., in the coordinates $\mathbf{y}$, are given by

$$\mathscr{E}_{\mathbf{y}} := (\ker d)^\perp \cap f^{-1}(\ker d). \tag{3.6}$$

Recall that $\ker d$ is the cycle space and $(\ker d)^\perp$ the cut space, which have strong topological interpretations and are easy to compute intuitively. In supplementary section SM2 we further develop this intuition and informally show why we should expect the set of equilibria to be at most $(m - n + 1)$-dimensional using (3.6).

The set of equilibrium points is better understood for specific systems, such as the Kuramoto model [34, 20], but when dealing with complex networks and functions $f_{ij}$, it will generally be infeasible to find all equilibria explicitly. Thus, on some occasions we will restrict our study to the detailed-balance stationary states, as done in [21], which are given by

$$\tilde{\mathscr{E}}_{\mathbf{y}} := (\ker d)^\perp \cap f^{-1}(\mathbf{0}).$$

In node coordinates, $\mathbf{x}$, we have

$$\mathscr{E}_{\mathbf{x}} := \langle \mathbf{1} \rangle^\perp \cap (d^\top)^{-1}(f^{-1}(\ker d))$$

and

$$\tilde{\mathscr{E}}_{\mathbf{x}} := \{\mathbf{x} \in \langle \mathbf{1} \rangle^\perp : f_{ij}(x_i - x_j) = 0 \text{ for all } \{i,j\} \in \mathcal{E}\}.$$

In particular, as odd functions have 0 as a root, $\mathbf{0}$ is an equilibrium point of (#) in $\langle \mathbf{1} \rangle^\perp$, and thus $\langle \mathbf{1} \rangle$ is a line of equilibria in $\mathbb{R}^n$. This is a desired property for generalizations of linear consensus, as in the linear case the equilibrium points are given by the *consensus states*, i.e., $\langle \mathbf{1} \rangle$ or, equivalently, $x_i = x_j$ for all $i, j$. Note that in our general context we may have other equilibria besides the consensus states. In general this is desirable as it allows for more flexibility; see, for instance, the animal group decision-making model with nonconsensus equilibria in [18]. In some instances, however, one may want a nonlinear model which always reaches consensus, and the following result adapted from [48] explains how to guarantee this.

Proposition 3.1. *Suppose that for all edges, $f_{ij}$ has 0 as its unique root with $f'_{ij}(0) < 0$. Then, the only equilibria of the system (#) in $\mathbb{R}^n$ are the consensus states $\langle \mathbf{1} \rangle$. Moreover, all forward orbits converge to one of these states.*

*Proof.* For each edge, as $f_{ij}$ is odd, we can define $k_{ij}(|y|) = -f(y)/y$ for $y \neq 0$ and $k_{ij}(0) = -f'(0)$. Then $f_{ij}(y) = -k_{ij}(|y|)y$ where $k_{ij}$ is positive and continuous by the assumptions of this theorem. Now denote by $L(\mathbf{x})$ the Laplacian matrix of the graph $\mathcal{G}$ with weights $k_{ij}(|x_i - x_j|) > 0$. If we define

$$E(\mathbf{x}) = \sum_{\{i,j\} \in \mathcal{E}} (x_i - x_j)^2,$$

one can show that $\dot{E}(\mathbf{x}) = -\mathbf{x}^\top L(\mathbf{x})\mathbf{x}$ (see [48, section IV]) and as Laplacians are positive semidefinite $\dot{E}(\mathbf{x}) \leq 0$. As always it is enough to restrict our study to the state space $\langle \mathbf{1} \rangle^\perp$

where the only consensus state is the origin. Then, as $\mathcal{G}$ is connected we have $\dot{E}(\mathbf{x}) = 0$ only at the origin, and $E(\mathbf{x}) \to \infty$ if $||\mathbf{x}|| \to \infty$. So applying LaSalle's invariance principle [45, Theorem 6.15] to $E$, we conclude that all forward orbits in $\langle \mathbf{1} \rangle^\perp$ converge to the origin. ■

### 3.4. Stability of equilibrium points.

To find the stability of an equilibrium point $\mathbf{x}^\star$ we consider the Jacobian matrix of (#), which we will denote by $J(\mathbf{x}^\star)$ or simply $J$. By the compact reformulation given in (3.4) and the chain rule we have

$$J(\mathbf{x}^\star) = d\, D_{d^\top \mathbf{x}^\star} f\, d^\top, \tag{3.7}$$

where $D_{\mathbf{y}} f$ denotes the Jacobian of $f$ at $\mathbf{y}$. Note that $\mathbf{1}$ is an eigenvector of eigenvalue 0 of $J$, and thus an equilibrium cannot be linearly stable in $\mathbb{R}^n$. However, it can be linearly stable in the dynamically invariant space $\langle \mathbf{1} \rangle^\perp$, which is also invariant under the symmetric matrix $J$. From now on, linear stability will always be restricted to the space $\langle \mathbf{1} \rangle^\perp$. Note that the eigenvalues of $J|_{\langle \mathbf{1} \rangle^\perp}$ are given by those in $\mathbb{R}^n$ while dropping one multiplicity of 0, which we will constantly use in our proofs. In particular, we have the following.

Proposition 3.2. *Let $\mathbf{x}^\star$ be an equilibrium point of (#). Then, if all eigenvalues of $J(\mathbf{x}^\star)$ (in $\langle \mathbf{1} \rangle^\perp$) are negative, $\mathbf{x}^\star$ is stable, whereas if $J(\mathbf{x}^\star)$ has a positive eigenvalue, $\mathbf{x}^\star$ is unstable.*

*Remark* 3.3. In the original state space $\mathbb{R}^n$ we will never have attractive points. Indeed, any equilibrium point in $\langle \mathbf{1} \rangle^\perp$ spans a line of equilibrium points in the direction $\mathbf{1}$ in $\mathbb{R}^n$. However, if a point is stable in $\langle \mathbf{1} \rangle^\perp$, it will also be stable in $\mathbb{R}^n$, as all parallel planes have the same dynamics.

The conditions in the proposition above determine the linear stability of the equilibria in $\langle \mathbf{1} \rangle^\perp$. Thus, the stability will persist under small perturbations of the functions $f_{ij}$ as long as they remain odd. This will hold for all stability results, since they follow from Proposition 3.2.

As $J$ is a symmetric matrix, we can use the Courant minmax principle [33] which, in particular, implies the following.

Proposition 3.4. *Let $M$ be a symmetric matrix, and let $\lambda_{\max}$ be its maximum eigenvalue in an invariant subspace $V$. Then, for all $\mathbf{x} \in V$,*

$$||\mathbf{x}||^2 \lambda_{\max} \geq \mathbf{x}^\top M \mathbf{x}.$$

*Moreover, there exists a unitary vector $\mathbf{z} \in V$ such that $\lambda_{\max} = \mathbf{z}^\top M \mathbf{z}$.*

In our context, $V = \langle \mathbf{1} \rangle^\perp$ and $M = J$. However, by the comment made above Proposition 3.2, if we find $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{x}^\top J \mathbf{x} > 0$, it will be enough to deduce that $\lambda_{\max} > 0$ in $\langle \mathbf{1} \rangle^\perp$. Similarly, if we find $\mathbf{x} \in \mathbb{R}^n \setminus \langle \mathbf{1} \rangle$ such that $\mathbf{x}^\top J \mathbf{x} \geq 0$, it will be enough to deduce that $\lambda_{\max} \geq 0$ in $\langle \mathbf{1} \rangle^\perp$.

A direct consequence of this result is the following simple criterion for instability.

Proposition 3.5. *Let $\mathcal{H}$ be a cut-set of $\mathcal{G}$, and let $\mathbf{x}^\star$ be an equilibrium point of (#). Then*[3]

$$\sum_{\{i,j\} \in \mathcal{H}} f'_{ij}(x_i^\star - x_j^\star) > 0 \Longrightarrow \mathbf{x}^\star \text{ is unstable.}$$

[3] Note that $f'_{ij} = f'_{ji}$ and is even, so the sum is well defined.

*In particular, for any $i \in \mathcal{V}$ we have*

$$\sum_{j\sim i} f'_{ij}(x_i^\star - x_j^\star) > 0 \Longrightarrow \mathbf{x}^\star \textit{ is unstable.}$$

*Proof.* Let $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ be a partition of the nodes such that $\mathcal{H}$ is the associated cut-set, and consider the vector $\mathbf{x} = \sum_{i\in\mathcal{V}_1} \mathbf{e}_i$. Now note that

$$\begin{aligned}\mathbf{x}^\top J\mathbf{x} &= \sum_{i\in\mathcal{V}_1} [J]_{i,i} + \sum_{\substack{i,j\in\mathcal{V}_1\\ j\sim i}} [J]_{i,j} = \sum_{i\in\mathcal{V}_1}\sum_{j\sim i} f'_{ij}(x_i^\star - x_j^\star) - \sum_{i\in\mathcal{V}_1}\sum_{\substack{j\in\mathcal{V}_1\\ j\sim i}} f'_{ij}(x_i^\star - x_j^\star)\\ &= \sum_{i\in\mathcal{V}_1}\sum_{\substack{j\notin\mathcal{V}_1\\ j\sim i}} f'_{ij}(x_i^\star - x_j^\star) = \sum_{\{i,j\}\in\mathcal{H}} f'_{ij}(x_i^\star - x_j^\star) > 0\end{aligned}$$

and apply Proposition 3.4 together with Proposition 3.2. ■

Informally the result above makes clear that if $\mathbf{x}^\star$ is stable, $f'_{ij}(x_i^\star - x_j^\star) \le 0$ must hold for "a substantial amount" of edges, as we need to have at least one of them in each cut-set. Moreover, their module has to be big enough such that all corresponding sums are nonpositive. We now proceed to show that "a substantial amount" can be expressed formally as containing a spanning tree. First, we introduce some notation.

