The Geometry and Stability of Fibrations

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A thesis submitted for the degree of
Doctor of Philosophy

Hilary 2022

For Kim.

Acknowledgements

First and foremost I would like to thank my supervisors, Frances Kirwan and Ruadhaí Dervan, for the countless interesting and helpful discussions we've had over the last few years. My knowledge of mathematics has grown immensely thanks to your contributions. I also thank Lars Sektnan, Julien Keller, Zakarias Sjöström Dyrefelt, and Eveline Legendre, for their helpful feedback and encouragement. I thank John McCarthy for many engaging discussions on optimal symplectic connections, and for help with an example in Chapter 6, which will appear in his PhD thesis. I thank Annamaria Ortu, for sharing her interesting research with me. Thank you to my family for your constant support, which was unfortunately too often via facetime. And thank you to Kim, for being the most amazing, funny and special person I've ever met, and for keeping me sane over lockdown.

Abstract

This thesis concerns two topics. First is the existence of optimal symplectic connections on holomorphic submersions, and links to algebro-geometric stability of fibrations. The main result is that a fibration admitting an optimal symplectic connection is stable with respect to a large class of fibration degenerations. The second topic is the deformation theory and families of Kähler metrics with weighted constant scalar curvature. Here we study the existence of such metrics when either the complex structures or the weight functions vary, and also prove the existence of a weighted Weil–Petersson metric associated to a family of these metrics. The results are new even in the fibrewise extremal and Kähler–Ricci soliton settings.

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Chapter 1

Introduction

Over the last sixty years there has been great progress in geometry, led by the discovery of deep interactions between differential and algebraic geometry. Perhaps the most famous among such interactions is the Kobayashi–Hitchin correspondence, which says that a holomorphic vector bundle on a compact Kähler manifold admits a Hermite–Einstein metric if and only if the vector bundle is slope polystable. Here a Hermite–Einstein metric is a solution to a canonical geometric partial differential equation (PDE) for metrics on the vector bundle [Kob82], and polystability is an algebro-geometric stability condition arising from Mumford's Geometric Invariant Theory (GIT) [Mum65].

The Kobayashi–Hitchin correspondence first made its appearance in the work of Narasimhan–Seshadri [NS65], who proved the theorem for vector bundles over algebraic curves. Then in the 1980s, Donaldson [Don85] showed that the theorem can be generalised to higher dimensions, by proving it holds for vector bundles over algebraic surfaces. Later, Uhlenbeck–Yau proved the theorem in full, for vector bundles over arbitrary compact Kähler manifolds [UY86].

The power of the Kobayashi–Hitchin correspondence lies in its ability to describe purely differential geometric concepts in terms of algebraic geometry, and vice versa. In one direction, proving existence of a solution to a PDE is typically difficult, so having a simple algebro-geometric criterion to verify instead is extremely useful. On the other hand, stable objects were originally introduced to form moduli spaces of objects in algebraic geometry, so having an equivalent differential geometric description of stable objects provides another way of viewing these moduli spaces.

Motivated by the Kobayashi–Hitchin correspondence, Yau conjectured the existence of an algebro-geometric stability condition on varieties that would be equivalent to the existence of a Kähler–Einstein metric [Yau93]. This led to the introduction of K-stability, a stability condition on varieties initially formulated for Fano manifolds

by Tian [Tia97], and later for arbitrary polarised varieties by Donaldson [Don02]. Nowadays, the conjecture that existence of a constant scalar curvature Kähler (cscK) metric on a polarised manifold is equivalent to K-polystability of the manifold is known as the Yau–Tian–Donaldson conjecture. That existence of a cscK metric on a manifold implies K-polystability was proven by Donaldson and Stoppa in the case of discrete automorphisms [Don01, Sto09a], and finally by Berman–Darvas–Lu in full generality [BDL20]. Both implications in the conjecture are known to be true for Fano manifolds, with one implication due to Tian [Tia97] and Berman [Ber16] and the other due to Chen–Donaldson–Sun [CDS15a, CDS15b, CDS15c]. Recent progress on the unsolved implication for arbitrary polarised varieties includes the works of Chen–Cheng [CC21a, CC21b], Berman–Boucksom–Jonsson [BBJ15], and Li [Li20].

Despite its original motivation being differential geometric, K-stability has also enjoyed great success in facilitating the construction of algebro-geometric moduli spaces of Fano varieties. There is a long list of authors who contributed to this theory over many years, and it is somewhat orthogonal to the contents of this thesis. So we will not go into further detail, however the interested reader may consult the following references for further information: [Oda15, Bir19, Bir21, Jia20, CP21, BLX19, Xu20, LWX19, BX19, ABHLX20, BHLLX21, XZ20, LXZ21].

The principal aim of this thesis is to study correspondences such as the Kobayashi–Hitchin correspondence and the Yau–Tian–Donaldson conjecture, but in different geometric contexts. The first context we consider is one recently introduced by Dervan–Sektnan [DS21b, DS21c, DS21a]. These authors consider holomorphic surjective submersions $\pi : X \to B$ whose fibres are all compact cscK manifolds, and define a natural notion of canonical relatively cscK metric on such a fibration called an *optimal symplectic connection*. Here a relatively cscK metric is a relatively Kähler metric whose restriction to each fibre is cscK. An important motivation for optimal symplectic connections is that the existence of an optimal symplectic connection allows one to construct a cscK metric on the total space of the fibration in certain circumstances. This was proven by Dervan–Sektnan [DS21b], providing a vast generalisation of previous work by Hong [Hon99] and Fine [Fin04]. The main conjecture is that an optimal symplectic connection exists if and only if the fibration is *polystable* in a suitable algebro-geometric sense.

Dervan–Sektnan have proven a partial result in one direction: they show that when the relatively Kähler class is the first Chern class of a relatively ample line bundle, the existence of an optimal symplectic connection implies *semistability* of the fibration [DS21a]. The approach taken is an asymptotic analysis of Bergman kernels, analogous

to Donaldson's proof that polarised cscK manifolds are K-semistable [Don05]. In this thesis we strengthen their result by instead analysing the geometry of the space of relatively cscK potentials $\mathcal{K}_E$, that is, the space of smooth functions φ such that $i\partial\bar{\partial}\varphi$ is the difference of two relatively cscK metrics. Not only does this refine their result to *polystability* with respect to a large class of fibration degenerations, but also the proof works in the transcendental Kähler setting, so extends their result in another direction. The results concerning optimal symplectic connections are also collected in the author's paper [Hal20].

First, we derive a geodesic equation arising from a natural Riemannian structure on the space $\mathcal{K}_E$, and then prove:

Theorem 1.0.1. *Any two points of $\mathcal{K}_E$ are joined by a unique smooth geodesic.*

It is true that in the Kähler case, one cannot expect geodesics of Kähler potentials to be smooth in general [Che00b, LV13]. However, in our case the fibrewise cscK assumption does give rise to smooth geodesics, essentially because the space of cscK metrics in a fixed Kähler class is a finite dimensional symmetric space. Using the explicit structure of geodesics, we compute a Riemannian decomposition of $\mathcal{K}_E$ into the product of a flat manifold and a non-positively curved manifold. This parallels the description of the space of Kähler potentials as a negatively curved space [Mab87, Sem92, Don99].

An important tool in the study of canonical metrics is to consider so-called log-norm functionals on spaces of metrics, such as Donaldson's functional on the space of hermitian metrics on a vector bundle [Don85], or the Mabuchi functional on the space of Kähler metrics in a fixed Kähler class [Mab86]. Dervan and Sektnan show that there exists a natural log-norm funtional $\mathcal{N} : \mathcal{K}_E \to \mathbb{R}$ on the space of relatively cscK potentials, whose critical points are precisely the optimal symplectic connections [DS21c]. We prove:

Theorem 1.0.2. *The functional $\mathcal{N}$ is convex along smooth geodesics in $\mathcal{K}_E$.*

This convexity immediately yields a greatly simplified proof of Dervan and Sektnan's recent uniqueness result for optimal symplectic connections, analogous to Donaldson's proof of uniqueness of Hermite–Einstein metrics:

Corollary 1.0.3 ([DS21c])**.** *Let ω_X and ω'_X be two optimal symplectic connections in the same cohomology class. Then there exists a holomorphic automorphism $g : X \to X$ preserving the projection $\pi : X \to B$, and a smooth function $f : B \to \mathbb{R}$ such that*

$$\omega'_X = g^*\omega_X + i\partial\bar{\partial}\pi^* f.$$

Using convexity, we also strengthen another result of Dervan–Sektnan, showing that existence of an optimal symplectic connection implies the functional $\mathcal{N}$ is bounded below.

A fibration is defined to be stable if a certain invariant associated to any nontrivial fibration degeneration is positive. A fibration degeneration is a $\mathbb{C}^*$-equivariant flat degeneration of the original fibration $X \to B$ to a possibly singular fibration $X' \to B$; see Chapter 2 for the precise definitions. The general fibre over B of a fibration degeneration may be assumed to be a product test configuration; here we restrict attention to a suitable class of "product-type" fibration degenerations whose fibres are *all* product test configurations. We can then associate to any product-type fibration degeneration a unique compatible geodesic ray in $\mathcal{K}_E$. The key to this is a reduction of the geodesic equation on a product test configuration to a "Wess–Zumino–Witten" equation by Donaldson [Don99]. We then calculate the limiting slope of the functional $\mathcal{N}$ along this geodesic ray, using a Chen–Tian style formula for the functional $\mathcal{N}$ which we derive, together with results of [PRS08, BHJ19, SD18]. Applying the convexity theorem above, we prove:

Theorem 1.0.4. *If $X \to B$ is a fibration admitting an optimal symplectic connection, then it is polystable with respect to product-type fibration degenerations.*

This strengthens the semistability result of Dervan–Sektnan in a large class of cases. In terms of stability of vector bundles, restricting attention to product-type fibration degenerations is analogous to considering slope stability only with respect to sub-vector bundles, rather than general coherent subsheaves. In particular, we expect that this should provide an essentially complete proof of polystability of fibrations over curves. We give a general class of examples of product-type fibration degenerations for both projectivised vector bundles and isotrivial fibrations. In particular, we also prove:

Corollary 1.0.5. *If $X \to B$ is an isotrivial fibration admitting an optimal symplectic connection, then it is polystable with respect to fibration degenerations that are smooth over $B \times \mathbb{C}$.*

Here smoothness is meant in the sense of scheme theory, so that $\mathcal{X} \to B \times \mathbb{C}$ is a holomorphic surjective submersion.

The general framework of ideas above has proven tremendously useful in the theory of cscK metrics and K-stability. For instance, the proof of Theorem 1.0.4 above is analogous to [BHJ19] and [BDL20] in the Kähler setting. One might hope that these

methods will eventually lead to a converse, yielding a full proof of the conjecture. Interestingly, these ideas have also fed back into the theory of vector bundles, particularly in [HK18] where the authors reprove one direction of the Kobayashi–Hitchin correspondence using convexity of Donaldson's functional and non-Archimedean limits.

The final aim of this thesis is to study deformations and families of weighted Kähler metrics of constant scalar curvature, or *weighted cscK metrics* for short. These metrics were recently introduced by Lahdili [Lah19], and provide a very general approach to canonical metrics in the presence of automorphisms. Namely, given a compact torus T in the group of automorphisms $\mathrm{Aut}(X)$ of a Kähler manifold, with moment polytope P, one can deform the cscK equation using weight functions $v, w : P \to \mathbb{R}$ on the polytope, to obtain a new weighted cscK condition depending on v and w. One calls a solution to this equation a (v, w)-weighted cscK metric. More generally one can study (v, w)-extremal metrics, whose weighted scalar curvature generates a holomorphic vector field. The importance of these metrics is that they encompass many well-known and common examples of canonical metrics in Kähler geometry, including cscK metrics, extremal metrics, Kähler–Ricci solitons, and extremal Sasaki metrics.

It is expected that the existence of a weighted cscK metric is equivalent to a suitable algebro-geometric stability notion, similar to K-stability. However, this stability notion has not yet been defined in the literature, due to difficulties formulating the appropriate stability invariant for singular test configurations. Despite this, there are partial results in the literature, including a definition of weighted K-stability with respect to smooth test configurations by Lahdili [Lah19], and a proof that existence of a weighted cscK metric implies weighted K-polystability with respect to smooth test configurations by Apostolov–Jubert–Lahdili [AJL21]. Inoue has also worked on a closely related class of so-called μ-cscK metrics, and obtained results on K-stability in this setting [Ino19a, Ino20].

First, we study deformations of weighted cscK metrics when the weight function $v : P \to \mathbb{R}$ is deformed. We show:

Theorem 1.0.6. *Let ω be a (v, w)-cscK metric. If v_t is a smooth 1-parameter family of weight functions such that $v_0 = v$, then for t sufficiently small there exists $\epsilon > 0$ such that for $|t| < \epsilon$ there exists a (v_t, w)-extremal metric in the class $[\omega]$.*

Next, we study deformations of complex structure, in the style of Brönnle [Brö11] and Székelyhidi [Szé10] for cscK metrics, and Inoue [Ino19b] for Kähler–Ricci solitons. Namely, we prove:

Theorem 1.0.7. *Let $X \to D$ be a family of compact complex manifolds over an open subset $D \subset \mathbb{C}^n$ containing $0 \in \mathbb{C}^n$, and assume that X_0 admits a (v, w)-weighted cscK metric. Then there exists a sufficiently small neighbourhood U of 0 such that if $b \in U$ then X_b admits a (v, w)-weighted cscK metric if and only if it is weighted K-polystable with respect to smooth T-equivariant test configurations.*

A corollary of this theorem is that given a family $X \to B$ of compact Kähler manifolds such that each fibre admits a (v, w)-weighted cscK metric in the same class, there exists a global relatively Kähler form ω on X whose restriction to each fibre is a (v, w)-cscK metric. In this situation, we then show:

Theorem 1.0.8. *There exists a natural real $(1, 1)$-form α on B, such that:*

1. *α is closed,*
2. *α is independent of the relatively (v, w)-cscK metric in the class $[\omega]$, and*
3. *α is semipositive, and strictly positive wherever the Kuranishi map $T_bB \to H^1(X_b, TX_b)$ giving the deformation of complex structure on the fibres is injective.*

We call the metric α the *weighted Weil–Petersson metric* of the fibration $X \to B$. It recovers the usual Weil–Petersson metric when the fibres are cscK, shows that the usual Weil–Petersson metric is a good choice of metric when the fibres are extremal Kähler manifolds, and gives a new metric when the fibres admit Kähler–Ricci solitons.

Outline

Here we give a brief outline of each chapter of the thesis.

Chapter 2 is a collection of essential background information on geometric invariant theory, Kähler geometry, K-stability, relatively cscK connections on fibrations, stability of fibrations, and weighted cscK metrics. The results and definitions summarised here will be used throughout the thesis.

Chapter 3 studies geodesics in the space of relatively cscK metrics. We first derive a geodesic equation for paths in the space of such metrics, by taking the variation of a natural energy functional on paths. We then show that any two relatively cscK metrics are joined by a unique smooth geodesic (Theorem 1.0.1). We end by considering the curvature of the infinite dimensional space of all such relatively Kähler metrics, which formally is a negatively curved space.

Chapter 4 then studies the log-norm functional for fibrations, which is the analogue of the Mabuchi functional on the space of Kähler potentials. Here we prove the log-norm functional is convex along smooth geodesics (Theorem 1.0.2), and use this to give a new simplified proof of uniqueness of optimal symplectic connections (Corollary 1.0.3). The chapter ends by deriving a Chen–Tian style formula for the log-norm functional, by expanding the usual Chen–Tian formula for the Mabuchi functional in the adiabatic limit.

Chapter 5 proves the main result on fibrations, that fibrations admitting optimal symplectic connections are polystable with respect to fibration degenerations that are of product-type (Theorem 1.0.4).

Chapter 6 is on weighted cscK metrics. We first show that one may deform the weight function v and obtain new weighted extremal metrics (Theorem 1.0.7). Next we study deformations of complex structures, and prove the equivalence of existence of weighted cscK metrics for small deformations to weighted K-polystability with respect to smooth test configurations (Theorem 1.0.7). Lastly, we consider families of weighted cscK metrics, and construct a natural weighted Weil–Petersson metric on the base space of such a family (Theorem 1.0.8).

Chapter 7 contains ideas for further research on optimal symplectic connections and families of weighted cscK metrics.

Chapters 3-5 are mostly the same as Sections 3-6 in the author's paper [Hal20]. Some sections in Chapter 2 are also expanded from Section 2 of that paper.

Chapter 2

Background

In this chapter we give the background required for the main results of the thesis. First, we give a brief account of basic concepts in Geometric Invariant Theory (GIT) including the Hilbert–Mumford criterion and the Kempf–Ness theorem. We will not need these results, although they give strong motivation for constructions later in the thesis. We next cover the links between constant scalar curvature and K-stability, which are by now well known. Then, we give background on the theory of stability and optimal symplectic connections on fibrations, introduced by Dervan and Sektnan [DS21b, DS21a]. Finally, we cover the weighted cscK metrics introduced by Lahdili [Lah19], with a view towards studying their deformations and behaviour in families in Chapter 6. One small contribution of this chapter is a refined definition of stability of fibrations, which prevents certain edge cases pointed out by M. Hattori from contradicting the main conjecture, namely that existence of an optimal symplectic connection is equivalent to polystability of the fibration.

2.1 Geometric Invariant Theory

In this section we give a brief review of Mumford's GIT, the Hilbert–Mumford criterion, and the Kempf–Ness theorem. By now these theorems are very well known, so we will not provide proofs and citations for specific results. The reader may instead consult the following references for more background: [MFK94, Hos12, Tho05].

The original motivation for GIT was to construct quotients of schemes by algebraic groups, with the aim of constructing moduli spaces of objects in algebraic geometry. Let G be a complex Lie group. We say G is *reductive* if it is the complexification of a maximal compact subgroup $K \subset G$. Common examples include $GL_n(\mathbb{C})$, $SL_n(\mathbb{C})$ and $\mathbb{C}^*$, which are the complexifications of the compact groups $U(n)$, $SU(n)$ and S^1 respectively. Mumford's definitions and results on GIT were originally for reductive

groups, although much recent work has gone into generalising this to non-reductive groups—see [BDHK16, BDHK18a, BDHK18b, BDK18]. In this chapter, we will restrict attention to actions of reductive groups, either on a projective variety or on a vector space.

Let (X, L) be a polarised variety, that is, X is a complex projective variety and $L \to X$ is an ample line bundle. Let G be a reductive group acting on X, with a lift of the action to L. In this situation, we say simply that G acts (linearly) on (X, L). Consider the graded coordinate ring

$$R(X) := \bigoplus_{i=0}^{\infty} H^0(X, L^i)$$

of X, where L^i denotes the i-th tensor power of L. Then G acts on each space of sections $H^0(X, L^i)$, so we may consider the subspace $H^0(X, L^i)^G$ of invariant sections. The ring of invariants

$$R(X)^G := \bigoplus_{i=0}^{\infty} H^0(X, L^i)^G$$

is then a graded $\mathbb{C}$-algebra, and we define the *GIT quotient* of X by the action of G to be

$$X /\!\!/ G := \operatorname{Proj}\left(R(X)^G\right).$$

Definition 2.1.1. A point $x \in X$ is:

1. *semistable* if there exist $i \geq 1$ and a non-zero invariant section $f \in H^0(X, L^i)^G$ such that $f(x) \neq 0$,
2. *polystable* if x is semistable and its orbit in X^{ss} is closed,
3. *stable* if x is polystable and its stabiliser is finite, and
4. *unstable* if x is not semistable.

We write X^{ss}, X^{ps}, X^s, and X^{us} for the sets of semistable, polystable, stable, and unstable points respectively. There is a canonical rational map $X \dashrightarrow X /\!\!/ G$ induced by the inclusion of graded $\mathbb{C}$-algebras $R(X)^G \hookrightarrow R(X)$. The set on which this map is well-defined is exactly the set of all semistable points X^{ss}. Two points $x, y \in X^{ss}$ have the same image in $X /\!\!/ G$ if and only if the closures of their orbits intersect in X^{ss}. This condition defines an equivalence relation on X^{ss} called S-equivalence, so the GIT quotient can be identified with the set of semistable points modulo S-equivalence. The orbit of a semistable point has a unique polystable orbit in its

closure. It follows that the GIT quotient $X /\!\!/ G$ can also be identified with the set X^{ps}/G of polystable orbits in X.

Let $\lambda : \mathbb{C}^* \to G$ be a 1-parameter subgroup, and let $x \in X$. By properness of X, the limit

$$x' := \lim_{t \to 0} \lambda(t)x$$

exists, and is fixed by $\lambda(\mathbb{C}^*)$. Since the G-action lifts to L, we then have an action of $\lambda(\mathbb{C}^*)$ on the fibre $L_{x'}$. We define $\mu(x, \lambda) \in \mathbb{Z}$ to be the weight of this action, so that

$$\lambda(t) \cdot \ell = t^{\mu(x,\lambda)}\ell$$

for all $\ell \in L_{x'}$.

Theorem 2.1.2 (Hilbert–Mumford criterion [MFK94])**.** *A point $x \in X$ is:*

1. *semistable if and only if $\mu(x, \lambda) \geq 0$ for all 1-parameter subgroups $\lambda : \mathbb{C}^* \to G$,*
2. *polystable if and only if x is semistable and whenever $\mu(x, \lambda) = 0$, the image of λ lies in the stabiliser of x, and*
3. *stable if and only if x is semistable and $\mu(x, \lambda) = 0$ only if λ is the trivial 1-parameter subgroup.*

Although we will not use the Hilbert–Mumford criterion in this particular form in the thesis, it provides strong motivation for the definition of the Donaldson–Futaki invariant later. We will however use another form of the Hilbert–Mumford criterion for actions of reductive groups on vector spaces. For all intents and purposes, we can take this as the definition of stability in a vector space:

Definition 2.1.3. Let G be a complex reductive group acting linearly on a complex vector space V. A point $x \in V$ is:

1. *stable* if its orbit in V is closed, and its stabiliser is finite,
2. *polystable* if its orbit in V is closed,
3. *semistable* if 0 is not contained in the closure of the orbit $G \cdot x$,
4. *unstable* if it is not semistable.

Remark 2.1.4. If (X, L) is a polarised variety, we may embed X into $\mathbb{P}^N$ using sections of L^i for some i. If G acts on (X, L), then it also acts on $\mathbb{C}^{N+1}$ via a map $G \to GL_{N+1}(\mathbb{C})$. We may therefore lift a point $x \in X$ to $\hat{x} \in \mathbb{C}^{N+1}$, and consider the action of G on this point. Then x is (semi/poly)stable in X if and only if $\hat{x}$ is (semi/poly)stable in $\mathbb{C}^{N+1}$.

We end this section with a discussion of symplectic quotients and the Kempf–Ness theorem. Let (X, ω) be a symplectic manifold, and K be a compact group acting on X by symplectomorphisms. Let $\mathfrak{k}$ be the Lie algebra of K. A *moment map* for the action of K on X is a map $\mu : X \to \mathfrak{k}^*$ such that

$$\langle d\mu(x), \xi \rangle = \omega_x(-, \xi_x)$$

for all $x \in X$ and $\xi \in \mathfrak{k}$. Here $\langle \, , \rangle$ is the natural pairing between $\mathfrak{k}^*$ and $\mathfrak{k}$, and $\xi_x \in T_x X$ denotes the infinitesimal action of ξ at x.

We define the *symplectic quotient* of X by K to be

$$X /\!\!/_s K := \mu^{-1}(0)/K.$$

If 0 is a regular value of μ, and K acts freely on $\mu^{-1}(0)$ then $X /\!\!/_s K$ is a smooth manifold, and inherits a natural symplectic form induced by ω.

When X is a smooth projective variety embedded in $\mathbb{P}^N$, we have a canonical symplectic form on X—the restriction of the Fubini–Study form ω_{FS} on $\mathbb{P}^N$ to X. If G acts linearly on X via $\mathcal{O}_{\mathbb{P}^N}(1)|_X$, so that a maximal compact subgroup $K \subset G$ acts through elements of $U(N+1)$, then K acts by symplectomorphisms on X. We therefore have two separate quotients available: the GIT quotient $X /\!\!/ G$ and the symplectic quotient $X /\!\!/_s K$. The Kempf–Ness theorem identifies these two objects.

Theorem 2.1.5 (Kempf–Ness [KN79])**.** *Let G be a reductive group with maximal compact subgroup K, and let G act linearly on a smooth projective variety $X \subset \mathbb{P}^N$ with moment map $\mu : X \to \mathfrak{k}^*$ for the K-action. A point $x \in X$ is polystable for the G-action if and only if the orbit $G \cdot x$ intersects $\mu^{-1}(0)$. Furthermore, if x is polystable then $G \cdot x \cap \mu^{-1}(0)$ is a single K-orbit.*

Since the GIT quotient parametrises the polystable orbits, the Kempf–Ness theorem identifies the GIT quotient with the symplectic quotient:

$$X /\!\!/ G \cong X /\!\!/_s K.$$

The main point of the proof is to consider the non-positively curved symmetric space G/K, and define the function $M : G/K \to \mathbb{R}$,

$$M(Kg) := \log \|\mu(g\hat{x})\|.$$

Here $\hat{x}$ is a lift of $x \in \mathbb{P}^N$ to $\mathbb{C}^{N+1}$, and $\|\cdot\|$ is an arbitrary K-invariant norm on $\mathbb{C}^{N+1}$. One then shows that:

1. Kg is a critical point of M if and only if $\mu(gx) = 0$, and
2. M is convex along geodesics in G/K.

The point x is polystable if and only if the orbit $G \cdot \hat{x}$ is closed in $\mathbb{C}^{N+1}$, if and only if the function $Kg \mapsto \|g\hat{x}\|$ is proper, if and only if $M : G/K \to \mathbb{R}$ is proper. Now, since G is the complexification of the maximal compact $K \subset G$, G/K is a complete simply connected symmetric space of non-positive curvature, so the Riemannian exponential map $T_{K\cdot 1}(G/K) \to G/K$ is a diffeomorphism by the Cartan–Hadamard theorem. But by the second point, since M is convex along geodesics, it follows that M is proper if and only if it has a critical point. By the first point, a critical point corresponds exactly to a zero of the moment map. Hence the orbit of x contains a zero of the moment map if and only if x is polystable. For more details on the proof, see [Szé14, Chapter 5].

The ideas behind this proof are precisely what go into certain parts of the proofs of the Kobayashi–Hitchin correspondence and the Yau–Tian–Donaldson conjecture. In fact, the Hermite–Einstein and cscK equations can be thought of as zeros of moment maps on infinite-dimensional Kähler manifolds. The analogue of M in the vector bundles setting is Donaldson's functional [Don85], and in the varieties setting is the Mabuchi functional [Mab86]. Both of these functionals satisfy the two important properties above: their critical points are precisely solutions to the equations of interest (either Hermite–Einstein or cscK), and they are convex along geodesics. However, properness of a function on an infinite-dimensional space is much more subtle—indeed the usual topological definition of properness does not suffice. Furthermore, in the cscK setting one must also enlarge the space of Kähler potentials on which the Mabuchi functional is defined to include singular Kähler potentials, so that geodesics exist. We will comment more on this in the next section.

2.2 Kähler geometry

Let (X, ω) be a compact Kähler manifold of dimension n. The *scalar curvature* of ω is the smooth function $S(\omega)$ on X given by contracting the Ricci curvature of ω,

$$\mathrm{Ric}(\omega) := -\frac{i}{2\pi}\partial\bar{\partial}\log\omega^n,$$

with the metric ω itself:

$$S(\omega) := \Lambda_\omega \mathrm{Ric}(\omega).$$

We will be particularly interested in Kähler metrics ω whose scalar curvature $S(\omega)$ is a constant function; such metrics will be abbreviated *cscK* (constant scalar curvature Kähler). The average of $S(\omega)$ is a topological constant, given by

$$\hat{S} = \frac{n\, c_1(X)\cdot[\omega]^{n-1}}{[\omega]^n}$$

where $[\omega]$ is the Kähler class of ω. Thus, the metric ω is cscK if and only if $S(\omega) = \hat{S}$. For links to algebraic geometry, we will later consider the case where $[\omega]$ is equal to the first Chern class $c_1(L)$ of an ample line bundle $L \to X$.

For a fixed Kähler metric ω, let

$$\mathcal{K} := \{\phi \in C^\infty(X, \mathbb{R}) : \omega + i\partial\bar{\partial}\phi > 0\}$$

be the space of Kähler potentials with respect to ω. We will use the notation $\omega_\phi := \omega + i\partial\bar{\partial}\phi$ whenever $\phi \in \mathcal{K}$. Any Kähler metric in the Kähler class $[\omega]$ may be written as ω_ϕ for some $\phi \in \mathcal{K}$ that is unique up to addition of a real constant. Since $\mathcal{K}$ is an open subset of the Fréchet space $C^\infty(X, \mathbb{R})$, it is naturally a Fréchet manifold with tangent space isomorphic to $C^\infty(X, \mathbb{R})$ at each point.

There is a natural L^2-inner product on each tangent space $T_\phi\mathcal{K} \cong C^\infty(X)$ defined by

$$\langle \psi_1, \psi_2\rangle_\phi := \int_X \psi_1\psi_2\, \omega_\phi^n.$$

This in turn gives rise to an energy functional μ on smooth paths $\phi : [0,1] \to \mathcal{K}$,

$$\mu(\phi) := \int_0^1 \langle \dot{\phi}_t, \dot{\phi}_t\rangle_{\phi_t}\, dt.$$

Definition 2.2.1. A smooth path $\phi : [0,1] \to \mathcal{K}$ is a *geodesic* if it is a critical point of the energy functional μ.

Taking the variation of the functional μ, one proves:

Proposition 2.2.2 (Mabuchi [Mab86])**.** *A smooth path $\phi : [0,1] \to \mathcal{K}$ is a geodesic if and only if it satisfies the PDE*

$$\ddot{\phi}_t = |\partial\dot{\phi}_t|_t^2$$

for all $t \in [0,1]$, where $|\cdot|_t$ is the hermitian metric on $\Lambda^{1,0}T^*X$ determined by $\omega_t := \omega + i\partial\bar{\partial}\phi_t$.

It is an important fact that any two points of $\mathcal{K}$ are joined by a unique geodesic [Che00b], although the geodesic may not be smooth and one must enlarge the space $\mathcal{K}$ to include Kähler potentials that are only $C^{1,1}$-regular.

A very useful class of geodesics can be constructed through holomorphic vector fields on X, which are in turn described by holomorphy potentials.

Definition 2.2.3. A *holomorphy potential* (with respect to ω) is a smooth function $u : X \to \mathbb{C}$ such that $\nabla^{1,0}u$ is a holomorphic vector field, where ∇ is the gradient operator determined by ω and $\nabla^{1,0}u := (\nabla u)^{1,0}$.

Equivalently, a holomorphy potential is a function in the kernel of the operator $\mathcal{D} : C^\infty(X,\mathbb{C}) \to \Omega^{0,1}(\Lambda^{1,0}TX)$ defined by

$$\mathcal{D}(u) = \bar{\partial}\nabla^{1,0}u.$$

The operator $\mathcal{D}^*\mathcal{D} : C^\infty(X,\mathbb{C}) \to C^\infty(X,\mathbb{C})$ is called the *Lichnerowicz operator*; it is a fourth order elliptic operator, and hence its kernel is finite-dimensional.

Let E be the finite-dimensional vector space of real-valued holomorphy potentials on X having zero integral with respect to ω, and let $\mathfrak{h}$ be the finite-dimensional vector space of holomorphic vector fields on X that are equal to $\nabla^{1,0}u$ for some holomorphy potential u. The operator $\nabla^{1,0} : E \oplus iE \to \mathfrak{h}$ is then an isomorphism. We remark that $\mathfrak{h}$ is exactly the space of holomorphic vector fields which vanish somewhere [LS94, Theorem 1], so is independent of the metric ω. It is also a Lie algebra under the Lie bracket of vector fields. For a proof of the following, see [Szé14, Example 4.26]

Proposition 2.2.4. *Let ω be a cscK metric, let $u \in E\backslash\{0\}$ and define $v := \nabla^{1,0}u \in \mathfrak{h}$. If $\exp(tv)$ is the time t flow of v, then*

$$\phi_t := \int_0^t \exp(sv)^*(u)\, ds$$

defines a non-trivial smooth geodesic in $\mathcal{K}$, and $\omega_t := \exp(tv)^*\omega = \omega + i\partial\bar{\partial}\phi_t$ has constant scalar curvature for every t.

We will see that the converse to this result holds: any two distinct cscK metrics in the same Kähler class are joined by a unique geodesic of the above kind. This follows from the following important uniqueness result. We denote by $\mathrm{Aut}(X)$ the complex Lie group of biholomorphisms of X, and by $\mathrm{Aut}_0(X)$ its identity component.

Theorem 2.2.5 ([Don01, BB17])**.** *If ω and ω' are two cscK metrics in the same Kähler class, then there exists a biholomorphism $g \in \mathrm{Aut}_0(X)$ so that $\omega' = g^*\omega$.*

If $K =: \mathrm{Isom}_0(\omega) \subset \mathrm{Aut}_0(X)$ is the set of isometries of ω in $\mathrm{Aut}_0(X)$, then clearly g is unique up to left multiplication by elements of K. Hence, if ω is cscK then the cscK metrics in $[\omega]$ are parametrised by the homogeneous space $\mathrm{Aut}_0(X)/\mathrm{Isom}_0(\omega)$. The following theorem reveals the structure of this space.

