Aspects of Nonelementary Stability Theory

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Dedicated to the memory of Arshaluys and Marine Haykazyans.

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Abstract

This thesis aims to contribute to neostability and in particular to the stability theory of nonelementary classes. The central themes of the thesis are quasiminimality and excellence. We explore both classical and geometric aspects of stability theory for such classes. In particular our methods aim to utilise the topology on the space of types in these settings.

Chapter 1 is introductory and sets the necessary background on nonelementary classes. In Chapter 2 we use infinitary methods to study quasiminimal excellent classes. We give a simplified proof of the Categoricity Theorem. In Chapter 3 we study quasiminimality from the first order perspective. We look for conditions on a first order theory that allow us to build quasiminimal models with various additional properties. In Chapter 4 we look at excellent groups. We aim to generalise various results from (geometric) stability theory to this nonelementary setting.

Contents

Introduction

Stability theory is a byproduct of Shelah's *classification program*. After generalising Morley's categoricity theorem from Morley [1965] to uncountable languages in Shelah [1974], he embarked on an ambitious program. The goal of the program was to produce tools to either classify models of a given first order theory or to prove that such a classification is impossible. Of course these terms are rather imprecise and need further clarification. The culmination of the program is the Main Gap theorem which is stated in terms of $I(T,\kappa)$ - the number of models of a complete theory T in cardinality κ up to isomorphism. According to it for a countable theory T the following dichotomy holds: either $I(T,\aleph_\alpha) < \beth_{\omega_1}(|\alpha|)$ for every ordinal α or $I(T,\aleph_\alpha) = 2^{\aleph_\alpha}$ for every ordinal α. For large ordinals α the first one is significantly below the second one (hence the word "gap"). The theories with few models are exactly those that are superstable, shallow and without DOP and OTOP and these models can be classified by trees of cardinal numbers. The full details of the classification program are presented in Shelah [1990] which has uncountably many ideas on first order model theory.

Shelah's approach to the classification program was through a series of dividing lines. These are properties of theories where he could prove a non-structure theorem on one side and a structure theorem on the other side. For example if a theory is stable then it has a canonical independence relation called nonforking. On the other hand if a theory is unstable, then it has a maximal number of models in each uncountable cardinality. Hence for the purposes of the classification program it is enough to consider only stable theories. Then the class of stable theories can be further refined to superstable theories and non-superstable theories (non-superstable theories still have the maximal number of models) and so on. As the reader can guess from the formulation of the Main Gap theorem, deep/shallow, DOP/NDOP and OTOP/NOTOP are further dividing lines.

The term stability theory refers to the tools and techniques developed by Shelah and others to analyse stable theories. Its uses however are by no means confined to the classification program. Stability theory is at the heart of applications of model theory to mathematics outside of logic. And this aspect has stimulated further developments in stability theory.

As nice as stable theories are, there are not many of them in mathematics. (Some examples are algebraically closed fields, differentially closed fields and free groups.) But over time it became clear that the main ideas apply in wider settings. The subject of generalising stability theory to wider settings is now called *neostability* (because it sounds better than generalised stability). There are two main directions in neostability. The first one is to study first order theories that are unstable but still have some structure properties (such as simple or NIP theories). The second one is to study nonelementary classes (i.e. classes that cannot be axiomatised in the first order logic). As the title suggests, this thesis is mainly concerned with the second direction. Although Chapter 3 is on the crossroads of the two.

Nonelementary classes have been studied since the beginning of model theory. In fact Shelah [2000] reports that the feeling in the late 1960s was that first order model theory was essentially done and that nonelementary model theory would be the future of model theory. Although there turned out to be more to first order model theory, Shelah considers classifying nonelementary classes as "the major problem of model theory". The stability theory of nonelementary classes has big potential for both inner developments of model theory and its applications to mathematics outside logic.

Overview

Now we briefly describe the contents and the main results of the thesis. Throughout the thesis we assume some familiarity with first order model theory and first order stability theory. Material from standard textbooks such as Tent and Ziegler [2012] and Marker [2002] is often used without further reference.

We give the full necessary background on nonelementary classes in Chapter 1. Everything in that chapter is more or less well known. A possible exception is Theorem 1.2.18, which we have not seen in the literature. We use it to give a low-tech proof of the categoricity transfer in one direction for excellent classes.

In Chapter 2 we study quasiminimal excellent classes. These are nonelementary classes introduced by Zilber to study pseudo-analytical structures such as the pseudo-exponential field. The uncountable categoricity of quasiminimal excellent classes was established in Zilber [2005b]. The original proof relied on the axiom of excellence which was inspired by the work of Shelah on categoricity in $L_{\omega_1,\omega}$. Later and to the great surprise of experts on the subject it was shown in Bays et al. [2014a] that the axiom of excellence follows from the other axioms. In Chapter 2 we give a direct proof of the categoricity theorem (Theorem 2.3.5) that bypasses the need for the axiom of excellence altogether. The key in our approach is to look at partial maps that preserve all $L_{\omega_1,\omega}$-formulas, possibly using infinitely many parameters. Previously such strong infinitary methods had little applications in mainstream model theory. The results of this chapter have been published in the paper Haykazyan [2016].

Chapter 3 is concerned with studying quasiminimality from the first order perspective. It is based on Pillay and Tanović [2011] and aims to fill the gap in the relationship between two concepts: quasiminimal structures and strongly regular types. An uncountable structure is quasiminimal if every first order definable subset is either countable or cocountable. And a global type is strongly regular if it is the generic type of a closure operator (see Definition 3.1.1). Pillay and Tanović [2011] show that if the type of cocountable sets of a quasiminimal structure is definable, then its global heir is strongly regular. Further if the quasiminimal structure is large (of cardinality strictly greater than $\aleph_1$), then the definability condition holds and moreover the strongly regular type is symmetric (i.e. the induced closure operator is a pregeometry).

Our aim in Chapter 3 is to establish the converse: to construct a quasiminimal model given a theory with a strongly regular type. The main results of the chapter are as follows. Given a definable strongly regular type we can construct a quasiminimal model of cardinality $\aleph_1$ (Theorem 3.2.3). To construct quasiminimal structures of larger cardinalities we at least need to assume that the regular type is symmetric. We present two constructions of arbitrarily large quasiminimal models: one assuming that there are definable Skolem functions (Theorem 3.3.4) and the other assuming that the theory is stable (Theorem 3.4.10). We also establish an upper bound on the Hanf number for the existence of quasiminimal models (Theorem 3.3.5). Finally we show that if a theory is ω-stable, then we can construct a quasiminimal excellent class from a strongly regular type (Theorem 3.5.6). The results of Chapter 3 have been published in the preprint Haykazyan [2015].

In the final Chapter 4 we study excellent groups both in the sense of Zilber and Shelah. The relationship between the two notions of excellence is not completely straightforward. Zilber's notion of a quasiminimal excellent class assumes existence of a quasiminimal pregeometry but is not in general $L_{\omega_1,\omega}$-axiomatisable (it can always be axiomatised in $L_{\omega_1,\omega}(Q)$). On the other hand, Shelah's notion of atomic excellence assumes $L_{\omega_1,\omega}$-axiomatisability but uses a much more general notion of independence. In terms of analogies, however, quasiminimal excellence is a nonelementary generalisation of strong minimality and atomic excellence is a nonelementary generalisation of ω-stability.

The question whether a quasiminimal excellent group has to be commutative has been around (in various forms) for some time. We answer this question positively assuming the pregeometry of the group is locally modular (Theorem 4.1.8).

In the subsequent sections of Chapter 4 we study atomic excellent groups. Our main tool is an analogue of Morley rank for an excellent class. We exploit the fact that the space of atomic types over a countable model is a complete metric space. For excellent structures it is moreover countable which allows us to use the Cantor-Bendixson rank to assign dimension to types and definable sets. Although this is only possible for countable models it turns out to be enough for a useful notion of dimension. Using this machinery we establish in Section 4.4 (usually weaker) analogues of some properties of ω-stable groups. We only scratch the surface of the theory of excellent groups but we expect there to be a rich theory which we hope will be as useful as the theory of stable groups.

Notation

Our notation is mainly standard. Here we spell out the notational conventions that we use. We try to follow these conventions throughout the thesis.

We look at finitary languages, i.e. languages that contain relation and function symbols of finite arity (and of course constant symbols). We often denote languages by L (here and throughout this section the notation may come with various subscripts and superscripts). The variables are usually denoted by x, y, z, constant symbols by c, function symbols by f, g and relation symbols by P, R. We have tried to keep the distinction between tuples and individual variables by putting a bar over the name, e.g. $\bar{x}$. Tuples are often finite, but in Chapter 2 we also deal with infinite tuples.

Theories, usually first order, are denoted by T. When looking at models of T we assume that they are substructures of a monster model $\mathfrak{C}$. In such a case all sets of parameters (usually denoted A, B, C) are assumed to be subsets of $\mathfrak{C}$ and all tuples (usually denoted $\bar{a}, \bar{b}, \bar{c}$) to consist of elements of $\mathfrak{C}$. We do not distinguish between a tuple and the set of its values. The concatenation of two tuples (or any sequences) $\bar{a}$ and $\bar{b}$ is denoted by $\bar{a}\bar{b}$ or by $\bar{a}^\frown\bar{b}$. We often use concatenation for a union, e.g. $A\bar{a}$ denotes the union of A with the set of values of $\bar{a}$. Sometimes, to emphasise that the monster model is a group, we denote it by $\mathfrak{G}$.

Formulas are denoted by ϕ, ψ, χ, types are denoted by p, q. Types are not assumed to be complete, but for emphasis we often explicitly mention that they are partial. We also consider types over $\mathfrak{C}$. Such types are called global types and denoted by $\mathfrak{p}, \mathfrak{q}$. The symbol $\mathop{\smile\!\!\!\!\!\mid}$ is used for various independence relations. If not specified, this refers to the forking independence in the first order sense.

We use $\subseteq$ for a subset. The symbol $\subset$ is reserved for a proper subset. If being a proper subset is important we use $\subsetneq$ for emphasis.

If X is a topological space and α is an ordinal, we denote α-th Cantor-Bendixson derivative by X^α. It is inductively defined as follows:

- $X^0 = X$;
- $X^{\alpha+1} = \{x \in X^\alpha : x \text{ is a limit point in } X^\alpha\}$;
- $X^\delta = \bigcap_{\alpha<\delta} X^\alpha$ for a limit ordinal δ.

Chapter 1

Background on Nonelementary Classes

In this chapter we provide the necessary background on the existing frameworks for dealing with nonelementary classes, their difficulties and differences compared to first order logic. Everything in this chapter is more or less well known. A possible exception is Theorem 1.2.18, which we have not seen in the literature. The result is not surprising in the context of the existing literature, but we use it for a simple low-tech proof of the categoricity transfer in one direction for excellent classes.

1.1 Infinitary Logic

The component "first" in first order logic seems to be the natural component to generalise. But second order logic (where one can quantify over subsets of the domain) is very expressive and its model theory is therefore intractable. Instead we can generalise first order logic by allowing infinite conjunctions and disjunctions. Such infinitary logics have been considered as early as the 1930s by Zermelo [1932]. But they rose in popularity in the 1960s when the techniques of first order model theory were being tested for stronger logics.

Definition 1.1.1. Let L be an ordinary finitary language (i.e. a collection of constant, function and relation symbols of finite arities) and κ, λ infinite cardinal numbers. The formulas of $L_{\kappa,\lambda}$ are defined as follows.

- Atomic L-formulas are $L_{\kappa,\lambda}$-formulas.
- If ϕ is an $L_{\kappa,\lambda}$-formula, then so is $\neg\phi$.
- If Φ is a set of $L_{\kappa,\lambda}$-formulas with $|\Phi| < \kappa$, then $\bigwedge_{\phi\in\Phi} \phi$ and $\bigvee_{\phi\in\Phi} \phi$ are $L_{\kappa,\lambda}$-formulas.
- If ϕ is an $L_{\kappa,\lambda}$-formula and $\bar{x}$ is a sequence of variables with $|\bar{x}| < \lambda$, then $\exists\bar{x}\phi$ and $\forall\bar{x}\phi$ are $L_{\kappa,\lambda}$-formulas.

Given an L-structure M, an $L_{\kappa,\lambda}$-formula ϕ and a variable assignment θ, the satisfaction relation $M \models_\theta \phi$ is defined naturally. In particular

$$M \models_\theta \bigwedge_{\phi\in\Phi} \phi \text{ if and only if } M \models_\theta \phi \text{ for all } \phi \in \Phi$$

and

$$M \models_\theta \exists\bar{x}\phi \text{ if and only if } M \models_{\theta[\bar{x}\mapsto\bar{a}]} \phi \text{ for some } \lambda\text{-sequence } \bar{a} \in M.$$

Similar rules holds for the $\bigvee$ connective and the $\forall$ quantifier. In this notation the ordinary first order logic has the same expressiveness as $L_{\omega,\omega}$.

The model theory of $L_{\kappa,\lambda}$ seems intractable for $\lambda > \omega$. So we will only consider logics $L_{\kappa,\omega}$. In addition we will look at $L_{\infty,\omega}$, where $L_{\infty,\omega}$-formulas are the union of $L_{\kappa,\omega}$-formulas for all κ. Of these logics $L_{\omega_1,\omega}$ has a special importance for us. It is the simplest and most studied infinitary logic. A systematic study of the model theory of $L_{\omega_1,\omega}$ was initiated by Keisler [1971], which also contains the first analogue of Morley's theorem for $L_{\omega_1,\omega}$ under additional assumptions. Next we present some background on infinitary logic. More details can be found e.g. in Marker [2007].

It is worth noting that we do not require $L_{\omega_1,\omega}$-formulas to have finitely many free variable. Although every subformula of an $L_{\omega_1,\omega}$-sentence has finitely many free variables. In general there are uncountably many $L_{\omega_1,\omega}$-formulas and this presents certain difficulties. However not all formulas are needed all the time. So often we only consider certain subsets.

Definition 1.1.2. A nonempty subset F of $L_{\omega_1,\omega}$-formulas is called a fragment if it is closed under taking a subformula, substituting a term for a variable and finitary connectives and quantifiers.

An example of a fragment, which we denote by "ω_1, ω", is the set of all $L_{\omega_1,\omega}$ sentences. Given a fragment F and L-structures M and N we say that they are equivalent in F and denote by $M \equiv_F N$ if they satisfy the same sentences in F. Similarly if $M \subseteq N$, then we say that M is an F-substructure and denote $M \preceq_F N$ if for every formula $\phi(\bar{x}) \in F$ and every $\bar{a} \in M$ we have $M \models \phi(\bar{a})$ if and only if $N \models \phi(\bar{a})$. Note that this also defines relations $\equiv_{\omega_1,\omega}$ and $\preceq_{\omega_1,\omega}$. Of course fragments also apply to $L_{\kappa,\lambda}$ more generally. In particular $M \equiv_{\infty,\omega} N$ means that M and N satisfy the same $L_{\infty,\omega}$ sentences.

Another way of obtaining a fragment is to start from an $L_{\omega_1,\omega}$ sentence ϕ and close it under the required operations.

Proposition 1.1.3. For every $L_{\omega_1,\omega}$-sentence ϕ, there is a countable fragment F containing it such that each formula in F has finitely many variables.

If the fragment F is such that every formula has finitely many variables, then we can add Skolem functions to witness quantifiers. If further F is countable we can do so without increasing the cardinality. This yields the analogue of the Downward Löwenheim-Skolem Theorem.

Theorem 1.1.4. Let F be a countable fragment of $L_{\omega_1,\omega}$ such that every formula in F has finitely many variables. Given a model M and a subset $A \subseteq M$, there is model $A \subseteq N \preceq_F M$ with $|N| = |A| + \aleph_0$. In particular if an $L_{\omega_1,\omega}$ sentence has a model, then it has a countable model.

Since the Downward Löwenheim-Skolem Theorem holds for $L_{\omega_1,\omega}$, by Lindström's Theorem, the Compactness Theorem and the Upward Löwenheim-Skolem Theorem must fail. (Recall that by Lindström's Theorem (see Lindström [1969]) first order logic is the most expressive logic that satisfies the Downward Löwenheim-Skolem Theorem and the Compactness Theorem and the most expressive logic that satisfies both the Downward and Upward Löwenheim-Skolem Theorems). Of course the failure of both theorems for $L_{\omega_1,\omega}$ is easy to see directly. The Compactness Theorem fails even on the propositional level: the set

$$\{P_i : i < \omega\} \cup \{\neg \bigwedge_{i<\omega} P_i\}$$

is finitely satisfiable but does not have a model. And the sentence $\forall x \bigvee_{i<\omega}(x = c_i)$ has only countable models. More precisely it can be shown that the Hanf number for the existence of models of an $L_{\omega_1,\omega}$ sentence is $\beth_{\omega_1}$. (I.e. $\beth_{\omega_1}$ is the least cardinal κ such that if an $L_{\omega_1,\omega}$ sentence ϕ has a model of cardinality κ, then it has arbitrarily large models.)

Next we give Karp's back and forth characterisation of $L_{\infty,\omega}$-equivalence, which will be very useful in the next chapter. For L-structures M and N a partial function f from M to N is called a (partial) embedding if it preserves quantifier free formulas. I.e. for every atomic $\phi(\bar{x})$ and $\bar{a} \in \mathrm{dom}(f)$ we have $M \models \phi(\bar{a})$ if and only if $N \models \phi(f(\bar{a}))$.

Definition 1.1.5. A back and forth system between L-structures M and N is a nonempty collection $\mathscr{F}$ of partial embeddings satisfying the following conditions.

- For all $f \in \mathscr{F}$ and $a \in M$ there is $g \in \mathscr{F}$ such that $f \subseteq g$ and $a \in \mathrm{dom}(g)$.
- For all $f \in \mathscr{F}$ and $b \in N$ there is $g \in \mathscr{F}$ such that $f \subseteq g$ and $b \in \mathrm{img}(g)$.

Theorem 1.1.6 (Karp [1965]). If M and N are L-structures, then $M \equiv_{\infty,\omega} N$ if and only if there is a back and forth system between them.

As a consequence if M and N are countable and $M \equiv_{\infty,\omega} N$, then they are isomorphic. (A back and forth system can be used to build an isomorphism between countable structures.) In fact the class of structures $L_{\infty,\omega}$-equivalent to a countable structure is determined by a single $L_{\omega_1,\omega}$-sentence.

Theorem 1.1.7 (Scott [1965]). For a countable structure M in a countable language L there is an $L_{\omega_1,\omega}$-sentence ϕ such that for every L-structure N we have $N \models \phi$ if and only if $N \equiv_{\infty,\omega} M$.

As a consequence the following are equivalent for a consistent $L_{\omega_1,\omega}$-sentence ϕ.

- For every $L_{\omega_1,\omega}$-sentence ψ either $\phi \models \psi$ or $\phi \models \neg\psi$.
- For every $L_{\infty,\omega}$-sentence ψ either $\phi \models \psi$ or $\phi \models \neg\psi$.
- The sentence ϕ is $\aleph_0$-categorical.

Such sentences are called *complete* and will play an important role later.

Actually if we are prepared to expand our language, we can view infinitary logic through the prism of first order logic.

Proposition 1.1.8. Given an $L_{\omega_1,\omega}$ sentence ϕ there is a first order theory T' in a countable expansion L' of L and a set Γ' of partial types such that the following conditions hold.

- If $M' \models T'$ and omits all types in Γ', then its L-reduct is a model of ϕ.
- If $M \models \phi$, then it has a unique L'-expansion to a model M' of T' that omits Γ'.

The proof is a variant of Morleyisation. First pick a countable fragment F containing ϕ such that every formula in F has finitely many free variables. For every formula $\psi \in F$ in n-free variables add to L' an n-ary relation symbol R_ψ (0-ary relation symbols being propositional constants). To form T' and Γ' we add "descriptions" of R_ψ for every ψ. We do this by case analysis.

- If $\psi(\bar{x})$ is atomic, then add $\forall \bar{x}(R_\psi(\bar{x}) \longleftrightarrow \psi(\bar{x}))$ to T'.

- If $\psi(\bar{x})$ is $\neg\chi(\bar{x})$, then add $\forall\bar{x}(R_\psi(\bar{x}) \leftrightarrow \neg R_\chi(\bar{x}))$ to T'.
- If $\psi(\bar{x})$ is $\exists y\chi(\bar{x}, y)$, then add $\forall\bar{x}(R_\psi(\bar{x}) \leftrightarrow \exists w R_\chi(\bar{x}, y))$ to T'.
- If $\psi(\bar{x})$ is $\forall y\chi(\bar{x}, y)$, then add $\forall\bar{x}(R_\psi(\bar{x}) \leftrightarrow \forall y R_\chi(\bar{x}, y))$ to T'.
- If $\psi(\bar{x})$ is $\bigvee_{\sigma\in\Sigma}\sigma(\bar{x})$, then add $\forall\bar{x}(R_\sigma(\bar{x}) \to R_\psi(\bar{x}))$ to T' for all $\sigma \in \Sigma$ and add $\{\neg R_\sigma(\bar{x}) : \sigma \in \Sigma\} \cup \{R_\psi(\bar{x})\}$ to Γ'.
- If $\psi(\bar{x})$ is $\bigwedge_{\sigma\in\Sigma}\sigma(\bar{x})$, then add $\forall\bar{x}(R_\psi(\bar{x}) \to R_\sigma(\bar{x}))$ to T' for all $\sigma \in \Sigma$ and add $\{R_\sigma(\bar{x}) : \sigma \in \Sigma\} \cup \{\neg R_\psi(\bar{x})\}$ to Γ'.

Finally to assert that ϕ must hold add R_ϕ to T'.

Thus instead of looking at $L_{\omega_1,\omega}$ we can fix a complete first order theory T, a subset $D \subseteq \mathrm{S}(T)$ of types and look at the models of T that only realise types from D. Such models are called *D-models*. One advantage of this approach is that we can now work in a monster model $\mathfrak{C}$ of T, which of course would not be a D-model, unless $D = \mathrm{S}(T)$. For a set A we denote by $D(A) = \{\mathrm{tp}(\bar{a}) : \bar{a} \in A\}$ the set of types realised in A. If $D(A) \subseteq D$, then A is called a D-set.

If $A \subseteq M$ is a D-set and $p \in \mathrm{S}(A)$ is a type then there is a natural obstruction to realising p in a D-model: if $\bar{c}$ realises p, we need $A\bar{c}$ to be a D-set. The type p would be called a *D-type* if $A\bar{c}$ is a D-set for some (equivalently every) realisation $\bar{c}$ of p. The set of all D-types over A is denoted by $\mathrm{S}_D(A)$.

Definition 1.1.9. A D-model M is called (D, λ)-homogeneous if M realises every $p \in \mathrm{S}_D(A)$ for every $A \subseteq M$ of cardinality $|A| < \lambda$.

Models that are (D, λ)-homogeneous are the analogue of λ-saturated models in first order logic. In particular they are λ-homogeneous and embed every D-model of cardinality $\leq \lambda$. Thus if (D, λ)-homogeneous models exist for arbitrarily large λ, they can serve the role of a monster model. The study of D-models under the assumption of existence of such models is called *homogeneous model theory*. Many aspects of first order (saturated) stability theory generalise naturally to homogeneous model theory. In particular in the pioneering paper on homogeneous model theory Shelah [1970] proves the analogue of Morley's theorem.

Theorem 1.1.10 (Shelah [1970])**.** Assume that the class $\mathscr{C}$ of D-models of a complete first order theory T in a countable language contains (D,λ)-homogeneous models for arbitrarily large λ. If $\mathscr{C}$ is categorical in one uncountable cardinality, then it is categorical in all uncountable cardinalities.

Keisler [1971] independently establishes the same result in the context of $L_{\omega_1,\omega}$. He further conjectured that the existence of arbitrarily large homogeneous models actually follows from categoricity in all uncountable cardinalities. But this conjecture has been disproved. Marcus [1972] and later Knight [1978] constructed a theory T whose prime model M has the following properties.

- It is minimal. I.e. M has no proper elementary substructures.
- There is an infinite set of indiscernibles in M.

Note that since its prime model is minimal, T does not have uncountable atomic models. To get a categorical class out of this, add a new predicate P to the language. Let T' assert that the elements that satisfy P form a model of T and the elements that satisfy $\neg P$ are just an infinite set. (We may assume without loss of generality that T does not use function symbols, so that we do not need to define them on $\neg P$.) Then consider the class $\mathscr{C}$ of atomic models T' (i.e. here D is the set of isolated types). Then $\mathscr{C}$ is categorical in every cardinality (including $\aleph_0$). However since there is a countable maximal set of indiscernible elements, no model is $\aleph_1$-homogeneous.

Later Zilber [2006] found a natural counterexample to Kiesler's conjecture.

1.2 Excellent Classes

The examples of Marcus [1972] and Knight [1978] suggest that homogeneous model theory is inadequate for handling categoricity in $L_{\omega_1,\omega}$. To rectify this Shelah [1983a,b] introduced excellent classes. But before looking at excellent classes let us discuss the connection of categoricity and completeness.

Since the Upward Löwenheim-Skolem Theorem fails for $L_{\omega_1,\omega}$, the standard argument that categoricity implies completeness fails as well. In fact, as we have seen, an $L_{\omega_1,\omega}$-sentence is complete if and only if it is $\aleph_0$-categorical. So if ϕ is the conjunction of axioms of a theory that is uncountably categorical but not $\aleph_0$-categorical (e.g. ACF), then ϕ is categorical in all uncountable cardinalities but is not complete. However for certain questions on the categoricity spectrum, arbitrary $L_{\omega_1,\omega}$-sentences can be replaced with categorical ones.

Proposition 1.2.1. Assume that an $L_{\omega_1,\omega}$-sentence ϕ has arbitrarily large models and is categorical in some cardinality κ. Then there is a complete $L_{\omega_1,\omega}$-sentence ψ that implies ϕ and such that for every $\lambda \geq \min(\kappa, \beth_{\omega_1})$ we have ϕ is λ-categorical if and only if ψ is λ-categorical.

Thus if we are interested in eventual categoricity of ϕ (i.e. what conditions imply that ϕ is κ-categorical for all sufficiently large κ), we can assume, without loss of generality, that ϕ is complete. The reason we are interested in complete $L_{\omega_1,\omega}$-sentences is because we can improve Proposition 1.1.8 by taking Γ to be the set of nonisolated types.

Proposition 1.2.2. Given a complete $L_{\omega_1,\omega}$ sentence ϕ there is a complete first order theory T' in a countable expansion L' of L such that the following conditions hold.

- If $M' \models T'$ is atomic then its L-reduct is a model of ϕ.
- If $M \models \phi$, then it has a unique L'-expansion to an atomic model M' of T'.

Thus instead of studying the class of models of an $L_{\omega_1,\omega}$ sentence we can study the class of atomic models of a countable first order theory. Such classes are called *atomic classes*. For atomic classes Shelah introduced the notion of *excellence* (see Definition 1.2.16 below) and proved the following result.

Theorem 1.2.3 (Shelah [1983a,b]). Let $\mathscr{C}$ be the class of atomic models of a countable complete theory.

