# Homogeneous Model Theory as a Framework for Algebraic Stability Theory

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Abstract. In this thesis we aim to persuade the readers to reconsider homogeneous model theory as a framework in which to study Algebraic Stability Theory. We argue that the tight link between homogeneous model theory and categorical amalgamation constructions gives it a flexibility beyond what the ordinary first-order context can offer while simultaneously it allows the development of a stability-theoretic machinery very close to classical first-order stability.

Thus we first introduce homogeneous model theory and categorical amalgamation theory separately (Chapters 1 and 2), laying out our own contributions alongside the known theory, before we utilise their interactions to prove related results pertinent to several different questions in algebraic stability theory.

In Chapter 3 we give a very general result on omitting (possibly isolated) types from homogeneous models, generalising an analogous theorem by Hrushovski and Itai [**18**] regarding differential fields. Our method makes it possible to omit any family of types of trivial geometry from a superstable (and simple) homogeneous model without impacting the dividing structure and thus the stability class of the model.

In Chapter 4 we apply homogeneous model theory to the study of Hrushovski constructions, enabling us to give a general treatment of the fundamental stability-theoretic properties beyond the combinatorial case to which such general accounts are usually confined. We obtain a general characterisation of dividing and by extension of U-Rank in the amalgamation limits of those constructions, and since we are not limited to finitary cases we can use our results for studying pseudo-exponentiation in the next two chapters.

In Chapters 5 and 6 we thus narrow our focus even more and apply the tools developed above to the single class of exponential fields, illustrating the power of our general results through application to a topic which has been under close investigation ever since Boris Zilber proposed pseudo-exponentiation as a model-theoretic approach to complex exponentiation in the 2000s [**42**]. In passing we will also construct a non-almost-exponentially-algebraically closed quasiminimal exponential field, answering a question asked by Bays and Kirby in their recent [**7**].

# Preliminaries and Introduction

## 0.1. Notation

In this thesis, $\mathcal{L}$ will always designate a signature. Cardinals and ordinals will be signified by their usual Greek and Hebrew letters, although $\omega$ is used for $\aleph_0$ where tradition so dictates (as in $\omega$-stability), and we write $\kappa^{<\lambda}$ for $\sup(\{\kappa^\alpha | \alpha < \lambda\})$.

Types are assumed complete, and the type spectrum of an $\mathcal{L}$-structure $M$ over a subset $A \subseteq M$ will be referred to as $\mathrm{Spec}_A(M) = \bigcup_{n\in\mathbb{N}} \mathrm{Spec}^n_A(M)$. $\mathrm{Spec}_\emptyset(M)$ will also be denoted as $\mathrm{Spec}(M)$. As is customary in this field, we will often write $AB$ for $A \cup B$ and even $A\vec{b}$ for $A \cup \{\mathrm{ran}(\vec{b})\}$.

Homogeneous model theory is the study of homogeneous $\mathcal{L}$-structures for a signature $\mathcal{L}$. As the terminology differs rather widely, we emphasise that we are following [**31**]:

Definition 0.1. An $\mathcal{L}$-structure $M$ is called *homogeneous* if whenever $A \subseteq M$ with $|A| < |M|$ and $f : A \to M$ is a partial elementary map, for any $a \in M$, there is a partial elementary extension of $f$ to $A \cup \{a\}$. Equivalently, $f$ extends to an automorphism of $M$.

## 0.2. The Subjects of Model Theory

We do not intend, in this section, to give a general account of how model theory has developed or what it covers; rather, we would like to draw attention to a subtle but potentially important point on what exactly model theoretic statements refer to.

When introducing model theory to undergraduates, it is usually presented as the study of the interplay between the syntax and semantics of (first-order) *theories*. As the exposition develops, one encounters saturated models as universal domains for first-order model theory, as a working model theorist will frequently describe a model-theoretic property as belonging to the universal domain rather than to the theory itself. Equally often, one will equate the theory with the class of its models, and attribute a property to a class of structures.

Strolling through the model theory section of any well-stocked mathematical library, one will see titles reflecting each of those attitudes - Frank Wagner's *Simple Theories* stands near Gerald Sacks' *Saturated Models*, and in between them the two volumes of Saharon Shelah's *Classification Theory for Abstract Elementary Classes* find their place.

In this thesis we are concerned with algebraic stability theory, which is seen, via Shelah's approach to stability as a measure of good behaviour, to be the study of well-behavedness in algebraic structures. As we are particularly concerned with finding the right framework within which to formulate this subject, we should determine whether we conceive of its questions as questions about the well-behavedness of individual fields, say, of theories of fields, or of a class of fields.

Here, we will adopt the first approach as being the one closest to that of algebra itself. In particular, the relationship between stability and geometric properties, especially important to Algebraic Stability Theory, strongly suggests this point of view.

To illustrate this choice, we would see Angus Macintyre's result **[30]** on algebraic closedness and total transcendentality as saying that certain fields with a good dimension notion (i. e. Morley rank) are algebraically closed rather than as a statement on theories or classes.

## 0.3. A Framework for Algebraic Stability Theory

Bearing in mind the discussion of the last section, typical questions of Algebraic Stability Theory can naïvely be formulated as "Given a class of algebraic structures, which of them have well-behaved notions of rank, independence or similar concepts?" When trying to make those questions precise, one must decide on a framework, that is, a unified way of interpreting them in a rigorous mathematical way.

There are two fundamental considerations to keep in mind when deciding on such a framework: Firstly, it should be adequate to our intuition, the precisification of our question expressing closely what we "meant" when posing the problem naïvely. Secondly, the resulting field of mathematical problems should admit a meaningful mathematical theory. We will briefly explain here how the framework of Homogeneous model theory can satisfy the former criterion, recalling the example mentioned in the last section.

The paradigmatic theorem by Macintyre seems to say that all fields with a Morley-rank-like dimension notion are algebraically closed.

However, as Macintyre's work was done within first-order model theory, he has really only shown that all *saturated* fields with a good dimension notion are algebraically closed. Saturated models are well known to be those that are both homogeneous as structures in their language (here $\mathcal{L}_{\text{ring}}$) and universal among those models of their first-order theory that are of lower cardinality. Note that those two conditions seem to be of a very different flavour: The former is an absolute property of the field itself, while the latter requires the logical notion of a first-order theory to formulate. Alternatively, saturation is also equivalent under homogeneity to compactness of its definable relations (See [**8**]).

It seems to us far more limiting to restrict the question "Which fields admit a good dimension notion?" to saturated fields than to restrict it to homogeneous fields. Homogeneity as a concept appears naturally throughout algebra and geometry, and to subsume this concept itself under "good geometric properties" seems reasonable. However, neither saturatedness, nor universality nor compactness have natural analogues in classical algebra, and the sophisticated stability theory developed for homogeneous model theory suggests that they are not necessary for good geometric behaviour. Therefore homogeneous model theory seems to be an intuitively rather more adequate framework for questions such as the above than classical saturated model theory.

## 0.4. Outline of the Thesis

The rest of this thesis is devoted to building machinery and proving results that hopefully convince the reader that applying homogeneous model theory to problems in algebraic stability leads to a very beautiful mathematical theory. The underlying cause for this we see in its amenability to categorical methods, which can be exploited far more systematically than in first-order algebraic stability theory.

In the first chapter, we will develop the techniques of homogeneous model theory as far as we require them. We will present our setting for homogeneous model theory and give the background on simple and stable homogeneous models that we will require later on. We will see that the relationship between stability and simplicity is much more involved than in the first-order setting: Unlike there, stability does not imply simplicity, and it is unclear in general whether the rank notions coming from stability are compatible with dividing. As well as spelling out the orthogonality calculus for simple homogeneous models, which has all the properties of its first-order analogue, we also modify the notion of Morley rank in Homogeneous Models as it has been developed by Kolesnikov and Krishnamurthi in **[28]**, aiming to provide a rank function for totally transcendental models for which a drop in rank might coincide with dividing by setting the rank of bounded sets to 0 by definition. We will see in a later chapter that for infinitary Hrushovski constructions, one of the main classes of application for non-elementary methods in recent years, we have indeed succeeded.

In the following chapter, we will introduce our most important tool, categorical amalgamation theory. This is a vastly generalised framework for amalgamation constructions modelled after the construction of universal homogeneous structures in classical model theory. It requires for its development a rather extensive library of notions, as not only do terms such as “universal” and “homogeneous” have to be transferred but also importantly a satisfactory analogue of cardinality in a general category must be found. After setting the scene and introducing both classical amalgamation results and the categorical formalism, we present the category-theoretic version of the construction under the General Continuum Hypothesis, due to Droste and Göbel in 1989. However, we are more interested in an unconditional construction in the stable case, due in the classical model-theoretical setting to Shelah. The categorical generalisation we derive here centres on the notion of a stability category (Here and for the remainder of this outline we refer to the relevant chapters for any details and an explanation of the notation used):

Definition 0.2. A category $\mathfrak{C}$ is called a $\lambda$-*stability category* if for all $K \in \mathfrak{C}$ the following holds:

(1) $\mathfrak{C}$ is closed under limits of $\mu$-chains for every $\mu \leq \lambda$.
(2) There is an initial object.
(3) For every $A \in \mathfrak{C}(K)$ there is a $\mu \leq \lambda$ such that $A$ is the colimit of a $\mu$-chain $(A_\alpha)_{\alpha<\mu}$ in $\mathfrak{C}(K)_{<\lambda}$ with $(A_{\alpha+1}, \to) \in \mathfrak{C}(A_\alpha)_{<\aleph_0}$, for every $\alpha < \mu$, $A_0 = K$ and $A_\kappa$ the colimit of $(A_\alpha)_{\alpha<\kappa}$ for any limit ordinal $\kappa$, where $\to$ is the morphism of the $\mu$-chain (*regular* $\mu$-chain).
(4) if $(B, \to) \in \mathfrak{C}(K)_{<\aleph_0}$, $K \to A$, there is a $D$ and arrows $B \dashrightarrow D$ and $A \dashrightarrow D$ such that $(D, \dashrightarrow) \in \mathfrak{C}(A)_{<\aleph_0}$ and the following square commutes:

$$\begin{tikzcd} K \arrow[r] \arrow[d, "<\aleph_0"] & A \arrow[d, dashed, "<\aleph_0"] \\ B \arrow[r, dashed] & D \end{tikzcd}$$

(5) $|\mathfrak{C}(K)_{<\aleph_0}| \leq \lambda$ (Counting only isomorphism types)

We prove that such categories have a unique amalgamation limit, that is, an object of cardinality $\lambda$ that is homogeneous and universal with respect to arrows and objects smaller than $\lambda$ from the category.

We also investigate the intricate relationship between homogeneous models and amalgamation constructions, proving that very mild conditions on the logical structure of the objects amalgamated (expressed in terms of concrete categories) are sufficient for homogeneity of the amalgamation limit also in the traditional model theoretic sense.

PROPOSITION 0.3. *Let $\mathfrak{C}$ be a $\lambda$-algebroidal $\mathcal{L}$-structure category. Let $M$ be an amalgamation limit for $\mathfrak{C}_{<\lambda}$ in $\mathfrak{C}$. Assume further that for any subsets $A, B$ of $M$ and any $a$ in the hull of $A$, the type of $a$ over $A$ is isolated and that, if* $\mathrm{tp}(Aa) = \mathrm{tp}(Bb)$, *then $a$ is in the hull of $A$ iff $b$ is in the hull of $B$. Then $M$ is $\lambda$-homogeneous.*

We also see that under suitable assumptions one can also characterise stability in the obvious categorical way.

This prepares us to prove our first major result in algebraic stability theory in Chapter 3. The problem of finding substructures of a model with a different theory but which share the fundamental stability properties has been tackled in different contexts in classical first-order stability theory. Macintyre's theorem above could be seen as a negative result on this as it shows that algebraically closed fields do not admit such substructures. Hrushovski's and Itai's work on differential fields [**18**], on the other hand, makes a positive existence statement on the types that can be omitted from differentially closed fields without impacting the stability structure, and as many of those types are isolated, this gives rise to $\omega$-stable theories of differential field not differentiall-algebraically closed. Their approach, however, relies on an intricate knowledge of heavy differential algebra to ensure saturatedness of those substructures, which is necessary to conclude stability of their first-order theory. We demonstrate that homogeneity, in contrast, is transferred downwards immediately through the collapsing construction iself, and that thus an analogous result, formulated in homogeneous model theory, can be proven without any reference to the algebraic characteristics of the individual setting. In fact, we obtain:

THEOREM 0.4. *Let $\mathbb{M}$ be a simply $\lambda$-stable homogeneous model of cardinality $\mu > \lambda$, $\lambda$ a regular cardinal, $C \subset \mathbb{M}$ of cardinality less than $\lambda$ and $P$ an admissible family of $C$-types. Then the category $\mathfrak{C}^{\mathbb{M}}_{(P)}$ of all substructures $A$ of $\mathbb{M}$ extending $C$ of cardinality $\leq \lambda$ orthogonal to every $p \in P$ over $C$ is a $\lambda$-stability category.*

Together with the theory on stability categories alluded to above, this allows us to construct substructures omitting such a family of types very generally, the original large structure even tightly controlling dividing in the substructure.

We will close that chapter with a discussion of the consequences our findings have for studying this type of question.

In the fourth chapter we are addressing a pertinent issue in model theory over the last years: providing a suitably general treatment for Hrushovski-style constructions beyond the combinatorial structures covered by works such as [**39**]. Here, homogeneous model theory is immensely useful, as amalgamation limits of infinitary constructions are rarely saturated, and, furthermore, the very fact that Hrushovski constructions are constructed via amlagamation makes them obvious candidates for applying our methods. We cast those general Hrushovski constructions in a categorical way, inserting them naturally into the framework we developed in Chapter 2. We then give combinatorial statements of the properties that are essential to the model-theoretic analysis of those constructions, inspired by the facts known about exponential fields and other $\Gamma$-fields (This notion of $\Gamma$-fields, introduced by Bays and Kirby in [**7**], brings together most of the known non-elementary Hrushovski constructions and gives a systematic exposition of their properties). We investigate which of those conditions lead to simplicity and superstability of the amalgamation limit and characterise dividing, and, following from that, SU-rank in Amalgamation-by-Predimension Constructions. The notation used in the theorem quoted below is explained in that chapter; the angular half-brackets represent a notion of closure and $\mathfrak{l}$ is a measure of the length of a minimal decomposition of extensions.

THEOREM 0.5. *Let $\mathbb{M}$ be the homogeneous amalgamation limit of a regular amalgamation-by-predimension category. Then $\mathbb{M}$ is supersimply stable and*

(1) *For any $A, B, C \subset \mathbb{M}$, $A \underset{C}{\perp} B$ if and only if*

$$\lceil (AC)^{\mathrm{hull}} \rceil \cap \lceil (BC)^{\mathrm{hull}} \rceil = \lceil C^{\mathrm{hull}} \rceil$$

*and*

$$\delta(\lceil (AC)^{\mathrm{hull}} \rceil \,/\, \lceil C^{\mathrm{hull}} \rceil) = \dim(\lceil (AC)^{\mathrm{hull}} \rceil \, \lceil (BC)^{\mathrm{hull}} \rceil \,/\, \lceil (BC)^{\mathrm{hull}} \rceil).$$

(2) *Let $A \subseteq B \subset \mathbb{M}$ with $\dim(B/A) < 1$. Then $\mathrm{SU}(B/A) = \mathfrak{l}(\lceil B \rceil \,/\, \lceil A \rceil)$ wherever $\mathfrak{l}(\lceil B \rceil \,/\, \lceil A \rceil)$ is finite.*

We then see how in those predimension constructions the modified notion of Morley rank defined in Chapter 2 does characterise dividing.

In the final two chapters of the thesis we apply everything that we have done so far in developing a novel account of the model theory of exponential fields. After the introduction of Pseudoexponentiation by Zilber's seminal [**42**], the model theory of such exponential fields has been formulated in the language of quasiminimal excellence, and the focus has been on those exponential fields that have a kernel isomorphic to the ring of integers. We will first present the necessary algebraic fundamentals for our study as well as give an account of the quasiminimal theory as it stands after [**7**]. In passing and in the spirit of the method used for homogeneous models in Chapter 3, we answer the question posed in Remark 11.10 at the end of that paper and construct an example of a quasiminimal excellent exponential field that is not almost exponentially-algebraically closed.

We then turn to constructing and investigating exponentially-algebraically closed exponential fields with $\aleph_0$-saturated kernels. Avoiding the number-theoretic rigidity of the definable integers and being closer to the saturated model, these are homogeneous, and we can apply the theory of the preceding chapters to prove that they are simply superstable and have SU-rank $\aleph_0$. Also, as in the case of quasiminimal excellent exponential fields, we can use the methods of Chapter 3 to prove that we can omit strongly minimal sets from exponentially-algebraically closed fields to obtain equally simply superstable homogneoeous exponential fields that are not closed in this way. This extends the work done by Kirby and Zilber in [**27**].