Recall (3.7) and that $D_{d^\top(\mathbf{x}^\star)}f$ is a diagonal matrix, with values in the diagonal $f'_{ij}(x_i^\star - x_j^\star)$ for each $\{j,i\} \in \mathcal{E}$. Grouping the positive and negative values into two terms, we can decompose the Jacobian matrix as

$$J(\mathbf{x}^\star) = J = L^+ - L^-, \tag{3.8}$$

where $L^-$ (resp., $L^+$) is the weighted Laplacian of $\mathcal{G}^-$ (resp., $\mathcal{G}^+$) defined as the network $\mathcal{G}$ with weights given by $|f'_{ij}(x_i^\star - x_j^\star)|$ if $f'_{ij}(x_i^\star - x_j^\star) < 0$ (resp., $f'_{ij}(x_i^\star - x_j^\star) > 0$) and 0 otherwise. Matrices of the form $L^+ - L^-$ are known as signed Laplacians and do not retain many of the properties of standard Laplacians; for instance, they may have both positive and negative eigenvalues. A general study on the signs of their spectrum can be found in [15], where an alternative proof of the following result is given.

Proposition 3.6. *Given an equilibrium point $\mathbf{x}^\star$ of (#) we have the following:*

- *If $L^+ = \mathbf{0}_{n,n}$ and $\mathcal{G}^-$ is connected, $\mathbf{x}^\star$ is stable.*
- *If $\mathcal{G}^-$ is disconnected, then $\mathbf{x}^\star$ is not linearly stable (in $\langle \mathbf{1}\rangle^\perp$). Moreover, if there is an edge in $\mathcal{G}^+$ between two distinct connected components of $\mathcal{G}^-$, $\mathbf{x}^\star$ is unstable.*

*Proof.* See Appendix A. ■

This proposition is useful for studying some simple situations. For instance, if a network has an edge $\{i,j\}$ with no cycles going through it, e.g., the connecting edge of the barbell graph, and $f'_{ij}(x_i^\star - x_j^\star) > 0$, then $\mathbf{x}^\star$ is unstable. To be able to deal with more complex situations, we will introduce the concept of effective resistances in subsection 3.4.2.

**3.4.1. Schur complement reduction.** In this section we show how to use the Schur complement to reduce the dimensionality of the stability problem. Essentially we will be able to remove all nodes which are not incident to any edge in $\mathcal{G}^+$. Clearly, this technique is very powerful when $\mathcal{G}^+$ contains few edges; see Theorem 3.10 for the case with a single edge, and see Appendix B for a concrete example.

We start by introducing the Schur complement for a symmetric matrix.

Definition 3.7. *Given an $n \times n$ symmetric matrix $M$, and $\mathcal{N}, \mathcal{M} \subset \{1, \dots, n\}$, we denote by $M_{\mathcal{N}\mathcal{M}}$ the submatrix of $M$ with rows indexed by $\mathcal{N}$ and columns by $\mathcal{M}$. Then, if $M_{\mathcal{N}\mathcal{N}}$ is invertible, we define the* Schur complement *of $M$ with respect to $\mathcal{N}$ by*

$$M/\mathcal{N} := M_{\mathcal{N}^c\mathcal{N}^c} - M_{\mathcal{N}^c\mathcal{N}} M_{\mathcal{N}\mathcal{N}}^{-1} \mathcal{M}_{\mathcal{N}\mathcal{N}^c}. \tag{3.9}$$

The Schur complement has many interesting properties and applications as reviewed in [51]. In our context, its two key features are that $\mathrm{In}(M) = \mathrm{In}(M/\mathcal{N}) + \mathrm{In}(M_{\mathcal{N}\mathcal{N}})$ (see [51, Theorem 1.6]) and that if $M$ is a weighted Laplacian, then $M/\mathcal{N}$ is also a Laplacian [23].

Recall that finding the stability of an equilibrium $\mathbf{x}^\star$ reduces to the study of the inertia of $J$. Here it will be convenient to assume $\mathcal{G}^-$ connected, and one can then use Proposition 3.6 to deduce the stability for the general case. Denote by $\mathcal{N}$ the set of nodes nonincident to any edge in $\mathcal{G}^+$, and assume that $\mathcal{N}$ and $\mathcal{N}^c$ are nonempty. Then reordering the nodes, we have

$$J = \begin{pmatrix} L^+_{\mathcal{N}^c\mathcal{N}^c} - L^-_{\mathcal{N}^c\mathcal{N}^c} & -L^-_{\mathcal{N}^c\mathcal{N}} \\ -L^-_{\mathcal{N}\mathcal{N}^c} & -L^-_{\mathcal{N}\mathcal{N}} \end{pmatrix}. \tag{3.10}$$

As $L^-_{\mathcal{N}\mathcal{N}}$ is a principal submatrix of a connected Laplacian, it is positive definite and thus invertible. Thus, we can consider the Schur complement $J/\mathcal{N}$ and get

$$\mathrm{In}(J) = \mathrm{In}\left(J/\mathcal{N}\right) + \mathrm{In}\left(J_{\mathcal{N}\mathcal{N}}\right) = \mathrm{In}\left(J/\mathcal{N}\right) + \mathrm{In}\left(-L^-_{\mathcal{N}\mathcal{N}}\right).$$

As $-L^-_{\mathcal{N}\mathcal{N}}$ is negative definite, the stability of $\mathbf{x}^\star$ is determined by $\mathrm{In}\left(J/\mathcal{N}\right)$. Moreover, from (3.10) it is clear that

$$J/\mathcal{N} = L^+_{\mathcal{N}^c\mathcal{N}^c} - L^-/\mathcal{N}, \tag{3.11}$$

where $L^-/\mathcal{N}$ is a Laplacian matrix, as it is the Schur complement of a Laplacian. Note that $L^+_{\mathcal{N}^c\mathcal{N}^c}$ is also a Laplacian matrix, as by the definition of $\mathcal{N}$ all other entries of $L^+$ are zeros. Thus, expression (3.11) is analogous to (3.8) but in dimension $|\mathcal{V}| - |\mathcal{N}|$, and we can interpret it as the stability problem for a smaller network. In summary we have shown the following.

Theorem 3.8. *Let $\mathbf{x}^\star$ be an equilibrium point of (#), assume that $\mathcal{G}^-$ is connected, and let $\mathcal{N}$ be the nodes not incident to any edge in $\mathcal{G}^+$. Then, the stability of $\mathbf{x}^\star$ is given by the linear stability of $J/\mathcal{N}$ in $\langle \mathbf{1} \rangle^\perp$.*

This theorem will only be useful when $|\mathcal{N}|$ is large, in particular if $\mathcal{G}^+$ is connected; then $|\mathcal{N}| = 0$, and we do not get any reduction. One small detail we have ignored so far is that in (3.11) one may have edges that are contained in both Laplacians $L^+_{\mathcal{N}^c\mathcal{N}^c}$ and $L^-/\mathcal{N}$ in contrast to (3.8). This can be used to apply the Schur complement to a new decomposition and reduce the dimensions even further; see Appendix B.

**3.4.2. Effective resistance criteria.** In this section we will apply ideas from electrical networks and, in particular, the concept of effective resistance, to determine the stability of the equilibrium points of (#). This application of effective resistance was introduced in the recent works by Devriendt and Lambiotte [21] for a system of the form (#) and by Tyloo, Pagnier, and Jacquod [46] for Kuramoto-like systems. While the analysis in [21] was based on unweighted networks, we will consider weights on $\mathcal{G}$ in our more general setting, which leads to improved results. We will show that when a pair of nodes are better connected[4] in $\mathcal{G}^+$ than in $\mathcal{G}^-$, then $\mathbf{x}^\star$ is unstable. Moreover, when $\mathcal{G}^+$ contains a single edge, this condition is tight.

Definition 3.9. *The* effective resistance $r_{ij}$ *between a pair of nodes $i$ and $j$ in a weighted network with Laplacian $L$ is defined as (see, e.g., [23, 29])*

$$r_{ij} := (\mathbf{e}_i - \mathbf{e}_j)^\top L^\dagger (\mathbf{e}_i - \mathbf{e}_j)$$

*if $i$ and $j$ are in the same connected component, and $r_{ij} = \infty$ otherwise.*

Note that the effective resistance between two adjacent nodes, with no other path between them, is given by the inverse of the edge weight. Moreover, this definition satisfies the well-known properties of resistors in series and in parallel; see supplementary section SM3. A generalization of this fact is given by the invariance of effective resistance by the Schur complement, also known as Kron reduction [23]. That is, the effective resistance between two nodes $i, j \notin \mathcal{N} \subset \mathcal{V}$ is the same in $L$ and in $L/\mathcal{N}$.

Recall that $J = L^+ - L^-$, and denote by $r_{ij}^+$, $r_{ij}^-$ the effective resistance between $i$ and $j$ of the corresponding networks $\mathcal{G}^+$, $\mathcal{G}^-$. With this notation we are prepared to state our first result, which gives tight conditions for the stability when $\mathcal{G}^+$ has a single edge.

Theorem 3.10. *Let $\mathbf{x}^\star$ be an equilibrium point of (#) such that there exists a unique edge $\{i,j\}$ with $f'_{ij}(x_i^\star - x_j^\star) \geq 0$, and assume that $f'_{ij}(x_i^\star - x_j^\star) > 0$. Then,*

$$f'_{ij}(x_i^\star - x_j^\star) r_{ij}^- < 1 \Longrightarrow \mathbf{x}^\star \text{ is stable;}$$
$$f'_{ij}(x_i^\star - x_j^\star) r_{ij}^- > 1 \Longrightarrow \mathbf{x}^\star \text{ is unstable.}$$

*Proof.* If $\mathcal{G}^-$ is disconnected, we have $r_{ij}^- = \infty$, and $\mathbf{x}^\star$ is unstable due to Proposition 3.6. If $\mathcal{G}^-$ is connected, by Theorem 3.8 the stability is determined by $J/\mathcal{N}$. As $\mathcal{N}^c = \{i,j\}$, we have

$$J/\mathcal{N} = L^+_{\{i,j\},\{i,j\}} - L^-/\mathcal{N} = \begin{pmatrix} f'_{ij}(x_i^\star - x_j^\star) & -f'_{ij}(x_i^\star - x_j^\star) \\ -f'_{ij}(x_i^\star - x_j^\star) & f'_{ij}(x_i^\star - x_j^\star) \end{pmatrix} - \begin{pmatrix} w_{ij}^- & -w_{ij}^- \\ -w_{ij}^- & w_{ij}^- \end{pmatrix}$$

for certain weight $w_{ij}^-$. Thus, $\mathbf{x}^\star$ is stable if $f'_{ij}(x_i^\star - x_j^\star) - w_{ij}^- < 0$ and unstable if $f'_{ij}(x_i^\star - x_j^\star) - w_{ij}^- > 0$. Finally, we use the fact that effective resistance is invariant under the Schur complement, so $r_{ij}^- = 1/w_{ij}^-$. ∎

[4]Better connected in the sense of effective resistance, i.e., the effective resistance between them in $\mathcal{G}^+$ is smaller than it is in $\mathcal{G}^-$.