- (Assume $2^{\aleph_n} < 2^{\aleph_{n+1}}$ for each $n < \omega$.) If $\mathscr{C}$ is $\aleph_n$-categorical for each $n < \omega$, then $\mathscr{C}$ is excellent.

- If $\mathscr{C}$ is excellent and categorical in some uncountable cardinality, then it is categorical in all uncountable cardinalities.

Hart and Shelah [1990] have shown that the categoricity may stop at $\aleph_n$ while still holding for $\aleph_0, \ldots, \aleph_{n-1}$ (for each $n < \omega$). Thus assuming categoricity below $\aleph_\omega$ is necessary to deduce excellence. Whether the set theoretic assumption of $2^{\aleph_n} < 2^{\aleph_{n+1}}$ for each $n < \omega$ (called Very Weak Generalised Continuum Hypothesis in Baldwin [2009]) is necessary is widely open as far as we are aware.

Here we present some background on excellence and deducing categoricity transfer for excellent classes. Our exposition is heavily influenced by Lessmann [2003], Lessmann [2005] and Baldwin [2009].

For the rest of the section fix a countable complete theory T and let $\mathscr{C}$ be the class of atomic models of T. Since we are interested in uncountable categoricity we assume that $\mathscr{C}$ contains uncountable models. We work in the monster model $\mathfrak{C}$ of T which is not in general in $\mathscr{C}$. All subsets and tuples are assumed to be from $\mathfrak{C}$. A subset $A \subseteq \mathfrak{C}$ is called *atomic* if A realises only isolated types (over $\emptyset$). Note that in particular $\mathscr{C}$ is $\aleph_0$-categorical and every model in $\mathscr{C}$ is ω-homogeneous.

As in homogeneous model theory, there is a natural obstruction for realising types in models of $\mathscr{C}$. If $\bar{c} \in M \in \mathscr{C}$ realises $p \in \mathrm{S}(A)$ for some $A \subseteq M$, then $A\bar{c}$ need to be atomic. This motivates the following definition.

Definition 1.2.4. Given a subset A let $S_{at}(A)$ be the set of types $p \in S(A)$ such that for some (all) $\bar{c} \models p$, the set $A\bar{c}$ is atomic.

We refer to types in $S_{at}(A)$ as *atomic types*. This however does not imply that they are isolated. Rather if $p \in S_{at}(A)$, then $p|_{A_0}$ is isolated for every finite $A_0 \subseteq A$. Conversely if $p \in S(A)$ is such that $p|_{A_0}$ is isolated for every finite $A_0 \subseteq A$, then $p \in S_{at}(A)$. Thus if $[\phi]$ denotes the set of types containing ϕ and $L_0(A_0)$ denotes the set of complete formulas over A_0 we have

$$S_{at}(A) = \bigcap_{A_0 \subseteq_{fin} A} \bigcup_{\phi \in L_0(A_0)} [\phi].$$

In particular if A is countable, then $S_{at}(A)$ is a dense G_δ-subset of a complete compact metric space $S(A)$. (Here we think of $S(A)$ and $S_{at}(A)$ as the space of types in a fixed n variables.) The density follows from the fact that A can be embedded in a model. Therefore $S_{at}(A)$ is itself a complete metric space. This will become useful in Chapter 4.

Note that in general a type $p \in S_{at}(A)$ need not be realised. The exception is again the case where A is countable. Indeed if $\bar{c} \models p$, then $A\bar{c}$ is a countable atomic set. Hence by ω-homogeneity it embeds in every model of $\mathscr{C}$. Another problem is that an atomic type over A may not extend to an atomic type over a superset $B \supseteq A$. This may fail even for countable A. In general in atomic classes types over arbitrary subsets are badly behaved, whereas we will find a way to deal with types over models.

Next we introduce a notion of stability for atomic classes.

Definition 1.2.5. The atomic class $\mathscr{C}$ is called λ-stable if for each $M \in \mathscr{C}$ of cardinality λ we have $|S_{at}(M)| \leq \lambda$.

The connection between categoricity and stability is more subtle here. We have the following result.

Proposition 1.2.6 (Shelah [1983a]). Assume that $2^{\aleph_n} < 2^{\aleph_{n+1}}$ for all $n < \omega$. If $\mathscr{C}$ has less than $2^{\aleph_1}$ models in cardinality $\aleph_1$, then $\mathscr{C}$ is ω-stable.

Thus ω-stability follows from categoricity in $\aleph_1$ (modulo additional set theoretic assumptions). So in what follows we assume that $\mathscr{C}$ is ω-stable. Note that this does not imply that T is ω-stable. (We can always add a sort with full arithmetic that does not interact with the rest of the structure. There will be a natural bijection between atomic models of the original and extended theories, since $\mathbb{N}$ is the only atomic model of its theory.) Neither does this imply that $\mathrm{S}_{\mathrm{at}}(A)$ is countable for all countable A. In fact this is very strong. Lessmann [2002] shows that if $\mathrm{S}_{\mathrm{at}}(A)$ is countable for every countable A, then $\mathscr{C}$ has homogeneous models of arbitrarily large cardinalities.

Recall that a model M is called *constructible* over $A \subseteq M$ if $M = A \cup \{b_i : i < \alpha\}$ such that $\mathrm{tp}(b_i/A \cup \{b_j : j < i\})$ is isolated. A constructible model (also called primary or strictly prime) is always prime over A. The converse is true if A is countable or the theory is ω-stable. In the latter case there is a constructible model over each subset A. In our setup this is not the case. So we introduce the notion of a *good* set. According to Shelah [1983a] the search for good sets is a central theme in excellence.

Definition 1.2.7. A subset A is good if isolated types are dense in $\mathrm{S}(A)$.

The Omitting Types Theorem gives us the following result about good sets.

Proposition 1.2.8. If A is a countable atomic good set, then there is a prime (and hence constructible) model over A (which will be in $\mathscr{C}$).

We give a criterion for a countable atomic set to be good.

Proposition 1.2.9. If A is a countable atomic set and $\mathrm{S}_{\mathrm{at}}(A)$ is countable, then A is good.

Proof. This follows from the fact that $\mathrm{S}_{\mathrm{at}}(A)$ is a countable dense G_δ subset of a complete metric space $\mathrm{S}(A)$.

Indeed more generally let X be a complete metric space, $Y = \bigcap_{i<\omega} U_i$ be a countable dense G_δ-subset (U_i open) and let Z be the set of isolated points in X. Then by the density of Y we have $Z \subseteq Y$. Also for each $y \in Y \setminus Z$ we have $X \setminus \{y\}$ is dense. Hence by the Baire Category Theorem

$$Z = \bigcap_{i<\omega} U_i \cap \bigcap_{y \in Y \setminus Z} X \setminus \{y\}$$

is also dense in X. $\square$

Next we introduce a rank for definable sets and types. In the ω-stable case, the rank will be bounded which will allow us to define a notion of independence.

Definition 1.2.10. Let $M \in \mathscr{C}$ be a model, $\phi(\bar{x})$ be a formula over M and α be an ordinal. We define $\mathrm{R}_M(\phi) \geq \alpha$ by induction on α as follows.

- $\mathrm{R}_M(\phi) \geq 0$ if ϕ is realised in M.
- If δ is a limit ordinal, then $\mathrm{R}_M(\phi) \geq \delta$ if $\mathrm{R}_M(\phi) \geq \alpha$ for all $\alpha < \delta$.
- $\mathrm{R}_M(\phi) \geq \alpha + 1$ if the following two conditions hold.
  - There is a formula $\psi(\bar{x})$ over M such that

    $$\mathrm{R}_M(\phi(\bar{x}) \wedge \psi(\bar{x})) \geq \alpha \text{ and } \mathrm{R}_M(\phi(\bar{x}) \wedge \neg\psi(\bar{x})) \geq \alpha.$$

  - For every $\bar{c} \in M$ there is a formula $\chi(\bar{x}, \bar{c})$ that isolates a complete type over $\bar{c}$ and

    $$\mathrm{R}_M(\phi(\bar{x}) \wedge \chi(\bar{x}, \bar{c})) \geq \alpha.$$

Before moving on let us calculate the rank of formulas in one variable in a simple example.

Example 1.2.11. Let T be the theory of infinite sets in the empty language. Since T is $\aleph_0$-categorical, all models of T are atomic. Let M be a model and $\phi(x)$ a formula in one variable. Since T is strongly minimal ϕ either defines a finite set or a cofinite set.

Let us start with finite sets. For every element $a \in M$ we have $\mathrm{R}_M(x = a) = 0$. Indeed for no formula ψ can both $x = a \wedge \psi(x)$ and $x = a \wedge \neg\psi(x)$ be realised. For the formula $x = a \vee x = b$ defining a two-element set $\{a, b\}$ we have $\mathrm{R}_M(x = a \vee x = b) = 1$. Indeed it can be split into two singletons which have rank 0 and it is easy to see that the second condition is also satisfied.

However this trend does not continue. Consider the formula $x = a \vee x = b \vee x = c$ defining the three-element set $\{a, b, c\}$. The only formulas that isolate a complete type over the tuple abc are $x = a$, $x = b$, $x = c$ and $x \neq a \wedge x \neq b \wedge x \neq c$. They define sets $\{a\}$, $\{b\}$, $\{c\}$ and $M \setminus \{a, b, c\}$ respectively. None of them has a two-element intersection with $\{a, b, c\}$. Therefore $\mathrm{R}_M(x = a \vee x = b \vee x = c) \ngeq 2$ and so $\mathrm{R}_M(x = a \vee x = b \vee x = c) = 1$. Similarly we see that all finite sets of more than one element have rank 1.

As for cofinite sets they get assigned rank 2. Indeed if $\phi(x)$ defines a cofinite set, it is easy to check that $\mathrm{R}_M(\phi) \geq 2$. However $\mathrm{R}_M(\phi) \ngeq 3$ since a cofinite set cannot be split into two definable cofinite sets.

If ϕ is not realised in M we define $\mathrm{R}_M(\phi) = -1$. If $\mathrm{R}_M(\phi) \geq \alpha$ for all ordinals α, we define $\mathrm{R}_M(\phi) = \infty$. Otherwise we define $\mathrm{R}_M(\phi) = \alpha$ if $\alpha + 1$ is the least ordinal such that $\mathrm{R}_M(\phi) \ngeq \alpha + 1$. For a partial type $p(\bar{x})$ over M we define

$$\mathrm{R}_M(p) = \min\{\mathrm{R}_M(\phi) : \phi(\bar{x}) \in L(M), p(\bar{x}) \vdash \phi(\bar{x})\}.$$

It can be seen that if a partial type p is over both M and N, then $\mathrm{R}_M(p) = \mathrm{R}_N(p)$. So we will drop the subscript for the model. Since the rank is monotone, i.e. $p \subseteq q$ implies $\mathrm{R}(p) \geq \mathrm{R}(q)$, we can use it to define an independence relation. We just need to show that ω-stability of $\mathscr{C}$ implies that the rank is always an ordinal number. First observe that $R(\phi(\bar{x}, \bar{a}))$ depends on ϕ and $\mathrm{tp}(\bar{a}/\emptyset)$. Hence there is a countable ordinal $\alpha < \omega_1$ such that $\mathrm{R}(p) \geq \alpha$ implies $\mathrm{R}(p) = \infty$.

Theorem 1.2.12. For every formula ϕ we have $\mathrm{R}(\phi) < \infty$.

Proof. Let $\phi = \phi(\bar{x}, \bar{b})$ be over a countable model M and assume for contradiction that $\mathrm{R}(\phi) = \infty$. We will construct uncountably many types in $\mathrm{S}_{\mathrm{at}}(M)$. First enumerate $M = \{a_n : n < \omega\}$. For every finite sequence $\eta \in 2^{<\omega}$ construct a formula $\psi_\eta(\bar{x}, \bar{c}_\eta)$ as follows.

We have $\mathrm{R}(\phi(\bar{x}, \bar{b})) \geq \omega_1 + 1$. Hence there is a formula $\psi_\emptyset(\bar{x}, \bar{b}a_0)$ that isolates a type over $\bar{c}_\emptyset = \bar{b}a_0$ and $\mathrm{R}(\phi \wedge \psi_\emptyset) \geq \omega_1$. Note that since $\psi_\emptyset$ isolates a complete type, we have $\forall \bar{x}(\psi_\emptyset(\bar{x}, \bar{b}a_0) \rightarrow \phi(\bar{x}, \bar{b}))$ and hence $\mathrm{R}(\psi_\emptyset) \geq \omega_1$.

Assume that $\psi_\eta(\bar{x}, \bar{c}_\eta)$ has been constructed and $\mathrm{R}(\psi_\eta) \geq \omega_1$. Then $\mathrm{R}(\psi_\eta) \geq \omega_1 + 2$. Hence there is a formula $\chi(\bar{x}, \bar{d})$ such that

$$\mathrm{R}(\psi_\eta(\bar{x}, \bar{c}_\eta) \wedge \chi(\bar{x}, \bar{d})) \geq \omega_1 + 1 \text{ and } \mathrm{R}(\psi_\eta(\bar{x}, \bar{c}_\eta) \wedge \neg\chi(\bar{x}, \bar{d})) \geq \omega_1 + 1.$$

Now let n be the length of η. Set $\bar{c}_{\eta^\frown 0} = \bar{c}_{\eta^\frown 1} = \bar{c}_\eta \bar{d} a_n$ and let $\psi_{\eta^\frown i}$ isolate a complete type over $\bar{c}_{\eta^\frown i}$ such that

$$\mathrm{R}(\psi_\eta(\bar{x}, \bar{c}_\eta) \wedge \chi(\bar{x}, \bar{d}) \wedge \psi_{\eta^\frown 0}(\bar{x}, \bar{c}_{\eta^\frown 0})) \geq \omega_1$$

and

$$\mathrm{R}(\psi_\eta(\bar{x}, \bar{c}_\eta) \wedge \neg\chi(\bar{x}, \bar{d}) \wedge \psi_{\eta^\frown 1}(\bar{x}, \bar{c}_{\eta^\frown 1})) \geq \omega_1.$$

And as before by completeness we have $\mathrm{R}(\psi_{\eta^\frown i}(\bar{x}, \bar{c}_{\eta^\frown i})) \geq \omega_1$.

Finally for each $\eta \in 2^\omega$ let $p_\eta = \{\psi_{\eta|_n} : n < \omega\}$. Then for each finite $A \subset M$, the type p_η contains a complete formula over A. So p_η has a unique extension to a type in $\mathrm{S}_{\mathrm{at}}(M)$. This shows that $|\mathrm{S}_{\mathrm{at}}(M)| = 2^{\aleph_0}$ contradicting ω-stability of $\mathscr{C}$. $\square$

The converse of this theorem also holds: if $\mathrm{R}(\phi) < \infty$ for every formula ϕ, then the class $\mathscr{C}$ is λ-stable for all λ. It follows from the next result.

Proposition 1.2.13. If $p \in \mathrm{S}_{\mathrm{at}}(M)$ is a type and $\phi(\bar{x}, \bar{b}) \in p$ is such that $\mathrm{R}(p) = \mathrm{R}(\phi) < \infty$, then p is the only type in $\mathrm{S}_{\mathrm{at}}(M)$ extending ϕ with the same rank.

Proof. Assume for contradiction that $q \in S_{at}(M)$ is another extension of $\phi(\bar{x}, \bar{b})$ of the same rank $R(\phi) = \alpha$. Assume that $\psi(\bar{x}) \in L(M)$ is a formula such that $\psi(\bar{x}) \in q$ and $\neg\psi(\bar{x}) \in p$. Then $R(\phi \wedge \psi) \geq \alpha$ and $R(\phi \wedge \neg\psi) \geq \alpha$. Also given $\bar{c} \in M$ we have that $p|_{\bar{b}\bar{c}}$ is isolated. So if χ isolates it, then $\chi \in p$ and so $R(\phi \wedge \chi) \geq \alpha$. This shows that $R(\phi) \geq \alpha + 1$ which is a contradiction. □

The previous two results combine to give a stability spectrum result.

Corollary 1.2.14. If $\mathscr{C}$ is ω-stable, then it is λ-stable for all λ.

Let A, B, C be sets such that $A \cup B \cup C$ is atomic. We say that A is *free* from B over C and write

$$A \underset{C}{\perp} B$$

if for every finite tuple $\bar{a} \in A$ we have $R(\mathrm{tp}(\bar{a}/BC)) = R(\mathrm{tp}(\bar{a}/C))$. In ω-stable classes this independence relation satisfies many of the properties of forking independence in stable theories. Some properties (such as symmetry) only hold if the base set is a model.

In the following definition we think of a natural number n as the set of its predecessors, i.e. $n = \{0, 1, ..., n-1\}$. Essentially we are using n as a canonical n-element set.

Definition 1.2.15. For a natural number n, an independent n-system is a collection $(M_s : s \subsetneq n)$ of models satisfying the following conditions.

- If $s \subseteq t \subsetneq n$, then $M_s \preceq M_t$.
- $A_s = \bigcup_{t \subsetneq s} M_t$ is atomic for every $s \subseteq n$.
- $M_s \perp_{A_s} B_s$ where $B_s = \bigcup_{t \not\supseteq s} M_t$.

Now we can give the main definition of this section.

Definition 1.2.16. An ω-stable class $\mathscr{C}$ is called excellent if every independent n-system $(M_s : s \subsetneq n)$ of countable model has a constructible model over $\bigcup_{s \subsetneq n} M_s$.

Shelah [1983a,b] takes a similar approach to Morley [1965] for deducing the categoricity transfer from excellence. Instead of saturation he uses fullness. A model $M \in \mathscr{C}$ is called *κ-full* if M realises every type $p \in S_{at}(N)$ for every $N \prec M$ with $|N| < \kappa$. The model M is called full if it is $|M|$-full. (This is not the definition used by Shelah, but it is equivalent for uncountable models). Now if $\mathscr{C}$ is excellent, then there is a unique full model in each cardinality. If there is a model that is not full (violating categoricity in some uncountable cardinality) then one can construct a model that is not full in each uncountable cardinality.

For a Baldwin-Lachlan style proof of the categoricity transfer, Lessmann [2003] noticed that certain model theoretical consequences of excellence suffice. These consequences are called **-excellence* in Baldwin [2009].

Definition 1.2.17. An atomic class $\mathscr{C}$ is called *-excellent if the following conditions hold.

- The class $\mathscr{C}$ is ω-stable.

- It has amalgamation over models. I.e. if $M_0, M_1, M_2 \in \mathscr{C}$ are models and maps $f_1 \colon M_0 \to M_1, f_2 \colon M_0 \to M_2$ are elementary, then there is $N \in \mathscr{C}$ and elementary maps $g_1 \colon M_1 \to N, g_2 \colon M_2 \to N$ such that $g_1 \circ f_1|_{M_0} = g_2 \circ f_2|_{M_0}$.

- Types over models are realised. If $M \in \mathscr{C}$ and $p \in S_{at}(M)$, then there is a model $N \succeq M$ that realises p.

- Existence of constructible models. If M is a model and $M\bar{a}$ is atomic, then there is a constructible model over $M\bar{a}$.

As noticed by Lessmann [2003] excellence implies *-excellence. We do not know if the converse is true. If not, *-excellence may be useful in proving the categoricity transfer in ZFC. Here we present one direction of the categoricity transfer for a *-excellent class $\mathscr{C}$: if $\mathscr{C}$ is categorical in some uncountable cardinality, then it is $\aleph_1$-categorical.

The key is the following omitting types theorem which is the analogue of Lachlan's Theorem from Baldwin and Lachlan [1971].

Theorem 1.2.18. Assume that $\mathscr{C}$ is *-excellent and $M \in \mathscr{C}$ is uncountable. Then M has arbitrarily large elementary extensions in $\mathscr{C}$ which omit every countable set of $L(M)$-formulas omitted in M.

Proof. Let $\phi(x)$ be an $L(M)$-formula of minimal rank that defines an uncountable set. Let $\mathrm{R}(\phi) = \alpha$. For every tuple $\bar{c} \in M$ we have countably many $L(\bar{c})$-formulas that isolate a complete type over $\bar{c}$. Since each element in $\phi(M)$ realises an isolated type over $\bar{c}$, there is a formula $\chi(x, \bar{c})$ isolating a complete type over $\bar{c}$ such that $\phi(x) \wedge \chi(x, \bar{c})$ defines an uncountable set. By minimality of α we have $\mathrm{R}(\phi(x) \wedge \chi(x, \bar{c})) \geq \alpha$. Thus for each formula $\psi(x) \in L(M)$ either $\phi(x) \wedge \psi(x)$ or $\phi(x) \wedge \neg\psi(x)$ define a countable set. Indeed otherwise $\mathrm{R}(\phi \wedge \psi) \geq \alpha$ and $\mathrm{R}(\phi \wedge \neg\psi) \geq \alpha$ showing that $\mathrm{R}(\phi) \geq \alpha + 1$.

Thus the set

$$p(x) = \{\psi(x) : \phi(M) \wedge \psi(M) \text{ is uncountable}\}$$

is a complete type over M. Further, each countable subset of $p(x)$ is realised in M. In particular for each finite $C \subset M$, the restriction $p|_C$ is realised in M showing that in fact $p \in \mathrm{S}_{\mathrm{at}}(M)$. Let a realise p and let N be a constructible (hence atomic) model over Ma (which exists by *-excellence). Note that N is a proper extension of M since p is omitted in M.

We now show that every countable set of formulas over M realised in N is already realised in M. Let $\bar{b} \in N$ and $\Sigma(\bar{y})$ be a countable subset of $\mathrm{tp}(\bar{b}/M)$. Let $\chi(x, \bar{y}) \in L(M)$ be a formula such that $\chi(a, \bar{y})$ isolates $\mathrm{tp}(\bar{b}/Ma)$. Thus for $\sigma(\bar{y}) \in \Sigma$ we have $\sigma'(x) = \forall \bar{y}(\chi(x, \bar{y}) \to \sigma(\bar{y})) \in p(x)$. Also $\exists \bar{y}\chi(x, \bar{y}) \in p(x)$. Let $a' \in M$ realise $\{\sigma'(x) : \sigma \in \Sigma\} \cup \{\exists \bar{y}\chi(x, \bar{y})\}$ which is a countable subset of $p(x)$. Now let $\bar{b}' \in M$ be such that $M \models \chi(a', \bar{b}')$. Then $\sigma(\bar{b}')$ holds for all $\sigma \in \Sigma$ and therefore $\bar{b}'$ realises $\Sigma(\bar{y})$.

We constructed a proper extension of M that omits all countable sets of $L(M)$-formulas omitted in M. By iterating this construction we get arbitrarily large models that omit all countable set of $L(M)$-formulas omitted in M. $\square$

Now we can use this omitting types result to deduce downwards categoricity transfer. First we show the existence of full models.

Proposition 1.2.19. If $\mathscr{C}$ is *-excellent, then for every κ and every regular $\lambda \leq \kappa$, there is a λ-full model of cardinality κ.

Proof. Let $M_0 \in \mathscr{C}$ be an arbitrary model of cardinality κ. By κ-stability $|\mathrm{S}_{\mathrm{at}}(M_0)| \leq \kappa$. (Recall that an ω-stable class is κ-stable for every κ.) So by type realisability and amalgamation there is $M_1 \in \mathscr{C}$ of cardinality κ that realises every type in $\mathrm{S}_{\mathrm{at}}(M_0)$. We iterate this process to construct a chain $(M_\alpha : \alpha < \lambda)$ of models of cardinality κ such that $M_{\alpha+1}$ realises all types in $\mathrm{S}_{\mathrm{at}}(M_\alpha)$. Then $M = \bigcup_{\alpha<\lambda} M_\alpha$ is λ-full.

Indeed, let $N \prec M$ and $|N| < \lambda$. Since λ is regular, $N \preceq M_\alpha$ for some $\alpha < \lambda$. Now by type realisability and amalgamation every type in $\mathrm{S}_{\mathrm{at}}(N)$ extends to a type in $\mathrm{S}_{\mathrm{at}}(M_\alpha)$ which is realised in $M_{\alpha+1}$. $\square$

This shows that a *-excellent class contains a full model of every regular cardinality. With heavy use of independence one can extend this to singular cardinalities as well. But the weaker result we have here will suffice for the next proposition.

Proposition 1.2.20. If $\mathscr{C}$ is *-excellent and κ-categorical, then the unique model of cardinality κ is full.

Proof. Let $M \in \mathscr{C}$ be of cardinality κ and $N \prec M$ be of cardinality $\lambda < \kappa$. Then $\lambda^+ \leq \kappa$ is regular. So there is a λ^+-full model of cardinality κ. By κ-categoricity, M must be λ^+-full and so every type in $\mathrm{S}_{\mathrm{at}}(N)$ is realised in M. $\square$

In analogy with the first order case full models of the same cardinality are isomorphic.

Proposition 1.2.21. If M, N are full models of the same cardinality in a *-excellent class, then they are isomorphic.

Proof. Since every atomic class is $\aleph_0$-categorical, assume $\kappa = |M| = |N|$ is uncountable. Enumerate $M = \{a_\alpha : \alpha < \kappa\}$ and $N = \{b_\alpha : \alpha < \kappa\}$. Construct partial isomorphisms $f_\alpha \colon M_\alpha \to N_\alpha$ as follows. Take $M_0 \prec M$ and $N_0 \prec N$ to be two countable models and $f_0 \colon M_0 \to N_0$ an isomorphism. For limit δ let $f_\delta = \bigcup_{\alpha<\delta} f_\alpha$. To construct $f_{\alpha+1}$ for even α pick $a \in M \setminus M_\alpha$ with the least index. Let $b \in N$ realise the type $f_\alpha(\mathrm{tp}(a/M_\alpha)) \in \mathrm{S}_{\mathrm{at}}(N_\alpha)$. If $M_{\alpha+1}$ is constructible over $M_\alpha a$, then $f_\alpha \cup \{(a, b)\}$ extends to an isomorphism $f_{\alpha+1} \colon M_{\alpha+1} \to N_{\alpha+1}$ for some $N_{\alpha+1} \prec N$. For odd α do the same construction backwards. □

Finally we are ready to prove the downward categoricity transfer in *-excellent classes.

Theorem 1.2.22. If $\mathscr{C}$ is *-excellent and κ-categorical for some uncountable κ, then $\mathscr{C}$ is $\aleph_1$-categorical.

Proof. If $\mathscr{C}$ is not $\aleph_1$-categorical, then it has a model M of cardinality $\aleph_1$ that is not full. Hence there is a countable $N \prec M$ and a type $p \in \mathrm{S}_{\mathrm{at}}(N)$ that is omitted in M. By Theorem 1.2.18 there is an extension M' of M of cardinality κ that omits p. Thus M' is not full and therefore $\mathscr{C}$ is not κ-categorical. □

1.3 Generalised Quantifiers

Generalised quantifiers were introduced in mathematical logic by Mostowski [1957]. They were studied and extended in the works of many mathematicians. A modern approach to generalised quantifiers is presented in the survey Väänänen [2004].

