SUPPLEMENTARY MATERIALS: Nonlinear Consensus on Networks: Equilibria, Effective Resistance, and Trees of Motifs[*]

Marc Homs-Dones[†], Karel Devriendt[‡], and Renaud Lambiotte[‡]

SM1. Potential dynamics.

We start by introducing some basic concepts of dynamical systems (we follow standard notation, see for instance [SM7]). We consider ODEs defined by $\mathcal{C}^1$ maps,

$$\dot{\mathbf{x}} = g(\mathbf{x}),$$

and thus, by Picard–Lindelöf theorem [SM7, Theorem 2.5] it will have existence and uniqueness of solution for any given initial condition. Moreover, to simplify some definitions of qualitative features, we will always assume that orbits are defined for all times. Given an ODE, $\dot{\mathbf{x}} = \tilde{g}(\mathbf{x})$, this can can be done by considering the equivalent system,

$$\dot{\mathbf{x}} = g(\mathbf{x}) := \frac{\tilde{g}(\mathbf{x})}{||\tilde{g}(\mathbf{x})|| + 1},$$

which solutions are reparametrizations of solutions of the original ODE.

Let $\mathbf{x}_0 \in \mathbb{R}^n$ and denote by $\varphi(t, \mathbf{x}_0)$ the solution of $\dot{\mathbf{x}} = g(\mathbf{x})$ at time t such that $\varphi(0, \mathbf{x}_0) = \mathbf{x}_0$. We denote its forward orbit and its ω-limit by

$$\gamma_+(\mathbf{x}_0) = \{\varphi(t, \mathbf{x}_0) : t > 0\}, \qquad \omega(\mathbf{x}_0) = \left\{\lim_{n\to\infty} \varphi(t_n, x) : t_n \xrightarrow[n\to\infty]{} +\infty\right\}.$$

We say that an equilibrium point $\mathbf{x}^\star$ is *stable* if for any neighbourhood U of $\mathbf{x}^\star$, there exists a neighbourhood $V \subset U$ of $\mathbf{x}^\star$ such that for all $\mathbf{x}_0 \in V$, $\gamma_+(\mathbf{x}_0) \subset U$. We say that $\mathbf{x}^\star$ is *attractive* if there exists a neighbourhood U of $\mathbf{x}^\star$ such that for all $\mathbf{x}_0 \in U$, $\omega(\mathbf{x}_0) = \mathbf{x}^\star$. Let $\mathbf{x}^\star$ be an equilibrium point (or infinity), we denote its basin of attraction, i.e., the set of points that converge to it, by $W^+(\mathbf{x}^\star)$. We define $\gamma_-(\mathbf{x}_0)$, $\alpha(\mathbf{x}_0)$ and $W^-(\mathbf{x}^\star)$ in the same manner but for the system $\dot{\mathbf{x}} = -g(\mathbf{x})$. We will need the following result regarding the basin of attraction.

Lemma SM1.1. *Consider $\dot{\mathbf{x}} = g(\mathbf{x})$ where $g \in \mathcal{C}^1$, and let $\mathbf{x}^\star$ be an unstable hyperbolic equilibrium point. Then, $W^+(\mathbf{x}^\star)$ has measure*[1] *zero.*

Proof. By the stable manifold theorem [SM7, Theorem 9.4], there is a neighbourhood U of $\mathbf{x}^\star$, such that the set

$$W_U^+(\mathbf{x}^\star) = \{\mathbf{x} \in W^+(\mathbf{x}^\star) : \gamma_+(\mathbf{x}) \cup \{\mathbf{x}\} \subset U\},$$

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[†]Mathematics Institute, University of Warwick, Coventry, UK (marc.homs-dones@warwick.ac.uk).
[‡]Mathematical Institute, University of Oxford, Oxford, UK (devriendt@maths.ox.ac.uk, renaud.lambiotte@maths.ox.ac.uk).

[1]Lebesgue measure.

is a $\mathcal{C}^1$ submanifold of $\mathbb{R}^n$ tangent to the stable manifold of the linearization of the system. Since $\mathbf{x}^\star$ is not stable, these submanifolds have at most dimension $n-1$ and thus $W_U^+(\mathbf{x}^\star)$ has measure zero (e.g., by Sard's theorem [SM6]). Then, note that,

$$W^+(\mathbf{x}^\star) = \bigcup_{m=0}^{\infty} \{\varphi(-m, \mathbf{x}) : \mathbf{x} \in W_U^+(\mathbf{x}^\star)\} = \bigcup_{m=0}^{\infty} \varphi(-m, W_U^+(\mathbf{x}^\star)),$$

and that, for all t the map $\varphi(t, \cdot)$ sends null sets to null sets, since it is a $\mathcal{C}^1$ diffeomorphism. So, $W^+(\mathbf{x}^\star)$ is the countable union of null measure sets and thus, it has measure zero. ■