*Remark* 3.11. If there is a unique edge with $f'_{ij}(x_i^\star - x_j^\star) > 0$, and $\mathcal{G}^- \cup \{i,j\}$ is connected, the result holds with the same arguments.

We can take an analogous approach when $\mathcal{G}^+$ contains two edges with a common node or, in fact, any particular small configuration. Then, using Sylvester's criterion [42], one gets a couple of inequalities on the entries of $J/\mathcal{N}$ (which can be related to effective resistance) that determine the stability. This operation gets quite messy so, instead, we expand by brute force Theorem 3.10 to the general case, but we lose the tightness of the bounds.

Proposition 3.12. *Let $\mathbf{x}^\star$ be an equilibrium point of (#). Then, if $\mathcal{G}^+ \cup \mathcal{G}^-$ is connected,*

$$\sum_{\{i,j\}\in\mathcal{G}^+} f'_{ij}(x_i^\star - x_j^\star) r_{ij}^- < 1 \Longrightarrow \mathbf{x}^\star \text{ is stable;}$$
$$\exists\{i,j\} \in \mathcal{G}^+ : f'_{ij}(x_i^\star - x_j^\star) r_{ij}^- > 1 \Longrightarrow \mathbf{x}^\star \text{ is unstable.}$$

*Proof.* See Appendix C. ∎

Both Theorem 3.10 and Proposition 3.12 are adapted results from [21] where a particular coupling function is considered. We have opted to give completely different proofs, but it is possible to generalize the arguments from [21] to our context with some work. Let us now present another approach which leads to a tighter instability condition. First, we need the following result; see Appendix D for the proof.

Lemma 3.13. *Let $A, B$ be symmetric positive semidefinite matrices of the same dimension. Then*[5]

$$\max_{\mathbf{x}\perp \ker B} \frac{\mathbf{x}^\top A \mathbf{x}}{\mathbf{x}^\top B \mathbf{x}} = \max_{\mathbf{x}\perp \ker A} \frac{\mathbf{x}^\top B^\dagger \mathbf{x}}{\mathbf{x}^\top A^\dagger \mathbf{x}}.$$

In supplementary section SM4 we discuss how this result could also be used to get stability conditions.

Theorem 3.14. *Let $\mathbf{x}^\star$ be an equilibrium point of our system, and let $i, j$ be a pair of nodes. Then*

$$r_{ij}^- > r_{ij}^+ \Longrightarrow \mathbf{x}^\star \text{ is unstable.}$$

*Proof.* Denote by $\lambda_{\max}$ the maximum eigenvalue of $J$ in $\langle \mathbf{1}\rangle^\perp$; then by Proposition 3.4 we have

$$\lambda_{\max} \geq \max_{\substack{\mathbf{x}\perp \ker L^- \\ ||\mathbf{x}||=1}} \mathbf{x}^\top L^+ \mathbf{x} - \mathbf{x}^\top L^- \mathbf{x} = \max_{\substack{\mathbf{x}\perp \ker L^- \\ ||\mathbf{x}||=1}} \mathbf{x}^\top L^- \mathbf{x} \left( \frac{\mathbf{x}^\top L^+ \mathbf{x}}{\mathbf{x}^\top L^- \mathbf{x}} - 1 \right)$$

and, as $L^-$ is positive semidefinite,

$$\operatorname{sign}(\lambda_{\max}) \geq \operatorname{sign}\left( \max_{\mathbf{x}\perp \ker L^-} \frac{\mathbf{x}^\top L^+ \mathbf{x}}{\mathbf{x}^\top L^- \mathbf{x}} - 1 \right). \tag{3.12}$$

[5] All maximums are taken over nonzero vectors.

Now if $r_{ij}^{+} = \infty$, the condition of this theorem is never satisfied and there is nothing to prove. If $r_{ij}^{-} = \infty$ and $r_{ij}^{+} < \infty$, $\mathbf{x}^\star$ is unstable by Proposition 3.6. So we may assume that $i$ and $j$ are in the same connected component of $\mathcal{G}^+$ and $\mathcal{G}^-$. Then, $(\mathbf{e}_i - \mathbf{e}_j) \perp \ker L^+$, and by the previous lemma,

$$\max_{\mathbf{x} \perp \ker L^-} \frac{\mathbf{x}^\top L^+ \mathbf{x}}{\mathbf{x}^\top L^- \mathbf{x}} = \max_{\mathbf{x} \perp \ker L^+} \frac{\mathbf{x}^\top (L^-)^\dagger \mathbf{x}}{\mathbf{x}^\top (L^+)^\dagger \mathbf{x}} \geq \frac{(\mathbf{e}_i - \mathbf{e}_j)^\top (L^-)^\dagger (\mathbf{e}_i - \mathbf{e}_j)}{(\mathbf{e}_i - \mathbf{e}_j)^\top (L^+)^\dagger (\mathbf{e}_i - \mathbf{e}_j)} = \frac{r_{ij}^-}{r_{ij}^+}.$$

So if $r_{ij}^- > r_{ij}^+$, then $\frac{r_{ij}^-}{r_{ij}^+} - 1 > 0$, and thus by (3.12), $\lambda_{\max} > 0$. ■

*Remark* 3.15. When there is a unique edge $\{i, j\}$ with positive derivative, then $r_{ij}^+ = 1/f'(x_i^\star - x_j^\star)$, and the instability condition coincides with Theorem 3.10. In general $r_{ij}^+ \leq 1/f'(x_i^\star - x_j^\star)$, and the new instability condition is tighter than Proposition 3.12.

It has been shown that the effective resistance is a distance function on the nodes of a network which, in some sense, encapsulates how well connected the different pairs of nodes are [29, 23]. With this point of view, the theorem above asserts that if there exist a pair of nodes which are better connected in $\mathcal{G}^+$ than in $\mathcal{G}^-$, then we have instability. For the case of a unique nonnegative edge, Theorem 3.10 asserts that this condition is tight and only needs to be checked for the nonnegative edge. Theorem 3.14 does not give tight conditions in general, but they might be tight for certain types of networks (see supplementary section SM8).

**3.5. Union of two networks in a node.** Assume that $\mathcal{G}$ contains a *cut-node*, i.e., a node that when "deleted" increases the number of connected components of $\mathcal{G}$. Equivalently, $\mathcal{G}$ is the union of two subnetworks $\mathcal{A}$, $\mathcal{B}$ that intersect in a single node, also known as a *coalescence* of $\mathcal{A}$ and $\mathcal{B}$ [5]. In this section we will show that the equilibrium points of $\mathcal{G}$ are exactly the combination of equilibrium points of $\mathcal{A}$ and $\mathcal{B}$. Moreover, we will show that the linear stability of an equilibrium point in $\mathcal{G}$ is determined by the stability of the corresponding equilibrium points in $\mathcal{A}$ and $\mathcal{B}$. These results are quite surprising, as there are several ways to coalesce two networks, which can produce significantly different structures. Our results show that from a basic dynamical point of view, the resulting networks will be equivalent. A comparison between the dynamical properties of networks and their coalescence was also considered in [5] for a system of coupled oscillators, where coalescence was shown to lead to an increased time to reach synchrony.

Without loss of generality we may assume that the nodes of $\mathcal{A}$ and $\mathcal{B}$ are $\{1, \ldots, \tilde{n}\}$ and $\{\tilde{n}, \ldots, n\}$, respectively, and let $n' = n - \tilde{n} + 1$. As all edges in $\mathcal{G}$ are in $\mathcal{A}$ or $\mathcal{B}$ exclusively, the edge state of $\mathcal{G}$ is given by the edge states of $\mathcal{A}$ and $\mathcal{B}$ combined. That is, reordering the components if necessary, we have $\mathbf{y} = (\mathbf{y}_\mathcal{A}, \mathbf{y}_\mathcal{B})$. Note that with these assumptions, $f(\mathbf{y}) = (f_\mathcal{A}(\mathbf{y}_\mathcal{A}), f_\mathcal{B}(\mathbf{y}_\mathcal{B}))$, and the generating cycles of $\ker d$ are entirely contained in $\mathcal{A}$ or $\mathcal{B}$. Then, by (3.6) it is straightforward to show that

$$\mathscr{E}_\mathbf{y} = \iota_\mathcal{A}(\mathscr{E}_\mathbf{y}^\mathcal{A}) \oplus \iota_\mathcal{B}(\mathscr{E}_\mathbf{y}^\mathcal{B}),$$

where $\mathscr{E}_\mathbf{y}^\mathcal{A}$ are the equilibria on $\mathcal{A}$ and $\iota_\mathcal{A}(\mathbf{y}_\mathcal{A}) = (\mathbf{y}_\mathcal{A}, \mathbf{0}_\mathcal{B})$ (resp., for $\mathcal{B}$); see supplementary section SM5 for a detailed exposition. Hence, $\mathbf{y} = (\mathbf{y}_\mathcal{A}, \mathbf{y}_\mathcal{B}) \in \mathscr{E}_\mathbf{y}$ if and only if $\mathbf{y}_\mathcal{A} \in \mathscr{E}_\mathbf{y}^\mathcal{A}$ and $\mathbf{y}_\mathcal{B} \in \mathscr{E}_\mathbf{y}^\mathcal{B}$.