Definition 1.3.1. Fix a language $(R_1, \ldots, R_n)$ with finitely many relations of the same arity and let $\mathscr{Q}$ be a class of structures closed under isomorphisms. For a logic L extend

L-formulas with formulas of the form $Q\bar{x}\phi_1(\bar{x}), \ldots, \phi_n(\bar{x})$ and interpret them as follows. Given a structure M we have

$$M \models Q\bar{x}\phi_1(\bar{x}), \ldots, \phi_n(\bar{x}) \text{ iff } (M, \phi_1(M), \ldots, \phi_n(M)) \in \mathcal{Q}.$$

Many common construction, such as defining schemes for types, can be viewed as generalised quantifiers. In that respect definability of types can be viewed as a quantifier elimination result. However this approach has not been extensively studied.

We are mainly going to be concerned with the generalised quantifier Q for the existence of uncountably many solutions. That is we extend the language with formulas of the form $Qx\phi(x)$ and interpret them as $M \models Qx\phi(x)$ iff $\phi(M)$ is uncountable. In the notation of the above definition we pick the language with one unary relation symbol R and the class $\mathcal{Q}$ of structures where R is interpreted by an uncountable set. The corresponding extensions of first order logic and $L_{\omega_1,\omega}$ are denoted by $L(Q)$ and $L_{\omega_1,\omega}(Q)$ respectively.

1.4 Abstract Elementary Classes

After his work on excellent classes, Shelah embarked on an ambitious generalisation of classification theory to the syntax free framework of abstract elementary classes. An abstract elementary class (AEC for short) is a class of structure in a fixed language together with a notion of strong embedding satisfying certain axioms. It was introduced in Shelah [1987] based on the earlier works of Fraïssé [1953] and Jónsson [1956, 1960].

Definition 1.4.1. Let $\mathcal{C}$ be a class of L-structures (for a fixed language L) and $\preceq_{\mathcal{C}}$ be a binary relation on $\mathcal{C}$ (called strong substructure). The pair $(\mathcal{C}, \preceq_{\mathcal{C}})$ is called an abstract elementary class (AEC) if the following conditions hold.

- *Closure under isomorphisms*

If $M, N \in \mathscr{C}$, $M \preceq_{\mathscr{C}} N$ and $f : N \to N'$ is an isomorphism, then $f(M), N' \in \mathscr{C}$ and $f(M) \preceq_{\mathscr{C}} N'$.

- *Partial order*

  The relation $\preceq_{\mathscr{C}}$ is a partial order on $\mathscr{C}$ and $M \preceq_{\mathscr{C}} N$ implies that M is a substructure of N.

- *Chain conditions*

  If $(M_\alpha : \alpha < \delta)$ is a $\preceq_{\mathscr{C}}$-increasing chain (i.e. $M_\alpha \preceq_{\mathscr{C}} M_\beta$ for $\alpha < \beta < \delta$), then

  - $M = \bigcup_{\alpha<\delta} M_\alpha \in \mathscr{C}$;
  - for each $\alpha < \delta$ we have $M_\alpha \preceq_{\mathscr{C}} M$;
  - If N is such that $M_\alpha \preceq_{\mathscr{C}} M$ for each $\alpha < \delta$, then $M \preceq_{\mathscr{C}} N$.

- *Coherence*

  If $M_1 \preceq_{\mathscr{C}} M_3$ and $M_2 \preceq_{\mathscr{C}} M_3$ and $M_1 \subseteq M_2$, then $M_1 \preceq_{\mathscr{C}} M_2$.

- *Löwenheim-Skolem axiom*

  There is a cardinal number $\mathrm{LS}(\mathscr{C})$ such that if $M \in \mathscr{C}$ and $A \subseteq M$ is a subset, then there is $N \in \mathscr{C}$ such that $A \subseteq N \preceq_{\mathscr{C}} M$ and $|N| \leq |A| + \mathrm{LS}(\mathscr{C})$.

Before proceeding any further let us give some examples of abstract elementary classes.

Example 1.4.2. The canonical example is of course the class of models of a first order theory together with elementary embeddings. The Löwenheim-Skolem number of this class is the cardinality of the language. More generally if $D \subseteq S(T)$ where T is a first order theory, then the class of D-models of T with elementary embeddings forms an AEC. In particular an excellent class is an AEC together with elementary embeddings.

The class of models of an $L_{\omega_1,\omega}$-sentence ϕ can be made into an AEC by carefully choosing the notion of a strong embedding. First construct a countable fragment F containing ϕ such that every formula in F has finitely many free variables (see Proposition

1.1.3). Now take $\preceq_F$ as the notion of a strong substructure. The Löwenheim-Skolem number of this class is $\aleph_0$ by Theorem 1.1.4.

Making the class of models of an $L_{\omega_1,\omega}(Q)$-sentence ϕ into an abstract elementary class requires more care. Similar to $L_{\omega_1,\omega}$ we can pick a countable fragment F containing ϕ where each formula has finitely many variables. (For $L_{\omega_1,\omega}(Q)$ we in addition require that a fragment be closed under Q-quantification). Then define the strong substructure relation $\preceq^*$ as follows $M \preceq^* N$ if $M \preceq_F N$ and $\psi(M,\bar{a}) = \psi(N,\bar{a})$ whenever $M \models \neg Qx\psi(x,\bar{a})$. The class of models of ϕ together with $\preceq^*$ forms an AEC with Löwenheim-Skolem number $\aleph_1$.

These examples show that the notion of abstract elementary classes generalises other frameworks discussed so far. However the notion of an AEC is much broader. For example the class of groups is an AEC where the relation of being a subgroup is the notion of a strong substructure. In the next chapter we will investigate quasiminimal excellent classes which are also AECs under a very general condition.

Notice that the definition of AEC does not mention any formulas. However by expanding the language one can still present an abstract elementary class as the class of reducts of models of a first order theory omitting certain types. This can be used to calculate the Hanf number for existence of models of an AEC.

Proposition 1.4.3. If an abstract elementary class $\mathscr{C}$ has a model of cardinality $H_{\mathscr{C}} = \beth_\kappa$ where $\kappa = (2^{|L|+\mathrm{LS}(\mathscr{C})})^+$ then $\mathscr{C}$ has arbitrarily large models.

The framework of abstract elementary classes is considered to be the most general framework for which there is a hope for an interesting classification theory. The first step in the classification theory would be the following analogue of Morley's theorem.

Conjecture 1.4.4 (Shelah's Categoricity Conjecture)**.** If an abstract elementary class $\mathscr{C}$ is categorical in some $\kappa > H_{\mathscr{C}}$, then it is categorical in all $\kappa > H_{\mathscr{C}}$.

As far as we know the conjecture is still open. Recently a weak form of the conjecture has been proven from large cardinal cardinal axioms in Boney [2014].

From the perspective of this thesis disregarding syntax restricts our ability to exploit the topological properties of the spaces of types. So we do not fully embrace abstract elementary classes here. Nevertheless abstract elementary classes are central to nonelementary stability theory and we present comparisons and connections where appropriate.

Chapter 2

Quasiminimal Excellent Classes

A quasiminimal excellent class is a nonelementary class of structures satisfying certain axioms. It was introduced by Zilber [2005b] to give canonical axiomatisations of pseudo-exponential fields (in Zilber [2005a]) and other related analytic structures. The notion has evolved through the works of Baldwin [2009] and Kirby [2010].

The original definition of Zilber [2005b] had an additional axiom called excellence. As the name suggests this axiom has some similarity to Shelah's notion of excellence. It played a central role in establishing categoricity in cardinalities above $\aleph_1$. In practice checking that excellence holds has been the most technically difficult part in applications of the categoricity theorem. Some of the original proofs of excellence had gaps, which have only recently been fixed.

In analogy with Shelah's theory of excellent classes, the axiom of excellence was thought to be the key to categoricity. However recently Bays et al. [2014a] showed that excellence actually follows from the rest of the axioms. Since excellence was no longer necessary to define quasiminimal excellent classes, they called an infinite dimensional structure in such a class a *quasiminimal pregeometry* structure. Following this we called the entire class a *quasiminimal pregeometry* class in Haykazyan [2016]. However this terminology was not very accurate. In particular in the next chapter we study structures that are quasiminimal and carry a pregeometry, but do not satisfy the axioms of Bays et al. [2014a]. So we have reverted the terminology back to *quasiminimal excellent* here.

In this chapter we present a direct proof of the categoricity result that bypasses the excellence axiom altogether. The main new idea is to look at (partial) embeddings that preserve all $L_{\omega_1,\omega}$ formulas possibly using infinitely many parameters. We call them *σ-embeddings*. Constructing σ-embeddings by a transfinite recursion presents additional challenges. Given an increasing chain $(f_\beta : \beta < \alpha)$ of σ-embeddings, their union $f = \bigcup_{\beta<\alpha} f_\beta$ need not be a σ-embedding. The problem is that f needs to preserve formulas over infinitely (countably) many parameters and if $\mathrm{cf}(\alpha) = \omega$ we cannot guarantee that these parameters all occur at an earlier stage. In case of quasiminimal excellent classes the axiom of $\aleph_0$-homogeneity over countable closed models provides a way around this problem in certain situations.

The results of this chapter have been published in Haykazyan [2016].

2.1 Quasiminimal Excellence

Let L be a countable first order language. We will study a class $\mathscr{C}$ of L-structures and prove its categoricity. (Technically $\mathscr{C}$ is defined as a class of pair.)

Recall that a partial function f from an L-structures M to an L-structure N (we use the notation $f : M \rightharpoonup N$ for partial functions) is called a (partial) *embedding* if it preserves quantifier-free formulas. Note that in particular f preserves the formula $x = x$ implying that it is injective. An isomorphism is just a bijective embedding. *Elementary* embeddings, on the other hand, preserve all first order formulas. We will deal with an even stronger notion of embedding. The function f is called a (partial) *σ-embedding* if it preserves all $L_{\omega_1,\omega}$-formulas (in general using infinitely many parameters from $\mathrm{dom}(f)$). In other words f is a σ-embedding if $(M, m)_{m\in\mathrm{dom}(f)} \equiv_{\omega_1,\omega} (N, f(m))_{m\in\mathrm{dom}(f)}$. The name $L_{\omega_1,\omega}$-embedding could be a better choice for our notion, but it is often used for functions that preserve all $L_{\omega_1,\omega}$-formulas using *finitely* many variable.

Now we introduce the notion of a quasiminimal excellent class from Zilber [2005b]. Our definition follows closely that of Kirby [2010].

Definition 2.1.1. A *quasiminimal excellent* class is a class $\mathscr{C}$ of pairs (H, cl_H) where H is an L-structure and cl_H is a pregeometry on H satisfying the following conditions.

- *Closure under isomorphisms*

  If $(H, \mathrm{cl}_H) \in \mathscr{C}$, H' is an L-structure and $f : H \to H'$ is an isomorphism, then $(H', \mathrm{cl}_{H'}) \in \mathscr{C}$, where $\mathrm{cl}_{H'}$ is defined as $\mathrm{cl}_{H'}(A') = f(\mathrm{cl}_H(f^{-1}(A')))$ for $A' \subseteq H'$.

- *Quantifier free theory*

  If $(H, \mathrm{cl}_H), (H', \mathrm{cl}'_H) \in \mathscr{C}$, then H and H' satisfy the same quantifier free sentences.

- *Pregeometry*

  - For each $(H, \mathrm{cl}_H) \in \mathscr{C}$ the closure of any finite set is countable.
  - If $(H, \mathrm{cl}_H) \in \mathscr{C}$ and $A \subseteq H$, then $\mathrm{cl}_H(A)$ is a substructure of H and together with the restriction of cl_H it is in $\mathscr{C}$.
  - If $(H, \mathrm{cl}_H), (H', \mathrm{cl}_{H'}) \in \mathscr{C}$, $A \subseteq H$, $b \in H$ and $f : H \rightharpoonup H'$ is a partial embedding defined on $A \cup \{b\}$, then $b \in \mathrm{cl}_H(A)$ if and only if $f(b) \in \mathrm{cl}_{H'}(f(A))$.

- *Uniqueness of the generic type over countable closed models*

  Let $(H, \mathrm{cl}_H), (H', \mathrm{cl}_{H'}) \in \mathscr{C}$, subsets $G \subseteq H, G' \subseteq H'$ be countable closed or empty and $g : G \to G'$ be an isomorphism. If $a \in H, a' \in H'$ are independent from G and G' respectively, then $g \cup \{(a, a')\}$ is a partial embedding.

- *$\aleph_0$-homogeneity over countable closed models*

  Let $(H, \mathrm{cl}_H), (H', \mathrm{cl}_{H'}) \in \mathscr{C}$, subsets $G \subseteq H, G' \subseteq H'$ be countable closed or empty and $g : G \to G'$ be an isomorphism. If $g \cup f : H \rightharpoonup H'$ is a partial embedding, $A =$

dom(f) is finite and $b \in \mathrm{cl}_H(A \cup G)$, then there is $b' \in H'$ such that $g \cup f \cup \{(b, b')\}$ is a partial embedding.

The definition is rather involved. So let us give some examples to illustrate it. If the closure operator is understood, we will abuse the notation and refer to a quasiminimal excellent class as a class of structures, rather than pairs.

Example 2.1.2. The class of models of a strongly minimal first order theory together with algebraic closure is a quasiminimal excellent class, provided that the closure of the empty set is infinite. The latter condition is needed to ensure that the closure of any subset of a given model is a model of the theory itself. This can be readily checked by the Tarski-Vaught test using the strong minimality of the theory.

Let the language L contain just one binary relation E. Consider the class of L-structures, where E is an equivalence relation and each equivalence class is countable. Define the closure of A to be the set of elements equivalent to some $a \in A$. This class is a quasiminimal excellent class. It can be realised as the class of models of an $L(Q)$-sentence.

Algebraically closed fields of characteristic 0 with a unary predicate symbol that is interpreted as the set of integers together with algebraic closure is quasiminimal excellent. This class is $L_{\omega_1,\omega}$ axiomatisable.

Finally we mention the mathematically interesting nonelementary classes of pseudo-exponential fields of Zilber [2005a] and group covers of Zilber [2006].

Before moving to detailed study of quasiminimal excellent classes let us explore its connection with abstract elementary classes and infinitary logic. To view a quasiminimal excellent class $\mathscr{C}$ as an abstract elementary class we need a notion of a strong embedding. The following easy proposition provides a place to look for.

Proposition 2.1.3. Let $\mathscr{C}$ be a quasiminimal excellent class, $H, H' \in \mathscr{C}$ and H is a closed substructure of H', i.e. $H \subseteq H'$ and $\mathrm{cl}_{H'}(H) = H$. Then for every $A \subseteq H$ we have

$\mathrm{cl}_H(A) = \mathrm{cl}_{H'}(A)$.

Proof. Indeed, consider the identity embedding from H to H'. Since the closure is determined by the language, for $b \in H$ we have $b \in \mathrm{cl}_H(A)$ if and only if $b \in \mathrm{cl}_{H'}(A)$. Thus $\mathrm{cl}_H(A) = \mathrm{cl}_{H'}(A) \cap H$. But since $A \subseteq H$ we have $\mathrm{cl}_{H'}(A) \subseteq \mathrm{cl}_{H'}(H) = H$. Thus $\mathrm{cl}_H(A) = \mathrm{cl}_{H'}(A)$. $\square$

For structures $H \subseteq H' \in \mathscr{C}$ define $H \preceq_{\mathscr{C}} H'$ if $\mathrm{cl}_{H'}(H) = H$. The structure H is called a *closed substructure* of H'. More generally a partial embedding is called closed if it takes closed subsets to closed subsets. In view of the previous proposition we drop the subscript from cl whenever no confusion arises.

If we check the axioms of abstract elementary classes for $(\mathscr{C}, \preceq_{\mathscr{C}})$, we can see that all axioms are satisfied (with Löwenheim-Skolem number $\aleph_0$) except possibly the chain axioms. In particular the union of a chain of closed substructures need not be in $\mathscr{C}$. However the following holds.

Proposition 2.1.4 (Kirby [2010]). Let $\mathscr{C}$ be a quasiminimal excellent class with arbitrarily large models and $(H_\alpha : \alpha < \delta)$ be a chain of structures in $\mathscr{C}$ with $H_\alpha \preceq_{\mathscr{C}} H_\beta$ for $\alpha < \beta < \delta$. Then $(H, \mathrm{cl}_H) \in \mathscr{C}$ where $H = \bigcup_{\alpha<\delta} H_\alpha$ and $\mathrm{cl}_H(A) = \bigcup_{\alpha<\delta} \mathrm{cl}_{H_\alpha}(A \cap H_\alpha)$.

Conversely if a quasiminimal excellent class contains an infinite dimensional model and is closed under unions of chains, then it has arbitrarily large models. This follows from uncountable categoricity (or more precisely from the fact that two models with the same dimension are isomorphic). Moreover any quasiminimal excellent class with an infinite dimensional model can be uniquely extended to a quasiminimal excellent class containing arbitrarily large models (just add unions of chains). Thus there is no harm in assuming that a quasiminimal excellent class is closed under unions of chains and therefore is an abstract elementary class. Since we have no further use of the chain axioms we do not assume them here.

The connection between quasiminimal excellent classes and infinitary logic is given in the next result.

Proposition 2.1.5 (Kirby [2010])**.** If a quasiminimal excellent class $\mathscr{C}$ has arbitrarily large models, then it is the class of models of an $L_{\omega_1,\omega}(Q)$-sentence.

The quantifier Q is necessary in general. For example the class of pseudo-exponential fields cannot be axiomatised in $L_{\omega_1,\omega}$ (see Kirby [2013]). So in general quasiminimal excellent classes are not covered by Shelah's theory of excellence.

Finally we present the axiom of excellence. Let B be an independent subset of $H \in \mathscr{C}$ and $b_1, ..., b_n \in B$. A set of the form $C = \bigcup_{i=1}^{n} \mathrm{cl}(B \setminus \{b_i\})$ is called a *crown*. Then excellence is the following axiom.

- *Quasiminimal excellence*
  Let $H, H' \in \mathscr{C}$, $g : H \rightharpoonup H'$ be a closed partial embedding whose domain $C = \mathrm{dom}(g)$ is a countable crown. Then for any finite subset $A \subseteq \mathrm{cl}(C)$ there is a finite subset $C_0 \subseteq C$ such that if $f : A \to H$ and $f \cup g|_{C_0}$ is an embedding, then $f \cup g$ is also an embedding.

As already mentioned this axiom follows from the rest of axioms by Bays et al. [2014a]. And our approach to categoricity does not use it at all. However we will compare our approach to the original one and highlight how we bypass the need for excellence.

2.2 Properties of Structures in Quasiminimal Excellent Classes

For the rest of this chapter fix a quasiminimal excellent class $\mathscr{C}$. As we have already done, we refer to $\mathscr{C}$ as a class of structures rather than pairs. Let us start by proving some direct consequences of $\aleph_0$-homogeneity and the uniqueness of the generic type.

Proposition 2.2.1. Let $H, H' \in \mathscr{C}$, subsets $G \subseteq H, G' \subseteq H'$ be countable closed or empty, $g\colon G \to G'$ be an isomorphism and $g \cup f : H \rightharpoonup H'$ be a partial embedding with $A = \mathrm{dom}(f)$, $A' = \mathrm{img}(f)$ finite.

- The mapping $g \cup f$ extends to an isomorphism $\hat{g}\colon \mathrm{cl}(A \cup G) \to \mathrm{cl}(A' \cup G')$.
- If $b \in H \setminus \mathrm{cl}(A \cup G)$ and $b' \in H' \setminus \mathrm{cl}(A' \cup G')$, then $g \cup f \cup \{(b, b')\}$ is a partial embedding.

Proof. For the first assertion note that by the countable closure property both $\mathrm{cl}(A \cup G)$ and $\mathrm{cl}(A' \cup G')$ are countable. Let $(a_n : n < \omega)$ and $(a'_n : n < \omega)$ enumerate $\mathrm{cl}(A \cup G)$ and $\mathrm{cl}(A' \cup G')$ respectively. Construct an increasing family $(f_n : n < \omega)$ of finite mappings such that $g \cup f_n$ is a partial embedding as follows. Let $f_0 = f$. For an odd n pick the least m such that a_m is not in the domain of f_{n-1}. By $\aleph_0$-homogeneity there is $a' \in \mathrm{cl}(A' \cup G')$ such that $g \cup f_{n-1} \cup \{(a_m, a')\}$ is a partial embedding. Put $f_n = f_{n-1} \cup \{(a_m, a')\}$. For an even n do the other way around. Then $\hat{g} = \bigcup_{n<\omega} f_n$ is an isomorphism between $\mathrm{cl}(A \cup G)$ and $\mathrm{cl}(A' \cup G')$.

For the second assertion extend $g \cup f$ to an isomorphism $\hat{g} : \mathrm{cl}(A \cup G) \to \mathrm{cl}(A' \cup G')$. Now $\hat{g} \cup \{(b, b')\}$ is a partial embedding by the uniqueness of the generic type. Hence $g \cup f \cup \{(b, b')\}$ is also a partial embedding. □

Next we introduce σ-types and prove σ-saturation of uncountable structures in $\mathscr{C}$.

Definition 2.2.2. Let $H \in \mathscr{C}$, $A \subseteq H$ and $\bar{x}$ be a finite tuple of variables. A σ-type p (in H) over A in variables $\bar{x}$ is a set of $L_{\omega_1,\omega}$-formulas with parameters from A and free variables among $\bar{x}$ such that every countable subset is consistent with H. That is for every countable $\Phi \subseteq p$ we have $H \models \exists \bar{x} \bigwedge_{\phi \in \Phi} \phi(\bar{x})$.

An important point is that formulas in a σ-type can use infinitely many parameters from A. If the length of the tuple $\bar{x}$ is n, then we call p an n-type. We can think semantically of a σ-type as a family of $L_{\omega_1,\omega}(A)$-definable subsets such that each countable subfamily has a nonempty intersection. A σ-type p is *complete* if for every $L_{\omega_1,\omega}(A)$-formula $\phi(\bar{x})$ we have either $\phi \in p$ of $\neg\phi \in p$. This corresponds to a σ-complete ultrafilter on the σ-algebra of $L_{\omega_1,\omega}(A)$-definable subsets. A priori σ-types do not necessarily extend to complete σ-types (σ-complete filters do not necessarily extend to σ-complete ultrafilters). Proposition 2.2.8 shows that in our case they do, but until then we need to be careful about the distinction.

A σ-type p over A is called *isolated* if there is a consistent $L_{\omega_1,\omega}(A)$-formula $\psi(\bar{x})$ such that $H \models \forall \bar{x}(\psi(\bar{x}) \to \phi(\bar{x}))$ for all $\phi \in p$. A σ-type p is *realised* in H if $\bigcap_{\phi\in p} \phi(H) \neq \emptyset$. Clearly each isolated σ-type is realised. Since $L_{\omega_1,\omega}$-definable sets are closed under countable intersections, we have the following

Proposition 2.2.3. If a σ-type contains a formula defining a countable set, then it is isolated and hence realised.

Proof. Indeed, let $p(\bar{x})$ be a σ-type in H containing a formula $\phi(\bar{x})$ that defines a countable set. For $\bar{a} \in \phi(H)$ let $\psi_{\bar{a}}(\bar{x}) \in p$ be a formula that is not satisfied by $\bar{a}$. If no such formula exists let $\psi_{\bar{a}}(\bar{x})$ be $\bar{x} = \bar{x}$. Then $\bigwedge_{\bar{a}\in\phi(H)} \psi_{\bar{a}}(\bar{x}) \wedge \phi(\bar{x})$ is consistent (since p is a σ-type) and isolates p. $\square$

Now let us study σ-types in quasiminimal excellent structures.

Proposition 2.2.4. Let $H \in \mathscr{C}$ and $A \subseteq H$ be countable. Let $a, b \in H \setminus \mathrm{cl}(A)$. Then a and b realise the same complete σ-type over A.

Proof. Let $G = \mathrm{cl}(A)$ and $g_0 = \mathrm{id}_G \cup \{(a, b)\}$. By the uniqueness of the generic type, g_0 is an embedding. Now consider the collection $\mathscr{F}$ of finite extensions of g_0 that are embeddings. It is not empty as $g_0 \in \mathscr{F}$. We claim that $\mathscr{F}$ is a back-and-forth system. Indeed if $g \in \mathscr{F}$ and $c \in H$, then $g = \mathrm{id}_G \cup f$, where f has finite domain. If $c \in \mathrm{cl}(\mathrm{dom}(g))$, then use $\aleph_0$-homogeneity to extend g to c. Otherwise there is $c' \in H \setminus \mathrm{cl}(\mathrm{img}(g))$. By Proposition 2.2.1 the map $g \cup \{(c, c')\}$ is an embedding. Similarly, we can extend g to an embedding with c in the image.

Thus by Karp's Theorem $(H, g, a)_{g\in G} \equiv_{\infty,\omega} (H, g, b)_{g\in G}$. This in particular implies $(H, g, a)_{g\in G} \equiv_{\omega_1,\omega} (H, g, b)_{g\in G}$. And every $L_{\omega_1,\omega}$-formula using parameters from G that is true of a is true of b. Since $A \subseteq G$ the same is true over A. $\square$

The next corollary establishes the analogy between quasiminimal excellent structures and minimality in the first order setting. It is also the motivation behind the term quasiminimality.

Corollary 2.2.5. Let $H \in \mathscr{C}$ and $\phi(x)$ be an $L_{\omega_1,\omega}$ formula in one variable (possibly using parameters). Then $\phi(H)$ is either countable or cocountable.

Proof. Suppose otherwise. Let $\bar{c}$ be the parameters used in ϕ. Then $\bar{c}$ is countable. Since both $\phi(H)$ and $\neg\phi(H)$ are uncountable, there are $a \in \phi(H)\backslash\mathrm{cl}(\bar{c})$ and $b \in \neg\phi(H)\backslash\mathrm{cl}(\bar{c})$. This contradicts the fact that a and b realise the same σ-type over $\bar{c}$. $\square$

And now we establish the analogy between the closure in quasiminimal excellent classes and the algebraic closure.

Corollary 2.2.6. Let $H \in \mathscr{C}$ be uncountable, and $A \subseteq H$ be a countable subset. Then $b \in \mathrm{cl}(A)$ if and only if it satisfies an $L_{\omega_1,\omega}$-formula over A that has countably many solutions.

Proof. Assume that b satisfies $\phi(x)$ which has countably many solutions. Since $\mathrm{cl}(A)$ is countable, there is $b' \in \neg\phi(H) \setminus \mathrm{cl}(A)$. Now b and b' do not satisfy the same σ-type over A. Hence $b \in \mathrm{cl}(A)$.

Conversely assume that $b \in \mathrm{cl}(A)$. Pick $b' \in H\backslash\mathrm{cl}(A)$. Since the closure is determined by the language, the map $\mathrm{id}_A \cup \{(b, b')\}$ is not an embedding. Hence there is a quantifier-free formula $\phi(x)$ over A satisfied by b but not b'. Now $\neg\phi(H)$ cannot be countable (as it implies that $b' \in \mathrm{cl}(A)$). Hence $\phi(H)$ is countable. $\square$

Next we introduce the infinitary analogue of saturation and prove this property for uncountable structures in quasiminimal excellent classes.

Definition 2.2.7. A structure H is called σ-saturated if for every $A \subset H$ with $|A| < |H|$ every partial σ-type over A is realised in H.

Proposition 2.2.8. Let $H \in \mathscr{C}$ be uncountable. Then H is σ-saturated.

