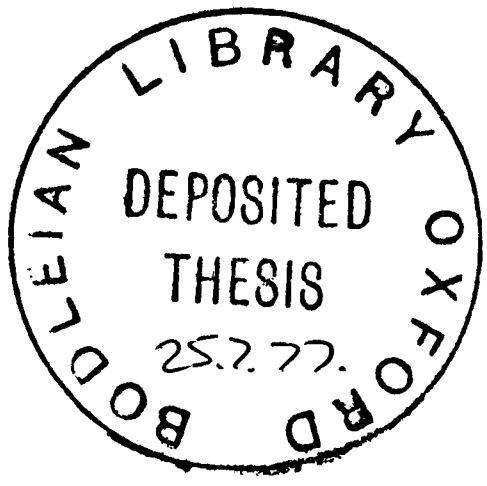
INFRARED PROBLEMS IN GAUGE THEORIES

Arturo Garcia Alvarez-Coque

Linacre College



A Thesis submitted for the Degree of Doctor of Philosophy at

the University of Oxford

April 1977

TO MY PARENTS, BROTHER AND SISTERS

ABSTRACT

After introductory remarks on the question of colour confinement in non-Abelian gauge theories of the strong interactions, Chapter 1 presents a brief review of the infrared (IR) phenomena in QED, with a comparative discussion of the non-Abelian case.

In Chapter 2, the singularities arising in a purely massless process (Compton scattering in massless QED) are studied in perturbation theory. The IR and mass singularities from soft and hard bremsstrahlung are computed, and the cancellation implied by the Kinoshita-Lee-Nauenberg theorem is verified. The appearance of mass singularities in the renormalization process is discussed in the context of dimensional regularization.

The first sections of Chapter 3 are devoted to the examination of general features of the IR divergence cancellation in the Yang-Mills theory. By making use of the Ward identities for on-shell amplitudes, soft virtual corrections and emission of soft and hard Yang-Mills quanta (gluons) are analyzed. In Section 3.4, the interest of considering a purely massless process in order to bring out characteristic features of the non-Abelian theory is emphasized, and gluon scattering in a colourless external potential is discussed. Soft and hard gluon emission corrections are computed, and the inclusive transition rate is found to be free of singularities, to lowest non-trivial order in perturbation theory. Thus we extend previous investigations restricted to massive fermions and soft gluon emission. Our treatment includes non-leading singularities in the order considered and questions of gauge independence.

Renormalization group-type equations which have been proposed, and which are presumed to sum up the singularities in perturbation theory, are described in Chapter 4. From the invariance of the S-matrix under the renormalization group, equations of the Callan-Symanzik type involving variations with respect to the IR cut-off are derived, and their significance is discussed. Questions related to the massive fermion decoupling theorem, zero-mass renormalization and dimensional regularization are also discussed.

ACKNOWLEDGEMENTS

I am grateful to Dr. J. C. Taylor for much invaluable assistance and encouragement at all stages of this work.

It is also a pleasure to thank:

- Dr. R. Meuldermans, for his permission to include part of the material in Chapter 2, which was published jointly with him.
- The many people with whom I had helpful discussions, in particular, M. Shailer, P. Cvitanović and I. Mohammad.
- Prof. R. H. Dalitz, for his attentions and hospitality in the Department of Theoretical Physics.
- Dr. Tomás Garza (IIMAS, University of México), for his interest and support.
- Mrs. Shirley Enoch, for her patient and accurate typing of the manuscript.

Financial support from the British Council, UNAM (México), Scott Fund and the Committee for Graduate Studies is gratefully acknowledged.

TABLE OF CONTENTS

Chapter 1

INTRODUCTION AND REVIEW

1.1 Quantum Chromodynamics 1

1.2 Soft Photons 7

1.3 Infrared Behaviour of QED 10

1.4 The Kinoshita-Lee-Nauenberg Theorem 15

1.5 Abelian vs. Non-Abelian 18

Appendix 0 The Renormalization Group 21

Chapter 2

MASSLESS QUANTUM ELECTRODYNAMICS

2.1 Introduction 25

2.2 Dimensional Regularization of IR Divergences 27

2.3 Vacuum Polarization 29

2.4 Compton Scattering of Massless Electrons 32

Appendix A-1 Some Hypergeometric Function Formulae 42

Appendix A-2 The Box Diagram 44

Appendix A-3 Narrow Angle e^+e^- Emission 50

Chapter 3

PURE YANG-MILLS THEORY

3.1 Introduction 54

3.2 Hard Gluon Emission 57

3.3 Soft Virtual Corrections 62

3.4 Gluon Scattering in an External Potential 65

3.5 Discussion 75

Appendix B-1 Virtual Corrections 78

Appendix B-2 Real Corrections 84

Appendix B-3 Gauge Independence 89

Chapter 4

BEYOND PERTURBATION THEORY?

4.1 Introduction 93

4.2 The Decoupling Theorem 97

4.3 The Near-Mass-Shell Coupling in QCD 99

4.4 The IR Differential Equation 104

4.5 Infrared Structure of the Quark Magnetic Moment 112

4.6 Discussion 114

Appendix C-1 The Cornwall and Tiktopoulos Mechanism 116

CONCLUSIONS 119

REFERENCES 122

Chapter 1

INTRODUCTION AND REVIEW

1.1 Quantum Chromodynamics

In the past few years there has been increasing hope that non-Abelian gauge theories will provide the framework for a theory encompassing all the elementary particle interactions (with the possible exception of gravity). Furthermore, it is widely believed that the infrared behaviour of these theories plays a central role in the dynamics of the fundamental fields of the strong interactions, corresponding to the elusive quarks. This thesis is concerned with the study of on-mass-shell amplitudes in gauge theories, in an attempt to acquire some understanding of their infrared behaviour.

The popularity enjoyed by non-Abelian gauge theories stemmed originally from the vast new possibilities they offered for the construction of renormalizable models unifying the weak and electromagnetic interactions (1). In these models, the intermediate gauge bosons acquire masses through the mechanism of spontaneous symmetry breaking and only the photon remains massless. The infrared (IR) divergences associated with soft photons have been extensively studied since the classic work of Bloch and Nordsieck and are well understood both physically and technically, as we shall describe in sections (1.2) and (1.3).

In the subsequent extension of the unified models to include semileptonic and non-leptonic weak processes, a quark model of the hadronic weak current inducing the required selection rules necessitates the introduction of a fourth (charmed) quark. The charm hypothesis seems to have been confirmed by the success of the

bound charm description of ψ-spectroscopy (2) and the recent discovery of states with properties expected of charmed particles (3). These discoveries as well as the general phenomenological success of the quark model, from hadron spectroscopy (1') to the quark-parton description of deep inelastic lepton-nucleon scattering (2'), lend strong support to the constituent picture of hadrons, with the quarks as the elementary fields.

Serious consideration of a field theory of strong interactions has been substantially encouraged since 1973 when it was discovered that non-Abelian gauge theories (and only them (5)) exhibit asymptotic freedom (4), the free field behaviour of Green's functions in the ultraviolet limit, which affords an understanding of the Bjorken scaling phenomenon in the deep inelastic scattering experiments (6). For asymptotic freedom to be possible, it appears that the strong gauge group should be unbroken* which implies massless Yang-Mills fields and the associated IR singular behaviour.

In order to specify the strong gauge group, one must specify the degrees of freedom assigned to the quarks. It appears that consideration of three times as many quarks, by giving them a further (colour) label, yields definite advantages over the original Gell-Mann-Zweig quark model (8). The SU(3) colour group transforms states with different colours into one another but leaves other quantum numbers and flavours (charge, isospin, strangeness, charm) unchanged. If, in order to avoid proliferation of coloured states, one proposes that all observable hadron states must be colour singlets, the baryon colour wavefunction $\epsilon_{ijk}q^i q^j q^k$ is antisymmetric

---

* Barring dynamical symmetry breaking (7).

under interchange of colour indices. Then the spin-statistics paradox for the low-lying baryons, which have a totally symmetric space-unitary spin wavefunction, is resolved (8). Apart from quark statistics, other motivations for colour are:

(a) The current algebra calculation of the rate for the decay $\pi^0 \to 2\gamma$ is experimentally correct if

$$\sum_i I_z^i Q_i^2 = \frac{1}{2}$$

where I_z^i and Q_i are the third component of isospin and charge of the quarks, respectively, and the sum is over all quark varieties. Without three colours, the predicted rate is in disagreement with experiment by a factor of 9.

(b) The predicted value for

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} \simeq \sum_i Q_i^2$$

resembles more the experimental value if three colours are included, giving R=2 (u, d, s triplets) and $R=\frac{10}{3}$ (u, d, s, c triplets).

(c) Adler-Bell-Jackiw anomalies (9), which spoil renormalizability, are cancelled between leptons and quarks in the four-triplet charm-colour scheme (provided there are four leptons).

All the above features hold for colour implemented as a <u>global</u> symmetry. Asymptotic freedom is a consequence of <u>local</u> colour gauge invariance. Quantum Chromodynamics (QCD) is an unbroken SU(3) colour gauge theory with the interaction mediated by an octet of gauge fields (gluons) (Fritzsch and Gell-Mann, Nambu, De Rújula, Georgi and Glashow (10)). The Lagrangian of what has come to be regarded as the standard model is then:

$$\mathcal{L} = -\frac{1}{4} F^{i}_{\mu\nu} F^{\mu\nu\, i} + \bar{\Psi}(i\not{D} - M)\Psi$$

the SU(4) flavour symmetry may be broken by a suitable mass matrix, but the colour local invariance is exact.

Beyond its several appealing features* (12), much remains unknown about QCD. The failure to detect free quarks indicates that the fundamental fields of the theory must be interpreted unconventionally, and that the spectrum of physical states cannot be read from the Lagrangian. In particular, the absence of non-colour singlets as free systems must be explained dynamically. The idea of quarks which are constrained by the dynamics so that they do not appear as asymptotic states was discussed in a field theoretic context by Johnson in 1972 (13). More recently, it has been conjectured by several authors (Weinberg, Fritzsch, Gell-Mann and Leutwyler (14), Gross and Wilczeck (6)) that the violent IR singularities of the theory may somehow provide the confinement mechanism. Although the precise connection between the IR singularities and the IR behaviour (in the sense of the renormalization group) is perhaps not immediately apparent, the plausibility of the conjecture is suggested by the asymptotically free behaviour itself which indicates that the coupling gets stronger at long distances and may cause the constituents and all states carrying the colour charge to remain eternally bound. In the language of the renormalization group, the alternative is that either there exists a finite IR stable fixed point or that the function $\beta(g(k^2))$ stays negative as $k^2 \to 0$, so that the coupling grows indefinitely

---

* For example, quark-quark forces mediated by an octet of gluons appear to favour zero triality non-exotic states (Lipkin (11)).

with the distance. Efforts to find insights or provide a workable framework for these ideas have gone in a variety of directions. Among these we mention only the study of two-dimensional models (15), lattice gauge theories (16) and more phenomenological approaches like the MIT bag (17).

The work in this thesis will be restricted to the conventional continuum theory and, to a large extent, will deal with the perturbative Feynman diagram expansion. This approach can be criticised from the standpoint that the IR (long distance) behaviour in the non-Abelian theory appears to be essentially a non-perturbative, strong coupling problem, and, in fact, the lack of non-perturbative methods is a motivation for unorthodox approaches like the lattice theory. However, there is so little known about the IR dynamics that we believe every possible avenue of investigation is worth pursuing. Moreover, in quantum electrodynamics (QED) it is also true that the IR problem is non-perturbative in nature. However, examination of the perturbation expansion leads to the elucidation of the all orders behaviour. That this is relatively easy to achieve in the Abelian case is related to the IR stability of the theory, but à priori this is no objection to attempting a similar analysis of the non-Abelian theory. It is still unclear whether the perturbative expansion can be treated systematically and lead to the elucidation of non-perturbative (all orders) features. On the other hand, independently of its ultimate relevance to answer the question of confinement, it would be desirable to have an understanding of the singularity structure in perturbation theory similar to that which we have in QED.

One specific proposal of a confinement mechanism is the

suggestion that IR singularities may cause production rates of coloured particles to vanish because of exponential factors $\exp(-\infty)$, like those arising from virtual photons in QED (Weinberg (14), Cornwall and Tiktopoulos (61)). Here it is argued that the non-Abelian theory possesses peculiarities which might cause an analogue of the Bloch-Nordsieck cancellation to fail. This failure may be manifested only after exact summation of singularities to infinite order* but it may in principle be signalled by a non-cancellation of divergences in finite order of perturbation theory. The latter possibility has motivated perturbative calculations of the IR singularities in transition rates for processes involving on-mass-shell coloured particles, this approach being equivalent to assuming that there is no confinement and then checking whether there is any pathological behaviour.

As it happens, the Bloch-Nordsieck programme turns out to be applicable in the non-Abelian theory, order by order in perturbation theory. Examples have been given (53,54,55) showing that in low order calculations, the IR divergences are not present for suitably defined cross-sections. In Chapter 3 we present in this connection a calculation of gluon scattering in an external scalar potential, to lowest non-trivial order in perturbation theory, and discuss peculiarities associated with on-shell gluons (as opposed to on-shell massive quarks). Chapter 2 contains a calculation of the IR and mass singularities arising in virtual and real corrections to Compton scattering in massless QED. In Chapter 4 we present a discussion on the application of the renormalization group equations

---

* This is the Cornwall and Tiktopoulos proposal (61), which we describe in more detail in Appendix (C-1)).

in the IR regime and on similar equations which are presumed to sum up the singularities in the perturbative expansion.

In this chapter, we continue by presenting in sections (1.2) and (1.3) a survey of the IR problem in QED. A thorough understanding of the IR behaviour of the Abelian theory may be of help in dealing with the non-Abelian case. In section (1.4) we describe the Kinoshita-Lee-Nauenberg theory as it will be relevant to our considerations in later chapters. Finally, a brief comparative review of what is known at present about the non-Abelian theory is presented in section (1.5).

1.2 Soft Photons

According to classical electromagnetic theory, if a charged particle undergoes an acceleration changing its velocity from an initial value $\underset{\sim}{v}$ (c=1) to a final value $\underset{\sim}{v}'$, in the zero frequency limit $\omega \to 0$, the spectrum of radiation with polarization $\underset{\sim}{\varepsilon}$ is given by (18):

$$I_\omega = \frac{d^2I}{d\omega\, d\Omega} \sim e^2 \left| \underset{\sim}{\varepsilon}^* \cdot \left(\frac{\underset{\sim}{v}'}{1 - \underset{\sim}{n}\cdot\underset{\sim}{v}'} - \frac{\underset{\sim}{v}}{1 - \underset{\sim}{n}\cdot\underset{\sim}{v}} \right) \right|^2 \qquad (1)$$

where e is the charge, I the intensity of radiation and $\underset{\sim}{n}$ a unit vector in the direction of the solid angle $d\Omega$. Thus, the intensity is independent of frequency as $\omega \to 0$. Making the transcription to the photon description, $I_\omega / h\omega$ is an estimate of the mean number of photons emitted per unit frequency range (the formula (1) is in fact also valid quantum mechanically). Therefore, the photon spectrum is of the form $\frac{d\omega}{\omega}$ which diverges as $\omega \to 0$.

In their 1937 classic paper, Bloch and Nordsieck (19) pointed out that the meaning of this IR divergence is that only the probability

of simultaneous emission of infinitely many photons can be finite, the probability of emission of any finite number of quanta must vanish. Consequently, the usual perturbation expansion in powers of α is inadequate since it assumes that the probability of multiple photon emission decreases with increasing number of emitted quanta. In fact, for a minimum frequency ω_0, the effective expansion parameter is $\alpha \ln \omega_0$ which is not small as $\omega_0 \to 0$. Therefore, it is necessary to consider the collective effect of soft photons.

In the long wavelength limit. the reaction of the electromagnetic field on the scattered electron must be negligible. Ignoring the electron recoil, Bloch and Nordsieck regarded the radiation as emitted from an external classical current. Then they were able to show that the cross-section for electron scattering in an external potential accompanied by emission of N soft photons with frequencies in the range $\omega_0 < \omega < \omega_1$ has the form:

$$\left(\frac{d\sigma}{d\Omega}\right) = \left(\frac{d\sigma}{d\Omega}\right)_0 \exp(-\hat{B}) \frac{\hat{B}^N}{N!} \tag{2}$$

where $(\frac{d\sigma}{d\Omega})_0$ is (approximately) the Born cross-section and,

$$\hat{B} = \frac{\alpha}{4\pi^2} \int_{\omega_0}^{\omega_1} \frac{d^3k}{2|\underset{\sim}{k}|} \left(\frac{p' \cdot \varepsilon}{p' \cdot k} - \frac{p \cdot \varepsilon}{p \cdot k} \right)^2 \tag{3}$$

where ε, $p(\varepsilon', p')$ are the polarization vector and momentum of the incoming (outgoing) electron. Since the integral (3) over photon momenta diverges at low frequencies, the exponential factor in (2) makes the cross-section for emission of any finite number of photons vanish. On the other hand, the cross-section for an arbitrary

number of emitted soft photons, obtained by carrying out the sum over N in (2), is finite and given essentially by the Born cross-section with all radiative corrections ignored. This is the well-known cancellation between virtual and real soft photons divergences. The mean energy of the radiation emitted in a frequency interval $d\omega$ into the solid angle $d\Omega$ can be calculated from eqs. (2) and (3), a finite result is obtained equal to the classical spectrum in eq.(1).

The first modern treatment of the IR divergence of QED using Feynman diagram techniques was given in 1954 by Jauch and Rorlich (20), which proved the cancellation of singularities to all orders in perturbation theory. In their treatment they considered the effect of electron recoil by introducing a parameter ΔE which defined the maximum energy carried by the emitted soft photons. Thus, they emphasized that in any scattering experiment for which the energy resolution is ΔE, an infinite number of soft photons escape detection and they must be included in order to construct the finite physical cross-sections. This is also understood from the fact that in the zero frequency limit there is essentially no distinction between real and virtual photons, so that all soft quanta must be treated on the same footing.

A refinement of this work with a more rigorous proof of the factorization and exponentiation of the soft photon contributions without neglect of hard photon corrections, as well as a discussion of various aspects of the IR problem, was given by Yennie, Frautschi and Suura (21). In particular, they pointed out that knowledge of the IR factors are of special interest since they give the dominant radiative corrections at high energies. We refer to this paper for a more complete bibliography. A more transparent treatment of

the extraction of the IR factors has been given by Eriksson (22) and, more recently, by Grammer and Yennie (23).

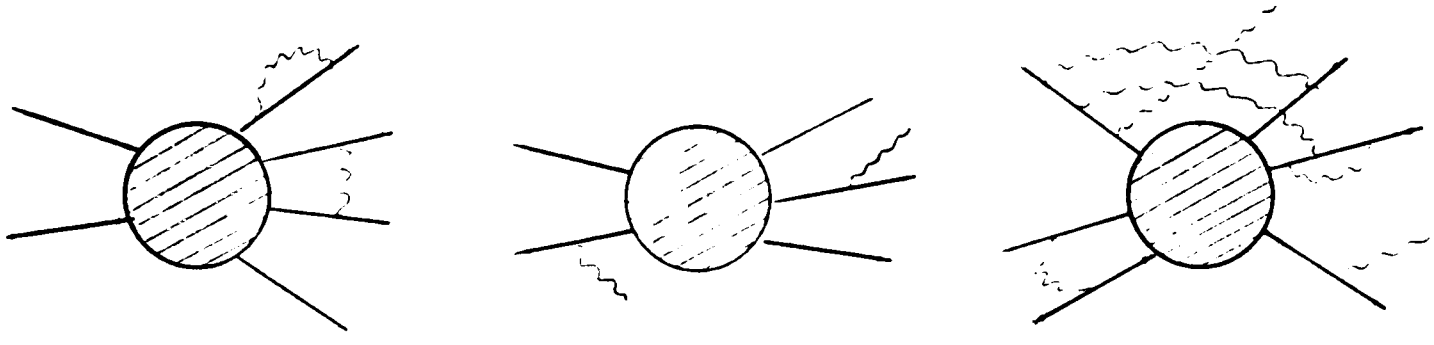
From the point of view of scattering theory, because of the IR (long distance) effects there is a difficulty in defining the asymptotic states. An electron is never completely free, apart from the self-interaction which contributes to its self-mass, the possibility of persistent emission of very soft quanta means that there are long range interactions with other charges. This is already manifest in non-relativistic Coulomb scattering where the asymptotic wavefunction is not a plane wave but is distorted by the long range Coulomb potential and acquires a divergent phase. In fact, in the QED calculation of a scattering process, in general the cancellation between real and virtual singularities is incomplete, and there remains an exponential factor with a divergent phase (21), as first conjectured by Dalitz (24). This divergence is not present for a screened potential or, equivalently, a finite photon mass. Such infinite phases have no observable consequences.

It has been possible to define scattering amplitudes (not merely transition rates or cross-sections) which are free of the IR divergence by considering asymptotic states with an indefinite number of photons (coherent states) (25). These formulations have a more sound theoretical basis than the conventional approach but offer no practical advantage.

1.3 Infrared Behaviour of QED

In this section we summarize the salient features of the IR phenomena in QED, elaborating briefly on some of the points raised in the previous section. This is to serve as a reference for a comparative description of the non-Abelian theory which we present in section 1.5.

(i) The IR divergent amplitudes in QED correspond to diagrams consisting of a "core" of hard virtual photons - surrounded by a cloud of virtual and real soft photons attached to external lines, of the type represented in the figure below:



Thus, in a given order of perturbation theory, the only radiative corrections of order α that are IR divergent are self-energy corrections to external lines or soft photon exchange between external lines. IR divergent terms due to virtual corrections to, or real emission from, internal lines cancel because of the QED Ward identity. Intuitively, this seems plausible since a long wavelength photon should sense only large scale features of the charge distribution, and the hard photon internal lines somehow correspond to a localized region where the main scattering takes place.

(ii) It is the separation of the soft photon contributions into factors depending only upon the external lines which permits the treatment to all orders in perturbation theory and, in particular, the demonstration of the virtual-real cancellation. The sum over all undetected photons provides the complete differential cross-section:

$$\frac{d\sigma}{d\Omega} = \exp\{2\alpha(B+\tilde{B})\}\,\frac{d\hat{\sigma}}{d\Omega} \qquad (4)$$

which is a form similar to eq.(2) but with the IR finite radiative corrections included, $\frac{d\hat{\sigma}}{d\Omega}$ contains the non-infrared factors (for details of these factors see Ref.(21)), and B and $\tilde{B}$ contain the virtual and real soft photon effects respectively. Gauge invariant expressions for these quantities are:

$$B = \frac{i}{(2\pi)^3}\int \frac{d^4k}{k^2-\lambda^2}\left(\frac{2p'_m - k_m}{2p'\cdot k - k^2} - \frac{2p_m - k_m}{2p\cdot k - k^2}\right)^2$$

$$\tilde{B} = -\frac{1}{8\pi^2}\int_0^{\Delta E}\frac{d^3k}{(k^2+\lambda^2)^{1/2}}\left(\frac{p'_m}{p'\cdot k} - \frac{p_m}{p\cdot k}\right)^2 \qquad (5)$$

where λ is a photon mass acting as IR cut-off.

The IR divergent parts of B and $\tilde{B}$ (terms in $\ln\lambda$) cancel between them. If the total energy loss by soft photon emission is ΔE, one is left with the form:

$$\frac{d\sigma}{d\Omega} \sim \left(\frac{\Delta E}{E}\right)^{\alpha A}\frac{d\hat{\sigma}}{d\Omega} \qquad (6)$$

where A is a simple function of the momentum transfer. The expression (6) vanishes as $\Delta E \to 0$, which is again a reflection of the impossibility of perfectly elastic scattering. The Bloch-Nordsieck cancellation is found to take place order by order in perturbation theory, leaving a finite part containing A $\ln\frac{\Delta E}{E}$ terms.

(iii) There is in QED a natural scale where to define the coupling constant, namely, on the mass-shell. The mass-shell electron wave-function and vertex renormalization constants, Z_1 and Z_2 respectively, are IR singular, but because of the Ward identity $Z_1 = Z_2$ and the fact that Z_3 is IR finite, the mass-shell renormalization introduces no IR divergence. In fact, there exists a gauge where the spurious

charge renormalization Z_1 is IR finite (21).

(iv) According to eq.(2), the emitted soft photons follow a Poisson distribution:

$$e^{-\langle x\rangle}\ \frac{\langle x\rangle^N}{N!} \tag{7}$$

where the average number of photons $\langle x\rangle$ in a frequency range is given by the integral (3), which is infinite if the frequency goes down to zero. Soft photons do not appreciably disturb the motion of the electron, so that they are emitted and absorbed independently, and a distribution of the type (7) results.

(v) In the appendix to this chapter we give a short description of the renormalization group, mainly in order to introduce some standard notation and terminology.

The renormalized photon propagator is of the form:

$$D_{\mu\nu}(k^2) = -\frac{g_{\mu\nu}}{k^2}\, d\left(\frac{k^2}{M^2}, \frac{m^2}{M^2}, g(M^2)\right) + (\text{gauge terms}) \tag{8}$$

where m is the electron mass and the normalization is done at $k^2=M^2$, so that

$$d\left(1, \frac{m^2}{M^2}, g(M^2)\right) = 1 \tag{9}$$

The quantity $g^2(M^2)d(\frac{k^2}{m^2}, \frac{m^2}{M^2}, g(m^2))$, called the invariant charge, is such that, for any M_1 and M_2:

$$g^2(M_1^2)\, d\left(\frac{k^2}{M_1^2}, \frac{m^2}{M_1^2}, g(M_1^2)\right) = g^2(M_2^2)\, d\left(\frac{k^2}{M_2^2}, \frac{m^2}{M_2^2}, g(M_2^2)\right) \tag{10}$$

Setting $k^2=M_1^2$, $M_2=M$ and using (9), we have:

$$g^2(k^2) = g^2(M^2)\, d\left(\frac{k^2}{M^2}, \frac{m^2}{M^2}, g(M^2)\right)$$

The behaviour of the RHS of this relation as $k^2 \to 0$ is known (26), the effect of renormalizations by electron loops vanishes in this limit so that:

$$g^2(k^2) = g^2(M^2)\left(1 + \mathcal{O}\left(\frac{k^2}{m^2}, \frac{M^2}{m^2}\right)\right) \tag{11}$$

Defining $\alpha = \frac{g^2(0)}{4\pi}$ we see that

$$g^2(k^2) = 4\pi\alpha + \mathcal{O}\left(\frac{k^2}{m^2}\right) \tag{12}$$

Therefore, at low momenta or, equivalently, long distances (long in the scale of the electron Compton wavelength), the coupling approaches $\alpha = \frac{1}{137}$ which is effectively a fixed point. In this sense QED is an IR stable theory, and the long distance limit is the natural one to define the "physical" coupling constant. Correspondingly, the β function vanishes at low momenta:

$$\beta(g(k^2)) = k^2 \frac{\partial g}{\partial k^2} \xrightarrow[k^2 \to 0]{} 0 \tag{13}$$

or

$$\beta\left(g(0) = \sqrt{4\pi\alpha}\right) = 0$$

The IR behaviour of $g(k^2)$ is opposite to that in the ultraviolet regime, the effective coupling constant grows at high energy and diverges if the bare coupling is infinite (31).

1.4 The Kinoshita-Lee-Nauenberg Theorem

This section is a digression intended to describe a result to which we shall refer repeatedly in later chapters. In theories with coupled massless fields, a new kind of singularity arises, different from the low frequency IR divergence of QED. In the figure below we represent a photon of momentum k emitted from an electron line. With p on-shell, as $k \to 0$ the propagator behaves as

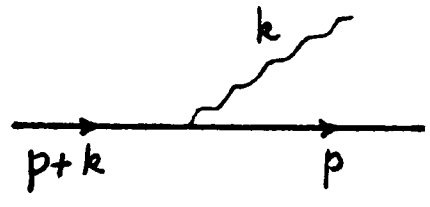


Fig. A

$\frac{1}{p.k}$ and this factor causes the IR divergence. However, if the electron is massless we have $\frac{1}{p.k} = \frac{1}{|\underset{\sim}{p}||\underset{\sim}{k}|(1-\cos\theta)}$ where θ is the angle between $\underset{\sim}{p}$ and $\underset{\sim}{k}$. There is now a double singularity as $|k| \to 0$ and $\theta \to 0$, the narrow angle divergence is present for arbitrary $|k|$, i.e. it appears even when the photon is hard if it is emitted parallel to the electron momentum. This singularity is already indicated by the classical formula (1) which shows strong peaking of the radiation emitted parallel to the scattered particle in the relativistic limit.* These "mass singularities" appear in perturbation theory as ln m terms, where m is the electron mass.

---

* The IR divergence which appears in the quantum theory because of the infinite number of soft photons, does not appear in the classical theory (in fact the coherent state formulation is related to a classical approximation). In contrast, the mass singularity appears in both the classical and quantum theories and cannot be eliminated by a classical approximation.

In his analysis of radiative corrections to weak decays, Kinoshita (27) observed that even if partial rates contained ln m terms, the limit $m \to 0$ was non-singular if total transition rates were considered, and proved this property in arbitrary finite order in perturbation theory. That the cancellation of mass singularities was a general consequence of the formal theory of scattering was shown by T. D. Lee and M. Nauenberg (28). Their analysis includes the usual IR divergence associated with soft photons. Lee and Nauenberg pointed out that divergences are present whenever there is a degeneracy of the initial or final states in a scattering process. A massless electron state may be degenerate with a state consisting of an electron and any number of photons and electron-positron pairs moving parallel to it; an electron state is also degenerate with a state containing an electron and a number of photons, in the limit of zero energy photons. The Kinoshita-Lee-Nauenberg (KLN) theorem is the statement that transition probabilities are free of IR and mass singularities order by order in perturbation theory when they are summed over ensembles of all possible degenerate initial and final states. The Bloch-Nordsieck cancellation is a special case of this result. The physical justification for summing over degenerate states is presumably that, for m=0, an electron travelling alone is physically indistinguishable from one moving together with a flux of photons and pairs emitted in the same direction, provided the total charge and energy is the same as the charge and energy of the electron alone.

At this point we would like to stress two aspects. First, it is important that the renormalization procedure does not introduce additional singularities. The theorem holds for the perturbation series expressed in terms of either the unrenormalized coupling or the coupling defined by off-shell renormalization. Second, the

elimination of the mass singularities requires ~~in~~ inclusion of degenerate initial states. For example, in the case of an incoming massless electron, one should consider the absorption of hard photons moving parallel to it (cf. our calculation of Chapter 2). This multiparticle initial state appears to be of difficult physical interpretation. The initial state must be specified rather by an ensemble of degenerate states consisting of different numbers of massless particles travelling in the same direction and somehow the probabilistic weights for the multiparticle states must be equal. We shall comment on this point a little further in connection with the gluon scattering process discussed in Chapter 3.

When there are only photonic IR divergences, the leading IR singularities in N-loop diagrams go like $\alpha^N \ln^N \lambda$. If there are coupled massless particles, for a graph with external massless particles, the leading singularities are doubly logarithmic, i.e. they are $\sim\alpha^N \ln^{2N} \lambda$ at the N-loop level. To illustrate this let us consider again Fig.A above, where a massless particle with momentum k is emitted by another massless particle and let θ be the angle between their three-momenta. The regions of phase space giving singularities are represented in Fig.B. Single logarithms arise when k is soft and θ arbitrary (soft "wide" region) and also when k is hard but θ vanishes (hard "narrow" region). The double logarithms (double poles in dimensional regularization) arise when k is soft and parallel to the other particle momentum (soft "narrow" region).