Now we focus on gradient dynamics. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a $\mathcal{C}^2$ function. We define its *gradient dynamics* as,

$$\dot{\mathbf{x}} = -\nabla V(\mathbf{x}), \tag{SM1.1}$$

so that V is a potential function of this system. The derivative of V along the trajectories is,

$$\dot{V}(\mathbf{x}) = \langle \nabla V(\mathbf{x}), \dot{\mathbf{x}} \rangle = -||\nabla V(\mathbf{x})||^2 \le 0, \tag{SM1.2}$$

and note that $\dot{V}(\mathbf{x}) = 0$ only in equilibrium points. Thus, $V(\mathbf{x}(t))$ is decreasing in solutions of (SM1.1) (except in equilibrium points).

Proposition SM1.2. *Assume that the set of equilibrium points $\mathscr{E}$ of the system (SM1.1) is countable. Then, if $\gamma_+(\mathbf{x}_0)$ (resp. $\gamma_-(\mathbf{x}_0)$) is bounded, $\omega(\mathbf{x}_0)$ (resp. $\alpha(\mathbf{x}_0)$) is an equilibrium point.*

Proof. As $\gamma_+(\mathbf{x}_0)$ is bounded, it is contained in a compact set so by LaSalle's invariance principle [SM5, SM7],

$$\omega(\mathbf{x}_0) \subset \{\mathbf{x} \in \mathbb{R}^n : \dot{V}(\mathbf{x}) = 0\} = \mathscr{E}.$$

Now $\gamma_+(\mathbf{x}_0)$ is contained in a compact set so $\omega(\mathbf{x}_0)$ is non-empty and connected (see [SM7, Lemma 6.6]). As $\mathscr{E}$ is countable the only non-empty connected subsets are singletons. ■

To be able to apply the previous result we need $\gamma_+(\mathbf{x}_0)$ to be bounded. One sufficient condition is for V to be *radially unbounded*, i.e., $V(\mathbf{x}) \to +\infty$ when $||\mathbf{x}|| \to +\infty$, as the potential is decreasing in orbits. Now we are prepared to show that, under mild conditions, any forward orbit of the dynamics (#) converges to a single equilibrium point.

Proposition SM1.3. *Let $\mathscr{E}_\mathbf{x}$ be the equilibrium points of (#) in the state space $\langle \mathbf{1} \rangle^\perp$. Assume that $\mathscr{E}_\mathbf{x}$ is countable and one of the following conditions holds:*
(a) *For all $\{i, j\} \in \mathcal{E}$, $-F_{ij}$ is radially unbounded.*
(b) *For all $\{i, j\} \in \mathcal{E}$, $\limsup_{y \to +\infty} y f_{ij}(y) < 0$.*
(c) *All orbits are bounded, i.e., contained in a compact set.*
Then,

$$\langle \mathbf{1} \rangle^\perp = \bigcup_{\mathbf{x}^\star \in \mathscr{E}_\mathbf{x}} W^+(\mathbf{x}^\star) = W^-(\infty) \cup \bigcup_{\mathbf{x}^\star \in \mathscr{E}_\mathbf{x}} W^-(\mathbf{x}^\star),$$

where if (c) *holds, we also have $W^-(\infty) = \emptyset$.*

Proof. By Proposition SM1.2 it is enough to show that forward orbits are bounded, and backwards unbounded ones tend to infinity. So condition (c) is clearly sufficient. Condition (a) is also sufficient, as then $V|_{\langle \mathbf{1}\rangle^\perp}$ defined by (3.1) is radially unbounded. Finally, condition (b) is a particular case of (a), as we proceed to show.

Assume that (b) holds, and consider $g(y) = \sup_{z\geq y} z f_{ij}(z)$, which is a non-increasing function. Then, let

$$a = \lim_{y\to+\infty} g(y),$$

and notice that by our assumption $a < 0$. Thus, there exists $M > 0$ such that $g(y) < \frac{a}{2}$ for all $y \geq M$. In particular, $f_{ij}(y) < \frac{a}{2}\frac{1}{y}$ for all $y \geq M$ and hence for certain constant c,

$$\lim_{y\to+\infty} -F_{ij}(y) = c + \int_M^\infty -f_{ij}(y) > c + \int_M^\infty -\frac{a}{2}\frac{1}{y} = +\infty,$$

as $-\frac{a}{2} > 0$. Finally, f_{ij} is odd so F_{ij} is even and thus, $\lim_{|y|\to\infty} -F_{ij}(y) = +\infty$. ■

Note that the coupled phase oscillators model with constants $w_i = 0$, trivially satisfies (c) when studied in $\mathbb{S}^1 \times \cdots \times \mathbb{S}^1$. Another important case is given when $f_{ij} = \hat{f}$ for all edges and $\hat{f}$ has finitely many roots. In this case (and changing sense of time if necessary), the system satisfies condition (b), as long as $\hat{f}$ does not tend to 0 at ∞.