By considering large homogeneous rather than quasiminimal exponential fields we believe that we can give a richer account of their model-theoretic structure, including an analysis of those types which are suppressed to countability in quasiminimal excellent models but strongly minimal in the full homogeneous exponential field. We hope that this might help bring forward the development of the model theory of exponential fields until it reaches a similar level as that, say, of differential fields, which has proven immensely fruitful from a logical perspective. At present we can already provide a characterisation of U-rank and dividing as well as a better understanding of the importance of exponential-algebraic closedness.

CHAPTER 1

# Homogeneous Model Theory

## 1.1. Previous Study of Homogeneous Model Theory

Homogeneous models were introduced into model theory in the late 1950s and early 1960s by the papers of Jónsson [**21**] and Morley and Vaught [**32**]. Those first origins of the study of homogeneous models already suggest their present usefulness to us, as they were discussed in the context of amalgamation constructions, which are also the crucial tool for our further work. In [**32**], the authors give a careful exposition of what we now know as the existence of saturated models of a complete first-order theory under GCH.

In the next years, Keisler and Morley [**23, 22**] undertook the systematic study of homogeneous models without the assumption of universality. Most of the elementary properties of homogeneous models covered in an elementary model theory class can be found there.

Then, in 1969, Shelah's seminal thesis work on stability [**33**] was to change the face of model theory for good. Intriguingly, already in his second paper [**34**] on the topic, Shelah extended his stability calculus to the homogeneous case, for which he used the term *good finite diagram* (This refers to the collection of types realised in the models under consideration). The fact that this was possible so early in the development, and in such a way that fundamental theorems of both the saturated and the homogeneous case could be treated uniformly, shows how closely analogous homogeneous and saturated model theory can be. This is particularly striking when compared to the extraordinary technical effort required to transfer results to the more general non-elementary setting in [**38**].

After those very promising beginnings, while model theorists throughout the world spent the next thirty years grappling with the elementary case and the fascinating prospects Shelah's visions opened to the subject, homogeneous stability theory was neglected by everyone except for Shelah himself, who continued to track the homogeneous analogues of his theorems in [**35, 36**] before his own interests moved on to far more general settings than this.

Thus it took until 2001 for Olivier Lessmann and his co-authors to rekindle interest in the subject with their series of papers on finite diagrams. In [**14**] he and Rami Grossberg collected in contemporary language and format what was known on homogeneous model theory from Shelah's three abovementioned papers, and in [**29**] Lessmann gave a characterisation of totally transcendental homogeneous models. Together with Tapani Hyttinen [**19**] he developed a rank function for the superstable case. In this work, they noticed the important fact that without saturation, the independence notions coming from stability-theoretic ranks might not coincide with dividing, and that indeed homogeneous stability does not imply the homogeneous analogue of simplicity. This more refined interplay between stability and simplicity will be crucial to us, and much of our work builds on [**8**], where Buechler and Lessmann provide a landmark analysis of simple homogeneous models. Since then, interest in the subject seems to have died down. We suspect that is because those authors who had worked on homogeneous model theory around the turn of the millenium tended to see it as a stepping stone towards a uniform general classification theory for non-elementary classes; and just like Shelah forty years earlier, they have since then moved on to more general territories. However, we hold that the greater potential for homogeneous model theory lies in algebraic applications, and we hope that this study might motivate some more interest in this direction and thus in developing more techniques especially in the area of simple homogeneous models.

## 1.2. Stable Homogeneous Models

In this section we collect the definitions and results we require on stable homogeneous models. Everything here is known since at least [**36**]; the references given with the individual statements refer to [**14**].

The first proposition is the reason why homogeneous models can be seen as generalisations of saturated models as universal domains.

Proposition 1.1 (Lemma 2.3). *Let $M$ be a homogeneous model and $N$ be a model of* $\mathrm{Th}(M)$ *with $|N| \leq |M|$. Let $A \subseteq N$ with $|A| < |M|$ and such that* $\mathrm{Spec}_A(N) \subseteq \mathrm{Spec}_A(M)$*. Then for every partial elementary embedding $f : A \to M$ there is an elementary embedding $g : N \to M$ extending $f$.*

The combinatorial definition of stability is exactly the same as in first-order model theory.

Definition 1.2 (Definition 3.1). Let $M$ be a homogeneous model of cardinality greater than a given infinite cardinal $\lambda$. Then $M$ is $\lambda$-*stable* if and only if over every subset of $M$ of cardinality at most $\lambda$ there are at most $\lambda$ types realised in $M$. $M$ is called *stable* if there is a $\lambda$ such that $M$ is $\lambda$-stable.

As in the first-order case, one can equivalently describe stability using the order property, but we will not need this description here.

In order to justify our focus on single models, we appeal to the stability spectrum theorem.

FACT 1.3 (Theorem 4.17). *Let $M$ be a stable homogeneous model, $|M| > \beth_{(2^{|\mathcal{L}|})^+}$. Then there are cardinals $\kappa \leq \lambda < \beth_{(2^{|\mathcal{L}|})^+}$ such that for every cardinal $\mu$, $M$ is stable in $\mu$ if and only if $\mu \geq \lambda$ and $\mu^{<\kappa} = \mu$.*

COROLLARY 1.4. *$M$ is $\omega$-stable if and only if $M$ is stable in all infinite cardinals.*

PROOF. $\lambda$ must be $\omega$, and thus $\kappa$ must be at most $\omega$. But then $\mu^{<\kappa} = \mu$ is vacuous. □

DEFINITION 1.5. We call $M$ superstable if $\kappa = 1$ (or equivalently $\kappa = \omega$) can be chosen in Fact 1.3.

Thus we can detect stability and superstability on any single homogeneous model of cardinality above $\beth_{(2^{|\mathcal{L}|})^+}$.

We will finish this short section with the homogeneity spectrum theorem.

DEFINITION 1.6. Let $M$ be a homogeneous model. We call an elementary submodel $N$ of $M$ *(relatively) saturated in $M$* if for all $A \subset N$ with $|A| < |N|$ holds that $\mathrm{Spec}_A(N) = \mathrm{Spec}_A(M)$.

FACT 1.7 (Theorem 5.9). *Let $M$ be a sufficiently large homogeneous model and $\kappa < |M|$ an infinite cardinal. Then there is a relatively saturated homogeneous elementary submodel of $M$ of cardinality $\kappa$ if and only if $\kappa \geq |\mathrm{Spec}(M)|$ and either $M$ is $\lambda$-stable or $\lambda^{<\lambda} = \lambda$.*

## 1.3. Simple Homogeneous Models

In first-order model theory, well-behaved notions of forking and dividing were long seen as central trademarks of stable theories. However, in the elementary case, already Shelah [**37**] realised that certain parts of this calculus carry over to a more general framework, and in 1998 Kim Byunghan [**24**] proved the equivalence of forking and dividing as well as the remaining properties of the independence relation for those so-called simple theories in a groundbreaking paper.

In the homogeneous setting, this notion is more subtle; in fact, dividing does not necessarily have all of those good properties even in a stable theory, while we can still profitably define simple theories for which it has them. This calculus is crucial to applying model theory to the algebraic setting, and thus we will set out here most of the results given in the standard reference paper on the subject [**8**], from which all references here are taken unless mentioned otherwise. The numbering in that paper is given in brackets alongside our numbering.

Throughout this section let $M$ be a sufficiently large homogeneous $\mathcal{L}$-structure. Also, for the remainder of this section, all subsets of $M$ are assumed to have smaller cardinality than $M$, a relationship we will denote by $\subset$. We will now give a string of definitions which are necessary to formulate simplicity in the homogeneous context.

Definition 1.8 (Definition 1.5). Let $X$ be an ordered set, $A \subset M$ and $I = a_i | i \in X$ a set of sequences from M indexed by X. Then, I is called *$A$-indiscernible* or *indiscernible over $A$* if for all $n < \omega$, and $i_1 < \ldots < i_n$ and $j_1 < \ldots < j_n$ from $X$, $\mathrm{tp}(a_{i_1}, ..., a_{i_n}/A) = \mathrm{tp}(a_{j_1}, ..., a_{j_n}/A)$.

DEFINITION 1.9 (Definition 2.1). A type $p(\vec{x}, \vec{b})$ with parameters $\vec{b}$ *divides over* $A \subset M$ if there is an infinite $A$-indiscernible sequence $\{\vec{b_i} | i \in X\}$ with $\mathrm{tp}(\vec{b_i}/A) = \mathrm{tp}(b/A)$ such that $\bigcup_{i \in X} p(\vec{x}, \vec{b_i})$ is inconsistent.

DEFINITION 1.10 (Definition 2.2). Given an infinite cardinal $\chi$, and subsets $A$, $B$ and $C$ of $M$, $A$ is *$\chi$-free from $B$ over $C$* if for all sequences $\vec{a}$ from $A$ and $\vec{b}$ from $BC$ with $|\vec{a}|, |\vec{b}| < \chi$, $\mathrm{tp}(\vec{a}/\vec{b})$ does not divide over $C$.

DEFINITION 1.11 (Definition 2.5). Let $\kappa$ be an infinite cardinal. Then $M$ is *$\kappa$-simple* if

(1) Whenever $A$ is $\aleph_0$-free from $B$ over $C$, it is $\kappa$-free from $B$ over $C$.
(2) For any sequence $\vec{a}$ of length less than $\kappa$ and every set $A$, there is $B \subset A$, $|B| < \kappa$, such that $\vec{a}$ is $\kappa$-free from $A$ over $B$.
(3) For any $A \subset M$ and any sequence $\vec{a}$ of length less than $\kappa$, any $B \subseteq A$ and any $A \subseteq C \subset M$ there is a $\vec{c}$ realising $\mathrm{tp}(\vec{a}/A)$ such that $\vec{c}$ is $\kappa$-free from $C$ over $B$.

REMARK 1.12. Our definition is marginally stronger than the one given in [**8**], which only demands extension for unbounded types. However, by the discussion in Remark 2.8 there, they are equivalent if the "character of smallness" (see there) is less than or equal to $\kappa$.

DEFINITION 1.13. If $M$ is a $\kappa$-simple homogeneous model and $A$, $B$ and $C$ are subsets of $\mathbb{M}$, then we will say that $A$ is *free from $B$ over $C$* if $A$ is $\chi$-free from $B$ over $C$.

We now list the properties of dividing that we will require in the further course of our work.

FACT 1.14. *$\kappa$-freeness in a $\kappa$-simple theory satisfies the following axioms, where $\vec{a}$ designates a sequence of less than $\kappa$ elements:*

1. *$\kappa$-Local Character: For any sequence $\vec{a}$ of length less than $\kappa$ and every set $A$, there is $B \subset A$, $|B| < \kappa$, such that $\vec{a}$ is free from $A$ over $B$.*
2. *Monotonicity: If $B \subseteq B_1 \subseteq C_1 \subseteq C$ and $\vec{a}$ is free from $C$ over $B$, then $a$ is free from $C_1$over $B_1$.*
3. *Extension: If $\vec{a}$ is free from $C$ over $B$ and $D$ contains $C$, then there is an $a' \models \mathrm{tp}(a/CB)$ that is free from $B$ over $D$.*
4. *Transitivity: If $B \subseteq C \subseteq D$, then $a$ is free from $D$ over $B$ if and only if it is free from $C$ over $B$ and free from $D$ over $C$.*
5. *Symmetry: If $A$ is free from $C$ over $B$ then $C$ is free from $A$ over $B$.*
6. *Finite Character: If all finite $\vec{a} \in A$ are free from all finite $\vec{c} \in C$ over $B$, then $A$ is free from $C$ over $B$.*
7. *Invariance: Freeness is invariant under automorphisms.*

PROOF. Finite and $\kappa$-local character are part of the definition of simplicity. Transitivity is given by (Corollary 2.15), Symmetry given by (Theorem 2.14).

Monotonicity: From $C$ to $C_1$ is a straightforward logical consequence of the definition of freeness, and from $B$ to $B_1$ is given by (Remark 2.1).

Extension: Follows from the third clause of the definition and from transitivity.

Invariance is clear. $\square$

Our own contribution consists of the development of an orthogonality calculus for simple homogeneous models.

From now on for the ret of the section, let $\mathbb{M}$ be a $\kappa$-simple homogeneous model for any $\kappa$, let all sets $A$, $B$, $C$ and $D$ be subsets of $\mathbb{M}$ and let all types be realised in $\mathbb{M}$.

The orthogonality relation will be derived from dividing in a simple homogeneous model. While it may seem curious that this has not been written out before, the main reason for that is that most work on Homogeneous Model Theory has been done in a stable not neccessarily simple context, where dividing is not so well-behaved, and indeed, in [**20**], Hyttinen and Shelah have developed the rudiments of orthogonality from "strong splitting", which in general can be strictly stronger than dividing and does not share its fundamental good properties.

Definition 1.15. Let $p$ and $q$ be complete types (of a not neccessarily finite sequence) over $C$ and $D$ respectively.

(1) If $C = D$, $p$ and $q$ are *almost orthogonal,* written $p \perp^a q$, if, for every pair of realisations $\vec{a}$ of $p$ and $\vec{b}$ of $q$, $\vec{a}$ is free from $\vec{b}$ over $C$.
(2) In general, $p \perp^a q$, if for every free extension $p'$ of $p$ and every free extension $q'$ of $q$, both to $C \cup D$, $p' \perp^a q'$ as defined above.
(3) If $C = D$, then $p$ and $q$ are *orthogonal,* written $p \perp q$, if for every $E \supseteq C$ and every $\vec{a}$ realising $p$, $\vec{b}$ realising $q$, if $\mathrm{tp}(\vec{a}/E)$ and $\mathrm{tp}(\vec{b}/E)$ are free extensions of $p$ and $q$ respectively, then $\vec{a}$ is free from $\vec{b}$ over $E$.
(4) In general, $p \perp q$ if for any free extension $p'$ of $p$ to $S(C \cup D)$ and any free extension $q'$ of $q$ to $S(C \cup D)$, $p \perp q$ as defined above.

NOTATION 1.16. If $A$,$B$ and $C$ are subsets of $\mathbb{M}$, then we write $A \perp_C B$ for $\text{tp}(\vec{A}/C) \perp \text{tp}(\vec{B}/C)$ (and analogously for almost-orthogonality). We will also frequently mix those notations, writing for instance $A \perp p$. This is not to be confused with the classical concept of orthogonality between a type and a set, which is not used in this thesis.

REMARK. Even in the presence of stability, the simplicity of the theory is required already here to ensure that there exists a nondividing extension to $S(C \cup D)$.

We will give the fundamental properties of orthogonality (symmetry, monotonicity, transitivity and the behaviour under free extensions) in the setting of simple homogeneous models. The statements and proofs for the first-order case can all be found in [**3**], from where we have also taken the terminology for the axioms of independence relations.

PROPOSITION 1.17. *(Finite Character) If $A' \perp_C B'$ for all finite $A' \subseteq A$ and finite $B' \subseteq B$, then $A \perp_C B$.*

PROOF. Follows directly from the finite character of freeness. $\square$

PROPOSITION 1.18. *(Symmetry) If $A \perp_C B$, then $B \perp_C A$.*

PROOF. Follows directly from the symmetry of freeness. $\square$

PROPOSITION 1.19. *(First Monotonicity Axiom) If* $\text{tp}(A/C) \perp \text{tp}(B/D)$ *and $A_0 \subseteq A$, then* $\text{tp}(A_0/C) \perp \text{tp}(B/D)$.

PROOF. We first show the special case $C = D$. Fix $A_0'$, $B'$ and $E$ such that $A_0' \vdash \text{tp}(A_0/C)$, $B' \vdash \text{tp}(B/C)$ and both are free from $E$ over $C$. By homogeneity,

$A_0$ is conjugated with $A'_0$ by an automorphism fixing $C$. Let $A''$ be the image of $A$ under this automorphism. By the free extension property, we can find an $A'$ with $\mathrm{tp}(A'/A'_0 \cup C) = \mathrm{tp}(A''/A'_0 \cup C)$ that is free from $E$ over $C$. $A \perp_C B$ implies that $A'$ is free from $B'$ over $E$ and we can now apply monotonicity of freeness to conclude.