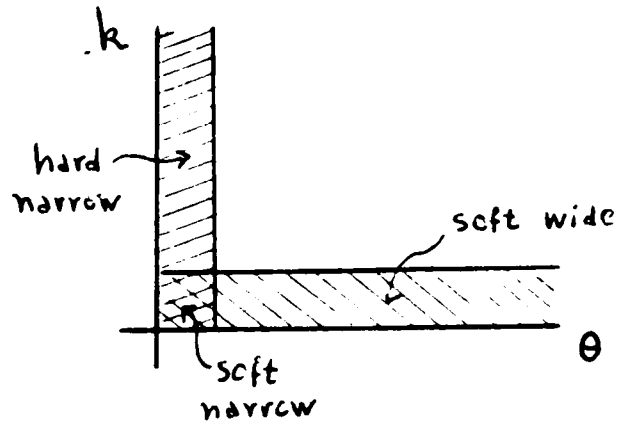


Fig. B

The limitations of the proofs of the KLN theorem in relation with Yang-Mills theory will be briefly mentioned in the introduction to Chapter 3.

1.5 Abelian vs. non-Abelian

We now discuss some of the facts known about the IR behaviour of non-Abelian theories, stressing the differences with the Abelian case described in section 1.3. Here we include the results of recent work on the subject, to which we shall refer again in later chapters, as well as some comments related to our own results in this thesis.

(i) Because of the Yang-Mills coupling, the mass singularities associated with coupled massless fields are present in non-Abelian gauge theories, and, to a large extent, their IR complexity is due to this feature. There is not a simple class of graphs which contain all the IR divergences* like in QED. Because of this, and since not only soft virtual momenta are relevant, the summation of IR singularities looks highly non-trivial.**

(ii) There is not a simple exponentiation of the IR singularities as that given by eq.(4) in QED. Instead, it is found that in QCD (62,63):

$$-\frac{\partial}{\partial \ln\lambda} T(q^2;\lambda) = \bar{g}^2(\ln\lambda)\, A(q^2)\, T(q^2;\lambda) \qquad (14)$$

---

* From now on we shall usually refer to IR and mass singularities generically as infrared.

** It has been possible to extract the IR singularities into exponential factors, much as in QED, for another theory with coupled massless fields, namely, gravitation (Weinberg (29)), but this is due to the peculiarity that the graviton self coupling vanishes in the IR limit.

where $T(q^2,\lambda)$ is the amplitude for quark scattering in an external potential and $A(q^2)$ is the same function as in eq.(6). The novel feature is that instead of a g^2 factor as in QED, there appears $\bar{g}^2$, the renormalization group effective coupling. Eq.(14) is probably true for all leading singularities (64,63) but is non-trivially modified if non-leading ones are included (65).

(iii) As we shall discuss more extensively in Chapter 3, the cancellation of all singularities has been verified in low-order calculations. For massive quarks in QCD, this entails summation over all soft gluons in the final state. However, if one incoming particle is a gluon itself, as in the scattering process illustrated in Fig.A below, finiteness of the cross section requires further incoming gluons (in addition to the outgoing ones) (Fig.B), for which the physical interpretation does not seem obvious. This particular

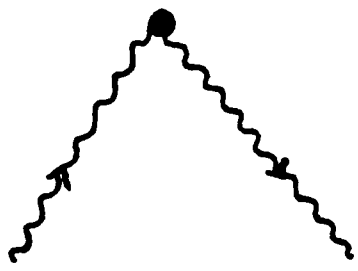


Fig. A

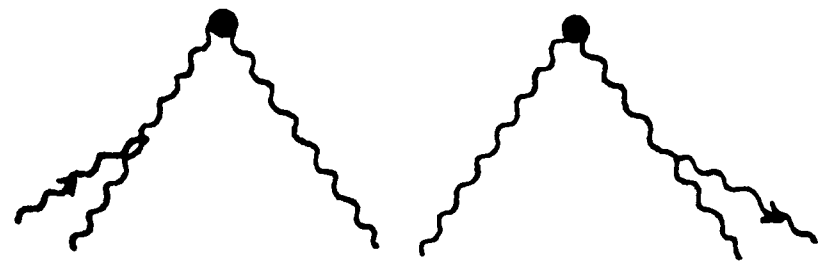


Fig. B

example of a gluon process (scattering in an external potential with gauge invariant coupling) is discussed in Chapter 3 where we calculate all the relevant virtual and real corrections.

(iv) Renormalization on the mass-shell introduces IR singularities because of the gluon self-energy contribution to Z_3. This happens even in the axial gauge (30) for which $Z_1=Z_2$.

(v) Gluon emission does not follow a Poisson distribution,

gluons affect one another as they are emitted or absorbed.

(vi) In QCD renormalization effects persist in the IR limit because of the gluon self coupling. In a manner similar to that described in point (v) of section (1.3), the contributions of massive fermions decouple at low momenta in QCD (26), which means:

$$\beta\left(g(\mu), \frac{\mu^2}{m^2}\right) \xrightarrow[\mu \to 0]{} \beta_{YM}(g(\mu))$$

where β_{YM} if the β function of the pure Yang-Mills theory, which governs the IR behaviour of the coupling. Asymptotic freedom suggests that $g(\mu)$ may blow up as $\mu^2 \to 0$.

Appendix 0

THE RENORMALIZATION GROUP (31-35)

This appendix briefly describes the renormalization group ideas and establishes notation and terminology to which we shall often refer, especially in Chapter 4.

The coupling constant which parametrizes a theory corresponds to some momentum scale at which, ideally, it would be determined experimentally. Thus, in QED the "natural" scale is the zero momentum transfer (mass-shell) limit. The mass scale is thus introduced even in theories which in principle did not involve dimensional parameters and determines also the normalization of the fields and related objects like the Green's functions. That the renormalization point μ need be introduced or that it should be an off-shell point is à priori not related to the need of avoiding ultraviolet or infrared divergences.

One-particle-irreducible Green's functions $\Gamma(p, \mu, g(\mu))$ normalized at various mass scales are all related to the bare functions in such a way that, for given μ_1 and μ_2:

$$Z(\mu_1)\Gamma(p, \mu_1, g(\mu_1)) = Z(\mu_2)\Gamma(p, \mu_2, g(\mu_2)) \qquad (15)$$

where $Z(\mu)$ is a product of renormalization constants. The differential statement of (15) is the renormalization group equation:

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \gamma\right]\Gamma = 0 \qquad (16)$$

where

$$\beta(g) = \mu \frac{\partial}{\partial \mu} g \tag{17a}$$

$$\gamma(g) = \mu \frac{\partial}{\partial \mu} \ln Z \tag{17b}$$

Thus (15) expresses the arbitrariness in choosing the scale at which couplings and fields are defined. In massless theories or in kinematical regimes where dimensional parameters other than μ are negligible (as specified by Weinberg's theorem (36) at asymptotic spacelike momenta) a simple analysis relates the scaling of external momenta to a variation in μ which is in turn correlated to a variation of $g(\mu)$ through eq. (16). Hence one can turn (16) into an equation governing the behaviour of Green's functions under scaling of external momenta:

$$\left[-\frac{\partial}{\partial t} + \beta(g) \frac{\partial}{\partial g} + \gamma + D \right] \Gamma(e^t p, \mu, g(\mu)) = 0 \tag{18}$$

where D is the canonical dimension of Γ.

The so-called effective coupling $\bar{g}(t, g)$ is just the coupling defined at μe^t whenever g is defined at μ and satisfies:

$$\left[-\frac{\partial}{\partial t} + \beta(g) \frac{\partial}{\partial g} \right] \bar{g}(t, g) = 0 \tag{19}$$

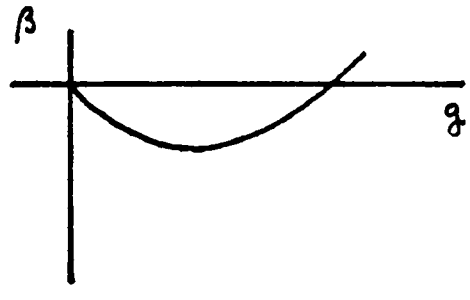
which can be used to write down a solution for eq.(18):

$$\Gamma(e^t p, \mu, g(\mu)) = e^{Dt} \Gamma(p, \mu, g(\mu e^t)) \exp \int_0^t dt' \gamma(g(\mu e^{t'})) \tag{20}$$

Limiting values of $\bar{g}(t, g)$ as $t \to \pm\infty$ are called fixed points. The remarkable feature of non-Abelian gauge theories is that the β function is negative near the origin in g-space (if there are not too many fermion multiplets) which by (17a) implies $g(\mu e^t) \to 0$ as

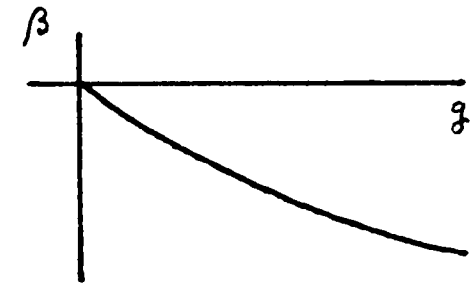
$t \to \infty$, the asymptotic freedom property. This happens provided the initial value $g(\mu)$ is in the domain of attraction of the origin, that is, lies to the left of the next zero of $\beta(g)$. The β function is known perturbatively to order g^5 (37).

If β remains negative for all g, the opposite behaviour ensues as $t \to -\infty$, that is, the coupling grows indefinitely in the long distance (near mass-shell) limit (Fig.B). Conjectures of colour confinement are largely based on this possibility.



(A)

Finite IR stable fixed point



(B)

Confining behaviour

Eq.(18) is exact at all momenta for massless theories but approximate when masses are only negligible. An equation which can be cast into the scaling form for arbitrary momenta in massive theories is that due to Callan and Symanzik (C-S) (33,34):

$$\left[m \frac{\partial}{\partial m} + \beta(g) \frac{\partial}{\partial g} + \gamma \right] \Gamma = \Gamma_{\Delta} \qquad (21)$$

which, however, is inhomogeneous and cannot be solved, except, again, at asymptotic space-like momenta where the "mass insertion" term in the right hand side can be neglected. In the C-S formulation, g is defined by renormalization at m, "the mass" of the theory and no further dimensional parameter is introduced from which follows that $\beta(g)$ is a function of g only. In contrast, the usual renormalization group β function in eq.(16) has in general a dependence on ratios of

dimensional parameters.

Improved methods for the treatment of mass parameters have also been developed (Weinberg (38),'t Hooft (39)).

Chapter 2

MASSLESS QUANTUM ELECTRODYNAMICS

2.1 Introduction

The theory of photons coupled to massless electrons is, so to speak, intermediate between QED and QCD as far as complexity of IR structure is concerned. It shares with the non-Abelian theory the feature of containing mass singularities but differs from it in that it is IR stable in the sense of the renormalization group (26,32). The leading singularities of massless QED are doubly logarithmic, and, in contrast with the massive theory, it is not known whether all singularities factorize and exponentiate (although the leading ones do exponentiate (42)). Massless QED is one of the simplest theories with coupled massless fields, and one might expect to learn from it something about the singularity structure of the Yang-Mills theory, even if the IR behaviours of the two theories are distinct in various ways.

Our interest in this Chapter will focus on applying dimensional regularization techniques to calculate IR and mass singularities in perturbation theory for a purely massless scattering process. The method of dimensional continuation for handling IR divergences, to be briefly described in the next section, has proved useful in low order calculations (50,53,54) in which a Bloch-Nordsieck type of treatment has turned out to be applicable order by order in the coupling constant to amplitudes involving massless Yang-Mills particles coupled to fermions. These calculations have so far dealt with massive fermions or else have considered slightly off-shell amplitudes. When the external lines are also massless, the full complexity of the theory with coupled massless fields is attained

and, in dimensional regularization, double poles at d=4 appear at the one-loop level (d being the space-time dimension). In order to provide an explicit example we consider (sec. 2.4) Compton scattering of massless electrons, including soft and hard real corrections, to lowest non-trivial order in perturbation theory. The Kinoshita-Lee-Nauenberg theorem (sec. 1.4) is fulfilled only after inclusion of mass singularities arising from proper subtractions of ultraviolet poles. However, it is not our purpose simply to verify the theorem on cancellation of mass singularities. When the space-time dimension is used as a common regularizing parameter for ultraviolet, infrared and mass singularities, the identification of these distinct types of singularities tends to be obscured. We illustrate this situation in section 2.3 by analyzing explicitly the appearance of the mass singularity after renormalization of the vacuum polarization correction to massless external lines. This analysis would be essentially the same for other theories with coupled massless fields, and therefore is relevant for pure Yang-Mills theory or for massless quantum chromodynamics where, moreover, the gauge invariance preserving dimensional regularization is the most convenient.

In section 2.4 the calculation of IR and mass singularities for massless Compton scattering is described, and the pole parts appearing in the real and virtual corrections to the lowest order cross-section are displayed, verifying cancellation when the remarks of section 2.3 are taken into account.

2.2 Dimensional Regularization of IR Divergences

Before the infinities of field theory are argued away on physical grounds, the undefined mathematics has to be given a meaning by means of some regularization prescription. The regularization methods are manifold, but the most powerful, especially so for non-Abelian gauge theories, is the dimensional regularization procedure (43, 44, 45). Apart from affording calculational simplicity, the method respects gauge invariance as well as other symmetries. The idea is to define Feynman integrals away from d=4, where d is the space-time dimension. The integrals are finite in some d-region from which they are analytically continued, and divergences appear thereby as poles at d=4. The basic formula reads:

$$\int d^d k \, \frac{(k^2)^r}{(k^2+C)^\alpha} = i(-\pi)^{\frac{d}{2}} \frac{\Gamma(r+\frac{d}{2})}{\Gamma(\frac{d}{2})} \frac{\Gamma(\alpha-r-\frac{d}{2})}{\Gamma(\alpha)} (C)^{\frac{d}{2}+r-\alpha} \qquad (2.1)$$

Since its invention by 't Hooft and Veltman (43) and Bollini and Giambiagi (44), it was pointed out that IR singularities are also controlled by the dimensional continuation. Hence it is not necessary to introduce a photon mass and gauge invariance is preserved at all stages.* Examples of the use of the dimensional

---

* The introduction of a vector meson mass is no real difficulty in the Abelian theory even if it leads to technical complications associated with the longitudinal polarization states. Although the difficulties in the non-Abelian theory may be more serious (46), several authors (61,68) have employed non-gauge invariant IR cut-offs (e.g. a gluon mass) in their computations of IR singularities. We shall employ dimensional regularization in our calculations of Chapters 2 and 3 but shall use a dimensional cut-off in our considerations of Chapter 4.

regularization for the treatment of IR singularities have been presented by several authors. Scattering of an electron by an external potential is considered by Gastmans and Meuldermans (47), while Marciano and Sirlin (48) examine the electron self-energy. Moreoever, mass singularities associated with coupled massless fields (sec. 1.4) are also controlled by the dimensional regularization (49,50).

Phase space integrals over the momenta of soft real quanta or, in general, over degenerate states (sec. 1.4), are also divergent at d=4. Hence, the integral over the element $(2\pi)^{-3} 2|k|^{-1} d^3\underset{\sim}{k}$ for energies below a given resolution ΔE must be continued to d dimensions by the expression:

$$\int \frac{d^d k}{(2\pi)^d}\ \delta(k^2)\,\theta(k^0)\,\theta(\Delta E - k^0) \tag{2.2}$$

After momentum integrals are carried out, the IR singularities show up in the Feynman parameter integrations. These are also defined by analytic continuation from their domain of convergence, a typical integral is defined as:

$$\int_0^1 dx\ x^{m-\beta-1} = \frac{1}{m-\beta} \tag{2.3}$$

There are, however, apparent difficulties with the use of dimensional regularization for both UV and IR divergences. The regions of the d-plane where Feynman integrals are UV or IR convergent are disjoint and there does not exist a domain from which the analytic continuation can be done. In practice, there seems to be no real difficulty, the regularized integrals split up in terms

which are either IR or UV singular, but not both. A consistent prescription is to start by setting $d < 4$ in off-shell (IR finite) Green's functions to make them UV convergent and continue to $d > 4$ after renormalization has been carried out.

Finally, we remark that the dimensions of fields and couplings of the theory depend on d. In order that renormalized quantities have the correct dimensionality, a mass scale μ must be introduced. This observation is the basis of 't Hooft's formulation of the renormalization group (39,40,41). The bare coupling g_B is thus related to the dimensionless renormalized coupling g_R as:

$$g_B = \mu^{d+4} Z g_R \tag{2.4}$$

where the renormalization constant Z is an infinite series in $\frac{1}{d-4}$.

2.3 Vacuum Polarization

The dimensional regularization enables one to carry out calculations setting the electron mass equal to zero from the start. Before considering the dimensionally regularized expressions, however, we would like to examine the behaviour as $m \to 0$ of the lowest order vacuum polarization graph in the massive theory, in order to have a better understanding of the origin of the mass singularity.*

Using dimensional continuation to control the UV divergence, but keeping $m \neq 0$, the unrenormalized vacuum polarization tensor is:

$$\Pi_{\mu\nu}(k) = -\int \frac{d^{4+\epsilon}p}{(2\pi)^{4+\epsilon}} \frac{Tr\{\gamma_\mu(\not{p}+m)\gamma_\nu(\not{p}-\not{k}+m)\}}{(p^2-m^2+i\epsilon)[(p-k)^2-m^2+i\epsilon]} \tag{2.5}$$

---

* We follow in part a discussion of K. Wilson (35).

where $\varepsilon=d-4<0$. After Wick-rotating to Euclidean space, the integral can diverge through vanishing denominators. However, the divergence comes about only if the integrand behaves as p^{-4} or $(p-k)^{-4}$ when $p\to 0$ or $(p-k)\to 0$, which is not the case. The mass singularity (a term $\sim \ln m$) is introduced by the renormalization procedure. The on-shell-renormalized function can be written as:

$$\Pi_{\mu\nu}^{ren}(k) = \Pi_{\mu\nu}(k) - \Pi_{\mu\nu}(0) - \tfrac{1}{2} k^{\alpha} k^{\beta} \left.\frac{\partial^2 \Pi_{\mu\nu}(k)}{\partial k^{\alpha} \partial k^{\beta}}\right|_{k=0} \tag{2.6}$$

The singularity as $m\to 0$ lies in the second derivative term, each differentiation with respect to k increases the power of the denominator by one, so that after differentiating twice and setting $k=0$, the integrand diverges as p^{-4} when $m=0$ and $p\to 0$. This situation is similar to that of the IR singularity in the electron propagator. The unrenormalized electron self-energy is IR finite, but the on-shell subtraction $(\not{p}-m)\left.\frac{\partial \Sigma}{\partial p^{\mu}}\right|_{\not{p}=m}$ is proportional to an IR divergent integral.

Performing the integral (2.5) one finds the transverse form:

$$\Pi_{\mu\nu}(k) = (k_{\mu}k_{\nu} - g_{\mu\nu}k^2)\,\Pi(k^2) \tag{2.7}$$

and it is seen that the second derivative term in (2.6) is just $\Pi(0)$.

It is interesting that it was, in fact, the realization that unrenormalized amplitudes have no mass singularity that prompted Gell-Mann and Low to devise a renormalization procedure that would introduce no such $m\to 0$ singularities and which led to their formulation of the renormalization group (31). Thus, it can be seen that the expression:

$$\Pi^{Ren}(k^2) = \Pi(k^2) - \Pi(-\mu^2) \tag{2.8}$$

for some momentum μ^2, contains no mass singularity, except at $k^2=0$. As a somewhat reversed situation, the on-shell-renormalized expression has no mass singularity only at $k^2=0$.

Setting m=0 from the outset and employing dimensional regularization, one finds for the vacuum polarization tensor the form (2.7) where:

$$\Pi(k^2) = -4\,\frac{i\alpha}{\pi}\;\frac{\Gamma(d/2)^2}{\Gamma(d)}\,\Gamma(3-d/2)\,\frac{1}{d-4}\left(-\frac{k^2}{\mu^2}\right)^{\frac{d}{2}-2} = \tag{2.9}$$

$$= \frac{S(d)}{d-4}\left(-\frac{k^2}{\mu^2}\right)^{\frac{d}{2}-2}$$

exhibiting the ultraviolet pole. Renormalizing, we get rid of the pole at d=4, on subtraction at the Euclidean point $k^2=-\mu^2$:

$$\Pi^{Ren}(k^2) = \frac{S(d)}{d-4}\left[\left(-\frac{k^2}{\mu^2}\right)^{\frac{d}{2}-2} - 1\right] \tag{2.10}$$

Putting k^2 on the mass-shell we are left back with a pole:

$$\Pi^{Ren}(0) = -\frac{S(4)}{d-4} + \text{finite terms} \tag{2.11}$$

It is clear that this is a mass singularity, just that appearing at $k^2=0$ after off-shell subtraction, as discussed above. Evidently, the limits $d\to4$, $k^2\to0$ are not interchangeable, but if (2.10) is a

self-energy correction to an external photon line, the limit $d \to 4$ is prescribed to be taken last. On the other hand, if we subtract on the mass-shell (i.e. at $k^2=0$, assuming the continuation to $d > 4$) in fact nothing is subtracted. For consistency with the analysis for $m \neq 0$ above, what is left must be regarded as the mass singularity. For the mass-shell renormalization constant, we then have $Z_3=1$ which may be interpreted as reflecting the cancellation between ultraviolet and mass singularity poles. These peculiar mass-shell renormalization constants have been discussed by Cvitanovič (52',53').

In our Compton scattering calculation, the pole (2.11) will be needed to cancel that one in the real corrections due to degenerate states containing electron-positron pairs with parallel three momentum. In fact, this qualification is made by Lee and Nauenberg (28), finiteness is guaranteed only avoiding spurious singularities introduced by on-shell renormalization. In this case, this spurious singularity would be manifested as the lack of the pole (2.11).

2.4 Compton Scattering of Massless Electrons (51)

Our expansion parameter $\alpha=e^2/4\pi$ will be defined by the off-shell renormalization employed in section 2.3, and therefore it is not the usual fine structure constant defined by the Thompson limit (20). The IR stability means that there exists a low momentum renormalization scale such that the coupling is weak. The object to be calculated is the inclusive differential cross-section for scattering with zero and one soft photons (energy less than a given resolution ΔE) in the final state as well as one hard photon or one electron-positron pair emitted (absorbed) nearly parallel to the final (initial) electron (Figs. 3a,b).

The lowest order centre-of-mass elastic differential cross-

section, calculated from the Born graphs (Fig. 1), is:

$$\left(\frac{d\sigma}{d\Omega}\right)_0 = \alpha^2 s^{-1}\left(\cos^2\theta/2 + \frac{1}{\cos^2\theta/2}\right) \tag{2.12}$$

with θ the centre-of-mass scattering angle and the standard notation (Fig. 1):

$$s = (p+q)^2 \ , \quad t = (p-p')^2 \ , \quad u = (p-q')^2 \tag{2.13}$$

with s+t+u=0. Eq. (2.12) can be written invariantly by setting $-\cos^2\theta/2 = \frac{u}{s}$. In the computations, the Dirac equation $\not{p}u(p)=0$ is used and normalization factors like $\sqrt{\frac{E}{m}}$ which involve the electron mass always cancel in the cross-section. Only s-channel diagrams are shown in Figs. 2, 3, the contribution of the crossed channel graphs is obtained simply by setting $s \leftrightarrow u$.

The renormalization mass μ is introduced in the manner of 't Hooft (39) to keep α dimensionless even at $d \neq 4$, as explained in section 2.2. Even the lowest order emission graphs then acquire a μ dependence which keeps dimensions right. For convenience we work in the Landau gauge, i.e. $\lambda=1$ in the covariant propagator $-\frac{i}{k^2}\left(g_{\mu\nu} - \lambda\frac{k_\mu k_\nu}{k^2}\right)$. The electron self-energy for general λ turns out to be:

$$\sum(p) = (\lambda-1)\frac{\alpha}{2\pi}\pi^{\frac{d}{2}-2}\Gamma\left(2-\frac{d}{2}\right)\frac{\Gamma^2\left(\frac{d}{2}\right)}{\Gamma(d-1)}\not{p}\,(-p^2)^{\frac{d}{2}-2} \tag{2.14}$$

so that in the Landau gauge it is identically zero. For consistency with the Ward identity, no renormalization of the vertex diagram

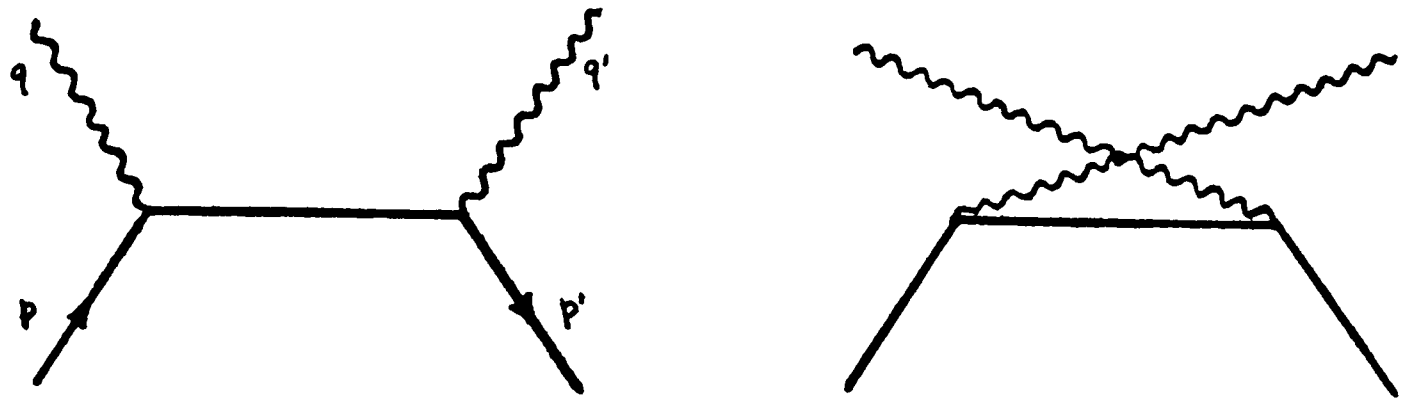
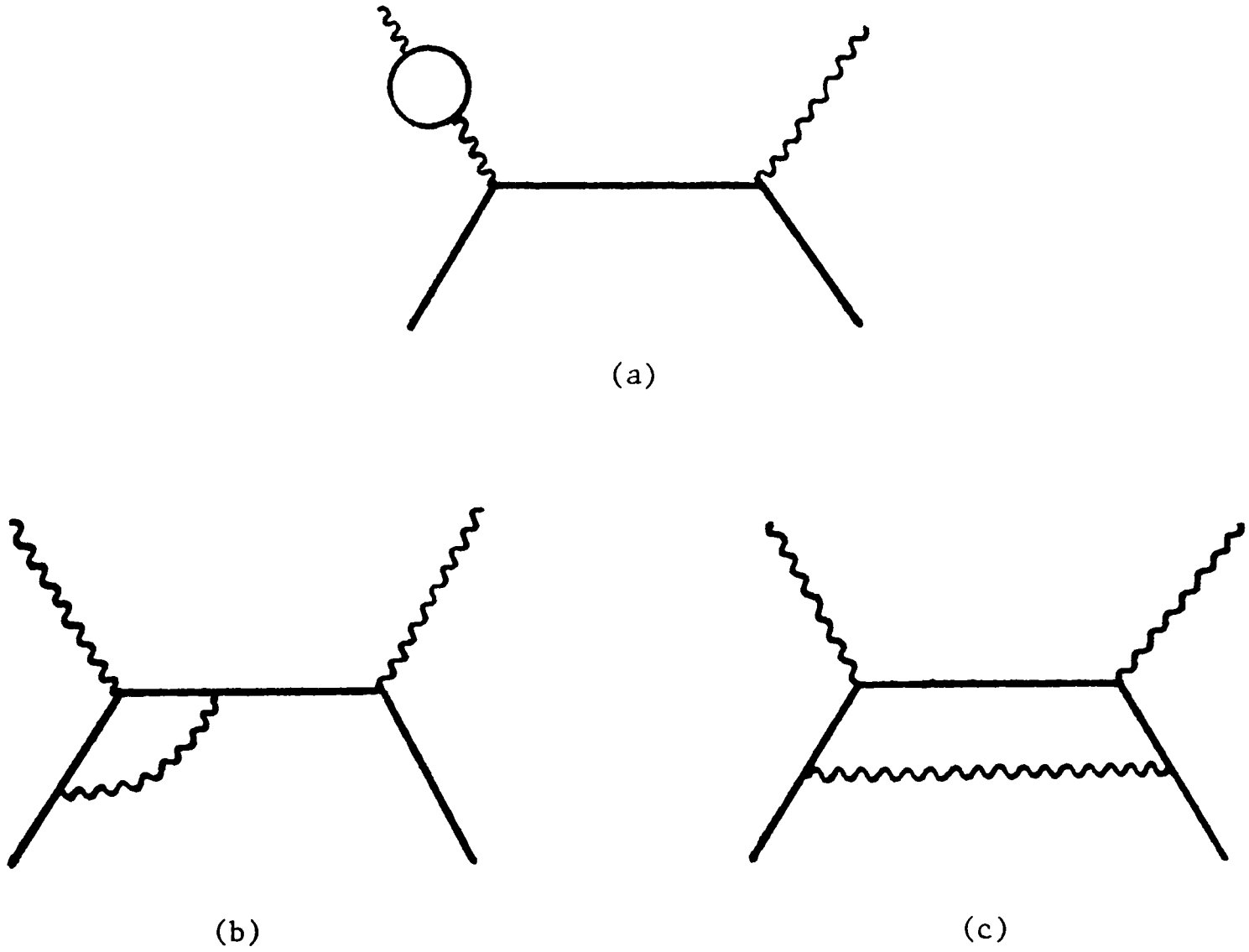


Fig. 1. Lowest order Compton scattering



(a)

(b)

(c)

Fig. 2. Virtual corrections relevant in the Landau gauge (only s-channel diagrams shown).