SM2. Dimension of the set of equilibrium points.

We first show that,

$$\dim(\ker d) = m - n + 1, \qquad \text{and} \qquad \dim\left((\ker d)^\perp\right) = n - 1. \tag{SM2.1}$$

where $n = |\mathcal{G}|$ and $m = |\mathcal{E}|$. To do so we find a basis of each space through a spanning tree[2] $\mathcal{T}$ of $\mathcal{G}$, which also gives us some insight on how to explicitly compute these spaces. Each edge in $\mathcal{G} \setminus \mathcal{T}$ determines a unique cycle with this edge and edges of $\mathcal{T}$. These cycles form a basis of the cycle space [SM1]. Similarly, each edge in $\mathcal{T}$ determines a unique cut-set with this edge and edges in $\mathcal{G} \setminus \mathcal{T}$. These cut-sets form a basis of the cut space [SM1]. In particular, we get Equation (SM2.1).

Assume now that for all edges, f_{ij} has null derivative in a finite set of points. Then, f_{ij} has finite changes of monotonicity, so the set $f_{ij}^{-1}(y)$ is finite for any y. Thus, the set $f^{-1}(\mathbf{y})$ is also finite for any point $\mathbf{y}$. Informally, we can expect then that $f^{-1}(\ker d)$ is $m - n + 1$ dimensional, as $\dim(\ker d) = m - n + 1$. Finally, $\mathscr{E}_{\mathbf{y}} = (\ker d)^\perp \cap f^{-1}(\ker d)$, so we expect the set of equilibria in $\langle \mathbf{1}\rangle^\perp$ to be at most $m - n + 1$ dimensional. In subsection 4.2 we will show that for the cycle graph this bound may be reached and we get a continuum of equilibrium points in the node space, $\langle \mathbf{1}\rangle^\perp$.

SM3. Resistors in series and in parallel.

The effective resistance as defined in Definition 3.9 is precisely the effective resistance in the electrical circuit sense, where the circuit is given by the network and the resistors by the inverse of the edge weights. Thus, this definition satisfies the well known properties of resistors in series and in parallel, see [SM2]. That is, if

[2]That is, a subnetwork of $\mathcal{G}$ with all of its nodes and no cycles.

we have a series of nodes $i_1, \ldots, i_k$ such that all paths from i_1 to i_k go through $i_1, \ldots, i_k$ in this order, then

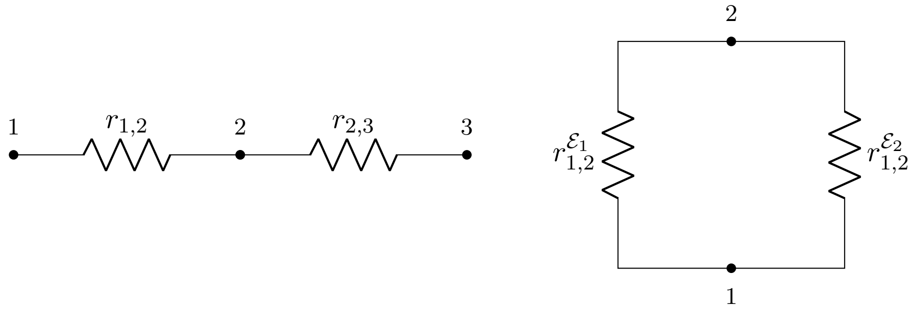
$$r_{i_1 i_k} = \sum_{j=1}^{k-1} r_{i_j i_{j+1}}. \tag{SM3.1}$$

For parallel resistors, if we have nodes i, j and a partition $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$ such that all paths from i to j are entirely contained in $\mathcal{E}_1$ or in $\mathcal{E}_2$ then,

$$\frac{1}{r_{ij}} = \frac{1}{r_{ij}^{\mathcal{E}_1}} + \frac{1}{r_{ij}^{\mathcal{E}_2}}, \tag{SM3.2}$$

where $r_{ij}^{\mathcal{E}_1}$ is the effective resistance in the network with only the edges $\mathcal{E}_1$ (resp. for $\mathcal{E}_2$).

The prototypical examples of resistances in series and parallel are depicted in the following diagrams.



Note that the underlying network on the second diagram is a multigraph and not a graph. This is not relevant as effective resistance can be defined similarly for these.

SM4. Using Lemma 3.13 for stability condition.