Theorem 2.2.6 (Partial Cartan decomposition / Matsushima–Lichnerowicz Theorem; see [Gau10, Sections 3.5–3.6] and [DR17a, Section 6.1])**.** *Let (X, ω) be a compact Kähler manifold with constant scalar curvature. Denote by $\mathfrak{g}$ the Lie algebra $H^0(X, TX)$ of holomorphic vector fields on X, and let $\mathfrak{h}$ be the Lie algebra of holomorphic vector fields which arise from holomorphy potentials. There is a splitting*

$$\mathfrak{g} = \mathfrak{a} \oplus \tilde{\mathfrak{k}} \oplus i \cdot \tilde{\mathfrak{k}}$$

of real vector spaces, where:

1. *$\mathfrak{a}$ is the complex abelian ideal in $\mathfrak{g}$ consisting of harmonic vector fields with respect to ω, which has fixed dimension $h^{1,0}(X)$,*
2. *$\tilde{\mathfrak{k}}$ is the space of vector fields generated by* purely imaginary *holomorphy potentials. Hence $\mathfrak{h} = \tilde{\mathfrak{k}} \oplus i \cdot \tilde{\mathfrak{k}}$,*
3. *$\mathfrak{a} \oplus \tilde{\mathfrak{k}}$ is the real Lie algebra of Killing fields on X with respect to ω.*

Furthermore, let $G := \mathrm{Aut}_0(X)$ and $K := \mathrm{Isom}_0(\omega)$. Then:

4. *$K \subset G$ is a maximal compact subgroup whose Lie algebra is $\mathfrak{a} \oplus \tilde{\mathfrak{k}}$.*
5. *The map $K \times \tilde{\mathfrak{k}} \to G$, $(k, v) \mapsto k \exp(iv)$ is surjective.*

Since $K \subset G$ is maximal compact, the homogeneous space G/K of right cosets of K is a complete simply connected symmetric space of non-positive curvature. By the Cartan–Hadamard Theorem from Riemannian geometry, the Riemannian exponential map $T_{K\cdot 1}(G/K) \to G/K$ is a diffeomorphism. Since geodesics from $K \cdot 1$ take the explicit form

$$t \mapsto K \exp(tiv)$$

for $v \in \mathfrak{k}$, we therefore have:

Lemma 2.2.7. *Let E denote the space of real holomorphy potentials having zero integral on (X, ω), and let $G/K = \mathrm{Aut}_0(X)/\mathrm{Isom}_0(\omega)$. Then the map*

$$u \mapsto \exp(\nabla^{1,0} u)$$

is a diffeomorphism $E \cong G/K$.

We have already remarked that the homogeneous space G/K parametrises the set of cscK metrics in the class $[\omega]$. Hence we now see that any two points of this space are joined by a unique smooth geodesic arising from a real holomorphy potential, and so the converse to Proposition 2.2.4 does hold.

The existence of a cscK metric in a fixed Kähler class is closely related to the properties of the following functional on the space of Kähler potentials:

Proposition 2.2.8 (Mabuchi functional [Mab86])**.** *Fix a Kähler metric ω on X. There exists a unique functional $\mathcal{M} : \mathcal{K} \to \mathbb{R}$ such that $\mathcal{M}(0) = 0$ and*

$$\frac{d}{dt}\mathcal{M}(\phi_t) = \int_X \dot{\phi}_t(\hat{S} - S(\omega_t))\, \omega_t^n, \tag{2.1}$$

for any smooth path $\phi : [0, 1] \to \mathcal{K}$, where $\omega_t := \omega + i\partial\bar{\partial}\phi_t$.

The functional $\mathcal{M}$ is called the *Mabuchi functional*. Note that critical points of the Mabuchi functional correspond to cscK metrics in the class $[\omega]$, essentially by definition.

Since geodesics between Kähler metrics only exist in a weak sense, much recent work has gone into extending the Mabuchi functional and its properties to singular potentials. Berman–Berndtsson [BB17] show using the Chen–Tian formula [Che00a, Tia00], that the Mabuchi functional is well-defined and convex along the weak geodesics whose existence was proven by Chen [Che00b]. They then use this to prove uniqueness of cscK metrics in a fixed Kähler class, a result first obtained by Donaldson [Don01] in the case of discrete automorphisms.

Further, Darvas–Rubinstein [DR17a] introduce a metric completion $\mathcal{E}^1$ of the space $\mathcal{K}$ with respect to an L^1-metric d^1, such that $(\mathcal{E}^1, d^1)$ is a geodesic metric space, and the Mabuchi functional admits a natural lower semi-continuous extension to $\mathcal{E}^1$. They then conjecture that any minimiser of the extended functional $\mathcal{M}$ is smooth, and hence corresponds to a cscK metric. Working with the space $\mathcal{E}^1$, Berman–Darvas–Lu [BDL20] use convexity to prove K-polystability of a manifold admitting a cscK metric.

Building on this work, Chen–Cheng [CC21a, CC21b] show that a minimiser of the extended Mabuchi functional is necessarily smooth, which by a result of Berman–Darvas–Lu implies that when X has discrete automorphisms, a cscK metric exists in the fixed Kähler class if and only if the Mabuchi functional is coercive. Here coercivity means there exist constants $C, D > 0$ and such that

$$\mathcal{M}(\varphi) \geq CJ(\varphi) - D$$

for all $\varphi \in \mathcal{K}$, where J is Aubin's J-functional [Aub84]. The functional J acts something like a norm or distance on the space $\mathcal{K}$, so this result implies coercivity is the correct notion of properness for the Mabuchi functional.

2.3 K-stability

Consider a pair (X, L), where X is a complex normal projective variety of dimension n and L is an ample line bundle on X. Such a pair (X, L) is called a *polarised variety*. It is known that when X is smooth and admits a cscK metric in the Kähler class $c_1(L)$, the polarised variety (X, L) is K-polystable [BDL20]. Here, K-polystability is an algebro-geometric stability condition, defined by analogy with the Hilbert-Mumford condition from Geometric Invariant Theory, which we will now describe.

Recall the Hilbert–Mumford criterion had two essential inputs: a notion of 1-parameter subgroup λ, and a notion of weight $\mu(x, \lambda) \in \mathbb{Z}$ for any λ and point x. The following definition corresponds to the notion of 1-parameter subgroup:

Definition 2.3.1. Let (X, L) be a polarised variety. A *test configuration* $(\mathcal{X}, \mathcal{L})$ for (X, L) consists of:

(i) a variety $\mathcal{X}$ with a $\mathbb{C}^*$-equivariant flat proper morphism $\mathcal{X} \to \mathbb{C}$, and

(ii) a $\mathbb{C}^*$-equivariant relatively ample line bundle $\mathcal{L} \to \mathcal{X}$,

such that there is an isomorphism $(\mathcal{X}_1, \mathcal{L}_1) \cong (X, L^r)$ for some $r > 0$ called the *exponent* of the test configuration.

Remark 2.3.2. We can construct many test configurations for a given polarised variety. To do this, we embed X into $\mathbb{P}^N$ using sections of a large tensor power L^r. Then, taking a 1-parameter subgroup $\lambda : \mathbb{C}^* \to GL_{N+1}(\mathbb{C})$, the flat limit $\lim_{t\to 0} \lambda(t)X$ exists, and is the central fibre of a test configuration $(\mathcal{X}, \mathcal{L})$ for (X, L) with exponent r, as $\mathcal{O}_{\mathbb{P}^N}(1)|_X \cong L^r$. Conversely, every test configuration for (X, L) arises in this manner; see [RT07] for a proof.

Two test configurations $(\mathcal{X}, \mathcal{L})$ and $(\mathcal{X}', \mathcal{L}')$ are *isomorphic* if there is a $\mathbb{C}^*$-equivariant isomorphism $(\mathcal{X}, \mathcal{L}) \xrightarrow{\sim} (\mathcal{X}', \mathcal{L}')$ preserving the projections to $\mathbb{C}$.

Example 2.3.3. Given a one-parameter subgroup $\rho : \mathbb{C}^* \to \mathrm{Aut}(X, L)$ we define a test configuration $(\mathcal{X}, \mathcal{L})_\rho$ by taking $\mathcal{X} := X \times \mathbb{C}$ with the diagonal $\mathbb{C}^*$-action and $\mathcal{L} := p^*L$ with the induced action from ρ, where $p : X \times \mathbb{C} \to X$ is the projection. Any test configuration isomorphic to one of the form $(\mathcal{X}, \mathcal{L})_\rho$ for some $\rho : \mathbb{C}^* \to \mathrm{Aut}(X, L)$ is called a *product test configuration*. If ρ is trivial, we call the test configuration *trivial*.

We thus have a notion corresponding to 1-parameter subgroups in the Hilbert–Mumford criterion, we must now find a suitable analogue of the weight. Given a test configuration $(\mathcal{X}, \mathcal{L})$, denote by A_k the infinitesimal generator of the $\mathbb{C}^*$-action on $H^0(\mathcal{X}_0, \mathcal{L}_0^k)$, and let $w_k := \mathrm{tr} A_k$. Then there are expansions

$$\begin{aligned} \dim H^0(\mathcal{X}_0, \mathcal{L}_0^k) &= a_0 k^n + a_1 k^{n-1} + \mathrm{O}(k^{n-2}), \\ w_k &= b_0 k^{n+1} + b_1 k^n + \mathrm{O}(k^{n-1}), \end{aligned}$$

which hold for all $k \gg 0$ by equivariant Grothendieck–Riemann–Roch. The *Donaldson–Futaki invariant* [Don02] of $(\mathcal{X}, \mathcal{L})$ is defined as

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) := \frac{a_1 b_0 - a_0 b_1}{a_0^2}.$$

If $(\mathcal{X}, \mathcal{L})$ is a test configuration, $\widetilde{\mathcal{X}}$ is the normalisation of $\mathcal{X}$ and $\widetilde{\mathcal{L}}$ is the pullback of $\mathcal{L}$ to $\widetilde{\mathcal{X}}$, then $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{L}})$ is also a test configuration for (X, L) called the *normalisation* of $(\mathcal{X}, \mathcal{L})$ [BHJ17].

Definition 2.3.4 ([Tia97, Don02])**.** A polarised variety (X, L) is:

(i) *K-semistable* if $\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq 0$ for all test configurations $(\mathcal{X}, \mathcal{L})$ for (X, L),

(ii) *K-polystable* if it is K-semistable and $\mathrm{DF}(\mathcal{X}, \mathcal{L}) = 0$ only if $(\mathcal{X}, \mathcal{L})$ normalises to a product test configuration for (X, L),

(iii) *K-stable* if it is K-semistable and $\mathrm{DF}(\mathcal{X}, \mathcal{L}) = 0$ only if $(\mathcal{X}, \mathcal{L})$ normalises to the trivial test configuration for (X, L),

(iv) *K-unstable* if it is not K-semistable.

Remark 2.3.5. By a result of Ross–Thomas [RT07], the normalisation $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{L}})$ of a test configuration $(\mathcal{X}, \mathcal{L})$ satisfies $\mathrm{DF}(\widetilde{\mathcal{X}}, \widetilde{\mathcal{L}}) \leq \mathrm{DF}(\mathcal{X}, \mathcal{L})$. Hence it suffices to consider only normal test configurations when testing K-(semi/poly)stability.

The central conjecture of the field is then:

Conjecture 2.3.6 (Yau–Tian–Donaldson [Yau93, Tia97, Don02])**.** *Let (X, L) be a smooth polarised variety. Then (X, L) admits a cscK metric in the class $c_1(L)$ if and only if (X, L) is K-polystable.*

The forward implication is a theorem, due to Berman–Darvas–Lu [BDL20]. The reverse implication is open, and it is also not known whether K-polystability as we've defined it is the correct notion. We remark that the scalar curvature can be seen as a moment map on an infinite-dimensional Kähler manifold, as shown by Donaldson [Don97].

When $(\mathcal{X}, \mathcal{L})$ is a normal test configuration, there is also an important intersection theoretic formula for the Donaldson–Futaki invariant. Since $(\mathcal{X}, \mathcal{L})|_{\mathbb{C}^*} \cong (X \times \mathbb{C}^*, L)$ via the $\mathbb{C}^*$-action, the test configuration $(\mathcal{X}, \mathcal{L})$ admits a trivial compactification over $\infty \in \mathbb{P}^1$. We denote this compactification by $(\overline{\mathcal{X}}, \overline{\mathcal{L}})$. There is also a relative canonical bundle $K_{\overline{\mathcal{X}}/\mathbb{P}^1}$ on $\overline{\mathcal{X}}$, which is only a Weil divisor in general.

Theorem 2.3.7 ([Wan12, Oda13])**.** *If $(\mathcal{X}, \mathcal{L})$ is a normal test configuration for (X, L) then*

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) = \frac{n}{n+1}\mu(X, L)(\overline{\mathcal{L}}^{n+1}) + (\overline{\mathcal{L}}^{n} \cdot K_{\overline{\mathcal{X}}/\mathbb{P}^1}),$$

where $\mu(X, L) := \frac{-K_X \cdot L^{n-1}}{L^n}$.

2.4 Optimal symplectic connections

In this section we cover the theory of optimal symplectic connections, introduced by Dervan–Sektnan in [DS21b], and further studied by Dervan–Sektnan in [DS21c, DS21a] and Ortu in [Ort22]. For reasons which will become apparent, optimal symplectic connections provide a natural notion of canonical relatively Kähler metric on a holomorphic submersion. Their existence on a fibration is conjecturally equivalent to a notion of stability of the fibration, to be defined in the next section.

In this section we consider holomorphic surjective submersions $\pi : X \to B$, where X and B are compact Kähler manifolds. By Ehresmann's theorem such a morphism π is then (smoothly) a locally trivial fibration; all the fibres X_b for $b \in B$ are diffeomorphic, although the complex structures may vary. We write n for the dimension of B and m for the dimension of each fibre X_b. We fix a Kähler metric ω_B on B and a relatively Kähler metric ω_X for $\pi : X \to B$. For each $b \in B$, we will denote by ω_b the Kähler metric $\omega_X|_{X_b}$ on the fibre X_b.

To study the differential geometry of such fibrations, we will further make the following assumptions:

1. Each fibre X_b admits a cscK metric in the class $[\omega_b] = [\omega_X]|_{X_b}$.

2. $\dim_{\mathbb{R}} E_b$ is independent of b, where E_b is the space of real holomorphy potentials on the fibre X_b having zero integral with respect to ω_b.

Example 2.4.1. (i) Let $V \to B$ be a holomorphic vector bundle, let $X := \mathbb{P}(V)$ be its projectivisation, and let $H := \mathcal{O}_{\mathbb{P}(V)}(1)$ be the relative $\mathcal{O}(1)$-line bundle. Then $(X, c_1(H))$ satisfies the hypotheses above.

(ii) More generally, let $P \to B$ be a holomorphic principal G-bundle, where G is a reductive group, and let (Y, L_Y) be a polarised manifold admitting a cscK metric in $c_1(L_Y)$. Given a representation $\rho : G \to \mathrm{Aut}(Y, L_Y)$, we can form the associated fibre bundle $X := P \times_\rho Y$, whose fibre over any point of b is Y. There is also an induced line bundle $H := P \times_\rho L_Y$ whose restriction to any fibre Y is L_Y. Hence $(X, c_1(H))$ satisfies the above hypotheses. If $G = \mathrm{GL}(N)$ and $(Y, L_Y) = (\mathbb{P}^{N-1}, \mathcal{O}(1))$ then we recover the previous example.

Definition 2.4.2. A *relatively cscK metric* is a relatively Kähler metric ω_X whose restriction ω_b to each fibre X_b has constant scalar curvature.

If every class $[\omega_b]$ admits a cscK metric, it is then a non-trivial fact that there exists a relatively cscK metric in $[\omega_X]$; see [DS21b, Lemma 3.8], which uses the deformation theory for cscK metrics of [Szé10, Brö11]. Thus, the first assumption is equivalent to assuming the existence of a relatively cscK metric in the relatively Kähler class.

By the partial Cartan decomposition, the second assumption is equivalent to assuming that the space of holomorphic vector fields $H^0(X_b, TX_b)$ on the fibre X_b has complex dimension independent of $b \in B$.

Proposition 2.4.3 ([DS21b, p. 13])**.** *If* $\dim E_b$ *is independent of* b*, then there exists a smooth vector bundle* $E \to B$ *whose fibre over* $b \in B$ *is* E_b*, such that a smooth section of* E *corresponds precisely to a smooth function on* X *restricting to a real holomorphy potential on each fibre with zero integral.*

The L^2-inner product on smooth real-valued functions on X is given by

$$\langle f, g \rangle := \int_X f g \, \omega_X^m \wedge \omega_B^n, \tag{2.2}$$

where $n = \dim B$, $m = \dim X - \dim B$, and we abusively write ω_B for what is really $\pi^*\omega_B$. There is then an orthogonal direct sum decomposition:

$$C^\infty(X) = C^\infty(B) \oplus C^\infty(E) \oplus C^\infty_R(X), \tag{2.3}$$

where $C^\infty(B)$ is the space of smooth functions on B pulled back to X, $C^\infty(E)$ denotes the smooth sections of $E \to B$, and $C^\infty_R(X)$ is the orthogonal complement of $C^\infty(B) \oplus C^\infty(E)$ in $C^\infty(X)$.

Definition 2.4.4. Given a fixed relatively cscK metric ω_X, define $\mathcal{K}_E \subset C^\infty(X, \mathbb{R})$ to be the space of smooth functions $\varphi : X \to \mathbb{R}$ such that $\omega_X + i\partial\bar{\partial}\varphi$ is also relatively cscK.

The notation $\mathcal{K}_E$ comes from the following computation of its tangent space:

Proposition 2.4.5 ([DS21c, Lemma 4.20])**.** *Let $\varphi_t : [0,1] \to \mathcal{K}_E$ be a smooth path. If E_t denotes the bundle of fibrewise holomorphy potentials corresponding to $\omega_t := \omega_X + i\partial\bar{\partial}\varphi_t$, then $\dot{\varphi}_t \in C^\infty(B) \oplus C^\infty(E_t)$ for all t.*

We often think of functions in $C^\infty(B) \subset C^\infty(X)$ as in some sense irrelevant, which is partly justified by Lemma 2.4.6 and Theorem 2.4.14 below. We will see in Chapter 3 that the smooth functions on B play a role analogous to that of the constant functions in the usual theory of Kähler metrics and potentials.

Lemma 2.4.6 ([DS21b, Lemma 3.9])**.** *Let η and ρ be two smooth real closed $(1,1)$-forms on X such that $[\rho] = [\eta]$ and $\rho|_{X_b} = \eta|_{X_b}$ for all $b \in B$. Then there exists a smooth function $f : B \to \mathbb{R}$ such that*

$$\rho = \eta + i\partial\bar{\partial}\pi^* f.$$

If ω_X and ω'_X are relatively cscK metrics with $[\omega_X] = [\omega'_X]$, then there exists a function $\varphi \in \mathcal{K}_E$ such that $\omega'_X = \omega_X + i\partial\bar{\partial}\varphi$. Note that $\omega_X + i\partial\bar{\partial}\varphi = \omega_X + i\partial\bar{\partial}\psi$ if and only if φ and ψ differ by a real constant. Hence $\mathcal{K}_E/\mathbb{R}$ parametrises all relatively cscK metrics. From the lemma, two relatively cscK metrics in the same class have the same restriction to every fibre if and only if they differ by $i\partial\bar{\partial}\pi^* f$ for some $f \in C^\infty(B)$.

In Chapter 3 we will show how to interpret a smoothly varying choice of cscK metrics on the fibres as a section of a certain smooth fibre bundle $H \to B$ whose fibres are homogeneous spaces. This determines the Fréchet manifold structure of $\mathcal{K}_E$, and further clarifies the structure of the space of relatively cscK metrics.

Recently, Dervan and Sektnan [DS21b] have introduced the notion of an *optimal symplectic connection*. This is a partial differential equation on relatively cscK metrics which plays the role for fibrations that the cscK equation does for varieties, and also the Hermite–Einstein equation for vector bundles. In fact, when $X = \mathbb{P}(V)$ is a projectivised vector bundle the optimal symplectic connection equation reduces to the Hermite–Einstein equation [DS21b, Proposition 3.17]. Most importantly, when an optimal symplectic connection exists it gives a canonical choice of relatively cscK metric in a fixed relatively Kähler class. To describe this equation, we must introduce some notation.

Firstly, any relatively Kähler metric ω_X defines a smooth splitting of the tangent bundle $TX \cong \mathcal{V} \oplus \mathcal{H}$, where $\mathcal{V}$ is the kernel of the projection $d\pi : TX \to TB$, and

$$\mathcal{H} := \{u \in TX : \omega_X(u, v) = 0 \text{ for all } v \in \mathcal{V}\}.$$

One then obtains a canonical splitting of all tensors on X into horizontal, vertical, and mixed components. The *symplectic curvature* of ω_X is the curvature of the Ehresmann connection $\mathcal{H} \subset TX$. Explicitly, this is a two-form $F_{\mathcal{H}}$ on the base B with values in the fibrewise hamiltonian vector fields, defined by

$$F_{\mathcal{H}}(v_1, v_2) := [v_1^{\#}, v_2^{\#}]_{\mathcal{V}}.$$

Here v_1, v_2 are vector fields on the base B, $v_i^{\#}$ denotes the horizontal lift of v_i to X, $[\,,]$ is the usual Lie bracket of vector fields, and the subscript $\mathcal{V}$ denotes the vertical component of a vector field.

Given a hamiltonian vector field v_b on a fibre X_b, we denote by $\mu_b^* v_b$ the corresponding hamiltonian function on (X_b, ω_b) with integral zero. That is, μ^* is the fibrewise co-moment map. Then $\mu^* F_{\mathcal{H}}$ is a 2-form on B with values in the fibrewise hamiltonian functions, which can be equivalently considered as a horizontal 2-form on X.

Given a horizontal 2-form α on X, we write

$$\Lambda_{\omega_B}\alpha := \frac{n\alpha \wedge \pi^*\omega_B^{n-1}}{\pi^*\omega_B^n},$$

where $n = \dim B$ and the division is taken in $\det \mathcal{H} = \Lambda^n \mathcal{H}$. When α is not necessarily horizontal, we shall sometimes write $\Lambda_{\omega_B}\alpha$ for what is really $\Lambda_{\omega_B}\alpha_{\mathcal{H}}$, where $\alpha_{\mathcal{H}}$ is the horizontal component of α. This is mostly to avoid the unsightly notation $\Lambda_{\omega_B}(\omega_X)_{\mathcal{H}}$.

The metric ω_X also determines a vertical Laplacian $\Delta_{\mathcal{V}}$, defined by

$$(\Delta_{\mathcal{V}} f)|_{X_b} := \Delta_b(f|_{X_b})$$

where Δ_b is the Kähler Laplacian on the fibre (X_b, ω_b).

Lastly, as ω_X is relatively Kähler it determines a hermitian metric on the vertical tangent bundle $\mathcal{V}$. This in turn determines a hermitian metric on the determinant bundle $\Lambda^m \mathcal{V} \cong -K_{X/B}$, where $m = \dim X - \dim B$. We denote the curvature of this metric by ρ, and write $\rho_{\mathcal{H}}$ for its horizontal component.

Definition 2.4.7 ([DS21b])**.** A relatively cscK metric ω_X is an *optimal symplectic connection* if it satisfies the equation

$$p(\Delta_{\mathcal{V}} \Lambda_{\omega_B} \mu^* F_{\mathcal{H}} + \Lambda_{\omega_B} \rho_{\mathcal{H}}) = 0, \tag{2.4}$$

where p denotes the L^2-orthogonal projection to $C^\infty(E) \subset C^\infty(X)$ with respect to the inner product (2.2), $F_{\mathcal{H}}$ is the symplectic curvature of the metric ω_X, and ρ is the curvature of the hermitian metric on $\Lambda^m \mathcal{V}$ determined by ω_X.

We will often denote by θ the smooth function $\Delta_{\mathcal{V}} \Lambda_{\omega_B} \mu^* F_{\mathcal{H}} + \Lambda_{\omega_B} \rho_{\mathcal{H}}$ appearing in the optimal symplectic connection equation. Hence the optimal symplectic connection equation may be written $p(\theta) = 0$.

One important motivation for this equation is that it is the obstruction to finding a cscK metric on the total space of the fibration in adiabatic classes $[\omega_X + k\omega_B]$ for $k \gg 0$. Let α be the Weil–Petersson metric of the fibration $X \to B$, defined by

$$\alpha := -\int_{X/B} \rho \wedge \omega_X^m + \frac{S(\omega_b)}{m+1} \int_{X/B} \omega_X^{m+1}.$$

This is a real closed semipositive $(1,1)$-form on B, measuring the variation in the complex structures of the fibres; see [FS90]. So one may consider the *twisted cscK equation*

$$S(\omega_B) = \Lambda_{\omega_B} \alpha + C,$$

where C is the appropriate topological constant. The term "twisted cscK" was coined by Stoppa [Sto09b], although the equation first appeared in earlier work of Fine [Fin04, Fin07].

Theorem 2.4.8 (Dervan–Sektnan [DS21b])**.** *Suppose B and X have discrete automorphism groups. If B admits an α-twisted cscK metric ω_B and X admits an optimal symplectic connection ω_X, then X admits cscK metrics in the classes $[\omega_X + k\omega_B]$ for all k sufficiently large.*

The first theorem of this kind was due to Hong [Hon99], who proved this result in the case where X is a projectivised vector bundle. Hong's work provided the first nontrivial general construction of cscK metrics that are not Kähler–Einstein, in higher dimensions. Dervan–Sektnan's theorem also generalises the work of Fine [Fin04, Fin07], who considered the case where X is a complex surface fibred over a curve.

To prove the theorem, Dervan–Sektnan derive an expansion of the scalar curvature:

$$S(\omega_X + k\omega_B) = S(\omega_b) + k^{-1}(S(\omega_B) + \theta) + \mathrm{O}(k^{-2}), \tag{2.5}$$

which holds as $k \to \infty$. The leading term $S(\omega_b)$ is the scalar curvature of any fibre X_b, which is a topological constant independent of the choice of b. In order to solve the cscK equation, they attempt to make the subleading order term constant, and then deform to a cscK metric for large k using an inverse function theorem argument. They show that there exists a smooth function $\psi_R \in C^\infty_R(X)$ such that the expansion becomes

$$S(\omega_X + k\omega_B + k^{-1}i\partial\bar{\partial}\psi_R) = S(\omega_b) + k^{-1}(S(\omega_B) - \Lambda_{\omega_B}\alpha + p(\theta)) + \mathrm{O}(k^{-2}),$$

the difference being that the $C^\infty_R(X)$ component of the subleading term has now vanished, and we're left with the $C^\infty(B)$-term $S(\omega_B) - \Lambda_{\omega_B}\alpha$ and the $C^\infty(E)$-term $p(\theta)$. The hypotheses of the theorem are then that the former is constant, and the latter vanishes, so that the scalar curvature in the adiabatic limit is indeed constant to $\mathrm{O}(k^{-2})$. Applying a quantitative version of the inverse function theorem, Dervan–Sektnan prove the theorem.

Example 2.4.9.

1. Let $V \to B$ be a holomorphic vector bundle, and $X := \mathbb{P}(V)$ be its projectivisation. Then a relatively cscK metric on X is a fibrewise Fubini–Study metric, which is induced by a hermitian metric on V. Then a relatively cscK metric is an optimal symplectic connection if and only if the corresponding hermitian metric on V is a Hermite–Einstein metric [DS21b, Proposition 3.17].

2. If all fibres of the fibration are Fano and the class of ω_X is $c_1(-K_{X/B})$, then the optimal symplectic connection equation simplifies to $p(\Lambda_{\omega_B}\mu^* F_{\mathcal{H}}) = 0$ [DS21c, Proposition 2.7].

Remark 2.4.10. In recent work, Ortu [Ort22] has extended the definition of optimal symplectic connection to fibrations whose fibres are not necessarily cscK, but may be strictly K-semistable. This should be an important step towards forming moduli of optimal symplectic connections, as we discuss in the next section.

Definition 2.4.11. An *automorphism* of $\pi : X \to B$ is a biholomorphism $g : X \to X$ satisfying $\pi \circ g = \pi$. We let $\mathrm{Aut}(\pi)$ be the complex Lie group of automorphisms of π, and let $\mathrm{Aut}_0(\pi)$ be its identity component.

Remark 2.4.12. One can easily show that if ω_X is an optimal symplectic connection then so is $\omega_X + \pi^* i\partial\bar{\partial} f$ for any $f \in C^\infty(B)$. Similarly, if $g \in \mathrm{Aut}(\pi)$ then $g^*\omega_X$ is also an optimal symplectic connection. If $g \in \mathrm{Aut}_0(\pi)$ is in the identity component, then $g^*\omega_X$ will be in the same relatively Kähler class as ω_X.

Dervan and Sektnan have proven various results which show these metrics are indeed canonical, including the following description of the automorphism group $\mathrm{Aut}_0(\pi)$ of $\pi : X \to B$ analogous to the Matsushima–Lichnerowicz theorem:

Theorem 2.4.13 (Dervan–Sektnan [DS21c, Theorem 4.3 and Corollary 4.4])**.** *Let $\mathfrak{isom}(\pi, \omega_X)$ be the Lie algebra of the group of automorphisms of π which leave ω_X invariant. Let $\mathfrak{aut}(\pi)$ be the Lie algebra of $\mathrm{Aut}_0(\pi)$. If ω_X is an optimal symplectic connection, there are decompositions*

$$\begin{aligned} \mathfrak{isom}(\omega_X) &= \mathfrak{a} \oplus \mathfrak{k}, \\ \mathfrak{aut}(\pi) &= \mathfrak{a} \oplus \mathfrak{k} \oplus J\mathfrak{k}. \end{aligned}$$

Here $\mathfrak{a}$ is the Lie algebra of harmonic vector fields, and $\mathfrak{k}$ is the compact Lie algebra of vertical Killing fields for ω_X.

We are especially interested in the following uniqueness result:

Theorem 2.4.14 ([DS21c])**.** *If ω_X and ω'_X are two optimal symplectic connections in the same relatively Kähler class, then there exist $g \in \mathrm{Aut}_0(\pi)$ and $f \in C^\infty(B)$ such that*

$$\omega'_X = g^*\omega_X + \pi^* i\partial\bar{\partial} f.$$

Dervan and Sektnan achieve this result by deforming to already known results for twisted cscK metrics. Using the method of geodesics, we will be able to provide a much simplified proof of this result, analogous to the proofs of Donaldson for uniqueness of Hermite–Einstein metrics [Don85] and of Berman–Berndtsson for uniqueness of cscK metrics [BB17].

The proof will require an analogue of the Mabuchi functional for the setting of fibrations, which is given by the following:

Proposition 2.4.15 ([DS21c, Lemma 4.28]). *There exists a well-defined functional $\mathcal{N} : \mathcal{K}_E \to \mathbb{R}$ whose derivative along any smooth path $\varphi : [0,1] \to \mathcal{K}_E$ is given by*

$$\frac{d\mathcal{N}(\varphi_t)}{dt} = -\int_X \dot{\varphi}_t p_t(\theta_t)\, \omega_t^m \wedge \omega_B^n.$$

Here $p_t(\theta_t)$ is the left-hand side of equation (2.4) for $\omega_t := \omega_X + i\partial\bar{\partial}\varphi_t$.

The critical points of $\mathcal{N}$ correspond to optimal symplectic connections in the class $[\omega_X]$. The minus sign in the definition ensures that the functional will be convex along smooth geodesics, rather than concave.

Our geodesic techniques will also allow us to generalise the following boundedness result of Dervan–Sektnan to arbitrary elements φ of $\mathcal{K}_E$:

Theorem 2.4.16 ([DS21c, Theorem 4.22]). *Suppose there exists an optimal symplectic connection $\omega_X + i\partial\bar{\partial}\psi$ in the class $[\omega_X]$. For any $\varphi \in \mathcal{K}_E$ that is invariant under the isometry group of $\omega_X + i\partial\bar{\partial}\psi$,*

$$\mathcal{N}(\varphi) \geq \mathcal{N}(\psi).$$

2.5 Stability of fibrations

In this section we review the theory of stability of fibrations, due to Dervan–Sektnan [DS21a]. Their definition of stability was created with optimal symplectic connections in mind, and indeed they conjecture that existence of an optimal symplectic connection on a fibration is equivalent to polystability of the fibration [DS21a, Conjecture 1.3]. Furthermore, it is expected that this definition of stability should be important for forming moduli of fibrations [DS21a, Conjecture 1.6].

In this section, we consider a flat projective morphism $\pi : X \to B$ of complex projective varieties. We let L be an ample line bundle on B and H be a relatively ample line bundle with respect to π. In the previous section we assumed that each fibre X_b admitted a cscK metric. In this section, we shall assume that the general fibre (X_b, H_b) is K-polystable, where $H_b := H|_{X_b}$.

Definition 2.5.1 (Dervan–Sektnan [DS21a]). Let $\pi : (X, H) \to (B, L)$ be a fibration. A *fibration degeneration* $(\mathcal{X}, \mathcal{H}) \to (B, L)$ of $(X, H) \to (B, L)$ consists of:

(i) a variety $\mathcal{X}$ together with a projective morphism $\mathcal{X} \to B \times \mathbb{C}$ that has connected fibres, is flat over $\mathbb{C}$, and is $\mathbb{C}^*$-equivariant over $\mathbb{C}$,

(ii) a $\mathbb{C}^*$-equivariant line bundle $\mathcal{H} \to \mathcal{X}$ that is relatively ample over $B \times \mathbb{C}$,

(iii) A fixed isomorphism $(\mathcal{X}_1, \mathcal{H}_1) \cong (X, H^r)$ as fibrations over B, for some $r > 0$ called the *exponent* of the fibration degeneration.

Example 2.5.2. We can produce many fibration degenerations as follows. Since the map $\pi : X \to B$ is flat and H is relatively ample, for all $k \gg 0$ the sheaf push-forward $\pi_* H^k$ is a vector bundle on B whose fibre over $b \in B$ is naturally isomorphic to $H^0(X_b, H_b^k)$. Since H is relatively ample, there is then a natural embedding of X into the projectivised bundle $\mathbb{P}((\pi_* H^k)^*) \to B$.

For fixed $k \gg 0$, write $V := (\pi_* H^k)^*$ so that $X \subset \mathbb{P}(V)$. Let $W \subset V$ be a saturated coherent subsheaf.[1] This gives rise in a well-known way to a degeneration of V into the splitting $W \oplus (V/W)$. That is, there is a sheaf $\mathcal{V}$ on $B \times \mathbb{C}$, flat and $\mathbb{C}^*$-equivariant over $\mathbb{C}$, whose general fibre is V and whose fibre over $0 \in \mathbb{C}$ is $W \oplus (V/W)$; see, e.g. [HK18, Lemma 1.10]. Note that $\mathcal{V}$ is not necessarily flat over B, but is flat over $\mathbb{C}$.