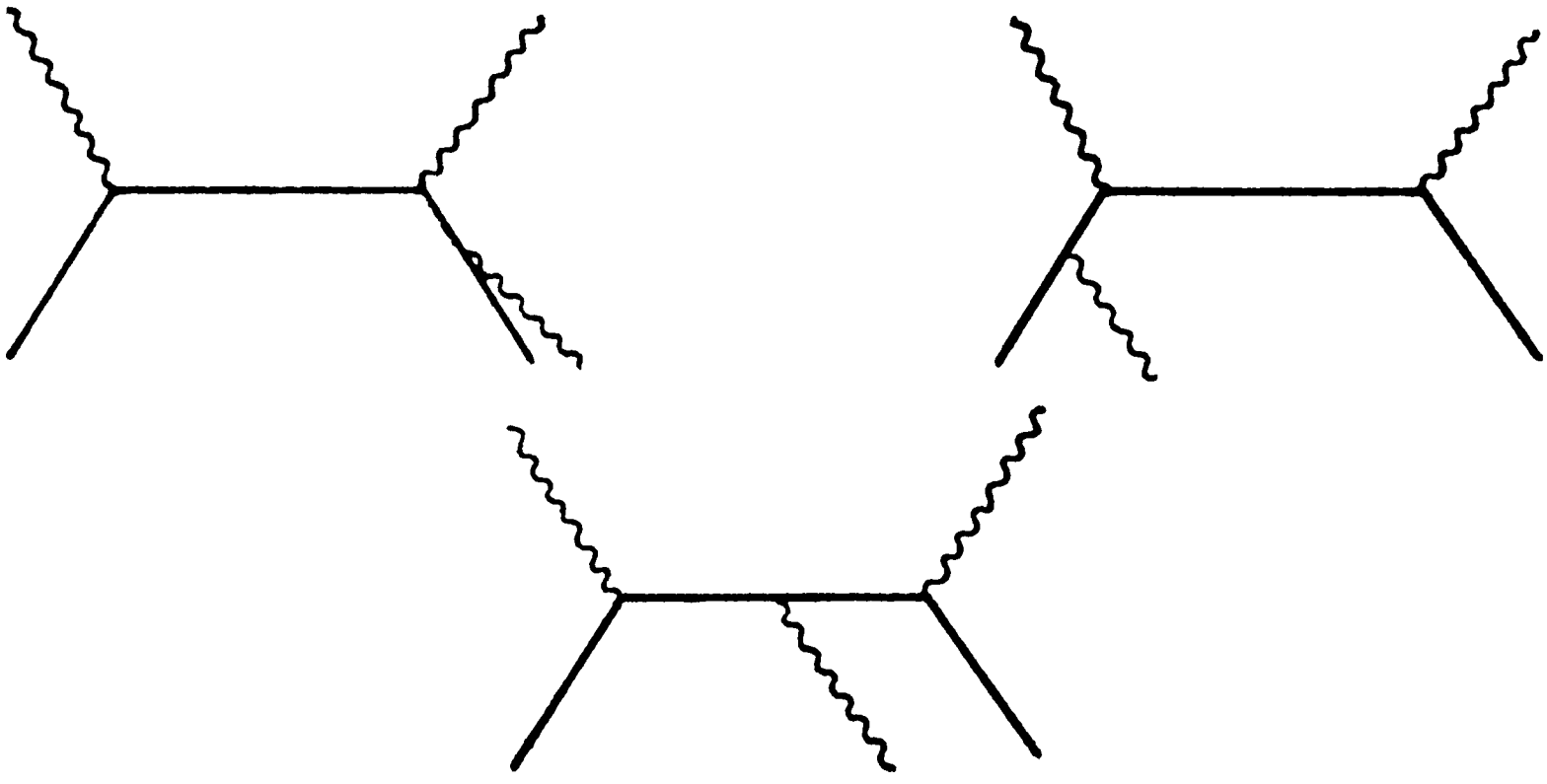


Fig. 3a. Bremsstrahlung

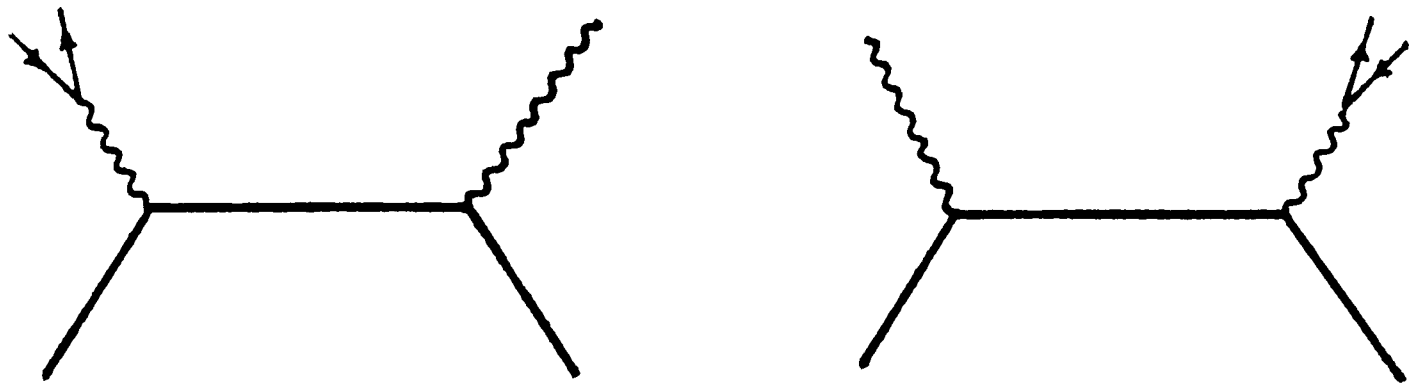


Fig. 3b. Electron-positron pairs in the initial and final states.

(Fig. 2b) is carried out. Hence, apart from the vacuum polarization correction described in section 2.3, the only virtual contributions are the vertex corrections (Fig. 2b) and the box diagram (Fig. 2c). The latter gives rise to double poles due to singular energy and angular integrations in the region where the virtual photon going across the diagram is both soft and parallel to either the incoming or outgoing electron. We sketch in Appendix A-2 the calculation of the box diagram contribution, which can be evaluated exactly in terms of hypergeometric functions. We also present in Appendix A-1 some formulae for derivatives of hypergeometric functions with respect to their parameters which we have derived in order to extract single poles associated to the double ones.* The pole parts from the box diagram contribution are, with $\varepsilon=d-4$:

$$\left(\frac{d\sigma}{d\Omega}\right)^{div}_{BOX} = B_0\left\{-\frac{1}{\epsilon^2}-\frac{1}{\epsilon}\left[\frac{1}{4}+\ln\frac{\sqrt{-t}}{\mu}\right]\right\}-4\pi\alpha^3 s^{-1}\frac{1}{\epsilon} \qquad (2.15)$$

where

$$B_0 = \frac{4\alpha}{\pi}\left(\frac{d\sigma}{d\Omega}\right)_0$$

Thus we see that, contrary to what happens in the massive case, not all the divergence is proportional to the Born cross-section. The single (non-leading) pole is partly IR (arising from the region

---

* The double poles appear multiplied by functions of d, so that:

$$\frac{f(d)}{(d-4)^2} = \frac{f(4)}{(d-4)^2} + \frac{f'(4)}{d-4} + \text{finite terms}$$

where the virtual photon momentum k is soft but not parallel to p or p') and partly due to the mass singularity (where k is not soft but $\underset{\sim}{k}||\underset{\sim}{p}$ or $\underset{\sim}{k}||\underset{\sim}{p}'$).

The vertex graph gives no pole proportional to $(\frac{d\sigma}{d\Omega})_0$ in this gauge, it only contributes a piece:

$$\left(\frac{d\sigma}{d\Omega}\right)_v^{div} = 8\pi\alpha^3 s^{-1} \frac{1}{\epsilon} \qquad (2.16)$$

According to the Kinoshita-Lee-Nauenberg theorem (sec. 1.4), to get a finite cross-section real corrections must be included, in order to construct the inclusive cross-section for all possible degenerate initial and final states. Soft photon emission (Fig. 3a) gives a contribution:

$$\left(\frac{d\sigma}{d\Omega}\right)_{SB}^{div} = B_0 \left\{ \frac{1}{\epsilon^2} + \frac{1}{\epsilon}\left[1 + \frac{1}{2}\ln\left(-\frac{4t}{s}\right) + \ln\frac{\Delta E}{\mu}\right]\right\} - 4\pi\alpha^3 s^{-1}\frac{1}{\epsilon} \qquad (2.17)$$

An energy resolution parameter ΔE defines the separation between the soft and hard momentum regions. Likewise, as described in section 1.4, a small angle δ corresponds to the angular resolution defining the "narrow" region in which two momenta are nearly light-like and parallel. In our example, hard bremsstrahlung photons parallel to the incoming or outgoing electrons are in this region and will give (single) pole contributions. The cross-section for such hard photons in the initial and final states is found to be:*

$$\left(\frac{d\sigma}{d\Omega}\right)_{HB}^{div} = B_0 \left\{ \frac{1}{\epsilon}\left[-\frac{3}{4} + \frac{1}{2}\ln\frac{s}{4\mu^2} - \ln\frac{\Delta E}{\mu}\right]\right\} \qquad (2.18)$$

---

* The contribution of <u>initial</u> degenerate states will be discussed in more detail in Chapter 3, for the case of gluon scattering.

Finally, to include all degenerate initial and final states, one must consider electron-positron pairs with parallel three-momentum (Fig. 3b), which adds:

$$\left(\frac{d\sigma}{d\Omega}\right)^{div}_{e^+e^-} = B_0 \left\{ \frac{1}{3} \frac{1}{\epsilon} \right\} \tag{2.19}$$

This pole is cancelled by the mass singularity in the vacuum polarization correction to the elastic amplitude, as pointed out in section 2.3. As a sample computation, we present in Appendix A-3 some details of the calculation of the narrow angle electron-positron production cross-section.

The pieces non-proportional to the Born cross-section in (2.15), (2.16) and (2.17) cancel off. As expected from the KLN theorem, the double and single poles in the box (2.15) are seen to compensate those in the soft and hard real corrections, (2.17) and (2.18). Thus we are left with a finite cross-section of the form:

$$\left(\frac{d\sigma}{d\Omega}\right) = \left(\frac{d\sigma}{d\Omega}\right)_0 \left\{ 1 + \frac{4\alpha}{\pi} \left[\ln \frac{\Delta E}{E} \left(\ln \sin \frac{\theta}{2} - \ln \delta \right) - \frac{5}{12} \ln \delta + \frac{1}{3} \ln \frac{E}{\mu} + f(\theta) \right] \right\} \tag{2.20}$$

where δ is the resolution angle between electron-photon in a state degenerate with a single electron or between electron-positron in a state degenerate with a single photon; the $\ln\mu$ term comes from charge renormalization and $f(\theta)$ is a function of the scattering angle only. One has the usual feature (sec. 1.3) that $(\frac{d\sigma}{d\Omega}) \to -\infty$ as $\Delta E \to 0$, to this order in perturbation theory, since the coefficient of $\ln\Delta E$ is positive ($\theta/2 > \delta$).

In view of the delicate cancellation involved, one should demonstrate that the result (2.20) is exactly the same obtained by the alternative procedure of starting with a non-zero mass. Here we shall partly rely on Ref. (49), where scattering of a massless electron in an external potential is considered, and in which the equivalence with the non-zero mass procedure has been checked. In fact, in that calculation, part of the ΔE and θ dependence is exactly the same and has the same origin as in ours, namely, the $\ln \frac{\Delta E}{E} \ln \sin\theta/2$ term, coming from the soft bremsstrahlung. As a further check we now consider the contribution from the vacuum polarization and electron-positron emission graphs.

In the massive theory, the unrenormalized vacuum polarization has the transverse form (2.7) where:

$$\Pi(k^2,m^2) = 4\frac{i\alpha}{\pi}\mu^{\epsilon}\frac{1}{\epsilon}\int_0^1 dx\,(1-x)x\left[k^2x(1-x)-m\right]^{\epsilon/2} \qquad (2.21)$$

Subtracting at $k^2=-\mu^2$ and going on-shell gives, in the limit $m\to 0$:

$$\Pi^{Ren}(0,m^2) = \frac{2}{3}\frac{i\alpha}{\pi}\frac{1}{\epsilon}\left[\left(\frac{m^2}{\mu^2}\right)^{\epsilon} - 6\,\frac{\Gamma(2+\epsilon/2)}{\Gamma(4+\epsilon)}\right] \qquad (2.22)$$

which is readily seen to give the same finite part and singular dependence as obtained from (2.11), with the substitution $\frac{1}{\epsilon} \to -\ln\frac{m}{\mu}$. In the real corrections due to electron-position creation, it is also easily found that the same $\ln E$, $\ln\delta$ and other finite contributions arise when the angular integration is performed with the correspondence:

$$\int_0^{\delta} \theta^{d-5} d\theta \longrightarrow \frac{1}{2} \int_0^{\delta} \frac{d\theta^2}{\theta^2 + \frac{m^2}{E^2}} \tag{2.23}$$

In the following appendices we present some details of the calculations leading to the results given above; the box diagram is considered in Appendix A-2 and e^+e^- emission in Appendix A-3.

Appendix A-1

SOME HYPERGEOMETRIC FUNCTION FORMULAE

To deal with the double poles we have derived the formulae:

$$\frac{d}{d\epsilon}\,{}_2F_1(1,\epsilon;a(\epsilon);z)\Big|_{\epsilon=0} = \frac{z}{a(0)}\,{}_2F_1(1,1;a(0)+1;z) \qquad (2.24a)$$

$$\frac{d}{d\epsilon}\,{}_2F_1(1,b;b+\epsilon;z)\Big|_{\epsilon=0} = \frac{z}{b(z-1)}\,{}_2F_1(1,b;b+1;z) \qquad (2.24b)$$

In the box diagram calculation we have repeatedly used (51',52):

$$\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}(1-\alpha x)^{\nu-1} = B(\lambda,\mu)\,{}_2F_1(\nu,\lambda;\lambda+\mu;\alpha) \qquad (2.25)$$

$$\mathrm{Re}\,\lambda>0\,,\ \mathrm{Re}\,\mu>0\,,\ |\alpha|<1$$

$$\int_0^1 dx\ x^{a-1}(1-x)^{b-1} = B(a,b) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)} \qquad (2.26)$$

and:

$$\begin{aligned} {}_2F_1(-a,b;b;-z) &= (1+z)^a \\ {}_2F_1(1,1;2;-z) &= \frac{1}{z}\ln(1+z) \end{aligned} \qquad (2.27)$$

Using the contiguity relations one can derive (52):

$$\begin{aligned} {}_2F_1(1,1;3;z) &= \frac{2}{z} + 2\frac{(1-z)}{z^2}\ln(1-z) \\ {}_2F_1(1,2;3;z) &= -\frac{2}{z} - \frac{2}{z^2}\ln(1-z) \\ {}_2F_1(1,3;4;z) &= -\frac{3}{2}\frac{1}{z} - \frac{3}{z^2} - \frac{3}{z^3}\ln(1-z) \\ {}_2F_1(1,1;4;z) &= \frac{3}{2}\frac{1}{z} - \frac{3}{2}\frac{(1-z)}{z}\,{}_2F_1(1,1;3;z) \end{aligned} \qquad (2.28)$$

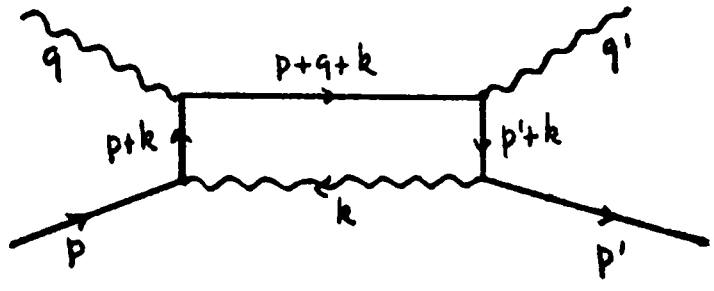
Gamma functions frequently appear in the combination $\frac{\Gamma(1+\varepsilon/2)^2}{\Gamma(1+\varepsilon)}$ and:

$$\frac{d}{d\epsilon}\left.\frac{\Gamma(1+\epsilon/2)^2}{\Gamma(1+\epsilon)}\right|_{\epsilon=0} = 0 \tag{2.29}$$

Appendix A-2

THE BOX DIAGRAM

We consider the s-channel diagram:



The contribution from the crossed channel is obtained directly from the substitution $s \leftrightarrow u$. Combining denominators by means of the Feynman parametrization:

$$\frac{1}{a_1 a_2 a_3 a_4} = 3! \int dx\,dy\,dz\,dw \; \frac{\delta(x+y+z+w-1)}{(a_1 x + a_2 y + a_3 z + a_4 w)} \tag{2.30}$$

dropping the IR finite k^2 terms in the numerator, shifting the integration variable and using the dimensional regularization formula (2.1) we obtain:

$$(-ie)^4 i^2 \frac{i\pi^{d/2}}{(2\pi)^4} \frac{\pi^{2-d/2}}{\Gamma(3-\frac{d}{2})} \frac{\Gamma(4-\frac{d}{2})}{\Gamma(4)} 6 \cdot \int \frac{dx\,dy\,dz\,dw\;\delta(x+y+z+w-1)}{[-(syw+txz)]^{4-\frac{d}{2}}} \tag{2.31}$$

$$\times N(p,p',q,q',x,y,z) \quad + \; G$$

where

$$N(p,p',q,q',x,y,z) = \gamma^\alpha (\not{p}' - \not{\alpha}) \not{\varepsilon}(q') (\not{p} + \not{q} - \not{\alpha}) \not{\varepsilon}(q) (\not{p} - \not{\alpha}) \gamma_\alpha$$

with

$$a = p'(x+y) + q'y + pz$$

The expression (2.31) is meant to be sandwiched between spinors $u(p)$, $\bar{u}(p')$ obeying $\not{p}u(p)=0=\bar{u}(p')\not{p}'$. We have attached a factor $\pi^{2-\frac{d}{2}}\Gamma(3-\frac{d}{2})^{-1}$ to the dimensional regularization formula (2.1). This analytical continuation avoids terms like γ and $\ln\pi$ which eventually cancel.

The term G in (2.31) stands for the contribution from the longitudinal term in the Landau gauge propagator. The momentum integral corresponding to these terms has the form:

$$\int d^dk \; \frac{\bar{u}(p')\, \not{k}\, (\not{p}'+\not{k})\, \not{q}(q')\, (\not{p}+\not{q}+\not{k})\, \not{q}(q)\, (\not{p}+\not{k})\, \not{k}\, u(p)}{k^4\,(p+q+k)^2\,(p+k)^2\,(p'+k)^2} \tag{2.32}$$

This expression is greatly simplified by noting that because of the Dirac equation we can write $\not{k} \to (\not{p}+\not{k})$ and $\not{k} \to (\not{p}'+\not{k})$ for the factors in the extreme right and left. Now $(\not{p}'+\not{k})(\not{p}'+\not{k})=(p'+k)^2$ and $(\not{p}+\not{k})(\not{p}+\not{k})=(p+k)^2$ so that two factors in the denominator are cancelled. The calculation of this term is then fairly simple, and we do not discuss it further.

The poles will result from parametric integrations of the type:

$$I(a,b,c) = \int dx\,dy\,dz\,dw\; \delta(x+y+z+w-1)\; \frac{x^a y^b z^c}{[syw+txz]^{4-\frac{d}{2}}} \tag{2.33}$$

It is convenient to make the change of variables:

$$x = uv$$
$$y = u(1-v) \quad , \quad x+y = u \quad , \quad \frac{\partial(x,y)}{\partial(u,v)} = u$$

under which (2.33) transforms to:

$$I(a,b,c) = \int_0^1 du\,dv\, u \int_0^{1-u} dz\; u^{\frac{d}{2}-3+a+b} v^a (1-v)^b z^c$$
$$\times \left[z(1-\alpha)v - (1-v)(1-u) \right]^{\frac{d}{2}-4} S^{\frac{d}{2}-4} \qquad (2.34)$$

where
$$\alpha = \frac{s+t}{s}$$

The z integration can be done by parts and the remaining integrals will yield a pole structure which depends on the powers of a, b and c.

As an example we consider I(a,b,1), which after the z-integration gives:

$$I(a,b,1) = \frac{1}{(\frac{d}{2}-2)} \int_0^1 du\,dv\, u^{\frac{d}{2}-3+a+b} (1-u)^{\frac{d}{2}-2} v^a (1-v)^b \frac{1}{(1-\alpha v)^2} \left\{ \left[(1-\alpha)v\right]^{\frac{d}{2}-2} - \left[-(1-v)\right]^{\frac{d}{2}-2} \right\}$$

$$+ \frac{1}{(\frac{d}{2}-3)} \int_0^1 du\,dv\, u^{\frac{d}{2}-3+a+b} (1-u)^{\frac{d}{2}-2} v^a (1-v)^{b+1} \frac{1}{(1-\alpha v)^2} \left\{ \left[(1-\alpha)v\right]^{\frac{d}{2}-3} - \left[-(1-v)\right]^{\frac{d}{2}-3} \right\}$$

$$= A_1(d) + A_2(d)$$

where

$$A_1(d) = \frac{1}{(\frac{d}{2}-2)} B(\tfrac{d}{2}-2+a+b, \tfrac{d}{2}-1) \left\{ (1-\alpha)^{\frac{d}{2}-2} B(a+\tfrac{d}{2}-1, b+1)\, {}_2F_1(2, a+\tfrac{d}{2}-1; a+b+\tfrac{d}{2}; \alpha) \right.$$
$$\left. - (-)^{\frac{d}{2}} B(a+1; b+\tfrac{d}{2}-1)\, {}_2F_1(2, a+1; a+b+\tfrac{d}{2}; \alpha) \right\}$$

$$A_2(d) = \frac{1}{(\frac{d}{2}-3)} B(\tfrac{d}{2}-2+a+b, \tfrac{d}{2}-1) \left\{ (1-\alpha)^{\frac{d}{2}-3} B(a+\tfrac{d}{2}-2, b+2)\, {}_2F_1(2, a+\tfrac{d}{2}-2; a+b+\tfrac{d}{2}; \alpha) \right.$$
$$\left. + (-)^{\frac{d}{2}} B(a+1, b+\tfrac{d}{2}-1)\, {}_2F_1(2, a+1; a+b+\tfrac{d}{2}; \alpha) \right\}$$

where B(a,b) is the beta function and ${}_2F_1(a,b;c;z)$ is the Gauss' hypergeometric function. It can be seen that $A_1(d)$ is singular at d=4 (single pole) only if a=b=0, whereas $A_2(d)$ has a double pole for a=b=0 and a single pole for a=0 and any b. Analyzing each I(a,b,c) in a similar way we find that the double poles are contained in I(a,0,0), I(0,b,0) and I(0,0,c). That the leading singularities arise in such terms can perhaps be ascertained by looking at the initial expression (2.33). However, it is necessary to calculate the complete expressions for I(a,b,c), since the differentiation of the coefficients of the double poles with respect to d will yield the non-leading single poles. Of some help in this task is the symmetry:

$$I(a,b,c) = I(c,b,a) \tag{2.35}$$

As an illustration, we compute the double and single poles of a typical term:

$$I(1,0,0) = \frac{1}{\epsilon} A_1(\epsilon) + \frac{1}{\epsilon^2} A_2(\epsilon)$$

where ε=d-4 and:

$$A_1(\epsilon) = -2\, S^{-2+\epsilon/2}\, \frac{\Gamma(1+\frac{\epsilon}{2})^2}{\Gamma(1+\epsilon)}\, \frac{1}{(1+\epsilon/2)}\, (1-\alpha)^{-1+\frac{\epsilon}{2}}\, {}_2F_1(1,1+\tfrac{\epsilon}{2};2+\tfrac{\epsilon}{2};\alpha)$$

$$A_2(\epsilon) = -4(-S)^{-2+\frac{\epsilon}{2}}\, \frac{\Gamma(1+\epsilon/2)^2}{\Gamma(1+\epsilon)}\, \frac{1}{(1+\epsilon/2)}\, {}_2F_1(1,2;2+\tfrac{\epsilon}{2};\alpha)$$

Using the formula (Appendix A-1):

$$\frac{d}{d\epsilon}\,{}_2F_1(1,b;b+\epsilon;z)\Big|_{\epsilon=0} = \frac{z}{b(z-1)}\,{}_2F_1(1,b;b+1;z)$$

we find:

$$\frac{d}{d\epsilon}A_1(\epsilon) = -2s^{-2}\ln(-s)\,{}_2F_1(1,2;2;\alpha) + 2s^{-2}(1-\alpha)^{-1}$$

$$+ 2s^{-2}\frac{\alpha}{1-\alpha}\,{}_2F_1(1,2;3;\alpha)$$

We also have:

$${}_2F_1(1,a;a;\alpha) = \frac{1}{1-\alpha}$$

$${}_2F_1(1,2;3;\alpha) = -\frac{2}{\alpha} - \frac{2}{\alpha^2}\ln(1-\alpha)$$

so that the total residue of the single pole is:

$$R^{(1)}(1,0,0) = A_1(0) + \frac{d}{d\epsilon}A_2(\epsilon)\Big|_{\epsilon=0} = \frac{2}{st}\ln(-s)$$

and $$R^{(2)}(1,0,0) = A_2(0) = \frac{4}{st}$$

One must also keep in mind that apart from the "pure" terms (only one of a,b,c, non-zero), there are "mixed" terms (two of a,b,c, non-zero) giving single poles. Moreover, in carrying out traces to give the cross-section, there appear terms proportional to d-4 which give additional single poles from the "pure" terms.

It is not possible to convert the numerator in (2.31) into a simple form proportional to the Born amplitude, so that it is necessary to construct the cross-section and calculate traces after summing over photon polarizations covariantly (omitting terms proportional to k_μ or k_ν which vanish by the Ward identities):

$$\sum_{\epsilon} \epsilon_\mu^* \epsilon_\nu = - g_{\mu\nu} \tag{2.36}$$

Then one can identify the terms proportional to $(\frac{d\sigma}{d\Omega})_0$.

The quantity of interest is therefore 2 $M_{BORN} M^*_{BOX}$ where:

$$M_{BORN} = (-ie)^2 i \left\{ \frac{1}{s} \bar{u}(p') \not{\epsilon}' (\not{p}+\not{q}) \not{\epsilon} u(p) + \frac{1}{u} \bar{u}(p') \not{\epsilon} (\not{p}'-\not{q}') \not{\epsilon}' u(p) \right\} \tag{2.37}$$

The resulting traces are carried out using the usual theorems and the dimensional regularization formulae:

$$\gamma^\mu \not{a} \not{b} \gamma_\mu = 4\, a\cdot b + (d-4)\, \not{a} \not{b}$$

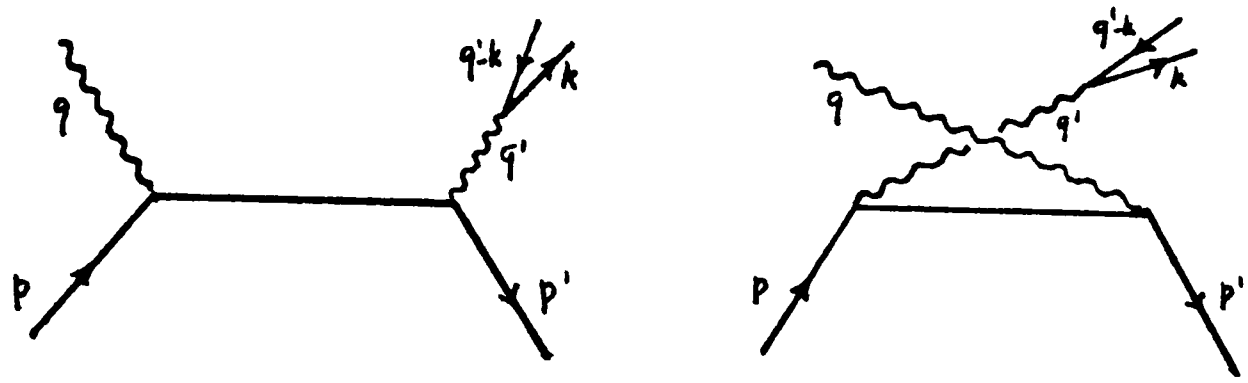
$$\gamma^\mu \not{a} \not{b} \not{c} \gamma_\mu = -2 \not{c} \not{b} \not{a} + (4-d)\, \not{a} \not{b} \not{c} \tag{2.38}$$

After a lengthy calculation, all the different sorts of I(a,b,c) terms are collected in order to compute the residues of double and single poles. This finally leads to the result (2.15).

Appendix A-3

NARROW ANGLE e^+e^- EMISSION

The relevant graphs for e^+e^- production from the outgoing photon are:



corresponding to the matrix element

$$M = (-ie)\,\bar{u}(k)\gamma_\mu v(q'-k)\,M^\mu \tag{2.39}$$

where

$$M^\mu = 4\pi\alpha\,\bar{u}(p')\left[\gamma_\nu(\not{p}+\not{q})\not{\epsilon}(q)\frac{1}{(p+q)^2} + \not{\epsilon}(q)(\not{p}-\not{q}')\gamma_\nu\frac{1}{(p'-q)^2}\right]u(p)$$

$$\times\frac{1}{q'^2}\left(g^{\mu\nu} - \frac{q'^\mu q'^\nu}{q'^2}\right) \tag{2.40}$$

Here the longitudinal terms in the propagator give a vanishing contribution. The (massive) electron and positron spinors satisfy:

$$\sum_s v(p)\,\bar{v}(p) = \frac{1}{2m}(\not{p}-m)$$

$$\sum_s u(p)\,\bar{u}(p) = \frac{1}{2m}(\not{p}+m) \tag{2.41}$$

We are interested in the case m=0, but the factors involving m will cancel in the cross-section as shown below. Squaring the matrix element and summing over fermion polarizations, we obtain:

$$\bar{M}^2 = -4\pi\alpha \frac{1}{4m^2} N^{\mu\nu} M_\mu M_\nu^* \frac{1}{q'^2} \tag{2.42}$$

where

$$\begin{aligned} N^{\mu\nu} &= T_\nu \gamma^\mu (\not{q}' - \not{k}) \gamma^\nu \not{k} \\ &= 4 (q'^\mu k^\nu + q'^\nu k^\mu - 2k^\mu k^\nu - g^{\mu\nu} q'\cdot k) \end{aligned} \tag{2.43}$$

Using the Ward identity

$$q'^\mu M_\mu = 0 \tag{2.44}$$

we find:

$$\bar{M}^2 = 4\pi\alpha \frac{1}{m^2} (2k\cdot M\, k\cdot M^* + q'\cdot k\, M_\mu^* M^\mu) \tag{2.45}$$

Now, $(q'-k)^2=0$, so that q' is slightly off-shell by an amount proportional to the angle θ between $\underset{\sim}{q}'$ and $\underset{\sim}{k}$. With the representation

$$\begin{aligned} q' &= \omega (1, \underset{\sim}{0}, 1 - \tfrac{1}{2}(\omega - k^0)^{-1} k^0 A^2) \\ k &= k^0 (1, A\underset{\sim}{u}, 1 - \tfrac{1}{2} A^2) \end{aligned} \tag{2.46}$$

where $\underset{\sim}{u}$ is a two-dimensional unit vector, we have:

$$q'^2 = 2q'\cdot k = \frac{k^0 \omega^2}{\omega - k^0} A^2 \tag{2.47}$$

and

$$k = \frac{k^0}{\omega} q' + k^0 \left(0 , A\underset{\sim}{u} , \tfrac{1}{2}(\omega - k^0)^{-1} \omega A^2\right) \tag{2.48}$$

Therefore, from the Ward identity (2.44):

$$k\cdot M\, k\cdot M^* = k^{0 2} A^2 u_i u_j M_i^* M_j + O(A^3) , \quad i,j = 1,2 \tag{2.49}$$

Averaging over directions we can substitute $u_i u_j \to \frac{1}{2}\delta_{ij}$. Also from (2.44) and the representation (2.46), one can show that $M_0^2 - M_3^2 = 0(\theta^2)$, so that:

$$k\cdot M\, k\cdot M^* = -\tfrac{1}{2} k^{0 2} A^2 M_\mu^* M^\mu + O(A^3) \tag{2.50}$$

The argument leading to (2.50) is essentially that used in section 3.2 for the case of hard gluon emission. The terms $0(\theta^3)$ do not give singular contributions. Thus, using (2.47), the singular part of the transition probability (2.45) can be written as:

$$\bar{M}^2 = \frac{4\pi\alpha}{m^2} \frac{1}{4q'\cdot k} \left(1 - 2\frac{k^0}{\omega} + 2\frac{k^{0 2}}{\omega^2}\right) M_\mu^* M^\mu \tag{2.51}$$

The factor $M_\mu^* M^\mu$ yields the Born cross-section, while the normalization factors for the fermions cancel the $\frac{1}{m^2}$ factor. Thus we have, with a dimensionally regularized phase space integral:

$$\left(\frac{d\sigma}{d\Omega}\right)_{e^+e^-}^{d\omega} = \left(\frac{d\sigma}{d\Omega}\right)_0 \frac{4\pi\alpha}{(2\pi)^3} \frac{4\omega}{} \int \frac{d^{d-1}k}{2k(\omega - k)} \frac{1}{4q'\cdot k} \left(1 - 2\frac{k_0}{\omega} + 2\frac{k_0^2}{\omega^2}\right) \tag{2.52}$$

where $k=|\underset{\sim}{k}|=k^0$, and, in the narrow region of interest, $\sin\theta\simeq\theta$. Again making use of (2.47), we can write:*

$$\left(\frac{d\sigma}{d\Omega}\right)^{div}_{e^+e^-} = \left(\frac{d\sigma}{d\Omega}\right)_0 \frac{4\pi\alpha}{\omega(2\pi)^3}\int_0^\omega k^{d-4}dk\,\left(1-2\frac{k}{\omega}+\frac{2k^2}{\omega^2}\right) \times 2\pi\int_0^\delta \theta^{d-5}d\theta \tag{2.53}$$

The angular integration gives a pole so that:

$$\left(\frac{d\sigma}{d\Omega}\right)^{div}_{e^+e^-} = \frac{2}{3}\,\frac{\alpha}{\pi}\,\frac{1}{d-4}\left(\frac{d\sigma}{d\Omega}\right)_0 + \text{finite terms} \tag{2.54}$$

With a factor of 2 due to the inclusion of electron-positron pairs in the initial state, we finally have the result (2.19).