The general case follows by applying the free extension property again and chasing the definition of orthogonality for types over different domains. □

In general, the Second Monotonicity Axiom does not hold for orthogonality. While we will assume it to hold in the model we will be investigating later, all we can prove at this stage is

PROPOSITION 1.20. *(Second Monotonicity Axiom for Almost-Orthogonality) For all $C \subseteq B \subseteq A \subset \mathbb{M}$ and $D \subset \mathbb{M}$, $A \perp^a_C D \Rightarrow A \perp^a_B D$.*

PROOF. Let $A'$ and $D'$ be realisations of the $B$-types of $A$ and $D$ respectively. Then $A'$ is free from $D'$ over $C$ by $A \perp^a_C D$. By monotonicity in the second coordinate for freeness, $A'$ is free from $D'$ even over $B$, which suffices to prove the proposition. □

PROPOSITION 1.21. *If $p \perp q$ and $p'$ and $q'$ are free extensions of $p$ and $q$ respectively, then $p' \perp q'$.*

PROOF. Follows from the transitivity of freeness. □

PROPOSITION 1.22. *(Transitivity) Let $B \subseteq C \subseteq D$ and $p$ be a type with domain a subset of $B$. Then if $p \perp_C D$ and $p \perp_B C$, $p \perp_B D$*

PROOF. Let $E \supseteq B$ and let $\mathrm{tp}(D'/E)$ be a free extension of $\mathrm{tp}(D/B)$. By homogeneity, we can assume w.l.o.g. (just as in the proof of monotonicity of orthogonality) that $D' = D$. Furthermore let $\mathrm{tp}(\vec{a}/E)$ be a free extension of $p$. We want to show that $\vec{a}$ is free from $D$ over $E$. By the monotonicity of freeness, $\mathrm{tp}(C/E)$ is a free extension of $\mathrm{tp}(C/B)$. As $p \perp_B C$, $\vec{a}$ is free from $C$ over $E$ and by the transitivity of freeness (*) $\mathrm{tp}(\vec{a}/CE)$ is a free extension of $p$.

By the second monotonicity axiom for freeness, $D$ is free from $E$ over $C$, and therefore (**) $\mathrm{tp}(D/CE)$ is a free extension of $\mathrm{tp}(D/C)$.

As $p \perp_C D$, (*) and (**) together imply that $\vec{a}$ is free from $D$ over $CE$. As $\vec{a}$ is also free from $C$ over $E$, it follows that $\vec{a}$ is free from $D$ over $E$ as required. □

PROPOSITION 1.23. *(Strong Triviality) For any family $I$ of tuples independent over $A$ such that $p \perp \mathrm{tp}(\vec{b}/A)$ for all $\vec{b} \in I$, $p \perp \mathrm{tp}(I/A)$ .*

PROOF. By finite character, we can assume $E$ to be finite without loss of generality and perform induction on $|I|$. The claim is vacuous for $|I| = 1$.

So assume true for $|I| = n$ and let $I_n := \bigcup_{i<n} \vec{b}_i$ We will derive the claim for $|I| = n + 1$. Let $E \supseteq A$ be arbitrary, $\mathrm{tp}(\vec{a}/E)$ be a free extension of $p$ and $\mathrm{tp}(I/E)$ a free extension of $\mathrm{tp}(I/A)$. We need to show that $\vec{a}$ is free from $I$ over $E$. By the induction hypothesis, $\vec{a}$ is free from $I_n$ over $E$. Since $I$ is independent over $A$ and $\mathrm{tp}(I/E)$ is a free extension of $\mathrm{tp}(I/A)$, $\mathrm{tp}(\vec{b}_{n+1}/I_nE)$ is a free extension of $\mathrm{tp}(\vec{b}_{n+1}/A)$. As $\mathrm{tp}(\vec{a}/I_nE)$ is also a free extension of $\mathrm{tp}(\vec{a}/A)$, orthogonality implies that $\vec{a}$ is free from $\vec{b}_{n+1}$ over $I_nE$. Therefore, by the Pairs Lemma, $I$ is free from $\vec{a}$ over $E$. □

## 1.4. Interactions between Stability and Simplicity

In first-order model theory, stability is strictly stronger than simplicity. In the homogeneous setting, however, they are incomparable, as the example given on pp. 1480f. of [**19**] shows.

In this short section we will state some criteria for a model being both simple and stable.

Proposition 1.24. *$M$ is supersimple and superstable if and only if there exists a freeness relation on $M$ satisfying the following axioms. In that case, freeness is $\aleph_0$-freeness.*

*(1) Local Character: For any finite sequence $\vec{a}$ and every set $A$, there is a finite $B \subset A$ such that $\vec{a}$ is free from $A$ over $B$.*

*(2) Monotonicity: If $B \subseteq B_1 \subseteq C_1 \subseteq C$ and $\vec{a}$ is free from $C$ over $B$, then $a$ is free from $C_1$ over $B_1$.*

*(3) Extension: If $\vec{a}$ is free from $C$ over $B$ and $D$ contains $C$, then there is an $a' \models \mathrm{tp}(a/CB)$ that is free from $B$ over $D$.*

*(4) Transitivity: If $B \subseteq C \subseteq D$, then $a$ is free from $D$ over $B$ if and only if it is free from $C$ over $B$ and free from $D$ over $C$.*

*(5) Symmetry: If $A$ is free from $C$ over $B$ then $C$ is free from $A$ over $B$.*

*(6) Finite Character: If all finite $\vec{a} \in A$ are free from all finite $\vec{c} \in C$ over $B$, then $A$ is free from $C$ over $B$.*

*(7) Invariance: Freeness is invariant under automorphisms.*

*(8) There exists a cardinal $\lambda$, such that, for each $A \subseteq C$, the size of $\{\mathrm{tp}(a/C) | a$ is free from $C$ ove is at most $\lambda + |A|$.*

PROOF. The direction "$\Leftarrow$" is Theorem 4.1 of [**19**]. The other direction is Fact 1.14, and the definition of superstability provides 8. $\square$

Under simplicity, we also have another characterisation of stability, familiar from the first-order setting.

DEFINITION 1.25 (Definition 3.1 of [**8**]). Define $\mathrm{SE}^{\mu}(A)$ as the collection of $A$-invariant equivalence relations on $M^{\mu}$ with a bounded (i. e. $< |M|$) number of equivalence relations.

Then tuples $a, b$ of the same length have the same *Lascar Strong Type* over $A \subset M$ if $E(a, b)$ whenever $E \in \mathrm{SE}^{|a|}(A)$

PROPOSITION 1.26 (Proposition 5.1 of [**8**]). *Let $M$ be simple. Then the following are equivalent:*

*(1) $M$ is stable.*

*(2) Lascar strong types are stationary in $M$.*

*(3) There exist cardinals $\nu \leq \mu < \beth_{(2^{|\mathcal{L}|})^+}$ such that for each $A$ and for each $B$ containing $A$, the set $\mathrm{tp}(a/B)|\mathrm{tp}(a/B)$ is free over $A$ has size at most $(|A| + \mu)^{<\nu}$.*

## 1.5. Rank Notions

While many ranks have been developed for (...)stable homogeneous models, we are interested in those that interact well with the freeness notion from dividing. The obvious candidate here is SU-rank, defined as in the first order setting.

DEFINITION 1.27. Let $\mathbb{M}$ be a supersimple large homogeneous $\mathcal{L}$-structure and let $p$ be an $\mathcal{L}$-type with parameters in $\mathbb{M}$. Then the *SU-rank* of $p$ is defined inductively as follows:

(1) $\mathrm{SU}(p) \geq 0$ if and only if $p$ is realised in $\mathbb{M}$.
(2) $\mathrm{SU}(p) \geq \alpha + 1$ if and only if there is a dividing extension $q$ of $p$ such that $\mathrm{SU}(q) \geq \alpha$.
(3) $\mathrm{SU}(p) \geq \delta$ for a limit ordinal $\delta$ if and only if $\mathrm{SU}(p) \geq \alpha$ for all $\alpha < \delta$

The standard properties of SU-rank in supersimple first-order theories still hold in the homogeneous context, the properties of dividing listed above being all that is needed in the proofs. We will just give a reminder of the Pairs Formula here:

FACT 1.28. *Let $A \subset \mathbb{M}$, $a, b \in \mathbb{M}$. Then*

*(1) If* $\mathrm{SU}(a/bA)$ *and* $\mathrm{SU}(\mathrm{b/A})$ *are finite, then* $\mathrm{SU}(ab/A) = \mathrm{SU}(a/bA) + \mathrm{SU}(\mathrm{b/A})$.
*(2) If $a \perp_A b$, then* $\mathrm{SU}(ab/A) = \mathrm{SU}(a/A) + \mathrm{SU}(\mathrm{b/A})$.
*(3) If $A \subseteq B$ and* $\mathrm{tp}(a/B)$ *does not divide over $B$, then* $\mathrm{SU}(a/A) = \mathrm{SU}(a/B)$.

PROOF. One can repeat verbatim e.g. the one given to clauses 1. and 4. of Thm. 5.1.6 and to Lemma 5.1.4 in [**40**]. $\square$

In the classical setting $\mathrm{SU}(a/A) = 0$ if and only if $\mathrm{tp}(a/A)$ is finite. As might be expected, the same statement is true in homogeneous model theory if one replaces "finite" with "bounded".

PROPOSITION 1.29. *For any $a \in \mathbb{M}$ and $A \subset \mathbb{M}$,* $\mathrm{SU}(a/A) = 0$ *if and only if the type of $a$ over $A$ is bounded.*

PROOF. This follows from the discussion on boundedness (referred to as "smallness" there) and dividing in Section 2.2 of [**8**] □

In [**28**], Alexei Kolesnikov and G. V. N. G. Krishnamurthi develop an analogue to first-order Morley rank for homogeneous models.

DEFINITION 1.30. Let $p$ be a potentially incomplete type over a finite $B \subset M$. Define $\mathrm{RM}(p) \geq A$ by induction.

(1) $\mathrm{RM}(p) \geq 0$ if $p$ is realised in $M$.
(2) $\mathrm{RM}(p) \geq \lambda$ for a limit ordinal $\lambda$ if $\mathrm{RM}(p) \geq \alpha$ for all $\alpha < \lambda$.
(3) $\mathrm{RM}(p) \geq \alpha + 1$ if there exist pairwise contradictory $\{\varphi_i(\vec{x}, \vec{a_i}) | i < \omega\}$ such that for each $i < \omega$ and each $\vec{b} \in M$ there is $q_i(\vec{x}, \vec{b})$ such that

$$\mathrm{RM}(p(\vec{x}) \cup \{\varphi_i(\vec{x}, \vec{b})\} \cup q_i(\vec{x}, \vec{b})) \geq \alpha.$$

The conventions regarding Morley ranks $\infty$ and $-\infty$ are standard, and for a type $p$ that is not over a finite $B$,

$$\mathrm{RM}(p) = \min(\{\mathrm{RM}(p') | p' \subseteq q, \mathrm{dom}(p') \text{ finite}\}).$$

In a stable (and thus simple) saturated model, all well-behaved rank notions are compatible with dividing in the sense that a type divides if and only if it drops in rank. We believe this to be key in the relationship between stability and simplicity and see this as an indicator of how well the stable and simple structure on a simply stable model fit together. Unfortunately, this definition of Morley rank assigns all infinite bounded types positive Morley rank, while none of those divide over $x = a$, which is allocated Morley rank 0. Thus, we can never hope to obtain "dividing coincides with a drop in Morley rank" for homogeneous models with bounded infinite type-definable sets.

We will see later that in homogeneous structures obtained by a Hrushovski construction, this is the only obstacle to compatibility with dividing.

In fact, we can analyse total transcendental behaviour in pseudo-exponentiation, for instance, using Morley rank *over bounded subsets*, the definition and fundamental properties of which we give at this point for future reference.

DEFINITION 1.31. Let $p(\vec{x})$ be a type over a finite $B \subset \mathbb{M}$. Define the *Morley rank over bounded subsets* or *b-Morley-rank* by defining $\mathrm{RM_b}(p) \geq \alpha$ by induction on $\alpha$:

(1) $\mathrm{RM_b}(p) \geq 0$ if $p$ is realised in $\mathbb{M}$.
(2) $\mathrm{RM_b}(p) \geq 1$ if $p$ is unbounded.
(3) $\mathrm{RM_b}(p) \geq \alpha$, where $\alpha$ is a limit ordinal, if $\mathrm{RM_b}(p) \geq \beta$ for every $\beta < \alpha$.
(4) $\mathrm{RM_b}(p) \geq \alpha+1$, where $\alpha \geq 1$, if there exist pairwise contradictory $\{\psi_i | i < \omega\}$ such that for each $i < \omega$ and each $\vec{b} \in \mathbb{M}$ there exists a complete type $q_i(\vec{x})$ over $\vec{b}$ such that

$$\mathrm{RM_b}(p(\vec{x}) \cup \psi_i \cup q_i(\vec{x})) \geq \alpha.$$

The conventions regarding b-Morley-rank $\pm\infty$ are again standard and $\mathbb{M}$ is called *b-totally-transcendental* if $\mathrm{RM_b}(\{x = x\}) < \infty$.

Just as in **[28]** we will introduce an auxiliary notion of "2-rank over bounded sets" to facilitate our technical results:

For a partial type $p(\vec{x}, \vec{b})$, where $\vec{b}$ is finite, define $\mathrm{R}^*_\mathrm{b} \geq \alpha$ by induction:

(1) $\mathrm{R}^*_{\mathrm{b}}(p) \geq 0$ if $p$ is realised in $\mathbb{M}$.

(2) $\mathrm{R}^*_{\mathrm{b}}(p) \geq 1$ if $p$ is unbounded.

(3) $\mathrm{R}^*_{\mathrm{b}}(p) \geq \alpha$, where $\alpha$ is a limit ordinal, if $\mathrm{R}^*_{\mathrm{b}}(p) \geq \beta$ for every $\beta < \alpha$.

(4) $\mathrm{R}^*_{\mathrm{b}}(p) \geq \alpha + n$, where $\alpha$ is a limit ordinal or 0 and $\alpha + n > 1$, if there exist pairwise contradictory $\{\psi_i | i < 2^n\}$ such that for each $i < 2^n$ and each $\vec{b} \in \mathbb{M}$ there exists a complete type $q_i(\vec{x})$ over $\vec{b}$ such that

$$\mathrm{R}^*_{\mathrm{b}}(p(\vec{x}) \cup \psi_i \cup q_i(\vec{x})) \geq \alpha.$$

Remark 1.32. Observe that now $\mathrm{R}^*_{\mathrm{b}}(p) = 0$ for any bounded $p$ and that $\mathrm{R}^*_{\mathrm{b}}(p) \geq \omega$ if and only if $p$ is unbounded. Therefore, $\mathrm{R}^*_{\mathrm{b}}$ does not take finite values.

Even more than in the original paper, the rank $\mathrm{R}^*_{\mathrm{b}}$ is for us merely a technical aid and not suggested as a reasonable rank notion for b-totally-transcendental theories.

Proposition 1.33. *There is a critical value for the boundedness of* $\mathrm{R}^*_{\mathrm{b}}$ *and* $\mathrm{RM}_{\mathrm{b}}$.

Proof. Follows immediately from Claim 3.4 in [**28**]. □

What is called the "weak ultrametric property" in [**28**] will again be used as the main technical result for our purposes.

Proposition 1.34. *Let $\alpha$ be a limit ordinal or* 0*. Then $\mathrm{R}^*_{\mathrm{b}}(p) \geq \alpha$ if and only if for any $\phi(\vec{x}, \vec{a})$ either $\mathrm{R}^*_{\mathrm{b}}(p \cup \{\phi(\vec{x}, \vec{a})\}) \geq \alpha$ or $\mathrm{R}^*_{\mathrm{b}}(p \cup \{\neg\phi(\vec{x}, \vec{a})\}) \geq \alpha$.*

Proof. For 0 and $\omega$, this is clear by 1.32. For the induction step, in which the additional clause in $\mathrm{R}^*_{\mathrm{b}}$ as opposed to $\mathrm{R}^*$ can be ignored, the proof of Prop. 3.7 in [**28**] transfers as written there. □

This implies the following fundamental relationship between $\mathrm{RM}_{\mathrm{b}}$ and $\mathrm{R}^*_{\mathrm{b}}$.

PROPOSITION 1.35. *For any type $p$ we have $\mathrm{R}^*_{\mathrm{b}}(p) \geq \omega * \alpha$ if and only if $\mathrm{RM}_{\mathrm{b}}(p) \geq \alpha$.*

PROOF. Just as above, the cases 0 and $\omega$ are clear, and the induction step is verbatim as in the proof of the analogous Lemma 3.8 of [**28**]. $\square$

As in [**28**], we can now define b-Morley degree and prove its finiteness. However, we naturally have to restrict ourselves to positive Morley rank.