Proof. Let $A \subset H$ be a subset with $|A| < |H|$ and let p be a partial σ-type over A in n variables. We prove by induction on n that p is realised in H.

Let $n = 1$. Put $G = \mathrm{cl}(A)$. Then $|G| = |A| + \aleph_0 < |H|$. So there is $b \in H \setminus G$. If b realises p, then we are done. So assume the opposite. Then there is a formula $\phi(x) \in p$ such that $H \models \neg\phi(b)$. Now since $b \notin \mathrm{cl}(A)$, we have that $\neg\phi(H)$ is uncountable. Hence $\phi(H)$ is countable. But then p is isolated and hence realised in H.

Assume the hypothesis for n. Let p be an $n+1$-type. As before let $G = \mathrm{cl}(A)$. We claim that there is $a \in H$ such that $q_a = \{\phi(\bar{x}, a) : \phi(\bar{x}, y) \in p\}$ is a σ-type. Assume the opposite. Then for every a there is a countable subset $p_a \subset p$ such that

$$H \models \neg\exists\bar{x} \bigwedge_{\phi \in p_a} \phi(\bar{x}, a).$$

Pick $b \in H \setminus G$ and let B be the set of parameters used in formulas of p_b. Then $B \subseteq A$ is countable and $b \notin \mathrm{cl}(B)$. But since any two elements outside $\mathrm{cl}(B)$ realise the same σ-type over B, for every $c \in H \setminus \mathrm{cl}(B)$ we have

$$H \models \neg\exists\bar{x} \bigwedge_{\phi \in p_b} \phi(\bar{x}, c).$$

Now let $W = p_b \cup \bigcup_{a \in \mathrm{cl}(B)} p_a$. Then W is countable and we have that

$$H \models \neg\exists y \exists\bar{x} \bigwedge_{\phi \in W} \phi(\bar{x}, y).$$

This contradicts the fact that p is a σ-type. Thus for some $a \in H$ we have that $q_a = \{\phi(\bar{x}, a) : \phi(\bar{x}, y) \in p\}$ is a σ-type. By induction hypothesis q_a is realised in H and hence p is also realised in H. $\square$

2.3 The Categoricity Theorem

In this section we prove that any two structures of the same uncountable cardinality in a quasiminimal excellent class are isomorphic. Let $H, H' \in \mathscr{C}$ be of the same uncountable cardinality. Since we can construct a back-and-forth system between H and H', we have that $H \equiv_{\omega_1,\omega} H'$. In other words the empty embedding is a partial σ-embedding. We would like to extend a partial σ-embedding to map H onto H'. By σ-saturation we can extend any σ-embedding to any one element (and recursively to any finite number of elements). At limit stages however, we need to take unions. But the union of a chain of σ-embeddings may not be a σ-embedding (since σ-embeddings need to preserve formulas with infinitely many parameters). However it is always an embedding and in some cases this is sufficient to get a σ-embedding.

Proposition 2.3.1. Let $H, H' \in \mathscr{C}$ be uncountable, subsets $G \subset H, G' \subset H'$ be countable closed and let $g : G \to G'$ be an isomorphism. Then g is a partial σ-embedding between H and H'.

Proof. By $\aleph_0$-homogeneity and Proposition 2.2.1 the set of embeddings between H and H' that are finite extensions of g is a back-and-forth system. Hence if we add constant symbols for G in H and for G' in H' the resulting structures will be $L_{\omega_1,\omega}$-equivalent. Therefore g is a σ-embedding. □

We can use this result to extend a σ-embedding to the closure of its domain provided the latter is countable.

Proposition 2.3.2. Let $H, H' \in \mathscr{C}$ be uncountable and let $g : H \rightharpoonup H'$ be a partial σ-embedding with $A = \mathrm{dom}(g), A' = \mathrm{img}(g)$ countable. Then g extends to a σ-embedding $\hat{g} : H \rightharpoonup H'$ with $\mathrm{dom}(\hat{g}) = \mathrm{cl}(A)$ and $\mathrm{img}(\hat{g}) = \mathrm{cl}(A')$.

Proof. Let $\mathrm{cl}(A) = \{a_n : n < \omega\}$ and $\mathrm{cl}(A') = \{a'_n : n < \omega\}$. Construct an increasing sequence $f_0 \subseteq f_1 \subseteq f_2 \subseteq \ldots$ of σ-embeddings as follows. Let $f_0 = g$. For an odd n, pick the least m not in the domain of f_n. Let p be the complete σ-type of a_m over $\mathrm{dom}(f_n)$. Consider the σ-type $p' = \{\phi(x, f_n(\bar{b})) : \phi(x, \bar{b}) \in p\}$. The set p' is a σ-type as f_n is a σ-embedding. By σ-saturation of H', the type p' is realised by some $c \in H'$. Let $f_{n+1} = f_n \cup \{(a_m, c)\}$. For an even n go the other direction.

Now take $\hat{g} = \bigcup_{n<\omega} f_n$. Then $\hat{g}$ is an embedding between the countable closed sets $\mathrm{cl}(A)$ and $\mathrm{cl}(A')$. By Proposition 2.3.1 the embedding $\hat{g}$ is a σ-embedding. □

In particular every countable embedding that extends to the closure of its domain must be a σ-embedding.

Corollary 2.3.3. Let $H, H' \in \mathscr{C}$ be uncountable, $G \subset H$, $G' \subset H'$ be countable closed or empty, $g : G \to G'$ an isomorphism, $a \in H \setminus G$ and $a' \in H' \setminus G'$. Then $g \cup \{(a, a')\}$ is a σ-embedding.

Let us now prove a technical lemma. The argument we use is a direct generalisation of an argument given in Baldwin [2009], Kirby [2010].

Lemma 2.3.4. Let $H, H' \in \mathscr{C}$ be uncountable, $D \subseteq H$ be countably infinite and independent and $f : \mathrm{cl}(D) \to H'$ be a closed embedding. Let $A \subset H$ be finite and independent over D. Assume that for every $B \subsetneq A$, we have $f_B : \mathrm{cl}(DB) \to H$ compatible σ-embeddings extending $f = f_\emptyset$ (i.e. if $B_1 \subseteq B_2$, then $f_{B_1} \subseteq f_{B_2}$). Define $g = \bigcup_{B \subsetneq A} f_B$. Then there is a cofinite subset $D' \subseteq D$ such that $g|_{\mathrm{cl}(D'A)}$ is a σ-embedding.

Proof. If $|A| \leq 1$ the assertion is trivial, so assume that $A = \{a_1, ..., a_n\}$ where $n > 1$. Note that if $B_1, B_2 \subsetneq A$ and $a \in \mathrm{cl}(DB_1) \cap \mathrm{cl}(DB_2)$, then $a \in \mathrm{cl}(D(B_1 \cap B_2))$. Hence by the compatibility condition $g = \bigcup_{B \subsetneq A} f_B$ is a well defined function.

Let $B_i = A \setminus \{a_i\}$ and $h_k = \bigcup_{i=1}^{k} f_{B_i}$, so that $g = h_n$. Thus the domain of h_k is $\bigcup_{i=1}^{k} \mathrm{cl}(DB_i)$. We prove by induction on k that there is a cofinite subset D_k of D such that the restriction $h_k|_{\bigcup_{i=1}^{k} \mathrm{cl}(D_k B_i)}$ is a σ-embedding. For $k = 1$, we have $h_k = f_{B_1}$ so we can simply take $D_1 = D$.

Assume that we have constructed D_{k-1}. Pick $d \in D_{k-1}$ arbitrary and let $D_k = D_{k-1} \setminus \{d\}$. Let $C_{k-1} = \bigcup_{i=1}^{k-1} \mathrm{cl}(D_k B_i)$. We should show that $h_k|_{C_{k-1} \cup \mathrm{cl}(D_k B_k)}$ is a σ-embedding.

Let θ be an automorphism of $\mathrm{cl}(D_{k-1}A)$ that fixes $\mathrm{cl}(D_{k-1}B_k)$ and swaps a_k with d. Let e be a σ-embedding from $\mathrm{cl}(D_{k-1}A)$ into H' extending $h_{k-1}|_{\bigcup_{i=1}^{k-1} \mathrm{cl}(D_{k-1}B_i)}$ that agrees with f_{B_k} on B_k. If $k \geq 3$ then e is simply an extension of $h_{k-1}|_{\bigcup_{i=1}^{k-1} \mathrm{cl}(D_{k-1}B_i)}$ to the closure of its domain (provided by Proposition 2.3.2). For $k = 2$ we have $A = B_1 \cup \{a_1\}$ and e is an extension of $h_1 \cup \{(a_1, f_{B_2}(a_1))\}$ to the closure of its domain. Finally consider $\tau = e\theta^{-1}e^{-1}f_{B_k}\theta$. The automorphism θ takes $C_{k-1} \cup \mathrm{cl}(D_k B_k)$ to $\mathrm{cl}(D_{k-1}B_k)$ and f_{B_k} is defined on it. Hence τ is well-defined on $C_{k-1} \cup \mathrm{cl}(D_k B_k)$. Since all these maps preserve $L_{\omega_1,\omega}$-formulas, so does τ.

We claim that on $C_{k-1} \cup \mathrm{cl}(D_k B_k)$ the mappings τ and h_k agree (and therefore h_k preserves $L_{\omega_1,\omega}$-formulas). Let $c \in C_{k-1}$. Denote $B_{ik} = B_i \cap B_k = A \setminus \{a_i, a_k\}$. We have

$$\theta(c) \in \bigcup_{i=1}^{k-1} \mathrm{cl}(D_{k-1}B_{ik}).$$

Since f_{B_k} agrees with f_{B_i} on $\mathrm{cl}(D_{k-1}B_{ik})$, it agrees with h_{k-1} on $\bigcup_{i=1}^{k-1} \mathrm{cl}(D_{k-1}B_{ik})$. Also e agrees with h_{k-1} on $\bigcup_{i=1}^{k-1} \mathrm{cl}(D_{k-1}B_i)$ and h_k agrees with h_{k-1} on C_{k-1}. Thus

$$\tau(c) = e\theta^{-1}e^{-1}f_{B_k}\theta(c) = e\theta^{-1}h_{k-1}^{-1}h_{k-1}\theta(c) = h_{k-1}(c) = h_k(c).$$

Now let $c \in \mathrm{cl}(D_k B_k)$. Then it is fixed by θ. Also e and f_{B_k} preserve the closure and agree on $D_k B_k$. Hence $e^{-1}f_{B_k}(c) \in \mathrm{cl}(D_k B_k)$ is fixed by θ. Hence

$$\tau(c) = e\theta^{-1}e^{-1}f_{B_k}\theta(c) = ee^{-1}f_{B_k}(c) = f_{B_k}(c) = h_k(c).$$

This completes the proof. $\square$

The above lemma holds if we substitute σ-embeddings with embeddings. Moreover the need for D' disappears (i.e. we can take $D' = D$). The reason is that embeddings only need to preserve formulas with finitely many parameters. So in the proof we can defer the choice of d until after we have fixed the parameters. With σ-embeddings we do not have such a luxury. So we have to pay the price of passing to a cofinite subset D' of D.

We can now prove the main result of this chapter.

Theorem 2.3.5. Let $H, H' \in \mathscr{C}$ be uncountable, let D, D' be bases of H, H' respectively and let $g\colon D \to D'$ be a bijection. Then g extends to an isomorphism $\hat{g}\colon H \to H'$.

Proof. Let $D = \{d_\alpha : \alpha < \kappa\}$, $D' = \{d'_\alpha : \alpha < \kappa\}$ and $g(d_\alpha) = d'_\alpha$. For $n < \omega$, let $G_n = \mathrm{cl}(\{d_m : n \leq m < \omega\})$ and $G'_n = \mathrm{cl}(\{d'_m : n \leq m < \omega\})$. By $\aleph_0$-homogeneity there is an isomorphism $f_0\colon G_0 \to G'_0$ extending g. By Proposition 2.3.1 the embedding f_0 is a partial σ-embedding from H to H'.

For each finite subset $A \subset D$ we construct a number n_A and a surjective σ-embedding $f_A : \mathrm{cl}(G_{n_A}A) \to \mathrm{cl}(G'_{n_A}A')$ that extends g and satisfies the following condition: whenever $A \subseteq B$, we have $n_A \leq n_B$ and f_A agrees with f_B on $\mathrm{cl}(G_{n_B}A)$.

Assume that we have constructed such embeddings. Define $\hat{g} : H \to H'$ as follows. For every $a \in H$, we have $a \in \mathrm{cl}(A)$ for some finite $A \subset D$. Define $\hat{g}(a) = f_A(a)$. By the assumption on the embeddings, the result does not depend on the choice of A. Now $\hat{g}$ is surjective. Indeed for $a' \in H'$ pick finite $A' \subset D'$ such that $a' \in \mathrm{cl}(A')$. Let $A = g^{-1}(A')$. Then $a' \in \mathrm{img}(f_A)$. Since f_A is an embedding $f_A^{-1}(a') \in \mathrm{cl}(A)$. Hence $\hat{g}(f_A^{-1}(a')) = a'$. Also if $\bar{a} \in H$ is a finite tuple, choose $A \subset D$ a finite set such that $\bar{a} \in \mathrm{cl}(A)$. Then $\hat{g}(\bar{a}) = f_A(\bar{a})$, preserves quantifier free formulas over $\bar{a}$. Thus $\hat{g}$ is an isomorphism.

We now proceed to the construction of σ-embeddings f_A by a well-founded induction on the partial order of subsets of D. Take $n_\emptyset = 0$ and $f_\emptyset = f_0$. If $A = \{d_\alpha\}$ is a singleton do the following. If $\alpha < \omega$, then take $n_A = \alpha + 1$, otherwise take $n_A = 0$. Then the restriction h_A of f_0 to G_{n_A} is an isomorphism onto G'_{n_A}. By Corollary 2.3.3, the map $h_A \cup \{(d_\alpha, d'_\alpha)\}$ is a σ-embedding. So by Proposition 2.3.2 it extends to an isomorphism $f_A : \mathrm{cl}(G_{n_A}A) \to \mathrm{cl}(G'_{n_A}A')$.

Assume that $|A| > 1$ and we have already constructed n_B and f_B for every $B \subsetneq A$. Let $n = \max\{n_B : B \subsetneq A\}$. Then each $g_B = f_B|_{\mathrm{cl}(G_n B)}$ is a σ-embedding of $\mathrm{cl}(G_n B)$ onto $\mathrm{cl}(G'_n B')$, where $B' = g(B)$. Now if $a \in \mathrm{dom}(g_{B_1}) \cap \mathrm{dom}(g_{B_2})$ for $B_1, B_2 \subsetneq A$, then $a \in \mathrm{dom}(g_{B_1 \cap B_2})$. Thus if we define $g_A = \bigcup_{B \subsetneq A} g_B$, then g_A is a well defined function. By the previous lemma there is some number $m \geq n$ such that $g_A|_{\bigcup_{B\subsetneq A} \mathrm{cl}(G_m B)}$ is a σ-embedding. Let f_A be its extension to the closure of the domain $\mathrm{cl}(\bigcup_{B \subsetneq A} \mathrm{cl}(G_m B)) = \mathrm{cl}(G_m A)$ and $n_A = m$. □

Now let us compare our approach to the that of Baldwin [2009], Kirby [2010]. The general outline of the proof is the same, except of course that one works with embeddings rather than σ-embeddings. As we have already mentioned, there is no longer a need to pass to a cofinite subset in Lemma 2.3.4. Hence in the proof there is no need for n_A for a finite subset $A \subset D$ (i.e. we can always take $n_A = 0$). Now given f_B for $B \subsetneq A$ we define $g = \bigcup_{B \subsetneq A} f_B$ which will be an embedding by the analogue of Lemma 2.3.4. Then we need to extend g to $\mathrm{cl}(G_0 A)$. If we have a σ-embedding, then we can extend it to an isomorphism from the closure of its domain to the closure of its image by Proposition 2.3.2. However this is no longer the case for embeddings. And this is exactly where we need to use excellence. The domain and image of g are countable crowns. The axiom of excellence together with $\aleph_0$-homogeneity can be used in the back and forth manner to extend g from the closure of its domain to the closure of its image.

Ever since its introduction by Shelah, excellence has been the key notion for categoricity in non-elementary classes. What our methods show is that in some very natural mathematical examples one can use infinitary logic instead. The fact that σ-embeddings and associated infinitary notions occur in natural mathematical contexts is remarkable in itself. This opens up a possibility of a broader use of infinitary logic both in elementary and non-elementary settings.

Chapter 3

Constructing Quasiminimal Structures

In this chapter we study quasiminimality from the first order perspective. An uncountable structure in a countable language is called *quasiminimal* if every (first order) definable subset is either countable or cocountable. Note that by Corollary 2.2.5 uncountable structures in a quasiminimal excellent class satisfy a stronger condition: every $L_{\omega_1,\omega}$-definable set is either countable or cocountable.

The study of quasiminimal structures from the first order perspective is pioneered in Pillay and Tanović [2011]. They isolate a key notion of a strongly regular type for an arbitrary theory (see Definition 3.1.1). An important result is that assuming the generic type (i.e. the type containing formulas defining cocountable sets) of a quasiminimal structure is definable, its unique global heir is strongly regular. In this chapter we establish the converse of this: if there is a definable strongly regular type, then we can construct a quasiminimal model. The definability condition holds in particular for groups. Thus given a regular group, there is a quasiminimal group elementarily equivalent to it. (The converse stating that the monster model of a quasiminimal group is regular is due to Pillay and Tanović [2011].) So the existence of non-commutative quasiminimal groups and quasiminimal fields that are not algebraically closed are reduced to the respective problems for regular groups and fields. This was also independently discovered by Moconja [2015].

The techniques of Pillay and Tanović [2011] provide more. If the cardinality of a quasiminimal structure is strictly greater than $\aleph_1$, then the global heir of the generic type is symmetric (i.e. Morley sequences are totally indiscernible). Thus to construct quasiminimal models of arbitrarily large cardinalities we need additional assumptions. Here we present two constructions of arbitrarily large quasiminimal models: one assuming the theory has definable Skolem functions and the other assuming the theory is stable. We also use a variant of Skolem functions to establish an upper bound on the Hanf number of having arbitrarily large quasiminimal models.

If we strengthen the stability assumption to ω-stability, we can use the existence of prime models. We can then easily get quasiminimal models by taking prime models over Morley sequences in the strongly regular type. Here we show that a stronger conclusion holds: the class of all such models is quasiminimal excellent. Since this class is clearly uncountably categorical, its excellence was expected. However the class of prime models over Morley sequences is not in general $L_{\omega_1,\omega}$-axiomatisable (it is $L_{\omega_1,\omega}(Q)$-axiomatisable). So Theorem 1.2.3 cannot be used to deduce excellence in this generality.

The results of this chapter are published in the preprint Haykazyan [2015].

3.1 Background on Strongly Regular Types

In this section we give the background on Pillay and Tanović [2011] on which this chapter is based. We present all necessary definitions and results. The proofs however are omitted since the results provide context for our work, but are not used in an essential way. For the proofs the reader can of course consult Pillay and Tanović [2011].

We work in a monster model $\mathfrak{C}$ of a first order theory T in a countable language L. Types over $\mathfrak{C}$ are called global types. A global type $\mathfrak{p}(x) \in S_1(\mathfrak{C})$ is an ultrafilter on the boolean algebra of definable subsets of $\mathfrak{C}$. If we think of sets in $\mathfrak{p}$ as "large" and sets outside $\mathfrak{p}$ as "small", we can, in analogy with the algebraic closure, define the closure of A to be the union of all small sets definable over A. More formally define

$$\mathrm{cl}_{\mathfrak{p}}(A) = \{b \in \mathfrak{C} : b \not\models \mathfrak{p}|_A\}.$$

A natural question is when is $\mathrm{cl}_{\mathfrak{p}}$ a closure operator (i.e. a pregeometry minus exchange) or even better a pregeometry. To answer that question we need the notion of a strongly regular type.

Definition 3.1.1. Let A be a subset, $\mathfrak{p}(\bar{x})$ be a global A-invariant type and $\phi(\bar{x}) \in \mathfrak{p}$ be a formula over A. The pair $(\mathfrak{p}, \phi)$ is called *strongly regular* if for all $B \supseteq A$ and $\bar{a}$ satisfying ϕ either $\bar{a} \models \mathfrak{p}|_B$ or $\mathfrak{p}|_B \vdash \mathfrak{p}|_{B\bar{a}}$.

Note that the following statements are all equivalent to $\mathfrak{p}|_B \vdash \mathfrak{p}|_{B\bar{a}}$.

- The type $\mathfrak{p}|_B$ has a unique extension to a type over $B\bar{a}$.
- If $\bar{b} \models \mathfrak{p}|_B$, then $\mathrm{tp}_{\bar{x}}(\bar{b}/B) \cup \mathrm{tp}_{\bar{y}}(\bar{a}/B)$ determines a complete type over B in $\bar{x}\bar{y}$.
- If $\bar{b} \models \mathfrak{p}|_B$, then $\mathrm{tp}(\bar{a}/B) \vdash \mathrm{tp}(\bar{a}/B\bar{b})$.

In our case $\mathfrak{p}$ will be a type in one variable, $\phi(x)$ will always be the formula $x = x$ (so we surpass ϕ from the notation) and we will assume $A = \emptyset$. So when we say $\mathfrak{p}$ is strongly regular, we mean that $(\mathfrak{p}, x = x)$ is strongly regular and $\mathfrak{p}$ is $\emptyset$-invariant. We deal with definable strongly regular types. So $\mathfrak{p}$ will be $\emptyset$-invariant after adding countably many parameters.

Proposition 3.1.2 (Pillay and Tanović [2011]). Let $\mathfrak{p} \in S_1(\mathfrak{C})$ be an $\emptyset$-invariant type. Then $\mathrm{cl}_{\mathfrak{p}}$ is a closure operator if and only if $\mathfrak{p}$ is strongly regular. Moreover, $\mathrm{cl}_{\mathfrak{p}}$ is a pregeometry if in addition all (equivalently some) Morley sequences in $\mathfrak{p}$ are totally indiscernible.

Here by a Morley sequence in $\mathfrak{p}$ we mean a sequence of $(a_i : i < \alpha)$ satisfying $a_i \models \mathfrak{p}|_{\{a_j : j<i\}}$. Since $\mathfrak{p}$ is $\emptyset$-invariant, a Morley sequence is always indiscernible. In this case $\mathfrak{p}$ is called *symmetric* and it is called *asymmetric* otherwise.

We can also start from the other end. Let M be a structure and cl an infinite dimensional pregeometry on it. Assume that the pregeometry is related to the language in the following way: for every finite $B \subset M$ the set of elements of M outside $\mathrm{cl}(B)$ is the set of realisations of a complete type $p_B(x)$ over B. Such a pregeometry is called *homogeneous* in Pillay and Tanović [2011]. In the context of quasiminimal excellent classes this property is referred as the *uniqueness of the generic type*. Now consider $p_{\mathrm{cl}}(x) = \bigcup_{B \subset_{fin} M} p_B(x)$. It can be seen that $p_{\mathrm{cl}}(x) \in \mathrm{S}_1(M)$ is a complete type over M. Results of Pillay and Tanović [2011] show that this type extends to a global strongly regular type.

Proposition 3.1.3 (Pillay and Tanović [2011])**.** Let (M, cl) be a homogeneous pregeometry. Then p_{cl} (defined above) is definable and after adding parameters its unique global heir $\mathfrak{p}$ is a symmetric strongly regular type. In particular $\mathrm{cl} = \mathrm{cl}_{\mathfrak{p}}|_M$.

Thus homogeneous pregeometries (such as those in quasiminimal excellent classes) come from symmetric regular types and conversely symmetric regular types induce homogeneous pregeometries. On the other hand we can also find strongly regular types if we start from quasiminimal structures.

Let M be a quasiminimal structure. Then

$$\{\phi(x) \in L(M) : \phi \text{ defines a cocountable set}\}$$

is a complete type over M. We call this type the *generic type* of M. Under suitable conditions we can also extend this type to a strongly regular global type.

Proposition 3.1.4 (Pillay and Tanović [2011])**.** Assume that the generic type p of a quasiminimal structure M is $\emptyset$-definable. Then its unique global heir $\mathfrak{p}$ is strongly regular. Moreover if $|M| > \aleph_1$, then the definability condition holds (after adding parameters) and $\mathfrak{p}$ is symmetric.

However unless $|M| > \aleph_1$, there are no guarantees that the strongly regular type $\mathfrak{p}$ is symmetric. For example the structure $(\aleph_1 \times \mathbb{Q}, <)$ in the lexicographic order is quasiminimal. Indeed it has quantifier elimination (as a dense linear order) and every formula of the form $x < a$ defines a countable set, while every formula of the form $x > a$ defines a cocountable set. Now the generic type is the completion of $\{x > a : a \in \aleph_1 \times \mathbb{Q}\}$, so is $\emptyset$-definable. But its unique global heir is clearly asymmetric (no indiscernible sequence is totally indiscernible because of the order).

Thus we have established a connection between homogeneous pregeometries and strongly regular types. On the other hand the generic type of a well behaved quasiminimal structure (certainly of large quasiminimal structures) also extends to a strongly regular type. To complete the picture we need to construct quasiminimal models starting from a strongly regular type. This will be our concern in this chapter. We present several constructions with increasing control of properties of the outcome using increasingly stronger assumptions on the theory.

3.2 Arbitrary Theories

In this section we describe a method for constructing quasiminimal structures which we then apply to construct a quasiminimal model of a theory with a strongly regular type. Our method is based on the original method of Morley and Vaught [1962] of constructing an $(\aleph_1, \aleph_0)$ model from a Vaughtian pair. The method is widely known among model theorists (although not used often outside of its original context) and expositions are available in many model theory texts.

Definition 3.2.1. A *special pair* for a theory T is a pair of models $M \prec N$ where N is a proper elementary extension and there is an $\emptyset$-definable type $p \in S_1(M)$ such that all

$a \in N \setminus M$ realise p.

Let $M \prec N$ be a special pair of models of T. Let $p \in S_1(M)$ be the type of elements in $N \setminus M$ and d_p be the schema defining p (note that all schemas defining p are equivalent). Add a new predicate symbol R to the language and consider the theory $\hat{T}$ of N where R is interpreted as M. The theory $\hat{T}$ in particular encodes the following

- The universe is a model of T;
- R defines a proper elementary substructure;
- $\{\phi(x,\bar{a}) : \bar{a} \in R \wedge d_p x \phi(x,\bar{a})\}$ is a complete type over R;
- $\forall x \notin R\ \forall \bar{y} \in R(\phi(x,\bar{y}) \longleftrightarrow d_p x \phi(x,\bar{y}))$ (every element outside R realises the above type).