One should actually introduce a mass scale μ in order to make the coupling constant dimensionless. Then the finite part of the cross-section is easily computed by expanding $\delta^{d-4} = 1+(d-u)\ln\delta+\ldots$ and similarly for $(\frac{\omega}{\mu})^{d-4}$. The result is:

$$\left(\frac{d\sigma}{d\Omega}\right)^{finite}_{e^+e^-} = \left(\frac{d\sigma}{d\Omega}\right)_0 \frac{2}{3}\,\frac{\alpha}{\pi}\left[\ln\delta+\ln\frac{\omega}{\mu}+f_1(\theta)\right] \tag{2.55}$$

where $f_1(\theta)$ is a function of θ only.

---

* Actually $\int d\Omega d = \frac{2\pi^{d/2}}{\Gamma(d/2)} = 2\pi f(d)$, but in extracting the finite part this factor only gives spurious terms like $\ln\pi$ and γ (Euler's constant). Thus we set directly $f(d)\to f(4)=1$.

Chapter 3

PURE YANG-MILLS THEORY

3.1 Introduction

As mentioned in earlier sections, recently several authors have computed perturbative corrections to processes involving on-shell coloured particles, in order to investigate the extension of the Bloch-Nordsiek analysis in non-Abelian gauge theories. Appelquist et al. (53) have considered $q\bar{q}$ (quark-antiquark) production by an external colourless source and demonstrated that the $O(g^4)$ transition rate corresponding to a finite-energy-resolution detector is free of IR singularities. Y. P. Yao (54) and Tyburski (55) have discussed qq scattering to lowest non-trivial order in perturbation theory and have shown that the transition probability summed and averaged over colour indices is finite if soft gluon emission is included. The average over colour states may be justified (55) since an indefinite number of soft gluons emitted by a fermion may carry indefinite colour so that the colour of the detected fermion is also indefinite.

The immediate conclusion derived from this investigation is therefore that for QCD processes involving massive quarks there are certain partial cross sections which are well defined order by order in perturbation theory. This is in exact analogy with the Bloch-Nordsieck result in QED, physical final state fermions are considered together with an indefinite number of soft gluons. There are, however, important aspects of the non-Abelian case which are left out in the analysis of these examples. The peculiarities associated with coupled massless fields are manifested in full only when initial and

final states involve (not necessarily soft) massless particles. This is because massive fermions have associated IR singularities only with emitted soft gluons whereas gluons can emit further soft and parallely movinghard gluons which produce the mass singularity peculiar to the Yang-Mills coupling (sec. 1.4). It is therefore of interest to consider the purely massless case, with its additional complications such as doubly logarithmic singularities per loop and degenerate final and initial states, corresponding to both soft and hard bremsstrahlung. In Yao's work (54) quark-gluon scattering in considered, but the complications are avoided by keeping the non-soft gluons off the mass-shell.

To provide an example of such massless process in this Chapter we discuss in detail Yang-Mills vector meson (gluon) scattering in an hypothetical external potential to which the gluon couples in a gauge invariant manner, computing all the relevant real and one-loop virtual corrections. We explicitly demonstrate that the inclusive transition rate is free of singularities and that the multigluon initial states are necessary for finiteness. To our knowledge, this is the only calculation of this type dealing with purely gluonic scattering and summation over degenerate states including the hard bremsstrahlung. Thus we treat both leading and non-leading singularities in the order considered.

The cancellation of singularities in suitable transition rates is of the general nature implied by the KLN theorem. One must note, however, the limitations of the theorem, whose validity has not been generally demonstrated in the non-Abelian theory. The Lee-Nauenberg (28) proof proceeds by formal arguments in non-relativistic perturbation theory. They assume a Hamiltonian formalism for which the

interaction is non-singular. There is a Hamiltonian formulation for gauge theories in the Coulomb gauge (83), but, even in QED, the Hamiltonian contains $\frac{1}{|\underset{\sim}{k}|^2}$ singularities.* The situation is more complicated in QCD, for which the interaction Hamiltonian in the Coulomb and axial gauges is more singular (82). Kinoshita's version (27) applies to <u>total</u> transition rates and not to partial rates corresponding to finite energy or angular resolution detectors.** Simplified proofs of a "weak" KLN theorem (finiteness of total cross-sections) that apply directly to Yang-Mills theory have been presented recently (Sterman, Poggio and Quinn (57)) on the basis of a power counting analysis. The straightforward extension of Kinoshita's method of cut graphs (27) to the Yang-Mills theory meets with the new feature that unitarity cuts of individual graphs do not have a direct interpretation in terms of intermediate physical states because of the presence of the Fadeev-Popov ghosts. In section 3.2 it is shown explicitly how, as a consequence of the Ward identities, summation over physical polarization states of real gluons give IR divergences in phase space integrals which correspond to the singularities from internal ghost loops. The hard narrow angle single gluon emission is then shown to factorize in the transition probability and compensate the mass singularity from self-energy corrections to external lines.

---

* For a discussion on this point, see Fleming and Frenkel (81).

** In particular, Kinoshita's theorem states that the 1PI two-point function is free of mass singularities for all non-zero values of the external momentum invariant q^2, a result used in discussions of the total e^+e^- annihilation cross-section (56').

In QED, the demonstration of IR finiteness relies on the systematics of factorization of the soft bremsstrahlung and of the divergences introduced by virtual corrections to external lines only (Sec. 1.3). In section 3.3 we examine this class of radiative corrections in the pure Yang-Mills theory,* finding that the contribution from soft virtual momentum in propagators linking two external lines does not factorize, as in the QED case, because of the more complicated tensor structure. However, the extra terms can be extracted by use of the Ward identities and are shown to compensate similar terms from soft real emission graphs.

In section 3.4 we discuss the gluon scattering process. Details of the calculations are given in the Appendices to this Chapter, a general covariant gauge is employed, and it is shown that cancellations occur graph by graph only in the Feynman gauge. Finally, in section 3.5, we discuss the physical significance of the results.

3.2 Hard Gluon Emission

Let us consider a general graph, represented by the shaded area in Fig. I, where one gluon with momentum k is emitted from an external line. Other external lines may be present, fermions included, but all are on-shell and vector lines have physical polarization vectors.

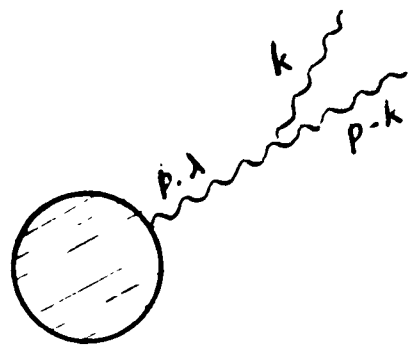


Fig. I

---

* Although we know that not only this kind of graphs are IR singular in the non-Abelian theory.

Let $M_\lambda(p,\ldots)$ represent the blob, where p,λ are the momentum and Lorentz index of the line shown, all other variables are suppressed. The 't Hooft-Ward identity tells us that

$$p^\lambda M_\lambda(p,\cdots) = 0 \tag{3.1}$$

even if p is off-shell. A real gluon can decay into two parallely moving gluons. If the two particles emerge with a narrow angle θ between them, p is slightly off-shell:

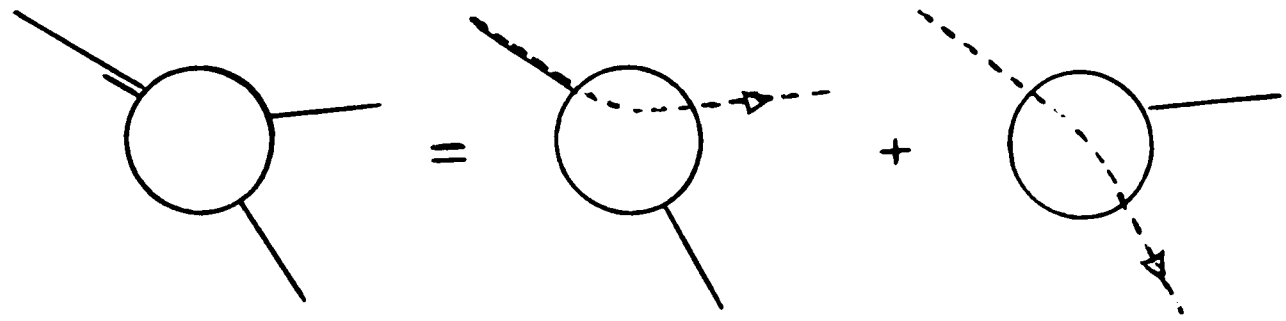
$$p^2 = 2p\cdot k = (E-\omega)^{-1} E^2 \omega\, \theta^2 \tag{3.2}$$

where we have adopted the representation:

$$p = E\left(1,\ \underset{\sim}{0},\ 1-\tfrac{1}{2}(E-\omega)^{-1}\omega\theta^2\right)$$
$$k = \omega\left(1,\ \theta\underset{\sim}{u},\ 1-\tfrac{1}{2}\theta^2\right) \tag{3.3}$$

to order θ^2, with θ the small angle between $\underset{\sim}{p}$ and $\underset{\sim}{k}$, $\omega = |\underset{\sim}{k}|$ and $\underset{\sim}{u}$ is a d-2 dimensional unit vector, with d the space-time dimension (we employ dimensional regularization throughout). The representation (3.3) satisfies (3.2) and $k^2 = (p-k)^2 = 0$.

In summing over polarization states of real particles, care is needed. One is not allowed to use a covariant Feynman spin sum because the Ward identities are more complicated than in QED. These involve, in general, amplitudes with open ghost lines, adding terms to the right hand side of eq. (3.1) if any of the external momenta are not transverse (58). In diagrammatic form, the 't Hooft-Ward identity is represented as:



(3.4)

for the case of two external lines which are on-shell but not transverse (with no polarization vector e such that e.p=0), the symbol ———═ means that the polarization vector e_λ is replaced by the corresponding momentum k_λ, while the arrow in --▷-- stands for multiplication by p_μ if p and μ are the momentum and Lorentz index of that line. For the trivial case of the bare Yang-Mills three vertex $V_{\lambda\mu\nu}(q,k,p)$ (all momenta going inwards):

$$q^\lambda V_{\lambda\mu\nu}(q,k,p) = k_\mu k_\nu - p_\mu p_\nu \tag{3.5}$$

if k and p are on-shell.

Now, in summing over the two physical polarization states one can choose a time-like unit vector n, obeying n.e=0 so that the polarization sum reads:*

$$\sum_e e_\mu(h)\, e_\nu(h) = -g_{\mu\nu} + \frac{1}{(k\cdot n)^2} k_\mu k_\nu + \frac{1}{k\cdot n}(k_\mu n_\nu + k_\nu n_\mu) \tag{3.6}$$

An $O(g^2)$ correction to the transition probability for the process represented by the basic blob is given by the square of the amplitude in Fig. I:

---

* One can have n space-like (axial gauge), but in that case k·n can vanish.

$$M_\lambda(p,\dots)\,M^*_{\lambda'}(p,\dots)\,V_{\lambda\mu\rho}(p,-p+k,-k)\,V_{\lambda'\mu'\rho'}(p,-p+h,-h)$$
$$\times\, e^\rho(h)\,e^{\rho'}(h)\,e^\mu(p-h)\,e^{\mu'}(p-h) \qquad (3.7)$$

Keeping in mind eq.(3.5) we can carry out the sum over $e_\mu(p-k)$ trivially because k is transverse. However, in summing over $e_\rho(k)$ now p-k is not transverse, and a term is left over:

$$M_\lambda(p,\dots)\,M^*_{\lambda'}(p,\dots)\,V_{\lambda\mu\rho}(p,-p+h,-h)\,V_{\lambda'\nu\rho}(p,-p+h,-h)\,k_\mu n_\nu \frac{1}{k\cdot n} \qquad (3.8)$$

Using (3.5) and (3.1), this gives:

$$2\,M_\lambda(p,\dots)\,M^*_{\lambda'}(p,\dots)\,(p-h)^\lambda k^{\lambda'} \qquad (3.9)$$

which formally corresponds to the "real ghost" graph in Fig. IIb. This argument follows essentially one of 't Hooft's, proving unitarity (58). Thus we see that the transition probability involves a contribution from Fig. IIa, where the Feynman spin sum is used and an additional ghost term (Fig. IIb). By gauge invariance, the remaining singular n-dependence must cancel when considering all contributions for a given process. In the calculation presented in section 3.4, we have checked that there are no other ghost terms apart from (3.9).



(a) (b)

Fig. II

The phase space integral associated with Figs. IIa,b is:

$$g^2 C_{YM} (2\pi)^{-3} \int \frac{d^{d-1}k}{2\omega 2(E-\omega)} \frac{4}{(p^2)^2} \left[p^2 g_{\lambda\lambda'} + (d-2) k_\lambda k_{\lambda'} \right] \times M_\lambda(p_1 \cdots) M^*_{\lambda'}(p_1 \cdots) \tag{3.10}$$

where $d^{d-1}k = \omega^{d-2} d\omega \theta^{n-3} d\theta d\Omega_{d-2}$ and C_{YM} is the Casimir eigenvalue in the adjoint representation. When the integrand is $\sim\theta^{-2}$, the angular integration goes as $\theta^{d-5}d\theta$, thus giving a pole. It is not quite obvious that the singular integral factorizes from the basic transition probability sinc $(k.M)^2$ may give terms in $p^2=2p.k$, which is $\sim\theta^2$ while the denominator is $\sim\theta^4$. However, the factorization occurs. Eqs. (3.1) and (3.3) imply:

$$M_0(p_1 \cdots) = M_3(p_1 \cdots)\left(1 - (E-\omega)^{-1}\omega\theta^2\right) + O(\theta^4) \tag{3.11}$$

from which it readily follows:

$$(k \cdot M)^2 = k_i k_j M_i M_j^* + O(\theta^4) \qquad i,j = 1, \ldots, d-2 \tag{3.12}$$

where terms involving k_0 and k_3 cancel, and terms like $k_i k_0 M_i M_0$ have been averaged out. We can set d=4 in the integrand of (3.10) since only a single pole appears and, with $k_i k_j \to \frac{1}{2}\omega^2\theta^2\delta_{ij}$, we write:

$$(k \cdot M)^2 = \tfrac{1}{2}\omega^2\theta^2 M_i M_i^* = -\tfrac{1}{2}\omega^2\theta^2 M_\mu M^{*\mu} + O(\theta^4) \tag{3.13}$$

Using $p^2 = (E-\omega)^{-1}E^2\omega\theta^2$, the integral (3.10) becomes:

$$\left[g^2 C_{YM} (2\pi)^{-3} E^{-4} \int \frac{d^{d-1}k}{\omega^2\theta^2} (E^2 + \omega^2 - E\omega) \right] M^2 \tag{3.14}$$

where the divergent integral is factorized, giving a pole:

$$g^2/4\pi^2 \quad \frac{5}{3} \quad \frac{1}{d-4} \quad \frac{M^2}{2E} \tag{3.15}$$

This pole cancels the mass singularity arising from the self-energy corrections in Fig. III* when the external line is put on-shell after subtraction of the ultraviolet pole at an off-shell point $p^2=-\mu^2$, in the same way as described in Sec. 2.2 for the case of massless QED.



(a) (b)

Fig. III

3.3 Soft Virtual Corrections

We now turn to the radiative correction obtained by attaching the ends of a propagator to different external lines in the amplitude represented by the shaded blob (Fig. IV) and

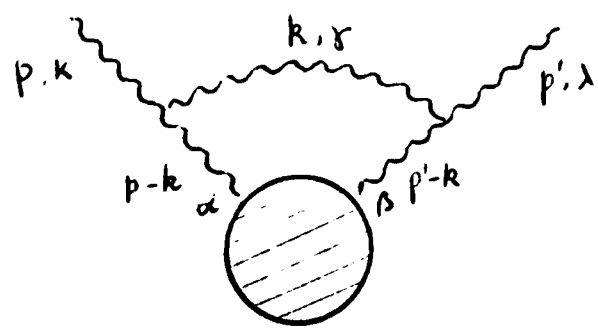


Fig. IV

* In the Feynman gauge.

denoted $M_{\alpha\beta}(p-k, p'-k)$. As before, all other external lines are on-shell and transverse and IR divergences are dimensionally regularized.

We shall restrict ourselves to the case when k is soft. Other singular regions, e.g. that in which k is hard but nearly on-shell and parallel to either one of the external momenta, are not considered. They give non-leading singularities, which are dealt with in Section 3.4. We therefore let $k\to 0$ in the numerator of the integrand, but shall keep k non-zero though arbitrarily small inside the blob in order to preserve its singularity structure. Doing the k_0 contour integration we pick up the pole from k^2 in the denominator, the other poles giving non-singular contributions as $k\to 0$. Taking the cross product of the amplitude in Fig. IV with the uncorrected amplitude, we have an $O(g^2)$ correction term to M^2 (in the Feynman gauge):

$$2g^2 C_{VM}(2\pi)^{-3}\int \frac{d^{d-1}k}{2E\,2|k|}\,\frac{1}{2p\cdot k\;2p'\cdot k}\Big\{V_{k\alpha\gamma}(-p,p,0)\,V_{\lambda\gamma\beta}(-p',0,p') \times M_{\alpha\beta}(p-k,p'-k)\,M^{*}_{\kappa'\lambda'}(p,p')\,e^{\kappa}(p)\,e^{\kappa'}(p)\,e^{\lambda}(p')\,e^{\lambda'}(p')\Big\} \tag{3.16}$$

where $k_0=|\underset{\sim}{k}|$ and $|\underset{\sim}{k}|$ has an upper bound ΔE which specifies what is meant by soft. We wish to compare expression (3.16) with the transition probability for soft bremsstrahlung (Figs. Va,b)*, a process which is of the same order as that corresponding to (3.16). However, this cannot be done directly since, when the cross product

---

* The graph obtained by interchanging k and p'-k in Fig. Va is not singular as $k\to 0$.



(a) (b)

Fig. V

of the soft emission amplitudes (Figs. Va,b) is written down, the Lorentz indices do not match. Note that the lines coming out of the blob in Fig. IV are on-shell as $k \to 0$ but not transverse, that is, they are internal lines and do not have polarization vectors attached. Here the Ward identities (3.4) imply:

$$p^\kappa M_{\kappa\lambda}(p,p') = p'_\lambda M_1 \quad , \quad (p-k)^\alpha M_{\alpha\beta}(p-k,p'-k) = (p'-k)_\beta M_1'$$

$$M_{\kappa\lambda}(p,p')\, p'^\lambda = p_\kappa M_2 \quad , \quad M_{\alpha\beta}(p-k,p'-k)\,(p'-k)^\beta = (p'-k)_\alpha M_2' \tag{3.17}$$

where M_1, M_2, M_1', M_2' are ghost amplitudes and terms vanishing as $k \to 0$ have been neglected. This is because, e.g., $p^\kappa M_{\kappa\lambda}(p, p')$ vanishes, except when p' is not transverse, in which case it is proportional to p'_λ.

Using (3.6) to carry out the polarization sums, applying (3.17) and dropping terms $0(k)$, we find that the quantity between brackets in (3.16) becomes:

$$V_{\kappa\alpha\gamma}(-p,p,0)\, V_{\lambda\gamma\beta}(-p',0,p')\, M_{\alpha\beta}(p-k,p'-k)\, M_{\kappa\lambda}(p,p') - p\cdot p'\,(M_1 M_1' + M_2 M_2') = \tag{3.18}$$

$$= 4p\cdot p'\, M_{\alpha\beta}(p-k,p'-k)\, M^{\alpha\beta}(p,p') - 2p\cdot p'\,(M_1 M_1' + M_2 M_2')$$

The expression (3.18) is exactly what is obtained from the cross product of the real emission terms in Fig. V, with a minus sign.* Thus, in the Feynman gauge, we find this graph by graph cancellation in the soft region, including the double pole (double logarithm), arising when the soft virtual momentum becomes parallel to external momenta.

It is possible that when considering the gauge invariant set of graphs for a given process, cancellations of ghost terms like those in eq. (3.18) could take place. However, we have shown that the same ghost terms appear in the real emission amplitudes, so that the cancellation proceeds graph by graph. This suggests that a general proof of virtual-real cancellation should be possible, at least for leading singularities, along the lines of the Jauch and Rorlich proof in QED (20), but we have not been able to implement it.

3.4 Gluon Scattering in an External Potential (56)

We shall now consider an example which illustrates the foregoing results but shall include in our analysis also the non-leading mass singularities arising from hard virtual momenta which were not generally dealt with in the previous sections. The interest of this calculation lies in that it provides a test of the Bloch-Nordsieck scheme for a process in which massless degenerate incoming and outgoing states are involved, a situation which has not been examined in the Yang-Mills case.

We treat a model process in which a real gluon scatters from an external source. In order to have gauge invariant amplitudes, we consider a scalar source Φ and the coupling taking place via

---

* The complete polarization sum (3.6) is also employed for Fig. V.

the gauge invariant operator $(F_{\mu\nu})^2$. The Lagrangian, including ghost and gauge fixing terms, is:

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2(1+\Phi) - \frac{1}{2\alpha}(\partial\cdot W)^2 + \partial^\mu \xi D_\mu \omega \qquad (3.19)$$

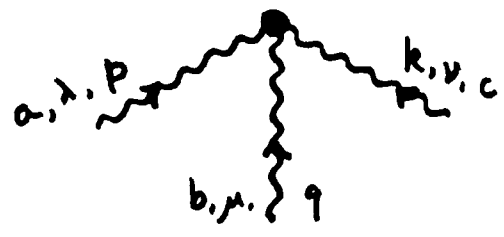
where

$$F^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g f^{abc} W^b_\mu W^c_\nu$$

D_μ is the covariant derivative, and ξ, ω are the ghost fields. The external source Φ is taken as not transforming. We discuss one-loop corrections to what can be envisaged either as gluon scattering in an external scalar (colourless) potential or as the on-shell two-gluon matrix element of the gauge invariant operator $(F_{\mu\nu})^2$ at non-zero momentum transfer. The coupling to Φ induces the vertices:



$$= \delta^{ab}\,\Phi\, M_{\mu\nu}(p,p')$$



$$= \epsilon^{abc}\, g\, \Phi\, \Gamma_{\lambda\mu\nu}(p,q,k)$$

where

$$M_{\mu\nu}(p,p') = i(p\cdot p' g_{\mu\nu} - p'_\mu p_\nu)$$

(3.20)

$$\Gamma_{\lambda\mu\nu}(p,q,k) = (p-q)_\nu g_{\lambda\mu} + (q-k)_\lambda g_{\mu\nu} + (k-p)_\mu g_{\lambda\nu}$$

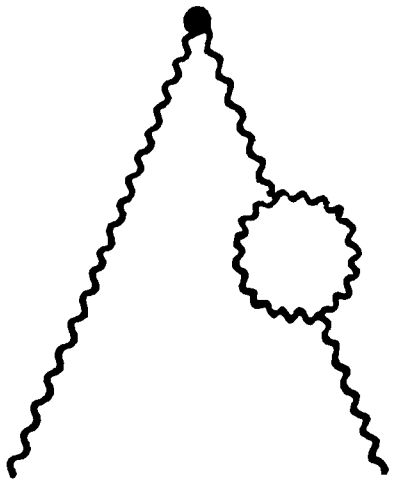
$$p+q+k \neq 0$$

The four-gluon coupling to Φ will not be relevant to our calculation. One can verify Ward identities of the type:

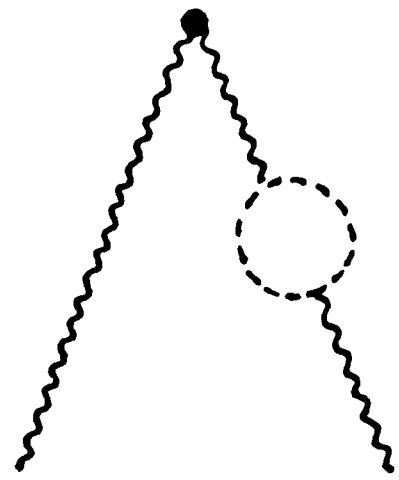
$$k^\nu \Gamma_{\lambda\mu\nu}(p,q,k) = i M_{\lambda\mu}(p-k,q-k) - i M_{\lambda\mu}(p,q) \qquad (3.21)$$

One-loop corrections are shown in Fig. VI. The graphs VIa,b and VId come together with their mirror images and have a combinatorial factor $\frac{1}{2}$ attached. The black dot indicates where the external potential acts. The Feynman rules are listed in Appendix B-1, we employ the general covariant gauge with propagator $-\frac{i}{k^2}\left(g_{\mu\nu}-(1-\alpha)\frac{k_\mu k_\nu}{k^2}\right)$. A renormalization scale μ is introduced by using a dimensionless renormalized coupling, μ serves as the subtraction point for ultraviolet poles.

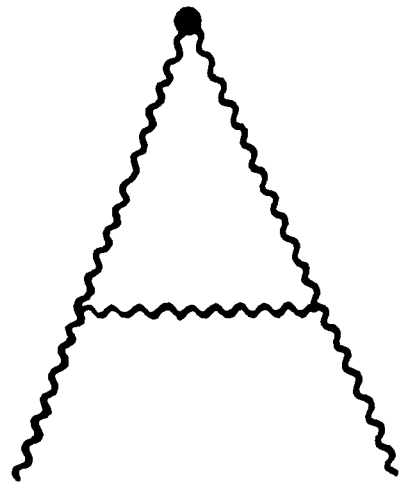
Real corrections with emission of one gluon are represented in Figs. VIIa'-d'. Each figure represents a term in the transition probability for two gluons in the final state, the lower graph in each term stands for a complex conjugate amplitude. In VIIa', the Feynman spin sum is understood and VIIb' is the ghost term discussed in section 3.2. We consider, in addition, transition rates for two incoming gluons, the corresponding graphs are obtained from the mirror images of those in Fig. VII by reversing the direction of momenta and interchanging p and p'. Two regions of the two-gluon phase space are considered. First, that in which one of the gluons is soft and is emitted isotropically, it turns out that only VIIc' is IR divergent in this case. Second, the region where each gluon can take any share of the momentum (hard bremsstrahlung) and they emerge nearly parallel to each other, within a narrow cone defining an angular resolution δ. All terms VIIa'-d', as well as the hard



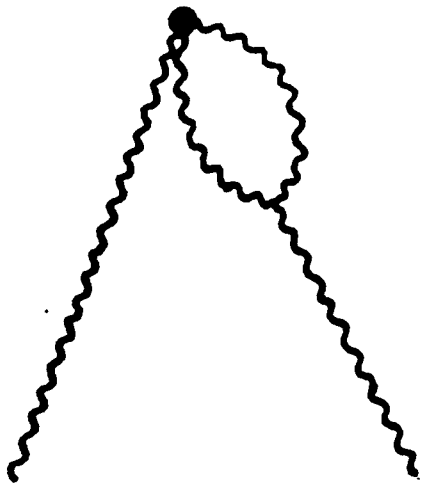
(a)



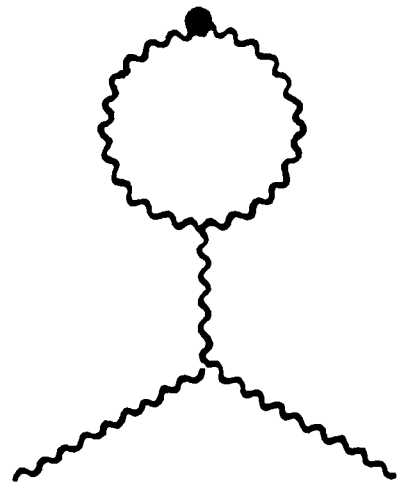
(b)



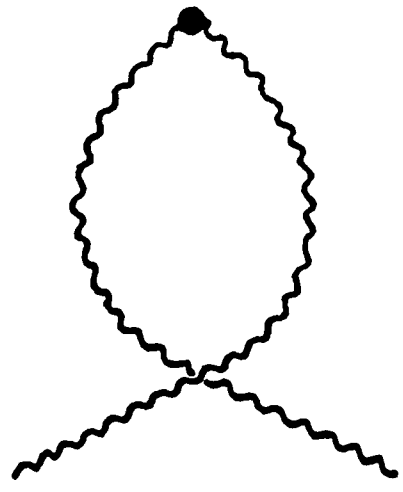
(c)



(d)

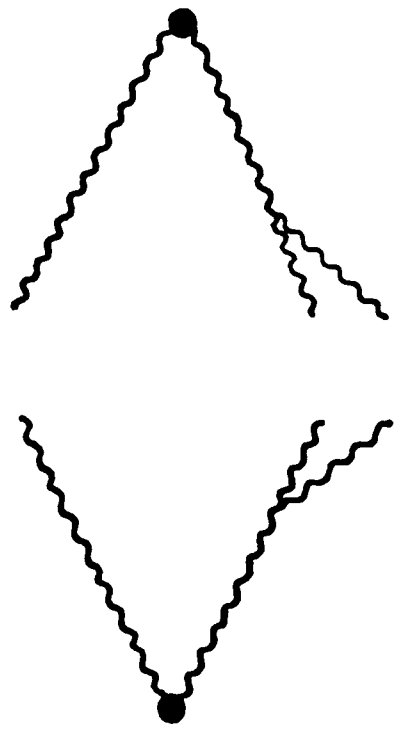


(e)

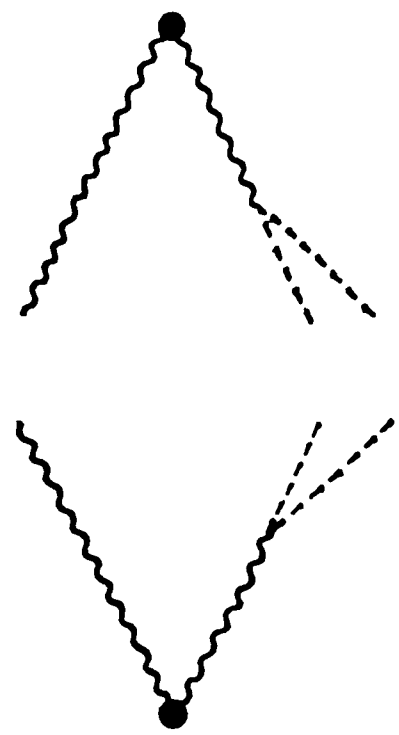


(f)

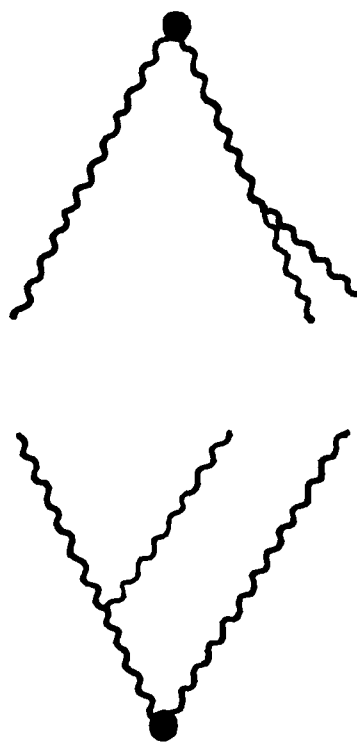
Fig. VI. One-loop corrections to gluon scattering.