THEOREM 1.36. *Suppose that $\mathrm{RM}_{\mathrm{b}}(p) = \alpha > 0$. Then there is an $n \in \mathbb{N}$ such that $p$ has exactly $n$ contradictory D-extensions of b-Morley rank $\alpha$.*

PROOF. Exactly as in the proof of Thm. 3.9 in [**28**]: Suppose not. Then there are pairwise contradictory $\{\psi_i^n(\vec{x}, \vec{a_i^n}) | i < 2^n\}$ extending $p$ of b-Morley rank $\alpha$. Furthermore, as $\alpha > 0$, all the $\psi_i^n$ are unbounded. Hence, we can conclude as in the theorem, invoking Proposition 1.35 twice. $\square$

CHAPTER 2

# Amalgamation Theory

In this section, we will give a systematic development of amalgamation theory from Fraïssé and Jónsson in the 1950s to contemporary categorical generalisations. As we feel that such a compilation is sorely lacking, we will attempt to fill that gap by giving a more comprehensive exposition than we did above for homogeneous model theory, although we will not include any proofs where they can easily be found in the literature.

## 2.1. Classical Amalgamation Theory

At the beginning, there were two distinct motivations to amalgamation in model theory. The first was to provide a framework in which universal models of large classes of structures could be found, serving as universal domains for this class. The other was to construct new countable structures that are homogeneous with respect to all of their substructures. We will follow Morley and Vaught [**32**] in our presentation of Jónsson's theory on the first topic.

We will define homogeneity and universality relative to a given class of $\mathcal{L}$-structures.

Definition 2.1. Let $\mathfrak{C}$ be a class of $\mathcal{L}$-structures closed under isomorphism and let $M \in \mathfrak{C}$.

(1) $M$ is called $\mathfrak{C}$-$\lambda$-*homogeneous* if, for all $N \in \mathfrak{C}$, $|N| < \kappa$, any embedding of $N$ into $M$ can be extended to an automorphism of $M$.
(2) $M$ is called $\mathfrak{C}$-*homogeneous* if $M$ is $\mathfrak{C}$-$|M|$-homogeneous.
(3) $M$ is called $\mathfrak{C}$-*universal* if all $N \in \mathfrak{C}$ with $|N| \leq |M|$ embed into $M$.
(4) $M$ is called $\mathfrak{C}$-*homogeneous-universal* if it is both $\mathfrak{C}$-homogeneous and $\mathfrak{C}$-universal.

Remark 2.2. $M$ is homogeneous (resp. universal) in the usual sense if an only if $M$ is $\mathfrak{C}$-homogeneous (universal) for $\mathfrak{C}$ the class of all models of $\mathrm{Th}(M)$ considered in a conservative expansion of $\mathcal{L}$ for which $\mathrm{Th}(M)$ has quantifier elimination. It is $\mathfrak{C}$-homogeneous-universal for this class if and only if it is saturated.

Amalgamation theory discusses conditions on $\mathfrak{C}$ that guarantee the existence and uniqueness of $\mathfrak{C}$-homogeneous and $\mathfrak{C}$-universal models.

We need the following notation.

Notation 2.3. A chain of structures in $\mathfrak{C}$ of order type $\kappa$ and ordered by inclusion is called a $\kappa$*-chain in* $\mathfrak{C}$.

Consider the following axioms:

Jónsson Axioms at $\kappa$ . *(1) $\mathfrak{C}$ contains an object of cardinality $\kappa$.*

*(2) For any $M_1$ and $M_2$ in $\mathfrak{C}$, $|M_1|, |M_2| < \kappa$, there are $N_1, N_2, N \in \mathfrak{C}$ such that $N_1, N_2 \subseteq N$, $M_1 \cong N_1$ and $M_2 \cong N_2$.*

*(3) If $M, M' \subseteq N$ in $\mathfrak{C}$, $|M_1|, |M_2| < \kappa$, and $f : M \xrightarrow{\sim} M'$ an isomorphism, then there are $P \supseteq N$ in $\mathfrak{C}$ such that $f$ extends to an embedding $g$ of $N$ into $P$.*

(4) *The union of any $\kappa$-chain in $\mathfrak{C}$ belongs to $\mathfrak{C}$.*

(5) *If $M \in \mathfrak{C}$ and $X \subseteq M$ with $|X| < \kappa$, then there is $N \in \mathfrak{C}$ with $X \subseteq N \subseteq M$ and $|N| < \kappa$ .*

REMARK 2.4. By clause 5 we can restrict clauses 2 and 3 to structures smaller than $\kappa$ on both sides of the proposition.

Under certain cardinality restrictions, the main result of [**21**], extended by [**32**], is the existence and uniqueness of $\mathfrak{C}$-homogeneous-universal structures of size $\kappa$ for all $\mathfrak{C}$ satisfying the axioms above. More precisely:

THEOREM 2.5 (Theorem 2.8 of [**32**]). *Let $\mathcal{L}$ be a relational language, $\mathfrak{C}$ be a class of $\mathcal{L}$-structures closed under isomorphism. Let $\kappa$ be a regular cardinal which satisfies $\kappa = \sup(\{2^\lambda | \lambda < \kappa\})$ and let the set of isomorphism types of $\mathfrak{C}$ smaller than $\kappa$ have at most cardinality $\kappa$itself. Let $\mathfrak{C}$ satisfy the Jónsson axioms at $\kappa$. Then there is a unique $\mathfrak{C}$-homogeneous-universal $M \in \mathfrak{C}$ of cardinality $\kappa$.*

Under the General Continuum Hypothesis, $\kappa = \sup(\{2^\lambda | \lambda < \kappa\})$ for each regular $\kappa$. However, the cardinality restrictions given here are very strong in general - in fact, assuming only the consistency of ZFC, there is a model of ZFC such that $\kappa < \sup(\{2^\lambda | \lambda < \kappa\})$ for all regular uncountable cardinals $\kappa$ (This is a special case of Easton's Theorem [**12**]). The limitation on $\mathcal{L}$ being relational is not as problematic in general, as $\mathcal{L}$ can be replaced by a relational language, and then the statements "$f$ is a function" for every function symbol $f$ and "There is exactly one $x$ such that $c(x)$" for every constant $c$ axiomatise the structures of the original language.

In the case that $\kappa = \omega$, we obtain the relational version of Fraïssé's theorem from the proof of Theorem 2.5.

This is the terminology usually associated with Fraïssé constructions

Definition 2.6. Let $M$ be a countable $\mathcal{L}$-structure.

(1) $M$ is called *ultrahomogeneous* if any embedding of a finite substructure $X \subset M$ into $M$ extends into an automorphism of $M$.
(2) The set of all isomorphism types of finite substructuires of $M$ is called the *age* of $M$.

Corollary 2.7 (Fraïssé's Theorem, Relational Version). *Let $\mathcal{L}$ be a relational language, and let $\mathfrak{C}$ be a countable class of finite $\mathcal{L}$-structures satisfying the following clauses. Then there is a countably infinite ultrahomogeneous $\mathcal{L}$-structure $L$, unique up to isomorphism, such that the age of $L$ is $\mathfrak{C}$.*

*(1) There are arbitrarily large finite structures in $\mathfrak{C}$.*
*(2) $\mathfrak{C}$ is closed under taking substructures.*
*(3) For any $M_1, M_2 \in \mathfrak{C}$ there is an $N \in \mathfrak{C}$ such that there are embeddings $g_1 : M_1 \to N$ and $g_2 : M_2 \to N$.*
*(4) If $f_1 : M_0 \to M_1$ and $f_2 : M_0 \to M_2$ are embeddings, then there is an $N \in \mathfrak{C}$ and embeddings $g_1 : M_1 \to N$ and $g_2 : M_2 \to N$ such that $g_1 \circ f_1 = g_2 \circ f_2$.*

Proof. Define $\mathfrak{C}'$ as the class of all unions of countable chains in $\mathfrak{C}$. As this includes trivial chains, $\mathfrak{C} \subseteq \mathfrak{C}'$. Then $\mathfrak{C}'$ satisfies all Jónsson axioms at $\omega$.

A countably infinite object is obtained as the union of a countable chain of ever larger objects generated by clauses 1 and 3.

Axiom 2 is just a reformulation of clause 3.

Axiom 3 follows from clause 4, taking $M_0$ as $M$ and $N$ as $M_1 = M_2$. Take $f_1$ to be the inclusion of $M$ in $N$ and $f_2$ to be the embedding coming from the isomorphism of $M$ to $M'$. Then clause 4 gives $P$ and an embedding with the required properties.

Axiom 4 is true by the definition of $\mathfrak{C}'$ and the fact that countable chains of countable chains can be replaced by single countable chains.

Axiom 5 follows from clause 2. $\square$

If we move away from relational languages, we can replace "finite" with "finitely generated" throughout the statement of Fraïssé's Theorem and in Definition 2.6 to obtain the version found in [**16**].

We will finish the section by mentioning some classical examples of Jónsson's and Fraïssé's results.

Example 2.8. As mentioned in Remark 2.2, saturated models can be seen as homogeneous-universal models of the class of models of their first-order theory. Jónsson's theorem then proves the existence of saturated models in regular cardinalities under GCH. In the same way, relatively saturated submodels of homogeneous models can be constructed in the $\lambda^{<\lambda} = \lambda$-case of Fact 1.7.

In the stable case, however, the methods of this section do not succeed in proving the existence of a (relatively) saturated model, and it will be the focus of a later section here to remedy that.

Fraïssé constructions have been widely used throughout model theory and wider mathematics, and we will just give two particularly important specimens. A good classical exposition of both can be found in [**16**].

EXAMPLE 2.9. The unique universal-homogeneous countable graph is precisely Rado's random graph.

EXAMPLE 2.10. The unique universal-homogeneous countable group (the limit of finitely generated groups) is known as Hall's universal locally finite group.

## 2.2. Categorical Generalisations: Generalities

In the remainder of this section we will present the advances made in amalgamation theory in the more than fifty years since Morley and Vaught published their account. As the process of amalgamating is essentially abstract, the language of category theory can very naturally be used to give streamlined, transparent and general results that are then available regardless of the area of application.

While we will make extensive use of this material throughout the remainder of the thesis, all our applications will take place in concrete categories, that is, categories of sets with morphisms that are actual functions. The reader is invited always to have the classical constructions from the last section in mind as paradigms.

In this section, we will fix the appropriate notation and state the categorical version of the uniqueness of homogeneous-universal models.

This terminology goes back essentially to Manfred Droste and Rüdiger Göbel [**11**], who in turn borrowed concepts from universal algebra.

DEFINITION 2.11. Let $\mathfrak{C}$ be a category, $\lambda$ an infinite regular cardinal. Then

(1) A *directed diagram in* $\mathfrak{C}$ is a functor from a directed partially ordered set into $\mathfrak{C}$.

(2) A sequence $(A_i, f_{ij})_{i\leq j<\lambda}$ of objects $A_i$ and morphisms $f_{ij} : A_i \to A_j$ is called a $\lambda$-*chain in* $\mathfrak{C}$ if $f_{ii} = 1_{A_i}$ and $f_{ik} = f_{jk} \circ f_{ij}$ for all $i, j, k < \kappa$, $i \leq j \leq k$.

(3) An object $A$ of $\mathfrak{C}$ is called $\lambda$-*small* if any morphism from $A$ into the colimit of a $\lambda$-directed diagram factors through one of the objects in the diagram. We denote the full subcategory of $\mathfrak{C}$ consisting of $\lambda$-small objects by $\mathfrak{C}_{<\lambda}$.

(4) $\mathfrak{C}$ is called $\lambda$-*algebroidal* if every directed diagram in $\mathfrak{C}_{<\lambda}$ with at most $\lambda$ objects and morphisms has a colimit in $\mathfrak{C}$ and every object of $\mathfrak{C}$ is the colimit of such a diagram.

Remark. [**11**] call similar categories "semi-$\lambda$-algebroidal", reserving the term "$\lambda$-algebroidal" for those categories satisfying all the requirements of their amalgamation theorem. A translation will be given below. Our use here corresponds more closely to the original terminology in [**4**], but is not equivalent to it.

Also, note that every $\lambda$-chain is a directed diagram, as it is just a functor from $\lambda$ to $\mathfrak{C}$.

While we are appealing to directed diagrams in clauses 2 and 3 of the definition, [**11**] use chains throughout. Our notion is in fact a slight narrowing of the definition, which will streamline proofs and results later.

Definition 2.12. Let $\mathfrak{C}$ be a category, $\lambda$ an infinite regular cardinal. Then

(1) An object $A$ of $\mathfrak{C}$ is called $\lambda$-*chain-small* if any morphism from $A$ into the colimit of a $\lambda$-chain factors through one of the objects in the diagram. We denote the full subcategory of $\mathfrak{C}$ consisting of $\lambda$-chain-small objects by $\mathfrak{C}^c_{<\lambda}$.

(2) $\mathfrak{C}$ is called $\lambda$-*chain-algebroidal* if for every $\mu \leq \lambda$ every $\mu$-chain in $\mathfrak{C}^c_{<\lambda}$ has a colimit in $\mathfrak{C}$ and every object of $\mathfrak{C}$ is the colimit of such a diagram. This is referred to as "semi-algebroidal" in [**11**].

The essential tool is the following lemma:

Lemma 2.13. *In a $\lambda$-algebroidal category $\mathfrak{C}$, every $c \in \mathfrak{C}$ is already the colimit of a $\lambda$-chain in $\mathfrak{C}_{<\lambda}$.*

Proof. By 1.A.1.6 in [**2**], every infinite directed diagram $D$ can be expressed as the union of a $\lambda$-chain $(D_\mu | \mu < \lambda)$ of directed subdiagrams, each of which have smaller cardinality than $D$ itself. Note also that for $\mu < \lambda$, the colimit of a $\mu$-chain of $\lambda$-small objects is itself $\lambda$-small. Therefore, the colimit of every diagram $D$ of cardinality $\lambda$ in $\mathfrak{C}_{<\lambda}$ is reached by the $\lambda$-chain made up of the colimits of the $D_\kappa$. □

Proposition 2.14. *Every $\lambda$-algebroidal category is $\lambda$-chain-algebroidal.*

Proof. It is easy to see that in a $\lambda$-algebroidal category (and indeed in any $\lambda$-chain-algebroidal category), *all* $\mu$-chains ($\mu \leq \lambda$) have colimits. By the above lemma, every $c \in \mathfrak{C}$ is the colimit of a $\lambda$-chain in $\mathfrak{C}_{<\lambda} \subseteq \mathfrak{C}^c_{<\lambda}$, which completes the proof. □

Remark. Although our analysis here concentrates on categories in which every arrow is monic, this is not essential to the notion of $\lambda$-algebroidality; in fact, it has originally been introduced and studied by [**4**] in the context of universal algebra, where epimorphisms rather than monomorphisms play a distinguished role.

We will now continue to present the terminology used in category-theoretical amalgamation constructions. They are all taken from [**11**].

Definition 2.15. Let $\mathfrak{C}$ be a category, all arrows of which are monic, $\mathfrak{C}'$ a full subcategory and $U \in \mathfrak{C}$. Then:

(1) We say that $U$ is $\mathfrak{C}'$-*universal* if for any $A \in \mathfrak{C}'$ there exists $g : A \to U$.
(2) We call $U$ $\mathfrak{C}'$-*homogeneous* if for any $A \in \mathfrak{C}'$ and $f, g : A \to U$ there exists an automorphism $h$ of $U$ such that $h \circ f = g$.
(3) We say that $U$ is $\mathfrak{C}'$-*saturated* if for any $A, B \in \mathfrak{C}'$ and arrows $f : A \to U$ and $g : A \to B$ there exists $h : B \to U$ with $h \circ g = f$.

The following theorem is the reason for the fundamental role that $\lambda$-algebroidal categories play in categorical amalgamation theory:

Theorem 2.16 (Proposition 2.2 in [**11**]). *Let $\lambda$ be a regular infinite cardinal and $\mathfrak{C}$ a $\lambda$-(chain)-algebroidal category in which all morphisms are monic.*

*(1) For any object $U$ of $\mathfrak{C}$ the following are equivalent:*
  *(a) $U$ is $\mathfrak{C}$-universal and $\mathfrak{C}_{<\lambda}$-homogeneous.*
  *(b) $U$ is $\mathfrak{C}_{<\lambda}$-universal and $\mathfrak{C}_{<\lambda}$-homogeneous.*
  *(c) $U$ is $\mathfrak{C}_{<\lambda}$-universal and $\mathfrak{C}_{<\lambda}$-saturated.*

  *Moreover, if $\mathfrak{C}_{<\lambda}$ contains an initial object, these are also equivalent to*
  *(d) $U$ is $\mathfrak{C}_{<\lambda}$-saturated.*
*(2) Any two objects satisfying these equivalent conditions are isomorphic.*
*(3) If there exists such an object, $\mathfrak{C}_{<\lambda}$ has the amalgamation property.*
*(4) If there exists a $\mathfrak{C}_{<\lambda}$-universal object, then $\mathfrak{C}_{<\lambda}$ has the joint embedding property.*
*(5) If $\mathfrak{C}_{<\lambda}$ contains an initial object and has the amalgamation property, then $\mathfrak{C}_{<\lambda}$ has the joint embedding property.*

Definition 2.17. A $U$ satisfying the equivalent statements of the theorem above will be called an *amalgamation limit for* $\mathfrak{C}_{<\lambda}$. By the theorem, if $\mathfrak{C}$ is a $\lambda$-algebroidal category in which all morphisms are monic, it is well-defined up to isomorphism in $\mathfrak{C}$.