Now we move on to proving that the stability can also be studied by restricting ourselves to $\mathcal{A}$ and $\mathcal{B}$. We need the following technical result proved in Appendix E.

Lemma 3.16. *Let $\mathcal{G}$ be a network with $m$ edges, let $d$ be its boundary map, and let $\tilde{d}$ be the same matrix with a row deleted. Let $M \in \mathbb{R}^{m\times m}$ be a diagonal matrix. Then, the number of positive (resp., negative) eigenvalues of $dMd^\top$ and $\tilde{d}M\tilde{d}^\top$ coincide.*

Recall from Proposition 3.2 that the stability only depends on the sign of the eigenvalues of $J$. We have

$$J = dD_{\mathbf{y}}fd^\top = \begin{pmatrix} d_{\mathcal{A}}D_{\mathbf{y}_{\mathcal{A}}}f_{\mathcal{A}}d_{\mathcal{A}}^\top & \mathbf{0}_{\tilde{n},n'-1} \\ \mathbf{0}_{n'-1,\tilde{n}} & \mathbf{0}_{n'-1,n'-1} \end{pmatrix} + \begin{pmatrix} \mathbf{0}_{\tilde{n}-1,\tilde{n}-1} & \mathbf{0}_{\tilde{n}-1,n'} \\ \mathbf{0}_{n',\tilde{n}-1} & d_{\mathcal{B}}D_{\mathbf{y}_{\mathcal{B}}}f_{\mathcal{B}}d_{\mathcal{B}}^\top \end{pmatrix},$$

where in the $\tilde{n}$th row/column we can have nonzero entries in both matrices. Now if we denote by $\tilde{d}$, $\tilde{d}_{\mathcal{A}}$, $\tilde{d}_{\mathcal{B}}$ the respective matrices when deleting the row corresponding to the node $\tilde{n}$, we find

$$\tilde{d}D_{\mathbf{y}}f\tilde{d}^\top = \begin{pmatrix} \tilde{d}_{\mathcal{A}}D_{\mathbf{y}_{\mathcal{A}}}f_{\mathcal{A}}\tilde{d}_{\mathcal{A}}^\top & \mathbf{0}_{\tilde{n}-1,n'-1} \\ \mathbf{0}_{n'-1,\tilde{n}-1} & \tilde{d}_{\mathcal{B}}D_{\mathbf{y}_{\mathcal{B}}}f_{\mathcal{B}}\tilde{d}_{\mathcal{B}}^\top \end{pmatrix}.$$

So the spectrum of $\tilde{d}D_{\mathbf{y}}f\tilde{d}^\top$ is the union of the spectrum of $\tilde{d}_{\mathcal{A}}D_{\mathbf{y}_{\mathcal{A}}}f_{\mathcal{A}}\tilde{d}_{\mathcal{A}}^\top$ and $\tilde{d}_{\mathcal{B}}D_{\mathbf{y}_{\mathcal{B}}}f_{\mathcal{B}}\tilde{d}_{\mathcal{B}}^\top$. Thus, by Lemma 3.16 the signs of the spectrum of $dD_{\mathbf{y}}fd^\top$ are the union of the signs in the spectrum of $d_{\mathcal{A}}D_{\mathbf{y}_{\mathcal{A}}}f_{\mathcal{A}}d_{\mathcal{A}}^\top$ and $d_{\mathcal{B}}D_{\mathbf{y}_{\mathcal{B}}}f_{\mathcal{B}}d_{\mathcal{B}}^\top$ (with a zero removed due to dimensionality). In particular, the linear stability of $\mathbf{y}$ is determined by the linear stability of $\mathbf{y}_{\mathcal{A}}$ and $\mathbf{y}_{\mathcal{B}}$.

**3.5.1. Tree of motifs.** Starting from a collection of "small" networks which are understood dynamically,[6] we can create new networks by recursively joining (coalescing) these smaller networks into single nodes. By our results above, the dynamics of this new network will be equally well understood if we simply compose the properties of the subsystems. We refer to these smaller networks as *motifs,* as they play the role of dynamical and structural building blocks, and refer to the larger network as a *tree of motifs,* as they consist of motifs interconnected in a tree (i.e., in a loopless tree-like way); see Figure 1.

Equivalently, our result can be stated as follows: given an arbitrary network, the task of finding equilibria and their stability can be reduced to the same task for each of its *blocks.*[7] While some networks will consist of a single block, for many other networks this procedure will drastically reduce the dimensions in which we are working. In [28] similar results are shown for Markovian SIR epidemic dynamics.

## 4. Applications to concrete networks.

In this section we focus on studying the equilibrium points of (#) and their stability in some specific types of networks. Concretely we will study tree, cycle, and complete graphs. Although these families of networks may seem quite specific, by using the main result from subsection 3.5 we will be able to tackle trees of motifs for the following motifs: trees, cycles, and complete graphs. See Figure 1 for an example and subsection 4.4 for explicit computations. Such trees of motifs appear as the constructs of certain growing graph models (see [6, section 3.5.2] and [49, section 2]), and the special case where all motifs are complete graphs is also known as block graphs [7].

[6]That is, we know their equilibrium points and their linear stability.
[7]A block of a network is a maximal connected subnetwork that does not contain any cut-nodes of itself.

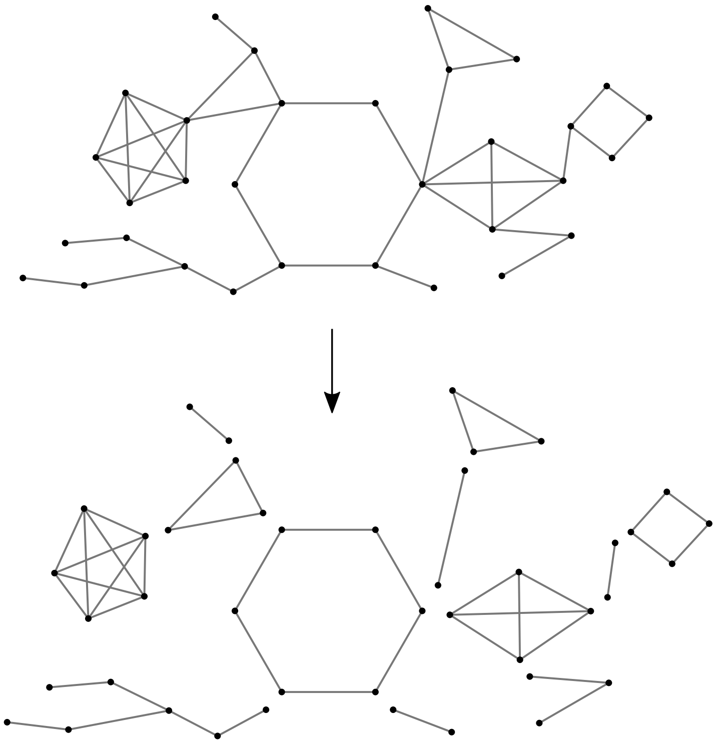

**Figure 1.** *In the upper half, a tree of motifs is shown where the motifs are trees, cycles, and complete graphs. In the lower half, the motifs are shown separated from one another. These are the subnetworks that we need to understand dynamically to comprehend the behavior of the whole network.*

**4.1. Tree graphs.** In this section we will study the case when $\mathcal{G}$ has the simplest topology, which in our context means that it does not contain any cycles, i.e., $\mathcal{G}$ is a tree. We will show explicit expressions of the equilibrium points of (#), given the roots of the functions $f_{ij}$, and a simple criterion for their stability. These results will directly follow from section 3.5.1, as tree graphs are, in particular, trees of motifs, where all motifs are simply pairs of connected nodes.

Proposition 4.1. *In a tree graph we have*

$$\mathscr{E}_{\mathbf{y}} = \tilde{\mathscr{E}}_{\mathbf{y}} = f^{-1}(\mathbf{0}) = \left\{(y_{\{i,j\}})_{\{i,j\}\in\mathcal{E}} \in \mathbb{R}^m \,:\, f_{ij}(y_{\{i,j\}}) = 0 \text{ for all } \{i,j\} \in \mathcal{E}\right\}.$$

*Moreover, the signs of the eigenvalues of $J(\mathbf{x}^\star)$ in $\langle \mathbf{1}\rangle^\perp$ are given by* $\mathrm{sign}(f'_{ij}(x_i^\star - x_j^\star))$ *for $\{i,j\} \in \mathcal{E}$. In particular, if there exists $\{i,j\} \in \mathcal{E}$ such that $f'_{ij}(x_i^\star - x_j^\star) > 0$, then $\mathbf{x}^\star$ is unstable, whereas $\mathbf{x}^\star$ is stable if $f'_{ij}(x_i^\star - x_j^\star) < 0$ for all $\{i,j\} \in \mathcal{E}$.*

*Proof.* First, as trees have no cycles, $\ker d = \mathbf{0}$, so from (3.6) and the definition of $\tilde{\mathscr{E}}_{\mathbf{y}}$ we get $\mathscr{E}_{\mathbf{y}} = \tilde{\mathscr{E}}_{\mathbf{y}} = f^{-1}(\mathbf{0})$. For the signs of the eigenvalues, apply subsection 3.5 inductively on edges. ■

In particular, in a tree the set of equilibria (on the edge space) and their linear stability only depend on the type of coupling functions $f_{ij}$ and not on the specific arrangement of the edges. For instance, the equilibria and stability in a line graph will be the same as in a star graph of the same size. Note that the result above not only gives us the stability of the equilibria but also finds the signs of all eigenvalues of $J$. This will be useful for the following result, which informally asserts that any random initial condition will converge to one of the stable equilibrium points or "infinity."