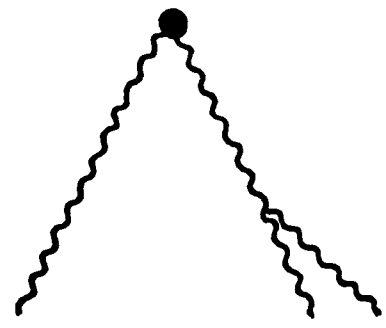
(a')

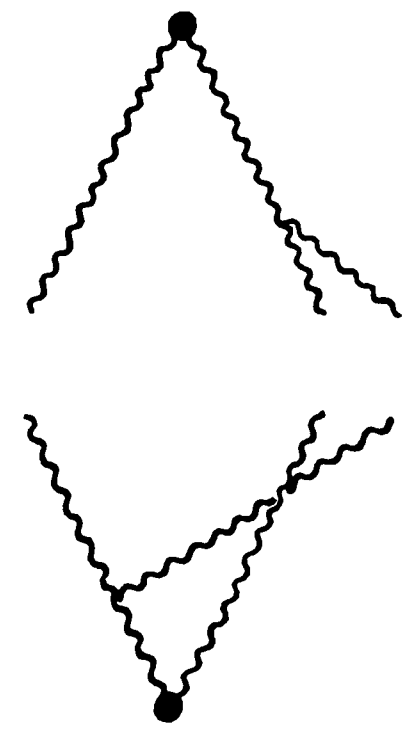


(b')

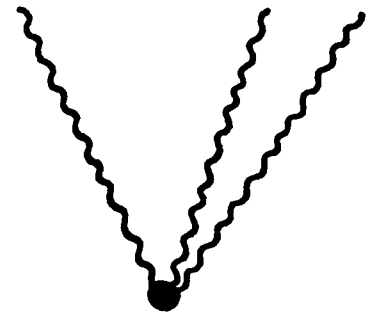


(c')





(c'')



(d')

Fig. VII. Gluon bremsstrahlung

gluon initial degenerate states ("absorption" terms), contribute in this region. An energy resolution parameter ΔE defines what is meant by hard and soft. The colour indices of the two-gluon states are summed over. There are some points that must be kept in mind when collecting the various contributions:

(i) In the computation of hard gluon emission, there is a factor $\frac{1}{2}$ in the phase space integral due to having two-identical particle states. This corresponds to the combinatorial factor $\frac{1}{2}$ in virtual graphs VIa,b and VId.

(ii) In the soft gluon emission the two gluons are treated unsymmetrically, so that the factor $\frac{1}{2}$ for identical particles is compensated when one allows the two possibilities that each particle be the soft one.

(iii) The incoming two-gluon states are treated symmetrically with respect to the outgoing ones. For the latter, the integration is over a narrow cone with $\underset{\sim}{p}'$ along the z-axis, whereas for the former, $\underset{\sim}{p}$ plays the same role and the two contributions are equal. In the next section, we shall comment on the meaning of these multigluon initial states.

Details of the calculations are given in the Appendices to this Chapter. In Table I, the individual contributions from virtual and real terms are listed, including the dependence of the finite parts on ΔE and δ. Here we simply describe the results and comment on some of the relevant features.

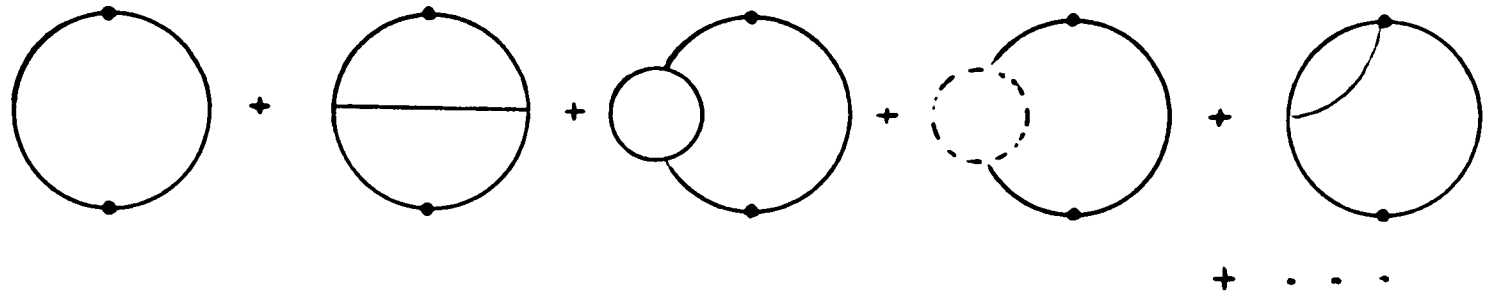
The contribution of virtual graphs (Fig. VI) to the cross-section is obtained by multiplication by the Born term. From the analysis of section 3.2 it follows that the mass singularity in the self-energy corrections (Figs. VIa,b and mirror image) is

$d\sigma_{a+b}=d\sigma_0 \frac{g^2}{\pi^2}C_{YM}[\frac{5}{12}\frac{1}{\varepsilon}]$	$d\sigma_{a'+b'}=d\sigma_0 \frac{g^2}{\pi^2}C_{YM}[-\frac{5}{12}\frac{1}{\varepsilon}-\frac{5}{12}\ln\frac{E\delta}{\mu}]$
$d\sigma_c=d\sigma_0 \frac{g^2}{\pi^2}C_{YM}[-\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}\{\frac{5}{4}-\ln\frac{\sqrt{-q^2}}{\mu}\}$ $+\frac{5}{4}\ln\frac{\sqrt{-q^2}}{\mu}-\frac{1}{2}\ln^2\frac{E}{\mu}-\ln\frac{E}{\mu}\ln\frac{\sqrt{-q^2}}{E}]$	$d\sigma_{c'+c''}^{soft}=d\sigma_0 \frac{g^2}{\pi^2}C_{YM}[\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}\{\ln\frac{\Delta E}{E}+\ln\frac{\sqrt{-q^2}}{\mu}\}$ $+\frac{1}{2}\ln^2\frac{2\Delta E}{\mu}+\ln\frac{\Delta E}{\mu}\ln\frac{\sqrt{-q^2}}{E}]$
	$d\sigma_{c'+c''}^{hard}=d\sigma_0 \frac{g^2}{\pi^2}C_{YM}[\frac{1}{\varepsilon}\{-\frac{5}{4}+\ln\frac{E}{\Delta E}\}$ $-\ln\delta\ln\frac{\Delta E}{E}-\frac{5}{4}\ln\frac{E\delta}{\mu}-\frac{1}{2}\ln^2\frac{\Delta E}{\mu}+\frac{1}{2}\ln^2\frac{2E}{\mu}]$
$d\sigma_d=d\sigma_0 \frac{g^2}{\pi^2}C_{YM}[-\frac{3}{4}\frac{1}{\varepsilon}]$	$d\sigma_{d'}=d\sigma_0 \frac{g^2}{\pi^2}C_{YM}[\frac{3}{4}\frac{1}{\varepsilon}+\frac{3}{4}\ln\frac{E\delta}{\mu}]$

Table 1.

Subscripts for each partial cross-section indicate the graph to which it corresponds in Figs. VIa,b,c,d and VIIa',b',c',c'',d'. Graphs VIa,b,d are considered together with their mirror images. The real correction terms in the right hand side of this Table include contributions from the incoming two-gluon states. $d\sigma_0$ stands for the Born cross-section, $\varepsilon=d-4$ and q^2 is the momentum transfer. The μ-dependence is that coming from the IR terms only.

compensated by Figs. VIIa',b' (plus absorption term) alone, when the real ghosts are properly included. Our calculation is in fact related to the gauge invariant two-point correlation function $\int d^4x\, e^{iq.x} \langle 0|T(F_{\mu\nu})^2(x)(F_{\mu\nu})^2(0)|0\rangle$* and the different unitarity cuts of its diagrammatic expansion:



with, however, the phase space in such cuts restricted by the resolution parameters ΔE and δ. The set of cuts corresponding to the virtual graphs (Fig. VI) are gauge invariant by themselves. The cuts giving the three-gluon intermediate states are then also gauge invariant. In a non-covariant, ghost-free gauge (axial gauge), the cuts correspond to transverse physical intermediate states, and Fig. A below, but not Fig. B, appears.



(A) (B)

The same results must ensue in the covariant (Feynman) gauge where both A and B are included. This is a heuristic way to look at the appearance of the "real ghost" terms. All the calculations

---

* More precisely, for q timelike, this function gives the cross-section for gluon production, rather than scattering.

have been carried out using the polarization sums over transverse states (eq. 3.6). The n dependence cancels amongst the different terms as it should by gauge invariance, but no further ghost terms arise apart from Fig. VIIb'. This may be expected from unitarity, but it is important to check it explicitly since it is crucial for the cancellations to go through. In Appendix B-3 we show the cancellation of all the gauge dependence with no extra ghost terms left.

The one-particle-reducible graph VIe is identically zero because the external source is colourless. The four-point vertex graph VIIf does not contribute either since it is finite after renormalization. In the graph VIc the infrared terms neatly separate from the ultraviolet ones (Appendix B-1). Due to the external massless lines, a double pole appears together with single poles. These are all compensated by VIIc', c''. This was generally described for the leading poles in section 3.3, but the non-leading single poles from hard virtual momenta are now also checked to cancel. The graph by graph cancellation is verified, as the mass singularity in VId plus mirror image is matched by that in VIId' plus absorption term. We remark that, although the cancellations, as well as the remaining finite parts, are, of course, gauge independent, the cancellation takes place graph by graph only in the Feynman gauge. It goes through in the general gauge only after the gauge dependence of the virtual graphs VIa,b and VId is cancelled between them, as shown in Appendix B-1.

The IR singular exclusive (elastic scattering) cross-section has the form (Table I):

$$\left(\frac{d\sigma}{d\Omega}\right)=\left(\frac{d\sigma}{d\Omega}\right)_0\left[1+\frac{g^2}{\pi^2}C_{YM}\left\{-\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\left(\frac{11}{12}+\ln 2\sin\frac{\theta}{2}+\ln\frac{E}{\mu}\right)+\text{finite terms}\right\}\right] \tag{3.22}$$

where $\epsilon=d-4$, $(\frac{d\sigma}{d\Omega})_0$ is the Born cross-section and θ is the scattering angle.

On the other hand, inclusion of the real corrections leads to the finite inclusive cross-section:

$$\left(\frac{d\sigma}{d\Omega}\right)'=\left(\frac{d\sigma}{d\Omega}\right)_0\left[1+\frac{g^2}{\pi^2}C_{YM}\left\{\ln\frac{\Delta E}{E}\left(\ln 2\sin\frac{\theta}{2}-\ln\delta\right)-\frac{11}{12}\ln\delta+f(E/\mu,\theta)\right\}\right] \tag{3.23}$$

The ΔE dependence is similar to that obtained in similar calculations in massless QED (Sec. 2.4), terms in $\ln^2\frac{\Delta E}{\mu}$ have cancelled between soft and hard emission. The dependence on E/μ is not given explicitly, it arises from the renormalization of the operator insertion considered, and has a coefficient related to the anomalous dimension of $(F_{\mu\nu})^2$. In this respect we mention that the non-gauge-invariant operators which mix with $(F_{\mu\nu})^2$ under renormalization (59) do not affect our results, since their contributions vanish on the mass-shell.

It is interesting that the coefficient $\frac{g^2}{4\pi^2}\frac{11}{3}$ of the single pole in (3.22) and of $\ln\delta$ in (3.23) appear to be numerically related to the renormalization group function $\beta(g)$. We do not know if this is a coincidence or if it is significant. A connection between the IR singularities in perturbation theory and the renormalization group effective coupling has been found in calculations of leading singularities in QCD, to order g^4 (62,63).

3.5 Discussion

Free quarks could in principle be detected through their electromagnetic coupling. A less obvious question is whether free gluons with only colour charge could actually be detected, at least indirectly, but we shall not discuss this point further. The example presented in the previous section is a theoretical model for studying the on-shell behaviour of gluon amplitudes. We have found that appropriately averaged transition rates for a process involving in and out gluonic states is free of singularities in perturbation theory. Thus we extend and give evidence supporting the general conclusion from similar investigations, although we emphasize that our example brings out features typical of the non-Abelian theory which are not dealt with in previous analyses.

One point of view in discussing the physical significance of the pertubation theoretic results is to naively take them at face value and, given that certain transition probabilities are well defined, discuss the sort of processes to which they may correspond. Perturbation theory seems to imply that massive quark production is not prevented, provided soft gluon emission is included, in exact analogy with QED. Not so straightforward is the case when scattered colour particles are massless (gluons, $m \to 0$ limit of very light quarks). If our result extends to higher orders, for having well defined transition probabilities, the initial and final asymptotic states must include an arbitrary number of soft and hard gluons*. Only the ensemble of multigluon states which are

---

* In higher orders, as the number of gluons in such states increases, all gluons may become soft since they share a fixed total momentum.

degenerate and assumed indistinguishable may have physical meaning. In other words, this would suggest that if gluons are not confined, only "jets" of gluons moving parallel to each other can be physical. In the actual calculation of the previous section, the incoming multigluon states are treated symmetrically with respect to the outgoing ones and there is no need to attempt defining flux factors for three-particle initial states, in particular, the spin average factors are required to be the same as for the one-gluon state. It may be then natural to suppose that the initial degenerate states should be interpreted as arising from the decay of some massive particle, an interpretation discussed by Lee and Nauenberg (28).

It should be noted that our example can be discussed as a gluon production process by taking the momentum transfer to be time like. Gluon production from similar hypothetical sources has been considered (Appelquist and Politzer (56)) in discussions of possible gluonic resonances in e^+e^- annihilation into hadrons.

From the more general point of view, barring the peculiarities mentioned above, it appears that perturbative studies have not given any insight into the confinement issue. In particular, it is known (Sterman, Poggio and Quinn (57)) that the total e^+e^- annihilation cross-section is free of singularities in perturbation theory, so that the process $\gamma \to q\bar{q}$ + gluons is in principle not prevented to occur, which does not support the conjecture that cross-sections may vanish due to the IR divergences, even allowing for finite energy resolution. In QED the order by order cancellation of IR logarithms is a reflection of the cancellation taking place after exact summation of IR logarithms to all orders. In QCD, the non-cancellation has to occur in such a way that there is no trace of

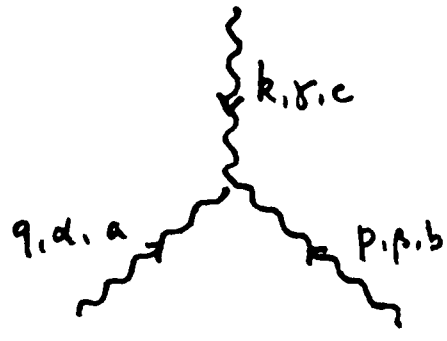
it in finite order of perturbation theory (Cornwall and Tiktopoulos (61), see Appendix C-1). Had we found a non-cancellation in low order, that might have been interpreted as a signal that it could occur also non-perturbatively.

On the other hand, it has been suggested (e.g. Gross (66), Yao (54)) that the problem of defining finite inclusive transition rates is a technical one, separate from that of elucidating the IR behaviour in the sense of the renormalization group. However, the two aspects seem to be related, the IR singularities in the perturbation expansion are governed by the β function (62,63) and defining the finite transition rates to all orders (the appropriate coherent states) looks highly non-trivial. The singularities that, at first sight, are no worse than in, e.g. massless QED, are indeed more complicated.

Appendix B-1

VIRTUAL CORRECTIONS

The Feynman rules are:

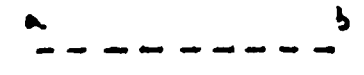


$$= g\epsilon^{abc}\{(p-k)_\alpha g_{\beta\gamma} + (k-q)_\beta g_{\gamma\alpha} + (q-p)_\gamma g_{\alpha\beta}\}$$

$$q + p + k = 0$$



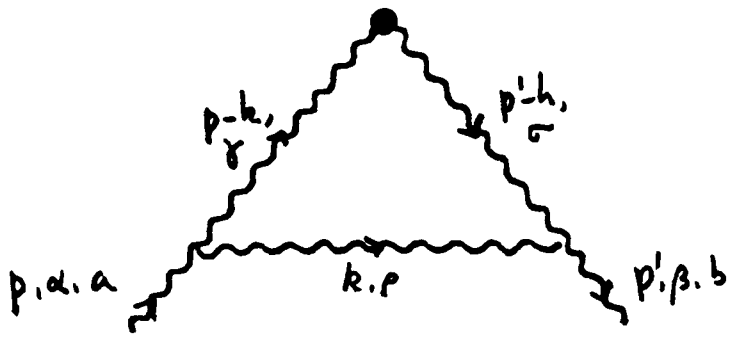
$$= -i\,\frac{1}{k^2}\left(g_{\mu\nu} - (1-\alpha)\,\frac{k_\mu k_\nu}{k^2}\right)$$



$$= -i\,\delta^{ab}\,\frac{1}{k^2}$$

The two- and three-point vertices involving the external source are given in eq. (20).

We sketch the computation of IR poles from the diagram:



Because of the identities $p^\alpha M_{\alpha\beta}(p, p') = M_{\alpha\beta}(p, p')p'\beta = 0$, and setting $\int d^n k (k^2)^{-2} = 0$, (60), the longitudinal terms in the propagator give vanishing contribution, so that we work effectively in the Feynman gauge. After combining denominators, we have the integral:

$$I_1 = \mu^{-\epsilon} g^2 \Phi C_{YM} \delta^{ab} \frac{1}{(2\pi)^4} \int_0^1 2y\,dy \int_0^1 dx \int d^4k \frac{1}{[(k-A)^2 + C]^3} \{$$

$$\times [-(p-k)\cdot(p'-k) g_{\rho\sigma} + (p'-k)_\rho (p-k)_\sigma]$$

$$\times [(2p-k)_\rho g_{\alpha\gamma} + (2k-p)_\alpha g_{\rho\gamma} - (k+p)_\gamma g_{\rho\alpha}] \qquad (3.24)$$

$$\times [-(p'+k)_\sigma g_{\beta\rho} + (2k-p')_\beta g_{\rho\sigma} + (2p'-k)_\rho g_{\beta\sigma}] \}$$

where $A = p\,x\,y + p'(1-y)$, $C = q^2\,x\,y(1-y)$ and $q^2 = (p-p')^2$, $p^2 = p'^2 = 0$.

This is somewhat reduced by dropping p_α and p'_β terms because of the conditions $e.p = e'.p' = 0$. Terms quartic, quadratic and of zero order in the integration momentum appear in the numerator after the shift $k \to k+pxy+p'(1-y)$. Using the dimensional regularization formula (Sec. 2.2) (with a factor $\pi^{\frac{d}{2}-2}\Gamma(3-\frac{d}{2})^{-1}$ to avoid spurious terms in $\ln\pi$ and Euler's constant γ), we find the form:

$$I_1 = \mu^{-\epsilon} g^2 \Phi C_{YM} \delta^{ab} \frac{1}{(2\pi)^4} \int_0^1 2y\,dy \int_0^1 dx\, [-q^2 x y(1-y)]^{\frac{d}{2}-2} \{$$

$$\Gamma(2-\tfrac{d}{2})\,\tilde{A}(x,y,p,p',d) + \Gamma(3-\tfrac{d}{2})[q^2 x y(1-y)]^{-1}\tilde{B}(x,y,p,p',d)\} \qquad (3.25)$$

The term proportional to $\Gamma(2-d/2)$ arises from the ultraviolet divergent terms (k^2, $(k^2)^2$ in the numerator), while the second term produces the IR divergence from the Feynman parameter integrations. In the expansion of (3.25), the typical term has the form:

$$I(a,b) = \int_0^1 y\,dy \int_0^1 dx\, [xy(1-y)]^{\frac{d}{2}-3} x^a (1-x)^b$$

$$= B(a+\tfrac{d}{2}-2, 1)\, B(\tfrac{d}{2}-1, b+\tfrac{d}{2}-2)$$

where B(m,n) is the beta function. There is a double pole when a=b=0 and single poles if a=0, b≠0, or a≠0, b=0. Therefore, terms proportional to x(1-y), xy(1-y) can be dropped and xy=x(1-(1-y))→x. The double pole terms yield further single poles whose residue is found by differentiating other d-dependent factors. These terms are of the form:

$$\left(-q^2/\mu^2\right)^{d/2-2} I(0,0) = \left(-q^2/\mu^2\right)^{\frac{d}{2}-2} B(d/2-2,1)\, B(d/2-1,d/2-2)$$

$$= \frac{4}{\epsilon^2}\,\frac{\Gamma(\epsilon/2+1)^2}{\Gamma(\epsilon+1)}\left(-q^2/\mu^2\right)^{\epsilon/2}$$

Now,

$$\frac{d}{d\epsilon}\,\frac{\Gamma(1+\epsilon/2)}{\Gamma(1+\epsilon)}\Big|_{\epsilon=0} = 0$$

so that the double pole terms contribute:

$$I(0,0) \longrightarrow \frac{4}{\epsilon^2} + \frac{2}{\epsilon}\ln\left(-q^2/\mu^2\right)$$

Likewise, the residues of single pole terms are:

$$I(a,0) \longrightarrow \frac{2}{a}\,\frac{1}{\epsilon}$$

$$I(0,b) \longrightarrow \frac{2}{b}\,\frac{1}{\epsilon}$$

Examining all terms arising from the shift of integration variables, and dropping terms not giving poles, we have for M_1 = (IR singular part of I_1):

$$M_1 = g^2\,\Phi\,C_{YM}\,\delta^{ab}\,i(\pi)^{d/2}\,\frac{\pi^{2-d/2}}{\Gamma(3-\frac{d}{2})}\,\frac{\Gamma(3-\frac{d}{2})}{\Gamma(3)}\left(-q^2/\mu^2\right)^{\frac{d}{2}-2}(q^2)^{-1}\frac{1}{(2\pi)^4} \tag{3.26}$$

$$\times\left(p\cdot p'\,p'_\alpha p_\beta - (p\cdot p')^2 g_{\alpha\beta}\right)\int_0^1 2y\,dy\int_0^1 dx\left[xy(1-y)\right]^{\frac{d}{2}-3}\left(4-6x+2x^2-6(1-y)+2(1-y)^2\right.$$

Collecting the residues as explained above, we have finally:

$$M_1 = -g^2 C_{YM}\,\delta^{ab}\,\frac{2}{(2\pi)^2}\,M_0\left[\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\left\{\ln\frac{\sqrt{-q^2}}{\mu}-\frac{5}{4}\right\}\right] \tag{3.27}$$

where the Born amplitude M_0 is:

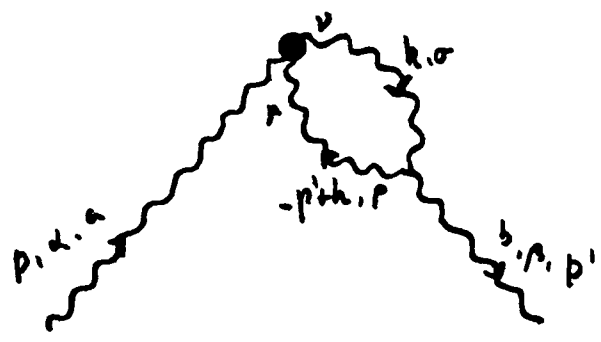
$$M_0 = i\,(p'_\alpha p_\beta - p\cdot p'\,g_{\alpha\beta})\,e^\alpha(p)\,e^\beta(p')\,\delta^{ab}\,\Phi \tag{3.28}$$

and the $O(g^2)$ correction to the transition probability is given by the product $2M_0^* M_1$ ($d\sigma_c$ in Table 1).

The finite parts corresponding to this contribution are obtained essentially from the ε-expansion of

$$\left(-q^2/\mu^2\right)^{\epsilon/2}\left[\frac{1}{\epsilon^2}-\frac{5}{4}\,\frac{1}{\epsilon}\right]$$

Next, we briefly describe the evaluation of IR terms from the diagram:



which gives (with combinatorial factor $\frac{1}{2}$):

$$\tilde{I}_2 = \tfrac{1}{2}\,g^2\,\Phi\,C_{YM}\,\delta^{ab}\,\frac{1}{(2\pi)^4}\int d^4k\,\frac{1}{k^2(p'-h)^2}\left[g_{\mu\rho}-(1-\alpha)\,\frac{(p'-h)_\mu (p'-h)_\rho}{(p'-h)^2}\right]\left[g_{\nu\sigma}-(1-\alpha)\,\frac{k_\nu h_\sigma}{k^2}\right]$$

$$\times\left[(2k-p')_\alpha\,g_{\mu\nu}-(p+h)_\mu\,g_{\alpha\nu}+(p+p'-h)_\nu\,g_{\alpha\mu}\right] \tag{3.29}$$

$$\times\left[(2k-p')_\beta\,g_{\rho\sigma}-(p'+h)_\rho\,g_{\beta\sigma}+(2p'-h)_\sigma\,g_{\beta\rho}\right]$$

which is to be evaluated keeping momenta off mass-shell. After combining denominators and shifting integration variables, in the momentum integration several terms coming from the $(1-\alpha)$ part of the propagators can be dropped by use of the prescription $\int d^n k (k^2)^{-1} = 0$ (60). Performing the parametric integrations, we find for the above diagram, plus mirror image:

$$I_2 = -i g^2 \Phi C_{YM} \delta^{ab} \frac{1}{(2\pi)^4} \left(\mu^{-\epsilon} \frac{1}{\epsilon}\right) \left\{ \frac{2}{2+\epsilon} B\left(\frac{\epsilon}{2}, \frac{\epsilon}{2}\right) \left[p^2 (-p^2)^{\epsilon/2} + p'^2 (-p'^2)^{\epsilon/2} \right] g_{\alpha\beta} \right.$$

$$\left. + \left[3 - \left(\frac{2+\epsilon}{4+\epsilon}\right)(1-\alpha) \right] B\left(\frac{\epsilon}{2}+1, \frac{\epsilon}{2}+1\right) \left[(-p^2)^{\epsilon/2} + (-p'^2)^{\epsilon/2} \right] (p \cdot p' g_{\alpha\beta} - p'_\alpha p_\beta) \right\} \tag{3.30}$$

The pole in this expression is of ultraviolet origin. The renormalization is carried out by subtracting at $p^2 = p'^2 = -\mu^2$. Thus, the ultraviolet pole is eliminated, but a pole reappears as a mass singularity when setting momentum on the mass-shell $p^2 = p'^2 = 0$. Explicitly, we have for the quantity between brackets:

$$\frac{2}{2+\epsilon} B\left(\frac{\epsilon}{2}, \frac{\epsilon}{2}\right) g_{\alpha\beta} \left[p^2 \left(-\frac{p^2}{\mu^2}\right)^{\epsilon/2} + p'^2 \left(-\frac{p'^2}{\mu^2}\right)^{\epsilon/2} - (p^2 + p'^2) \right]_{p^2 = p'^2 = 0} +$$

$$+ \left(3 - \frac{2+\epsilon}{4+\epsilon}(1-\alpha)\right) B\left(\frac{\epsilon}{2}+1, \frac{\epsilon}{2}+1\right) (p \cdot p' g_{\alpha\beta} - p'_\alpha p_\beta) \left[\left(-\frac{p^2}{\mu^2}\right)^{\epsilon/2} + \left(-\frac{p'^2}{\mu^2}\right)^{\epsilon/2} - 2 \right]_{p^2 = p'^2 = 0}$$

so that only the part proportional to $(p \cdot p' g_{\alpha\beta} - p'_\alpha p_\beta)$ remains. Note that the mass-shell limit must be taken before the limit $\epsilon \to 0$. Thus, the residue M_2 = (IR divergent part of I_2) is:

$$M_2 = \frac{g^2}{\pi^2} C_{YM} \left(-\frac{3}{8} \frac{1}{\epsilon} - \frac{1}{16}(1-\alpha) \frac{1}{\epsilon} \right) M_0 \tag{3.31}$$

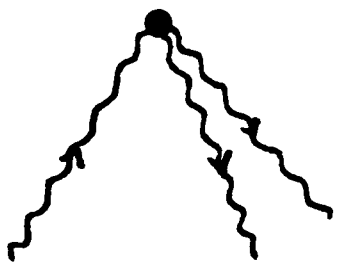
with M_0 defined in (3.28). Eq. (3.31) leads to $d\sigma_d$ in Table 1.

On the other hand, the self-energy graphs VIa,b give a contribution:

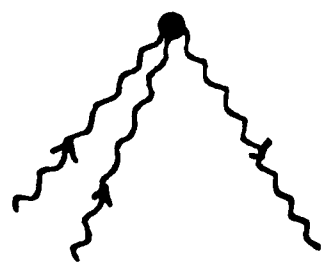
$$M_3 = \frac{g^2}{\pi^2} C_{VM} \left\{ \frac{5}{24} \frac{1}{\epsilon} + \frac{1}{16} (1-\alpha) \frac{1}{\epsilon} \right\} M_0 \qquad (3.32)$$

so that the α dependence cancels between (3.31) and (3.32).

It should be noted that only half of (3.31) is cancelled by the real emission term VIId' (Fig. A below), the incoming gluon term (Fig. B) is necessary.



(A)

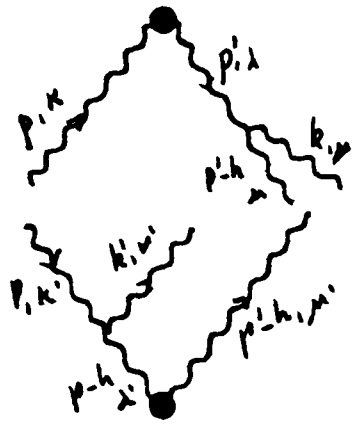


(B)

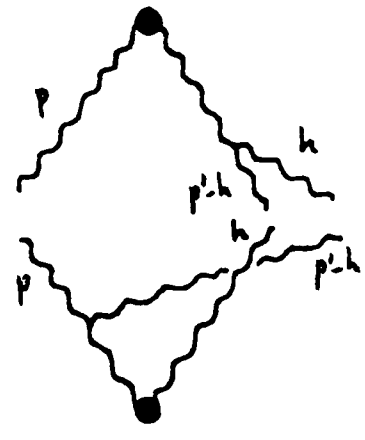
Appendix B-2

REAL CORRECTIONS

Let us consider the contribution to the transition probability from the real gluon emission terms represented in Figs. A, B below (Figs. VIIc', c'' of page 69).