In order to understand those notions better in examples and to simplify applications later, we will introduce the terminology we use for concrete categories.

Definition 2.18. ([**1**]) A *construct* is a category $\mathfrak{C}$ equipped with a faithful functor $|\cdot| : \mathfrak{C} \to \mathbf{Set}$.

(1) An *embedding* is a morphism $\iota : A \to B$ in $\mathfrak{C}$ such that
   (a) For any $C \in \mathfrak{C}$ and set map $g : |C| \to |A|$, $g$ is the image of a morphism in $\mathfrak{C}$ whenever $|\iota| \circ g$ is.
   (b) $|\iota|$ is injective.
(2) A *subobject* of a $B \in \mathfrak{C}$ is an $A \in \mathfrak{C}$ together with an embedding $\iota : A \to B$.
(3) An *intersection* of a family of subobjects $(A_i, \iota_i)$ of a $B \in \mathfrak{C}$ is a subobject $(A, \iota)$ of $B$ satisfying the following properties;
   (a) $\iota$ factors through each $\iota_i$.
   (b) If a subobject $(C, \tau)$ of $B$ factors through each $\iota_i$, then it factors through $\iota$.
(4) An (intersection, colimit, etc.) is called *concrete* if it is preserved by $|\cdot|$.

A construct $(\mathfrak{C}, |\cdot|)$ will be called a *hull category* if the following holds:

(1) Every arrow is an embedding.
(2) $\mathfrak{C}$ has (arbitrary) concrete intersections.
(3) $\mathfrak{C}$ has concrete colimits of directed diagrams.

Remark. By 1.A.1.7 in **[2]**, the third clause could equivalently be restricted to colimits of $\lambda$-chains for every $\lambda$.

In a hull category, for every $A \subseteq B$, one can define $\hat{A}$ to be the intersection of all subobjects of $B$ containing $A$ and this will be referred to as the *hull of $A$ in $B$*. Conversely, we will call $\hat{A}$ *generated by $A$* and say that an object $B$ is *$\kappa$-generated* for a cardinal $\kappa$ if it is generated by an $A \subseteq B$ with $|A| < \kappa$.

The notions of $\kappa$-small and $\kappa$-generated should properly be read as "small*er* than $\kappa$" and "generated by *less than* $\kappa$ elements". Our terminology here conforms to common usage and enables easier typesetting.

Proposition 2.19. *(1) Let $\mathfrak{C}$ be a hull category. Then $B$ is $\lambda$-generated iff it is $\lambda$-small.*

*(2) If $\mathfrak{C}$ is a concrete category satisfying only the 1st and 3rd clause of the definition of a hull category, then still every object $B$ with $|B| < \lambda$ is $\lambda$-small.*

Proof. 1. "$\Rightarrow$": Let $g : B \to C$, $C$ be the colimit of the $\lambda$-directed diagram $(C_\alpha, f_\beta)$ with induced embeddings $f'_\alpha : C_\alpha \to C$ and let $A$ be a generating set for $B$ with $|A| < \lambda$. Since $C$ is a concrete colimit, it is set-theoretically the union of the $C_i$. Therefore, for every $a \in A$ there is an $\alpha$ such that $g(a) \in C_\alpha$. Since $(C_\alpha, f_\beta)$ is $\lambda$-directed, this implies that there is a $\mu$ such that $g(A) \subseteq C_\mu$. Form the concrete intersection $(D, h)$ of $(C_\mu, f'_\mu)$ and $(B, g)$ in $C$. Then this factors through $B$ as a subobject containing $A$. Therefore, $(D, h) = (B, g)$, $B$ is set-theoretically contained in $C_\mu$ and, as $g$ is an embedding (in the construct sense), $g$ factors through $(C_\mu, f_\mu)$.
"$\Leftarrow$": Let $B$ not be $\lambda$-generated and let $\mathfrak{A}$ be the set of all $\lambda$-generated subobjects of

$B$. Furthermore, let $\mathfrak{F} := \{g | \exists (A, f), (A', f') \in \mathfrak{A}$ such that $g : A \to A'$ and $f' \circ g = f\}$. Then $(\mathfrak{A}, \mathfrak{F})$ forms a $\lambda$-directed diagram by the regularity of $\lambda$. However, the identity morphism does not factor through this chain as $B$ is not $\lambda$-generated. Therefore, $B$ is not $\lambda$-small.

2. Just as "$\Rightarrow$" above, setting $A = B$ and noting that intersections have nowhere else been used. □

We close this section with some examples of $\lambda$-algebroidal categories:

EXAMPLE 2.20. (1) The category $\mathbf{Set}^{\hookrightarrow}_{\leq\lambda}$ of sets of cardinality not exceeding $\lambda$ and with only the injective morphisms is a $\lambda$-algebroidal category.

(2) The class of all models of cardinality not exceeding $\lambda$ of a complete first-order theory $T$ with $|T| < \lambda$ with elementary embeddings between them is also $\lambda$-algebroidal.

(3) Let $U$ be the generic structure of a Fraïssé amalgamation class. Then the age of $U$ along with embeddings between them is $\aleph_0$-algebroidal.

PROOF. 1. This can be seen using Proposition 2.14 and the fact that every set $\{a_i\}_{i<\lambda}$ is the limit of a $\lambda$-chain $(A_\alpha, \subseteq)$, where $A_\alpha = \{a_i\}_{i\leq\alpha}$.

2. First we note that every model of cardinality less than $\lambda$ is $\lambda$-small in this category $\mathfrak{C}^T$ by the second part of 2.19: That clause 1 of the definition of a hull category holds is clear and that clause 3 holds is the Tarski-Vaught theorem on unions of elementary chains (vide e.g. [**16**], 2.5.2). To see that every structure $M$ in $\mathfrak{C}_T$ is the limit of a $\lambda$-chain in $\mathfrak{C}^T_{<\lambda}$, enumerate $M$ and use the downward Löwenheim-Skolem Theorem.

3. Follows from the discussion in Section 2.1 and clause 2. of Proposition 2.19. □

## 2.3. Droste and Göbel's Amalgamation Theory

In their 1989 conference paper [**11**], Droste and Göbel gave a categorical approach to Jónsson Theory that seperates the combinatorial core of the argument from the model-theoretic setting of the original.

To formulate this, we first need category-theoretic analogues of clauses 3 and 4 of Corollary 2.7, the amalgamation property and the joint embedding property, clauses 3 and 4 of Corollary 2.7.

DEFINITION 2.21. Let $\mathfrak{C}$ a category in which all morphisms are monic. Then

$\mathfrak{C}$ has the *joint embedding property* if for any $A, B \in \mathfrak{C}$ there exists $C \in \mathfrak{C}$and arrows $f : A \to C$ and $g : B \to C$.

$\mathfrak{C}$ has the *amalgamation property* if for any $A, B_1, B_2 \in \mathfrak{C}$ and arrows $f_i : A \to B_i$ there exists $C \in \mathfrak{C}$ and arrows $g_i : B_i \to C$ such that the following diagram commutes:

$$\begin{tikzcd} & B_1 \arrow[dr, dashed] & \\ A \arrow[ur] \arrow[dr] & & B \\ & B_2 \arrow[ur, dashed] & \end{tikzcd}$$

THEOREM 2.22 (Theorem 1.1 of [**11**]). *Let $\lambda$ be a regular cardinal, and let $\mathfrak{C}$ be a $\lambda$-algebroidal category in which all morphisms are monic such that $\mathfrak{C}_{<\lambda}$ contains only at most $\lambda$ objects up to isomorphism and between any two objects of $\mathfrak{C}_{<\lambda}$ there* *are at most $\lambda$ morphisms. Then there exists a $\mathfrak{C}$-universal $\mathfrak{C}_{<\lambda}$-homogeneous object in $\mathfrak{C}$ if and only if $\mathfrak{C}_{<\lambda}$ has the joint embedding and the amalgamation property.*

REMARK 2.23. As noticed by Jonathan Kirby **[25]**, one can replace " between any two objects of $\mathfrak{C}_{<\lambda}$ there are at most $\lambda$ morphisms" with "Every object $A$ of $\mathfrak{C}_{<\lambda}$ has at most $\lambda$ extensions up to $A$-isomorphism", which is more general and more natural, while leaving the proof unchanged.

This theorem generalises Jónsson's result beyond the context of relational structures and embeddings. By using the more general notion of smallness, it also enables us to deal much more freely with structures that are in some way generated by a subset of a certain cardinality rather than being of that cardinality themselves.

However, it still leaves open a generalisation of the stable case of the unconditional existence of (relatively) saturated models, and, in fact, no such generalisation can be found in the literature.

As this is what we require in applications, we will formulate and prove such a theorem ourselves in the next section.

## 2.4. The stable case

First we need some more notation:

NOTATION 2.24. For every $K \in \mathfrak{C}$, define $\mathfrak{C}(K)$ as the category whose objects consist of all pairs $(X, \to)$, where $\to$ is an arrow from $K$ to $X$ in $\mathfrak{C}$ and

$$\mathrm{Mor}_{(A,\to_A),(B,\to_B)}(\mathfrak{C}(K)) := \{\to \in \mathrm{Mor}_{A,B}(\mathfrak{C})\ \mathrm{s.\,t.}\ \to\ \text{commutes with}\ \to_{\mathrm{A}}\ \text{and}\ \to_B\}.$$

DEFINITION 2.25. A category $\mathfrak{C}$ is called a *$\lambda$-stability category* if for all $K \in \mathfrak{C}$ the following holds:

(1) $\mathfrak{C}$ is closed under limits of $\mu$-chains for every $\mu \leq \lambda$.
(2) There is an initial object.
(3) For every $A \in \mathfrak{C}(K)$ there is a $\mu \leq \lambda$ such that $A$ is the colimit of a $\mu$-chain $(A_\alpha)_{\alpha<\mu}$ in $\mathfrak{C}(K)_{<\lambda}$ with $(A_{\alpha+1}, \rightarrow) \in \mathfrak{C}(A_\alpha)_{<\aleph_0}$, for every $\alpha < \mu$, $A_0 = K$ and $A_\kappa$ the colimit of $(A_\alpha)_{\alpha<\kappa}$ for any limit ordinal $\kappa$, where $\rightarrow$ is the morphism of the $\mu$-chain (*regular* $\mu$-chain).
(4) If $(B, \rightarrow) \in \mathfrak{C}(K)_{<\aleph_0}$, $K \rightarrow A$, there is a $D$ and arrows $B \dashrightarrow D$ and $A \dashrightarrow D$ such that $(D, \dashrightarrow) \in \mathfrak{C}(A)_{<\aleph_0}$ and the following square commutes:

$$\begin{tikzcd} K \arrow[r] \arrow[d, "<\aleph_0"] & A \arrow[d, dashed, "<\aleph_0"] \\ B \arrow[r, dashed] & D \end{tikzcd}$$

(5) $|\mathfrak{C}(K)_{<\aleph_0}| \leq \lambda$ (Counting only isomorphism types)

REMARK. Every $\lambda$-stability category is $\lambda$-algebroidal.

In the cases in which we will be interested, demanding decomposition into regular $\mu$-chains is no additional restriction.

PROPOSITION 2.26. *Let $\mathfrak{C}$ be a hull category . Then $\mathfrak{C}_{<\lambda^+}$ satifies clause 3 of 2.25.*

PROOF. First we show that $\mathfrak{C}(K)$ is again a hull category under the natural forgetful functor inherited from $\mathfrak{C}$.

Morphisms are embeddings: (b) is inherited directly from $\mathfrak{C}$. To check (a), consider the following diagram

$$\begin{tikzcd}
K \arrow[r, "\tau_B"] \arrow[d, "\tau_A"'] \arrow[dr, "\tau_C"] & B \arrow[d, "f \circ g"] \\
A \arrow[r, "f"'] & C
\end{tikzcd}$$

Both triangles commute by the definition of $\mathfrak{C}(K)$, and therefore the whole square commutes. As $\mathfrak{C}$ is a hull category, there is a morphism $g$ such that in

$$\begin{tikzcd}
K \arrow[r, "\tau_B"] \arrow[d, "\tau_A"'] & B \arrow[d, "f \circ g"] \\
A \arrow[r, "f"'] \arrow[ur, "g"'] & C
\end{tikzcd}$$

the lower right triangle commutes. Since the whole square commutes too, the upper left triangle commutes, and $g$ is a morphism in $\mathfrak{C}(K)$.

Concrete intersections: As $(K, \tau_B)$ factors through each $\iota_i$, it factors through the intersection as $(K, \tau_A)$.

Concrete colimits of directed diagrams: Clear.

Therefore it suffices to show the conclusion of clause 3 for $\mathfrak{C}$ itself. For this, consider an enumeration $\{a_i\}_{i<\mu}$ of a $\lambda$-generator of $A \in \mathfrak{C}$. Let $A_\alpha := \widehat{\{a_i\}_{i<\mu}}$. Then the chain of $(A_\alpha)_{\alpha<\mu}$ is regular with colimit $A$. $\square$

THEOREM 2.27. *Every $\lambda$-stability category $\mathfrak{C}$ contains an amalgamation limit.*

PROOF. We give the construction of the desired element as a direct limit of a $\lambda$-chain in $\mathfrak{C}$.

Let $A_0$ be the initial object in $\mathfrak{C}$. (Exists by 2.)

Assume $A_\alpha$ has been constructed. Then we will give $A_{\alpha+1}$ as a direct limit of a $\lambda$-chain itself:

Enumerate $\mathfrak{C}(A_\alpha)_{<\aleph_0}$ as $(A_\alpha^\gamma)_{\gamma<\lambda}$ using 5.

Let $A_{\alpha,0} := A_\alpha$.

Let $A_{\alpha,\beta+1}$ complete the following square using 4.:

$$\begin{tikzcd}
A_\alpha \arrow[r] \arrow[d, "<\aleph_0"'] & A_{\alpha,\beta} \arrow[d, dashed, "<\aleph_0"] \\
A_\alpha^\beta \arrow[r, dashed] & A_{\alpha,\beta+1}
\end{tikzcd}$$

At limit ordinals, take the limit.

Then $A_{\alpha+1} := A_{\alpha,\lambda} \in \mathfrak{C}$ using 1.

At limit steps in the outer construction, also take the limit, again invoking 1.

Let $U := A_\lambda$. We claim that $U$ is an amalgamation limit for $\mathfrak{C}_{<\lambda}$.

As clauses 1. and 3. imply that $\mathfrak{C}$ is $\lambda$-algebroidal, it suffices by 2.16 to show that $U$ is $\mathfrak{C}_{<\lambda}$-saturated.

Let $B, C \in \mathfrak{C}$, $B \to C$ and $B \to U$. We need to show that there is an arrow $C \to U$ such that $B \to C$ factors into the other two.

Let $(C, \to)$ be the limit of the regular $\lambda$-chain $(C_\nu)_{\nu<\lambda}$, where $C_\nu \in \mathfrak{C}(B)$ and $C_0 \in$, and let $C_\lambda := C$. (Possible by 3.)

We give an arrow from $C_\nu \to U$ by induction on $\nu$.

Let $C_0 = B \to U$ be the arrow given in the data.

For the successor step, consider the following diagram:

$$\begin{tikzcd}
C_\nu \arrow[r] \arrow[d, "<\aleph_0"] & A_\alpha \arrow[d, dashed, "<\aleph_0"] & & \\
C_{\nu+1} \arrow[r, dashed] & A_\alpha^\gamma \arrow[r] & A_{\alpha+1} \arrow[r] & U
\end{tikzcd}$$

Here, the $A_\alpha$ is the object occurring in the chain constructed in the first part of the proof through which the arrow $C_\nu \to U$ factorises; such an $A_\alpha$ exists as $C_\nu \in \mathfrak{C}_{<\lambda}$. In particular, all the arrows to $U$ commute.