We assume that $\mathcal{G}^-$ is connected, as by Proposition 3.6 it is a necessary condition to have linear stability. Then, by 3.4 let $\mathbf{z} \in \langle \mathbf{1} \rangle^\perp$ be unitary such that,

$$\lambda_{\max} = \mathbf{z}^\top L^+ \mathbf{z} - \mathbf{z}^\top L^- \mathbf{z} = \mathbf{z}^\top L^- \mathbf{z} \left(\frac{\mathbf{z}^\top L^+ \mathbf{z}}{\mathbf{z}^\top L^- \mathbf{z}} - 1 \right),$$

where we have used that, L^- is positive definite in our space $\langle \mathbf{1} \rangle^\perp$ (as $\mathcal{G}^-$ is connected), so $\mathbf{z}^\top L^- \mathbf{z} > 0$. Thus[3],

$$\operatorname{sign}(\lambda_{\max}) = \operatorname{sign}\left(\frac{\mathbf{z}^\top L^+ \mathbf{z}}{\mathbf{z}^\top L^- \mathbf{z}} - 1 \right) = \operatorname{sign}\left(\max_{\mathbf{x} \perp \mathbf{1}} \frac{\mathbf{x}^\top L^+ \mathbf{x}}{\mathbf{x}^\top L^- \mathbf{x}} - 1 \right) = \operatorname{sign}\left(\max_{\mathbf{x} \perp \ker L^+} \frac{\mathbf{x}^\top (L^-)^\dagger \mathbf{x}}{\mathbf{x}^\top (L^+)^\dagger \mathbf{x}} - 1 \right),$$

where in the last equality we use Lemma 3.13. We believe this last expression can lead to stability criteria. For instance, if i is an isolated node in $\mathcal{G}^+$ and $\mathbf{x} \perp \ker L^+$, then $[\mathbf{x}]_i = 0$ (as $\ker L^+$ is generated by the connected components of $\mathcal{G}^+$). This gives rise to a different proof of the stability reduction in Theorem 3.8. Moreover, $\mathbf{x} \perp \ker L^+$ imposes more conditions

[3] As in the main article, all maximums are taken over non-null vectors.

on $\mathbf{x}$, coming from the connected components of $\mathcal{G}^+$ with more than one node, which could potentially be used to reduce our problem even further.

Another approach is to use the fact that the pseudoinverse of a Laplacian is closely related to the effective resistance, even more than the Laplacian itself. This might lead to stability criteria with effective resistance, as we did for instability in Theorem 3.14.

SM5. Equilibrium points of the coalescence of two networks. Let $\mathcal{G}$ be the coalescence of the networks $\mathcal{A}$ and $\mathcal{B}$. Recall that the equilibrium points are given by the equation,

$$\mathscr{E}_{\mathbf{y}} = (\ker d)^{\perp} \cap f^{-1}(\ker d), \tag{SM5.1}$$

for $\mathcal{G}$ and by

$$\mathscr{E}_{\mathbf{y}}^{\mathcal{A}} = (\ker d_{\mathcal{A}})^{\perp} \cap f_{\mathcal{A}}^{-1}(\ker d_{\mathcal{A}}),$$

for $\mathcal{A}$, where the superindex $\mathcal{A}$ denotes the respective maps for the network $\mathcal{A}$ (similarly for $\mathcal{B}$). Now note that any generating cycle of $\ker d$ is entirely contained in $\mathcal{A}$ or in $\mathcal{B}$ as the only edges from $\mathcal{A}$ to $\mathcal{B}$ are the ones with n' as a node. Thus,

$$\ker d = \iota_{\mathcal{A}}(\ker d_{\mathcal{A}}) \oplus \iota_{\mathcal{B}}(\ker d_{\mathcal{B}}),$$

where, $\iota_{\mathcal{A}}(\mathbf{y}_{\mathcal{A}}) = (\mathbf{y}_{\mathcal{A}}, \mathbf{0}_{\mathcal{B}})$ (resp. for $\iota_{\mathcal{B}}$). Moreover, it is clear that,

$$(\ker d)^{\perp} = \iota_{\mathcal{A}}((\ker d_{\mathcal{A}})^{\perp}) \oplus \iota_{\mathcal{B}}((\ker d_{\mathcal{B}})^{\perp}).$$

Now, as f is defined in each component by one variable functions f_{ij}, we have $f(\mathbf{y}) = (f_{\mathcal{A}}(\mathbf{y}_{\mathcal{A}}), f_{\mathcal{B}}(\mathbf{y}_{\mathcal{B}}))$, and thus,

$$f^{-1}(\ker d) = \iota_{\mathcal{A}}(f_{\mathcal{A}}^{-1}(\ker d_{\mathcal{A}})) \oplus \iota_{\mathcal{B}}(f_{\mathcal{B}}^{-1}(\ker d_{\mathcal{B}})).$$