(A)



(B)

We deal for the moment with the evaluation of (A). Calling the upper and lower graphs in A, M_1 and M_2^*, respectively,

$$2 M_1 M_2^* = -\frac{2g^2}{2p\cdot h\, 2p'\cdot h} C_{rm}\, e^{\kappa}(p)\, M_{\kappa\lambda}(p,-p')\, V_{\lambda\mu\nu}(p', h-p', -h)\, e^{\mu}(p-h)\, e^{\nu}(h)$$

$$\times e^{\kappa'}(p)\, V_{\kappa'\lambda'\nu'}(p, h-p, -h)\, M^*_{\lambda'\mu'}(p-h, -p'+h)\, e^{\mu'}(p'-h)\, e^{\nu'}(h) \qquad (3.33)$$

with the mass-shell conditions $k^2=(p'-k)^2=0$, $M_{\kappa\lambda}(p,-p')$ is defined in eq. (3.20), $V_{\lambda\mu\nu}$ is the Yang-Mills three-vertex and the sum over colour indices of the final state particles is carried out. In summing over physical polarization states, eq. (3.6) must be used. However, the dependence on the gauge fixing vector n cancels amongst the various contributions, leaving only the ghost emission term VIIb' (Appendix B-3), and therefore one can use effectively the covariant Feynman spin sum:

$$\sum_e e_\mu(h)\, e_\nu(h) = -g_{\mu\nu} \qquad (3.34)$$

However, one must note that if (3.34) is employed, one is not allowed to use the transversality condition e.p=0 in (3.33).

Thus we have:

$$2M_1M_2^* = 2g^2C_{VM}\frac{1}{(2p_1.k\,2p'.k)}M_{k\lambda}(p_1-p')M^*_{\lambda'\mu}(p-k,-p'+k)$$
$$\times\left[(2p'-k)_\nu g_{\lambda\mu} + (2k-p')_\lambda g_{\mu\nu} - (p'+k)_\mu g_{\lambda\nu}\right] \quad (3.35)$$
$$\times\left[(2p-k)_\kappa g_{k\lambda'} + (2k-p)_\kappa g_{\lambda'\nu} - (p+k)_{\lambda'} g_{k\nu}\right]$$

We are interested in the region where k is nearly parallel to $\underset{\sim}{p}'$, in which case the factor 2p'.k in the denominator is vanishing and gives a mass singularity. Consequently, terms proportional to p'.k in the numerator cancel the singularity. Since these terms can be seen to be non-divergent either when k is soft, they can be dropped altogether.

Writing the explicit form of $M_{\kappa\lambda}$ and $M^*_{\lambda'\mu}$ above, one eventually finds that the singular part of (3.35) is:

$$2(M_1M_2^*)_{IR} = 2g^2C_{VM}\Phi\frac{1}{2p.k\,2p'.k}4p.p'(p.p'-p.k)(2p.p'-p.k) \quad (3.36)$$

If the emitted gluon is soft, we have:

$$2(M_1M_2^*)^{SB} = g^2C_{VM}\frac{2p.p'}{p.k\;p'.k}M_0^2 \quad (3.37)$$

where $M_0^2=2(p.p')^2\Phi^2$ is the square of the Born amplitude in eq. (3.28). The phase space integral over all directions of the soft gluon with energy less than the detector resolution ΔE is then:

$$g^2 C_{YM} (2E)^{-1} (2\pi)^{-3} \int \frac{d^3k}{2k_0} \frac{2p\cdot p'}{p\cdot k\, p'\cdot k} M_0^2 \qquad (3.38)$$

$$\xrightarrow{\text{dimensional regularization}} g^2 C_{YM} M_0^2 (2E)^{-1} (2\pi)^{-3} \mu^{-\epsilon} \int d^{4+\epsilon}k\, \delta(k^2)\theta(k^0) \frac{2p\cdot p'}{p\cdot k\, p'\cdot k}$$

where the k_0 integration is restricted to $0<k_0<\Delta E$. There are double poles from the region where the soft gluon is emitted parallel to $\underset{\sim}{p}$ or $\underset{\sim}{p}'$. These leading poles are easily extracted from (3.38). However, we need also the non-leading single poles obtained by integration over all directions which means that exact evaluation of the integral is necessary. This involves manipulations of hypergeometric functions and is rather complicated. Here we just state the result (49), which is simply:

$$g^2 \mu^{-\epsilon} (2\pi)^{-3} \int_{k_0<\Delta E} d^{4+\epsilon}k \frac{2p\cdot p'}{p\cdot k\, p'\cdot k} \delta(k^2)\theta(k^0)$$

$$= \frac{g^2}{\pi^2} \left(\frac{\Delta E}{\mu}\right)^{\epsilon} \left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon} \ln 2\sin\theta/2 + O(\epsilon)\right] \qquad (3.39)$$

Therefore, with $q^2=-2p.p'=-4E^2\sin^2\theta/2$,

$$(2\pi)^{-3}\int d^{3+\epsilon}k \frac{1}{2k_0} 2(M_1 M_2^+)^{SB} = \frac{g^2}{\pi^2} M_0^2 \left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\left\{\ln\frac{\Delta E}{E} + \ln\frac{\sqrt{-q^2}}{\mu}\right\}\right.$$

$$\left. + \frac{1}{2}\ln^2\frac{\Delta E}{\mu} + \ln\frac{\Delta E}{\mu}\ln\frac{\sqrt{-q^2}}{E} + O(\epsilon)\right] \qquad (3.40)$$

which is denoted $d\sigma^{soft}_{c'+c''}$ in Table I on page 71.

Now, in the region where the emitted gluon is hard and has momentum $\underset{\sim}{k}$ making a small angle ζ with $\underset{\sim}{p}'$ (hard bremsstrahlung), we have:

$$p\cdot k = k_0 E\,(1-\cos\theta) + O(\zeta^2)$$
$$\simeq \frac{k_0}{E} E^2(1-\cos\theta) = \frac{k_0}{E}\, p\cdot p' \qquad (3.41)$$

with $|\underset{\sim}{p}|=|\underset{\sim}{p}'|=E$, $|\underset{\sim}{k}|=k_0$, and θ is the scattering angle. Using relation (3.41), we can write from (3.36)

$$2(M_1 M_2^*)^{HB} = 2g^2 C_{YM} M_0^2 \frac{1}{2p'\cdot k} \frac{E}{k_0}\left(2-3\frac{k_0}{E}+\frac{k_0^2}{E^2}\right) \qquad (3.42)$$

We take $\underset{\sim}{p}'$ along the z-axis and use the representation (3.3) so that:

$$2p'\cdot k = (E-k_0)^{-1} E^2 k_0 \zeta^2 \qquad (3.43)$$

The phase space integration in the hard "narrow" region is then:

$$I^{HR} = (2\pi)^{-3}\int d^{3+\epsilon}k \frac{1}{2k_0\, 2(E-k_0)}\, 2(M_1 M_2^*)^{HR} =$$
$$= 2g^2 C_{YM} M_0^2 (2\pi)^{-3}\int \frac{d^{3+\epsilon}k}{2k_0\, 2(E-k_0)} \frac{(E-k_0)}{E^2 k_0 \zeta^2} \frac{E}{k_0}\left(2-3\frac{k_0}{E}+\frac{k_0^2}{E^2}\right) \qquad (3.44)$$

The $3+\epsilon$-dimensional volume element is:

$$d^{3+\epsilon}k = k^{2+\epsilon}\, dk\, \sin^{1+\epsilon}\zeta\, d\zeta\, d\Omega_{2+\epsilon}$$

where $|k|=k_0\equiv k$. In our case $\sin\zeta\simeq\zeta$ and we consider the region $\Delta E<k<E$, $0<\zeta<\delta$. Thus we have for (3.44):

$$I^{HB} = g^2 C_{YM} M_0^2 (2\pi)^{-3} \frac{2\pi^{1+\epsilon/2}}{\Gamma(1+\epsilon/2)} \mu^{-\epsilon} \int_0^{\delta} \zeta^{-1+\epsilon} d\zeta \int_{\Delta E}^{E} k^{-1+\epsilon} dk \left[1 - \tfrac{3}{2} E^{-1} k + \tfrac{1}{2} E^{-2} k^2\right] \tag{3.45}$$

where we set $\int d\Omega_{2+\epsilon} = \frac{2\pi^{1+\epsilon/2}}{\Gamma(1+\epsilon/2)}$, the area of the sphere in $2+\epsilon$ dimensions. This factor gives spurious finite terms that can be avoided with an analytic continuation factor $\pi^{-\epsilon/2}\Gamma(1-\epsilon/2)^{-1}$, as indicated for virtual diagrams in the previous Appendix. Performing the integrals in (3.45) and expanding in ϵ as $\delta^{\epsilon} = 1+\epsilon \ln\delta + \ldots$, $(\frac{\Delta E}{\mu})^{\epsilon} = 1+\epsilon \ln(\frac{\Delta E}{\mu}) + \ldots$, we get:

$$I^{HB} = g^2 C_{YM} \frac{M_0^2}{2E} \frac{1}{2\pi^2} \left[\frac{1}{\epsilon} \ln\frac{\Delta E}{E} - \frac{5}{4}\frac{1}{\epsilon} + \frac{1}{2}\ln^2\frac{E}{\mu} - \frac{1}{2}\ln^2\frac{\Delta E}{\mu} + \ln\delta \ln\frac{E}{\Delta E} - \frac{5}{4}\ln\delta - \frac{5}{4}\ln\frac{E}{\mu} + O(\epsilon) \right] \tag{3.46}$$

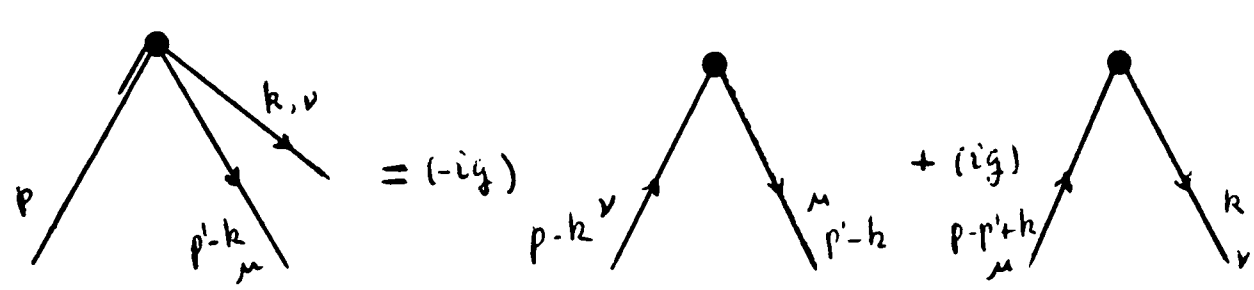
The complete contribution is given with some extra factors: one factor of 2 to take into account term (B) (drawn at the beginning of this Appendix), in which k and p'-k are interchanged; one factor $\frac{1}{2}$ due to having two-identical particle states; a factor of 2 to include the incoming two-gluon states. With these factors, we finally have the contribution $d\sigma_{c'+c''}$ of Table 1.

The result (3.40) turns out to be the only divergent term from soft bremsstrahlung. Apart from (3.46), there are other hard emission terms from Figs. VIIa',b',d'. These are calculated using the same techniques leading to (3.46).

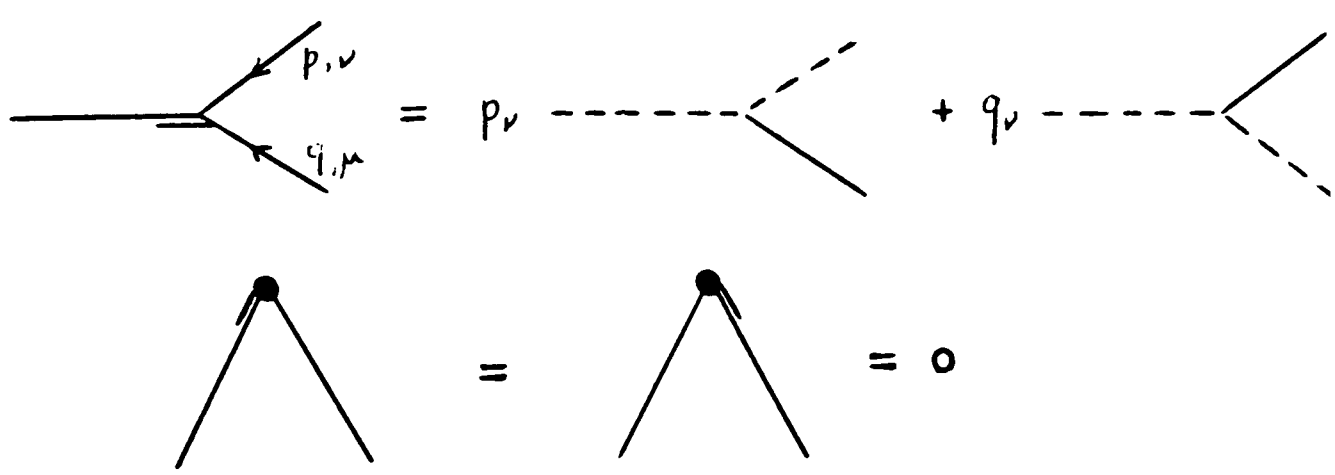
Appendix B-3

GAUGE INDEPENDENCE

In this Appendix we show explicitly how the dependence on the gauge fixing vector n in the polarization sum (3.6) is cancelled in the squared matrix elements without inducing ghost terms in addition to VIIb'. We shall use repeatedly Ward identities of the type (3.21), in their diagrammatic form, since otherwise the explicit expressions are exceedingly lengthy. For instance, with the two- and three-vertices defined by (3.20), we have (continuous lines for the gluons are used)

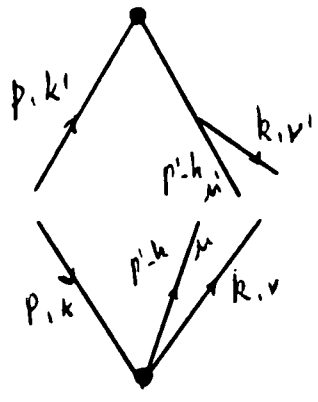


where ——═══ indicates contraction with p^K and ——▶—— denotes a line to which n^μ is contracted. Furthermore:



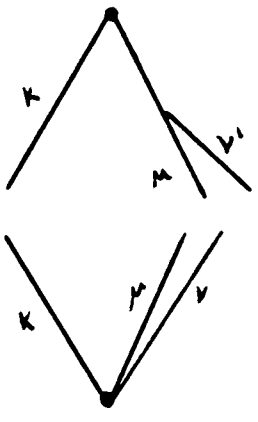
Let us consider the gluon emission term in Fig.VIId'. Since we are interested only in the singular terms, we drop everywhere terms with $p'^2=2p'.k$ in the numerator.

Thus:

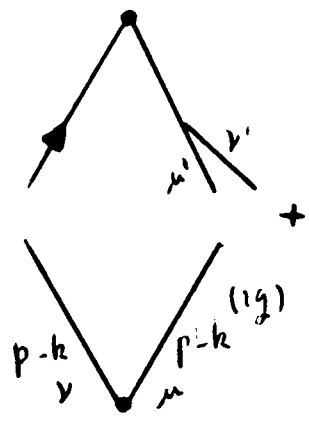


$$\left[\sum_{e} \varepsilon^{\kappa}(p)\, \varepsilon^{\kappa'}(p)\, \varepsilon^{\mu}(p'-h)\, \varepsilon^{\mu'}(p'-h)\, \varepsilon^{\nu}(h)\, \varepsilon^{\nu'}(h) \right] =$$

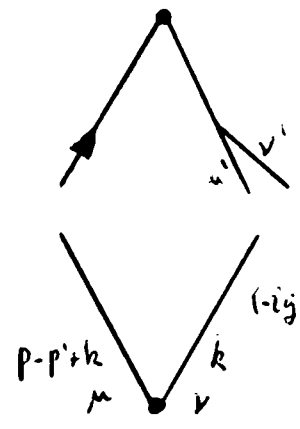
$$= \Bigg[-$$



$$+ \frac{1}{p\cdot n} \Bigg\{$$



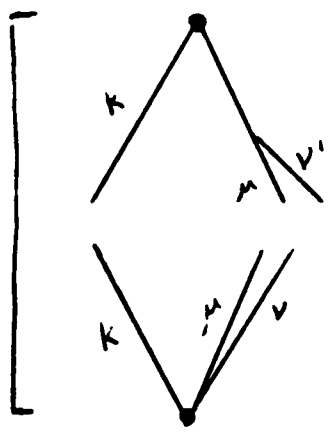
$+$



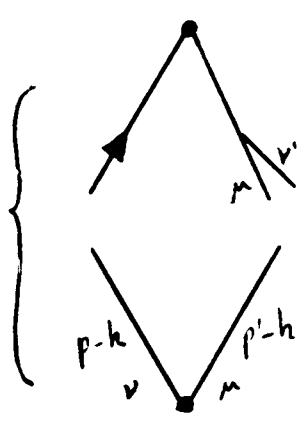
$$\Bigg\} \Bigg]$$

$$\times \left[-g^{\mu\nu'} + \frac{1}{((p'-h)\cdot n)^2} (p'-h)^{\mu} (p'-h)^{\mu'} + \frac{1}{(p'-h)\cdot n} \left((p'-h)^{\mu} n^{\mu'} + (p'-h)^{\mu'} n^{\mu} \right) \right] \sum_{e(h)} \varepsilon^{\nu}(h)\, \varepsilon^{\nu'}(h)$$

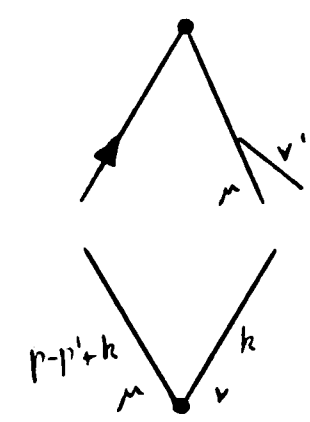
$$= \Bigg[$$



$$- \frac{1}{p\cdot n} \Bigg\{$$

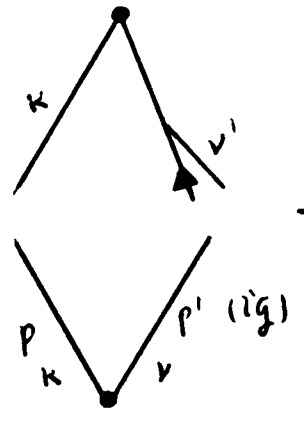


$+$

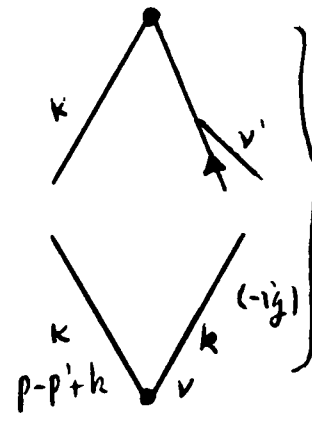


$$\Bigg\}$$

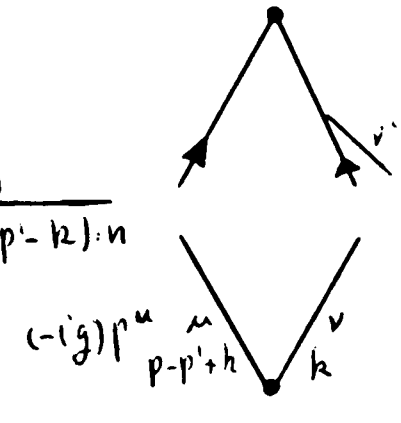
$$- \frac{1}{(p'-k)\cdot n} \Bigg\{$$



$+$

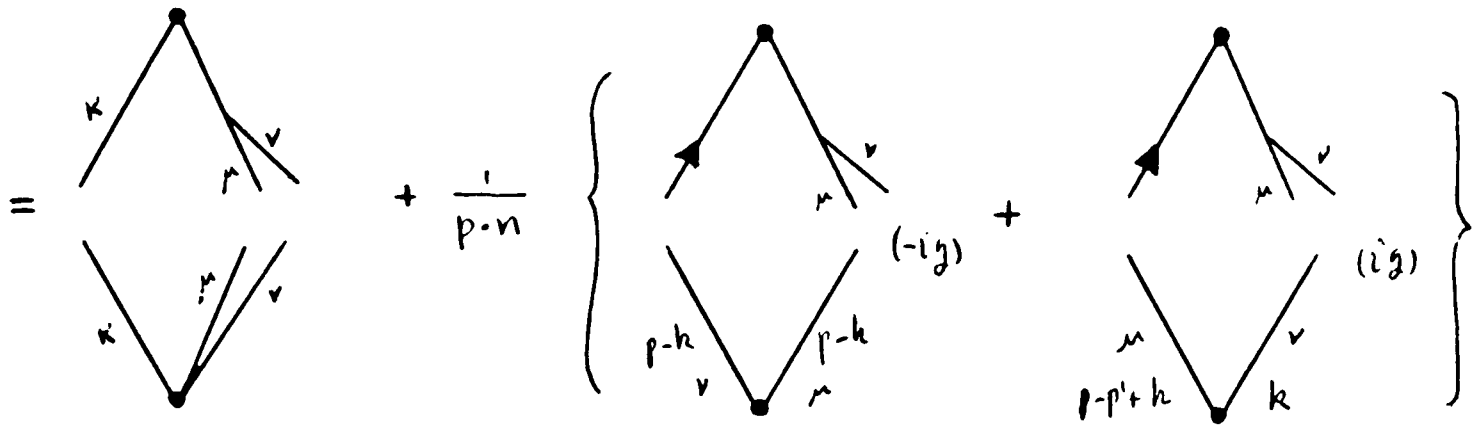


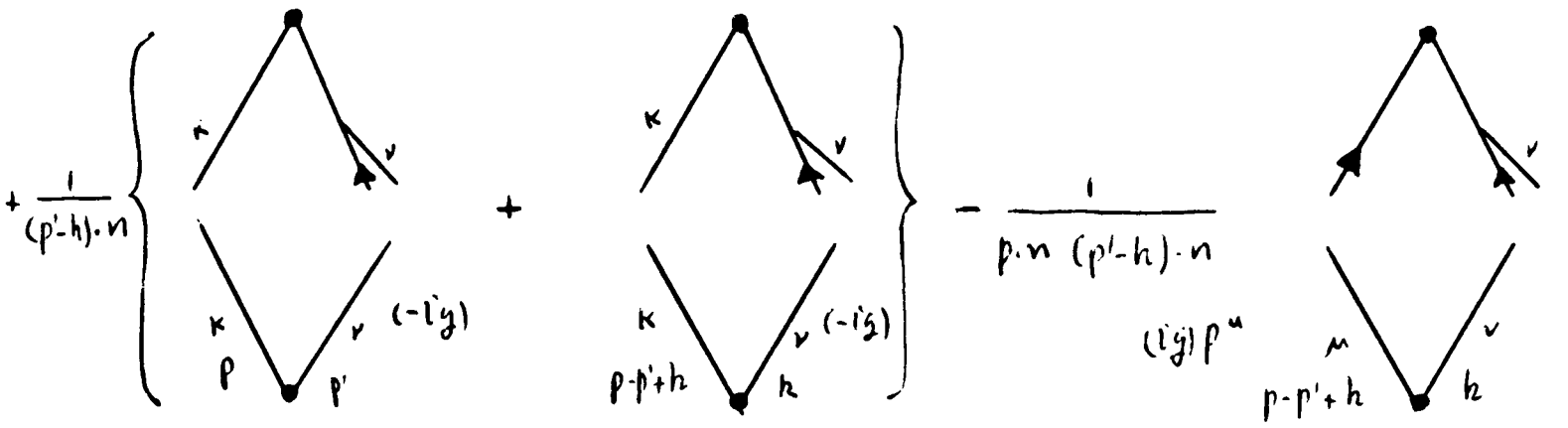
$$\Bigg\} + \frac{1}{p\cdot n\,(p'-k)\cdot n}$$

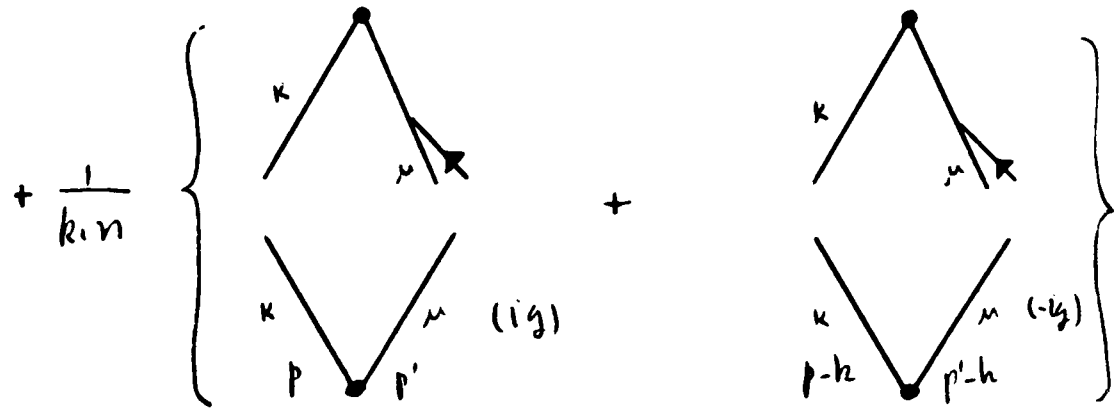


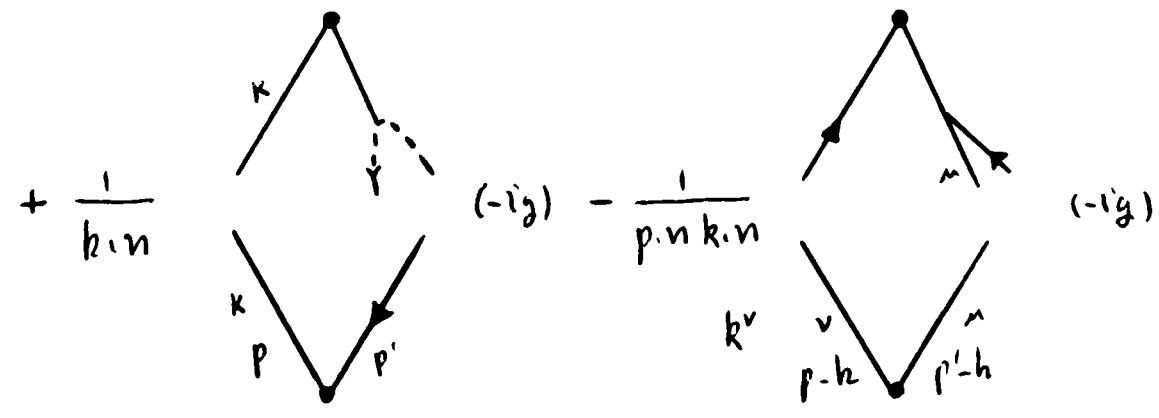
$$\Bigg]$$

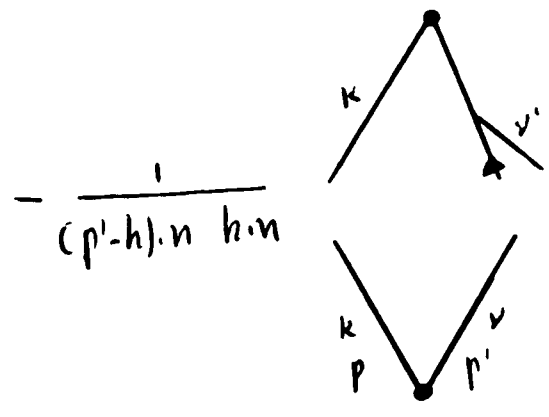
$$\times \left[-g^{\nu\nu'} + \frac{1}{(k\cdot n)^2} k^{\nu} k^{\nu'} + \frac{1}{k\cdot n} \left(k^{\nu} n^{\nu'} + k^{\nu'} n^{\nu} \right) \right]$$





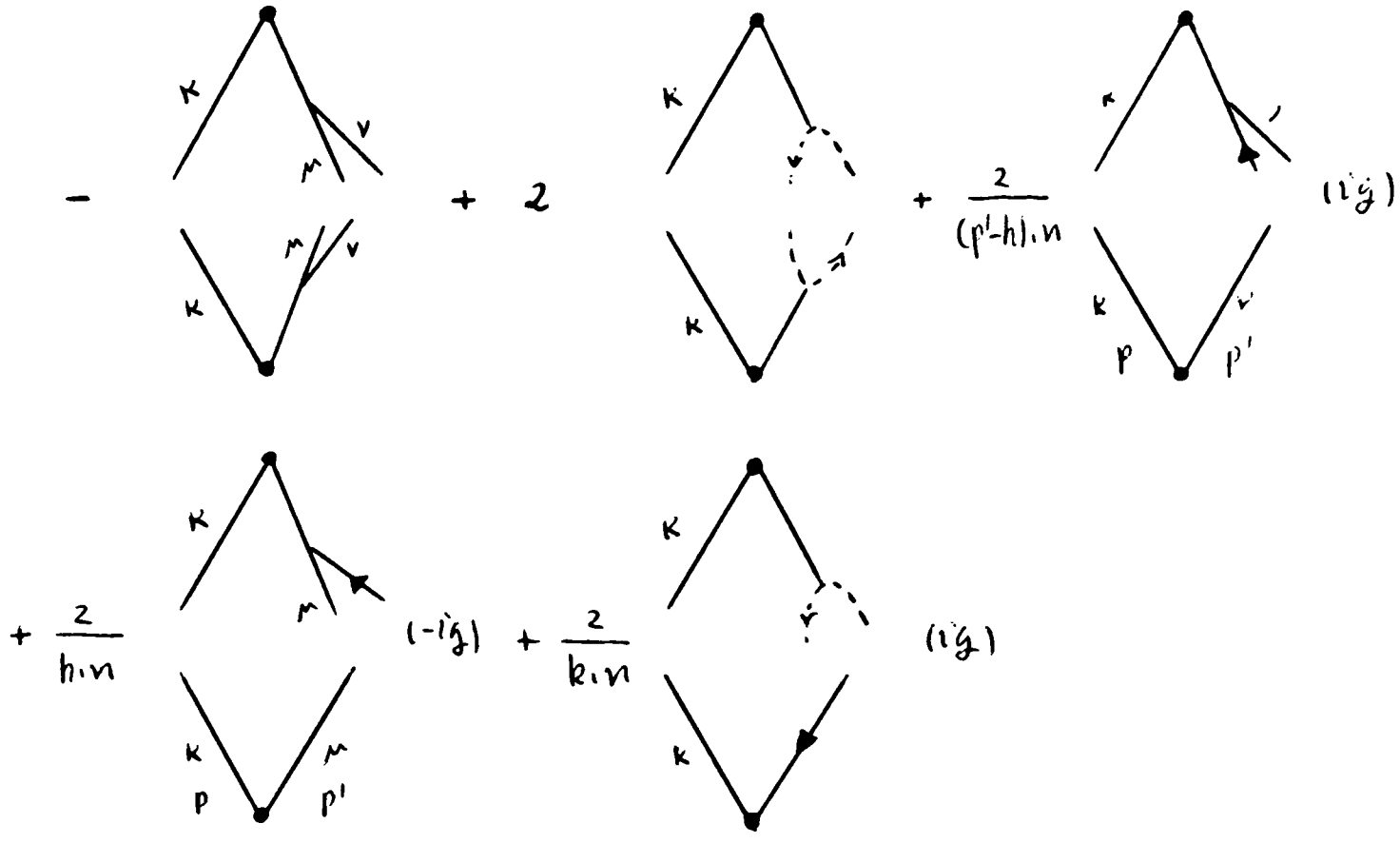


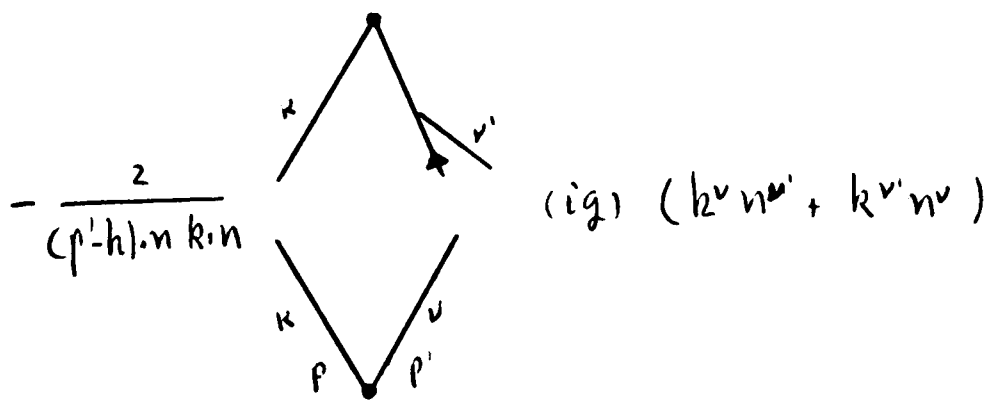




$$(-ig)\,(k_\nu n_{\nu'} + k_{\nu'} n_\nu)$$

On the other hand, similar manipulations for Fig. VIIa' eventually lead to





The second term above is the real ghost term discussed in section 3.2. The third to sixth terms cancel the first terms of the second to fourth lines in the previous page (with a factor of 2 because it is a cross term). Similarly, the second terms of the first, third and fourth lines in the previous page compensate the n-dependence from Fig. VIIc', while the rest corresponds to VIIc''. In this way, all the n-dependence vanishes, but no extra ghost terms remain.