Thus any model of $\hat{T}$ provides a special pair for T. So if we have a special pair for T, we can construct a countable one. Therefore without loss of generality assume that N is countable. Further by iteratively realising types in M and N we may assume that both M and N are homogeneous and realise the same types over $\emptyset$. Indeed we can construct a sequence

$$(N_0, M_0) \preceq (N_1, M_1) \preceq (N_2, M_2) \preceq \dots$$

of countable models of $\hat{T}$ such that

- if $q \in S_n(T)$ is realised in N_{3i}, then it is realised in M_{3i+1};
- if $\bar{a}, \bar{b}, c \in M_{3i+1}$ and $\mathrm{tp}^{M_{3i+1}}(\bar{a}) = \mathrm{tp}^{M_{3i+1}}(\bar{b})$, then there is $d \in M_{3i+2}$ such that $\mathrm{tp}^{M_{3i+2}}(\bar{a}, c) = \mathrm{tp}^{M_{3i+2}}(\bar{b}, d)$;
- if $\bar{a}, \bar{b}, c \in N_{3i+2}$ and $\mathrm{tp}^{N_{3i+2}}(\bar{a}) = \mathrm{tp}^{N_{3i+2}}(\bar{b})$, then there is $d \in N_{3i+3}$ such that $\mathrm{tp}^{N_{3i+3}}(\bar{a}, c) = \mathrm{tp}^{N_{3i+3}}(\bar{b}, d)$;

Then take $(N, M) = \bigcup_{i<\omega}(N_i, M_i)$. Now M and N would be homogeneous by the second and third clauses and realise the same types over $\emptyset$ by the first clause. In particular M and N would be isomorphic. This is a standard step in Vaught's two cardinal theorem and more details can be found in many textbooks such as Marker [2002].

Proposition 3.2.2. If T has a special pair, then it has a quasiminimal model of cardinality $\aleph_1$.

Proof. Let $M \prec N$ be a special pair, p the type of elements in $N \setminus M$ and d_p the defining scheme. By the above discussion we may assume that M and N are countable, homogeneous and isomorphic. We construct an elementary sequence $(A_\alpha : \alpha < \aleph_1)$ of structures, each isomorphic to M (and hence N) such that for each $\beta < \alpha$ every element of $A_\alpha \setminus A_\beta$ realises the unique heir of p over A_β. The construction is by transfinite recursion.

- Take $A_0 = M$.
- If δ is a limit ordinal, then take $A_\delta = \bigcup_{\alpha<\delta} A_\alpha$. Note that A_δ is homogeneous, countable and realises the same types (over $\emptyset$) as M. Hence $A_\delta \cong M$. For $\beta < \delta$ every element in $A_\delta \setminus A_\beta$ is in some A_α with $\beta < \alpha < \delta$. Therefore it realises the unique heir of p over A_β.
- Assume A_α is constructed. Let $f : M \to A_\alpha$ be an isomorphism. Then f extends to an isomorphism $\hat{f} : N \to B$ for some structure B. Take $A_{\alpha+1} = B$. Then $A_\alpha \prec A_{\alpha+1}$ and $A_{\alpha+1} \cong N \cong M$. It remains to show that every $c \in A_{\alpha+1} \setminus A_\alpha$ realises the unique heir of p over A_α. Let $\bar{a} \in A_\alpha$ and $A_{\alpha+1} \models d_p x \phi(x, \bar{a})$. Let $c' = \hat{f}^{-1}(c), \bar{a}' = \hat{f}^{-1}(\bar{a})$. Note that $c' \in N \setminus M$. Since $\hat{f}$ is an isomorphism, we have $N \models d_p x \phi(x, \bar{a}')$. But $\hat{f}$ extends f and since $\bar{a} \in A_\alpha$, we have $\bar{a}' = f^{-1}(\bar{a}) \in M$. Therefor $N \models \phi(c', \bar{a}')$ and so $A_{\alpha+1} \models \phi(c, \bar{a})$.

Now take $A = \bigcup_{\alpha<\aleph_1} A_\alpha$. If $\bar{a} \in A$, then $\bar{a} \in A_\alpha$ for some countable α. Now if $d_p x \phi(x, \bar{a})$ is satisfied, then every element in $A \setminus A_\alpha$ satisfies $\phi(x, \bar{a})$ and hence it defines a cocountable set. Otherwise $\phi(x, \bar{a})$ defines a countable set. Thus A is quasiminimal. Also observe that countability is weakly definable in A in the sense of Zilber [2003]. $\square$

Now we can apply this method to construct a quasiminimal model from a strongly regular type.

Theorem 3.2.3. Let T be a theory with $\mathfrak{p}$ a definable strongly regular type. Then there is a quasiminimal model of T.

Proof. Let $M \prec \mathfrak{C}$ be a small model. Consider $N = \mathrm{cl}_{\mathfrak{p}}(M)$. Since $\mathfrak{p}|_M$ must have a realisation, we have $N \subsetneq \mathfrak{C}$. We claim that N is an elementary substructure. We use the Tarski-Vaught test. Assume that $\bar{a} \in N$ and $\phi(b, \bar{a})$ holds for some $b \in \mathfrak{C}$. If $b \in N$, then we are done. So assume that $b \notin N$. Then b realises $\mathfrak{p}|_M$ and hence $\mathfrak{p}|_N$. Now since $\bar{a} \in N$, we have that $\mathfrak{p}|_M \vdash \mathfrak{p}|_N$ and therefore some formula $\psi(x) \in \mathfrak{p}|_M$ implies $\phi(x, \bar{a})$. Since M is a model, ψ must be realised in M and hence in N. Thus ϕ is realised in N and $N \prec \mathfrak{C}$. Now every element of $\mathfrak{C} \setminus N$ realises $\mathfrak{p}|_N$ and hence N and $\mathfrak{C}$ are a special pair. $\square$

Without additional assumptions we cannot construct quasiminimal models of cardinalities larger than $\aleph_1$. Indeed by Proposition 3.1.4 the generic type of such a structure is symmetric. So we at least need to assume that $\mathfrak{p}$ is symmetric.

We can apply Theorem 3.2.3 to regular groups. Let $\mathfrak{G}$ be the monster model of a structure where there is an $\emptyset$-definable binary function $\cdot : \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}$ making it a group. The structure $\mathfrak{G}$ is called a *regular group* if there is a strongly regular type $\mathfrak{p} \in S_1(\mathfrak{G})$. Similarly an uncountable structure G is called a *quasiminimal group* if there is an $\emptyset$-definable binary function $\cdot : G \times G \to G$ making it a group. Note that being a quasiminimal group is a property of a structure whereas being a regular group is a property of a theory.

In analogy with minimal groups, one can ask whether all quasiminimal groups are commutative and similarly if all regular groups are commutative, see Hyttinen [2002], Hyttinen et al. [2005], Pillay and Tanović [2011]. Similarly for regular and quasiminimal fields, one can ask whether all such fields are algebraically closed. This is known to be true for fields with a *symmetric* strongly regular type and for quasiminimal fields of cardinality $> \aleph_1$, see Hyttinen et al. [2005], Gogacz and Krupiński [2014]. The connections we have between quasiminimality and regularity help to clarify the relationship between this questions.

Firstly observe that the regular type in a quasiminimal group G is definable. Indeed $X \subseteq G$ is cocountable if and only if $X \cdot X = G$. To see this note that if X is countable, then so is $X \cdot X$ and hence it cannot be the whole group G. While if X is cocountable, the so is gX^{-1} for any $g \in G$. Hence $gX^{-1} \cap X$ is nonempty showing that $g \in X \cdot X$. Thus using Proposition 3.1.4 we see that the monster model $\mathfrak{G}$ of G has a strongly regular type.

Conversely if $\mathfrak{p}$ is strongly regular in the group $\mathfrak{G}$, then the results of Pillay and Tanović [2011] imply that $\mathfrak{p}$ is definable. To see this first observe that $\mathfrak{p}$ is invariant under definable bijections. Indeed if $f : \mathfrak{G} \to \mathfrak{G}$ is B-definable and $a \models p|_B$, then $a \not\models p|_{Bf(a)}$ (as a is definable from $Bf(a)$). Therefore $p|_B \not\vdash p|_{Bf(a)}$ and by the definition of strong regularity $f(a) \models p|_B$. Now let $D \subseteq \mathfrak{G}$ be A-definable. We claim that $D \in \mathfrak{p}$ if and only if two translates of D cover $\mathfrak{G}$. Indeed if two translates of D cover $\mathfrak{G}$, then one of them must be in $\mathfrak{p}$ and since it is preserved by definable bijections D must also be in $\mathfrak{p}$. Conversely if $D \in \mathfrak{p}$ and $g \models p|_A$ then $\mathfrak{G} = D \cup gD$. To see this let $b \in \mathfrak{G} \setminus D$, then by the strong regularity $g \models p|_{Ab}$ and by the above $g^{-1}b \models p|_{Ab}$. This shows that $g^{-1}b \in D$ and so $b \in gD$.

This together with Theorem 3.2.3 gives us the next result.

Corollary 3.2.4. If $\mathfrak{G}$ is a regular group, then there is a quasiminimal group H elementarily equivalent to $\mathfrak{G}$.

Thus we see that there is a non-commutative regular group if and only if there is a non-commutative quasiminimal group. Similarly there is a regular field that is not algebraically closed if and only if there is a quasiminimal field that is not algebraically closed.

As mentioned in the introduction, the above corollary is independently due to Moconja [2015]. His approach is similar to ours, but Theorem 3.2.3 is proved more generally without requiring $\mathfrak{p}$ to be definable. At the time when this work was completed, we were not aware of Moconja's work. An earlier draft of Haykazyan [2015] is referenced in Moconja [2015].

3.3 Theories with Definable Skolem Functions

We say that a theory T has *definable Skolem functions* if for every L-formula $\phi(y, \bar{x})$ there is a $\emptyset$-definable function $f_\phi(\bar{x})$ such that $T \models \forall \bar{x}(\exists y \phi(y, \bar{x}) \to \phi(f_\phi(\bar{x}), \bar{x}))$. In this section we study symmetric strongly regular types in theories with definable Skolem functions. So let T be such a theory and let $\mathfrak{p}$ be a symmetric strongly regular type. Note that by Proposition 3.1.3 the type $\mathfrak{p}$ is definable.

We will show that T has a quasiminimal model of arbitrary uncountable cardinality κ. Since there are definable Skolem functions, for every subset A there is the least model containing it, namely the submodel with universe $\mathrm{dcl}(A)$. Thus if there is a quasiminimal model containing a Morley sequence $A = (a_\alpha : \alpha < \kappa)$, then $\mathrm{dcl}(A)$ would be one. Thus we need to show that $\mathrm{dcl}(A)$ is quasiminimal. Further since $\mathrm{dcl}(A)$ embeds in every model containing A, all we need is to construct a model M containing A such that for a fixed countable $A_0 \subseteq A$ we have that $\mathrm{cl}_\mathfrak{p}(A_0) \cap M$ is countable. For that we will use the technique of self-extending models originally due to Vaught [1965].

The type $\mathfrak{p}$ is an ultrafilter on the boolean algebra of definable subsets. We can use this ultrafilter for a definable analogue of the ultrapower construction. More specifically given a model M of T we can associate a canonical extension M^* defined as follows. The elements of M^* are all definable (with parameters) functions $f : M \to M$ modulo agreeing on a large set (i.e. a member of $\mathfrak{p}$). That is $f, g: M \to M$ are considered equal if $f(x) = g(x) \in \mathfrak{p}$. The nonlogical symbols of L are interpreted as follows. Given an n-ary relation R and elements $f_1, ..., f_n$ of M^*, the formula $R(f_1, ..., f_n)$ holds in M^* if and only if $R(f_1(x), ..., f_n(x)) \in \mathfrak{p}$. Just as in the regular ultrapower construction, one sees that this does not depend on the representatives $f_1, ..., f_n$. Similar definitions are applied to the interpretation of constant and function symbols. We have the analogue of Łoś's Theorem.

Lemma 3.3.1. Given an L-formula $\phi(y_1, ..., y_n)$ and elements $f_1, ..., f_n \in M^*$ we have $M^* \models \phi(f_1, ..., f_n)$ if and only if $\phi(f_1(x), ..., f_n(x)) \in \mathfrak{p}$.

Proof. By induction on ϕ. The only nontrivial case is when the formula ϕ is of the form $\exists y_0 \psi(y_0, y_1, ..., y_n)$. Assuming $M^* \models \exists y_0 \psi(y_0, f_1, ..., f_n)$, there exists $f_0 \in M^*$ such that $M^* \models \psi(f_0, f_1, ..., f_n)$. Then by the induction hypothesis $\psi(f_0(x), f_1(x), ..., f_n(x)) \in \mathfrak{p}$. And since it implies $\exists y_0 \psi(y_0, f_1(x), ..., f_n(x))$, the latter is also in $\mathfrak{p}$. Conversely assume that $\exists y_0 \psi(y_0, f_1(x), ..., f_n(x)) \in \mathfrak{p}$. Since T has definable Skolem functions, there is a definable function $f_\psi : M \to M$ such that $\forall x(\exists y_0 \psi(y_0, f_1(x), ..., f_n(x)) \to \psi(f_\psi(x), f_1(x), ..., f_n(x)))$. Therefore $\psi(f_\psi(x), f_1(x), ..., f_n(x)) \in \mathfrak{p}$ and by the induction hypothesis $M^* \models \exists y_0 \psi(y_0, f_1, ..., f_n)$. $\square$

Corollary 3.3.2. If we identify each element of M with the constant function in M^*, then M^* is an elementary extension of M.

Using the fact that $\mathfrak{p}$ is a strongly regular type, the result of the previous lemma can also be expressed as follows. Assume that $a \in M$ is generic over parameters defining $f_1, ..., f_n$ in M, then $M^* \models \phi(f_1, ..., f_n)$ if and only if $M \models \phi(f_1(a), ..., f_n(a))$.

We call M^* the canonical extension of M. (This explains the terminology self-extending.) The following property of the canonical extension is crucial for our purposes.

Proposition 3.3.3. If M is an infinite dimensional model, then its canonical extension M^* is a proper extension where every new element realises $\mathfrak{p}|_M$.

Proof. First note that since $\mathfrak{p}|_M$ is not isolated, the identity function in M^* is different from all constant functions modulo $\mathfrak{p}$. Hence M^* is a proper extension of M.

On the other hand, let $f \in M^*$ be an element that does not realise $\mathfrak{p}|_M$. Then there is an $L(M)$-formula $\psi(\bar{a}, y) \notin \mathfrak{p}$ such that $M^* \models \psi(\bar{a}, f)$. Let $\phi(\bar{b}, x, y)$ define f in M. Pick an element $c \in M$ generic over $\bar{a}\bar{b}$. By the assumption $d = f(c)$ satisfies $\psi(\bar{a}, y)$ in M. But then $d \in \mathrm{cl}_{\mathfrak{p}}(\bar{a})$ and since c is generic over $\bar{a}\bar{b}$ we see that $f(x) = d$ for every generic x. Thus f is a constant function (modulo $\mathfrak{p}$) and so is already in M. $\square$

Theorem 3.3.4. Let $\mathfrak{p}$ be a symmetric strongly regular type in a theory T with definable Skolem functions. Then T has quasiminimal models of arbitrarily large cardinalities.

Proof. We follow the approach outlined in the beginning of the section. Let κ be an uncountable cardinal and let $A = (a_\alpha : \alpha < \kappa)$ be a Morley sequence in $\mathfrak{p}$. We show that $\mathrm{dcl}(A)$ is quasiminimal. Let $A_0 \subseteq A$ be countably infinite. Let M be a countable model containing A_0. Iterating the previous proposition we obtain an extension N such that $\dim_{\mathfrak{p}}(N) = \kappa$ and $\mathrm{cl}_{\mathfrak{p}}(A_0) \cap N = \mathrm{cl}_{\mathfrak{p}}(A_0) \cap M$ is countable. By conjugating N with an automorphism if necessary we may assume that $A \subseteq N$. But then $\mathrm{dcl}(A) \subseteq N$ and hence $\mathrm{cl}_{\mathfrak{p}}(A) \cap \mathrm{dcl}(A)$ is also countable. This shows that $\mathrm{dcl}(A)$ is quasiminimal. $\square$

In the rest of this section we do not assume that T has definable Skolem functions. Instead we add function symbols to the language that will resemble Skolem functions. We use this to prove that if T has a quasiminimal model of cardinality $\beth_{\omega_1}$, then it has quasiminimal models of all uncountable cardinalities. The existence of such a cardinal follows from a very general argument due to Hanf, and so the least such cardinal is often called the Hanf number. In this terminology we show that the Hanf number of the existence of quasiminimal models is at most $\beth_{\omega_1}$.

Theorem 3.3.5. If T has a quasiminimal model of cardinality $\beth_{\omega_1}$, then it has quasiminimal models of arbitrarily large cardinalities.

Proof. Let $M \models T$ with $|M| = \beth_{\omega_1}$. Then by Proposition 3.1.4 (after adding parameters) there is a global symmetric regular type $\mathfrak{p}$ that extends the type of cocountable subsets of $|M|$. Since M is quasiminimal, its dimension (with respect to $\mathrm{cl}_{\mathfrak{p}}$) is $\beth_{\omega_1}$ and so there is a Morley sequence $(a_\alpha : \alpha < \beth_{\omega_1})$ in $\mathfrak{p}$.

Expand the language by n-ary function symbols f_n^i for each $i, n < \omega$ (nullary function symbols being constant symbols). Interpret f_n^i in M such that for every $\alpha_1 < \ldots < \alpha_n < \kappa$ the set $\{f_n^i(a_{\alpha_1}, \ldots, a_{\alpha_n}) : i < \omega\}$ enumerates the closure $\mathrm{cl}_{\mathfrak{p}}(a_{\alpha_1}, \ldots., a_{\alpha_n})$ in M. Denote the resulting language, theory and structure by L', T' and M' respectively.

Given a cardinal κ, there is an indiscernible sequence $B = (b_\beta : \beta < \kappa)$ in the monster model $\mathfrak{C}'$ of T' such that for every m there are $\alpha_1 < \ldots < \alpha_m$ satisfying

$$\mathrm{tp}^{\mathfrak{C}'}(b_0, \ldots, b_{m-1}) = \mathrm{tp}^{M'}(a_{\alpha_1}, \ldots, a_{\alpha_m}).$$

(This is commonly known as "Morley's method". A proof can be found for example in Tent and Ziegler [2012] or Casanovas [2011] where the length of the sequence is assumed to be $\beth_{(2^{|T|})^+}$. The better bound of $\beth_{\omega_1}$ for countable theories is from Grossberg et al. [2002].)

Now let $N = \{f_n^i(b_{\beta_1}, \ldots, b_{\beta_n}) : i, n < \omega, \beta_1 < \ldots < \beta_n < \kappa\} \subseteq \mathfrak{C}'$ be the closure of this indiscernible sequence under all f_n^i-s. Note that $B \subset N$ is a Morley sequence in $\mathfrak{p}$. Also each $f_n^i(b_{\beta_1}, \ldots, b_{\beta_n}) \in \mathrm{cl}_{\mathfrak{p}}(b_{\beta_1}, \ldots, b_{\beta_n})$ (since this is encoded in the type of the corresponding sequence in M'). Hence $N \subseteq \mathrm{cl}_{\mathfrak{p}}(B)$ has dimension κ.

We claim that N is a quasiminimal model of T. This essentially follows from the fact that for every $\beta_1 < \ldots < \beta_n < \kappa$ there are $\alpha_1 < \ldots < \alpha_n < \beth_{\omega_1}$ such that the mapping sending $f_n^i(b_{\beta_1}, \ldots, b_{\beta_n}) \mapsto f_n^i(a_{\alpha_1}, \ldots, a_{\alpha_n})$ is elementary in L. Now by embedding a suitable fragment of N inside M in this way we see that the elements of

$N \setminus \{f_n^i(b_{\beta_1}, \ldots, b_{\beta_n}) : i < \omega\}$ are generic over $b_{\beta_1}, \ldots, b_{\beta_n}$. This shows that N is quasiminimal. Similarly each satisfiable L-formula over $f_n^{i_1}(b_{\beta_1}, \ldots, b_{\beta_n}), \ldots, f_n^{i_m}(b_{\beta_1}, \ldots, b_{\beta_n})$ is either in $\mathfrak{p}$ and hence realised in B (since B is infinite dimensional), or not in $\mathfrak{p}$ and realised in $\{f_n^i(b_{\beta_1}, \ldots, b_{\beta_n}) : i < \omega\}$. This shows that N is an L-elementary substructure of $\mathfrak{C}'$ and hence a model of T. $\square$

3.4 Stable Theories

In this section we assume that apart from having a strongly regular type $\mathfrak{p}$, the theory T is stable. This automatically makes $\mathfrak{p}$ symmetric (all indiscernible sequence are totally indiscernible in stable theories). And since all symmetric global regular types are definable (also all types in stable theories are definable), the results of Section 3.2 apply here.

Some standard background on stable theories would be assumed. To construct a quasiminimal model we use the technology of local isolation and local atomicity. It was first introduced by Lachlan [1972] in order to prove a two cardinal theorem for stable theories. Here we present the background on local isolation in order to make our account more complete. More details can be found in advanced books on stability theory such as Shelah [1990], Baldwin [1988]. Our notion of local isolation coincides with $\mathbf{F}^l_{\aleph_0}$-isolation of Shelah [1990].

Definition 3.4.1. A type $p(\bar{x}) \in \mathrm{S}(A)$ is called *locally isolated* if for every L-formula $\phi(\bar{x}, \bar{y})$ there is a formula $\psi_\phi(\bar{x}) \in p$ such that $\psi_\phi(\bar{x}) \vdash p(\bar{x})|_\phi$. (Here $p(\bar{x})|_\phi = \{\phi(\bar{x}, \bar{a}) : \phi(\bar{x}, \bar{a}) \in p\} \cup \{\neg\phi(\bar{x}, \bar{a}) : \neg\phi(\bar{x}, \bar{a}) \in p\}$ is the ϕ-part of p.) A set B is called *locally atomic* (over A) if every type (over A) realised in B is locally isolated.

The notion of local isolation can be seen through the topology of ϕ-types as follows. Let $S_\phi(A)$ be the space of all complete ϕ-types over A. Topologise $S_\phi(A)$ by taking the basic clopen sets to be of the form $[\psi] = \{p \in S_\phi(A) : p \vdash \psi\}$ where ψ is any boolean combination of ϕ-formulas over A. Then $S_\phi(A)$ is a boolean topological space (compact, Hausdorff and totally disconnected) and the canonical restriction $\sigma_\phi : \mathrm{S}(A) \to S_\phi(A)$ is continuous (and hence also closed). Now a type $p \in \mathrm{S}(A)$ will be locally isolated if and only if $\sigma_\phi^{-1}(\sigma_\phi(p))$ is a neighbourhood of p for every ϕ (i.e. contains an open set containing p).

Recall that in stable theories $S_\phi(\mathfrak{C})$ is a scattered topological space (i.e. every nonempty subset has an isolated point) of finite Cantor-Bendixson rank (see Casanovas [2011]). For a partial type p we denote by $\mathrm{CB}_\phi(p)$ and $\mathrm{Mlt}_\phi(p)$ the Cantor-Bendixson rank and degree of the set $\sigma_\phi([p]) = \{q \in S_\phi(\mathfrak{C}) : p \cup q \text{ is consistent}\}$ in $S_\phi(\mathfrak{C})$.

We would like to show that for every subset A, there is a model $M \supseteq A$ that is locally atomic over A. For that we prove

Lemma 3.4.2. For any set A, locally isolated types are dense in $\mathrm{S}(A)$.

Proof. Given a formula $\chi(\bar{x})$ over A, we need to find a locally isolated type that contains χ. Enumerate $(\phi_n(\bar{x}, \bar{y}_n) : n < \omega)$ all formulas in L. Put $p_0 = \{\chi(\bar{x})\}$ and $p_{n+1} = p_n \cup \{\psi_n(\bar{x}, \bar{a}_n)\}$ where $\psi_n(\bar{x}, \bar{a}_n)$ is chosen such that $\mathrm{CB}_{\phi_n}(p_{n+1})$ is the least possible and among those $\mathrm{Mlt}_{\phi_n}(p_{n+1})$ is the least possible. We claim that $p = \bigcup_{n<\omega} p_n$ has a unique completion which is locally isolated. Indeed given $\phi_n(\bar{x}, \bar{y}_n)$ and $\bar{b}_n \in A$, we cannot have both $\phi_n(\bar{x}, \bar{b}_n)$ and $\neg\phi_n(\bar{x}, \bar{b}_n)$ consistent with p_{n+1} as it contradicts the choice of ψ_n. Hence p_{n+1}, which is finite isolates the ϕ_n-part of the completion of p. □

This allows us to iteratively realise formulas by locally isolated types similar to the constructible models in ω-stable theories. To make the whole construction locally atomic over A we need

Lemma 3.4.3. If A is locally atomic over BC and B is locally atomic over C, then AB is locally atomic over C.

Proof. Let $\bar{a} \in A$ and $\bar{b} \in B$ and $\phi(\bar{x}, \bar{y}, \bar{z})$ be a formula. It is enough to find a formula $\psi(\bar{x}, \bar{y}) \in \mathrm{tp}(\bar{a}\bar{b}/C)$ such that for every $\bar{c} \in C$ for which $\phi(\bar{a}, \bar{b}, \bar{c})$ holds we have

$\psi(\bar{x},\bar{y}) \to \phi(\bar{x},\bar{y},\bar{c})$. Since $\mathrm{tp}(\bar{a}/BC)$ is locally isolated, we have a formula $\chi(\bar{x},\bar{b}')$ over C in $\mathrm{tp}(\bar{a}/BC)$ such that $\chi(\bar{x},\bar{b}') \to \phi(\bar{x},\bar{b},\bar{c})$ whenever $\phi(\bar{a},\bar{b},\bar{c})$ holds. Now we have $\forall\bar{x}(\chi(\bar{x},\bar{v}) \to \phi(\bar{x},\bar{y},\bar{c})) \in \mathrm{tp}(\bar{b}\bar{b}'/C)$. By the local isolation of the latter there is a formula $\sigma(\bar{y},\bar{v}) \in \mathrm{tp}(\bar{b}\bar{b}'/C)$ such that

$$\sigma(\bar{y},\bar{v}) \to \forall\bar{x}(\chi(\bar{x},\bar{v}) \to \phi(\bar{x},\bar{y},\bar{c}))$$

whenever $\forall x(\chi(\bar{x},\bar{b}') \to \phi(\bar{x},\bar{b},\bar{c}))$ holds. Or equivalently

$$(\exists\bar{v}(\sigma(\bar{y},\bar{v}) \wedge \chi(\bar{x},\bar{v}))) \to \phi(\bar{x},\bar{y},\bar{c}).$$

Thus $\exists\bar{v}(\sigma(\bar{y},\bar{v}) \wedge \chi(\bar{x},\bar{v}))$ is the required formula $\psi(\bar{x},\bar{y})$. □

Now the two combine to give

Proposition 3.4.4. For every A in a stable theory there is a model $M \supseteq A$ that is locally atomic over A and $|M| = |A| + \aleph_0$.

Proof. We imitate the construction of prime models in ω-stable theories. Given B construct B' as follows: enumerate $(\phi_\alpha(\bar{x}_\alpha) : \alpha < \kappa)$ all consistent $L(B)$-formulas that are not realised in B. Then add realisations b_α to B' such that $\mathrm{tp}(b_\alpha/B \cup \{b_\beta : \beta < \alpha\})$ is locally isolated. By Lemma 3.4.3, B' is locally atomic over B. Now take $A_0 = A$ and $A_{n+1} = A_n'$. Finally $M = \bigcup_{n<\omega} A_n$ is the required model. □

The final tool that we need for working with local isolation is the analogue of the Open Mapping Theorem.