In conclusion, from (SM5.1) we have,

$$\begin{aligned}
\mathscr{E}_{\mathbf{y}} &= \Big(\iota_{\mathcal{A}}((\ker d_{\mathcal{A}})^{\perp}) \oplus \iota_{\mathcal{B}}((\ker d_{\mathcal{B}})^{\perp})\Big) \cap \big(\iota_{\mathcal{A}}(f_{\mathcal{A}}^{-1}(\ker d_{\mathcal{A}})) \oplus \iota_{\mathcal{B}}(f_{\mathcal{B}}^{-1}(\ker d_{\mathcal{B}}))\big) \\
&= \iota_{\mathcal{A}}\Big((\ker d_{\mathcal{A}})^{\perp} \cap f_{\mathcal{A}}^{-1}(\ker d_{\mathcal{A}})\Big) \oplus \iota_{\mathcal{B}}\Big((\ker d_{\mathcal{B}})^{\perp} \cap f_{\mathcal{B}}^{-1}(\ker d_{\mathcal{B}})\Big) \\
&= \iota_{\mathcal{A}}(\mathscr{E}_{\mathbf{y}}^{\mathcal{A}}) \oplus \iota_{\mathcal{B}}(\mathscr{E}_{\mathbf{y}}^{\mathcal{B}}).
\end{aligned}$$

SM6. Phase portrait of a tree in low dimension. We study the qualitative dynamics of (#) in a line graph with three nodes where $f_{ij} = \hat{f}$ for all edges and all roots of $\hat{f}$ are simple. To simplify our study we also assume that the distance between consecutive roots of $\hat{f}$ are similar. That is, let $\alpha_0 = 0$, and $\{\alpha_i\}_{i=1}^{k}$ be the positive roots of $\hat{f}$ in increasing order, where k may be infinity. Then, we assume that,

$$\sup_{0 \le i < k} (\alpha_{i+1} - \alpha_i) \le 2 \inf_{0 \le i < k} (\alpha_{i+1} - \alpha_i). \tag{SM6.1}$$

If we let 1 be the middle node and 2, 3 the leaf nodes, the system (#) in edge coordinates $y_1 = y_{\{1,2\}} = x_2 - x_1$ and $y_2 = y_{\{1,3\}} = x_3 - x_1$ is given by,

$$\begin{aligned}
\dot{y}_1 &= 2\hat{f}(y_1) + \hat{f}(y_2), \\
\dot{y}_2 &= \hat{f}(y_1) + 2\hat{f}(y_2),
\end{aligned} \tag{SM6.2}$$

or compactly $\dot{\mathbf{y}} = g(\mathbf{y})$. Equation (SM6.2) is very symmetric, concretely if we consider a map that swaps the first and second coordinates, i.e., $p_{1,2}(y_1, y_2) = (y_2, y_1)$, we have,

$$g \circ p_{1,2} = p_{1,2} \circ g, \qquad \text{and} \qquad g \circ -\mathrm{id} = -\mathrm{id} \circ g.$$

Thus the flow in $\mathbf{y}$ determines the flow in $\pm\mathbf{y}$ and $\pm(y_2, y_1)^\top$. In particular, we may restrict the study of the ODE in the region $y_2 \geq |y_1|$. This situation immediately generalises when studying any star graph where we will have more possible permutations of variables. This is very reminiscent of the work done in [SM3] as the symmetry of the star graph restricts the dynamics of the system.

Reparametrising time we can assume $\hat{f}'(0) > 0$. Then, by Proposition 4.1, the equilibrium points of (SM6.2) are of the form $(\pm\alpha_i, \pm\alpha_j)^\top$ and their stability is given by $\hat{f}'(\pm\alpha_i)$ and $\hat{f}'(\pm\alpha_j)$. Note that,

$$\hat{f}'(\pm\alpha_i) = \begin{cases} > 0 & \text{if } i \text{ is even,} \\ < 0 & \text{if } i \text{ is odd,} \end{cases}$$

as all roots of $\hat{f}$ are simple so the signs alternate. We first compute,

$$(\dot{y}_1 + \dot{y}_2)|_{y_1+y_2=0} = 3(\hat{f}(y_1) + \hat{f}(y_2))\Big|_{y_1+y_2=0} = 3(\hat{f}(y_1) + \hat{f}(-y_1)) = 0,$$
$$(\dot{y}_1 - \dot{y}_2)|_{y_1-y_2=0} = (\hat{f}(y_1) - \hat{f}(y_2))\Big|_{y_1-y_2=0} = \hat{f}(y_1) - \hat{f}(-y_1) = 0,$$