Chapter 4

BEYOND PERTURBATION THEORY?

4.1 Introduction

As discussed in the preceding chapter, the example we presented there, as well as other perturbation theoretic investigations, does not show peculiarities drastically different from the QED results.* Order by order in the coupling constant, suitably defined cross sections for in principle measurable QCD processes are finite. The possibility has been suggested (61,67) that the perturbation expansion would still give a signal for confinement if properly reinterpreted in such a way that non-perturbative effects are incorporated. In this respect we describe in Appendix C-1 a definite proposal of Cornwall and Tiktopoulos (61).

However, even without abandoning hope of gaining some insight from the study of the perturbation expansion, it is likely that understanding of confinement will require more powerful methods. To date there are in quantum field theory hardly any means of transcending perturbation theory. The renormalization group is a powerful tool insofar as one has at least qualitative information on the functions $\beta(g)$ and $\gamma(g)$; this is possible in the short distance limit because of asymptotic freedom. No such perturbative estimate of $\beta(g)$ can be made with a presumably very strong long distance effective coupling. On the other hand, the usefulness of the renormalization group equation in the ultraviolet region depends crucially on being able to turn it into a scaling equation as the effect of the masses is neglected. Then the effect of scaling up a set of momenta far off their mass-shells can be studied by isolating the dependence on the kinematical invariants

---

* Except those associated with real gluons, as discussed in Sec. (3.5).

which are getting large. In contrast, in the IR limit, momenta approach the mass-shell and, although the effective β-function is that of a massless theory (Sec. 4.3), the mass dependence of Green's functions remains. This sort of difficulty precludes the summation of the large IR logarithms appearing in perturbative calculations and which might give rise to a confining potential.

Several authors (Cornwall and Tiktopoulos (61), Frenkel et al. (62,64), Korthals Altes and De Rafael (68,75)) have performed computations of on-shell infrared-regulated Feynman amplitudes in QCD, in the leading logarithm approximation, up to order g^4. Despite the complexity of the calculations, remarkably simple factorization and exponentiation properties have emerged, not only in the IR regime (infrared cut-off λ << other masses or momentum transfers), but also in the wide-angle limit of quark scattering off a colour singlet current ($q^2 >> p^2-m^2$, where q, p and m are momentum transfer, quark momentum and mass, respectively) and wide fixed-angle limit of quark-quark scattering ($s, t >> m^2 >> p^2-m^2$, $\frac{s}{t}$ fixed). Moreover, it has been found that the infrared behaviour is governed by the renormalization group β-function of the pure Yang-Mills theory.

The results in the IR regime are expressed by an equation of the type:

$$\lambda \frac{\partial}{\partial \lambda} T(p_i, \lambda, g) = \left[\beta(g) \frac{\partial}{\partial g} + \Gamma(p_i \cdot p_j, g)\right] T(p_i, \lambda, g) \quad (4.1)$$

where $T(p_i, \lambda, g)$ is the on-shell amplitude, λ is the IR cut-off (either gluon mass or distance from the mass-shell), and $\Gamma(p_i \cdot p_j, g)$ is a "momentum dependent anomalous dimension" depending on the invariants $p_i \cdot p_j$. Eq. (4.1) is consistent with fourth-order

calculations of leading singularities. In the case of quark scattering in an external colourless potential (J. Frenkel, R. Meuldermans, I. Mohammad and J. C. Taylor (62):

$$\Gamma\left(q^2/m^2, g\right) = \frac{+ig^2}{(2\pi)^4} C_F \frac{\partial}{\partial(1/\epsilon)} \int \frac{d^{4+\epsilon}k}{k^2} \left[\frac{p\cdot p'}{p\cdot k\, p'\cdot k} - \frac{1}{2}\frac{m^2}{(p\cdot k)^2} - \frac{1}{2}\frac{m^2}{(p'\cdot k)^2} \right] \quad (4.2)$$

where C_F is the Casimir eigenvalue in the fermion representation.

Plausible arguments (63,64) indicate that (4.1) is true for leading singularities to all orders, with the Γ term given precisely by (4.2). The hope was originally expressed that eq. (4.1) would include also non-leading singularities. However, examination of non-leading contributions to order g^4 (J. Frenkel and J. C. Taylor (65)) shows that the function (4.2) is non-trivially modified by terms of higher order in g. It may be expected that Γ is, in general, λ dependent. Then eq. (4.1) contains useful information or not, according to the form of the dependence on λ. If Γ is λ independent, the solution can be obtained in the usual manner and is a function of $\bar{g}(\lambda)$ and λ only, where $\bar{g}(\lambda)$ is the renormalization group effective coupling. More complicated dependence of Γ on λ will imply solutions with non-trivial dependence on g as well as $\bar{g}(\lambda)$ and λ. We mention that equations of the type (4.1) have already been applied to models of high energy scattering (70).

In the QED case, it has been shown by Korthals Altes and De Rafael (71) that the on-shell vertex function obeys the differential equation:

$$\left[\lambda\frac{\partial}{\partial\lambda} - \Gamma\left(q^2/m^2, g\right)\right] T\left(q^2/m^2, \lambda^2/m^2, g\right) = 0 \quad (4.3)$$

Here Γ is given exactly by (4.2) for $C_F=1$, with no higher order modifications and no β-term appearing. In contrast to the non-abelian theory, all the IR singularities are contained in this equation which provides an alternative proof of the factorization and exponentiation of the IR logarithms in QED. The proof proceeds by straightforward examination of the effect of the photon mass insertion operator $\lambda\frac{\partial}{\partial\lambda}$ in the perturbative expansion of the vertex functions and systematic cancellations are shown to take place due to the simple Ward identities.

In sections 3 and 4 of this Chapter we briefly review the decoupling theorem and examine in some detail its consequences for the near mass-shell coupling of QCD. Subsequent sections are devoted to the discussion of possible renormalization group-type equations that can give information on the near mass-shell behaviour of amplitudes. Because of the difficulty in applying renormalization group scaling arguments to QCD Green's functions in the IR region, one is led to consider equations involving the IR cut-off which, it is argued, are of the Callan-Symanzik type and are related to the IR equations presented above. General formal expressions for such equations will be derived and discussed.

As a further motivation for this discussion, we give in Appendix C-1 a brief description of the sort of nonperturbative effect that can be extracted from the IR equations and which has led (Cornwall and Tiktopoulos (61)) to the suggestion of a specific confinement mechanism.

4.2 The Decoupling Theorem and Zero Mass Renormalization

As a preparation for forthcoming developments we now discuss a result of Appelquist and Carazzone (26) on the behaviour of Feynman graphs at low external momenta. For definiteness, we consider a theory like QCD in which a set of massless vector fields is coupled to a set of massive fermions. The following important result can be proved in perturbation theory (26): For any one-particle-irreducible Feynman graph with external vector meson lines only but containing internal fermions, when all the external momenta are small relative to m (the fermion mass), the graph will be suppressed by some power of momentum relative to a graph with the same number of external vector mesons but no internal fermions. One important qualification is that the mass scale μ defining the subtraction points must satisfy $\mu << m$. Then, graphs with internal fermion loops are suppressed as $O(\frac{k^2}{m^2}, \frac{\mu^2}{m^2})$ when the external momenta $k \to 0$.

This result is rather general and in fact applies to any renormalizable theory with different mass scales, at momenta small compared to the large masses, the dynamics is determined by the interactions of the light quanta. Consequently, in the low momentum limit, the β-function of QCD reduces to the β-function of the pure Yang-Mills theory, order by order in perturbation theory.

We have stated the decoupling theorem here as it will prove relevant to the discussion in the ensuing sections. However, we now wish to point out that in the actual application of the theorem care is needed as to the renormalization conventions employed. In this context, we examine Weinberg's zero mass renormalization method (38). The method has been devised in order to elucidate

certain assumptions involved in the use of the renormalization group and which are based on perturbation theory estimates, such as mass independence in the deep euclidean region. In this scheme all counterterms are defined at zero mass in order that renormalization constants as well as $\beta(g)$ and $\gamma(g)$ be mass independent.

As a simple example we consider the lowest order fermion contribution to the vacuum polarization tensor:

$$\Pi_{\mu\nu}(k^2) = (k_\mu k_\nu - g_{\mu\nu} k^2)\,\Pi(k^2) \tag{4.4}$$

where, using dimensional regularization, one has:

$$\Pi(k^2) = i\frac{g^2}{\pi^2}\mu^{4-d} C_F \frac{1}{d-4}\int_0^1 dx\,(1-x)x\left[k^2 x(1-x) - m^2\right]^{\frac{d}{2}-2} \tag{4.5}$$

If the counterterm is defined at $k^2=-\mu^2$ <u>and</u> at zero mass, one finds:

$$\Pi^{Ren}(k^2) = \Pi(k^2) - \Pi(-\mu^2)\Big|_{m=0} \underset{k^2\to 0}{\sim} \Theta\left(\ln\frac{\mu^2}{m^2}\right) \tag{4.6}$$

which is by no means small if $\mu \ll m$ and therefore the effect of the fermion loop is not negligible at low momenta.* One notes here that zero mass renormalization is equivalent to the "minimal subtraction" in dimensional regularization whereby one subtracts off only the poles (43). We see, therefore, that this prescription is incompatible with a straightforward application of the decoupling theorem.

---

* This point is discussed by Olesen (84), but not in the dimensional regularization scheme.

However, if the subtraction is performed by a counterterm defined at $k^2=-\mu^2$ but $m^2\neq 0$, then

$$\Pi^{Ren}(k^2) = \Pi(k^2) - \Pi(-\mu^2) \sim \int_0^1 dx\,(1-x)x \ln\left[\frac{k^2(1-x)x - m^2}{-\mu^2(1-x)x - m^2}\right] \sim \mathcal{O}\left(\frac{k^2}{m^2}, \frac{\mu^2}{m^2}\right) \tag{4.7}$$

with $k \to 0$ and $\mu^2 << m^2$. Thus, even in dimensional regularization, there seems to exist a consistent procedure such that the decoupling theorem is valid, although this is not the conventional minimal subtraction.

4.3 The Near Mass-Shell Coupling in QCD

We now consider QCD again, a set of coloured massless gauge fields A_μ^a (gluons) interacting with massive fermion fields ψ^a (quarks). The fermions have mass m defined by the pole of the complete fermion propagator, i.e. mass renormalization is carried out so that the one-particle-irreducible fermion self-energy vanishes at $\not{p}=m$.

The unrenormalized gluon propagator defines the constant Z_3 by the renormalization condition:

$$k^2 D_{UR\,\mu\nu}(k^2)\Big|_{k^2=-\mu^2} = Z_3\, g_{\mu\nu} + \text{gauge terms} \tag{4.8}$$

for a given mass scale μ. The complications associated with the gauge dependence are unessential and can be avoided by working in the transverse Landau gauge (32) which can always be assumed.

The renormalized gluon propagator will have the form:

$$D_{\mu\nu}(k^2) = Z_3^{-1} D_{UR\,\mu\nu}(k^2) = \frac{1}{k^2} g_{\mu\nu}\, d\left(\frac{k^2}{\mu^2}, \frac{m^2}{\mu^2}, g(\mu)\right) + \text{gauge terms} \tag{4.9}$$

with the normalization:

$$d(-1, \frac{m^2}{\mu^2}, g(\mu)) = 1 \tag{4.10}$$

We now wish to consider the theory renormalized at two different mass scales, λ and μ, such that $0 < \lambda < \mu$; λ^2, $\mu^2 << m^2$. One may, for instance, keep μ fixed as a reference mass scale and let λ be a running parameter which can be arbitrarily small. The scale parameter may be taken to be $t = \ln\frac{\lambda}{\mu}$ so that $t \to -\infty$ corresponds to the mass-shell limit.

Let $\tilde{Z}_3$ correspond to the propagator renormalization at $k^2=-\lambda^2$. By reference to the unrenormalized propagator it is readily seen that:

$$\tilde{Z}_3\, d(\frac{k^2}{\lambda^2}, \frac{m^2}{\lambda^2}, g(\lambda)) = Z_3\, d(\frac{k^2}{\mu^2}, \frac{m^2}{\mu^2}, g(\mu)) \tag{4.11}$$

where quantities in the LHS are defined at a renormalization point λ^2 near the mass shell. The finite renormalization group transformation then reads:

$$d(\frac{k^2}{\lambda^2}, \frac{m^2}{\lambda^2}, g(\lambda)) = z_3\, d(\frac{k^2}{\mu^2}, \frac{m^2}{\mu^2}, g(\mu)) \tag{4.12}$$

where, from the normalization condition (4.10):

$$z_3^{-1} = Z_3^{-1}\tilde{Z}_3 = d(-\frac{\lambda^2}{\mu^2}, \frac{m^2}{\mu^2}, g(\mu)) \tag{4.13}$$

Similarly, the renormalized three-point vector vertex is given by:

$$\Gamma^{abc}_{\alpha\beta\gamma}(k,p,q,\mu,g(\mu)) = Z_1 \Gamma^{abc}_{\mu\alpha\beta\gamma}(k,p,q,g_{\mu\alpha}) \tag{4.14}$$

where the upper and lower indices are internal and Lorentz indices, respectively. In analogy with (4.12), we have:

$$\Gamma^{abc}_{\alpha\beta\gamma}(k,p,q,\mu,g(\mu)) = z_1 \Gamma^{abc}_{\alpha\beta\gamma}(k,p,q,\lambda,g(\lambda)) \tag{4.15}$$

and

$$z_1 = Z_1 \tilde{Z}_1^{-1}$$

It is possible to fix the counterterm so that the general structure of the three-point vector vertex at the symmetric point $p^2=q^2=k^2=-\lambda^2$ is (Lee and Zinn-Justin (72)):

$$i\Gamma^{abc}_{\alpha\beta\delta}(k,p,q,\mu,g(\mu))\Big|_{p^2=q^2=k^2=-\lambda^2} =$$

$$= G\left(-\frac{\lambda^2}{\mu^2}, \frac{m^2}{\mu^2}, g(\mu)\right)\left[(k-p)_\alpha g_{\beta\gamma} + (p-q)_\gamma g_{\alpha\beta} + (q-k)_\beta g_{\alpha\beta}\right] \tag{4.16}$$

with the normalization:

$$G\left(-1, \frac{m^2}{\mu^2}, g(\mu)\right) = 1 \tag{4.17}$$

Other possible tensor structures in the RHS of (4.16) involve three powers of the momentum and are superficially convergent (72). When p, q, k are $O(\lambda)$, they can be neglected as $\lambda \to 0$.

Because of the normalization of $\Gamma^{abc}_{\alpha\beta\gamma}$ at $p^2=q^2=k^2=-\lambda^2$ and (4.17), on comparing (4.15) and (4.16) it can be seen that:

$$\mathcal{Z}_1 = G\left(-\frac{\lambda^2}{\mu^2}, \frac{m^2}{\mu^2}, g(\mu)\right) \tag{4.18}$$

Now we make the assumption that the decoupling theorem discussed in the foregoing section holds beyond perturbation theory. We know that for the Feynman graphs in the perturbative expansion of $D_{\mu\nu}(k^2)$ and $\Gamma^{abc}_{\alpha\beta\gamma}(k, p, q, g(\mu))$, the contributions of massive fermions are negligible when the external momenta are $\sim\lambda^2 << m^2$ (and $\mu^2 << m^2$). The assumption that this holds for the summed up perturbation series means that $\mathcal{Z}_3$ and $\mathcal{Z}_1^{-1}$, as defined by (4.13) and (4.18), become independent of the fermion mass in the IR limit $\lambda^2 << m^2$.

The renormalized coupling is defined by the rescaling

$$g(\mu) = Z_3^{3/2} Z_1^{-1} g_{un} \tag{4.19}$$

and similarly for $g(\lambda)$.

Hence we find:

$$\begin{aligned} g(\lambda) &= \mathcal{Z}_3^{-3/2} \mathcal{Z}_1 g(\mu) \\ &= g(\mu) G\left(-\frac{\lambda^2}{\mu^2}, g(\mu)\right) d^{3/2}\left(-\frac{\lambda^2}{\mu^2}, g(\mu)\right) \end{aligned} \tag{4.20}$$

where we have explicitly indicated that $g(\lambda)$ is a function of $g(\mu)$ and the ratio $\frac{\lambda}{\mu}$ only, as λ and μ become small compared to m. This sort of m-independent relation between $g(\lambda)$ and $g(\mu)$ is equivalent to the statement that the β function of the massive theory becomes effectively the β function of the pure Yang-Mills theory in the IR limit. We have preferred to present the argument in detail for the sake of clarity and to exhibit the nature of the assumptions involved.

Defining $t=\ln\frac{\lambda}{\mu}$, $g(\mu)\equiv g$, $g(\lambda)\equiv\bar{g}(t,g)$ and noting that $g(\lambda)$ in (4.20) must be independent of μ, we have:

$$\left[\mu\frac{\partial}{\partial\mu}+\mu\frac{\partial g}{\partial\mu}\frac{\partial}{\partial g}\right]\bar{g}\left(t=\ln\frac{\lambda}{\mu},g(\mu)\right)=0 \qquad (4.21)$$

With the replacement $\mu\frac{\partial}{\partial\mu}\big|_\lambda=-\lambda\frac{\partial}{\partial\lambda}\big|_\mu=-\frac{\partial}{\partial t}$ we find:

$$\left[-\frac{\partial}{\partial t}+\beta(g)\frac{\partial}{\partial g}\right]\bar{g}(t,g)=0 \qquad (4.22)$$

with $\beta(g)=\mu\frac{\partial}{\partial\mu}g$ and $\bar{g}(0,g)=g$. Eq. (4.22) characterizes $g(\lambda)=\bar{g}(t,g)$ as the effective coupling, the IR limit being $t\to-\infty$. This derivation of eq. (4.22) is simpler than those found in the literature (34).

There is an alternative way to introduce the scale parameter λ in order to define the near mass-shell coupling. Instead of normalizing the Green's functions at λ^2, as we have done, one can normalize them strictly on the mass shell, and control the IR divergences by introducing λ as a gluon mass put by hand in, say, the Feynman gauge gluon propagator. The on-shell coupling so defined acquires a λ dependence and all the considerations above apply to it, in particular eq. (4.22) holds. In the Abelian theory, the gauge dependent and IR divergent renormalization constants Z_1 and Z_2 do not renormalize the charge because of the simple Ward identity $Z_1=Z_2$, and since Z_3 is IR finite, the usual on-shell defined coupling has no λ dependence. In the non-Abelian theory, there is no such simple Ward identity, and Z_3 is λ dependent because of the gluon self-interactions. Therefore, from this point of view, the question of whether the on-shell-defined coupling is IR singular is equivalent to that of the presumed divergent behaviour of the renormalization group effective coupling at long distances.

4.4 The IR Differential Equation

The renormalization group equation cannot be usefully applied before one makes use of supplementary information in order to eliminate some variables. In the ultraviolet (deep euclidean) region, namely, as momenta $p_i^2 >> m^2$ and renormalization mass scale $\mu^2 >> m^2$, the masses drop out and the scaling analysis can be applied. This means that the only dependence is essentially on the ratios $\frac{p_i \cdot p_j}{\mu^2}$ so that $\mu\frac{\partial}{\partial\mu} \to -\frac{\partial}{\partial t}$ where t is the parameter that scales momenta as $p \to e^t p$. By the same token one has:

$$\beta\left(\frac{\mu^2}{m^2}, g(\mu)\right) \xrightarrow[\mu^2/m^2 \to \infty]{} \beta(g(\mu)) + \mathcal{O}\left(\frac{m^2}{\mu^2}\right) \qquad (4.23)$$

In the Callan-Symanzik (C-S) approach (Appendix O), the renormalization mass scale is "the mass" of the theory, and no extra parameter μ is introduced. As a reflection of this, the C-S β function only depends on the coupling g, but not on the mass.

As a consequence of the decoupling theorem, in the infrared regime:

$$\beta\left(\frac{\mu^2}{m^2}, g(\mu)\right) \xrightarrow[\mu^2/m^2 \to 0]{} \beta_{YM}(g(\mu)) + \mathcal{O}\left(\frac{\mu^2}{m^2}\right) \qquad (4.24)$$

where β_{YM} is the β function of the pure Yang-Mills theory. In the infrared case, the relevant variable that should be scaled by the subtraction point μ is, typically, $p^2 - m^2$, the distance of external momenta from the mass-shell. However, the scaling analysis cannot be applied in general because there are too many variables. There is no analogue of Weinberg's theorem in the IR region, and the mass

dependence of Green's functions cannot be ignored, besides $\frac{p^2-m^2}{\mu^2}$ there will be a dependence on $\frac{\mu^2}{m^2}$.

Poggio and Quinn (73) have analysed this situation in the more familiar case of the fermion-fermion-photon vertex $\Gamma_\mu(p, p')$ in QED. They show that for a choice of subtractions at $p^2-m^2=p'^2-m^2=q^2=-\mu^2$, then:

$$\Gamma\left(\frac{p_i^2-m^2}{\mu^2}, \frac{m^2}{\mu^2}, \frac{q^2}{\mu^2}, g(\mu)\right) \longrightarrow \Gamma_{IR}\left(\frac{p_i^2-m^2}{\mu^2}, g(\mu)\right) + O\left(\frac{\mu^2}{m^2}, \frac{p_i^2-m^2}{m^2}, \frac{q^2}{m^2}\right)$$

whenever

$$p_i^2-m^2 \lesssim \mu^2 \ll m^2$$
$$q^2 \lesssim \mu^2 \ll m^2 \qquad (4.25)$$

Thus, only the dependence on the interesting variable $\frac{p^2-m^2}{\mu^2}$ is left. The renormalization group equation can then be solved (with $q=p-p'=0$) to obtain the near mass-shell structure of the electron propagator, with the usual factorization and exponentiation result.*

No such simplicity can be expected in the non-Abelian theory. The result (4.25) is a consequence of the fact that the photon does not interact with other massless particles. In the non-Abelian case, leading singularity terms will also have no $\frac{\mu^2}{m^2}$ dependence, but, as examination of low order computation shows (62,68,75), non-leading singularities have associated factors of $\ln\frac{\mu^2}{m^2}$. From these considerations one can see that the conventional renormalization group equation cannot be expected to yield useful information on the nearly on-shell behaviour

---

* Poggio and Quinn argue that previous applications of the renormalization group to the QED IR problem (32,26,74) have relied on implicit unjustified assumptions.

of Green's functions.

One alternative is to investigate equations where the variations $\lambda\frac{\partial}{\partial\lambda}$ with respect to the IR cut-off are introduced directly. These are of the type of the conjectured IR equations which have been studied in two-loop calculations and which we have presented in section (4.1). If the IR cut-off is at the same time used as the renormalization scale defining the coupling $g(\lambda)$, it can be expected that the resulting equation should have the Callan-Symanzik form:

$$\left[\lambda\frac{\partial}{\partial\lambda} + \beta(g)\frac{\partial}{\partial g}\right]T = \Delta_\lambda \qquad (4.26)$$

where Δ_λ is the "mass insertion" term. This is because λ is being used both as a mass parameter and as the renormalization point, hence playing the same role as the renormalized mass in the C-S equation, and also because β (by the decoupling theorem) depends only in $g(\lambda)$. We shall later elaborate on and make this statement more precise.

There is more than one way to formulate the IR differential equations of QCD, according to the details of the cut-off procedure and renormalization prescription employed. Some authors have used dimensional regularization (62) in other cases (61,63,68) a gluon mass is introduced. We shall consider two particular schemes: (i) Renormalization is performed nearly on-shell, e.g. the gluon self-energy is subtracted at $k^2=\lambda^2$ with $0 \lesssim \lambda^2 << m^2$ (in the manner of Poggio and Quinn explained above). This same parameter λ is used as the distance of external momenta from the mass-shell, e.g. for the fermion-fermion-gluon vertex $\Gamma_{\mu a}(p,p',q)$, $p^2-m^2=-\lambda^2$, $p^2-m^2=-\lambda^2$, $q^2=-\lambda^2$.

(ii) Renormalization is performed at a mass scale μ (which may be taken as $\mu^2 << m^2$), and the IR cut-off λ just keeps external momenta off the mass-shell as in (i); μ may be kept fixed while $\lambda \to 0$ in the mass-shell limit.

Prescriptions (i) and (ii) have been called the λ-scheme and μ-scheme, respectively (Korthals Altes and De Rafael (68)).

In any case, whatever the subtraction point considered, physical quantities should not be affected. This is the statement of invariance of the S-matrix $S(p_i, \mu, g(\mu))$ under the renormalization group. Thus, we can write:

$$S(p_i, \lambda, g(\lambda)) = S(p_i, \mu, g(\mu)) \tag{4.27}$$

where p_i stands for the set of momentum variables. The LHS and RHS are the near-mass-shell, renormalized amplitudes parametrized by couplings defined at λ and μ, respectively, corresponding to prescriptions (i) and (ii) above. Since the IR divergences are controlled by the cut-off λ on both sides, the RHS of (4.27) is also λ dependent. We indicate this by writing henceforth $S(p, \mu, g(\mu))_{p \wr \lambda}$. The explicit λ dependence is, of course, different in each side. We shall illustrate (4.27) later (eq. 4.38a,b) by presenting a particular $O(g^4)$ calculation.

Eq. (4.27) is similar to the renormalization group equation relating Green's functions renormalized at two different mass scales, although in this case there is no wavefunction renormalization factor because we are dealing with S-matrix elements.

One can normalize dividing by the Born amplitude; the dependence on dimensionless quantities will then be:

$$S(p, \lambda, g(\lambda))_{p\sim\lambda} = S\left(\frac{p_i\cdot p_j}{\lambda^2}, \frac{m^2}{\lambda^2}, 1, g(\lambda)\right)$$

$$= S\left(\frac{p_i\cdot p_j}{\mu^2}, \frac{m^2}{\mu^2}, \frac{\lambda^2}{\mu^2}, g(\mu)\right) = S(p, \mu, g(\mu))_{p\sim\lambda} \quad (4.28)$$

where $p_i \cdot p_j$ are the kinematical invariants.

If both λ and μ are much smaller than other masses in the theory (which would not preclude $\lambda \ll \mu$), the developments in the previous sections (eq. 4.20) imply that $g(\lambda)$ is a function of λ/μ and $g(\mu)$ only:

$$g(\lambda) \equiv \tilde{g}\left(\frac{\lambda}{\mu}, g(\mu)\right) \quad (4.29)$$

Moreover, $\tilde{g}(\frac{\lambda}{\mu}, g(\mu))$ is not overall μ dependent (eq. 4.21), so that:

$$\left[\mu\frac{\partial}{\partial\mu} + \mu\frac{\partial g}{\partial\mu}\frac{\partial}{\partial g}\right]\tilde{g}\left(\frac{\lambda}{\mu}, g(\mu)\right) = 0 \quad (4.30)$$

Now,

$$\lambda\frac{\partial}{\partial\lambda}\, g(\lambda) = \lambda\frac{\partial}{\partial\lambda}\,\tilde{g}\left(\frac{\lambda}{\mu}, g(\mu)\right)\Big|_{\mu}$$

$$= -\mu\frac{\partial}{\partial\mu}\,\tilde{g}\left(\frac{\lambda}{\mu}, g(\mu)\right)\Big|_{\lambda} \quad (4.31)$$

Therefore, defining $\beta(g) = \mu\frac{\partial}{\partial\mu}g(\mu)$, and using (4.30), we have:

$$\lambda\frac{\partial}{\partial\lambda}\, g(\lambda) = \beta(g)\frac{\partial}{\partial g}\,\tilde{g}\left(\frac{\lambda}{\mu}, g(\mu)\right) \quad (4.32)$$

Furthermore, defining:

$$\beta(\tilde{g}) = \lambda \frac{\partial}{\partial \lambda} \tilde{g}\left(\frac{\lambda}{\mu}, g(\mu)\right) = \lambda \frac{\partial}{\partial \lambda} g(\lambda) \tag{4.33}$$

we have from (4.32):

$$\beta(\tilde{g}) = \beta(g) \frac{\partial \tilde{g}}{\partial g} \tag{4.34}$$

where, for brevity here and henceforth, $g \equiv g(\mu)$ and $\tilde{g} \equiv g(\lambda) = \tilde{g}(\frac{\lambda}{\mu}, g(\mu))$.