Proposition 4.2. *Let $\mathcal{G}$ be a tree where for all edges the roots of $f_{ij}$ are simple. Then, for Lebesgue almost every point $\mathbf{x} \in \langle \mathbf{1} \rangle^{\perp}$, its forward orbit converges to a stable equilibrium or is unbounded. Moreover, the stable equilibrium points are given by*

$$\left\{ \mathbf{x} \in \langle \mathbf{1} \rangle^{\perp} : f_{ij}(x_i - x_j) = 0 \text{ and } f'_{ij}(x_i - x_j) < 0 \text{ for all } \{i,j\} \in \mathcal{E} \right\}.$$

*Proof.* By the previous proposition, all equilibrium points are hyperbolic and thus isolated, and the stable ones are given by the expression above. Then, in gradient dynamics, LaSalle's invariance principle [45, Theorem 6.15] guarantees that all forward bounded orbits converge to a set of equilibrium points. As they are isolated, orbits converge to a single point. It also follows that there are countably many equilibrium points, so it is enough to prove that the set of points converging to an unstable equilibrium has null measure. This holds, as its stable manifold has at least codimension 1 (see supplementary Lemma SM1.1 for details). ∎

*Remark* 4.3. In the previous result, if for all edges, $f_{ij}$ has three simple roots with $f'_{ij}(0) < 0$, then the only stable equilibrium is the origin. So almost every forward bounded orbit converges to it.

Imposing more conditions on $f_{ij}$, one can show that all forward orbits are bounded; see supplementary Proposition SM1.3. To get a flavor of how a complete description of the dynamics of (#) may look, we give the phase portrait of trees in low dimension; see supplementary section SM6.

**4.2. Cycle graphs.** Let $\mathcal{C}_n$ be the cycle graph with $n$ nodes, i.e., a line graph where the end nodes have been joined, and let $\mathbf{x}^\star$ be an equilibrium point of (#) on this network. Then if $f'_{ij}(x_i^\star - x_j^\star) < 0$ for all edges, $\mathbf{x}^\star$ is stable by Proposition 3.6. If, instead, there exist two edges with nonnegative derivatives and one of them is positive, $\mathbf{x}^\star$ is unstable by Proposition 3.6. Assume now that there is a unique edge $\{i,j\}$ with nonnegative derivative and, moreover, that $f'_{ij}(x_i^\star - x_j^\star) \neq 0$. Then, by Theorem 3.10 we have

$$f'_{ij}(x_i^\star - x_j^\star) \cdot \sum_{\{i',j'\} \in \mathcal{E} \setminus \{\{i,j\}\}} \frac{1}{|f'_{i'j'}(x_{i'}^\star - x_{j'}^\star)|} < 1 \Longrightarrow \mathbf{x}^\star \text{ is stable;}$$

$$f'_{ij}(x_i^\star - x_j^\star) \cdot \sum_{\{i',j'\} \in \mathcal{E} \setminus \{\{i,j\}\}} \frac{1}{|f'_{i'j'}(x_{i'}^\star - x_{j'}^\star)|} > 1 \Longrightarrow \mathbf{x}^\star \text{ is unstable,}$$

where we have used the formula for effective resistances in series; see supplementary equation (SM3.1). Thus, we have tight conditions for the stability of $\mathbf{x}^\star$ except for the case when $\max_{\{i,j\} \in \mathcal{E}} f'_{ij}(x_i^\star - x_j^\star) = 0$ or when there is a unique edge with nonnegative derivative and the expression above equals 1.

In contrast with trees, where $\tilde{\mathscr{E}}_{\mathbf{x}} = \mathscr{E}_{\mathbf{x}}$, this equality will not always hold for cycle graphs (which may be considered the next simplest topology). This was already identified in [43] for $\mathcal{C}_3$ taking $f_{ij}(y) = \hat{f}(y) = \lambda y - y^3$ with $\lambda > 0$ for all $\{i,j\} \in \mathcal{E}$. There, it was noted that in the original node space $\mathbb{R}^3$, the subspace $\langle \mathbf{1} \rangle$ consists of unstable equilibrium points, and that there is a cylinder with axis $\langle \mathbf{1} \rangle$ consisting of stable equilibria. We note that

$$H(x_1, x_2, x_3) = \frac{x_1 - x_2}{x_1 - x_3}$$

is a first integral of this system, so that the phase portrait in the parallel planes to $\langle \mathbf{1} \rangle^{\perp}$ is as depicted in Figure 2. Notice that we get a continuum of equilibrium points.

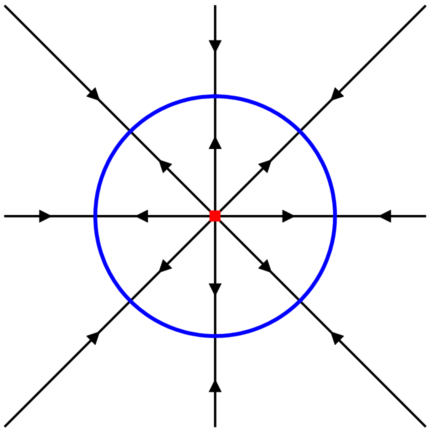

**Figure 2.** *Phase portrait of (#) for $\mathcal{C}_3$ and $\hat{f}(y) = \lambda y - y^3$ with $\lambda > 0$ in a plane perpendicular to $\mathbf{1}$. A circle of stable equilibrium points of radius $\sqrt{2\lambda/3}$ is depicted in blue, and a source equilibrium is shown by a red square. All other orbits are contained in lines going through the source point.*

We now study when and why this phenomenon of a curve of equilibria in the node space $\langle \mathbf{1} \rangle^{\perp}$ occurs for a general function $\hat{f}$. As $n = m$ and following the comments made in subsection 3.3, we suspect that this set is at most one-dimensional. In this context it will be convenient to work with a slight modification of the edge coordinates,

$$z_1 = y_{\{1,2\}}, \ \ldots, \ z_{n-1} = y_{\{n-1,n\}}, \ z_n = -y_{\{1,n\}}.$$

Recall that $\ker d$ is generated by the cycles, so in $\mathbf{z}$ coordinates, $\ker d = \langle \mathbf{1} \rangle$. Thus from (3.6) we get

$$\mathscr{E}_{\mathbf{z}} = \langle \mathbf{1} \rangle^{\perp} \cap f^{-1}(\langle \mathbf{1} \rangle) = \bigcup_{\lambda \in \mathbb{R}} \langle \mathbf{1} \rangle^{\perp} \cap f^{-1}(\lambda \mathbf{1}), \tag{4.1}$$

where $\mathscr{E}_{\mathbf{z}}$ are the equilibria in the $\mathbf{z}$ coordinates, and

$$\langle \mathbf{1} \rangle^{\perp} \cap f^{-1}(\lambda \mathbf{1}) = \left\{ \mathbf{z} \in \mathbb{R}^n : \sum_{i=1}^{n} z_i = 0 \text{ and } z_i \in \hat{f}^{-1}(\lambda) \right\}. \tag{4.2}$$

Now if the sets $\langle \mathbf{1}\rangle^\perp \cap f^{-1}(\lambda\mathbf{1})$ are nonempty for all $\lambda$ and they do not coincide, we expect to have a curve of equilibrium points parametrized by $\lambda$. Note that $\langle \mathbf{1}\rangle^\perp \cap f^{-1}(\lambda\mathbf{1}) \neq \emptyset$ is equivalent to the existence of $\alpha_1^\lambda, \dots, \alpha_n^\lambda$ roots of $\hat{f} - \lambda$ such that $\sum_{i=1}^n \alpha_i^\lambda = 0$. The following polynomials are an important class of functions that satisfy this condition.

Proposition 4.4. *Consider (#) for the network $\mathcal{C}_n$ with $f_{ij} = \hat{f}$ for all $\{i,j\} \in \mathcal{E}$. Assume that $\hat{f}$ is an odd polynomial of degree $k \neq 1$ with $k$ distinct real roots and $k|n$. Then, (#) has a continuum of equilibrium points.*

*Proof.* Denote by $a_0, \dots, a_k$ the coefficients of $\hat{f}$. As $\hat{f}$ has $k$ distinct roots and all of them are simple, there exists $\epsilon > 0$ such that for all $\lambda \in (-\epsilon, \epsilon)$, $\hat{f} - \lambda$ has $k$ real roots, which we denote by $\beta_0^\lambda, \dots, \beta_{k-1}^\lambda$. Moreover, for each $i$, $\beta_i^\lambda$ takes a continuum of values parametrized by $\lambda$. As $\hat{f}$ is an odd function, its even coefficients are null, and in particular, $a_{k-1} = 0$. By the well-known Vieta's formulas [47, p. 99] we have for all $\lambda$,

$$0 = -\frac{a_{k-1}}{a_k} = \sum_{i=0}^{k-1} \beta_i^\lambda.$$

Now for $i \in \{1, \dots, n\}$ define $\alpha_i^\lambda = \beta_{i \bmod k}^\lambda$, and note that $\sum_{i=1}^n \alpha_i^\lambda = \frac{n}{k}(\sum_{i=0}^{k-1} \beta_i^\lambda) = 0$ as $k|n$. So from (4.2) and (4.1), we deduce that

$$(\alpha_i^\lambda)_{i=1}^n \in \langle \mathbf{1}\rangle^\perp \cap f^{-1}(\lambda\mathbf{1}) \subset \mathscr{E}_\mathbf{z}$$

for all $\lambda \in (-\epsilon, \epsilon)$. Hence, $(\alpha_i^\lambda)_{i=1}^n$ gives a continuum of equilibria parametrized by $\lambda$. ■

### 4.3. Complete graphs or cliques.

In this subsection we will deal with the complete graph of $n$ nodes $\mathcal{K}_n$. We will limit our study to the subset of equilibrium points $\tilde{\mathscr{E}}_\mathbf{x}$ with $f_{ij} = \hat{f}$ for all $\{i,j\} \in \mathcal{E}$. Moreover, it will be more convenient to think of the state space as $\mathbb{R}^n/\langle \mathbf{1}\rangle$ rather than $\langle \mathbf{1}\rangle^\perp$.