Proposition 3.4.5. (Local Open Mapping Theorem) If $p \in \mathrm{S}(A)$ does not fork over $B \subseteq A$ and is locally isolated, then so is $p|_B$.

Proof. Let $N(A/B)$ be the set of types in $\mathrm{S}(A)$ that do not fork over B. By the usual Open Mapping Theorem the restriction map $\pi\colon N(A/B) \to \mathrm{S}(B)$ is open. Now fix a formula

$\phi(\bar{x}, \bar{y})$. By the local isolation of p we have that $\sigma_\phi^{-1}(\sigma_\phi(p)) \cap N(A/B)$ is a neighbourhood of p in $N(A/B)$. Hence its image under π, which is contained in $\sigma_\phi^{-1}(\sigma_\phi(p|_B))$ is a neighbourhood of $p|_B$. $\square$

We can use locally atomic models to construct a proper extension of a given model M where all the new elements realise the type $\mathfrak{p}|_M$. (In the previous section we used the canonical extension of a model for theories with definable Skolem functions for this purpose.) Indeed let a realise $\mathfrak{p}|_M$ and let N be locally atomic over Na. Now let $b \in N \setminus M$. Then $\mathrm{tp}(b/Ma)$ is locally isolated and $\mathrm{tp}(b/M)$ is not (otherwise it would be realised). Hence by the Local Open Mapping Theorem $b \not\perp_M a$. But then by symmetry we have $a \not\perp_M b$, which means that $a \not\models \mathfrak{p}|_{Mb}$, i.e. $a \in \mathrm{cl}_\mathfrak{p}(Mb)$. Hence $b \notin \mathrm{cl}_\mathfrak{p}(M)$, as otherwise $a \in \mathrm{cl}_\mathfrak{p}(M)$.

However the situation is a bit more delicate here. The problem is that there is no equivalent notion of local primeness. That is we cannot in general embed a locally atomic or a locally constructible model over A in an arbitrary model containing A. So we cannot use the same idea as in the previous section to construct arbitrarily large quasiminimal models.

As an alternative we could try to construct a locally atomic model over a Morley sequence A in the following way. Enumerate $A = (a_\alpha : \alpha < \kappa)$. Then build a sequence $(M_\alpha : \alpha < \kappa)$ such that $M_{\alpha+1}$ is locally atomic over $M_\alpha a_\alpha$. Then all the elements in $M_{\alpha+1} \setminus M_\alpha$ will be generic over M_α, so we do not extend the closure. The problem with this, however, is that for $\alpha \geq \aleph_1$ the models M_α will be uncountable. And so in general $M_{\alpha+1}$ adds uncountably many elements.

To remedy this we employ a construction along the lines of Excellence/NOTOP. The technique was adapted for local atomicity in Bays et al. [2014b]. Some of our arguments are adapted from that paper. The following definition is the first order analogue of Definition 1.2.15.

Definition 3.4.6. Let I be a downward-closed set of subsets (i.e. $s \subseteq t \in I$ implies

$s \in I$). An *I-system* is a collection $(M_s : s \in I)$ of models such that $M_s \preceq M_t$ whenever $s \subseteq t$. If $J \subseteq I$ we denote $M_J = \bigcup_{s \in J} M_s$.

For $s \in I$ denote $< s := \mathscr{P}(s) \setminus \{s\}$ and $\not\geq s := \{t \in I : t \not\supseteq s\}$.

The system is *independent* if $M_s \mathop{\smile\!\!\!\!\!\!|}_{M_{<s}} M_{\not\geq s}$.

An *enumeration* of I is an ordering $(s_\alpha : \alpha \in \kappa)$ of I such that $\alpha \leq \beta$ whenever $s_\alpha \subseteq s_\beta$. If an enumeration is fixed we write $< \alpha$ for $\{s_\beta : \beta < \alpha\}$. I.e. $M_{<\alpha} = \bigcup_{\beta<\alpha} M_{s_\beta}$.

Definition 3.4.7. Given two subsets $A \subseteq B$ we say that A is Tarski-Vaught in B (in symbols $A \subseteq_{\mathrm{TV}} B$) if every formula over A realised in B is already realised in A.

We use the Tarski-Vaught condition to lift local isolation to larger sets.

Proposition 3.4.8. If $A \subseteq_{\mathrm{TV}} B$ and $\bar{c}$ is locally isolated over A, then it is locally isolated over B.

Proof. Let $\phi(\bar{x}, \bar{y})$ be a formula without parameters and assume that $\psi(\bar{x}) \in \mathrm{tp}(\bar{c}/A)$ isolates the ϕ-part of $\mathrm{tp}(\bar{c}/A)$. Then $\psi(\bar{x})$ isolates a ϕ-type over B. Indeed if for some $\bar{b} \in B$ we have $\exists \bar{x}(\psi(\bar{x}) \wedge \phi(\bar{x}, \bar{b})) \wedge \exists \bar{x}(\psi(\bar{x}) \wedge \neg\phi(\bar{x}, \bar{b}))$, then the same should be true for some $\bar{a} \in A$. Thus $\psi(\bar{x})$ isolates the ϕ-part of $\mathrm{tp}(\bar{c}/B)$. □

Finally we need the following lemma from Shelah [1990].

Lemma 3.4.9 (TV Lemma, [Shelah, 1990, XII.2.3(2)])**.** Let $(M_s : s \in I)$ be an independent system in a stable theory. Assume that $J \subseteq I$ is such that for all $s \in I$ if $s \subseteq \bigcup J$, then $s \in J$. Then $M_J \subseteq_{\mathrm{TV}} M_I$.

Theorem 3.4.10. Let $\mathfrak{p}$ be a strongly regular type in a stable theory T. Then T has quasiminimal models of arbitrarily large cardinalities.

Proof. Let I be the set of all finite subsets of some uncountable cardinal κ. We inductively build an independent I-system of countable models as follows. Enumerate $I = \{s_\alpha : \alpha < \kappa\}$ so that $s_\alpha \subseteq s_\beta$ implies $\alpha \leq \beta$. Note that this implies $s_0 = \emptyset$. For $M_\emptyset$ pick a countable infinite dimensional model. Given an ordinal α, assume that M_{s_β} has been constructed for $\beta < \alpha$. If $s_\alpha = \{\delta\}$ is a singleton, pick a_δ a realisation of $\mathfrak{p}|_{M_{<\alpha}}$ and let $M_{\{\delta\}}$ be a countable locally atomic model over $M_\emptyset a_\delta$ that is independent from $M_{<\alpha}$ over $M_\emptyset$. Otherwise let M_s be a countable locally atomic model over $M_{<s}$ that is independent from $M_{<\alpha}$ over $M_{<s}$. Then $(M_s : s \in I)$ is an independent system (this is [Shelah, 1990, Lemma XII.2.3(1)], whose proof is "an exercise in non-forking").

Now M_I is a model (by the Tarski-Vaught test) and $|M_I| = \kappa$. We claim that M_I is quasiminimal. Fix $s = \{\alpha_1, ..., \alpha_n\} \subset \kappa$. We claim that all elements of $M_I \setminus M_s$ realise $\mathfrak{p}|_{M_s}$ over M_s. Since the later is countable, this implies that every subset of M_I definable over M_s is either countable or cocountable. Since s was arbitrary we conclude that M_I is quasiminimal.

Let $A = \{a_\alpha : \alpha < \kappa\}$. We show that M_I is locally atomic over $M_s A$. With the above enumeration of I we show by induction on α that $M_{<\alpha}$ is locally atomic over $M_s A$. This is clear if $\alpha = 0$ or α is a limit ordinal. So assume that it holds for α and let us prove it for $\alpha + 1$. Consider several cases.

- If $s_\alpha \subseteq s$, then clearly M_{s_α} is locally atomic over $M_{<\alpha} M_s A$.
- If $|s_\alpha| > 1$, then M_{s_α} is locally atomic over $M_{<s_\alpha}$. By the first clause we may assume that $s_\alpha \nsubseteq s$. But by the TV Lemma (with $I = \{s_\beta : \beta < \alpha\} \cup \mathscr{P}(s)$ and $J = \mathscr{P}(s_\alpha) \setminus \{s_\alpha\}$) we have $M_{<s_\alpha} \subseteq_{\text{TV}} M_{<\alpha} M_s$. Also $M_{<\alpha} M_s \subseteq_{\text{TV}} M_{<\alpha} M_s A$ since $M_\emptyset$ is infinite dimensional. Hence M_{s_α} is locally atomic over $M_{<\alpha} M_s A$.
- If $s_\alpha = \{\delta\}$ is a singleton, then M_{s_α} is locally atomic over $M_\emptyset a_\delta$. By the first clause we may assume that $\delta \notin s$ and hence a_δ realises $\mathfrak{p}$ over $M_{<\alpha} M_s$. Now since $M_\emptyset$ is a model, we have $M_\emptyset \subseteq_{\text{TV}} M_{<\alpha} M_s$. Using the definability of $\mathfrak{p}$ we can show that $M_\emptyset a_\delta \subseteq_{\text{TV}} M_{<\alpha} M_s a_\delta$ and as before $M_{<\alpha} M_s a_\delta \subseteq_{\text{TV}} M_{<\alpha} M_s A$. Thus we conclude that M_{s_α} is locally atomic over $M_{<\alpha} M_s A$.

Thus in all cases we have M_{s_α} is locally atomic over $M_{<\alpha} M_s A$. But by the induction hypothesis $M_{<\alpha}$ is locally atomic over $M_s A$. Hence $M_{s_\alpha} M_{<\alpha} = M_{<\alpha+1}$ is locally atomic over $M_s A$. This completes the induction.

Now given $b \in M \setminus M_s$, it is locally atomic over $M_s A$. Since b is not locally atomic over M_s (otherwise its type would be realised in M_s), we conclude by the Local Open Mapping Theorem that $b \not\perp_{M_s} A$. Pick a finite subset $A' \subset A$ such that $b \not\perp_{M_s} A'$. We can also assume that A' is disjoint from M_s. Now by symmetry we have $A' \not\perp_{M_s} b$. Since A' is a Morley sequence in $\mathfrak{p}$ over M_s, we conclude that $\dim(A'/M_s b) < \dim(A'/M_s)$ (where dim is in the sense of pregeometry $\mathrm{cl}_\mathfrak{p}$). Hence $b \notin \mathrm{cl}_\mathfrak{p}(M_s)$. This finishes the proof. $\square$

3.5 ω-stable Theories

In this section we assume that the theory T is ω-stable. This allows us to construct prime models over every subset. It is easy to see that a prime model over an uncountable Morley sequence in $\mathfrak{p}$ must be quasiminimal. Here we show more: the class $\mathscr{C}$ of prime models over Morley sequences in $\mathfrak{p}$ is a quasiminimal excellent class (see Definition 2.1.1). Since $\mathscr{C}$ is clearly uncountably categorical, its excellence was expected. However as pointed out earlier, Theorem 1.2.3 cannot be used to deduce excellence and we are not aware of a proof in the literature.

The following simple observation will be used repeatedly, so it is worth stating explicitly. We have already used a variant of it for locally atomic models in the previous section.

Proposition 3.5.1. Let M be a model, $a \notin \mathrm{cl}_\mathfrak{p}(M)$ and N prime over Ma. Then for every $b \in N \setminus M$ we have $a \in \mathrm{cl}_\mathfrak{p}(Mb)$. Hence by the exchange property also $b \in \mathrm{cl}_\mathfrak{p}(Ma) \setminus \mathrm{cl}_\mathfrak{p}(M)$.

Proof. Since $\mathrm{tp}(b/Ma)$ is isolated, whereas $\mathrm{tp}(b/M)$ is not, by the Open Mapping Theorem we have $b \not\perp_M a$. Hence $a \not\perp_M b$. So that $a \in \mathrm{cl}_\mathfrak{p}(Mb)$. $\square$

Let us first show that each uncountable model in $\mathscr{C}$ is quasiminimal. One consequence of quasiminimality is that there are no uncountable indiscernible sequences except in $\mathfrak{p}$. This will help us establish primeness of models in some cases. Recall that a model M in an ω-stable theory is prime over A if and only if M is atomic over A and does not contain an uncountable set of indiscernibles over A.

Lemma 3.5.2. If A is an uncountable Morley sequence in $\mathfrak{p}$, then the prime model over A is quasiminimal.

Proof. This follows the standard pattern we have used already in previous sections. For a fixed countable $A_0 \subseteq A$ we can construct a model N containing A such that $\mathrm{cl}_{\mathfrak{p}}(A_0) \cap N$ is countable. To do so enumerate $A \setminus A_0 = (a_\alpha : \alpha < \kappa)$ and take $N_{\alpha+1}$ to be prime over $N_\alpha a_\alpha$. Let N_0 be an arbitrary countable model containing A_0 and $N_\delta = \bigcup_{\alpha<\delta} N_\delta$ for a limit ordinal δ. Then take $N = N_\kappa$. Finally M embeds into N over A showing that $\mathrm{cl}_{\mathfrak{p}}(A_0) \cap M$ is also countable. □

In order to satisfy some technical conditions of Definition 2.1.1, we make two further assumptions, which can be achieved by expanding the language. Firstly we assume that the theory T has quantifier elimination. Otherwise we can consider its Morleyisation, i.e. its expansion with predicate symbols for all $\emptyset$-definable relations. Secondly we assume that $\mathfrak{p}|_A$ is not isolated for all finite A. (Note that this implies that $\mathfrak{p}|_A$ is not isolated for all A.) Otherwise we can add a countably infinite Morley sequence in $\mathfrak{p}$ to the language. The result of the second assumption is

Lemma 3.5.3.

1. If M is a model and $A \subseteq M$, then $\mathrm{cl}_{\mathfrak{p}}(A) \cap M$ is a model.
2. If M is prime over a Morley sequence A in $\mathfrak{p}$, then A is a basis for M (i.e. $M \subseteq \mathrm{cl}_{\mathfrak{p}}(A)$).

Proof.

1. Let $N = \mathrm{cl}_{\mathfrak{p}}(A) \cap M$. We use the Tarski-Vaught test. Let $\phi(x, \bar{b})$ be a formula over N. Since $\mathfrak{p}|_{\bar{b}}$ is not isolated, there is a consistent formula $\psi(x, \bar{b})$ that implies

ϕ and is not in $\mathfrak{p}$. Now every element in $M \setminus N$ realises $\mathfrak{p}|_{\bar{b}}$. Hence a realisation of ψ must be in N.

2. Given $b \in M$, we have that $\mathrm{tp}(b/A)$ is isolated. Hence $b \not\models \mathfrak{p}|_A$ and so $b \in \mathrm{cl}_{\mathfrak{p}}(A)$.

□

Note that both assertions can fail without assuming that $\mathfrak{p}|_A$ is isolated for all finite A. For example both assertions fail in the theory of an infinite set in the empty language, where $\mathfrak{p}$ is the unique non-algebraic type (so the theory is strongly minimal).

Before proceeding further the reader is invited to revise the definition of a quasiminimal excellent class (Definition 2.1.1).

Note that given a strongly regular type $\mathfrak{p}$ in an arbitrary theory, the class of elementary submodels of the monster model $\mathfrak{C}$ (closed under isomorphisms) together with the restriction of $\mathrm{cl}_{\mathfrak{p}}$ satisfies the first two axioms (Closure under isomorphisms and Quantifier free theory). If further the theory has quantifier elimination and $\mathfrak{p}|_A$ is not isolated for all finite A, then the class satisfies the Pregeometry and the Uniqueness of the generic type axioms, except possibly the countable closure property. To satisfy the countable closure property we need to take the class of elementary submodels of a large quasiminimal structure (if such a structure exists).

Finally satisfying $\aleph_0$-homogeneity over countable closed models is the real challenge. For that we need to show that in a prime model over a Morley sequence, if we pick a different Morley sequence, then the respective closure is prime over it. We use the following lemma extracted from Makkai [1984].

Lemma 3.5.4. Assume M is a model $a \models \mathfrak{p}|_M$ and N is a prime model over Ma. Let $b \in N \setminus M$. Then N is prime over Mb. Consequently there is an automorphism in $\mathrm{Aut}(N/M)$ taking a to b.

Proof. By Proposition 3.5.1 we have $a \not\perp_M b$. Let $\theta(x, b) \in \mathrm{tp}(a/Mb)$ be a formula over Mb such that any type containing it forks over M. Let $\phi(a, y)$ be a formula over Ma

that isolates $\mathrm{tp}(b/Ma)$. We claim that $\theta(x,b)\wedge\phi(x,b)$ isolates $\mathrm{tp}(a/Mb)$. It is enough to show that for every $a' \in N$ such that $\theta(a',b)\wedge\phi(a',b)$ we have $a \equiv_{Mb} a'$. Given such an a' we have $a' \not\perp_M b$. Then $a' \in N \setminus M$. Hence by Proposition 3.5.1 again $a' \models \mathfrak{p}_M$. Now we can get the desired conclusion from $\phi(a',b)$. Indeed let $f \in \mathrm{Aut}(\mathfrak{C}/M)$ be an automorphism that maps a' to a. Then $\phi(a,f(b))$ holds. Since $\phi(a,y)$ isolates $\mathrm{tp}(b/Ma)$ we have $ba \equiv_M f(b)a \equiv_M ba'$. Hence $a \equiv_{Mb} a'$.

Now since N is constructible over Ma and $\mathrm{tp}(a/Mb)$ is isolated, we conclude that N is constructible and hence prime and atomic over Mb. Indeed, if $(c_\alpha : \alpha < \delta)$ is a construction of N over Ma, then $(a)^\frown(c_\alpha : \alpha < \delta)$ is a construction of N over Mb. □

In the next proposition we use the fact that if M is ω-stable, atomic over A and $A \subseteq B \subseteq M$ is normal, then M is atomic over B. Recall that $B \supseteq A$ is called *normal* in M over A if for every $b \in B$ all realisations of $\mathrm{tp}(b/A)$ in M are contained in B.

Proposition 3.5.5. Assume that M is prime over a Morley sequence $A = (a_\alpha : \alpha < \kappa)$ in $\mathfrak{p}$. Let $B = (b_\alpha : \alpha < \mu) \subseteq M$ be another Morley sequence in $\mathfrak{p}$. Then $\mathrm{cl}_{\mathfrak{p}}(B) \cap M$ is prime over B.

Proof. By Lemma 3.5.3, the closure $\mathrm{cl}_{\mathfrak{p}}(B) \cap M$ is a model. Also by the countable closure property it does not contain an uncountable indiscernible sequence over B. Thus it remains to show that $\mathrm{cl}_{\mathfrak{p}}(B) \cap M$ is atomic over B.

We can assume that B is finite. Indeed every $a \in \mathrm{cl}_{\mathfrak{p}}(B) \cap M$ is in the closure of a finite subset $B_0 \subseteq B$. Assume we prove that a is atomic over B_0. Then given $b \in B \setminus B_0$, we have that $\mathrm{tp}(a/B_0) \vdash \mathrm{tp}(a/B_0 b)$ (see the discussion after Definition 3.1.1). Hence $\mathrm{tp}(a/B_0 b)$ is also isolated. Iterating this we obtain that $\mathrm{tp}(a/B)$ is isolated. So assume that $B = \{b_0, \ldots, b_{n-1}\}$ is finite.

Further we may assume that B is a subset of A. Indeed by the exchange property there is $a \in A$ such that for $A' = A \setminus \{a\}$ we have that $A' \cup \{b_0\}$ is a basis of M. Now $N = \mathrm{cl}_{\mathfrak{p}}(A') \cap M$ is a model by Lemma 3.5.3. Further M is prime over Na (it is atomic over Na as Na is normal over A) and $b_0 \in M \setminus Na$. Thus by Lemma 3.5.4 there is an automorphism of M fixing A' and taking a to b_0. Thus we can assume that $b_0 \in A$. Iterating this construction we can assume that $B \subseteq A$.

But now the claim follows from the Open Mapping Theorem. Indeed if $c \in \mathrm{cl}_{\mathfrak{p}}(B) \cap M$, then $A \mathop{\smile\!\!\!\!\!\!\mid}_B c$ and hence $c \mathop{\smile\!\!\!\!\!\!\mid}_B A$. Since $\mathrm{tp}(c/A)$ is isolated so is $\mathrm{tp}(c/B)$. □

Now we are ready to prove the main result of this section.

Theorem 3.5.6. Let $\mathscr{C}$ be the class of prime models over Morley sequences in $\mathfrak{p}$. Then $\mathscr{C}$ together with restrictions of $\mathrm{cl}_{\mathfrak{p}}$ is a quasiminimal excellent class.

Proof. The only axiom not covered so far is the $\aleph_0$-homogeneity over countable closed models. Let $H, H' \in \mathscr{C}$ be prime over Morley sequences A and A' respectively. Let $G \subseteq H, G' \subseteq H'$ be countable closed or empty. Let $g : G \to G'$ be an isomorphism. If G (and hence G') is nonempty, we may assume by Proposition 3.5.5 that $G = \mathrm{cl}_{\mathfrak{p}}(A_0) \cap H$, $G' = \mathrm{cl}_{\mathfrak{p}}(A_0') \cap H'$ where $A_0 \subseteq A$, $A_0' \subseteq A$ and g maps A_0 to A_0'.

Let $f : H \to H'$ be a partial embedding with a finite domain. Assume that $\mathrm{dom}(f) = \bar{a}\bar{b}$ where $\bar{a}$ is independent over G and $\bar{b} \in \mathrm{cl}_{\mathfrak{p}}(G\bar{a})$. Then $\mathrm{cl}_{\mathfrak{p}}(G\bar{a}) \cap H$ is prime and constructible over $G\bar{a}$. (In case if G is nonempty, $\mathrm{cl}_{\mathfrak{p}}(G\bar{a}) \cap H$ is prime over $A_0\bar{a}$ by Proposition 3.5.5 and $G\bar{a}$ is normal over $A_0\bar{a}$.) Since $\mathrm{tp}(\bar{b}/G\bar{a})$ is isolated, $\mathrm{cl}_{\mathfrak{p}}(G\bar{a}) \cap H$ is countable and atomic and so prime over $G\bar{a}\bar{b}$. Thus the elementary mapping $g \cup f$ extends to any element of $\mathrm{cl}_{\mathfrak{p}}(G\bar{a}\bar{b}) \cap H$. □

3.6 Further Work

We have shown that $\beth_{\omega_1}$ is an upper bound for the Hanf number of the existence of quasiminimal models. We do not have an example to show that it is sharp, we do not even know if the Hanf number is $\aleph_2$. One way of proving it would be to show that the existence of a symmetric strongly regular type is enough to construct arbitrarily large quasiminimal models. We have shown so using additional assumptions: either stability or presence of definable Skolem functions. One could try to expand the language with Skolem functions, but doing so naively will destroy the regular type. Detailed examination of the proof however reveals that not all Skolem functions are necessary, which gives this approach some hope.

On the other direction one could ask whether the stability assumptions we use for our constructions are sharp. For Theorem 3.4.10 one can ask whether there is a simple theory with a strongly regular type and no arbitrarily large quasiminimal models. A candidate for such a theory could be ACFA where the type of a transformally transcendental element is strongly regular. By Theorem 3.2.3 there is a quasiminimal model of ACFA of cardinality $\aleph_1$, but we do not know what happens in other cardinalities.

One can also try to construct a quasiminimal excellent class from a weaker assumption than ω-stability. A reasonable assumption could be a strongly regular type in a superstable theory with NOTOP. There is a deep analogy between NOTOP and excellence and one could try to exploit it. The methods of Bays et al. [2014a] allow one to construct a quasiminimal excellent class if the theory is superstable and all types are definable over finite subsets. This is however not very far from an ω-stable theory. Every such theory which is in addition small (i.e. has countably many pure types) is ω-stable.

Chapter 4

Excellent Groups

In this chapter we study excellent groups both in the sense of Zilber (see Definition 2.1.1) and in the sense of Shelah (see Definitions 1.2.16 and 1.2.17). The relation between the two notions of excellence is a bit subtle. Zilber's notion requires presence of a homogeneous pregeometry but is not necessarily $L_{\omega_1,\omega}$-axiomatisable (it is always $L_{\omega_1,\omega}(Q)$-axiomatisable). On the other hand Shelah's notion uses a general independence notion but requires $L_{\omega_1,\omega}$-axiomatisability. However in terms of analogies quasiminimal excellent groups correspond to strongly minimal groups whereas excellent atomic groups correspond to ω-stable groups.

The outline of the chapter is as follows. In the first section we look at quasiminimal excellent groups. We prove that a quasiminimal excellent group whose pregeometry is locally modular must be commutative. The next three sections are dedicated to excellent atomic groups. In the first two sections, which do not assume the presence of a groups structure, we set up the necessary machinery and in the final section we establish some (usually weaker) analogues of fundamental results in ω-stable groups.

The approach we take for studying excellent atomic groups is to look at the Cantor-Bendixson rank on the space of atomic types over a countable model. This gives us a rank notion on definable subsets that is different from the R-rank of Chapter 1. The two rank notions also induce different independence relations (although they agree if the base is a model). One advantage of our approach is that the rank and the independence relation agree with Morley rank and forking independence for the class of models of an ω-stable $\aleph_0$-categorical theory.

4.1 Quasiminimal Excellent Groups

Strongly regular types, quasiminimality and quasiminimal excellence are all generalisations of strong minimality. It is natural to ask how much of geometric stability theory is preserved in these wider contexts. A basic result of geometric stability theory is by Reineke [1975] stating that a strongly minimal group must be commutative. Reineke's argument can be modified to show that a non-commutative group with one of the properties above must be unstable, see e.g. Hyttinen [2002] or Maesono [2007]. However not much is known beyond this. In this section we show that a quasiminimal excellent group, whose pregeometry is locally modular, must be commutative.

By a *quasiminimal excellent structure* we mean an infinite dimensional structure in a quasiminimal excellent class. Such a structure uniquely determines the quasiminimal excellent class by closing it under isomorphisms, closed substructures and unions of closed chains (see Bays et al. [2014a], Kirby [2010]). Actually apart from the existence of an infinite dimensional pregeometry, the only other property we need of quasiminimal excellent structures is the following homogeneity condition.

Proposition 4.1.1. If M is a quasiminimal excellent structure, $H \subset M$ countable dimensional closed subset and $a, b \in M \setminus H$, then there is an automorphism $\alpha \in \mathrm{Aut}(M/H)$ with $\alpha(a) = b$.

It follows from the proof of the Categoricity Theorem 2.3.5. (See also Bays et al. [2014a].) Note that in the monster model of a regular group, closed sets are large (of the size of the monster model). Hence this homogeneity condition fails in general.

From now on let (G, cl) be a quasiminimal excellent structure, where there is a 0-definable operation $\cdot\colon G \times G \to G$ making $(G, \cdot)$ into a group. The following simple observations are from Hyttinen et al. [2005] and hold in settings with milder homogeneity assumptions.

Proposition 4.1.2. If a is (first order) definable over a finite set $A \subset G$, then $a \in \mathrm{cl}(A)$.