so the lines $y_1 = \pm y_2$ are invariant in our system. Moreover for all i,

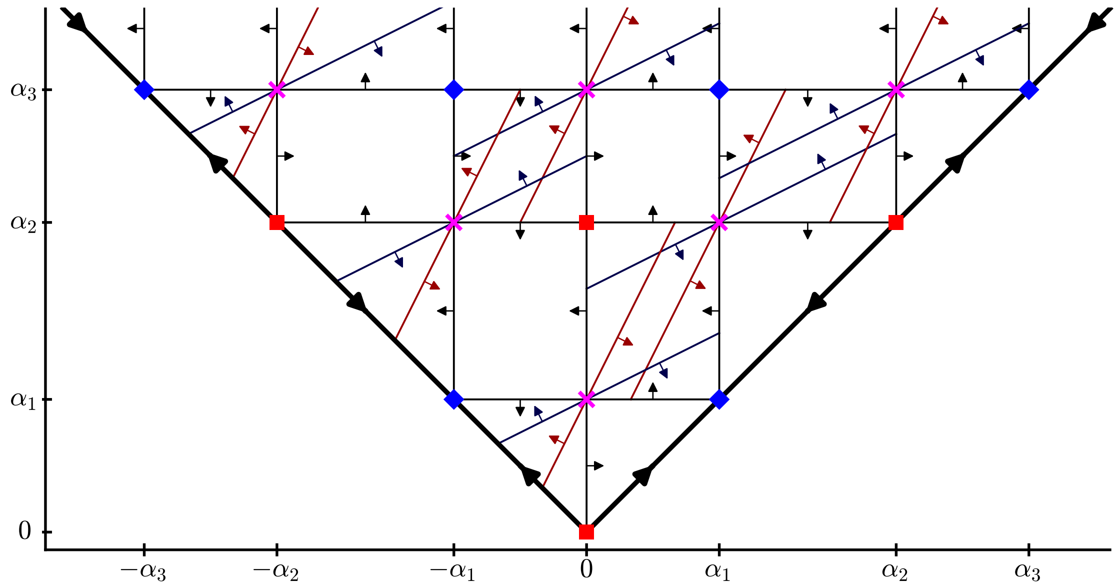


Figure SM1. *Field* (SM6.2) *in the invariant region* $y_2 \geq |y_1|$*.* Red squares, blue diamonds *and* magenta exes *represent respectively source, sink and saddle equilibrium points. The arrows represent the sense in which the field crosses the different color coded segments.*

$$\dot{y}_1\big|_{y_1=\pm\alpha_i} = (2\hat{f}(y_1) + \hat{f}(y_2))\Big|_{y_1=\pm\alpha_i} = \hat{f}(y_2),$$
$$\dot{y}_2\big|_{y_2=\pm\alpha_i} = (\hat{f}(y_1) + 2\hat{f}(y_2))\Big|_{y_2=\pm\alpha_i} = \hat{f}(y_1),$$

so the horizontal and vertical segments between different equilibria are transversal to the field, i.e., orbits always cross them in the same sense, as depicted in black in Figure SM1. Finally,

$$2\dot{y}_1 - \dot{y}_2 = 3\hat{f}(y_1),$$
$$2\dot{y}_2 - \dot{y}_1 = 3\hat{f}(y_2),$$

and in particular the segments parallel to $2y_1 = y_2$ (resp. $2y_2 = y_1$), that go through the saddle equilibrium points, are transverse to the field for y_1 (resp. y_2) between roots, see in blue and red in Figure SM1. By (SM6.1), these segments are contained in the regions delimited by the horizontal/vertical lines mentioned above.