First, assume that $\hat{f}$ only has three distinct roots, which we denote by 0, $\pm\alpha$, and let $\mathbf{x}^\star \in \tilde{\mathscr{E}}_\mathbf{x}$. Then, by definition of $\tilde{\mathscr{E}}_\mathbf{x}$ on a complete graph, $x_i^\star - x_j^\star \in \{0, \pm\alpha\}$ for all $i,j$. With elementary arguments (see supplementary section SM7) one can show that for an appropriate representative and ordering of the nodes,

$$\mathbf{x}^\star = (0, \dots, 0, \alpha, \dots, \alpha)^\top, \tag{4.3}$$

with 1 up to $n$ null entries, depending on the equilibrium point. In fact, the equilibria in $\tilde{\mathscr{E}}_\mathbf{x}$ will be of the form (4.3) even if $\hat{f}$ is an arbitrary odd function, as long as $\hat{f}^{-1}(0) \cap (0, \infty)$ is a sum-free set, i.e., $a + b \neq c$ for all $a, b, c \in \hat{f}^{-1}(0) \cap (0, \infty)$; see supplementary section SM7. We characterize the stability of these equilibrium points.

Proposition 4.5. *Let $\alpha$ be a root of $\hat{f}$, and consider (#) in $\mathcal{K}_n$ with $n \geq 3$. Assume that $\hat{f}'(0) > 0$, $\hat{f}'(\alpha) < 0$, and denote*

$$a = \frac{\hat{f}'(0)}{\hat{f}'(0) + |\hat{f}'(\alpha)|}, \qquad b = \frac{|\hat{f}'(\alpha)|}{\hat{f}'(0) + |\hat{f}'(\alpha)|}.$$

*Then, the stability of* $\mathbf{x}^\star$ *as in* (4.3) *with* $n_0$ *zero entries is given by*

$$n_0/n \in (a,b) \implies \mathbf{x}^\star \text{ is stable;}$$
$$n_0/n \notin [a,b] \implies \mathbf{x}^\star \text{ is unstable.}$$

Our proof follows from the direct computation of the Jacobian eigenvalues for a given number of zero entries in $\mathbf{x}^\star$; see Appendix F. Moreover, one gets the same instability condition by applying Theorem 3.14 in this context (see supplementary section SM8), which shows how powerful this theorem can be.

*Remark* 4.6. With the notation of the previous proposition, if $\hat{f}'(0), \hat{f}'(\alpha) > 0$ (resp., $\hat{f}'(0), \hat{f}'(\alpha) < 0$), we get that $\mathbf{x}^\star$ is unstable (resp., stable) by Proposition 3.6. If $\hat{f}'(0) < 0$ and $\hat{f}'(\alpha) > 0$, then, again by Proposition 3.6, $\mathbf{x}^\star$ is stable if $n_0 \in \{0, n\}$ and unstable otherwise.

**4.4. Example of the study of a tree of motifs.** Consider the system (#) with $f_{ij}(y) = \hat{f}(y) = y - y^3$ for all $i, j$, in the network $\mathcal{G}$ depicted in Figure 3. By the main result of subsection 3.5 we can find its equilibria and their stability by separately studying the subnetworks shown on the right-hand side of Figure 3, i.e., a cycle graph $\mathcal{C}_3$, a star graph with three edges, and a complete graph $\mathcal{K}_4$.

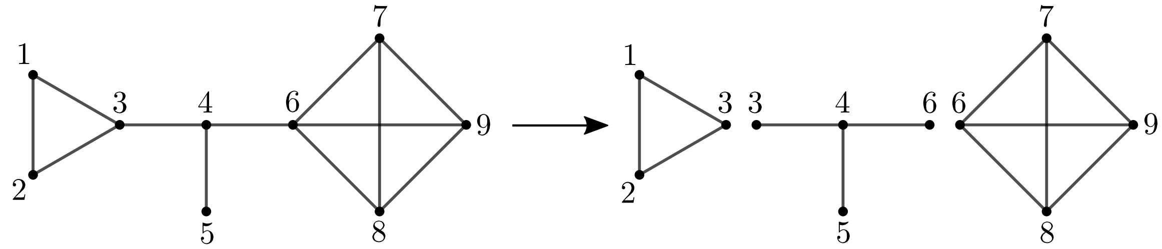


**Figure 3.** *On the left-hand side the network* $\mathcal{G}$ *is depicted with its nodes labeled. On the right-hand side we depict the relevant subnetworks to study by subsection 3.5.*

**Cycle graph $\mathcal{C}_3$.** The results from Figure 2 with $\lambda = 1$ claim that in $\langle \mathbf{1} \rangle^\perp$ this system has an unstable equilibrium at $\mathbf{0}$ and a circle of stable equilibria, which can be parametrized by

$$(x_1^\star, x_2^\star, x_3^\star)^\top = \frac{\sqrt{2}}{\sqrt{3}} \left( \frac{1}{\sqrt{2}} \cos(\theta)(-1,1,0) + \frac{1}{\sqrt{6}} \sin(\theta)(1,1,-2) \right)^\top$$

for $\theta \in [0, 2\pi)$. In the edge space, the set of stable equilibria is given by

$$\mathcal{P}_1^- = \left\{ \frac{1}{\sqrt{3}} \left( 2\cos(\theta), \cos(\theta) - \sqrt{3}\sin(\theta), -\cos(\theta) - \sqrt{3}\sin(\theta) \right)^\top : \theta \in [0, 2\pi) \right\},$$

and the unstable equilibrium is $\mathcal{P}_1^+ = \{(0,0,0)^\top\}$.

**Star graph with three edges.** Note that the roots of $\hat{f}(y) = y - y^3$ are $\{0, \pm 1\}$ and that $\hat{f}'(0) = 1$, $\hat{f}'(\pm 1) = -2$. So by Proposition 4.1 the stable equilibria in the edge space are given by

$$\mathcal{P}_2^- = \{\pm 1\}^3 := \{\pm 1\} \times \{\pm 1\} \times \{\pm 1\},$$

and the unstable ones are given by

$$\mathcal{P}_2^+ = \left\{ \left( y_{\{3,4\}}, y_{\{4,5\}}, y_{\{4,6\}} \right)^\top \in \{0, \pm 1\}^3 \,:\, y_{\{3,4\}} y_{\{4,5\}} y_{\{4,6\}} = 0 \right\}.$$

**Complete graph $\mathcal{K}_4$.** Using the results from subsection 4.3 and imposing the condition of orthogonality to $\mathbf{1}$, we find that $\mathbf{x}^\star = (x_6^\star, x_7^\star, x_8^\star, x_9^\star)^\top \in \tilde{\mathscr{E}}_{\mathbf{x}}$ if and only if

$$\{x_6^\star, x_7^\star, x_8^\star, x_9^\star\} \in \left\{ \{0,0,0,0\}, \left\{\tfrac{-1}{4}, \tfrac{-1}{4}, \tfrac{-1}{4}, \tfrac{3}{4}\right\}, \left\{\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{-3}{4}\right\}, \left\{\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{-1}{2}, \tfrac{-1}{2}\right\} \right\}, \tag{4.4}$$

where we use the brackets {} to denote multisets. To find the stability of these equilibria recall the notation of Proposition 4.5 and note that in our case,

$$a = \frac{\hat{f}'(0)}{\hat{f}'(0) + |\hat{f}'(1)|} = \frac{1}{3}, \qquad b = \frac{|\hat{f}'(1)|}{\hat{f}'(0) + |\hat{f}'(1)|} = \frac{2}{3}.$$

Now, note that the representative in the form of (4.3) of the first three types of equilibria in (4.4) have, respectively, four, three, and one null entries. Hence, for these points $n_0/n \notin [a, b]$, and thus they are unstable. We denote by $\tilde{\mathcal{P}}_3^+$ the set of these points in edge coordinates. For the fourth type of equilibria in (4.4) the representative has two null entries, so $n_0/n = 1/2 \in (a, b)$, and thus these points are stable. We denote by $\mathcal{P}_3^-$ the set of these points in the edge coordinates.

Recall that for the complete graph there may be equilibria outside of $\tilde{\mathscr{E}}_{\mathbf{x}}$. As we are working with polynomials over the integers, we can hope to find all equilibria using Gröbner bases (see [9]). Using `Singular`,[8] we find that there is only one other type of equilibrium point,

$$\{x_6^\star, x_7^\star, x_8^\star, x_9^\star\} = \left\{ 0, 0, \tfrac{\sqrt{2}}{\sqrt{5}}, \tfrac{-\sqrt{2}}{\sqrt{5}} \right\}.$$

As this type of equilibrium point is not in $\tilde{\mathscr{E}}_{\mathbf{x}}$, we cannot use Proposition 4.5 to determine its stability. However, if we let $i$ and $j$ be the nodes with value 0, we find that $r_{ij}^+ = 1$ and $r_{ij}^- = 5$. So we have $r_{ij}^- > r_{ij}^+$, and by Theorem 3.14 we conclude that these points are unstable. Denote by $\mathcal{P}_3^+$ the union of these points in edge coordinates with $\tilde{\mathcal{P}}_3^+$.

We are now prepared to tackle the whole network $\mathcal{G}$. For convenience, we denote

$$\mathbf{y} = (\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3)^\top,$$

where

$$\mathbf{y}_1 = \left( y_{\{1,2\}}, y_{\{1,3\}}, y_{\{2,3\}} \right)^\top, \qquad \mathbf{y}_2 = \left( y_{\{3,4\}}, y_{\{4,5\}}, y_{\{4,6\}} \right)^\top,$$

and

$$\mathbf{y}_3 = \left( y_{\{6,7\}}, y_{\{6,8\}}, y_{\{6,9\}}, y_{\{7,8\}}, y_{\{7,9\}}, y_{\{8,9\}} \right)^\top.$$

[8] https://www.singular.uni-kl.de:8003/

Let $\mathcal{P}_i = \mathcal{P}_i^+ \cup \mathcal{P}_i^-$ for $i = 1, 2, 3$. Then, by the results from subsection 3.5, the stable equilibrium points in edge coordinates are

$$\mathcal{P}_- = \{(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3)^\top : \mathbf{y}_i \in \mathcal{P}_i^-\},$$

and the unstable ones are

$$\mathcal{P}_+ = \{(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3)^\top : \mathbf{y}_i \in \mathcal{P}_i \text{ and } \mathbf{y} \notin \mathcal{P}_-\}.$$

One can use the map $d$ to find the node coordinates of these states. Note that the node coordinates in the whole network will not coincide with the node coordinates found for the different subnetworks.