Since $X \subset \mathbb{P}(V) = \mathbb{P}(\mathcal{V}|_1)$, we can take the closure $\overline{\mathbb{C}^* X} \subset \mathbb{P}(\mathcal{V})$ to obtain a scheme $\mathcal{X} \subset \mathbb{P}(\mathcal{V})$ which is the total space of a fibration degeneration for X. The line bundle $\mathcal{H}$ is simply given by restricting the $\mathcal{O}(1)$-line bundle on $\mathbb{P}(\mathcal{V})$ to $\mathcal{X}$.

We will often denote fibration degenerations simply by $(\mathcal{X}, \mathcal{H})$. As for test configurations, there are certain special classes of fibration degenerations. If there is a $\mathbb{C}^*$-equivariant isomorphism $\mathcal{X} \cong X \times \mathbb{C}$ over $B \times \mathbb{C}$ with $X \times \mathbb{C}$ having the trivial $\mathbb{C}^*$-action, we call the fibration degeneration *trivial*. If $\mathcal{X} \cong X \times \mathbb{C}$ with the $\mathbb{C}^*$-action induced from a one-parameter subgroup of $\mathrm{Aut}_0(\pi)$, we call $\mathcal{X}$ a *product fibration degeneration*.

Note that for every $j \gg 0$, we have a test configuration $(\mathcal{X}, jL + \mathcal{H})$ for $(X, jL + H^k)$; here we have used additive notation for tensor products of line bundles, and omitted pullbacks. The Donaldson–Futaki invariant then admits an expansion

$$\mathrm{DF}(\mathcal{X}, jL + \mathcal{H}) = j^n W_0(\mathcal{X}, \mathcal{H}) + j^{n-1} W_1(\mathcal{X}, \mathcal{H}) + \mathrm{O}(j^{n-2}), \tag{2.6}$$

which is most easily observed from the intersection theory formula for the Donaldson–Futaki invariant.

Definition 2.5.3 (Dervan–Sektnan [DS21a])**.** The fibration $\pi : (X, H) \to (B, L)$ is:

(i) *semistable* if $W_0(\mathcal{X}, \mathcal{H}) \geq 0$ for all fibration degenerations and $W_1(\mathcal{X}, \mathcal{H}) \geq 0$ whenever $W_0(\mathcal{X}, \mathcal{H}) = 0$,

[1] Recall a subsheaf $W \subset V$ is *saturated* if the quotient sheaf V/W is torsion-free.

(ii) *polystable* if it is semistable and whenever $W_0(\mathcal{X}, \mathcal{H}) = W_1(\mathcal{X}, \mathcal{H}) = 0$, there exists an open subset $U \subset B$ whose complement has codimension at least 2, such that $(\mathcal{X}, \mathcal{H})|_U$ normalises to a product fibration degeneration over U,

(iii) *stable* if it is semistable and whenever $W_0(\mathcal{X}, \mathcal{H}) = W_1(\mathcal{X}, \mathcal{H}) = 0$, there exists an open subset $U \subset B$ whose complement has codimension at least 2, such that degeneration $(\mathcal{X}, \mathcal{H})|_U$ is trivial,

(iv) *unstable* if it is not semistable.

Remark 2.5.4. The definitions of stability and polystability here are slightly different to those in [DS21a]. The reason for this is to exclude certain counterexamples due to M. Hattori from preventing any fibration from being polystable. Hattori's example is to take a non-trivial product fibration degeneration of X and to blow up a single fixed point in the central fibre $\mathcal{X}_0$. If B has dimension at least 2, this will leave the invariants W_0 and W_1 of the degeneration, and the minimum norm considered by Dervan–Sektnan [DS21a, Definition 2.27] unchanged. It follows that a fibration with nontrivial automorphisms can never be polystable with respect to the definition of Dervan–Sektnan. Including the codimension 2 hypotheses in the definitions circumvents this kind of counterexample, since two degenerations that are isomorphic away from codimension 2 will have the same invariants W_0 and W_1. This invariance will be made clear from the intersection theoretic formulae in Proposition 2.5.6, as they all involve intersections with either L^n or L^{n-1}. The definition is inspired by the vector and principal bundle setting, where it suffices to define stability by considering reductions of the structure group away from codimension 2; see [AB01, Definition 1.1] for example.

Given our assumption that the general fibre (X_b, H_b) is K-polystable, the condition that $W_0(\mathcal{X}, \mathcal{H}) \geq 0$ in the definition of stability is in fact superfluous:

Proposition 2.5.5 ([DS21a, Lemma 2.29])**.** *Let $(\mathcal{X}, \mathcal{H})$ be a fibration degeneration of (X, H). Then for general $b \in B$,*

$$W_0(\mathcal{X}, \mathcal{H}) = \binom{m+n}{n} L^n \cdot \mathrm{DF}(\mathcal{X}_b, \mathcal{H}_b).$$

As we have assumed the general fibre of $X \to B$ is K-polystable, it is automatic that $\mathrm{DF}(\mathcal{X}_b, \mathcal{H}_b) \geq 0$ for general b. Hence $W_0(\mathcal{X}, \mathcal{H}) \geq 0$ for any fibration degeneration. Note that if $W_0(\mathcal{X}, \mathcal{H}) = 0$ then the general fibre $\mathcal{X}_b$ of $\mathcal{X}$ normalises to a product test configuration for X_b by definition of K-polystability.

For $k \gg 0$, we define

$$\mu_k(X,H) := \frac{-K_X \cdot (kL+H)^{m+n-1}}{(kL+H)^{m+n}}.$$

There is then an expansion

$$\mu_k(X,H) = \frac{m}{m+n}A_0(X,H) + \frac{n}{m+n}A_1(X,H)k^{-1} + \mathrm{O}(k^{-2}), \tag{2.7}$$

where

$$A_0(X,H) = \frac{-K_X \cdot L^n \cdot H^{m-1}}{L^n \cdot H^m},$$
$$A_1(X,H) = \frac{(-K_X \cdot L^{n-1} \cdot H^m)(L^n \cdot H^m) - \frac{m}{m+1}(-K_X \cdot L^n \cdot H^{m-1})(L^{n-1} \cdot H^{m+1})}{(L^n \cdot H^m)^2}.$$

Just as for test configurations, we may $\mathbb{C}^*$-equivariantly trivialise $(\mathcal{X},\mathcal{H})|_{\mathbb{C}^*} \cong (X \times \mathbb{C}^*, p^*H)$, and compactify trivially over $\mathbb{P}^1$ to obtain $(\overline{\mathcal{X}},\overline{\mathcal{H}})$.

Proposition 2.5.6 ([DS21a, pp. 16–17])**.** *Let $(\mathcal{X},\mathcal{H})$ be a normal fibration degeneration for (X,H), compactified trivially over $\mathbb{P}^1$. Then*

$$W_0(\mathcal{X},\mathcal{H}) = \binom{m+n}{n}\left(\frac{m}{m+1}A_0(X,H)(L^n \cdot \overline{\mathcal{H}}^{m+1}) + (L^n \cdot \overline{\mathcal{H}}^m \cdot K_{\overline{\mathcal{X}}/B\times\mathbb{P}^1})\right)$$

and

$$W_1(\mathcal{X},\mathcal{H}) = \binom{m+n}{n-1}(C_1(\mathcal{X},\mathcal{H}) + C_2(\mathcal{X},\mathcal{H}) + C_3(\mathcal{X},\mathcal{H})),$$

where

$$C_1(\mathcal{X},\mathcal{H}) = \frac{m}{m+2}A_0(X,H)(L^{n-1} \cdot \overline{\mathcal{H}}^{m+2}),$$
$$C_2(\mathcal{X},\mathcal{H}) = A_1(X,H)(L^n \cdot \overline{\mathcal{H}}^{m+1}),$$
$$C_3(\mathcal{X},\mathcal{H}) = (L^{n-1} \cdot \overline{\mathcal{H}}^{m+1} \cdot K_{\overline{\mathcal{X}}/\mathbb{P}^1}).$$

Conjecturally, when $X \to B$ is a holomorphic surjective submersion the existence of an optimal symplectic connection in $c_1(H)$ is equivalent to polystability of $\pi : (X,H) \to (B,L)$. A partial result in this direction has been obtained by Dervan and Sektnan: if $c_1(H)$ admits an optimal symplectic connection then the fibration $(X,H) \to (B,L)$ is semistable [DS21a]. A main result in this thesis is a refinement of their theorem to polystability for a certain class of test configurations. Furthermore, their techniques rely on algebraic geometry, namely the expansion of the Bergman kernel. Since our techniques are purely analytic, the results are also completely new in the non-algebraic Kähler setting.

It is also expected this notion of stability of fibrations should be an important ingredient for forming moduli spaces of fibrations. In particular, Dervan–Sektnan [DS21a] conjecture the existence of a proper separated moduli space of polystable fibrations over a fixed base. An important step towards this on the differential geometric side is the recent work of Ortu [Ort22], who extends the notion of optimal symplectic connection to certain fibrations with strictly semistable fibres. She then conjectures the existence of a Hausdorff complex space that is a moduli space for these generalised optimal symplectic connections [Ort22, Conjecture 1.3]. The importance of the more general definition for optimal symplectic connections is that when one deforms the complex structures of the fibres, there is no guarantee that the new fibres will be K-polystable. So the more general theorem should allow for a much better deformation theory to construct moduli.

2.6 Weighted cscK metrics

In this section, we cover the weighted cscK metrics introduced by Lahdili [Lah19], and links with algebro-geometric stability. The main motivation for weighted cscK metrics is that they encompass many important examples of canonical metrics in Kähler geometry, and thus provide a very general approach to studying them. At present, little is known about these metrics in full generality. However there is much known about certain special cases, such as Kähler–Ricci solitons, and the μ-cscK metrics of Inoue [Ino19a].

Let:

- (X, ω) be a compact Kähler manifold,
- $T \subset \mathrm{Aut}_0(X)$ a compact torus in the holomorphic automorphism group of X, acting on (X, ω) by isometries,
- $\mu : X \to \mathfrak{t}^*$ a moment map for the T-action,
- $P := \mu(X) \subset \mathfrak{t}^*$ the moment polytope,
- $v, w : X \to \mathbb{R}_{>0}$ positive smooth functions.

Note $\mu(X)$ is indeed a convex polytope by a theorem of Atiyah [Ati82] and Guillemin–Sternberg [GS82]. Our convention for moment maps is the following: given an element $\xi \in \mathfrak{t}$,

$$\langle d\mu, \xi \rangle = -\omega(\xi, -),$$

where we abuse notation by conflating the element ξ of $\mathfrak{t}$ with the vector field it generates on X. Given the above data, we define the v-weighted scalar curvature of ω to be

$$S_v(\omega) := 2v(\mu)S(\omega) - 4\Delta(v(\mu)) + \mathrm{Tr}(g \circ \mathrm{Hess}(v)(\mu)).$$

Here:

- $S(\omega)$ is the scalar curvature,
- $\Delta = -\partial^*\partial$ is the Kähler Laplacian of ω,
- g is the Riemannian metric determined by ω.

It follows that the term $\mathrm{Tr}(g \circ \mathrm{Hess}(v)(\mu))$ may be written

$$\sum_{i,j} v_{,ij}(\mu) g(\xi_i, \xi_j),$$

where $\xi_1, \ldots, \xi_r$ is a basis of $\mathfrak{t}$, and $v_{,ij}$ denotes the ij-partial derivative of v with respect to the dual basis of $\mathfrak{t}^*$.

Remark 2.6.1. While the definition of $S_v(\omega)$ may seem arbitrary at first, the formula arises naturally as an infinite-dimensional moment map on the space $\mathcal{J}^T$ of T-invariant almost complex structures when one perturbs the metric on this space using the weight function v; see [Lah19, Section 4]. What makes this curvature worth studying is that it can further recover many well-known and important examples of canonical metrics in Kähler geometry—see Example 2.6.5 below.

Remark 2.6.2. In Lahdili's paper [Lah19], the v-weighted scalar curvature is written

$$S_v(\omega) = S(\omega) + 2\Delta(v(\mu)) + \mathrm{Tr}(g \circ \mathrm{Hess}(v)(\mu)).$$

The differences in signs and constants are due to Lahdili's use of the Riemannian scalar curvature and the Riemannian Laplacian, rather than their Kähler variants.

Definition 2.6.3 ([Lah19])**.** The metric ω is:

1. a (v, w)-*weighted cscK metric*, if

$$S_v(\omega) = c_{v,w} w(\mu),$$

where $c_{v,w}$ is a constant;

2. a (v, w)-*weighted extremal metric* if the function

$$S_v(\omega)/w(\mu)$$

is a holomorphy potential.

Sometimes we shorten the full name to just a (v, w)-cscK metric, or a (v, w)-extremal metric. If the weight functions v and w are understood or irrelevant, we may also refer to such a metric simply as a weighted cscK metric, or a weighted extremal metric.

Remark 2.6.4. Our definition of weighted extremal metric is slightly more general than Lahdili's. Lahdili [Lah19] requires that the function $S_v(\omega)/w(\mu)$ is of the form $w_{\text{ext}}(\mu)$, where $w_{\text{ext}} : P \to \mathbb{R}$ is an affine linear function. Here we do not require that $S_v(\omega)/w(\mu)$ can be described as such, but note that since this function is a T-invariant holomorphy potential, we can enlarge the torus to $T' \supset T$ by taking the torus generated by a basis $\xi_1, \ldots, \xi_r$ for $\mathfrak{t}$ together with the holomorphic vector field ξ determined by $S_v(\omega)/w(\mu)$. If P' is the moment polytope of T', $S_v(\omega)/w(\mu)$ may then be described as an affine linear function on this polytope. When the torus T is maximal, our two definitions therefore coincide.

Example 2.6.5. Fix an element $\xi \in \mathfrak{t}$, and denote by $\ell_\xi : \mathfrak{t}^* \to \mathbb{R}$ the corresponding element of $(\mathfrak{t}^*)^*$. Let a be a constant such that $a + \ell_\xi > 0$ on P. Many standard canonical metrics can be obtained from certain choices of the functions v, w [Lah19, Section 3]:

1. **cscK:** $v = w = 1$, gives the equation

$$S(\omega) = c,$$

which is the usual cscK equation.

2. **extremal:** $v = 1$, $w = a + \ell_\xi$, gives the equation

$$\nabla^{1,0} S(\omega) = \xi,$$

which is the extremal metric equation.

3. **Kähler–Ricci soliton:** For X Fano, $v = e^{\ell_\xi}$, $w = v \cdot (a + \ell_\xi)$, gives

$$\mathrm{Ric}(\omega) - \omega = \mathcal{L}_\xi \omega,$$

where $\mathcal{L}_\xi$ is the Lie derivative with respect to the holomorphic vector field generated by ξ. This is the Kähler–Ricci soliton equation, which has an extensive literature. One important context in which they arise is as the Gromov–Hausdorff limits of solutions to the Kähler–Ricci flow, which is a powerful theorem of Chen–Wang [CW20] and Chen–Sun–Wang [CSW18].

4. **extremal Sasaki:** $v = (a + \ell_\xi)^{-n-1}$, $w = (a + \ell_\xi)^{-n-3}$. The equation takes some set-up to describe, so we refer the reader to [AC18].

5. **conformally Kähler–Einstein Maxwell:** $v = (a + \ell_\xi)^{-2n+1}$, $w = (a + \ell_\xi)^{-2n-1}$. Again, we refer the reader to [Lah20] for more details.

6. **v-soliton:** Taking v arbitrary and defining $w(p) = 2v(p)(n + \langle d \log v(p), p\rangle)$ gives the equation
$$\mathrm{Ric}(\omega) - \omega = i\partial\bar{\partial} \log v(\mu),$$
which is the v-soliton equation. This generalises the Kähler–Ricci solitons, which are recovered by choosing $v = e^{\ell_\xi}$.

7. **μ-cscK:** In [Ino19a, Ino20], Inoue has introduced and studied a class of so called μ-cscK metrics. These are a special case of Lahdili's weighted cscK metrics, given by the same weight functions as for Kähler–Ricci solitons, only one drops the condition of X being Fano [Ino19a, Section 2.1.6]. By restricting to this class, Inoue is able to obtain a stronger notion of K-stability than for weighted cscK metrics [Ino20]; see the end of this chapter for more discussion on this.

For an element $\xi \in \mathfrak{t}$, we will write $\mu^\xi := \langle \mu, \xi\rangle = \ell_\xi \circ \mu$, where the pairing is the natural one on $\mathfrak{t}^* \otimes \mathfrak{t}$. This is then a moment map for the infinitesimal action of ξ on X. We are interested in finding a weighted cscK metric in the class $[\omega]$. Given a T-invariant Kähler potential $\varphi \in \mathcal{K}^T$, we let

$$\mu_\varphi := \mu + d^c\varphi.$$

That is, for any $\xi \in \mathfrak{t}$,

$$\mu^\xi_\varphi := \mu^\xi + d^c\varphi(\xi),$$

where we abuse notation by writing ξ for the vector field it generates on X.

Lemma 2.6.6 ([Lah19, Lemma 1])**.** *With the above definition of μ_φ, we have $\mu_\varphi(X) = P$, where P is the moment polytope for $\mu = \mu_0$.*

With this lemma in mind, it then makes sense to search for a (v, w)-cscK metric in the class $[\omega]$, which is T-invariant by fiat.

Lemma 2.6.7 ([Lah19, Lemma 2])**.** *With the above choice of moment map μ_φ, the following quantities are independent of the choice of $\varphi \in \mathcal{K}^T$:*

1. $\int_X v(\mu_\varphi)\, \omega_\varphi^n$,
2. $\int_X v(\mu_\varphi)\, \mathrm{Ric}(\omega_\varphi) \wedge \omega_\varphi^{n-1} + \int_X \langle dv(\mu_\varphi), \Delta_\varphi \mu_\varphi \rangle\, \omega_\varphi^n$,
3. $\int_X S_v(\omega_\varphi)\, \omega_\varphi^n$.

It follows that the constant $c_{v,w}$ is fixed, given by

$$c_{v,w} = \frac{\int_X S_v(\omega)\, \omega^n}{\int_X w(\mu)\, \omega^n}.$$

Remark 2.6.8. The significance of $\Delta_\varphi \mu_\varphi$ is that it is a moment map for the Ricci curvature $\mathrm{Ric}(\omega_\varphi)$, see [Lah19, Lemma 5]. That is, for any $\xi \in \mathfrak{t}$,

$$\langle d\Delta_\varphi \mu_\varphi(x), \xi \rangle = \mathrm{Ric}(\omega_\varphi)(x)(-, \xi_x).$$

Conjecturally, the existence of a weighted cscK metric is equivalent to a suitable notion of weighted K-polystability, which has not yet been determined in the literature.

Let $(\mathcal{X}, \mathcal{A})$ be a smooth T-equivariant Kähler test configuration for (X, ω). That is, $\mathcal{X} \to \mathbb{P}^1$ is a test configuration for X in the usual sense with smooth total space $\mathcal{X}$, and $\mathcal{A}$ is an S^1-invariant Kähler class on X; see [SD18] or [DR17b] for the definition of K-stability in the non-algebraic Kähler setting. By T-equivariance, we mean that $(\mathcal{X}, \mathcal{A})$ admits an action of T by biholomorphisms, preserving the projection $\mathcal{X} \to \mathbb{P}^1$.

Lemma 2.6.9. *Let m_Ω be the moment map for the T-action on $\mathcal{X}$, normalised so that $m_\Omega(X_1) = P$. Then $m_\Omega(X_\tau) = P$ for all $\tau \in \mathbb{C}$, and hence $m_\Omega(\mathcal{X}) = P$.*

Definition 2.6.10. Let $(\mathcal{X}, \mathcal{A})$ be a smooth T-equivariant Kähler cohomological test configuration for (X, ω), and let Ω be a Kähler form in the class $\mathcal{A}$. The *weighted Donaldson–Futaki invariant* of $(\mathcal{X}, \mathcal{A})$ is defined to be

$$\mathrm{DF}_{v,w}(\mathcal{X}, \mathcal{A}) := -\int_{\mathcal{X}} (S_v(\Omega) - c_{v,w} w(m_\Omega)) \Omega^{n+1} + 2 \int_{\mathcal{X}} \pi^*(\omega_{FS}) \wedge \Omega^n.$$

Definition 2.6.11. The manifold (X, ω) is:

1. (v, w)-*weighted K-semistable* with respect to smooth degenerations if for all smooth T-equivariant Kähler test configurations for (X, ω), $\mathrm{DF}_{v,w}(\mathcal{X}, \mathcal{A}) \geq 0$;

2. (v, w)-*weighted K-polystable* with respect to smooth degenerations if it is (v, w)-weighted K-semistable, and whenever $(\mathcal{X}, \mathcal{A})$ is a smooth T-equivariant Kähler test configuration for (X, ω) with $\mathrm{DF}_{v,w}(\mathcal{X}, \mathcal{A}) = 0$, the test configuration is T-equivariantly isomorphic to a product test configuration.

It is now known that the existence of a (v, w)-cscK metric implies (v, w)-weighted K-polystability with respect to smooth degenerations [AJL21]. However, it is expected that the converse should not hold in general, and that singular test configurations are required to destabilise certain manifolds which do not admit weighted cscK metrics. This definition should suffice for weighted K-semistability however, as given a test configuration one may always resolve singularities and perturb the cohomology class to a Kähler class. We note that for Inoue's μ-cscK equation, there is indeed a good notion of singular test configuration, and Inoue has proven μK-semistability of varieties admitting μ-cscK metrics [Ino20].

Chapter 3

Geodesics in the space of relatively cscK metrics

In this chapter we derive and study the geodesic equation for relatively Kähler metrics. It transpires that a smooth path in the space of relatively Kähler metrics is a geodesic if and only if its restriction to every fibre is a geodesic in the space of Kähler potentials in the fibre, in the usual sense of Mabuchi [Mab86]. The bulk of the chapter is dedicated to proving the existence of a unique smooth geodesic joining any two relatively cscK metrics. That geodesics exist fibrewise is already known from the description of the space of cscK metrics given in Chapter 2, what must be checked is that these geodesics vary smoothly with the basepoint. To prove this, we apply the inverse function theorem. Along the way, we give a description of the space of relatively cscK metrics in a fixed Kähler class as the space of sections of a fibre bundle whose fibres are non-positively curved symmetric spaces. We finish by discussing the formal structure of the space of relatively cscK metrics as an infinite-dimensional symmetric space of non-positive curvature, a result which in the Kähler setting is due independently to Donaldson [Don99], Mabuchi [Mab87] and Semmes [Sem92].

3.1 The geodesic equation

Let $X \to B$ be a holomorphic surjective submersion of compact Kähler manifolds, where B has complex dimension n and the fibres have complex dimension m. Fix a relatively Kähler metric ω_X on $\pi : X \to B$, not assumed to be relatively cscK. Denote by $\mathcal{K}$ the set of smooth functions $\varphi \in C^\infty(X)$ such that $\omega_X + i\partial\bar{\partial}\varphi$ is also relatively Kähler. Recall that if ω_X is relatively cscK, then $\mathcal{K}_E$ denotes the space of smooth functions $\varphi : X \to \mathbb{R}$ such that $\omega_X + i\partial\bar{\partial}\varphi$ is also relatively cscK, and we have $\mathcal{K}_E \subset \mathcal{K}$.

Let $\varphi : [0,1] \to \mathcal{K}$ be a smooth path. We define the *energy* of φ to be

$$\mu(\varphi) := \int_0^1 \int_X \dot{\varphi}_t^2 \, \omega_t^m \wedge \omega_B^n \, dt,$$

where $n = \dim B$, $m = \dim X - \dim B$, and $\omega_t := \omega_X + i\partial\bar{\partial}\varphi_t$. The path φ is called a *geodesic* if it is a critical point for the energy functional μ.

Proposition 3.1.1. *A smooth path $\varphi : [0,1] \to \mathcal{K}$ is a geodesic if and only if it satisfies the equation*

$$\ddot{\varphi}_t = |\partial_{\mathcal{V}} \dot{\varphi}_t|^2_{\mathcal{V},t}$$

for all $t \in [0,1]$.

Here we define $\partial_{\mathcal{V}} f$ by restricting f to a fibre X_b and then taking $\partial(f|_{X_b}) \in \Lambda^{1,0}(T^*X_b)$. The hermitian metric $|\cdot|_{\mathcal{V},t}$ on T^*X_b is that determined by $\omega_{b,t}$. Note that the above geodesic equation is simply the usual geodesic equation for Kähler metrics but fibrewise. That is, φ is a geodesic if and only if $\varphi|_{X_b}$ is a geodesic in the space of Kähler potentials for (X_b, ω_b) for all $b \in B$.

Proof. Suppose we vary φ by a path ψ with $\psi_0 = \psi_1 = 0$. Then

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \omega^m_{\varphi_t + \epsilon\psi_t} \wedge \omega_B^n = m\, i\partial\bar{\partial}\psi_t \wedge \omega_t^{m-1} \wedge \omega_B^n = (\Delta_{\mathcal{V},t}\psi_t)\, \omega_t^m \wedge \omega_B^n,$$

where $\Delta_{\mathcal{V},t}$ is the vertical Kähler Laplacian with respect to ω_t. The variation of the energy is then

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \mu(\varphi + \epsilon\psi) = \int_0^1 \int_X (2\dot{\varphi}_t\dot{\psi}_t + \dot{\varphi}_t^2(\Delta_{\mathcal{V},t}\psi_t))\, \omega_t^m \wedge \omega_B^n \, dt.$$

Splitting the integrand into two terms, for the second term we have

$$\begin{aligned}
\int_X \dot{\varphi}_t^2(\Delta_{\mathcal{V},t})\, \omega_t^m \wedge \omega_B^m &= \int_B \left(\int_{X/B} \dot{\varphi}_t^2(\Delta_{\mathcal{V},t}\psi_t)\, \omega_t^m \right) \omega_B^n \\
&= \int_B \left(\int_{X/B} \Delta_{\mathcal{V},t}(\dot{\varphi}_t^2)\psi_t\, \omega_t^m \right) \omega_B^n \\
&= \int_X \psi_t \Delta_{\mathcal{V},t}(\dot{\varphi}_t^2)\, \omega_t^m \wedge \omega_B^n.
\end{aligned}$$

For the first term, consider

$$\frac{d}{dt}(\dot{\varphi}_t\psi_t\, \omega_t^m \wedge \omega_B^n) = (\ddot{\varphi}_t\psi_t + \dot{\varphi}_t\dot{\psi}_t + \dot{\varphi}_t\psi_t(\Delta_{\mathcal{V},t}\dot{\varphi}_t))\, \omega_t^m \wedge \omega_B^n.$$

Integrating this first over X and then over $t \in [0,1]$, the left hand side vanishes since $\psi_0 = \psi_1 = 0$. Hence the first term in the variation of the energy becomes

$$\int_0^1 \int_X -2\psi_t(\ddot{\varphi}_t + \dot{\varphi}_t(\Delta_{\mathcal{V},t}\dot{\varphi}_t))\,\omega_t^m \wedge \omega_B^n\,dt.$$

Putting this all together, the variation is equal to

$$\int_0^1 \int_X \psi_t[-2\ddot{\varphi}_t - 2\dot{\varphi}_t(\Delta_{\mathcal{V},t}\dot{\varphi}_t) + \Delta_{\mathcal{V},t}(\dot{\varphi}_t^2)]\,\omega_t^m \wedge \omega_B^n\,dt.$$

Using the fact that

$$\Delta_{\mathcal{V},t}(\dot{\varphi}_t^2) = 2\dot{\varphi}_t(\Delta_{\mathcal{V},t}\dot{\varphi}_t) + 2\langle \partial_{\mathcal{V}}\dot{\varphi}_t\,,\partial_{\mathcal{V}}\dot{\varphi}_t\rangle_{\mathcal{V},t},$$

where $\partial_{\mathcal{V}}$ is the vertical ∂-operator and $\langle\,,\rangle_{\mathcal{V},t}$ is the vertical inner product given by restriction of ω_t to the fibres, the variation is equal to

$$\int_0^1 \int_X \psi_t(-2\ddot{\varphi}_t + 2|\partial_{\mathcal{V}}\dot{\varphi}_t|^2_{\mathcal{V},t})\,\omega_t^m \wedge \omega_B^n\,dt.$$

Hence the geodesic equation is

$$\ddot{\varphi}_t = |\partial_{\mathcal{V}}\dot{\varphi}_t|^2_{\mathcal{V},t}. \qquad \square$$

Remark 3.1.2. If ω_X is relatively cscK, the above proof shows that a smooth path $\varphi_t : [0,1] \to \mathcal{K}_E$ is a geodesic if and only if it is a geodesic in the ambient space $\mathcal{K}$; the same calculation holds if we restrict to variations of paths within $\mathcal{K}_E$.

3.2 Existence of geodesics

We now work towards proving the following existence result:

Theorem 3.2.1. *Let ω_X be a relatively cscK metric. For any $\varphi_0, \varphi_1 \in \mathcal{K}_E$, there exists a unique geodesic $\varphi : [0,1] \to \mathcal{K}_E$ joining φ_0 to φ_1.*

We will assume $\varphi_0 = 0$ for simplicity, and write $\omega'_X := \omega_X + i\partial\bar{\partial}\varphi_1$. For each $b \in B$, denote by E_b the space of real holomorphy potentials on (X_b, ω_b) having zero integral. Recalling Chapter 2.2, on any fibre X_b there exists a unique $u_b \in E_b$ which relates the metrics ω_b and ω'_b in the following way. Letting $v_b := \nabla_b^{1,0} u_b$ be the holomorphic vector field generated by u_b, and letting $\exp(tv_b)$ be the time t flow of v_b, we have that

$$\psi_{b,t} := \int_0^t \exp(sv_b)^* u_b\,ds \tag{3.1}$$

is a smooth geodesic from 0 to $\psi_{b,1}$ in the space $\mathcal{K}_b$ of Kähler potentials for (X_b, ω_b), and $\omega_b' = \omega_b + i\partial\bar{\partial}\psi_{b,1}$.

Let us now *assume* that the u_b vary smoothly with b, so as to define a global smooth function $u : X \to \mathbb{R}$. Then the $\psi_{b,t}$ also depend smoothly on b and t, and so define a global smooth geodesic $\psi : [0,1] \to \mathcal{K}_E$. If we let

$$\omega_{X,t} := \omega_X + i\partial\bar{\partial}\psi_t$$

then we have a smooth path of relatively cscK metrics such that $\omega_{X,0} = \omega_X$ and $\omega_{X,1}|_{X_b} = \omega_X'|_{X_b}$ for any $b \in B$. By Lemma 2.4.6 there exists a smooth function $f : B \to \mathbb{R}$ such that $\omega_X' - \omega_{X,1} = i\partial\bar{\partial}\pi^* f$. Since

$$\omega_X + i\partial\bar{\partial}\varphi_1 = \omega_X + i\partial\bar{\partial}(\psi_1 + f),$$

by adding a constant to f we may assume that $\varphi_1 = \psi_1 + f$. Putting this all together, the path $\varphi_t := \psi_t + tf$ is a smooth geodesic in $\mathcal{K}_E$ joining $\varphi_0 = 0$ to φ_1.

Thus, for the existence of smooth geodesics it only remains to show that the holomorphy potentials $u_b \in E_b$ depend smoothly upon b. To prove this, we will introduce two fibre bundles: a smooth fibre bundle $H \to B$ whose fibre H_b is the homogeneous space of cscK metrics on $(X_b, [\omega_b])$, and a fibre bundle $\mathcal{K}_0$ whose fibre $\mathcal{K}_{0,b} \subset \mathcal{K}_b$ consists of suitably normalised Kähler potentials on (X_b, ω_b). The idea will be to show that H embeds as a smooth subbundle of $\mathcal{K}_0$, so that a smooth choice of fibrewise cscK metric will then determine a smooth section of H.

Given the fixed relatively cscK metric ω_X, for each $b \in B$ write $G_b := \mathrm{Aut}_0(X_b)$ and let $K_b := \mathrm{Isom}_0(\omega_b) \subset G_b$ be the maximal compact subgroup of isometries of ω_b. By Lemma 2.2.7 we have a diffeomorphism $E_b \cong G_b/K_b$, and the homogeneous space G_b/K_b parametrises the cscK metrics on X_b in the class $[\omega_b]$. Letting H be the disjoint union of the H_b over $b \in B$, we have a set-theoretic bijection $H \cong E$. Proposition 2.4.3 tells us that E is a smooth vector bundle, so we may endow H with a smooth structure by declaring the set-theoretic bijection between H and E to be a diffeomorphism.

Let us now consider the non-relative case, so we have a compact Kähler manifold (M, η), with space of Kähler potentials $\mathcal{K}$. For any $\phi \in \mathcal{K}$, there is a splitting of the tangent space

$$T_\phi\mathcal{K} = \left\{ \psi \in C^\infty(M) : \int_M \psi\, \omega_\phi^n = 0 \right\} \oplus \mathbb{R},$$

where $\mathbb{R}$ denotes the constant functions. This splitting integrates to a decomposition

$$\mathcal{K} \cong \mathcal{K}_0 \times \mathbb{R},$$

see [Che00b, Section 2.4]. In fact, $\mathcal{K}_0 = I^{-1}(0)$, where $I : \mathcal{K} \to \mathbb{R}$ is the *Monge–Ampère energy* (also called the *Aubin–Mabuchi functional*):

$$I(\phi) := \frac{1}{n+1} \sum_{i=0}^{n} \int_M \phi \, \omega_\phi^i \wedge \omega^{n-i}.$$

Since $I(\phi + c) = I(\phi) + c$, for any $\omega' \in [\omega]$ there exists a unique $\phi \in \mathcal{K}_0$ such that $\omega' = \omega_\phi$.