Proof. Indeed, pick $c \in G$ generic over $A \cup a$. This is possible since G is infinite dimensional. (Here and throughout this section generic means in the sense of the pregeometry, i.e. $c \notin \mathrm{cl}(A, a)$. If $a \notin \mathrm{cl}(A)$, then there is an automorphism fixing A and taking a to c. This contradicts the definability of a. $\square$

The proof also works if a is $L_{\omega_1,\omega}$-definable, or more generally a is definable in a logic where the truth is preserved by automorphisms. But the weaker statement is all we need.

Corollary 4.1.3. Let $A \subset G$ be a finite subset. If a is generic over $A \cup b$, then $a \cdot b$ is also generic over $A \cup b$.

Proof. Indeed, a is definable from b and $a \cdot b$. Hence $a \in \mathrm{cl}(A, b, a \cdot b)$. So in particular $a \cdot b \notin \mathrm{cl}(A, b)$. $\square$

As another corollary we can exclude one type of pregeometry. Recall that a pregeometry cl is said to be *trivial* if $\mathrm{cl}(A) = \bigcup_{a \in A} \mathrm{cl}(a)$. (In the associated geometry this means that every set is closed.)

Corollary 4.1.4. The pregeometry (G, cl) is not trivial.

Proof. Pick a generic pair a, b (i.e. $\dim(a, b) = 2$) and consider $a \cdot b$. By the above $a \cdot b \notin \mathrm{cl}(a) \cup \mathrm{cl}(b)$. However $a \cdot b \in \mathrm{cl}(a, b)$ demonstrating that cl is not trivial. $\square$

Another consequence is that proper definable subgroups are small.

Proposition 4.1.5. Let A be a finite subset and $H \leq G$ an A-definable subgroup. If H contains an element generic over A, then $H = G$.

Proof. Let $b \in G$ be an arbitrary element. Pick a generic over $A \cup b$. Since H is A-definable (and hence A-invariant) and contains a generic element, we have that $a \in H$. (Recall that there is an automorphism over A mapping one generic element to the other.) By Proposition 4.1, $a \cdot b$ is also generic over A. Hence as above $a \cdot b \in H$. Finally since H is a subgroup, we have that $b \in H$. □

As a corollary we get the following criterion for a quasiminimal excellent group to be commutative.

Proposition 4.1.6. If a generic pair commutes in a quasiminimal pregeometry group, then the group is commutative.

Proof. Let (a, b) be a generic pair (i.e. $\dim(a, b) = 2$) and assume that a and b commute. Consider the centraliser $C_G(a)$ of a. It is definable over a and contains a generic element b. So by the previous proposition, we have $C_G(a) = G$. By homogeneity the same is true for every generic element. Now given $c \in \mathrm{cl}(\emptyset)$, by the above it commutes with a. Hence $C_G(c)$ is c-definable and contains a generic element and therefore must be the whole group G. □

We will apply the following improved criterion.

Proposition 4.1.7. Let (a, b) be a generic pair over some finite A. If $b \cdot a \in \mathrm{cl}(A, a \cdot b)$, then G is commutative.

Proof. Assume that $b \cdot a \in \mathrm{cl}(A, a \cdot b)$. By homogeneity it is true for every generic pair over A. Now $(b, b^{-1} \cdot a)$ is a generic pair over A. And so $b^{-1}ab \in \mathrm{cl}(A, a)$. Now consider the set $\{x \in G : x^{-1}ax = b^{-1}ab\}$. It is definable over $A, a, b^{-1}ab$ and contains a generic element b. Hence it contains all generic elements over A, a. Pick c generic over A, a, b.

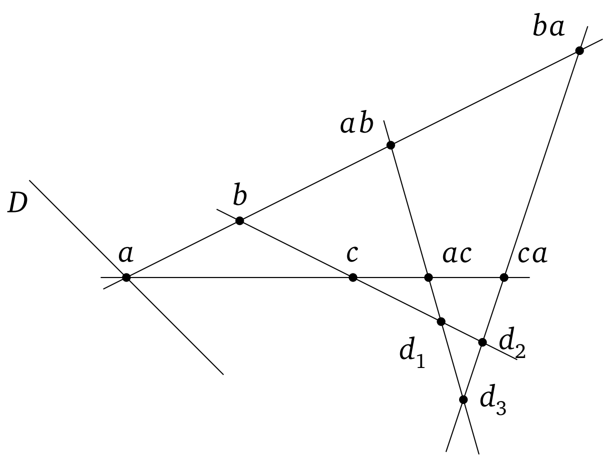
Then $c^{-1}ac = b^{-1}ab$ and therefore $(bc^{-1})a = a(bc^{-1})$. Since (a, bc^{-1}) is a generic pair, by the above G is commutative. $\square$

So far we have used very little homogeneity. So everything said so far also applies to regular groups. We now use the full strength of our homogeneity condition and local modularity of the pregeometry for the main result.

Theorem 4.1.8. If a quasiminimal excellent group has a locally modular pregeometry then it is commutative.

Proof. Assume that (G, cl) is a quasiminimal pregeometry group with a locally modular pregeometry. Pick a finite $A \not\subseteq \mathrm{cl}(\emptyset)$. Then the localisation cl_A of cl at A is modular. (Recall that cl_A is defined as $\mathrm{cl}_A(X) = \mathrm{cl}(A \cup X)$.) Consider a generic pair (a, b) over A. By the previous proposition $b \cdot a \in \mathrm{cl}_A(a \cdot b)$ implies that G is commutative. So assume that $b \cdot a \notin \mathrm{cl}_A(a \cdot b)$. Pick an element $c \notin \mathrm{cl}_A(a, b)$. Similarly we assume that $c \cdot a \notin \mathrm{cl}_A(a \cdot c)$. Consider the geometry of the operator cl_A on $\mathrm{cl}_A(a, b, c)$. Since the geometry is modular but not trivial (Corollary 4.1.4), it is a projective plane. And since it is embedded in a projective space of a larger dimension it is a projective plane over a division ring.

We claim that the lines (b, c), (ab, ac) and (ba, ca) meet in a single point. Assume the opposite. Then these three lines meet in three different points d_1, d_2 and d_3.



Now consider a line D through a that does not pass through b and c. By Proposition 4.1.1 there is an automorphism α of $\mathrm{cl}_A(a,b,c)$ fixing D pointwise and exchanging b and c. Since it is an automorphism of the group structure, it exchanges ab with ac and ba with ca. But then α fixes the lines (b,c), (ab,ac) and (ba,ca) setwise. Hence it fixes the points d_1, d_2 and d_3. But the only automorphism of a projective plane (over a division ring) that fixes all pints on a line and four points in a general position is the identity (see e.g. Beutelspacher and Rosenbaum [1998]). This contradiction shows that $d_1 = d_2 = d_3$ as points of the projective plane.

But now note that $c^{-1}b \in \mathrm{cl}_A(b,c) \cap \mathrm{cl}_A(ab,ac)$ and $bc^{-1} \in \mathrm{cl}_A(b,c) \cap \mathrm{cl}_A(ba,bc)$. Thus $c^{-1}b \in \mathrm{cl}_A(bc^{-1})$. Since (b,c^{-1}) is a generic pair over A, by Proposition 4.1.7 the group G is commutative. □

4.2 Cantor-Bendixson Rank in Excellent Classes

In the rest of the this chapter we present foundations for the theory of excellent groups. We only go as far as generalising some results from the theory of stable groups. But we expect the theory of excellent groups to be as rich and hopefully as useful as the theory of stable groups has been. We start by introducing our main tool, which will give a notion of dimension to definable sets. It is completely general and does not assume presence of a group.

Let $\mathscr{C}$ be an excellent atomic class in a countable language L and let $\mathfrak{M} \in \mathscr{C}$ be a large full model. The structure $\mathfrak{M}$ plays the role of the monster. We wish to assign ranks to definable subsets of $\mathfrak{M}$. We start by looking at countable models.

Let $M \in \mathscr{C}$ be a countable model. As noted in Chapter 1, the space of atomic types $\mathrm{S}_{\mathrm{at}}(M)$ is a countable dense G_δ-subset of $\mathrm{S}(M)$ (see the discussion after Definition 1.2.4). As such it is a Polish space, and since it is countable, we can assign Cantor-Bendixson rank to its elements.

Definition 4.2.1. For a countable model $M \in \mathscr{C}$ and $p \in \mathrm{S}_{\mathrm{at}}(M)$, let $\mathrm{CB}_M(p)$ denote its Cantor-Bendixson rank. For a formula $\phi(\bar{x}) \in L(M)$ define its Cantor-Bendixson rank by

$$\mathrm{CB}_M(\phi) = \sup\{\mathrm{CB}_M(p) : p \in \mathrm{S}_{\mathrm{at}}(M), \phi \in p\},$$

its Cantor-Bendixson degree (or multiplicity) by

$$\mathrm{Mlt}_M(\phi) = |\{p \in \mathrm{S}_{\mathrm{at}}(M) : \phi \in p, \mathrm{CB}_M(\phi) = \mathrm{CB}_M(p)\}|.$$

As defined Cantor-Bendixson rank and degree of a formula depend on the model where it is calculated. We now show that the actual values are independent of the model. This is similar to the first order setting. In some sense it is even simpler as every model in $\mathscr{C}$ is $\aleph_0$-saturated with respect to atomic types.

Proposition 4.2.2. The Cantor-Bendixson rank and degree of a formula do not depend on the model. That is, if $\phi = \phi(\bar{x}, \bar{a})$ and $a \in M, N$ for countable atomic models M and N, then $\mathrm{CB}_M(\phi) = \mathrm{CB}_N(\phi)$ and $\mathrm{Mlt}_M(\phi) = \mathrm{Mlt}_N(\phi)$.

Proof. Since M and N are both countable and $\aleph_0$-homogeneous, there is an isomorphism $f : M \to N$ fixing $\bar{a}$. This isomorphism induces a homeomorphism between $\mathrm{S}_{\mathrm{at}}(M)$ and $\mathrm{S}_{\mathrm{at}}(N)$ taking the clopen set $\{p \in \mathrm{S}_{\mathrm{at}}(M) : \phi(\bar{x}, \bar{a}) \in p\}$ to the clopen set $\{q \in \mathrm{S}_{\mathrm{at}}(N) : \phi(\bar{x}, \bar{a}) \in q\}$. Therefore they are assigned the same Cantor-Bendixson rank and degree. $\square$

In view of this proposition we will drop the subscript for the model unless we want to emphasise it. The same argument shows that $\mathrm{CB}(\phi(\bar{x}, \bar{a}))$ depends only on ϕ and $\mathrm{tp}(\bar{a}/\emptyset)$. Note that unlike first order case, the Cantor-Bendixson degree of a formula can be 0 or infinite.

We extend the notion of the Cantor-Bendixson rank and degree to partial types over

$\mathfrak{M}$ as follows. If π is a partial type, then define its Cantor-Bendixson rank by

$$\mathrm{CB}(\pi) = \min\{\mathrm{CB}(\phi_1 \wedge \ldots \wedge \phi_n) : n \geq 1, \phi_1, \ldots, \phi_n \in \pi\}$$

and its Cantor-Bendixson degree (or multiplicity) by

$$\mathrm{Mlt}(\pi) = \min\{\mathrm{Mlt}(\phi_1 \wedge \ldots \wedge \phi_n) : n \geq 1, \phi_1, \ldots, \phi_n \in \pi \text{ and } \mathrm{CB}(\phi_1 \wedge \ldots \wedge \phi_n) = \mathrm{CB}(\pi)\}.$$

Note that this assigns a rank to every type in $\mathrm{S}(\mathfrak{M})$, not just atomic types. However $\mathrm{CB}(\pi) \geq 0$ does not mean it can be realised in $\mathfrak{M}$.

But now given a type $p \in \mathrm{S}_{\mathrm{at}}(M)$ for a countable model M we have two ways of calculating $\mathrm{CB}(p)$: one as its Cantor-Bendixson rank in the Polish space $\mathrm{S}_{\mathrm{at}}(M)$ and another as $\mathrm{CB}(p) = \min\{\mathrm{CB}(\phi) : \phi \in p\}$.

Proposition 4.2.3. The above two ways of calculating $\mathrm{CB}(p)$ agree.

Proof. Let α denote the Cantor-Bendixson rank of p in the topological space $\mathrm{S}_{\mathrm{at}}(M)$. We should show that $\alpha = \min\{\mathrm{CB}(\phi) : \phi \in p\}$. If $\phi \in p$, then $\mathrm{CB}(\phi) = \sup\{\mathrm{CB}_M(q) : q \in \mathrm{S}_{\mathrm{at}}(M), \phi \in q\} \geq \alpha$. One the other hand if $\phi \in p$ isolates p in $\mathrm{S}_{\mathrm{at}}(M)$ among types of rank at least α, then $\mathrm{CB}(\phi) = \alpha$. $\square$

For a type $p \in \mathrm{S}_{\mathrm{at}}(M)$ over a model M (not necessarily countable) we have $\mathrm{Mlt}(p) = 1$. Indeed, first note that there is a countable model $M_0 \subseteq M$ such that $\mathrm{CB}(p) = \mathrm{CB}(p|_{M_0})$. Now pick $\phi(\bar{x}, \bar{a}) \in L(M_0)$ that isolates $p|_{M_0}$ among the types of the same Cantor-Bendixson rank in $\mathrm{S}_{\mathrm{at}}(M_0)$. This shows $\mathrm{Mlt}(p) \leq 1$. On the other hand if $\psi(\bar{x}, \bar{b}) \in p$ is a formula with $\mathrm{CB}(\psi) = \mathrm{CB}(p)$, then pick a countable model M_1 containing $\bar{b}$. Now $\mathrm{CB}(p|_{M_1}) = \mathrm{CB}(p)$ showing that $\mathrm{Mlt}(\psi) \geq 1$.

Proposition 4.2.4. Let $p \in \mathrm{S}_{\mathrm{at}}(M)$ and let $\phi(\bar{x}, \bar{a}) \in p$ be a formula with $\mathrm{CB}(p) = \mathrm{CB}(\phi)$ and $\mathrm{Mlt}(\phi) = 1$. Then p does not split over $\bar{a}$ and furthermore p is the only type in $\mathrm{S}_{\mathrm{at}}(M)$ extending ϕ with the same rank.

Proof. Assume for contradiction that p splits over $\bar{a}$. Let $\bar{b}, \bar{c} \in M$ be such that $\mathrm{tp}(\bar{b}/\bar{a}) = \mathrm{tp}(\bar{c}/\bar{a})$ and there is a formula $\psi(\bar{x}, \bar{y}) \in L$ such that $\psi(\bar{x}, \bar{b}) \wedge \neg\psi(\bar{x}, \bar{c}) \in p$. Let M_0 be a countable model containing $\bar{a}\bar{b}\bar{c}$. Let $f \in \mathrm{Aut}(M_0/\bar{a})$ be an automorphism taking $\bar{b}$ to $\bar{c}$. Then $f(p) \neq p$, $\mathrm{CB}(f(p)) = \mathrm{CB}(p)$ and $\phi(\bar{x}, \bar{a}) \in f(p)$. This contradicts the fact that $\mathrm{Mlt}(\phi(\bar{x}, \bar{a})) = 1$.

For uniqueness assume that $q \in \mathrm{S}_{\mathrm{at}}(M)$ is another type of rank $\mathrm{CB}(\phi(\bar{x}, \bar{a}))$. Then there is a countable model M_0 containing $\bar{a}$ such that $p|_{M_0} \neq q|_{M_0}$ again contradicting $\mathrm{Mlt}(\phi(\bar{x}, \bar{a})) = 1$. $\square$

Now we show that types over models can be extended without increasing the rank. In the first order setting this follows from compactness and is true for types over arbitrary sets (and even for partial types).

Proposition 4.2.5. Let $p \in \mathrm{S}_{\mathrm{at}}(M)$ be a type and let $N \supseteq M$ be an atomic model extending M. Then there is a unique type $q \in \mathrm{S}_{\mathrm{at}}(N)$ that extends p such that $\mathrm{CB}(q) = \mathrm{CB}(p)$.

Proof. Let $\phi(\bar{x}, \bar{a}) \in p$ be a formula with $\mathrm{CB}(\phi) = \mathrm{CB}(p)$ and $\mathrm{Mlt}(\phi) = 1$. The uniqueness of q follows as in Proposition 4.2.4. Indeed if $q' \in \mathrm{S}_{\mathrm{at}}(N)$ is another such extension, then there is a countable model M_0 containing $\bar{a}$ such that $q|_{M_0} \neq q'|_{M_0}$ contradicting $\mathrm{Mlt}(\phi) = 1$.

For the existence take

$$q = \{\psi(\bar{x}, \bar{n}) : \text{there is } \bar{m} \in M \text{ with } \mathrm{tp}(\bar{n}/\bar{a}) = \mathrm{tp}(\bar{m}/\bar{a}) \text{ and } \psi(\bar{x}, \bar{m}) \in p\}.$$

Since the type p does not split over $\bar{a}$, the type q is well defined and $q \in \mathrm{S}_{\mathrm{at}}(N)$. Also if $\mathrm{tp}(\bar{n}/\bar{a}) = \mathrm{tp}(\bar{m}/\bar{a})$ then $\mathrm{CB}(\psi(\bar{x}, \bar{n})) = \mathrm{CB}(\psi(\bar{x}, \bar{m}))$ and therefore $\mathrm{CB}(q) = \mathrm{CB}(p)$. $\square$

This shows that for $\phi \in L(M)$ we have

$$\mathrm{CB}(\phi) = \sup\{\mathrm{CB}(p) : p \in \mathrm{S}_{\mathrm{at}}(M), \phi \in p\}$$

and

$$\mathrm{Mlt}(\phi) = |\{p \in \mathrm{S}_{\mathrm{at}}(M) : \phi \in p, \mathrm{CB}(\phi) = \mathrm{CB}(p)\}|$$

for an arbitrary (not necessarily countable) model M. We can show more.

Proposition 4.2.6. Let $M \in \mathscr{C}$ be a model (not necessarily countable) and $p \in \mathrm{S}_{\mathrm{at}}(M)$. Then $\mathrm{CB}(p)$ is the Cantor-Bendixson rank of p in the space $\mathrm{S}_{\mathrm{at}}(M)$.

Proof. We show by induction on α that $\mathrm{S}_{\mathrm{at}}(M)^{\alpha} = \{p \in \mathrm{S}_{\mathrm{at}}(M) : \mathrm{CB}(p) \geq \alpha\}$. (See page 12 for the notation $\mathrm{S}_{\mathrm{at}}(M)^{\alpha}$.) This is clear if $\alpha = 0$ or is a limit ordinal. So assume that it is true for α and let us prove it for $\alpha + 1$.

Let $p \in \mathrm{S}_{\mathrm{at}}(M)^{\alpha}$ be isolated by say a formula $\phi(\bar{x}, \bar{a}) \in p$. Let $M_0 \subseteq M$ be a countable model containing $\bar{a}$. Now consider $p|_{M_0}$. We have $\mathrm{CB}(p|_{M_0}) \geq \mathrm{CB}(p) \geq \alpha$, so $p|_{M_0} \in \mathrm{S}_{\mathrm{at}}(M_0)^{\alpha}$. We claim that $\phi(\bar{x}, \bar{a})$ isolates $p|_{M_0}$ in $\mathrm{S}_{\mathrm{at}}(M_0)^{\alpha}$. Indeed if $q_0 \in \mathrm{S}_{\mathrm{at}}(M_0)^{\alpha}$ is another type containing ϕ, then we can extend q_0 to a type $q \in \mathrm{S}_{\mathrm{at}}(M)$ of the same rank. Then $\mathrm{CB}(q) = \mathrm{CB}(q_0) \geq \alpha$ showing that $q \in \mathrm{S}_{\mathrm{at}}(M)^{\alpha}$. This contradiction shows that $\mathrm{CB}(\phi(\bar{x}, \bar{a})) = \alpha$ and so $\mathrm{CB}(p) = \alpha$.

Conversely assume that $\mathrm{CB}(p) = \alpha$. We can find a countable model $M_1 \subseteq M$ such that $\mathrm{CB}(p|_{M_1}) = \alpha$ and let $\phi(\bar{x}, \bar{a})$ isolate $p|_{M_1}$ in $\mathrm{S}_{\mathrm{at}}(M_1)^{\alpha}$. We claim that ϕ isolates $p \in \mathrm{S}_{\mathrm{at}}(M)^{\alpha}$. Indeed if $q \in \mathrm{S}_{\mathrm{at}}(M)^{\alpha}$ is another type containing ϕ, then there is a countable model $M_2 \subseteq M$ containing $\bar{a}$ such that $p|_{M_2} \neq q|_{M_2}$. This contradicts the fact that $\mathrm{Mlt}(\phi(\bar{x}, \bar{a})) = 1$. $\square$

The following simple observations will be useful for studying excellent groups.

Proposition 4.2.7.

- $\mathrm{CB}(\phi \vee \psi) = \max(\mathrm{CB}(\phi), \mathrm{CB}(\psi))$.
- If there is a definable bijection between $\phi(\bar{x})$ and $\psi(\bar{x})$, then $\mathrm{CB}(\phi) = \mathrm{CB}(\psi)$.

Proof. The first statement follows directly from the definition of Cantor-Bendixson rank for formulas. For the second one note that if M is a model containing the parameters of ϕ, ψ and the formula defining a bijection between them, then the bijection induces a homeomorphism between clopen sets $\{p \in \mathrm{S}_{\mathrm{at}}(M) : \phi \in p\}$ and $\{p \in \mathrm{S}_{\mathrm{at}}(M) : \psi \in p\}$. Since the Cantor-Bendixson rank of a clopen set does not depend on its embedding in $\mathrm{S}_{\mathrm{at}}(M)$ we conclude that $\mathrm{CB}(\phi) = \mathrm{CB}(\psi)$. □

Now that we have established the basic properties of the Cantor-Bendixson rank we can define a notion of independence using it. It would be natural to consider $\bar{a}$ independent of B over C if $\mathrm{CB}(\bar{a}/BC) = \mathrm{CB}(\bar{a}/C)$. (Here of course we shorten $\mathrm{CB}(\mathrm{tp}(\bar{a}/B))$ to $\mathrm{CB}(\bar{a}/B)$.) However it is possible that $\mathrm{Mlt}(\bar{a}/BC) = 0$ while $\mathrm{Mlt}(\bar{a}/C) > 0$. This creates some problems, e.g. $\mathrm{tp}(\bar{a}/BC)$ cannot be extended to a type of the same rank over a model. So instead we consider $\bar{a}$ independent of B over C if $\mathrm{CB}(\bar{a}/BC) = \mathrm{CB}(\bar{a}/C)$ and if $\mathrm{Mlt}(\bar{a}/C) > 0$, then $\mathrm{Mlt}(\bar{a}/BC) > 0$.

Recall that in Chapter 1 we used a different rank (Definition 1.2.10) to define a notion of independence. It is natural to ask if the two ranks induce the same independence relation. The answer to this question is negative in general.

Example 4.2.8. Let L be the language containing one binary relation symbol E and let T be the theory asserting that E is an equivalence relation with two infinite equivalence classes. Since T is $\aleph_0$-categorical the class of atomic models coincides with the class of all models. And since T is ω-stable, this class is excellent. In particular the Cantor-Bendixson rank and degree defined in this section coincides with the familiar Morley rank and degree and the independence relation induced by it coincides with forking independence.

As for the R-rank, note that each equivalence class is an infinite set with no additional structure. We have worked out the R-rank for an infinite set in Example 1.2.11 and a similar analysis shows that rank of a singleton is 0, the rank of other finite sets is 1 and each equivalence class has rank 2. Since the whole structure is split into two equivalence classes we get that $\mathrm{R}(x = x) = 3$ (it is easy to see that the second condition of the rank is also satisfied).

Now let a, b be two elements such that $a \neq b$ and $E(a, b)$. Then $\mathrm{R}(a/\emptyset) = 3 >$

$2 = \mathrm{R}(a/b)$ (since $\mathrm{tp}(a/b)$ contains the formula $E(x,b)$) but $\mathrm{CB}(a/\emptyset) = \mathrm{CB}(a/b) = 1$. (Multiplicity of any type here is positive, since it coincides with Morley degree.)

However as we will now see, both notions of independence agree over models.

Proposition 4.2.9. Let $\bar{a}$ be a tuple, M a model and $M \subseteq B$ an atomic extension. Then the following are equivalent:

1. $\mathrm{CB}(\bar{a}/B) = \mathrm{CB}(\bar{a}/M)$ and $\mathrm{Mlt}(\bar{a}/B) > 0$.

2. $\mathrm{R}(\bar{a}/B) = \mathrm{R}(\bar{a}/M)$.

3. $\mathrm{tp}(\bar{a}/B)$ does not split over a finite subset of M.

Proof. The equivalence of 2 and 3 can be found e.g. in Lessmann [2005]. We show the equivalence of 1 and 3.

Assume 1. Let $\phi(\bar{x}, \bar{b}) \in \mathrm{tp}(\bar{a}/M)$ be a formula of the same Cantor-Bendixson rank and $\mathrm{Mlt}(\phi) = 1$. We claim that $\mathrm{tp}(\bar{a}/B)$ does not split over $\bar{b}$. Indeed assume for contradiction that $\bar{c}, \bar{d} \in B$ are such that $\mathrm{tp}(\bar{c}/\bar{b}) = \mathrm{tp}(\bar{d}/\bar{b})$ and $\psi(\bar{a}, \bar{c}) \wedge \neg\psi(\bar{a}, \bar{d})$. This shows that $\mathrm{CB}(\phi(\bar{x}, \bar{b}) \wedge \psi(\bar{x}, \bar{c})) = \mathrm{CB}(\phi(\bar{x}, \bar{b}) \wedge \neg\psi(\bar{x}, \bar{d})) = \mathrm{CB}(\phi(\bar{x}, \bar{b}))$. But since $\mathrm{tp}(\bar{c}/\bar{b}) = \mathrm{tp}(\bar{d}/\bar{b})$ we also have that $\mathrm{CB}(\phi(\bar{x}, \bar{b}) \wedge \neg\psi(\bar{x}, \bar{c})) = \mathrm{CB}(\phi(\bar{x}, \bar{b}))$. Also $\mathrm{Mlt}(\phi(\bar{x}, \bar{b}) \wedge \psi(\bar{x}, \bar{c})) > 0$ and $\mathrm{Mlt}(\phi(\bar{x}, \bar{b}) \wedge \neg\psi(\bar{x}, \bar{c})) > 0$. Therefore $\mathrm{Mlt}(\phi(\bar{x}, \bar{b})) \geq 2$ which is a contradiction.