## 5. Conclusion.

In this paper, we have presented an in-depth study of a general nonlinear model of consensus dynamics, which shows a rich phenomenology depending on the structural properties of its underlying network. The dynamical model can be viewed as a gradient system that conserves the mean state. Furthermore, it does not produce complex dynamical structures such as periodic orbits, as all orbits converge to equilibria. Notably, for a high-dimensional nonlinear system with arbitrary topology, we find a compact expression for the equilibrium points that highlights how these equilibria are determined by the interplay between the coupling function and the underlying network. Moreover, in line with previous results in the literature, we find that the stability of certain equilibria depends on effective resistances in the network, where the influence of general coupling functions can be taken into account using appropriate link weights.

Our analysis provides insight into simple networks, such as trees, cycles, and cliques, and also for any combination of them in a tree of motifs, as we show that knowledge of the equilibria and their stability for individual motifs is, in that case, sufficient for characterizing the whole-network dynamics. Although these conditions may seem restrictive, the resulting structures may be seen as a generalization of trees to the case of higher-order networks [31]. Moreover, locally tree-like structures of cliques are expected to appear in *projections of bipartite graphs* [26], which include coauthor networks, and when modeling *pervasive overlap* [1, 14] in social networks.

As a perspective for future research, we believe that a more careful deduction in line with Theorem 3.10 and Lemma 3.13 could lead to better stability conditions for arbitrary equilibrium points compared to those now presented in Proposition 3.12. Also in the spirit of Theorem 3.10, one could specialize the Schur complement reduction to particular configurations of positive edges, which would lead to tight stability conditions. As a third extension, it might be possible to obtain stability results for large dense networks (which are "approximately" complete) based on the exact Jacobian eigenvalues for complete graphs (as in Appendix F) and invoking perturbation arguments on the Jacobian [33]. A similar approximate setting where some of our exact results could be used as a starting point is the study of networks whose local structure may be approximated by a tree of motifs, which would generalize standard approximations based on a locally tree-like structure [35]. Finally, while this work is theoretical in nature, we believe that our developed insights can be used for the application and specialization of system (#) in a practical context.

**Appendix A. Proof of Proposition 3.6.** For the stability condition, note that we have $J = -L^-$. Now as mentioned below (2.3), $L^-$ is positive semidefinite and, as $\mathcal{G}^-$ is connected, 0 has multiplicity one in $\mathbb{R}^n$. Thus, in the state space $\langle \mathbf{1} \rangle^\perp$ all eigenvalues of $-L^-$ are negative.

If $\mathcal{G}^-$ is disconnected, consider a connected component $\mathcal{V}_1$ and the vector $\mathbf{x} = \sum_{i \in \mathcal{V}_1} \mathbf{e}_i$. Then, follow the proof of Proposition 3.5 to deduce that $\lambda_{\max} \geq 0$. If there is an edge between $\mathcal{V}_1$ and $(\mathcal{V} \setminus \mathcal{V}_1)$ in $\mathcal{G}^+$, we may directly apply Proposition 3.5 to the associated cut-set.

**Appendix B. Further Schur complement reduction.** In this appendix we explain how to apply the Schur complement reduction recursively through an example. Consider a Jacobian matrix $J = L^+ - L^-$, where the weights of $L^+$ and $L^-$ are depicted in Figure 4.a. Note that there is a unique node not incident to any edge in $\mathcal{G}^+$, i.e., $\mathcal{N} = \{5\}$. By Theorem 3.8, the linear stability of $J$ is determined by $J/\mathcal{N} = L^+_{\mathcal{N}^c,\mathcal{N}^c} - L^-/\mathcal{N}$, where the weights of these Laplacians are depicted in Figure 4.b. When we have multiple edges between a pair of nodes, in Figure 4.b we can simply add the (signed) weights to get Figure 4.c. This corresponds to finding another decomposition $J/\mathcal{N} = \tilde{L}^+ - \tilde{L}^-$ with Laplacians which do not have common edges. We observe that in Figure 4.c, the node 4 is not incident to any edge in $\tilde{L}^+$. Thus, we can apply our result again with $\tilde{\mathcal{N}} = \{4\}$ to get Figure 4.d, and by adding edges we get Figure 4.e. In summary, we have shown that the linear stability of $J$ is given by the linear stability of $(J/\mathcal{N})/\tilde{\mathcal{N}}$ depicted in Figure 4.e, which is unstable by Proposition 3.6.

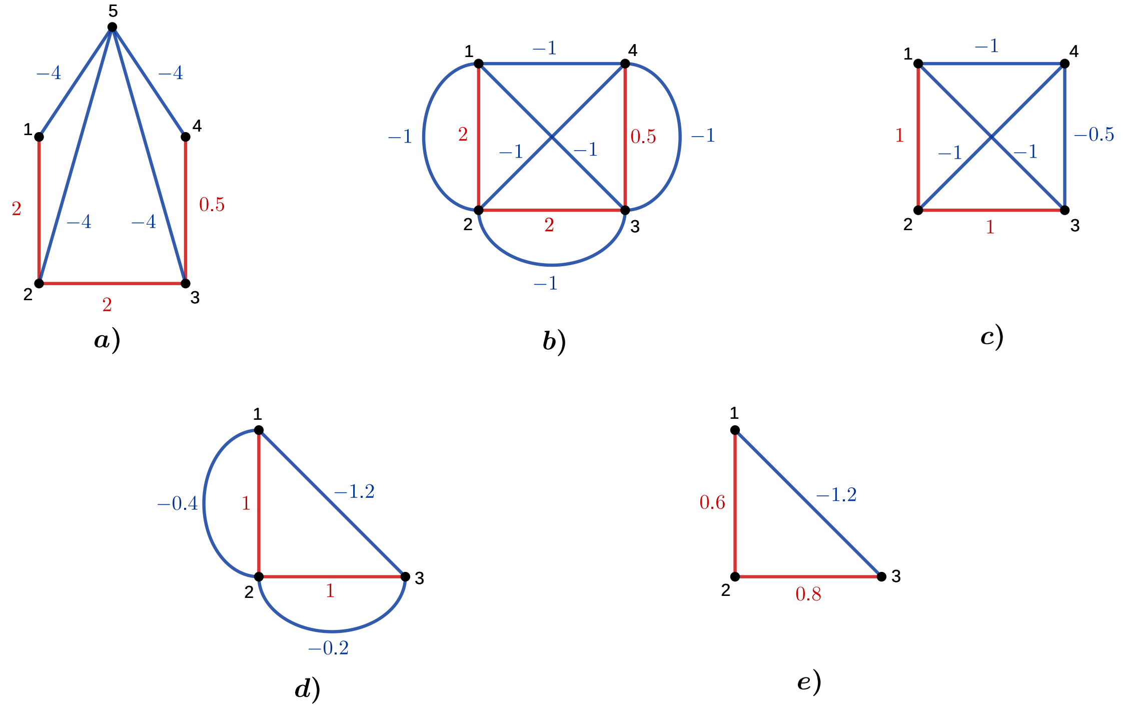


**Figure 4.** *Networks obtained by the recursive application of Schur complement. We depict edges from the corresponding $L^+$ in* red. *Edges from $L^-$ are depicted in* blue *and with a negative sign added to their weights.*

Note that in this example we have considered $(J/\mathcal{N})/\tilde{\mathcal{N}}$. The quotient formula of the Schur complement [51, equation 6.0.26] states that if $\mathcal{N}$ and $\tilde{\mathcal{N}}$ are disjoint sets of nodes, then

$$(J/\mathcal{N})/\tilde{\mathcal{N}} = J/(\mathcal{N} \cup \tilde{\mathcal{N}})$$

when the left-hand side is well defined. Given a Jacobian of our system, it would be interesting to determine which nodes we would be able to remove by applying the Schur complement process recursively. Then, by the quotient formula we could reduce all of them at once. One can sideline this problem by taking the opposite approach. That is, in each iteration of the Schur reduction, only reduce one node until no node is isolated in the corresponding $\mathcal{G}^+$. This approach has the advantage that each iteration only requires inverting a scalar instead of a matrix; see (3.9).

**Appendix C. Proof of Proposition 3.12.** For the instability, note that if $\mathcal{G}^-$ is disconnected, the result follows from Proposition 3.6. Otherwise, we can write $J = L^+_{ij} + \tilde{L}^+ - L^-$, where $L^+_{ij}$ is the Laplacian with only edge $\{i,j\}$ with weight coming from $\mathcal{G}^+$, and $\tilde{L}^+$ is the Laplacian of the rest of the edges from $\mathcal{G}^+$. Now as $\tilde{L}^+$ is positive semidefinite, it is enough to show that $L^+_{ij} - L^-$ is linearly unstable, which follows from Remark 3.11.

For the stability result, as $\sum_{\{i,j\}\in\mathcal{G}^+} f'_{ij}(x^\star_i - x^\star_j) r^-_{ij} < 1$ we have $r^-_{ij} < \infty$ for all edges in $\mathcal{G}^+$, and thus $\mathcal{G}^-$ is connected. Moreover, we can take weights $\alpha_{ij} > f'_{ij}(x^\star_i - x^\star_j) r^-_{ij}$ such that $\sum_{\{i,j\}\in\mathcal{G}^+} \alpha_{ij} = 1$. Then, using the notation $L^+_{ij}$ from above, we have

$$J = L^+ - L^- = \sum_{\{i,j\}\in\mathcal{G}^+} L^+_{ij} - \alpha_{ij} L^- = \sum_{\{i,j\}\in\mathcal{G}^+} \alpha_{ij}(\alpha^{-1}_{ij} L^+_{ij} - L^-).$$

Now apply Remark 3.11 to each $\alpha^{-1}_{ij} L^+_{ij} - L^-$ to conclude that all of them are linearly stable matrices; thus so is $J$, and $\mathbf{x}^\star$ is stable.