Now, consider the above splitting for the fibre (X_b, ω_b): let $\mathcal{K}_b$ to be the space of Kähler potentials for (X_b, ω_b) with decomposition $\mathcal{K}_b \cong \mathcal{K}_{0,b} \times \mathbb{R}$. This decomposition depends smoothly upon b since the functionals $I_b : \mathcal{K}_b \to \mathbb{R}$ do, and so there is a smooth Fréchet bundle $\mathcal{K}_0 \to B$ whose fibre over b is $\mathcal{K}_{0,b}$. Notice that the path of Kähler potentials $\psi_{b,t}$ defined in (3.1) lies within $\mathcal{K}_{0,b}$. The following Lemma will tell us that $\psi_{b,1}$ is smooth in b.

Lemma 3.2.2. *Let ω'_X be a relatively Kähler metric, and for each $b \in B$ denote by ϕ_b the unique Kähler potential in $\mathcal{K}_{0,b}$ such that $\omega'_b = \omega_b + i\partial\bar{\partial}\phi_b$. Then the ϕ_b depend smoothly upon b, so piece together to a global smooth section $\phi : B \to \mathcal{K}_0$.*

Proof. Write $\mathcal{K}^k$ for the Banach fibre bundle on B whose fibre $\mathcal{K}^k_b$ consists of C^k-regular Kähler potentials for (X_b, ω_b). The fibrewise functionals $I_b : \mathcal{K}^k_b \to \mathbb{R}$ piece together to define a global smooth map $I : \mathcal{K}^k \to B \times \mathbb{R}$ whose derivative is everywhere surjective. By the implicit function theorem for Banach manifolds, the inverse image $I^{-1}(0)$ is a smooth Banach subbundle $\mathcal{K}^k_0 \subset \mathcal{K}^k$, consisting of the normalised C^k-regular Kähler potentials on each fibre.

Similarly, denote by $\mathcal{L}^{k-2}$ the Banach fibre bundle on B whose fibre $\mathcal{L}^{k-2}_b$ is the space of C^{k-2}-regular Kähler metrics in the class $[\omega_b]$. There is a smooth morphism of Banach fibre bundles $\mathcal{K}^k_0 \to \mathcal{L}^{k-2}$ given over $b \in B$ by

$$\psi_b \mapsto \omega_b + i\partial\bar{\partial}\psi_b.$$

The derivative of this morphism on any fibre is an isomorphism, hence by the inverse function theorem for Banach manifolds, we have an isomorphism $\mathcal{K}^k_0 \cong \mathcal{L}^{k-2}$.

For any k, the relatively cscK metric ω'_X determines a smooth section of $\mathcal{L}^{k-2}$. It follows that the corresponding section ϕ of normalised Kähler potentials is a smooth section of $\mathcal{K}^k_0$ for any k. Hence the ϕ_b depend smoothly upon b. □

Letting $\mathcal{L}$ denote the fibre bundle whose fibre $\mathcal{L}_b$ is the space of Kähler metrics in $[\omega_b]$ as in the above proof, we have the following commutative diagram of morphisms of fibre bundles over B:

$$\begin{array}{ccc} H & \xleftarrow{\sim} & E \\ \downarrow & & \downarrow \\ \mathcal{L} & \xleftarrow{\sim} & \mathcal{K}_0 \end{array}.$$

The right-most vertical arrow $E \to \mathcal{K}_0$ is simply defined by composing the other three arrows. The smooth map of fibre bundles $H \to \mathcal{L}$ is given by

$$(b, K_b g) \mapsto (b, g^* \omega_b).$$

There are right G_b-actions on H_b and $\mathcal{L}_b$, and the morphism $H_b \to \mathcal{L}_b$ is G_b-equivariant by construction. We can therefore consider the associated morphism $E_b \to \mathcal{K}_{0,b}$ as G_b-equivariant as well.

Theorem 3.2.3. *The map of fibre bundles $E \to \mathcal{K}_0$ satisfies the following universal property: given any relatively cscK metric ω'_X on X with associated smooth section $B \to \mathcal{K}_0$, there exist a unique smooth section $u : B \to E$ such that the following diagram commutes*

$$\begin{array}{ccc} E & \longrightarrow & \mathcal{K}_0 \\ & \nwarrow \quad \nearrow & \\ & B & \end{array}$$

The section $u : B \to E$ is such that $u_b \in E_b$ is the unique real holomorphy potential relating ω_b and ω'_b as in Proposition 2.2.4.

Proof. Existence and uniqueness of the map as a map of sets is clear. The only question is smoothness. For this, we will show that $E \to \mathcal{K}_0$ is an injective immersion, and that the preimage of any closed bounded set in $\mathcal{K}_0$ (with respect to any C^k-norm) is compact in E. Then smoothness will follow immediately from the commuting diagram.

To see that $E \to \mathcal{K}_0$ is an immersion, it suffices to show that for each $b \in B$ the map $E_b \to \mathcal{K}_{0,b}$ is an immersion. We thus drop the subscript b and work on a single fixed Kähler manifold (X, ω) of constant scalar curvature. Given $u \in E$, let $v_u := \nabla^{1,0} u$ be the associated holomorphic vector field. The map $E \to \mathcal{K}_0$ is then given by

$$u \mapsto \int_0^1 \exp(t v_u)^* u \, dt.$$

We will show that the derivative of this map is injective at the origin. By G-equivariance of the map $E \to \mathcal{K}_0$ and transitivity of the G-action on $E \cong G/K$,

this will imply the derivative is everywhere injective. We compute

$$\left.\frac{d}{ds}\right|_{s=0} \int_0^1 \exp(stv_\varphi)^*(s\varphi)\, dt = \int_0^1 \varphi\, dt = \varphi.$$

Hence the derivative is injective at the origin.

To show the inverse image of any closed bounded set is compact, we take a maximum $x \in X_b$ of $\varphi \in E_b \backslash \{0\}$. Then $\nabla_b^{1,0}\varphi(x) = 0$, and the flow $\exp(v_\varphi)$ fixes x. Since $\varphi(x) > 0$,

$$\int_0^t \exp(sv_\varphi)^*\varphi(x)\, ds = \int_0^t \varphi(x)\, ds = t\varphi(x) \to \infty$$

as $t \to \infty$. A compactness argument on the unit sphere bundle of E shows this divergence is uniform over compact subsets, and so the preimage of a closed bounded set must be compact. □

Proof of Theorem 3.2.1. We have shown that a geodesic exists, since the previous theorem shows the u_b depend smoothly upon b. For uniqueness, note that we already have fibrewise uniqueness from the usual theory of Kähler geodesics. Thus, two geodesics could only potentially differ by a smooth path ψ_t in $C^\infty(B)$. By the geodesic equation, this path must then satisfy $\ddot{\psi}_t = 0$. Hence $\psi_t = tf + g$ for some smooth functions $f, g : B \to \mathbb{R}$. The boundary conditions then imply $f = g = 0$. □

3.3 Negative curvature

We end this chapter with a discussion of the structure of the space $\mathcal{K}_E$. First, the following decomposition is clear from the above:

Proposition 3.3.1. *There is a Riemannian decomposition*

$$\mathcal{K}_E \cong C^\infty(H) \times C^\infty(B),$$

where $H \to B$ is the fibre bundle whose fibre is G_b/K_b and $C^\infty(H)$ denotes the smooth sections of H. This decomposition integrates the infinitesimal decomposition of Proposition 2.4.5.

This is the relative version of the decomposition $\mathcal{K} \cong \mathcal{K}_0 \times \mathbb{R}$ of the space of Kähler potentials described above, and thus the smooth functions on B play the role of the constant functions in this setting.

One can go further with this; the space of Kähler potentials $\mathcal{K}$ on a compact Kähler manifold (M, η) has a natural connection defined as follows. Let $\phi : [0, 1] \to \mathcal{K}$ be a

smooth path, and let $\psi : [0,1] \to C^\infty(M)$ be a tangent vector along ϕ. The covariant derivative of ψ along ϕ is then given by

$$\nabla_{\dot{\phi}_t}\psi = \dot{\psi}_t - \langle \partial\psi_t, \partial\dot{\phi}_t\rangle_t.$$

Taking $\psi_t = \dot{\phi}_t$, one recovers the geodesic equation. The curvature of this connection at $\phi \in \mathcal{K}$ is

$$R_\phi(\psi,\eta)\xi = -\frac{1}{4}\{\{\psi,\eta\}_\phi,\xi\}_\phi,$$

where $\psi, \eta, \xi \in T_\phi\mathcal{H} \cong C^\infty(X)$ and $\{\,,\}_\phi$ is the Poisson bracket of the symplectic form $\omega_\phi = \omega + i\partial\bar{\partial}\phi$; [Mab87, Sem92, Don99]. The sectional curvature is therefore

$$K_\phi(\psi,\eta) = -\frac{1}{4}\|\{\psi,\eta\}_\phi\|^2_\phi,$$

so $\mathcal{K}$ is non-positively curved. Note the slices $\{\phi\} \times \mathbb{R} \subset \mathcal{K}_0 \times \mathbb{R} \cong \mathcal{K}$ are all flat geodesic submanifolds.

The exact same calculations in the relative setting (compare to [Bło13, Section 4]) show the following:

Proposition 3.3.2. *There is a connection on $\mathcal{K}_E$ given by*

$$\nabla_{\dot{\varphi}_t}\psi = \dot{\psi}_t - \langle \partial_\mathcal{V}\psi_t, \partial_\mathcal{V}\dot{\varphi}_t\rangle_{\mathcal{V},t},$$

where $\varphi : [0,1] \to \mathcal{K}_E$ and $\psi_t \in T_{\varphi_t}\mathcal{K}_E$ is a tangent vector along φ_t. The curvature of this connection at $\varphi \in \mathcal{K}_E$ is

$$R_\varphi(\psi,\eta)\xi = -\frac{1}{4}\{\{\psi,\eta\}_{\mathcal{V},\varphi},\xi\}_{\mathcal{V},\varphi},$$

where $\psi, \eta, \xi \in T_\varphi\mathcal{K}_E \subset C^\infty(X)$ and $\{\,,\}_{\mathcal{V},\varphi}$ is the fibrewise Poisson bracket corresponding to the relatively symplectic form $\omega_{X,\varphi} = \omega_X + i\partial\bar{\partial}\varphi$. The sectional curvature is

$$K_\varphi(\psi,\eta) = -\frac{1}{4}\|\{\psi,\eta\}_{\mathcal{V},\varphi}\|^2_\varphi,$$

so $\mathcal{K}_E$ is non-positively curved. The subspace $C^\infty(B) \subset \mathcal{K}_E$ is a flat geodesic submanifold.

Note that the fibres H_b have their own curvatures as Riemannian symmetric spaces: if $u, v, w \in T_{K_b}H_b = i \cdot \mathfrak{h}_b$, then

$$R_{K_b}(u,v)w = -\frac{1}{4}[[u,v],w],$$

where $[\,,]$ is the Lie bracket of the Lie algebra of G_b. Since the isomorphism $E_b \cong T_{K_b}H_b$ is essentially the hamiltonian vector field construction, this corresponds precisely to the Poisson bracket of functions in E_b. So the natural curvatures of the spaces H_b gives rise to the curvature on $C^\infty(H)$ of Proposition 3.3.2.

Chapter 4

The log-norm functional for fibrations

In this chapter we introduce the analogue of the Mabuchi, or log-norm, functional $\mathcal{N} : \mathcal{K}_E \to \mathbb{R}$ for relatively cscK potentials on fibrations. In the Kähler setting, the Mabuchi functional is a key tool used in the proofs of many important results, including uniqueness of cscK metrics [BB17], and K-polystability of manifolds admitting cscK metrics [BDL20]. This is our main motivation for introducing the functional $\mathcal{N}$ – to extend these results to the setting of fibrations. In proving uniqueness of cscK metrics and K-polystability of cscK manifolds, a key tool is convexity of the Mabuchi functional along weak geodesics in the space of Kähler metrics. In our setting we are fortunate to not have to consider weak geodesics – our first result is that the functional $\mathcal{N}$ is convex along smooth geodesics in $\mathcal{K}_E$. This convexity is then applied to give a new streamlined proof of uniqueness of optimal symplectic connections, originally due to Dervan and Sektnan [DS21c]. Polystability of fibrations will be considered in the next chapter. With a view towards this, we finish this chapter by deriving a Chen–Tian style formula for the functional $\mathcal{N}$, by computing the asymptotic expansion of the Chen–Tian formula for the usual Mabuchi functional in the adiabatic limit.

4.1 Convexity of $\mathcal{N}$ along geodesics

Recall the log-norm functional for fibrations is defined through its first derivative along smooth paths φ_t in $\mathcal{K}_E$:

$$\frac{d}{dt}\mathcal{N}(\varphi_t) = -\int_X \dot{\varphi}_t \, p_t(\theta_t) \, \omega_t^m \wedge \omega_B^n,$$

where $p_t(\theta_t)$ is the optimal symplectic connection operator applied to the relatively cscK metric ω_t, as defined in equation (2.4).

Theorem 4.1.1 (Convexity)**.** *The functional $\mathcal{N} : \mathcal{K}_E \to \mathbb{R}$ is convex along smooth geodesics in $\mathcal{K}_E$. If φ_t is a smooth geodesic in $\mathcal{K}_E$, the function $t \mapsto \mathcal{N}(\varphi_t)$ is affine linear if and only if there exist $u \in C^\infty(E)$ and $f \in C^\infty(B)$ such that $v := \nabla^{1,0}_{\mathcal{V}} u$ is a holomorphic vector field, and*

$$\varphi_t = \psi_t + tf,$$

where ψ_t is defined as in (3.1)*. Otherwise, $\mathcal{N}(\varphi_t)$ is strictly convex.*

Before proving this, we must first linearise the optimal symplectic connection operator $\varphi \mapsto p_\varphi(\theta_\varphi)$. This is achieved through the following expansion:

Lemma 4.1.2 ([DS21a, Proposition 2.4])**.** *Let ω_X be relatively cscK, and for $k \gg 0$ write $\omega_k := \omega_X + k\omega_B$, which is a Kähler metric on X. Then there exists a smooth function $\psi_R \in C^\infty_R(X)$ such that there is a C^∞-expansion*

$$S(\omega_k + k^{-1} i\partial\bar\partial \psi_R) = S(\omega_b) + k^{-1}(S(\omega_B) - \Lambda_{\omega_B}\alpha + p(\theta)) + \mathrm{O}(k^{-2}),$$

where $S(\omega_b)$ is the smooth function on X whose restriction to the fibre X_b is the scalar curvature of ω_b, α is the Weil–Petersson metric of the fibration, and $\theta := \Delta_{\mathcal{V}} \Lambda_{\omega_B} \mu^* F_{\mathcal{H}} + \Lambda_{\omega_B} \rho_{\mathcal{H}}$.

Notice that since ω_X is relatively cscK, the leading term $S(\omega_b)$ is a topological constant independent of b.

For any $\varphi \in \mathcal{K}_E$, define the operator $\mathcal{R}_\varphi : C^\infty(X) \to \Omega^{0,1}(\Lambda^{1,0}\mathcal{V})$ by

$$\mathcal{R}_\varphi \psi := \bar\partial \nabla^{1,0}_{\mathcal{V},\varphi} \psi,$$

where $\nabla_{\mathcal{V},\varphi}$ is the vertical fibrewise gradient operator of $\omega_{X,\varphi}$. When $\psi \in C^\infty(B) \oplus C^\infty(E_\varphi) = T_\varphi \mathcal{K}_E$ is a fibrewise holomorphy potential, $\nabla^{1,0}_{\mathcal{V},\varphi}\psi$ is a fibrewise holomorphic vector field on X that is globally holomorphic if and only if $\mathcal{R}_\varphi \psi = 0$.

Proposition 4.1.3. *The linearisation of the operator $\varphi \mapsto p(\theta)$ at $\varphi \in \mathcal{K}_E$ in the direction $\psi \in C^\infty(B) \oplus C^\infty(E_\varphi) = T_\varphi \mathcal{K}_E$ is given by*

$$-\mathcal{R}^*_\varphi \mathcal{R}_\varphi \psi + \langle \partial_{\mathcal{V}} p_\varphi(\theta_\varphi), \partial_{\mathcal{V}} \psi \rangle_{\mathcal{V},\varphi}.$$

Proof. For ease of notation, we fix $\omega_{X,\varphi}$ as our new reference metric ω_X, so that $\varphi = 0$. Consider the expansion

$$S(\omega_k + k^{-1} i\partial\bar\partial \psi_R) = S(\omega_b) + k^{-1}(S(\omega_B) - \Lambda_{\omega_B}\alpha + p(\theta)) + \mathrm{O}(k^{-2}), \tag{4.1}$$

and write $\omega_k' := \omega_k + k^{-1} i\partial\bar{\partial}\psi_R$. Suppose we vary ω_X by $i\partial\bar{\partial}\psi$, where $\psi \in C^\infty(B) \oplus C^\infty(E)$. Then the linearisation of the left hand side of equation (4.1) is

$$-(\mathcal{D}_k')^*\mathcal{D}_k'\psi + \langle \partial S(\omega_k'), \partial\psi\rangle_{\omega_k'}$$

where $(\mathcal{D}_k')^*\mathcal{D}_k'$ is the Lichnerowicz operator of ω_k'; see [Szé14, Lemma 4.4].

Consider the term $\langle \partial S(\omega_k'), \partial\psi\rangle_{\omega_k'}$. A simple calculation in local coordinates shows that the metrics on T^*X satisfy $\langle\, ,\rangle_{\omega_k'} = \langle\, ,\rangle_{\omega_k} + \mathrm{O}(k^{-1})$ in the limit, and since $\partial S(\omega_k')$ is also of order k^{-1} by (4.1), we may instead consider $\langle \partial S(\omega_k'), \partial\psi\rangle_{\omega_k}$. Splitting this into its horizontal and vertical components with respect to ω_k gives

$$\langle \partial_{\mathcal{H}} S(\omega_k'), \partial_{\mathcal{H}}\psi\rangle_{\omega_k} + \langle \partial_{\mathcal{V}} S(\omega_k'), \partial_{\mathcal{V}}\psi\rangle_{\omega_k}.$$

The horizontal term is uniformly of order k^{-2}; the leading coefficient of $S(\omega_k)$ is constant, so $\partial_{\mathcal{H}} S(\omega_k)$ is order k^{-1}, and another k^{-1} factor comes from the inverse of ω_k in the horizontal direction, as may again be seen in local coordinates. From the expansion of $S(\omega_k')$, the order k^{-1} part of the vertical term is given by

$$\langle \partial_{\mathcal{V}} p(\theta), \partial_{\mathcal{V}}\psi\rangle_{\mathcal{V}}.$$

The order k^{-1} term of $(\mathcal{D}_k')^*\mathcal{D}_k'\psi$ is given by $\mathcal{R}^*\mathcal{R}\psi$; see [DS21b, Section 4.4]. The linearisation of the optimal symplectic connection operator in the direction ψ is then

$$-\mathcal{R}^*\mathcal{R}\psi + \langle \partial_{\mathcal{V}} p(\theta), \partial_{\mathcal{V}}\psi\rangle_{\mathcal{V}}. \qquad \square$$

Proof of Theorem 4.1.1. We compute

$$-\frac{d}{dt}\int_X \dot{\varphi}_t\, p_t(\theta_t)\, \omega_t^m \wedge \omega_B^n = -\int_X [\ddot{\varphi}_t\, p_t(\theta_t) + \dot{\varphi}_t(-\mathcal{R}_t^*\mathcal{R}_t\dot{\varphi}_t + \langle \partial_{\mathcal{V}} p_t(\theta_t), \partial_{\mathcal{V}}\dot{\varphi}_t\rangle_{\mathcal{V},t}) + \dot{\varphi}_t p_t(\theta_t)\Delta_{\mathcal{V},t}\dot{\varphi}_t]\, \omega_t^m \wedge \omega_B^n.$$

Breaking this up, we have

$$\begin{aligned}\int_X \dot{\varphi}_t \langle \partial_{\mathcal{V}} p_t(\theta_t), \partial_{\mathcal{V}}\dot{\varphi}_t\rangle_{\mathcal{V},t}\, \omega_t^m \wedge \omega_B^n &= \int_B \left(\int_{X/B} \frac{1}{2}\langle \partial_{\mathcal{V}} p_t(\theta_t), \partial_{\mathcal{V}}(\dot{\varphi}_t^2)\rangle_{\mathcal{V},t}\, \omega_t^m\right)\omega_B^n \\ &= \int_B \left(\int_{X/B} -\frac{1}{2} p_t(\theta_t)\Delta_{\mathcal{V},t}(\dot{\varphi}_t^2)\, \omega_t^m\right)\omega_B^n \\ &= \int_X -\frac{1}{2} p_t(\theta_t)\Delta_{\mathcal{V},t}(\dot{\varphi}_t^2)\, \omega_t^m \wedge \omega_B^n.\end{aligned}$$

Also,

$$\int_X \dot{\varphi}_t p_t(\theta_t)\Delta_{\mathcal{V},t}\dot{\varphi}_t\, \omega_t^m \wedge \omega_B^n = \int_X p_t(\theta_t)\left(\frac{1}{2}\Delta_{\mathcal{V},t}(\dot{\varphi}_t^2) - \langle \partial_{\mathcal{V}}\dot{\varphi}_t, \partial_{\mathcal{V}}\dot{\varphi}_t\rangle_{\mathcal{V},t}\right)\omega_t^m \wedge \omega_B^n.$$

Hence the second variation is equal to

$$\int_X |\mathcal{R}_t\dot{\varphi}_t|^2\omega_t^m \wedge \omega_B^n - \int_X (\ddot{\varphi}_t - |\partial_{\mathcal{V}}\dot{\varphi}_t|^2_{\mathcal{V},t})\, p_t(\theta_t)\, \omega_t^m \wedge \omega_B^n.$$

If φ_t satisfies the geodesic equation then this is non-negative, and the functional is convex. It is strictly convex unless $R_t\dot{\varphi}_t = 0$, which holds if and only if the fibrewise-holomorphic vector field $\nabla_t^{1,0}\dot{\varphi}_t$ is globally holomorphic on X. □

4.2 Applications of convexity to uniqueness and boundedness

Theorem 4.2.1 (Uniqueness [DS21c]). *If ω_X and ω'_X are two optimal symplectic connections in the same relatively Kähler class, then there exist $g \in \mathrm{Aut}_0(\pi)$ and $f \in C^\infty(B)$ such that*

$$\omega'_X = g^*\omega_X + i\partial\bar{\partial}\pi^* f.$$

Proof. Suppose $\omega'_X = \omega_X + i\partial\bar{\partial}\varphi_1$ and let $\varphi : [0,1] \to \mathcal{K}_E$ be the unique geodesic joining $\varphi_0 = 0$ to φ_1. Let

$$\varphi_t = \psi_t + tf$$

be the decomposition so that $\dot{\psi}_t \in C^\infty(E_t)$ for all t and $f \in C^\infty(B)$. Writing $v := \nabla_{\mathcal{V}}^{1,0}\dot{\psi}_0$, we have

$$\psi_t = \int_0^t \exp(sv)^*\dot{\psi}_0\, ds,$$

and $\omega_{X,t} = \exp(sv)^*\omega_X$ where $\omega_{X,t} := \omega_X + i\partial\bar{\partial}\psi_t$. Since ω_X and $\omega_{X,1}$ are optimal symplectic connections,

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{N}(\psi_t) = \left.\frac{d}{dt}\right|_{t=1} \mathcal{N}(\psi_t) = 0.$$

By convexity along geodesics, this implies $\mathcal{N}(\psi_t) = 0$ for all t. Hence $\mathcal{R}_t\dot{\psi}_t = 0$ for all t, and in particular $\mathcal{R}_0\dot{\psi}_0 = 0$ so that v is globally holomorphic. Finally, we get $\omega'_X = \exp(v)^*\omega_X + i\partial\bar{\partial}\pi^* f$. □

The following result generalises the boundedness result 2.4.16 of Dervan–Sektnan [DS21c].

Theorem 4.2.2 (Boundedness). *If $X \to B$ admits an optimal symplectic connection ω'_X, then the functional $\mathcal{N} : \mathcal{K}_E \to \mathbb{R}$ is bounded below (with respect to any choice of reference metric ω_X).*

Proof. Let $\varphi : [0,1] \to \mathcal{K}_E$ be a geodesic with $\omega'_X = \omega_X + i\partial\bar{\partial}\varphi_0$. Then

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{N}(\varphi_t) = 0,$$

and by convexity of $\mathcal{N}$ along φ we have

$$\mathcal{N}(\varphi_1) \geq \mathcal{N}(\varphi_0).$$

By geodesic connectedness of $\mathcal{K}_E$, we have $\mathcal{N}(\varphi) \geq \mathcal{N}(\varphi_0)$ for all $\varphi \in \mathcal{K}_E$. □

4.3 A Chen–Tian style formula

Recall that for a compact Kähler manifold (X, ω), the Mabuchi functional is defined on the space of Kähler potentials $\mathcal{K}$ with respect to ω, and its derivative along a smooth path φ_t in $\mathcal{K}$ is

$$\frac{d}{dt}\mathcal{M}(\varphi_t) := \int_X \dot{\varphi}_t(\hat{S} - S(\omega_\varphi))\,\omega_\varphi^n.$$

The Mabuchi functional has an explicit formula, which may be derived by taking a straight line path $\varphi_t := t\varphi$ from the origin to φ:

Proposition 4.3.1 (Chen–Tian formula, [Che00a, Tia00])**.** *For any $\varphi \in \mathcal{K}$,*

$$\begin{aligned}\mathcal{M}(\varphi) = \int_X \log\left(\frac{\omega_\varphi^n}{\omega^n}\right)\omega_\varphi^n + \sum_{i=0}^{n-1}\int_X \varphi\,\mathrm{Ric}(\omega)\wedge\omega^i\wedge\omega_\varphi^{n-i} \\ + \frac{n}{n+1}\mu(X)\sum_{i=0}^{n}\int_X \varphi\,\omega^i\wedge\omega_\varphi^{n-i},\end{aligned}$$

where

$$\mu(X) := \frac{c_1(X)\cdot[\omega]^{n-1}}{[\omega]^n}.$$

It is often useful to define the functionals appearing in the formula:

$$\begin{aligned}H(\varphi) &:= \int_X \log\left(\frac{\omega_\varphi^n}{\omega^n}\right)\omega_\varphi^n, \\ R(\varphi) &:= \sum_{i=0}^{n-1}\int_X \varphi\,\mathrm{Ric}(\omega)\wedge\omega^i\wedge\omega_\varphi^{n-1-i}, \\ I(\varphi) &:= \frac{1}{n+1}\sum_{i=0}^{n}\int_X \varphi\,\omega^i\wedge\omega_\varphi^{n-i}.\end{aligned}$$

The Chen–Tian formula then states

$$\mathcal{M}(\varphi) = H(\varphi) + R(\varphi) + n\mu(X)I(\varphi).$$

The purpose of this section is to prove an analogue of the Chen–Tian formula for the functional $\mathcal{N}$. This will be achieved through the following result:

Proposition 4.3.2 ([DS21c, Lemma 4.26])**.** *For $k \gg 0$, write $\omega_k := \omega_X + k\omega_B$, and let $\mathcal{M}_k$ be the corresponding Mabuchi functional restricted to $\mathcal{K}_E$. Then there is an expansion*

$$\mathcal{M}_k = k^n\mathcal{F} + k^{n-1}\mathcal{N} + \mathrm{O}(k^{n-2}).$$

The proof in [DS21c] is strictly speaking for the twisted Mabuchi functional, however an entirely analogous proof yields the above result. We also remark that the functional $\mathcal{F}$ is constant along $\mathcal{K}_E$.

Our goal is now to expand the usual Chen–Tian formula in k and identify the k^{n-1}-coefficient to produce a similar formula for the functional $\mathcal{N}$. Recall the functionals H, R and I from the Chen–Tian formula; we will denote by H_k, R_k and I_k their counterparts for ω_k, and expand each of these separately. We remark that the slope

$$\mu_k(X) := \frac{c_1(X)\cdot[\omega_X + k\omega_B]^{m+n-1}}{[\omega_X + k\omega_B]^{m+n}}$$

has an expansion

$$\mu_k(X) = \frac{m}{m+n}A_0(X) + \frac{n}{m+n}A_1(X)k^{-1} + \mathrm{O}(k^{-2})$$

which we shall not write down explicitly, but is essentially the same as (2.7).

Proposition 4.3.3. *There are expansions*

$$\begin{aligned}
I_k(\varphi) =& k^n \frac{1}{m+1}\binom{m+n}{n}\sum_{i=0}^{m}\int_X \varphi\,\omega_B^n\wedge\omega_X^i\wedge\omega_{X,\varphi}^{m-i}\\
&+k^{n-1}\frac{1}{m+2}\binom{m+n}{n-1}\sum_{i=0}^{m+1}\int_X \varphi\,\omega_B^{n-1}\wedge\omega_X^i\wedge\omega_{X,\varphi}^{m+1-i}\\
&+\mathrm{O}(k^{n-2}),
\end{aligned}$$

$$\begin{aligned}
R_k(\varphi) =& k^n\binom{m+n}{n}\sum_{i=0}^{m-1}\int_X \varphi\,\rho\wedge\omega_B^n\wedge\omega_X^i\wedge\omega_{X,\varphi}^{m-1-i}\\
&+k^{n-1}\binom{m+n}{n-1}\sum_{i=0}^{m}\int_X \varphi\,(\rho+\mathrm{Ric}(\omega_B))\wedge\omega_B^{n-1}\wedge\omega_X^{m-i}\wedge\omega_{X,\varphi}^i\\
&+k^{n-1}\binom{m+n}{n}\sum_{i=0}^{m-1}\int_X \varphi\, i\partial\bar\partial(\Lambda_{\omega_B}\omega_X)\wedge\omega_B^n\wedge\omega_X^{m-1-i}\wedge\omega_{X,\varphi}^i\\
&+\mathrm{O}(k^{n-2}),
\end{aligned}$$

and

$$\begin{aligned}H_k(\varphi) =& k^n\binom{m+n}{n}\int_X \log\left(\frac{\omega_B^n\wedge\omega_{X,\varphi}^m}{\omega_B^n\wedge\omega_X^m}\right)\omega_B^n\wedge\omega_{X,\varphi}^m\\ &+k^{n-1}\binom{m+n}{n-1}\int_X \log\left(\frac{\omega_B^n\wedge\omega_{X,\varphi}^m}{\omega_B^n\wedge\omega_X^m}\right)\omega_B^{n-1}\wedge\omega_{X,\varphi}^{m+1}\\ &-k^{n-1}\binom{m+n}{n}\sum_{i=0}^{m-1}\int_X \varphi\, i\partial\bar\partial(\Lambda_{\omega_B}\omega_X)\wedge\omega_B^n\wedge\omega_X^{m-1-i}\wedge\omega_{X,\varphi}^i\\ &+\mathrm{O}(k^{n-2}).\end{aligned}$$

Proof. An important tool in the proofs is the combinatorial formula

$$\binom{n+1}{p+1} = \binom{n}{p} + \binom{n-1}{p} + \cdots + \binom{p}{p}. \tag{4.2}$$

This, for example, lets one write the functional I instead as

$$I(\varphi) = \frac{1}{n+1}\sum_{p=0}^{n}\binom{n+1}{p+1}\int_X \varphi\,\omega^{n-p}\wedge(i\partial\bar\partial\varphi)^p.$$

This is achieved by expanding $\omega_\varphi^i = (\omega + i\partial\bar\partial\varphi)^i$, and reindexing the sum over powers of $i\partial\bar\partial\varphi$. Conversely, if one obtains a formula of this form, then one can factorise it using the same combinatorial formula.