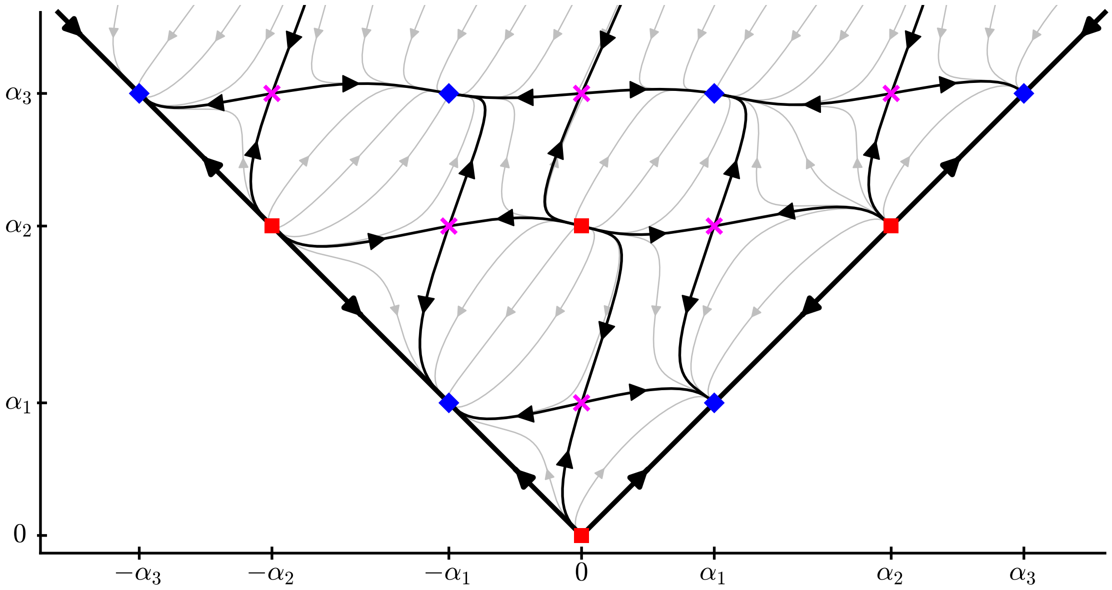


Figure SM2. *Phase portrait of (SM6.2) in the invariant region $y_2 \geq |y_1|$.* Red squares, blue diamonds *and* magenta exes *represent respectively source, sink and saddles equilibrium points. Lines in* black *show the heteroclinic connections between saddles and other equilibrium points.*

It is not hard to see from Figure SM1, that the heteroclinic connections between the saddle and other equilibrium points are the ones depicted in Figure SM2. Moreover, under mild conditions on $\hat{f}$, Proposition SM1.3 guarantees that the ω and α-limit of any orbit is an equilibrium point or infinity, thus Figure SM2 gives a complete description of the qualitative behaviour of the system.

SM7. Equilibrium points for complete graphs. As in the main article we are working in the state space $\mathbb{R}^n/\langle\mathbf{1}\rangle$ and we restrict our study to the equilibria $\tilde{\mathscr{E}}_{\mathbf{x}}$ with $f_{i,j} = \hat{f}$ for all $\{i,j\} \in \mathcal{E}$. For each $\alpha \in \hat{f}^{-1}(0)$, if $x_i^\star - x_j^\star \in \{0, \pm\alpha\}$ for all $i \neq j$, then $\mathbf{x}^\star \in \tilde{\mathscr{E}}_{\mathbf{x}}$. In this case, we may take the representative of $\mathbf{x}^\star$ such that $x_1^\star = 0$. Then, for all i, $x_i^\star = x_i^\star - x_1^\star \in \{0, \pm\alpha\}$. Note that if $x_i = \alpha$ and $x_j = -\alpha$ for some i, j, then $x_i - x_j = \pm 2\alpha$ which is a contradiction. Hence, either $x_i^\star \in \{0, \alpha\}$ for all i or $x_i^\star \in \{0, -\alpha\}$ for all i. In either case, choosing appropriate representative and ordering of the nodes,

$$\mathbf{x}^\star = (0, \ldots, 0, \alpha, \ldots, \alpha)^\top. \tag{SM7.1}$$

We show that generically these are all elements of $\tilde{\mathscr{E}}_{\mathbf{x}}$. Let $\mathbf{x}^\star \in \tilde{\mathscr{E}}_{\mathbf{x}}$, then $x_i^\star - x_j^\star \in \hat{f}^{-1}(0)$ for all $i \neq j$. So if i, j, k are pairwise different, and we denote $x_i^\star - x_j^\star = \alpha_1$, $x_j^\star - x_k^\star = \alpha_2$ and $x_k^\star - x_i^\star = \alpha_3$ we have $\alpha_1, \alpha_2, \alpha_3 \in \hat{f}^{-1}(0)$ and

$$\alpha_1 + \alpha_2 + \alpha_3 = (x_i^\star - x_j^\star) + (x_j^\star - x_k^\star) + (x_k^\star - x_i^\star) = 0. \tag{SM7.2}$$

Now assume[4] that $\hat{f}^{-1}(0) \cap (0, \infty)$ is a sum-free set, i.e., $a + b \neq c$ for all $a, b, c \in \hat{f}^{-1}(0) \cap (0, \infty)$. Note that in (SM7.2) we may assume without loss of generality that $\alpha_3 \leq 0$. Moreover, (SM7.2) can be rewritten as $\alpha_1 + \alpha_2 = -\alpha_3$ where $\alpha_1, \alpha_2, -\alpha_3 \geq 0$ and as $\hat{f}$ is odd, $-\alpha_3$ is also a root of it. By the sum-free condition, this can only be satisfied if $\alpha_l = 0$ for some $l \in \{1, 2, 3\}$. Then, clearly the terms in the left hand of (SM7.2) are $0, \alpha, -\alpha$ for some root α of $\hat{f}$ which in principle could change between different triplets of nodes. Now fix α and i, j two nodes such that $x_i^\star - x_j^\star = \pm\alpha$. Letting k run over all other nodes we conclude that $x_i^\star - x_k^\star \in \{0, \pm\alpha\}$ and $x_j^\star - x_k^\star \in \{0, \pm\alpha\}$. Repeating this process for other choices of i, j we find that $x_{l_1}^\star - x_{l_2}^\star \in \{0, \pm\alpha\}$ for all $l_1 \neq l_2$. Thus, by the argument made at the beginning of this section, $\mathbf{x}^\star$ can be represented by (SM7.1).