Therefore we can pass to the limit at limit steps and obtain an arrow $C = C_\lambda \to U$ with the desired properties. $\square$

Before we will carry on to investigate the model-theoretic implications of amalgamation in such a stability category, we will show that we have actually captured the case we were meaning to generalise:

Example 2.28. Every $\lambda$-stable theory $T$ (with $|T| < \lambda$) has a saturated model of cardinality $\lambda$.

Proof. We consider a larger category than the one in Examples 2.20 which satisfies the hypothesis of Proposition 2.26. Let $\mathbb{M}$ be a monster model for the theory $T$ and let $\mathfrak{C}$ be the category of sub*structures* of $\mathbb{M}$ with partial elementary embeddings between them (allowing the empty substructure as an intial object) Alternatively, we could also consider $\mathbb{M}$ as an $\mathcal{L}'$-structure for an expanded language $\mathcal{L}'$ which enforces quantifier elimination. Then the union of a $\kappa$-chain of embeddings is itself an embedding and therefore the third clause of the definition of a hull category is satisfied. The first clause is trivially true and the second follows from our consideration of substructures rather than submodels. Therefore $\mathfrak{C}_{\leq\lambda}$

satisfies clause 3. of Definition 2.25. As the theory is $\lambda$-stable, it also satisfies the other requirements of a $\lambda$-stability category. By induction on quantifier depth, for instance, one can easily see that the amalgamation limit is itself a submodel of $\mathbb{M}$, and it is clearly also an amalgamation limit of the full subcategory of all submodels of cardinality less than $\lambda$. Therefore, it is $\lambda$-saturated. $\square$

The reason the tools we presented here make particularly homogeneous model theory into a powerful framework is that there is a straightforward translation between category-theoretic and model-theoretic homogeneoity, while trying to find any such abstract characterisation of saturation is hopeless.

## 2.5. Category Theory and Homogeneous Model Theory

We will formulate sufficient criteria for the amalgamation limit of a hull category to be homogeneous. The definition below gives the setting within which we will work in the rest of this section:

Definition 2.29. A hull category $\mathfrak{C}$ is called an *$\mathcal{L}$-structure category* if

(1) Every object of $\mathfrak{C}$ is an $\mathcal{L}$-structure with domain $|\mathfrak{C}|$.
(2) Every morphism is an $\mathcal{L}$-embedding.
(3) Every $\mathcal{L}$-isomorphism between objects of $\mathfrak{C}$ is a morphism in $\mathfrak{C}$.

Proposition 2.30. *Let $\mathfrak{C}$ be a $\lambda$-algebroidal $\mathcal{L}$-structure category. Let $M$ be an amalgamation limit for $\mathfrak{C}_{<\lambda}$ in $\mathfrak{C}$. Assume further that for any subsets $A, B$ of $M$ and any $a$ in the hull of $A$, the type of $a$ over $A$ is isolated and that, if* $\mathrm{tp}(Aa) = \mathrm{tp}(Bb)$, *then $a$ is in the hull of $A$ iff $b$ is in the hull of $B$. Then $M$ is $\lambda$-homogeneous.*

PROOF. Let $|A| < \lambda$ and $f$ be a partial elementary embedding from $A \subseteq M$ into $M$ and let $a \in M$. We need to extend $f$ to an automorphism of $M$. We will do this in two steps. First, we will extend $f$ to an isomorphism between the hulls of $A$ and its image. This is a standard back-and-forth construction, using the condition on the types and the fact that one can extend partial elementary embeddings to elements whose types are isolated over the domain of the embedding. Note that since $\mathfrak{C}$ is a hull category, the hull of $A$ is $\lambda$-small. By clause 3. in the definition of an $\mathcal{L}$-structure category, the resulting isomorphism is in $\mathfrak{C}$ and by $\mathfrak{C}_{<\lambda}$-homogeneity it can be exended to a $\mathfrak{C}$-automorphism of $M$, which is an embedding by clause 2. $\square$

REMARK. As the type of every algebraic element is isolated, it suffices for instance if the hull of a subset is contained in its algebraic closure.

Observe that the further conditions demanded by this proposition are very benign and can usually be read of the construction rather easily. On the other hand, would we restrict ourselves to traditional first-order model theory as our setting and demand saturation of the limit model, there are no such properties that could guarantee this; the great technical effort required to prove that the classes constructed by **[18]** are first-order illustrate that point.

We will now give a combinatorial condition characterising stability of the amalgamation limits, starting with a helpful category-theoretic lemma:

LEMMA 2.31. *Let $\mathfrak{C}$ be a hull category, $M, N \in \mathfrak{C}$ and $A \subseteq M$. Let $f : M \to N$ be an isomorphism. Then $f$ restricts to an isomorphism between the hull of $A$ and the hull of $f(A)$.*

PROOF. Consider the following two commuting diagrams, obtained from the definition of an intersection:

$$\begin{tikzcd} & M & \\ \widehat{f(A)} \arrow[ur, "f^{-1}\circ\tau"] & & \widehat{A} \arrow[ul, "\iota"'] \\ & \widehat{A} \arrow[ul, "f'"] \arrow[ur, "\mathrm{id}_{\widehat{A}}"'] & \end{tikzcd} \qquad \begin{tikzcd} & N & \\ \widehat{A} \arrow[ur, "f\circ\iota"] & & \widehat{f(A)} \arrow[ul, "\tau"'] \\ & \widehat{f(A)} \arrow[ul, "f''"] \arrow[ur, "\mathrm{id}_{\widehat{f(A)}}"'] & \end{tikzcd}$$

They give us the equations $f \circ \iota \circ f'' = \tau$ and $f^{-1} \circ \tau \circ f' = \iota$, which, taken together, simplify to $\tau \circ f' \circ f'' = \tau$ and $\iota \circ f'' \circ f' = \iota$. As all arrows in a hull category are monic, $f' \circ f'' = \mathrm{id}_{\widehat{f(A)}}$ and $f'' \circ f' = \mathrm{id}_{\widehat{A}}$. The equation $f^{-1} \circ \tau \circ f' = \iota$ also implies $f \circ \iota = \tau \circ f'$ and therefore that $f'$ really is the restriction of $f$. $\square$

PROPOSITION 2.32. *Let $\mathfrak{C}$ be a $\mu$-algebroidal $\mathcal{L}$-structure category and $M$ be a homogeneous amalgamation limit for $\mathfrak{C}_{<\mu}$ in $\mathfrak{C}$. Let $\lambda < \mu$ be a cardinal such that $\mathfrak{C}_{<\lambda^+} = \{K \in \mathfrak{C} | |K| < \lambda^+\}$. Then $M$ is $\lambda$-stable iff for every $K \in \mathfrak{C}_{<\lambda^+}$, $|\mathfrak{C}(K)_{<\aleph_0}| \leq \lambda$.*

PROOF. "$\Rightarrow$": Let $K \in \mathfrak{C}_{<\lambda^+}$, $\{\vec{a^n_\kappa} | \kappa < \lambda, n < \omega\}$ be an enumeration of realisations of the $n$-types over $K$ realised in $M$ (This exists as $M$ is $\lambda$-stable and $|K| < \lambda^+$). Let $A^n_\kappa$ be the hull of $K\vec{a^n_\kappa}$. We claim that the $A^n_\kappa$ enumerate the isomorphism types in $\mathfrak{C}(K)_{<\aleph_0}$. So let $B \in \mathfrak{C}(K)_{<\aleph_0}$. By $\mathfrak{C}_{<\mu}$-saturatedness there is a $K$-embedding of $B$ into $M$ and a finite tuple $\vec{b} \in M$ generating $B$. By homogeneity, there is now a $K$-automorphism of $M$ mapping $\vec{b}$ to an $\vec{a^n_\kappa}$. By the preceding lemma, this restricts to a $K$-automorphism from $B$ to $A^n_\kappa$.

"$\Leftarrow$": Let $A$ be a subset of $M$ of cardinality $\leq \lambda$ . Then $K$, the hull of $A$ as a subset of $M \in \mathfrak{C}$, lies in $\mathfrak{C}_{<\lambda^+}$ by Proposition 2.19. Now let $\{B_\kappa | \kappa < \lambda\}$ be an enumeration of $\mathfrak{C}(K)_{<\aleph_0}$ and let $c \in M$ be arbitrary. Again by Proposition 2.19, $C$, the hull of $Kc$, lies in $\mathfrak{C}(K)_{<\aleph_0}$. Therefore there is an isomorphism between $C$ and $B_\kappa$ for a $\kappa \in \lambda$. Now as $M$ is $\mathfrak{C}_{<\mu}$-homogeneous, it can be lifted to an automorphism of $M$. This means that the union of the $\{B_\kappa | \kappa < \lambda\}$ contains representatives for all $n$-types over $K \supseteq A$. As every structure in $\mathfrak{C}(K)_{<\aleph_0}$ also lies in $\mathfrak{C}_{<\lambda^+}$ , this is a union of at most $\lambda$ structures, each of which have cardinality at most $\lambda$. Therefore the whole union has cardinality not exceeding $\lambda$ as required. $\square$

CHAPTER 3

# Stability-Preserving Substructures

## 3.1. Introduction

When looking for structures with favourable model-theoretic behaviour within a class of elementary structures, it is often promising to consider the model completion. Examples include the strongly minimal theory of algebraically closed fields, the model completion of the theory of fields and the $\omega$-stable theory of differentially closed fields, the model completion of the theory of differential fields, but also the supersimple theory of algebraically closed fields with an automorphism as the model completion of the theory of difference fields [**9**] and the metastable theory of algebraically closed valued fields as the model completion of the theory of valued fields [**15**]. One might also include in this list the exponentially-algebraically closed fields that arise from amalgamating exponential fields; they will be the focus of a later chapter of this thesis.

In all these cases the question arises of whether the model completion is the *only* well-behaved theory within this class of structures. When restricting oneself to saturated universal domains (i. e. first-order theories), this question can generally only be answered with an extensive knowledge of the algebraic structure at hand; while the "positive" result for pure fields is classical [**30**], the "negative" result (i. e. constructing a class of counterexamples) for differential fields by Hrushovski and Itai [**18**] requires substantial non-trivial differential algebra.

In this section we will show that if one changes the setting to homogeneous model theory, more general results will be possible accross very different settings and structures and furthermore, the transparent categorical construction underpinning the proofs allow direct access to the dividing structure of the counterexamples.

## 3.2. The Construction

We will bring together the tools of the preceding chapters to prove that for any simply stable homogeneous structure $\mathbb{M}$ and any suitable collection of complete types $P$ over a common $C \subset \mathbb{M}$ there is a substructure $\mathbb{M}_{(P)}$ in a language extended merely by constants for all $c \in C$ in which none of the types in $P$ are realised. Furthermore, dividing and stability in this substructure will be taken from $\mathbb{M}$.

In the following, we will assume without loss of generality that $\mathbb{M}$ has *quantifier elimination for types*, i. e. that every type realised in $\mathbb{M}$ is completely determined by the quantifier free formulas it implies. This can of course be achieved by expanding the language and without impacting any (neo)stability properties of $\mathbb{M}$. If $\mathbb{M}$ is saturated, it coincides with full quantifier elimination.

In order to carry out our construction, we will require the second monotonicity law to hold for full orthogonality. This is indeed a serious limitation, and sufficient conditions for this will be given below.

ASSUMPTION 3.1. *(Second Monotonicity Law for orthogonality) Let $C$ be the domain of the complete type $p$. Then for all $C \subseteq B \subseteq A \subset \mathbb{M}$, $A \perp_C p \Rightarrow A \perp_B p$.*

PROPOSITION 3.2. *In the following, 1.⇒2.⇒3.*

*(1) $p$ is a regular type with trivial geometry [and the model is saturated].*

*(2) For all $\mathrm{dom}(p) \subseteq B \subseteq A \subset \mathbb{M}$, $A \perp^a p \Rightarrow A \perp p$.*

*(3) $p$ satisfies 3.1.*

PROOF. 1.⇒2.: Follows from Lemma XVI.2.4 in [**3**].

2.⇒3.: By $A \perp_C p$, $A \perp_C^a p$ and therefore by the Second Monotonicity Law for almost-orthogonality $A \perp_B^a p$. By 2., this is equivalent to $A \perp_B p$. $\square$

The new structure will be obtained as the amalgamation limit of a full subcategory of the category of all substructures of $\mathbb{M}$.

REMARK. We believe that 1.⇒2. to be equally true by the same arguments without the additional hypothesis that the model is saturated. As the proof given in [**3**] is based on a substantial amount of background results that have yet to be transferred to homogeneous model theory, however, we only claim it here as stated.

DEFINITION 3.3. A collection $P$ of complete types over a set $C \subset \mathbb{M}$ satisfying 3.1 and such that there exists an unbounded complete type over $C$ that is orthogonal to every $p \in P$ will be referred to as an *admissible family of $C$-types.*

THEOREM 3.4. *Let $\lambda$ be a regular cardinal, $\mathbb{M}$ a simply $\lambda$-stable homogeneous model of cardinality $\mu > \lambda$,$C \subset \mathbb{M}$ of cardinality less than $\lambda$ and $P$ an admissible family of $C$-types. Let $\mathfrak{C}^{\mathbb{M}}_{(P)}$ be the category of all substructures $A$ of $\mathbb{M}$ such that*

*(1) $A \supseteq C$.*

*(2) $|A| \leq \lambda$.*

*(3) $A$ is orthogonal to every $p \in P$ over $C$.*

*Then $\mathfrak{C}^{\mathbb{M}}_{(P)}$ is a $\lambda$-stability category.*

PROOF. From 2.25 we know that the category $\mathfrak{C}^{\mathbb{M}}$ of all substructures of $\mathbb{M}$ is a hull category. The first clause of that definition immediately goes down, and the second clause transfers since $\mathfrak{C}^{\mathbb{M}}_{(P)}$ is closed under taking substructures of objects already in $\mathfrak{C}^{\mathbb{M}}_{(P)}$. The third clause follows by the finite character of orthogonality. So $\mathfrak{C}^{\mathbb{M}}_{(P)}$ is a hull category. Since the empty set is an initial object and $\mathfrak{C}^{\mathbb{M}}$ satisfies Axiom 5 from the definition of a $\lambda$-stability category by 2.32, which goes down immediately, it remains only to prove the amalgamation property in the form of Axiom 4. Let $K$, $A$, $B$ be as in the statement of that clause and let $p \in P$ be arbitrary. By the free extension property, we can find $\tilde{A} \vdash \mathrm{tp}(A/K)$ free from $B$ over $K$. We need to show that $\tilde{A}B \perp_C p$ . By 3.1, $B \perp_K p$ and $A \perp_K p$ (This is the same as saying that $\tilde{A} \perp_K p$). As $\tilde{A}$ is free from $B$ over $K$, strong triviality implies that $\tilde{A}B \perp_K p$. By transitivity, this together with $K \perp_C p$ implies $\tilde{A}B \perp_C p$. Setting $D$ to be the substructure generated by $\tilde{A}B$, we see that the canonical embeddings of $A$ and $B$ into $D$ complete the diagram as required, noting that any generating set for $B$ over $K$ is also a generating set for $D$ over $A$. □

Corollary 3.5. *Let $\mathbb{M}$ be a simply $\lambda$-stable homogeneous model of cardinality $\mu > \lambda$, $\lambda$ a regular cardinal, $C \subset \mathbb{M}$ of cardinality less than $\lambda$ and $P$ an admissible family of $C$-types. Then there is an amalgamation limit $\mathbb{M}_{(P)}$ of $\mathfrak{C}^{\mathbb{M}}_{(P),<\lambda}$ in $\mathfrak{C}^{\mathbb{M}}_{(P)}$.*

Proof. Follows from the proposition above and 2.27. □

We will now go on to show that $\mathbb{M}_{(P)}$ is itself a simply stable homogeneous model and furthermore the very strong statement that dividing in $\mathbb{M}_{(P)}$ is the same as dividing in $\mathbb{M}$. We will make heavy use of the amalgamation-theoretic conditions we provided in the preceding section. Although we will have to make some assumptions on the cardinalities involved, this is unproblematic, as any stable homogeneous model is stable in arbitrarily large cardinalities (cf. [**14**]). Thus we could always move to a larger model if necessary, in complete analogy to the use of monster models in classical model theory.

PROPOSITION 3.6. *(Homogeneity of $\mathbb{M}_{(P)}$) $\mathbb{M}_{(P)}$ is homogeneous in the language $\mathcal{L}' := \mathcal{L} \cup \{c | c \in C\}$, where the new constants are interpreted canonically.*

PROOF. Note first that $\mathfrak{C}^{\mathbb{M}}_{(P)}$ is an $\mathcal{L}'$-structure category and that the hull of a subset $A$ of $\mathbb{U}$ is just the $\mathcal{L}'$-substructure of $\mathbb{U}$ generated by $A$. Then 2.30 shows that $\mathbb{M}_{(P)}$ is homogeneous. $\square$

From now on, we will investigate $\mathbb{M}_{(P)}$ in the extended language $\mathcal{L}'$.