First, for the expansion of I_k we have

$$I_k(\varphi) = \frac{1}{m+n+1}\sum_{p=0}^{m+n}\binom{m+n+1}{p+1}\int_X \varphi\,\omega_k^{m+n-p}\wedge(i\partial\bar\partial\varphi)^p.$$

Expanding this expression in k, we find the leading coefficient of k^n is

$$\frac{1}{m+n+1}\sum_{p=0}^{m}\binom{m+n+1}{p+1}\int_X \varphi\binom{m+n-p}{n}\omega_B^n\wedge\omega_X^{m-p}\wedge(i\partial\bar\partial\varphi)^p.$$

Now note that

$$\begin{aligned}\frac{1}{m+n+1}\binom{m+n+1}{p+1}\binom{m+n-p}{n} &= \frac{(m+n)!}{(p+1)!(m+n-p)!}\,\frac{(m+n-p)!}{n!(m-p)!}\\ &= \frac{(m+n)!}{(m+1)!\,n!}\,\frac{(m+1)!}{(p+1)!(m-p)!}\\ &= \frac{1}{m+1}\binom{m+n}{n}\binom{m+1}{p+1}.\end{aligned}$$

Hence the leading coefficient of $I_k(\varphi)$ is

$$\frac{1}{m+1}\binom{m+n}{n}\sum_{p=0}^{m}\binom{m+1}{p+1}\int_X \varphi\,\omega_B^n\wedge\omega_X^{m-p}\wedge(i\partial\bar\partial\varphi)^p$$

which by the combinatorial formula (4.2) can be factorised as

$$\frac{1}{m+1}\binom{m+n}{n}\sum_{i=0}^{m}\int_X \varphi\,\omega_B^n\wedge\omega_X^i\wedge\omega_{X,\varphi}^{m-i}.$$

The subleading order term is

$$\frac{1}{m+n+1}\sum_{p=0}^{m+1}\binom{m+n+1}{p+1}\binom{m+n-p}{n-1}\int_X \varphi\,\omega_B^{n-1}\wedge\omega_X^{m-p+1}\wedge(i\partial\bar\partial\varphi)^p.$$

Calculating, we get

$$\frac{1}{m+n+1}\binom{m+n+1}{p+1}\binom{m+n-p}{n-1}=\frac{1}{m+2}\binom{m+n}{n-1}\binom{m+2}{p+1}.$$

Hence the coefficient of k^{n-1} in the expansion of $I_k(\varphi)$ is

$$\frac{1}{m+2}\binom{m+n}{n-1}\sum_{p=0}^{m+1}\binom{m+2}{p+1}\int_X \varphi\,\omega_B^{n-1}\wedge\omega_X^{m+1-p}\wedge(i\partial\bar\partial\varphi)^p,$$

and applying (4.2) we get

$$\frac{1}{m+2}\binom{m+n}{n-1}\sum_{i=0}^{m+1}\int_X \varphi\,\omega_B^{n-1}\wedge\omega_X^{i}\wedge\omega_{X,\varphi}^{m+1-i}.$$

For the R-functional, we use the expansion of the Ricci curvature given by

$$\mathrm{Ric}(\omega_k)=\rho+\mathrm{Ric}(\omega_B)+k^{-1}i\partial\bar\partial(\Lambda_{\omega_B}\omega_X)+\mathrm{O}(k^{-2});$$

see [DS21b, Lemma 4.6]. Applying (4.2) we have

$$R_k(\varphi)=\sum_{p=0}^{m+n-1}\binom{m+n}{p+1}\int_X \varphi\,\mathrm{Ric}(\omega_k)\wedge\omega_k^{m+n-1-p}\wedge(i\partial\bar\partial\varphi)^p.$$

The coefficient of k^n in the expansion of $R_k(\varphi)$ is

$$\begin{aligned}
&\sum_{p=0}^{m-1}\binom{m+n}{p+1}\binom{m+n-1-p}{n}\int_X \varphi\,\rho\wedge\omega_B^n\wedge\omega_X^{m-1-p}\wedge(i\partial\bar\partial\varphi)^p\\
=&\binom{m+n}{n}\sum_{p=0}^{m-1}\binom{m}{p+1}\int_X \varphi\,\rho\wedge\omega_B^n\wedge\omega_X^{m-1-p}\wedge(i\partial\bar\partial\varphi)^p\\
=&\binom{m+n}{n}\sum_{i=0}^{m-1}\int_X \varphi\,\rho\wedge\omega_X^{i}\wedge\omega_{X,\varphi}^{m-1-i}.
\end{aligned}$$

The coefficient of k^{n-1} is

$$\begin{aligned}
&\sum_{p=0}^{m}\binom{m+n}{p+1}\binom{m+n-1-p}{n-1}\int_X \varphi(\rho+\mathrm{Ric}(\omega_B))\wedge\omega_B^{n-1}\wedge\omega_X^{m-p}\wedge(i\partial\bar\partial\varphi)^p\\
+&\sum_{p=0}^{m-1}\binom{m+n}{p+1}\binom{m+n-1-p}{n}\int_X \varphi\, i\partial\bar\partial(\Lambda_{\omega_B}\omega_X)\wedge\omega_B^{n}\wedge\omega_X^{m-1-p}\wedge(i\partial\bar\partial\varphi)^p\\
=&\binom{m+n}{n-1}\sum_{p=0}^{m}\binom{m+1}{p+1}\int_X \varphi(\rho+\mathrm{Ric}(\omega_B))\wedge\omega_B^{n-1}\wedge\omega_X^{m-p}\wedge(i\partial\bar\partial\varphi)^p\\
+&\binom{m+n}{n}\sum_{p=0}^{m-1}\binom{m}{p+1}\int_X \varphi\, i\partial\bar\partial(\Lambda_{\omega_B}\omega_X)\wedge\omega_B^{n}\wedge\omega_X^{m-1-p}\wedge(i\partial\bar\partial\varphi)^p\\
=&\binom{m+n}{n-1}\sum_{i=0}^{m}\int_X \varphi(\rho+\mathrm{Ric}(\omega_B))\wedge\omega_B^{n-1}\wedge\omega_X^{i}\wedge\omega_{X,\varphi}^{m-i}\\
+&\binom{m+n}{n}\sum_{i=0}^{m-1}\int_X \varphi\, i\partial\bar\partial(\Lambda_{\omega_B}\omega_X)\wedge\omega_B^{n}\wedge\omega_X^{i}\wedge\omega_{X,\varphi}^{m-1-i}.
\end{aligned}$$

Finally we expand the H-functional,

$$H_k(\varphi)=\int_X \log\left(\frac{\omega_{k,\varphi}^{m+n}}{\omega_k^{m+n}}\right)\omega_{k,\varphi}^{m+n}.$$

We first expand the logarithm. This is given by

$$\log\left(\frac{\omega_B^n\wedge\omega_{X,\varphi}^m}{\omega_B^n\wedge\omega_X^m}\right)+\log\left(1+k^{-1}\frac{n}{m+1}\left(\frac{\omega_B^{n-1}\wedge\omega_{X,\varphi}^{m+1}}{\omega_B^n\wedge\omega_{X,\varphi}^m}-\frac{\omega_B^{n-1}\wedge\omega_X^{m+1}}{\omega_B^n\wedge\omega_X^m}\right)+\mathrm{O}(k^{-2})\right).$$

Using the power series formula

$$\log(1+x)=\sum_{i=1}^{\infty}(-1)^{i+1}\frac{x^i}{i}$$

we get

$$\begin{aligned}
\log\left(\frac{\omega_{k,\varphi}^{m+n}}{\omega_k^{m+n}}\right)&=\log\left(\frac{\omega_B^n\wedge\omega_{X,\varphi}^m}{\omega_B^n\wedge\omega_X^m}\right)\\
&\quad+k^{-1}\frac{n}{m+1}\left(\frac{\omega_B^{n-1}\wedge\omega_{X,\varphi}^{m+1}}{\omega_B^n\wedge\omega_{X,\varphi}^m}-\frac{\omega_B^{n-1}\wedge\omega_X^{m+1}}{\omega_B^n\wedge\omega_X^m}\right)+\mathrm{O}(k^{-2}).
\end{aligned}$$

The k^n-coefficient of $H_k(\varphi)$ is therefore

$$\binom{m+n}{n}\int_X \log\left(\frac{\omega_B^n\wedge\omega_{X,\varphi}^m}{\omega_B^n\wedge\omega_X^m}\right)\omega_B^n\wedge\omega_{X,\varphi}^m.$$

The k^{n-1}-coefficient is

$$\binom{m+n}{n-1}\int_X\left(\frac{\omega_B^{n-1}\wedge\omega_{X,\varphi}^{m+1}}{\omega_B^n\wedge\omega_{X,\varphi}^m}-\frac{\omega_B^{n-1}\wedge\omega_X^{m+1}}{\omega_B^n\wedge\omega_X^m}\right)\omega_B^n\wedge\omega_{X,\varphi}^m$$
$$+\binom{m+n}{n-1}\int_X\log\left(\frac{\omega_B^n\wedge\omega_{X,\varphi}^m}{\omega_B^n\wedge\omega_X^m}\right)\omega_B^{n-1}\wedge\omega_{X,\varphi}^{m+1}. \tag{4.3}$$

We can reduce the first term of (4.3) as follows:

$$\begin{aligned}
&\int_X\left(\frac{\omega_B^{n-1}\wedge\omega_{X,\varphi}^{m+1}}{\omega_B^n\wedge\omega_{X,\varphi}^m}-\frac{\omega_B^{n-1}\wedge\omega_X^{m+1}}{\omega_B^n\wedge\omega_X^m}\right)\omega_B^n\wedge\omega_{X,\varphi}^m\\
=&\int_X\left(\omega_B^{n-1}\wedge\omega_{X,\varphi}^{m+1}-\frac{\omega_B^n\wedge\omega_{X,\varphi}^m}{\omega_B^n\wedge\omega_X^m}\,\omega_B^{n-1}\wedge\omega_X^{m+1}\right)\\
=&\int_X\left(1-\frac{\omega_B^n\wedge\omega_{X,\varphi}^m}{\omega_B^n\wedge\omega_X^m}\right)\omega_B^{n-1}\wedge\omega_X^{m+1}\\
=&\int_X\frac{\omega_B^n\wedge\omega_X^m-\omega_B^n\wedge\omega_{X,\varphi}^m}{\omega_B^n\wedge\omega_X^m}\,\omega_B^{n-1}\wedge\omega_X^{m+1}\\
=&\frac{m+1}{n}\int_X(\Lambda_{\omega_B}\omega_X)(\omega_B^n\wedge\omega_X^m-\omega_B^n\wedge\omega_{X,\varphi}^m)\\
=&-\frac{m+1}{n}\sum_{p=1}^m\binom{m}{p}\int_X(\Lambda_{\omega_B}\omega_X)\omega_B^n\wedge\omega_X^{m-p}\wedge(i\partial\bar\partial\varphi)^p\\
=&-\frac{m+1}{n}\sum_{p=1}^m\binom{m}{p}\int_X\varphi\,i\partial\bar\partial(\Lambda_{\omega_B}\omega_X)\wedge\omega_B^n\wedge\omega_X^{m-p}\wedge(i\partial\bar\partial\varphi)^{p-1}\\
=&-\frac{m+1}{n}\sum_{p=0}^{m-1}\binom{m}{p+1}\int_X\varphi\,i\partial\bar\partial(\Lambda_{\omega_B}\omega_X)\wedge\omega_B^n\wedge\omega_X^{m-1-p}\wedge(i\partial\bar\partial\varphi)^p\\
=&-\frac{m+1}{n}\sum_{i=0}^{m-1}\int_X\varphi\,i\partial\bar\partial(\Lambda_{\omega_B}\omega_X)\wedge\omega_B^n\wedge\omega_X^{m-1-i}\wedge\omega_{X,\varphi}^i.
\end{aligned}$$

□

Combining all of these expansions together with Proposition 4.3.2, we get the following. Note there is a miraculous cancellation that occurs between the k^{n-1}-coefficients of H_k and R_k, following from the lengthy calculation at the end of the last proof. This suggests there is perhaps another, more natural method for deriving this formula.

Theorem 4.3.4. *The functional $\mathcal{N}$ satisfies the formula*

$$\begin{aligned}\mathcal{N}(\varphi) = \binom{m+n}{n-1}\Bigg[&\int_X \log\left(\frac{\omega_B^n \wedge \omega_{X,\varphi}^m}{\omega_B^n \wedge \omega_X^m}\right)\omega_B^{n-1}\wedge\omega_{X,\varphi}^{m+1}\\ &+\sum_{i=0}^{m}\int_X \varphi\,(\rho+\mathrm{Ric}(\omega_B))\wedge\omega_B^{n-1}\wedge\omega_X^{m-i}\wedge\omega_{X,\varphi}^{i}\\ &+\frac{m}{m+2}A_0(X)\sum_{i=0}^{m+1}\int_X \varphi\,\omega_B^{n-1}\wedge\omega_X^i\wedge\omega_{X,\varphi}^{m+1-i}\\ &+A_1(X)\sum_{i=0}^{m}\int_X \varphi\,\omega_B^n\wedge\omega_X^i\wedge\omega_{X,\varphi}^{m-i}\Bigg],\end{aligned}$$

where ρ is the curvature of the metric on $\Lambda^m\mathcal{V}$ determined by ω_X, and $A_0(X)$ and $A_1(X)$ are the topological constants appearing in the expansion

$$\mu_k(X) = \frac{m}{m+n}A_0(X) + \frac{n}{m+n}k^{-1}A_1(X) + \mathrm{O}(k^{-2}).$$

As with the standard Chen–Tian formula, it will be useful to name the functionals appearing in the formula for later use:

$$\begin{aligned}\widetilde{H}(\varphi) &:= \int_X \log\left(\frac{\omega_B^n \wedge \omega_{X,\varphi}^m}{\omega_B^n \wedge \omega_X^m}\right)\omega_B^{n-1}\wedge\omega_{X,\varphi}^{m+1}\\ \widetilde{R}(\varphi) &:= \sum_{i=0}^{m}\int_X \varphi\,(\rho+\mathrm{Ric}(\omega_B))\wedge\omega_B^{n-1}\wedge\omega_X^{m-i}\wedge\omega_{X,\varphi}^{i}\\ \widetilde{I}(\varphi) &:= \sum_{i=0}^{m}\int_X \varphi\,\omega_B^n\wedge\omega_X^i\wedge\omega_{X,\varphi}^{m-i}\\ \widetilde{J}(\varphi) &:= \sum_{i=0}^{m+1}\int_X \varphi\,\omega_B^{n-1}\wedge\omega_X^i\wedge\omega_{X,\varphi}^{m+1-i}\end{aligned} \tag{4.4}$$

The formula then takes the more notationally compact form:

$$\mathcal{N}(\varphi) = \binom{m+n}{n-1}\left(\widetilde{H}(\varphi) + \widetilde{R}(\varphi) + A_1(X)\widetilde{I}(\varphi) + \frac{m}{m+2}A_0(X)\widetilde{J}(\varphi)\right).$$

Chapter 5

Polystability of fibrations

In this chapter we prove our main result on fibrations—that a fibration admitting an optimal symplectic connection is polystable with respect to a class of degenerations whose fibres are all product test configurations. The ideas are similar to the proof that cscK manifolds are K-polystable, by Berman–Darvas–Lu [BDL20]. In the Kähler setting, one associates to any test configuration a unique weak geodesic ray in the space of Kähler potentials. Using the theory of Deligne pairings, the limiting slope of the Mabuchi functional along this ray is calculated to be the Donaldson–Futaki invariant of the test configuration. Applying convexity, this slope is non-negative, and one can further show that it is strictly positive if the test configuration is not a product test configuration. In the case of varieties, the ideas used here go back to Tian [Tia97], and have been carried out by various authors in [PS10], [PRS08], [BHJ19], with extensions to the non-algebraic Kähler case in [DR17b] and [SD18, SD20].

In our fibrations setting, we follow a similar path. The first section in this chapter reviews the theory of Deligne pairings, and we show how one may describe the functional $\mathcal{N}$ in terms of Deligne pairings. The idea of using Deligne pairings in this way goes back to Phong–Ross–Sturm [PRS08], and has been carried out thoroughly in the work of Boucksom–Hisamoto–Jonsson [BHJ19].

We next show that to any product-type fibration degeneration, we can associate a unique smooth geodesic ray in the space $\mathcal{K}_E$ compatible with this fibration degeneration. In the Kähler setting, the analogous result is due to Phong–Sturm [PS10], who build on the work of Chen [Che00b]. Once again, that geodesics exist fibrewise is immediate, and the crux is to show they depend smoothly on the basepoint. For this, we use the characterisation of geodesics by Donaldson [Don99] as solutions of the Wess–Zumino–Witten equation for maps from the complex disc to a non-positively curved symmetric space. Showing that the linearisation of this equation at a solu-

tion is an isomorphism and applying the inverse function theorem, we get smoothly varying geodesics.

We finish the chapter by calculating the limiting slope of the functional $\mathcal{N}$ along the geodesic ray associated to the fibration degeneration, using the description of $\mathcal{N}$ in terms of Deligne pairings, which ultimately comes from the Chen–Tian style formula for $\mathcal{N}$. Applying convexity of $\mathcal{N}$ along geodesics, we get polystability of the fibrations with respect to product-type degenerations.

Before beginning, we recall our conventions from Chapter 2 on fibrations. We consider a smooth projective morphism $\pi : X \to B$ of non-singular projective varieties, together with a choice of ample line bundle $L \to B$ and relatively ample line bundle $H \to X$. We further assume that $\dim H^0(X_b, TX_b)$ is independent of b, which by the partial Cartan decomposition is equivalent to assuming that $\dim E_b$ is independent of b, and assume that each fibre X_b admits a cscK metric in the class $c_1(H_b)$. Even though we have chosen to work in the algebraic setting with line bundles, the results will carry over easily to the transcendental Kähler setting, as we comment on in Remark 5.3.5.

5.1 Deligne pairings

Towards calculating the limiting slope of the functional $\mathcal{N}$, we first review the theory of Deligne pairings; see [BHJ19, Section 1.2] for further information. Let Y and T be smooth varieties, and let $\pi : Y \to T$ be a flat projective morphism of relative dimension m. Given line bundles $L_1, \dots, L_{m+1}$ on Y, integration over the fibres gives a well-defined cohomology class

$$\int_{Y/T} c_1(L_1) \cdots c_1(L_{m+1})$$

on the base T. The theory of Deligne pairings gives a canonical choice of line bundle $\langle L_1, \dots, L_{m+1} \rangle_{Y/T}$ whose first Chern class is this cohomology class. Moreover, if $h_1, \dots, h_{m+1}$ are smooth hermitian metrics on $L_1, \dots, L_{m+1}$ respectively, then the theory also produces a canonical *continuous* hermitian metric $\langle h_1, \dots, h_{m+1} \rangle_{Y/T}$ on $\langle L_1, \dots, L_{m+1} \rangle_{Y/T}$ that is smooth over the smooth locus of π. The Deligne pairings are symmetric in the factors, multilinear, and functorial. Most importantly, they satisfy:

1. The curvature of $\langle h_1, \dots, h_{m+1} \rangle_{Y/T}$ is given by

$$\int_{Y/T} \omega_1 \wedge \cdots \wedge \omega_{m+1},$$

where ω_i is the curvature of h_i.

2. If h_1' is another smooth hermitian metric on L_1, then

$$\log\left(\frac{\langle h_1', h_2, \ldots, h_{m+1}\rangle_{Y/T}}{\langle h_1, h_2, \ldots, h_{m+1}\rangle_{Y/T}}\right) = \int_{Y/T} \phi\, \omega_2 \wedge \cdots \wedge \omega_{m+1}, \tag{5.1}$$

where $\phi := \log(h_1'/h_1)$.

To simplify notation, we shall abusively write

$$\langle h_1', h_2, \ldots, h_{m+1}\rangle_{Y/T} - \langle h_1, h_2, \ldots, h_{m+1}\rangle_{Y/T} \tag{5.2}$$

for what is really

$$\log\left(\frac{\langle h_1', h_2, \ldots, h_{m+1}\rangle_{Y/T}}{\langle h_1, h_2, \ldots, h_{m+1}\rangle_{Y/T}}\right).$$

This is in line with using additive notation for line bundles and metrics rather than multiplicative notation. Furthermore, when T is a point we shall simply write $\langle\,,\rangle_Y$ in place of $\langle\,,\rangle_{Y/\mathrm{pt}}$.

Using the change of metric formula (5.1), we can rewrite the functionals appearing in the Chen–Tian style formula in Theorem 4.3.4 in terms of Deligne pairings. Let $\pi : X \to B$ be a smooth projective morphism of projective varieties, let $L \to B$ be ample and $H \to X$ be relatively ample, let h_X be a smooth metric on H with curvature a relatively Kähler form ω_X, and let h_B be a smooth metric on L with curvature a Kähler form ω_B. Denote by η_X the metric on $-K_X$ corresponding to the volume form $\omega_X^m \wedge \omega_B^n$. Lastly, write $f = \Lambda_{\omega_B}\omega_X$ which we consider as a hermitian metric on the trivial line bundle on X with curvature $i\partial\bar\partial(\Lambda_{\omega_B}\omega_X)$.

Proposition 5.1.1. *Let $h_{X,\varphi}$ be another smooth metric on H with relatively Kähler curvature $\omega_{X,\varphi}$, where $\varphi = \log(h_{X,\varphi}/h)$, and let $\eta_{X,\varphi}$ be the metric on $-K_X$ corresponding to the volume form $\omega_{X,\varphi}^m \wedge \omega_B^n$. Then:*

(i) $\widetilde{H}(\varphi) = \langle \eta_{X,\varphi}, h_B^{n-1}, h_{X,\varphi}^{m+1}\rangle_X - \langle \eta_X, h_B^{n-1}, h_{X,\varphi}^{m+1}\rangle_X$,

(ii) $\widetilde{R}(\varphi) = \langle \eta_X, h_B^{n-1}, h_{X,\varphi}^{m+1}\rangle_X - \langle \eta_X, h_B^{n-1}, h_X^{m+1}\rangle_X$,

(iii) $\widetilde{I}(\varphi) = \langle h_B^n, h_{X,\varphi}^{m+1}\rangle_X - \langle h_B^n, h_X^{m+1}\rangle_X$,

(iv) $\widetilde{J}(\varphi) = \langle h_B^{n-1}, h_{X,\varphi}^{m+2}\rangle_X - \langle h_B^{n-1}, h_X^{m+2}\rangle_X$,

where the functionals $\widetilde{H}, \widetilde{R}, \widetilde{I}, \widetilde{J}$ are defined in (4.4)*, and we abuse notation as in* (5.2)*.*

Proof. This is immediate from the change of metric formula (5.1) for Deligne pairings; we only note that

$$\log(\eta_{X,\varphi}/\eta_X) = \log\left(\frac{\omega_{X,\varphi}\wedge\omega_B^n}{\omega_X\wedge\omega_B^n}\right),$$

which is the integrand appearing in the functional $\widetilde{H}$. □

5.2 Geodesics associated to fibration degenerations, and the Wess–Zumino–Witten equation

Now, let us turn to fibration degenerations. First, notice that any fibration degeneration with $W_0(\mathcal{X},\mathcal{H}) > 0$ may be ignored, as these by definition do not destabilise the fibration. We may therefore assume $W_0(\mathcal{X},\mathcal{H}) = 0$, which by Proposition 2.5.5 means that the general fibre $(\mathcal{X}_b,\mathcal{H}_b)$ normalises to a product test configuration for (X_b,H_b). If we normalise $(\mathcal{X},\mathcal{H})$, then the general fibre $(\mathcal{X}_b,\mathcal{H}_b)$ of $(\mathcal{X},\mathcal{H})$ over B will also be normal, and hence will be a genuine product test configuration for (X_b,H_b) by K-polystability of the fibres.

It follows that it suffices to consider fibration degenerations whose general fibre is a genuine product test configuration. In what follows, we will make a more restrictive assumption that implies all fibres are product test configurations:

Definition 5.2.1. A fibration degeneration $(\mathcal{X},\mathcal{H})$ of (X,H) is called *product-type* if for every $b\in B$ there is an open neighbourhood U of b and a holomorphic isomorphism

$$(\mathcal{X}|_{U\times\mathbb{C}},\mathcal{H}|_{U\times\mathbb{C}}) \cong (X|_U\times\mathbb{C}, p^*H|_U),$$

where $p : X|_U\times\mathbb{C}\to X|_U$ is the projection, and the right-hand side may have a non-trivial $\mathbb{C}^*$-action.

Restricting attention to product-type fibration degenerations is analogous to considering vector subbundles instead of subsheaves when studying stability of vector bundles; see Examples 5.3.6 and 5.3.7. Thus, when the base B of the fibration is a curve, we expect such fibration degenerations to play an important role, perhaps determining stability completely.

A product-type fibration degenerations has all fibres $\mathcal{X}_b$ product test configurations. When the fibration is isotrivial, the converse holds:

Lemma 5.2.2. *Suppose that all fibres X_b are isomorphic as complex manifolds, and let $\mathcal{X}$ be a fibration degeneration of X. If all fibres $\mathcal{X}_b$ are product test configurations, then $\mathcal{X}$ is product-type.*

Proof. This is a consequence of the Fischer–Grauert theorem [FG65], which states that if $Y \to T$ is a proper holomorphic surjective submersion with all fibres Y_t isomorphic as complex manifolds, then the fibration is holomorphically locally trivial. Applying this to $\mathcal{X}$ in a neighbourhood of $(b,0) \in U \times \mathbb{C}$ and applying $\mathbb{C}^*$-equivariance gives that $\mathcal{X}|_U \cong X|_U \times \mathbb{C}$.

It remains to establish the isomorphism of line bundles $\mathcal{H}_{U\times\mathbb{C}} \cong p^*H|_U$. This kind of result is well established, see for example [New78, Lemma 5.10] which applies in the algebraic case but extends equally well to the holomorphic setting. □

We also have the following, which tells us that for isotrivial fibrations, our result applies to all fibration degenerations that are smooth over $B \times \mathbb{C}$.

Lemma 5.2.3. *Let $(\mathcal{X}, \mathcal{H})$ be a fibration degeneration of (X, H) that is smooth over $B \times \mathbb{C}$, with $W_0(\mathcal{X}, \mathcal{H}) = 0$. Then all fibres $(\mathcal{X}_b, \mathcal{H}_b)$ are product test configurations.*

Here smoothness is in the sense of scheme theory, so that $\mathcal{X} \to B \times \mathbb{C}$ is a holomorphic submersion.

Proof. Since the map $\mathcal{X} \to B \times \mathbb{C}$ is smooth, all fibres $\mathcal{X}_b$ will be normal varieties. Since $W_0(\mathcal{X}, \mathcal{H}) = 0$, by Proposition 2.5.5 the general fibre $(\mathcal{X}_b, \mathcal{H}_b)$ satisfies $\mathrm{DF}(\mathcal{X}_b, \mathcal{H}_b) = 0$. Now, since the morphism $\mathcal{X} \to B \times \mathbb{C}$ is smooth it is in particular flat. The Donaldson–Futaki invariant is constant in flat families, and therefore every fibre satisfies $\mathrm{DF}(\mathcal{X}_b, \mathcal{H}_b) = 0$. Since every fibre is also normal, it follows that all fibres $(\mathcal{X}_b, \mathcal{H}_b)$ are product test configurations, by K-polystability of the fibres (X_b, H_b). □

Our aim is now to associate to any product-type fibration degeneration a unique geodesic ray $\varphi : [0,\infty) \to \mathcal{K}_E$ that is compatible with this fibration degeneration. Firstly, let $D \subset \mathbb{C}$ be the closed unit disc, and write $D^\times := D\backslash\{0\}$. Given a fibration degeneration $(\mathcal{X}, \mathcal{H})$, we will denote by $\mathcal{X}|_D$ the restriction of $\mathcal{X}$ to $D \subset \mathbb{C}$, and similarly for $\mathcal{H}|_D$. Using the $\mathbb{C}^*$-action on $\mathcal{X}$, we consider $(X \times D^\times, p^*H) \subset (\mathcal{X}|_D, \mathcal{H}|_D)$ as an open subset, again writing $p : X \times D^\times \to X$ for the projection.

Definition 5.2.4. Let $(\mathcal{X}, \mathcal{H})$ be a product-type fibration degeneration of (X, H), and let h_X be a fixed hermitian metric on H with curvature ω_X a relatively cscK metric on X. A smooth geodesic ray $\varphi : [0,\infty) \to \mathcal{K}_E$ is said to be *compatible* with $(\mathcal{X}, \mathcal{H})$ if there exists a smooth S^1-invariant hermitian metric $h_\mathcal{X}$ on $\mathcal{H}|_D$ such that the function $\Phi : X \times D^\times \to \mathbb{R}$ defined by

$$h_\mathcal{X}|_{D^\times} = e^\Phi p^* h_X$$

satisfies:

(i) $\Phi|_{\partial\mathcal{X}} = 0$, and

(ii) $\Phi(x,\tau) := \varphi_{-\log|\tau|}(x)$ for all $\tau \in D^\times$.

This compatibility condition will allow us to calculate the asymptotic slope of the log-norm functional $\mathcal{N}$ in the following way. First, considering the geodesic ray as a metric on the line bundle $\mathcal{H}$ via (ii), we can take its curvature, which gives rise to intersection numbers on $\mathcal{X}$ via integration. Applying a theorem of Phong–Ross–Sturm [PRS08] and Boucksom–Hisamoto–Jonsson [BHJ19], the Deligne pairing description of the functional $\mathcal{N}$ in Proposition 5.1.1 via the Chen–Tian style formula from Theorem 4.3.4 then allows us to compute the asymptotic slope of $\mathcal{N}$ along the geodesic ray in terms of these intersection numbers. When the reference relatively cscK metric is an optimal symplectic connection, condition (i) implies the geodesic ray begins at a critical point of $\mathcal{N}$, and convexity of $\mathcal{N}$ along geodesics then yields non-negativity of the asymptotic slope of $\mathcal{N}$.

Remark 5.2.5. In the non-relative case, there is already a suitable notion of a geodesic compatible with a given test configuration, see for example [PS10]. Since such a geodesic solves a degenerate Monge–Ampère equation with boundary condition, the geodesic is necessarily unique. For arbitrary test configurations, one must first resolve singularities on the total space, and even then the geodesic is not guaranteed to be smooth but only $C^{1,1}$ in general. When the test configuration is a product however, we shall see that the unique associated geodesic ray is smooth. Moreover if the initial condition h_X is a cscK metric, then the geodesic ray will also be through cscK metrics.

We wish to associate a unique smooth geodesic ray to any product-type fibration degeneration. As for the geodesic equation in Chapter 3, the notion of geodesic ray compatible with a fibration degeneration is simply the usual notion of compatibility with test configurations but fibrewise over B. So we will again construct the geodesics fibrewise and show they vary smoothly with the base point.

First, the fibrewise geodesics must be shown to exist, be smooth, and be through cscK potentials. This is the content of the following proposition, which is a trivial consequence of results by Donaldson, as explained below.

Proposition 5.2.6 ([Don92, pp. 100–101], [Don99, Propositions 3, 4])**.** *Take a product test configuration $(X \times \mathbb{C}, p^*L)_\rho$ for (X, L), and let ω be a cscK metric in $c_1(L)$. Then there exists a unique geodesic ray $\varphi : [0,\infty) \to \mathcal{K}_\omega$ compatible with the product*

test configuration having initial condition $0 \in \mathcal{K}_\omega$. *Furthermore,* $\omega + i\partial\bar{\partial}\varphi_t$ *is cscK for all* t.

Proof. This essentially follows from Donaldson's reduction of the geodesic equation on the disc to a so-called "Wess–Zumino–Witten" (WZW) equation. Denote by D the disc and let G/K be the symmetric space $\mathrm{Aut}_0(X)/\mathrm{Isom}_0(\omega)$, where ω is cscK. Then G/K is a non-positively curved symmetric space, embedded in the space $\mathcal{K}_\omega$ of Kähler potentials as a totally geodesic submanifold. We wish to construct a smooth map $f : D \to G/K$ satisfying the geodesic equation in the following sense: the ray $\varphi : [0, \infty) \to \mathcal{K}_\omega$ given by

$$f(e^{-t})^*\omega - \omega = i\partial\bar{\partial}\varphi_t$$

is a geodesic in $\mathcal{K}_\omega$, where we normalise so that $\int_X \dot{\varphi}_t f(e^{-t})^*\omega = 0$. The initial condition $\varphi_0 = 0$ will correspond to the boundary condition $f(e^{i\theta}) = K \cdot \rho(e^{i\theta})$, where $\rho : \mathbb{C}^* \to \mathrm{Aut}(X, L)$ is the action determining the product test configuration.

By [Don99, Propositions 3, 4], $f : D \to G/K$ defines a geodesic ray in the above manner if and only if f solves the WZW equation

$$\tilde{\tau}(f) := \tau(f) + [D_u f, D_v f] = 0. \tag{5.3}$$

Here $\tau(f) := \mathrm{tr}(\nabla df)$ is the usual tension field appearing in the harmonic map equation, and $[\,,\,] : T(G/K) \otimes T(G/K) \to T(G/K)$ is the Lie bracket given by the canonical isomorphism of the tangent space with the Lie algebra of K. Also, $u + iv$ is the standard complex coordinate on $D \subset \mathbb{C}$, and $D_u f := df(\partial_u)$, $D_v f := df(\partial_v)$. As is proven in [Don92, pp. 100–101] (see also the comments in Section 7 of [Don99]), the WZW equation has a unique smooth solution for any fixed boundary condition. It follows that the geodesic equation on a product test configuration is uniquely soluble, and the solution is smooth and lies within the space of cscK metrics. □

Remark 5.2.7. When the circle action of ρ preserves ω, one may take the trivial solution $f = const.$ If the circle action does not preserve ω, the solution f will be nontrivial to account for the nontrivial boundary condition. The above proposition could also be proven by conjugating the circle action to lie in the isometry group of ω, which is possible since this group is maximal compact. However, we have chosen the WZW approach so as to more easily show that solutions vary smoothly in families.

Hence, on a product-type fibration degeneration, each fibre $(\mathcal{X}_b, \mathcal{H}_b)$ has a unique geodesic ray $\varphi_b : [0, \infty) \to \mathcal{K}_{E,b}$ associated to it. It once again falls to proving these geodesic rays depend smoothly on the base point b. This will be done by linearising

the WZW equation at a solution, and showing the derivative is an isomorphism. This has been carried out by Goldstein [Gol72] for the standard harmonic map equation; adapting his proof, we show:

Proposition 5.2.8. *For $r \geq 2$ and $\alpha \in [0,1]$, let*

$$F : C^{r,\alpha}(D, G/K) \to C^{r-2,\alpha}(D, T(G/K)) \times C^{r,\alpha}(\partial D, G/K)$$

be given by

$$F(f) = (\tilde{\tau}(f), f|_{\partial D}),$$

where $\tilde{\tau}(f) := \mathrm{tr}(\nabla df) + [D_u f, D_v f]$ is the WZW operator (5.3). If $f : D \to G/K$ is a solution to the WZW equation, then $dF|_f$ is an isomorphism.

Proof. Following [Gol72], we show the derivative is both injective and surjective at a solution. In fact, surjectivity follows from abstract index theory, given that the leading order term in the linearisation of the WZW equation agrees with that of the harmonic map equation; see [Gol72, pp. 353–355]. Hence we only need to show that the derivative is injective at a solution.