**Appendix D. Proof of Lemma 3.13.** First, assume that $A$ and $B$ are nonsingular, and let $B^{1/2}$ be the positive definite matrix such that $B = B^{1/2}B^{1/2}$. Then,

$$\begin{aligned} \max_{\mathbf{x}} \frac{\mathbf{x}^\top A\mathbf{x}}{\mathbf{x}^\top B\mathbf{x}} &= \max_{\mathbf{x}} \frac{\mathbf{x}^\top A\mathbf{x}}{(B^{1/2}\mathbf{x})^\top (B^{1/2}\mathbf{x})} = \max_{\mathbf{y}} \frac{\mathbf{y}^\top B^{-1/2}AB^{-1/2}\mathbf{y}}{\mathbf{y}^\top\mathbf{y}} \\ &= \max_{||\mathbf{y}||=1} \mathbf{y}^\top B^{-1/2}AB^{-1/2}\mathbf{y} = \mu_{\max}, \end{aligned}$$

where $\mathbf{y} = B^{1/2}\mathbf{x}$, $\mu_{\max}$ is the largest eigenvalue of $B^{-1/2}AB^{-1/2}$, and in the last equality we use the Courant minmax principle [33]. Now, note that $\mu^{-1}_{\max} = \tilde{\mu}_{\min}$ where $\tilde{\mu}_{\min}$ is the minimum eigenvalue of $B^{1/2}A^{-1}B^{1/2}$, and so

$$\begin{aligned} \max_{\mathbf{x}} \frac{\mathbf{x}^\top A\mathbf{x}}{\mathbf{x}^\top B\mathbf{x}} = \mu_{\max} = \frac{1}{\mu^{-1}_{\max}} = \frac{1}{\tilde{\mu}_{\min}} &= \frac{1}{\min_{||\mathbf{y}||=1} \mathbf{y}^\top B^{1/2}A^{-1}B^{1/2}\mathbf{y}} \\ &= \max_{||\mathbf{y}||=1} \frac{1}{\mathbf{y}^\top B^{1/2}A^{-1}B^{1/2}\mathbf{y}} = \max_{\mathbf{y}} \frac{\mathbf{y}^\top\mathbf{y}}{\mathbf{y}^\top B^{1/2}A^{-1}B^{1/2}\mathbf{y}} = \max_{\mathbf{z}} \frac{\mathbf{z}^\top B^{-1}\mathbf{z}}{\mathbf{z}^\top A^{-1}\mathbf{z}}. \end{aligned}$$

For the general case, let $V = (\ker A + \ker B)^\perp$, and note that $A$ and $B$ are positive definite in this space. Recall from the preliminaries that $(A|_V)^{-1} = A^\dagger|_V$ (resp., for B), so from the argument above we get

$$\max_{\mathbf{x}\in V} \frac{\mathbf{x}^\top A\mathbf{x}}{\mathbf{x}^\top B\mathbf{x}} = \max_{\mathbf{x}\in V} \frac{\mathbf{x}^\top B^\dagger\mathbf{x}}{\mathbf{x}^\top A^\dagger\mathbf{x}}.$$

Now as $A, B$ are positive semidefinite, and $\ker A = \ker A^\dagger$ (resp., for $B$), the previous expression is equivalent to

$$\max_{\mathbf{x}\perp \ker B} \frac{\mathbf{x}^\top A\mathbf{x}}{\mathbf{x}^\top B\mathbf{x}} = \max_{\mathbf{x}\perp \ker A} \frac{\mathbf{x}^\top B^\dagger\mathbf{x}}{\mathbf{x}^\top A^\dagger\mathbf{x}}.$$

**Appendix E. Proof of Lemma 3.16.** Reordering the nodes, we can assume that the deleted row in $\tilde{d}$ is the first one. Denote by $\lambda_1 \le \cdots \le \lambda_n$ the eigenvalues of $dMd^\top$ and by $\tilde\lambda_1 \le \cdots \le \tilde\lambda_{n-1}$ the ones of $\tilde{d}M\tilde{d}^\top$. As $dMd^\top$ is a symmetric matrix and $\tilde{d}M\tilde{d}^\top$ is the same matrix with the first row and column deleted, we have

$$dMd^\top = \begin{pmatrix} a & \mathbf{b}^\top \\ \mathbf{b} & \tilde{d}M\tilde{d}^\top \end{pmatrix}. \tag{E.1}$$

Thus we can apply Cauchy's interlace theorem [27], and we get

$$\lambda_1 \le \tilde\lambda_1 \le \lambda_2 \le \cdots \le \lambda_{n-1} \le \tilde\lambda_{n-1} \le \lambda_n. \tag{E.2}$$

Recall that $d^\top \mathbf{1} = 0$, so the multiplicity of the eigenvalue 0 for $dMd^\top$, which we denote by $k$, is at least one. By (E.2) it is clear that the multiplicity of 0 for $\tilde{d}M\tilde{d}^\top$, which we denote by $\tilde{k}$, is $k-1$, $k$, or $k+1$. If $\tilde{k} = k-1$, using (E.2) makes it clear that the number of positive (resp., negative) eigenvalues of each matrix coincide. We now show by contradiction that $\tilde{k} \ne k, k+1$.

First, note that by (E.1) for all $\mathbf{y} \in \ker \tilde{d}M\tilde{d}^\top \cap \langle \mathbf{b}\rangle^\perp$, $(0, \mathbf{y}^\top)^\top \in \ker dMd^\top$. Moreover, $\mathbf{1} \in \ker dMd^\top$, so

$$k = \dim(\ker dMd^\top) \ge \dim\left(\ker \tilde{d}M\tilde{d}^\top \cap \langle \mathbf{b}\rangle^\perp\right) + 1. \tag{E.3}$$

Now, if $\tilde{k} = k+1$, then $\ker \tilde{d}M\tilde{d}^\top \cap \langle \mathbf{b}\rangle^\perp$ has at least dimension $k$, so we get a contradiction with (E.3). If $\tilde{k} = k$, the multiplicity of 0 in both matrices coincide, and thus we can apply an extension of the Cauchy interlace theorem (see [27, Theorem 2]), which claims that $\langle \mathbf{b}\rangle \perp \ker \tilde{d}M\tilde{d}^\top$. Then, $\ker \tilde{d}M\tilde{d}^\top \cap \langle \mathbf{b}\rangle^\perp = \ker \tilde{d}M\tilde{d}^\top$, and again we get a contradiction with (E.3).

**Appendix F. Proof of Proposition 4.5.** Our argument closely resembles the one given in [21] for a specific function $\hat{f}$.

Note that $a, b \in (0,1)$, so if $n_0 \in \{0, n\}$, then $n_0/n \notin [a,b]$. In these cases, for all edges $x_i^\star - x_j^\star = 0$ and as $f'(0) > 0$, $\mathbf{x}^\star$ is unstable by Proposition 3.6. Otherwise, recall that $J = L^+ - L^-$, and it is easy to check that

$$L^+ = f'(0)\begin{pmatrix} n_0 P_{n_0} & \mathbf{0}_{n_0, n-n_0} \\ \mathbf{0}_{n-n_0, n_0} & (n-n_0)P_{n-n_0} \end{pmatrix}, \qquad L^- = -f'(\alpha)\begin{pmatrix} (n-n_0)I_{n_0} & -\mathbf{1}_{n_0}\mathbf{1}_{n-n_0}^\top \\ -\mathbf{1}_{n-n_0}\mathbf{1}_{n_0}^\top & n_0 I_{n-n_0} \end{pmatrix},$$

where $P_k = I_k - \mathbf{1}_k\mathbf{1}_k^\top/k$, i.e., the orthogonal projection onto $\langle \mathbf{1}_k\rangle^\perp$. Note that these two matrices commute, and hence they can be simultaneously diagonalized. We find the following four types of eigenvectors of $J$:

*Type* 1. $\mathbf{1}_n$ is an eigenvector of eigenvalue 0, which does not affect the stability in the state space $\langle \mathbf{1}\rangle^\perp$.

*Type* 2. $(-\mathbf{1}_{n_0}^\top/n_0, \mathbf{1}_{n-n_0}^\top/(n-n_0))^\top$ is an eigenvector of eigenvalue $\hat{f}'(\alpha)n$.

*Type* 3. Any vector of the form $\mathbf{z} = (\mathbf{z}_{n_0}^\top, \mathbf{0}_{n-n_0})^\top$ such that $\mathbf{1}_{n_0}^\top \mathbf{z}_{n_0} = 0$ is an eigenvector of eigenvalue $(\hat{f}'(0) - \hat{f}'(\alpha))n_0 + \hat{f}'(\alpha)n$.

*Type* 4. Any vector of the form $\mathbf{z} = (\mathbf{0}_{n_0}, \mathbf{z}_{n-n_0}^\top)^\top$ such that $\mathbf{1}_{n-n_0}^\top \mathbf{z}_{n-n_0} = 0$ is an eigenvector of eigenvalue $(\hat{f}'(\alpha) - \hat{f}'(0))n_0 + \hat{f}'(0)n$.

As there are $n_0 - 1$ linear independent eigenvectors of Type 3 and $n - n_0 - 1$ eigenvectors of Type 4, these are all eigenvectors of $J$.

As $\hat{f}'(\alpha)n < 0$, the stability of the equilibrium points is given by the signs of the other eigenvalues. Now assume that $n_0 \notin \{1, n-1\}$, so that there is at least one eigenvector of each type. The Type 3 eigenvalue is positive if $b < n_0/n$ and negative if $n_0/n < b$. Similarly, the Type 4 eigenvalue is positive if $n_0/n < a$ and negative if $a < n_0/n$. Thus, using Proposition 3.2 we get the desired result.

If $n_0 = 1$, then we do not have eigenvectors of Type 3, so $\mathbf{x}^\star$ is stable if $a < n_0/n$ and unstable if $n_0/n < a$. Using the facts that $n_0/n = 1/n < 1/2$ and that $[a,b] \neq \emptyset$ if and only if $a \leq 1/2$, it follows that

$$a < n_0/n \Longleftrightarrow n_0/n \in (a,b) \qquad \text{and} \qquad n_0/n < a \Longleftrightarrow n_0/n \notin [a,b].$$

Similar arguments work for the case $n_0 = n - 1$.

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