PROPOSITION 3.7. *(Stability of $\mathbb{M}_{(P)}$) Let $|C| \leq \lambda' < \lambda$ be a cardinal and let $\mathbb{M}$ be $\lambda'$-stable. Then $\mathbb{M}_{(P)}$ is also $\lambda'$-stable.*

PROOF. By 2.32 and as $\mathfrak{C}^{\mathbb{M}}$ is just the category of substructures, $\lambda'$-stability is equivalent to $|\mathfrak{C}^{\mathbb{M}}(K)_{<\aleph_0}| \leq \lambda$ for all $K \in \mathfrak{C}^{\mathbb{M}}_{<\lambda'^+}$. As for any substructure $K \subset \mathbb{M}_{(P)}$, $\mathfrak{C}^{\mathbb{M}}_{(P)}(K) \subseteq \mathfrak{C}^{\mathbb{M}}$ and $\kappa$-smallness is conserved for all cardinals $\kappa > |C|$, $|\mathfrak{C}^{\mathbb{M}}_{(P)}(K)_{<\aleph_0}| \leq \lambda$ for all $K \in \mathfrak{C}^{\mathbb{M}}_{(P),<\lambda'^+}$ and therefore $\mathbb{M}_{(P)}$ is $\lambda'$-stable $\square$

The following lemma and theorem demonstrate the tight interdependence between $\mathbb{M}$ and $\mathbb{M}_{(P)}$ and by extension the relationship of $\mathbb{M}_{(P)}$ and $\mathbb{M}_{(Q)}$ for different admissible families $P$ and $Q$.

LEMMA 3.8. *Let $C \subseteq B \subset \mathbb{M}_{(P)}$ and $A, A' \subset \mathbb{M}_{(P)}$. Consider $\mathbb{M}_{(P)}$ as a substructure of $\mathbb{M}$. Then $\mathrm{tp}^{\mathbb{M}}(A/B) = \mathrm{tp}^{\mathbb{M}}(A'/B)$ if and only if $\mathrm{tp}^{\mathbb{M}_{(P)}}(A/B) = \mathrm{tp}^{\mathbb{M}_{(P)}}(A'/B)$.*

*In particular, $\mathbb{M}_{(P)}$ has quantifier elimination for types as an $\mathcal{L}'$-structure.*

PROOF. "$\Rightarrow$": If $\mathrm{tp}^{\mathbb{M}}(A/B) = \mathrm{tp}^{\mathbb{M}}(A'/B)$, $\mathrm{tp}^{\mathbb{M}}(A/B) = \mathrm{tp}^{\mathbb{M}}(A'/B)$ then there is a $B$-automorphism of $\mathbb{M}$ sending $A$ to $A'$. This restricts to a $B$-isomorphism between the substructures generated by $A$ and $A'$ over $B$. By $\mathfrak{C}^{\mathbb{M}}_{(P),<\lambda}$-homogeneity, this extends to a $B$-automorphism of $\mathbb{M}_{(P)}$, whence the statement.

"$\Leftarrow$": If $\mathrm{tp}^{\mathbb{M}_{(P)}}(A/B) = \mathrm{tp}^{\mathbb{M}_{(P)}}(A'/B)$, then in particular $\mathrm{qftp}(A/B) = \mathrm{qftp}(A'/B)$. By quantifier elimination for types in $\mathbb{M}$, this implies $\mathrm{tp}^{\mathbb{M}}(A/B) = \mathrm{tp}^{\mathbb{M}}(A'/B)$.

Quantifier elimination for types: $\mathrm{qftp}(A/B) = \mathrm{qftp}(A'/B)$ implies that $\mathrm{tp}^{\mathbb{M}}(A/B) = \mathrm{tp}^{\mathbb{M}}(A'/B)$, which by the first part of the lemma implies $\mathrm{tp}^{\mathbb{M}_{(P)}}(A/B) = \mathrm{tp}^{\mathbb{M}_{(P)}}(A'/B)$.

$\square$

THEOREM 3.9. *Let $C \subseteq B \subseteq A \subset \mathbb{M}_{(P)}$ and $D \subset \mathbb{M}_{(P)}$. Consider $\mathbb{M}_{(P)}$ as a substructure of $\mathbb{M}$ and assume that $\mathbb{M}$ is $\kappa$-simple for a $\kappa$ with $2^\kappa < |\mathbb{M}_{(P)}|$. Then $\mathrm{tp}^{\mathbb{M}_{(P)}}(D/A)$ divides over $B$ if and only if $\mathrm{tp}^{\mathbb{M}}(D/A)$ divides over $B$.*

PROOF. "$\Rightarrow$": We show that if $\mathrm{tp}^{\mathbb{M}}(D/A)$ does not divide over $B$, then neither does $\mathrm{tp}^{\mathbb{M}_{(P)}}(D/A)$.

Let $I$ be any sequence of $B$-indiscernibles in $\mathbb{M}_{(P)}$ Then by Lemma 3.8, $I$ is also a sequence of $B$-indiscernibles in $\mathbb{M}$. We will now work in $\mathbb{M}$, which we know to be simply stable. By homogeneity, we can assume w. l. o. g. that $A \in I$. By the strong extension property for simple homogeneous models (Lemma 3.7 in [**8**]), the Lascar strong type of $D$ over $B$ extends freely to a type over $I$. Since $\mathbb{M}$ is stable, Lascar strong types in $\mathbb{M}$ are stationary (Prop. 5.1 of [**8**]) and therefore any $\tilde{D}$ realising this free extension will have the same type as $D$ over $A$. However, as $I$ is an indiscernible sequence, any $A' \in I$ can be sent to $A$ by a strong $B$-automorphism $f$. (Lemmata 3.3ff. in [**8**]). Then by (strong) invaririance,

$\mathrm{lstp}(f(\tilde{D})/B) = \mathrm{lstp}(\tilde{D}/B) = \mathrm{lstp}(D/B)$, $\mathrm{lstp}(f(\tilde{D})/A)$ is free over $B$ and thus $\mathrm{tp}(\tilde{D}/A') = \mathrm{tp}(f(\tilde{D})/A) = \mathrm{tp}(D/A)$. Therefore, the type of $\tilde{D}$ over *any* $A' \in I$ will be the type of $D$ over $A$. As orthogonality is conserved by free extensions, $\tilde{D} \perp_{BI} p$ as $D \perp_B p$, and therefore by the $\mathfrak{C}^{\mathbb{M}}_{(P),<\lambda}$-saturation of the amalgamation limit there is a copy of $\mathrm{tp}(\tilde{D}/BI)$ in $\mathbb{M}_{(P)}$. By Lemma 3.8 and since $I$ was arbitrary, this suffices.

"$\Leftarrow$": We show that if $\mathrm{tp}^{\mathbb{M}}(D/A)$ divides over $B$, then so does $\mathrm{tp}^{\mathbb{M}_{(P)}}(D/A)$.

By Prop. 2.5 of [**8**], dividing in $\mathbb{M}$ is witnessed by a Morley Sequence $I$ over $B$, which we can choose to be of cardinality $\leq 2^\kappa$. By Rem. 2.9 of [**8**] $I$ is independent. Therefore by strong triviality of orthogonality, as the type of every $A \in I$ is orthogonal to $p$, so is the type of $I$ itself, and by $\mathfrak{C}^{\mathbb{M}}_{(P),<\lambda}$-saturation of $\mathbb{M}_{(P)}$ there is a copy of $\mathrm{tp}(I/B)$ in $\mathbb{M}_{(P)}$. As the union of types referenced in the definition of dividing is not even realised in $\mathbb{M}$, it is a fortiori (and by Lemma 3.8) not realised in $\mathbb{M}_{(P)}$. $\square$

COROLLARY 3.10. *Let $\kappa$ be a cardinal such that $2^\kappa < |\mathbb{M}_{(P)}|$. If $\mathbb{M}$ is $\kappa$-simple, then $\mathbb{M}_{(P)}$ is also $\kappa$-simple.*

PROOF. All the properties in the definition of simplicity depend only on the behaviour of dividing and therefore transfer immediately by the preceding theorem. $\square$

## 3.3. Consequences for Classifying (Simply) Stable Models

We will now return to the original problem of classifying the (saturated or homogeneous) stable models of an incomplete universal theory. In so far as we are interested only in finding stable homogeneous models, it seems as if we have reduced the problem of finding several non-elementarily-equivalent (simply) stable homogeneous models of a given cardinality to finding sufficiently many pairwise orthogonal isolated types with trivial geometry. Then we can apply the collapsing technique above to any subfamily of those to obtain the required models. If we are interested in first-order stability, we will also have to take into account axiomatisability of the $\mathbb{M}_{(P)}$. This is certainly non-trivial and can require substantial information about the universal class under investigation (cf. the heavy differential-algebraic machinery employed in [**18**]). However, even if the homogeneous $\mathbb{M}_{(P)}$ we can construct using the results here do not turn out to be saturated and the complete theory of the original model is the only complete stable theory with models within the class, we believe that being the unique stable saturated model due to non-axiomatisability of possible counterexamples would be substantially different to being the unique stable (homogeneous) model for reasons inherent in the forking structure of $\mathbb{M}$ itself.

However, we have also seen that the models we construct are very close to their originals - in fact, dividing (and therefore forking, if both models are saturated) concide. This means that they have the same underlying geometric structure. Returning to the class of fields, it is well-known that in positive characteristic algebraically closed fields are still the unique superstable fields, but that the theory of separably closed fields is strictly stable. Therefore there are non-elementarily-equivalent stable fields, and the structure of dividing in them is very different (Indeed, if dividing in algebraically closed fields would specialise to dividing in separably closed fields, the latter would also be superstable by the theory developed above). This means that the way in which there are different stable theories of fields is of a completely distinct nature to the way in which we know there to be different superstable differential fields; the former have completely different forking geometries, while the latter are merely structures derived from $\mathrm{DCF}_0$whose dividing notion simply specialises the one from there.

While we do not have any results on this, we suggest a definitional framework to make the question of whether there are non-isomorphic models of a given universal theory more meaningul.

DEFINITION 3.11. Let $M \subseteq N$ be simple homogeneous $\mathcal{L}$-structures. Then $M$ is a *stability-preserving substructure of* $N$, written $M \sim N$, if there is an expansion $\mathcal{L}'$ of $\mathcal{L}$ by $\emptyset$-definable relations (with the same definitions in both $M$ and $N$) and a set of constants (of lower cardinality than the cardinalities of $M$ and $N$) such that $M$ and $N$ have quantifier elimination of types in $\mathcal{L}'$ and for all $C \subseteq B \subseteq A \subseteq M$, the type of $A$ over $B$ divides over $C$ iff the type of $A$ over $B$ divides over $C$ in $N$.

This leads us to the following definitions

DEFINITION 3.12. Let $T$ be a universal $\mathcal{L}$-theory and $\mathfrak{P}$ be a stability/simplicity property implying "simply stable". Then

(1) *Models with* $\mathfrak{P}$ *are essentially unique in* $T$ if, for arbitrarily large cardinalities, there is a homogeneous model $\mathbb{M}$ in $T$ of that cardinality such that all homogeneous models $\mathbb{N}$ with $\mathfrak{P}$ in $T$ and $|\mathbb{M}| = |\mathbb{N}|$ are stability-preserving substructures of $\mathbb{M}$.

(2) *Models with* $\mathfrak{P}$ *are essentially in* $T$*'* for any theory $T' \supseteq T$ if for any homogeneous model $M$ with $\mathfrak{P}$ in $T$ there is a homogeneous model $N$ in $T'$ such that $M$ is a stability-preserving substructure of $N$.

(3) *Saturated models with $\mathfrak{P}$ are essentially in $T$'* for any theory $T' \supseteq T$ if for any saturated model $M$ with $\mathfrak{P}$ in $T$ there is a saturated model $N$ in $T'$ such that $M$ is a stability-preserving substructure of $N$.

As we had mentioned above, being of compatible dividing type is most informative when one structure is a subset of the other and therefore the notions above are especially interesting in the frequent cases in which $T$ has a model completion that is known to have $\mathfrak{P}$.

In particular, one could conjecture that (saturated or simply homogeneous) superstable differential fields are essentially differentially-algebraically closed as an analogue to the statement that saturated superstable fields are algebraically closed, proven in [**30, 10**].

## 3.4. Open Questions

The first step to understanding the true scope of the theorem is to understand the scope of 3.1. This could be done in both directions:

QUESTION 3.13. *Is there a general condition beyond being regular with trivial geometry that forces a type to satisfy 3.1? Maybe in conjunction with a condition on the structure?*

*Conversely, are there conditions on the types (or structures) that allow us to show that the assumption must fail for them?*

In a possibly interesting generalisation, one can also omit types that do not satisfy the standing assumption. Then, however, we must restrict ourselves to amalgamating embeddings that are in a certain way strong and we do not get the (immediate) transfer of properties that we obtain under the assumption. Therefore:

QUESTION 3.14. *If we amalgamate a more restricted class of embeddings to avoid reliance on 3.1, is there anything we can say about the resulting structure?*

Of course, the direction indicated in the preceding section is potentially the most fruitful for further study, so

QUESTION 3.15. *Are superstable differential fields essentially differentially closed? How about the many other expansions of the field language? In particular, as a truly non-first-order case, how about exponential fields?*

Lastly, this work motivates the search for many types with trivial geometries in algebraic structres, especially if one remains comitted to the original formulation of the question (with or without axiomatisability constraints).

CHAPTER 4

# Analysing Infinitary Hrushovski Constructions

## 4.1. Introduction

When Ehud Hrushovski refuted Zilber's Trichotomy Conjecture in his groundbreaking [**17**], he relied on a combinatorial amalgamation construction of finite relational structures. However, unlike the classical cases of Fraïssé's construction, which were well-known usually not to produce interesting stable amalgams, he only amalgamated structures in his class with respect to certain embeddings, subsequently termed *strong*. They were governed by a predimension function $\delta$, that is, a non-decreasing submodular function on the lattice of structures.

Subsequently, constructions of this kind appeared throughout model theory, soon expanding beyond the original case of finite relational structures to include fields with additional predicates (giving rise to so-called *coloured fields*, see e. g. [**5**]). At the end of the last milennium, Zilber announced a program of finding analytic prototypes for the "artificial" amalgamation constructions considered earlier. This led most famously to the development of pseudo-exponentiation [**42**], the focus of the following chapters, but also to a model-theoretic study of the differential exponential equation by his then-student Jonathan Kirby [**25**]. While most of the first-order axiomatisable cases are referenced in [**5**], the non-elementary constructions based on expansions of the field language have been collected and treated in a uniform manner by Bays and Kirby in [**7**].

While there continues to be much work on the general model theory of combinatorial (i. e. finite relational) amalgamation-by-predimension constructions, this chapter aims to give a unified and coherent account of this algebraic case, utilising the general categorical methods developed in the beginning of this thesis.

## 4.2. Categorical Amalgamation-by-Predimension Constructions

Definition 4.1. A *predimension category* consists of the following data:

(1) A hull category $\mathfrak{C}$ with an initial object $I$ (not necessarily based on the empty set).

(2) A *predimension* $\delta$:$\mathrm{Mor}_{<\aleph_0}(\mathfrak{C}) \to \mathbb{Z}$, that is a function such that
   - (a) $\delta(I \to A) \geq 0$ for all finitely generated $A \in \mathfrak{C}$.
   - (b) $\delta(A \xrightarrow{\sim} B) = 0$ for all isomorphisms $A \xrightarrow{\sim} B$.
   - (c) For all $C \in \mathfrak{C}$ and $A, B \to C$,

$$\delta(A \to (A \cup B)^{\mathrm{hull}}_C) \leq \delta(A \cap B \to B)$$

     whenever the two arrows in this expression are finitely generated.
   - (d) If $A \to B \to C$ are both finitely generated, then $\delta(A \to C) = \delta(A \to B) + \delta(B \to C)$.

We will note an immediate consequence of this.