So, suppose that $W \in T_f C^{r,\alpha}(D, G/K) = C^{r,\alpha}(D, f^*T(G/K))$ lies in the kernel of $dF|_f$. Then $W|_{\partial D} = 0$, and W arises from a variation f_t of f with $f_0 = f$ and $f_t|_{\partial D} = f|_{\partial D}$ for all t. Let $u+iv$ be the standard complex coordinate on $\mathbb{C}$ restricted to the closed unit disc $D \subset \mathbb{C}$. In these coordinates, the Wess–Zumino–Witten equation takes the form

$$\tilde{\tau}(f) = D_u D_u f + D_v D_v f + [D_u f, D_v f] = 0,$$

where for a vector field V along f, $D_u V$ (resp. $D_v V$) is the covariant derivative of V along $f_*(\partial_u)$ (resp. $f_*(\partial_v)$). In [Gol72, "Lemma"], Goldstein shows that that W lying in the kernel of $dF|_f$ is equivalent to the condition[1]

$$0 = D_t(\tau(f_t) + [D_u f_t, D_v f_t])|_{t=0}.$$

Omitting the evaluation at $t = 0$, the WZW equation then differentiates to give

$$\begin{aligned} 0 =& D_t D_u D_u f + D_t D_v D_v f + D_t[D_u f, D_v f] \\ =& D_u D_t D_u f + R(D_t f, D_u f) D_u f \\ &+ D_v D_t D_v f + R(D_t f, D_v f) D_v f + D_t[D_u f, D_v f], \end{aligned}$$

where R is the curvature tensor of G/K. Next, we use that $[\,,]$ is covariantly constant on G/K, so that $D_t[D_u f, D_v f] = [D_t D_u f, D_v f] + [D_u f, D_t D_v f]$. Also, the connection

[1] It is here that one uses the condition that f solves the WZW equation.

on G/K is torsion-free, so we have $D_tD_uf = D_uD_tf$. Lastly, the curvature operator R on G/K takes the explicit form $R(U,V)Z = -\frac{1}{4}[Z,[U,V]]$. Putting this all together, we get

$$\begin{aligned}0 =& D_uD_uW + D_vD_vW - \frac{1}{4}[D_uf,[W,D_uf]] - \frac{1}{4}[D_vf,[W,D_vf]] \\ &+[D_uW,D_vf] + [D_uf,D_vW].\end{aligned}$$

Taking the L^2-inner product with $-W$,

$$\begin{aligned}0 =& -\langle W, D_uD_uW\rangle - \langle W, D_vD_vW\rangle \\ &+ \frac{1}{4}\langle W,[D_uf,[W,D_uf]]\rangle + \frac{1}{4}\langle W,[D_vf,[W,D_vf]]\rangle \\ &- \langle W,[D_uW,D_vf]\rangle - \langle W,[D_uf,D_vW]\rangle.\end{aligned}$$

Since $W|_{\partial D} = 0$, we can integrate D_u and D_v by parts without picking up boundary terms, hence $-\langle W, D_uD_uW\rangle = \langle D_uW, D_uW\rangle$ and similarly for v. Furthermore, the inner product on G/K is ad-invariant, so that $\langle [U,V],W\rangle + \langle V,[U,W]\rangle = 0$. Hence we get

$$\begin{aligned}0 =& \langle D_uW, D_uW\rangle + \langle D_vW, D_vW\rangle \\ &+\frac{1}{4}\langle [D_uf,W],[D_uf,W]\rangle + \frac{1}{4}\langle [D_vf,W],[D_vf,W]\rangle \\ &+\langle [D_uf,W],D_vW\rangle - \langle [D_vf,W],D_uW\rangle \\ =& \|D_uW - \frac{1}{2}[D_vf,W]\|^2 + \|D_vW + \frac{1}{2}[D_uf,W]\|^2.\end{aligned}$$

Since all terms in the final line are non-negative, they must vanish. Hence $D_uW = \frac{1}{2}[D_vf,W]$ and $D_vW = -\frac{1}{2}[D_uf,W]$. Now, letting $(\,,\,)$ denote the pointwise inner product on $T(G/K)$, we have

$$\partial_u(W,W) = 2(D_uW,W) = ([D_vf,W],W) = (D_vf,[W,W]) = 0,$$

and similarly $\partial_v(W,W) = 0$. It follows that (W,W) is constant, and since $W|_{\partial D} = 0$ we have $W = 0$. □

Theorem 5.2.9. *Let $(\mathcal{X},\mathcal{H})$ be a product-type fibration degeneration of (X,H). Given any relatively cscK metric ω_X on X, there exists a unique smooth geodesic ray $\varphi : [0,\infty) \to \mathcal{K}_E$ compatible with $(\mathcal{X},\mathcal{H})$.*

Proof. Over an open subset $U \subset B$, we trivialise the fibration degeneration,

$$(\mathcal{X}_{U\times\mathbb{C}}, \mathcal{H}|_{U\times\mathbb{C}}) \cong (X|_U \times \mathbb{C}, p^*H).$$

On U, construct the fibre bundles:

- $\mathcal{D} := U \times D$,
- $\partial\mathcal{D} := U \times \partial D \subset \mathcal{D}$,
- $\mathcal{G}/\mathcal{K}$ the smooth fibre bundle whose fibre over $b \in U$ is the homogeneous space G_b/K_b of cscK metrics in $c_1(H_b)$, as in Chapter 3,
- $T(\mathcal{G}/\mathcal{K})$ the vertical tangent bundle of the fibre bundle $\mathcal{G}/\mathcal{K}$.

For any of these fibre bundles $\mathcal{F}_1, \mathcal{F}_2$, denote by $C^\infty_U(\mathcal{F}_1, \mathcal{F}_2)$ the smooth morphisms $\mathcal{F}_1 \to \mathcal{F}_2$ compatible with the projections to U, which is also a fibre bundle over U.

The fibrewise WZW operator is then a smooth map

$$\tilde{\tau} : C^\infty_U(\mathcal{D}, \mathcal{G}/\mathcal{K}) \to C^\infty_U(\mathcal{D}, T(\mathcal{G}/\mathcal{K})) \times_U C^\infty_U(\partial\mathcal{D}, \mathcal{G}/\mathcal{K}).$$

The fibrewise S^1-action on $(X|_U \times \mathbb{C}, p^*H)$ determined by the fibration degeneration yields a smooth section $s : U \to C^\infty_U(\partial\mathcal{D}, \mathcal{G}/\mathcal{K})$, by pulling back the initial relatively cscK metric by the action. For each point $b \in U$ there is then a unique solution $f_b : D \to G_b/K_b$ of the WZW equation with boundary value $s(b)$. Proposition 5.2.8 together with the implicit function theorem for Banach manifolds implies the solutions f_b vary smoothly with b. □

5.3 Polystability with respect to product-type fibration degenerations

Theorem 5.3.1. *Let $(\mathcal{X}, \mathcal{H}) \to (B, L) \times \mathbb{C}$ be a product-type fibration degeneration of $(X, H) \to (B, L)$. Let ω_X be a fixed choice of relatively cscK metric on X, and let $\varphi : [0, \infty) \to \mathcal{K}_E$ be the unique geodesic ray associated to $(\mathcal{X}, \mathcal{H})$. Then*

$$\lim_{t\to\infty} \frac{\mathcal{N}(\varphi_t)}{t} = W_1(\mathcal{X}, \mathcal{H}).$$

To prove this, we will need the following lemma, which is a special case of a result by Boucksom–Hisamoto–Jonsson:

Lemma 5.3.2 ([BHJ19, Lemma 3.9])**.** *Let $(\mathcal{X}, \mathcal{H})$ be a product-type fibration degeneration of (X, H). Let $\mathcal{L}_0, \ldots, \mathcal{L}_{m+n}$ be line bundles on $\mathcal{X}$ with restrictions $L_0, \ldots, L_{m+n}$ to $X = \mathcal{X}_1$, and choose fixed hermitian metrics $h_0, \ldots, h_{m+n}$ on $L_1, \ldots, L_{m+n}$. We compactify $\mathcal{X}$ and each of the $\mathcal{L}_i$ trivially over $\mathbb{P}^1$, and denote them by the $\overline{\mathcal{X}}$ and $\overline{\mathcal{L}}_i$. For each i, let H_i be a smooth S^1-invariant metric on $\mathcal{L}_i$ defined in a neighbourhood of $0 \in \mathbb{C}$, and write h_i^t for the corresponding ray of smooth metrics on L_i. Then*

$$\langle h_0^t, \ldots, h_{m+n}^t \rangle_X - \langle h_0, \ldots, h_{m+n} \rangle_X = t\,(\overline{\mathcal{L}}_0 \cdot \ldots \cdot \overline{\mathcal{L}}_{m+n}) + \mathrm{O}(1).$$

Proof of Theorem 5.3.1. Let $\varphi : [0, \infty) \to \mathcal{K}_E$ be the unique geodesic ray associated to the fibration degeneration; this corresponds to a smooth hermitian metric $h_{\mathcal{X}}$ on $\mathcal{H}|_D$ whose restriction to $\mathcal{H}|_{\partial D}$ is isomorphic to $p^* h_X$. Denote by h_X^t the ray of smooth metrics on X determined by $h_{\mathcal{X}}$. Then

$$\varphi_t = \log(h_X^t / h_X)$$

is a smooth geodesic ray in $\mathcal{K}_E$. The metric $h_{\mathcal{X}}$ also determines a smooth metric $\eta_{\mathcal{X}}$ on the relative canonical bundle $K_{\mathcal{X}/D}$, which is a line bundle in the case of product-type fibration degenerations. We then obtain a ray of smooth metrics η_X^t on K_X. Proposition 5.1.1 then gives

$$\begin{aligned}
\widetilde{H}(\varphi_t) + \widetilde{R}(\varphi_t) &= \langle \eta_X^t, h_B^{n-1}, (h_X^t)^{m+1} \rangle_X - \langle \eta_X, h_B^{n-1}, h_X^{m+1} \rangle_X, \\
\widetilde{I}(\varphi_t) &= \langle h_B^n, (h_X^t)^{m+1} \rangle_X - \langle h_B^n, h_X^{m+1} \rangle_X, \\
\widetilde{J}(\varphi_t) &= \langle h_B^{n-1}, (h_X^t)^{m+2} \rangle_X - \langle h_B^{n-1}, h_X^{m+2} \rangle_X.
\end{aligned}$$

Lemma 5.3.2 together with the Chen–Tian style formula Theorem 4.3.4 and the intersection theory formula Proposition 2.5.6 for $W_1(\mathcal{X}, \mathcal{H})$ then immediately imply the result. For example, applying Lemma 5.3.2 to the formula for $\tilde{J}(\varphi_t)$ in terms of Deligne pairings, we get

$$\lim_{t \to 0} \frac{\widetilde{J}(\varphi_t)}{t} = L^{n-1} \cdot \mathcal{H}^{m+2}.$$

From the Chen–Tian formula for $\mathcal{N}$, we have

$$\mathcal{N}(\varphi_t) = \binom{m+n}{n-1} \left(\widetilde{H}(\varphi_t) + \widetilde{R}(\varphi_t) + A_1(X, H) \widetilde{I}(\varphi_t) + \frac{m}{m+2} A_0(X, H) \widetilde{J}(\varphi_t) \right),$$

and from Proposition 2.5.6,

$$W_1(\mathcal{X}, \mathcal{H}) = \binom{m+n}{n-1} (C_1(\mathcal{X}, \mathcal{H}) + C_2(\mathcal{X}, \mathcal{H}) + C_3(\mathcal{X}, \mathcal{H})),$$

where

$$\begin{aligned}
C_1(\mathcal{X}, \mathcal{H}) &= \frac{m}{m+2} A_0(X, H) (L^{n-1} \cdot \overline{\mathcal{H}}^{m+2}), \\
C_2(\mathcal{X}, \mathcal{H}) &= A_1(X, H) (L^n \cdot \overline{\mathcal{H}}^{m+1}), \\
C_3(\mathcal{X}, \mathcal{H}) &= (L^{n-1} \cdot \overline{\mathcal{H}}^{m+1} \cdot K_{\overline{\mathcal{X}}/\mathbb{P}^1}).
\end{aligned}$$

It follows that

$$\frac{m}{m+2} A_0(X, H) \frac{\widetilde{J}(\varphi_t)}{t} \to C_1(\mathcal{X}, \mathcal{H})$$

as $t \to \infty$. Applying this same argument to $\widetilde{H}(\varphi_t) + \widetilde{R}(\varphi_t)$ and $\widetilde{I}(\varphi_t)$, and comparing terms in the Chen–Tian formula to terms in $W_1(\mathcal{X}, \mathcal{H})$, we get

$$\lim_{t\to\infty} \frac{\mathcal{N}(\varphi_t)}{t} = W_1(\mathcal{X}, \mathcal{H}). \qquad \square$$

Theorem 5.3.3. *Let $(X, H) \to (B, L)$ be a fibration that admits an optimal symplectic connection in the class $c_1(H)$, and let $(\mathcal{X}, \mathcal{H})$ be a fibration degeneration of (X, H) that is product-type. Then $W_0(\mathcal{X}, \mathcal{H}) \geq 0$, and if $W_0(\mathcal{X}, \mathcal{H}) = 0$ then $W_1(\mathcal{X}, \mathcal{H}) \geq 0$ with equality if and only if $(\mathcal{X}, \mathcal{H})$ is a product fibration degeneration.*

Proof. We may assume that $W_0(\mathcal{X}, \mathcal{H}) = 0$, since the fibres (X_b, H_b) are K-polystable. Noting that any optimal symplectic connection is a critical point of the functional $\mathcal{N}$, the fact that $W_1(\mathcal{X}, \mathcal{H}) \geq 0$ follows from the previous theorem and convexity of the functional $\mathcal{N}$ along geodesics.

If $W_1(\mathcal{X}, \mathcal{H}) = 0$ then the geodesic ray φ_t must originate from a one-parameter subgroup of $\mathrm{Aut}_0(\pi)$, by Theorem 4.1.1. The fibration degeneration is then isomorphic to the product fibration degeneration determined by the one-parameter subgroup. This can be seen by the result [SD20, Lemma 3.11]; note that in Sjöström Dyrefelt's terminology, our geodesic ray determines a subgeodesic ray on the test configuration $(\mathcal{X}, kL + \mathcal{H})$ for all k sufficiently large. The result then says that there can be at most one normal test configuration compatible with our subgeodesic ray, and hence it must be the induced product test configuration. $\square$

Corollary 5.3.4. *Let $(X, H) \to (B, L)$ be a fibration whose fibres X_b are all isomorphic as complex manifolds. Then (X, H) is polystable with respect to fibration degenerations that are smooth over $B \times \mathbb{C}$.*

Proof. This follows immediately from the previous theorem, together with Lemmas 5.2.2 and 5.2.3. $\square$

Remark 5.3.5. We have chosen here to work in the algebraic setting, however the results carry to the Kähler setting with minimal effort. Namely, in place of Lemma 5.3.2, we instead use the result of Sjöström Dyrefelt [SD18, Theorem 3.6], and work with Kähler forms instead of hermitian metrics.

We end by giving a large class of examples of product-type fibration degenerations.

Example 5.3.6. First, consider the case where $X = \mathbb{P}(V)$ is a projectivised vector bundle over B. If $W \subset V$ is a vector subbundle, then there is a degeneration of vector bundles over B from V to $W \oplus (V/W)$. That is, there is a sheaf $\mathcal{V}$ on $B \times \mathbb{C}$ that is

flat over $\mathbb{C}$, and whose general fibre over $\mathbb{C}$ is the bundle V and whose central fibre is $W \oplus (V/W)$. This degeneration of vector bundles naturally induces a fibration degeneration of X simply by taking $\mathcal{X} := \mathbb{P}(\mathcal{V})$ and $\mathcal{H} := \mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)$. More generally, one could take a flag $0 = W_0 \subset W_1 \subset \cdots \subset W_k = V$ of vector subbundles, and this will similarly induce a product-type degeneration.

Example 5.3.7. This example is due to J. McCarthy, and will appear in his upcoming PhD thesis. Let $Q \to B$ be a holomorphic principal G-bundle, where G is a reductive group, let (Y, L_Y) be a polarised variety, and let $\rho : G \to \mathrm{Aut}(Y, L_Y)$ be a representation. Letting $X := Q \times_\rho Y$ be the associated bundle with induced relatively ample line bundle $H := Q \times_\rho L_Y$, we have a fibration $(X, H) \to (B, L)$ all of whose fibres are isomorphic to (Y, L_Y).

Recall a subgroup $P \subset G$ is *parabolic* if it contains a maximal solvable subgroup, called a *Borel subgroup*. Parabolic subgroups can be described in the following way: a subgroup $P \subset G$ is parabolic if and only if there exists a one-parameter subgroup $\lambda : \mathbb{C}^* \to P$ such that

$$P = \{g \in G : \lim_{t \to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists}\}.$$

In concrete terms, we can always embed the reductive group G in some $\mathrm{GL}(N, \mathbb{C})$, such that P consists of matrices in G of the form

$$\begin{pmatrix} A_1 & * & * & \cdots & * \\ 0 & A_2 & * & \cdots & * \\ 0 & 0 & A_3 & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_k \end{pmatrix}$$

where the A_i are invertible square matrices of fixed size $a_i \times a_i$, the $*$'s denote arbitrary matrices of the appropriate sizes, and $\sum_{i=1}^k a_i = N$. The one-parameter subgroup λ can be taken to be of the form

$$\lambda(t) := \begin{pmatrix} t^{r_1} I_{a_1} & 0 & 0 & \cdots & 0 \\ 0 & t^{r_2} I_{a_2} & 0 & \cdots & 0 \\ 0 & 0 & t^{r_3} I_{a_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & t^{r_k} I_{a_k} \end{pmatrix}$$

where $r_1 > r_2 > \cdots > r_k$ and I_{a_i} is the identity matrix of size a_i. Writing L for the set of block-diagonal matrices with block sizes $(a_1, \ldots, a_k)$, and writing U for the set of strictly upper block triangular matrices in P with diagonal entries all 1, we have

$P = U \rtimes L$. The subset U is the *unipotent radical* of P, and is independent of the choice of λ. The reductive subgroup L is called a *Levi subgroup* of P; it depends on the choice of λ but is unique up to conjugation.

Given a parabolic subgroup $P \subset G$, a *reduction of the structure group* of Q to P is by definition a choice of open cover $\{U_\alpha\}$ of B together with holomorphic transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to P$ for the principal bundle Q which take values in P. Such reductions notably arise in the definition of stability for principal bundles, given in [AB01]. Given a reduction of structure group to P, we can define a degeneration of the principal bundle as follows. On $B \times \mathbb{C}$, take the open cover $\{U_\alpha \times \mathbb{C}\}$, and define transition functions $\tilde{g}_{\alpha\beta} : (U_\alpha \cap U_\beta) \times \mathbb{C} \to P$ by

$$\tilde{g}_{\alpha\beta}(b,t) := \begin{cases} \lambda(t) g_{\alpha\beta}(b) \lambda(t)^{-1} & \text{for } t \neq 0, \\ \lim_{t\to 0} \lambda(t) g_{\alpha\beta}(b) \lambda(t)^{-1} & \text{for } t = 0. \end{cases}$$

Let $\mathcal{Q}$ be the principal P-bundle on $B \times \mathbb{C}$ determined by these transition functions. The fibre of $\mathcal{Q}$ over $t \neq 0$ isomorphic to the original principal bundle Q, however the fibre over $t = 0$ is a principal P-bundle Q_0 that may be different to Q. Note that Q_0 comes equipped with a reduction of the structure group from P to L.

Taking the associated bundle $(\mathcal{X}, \mathcal{H}) := \mathcal{Q} \times_\rho (Y, L_Y)$, we obtain a fibration degeneration of (X, H) that is of product-type. To write down the $\mathbb{C}^*$-action on this fibration degeneration, let us trivialise $\mathcal{Q}$ over $U_\alpha \times \mathbb{C}$, so that $\mathcal{Q}|_{U_\alpha \times \mathbb{C}} \cong U_\alpha \times \mathbb{C} \times P$. An element of $(\mathcal{X}, \mathcal{H})|_{U_\alpha \times \mathbb{C}}$ can then be written as $(b, t, [p, (y, \ell)])$, where $b \in U_\alpha$, $t \in \mathbb{C}$, and $[p, (y, \ell)]$ is an equivalence class in the Cartesian product $P \times (Y, L_Y)$. The $\mathbb{C}^*$-action is defined by

$$s \cdot (b, t, [p, (y, \ell)]) := (b, st, [\lambda(t) p \lambda(t)^{-1}, \rho(\lambda(t))(y, \ell)]).$$

It is straightforward to check that these actions respect the transition functions $\tilde{g}_{\alpha\beta}$, and so glue to a well-defined $\mathbb{C}^*$-action.

We remark that any fibration whose fibres are all isomorphic arises from a holomorphic principal bundle; this is essentially a consequence of the Fischer–Grauert theorem [FG65], which states that a proper holomorphic fibration whose fibres are all isomorphic is locally trivial. For more information on fibrations associated to principal bundles, see the upcoming PhD thesis of J. McCarthy.

Chapter 6

Deformations and families of weighted cscK metrics

In this chapter we study the weighted cscK metrics introduced by Lahdili [Lah19]. First we consider deformations of weighted cscK manifolds in two different contexts. We begin by showing that given a (v, w)-cscK manifold, one can deform the weight functions $v, w : P \to \mathbb{R}_{>0}$ to produce many examples of weighted extremal manifolds. We then show that for a given family of compact Kähler manifolds with a fibrewise T-action, if a fibre admits a weighted (v, w)-cscK metric, then neighbouring fibres admit weighted (v, w)-cscK metrics if and only if they are weighted (v, w)-K-polystable with respect to smooth degenerations. This generalises similar results on cscK metrics by Brönnle [Brö11] and Székelyhidi [Szé10], and also on Kähler–Ricci solitons by Inoue [Ino19b]. This relies on the fact that existence of a weighted cscK metric implies weighted K-polystability with respect to smooth degenerations, which is a recent result of Apostolov–Jubert–Lahdili [AJL21].

We next introduce and study a suitable class of relatively weighted cscK fibrations. Our definition is a holomorphic proper surjective submersion $X \to B$, admitting a fibrewise holomorphic T-action and a relatively Kähler metric ω such that the moment polytope for each fibre is the same polytope P, and the restriction of ω to every fibre is (v, w)-cscK. Our main result is that any such fibration determines a unique *weighted Weil–Petersson metric*, compatible with and determined by the weighted cscK metrics on the fibres. It is a real closed semipositive $(1, 1)$-form on the base, and is strictly positive whenever the family of complex manifolds is effective. This construction generalises the classical Weil–Petersson metric in the cscK setting due to Fujiki–Schumacher [FS90], shows that the classical Weil–Petersson metric is the correct choice when the fibres are extremal, and gives a completely new metric when the fibres are Kähler–Ricci solitons. We end the section with examples of families of

extremal metrics and Kähler–Ricci solitons, whose weighted Weil–Petersson metrics are nontrivial.

6.1 Deformations of weighted cscK metrics

We recall the notations and conventions of Chapter 2.6. We let:

- (X, ω) be a compact Kähler manifold,
- T be a compact real torus acting on (X, ω) by holomorphic isometries,
- $\mu : X \to \mathfrak{t}^*$ be a moment map for the T-action, meaning

  $$d\langle \mu, \xi \rangle = \omega(-, \xi)$$

  for all $\xi \in \mathfrak{t}$, where $\langle \, , \rangle$ denotes the canonical pairing on $\mathfrak{t}^* \otimes \mathfrak{t}$ and we abuse notation by identifying ξ with the vector field it generates on X,
- $P := \mu(X) \subset \mathfrak{t}^*$ be the moment polytope, and
- $v, w : P \to \mathbb{R}_{>0}$ be smooth positive functions.

Given a Kähler potential $\varphi \in \mathcal{K}$ with respect to ω, the map $\mu_\varphi : X \to \mathfrak{t}^*$ is defined by $\mu_\varphi(x)(\xi) := \mu(x) + d^c\varphi(\xi)$ for any $x \in X$ and $\xi \in \mathfrak{t}$, where we again identify ξ with a vector field on X. The map μ_φ is then a moment map with respect to the Kähler metric ω_φ, having the property that $\mu_\varphi(X) = P$. We always choose this normalisation for the moment map, so that the moment polytope is fixed and the weight functions v, w are defined.

6.1.1 Variation of weights

In this section we consider the existence of weighted cscK metrics when one perturbs the weight functions $v, w : P \to \mathbb{R}_{>0}$. Let $v_t : (-\epsilon, \epsilon) \to C^\infty(P, \mathbb{R}_{>0})$ be any smooth path with $v_0 = v$.

Let $\mathcal{K}^T$ denote the space of T-invariant Kähler potentials with respect to ω. The tangent space to $\mathcal{K}^T$ is then canonically identified with the space $C^\infty(X, \mathbb{R})^T$ of T-invariant smooth functions. At each tangent space $T_\varphi\mathcal{K}^T \cong C^\infty(X, \mathbb{R})^T$, we have the inner product

$$\langle f, g \rangle_{w,\varphi} := \int_X f g \, w(\mu_\varphi) \, \omega_\varphi^n.$$

In $T_\varphi \mathcal{K}^T$, let E_φ denote the space of T-invariant holomorphy potentials with respect to the metric ω_φ. The spaces E_φ form a finite-dimensional vector subbundle E of $T\mathcal{K}^T$. Consider the subbundle of finite codimension $E^\perp$, where the orthogonal complement is taken with respect to the L^2-inner product above.

Lemma 6.1.1. *The distribution $E^\perp$ is integrable, so there exists a submanifold $\mathcal{V}$ of finite codimension in $\mathcal{K}^T$ containing the origin, whose tangent space at φ is $E^\perp_\varphi$.*

Proof. Let $f_1, \dots, f_r$ be a basis of E_0. This extends naturally to a frame of the vector bundle E, noting that $E_0 \cong E_\varphi$ via the map

$$f \mapsto f + \nabla_0 f(\varphi),$$

where ∇_0 is the $(1,0)$-gradient with respect to $\omega = \omega_0$; see [Szé14, Lemma 4.10]. In fact, if ∇_φ is the $(1,0)$-gradient operator with respect to ω_φ, and $\nabla_0 f =: \xi$, then this map satisfies

$$\nabla_0 f = \nabla_\varphi(f + \nabla_0 f(\varphi)) = \nabla_\varphi(f + \xi(\varphi)) = \xi. \tag{6.1}$$

That is, the holomorphy potential for ξ with respect to ω_φ is $f + \nabla_0 f(\varphi) = f + \xi(\varphi)$.

Now $E^\perp_\varphi$ is the kernel of $\alpha_\varphi : T_\varphi \mathcal{K}^T \to \mathbb{R}^r$,

$$\alpha_\varphi(\psi) := \left(\int_X \psi(f_1 + \nabla_0 f_1(\varphi))\, w(\mu_\varphi)\, \omega_\varphi^n, \dots, \int_X \psi(f_r + \nabla_0 f_r(\varphi))\, w(\mu_\varphi)\, \omega_\varphi^n \right).$$

We claim that α is a closed 1-form on $\mathcal{K}^T$ with values in $\mathbb{R}^r$, and it is pointwise surjective. Since $\mathcal{K}^T$ is contractible, it will integrate to a smooth function $\mathcal{K}^T \to \mathbb{R}^r$, whose inverse image of 0 is the required integral submanifold by the inverse function theorem applied to Sobolev space approximations of $\mathcal{K}^T$.

Surjectivity is immediate from the fact $f_1, \dots, f_r$ is a basis of E_φ, and $\langle\, , \rangle_{w,\varphi}$ is non-degenerate. For closedness, it suffices to consider a single component 1-form $\beta_\varphi(\psi) := \int_X \psi(f + \nabla_0 f(\varphi))\, w(\mu_\varphi)\, \omega_\varphi^n$. We then need to show $\frac{d}{dt}\big|_{t=0} \beta_{\varphi + t\psi_2}(\psi_1)$ is symmetric in ψ_1 and ψ_2. Let ξ be the vector field generated by f, so that $\nabla_0 f = \xi = \nabla_\varphi(f + \nabla_0 f(\varphi))$ by (6.1). Then

$$\beta_{\varphi + t\psi_2}(\psi_1) = \int_X \psi_1(f + \xi(\varphi + t\psi_2)) w(\mu_{\varphi + t\psi_2}) \omega^n_{\varphi + t\psi_2},$$

and, differentiating under the integral with respect to t,

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} \beta_{\varphi+t\psi_2}(\psi_1) &= \int_X \psi_1 \xi(\psi_2) w(\mu_\varphi)\omega_\varphi^n \\
&\quad + \int_X \psi_1 (f+\xi(\varphi))(\Delta_{\varphi,w}\psi_2) w(\mu_\varphi)\omega_\varphi^n \qquad (6.2)\\
&= \int_X \psi_1 \xi(\psi_2) w(\mu_\varphi)\omega_\varphi^n \\
&\quad - \int_X \langle \nabla_\varphi \psi_1, \nabla_\varphi \psi_2\rangle_\varphi (f+\xi(\varphi)) w(\mu_\varphi)\omega_\varphi^n \\
&\quad - \int_X \psi_1 \langle \nabla_\varphi (f+\xi(\varphi)), \nabla_\varphi \psi_2\rangle_\varphi w(\mu_\varphi)\omega_\varphi^n \\
&= -\int_X \langle \nabla_\varphi \psi_1, \nabla_\varphi \psi_2\rangle_\varphi (f+\xi(\varphi)) w(\mu_\varphi)\omega_\varphi^n \\
&\quad + \int_X \psi_1 \xi(\psi_2) w(\mu_\varphi)\omega_\varphi^n - \int_X \psi_1 \xi(\psi_2) w(\mu_\varphi)\omega_\varphi^n \\
&= -\int_X \langle \nabla_\varphi \psi_1, \nabla_\varphi \psi_2\rangle_\varphi (f+\xi(\varphi)) w(\mu_\varphi)\omega_\varphi^n. \qquad (6.3)
\end{aligned}$$

Here $\Delta_{\varphi,w}$ is the weighted Laplacian, defined by

$$\Delta_{\varphi,w} h := \frac{-1}{w(\mu_\varphi)} \nabla_\varphi^* (w(\mu_\varphi)\nabla_\varphi h) = \Delta_\varphi h + \frac{1}{w(\mu_\varphi)} \nabla_\varphi h(w(\mu_\varphi)),$$

and the second line (6.2) follows from

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} (w(\mu_{\varphi+t\psi_2})\omega_{\varphi+t\psi_2}^n) &= \left.\frac{d}{dt}\right|_{t=0} (w(\mu_\varphi + t d^c\psi_2)\omega_{\varphi+t\psi_2}^n) \\
&= \sum_{i=1}^r w_{,i}(\mu_\varphi) d^c\psi_2(\xi_i)\omega_\varphi^n + n w(\mu_\varphi) i\partial\bar\partial\psi_2 \wedge \omega_\varphi^{n-1} \\
&= \sum_{i=1}^r w_{,i}(\mu_\varphi)\omega_\varphi(\nabla_\varphi\psi_2, \xi_i)\omega_\varphi^n + w(\mu_\varphi)(\Delta_\varphi\psi_2)\omega_\varphi^n \\
&= \left(\sum_{i=1}^r w_{,i}(\mu_\varphi) d\mu_\varphi^{\xi_i}(\nabla_\varphi\psi_2) + w(\mu_\varphi)\Delta_\varphi\psi_2\right)\omega_\varphi^n \\
&= w(\mu_\varphi)\left(\frac{1}{w(\mu_\varphi)}\nabla_\varphi\psi_2(w(\mu_\varphi)) + w(\mu_\varphi)\Delta_\varphi\psi_2\right)\omega_\varphi^n \\
&= w(\mu_\varphi)\Delta_{\varphi,w}\psi_2\,\omega_\varphi^n.
\end{aligned}$$

Here we have chosen a basis $\xi_1, \ldots, \xi_r$ for $\mathfrak{t}$ with dual basis $\xi^1, \ldots, \xi^r$, and $w_{,i}$ is the partial derivative of w in the ξ^i-direction. Since the final line (6.3) is symmetric in ψ_1 and ψ_2, β is closed. □

Consider the map $F : (-\epsilon, \epsilon) \times \mathcal{V} \to T\mathcal{V}$, given by

$$(t, \varphi) \mapsto \left(\varphi, p_\varphi \left(\frac{S_{v_t}(\omega_\varphi)}{w(\mu_\varphi)} \right) \right),$$

where p_φ is the orthogonal projection from $T_\varphi \mathcal{K}^T \cong C^\infty(X)^T$ to $E_\varphi^\perp$. We write F_t for the restriction of F to $\{t\} \times \mathcal{V}$, which is then a section of the vector bundle $T\mathcal{V} \to \mathcal{V}$. By [Lah19, Lemma B.1] the derivative of F_0 at the origin in $C^\infty(X, \mathbb{R})^T$ is the elliptic operator $\frac{v(\mu)}{w(\mu)} \mathcal{D}_v^* \mathcal{D}$, whose kernel is the finite-dimensional space E_0 of T-invariant holomorphy potentials. Here $\mathcal{D} := \bar{\partial} \nabla^{1,0}$, and $\mathcal{D}_v^* f := \frac{1}{v(\mu)} \mathcal{D}^*(v(\mu) f)$. The restriction of $\frac{v(\mu)}{w(\mu)} \mathcal{D}_v^* \mathcal{D}$ to $E_0^\perp$ has image in $E_0^\perp$, and is an isomorphism since it is a self-adjoint elliptic operator with trivial kernel. It follows from the inverse function theorem (applied to successive Banach manifold approximations of $\mathcal{V}$), we can solve the equation $p_\varphi \left(\frac{S_{v_t}(\omega_\varphi)}{w(\mu_\varphi)} \right) = 0$ for all t sufficiently small. So the function $\frac{S_{v_t}(\omega_\varphi)}{w(\mu_\varphi)}$ is a T-invariant holomorphy potential, and hence we have proven:

Theorem 6.1.2. *Let (X, ω) be a (v, w)-cscK manifold, and let v_t be a 1-parameter deformation of the weight function v. Then for t sufficiently small, there exists a (v_t, w)-extremal metric in the class $[\omega]$.*

Remark 6.1.3. We recall our notion of weighted extremal metric is different to Lahdili's: see Definition 2.6.3 and Remark 2.6.4. Namely, we do not require for a weighted extremal metric that $S_v(\omega)/w(\mu)$ is given by an affine linear function on the moment polytope, however this will be the case if the torus T is maximal.

6.1.2 Variation of complex structure

In this section we consider the existence of weighted cscK metrics when one deforms the complex structure of a weighted cscK manifold. Such results were originally obtained by Brönnle [Brö11] and Székelyhidi [Szé10] in the cscK case, and generalised to the Kähler–Ricci soliton setting by Inoue [Ino19b]. Our exposition here will follow these works closely, as the proofs are very similar and require only minor modifications.