SM8. Another deduction of the instability criterion in $\mathcal{K}_n$.

Let $\mathbf{x}^\star$ be as in (SM7.1), denote by n_0 the number of 0's of $\mathbf{x}^\star$ and by $n_\alpha = n - n_0$ the number of α's. Then,

$$L^+ = \hat{f}'(0) \begin{pmatrix} n_0 P_{n_0} & \mathbf{0}_{n_0, n_\alpha} \\ \mathbf{0}_{n_\alpha, n_0} & n_\alpha P_{n_\alpha} \end{pmatrix}, \qquad L^- = |\hat{f}'(\alpha)| \begin{pmatrix} n_\alpha I_{n_0} & -\mathbf{1}_{n_0}\mathbf{1}_{n_\alpha}^\top \\ -\mathbf{1}_{n_\alpha}\mathbf{1}_{n_0}^\top & n_0 I_{n_\alpha} \end{pmatrix},$$

where $P_k = I_k - \mathbf{1}_k \mathbf{1}_k^\top / k$, i.e., the orthogonal projection onto $\langle \mathbf{1}_k \rangle^\perp$. Note that L^+ is the Laplacian of two disconnected complete graphs which we denote by $\mathcal{K}_0^+$ and $\mathcal{K}_\alpha^+$. Also note that L^- is the Laplacian of a complete bipartite graph $\mathcal{K}_{n_0, n_\alpha}$. The effective resistances between nodes in these graphs are well known and can be deduced from [SM4, Corollary C2]. In our case we have,

$$r_{ij}^+ = \begin{cases} \frac{2}{\hat{f}'(0) n_0} & \text{if } i, j \in \mathcal{K}_0^+, \\ \frac{2}{\hat{f}'(0) n_\alpha} & \text{if } i, j \in \mathcal{K}_\alpha^+, \\ \infty & \text{else}, \end{cases} \qquad \text{and} \qquad r_{ij}^- = \begin{cases} \frac{2}{|\hat{f}'(\alpha)| n_\alpha} & \text{if } i, j \in \mathcal{K}_0^+, \\ \frac{2}{|\hat{f}'(\alpha)| n_0} & \text{if } i, j \in \mathcal{K}_\alpha^+, \\ \neq \infty & \text{else}. \end{cases}$$

By Theorem 3.14, we have instability if $r_{ij}^- > r_{ij}^+$ for some edge $\{i, j\}$, which in or context reduces to,

$$\frac{n_0}{n_\alpha} < \frac{\hat{f}'(0)}{|\hat{f}'(\alpha)|} \qquad \text{or} \qquad \frac{n_0}{n_\alpha} > \frac{|\hat{f}'(\alpha)|}{\hat{f}'(0)},$$

[4] This condition is generically satisfied as its negation implies a specific non-trivial linear relation between some roots of $\hat{f}$.

or equivalently,

$$\frac{n_0}{n_\alpha} \notin \left[\frac{\hat{f}'(0)}{|\hat{f}'(\alpha)|}, \frac{|\hat{f}'(\alpha)|}{\hat{f}'(0)} \right]. \tag{SM8.1}$$

Now we need the following elementary relation between inequalities. Given positive numbers $a, b, c, d > 0$, then

$$\frac{a}{b} > \frac{c}{d} \Leftrightarrow \frac{b}{a} < \frac{d}{c} \Leftrightarrow \frac{b}{a} + 1 < \frac{d}{c} + 1 \Leftrightarrow \frac{a+b}{a} < \frac{c+d}{c} \Leftrightarrow \frac{a}{a+b} > \frac{c}{c+d},$$

and similarly,

$$\frac{a}{b} < \frac{c}{d} \Leftrightarrow \frac{a}{a+b} < \frac{c}{c+d}.$$

Thus, condition (SM8.1) is equivalent to,

$$\frac{n_0}{n_0 + n_\alpha} = \frac{n_0}{n} \notin \left[\frac{\hat{f}'(0)}{\hat{f}'(0) + |\hat{f}'(\alpha)|}, \frac{|\hat{f}'(\alpha)|}{\hat{f}'(0) + |\hat{f}'(\alpha)|} \right],$$

which is precisely the one we got in Proposition 4.5 by computing the eigenvalues, and hence it is also tight.

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