Conversely assume that $\mathrm{tp}(\bar{a}/B)$ does not split over a finite subset $\bar{b}$ of M. Then given any formula $\phi(\bar{x}, \bar{c})$ we can find $\bar{d} \in M$ such that $\mathrm{tp}(\bar{c}/\bar{b}) = \mathrm{tp}(\bar{d}/\bar{b})$. Now $\mathrm{CB}(\phi(\bar{x}, \bar{c})) = \mathrm{CB}(\phi(\bar{x}, \bar{d}))$ and $\mathrm{Mlt}(\phi(\bar{x}, \bar{c})) = \mathrm{Mlt}(\phi(\bar{x}, \bar{d}))$. But by non-splitting we have $\phi(\bar{x}, \bar{d}) \in \mathrm{tp}(\bar{a}/M)$. $\square$

Thus for this chapter we redefine the independence relation. For atomic sets A, B, C we say that A is independent (or free) of B from C and write

$$A \underset{C}{\downarrow} B$$

if for every finite tuple $\bar{a} \in A$ we have $\mathrm{CB}(\bar{a}/BC) = \mathrm{CB}(\bar{a}/C)$ and if $\mathrm{Mlt}(\bar{a}/C) > 0$, then $\mathrm{Mlt}(\bar{a}/BC) > 0$. We summarise the properties of the independence relation (this is the analogue of Proposition 1.13 of Lessmann [2005] for our independence relation).

Proposition 4.2.10. Let A, B, C, D be subsets, $\bar{a}, \bar{a}_1, \bar{a}_2$ tuples and $M \in \mathscr{C}$. (As usual we assume that $A, B, C, D, M, \bar{a}, \bar{a}_1, \bar{a}_2 \subset \mathfrak{M}$ so their union is atomic).

- *Invariance.* If f is elementary, then $A \mathrel{\raise0.2ex\hbox{$\smile$}\!\!\!\!\!\lower0.4ex\hbox{$\bigcup$}}$ is not used; $A \downarrow_B C$ if and only if $f(A) \downarrow_{f(B)} f(C)$.
- *Monotonicity.* If $A' \subseteq A$ and $C' \subseteq C$, then $A \downarrow_B C$ implies $A' \downarrow_B C'$.
- *Finite Character.* We have $A \downarrow_B C$ if and only if $A' \downarrow_B C'$ for every finite $A' \subseteq A$ and finite $C' \subseteq C$.
- *Transitivity.* If $B \subseteq C \subseteq D$, then $A \downarrow_B D$ if and only if $A \downarrow_B C$ and $A \downarrow_C D$.
- *Local character.* For a tuple $\bar{a}$ there exists a finite $B \subseteq C$ such that $\bar{a} \downarrow_B C$.
- *Extension over models.* If $M \subseteq C$ then for a tuple $\bar{a}$ there exists $\bar{a}' \in \mathfrak{M}$ such that $\bar{a} \equiv_M \bar{a}'$ and $a' \downarrow_M C$.
- *Stationarity over models.* If $M \subseteq C$, $\bar{a}_1 \equiv_M \bar{a}_2$ and $\bar{a}_i \downarrow_M C$ for $i = 1, 2$, then $\bar{a}_1 \equiv_C \bar{a}_2$.
- *Symmetry over models.* We have $A \downarrow_M B$ if and only if $B \downarrow_M A$.

Proof. The only property that is not immediate and has not been proved so far is symmetry over models. This follows from the symmetry over models of the independence relation defined from R (see Proposition 1.12 in Lessmann [2005]) and the fact that it agrees with the one defined from CB over models. $\square$

Finally we mention that excellence of $\mathscr{C}$ is not essential for the developments of this section. (Just ω-stability of $\mathscr{C}$ would be enough.) So the whole theory of excellence can be developed using the independence relation defined from CB which has the advantage of agreeing with nonforking if $\mathscr{C}$ is the class of models of an ω-stable $\aleph_0$-categorical theory.

4.3 Definability of Types

Unlike first order ω-stability the types over models may not be definable in excellent classes. To remedy this we will expand the language with such definitions and show that this expansion is harmless. We follow the approach sketched in Lemma 25.10 in Baldwin [2009].

The key result is that an expansion of the language with a predicate symbol for a type definable set (over $\emptyset$) gives us an excellent class. This seems a very strong result and at odds with the intuition in elementary classes. It becomes much more intuitive if we view the excellent class as the class of models of some $L_{\omega_1,\omega}$-sentence ϕ (the Scott sentence of its countable model). To get an atomic class from such a sentence we choose a countable fragment F containing ϕ and expand the language with predicate symbols for formulas of F (see Proposition 1.1.8 and the subsequent discussion). Now expanding the language with a type definable set corresponds to choosing a larger fragment $F' \supseteq F$. So our assertion simply states that an excellent class remains excellent if we choose a larger fragment.

Let T be a complete theory in a countable language L and assume that the class $\mathscr{C}$ of atomic models of T is excellent. Let $\pi(\bar{x})$ be a partial type (over $\emptyset$) in n variables. Let L' be an expansion of L with a new n-ary predicate symbol P. For each $M \in \mathscr{C}$, let M' be the L' expansion of M where P is interpreted as $\pi(M)$.

Lemma 4.3.1. The class $\mathscr{C}' = \{M' : M \in \mathscr{C}\}$ is an atomic class and moreover the elementary embeddings in $\mathscr{C}$ and $\mathscr{C}'$ are the same.

Proof. If $M, N \in \mathscr{C}$, then $M \equiv_{\infty,\omega} N$ (since there is a back and forth system between M and N). Since P is $L_{\omega_1,\omega}$-definable, we conclude that $M' \equiv_{\infty,\omega} N'$. In particular M'

and N' are elementarily equivalent. Actually we can say even more. Let $M \preceq N$ and let $\bar{m} \in M$ be a tuple. We have that $(M, \bar{m}) \equiv_{\infty,\omega} (N, \bar{m})$. In particular every $L_{\omega_1,\omega}$-formula true of $\bar{m}$ in M is true of $\bar{m}$ in N. Therefore $M' \preceq N'$.

Let T' be the first order theory of M' for some (any) $M \in \mathscr{C}$ (in the language L'). We claim that $\mathscr{C}' = \{M' : M \in \mathscr{C}\}$ is the class of atomic models of T'. First pick $M \in \mathscr{C}$ and let us show that M' is atomic. Let $q \in \mathrm{S}(T')$ be a non-principal type. For each $n \geq 1$, let $\Sigma_n = \{\neg\phi(x_1, ..., x_n) : \phi \in L \text{ is complete}\}$ and let $\Pi = \{\neg P(\bar{x})\} \cup \pi(\bar{x})$. Not that Π and each Σ_n are non-principal in T' as there is a model that omits them (any N' for $N \in \mathscr{C}$). By the omitting types theorem there is a model $K' \models T'$ that omits q, Π and each Σ_n. Then the restriction K of K' to L is an atomic model of T. Thus $K \equiv_{\infty,\omega} M$ and so $K' \equiv_{\infty,\omega} M'$ showing that M' also omits q. Conversely if K' is an atomic model of T', then it omits Π. So it is obtained from its restriction K to L by interpreting P with $\pi(K)$. Pick $M \in \mathscr{C}$. By the above M' is also atomic. Hence $K' \equiv_{\infty,\omega} M'$ and therefore $K \equiv_{\infty,\omega} M$ showing that $K \in \mathscr{C}$. $\square$

Thus there is a bijection between the atomic classes $\mathscr{C}$ and $\mathscr{C}'$ which moreover preserves elementary embeddings. In the language of category theory the two categories (with arrows being elementary embeddings) are equivalent.

Corollary 4.3.2. A subset A' of the monster model of T' is atomic in L' if and only if it is atomic in L.

Proof. Let $\bar{a} \in A'$ be a finite tuple. If $\bar{a}$ realises a principal type in L', then there is a countable atomic model M' containing it. The L-restriction of M is then an atomic model containing $\bar{a}$ showing that $\bar{a}$ realises a principal type in L. Similarly if $\bar{a}$ realises a principal type in L, then it realises a principal type in L'. $\square$

Thus for atomic sets it does not matter if we work in the monster model of T or T'. Now let us look at what happens to atomic types.

Lemma 4.3.3. Let A be an atomic set. Each atomic type over A in L can be uniquely extended to an atomic type in L'. Furthermore this defines a homeomorphism between topologies induced from the Stone spaces.

Proof. If $\bar{a}$ realises an atomic type over A in L, then $\bar{a}A$ is atomic in L and hence in L'. So $\mathrm{tp}_{L'}(\bar{a}/A)$ is an extension of $\mathrm{tp}_L(\bar{a}/A)$. To show the uniqueness assume that $\mathrm{tp}_L(\bar{a}/A) = \mathrm{tp}_L(\bar{b}/A)$ but $\mathrm{tp}_{L'}(\bar{a}/A) \neq \mathrm{tp}_{L'}(\bar{b}/A)$. Then there is a finite subset $A_0 \subseteq A$ such that $\mathrm{tp}_{L'}(\bar{a}/A_0) \neq \mathrm{tp}_{L'}(\bar{a}/A_0)$. Pick M a countable atomic model containing $\bar{a}\bar{b}A_0$. And let M' be its expansion to the atomic model of T'. Since M is homogeneous there is an automorphism of M that fixes A_0 pointwise and takes $\bar{a}$ to $\bar{b}$. But an automorphism of M is also an automorphism of M' (since its expansion is type definable). This contradicts the fact that $\bar{a}$ and $\bar{b}$ realise different types over A_0 in M'.

To see that this defines a homeomorphism note that the new definable sets in L' are type definable in L. But every type definable set in an atomic class is also definable by a (possibly infinite) disjunction of formulas. (For example if $\{p_i(\bar{x}) : i \in I\}$ is the set of complete principal extensions of $\pi(\bar{x})$ and $\phi_i(\bar{x})$ isolates p_i, then $\pi(\bar{x})$ is defined by $\bigvee_{i \in I} \phi_i(\bar{x})$.) Thus the set of types containing such a formula is already open in the topology of L-formulas. $\square$

But this means that we also do not change the independence relation and so the class $\mathscr{C}'$ is excellent provided that $\mathscr{C}$ is. Let us summarise this result.

Proposition 4.3.4. If $\mathscr{C}$ is an excellent class in the language L and $\pi(\bar{x})$ is a partial type in L, then the class of expansions of models in $\mathscr{C}$ with a predicate for $\pi(\bar{x})$ is excellent.

Now we can obtain an expansion of the language such that types over models are definable.

Proposition 4.3.5. Let $\mathscr{C}$ be an excellent class in the language L. There is an expansion L' of L and an excellent class $\mathscr{C}'$ whose models are unique expansions of models of $\mathscr{C}$ and each atomic type over a model is definable in $\mathscr{C}'$.

Proof. Let $\mathscr{C}_0 = \mathscr{C}$ and let $\mathscr{C}_{i+1}$ be obtained from $\mathscr{C}_i$ by expanding the language as follows. For each formula $\phi(\bar{x}, \bar{y})$ and for each countable ordinal α, add a new predicate symbol defining $\{\bar{a} : \mathrm{CB}(\phi(\bar{x}, \bar{a})) \geq \alpha$ and $\mathrm{Mlt}(\phi(\bar{x}, \bar{a})) > 0\}$. Since $\mathrm{CB}(\phi(\bar{x}, \bar{a}))$ and $\mathrm{Mlt}(\phi(\bar{x}, \bar{a}))$ depend only on ϕ and $\mathrm{tp}(\bar{a}/\emptyset)$, it is defined by a disjunction of formulas and so is also type definable. Let L' be the expansion of L with all these formulas and let $\mathscr{C}'$ be the class of corresponding expansion of models of $\mathscr{C}$.

Now let $M \in \mathscr{C}'$ and $p \in \mathrm{S}_{\mathrm{at}}(M)$. Let $\phi(\bar{x}, \bar{a}) \in p$ be a degree 1 formula of the same rank. Then for every formula $\psi(\bar{x}, \bar{b}) \in L(M)$ we have that $\psi(\bar{x}, \bar{b}) \in p$ if and only if $\mathrm{CB}(\phi(\bar{x}, \bar{a}) \wedge \psi(\bar{x}, \bar{b})) = \mathrm{CB}(p)$ and $\mathrm{Mlt}(\phi(\bar{x}, \bar{a}) \wedge \psi(\bar{x}, \bar{b})) > 0$ which is definable. $\square$

Finally note that if $N \supseteq M$ and $p' \in \mathrm{S}_{\mathrm{at}}(N)$ is the free extension of $p \in \mathrm{S}_{\mathrm{at}}(M)$, then p' is defined by the same formulas that define p.

4.4 Properties of Excellent Groups

Let $\mathscr{C}$ be an excellent atomic class in a countable language L. In this section we assume that L has a binary function symbol $\cdot$ making each structure in $\mathscr{C}$ into a group. We call a structure in $\mathscr{C}$ an *excellent group*. To highlight that $\mathscr{C}$ is a class of groups, we denote the large full model of $\mathscr{C}$ by $\mathfrak{G}$. Some of the results we prove in this section apply more generally to groups definable in excellent structures, but we chose this simpler context to have a smoother exposition. (We do not know if a structure definable in an excellent structure would itself be excellent.) We further assume that every type over a model is definable. Our treatment of excellent groups owes a great deal to the treatment of ω-stable groups in Lascar [1998].

Example 4.4.1. If we have an ω-stable $\aleph_0$-categorical theory, then the class of its models is excellent. So a familiar example of an excellent group is an ω-stable $\aleph_0$-categorical group. Any $\aleph_0$-saturated ω-stable group can also be treated in this context. We look at the class of models of the Scott sentence of the countable saturated model. This class will exactly be the class of $\aleph_0$-saturated models which can be made into an atomic class by expanding the language. (More explicitly by adding a predicate for the realisations of each type over $\emptyset$). A similar construction works for every homogeneous ω-stable group. However our theory does not have anything to add to the theory of ω-stable groups and these examples do not illustrate difficulties with excellent groups.

Example 4.4.2. Zilber's pseudo-analytic structures (which are quasiminimal excellent) provide interesting and important examples of excellent groups. We mention two such structures: group covers of the multiplicative group of an algebraically closed field (see Zilber [2006]) and the pseudo-exponential fields (see Zilber [2005a]). The first one has a stable first order theory, but the second one interprets arithmetic and so is highly unstable. Note also that the countable closure property in the pseudo-exponential fields is not $L_{\omega_1,\omega}$-axiomatisable (see Kirby [2013]), so to treat them in our context we need to consider models without the countable closure property.

Example 4.4.3. Let G be any countable group. By expanding the language if necessary we can make sure G is the only atomic model of $\mathrm{Th}(G)$. (For example we can add a constant symbol for each element of G.) The class of atomic models of $\mathrm{Th}(G)$ then consists of only G and is therefore ω-stable. The group G can bring all sorts of difficulties, such as infinite descending chains of definable groups. It is easy to exclude this example by requiring an excellent atomic class to contain an uncountable model (which we often do). However the same kind of difficulties can be found in the class of infinite direct sums of G.

Consider an infinite dimensional $\mathbb{Q}$-vector space V (or equivalently a divisible torsion free abelian group). Add a unary predicate symbol B for a basis of V and a binary predicate symbol S for the span. (More precisely $S(a, b)$ holds if and only if $a \in B$ and $b \in \langle a \rangle$.) Name an element c in B, which would define a copy of $\mathbb{Q}$ by $S(c, x)$. Identify this set with G. (That is expand the language with symbols of $\mathrm{Th}(G)$ and interpret them in $S(c, x)$ so that it is a model of $\mathrm{Th}(G)$. We can assume that this expansion is relational to avoid complications.) Further for $b \in B$ and $v \in V$ the coordinate (i.e. the element of $S(c, x)$) of b in the B-expansion of v is $L_{\omega_1,\omega}$-definable. Thus we can define (in $L_{\omega_1,\omega}$) a binary operation $\cdot : V \times V \to V$ making $(V, \cdot)$ an isomorphic copy of the direct sum $\bigoplus_{i \in B} G$ of $|B|$-many copies of G. All this can be encoded in an $L_{\omega_1,\omega}$-sentence which would be uncountably categorical implying that the class of its models is excellent in an appropriate language. (Strictly speaking this depends on the set-theoretic assumption $2^{\aleph_n} < 2^{\aleph_{n+1}}$.) But now if H is a definable subgroup of G, then $\bigoplus_{i \in B} H$ would be a definable subgroup of V. This shows that there can be infinite descending chains of definable subgroups of excellent groups.

Let $G \in \mathscr{C}$ be an excellent group. We can show the following weak version of the descending chain condition.

Proposition 4.4.4. There is no chain $(G_\alpha : \alpha < \beth_{\omega_1})$ of definable subgroups such that $G_\alpha \gneqq G_\beta$ for $\alpha < \beta$.

Proof. Assume for contradiction that there is such a chain. Since $\mathrm{cf}(\beth_{\omega_1}) > \omega$ by dropping to a subchain we may assume that there is a formula $\phi(x, \bar{y})$ such that G_α is defined by $\phi(x, \bar{c}_\alpha)$ for some $\bar{c}_\alpha \in G$. But then the formula $\forall x(\phi(x, \bar{y}) \to \phi(x, \bar{z}))$ orders the sequence $(\bar{c}_\alpha : \alpha < \beth_{\omega_1})$ which contradicts excellence (see Proposition 1.11 in Lessmann [2005]). □

Now let us consider stabilisers in G. We have an action of G on $\mathrm{S}_1(G)$ defined as follows: if $g \in G$ and $p \in S_1(G)$, then

$$g \cdot p = \{\phi(x) \in L(G) : \phi(g \cdot x) \in p\}.$$

If a is an element realising p in an elementary extension $G' \succeq G$, then $g \cdot a$ realises $g \cdot p$. Now if p is an atomic type, then G' can be chosen in $\mathscr{C}$ and so $g \cdot a$ would realise an atomic type. It follows that $\mathrm{S}_{\mathrm{at},1}(G)$ - the space of atomic 1-types, is invariant under the action of G. Note also that $\mathrm{CB}(\phi(x)) = \mathrm{CB}(\phi(g \cdot x))$ because there is a definable bijection between them. It follows that $\mathrm{CB}(g \cdot p) = \mathrm{CB}(p)$.

Definition 4.4.5. Let $p \in \mathrm{S}_{\mathrm{at},1}(G)$ be an atomic 1-type. The stabiliser of p, denoted stab_p, is the subgroup

$$\{g \in G : g \cdot p = p\}.$$

Proposition 4.4.6. For any $p \in \mathrm{S}_{\mathrm{at},1}(G)$ the group stab_p is a definable subgroup of G.

Proof. Let $\phi(x) \in p$ be a formula with $\mathrm{CB}(\phi) = \mathrm{CB}(p)$ and $\mathrm{Mlt}(\phi) = 1$. By Proposition 4.2.4, the type p is the only type in $\mathrm{S}_{\mathrm{at}}(G)$ of rank $\mathrm{CB}(\phi)$ containing ϕ. Because $\mathrm{CB}(g \cdot p) = \mathrm{CB}(p)$ we have that $g \in \mathrm{stab}_p$ if and only if $\phi(x) \in g \cdot p$. But by definition of $g \cdot p$ we have $\phi(x) \in g \cdot p$ if and only if $\phi(g \cdot x) \in p$. But now by the definability of types there is a formula $\psi(y)$ such that $\psi(g)$ if and only if $\phi(g \cdot x) \in p$. $\square$

The proof actually give something more.

Corollary 4.4.7. If $G' \in \mathscr{C}$ is an extension of G and $p' \in \mathrm{S}_{\mathrm{at}}(G')$ is the unique free extension of p to G', then $\mathrm{stab}_{p'} \leq G'$ is defined by the same formula as $\mathrm{stab}_p \leq G$.

Proof. Indeed if $\phi(x) \in p$ is a degree 1 formula of rank $\mathrm{CB}(p)$, then p' is the unique extension of ϕ to G' of the same rank and further p and p' are definable using the same defining scheme. $\square$

Proposition 4.4.8. For any $p \in \mathrm{S}_{\mathrm{at},1}(G)$ we have $\mathrm{CB}(\mathrm{stab}_p) \leq \mathrm{CB}(p)$.

Proof. Let $q \in \mathrm{S}_{\mathrm{at}}(G)$ be an extension of $\mathrm{stab}_p(x)$. We need to show that $\mathrm{CB}(q) \leq \mathrm{CB}(p)$. Let b realise q in an atomic extension G' of G. Let p' be the free extension of p to G' and let a realise p' (in the large full model of $\mathscr{C}$).

By the choice of a we have $a \perp_G b$ and by symmetry over models we have $b \perp_G a$. Hence $\mathrm{CB}(b/G) = \mathrm{CB}(b/G \cup a)$. But since there are definable bijections between $G \cup a$-definable sets containing b and $b \cdot a$ we have that $\mathrm{CB}(b/G \cup a) = \mathrm{CB}(b \cdot a/G \cup a)$.

On the other hand $\mathrm{CB}(b \cdot a/G \cup a) \leq \mathrm{CB}(b \cdot a/G) = \mathrm{CB}(p)$, since $\mathrm{tp}(b \cdot a/G) = p$. Thus $\mathrm{CB}(q) = \mathrm{CB}(b/G) \leq \mathrm{CB}(p)$ finishing the proof. □

We say that a definable subgroup H has *bounded index* in G if the formula $\phi(x)$ defining H defines a subgroup $\mathfrak{H}$ in the large full model $\mathfrak{G}$ of index $[\mathfrak{G} : \mathfrak{H}] < |\mathfrak{G}|$. Equivalently there is an atomic extension G' of G such that every coset $\mathfrak{G}/\mathfrak{H}$ has a representative in G'.

Proposition 4.4.9. If H is a definable subgroup of G of bounded index and $p \in \mathrm{S}_{\mathrm{at},1}(G)$, then $\mathrm{stab}_p \leq H$.

Proof. Let $\phi(x)$ define H and let $\mathfrak{H} = \phi(\mathfrak{G})$. Assume without loss of generality that every coset of $\mathfrak{G}/\mathfrak{H}$ has a representative in G. Let b realise p in $\mathfrak{G}$. Then $b \cdot c^{-1} \in \mathfrak{H}$ for some $c^{-1} \in G$. Hence $\phi(x \cdot c^{-1}) \in p(x)$. Pick $a \in \mathrm{stab}_p$. Then $\mathrm{tp}(a \cdot b/G) = \mathrm{tp}(b/G)$. Hence $a \cdot b \cdot c^{-1} \in \mathfrak{H}$. But since $\mathfrak{H}$ is a group we have $a \in \mathfrak{H}$ and therefore $a \in H$. □

Now let us move on to generic types. Recall that a definable set A of a group G is called *left (right) generic* if finitely many left (right) translates of A cover G, i.e. $G = B \cdot A$ (respectively $G = A \cdot B$) for some finite $B \subset G$. A type $p \in \mathrm{S}_1(G)$ is called *left (right) generic* if all formulas in p define left (right) generic subsets. Note that in a stable group a definable subset is left generic if and only if it is right generic and generic types always exist (see Poizat [2001]).

Since we have a rank on atomic types we also have an alternative notion of a generic type.

Definition 4.4.10. A type $p \in \mathrm{S}_{\mathrm{at},1}(G)$ is called CB-generic if $\mathrm{CB}(p) = \mathrm{CB}(G)$.

Proposition 4.4.11. If a type $p \in \mathrm{S}_{\mathrm{at},1}(G)$ is left generic or right generic, then it is CB-generic.

Proof. Assume that p is left generic. If $\phi(x) \in p$ and $A = \phi(G)$, then finitely many translates of A cover G. Let $b_1, ..., b_n \in G$ be such that $G = b_1 \cdot A \cup ... \cup b_n \cdot A$. Then by Proposition 4.2.7 we have $\mathrm{CB}(G) = \max(\mathrm{CB}(b_1 \cdot A), \ldots, \mathrm{CB}(b_n \cdot A)) = \mathrm{CB}(A)$. Hence $\mathrm{CB}(p) = \mathrm{CB}(G)$ proving that p is CB-generic. Similarly right generic types are CB-generic. □

As in ω-stable groups there are exactly $\mathrm{Mlt}(G)$-many CB-generic types. Unlike ω-stable groups however, $\mathrm{Mlt}(G)$ may be 0 in which case there are no CB-generic types (and hence no left or right generic types as well). This can happen only if $\mathrm{CB}(G)$ is a limit ordinal. We now show that this is possible even when the atomic class containing G is uncountably categorical.

Example 4.4.12. Consider the class of free groups in infinitely many generators in the language of groups together with a unary predicate P for a basis of the group. This is an excellent atomic class (it contains large homogeneous models) and is uncountably categorical. Let F_ω be the countable model and let us calculate the Cantor-Bendixson rank of types in $\mathrm{S}_{\mathrm{at},1}(F_\omega)$. The types realised in F_ω would obviously be isolated and so have rank 0. Excluding those, the type of a new element a in the basis would be isolated by the formula $P(x)$ and so has rank 1. Similarly the type of any element generated by $F_\omega \cup \{a\}$ would be isolated and so has rank 1. Excluding those, the type of an element generated by 2 new basis elements a, b and F_ω would have rank 2. Continuing in this way we see that the type of an element generated by n new basis elements over F_ω has rank n. Thus there are atomic 1-types of rank n for every finite n. But this exhausts all types and so $\mathrm{CB}(F_\omega) = \omega$ and $\mathrm{Mlt}(F_\omega) = 0$. In particular there are no generic types in this class.

However if CB-generic types exist we can characterise them using stabilisers.

Proposition 4.4.13. A type $p \in \mathrm{S}_{\mathrm{at},1}(G)$ is CB-generic if and only if stab_p has bounded index. In such a case stab_p is the least definable subgroup of bounded index.

Proof. If p is CB-generic, then the orbit of p consists of only CB-generic types. But there are at most $\aleph_0$-many CB-generic types. Therefore $[G : \mathrm{stab}_p] \leq \aleph_0$ in all models G.

Conversely assume that stab_p has bounded index. But then $\mathrm{CB}(\mathrm{stab}_p) = \mathrm{CB}(G)$. Indeed assume without loss of generality that every coset of stab_p in $\mathfrak{G}$ is represented in G. Now for every $a \in \mathfrak{G}$ we have $a \in b \cdot \mathrm{stab}_p$ for some $b \in G$. Hence $\mathrm{CB}(a/G) \leq \mathrm{CB}(b \cdot \mathrm{stab}_p) = \mathrm{CB}(\mathrm{stab}_p)$. This shows that $\mathrm{CB}(G) = \mathrm{CB}(\mathrm{stab}_p)$. But we have $\mathrm{CB}(p) \geq \mathrm{CB}(\mathrm{stab}_p)$ showing that p is indeed CB-generic. $\square$

To finish let us prove the following proposition about excellent groups with a unique CB-generic type.

Proposition 4.4.14. Let G be an excellent group and assume that $\mathrm{Mlt}(G) = 1$. If $A \subseteq G$ is definable, $\mathrm{CB}(A) = \mathrm{CB}(G)$ and $\mathrm{Mlt}(A) > 0$, then $G = A \cdot A$.

Proof. Let $\phi(x)$ define A. Let $p \in \mathrm{S}_{\mathrm{at},1}(G)$ be the unique CB-generic type of G. Pick $g \in G$ and let b realise p (in an atomic extension). Then $\mathrm{CB}(g \cdot b^{-1}/G) = \mathrm{CB}(b/G)$ and $g = g \cdot b^{-1} \cdot b$. Since $\phi \in p$ in the large full model we have $\models \exists x, y (g = x \cdot y \wedge \phi(x) \wedge \phi(y))$. So the same is true in G and so $G = A \cdot A$. $\square$

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