Theorem 6.1.4. *Let $X \to D$ be a family of compact complex manifolds, where $D \subset \mathbb{C}^n$ is a neighbourhood of 0. We assume X is equipped with a fibrewise holomorphic T-action and a T-invariant relatively Kähler metric ω. Let $v, w : P \to \mathbb{R}$ be positive smooth functions, and suppose the central fibre X_0 admits a (v, w)-weighted cscK metric. Then there is a neighbourhood U of 0 in D such that if $t \in U$ then X_t admits a (v, w)-weighted cscK metric if and only if X_t is (v, w)-weighted K-polystable with respect to smooth test configurations.*

The forward implication is due to the recent work of Apostolov–Jubert–Lahdili [AJL21], who showed that existence of a weighted cscK metric implies weighted polystability with respect to smooth degenerations, under any circumstances.

To prove the main theorem, we fix a Kähler form ω in the class α, and apply Ehresmann's theorem [Ehr51] to smoothly trivialise the fibration so that $X \cong D \times M$, where M is the underlying smooth manifold of X_0. In the proof of Ehresmann's theorem, one takes a Riemannian metric on X, uses it to lift vector fields on D to horizontal vector fields on X, then flows along these vector fields to trivialise the fibration. By taking a Riemannian metric which is T-invariant, this can be done T-equivariantly, so the induced T-action on $D \times M$ is $t \cdot (d, y) = (d, ty)$. Now, define

$$\mathcal{J}^T := \{T\text{-invariant almost complex structures on } M \text{ compatible with } \omega\}.$$

Here, an almost complex structure J is compatible with ω if

$$\omega(-, J-)$$

is a hermitian metric. The tangent space to $J \in \mathcal{J}^T$ is the set of $A \in \Omega^{1,1}(M, J)^T$ such that

$$\omega(Av, w) + \omega(v, Aw) = 0$$

fo all $v, w \in TM$. The vector space $T_J\mathcal{J}^T$ has a natural complex structure, given by left multiplication by J.

Let $\mathcal{G}^T$ be the group of T-commuting hamiltonian symplectomorphisms of (M, ω). There is a natural action of $\mathcal{G}^T$ on the space $\mathcal{J}^T$ given by pullback. The Lie algebra of $\mathcal{G}^T$ is the set $C_0^\infty(M, \mathbb{R})^T$ of T-invariant integral zero smooth functions on Y, and the infinitesimal action is $\mathcal{D} : C_0^\infty(M, \mathbb{R})^T \to T_J\mathcal{J}^T$, where

$$\mathcal{D}f := \bar{\partial}\nabla_J^{1,0} f.$$

We equip the Lie algebra $C_0^\infty(M)^T$ with the L^2-inner product

$$\langle f, g\rangle = \int_M fg\, w(\mu)\omega^n.$$

We similarly define the L_k^2-Sobolev inner product by

$$\langle f, g\rangle_k := \sum_{i \le k} \int_M \langle \nabla^i f, \nabla^i g\rangle\, w(\mu)\omega^n.$$

The action of $\mathcal{G}^T$ has a "formal complexification", meaning there exists a foliation of $\mathcal{J}^T$ whose tangent space at each J is given by the image of the complexified

infinitesimal action $\mathcal{D} : C_0^\infty(M,\mathbb{C})^T \to T_J\mathcal{J}^T$. We define the *complex orbit* of a point $J \in \mathcal{J}^T$ to be the leaf of the foliation passing through J. If J is integrable, its complex orbit can be identified with the space of T-invariant Kähler metrics in the Kähler class $[\omega]$ [Szé10, p. 1083]. We equip the tangent space $T_J\mathcal{J}^T$ with the L^2-inner product

$$\langle A, B\rangle_v := \int_M \mathrm{Tr}(AB^*)v(\mu)\,\omega^m,$$

which in turn gives rise to the Kähler form

$$\Omega_v(A,B) := \langle JA, B\rangle_v.$$

The formal complexified action then admits a moment map $m : \mathcal{J}^T \to C_0^\infty(M)^T$, given by:

$$m(J) := \frac{S_v(\omega,J)}{w(\mu)} - c_{v,w}, \tag{6.4}$$

where $S_v(\omega,J)$ is the hermitian v-scalar curvature [Lah19, Theorem 4], and $c_{v,w}$ is the topological constant in the weighted cscK equation; see Lemma 2.6.7. That is to say, for any $A \in T_J\mathcal{J}^T$ and $h \in C_0^\infty(M)^T$,

$$\langle dm|_J(A), h\rangle_v = \Omega_v(A, \mathcal{D}h).$$

The operator $\mathcal{D}$ fits into an elliptic complex

$$C_0^\infty(M,\mathbb{C})^T \xrightarrow{\mathcal{D}} T_J\mathcal{J}^T \xrightarrow{\bar\partial} \Omega^{1,2}(M,J)^T. \tag{6.5}$$

We consider the harmonic theory of this complex with respect to the hermitian metric $v(\mu)^{1/m}\omega(-,J-)$. For example, the inner product on $C_0^\infty(M,\mathbb{C})^T$ is

$$\langle f, g\rangle := \int_M f\bar g\, v(\mu)\,\omega^m.$$

By standard elliptic complex theory, the harmonic space

$$\tilde H_v^1 := \{A \in T_J\mathcal{J}^T : \bar\partial A = 0 \text{ and } \mathcal{D}_v^* A = 0\}$$

is finite-dimensional, and naturally isomorphic to the cohomology group H^1 of the complex. Here $\mathcal{D}_v^* A = \frac{1}{v}\mathcal{D}(vA)$ is the adjoint of $\mathcal{D}$ with respect to the inner products depending on v.

We let J_0 be the underlying almost complex structure of the complex manifold X_0 above $0 \in D$, and write K^T for the group of T-commuting hamiltonian isometries of (M,ω,J_0), with Lie algebra $\mathfrak{k}^T$. This group acts naturally on the space $\tilde H_v^1$, and the complexification G^T of K^T is a complex Lie group acting on both $\tilde H_v^1$ and $\mathcal{J}^T$.

Proposition 6.1.5. *There exists a ball $B \subset \tilde{H}^1_v$ containing the origin and a K^T-equivariant map*

$$\Phi : B \to \mathcal{J}^T$$

satisfying the following properties:

1. *The image $\Phi(B)$ intersects the complex orbit of every integrable $J \in \mathcal{J}^T$ sufficiently near J_0,*

2. *If $x, x' \in B$ are in the same G^T-orbit and $\Phi(x)$ is integrable, then $\Phi(x)$ and $\Phi(x')$ are in the same complex orbit,*

3. *For every $x \in B$, $S_v(\Phi(x), \omega)/w(\mu) - c_{v,w} \in \mathfrak{k}^T$.*

We already know m in (6.4) is a moment map for the infinite-dimensional picture, so the third point tells us we can reduce the problem to a finite-dimensional GIT question by taking a slice of the formal complexified action of the gauge group of hamiltonian symplectomorphisms.

Proof. According to Székelyhidi [Szé10], for $v = 1$ there exists a ball $B_1 \subset \tilde{H}^1_1$ and a K-equivariant map $\Phi_1 : B_1 \to \mathcal{J}^T$ satisfying the first two conditions. In our situation, we note there is a canonical equivariant isomorphism of the harmonic spaces $\tilde{H}^1_v \cong \tilde{H}^1_1$, given by their identifications with the first cohomology group of the complex (6.5). Hence we can equivalently consider $\Phi_1 : B_1 \to \mathcal{J}^T$, where $B_1 \subset \tilde{H}^1_v$.

We now need to deform our map Φ_1 so as to satisfy the third property. We work with the L^2_k-Sobolev spaces in what follows. Let ϕ be an element in a small neighbourhood U of $0 \in L^2_k(M)^T$, and let $J \in \mathcal{J}^T$. For $t \in [0, 1]$, let

$$\omega_t := \omega + t dd^c \phi,$$

where the d^c operator is with respect to the fixed J. Then $\frac{d}{dt}\omega_t = d\alpha$, where $\alpha = d^c\phi$ is a fixed 1-form with coefficients in L^2_{k-1}. Let X_t be the L^2_{k-2}-vector field[1] on M dual to $-\alpha$ with respect to the symplectic form ω_t, and let f_t be the time t flow of this time-dependent vector field. Then $f_t^*\omega_t = \omega$ for all t, and we define $F_\phi(J) := f_1^*J$, which for ϕ small is a L^2_{k-2}-perturbation of the complex structure J, lying in the same complex orbit.

[1]Here we lose two derivatives when taking the dual of α, since ω_t is L^2_{k-2}.

We'll denote by $\mathcal{J}_k^T$ the space of T-invariant almost complex structures compatible with ω and having L_k^2-coefficients. Note the derivative of the map $P : U \to \mathcal{J}_{k-2}^T$, $P(\phi) := F_\phi(J)$ at 0 is given by

$$dP|_0(\psi) = J\mathcal{D}\psi, \tag{6.6}$$

where $\mathcal{D} = \bar{\partial}\nabla^{1,0}$ is half the Lichnerowicz operator of J. We note P maps to L_{k-2}^2 since it is the flow of an L_{k-2}^2-vector field; see for example [Ino19b, p. 20].

Now, let us consider $(\mathfrak{k}^T)_k^\perp$, the orthogonal complement of $\mathfrak{k}^T \subset L_k^2(M)^T$, and let $U_k \subset (\mathfrak{k}^T)_k^\perp$ be a small neighbourhood of the origin. Define the map $G : B_1 \times U_k \to \mathfrak{k}_{k-4}^\perp$,

$$G(x, \phi) := (p \circ m)(F_\phi(\Phi_1(x))),$$

where p is the orthogonal projection $p : L_{k-4}^2(M)^T \to (\mathfrak{k}^T)_{k-4}^\perp$ and m is the moment map (6.4). To see that G maps to L_{k-4}^2, note the hermitian scalar curvature of (M, ω, J) is given by first taking the curvature ρ^∇ of the hermitian connection $\nabla_X Y := D_X^g Y - \frac{1}{2}J(D_X^g J)Y$ on TM, where D^g is the Levi–Civita connection of $g(-,-) = \omega(-, J-)$, and then contracting with ω:

$$S(\omega)\omega^n = n\rho^\nabla \wedge \omega^{n-1}.$$

In our situation, g is L_{k-2}^2 since $J = F_\phi(\Phi_1(x))$ is, so its Christoffel symbols are L_{k-3}^2 as they involve first derivatives of the metric coefficients. Hence the connection ∇ also has L_{k-3}^2-coefficients, and finally taking its curvature produces a 2-form in L_{k-4}^2.

Writing G_x for the restriction of G to $\{x\} \times U_k$, by (6.6) and the moment map equation we have

$$dG_0|_0(\psi) = \frac{v(\mu)}{w(\mu)}\mathcal{D}_v^*\mathcal{D}\psi,$$

which is an isomorphism $(\mathfrak{k}^T)_k^\perp \to (\mathfrak{k}^T)_{k-4}^\perp$. It follows from the inverse function theorem that we may find a neighbourhood B of 0 in B_1 and a function $\Phi : B \to \mathcal{J}_{k-2}^T$ such that $p(m(\Phi(x))) = 0$ for all $x \in B$. Since this perturbation is unique and compatible with the inclusions of Sobolev spaces, we have proven there is a smooth map $\Phi : B \to \mathcal{J}^T$ satisfying the third condition. Note the first two conditions of the proposition are unchanged by the perturbation, since we are perturbing within the same complex orbit of the original map Φ_1. □

Given our slice $\Phi : B \to \mathcal{J}^T$ of the formal complexified action, let us pull back the symplectic structure on $\mathcal{J}^T$ to a symplectic form Ω on B. Since Φ is K^T-equivariant, the moment map m on $\mathcal{J}^T$ also pulls back to a moment map

$$m(x) = \frac{S_v(\Phi(x), \omega)}{w(\mu)} - c_{v,w}.$$

Lemma 6.1.6. *We may shrink B so that the following holds. If $y \in B$ is polystable for the G^T-action on $\tilde{H}^1_v$ then there exists a $y_0 \in B$ lying in the G^T-orbit of y such that $m(v_0) = 0$.*

The Kempf–Ness theorem already tells us that we may find $y_0 \in \tilde{H}^1_v$ in the G^T-orbit of y such that $m(y_0) = 0$, so the content of the lemma is that we may take this y_0 to lie in B.

To prove this lemma, the main point is to apply the result of Donaldson below, for which we need to introduce some notation. First, for elements $x \in B$ and $\xi \in \mathfrak{k}^T$, denote by $\sigma_x(\xi) \in T_xB$ the infinitesimal action of ξ at x. Next, let $\mathfrak{k}^T_x$ be the stabiliser of x in $\mathfrak{k}^T$, which is the set of all $\xi \in \mathfrak{k}^T$ for which $\sigma_x(\xi) = 0$. One easily verifies that $Q_x := \sigma_x^* \sigma_x : (\mathfrak{k}^T_x)^\perp \to (\mathfrak{k}^T_x)^\perp$ is a positive self-adjoint operator, where the adjoint $\sigma_x^* : T_xB \to \mathfrak{k}^T_x$ is defined using the metric determined by Ω.

Lemma 6.1.7 (Donaldson [Don01, Proposition 17])**.** *Suppose that $x_0 \in B$ satisfies $m(x_0) \in (\mathfrak{k}^T_{x_0})^\perp$, and let $\delta, \lambda > 0$ be such that $\|Q_x^{-1}\| \leq \lambda$ for all $x = e^{i\xi} x_0$ with $\|\xi\| < \delta$. If $\lambda \|m(x_0)\| < \delta$, then there exists $\eta \in (\mathfrak{k}^T_{x_0})^\perp$ with $\|\eta\| \leq \lambda \|m(x_0)\|$, such that $m(e^{i\eta} x_0) = 0$.*

This result gives us suitable criteria under which we may deform a point in B to a zero of the moment map m. Note the first criterion is satisfied for all $x \in B$, since for any $\xi \in \mathfrak{k}^T_x$ we have

$$\frac{d}{dt} \langle m(tx), \xi \rangle = \Omega_{tx}(x, \sigma_{tx} \xi) = 0,$$

since $\xi \in \mathfrak{k}^T_x$ if and only if $\xi \in \mathfrak{k}^T_{tx}$ for all t. Since $\langle m(tx), \xi \rangle = 0$ at $t = 0$, setting $t = 1$ we get $\langle m(x), \xi \rangle = 0$, so $m(x) \in (\mathfrak{k}^T_x)^\perp$ as claimed. Hence to apply Donaldson's result, we just need suitable estimates on the norms $\|Q_x^{-1}\|$ and $\|m(x)\|$.

Proof of Lemma 6.1.6. Identifying $T_0B \cong \tilde{H}^1_v$, we let Ω_0 be the constant symplectic form on $\tilde{H}^1_v$ determined by Ω. Since y is polystable for the G^T-action, there exists an element of its orbit in $\tilde{H}^1_v$ having minimum norm with respect to the metric determined by Ω_0. Shrinking B so that it is a ball with respect to this metric, we may replace y with such a point.

Expanding the moment map, we have

$$m(ty) = m(0) + t dm|_0(y) + \frac{t^2}{2} \frac{d^2}{dt^2}\bigg|_{t=0} m(ty) + \mathrm{O}(t^3).$$

By assumption the origin maps to a weighted cscK metric, so $m(0) = 0$. Since 0 is a fixed point of the K^T-action, $dm|_0(y) = 0$ as well. Finally, note that for any $\xi \in \mathfrak{k}^T$,

$$\begin{aligned}\left.\frac{d^2}{dt^2}\right|_{t=0} m^\xi(ty) &= \left.\frac{d}{dt}\right|_{t=0} dm^\xi|_{ty}(y) \\ &= \left.\frac{d}{dt}\right|_{t=0} \Omega_{ty}(y, \xi_{ty}) \\ &= \Omega_0(y, \xi_y).\end{aligned}$$

The final line follows since $\xi_0 = 0$ as the origin is a fixed point, and $\xi_{ty} = t\xi_y$. Now, since y minimises the norm with respect to Ω_0 in its orbit, the infinitesimal action $J\xi_y$ of $J\xi$ at y must be orthogonal to y. Hence $\Omega_0(y, \xi_y) = 0$, and therefore $m(ty)$ is $\mathrm{O}(t^3)$ near $t = 0$.

We have obtained a good estimate on $\|m(ty)\|$ for t small, so we must now estimate the norms $\|Q_{ty'}^{-1}\|$, where $y' = e^{i\xi}y$. First note that for small $\delta > 0$, there exists a constant C such that

$$\|\sigma_{y'}(\eta)\|_{\Omega_0}^2 \geq C\|\eta\|^2$$

for all $\eta \in (\mathfrak{k}_{y'}^T)^\perp$ with $\|\eta\| < \delta$. Next, since $\sigma_{ty'} = t\sigma_{y'}$ and since by shrinking B we can bound the metric Ω below by $\frac{1}{2}\Omega_0$, we have

$$\|\sigma_{ty'}(\eta)\|_\Omega \geq \frac{1}{2}t\|\sigma_{y'}(\eta)\|_{\Omega_0}.$$

Hence

$$\langle \sigma_{ty'}^* \sigma_{ty'}(\eta), \eta\rangle = \|\sigma_{ty'}(\eta)\|_\Omega^2 \geq \frac{1}{4}t^2 C\|\eta\|^2,$$

and so $\|Q_{ty'}^{-1}\| < C't^{-2}$. Since $\|m(ty)\| = \mathrm{O}(t^3)$, for t sufficiently small we have $C't^{-2}\|m(ty)\| < \delta$ so by Lemma 6.1.7, there exists an $\eta \in (\mathfrak{k}_y^T)^\perp$ such that $m(e^{i\eta}y) = 0$. □

At last we can prove Theorem 6.1.4.

Proof of Theorem 6.1.4. Given the family $X \to D$, after shrinking D there is an induced map $f : D \to B$ such that $(M, \Phi(f(z))) \cong X_z$ for all $z \in D$, where B is as in Lemma 6.1.6. Note this implies the complex structure $\Phi(f(z))$ is integrable. Let us fix a $z \in D\backslash\{0\}$, and let $x = f(z) \in B$. If x is polystable for the G^T-action on $\tilde{H}_v^1$, then by Lemma 6.1.6 the orbit of x has a zero of the moment map in B. It follows that X_z admits a weighted cscK metric. Hence X_z is weighted K-polystable with respect to smooth degenerations by Apostolov–Jubert–Lahdili [AJL21].

It remains to show that if X_z does not admit a weighted cscK metric, then X_z is not K-polystable with respect to smooth degenerations, and so does not admit a weighted

cscK metric. Since X_z does not admit a weighted cscK metric, x is not polystable. By the Hilbert–Mumford criterion, there exists a 1-parameter subgroup $\rho : \mathbb{C}^* \to G^T$ such that $x' := \lim_{t\to 0} \rho(t)x$ is polystable. Furthermore $x' \in B$ and $\Phi(x')$ is integrable, since integrability of almost complex structures is a closed condition by the Newlander–Nirenberg theorem. We denote by X' the complex manifold $(M, \Phi(x'))$. Since x' is integrable and polystable, X' admits a weighted cscK metric.

Consider the S^1-equivariant holomorphic map $F : \Delta \xrightarrow{t\mapsto\rho(t)\cdot x} B$, where Δ is a small neighbourhood of 0 in $\mathbb{C}$, and we have defined $\rho(0)x := x'$. Pulling back the Kuranishi family [Kur62, Kur65] on B via F, we obtain a complex manifold $\mathcal{X}$ together with a morphism $\mathcal{X} \to \Delta$ such that $\mathcal{X}_t \cong X_z$ for all $t \neq 0$, and $\mathcal{X}_0 \cong X'$. By T-equivariance, $\mathcal{X}$ has a canonical holomorphic fibrewise T-action, and a further local T-commuting action of $\mathbb{C}^*$ by biholomorphisms. This local $\mathbb{C}^*$-action allows us to extend $\mathcal{X}$ to a family over $\mathbb{C}$, which we also denote by $\mathcal{X}$. The family $\mathcal{X} \to \mathbb{C}$ can be smoothly S^1-equivariantly trivialised, $\mathcal{X} \cong M \times \mathbb{C}$, so that the pullback of ω to $\mathcal{X}$ via the smooth projection $\mathcal{X} \to M$ is a relatively Kähler form Ω. Hence we have a T-equivariant Kähler test configuration $(\mathcal{X}, \Omega)$ for the Kähler manifold (X_z, ω) with central fibre (X', ω). Since X' admits a weighted cscK metric, its weighted Futaki invariant vanishes, hence the weighted Donaldson–Futaki invariant of this Kähler test configuration is zero. Since X_z does not admit a weighted cscK metric, this test configuration is not a product configuration, so (X_z, ω) is not weighted K-polystable with respect to smooth degenerations. □

6.2 A v-twisted Weil–Petersson metric

Let $X \to B$ be a holomorphic surjective submersion of compact Kähler manifolds, T a compact torus acting holomorphically on X and preserving the fibres X_b, which all admit (v, w)-cscK metrics. We assume there exists a relatively Kähler metric ω, such that for every $b \in B$ the moment map $\mu_b : X_b \to \mathfrak{t}^*$ has the same image P, and ω_b is (v, w)-cscK for all $b \in B$. As in previous chapters, we denote by ρ the curvature of the metric on the holomorphic vector bundle $\Lambda^m\mathcal{V}$ induced by ω, where $\mathcal{V}$ is the vertical tangent bundle, and by $\rho_{\mathcal{H}}$, ρ_{mixed} and $\rho_{\mathcal{V}}$ its horizontal, mixed and vertical components respectively. We write n for the dimension of B and m for the dimension of a fibre X_b.

Theorem 6.2.1. *Let α be the real $(1,1)$-form on B defined by*

$$\alpha := -2\int_{X/B} v(\mu)\rho_{\mathcal{H}} \wedge \omega^m - \frac{2}{m+1}\int_{X/B} \Delta_{\mathcal{V}}(v(\mu))\,\omega^{m+1}.$$

Then α satisfies the following conditions:

1. *α is closed,*
2. *α is independent of the relatively (v, w)-cscK metric in the class $[\omega]$, and*
3. *α is semipositive, and strictly positive wherever the Kuranishi map $T_bB \to H^1(X_b, TX_b)$ is injective.*

Definition 6.2.2. We call α the *weighted Weil–Petersson metric* of the relatively (v, w)-cscK fibration $X \to B$.

Remark 6.2.3. As mentioned in the Introduction to this chapter, when $v = 1$ and we are in the relatively cscK setting, note that the weighted Weil–Petersson metric reduces to the classical Weil–Petersson metric defined by Fujiki and Schumacher [FS90]. Note that $v = 1$ also encompasses the fibrewise extremal case, so the theorem shows the usual Weil–Petersson metric is a natural choice of base metric in this setting as well. In the fibrewise Kähler–Ricci soliton setting, the construction provides a completely new metric.

Lemma 6.2.4. *The form α is closed.*

Proof. We have

$$\alpha = -2\int_{X/B} v(\mu)\rho \wedge \omega^m + 2\int_{X/B} v(\mu)\rho_{\mathcal{V}} \wedge \omega^m - \frac{2}{m+1}\int_{X/B} \Delta_{\mathcal{V}}(v(\mu))\,\omega^{m+1}.$$

The mixed $(\mathcal{V}, \mathcal{H})$-component of ρ does not appear in these integrals, since ω only has $(\mathcal{H}, \mathcal{H})$ and $(\mathcal{V}, \mathcal{V})$ components. Next,

$$2\int_{X/B} v(\mu)\rho_{\mathcal{V}} \wedge \omega^m = \frac{2}{m+1}\int_{X/B} v(\mu)\mathrm{Scal}(\omega_b)\omega^{m+1}$$

and

$$\begin{aligned} 2v(\mu)\mathrm{Scal}(\omega_b) &= \mathrm{Scal}_v(\omega_b) + 4\Delta_{\mathcal{V}}(v(\mu)) - \mathrm{Tr}(g \circ \mathrm{Hess}(v)) \\ &= c_{v,w}w(\mu) + 4\Delta_{\mathcal{V}}(v(\mu)) - \mathrm{Tr}(g \circ \mathrm{Hess}(v)), \end{aligned}$$

from the fibrewise weighted cscK condition, where g is the Riemannian metric associated to the Kähler metric ω. We get

$$\begin{aligned}\alpha &= -2\int_{X/B} v(\mu)\rho\wedge\omega^m + \frac{c_{v,w}}{m+1}\int_{X/B} w(\mu)\omega^{m+1}\\ &\quad + \frac{2}{m+1}\int_{X/B}\Delta_{\mathcal{V}}(v(\mu))\omega^{m+1} - \frac{1}{m+1}\int_{X/B}\mathrm{Tr}(g\circ\mathrm{Hess}(v))\,\omega^{m+1}\\ &= -2\int_{X/B} v(\mu)\rho\wedge\omega^m + \frac{c_{v,w}}{m+1}\int_{X/B} w(\mu)\omega^{m+1}\\ &\quad + \frac{2}{m+1}\int_{X/B}\sum_i v_{,i}(\mu)\Delta_{\mathcal{V}}(\mu^{\xi_i})\,\omega^{m+1}.\end{aligned} \tag{6.7}$$

Here ξ_i is a choice of basis for the Lie algebra $\mathfrak{t}$ of T, and $v_{,i}$ is the i-th partial derivative of v with respect to the dual basis. The final line follows from the calculation

$$\begin{aligned}&2\Delta_{\mathcal{V}}(v(\mu)) - \mathrm{Tr}(g\circ\mathrm{Hess}(v))\\ =&\sum_{k=1}^{2n}\frac{\partial^2}{\partial x_k^2}(v(\mu)) - \sum_{i,j=1}^{r} v_{,ij}(\mu)g(\xi_i,\xi_j)\\ =&\sum_{k=1}^{2n}\frac{\partial}{\partial x_k}\left(\sum_{i=1}^{r} v_{,i}(\mu)\frac{\partial\mu^{\xi_i}}{\partial x_k}\right) - \sum_{i,j=1}^{r} v_{,ij}(\mu)g(\xi_i,\xi_j)\\ =&\sum_{k=1}^{2n}\sum_{i,j=1}^{r} v_{,ij}(\mu)\frac{\partial\mu^{\xi_i}}{\partial x_k}\frac{\partial\mu^{\xi_j}}{\partial x_k} + \sum_{k=1}^{2n}\sum_{i=1}^{r} v_{,i}(\mu)\frac{\partial^2\mu^{\xi_i}}{\partial x_k^2} - \sum_{i,j=1}^{r} v_{,ij}g(\nabla\mu^{\xi_i},\nabla\mu^{\xi_j})\\ =&2\sum_{i=1}^{r} v_{,i}(\mu)\Delta_{\mathcal{V}}\mu^{\xi_i},\end{aligned}$$

where the calculation is carried out at the centre of a Riemannian normal coordinate system $x_1,\ldots,x_{2n}$ for g. First note the second term of (6.7) is closed, since

$$d(w(\mu)) = -\omega(dw(\mu),-)$$

is vertical and ω has only pure horizontal and vertical parts.[2] Similarly

$$\int_{X/B} d(v_{,i}(\mu))\wedge\Delta_{\mathcal{V}}(\mu^{\xi_i})\omega^{m+1} = 0$$

since $d(v_{,i}(\mu))$ is vertical and ω has no mixed components. We are left with

$$d\alpha = 2\int_{X/B}\sum_i v_{,i}(\mu)\omega(\xi_i,-)\wedge\rho\wedge\omega^m + \frac{2}{m+1}\int_{X/B}\sum_i v_{,i}(\mu)d(\Delta_{\mathcal{V}}(\mu^{\xi_i}))\wedge\omega^{m+1}.$$

[2]Here we have abused notation, identifying $dw(\mu)\in\mathfrak{t}^{**}\cong\mathfrak{t}$ with the vector field it generates on X. We will continue with such notation.

Now

$$d(\Delta_{\mathcal{V}}(\mu^{\xi_i})) = \rho(\xi_i, -),$$

by [Lah19, Lemma 5]. Only the horizontal component of this form contributes to the right-hand integral. In the left-hand integral, the form $\omega(\xi_i, -)$ is vertical, so only the mixed $(\mathcal{V}, \mathcal{H})$ component of ρ contributes. Note that $m\omega(\xi_i, -) \wedge \rho \wedge \omega_{X,\mathcal{V}}^{m-1} = -\rho(\xi_i, -) \wedge \omega_{X,\mathcal{V}}^m$ (the minus sign comes from commuting the horizontal component of ρ_{mixed} with $\omega(\xi_i, -)$, before the contraction is taken). It follows that the two integrals cancel, and $d\alpha = 0$. □

Remark 6.2.5. In the course of the proof, we derived the useful formula

$$\alpha = -2\int_{X/B} v(\mu)\rho \wedge \omega^m - \frac{2}{m+1}\int_{X/B} \langle dv(\mu), \mu_\rho \rangle \omega^{m+1} + \frac{c_{v,w}}{m+1}\int_{X/B} w(\mu)\omega^{m+1},$$

where the pairing $\langle\, , \rangle$ is the natural one on $\mathfrak{t} \otimes \mathfrak{t}^*$, and $\mu_\rho := \Delta_{\mathcal{V}}\mu$ is the ρ-moment map.

Lemma 6.2.6. *α is independent of the choice of relatively weighted cscK metric in the fixed class.*

In the fibrewise cscK case this is due to Dervan–Naumann [DN18].

Remark 6.2.7. Lahdili [Lah19, Section 7] has a similar result in the weighted setting where the base is $\mathbb{P}^1$. S. Donaldson and J. Lotay have pointed out that formula (6.8) below holds if and only if its restriction to every embedded complex disc in B holds. This follows from Lahdili's calculation over $\mathbb{P}^1$, so the Lemma can also be derived through Lahdili's work.

Proof. We will show that

$$\alpha_\varphi - \alpha = dd^c \mathcal{M}_{X/B}(\varphi). \tag{6.8}$$

Here

$$\mathcal{M}_{X/B} := \int_{X/B} \mathcal{M}_{v,w}(\varphi_b)\, \omega^m$$

is the fibrewise weighted Mabuchi functional. Since the weighted Mabuchi functional is constant along the set of weighted cscK metrics in a fixed class, this will give the result.

We use the Chen–Tian formula for $\mathcal{M}_{v,w}$, proved by Lahdili [Lah19, Theorem 5]:

$$\mathcal{M}_{X/B}(\varphi) = 2\mathcal{H}_v(\varphi) - 2\mathcal{E}_v^\rho(\varphi) + c_{v,w}\mathcal{E}_w(\varphi),$$

where

$$\begin{aligned}\mathcal{H}_v(\varphi) &:= \int_{X/B} v(\mu_\varphi) \log\left(\frac{\omega_\varphi^m \wedge \omega_B^n}{\omega^m \wedge \omega_B^n}\right) \omega_\varphi^m,\\ \frac{d}{dt}\mathcal{E}_v^\rho(\varphi_t) &:= \int_{X/B} \dot\varphi_t (v(\mu_t)\rho \wedge \omega_t^{m-1} + \langle dv(\mu_t), \mu_\rho\rangle\, \omega_t^m),\\ \frac{d}{dt}\mathcal{E}_w(\varphi_t) &:= \int_{X/B} \dot\varphi_t w(\mu_t)\, \omega_t^m,\end{aligned}$$

for any smooth path φ_t in $\mathcal{K}^T$, and we normalise so that $\mathcal{E}_v^\rho(0) = \mathcal{E}_w(0) = 0$.

First consider

$$\mathcal{E}_w(\varphi) = \int_0^1 \int_{X/B} \varphi\, w(\mu_t)\, \omega_t^m\, dt.$$

Here we have chosen the straight line path $\varphi_t := t\varphi$ from 0 to φ. Taking dd^c on the base B and differentiating under the integral,

$$\begin{aligned}dd^c\mathcal{E}_w(\varphi) &= d\left(\int_0^1 \int_{X/B} w(\mu_t) d^c\varphi \wedge \omega_t^m\, dt + \int_0^1 \int_{X/B} \varphi\, d^c(w(\mu_t)) \wedge \omega_t^m\, dt\right)\\ &= \int_0^1 \int_{X/B} d(w(\mu_t)) \wedge d^c\varphi \wedge \omega_t^m\, dt + \int_0^1 \int_{X/B} w(\mu_t) dd^c\varphi \wedge \omega_t^m\, dt.\end{aligned}$$

Note the right-most term in the first line vanishes since $d^c(w(\mu_{t\varphi}))$ is vertical. The first term in the next line becomes

$$\begin{aligned}\int_0^1 \int_{X/B} \langle dw(\mu_t), d\mu_t\rangle_{\mathfrak{t}} \wedge d^c\varphi \wedge \omega_t^m &= \int_0^1 \int_{X/B} -m\omega_t(dw(\mu_t), -) \wedge d^c\varphi \wedge \omega_{\mathcal{V},t}^{m-1} \wedge \omega_{\mathcal{H},t}\\ &= \frac{1}{m+1}\int_0^1 \int_{X/B} d^c\varphi(dw(\mu_t))\omega_t^{m+1}\end{aligned}$$

On the other hand, using the fact that $\mu_t = \mu + td^c\varphi$,

$$\begin{aligned}&\frac{1}{m+1}\int_{X/B} w(\mu_\varphi)\, \omega_\varphi^{m+1} - \frac{1}{m+1}\int_{X/B} w(\mu)\, \omega^{m+1}\\ =&\frac{1}{m+1}\int_0^1 \left(\frac{d}{dt}\int_{X/B} w(\mu_t)\, \omega_t^{m+1}\right) dt\\ =&\frac{1}{m+1}\int_0^1 \int_{X/B} d^c\varphi(dw(\mu_t))\, \omega_t^{m+1}\, dt + \int_0^1 \int_{X/B} w(\mu_t) dd^c\varphi \wedge \omega_t^m\, dt.\end{aligned}$$

This agrees exactly with $dd^c\mathcal{E}_w(\varphi)$.