Consequently, we have from (4.27) and (4.34),

$$\beta(\tilde{g}) \frac{\partial}{\partial \tilde{g}} S(p_i, \lambda, \tilde{g})_{p\sim\lambda} = \beta(g) \frac{\partial \tilde{g}}{\partial g} \frac{\partial}{\partial \tilde{g}} S(p_i, \lambda, \tilde{g})_{p\sim\lambda} \tag{4.35}$$

$$= \beta(g) \frac{\partial}{\partial g} S(p_i, \mu, g)_{p\sim\lambda}$$

Operating with $\lambda\frac{\partial}{\partial\lambda}$ on both sides of (4.27), and using (4.33) and (4.35), we obtain immediately:

$$\left[\lambda \frac{\partial}{\partial \lambda} + \beta(\tilde{g}) \frac{\partial}{\partial \tilde{g}}\right] S(p, \lambda, \tilde{g})_{p\sim\lambda} = \lambda \frac{\partial}{\partial \lambda} S(p, \mu, g)_{p\sim\lambda} \tag{4.36a}$$

and

$$\left[\lambda \frac{\partial}{\partial \lambda} - \beta(g) \frac{\partial}{\partial g}\right] S(p, \mu, g)_{p\sim\lambda} = \lambda \frac{\partial}{\partial \lambda} S(p, \lambda, \tilde{g})_{p\sim\lambda} \tag{4.36b}$$

It should be remarked that the physical input used in deriving (4.36a,b) is the renormalization group invariance of the S-matrix elements and the decoupling theorem. We can also write, using (4.27) again:

$$\left[\lambda\frac{\partial}{\partial\lambda} - \beta(g)\frac{\partial}{\partial g} + \gamma\right] S(p,\mu,g)_{p\sim\lambda} = 0 \qquad (4.37)$$

where:

$$\gamma\left(\frac{p_i\cdot p_j}{m^2}, \frac{\lambda^2}{m^2}, \tilde{g}\left(\frac{\lambda}{\mu},g\right)\right) = -\lambda\frac{\partial}{\partial\lambda}\ln S(p,\lambda,\tilde{g})_{p\sim\lambda}$$

which is a general expression for the "momentum dependent anomalous dimension" term.

Now we check the validity of (4.36a,b), making use of the $O(g^4)$ calculation of leading singularities for (exclusive) quark scattering in an external colourless potential (Frenkel et al. (62), Korthals Altes and De Rafael (68)), according to which (dividing by the Born amplitude):

$$S(p,\mu,g)_{p\sim\lambda} = 1 + \left\{1 - \frac{1}{2}\frac{11}{24\pi^2}g^2 C_{YM}\ln\frac{\lambda}{m}\right\} g^2 C_F \gamma(q^2)\ln\frac{\lambda}{m} + \frac{1}{2}\left[g^2 C_F\gamma(q^2)\ln\frac{\lambda}{m}\right]^2 + O\left(g^4\ln\frac{\lambda}{m}\ln\frac{\mu}{m}\right) \qquad (4.38a)$$

$$S(p,\lambda,\tilde{g})_{p\sim\lambda} = 1 + \left\{1 + \frac{1}{2}\frac{11}{24\pi^2}\tilde{g}^2 C_{YM}\ln\frac{\lambda}{m}\right\}\tilde{g}^2 C_F\gamma(q^2)\ln\frac{\lambda}{m} + \frac{1}{2}\left[\tilde{g}^2 C_F\gamma(q^2)\ln\frac{\lambda}{m}\right]^2 + \ldots \qquad (4.38b)$$

These results assume $\left|\ln\frac{\lambda}{m}\right| >> \left|\ln\frac{\mu}{m}\right|$ (i.e. $\ln\frac{\lambda}{\mu} \simeq \ln\frac{\lambda}{m}$), which is consistent with keeping only leading singularities. The function $\gamma(q^2)$ is given by (4.2). We note the change of sign from (4.38a)

to (4.38b) in the $C_F C_{YM} \ln^2 \frac{\lambda}{m}$ contribution. The coefficient of this term appears to be related to the β function of the pure Yang-Mills theory:

$$\beta(g) = -\frac{11}{48\pi^2} C_{YM} g^3 + \cdots \tag{4.39}$$

On the other hand, the connection between $\tilde{g}$ and g is:

$$\tilde{g}^2 = \tilde{g}^2\left(\frac{\lambda}{\mu}, g\right) = g^2\left(1 - \frac{11}{24\pi^2} g^2 C_{YM} \ln\frac{\lambda}{\mu} + \cdots\right) \tag{4.40}$$

and $\tilde{g}^4 = g^4 + O(g^6)$. This relation can be obtained by considering (4.20) in the axial gauge (64,76,30). In that gauge $Z_1 = Z_2$, and the number $\frac{11}{24\pi^2}$ is essentially the coefficient of the covariant part of the gluon propagator. The non-covariant part does not contribute to the leading singularities (64). Eq. (4.40) can also be got by realizing that $g(\lambda) = \tilde{g}(\frac{\lambda}{\mu}, g)$ has, because of (4.30) and (4.33), the form of the renormalization group effective coupling with $\tilde{g}(1, g) = g$.

Using (4.38b) and (4.40) we find

$$\lambda\frac{\partial}{\partial\lambda} S(p,\mu,g)_{p\sim\lambda} - \lambda\frac{\partial}{\partial\lambda} S(p,\lambda,\tilde{g})_{p\sim\lambda} =$$

$$= -\frac{11}{24\pi^2} g^4 C_{YM} C_F \gamma(q^2) \ln\frac{\lambda}{m} \tag{4.41}$$

$$= \beta(g)\frac{\partial}{\partial g} S(p,\mu,g)_{p\sim\lambda}$$

Thus, (4.38b) is verified to order g^4 in the leading logarithm approximation. Eq. (4.36a) is readily checked in a similar way.

4.5 Infrared Structure of the Quark Magnetic Moment

It is known that the electron magnetic moment is IR finite to all orders of perturbation theory (21,71), as it should be, being a measurable quantity. The on-mass-shell vertex function in QED has the structure:

$$-ie^2\bar{u}(p')\left(\gamma^\mu F_1(q^2)+\frac{i}{2m}\sigma^{\mu\nu}q_\nu F_2(q^2)\right)u(p) \qquad (4.42)$$

where q is the momentum transfer. In perturbation theory, the form factors F_1 and F_2 are power series in α with IR singular coefficients. These singularities exponentiate, as shown by Yennie et al. (21):

$$F_i\left(q^2/m^2,\lambda^2/m^2,\alpha\right)=\exp\left\{\frac{\alpha}{\pi}B(q^2/m^2)\ln\frac{\lambda}{m}\right\}\mathcal{F}_i\left(q^2/m^2,\alpha\right) \qquad (4.43)$$

where $\mathcal{F}_i$ is IR finite and $B(\frac{q^2}{m^2})$ is obtained from the one-loop calculation. Singe $B(0)=0$, the static quantities $F_1(q^2=0)$ and $F_2(q^2=0)$, which correspond to the electric charge and the anomalous magnetic moment of the electron, respectively, have no IR singularities.

In the search for insights into the meaning of mass-shell quantities in QCD, it is natural to investigate the analogue of (4.42) and the quark magnetic moment. This object is defined by scattering of a quark carrying both charge and colour in a weak external magnetic field. The form factors in (4.42) arise from all possible higher order 1PI corrections to the quark-photon-quark vertex, including gluonic exchanges. It turns out from explicit two-loop calculations (Korthals Altes and De Rafael (75)) that the magnetic moment amplitude is IR singular when expanded in terms of the coupling constant defined

on the mass-shell, in contrast with the QED result. Further, it is found that the IR divergence is governed (to that order) by the renormalization group function $\beta(g)$. The one-loop contribution is QED-like and yields the lowest order g-2 correction. However, the IR singularity appears in $O(g^4)$ from the gluon self-couplings and the lack of simple cancellations between vertices and self-energies (as implied by the $Z_1=Z_2$ identity in QED).

On the other hand, on the basis of a method for separating the IR divergent parts (77), Cvitanović (53') has shown that the unrenormalized amplitude for the magnetic moment is free of IR divergences in both QED and QCD, and therefore that in this quantity all IR singularities are introduced by the renormalization on the mass-shell.

If the unrenormalized amplitude is IR finite, further scaling by off-shell renormalization constants will not introduce any IR singularity. This situation can be understood by a simple physical argument. The static quantities $F_1(0)$ and $F_2(0)$ are defined at zero momentum transfer so that they correspond to processes for which there is no bremsstrahlung. If the Bloch-Nordsieck cancellation (or the KLN theorem of section 1.4) is at work in QCD, as seems to be the case, there can be no virtual singularities either, so that $F_1(0)$ and $F_2(0)$ must be IR finite. Moreover, this argument goes through for unrenormalized, or off-shell renormalized, amplitudes, as specified by the KLN theorem.* This means that for the magnetic moment amplitude $a(\mu, g(\mu))_\lambda$ (in our notation of the previous section) we have $\lambda\frac{\partial}{\partial\lambda} a(\mu, g(\mu))_\lambda = 0$. Then the RHS of eq. (4.36a) is zero, and we find:

---

* A related argument is given by Cvitanović (53').

$$\left[\lambda\frac{\partial}{\partial\lambda} + \beta(\tilde{g})\frac{\partial}{\partial\tilde{g}}\right] a(\lambda,\tilde{g})_\lambda = 0 \qquad (4.44)$$

where $\tilde{g}$ is the coupling constant defined in the mass-shell (more precisely, at an arbitrarily small distance λ from the mass-shell).

This equation is that proposed by Korthals Altes and De Rafael on the basis of the $O(g^4)$ calculation of Ref.(75): Eq. (4.44) is thus generally true (order by order) for the QCD magnetic moment, and it is essentially the precise statement of the fact that IR singularities are introduced into this quantity only by on-mass-shell renormalization.

4.6 Discussion

In connection with eqs. (4.36a,b) we would like to make the following remarks:

(i) In deriving (4.36a) and (4.36b), we have exhibited the general structure of the IR equations obtained in explicit low order calculations and conjectured to hold in general. The term in the RHS is analogous to the mass insertion term in the C-S equation. As we remarked in the section 4.4, the equations governing the approach to the mass-shell cannot be expected to be, in general, of the ordinary renormalization group type, but to be rather of the C-S type. The parallel of eqs. (4.36) with the C-S equation (as derived by Coleman (34)) can be made exact by recalling that the C-S mass insertion term is defined by applying $\lambda\frac{\partial}{\partial\lambda}$, where λ is the renormalized mass parameter, to the unrenormalized Green's functions. This corresponds precisely to RHS of (4.36a), if we consider instead finite renormalizations from the "unrenormalized" coupling $g(\mu)$ to the renormalized

one $g(\lambda) = \tilde{g}(\frac{\lambda}{\mu}, g(\mu))$ and correspondingly from $S(p, \mu, g)$ to $S(p, \lambda, \tilde{g})$. No wavefunction renormalization comes into the analogy, since we are dealing with S-matrix elements. In the ultraviolet limit, the C-S mass insertion term can be neglected. Here the equivalent term is necessarily present and can be expected to be rather complicated. The analysis of the term in the RHS could, in principle, follow that by Korthals Altes and De Rafael (71) in the derivation of the corresponding equation in QED (eq. 4.3). This analysis would be much more involved because of the non-commutative nature of the vertices and the gluon self-coupling.

Eqs. (4.36) are formal expressions which are not useful as they stand, but we believe they give some insight and perhaps can motivate alternative approaches to the study of the mass-shell behaviour.

(ii) Eqs. (4.36) are essentially associated with the non-Abelian structure. In QED, $\beta \backsim 0$ and $\tilde{g}^2(\lambda) \to 4\pi\alpha + 0(\frac{\lambda}{m})$ in the IR limit, so that (4.36) becomes a triviality.

(iii) The general relation between the "λ-scheme" and the "μ-scheme" (sec. 4.4 and Ref. (68)) is given by (4.36a,b). In particular, the relative minus sign between the β terms in (4.36a) and (4.36b) has been noted in $0(g^4)$ calculations.

(iv) It would be of interest to formulate our considerations in the dimensional regularization scheme. Apart from gauge invariance, there may be additional advantages as it is found in the dimensional regularization version of the C-S equation (39,40,41). Using the dimensional scheme, Cvitanović (52, 78) has shown that the identity $Z_1 = Z_2$ (and $Z_3 = 1$, for the pure Yang-Mills theory Z_3 to one loop order) hold, independent of the gauge, if renormalization is done on the mass-shell.

Appendix C-1

THE CORNWALL AND TIKTOPOULOS MECHANISM

The IR equation (4.1) suggests that QCD amplitudes for processes involving coloured particles have exponential factors (in the leading logarithm approximation) which make them vanish in the IR limit $\lambda \to 0$. This is qualitatively just what happens in QED, where the singularities due to soft virtual photons exponentiate to give a vanishing cross-section.

Now let us consider soft gluon emission from a coloured process such as quark-quark scattering. Let $T^{(N)}$ be the amplitude for scattering with emission of N soft gluons. Fig. A below shows the elastic amplitude $T^{(0)}$ corrected by lowest order one-gluon emission. The full gluon emission vertex is represented by the blob in Fig. B.

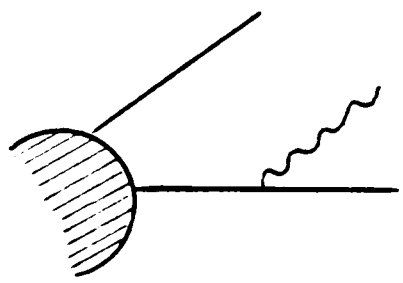


Fig. A

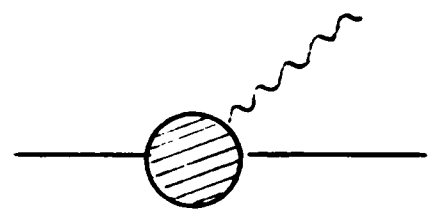


Fig. B

The cross-section for emission of one soft gluon with momentum k in the region $\lambda < |\underset{\sim}{k}| < \Delta$ has the form $\sim g^2 |T^{(0)}|^2 \ln \frac{\Delta}{\lambda}$ in lowest order. In the Abelian theory this logarithmic divergence is not altered by higher order corrections to the photon vertex. Similarly, phase space integrations in the N-photon emission cross-section diverge as $\frac{1}{N!} [g^2 \ln \frac{\Delta}{\lambda}]^N |T^{(0)}|^2$. The sum over N yields an exponential which exactly cancels the vanishing IR factor due to virtual photon singularities in the well-known manner. Thus, QED cross-sections for emission of an indefinite number of soft photons turn out to be finite.

In the non-abelian theory, however, there are higher order corrections due to virtual gluon exchange between the on-shell gluon and other coloured particles in the process. These are depicted by the blob in the quark-quark-gluon vertex in Fig. B. The IR equation (4.1) can be applied to sum up these exchanges (leading logarithm approximation) and give the result:

$$T^{(1)} \sim T^{(0)} \exp\left\{ -\frac{g^2}{32\pi^2} C_A \ln^2 \frac{|\underset{\sim}{k}|}{\lambda} \right\} \tag{4.45}$$

where C_A is the Casimir eigenvalue for the adjoint representation of the colour gauge group. The double logarithm appears because of the exchanges between a massless (gluon) line and a massive (quark) line. In QED, $C_A=0$, and there is no such exponential damping factor.

The cross-section is given by the sum of the squares of the $T^{(N)}$ integrated over gluon phase space:

$$\sigma \sim |T^{(0)}|^2 \left[1 + \text{const.} \int_\lambda \frac{d^3\underset{\sim}{k}}{2|\underset{\sim}{k}|} \frac{1}{(p\cdot k)^2} \exp\left\{ -\frac{g^2}{16\pi^2} C_A \ln^2 \frac{|\underset{\sim}{k}|}{\lambda} \right\} + \cdots \right] \tag{4.46}$$

If the exponential is expanded in powers of g^2, the resulting divergent integrals cancel the singularities from virtual gluons, order by order in g, so that there is no perturbative breakdown of the KLN theorem (section 1.4). On the other hand, if the complete integral is carried out the result is finite as $\lambda \to 0$, the exponential factor has suppressed the original logarithmic divergence. Therefore nothing is left to cancel the virtual IR singularities in $|T^0|^2$, so that σ will vanish when $\lambda \to 0$.

Thus, from this non-perturbative non-cancellation of the IR singularities, a confinement mechanism is implied, assuming that hadrons are group singlet bound states of non-singlet constituents. No particles carrying the group charge (quarks, gluons or coloured bound states) would ever be produced in a hadronic collision.

This proposal of Cornwall and Tiktopoulos has been examined and discussed by several authors (B. W. Lee (79), E. Poggio and H. Quinn (57), J. C. Taylor (80)). It has been pointed out that the scheme entails a particular order of limits: the IR limit $\lambda \to 0$ is carried out after summing up all orders of perturbation theory for a fixed number of emitted gluons. A finite cross-section is obtained, in accord with the KLN theorem, if the limit $\lambda \to 0$ is taken order by order, with an arbitrary number of real gluons included. Another point is that only leading singularities have been considered since the IR equations are presumed to be known only for such leading singularities. The effect of the non-leading ones is unknown.

CONCLUSIONS

The IR and mass singularities in perturbation theory in an Abelian massless theory were studied in Chapter 2. By calculating the soft and hard bremsstrahlung in Compton scattering of a massless electron we have extended previous investigations (decay of massive particles (86,50), scattering in an external potential (27,49)) and given an explicit verification of the KLN theorem for a purely massless scattering process. The appearance of the mass singularity in the renormalization process was described in detail in the case when a common (dimensional) regularization is employed for the various types of divergences, in particular, showing how ultraviolet poles are turned into mass singularities. With appropriate formulae (Appendix A-1), the simplicity of calculation that ensues from setting the electron mass equal to zero from the start suggests the possibility that computation of leading radiative corrections associated with IR and mass singularities (50,21) would be simplified by the use of dimensional regularization with the mass parameters set equal to zero.

Further study of massless QED, in particular by using non-perturbative renormalization group methods, to which the Abelian theory might be amenable, should serve to clarify the distinction between those features due to masslessness and those due to the Abelian or non-Abelian nature.

In Chapter 2, general features of the IR divergence cancellation in the Yang-Mills theory have been studied, the complicated non-Abelian structure can, in principle, be analyzed by systematic use of the Ward identities. We have shown how the Ward identities can

serve to extract the singularities on a graph by graph basis and to show that appropriate sums over physical polarization states yield the "real ghost" singularities compensating those from virtual ghost loops. This suggests that general proofs of the cancellation should be possible. However, apart from the single hard gluon emission case in Sec. (3.2) and the one-loop gluon scattering example of Sec. (3.4), we have limited our considerations to leading singularities; the non-leading ones due to hard virtual momenta may be much more difficult to analyze.

It is often stated that the IR singularities in non-Abelian theories are severe; perturbative investigations are interesting at least because they exhibit these singularities explicitly and are a first modest attempt at their understanding. In Sec. (3.4) we have given evidence that a Bloch-Nordsieck analysis is, in principle, possible even when a purely gluonic process is involved. By calculating the soft and hard bremsstrahlung, we have included non-leading singularities in the order considered. A qualification in order is that even if for processes involving massive quarks the analogue of the Bloch-Nordsieck scheme is exact, the gluonic processes show peculiarities associated with multigluon "jets" in the initial and final states and the need to include both soft and hard bremsstrahlung. Whether these aspects have any physical significance is still unclear. It appears that, generally speaking, low order calculations have not given any insight into the confinement problem.

However, there are two aspects that we believe deserve further study. First, the clarification of the Cornwall and Tiktopoulos mechanism, that is, genuinely non-perturbative effects abstracted

from perturbation theory. It would be of interest to consider massless QED and check whether the order by order cancellation takes place after an exact summation of singularities which perhaps could be achieved by renormalization group methods as poin pointed out above. A second aspect deserving attention is the study of the general behaviour of singularities in the perturbation expansion in higher orders. Recently, a method for determining the power of IR logarithms for arbitrary diagrams has been developed (85).

Our formal considerations of Chapter 4 aimed at the understanding of renormalization group-type equations proposed in order to sum up the singularities to all orders μ of perturbation theory. We remarked that the ordinary renormalization group equations face difficulties which hamper their usefulness in the IR regime. The Callan-Symanzik type equations which we have derived cannot be expected to be useful either as they stand, but we believe they offer an alternative point of view. In the study of near-mass-shell amplitudes the particular renormalization conventions and subtraction procedure employed appear to be crucial, as we illustrated in the discussion of the decoupling theorem in Sec. (4.2). Eqs. (4.36) may be relevant in the analysis of these questions since they relate the IR singularities in the on-shell and off-shell schemes.

REFERENCES

1. C. N. Yang and R. Mills, Phys. Rev. 96 (1954) 191;

2. E. S. Abers and B. W. Lee, Phys. Rep. 9C (1973) 1;
J. C. Taylor, "Gauge Theories of Weak Interactions", Cambridge University Press (1976).

2. T. Appelquist et al., Phys. Rev. Lett. 34 (1975) 365;
E. Eichten et al., Phys. Rev. Lett. 34 (1975) 369; Review of Particle Properties, Rev. Mod. Phys. 48, S1 (1976).

1'. J. Rosner, Proc. XVII Intl. Conf. on High Energy Physics, London, 1974.

2'. D. H. Perkins, Contemporary Physics 16, No.2 (1975).

3. G. Goldhaber et al., Phys. Rev. Lett. 37 (1976) 255;
M. K. Gaillard, B. W. Lee and J. Rosner, Rev. Mod. Phys. 47 (1975) 277.

4. H. D. Politzer, Phys. Rev. Lett. 30 (1973) 1346; D. J. Gross and F. Wilczek, Phys. Rev. Lett. 30 (1973) 1343; G. 't Hooft, Marseille Conf. 1972 (unpublished).

5. S. Coleman and D. J. Gross, Phys. Rev. Lett. 31 (1973) 851.

6. D. J. Gross and F. Wilczek, Phys. Rev. D8 (1973) 3633 and D9 (1974) 980; H. Georgi and H. D. Politzer, Phys. Rev. D9 (1974) 416; H. D. Politzer, Phys. Rep. 14C (1974) 129.

7. J. M. Cornwall and R. E. Norton, Phys. Rev. D8 (1973) 3338;
R. Jackiw and K. Johnson, Phys. Rev. D8 (1973) 2368.

8. M. Gell-Mann in Elementary Particle Physics (ed. P. Urban, Springer-Verlag, Vienna, 1972) 145; W. Bardeen, H. Fritzsch and M. Gell-Mann in "Scale and Conformal Symmetry in Hadron Physics", ed. R. Gatto (Wiley, New York, 1973).

9. S. L. Adler, Phys. Rev. 177 (1969) 2426; J. S. Bell and R. Jackiw, Nuovo Cimento 51A (1969) 47; W. Bardeen, Phys. Rev. 184 (1969) 1848.

10. H. Fritzsch and M. Gell-Mann, Proc. XVI Intl. Conf. on High Energy Physics (1972) Vol.2, 135; Y. Nambu in "Preludes in Theoretical Physics", ed. A. De Shalit, H. Feshbach and L. Van Hove (North Holland) 133; A. De Rujula, H. Georgi and S. L. Glashow, Phys. Rev. D12 (1975) 147.

11. H. J. Lipkin, Phys. Lett. 45B (1973) 147.

12. H. Fritzsch, M. Gell-Mann and H. Leutwyler, Phys. Lett. 47B (1973) 365.

13. K. Johnson, Phys. Rev. D6 (1972) 1101.

14. S. Weinberg, Phys. Rev. Lett. 31 (1973) 494 and Phys. Rev. D8 (1973) 4482; H. Fritzsch, M. Gell-Mann and H. Leutwyler (unpublished).

15. A. Casher, J. Kogut and L. Susskind, Phys. Rev. D10 (1974) 732 and Phys. Rev. Lett. 31 (1973) 792; G. 't Hooft, Nucl. Phys. B75 (1974) 461; C. G. Callan, N. Coote and D. J. Gross, Phys. Rev. D13 (1976) 1649.

16. K. G. Wilson, Phys. Rev. D10 (1974) 2445; J. Kogut and L. Susskind, Phys. Rev. D11 (1975) 395.

17. A. Chodos, R. L. Jaffe, K. Johnson, C. B. Thorn and V. F. Weisskopf, Phys. Rev. D9 (1974) 3471; T. DeGrand, R. L. Jaffe, K. Johnson and J. Kiskis, Phys. Rev. D12 (1975) 2060.

18. J. D. Jackson, "Classical Electrodynamics" (Wiley, New York).

19. F. Bloch and A. Nordsieck, Phys. Rev. 52 (1937) 54.

20. J. M. Jauch and F. Rorlich, Helv. Phys. Acta 27 (1954) 613 and "The Theory of Photons and Electrons" (Addison-Wesley Pub. Co.

(1955) Chap. 16).

21. D. R. Yennie, S. C. Frautschi and H. Suura, Ann. Phys. 13 (1961) 379.

22. K. E. Eriksson, Nuovo Cimento 19 (1961) 1010.

23. G. Grammer and D. R. Yennie, Phys. Rev. D8 (1973) 4332.

24. R. H. Dalitz, Proc. Roy. Soc. A206 (1951) 509.

25. V. Chung, Phys. Rev. 140B (1965) 1110; T. W. B. Kibble, Phys. Rev. 173, 1527; 174, 1882 (1968); P. R. Kulish and L. D. Fadeev, Theor. Math. Phys. 4 (1971) 745; N. Papanicolau, Phys. Rep. 24C (1976) 229.

26. T. Appelquist and J. Carazzone, Phys. Rev. D11 (1975) 2856.

27. T. Kinoshita, J. Math. Phys. 3 (1962) 650; T. Kinoshita and A. Ukawa, Phys. Rev. D13 (1976) 1573.

28. T. D. Lee and M. Nauenberg, Phys. Rev. 133 (1964) B1549.

29. S. Weinberg, Phys. Rev. 140 (1965) B516.

30. See, for example, W. Konetschny and W. Kummer, Nucl. Phys. B100 (1975) 106.

31. M. Gell-Mann and F. Low, Phys. Rev. 95 (1954) 1300.

32. N. N. Bogoliubov and D. V. Shirkov, "Introduction to the Theory of Quantized Fields" (Wiley, New York).

33. C. G. Callan, Phys. Rev. D2 (1970) 1542; K. Symanzik, Comm. Math. Phys. 18 (1970) 227.

34. S. Coleman, "Dilations", Proc. 1971 Intl. Summer School "Ettore Majorana".

35. K. G. Wilson, Phys. Rev. D3 (1971) 1818.

36. S. Weinberg, Phys. Rev. 118 (1960) 838.

37. D. R. T. Jones, Nucl. Phys. B75 (1974) 531; W. E. Caswell, Phys. Rev. Lett. 33 (1974) 244; O. Tarasov and A. Vladimirov, Dubna

preprint E2-10079 (1976).

38. S. Weinberg, Phys. Rev. D8 (1973) 3497.

39. G. 't Hooft, Nucl. Phys. B61 (1973) 455.

40. J. C. Collins and A. J. MacFarlane, Phys. Rev. D10 (1974) 1201.

41. M. Holwerda, W. Van Neerven and R. Van Royen, Nucl. Phys. B75 (1974) 302.

42. J. C. Frenkel and J. C. Taylor, Nucl. Phys. B116 (1976) 185.

43. G. 't Hooft and M. Veltman, Nucl. Phys. B44 (1972) 189.

44. C. G. Bollini and J. J. Giambiagi, Phys. Lett. 40B (1972) 566.

45. R. Delbourgo, Rep. Prof. Phys. 39 (1976) 345.

46. Van Dam and M. Veltman, Nucl. Phys. B22 (1970) 397.

47. R. Gastmans and R. Meuldermans, Nucl. Phys. B63 (1973) 277.

48. W. J. Marciano and A. Sirlin, Nucl. Phys. B88 (1975) 86.

49. R. Gastmans and R. Meuldermans, Nucl. Phys. B105 (1976) 454.

50. W. J. Marciano, Phys. Rev. D12 (1975) 3861.

51. A. Garcia-Alvarez and R. Meuldermans, Nucl. Phys. B119 (1977) 539.

51'. I. S. Gradshteyn and J. M. Ryzhnik: Table of Integrals, Series and Products (Academic Press, New York and London).

52. A. Erdélyi et al., "Higher Transcendental Functions", (McGraw-Hill, New York, 1953) (Bateman Project).

53. T. Appelquist, J. Carazzone, H. Kluberg-Stern and M. Roth, Phys. Rev. Lett. 36 (1976) 768.

53'. P. Cvitanović, Phys. Lett. 65B (1976) 272.

54. Y. P. Yao, Phys. Rev. Lett. 36 (1976) 653.

55. L. Tyburski, Phys. Rev. Lett. 37 (1976) 319.

56. A. Garcia-Alvarez, Nucl. Phys. B120 (1977) 355.

56'. T. Appelquist and H. D. Politzer, Phys. Rev. D12 (1975) 1404; E. Poggio, S. Weinberg and H. Quinn, Phys. Rev. D13 (1976) 1958.

57. G. Sterman, Phys. Rev. D14 (1976) 2123; E. Poggio and H. Quinn, Phys. Rev. D14 (1976) 578.

58. G. 't Hooft, Nucl. Phys. B33 (1971) 173; J. C. Taylor, Nucl. Phys. B33 (1971) 436.

59. H. Kluberg-Stern and J. B. Zuber, Phys. Rev. D12 (1975) 467.

60. D. M. Capper and G. Leibbrandt, J. Math. Phys. 15 (1974) 82.

61. J. M. Cornwall and G. Tiktopoulos, Phys. Rev. D13 (1976) 3370.

62. J. Frenkel, R. Meuldermans, I. Mohammad and J. C. Taylor, Phys. Lett. 64B (1976) 211, and Oxford preprint 67 (1976), to be published in Nucl. Phys.

63. J. M. Cornwall and G. Tiktopoulos in "New Pathways in High Energy Physics", (ed. A. Perlmutter, Proc. of Orbis Scientiae 1976, Coral Gables); J. M. Cornwall, University of California, Los Angeles, report, UCLA/76/TEP/10 (June 1976).

64. J. Frenkel, Phys. Lett. 65B (1976) 383.

65. J. Frenkel and J. C. Taylor, Oxford preprint 88 (1976).

66. D. J. Gross in "Methods in Field Theory", Les Houches (1975) 141.

67. E. Poggio, Harvard preprint, invited talk at Rochester Workshop on Quark Confinement, HUTP-76/A163 (June 1976).

68. C. P. Korthals-Altes and E. de Rafael, Marseille preprint 76/p.877 (December 1976).

70. F. Zachariasen and P. Carruthers in "New Pathways in High Energy Physics" (ed. A. Perlmutter, Proc. of Orbis Scientiae 1976, Coral Gables).

71. C. P. Korthals-Altes and E. de Rafael, Nucl. Phys. B106 (1976) 237.

72. B. W. Lee and J. Zinn-Justin, Phys. Rev. D5 (1972) 3121.

73. E. Poggio, H. Quinn and J. Zuber, Harvard preprint HUTP-76/A171.

74. N. Papanicolau, Phys. Rep. 24C (1976) 229.

75. C. P. Korthals-Altes and E. de Rafael, Phys. Lett. 62B (1976) 320.

76. J. Frenkel and R. Meuldermans, Oxford preprint 50 (1976).

77. P. Cvitanović and T. Kinoshita, Phys. Rev. D10 (1974) 3991.

78. P. Cvitanović, Phys. Rev. Lett. 37 (1976) 1528.

79. B. W. Lee in "New Pathways in High Energy Physics" (ed. A. Perlmutter, Orbis Scientiae 1976, Coral Gables).

80. J. C. Taylor, Oxford preprint 99 (1976).

81. H. Fleming and J. Frenkel, Universidade de Sao Paulo preprint, IFUSP/p.114.

82. I. B. Kriplovich, Soviet J. of Nucl. Phys. 10 (1970) 235; R. L. Arnowitt and S. I. Fickler, Phys. Rev. 127 (1962) 1821.

83. R. N. Mohapatra, Phys. Rev. D4 (1971) 378.

84. P. Olesen, Nucl. Phys. B104 (1976) 125.

85. T. Kinoshita and A. Ukawa, Cornell preprint, CLNS-349 (1976).

86. T. Kinoshita and A. Sirlin, Phys. Rev. 113 (1959) 1652.