Next, we look at the term

$$\mathcal{H}_v(\varphi) := \int_{X/B} v(\mu_\varphi) \log\left(\frac{\omega_\varphi^m \wedge \omega_B^n}{\omega^m \wedge \omega_B^n}\right) \omega_\varphi^m.$$

We will write $\Phi_\varphi := \log\left(\frac{\omega_\varphi^m \wedge \omega_B^n}{\omega^m \wedge \omega_B^n}\right)$ to simplify calculations. Taking dd^c gives

$$\begin{aligned} dd^c\mathcal{H}_v(\varphi) &= d\left(\int_{X/B} v(\mu_\varphi) d^c\Phi_\varphi \wedge \omega_\varphi^m\right) \\ &= \int_{X/B} d(v(\mu_\varphi)) \wedge d^c\Phi_\varphi \wedge \omega_\varphi^m + v(\mu_\varphi) dd^c\Phi_\varphi \wedge \omega_\varphi^m. \end{aligned}$$

Now, by [Lah19, Lemma 5], $dd^c\Phi_\varphi = \rho - \rho_\varphi$ and $d^c\Phi_\varphi(\xi) = \mu_\rho^\xi - \mu_{\rho_\varphi}^\xi$. So

$$\begin{aligned} dd^c\mathcal{H}_v(\varphi) &= \frac{1}{m+1}\int_{X/B} (\langle dv(\mu_\varphi), \mu_\rho\rangle - \langle dv(\mu_\varphi), \mu_{\rho_\varphi}\rangle)\omega_\varphi^{m+1} \\ &\quad + \int_{X/B} v(\mu_\varphi)(\rho - \rho_\varphi) \wedge \omega_\varphi^m \\ &= \left(\frac{1}{m+1}\int_{X/B} \langle dv(\mu_\varphi), \mu_\rho\rangle \omega_\varphi^{m+1} + \int_{X/B} v(\mu_\varphi)\rho \wedge \omega_\varphi^m\right) \\ &\quad - \left(\frac{1}{m+1}\int_{X/B} \langle dv(\mu_\varphi), \mu_{\rho_\varphi}\rangle \omega_\varphi^{m+1} + \int_{X/B} v(\mu_\varphi)\rho_\varphi \wedge \omega_\varphi^m\right), \end{aligned}$$

The remaining terms in $\alpha_\varphi - \alpha$ are

$$\begin{aligned} &- 2\int_{X/B} v(\mu_\varphi)\rho_\varphi \wedge \omega_\varphi^m - \frac{2}{m+1}\int_{X/B} \langle dv(\mu_\varphi), \mu_{\rho_\varphi}\rangle \omega_\varphi^{m+1} \\ &+ 2\int_{X/B} v(\mu)\rho \wedge \omega^m + \frac{2}{m+1}\int_{X/B} \langle dv(\mu), \mu_\rho\rangle \omega^{m+1} \\ = &- 2\int_{X/B} v(\mu_\varphi)\rho_\varphi \wedge \omega_\varphi^m - \frac{2}{m+1}\int_{X/B} \langle dv(\mu_\varphi), \mu_{\rho_\varphi}\rangle \omega_\varphi^{m+1} \\ &+ 2\int_{X/B} v(\mu_\varphi)\rho \wedge \omega_\varphi^m + \frac{2}{m+1}\int_{X/B} \langle dv(\mu_\varphi), \mu_\rho\rangle \omega_\varphi^{m+1} \\ &- 2\int_{X/B} v(\mu_\varphi)\rho \wedge \omega_\varphi^m - \frac{2}{m+1}\int_{X/B} \langle dv(\mu_\varphi), \mu_\rho\rangle \omega_\varphi^{m+1} \\ &+ 2\int_{X/B} v(\mu)\rho \wedge \omega^m + \frac{2}{m+1}\int_{X/B} \langle dv(\mu), \mu_\rho\rangle \omega^{m+1}. \end{aligned}$$

We can see the first four terms are exactly $2dd^c\mathcal{H}_v(\varphi)$, so we must show the last four

terms equate to $-2dd^c\mathcal{E}^\rho_v(\varphi)$. Removing the factor of 2, the last four terms are

$$\begin{aligned}
&-\int_{X/B} v(\mu_\varphi)\rho\wedge\omega_\varphi^m - \frac{1}{m+1}\int_{X/B}\langle dv(\mu_\varphi),\mu_\rho\rangle\omega_\varphi^{m+1}\\
&+\int_{X/B} v(\mu)\rho\wedge\omega^m + \frac{1}{m+1}\int_{X/B}\langle dv(\mu),\mu_\rho\rangle\omega^{m+1}\\
=&-\int_0^1\frac{d}{dt}\left(\int_{X/B} v(\mu_t)\rho\wedge\omega_t^m + \frac{1}{m+1}\int_{X/B}\langle dv(\mu_t),\mu_\rho\rangle\omega_t^{m+1}\right)dt\\
=&-\left[\int_0^1\int_{X/B} d^c\varphi(dv(\mu_t))\rho\wedge\omega_t^m + m\int_0^1\int_{X/B} v(\mu_t)\rho\wedge dd^c\varphi\wedge\omega_t^{m-1}\right.\\
&\left.+\frac{1}{m+1}\int_0^1\int_{X/B}\sum_{i,j} v_{,ij}(\mu_t)d^c\varphi(\xi_j)\mu_\rho^{\xi_i}\omega_t^{m+1} + \int_0^1\int_{X/B}\langle dv(\mu_t),\mu_\rho\rangle\, dd^c\varphi\wedge\omega_t^m\right],
\end{aligned}\tag{6.9}$$

where the final line follows from differentiation under the integral, and the calculation

$$\begin{aligned}
\frac{d}{dt}\langle dv(\mu_t),\mu_\rho\rangle &= \frac{d}{dt}\left(\sum_{i=1}^r v_{,i}(\mu_t)\mu_\rho^{\xi_i}\right)\\
&= \sum_{i,j=1}^r v_{,ij}(\mu)d^c\varphi(\xi_j)\mu_\rho^{\xi_i}.
\end{aligned}$$

The final term in the Chen–Tian formula is

$$\mathcal{E}^\rho_v(\varphi) := \int_0^1\int_{X/B}\varphi\left[mv(\mu_t)\rho\wedge\omega_t^{m-1} + \langle dv(\mu_t),\mu_\rho\rangle\,\omega_t^m\right]\,dt.$$

Differentiating under the integral,

$$\begin{aligned}
dd^c\mathcal{E}^\rho_v(\varphi) &= d\int_0^1\int_{X/B} d^c\varphi\wedge\left[mv(\mu_t)\rho\wedge\omega_t^{m-1} + \langle dv(\mu_t),\mu_\rho\rangle\,\omega_t^m\right]\,dt\\
&= \int_0^1\int_{X/B} dd^c\varphi\wedge\left[mv(\mu_t)\rho\wedge\omega_t^{m-1} + \langle dv(\mu_t),\mu_\rho\rangle\,\omega_t^m\right]\,dt\\
&\quad+\int_0^1\int_{X/B} md(v(\mu_t))\wedge d^c\varphi\wedge\rho\wedge\omega_t^{m-1}\,dt \qquad (6.10)\\
&\quad+\int_0^1\int_{X/B} d(\langle dv(\mu_t),\mu_\rho\rangle)\wedge d^c\varphi\wedge\omega_t^m\,dt. \qquad (6.11)
\end{aligned}$$

Now, the second line (6.10) in this expression for $dd^c\mathcal{E}^\rho_v(\varphi)$ is

$$\begin{aligned}
&\int_0^1\int_{X/B} m\langle dv(\mu_t),d\mu_t\rangle_{\mathfrak{t}}\wedge d^c\varphi\wedge\rho\wedge\omega_t^{m-1}\,dt\\
=&\int_0^1\int_{X/B} m\langle dv(\mu_t),d\mu_t\rangle_{\mathfrak{t}}\wedge d^c\varphi\wedge\rho\wedge\omega_{\mathcal{V},t}^{m-1}\,dt
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^1\int_{X/B} m(m-1)\langle dv(\mu_t), d\mu_t\rangle_{\mathfrak{t}}\wedge d^c\varphi\wedge\rho\wedge\omega_{\mathcal{V},t}^{m-2}\wedge\omega_{\mathcal{H},t}\,dt\\
=&\int_0^1\int_{X/B} m\langle dv(\mu_t), d\mu_t\rangle_{\mathfrak{t}}\wedge(d^c\varphi)_{\mathcal{H}}\wedge\rho_{\text{mixed}}\wedge\omega_{\mathcal{V},t}^{m-1}\,dt\\
&+\int_0^1\int_{X/B} m\langle dv(\mu_t), d\mu_t\rangle_{\mathfrak{t}}\wedge(d^c\varphi)_{\mathcal{V}}\wedge\rho_{\mathcal{H}}\wedge\omega_{\mathcal{V},t}^{m-1}\,dt\\
&+\int_0^1\int_{X/B} [d^c\varphi(dv(\mu_t))S(\omega_b)-g_t(\langle dv(\mu_t), d\mu_t\rangle_{\mathfrak{t}}\wedge(d^c\varphi)_{\mathcal{V}},\rho_{\mathcal{V}})]\omega_{\mathcal{V},t}^m\wedge\omega_{\mathcal{H},t}\,dt\\
=&-\int_0^1\int_{X/B}(d^c\varphi)_{\mathcal{H}}\wedge\rho(dv(\mu_t),-)_{\mathcal{H}}\wedge\omega_{\mathcal{V},t}^m\,dt\\
&+\int_0^1\int_{X/B} d^c\varphi(dv(\mu_t))\rho_{\mathcal{H}}\wedge\omega_{\mathcal{V},t}^m\,dt+\int_0^1\int_{X/B} md^c\varphi(dv(\mu_t))\rho_{\mathcal{V}}\wedge\omega_{\mathcal{V},t}^{m-1}\wedge\omega_{\mathcal{H},t}\,dt\\
&+\int_0^1\int_{X/B}\rho(dv(\mu_t),-J\nabla_{\mathcal{V}}\varphi)\,\omega_{\mathcal{V},t}^m\wedge\omega_{\mathcal{H},t}\,dt\\
=&\int_0^1\int_{X/B}\rho(dv(\mu_t),-)_{\mathcal{H}}\wedge(d^c\varphi)_{\mathcal{H}}\wedge\omega_{\mathcal{V},t}^m\,dt\\
&+\int_0^1\int_{X/B} d^c\varphi(dv(\mu_t))\rho\wedge\omega_t^m\,dt\\
&+\int_0^1\int_{X/B} m\rho(dv(\mu_t),-)_{\mathcal{V}}\wedge(d^c\varphi)_{\mathcal{V}}\wedge\,\omega_{\mathcal{V},t}^{m-1}\wedge\omega_{\mathcal{H},t}\,dt\\
=&\int_0^1\int_{X/B} d^c\varphi(dv(\mu_t))\rho\wedge\omega_t^m\,dt\\
&+\int_0^1\int_{X/B}\rho(dv(\mu_t),-)\wedge d^c\varphi\wedge\omega_t^m\,dt.
\end{aligned}
$$

On the other hand, the third line (6.11) is

$$
\begin{aligned}
&\int_0^1\int_{X/B} d(\langle dv(\mu_t),\mu_\rho\rangle)\wedge d^c\varphi\wedge\omega_t^m\,dt\\
=&\int_0^1\int_{X/B}\sum_{i,j} v_{,ij}(\mu_t)d\mu_t^{\xi_j}\mu_\rho^{\xi_i}\wedge d^c\varphi\wedge\omega_t^m\,dt\\
&+\int_0^1\int_{X/B}\langle dv(\mu_t), d\mu_\rho\rangle_{\mathfrak{t}}\wedge d^c\varphi\wedge\omega_t^m\,dt\\
=&\frac{1}{m+1}\int_0^1\int_{X/B}\sum_{i,j} v_{,ij}(\mu_t)d^c\varphi(\xi_j)\mu_\rho^{\xi_i}\omega_t^{m+1}\,dt\\
&-\int_0^1\int_{X/B}\rho(dv(\mu_t),-)\wedge d^c\varphi\wedge\omega_t^m\,dt.
\end{aligned}
$$

Combining these, we get

$$\begin{aligned} dd^c\mathcal{E}_v^\rho(\varphi) = &\int_0^1 \int_{X/B} dd^c\varphi \wedge \left[mv(\mu_t)\rho \wedge \omega_t^{m-1} + \langle dv(\mu_t), \mu_\rho \rangle \, \omega_t^m \right] dt \\ &+ \int_0^1 \int_{X/B} d^c\varphi(dv(\mu_t))\rho \wedge \omega_t^m \, dt \\ &+ \int_0^1 \int_{X/B} \rho(dv(\mu_t), -) \wedge d^c\varphi \wedge \omega_t^m \, dt \\ &+ \frac{1}{m+1} \int_0^1 \int_{X/B} \sum_{i,j} v_{,ij}(\mu_t) d^c\varphi(\xi_j) \mu_\rho^{\xi_i} \omega_t^{m+1} \, dt \\ &- \int_0^1 \int_{X/B} \rho(dv(\mu_t), -) \wedge d^c\varphi \wedge \omega_t^m \, dt \end{aligned}$$

which is minus equation (6.9). □

Corollary 6.2.8. *If the fibration $X \to B$ is locally trivial then $\alpha_\varphi = 0$ for any φ.*

Proof. Locally, we can trivialise $X|_U \cong U \times F$, and assume ω is pulled back from the fibre F. But then both $\rho_\mathcal{H}$ and ω^{m+1} vanish, so $\alpha = 0$. □

Proposition 6.2.9. *The form α is semipositive, and strictly positive if the family of complex manifolds given by the fibration is effective.*

Proof. The result is local, so we assume $B = D \subset \mathbb{C}^n$ is an open subset of $\mathbb{C}^n$ containing the origin, and we prove the result at $0 \in D$. It will be convenient to write ω_X for the relatively Kähler metric on X, and ω for the restriction of ω_X to X_0. By Fujiki–Schumacher [FS90, Lemma 8.6], there is a C^∞-trivialisation $\psi : X \cong D \times M$ of the family $X \to D$ such that:

1. The horizontal distribution determined by ω_X satisfies $\mathcal{H}|_{X_0} \cong d\psi^{-1}|_{\{0\}\times M}(TD)$, and

2. For all $m \in M$, $D \times \{m\} \subset D \times M \cong X$ is a complex submanifold.

Let $z^1, \ldots, z^n$ be the standard coordinates on $\mathbb{C}^n$, and denote by ω_z the restriction of ω_X to the fibre $X_z \cong \{z\} \times M$, and ω for ω_0 extended trivially over $D \times M$. Then near the origin of D, we have [FS90, Equation 8.6]

$$\omega_z = \omega + z^\beta \bar{z}^\gamma \omega_{\beta\bar{\gamma}} + O(|z|^3),$$

for some unspecified forms $\omega_{\beta\bar{\gamma}}$. Furthermore, the fibrewise volume forms ω_z^m and ω^m determine hermitian metrics h_z and h on $-K_{X/D} \cong \Lambda^m\mathcal{V}$. We denote by $c_1(X/D, h_z)$ and $c_1(X/D, h)$ the associated first Chern forms for these metrics.

Now, writing

$$\omega_z^m = \omega^m + \sum_{\beta,\gamma} \nu_{\beta\bar{\gamma}} z^\beta \overline{z}^\gamma + \mathrm{O}(|z|^3),$$

we have

$$\int_{X_0} v(\mu) c_1(X/D, h_z) \wedge \omega^n = \int_{X_0} v(\mu) c_1(X/D, h) \wedge \omega^n - \sum_{\beta,\gamma} \int_{X_0} v(\mu) \nu_{\beta\bar{\gamma}}\, dz^\beta \wedge d\bar{z}^\gamma.$$

Here by $\int_{X_0}$ we mean the restriction of $\int_{X/D}$ to X_0. Now, since $\int_{X/D} v(\mu)\omega_z^m$ is constant,

$$0 = i\partial_z\bar{\partial}_z \int_{X/D} v(\mu)\omega_z^m \Big|_{X_0} = \int_{X_0} i\partial_z\bar{\partial}_z(v(\mu)) \wedge \omega^m + \sum_{\beta,\gamma} \int_{X_0} v(\mu)\nu_{\beta\bar{\gamma}} \wedge dz^\beta \wedge d\bar{z}^\gamma.$$

The left-hand term is equal to $-\frac{1}{m+1}\int_{X_0} \Delta_{\mathcal{V}}(v(\mu))\omega_X^{m+1}$, which is a term of α. So

$$\begin{aligned}
\alpha|_0 &= -2\int_{X_0} v(\mu)\rho_{\mathcal{H}} \wedge \omega_{X,\mathcal{V}}^m - \frac{2}{m+1}\int_{X_0} \Delta_{\mathcal{V}}(v(\mu))\, \omega_X^{m+1} \\
&= -2\int_{X_0} v(\mu) c_1(X/D, h_z)\, \omega_z^m - 2\sum_{\beta,\gamma} \int_{X_0} v(\mu)\nu_{\beta\bar{\gamma}}\, dz^\beta \wedge d\bar{z}^\gamma \\
&= -2\int_{X_0} v(\mu) c_1(X/B, h)\omega^m.
\end{aligned}$$

Now, let φ_t be a C^∞ family of elements of $A^1(X_0) := \Gamma(TX_0 \otimes \overline{T^*X_0})$ representing the deformation of complex structure, and such that $\varphi_0 = 0$. Then on each almost complex submanifold $D \times \{m\}$,

$$\begin{aligned}
c_1(X/D, h)|_0 &= i\partial_z\bar{\partial}_z \log\det(I + \varphi_z\overline{\varphi}_z) \\
&= -\mathrm{Tr}(\partial_z\varphi_z(0)\bar{\partial}_z\overline{\varphi}_z(0)).
\end{aligned}$$

It follows that

$$\alpha|_0 = 2\int_{X_0} \mathrm{Tr}(\partial_z\varphi\overline{\partial_z\varphi_z})v(\mu)\omega^m.$$

Since $v(\mu)$ is strictly positive, this is a semipositive form on T_0D, which is strictly positive if the family is effective since $\partial_z\varphi$ is the Kuranishi map $T_0D \to H^1(X_0, \Theta)$. □

We have thus proven all the properties of α in Theorem 6.2.1. We will end with some examples of families of weighted cscK metrics.

Example 6.2.10. First, an example where the fibres are extremal. Let C be a smooth projective curve, with genus $g \geq 2$. Denote by J the Jacobian of line bundles of this curve. Then there is a universal family

$$\mathcal{L} \to J \times C$$

of degree zero line bundles on C; see [GH94, p. 328]. Taking the projectivisation $X := \mathbb{P}(\mathcal{L} \oplus \mathcal{O}_{J\times\mathbb{C}})$ and composing with the projection $J \times C \to J$, we get a family of compact complex manifolds $X \to J$.

A line bundle always admits a Hermite–Einstein metric by Hodge theory, so let h be a Hermite–Einstein metric on $\mathcal{L}$ and consider the metric $h \oplus 1$ on $\mathcal{L} \oplus \mathcal{O}$, where 1 denotes the trivial metric on $\mathcal{O}$. Taking a metric ω_C on C together with the fibrewise Fubini–Study metric on X_b induced by $h_b \oplus 1$, we get induced metrics on each fibre X_b in the classes $c_1(\mathcal{O}_{X_b}(1)) + k[\omega_C]$ for $k \gg 0$. By the work of Brönnle [Brö11], on each fibre X_b there exists an extremal metric in the class $c_1(\mathcal{O}_{X_b}(1)) + k[\omega_C]$ for $k \gg 0$, and by the proof which is via an inverse function theorem argument, these can be taken to vary smoothly with b. The extremal field generates the $\mathbb{C}^*$-action on X_b given by $t \cdot [\ell, k] := [\ell, tk]$, where $\ell \in \mathcal{L}_b$, $k \in \mathcal{O}_{X_b}$, and $[\,,]$ denotes the equivalence class in the projectivisation. The extremal vector fields on the fibres then clearly piece together to form a globally holomorphic vertical vector field on X induced by a fibrewise holomorphic S^1-action. By algebraicity, the moment polytopes of the fibres have integer vertices, and hence do not vary with b.

In the terminology of Dervan–Sektnan, the metric $h_b \oplus 1$ induces an extremal symplectic connection on $X_b \to C$; see [DS21b, Remark 3.18(ii)]. That X_b admits an extremal metric in adiabatic classes is then an application of their Theorem 1.3 in [DS21b].

Example 6.2.11. In this example due to Inoue [Ino19b, Section 5], the fibres are Kähler–Ricci solitons. First, let X be a del Pezzo surface of degree $1 \leq d \leq 4$, and let $Y = \mathrm{Bl}_p\mathbb{P}^2$. Then X and Y both admit Kähler–Ricci solitons, hence so does their product $X \times Y$. The manifold X admits deformations of complex structure, one may see this by considering X as the blow up of $\mathbb{P}^2$ at a number of suitably general points, and then allowing these points to vary. For the torus T we take S^1 acting only on Y, with the induced vector field ξ being the one appearing in the Kähler–Ricci soliton equation. Letting $\mathcal{X}$ be a variation of complex structure of X, we obtain a local family $\mathcal{X} \times Y$ of complex manifolds admitting Kähler–Ricci solitons. Again by algebraicity, the moment polytopes of the fibres are constant.

Chapter 7

Further directions

Here we outline some ideas for further research.

7.1 Polystability of fibrations

It would be desirable to strengthen the theorem on polystability of fibrations with respect to product-type degenerations, to full polystability. There is a general strategy for how this might be done, which we outline here:

1. First show that it suffices to consider fibration degenerations $\mathcal{X} \to B \times \mathbb{C}$ such that the following holds: there exists an open subset $U \subset B$ whose complement has codimension at least 2, such that for every $b \in B$, $\mathcal{X}_b \to \mathbb{C}$ normalises to a product test configuration for X_b. To motivate this condition, recall in the vector bundle setting that it suffices to consider *saturated subsheaves* $W \subset V$ of a vector bundle V, meaning the quotient sheaf V/W is torsion-free [Kob82, Chapter 5]. These subsheaves are vector bundles away from codimension 2, so the associated degeneration of $\mathbb{P}(V)$ to $\mathbb{P}(W \oplus (V/W))$ is flat away from codimension 2. Wherever the fibration $\mathcal{X} \to B$ is flat, the Donaldson–Futaki invariant of the fibre $\mathcal{X}_b$ will be constant in b. Since we may assume the general fibre is a product test configuration and has Donaldson–Futaki invariant zero, the fibres in the flat locus will all have vanishing Donaldson–Futaki invariant and so normalise to product test configurations. This suggests it may suffice to restrict to fibration degenerations whose fibres normalise to product test configurations away from codimension 2.

2. Next, show that one can find a smooth curve $C \subset B$ such that the restriction $\mathcal{X}|_C$ has the same invariants W_0 and W_1 as $\mathcal{X}$, up to multiplying by a positive constant. This can indeed be done, since the invariants involve intersecting

with L^{n-1} we can restrict to a smooth curve dual to L^{n-1} as claimed. If the complement of the flat locus has codimension at least 2, we can take C to have zero intersection with this complement, so $\mathcal{X}|_C \to C$ will be flat, and all fibres will normalise to product test configurations.

3. Show that if $X \to B$ admits an optimal symplectic connection then the restriction $X|_C \to C$ also admits an optimal symplectic connection. In the vector bundles setting, the restriction of a stable vector bundle to a divisor dual to the ample line bundle is again stable [MR84], so by the Kobayashi–Hitchin correspondence, if the vector bundle admits a Hermite–Einstein connection then so too does its restriction. This motivates the possibility of a similar result holding for optimal symplectic connections.

4. Finally, prove that fibrations over curves which admit optimal symplectic connections are polystable. Given the first point, we can assume that every fibre of the fibration degeneration normalises to a product test configuration. The general fibre $\mathcal{X}_b$ will be smooth, however there may be finitely many singular fibres $\mathcal{X}_{b_1}, \ldots, \mathcal{X}_{b_k}$. It seems uncertain how the geodesic rays constructed in Chapter 5 will behave near the singularities, and some analysis may be required.

7.2 The OSC flow

It should be possible to develop a geometric flow for the optimal symplectic connection equation, comparable to the Calabi flow for the cscK equation [Cal82], or the Donaldson flow for the Hermite–Einstein equation [Don85]. Some natural conjectures relating to this flow would be:

1. The flow exists for all time.

2. Existence of an optimal symplectic connection implies the flow converges to an optimal symplectic connection.

3. If the flow does not converge, it limits to a destabilising fibration degeneration.

Potential applications of the flow would include:

i. Local existence of optimal symplectic connections where the complex structures of the fibres vary. This is currently unknown, and a natural guess is that the flow with imposed boundary conditions will converge to an optimal symplectic connection, as in the vector bundles case [Don92].

ii. Long time existence of the Calabi flow on the total space in adiabatic classes $[\omega_X + k\omega_B]$.

Here we write down the flow, explain its short time existence, and prove that the L^2-norm of the optimal symplectic connection decreases along the flow.

Let $(X, \omega_X) \to (B, \omega_B)$ be a relatively cscK fibration. Denote by $\mathcal{K}_E$ the space of relatively cscK potentials with respect to ω_X, and let $\varphi_t : [0, T) \to \mathcal{K}_E$ be a smooth path.

Definition 7.2.1. The path φ_t satisfies the *OSC flow* if

$$\frac{\partial \varphi_t}{\partial t} = p_t(\theta_t).$$

The linearisation of the optimal symplectic connection is a negative elliptic operator, so the flow has short time existence and uniqueness by standard PDE theory; see [CH08] for example.

Lemma 7.2.2. *Let φ_t be a smooth solution of the OSC flow. Then $\|p_t(\theta_t)\|^2_{L^2,t}$ is non-increasing in t. It is strictly decreasing unless φ_t is an extremal symplectic connection, in which case the flow is along a standard geodesic.*

Proof. We have

$$\begin{aligned}
\frac{d}{dt}\|p_t(\theta_t)\|^2_{L^2,t} &= \frac{d}{dt}\int_X p_t(\theta_t)^2\, \omega_t^m \wedge \omega_B^n \\
&= 2\int_X p_t(\theta_t)(-\mathcal{R}_t^*\mathcal{R}_t\dot{\varphi}_t + \langle \partial_{\mathcal{V}} p_t(\theta_t), \partial_{\mathcal{V}}\dot{\varphi}_t\rangle_{\mathcal{V},t})\, \omega_t^m \wedge \omega_B^n \\
&+ \int_X p_t(\theta_t)^2 \Delta_{\mathcal{V},t}\dot{\varphi}_t\, \omega_t^m \wedge \omega_B^n \\
&= -2\int_X |\mathcal{R}_t p_t(\theta_t)|_t^2\, \omega_t^m \wedge \omega_B^n
\end{aligned}$$

which is non-positive, and zero if and only if $\mathcal{R}_t p_t(\theta_t) = 0$ (i.e. φ_t is extremal). If φ_t is extremal, differentiating the OSC flow in t we get

$$\ddot{\varphi}_t = |\partial_{\mathcal{V}}\dot{\varphi}_t|^2_{\mathcal{V},t},$$

which is the geodesic equation. □

7.3 Optimal relatively weighted cscK metrics

One question is whether there exists a notion of optimal symplectic connection where the fibres are weighted cscK, rather than just cscK. Let $X \to B$ be a holomorphic surjective submersion of compact Kähler manifolds, T a compact torus acting holomorphically on X and preserving the fibres X_b, which all admit (v, w)-cscK metrics. We assume there exists a relatively Kähler metric ω_X, such that for every $b \in B$ the moment map $\mu_b : X_b \to \mathfrak{t}^*$ has the same image P, and ω_b is (v, w)-cscK for all $b \in B$.

We then want to construct a (v, w)-cscK metric on the total space X. That is, we wish to solve

$$\frac{S_v(\omega)}{w(\mu)} = c_{v,w}$$

for some Kähler metric ω on X. Letting $\omega_k := \omega_X + k\omega_B$ for $k \gg 0$, we expand the above equation in the adiabatic limit. Note that $\mu_{\omega_k} = \mu_{\omega_X}$ for all k, since the T-action is vertical. Similarly, $g_{\omega_k} \circ \mathrm{Hess}(v)$ is independent of k, where g_{ω_k} is the Riemannian metric associated to ω_k. Hence, recalling the expansion (2.5), where $\theta := \Delta_{\mathcal{V}} \Lambda_{\omega_B} \mu^* F_{\mathcal{H}} + \Lambda_{\omega_B} \rho_{\mathcal{H}}$,

$$\begin{aligned} S_v(\omega_k) &= 2v(\mu)S(\omega_k) - 4\Delta_k(v(\mu)) + \mathrm{Tr}(g_{\omega_k} \circ \mathrm{Hess}(v)) \\ &= 2v(\mu)(S(\omega_b) + k^{-1}(S(\omega_B) + \theta)) - 4\Delta_{\mathcal{V}}(v(\mu)) \\ &\quad - k^{-1}4\Delta_{\mathcal{H}}(v(\mu)) + \mathrm{Tr}(g_{\omega_X} \circ \mathrm{Hess}(v)) + \mathrm{O}(k^{-2}) \\ &= S_v(\omega_b) + k^{-1}[2v(\mu)(S(\omega_B) + \theta) - 4\Delta_{\mathcal{H}}(v(\mu))] + \mathrm{O}(k^{-2}). \end{aligned}$$

Here $\Delta_{\mathcal{H}}$ is the horizontal Laplacian, defined by

$$\Delta_{\mathcal{H}}\varphi := n\frac{i\partial_{\mathcal{H}}\bar{\partial}_{\mathcal{H}}\varphi \wedge \omega_B^{n-1}}{\omega_B^n},$$

and the contraction is taken in $\det(\mathcal{H})$. The Laplacian Δ_k then satisfies $\Delta_k = \Delta_{\mathcal{V}} + k^{-1}\Delta_{\mathcal{H}} + \mathrm{O}(k^{-2})$, by [DS21b].

It follows that the leading order term of $S_v(\omega_k)/w(\mu_k)$ will be constant, and we wish to solve the subleading order term

$$\frac{2}{w(\mu)}(v(\mu)(S(\omega_B) + \theta) - 2\Delta_{\mathcal{H}}(v(\mu)))$$

to be constant, so that we can deform to a (v, w)-cscK metric for large k.

Consider the L^2-inner product

$$\langle f, g \rangle := \int_X f g \, w(\mu)\omega_X^m \wedge \omega_B^n$$

on $C^\infty(X)^T$. This provides a decomposition

$$C^\infty(X)^T = C^\infty(B) \oplus C^\infty(E)^T \oplus C^\infty(R)^T,$$

where $C^\infty(B)$ are the functions pulled back from B to X, $C^\infty(E)^T$ are the torus invariant fibrewise holomorphy potentials having fibrewise integral zero with respect to $w(\mu)\omega_X^m$, and $C^\infty(R)^T$ is the orthogonal complement of the first two summands in $C^\infty(X)^T$.

The $C^\infty(B)$-component of the subleading order term in the expansion is

$$\begin{aligned}
&2\int_{X/B} \frac{1}{w(\mu)}(v(\mu)(S(\omega_B)+\theta) - 2\Delta_{\mathcal{H}}(v(\mu)))\, w(\mu)\omega_X^m \\
=&2\int_{X/B} (v(\mu)(S(\omega_B) + \Delta_{\mathcal{V}}\Lambda_{\omega_B}\omega_{X,\mathcal{H}} + \Lambda_{\omega_B}\rho_{\mathcal{H}}) - 2\Delta_{\mathcal{H}}(v(\mu)))\, \omega_X^m.
\end{aligned}$$

Now,

$$\begin{aligned}
\int_{X/B} \Delta_{\mathcal{H}}(v(\mu))\, \omega_X^m =& \frac{1}{\omega_B^n}\int_{X/B} \Delta_{\mathcal{H}}(v(\mu))\, \omega_X^m \wedge \omega_B^n \\
=& \frac{n}{\omega_B^n}\int_{X/B} i\partial\bar\partial(v(\mu)) \wedge \omega_B^{n-1} \wedge \omega_{X,\mathcal{V}}^m \\
=& \frac{n}{\omega_B^n}\int_{X/B} i\partial\bar\partial(v(\mu)) \wedge \omega_B^{n-1} \wedge (\omega_X - \omega_{X,\mathcal{H}})^m \\
=& \frac{-mn}{\omega_B^n}\int_{X/B} i\partial\bar\partial(v(\mu)) \wedge \omega_B^{n-1} \wedge \omega_{X,\mathcal{H}} \wedge \omega_X^{m-1} \\
=& \frac{-m}{\omega_B^n}\int_{X/B} (\Lambda_{\omega_B}\omega_{X,\mathcal{H}}) i\partial\bar\partial(v(\mu)) \wedge \omega_B^n \wedge \omega_X^{m-1} \\
=& -\int_{X/B} (\Lambda_{\omega_B}\omega_{X,\mathcal{H}})\Delta_{\mathcal{V}}(v(\mu))\, \omega_X^m.
\end{aligned}$$

So the $C^\infty(B)$-component is

$$\begin{aligned}
&\int_{X/B} 2v(\mu)(S(\omega_B) + \Lambda_{\omega_B}\rho_{\mathcal{H}})\, \omega_X^m + 6\Delta_{\mathcal{V}}(v(\mu))(\Lambda_{\omega_B}\omega_{X,\mathcal{H}})\, \omega_X^m \\
=& c_v S(\omega_B) + \Lambda_{\omega_B}\int_{X/B} 2v(\mu)\rho_{\mathcal{H}} \wedge \omega_X^m + 6\Delta_{\mathcal{V}}(v(\mu))\omega_{X,\mathcal{H}} \wedge \omega_X^m \\
=& c_v S(\omega_B) - \Lambda_{\omega_B}\alpha + 4\Lambda_{\omega_B}\int_{X/B} \Delta_{\mathcal{V}}(v(\mu))\omega_{X,\mathcal{H}} \wedge \omega_X^m.
\end{aligned}$$

Note that we could almost solve this to be constant by taking ω_B to be twisted cscK with respect to the weighted Weil–Petersson metric α. However this doesn't quite work due to the presence of the extra term involving $\Delta_{\mathcal{V}}(v(\mu))$. We are currently unsure how to resolve this issue. Note however, that in the extremal case, $v = 1$ and so the extra term vanishes. Hence there should be a perfectly good notion of optimal symplectic connection when the fibres admit extremal metrics.

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