Proposition 4.2. *If $(\mathfrak{C}, \delta)$ is a predimension category, then $\delta(A \xrightarrow{\iota} B) = \delta(A \xrightarrow{\tau} C)$ if $\iota$ and $\tau$ are isomorphic in $\mathfrak{C}(A)$.*

Proof. Let $(\mathfrak{C}, \delta)$ be a predimension category. Let $\sigma : B \xrightarrow{\sim} C$ be the isomorphism such that $\sigma \circ \iota = \tau$. Then $\delta(\tau) = \delta(\sigma) + \delta(\iota) = 0 + \delta(\iota) = \delta(\iota)$. □

Amalgamating *by* a predimension means amalgamating only those extensions for which the predimension function does not go down. Precisely:

Definition 4.3. Let $(\mathfrak{C}, \delta)$ be a predimension category. Then:

(1) If $A \to B \to C$ are embeddings, $A \to B$ finitely generated, then the *$C$-dimension of $B$ over $A$,* written $\dim_C(A \to B)$ or $\dim_C(B/A)$, is the minimum of the predimensions of $A \to B'$ for all finitely generated $B \to B' \to C$.

(2) An embedding $\iota : A \to B$ in $\mathfrak{C}$ is called *strong* if all finitely generated $\iota' : A \to B'$ through which $\iota$ factors (i.e. such that there exists a $\tau : B' \to B$ with $\tau \circ \iota' = \iota$) have non-negative predimension.

As we are interesting in describing in general the model-theoretic properties of the amalgamation limits, we will set out additional conditions under which the combinatorial properties of the predimension govern the behaviour of the limit.

Definition 4.4. Let $(\mathfrak{C}, \delta)$ be a predimension category. Then $(\mathfrak{C}, \delta)$ is called *regular* if it satisfies the following additional clauses:

(e): For any finitely generated extension $A \to B$ there are objects and morphisms as in the commuting diagram below such that $\delta(\iota) = \delta(\iota')$.

$$\begin{tikzcd}
A \arrow[r, "<\aleph_0", "\iota"'] & B = (AB')^{\mathrm{hull}}_B \\
A' \arrow[u] \arrow[r, "<\aleph_0", "\iota'"'] & B' \arrow[u] \\
I \arrow[u, "<\aleph_0"]
\end{tikzcd}$$

(f): The hull operator satisfies the exchange principle, i. e. if any $b \in (A \cup \{c\})^{\mathrm{hull}}_C \setminus A^{\mathrm{hull}}_C$, then $c \in (A \cup \{b\})^{\mathrm{hull}}_C$.

(g): If $A \to B$ and $A \to C$ are strong embeddings, then there is a *free amalgam of* $B$ *and* $C$ *over* $A$, written $B \oplus_A C$, that is, a completion of the commuting diagram

$$\begin{tikzcd} & B \arrow[dr, dashed] & \\ A \arrow[ur] \arrow[dr] & & B \oplus_A C \\ & C \arrow[ur, dashed] & \end{tikzcd}$$

such that $B \cup C$ generates $B \oplus_A C$,

$$\delta(B \oplus_A C/B) = \delta(C/A),$$

$$\delta(B \oplus_A C/C) = \delta(B/A),$$

and, as substructures of $B \oplus_A C$,

$$B \cap C = A.$$

In the following, all predimension categories are assumed to be regular unless stated otherwise. In major propositions, this may be repeated for emphasis.

Proposition 4.5. *Let*$(\mathfrak{C}, \delta)$ *be a regular predimension category. Then every embedding* $\iota : A \to C$ *in* $\mathfrak{C}$ *factors uniquely into* $A \xrightarrow{\iota_1} B \xrightarrow{\iota_2} C$ *such that*

*(1)* $\iota_2$ *is strong.*

*(2) For any other factorisation* $A \xrightarrow{\iota_1'} B' \xrightarrow{\iota_2'} C$ *such that* $\iota_2'$ *is strong,* $\iota_2$ *factors through* $\iota_2'$*.*

Proof. As $C \to C$ is trivially strong, it suffices to show that pairwise intersections of strong substructures and intersections of descending chains of strong substructures are strong (dual Zorn's Lemma). So consider $B_1 \xrightarrow{\iota^1} C$ and $B_2 \xrightarrow{\iota^2} C$ strong embeddings and let $B \xrightarrow{\iota} C$ be their intersection. Let $B \xrightarrow{\tau} B'$ be a finitely generated extension. We will show that $\delta(B \xrightarrow{\tau} B') \geq 0$. Observe that $B \xrightarrow{\tau} B'$ factors as $B \to B_1 \cap B' \to B'$. We will show that both factors are strong. By clause (c) of the definition of a predimension,

$$0 \leq \delta(B_1 \to (B_1 \cup B')^{\mathrm{hull}}_C) \leq \delta(B_1 \cap B' \to B').$$

Again by that clause,

$$0 \leq \delta(B_2 \to (B_2 \cup (B_1 \cap B'))^{\mathrm{hull}}_C) \leq \delta(B_1 \cap B_2 \cap B' \to B_1 \cap B').$$

Since $B = B_1 \cap B_2 \cap B'$, this is exactly what was to be proven.

It only remains to see that intersections of descending chains are strong. So let $(B_i | i \in \mathbb{N})$ be a descending chain of strong subobjects of $C$, $B := \cap B_i$. Then let $B \to B'$ be finitely generated.

Using the exchange principle, we see that $B_n \cap B'$ must eventually become stationary, in other words that eventually $B_n \cap B' = B$. But for such an $n$,

$$0 \leq \delta(B_n \to (B' B_n)^{\mathrm{hull}}_C) \leq \delta(B_n \cap B' \to B') = \delta(B \to B').$$

$\square$

Definition 4.6. Let $A \to C$ be an embedding. Then the $B$ from the proposition above is known as the *strong hull of $A$ in $C$*, written $\lceil A \rceil_C$.

Proposition 4.7. *Let $A \to B \to C$ be embeddings, $A \to B$ finitely generated. Then the induced embedding $\lceil A \rceil_C \to \lceil B \rceil_C$ is also finitely generated.*

Proof. As $B$ is finitely generated over $A$, $\dim_C((B\lceil A \rceil_C)^{\text{hull}}_C / \lceil A \rceil_C)$ is well-defined. Therefore there is a finitely generated extension realising this dimension, and this is the strong hull of $B$ in $C$. □

We will now give a decomposition result for finitely generated strong embeddings.

Definition 4.8. Let $\iota : A \to C$ be a strong embedding. Then $\iota$ is called *minimal* if $\iota$ is not an isomorphism and for any $B$ and strong embeddings $A \xrightarrow{\iota_1} B \xrightarrow{\iota_2} C$ concatenating to $\iota$ either $\iota_1$ or $\iota_2$ is an isomorphism.

Proposition 4.9. *Let $\iota : A \to C$ be a finitely generated strong embedding that is not an isomorphism. Then $\iota$ decomposes into minimal strong extensions $A \xrightarrow{\iota_1} B_1 \xrightarrow{\iota_2} \dots \xrightarrow{\iota_n} C$.*

Proof. We will prove the result by induction on the number $n$ of generators of $C$ over $A$.

$n = 1$: We claim that $n$ is already minimal. So let $C = (Aa)^{\text{hull}}$ and take $A \to B \to C$ strong embeddings. Then let $b \in B \backslash A$. Then $b \in (Aa)^{\text{hull}} \backslash A$ and thus by exchange $a \in B$. Thus, if $A \to B$ is not an isomorphism, $B \to C$ is, which was to be shown.

$n+1$: Let $C = (Aa_1 \dots a_{n+1})^{\text{hull}}$. If $A \to C$ is minimal, we are done. So assume not and let $B$ be such that $A \to B \to C$ are strong embeddings, but not isomorphisms. By repeated use of exchange, $B$ is found to be generated by at most $n$ generators over $A$, and similarly $C$ is generated by at most $n$ generators over $B$. The induction hypothesis can now be applied to both extensions. $\square$

NOTATION 4.10. The length of such a decomposition will simply be referred to as the *length* of the extension $\iota : A \to C$ and will be denoted $\mathfrak{l}(\iota)$ (or $\mathfrak{l}(C/A)$ if the embedding is clear from the context).

The final definition of this section will give sufficient conditions under which a (regular) predimension category is a $\lambda$-stability category for all sufficiently large $\lambda$:

DEFINITION 4.11. Let $(\mathfrak{C}, \delta)$ be a regular predimension category. It is called $\lambda$-*superstable* for a cardinal $\lambda$ if the following two clauses hold:

(1) For any $\kappa \geq \lambda$ and any $\kappa^+$-small object $K$ of $\mathfrak{C}$, there are at most $\kappa$ many isomorphism classes of finitely generated strong extensions.
(2) The hull of a subset $A \subseteq C$ of a $C \in \mathfrak{C}$ has cardinality at most $|A| + \lambda$.

PROPOSITION 4.12. *For any $\lambda$-superstable regular predimension category $(\mathfrak{C}, \delta)$, the category $\mathfrak{C}_{\text{strong}}$ consisting of the objects of $\mathfrak{C}$ with strong embeddings as morphisms has a $\kappa$-amalgamation limit for any cardinal $\kappa \geq \lambda$.*

PROOF. By Fact 2.27 it suffices to check the terms of Definition 2.25. Clause 1 there follows immediately from free amalgamation of strong embeddings, and clause 2 is just a rephrasing of Definition 4.11. $\square$

We will refer to this amalgamation limit as the *strong* amalgamation limit of $(\mathfrak{C}, \delta)$.

NOTE. Clearly, the above proposition requires much less than regularity; in fact, replacing free amalgamation directly with Clause 1 of Definition 2.25 suffices and is weaker.

This general result allows us to investigate algebraic Hrushovski constructions of arbitrarily large cardinality and to provide an abstract account of their model theory from the combinatorial data given by the predimension category itself. As outlined in the introduction, this requires us to leave the realm of first-order or saturated model theory and move to homogeneous model theory, which nonetheless is close enough to the classical case to allow for a meaningful theory of dividing and simplicity and to develop good analogues to the fundamental notions of traditional stability theory.

In order to deduce model-theoretic homogeneity from the category-theoretic one that holds for the amalgamation limit, we will make the following assumption on the language in which we work. This is generally harmless, as we can always extend the language by type-definable sets to satisfy the assumption.

ASSUMPTION 4.13. *Let $A$ be a subset of the strong amalgamation limit, $b \in \lceil A \rceil$. Then the $\mathcal{L}$-type of $b$ over $A$ is isolated.*

## 4.3. Dividing and U-Rank in Amalgamation-by-Predimension Constructions

In section we will show that combinatorially well-behaved limits of amalgamation-by-predimension constructions are supersimply stable and give a concrete combinatorial characterisation of dividing and U-rank in terms of their predimension function.

THEOREM 4.14. *Let $\mathbb{M}$ be the amalgamation limit of a regular amalgamation-by-predimension category. Then $\mathbb{M}$ is homogeneous when considered in a language that satisfies Assumption 4.13. It is supersimply stable and for any $A, B, C \subset \mathbb{M}$, $A \mathop{\smile\!\!\!\!\!\!|}_C B$ if and only if*

$$\left\lceil (AC)^{\text{hull}} \right\rceil \cap \left\lceil (BC)^{\text{hull}} \right\rceil = \left\lceil C^{\text{hull}} \right\rceil$$

*and*

$$\delta(\left\lceil (AC)^{\text{hull}} \right\rceil / \left\lceil C^{\text{hull}} \right\rceil) = \dim(\left\lceil (AC)^{\text{hull}} \right\rceil \left\lceil (BC)^{\text{hull}} \right\rceil / \left\lceil (BC)^{\text{hull}} \right\rceil).$$

PROOF. Homogeneity follows immediately from Proposition 2.30.

To prove the remainder of the theorem, we will apply Fact 1.24. For simplicity and without loss of generality, we will assume that $A$, $B$ and $C$ are structures and that $C \subseteq A, B$ wherever only one base $C$ is referenced.

(1) Local Character:

This follows from the local character of the predimension.

(2) Monotonicity in $B$ and $C$:

In $B$:

If $C \subseteq B_1 \subseteq B$, then $\lceil C \rceil \subseteq \lceil A \rceil \cap \lceil B_1 \rceil \subseteq \lceil A \rceil \cap \lceil B \rceil = \lceil C \rceil$.

Furthermore, by condition (c) of the definition of a predimension category,

$$\delta(\lceil A\rceil / \lceil C\rceil) = \delta(\lceil A\rceil \lceil B\rceil / \lceil B\rceil) \leq \delta(\lceil A\rceil \lceil B_1\rceil X / \lceil B_1\rceil) \leq \delta(\lceil A\rceil / \lceil C\rceil).$$

Furthermore, for any $X \subseteq \mathbb{M}$,

$$\delta(\lceil A\rceil \lceil B_1\rceil X / \lceil B_1\rceil) = \delta(\lceil A\rceil \lceil B_1\rceil X / \lceil B_1\rceil X \cap B) + \delta(\lceil B_1\rceil X \cap B / \lceil B_1\rceil)$$

and

$$\delta(\lceil A\rceil \lceil B_1\rceil X / \lceil B_1\rceil X \cap B) \geq \delta(\lceil A\rceil \lceil B\rceil X / \lceil B\rceil) \geq \delta(\lceil A\rceil \lceil B\rceil / \lceil B\rceil) = \delta(\lceil A\rceil / \lceil C\rceil)$$

while $\delta(\lceil B_1\rceil X \cap B / \lceil B_1\rceil) \geq 0$ by definition of the strong hull.

In $C$:

Let $C \subseteq C_1 \subseteq B$. By the previous, $A \mathop{\smile\!\!\!\!\!\!\mid}_C C_1$, and thus $\lceil AC_1\rceil = \lceil A\rceil \lceil C_1\rceil$. It therefore suffices to show that $\lceil A\rceil \lceil C_1\rceil \cap \lceil B\rceil = \lceil C_1\rceil$. So for the sake of contradiction, let $x \in \lceil A\rceil \lceil C_1\rceil \cap \lceil B\rceil \setminus \lceil C_1\rceil$. By exchange, there is an $a \in \lceil A\rceil \setminus \lceil C_1\rceil$ that lies in $\lceil B\rceil$. This contradicts $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$.

Furthermore, we know that

$$\delta(\lceil A\rceil \lceil C_1\rceil / \lceil C_1\rceil) \leq \delta(\lceil A\rceil / \lceil C\rceil) = \dim(\lceil A\rceil \lceil B\rceil / \lceil B\rceil),$$

and that $\delta(\lceil A\rceil \lceil C_1\rceil / \lceil C_1\rceil) = \delta(\lceil AC_1\rceil / \lceil C_1\rceil)$. This suffices.

(3) Extension

Let $A \mathop{\smile\!\!\!\!\!\!\mid}\limits_{C} B$ and let $B \subseteq D$. By free amalgamation of strong extensions and the homogeneity properties of $\mathbb{M}$ as the amalgamation limit, there is $\lceil AB\rceil \oplus_{\lceil B\rceil} \lceil B_1\rceil$ and strong extensions such that the following diagram commutes:

$$\begin{tikzcd}
 & \lceil AB \rceil \arrow[rr] \arrow[dr, dashed] & & \mathbb{M} \arrow[dd, dotted, "\sigma"] \\
\lceil B \rceil \arrow[ur] \arrow[dr] & & \lceil AB \rceil \oplus_{\lceil B \rceil} \lceil B_1 \rceil \arrow[dr, dotted, "\iota"] & \\
 & \lceil B_1 \rceil \arrow[rr] \arrow[ur, dashed] & & \mathbb{M}
\end{tikzcd}$$

The automorphism formulation of the definition of a type implies that the copy $A'$ of $A$ under the automorphism $\sigma$ satisfies $A' \models \mathrm{tp}(A/B)$, and the other conditions are a reformulation of the definition of a free amalgam,observing the strongness of the embedding $\iota$.

(4) Transitivity

Assume without loss of generality that $D \subseteq A$. Then

$$\lceil A \rceil \cap \lceil B \rceil \subseteq \lceil AC \rceil \cap \lceil B \rceil = \lceil C \rceil .$$

Thus

$$\lceil A \rceil \cap \lceil B \rceil \subseteq \lceil A \rceil \cap \lceil C \rceil = \lceil D \rceil .$$

Also,

$$\dim(\lceil A \rceil \lceil B \rceil / \lceil B \rceil) = \delta(\lceil AC \rceil / \lceil C \rceil) = \dim(\lceil A \rceil \lceil C \rceil / \lceil C \rceil) = \delta(\lceil A \rceil / \lceil D \rceil).$$

(5) Symmetry

Trivially,$\lceil A \rceil \cap \lceil B \rceil = \lceil C \rceil$ implies$\lceil B \rceil \cap \lceil A \rceil = \lceil C \rceil$.

So let $X \subseteq \mathbb{M}$ be arbitrary. Then by $A \mathop{\smile\!\!\!\!\!\!|}_C B$, $\delta(\lceil A\rceil \lceil B\rceil X / \lceil B\rceil) \geq \delta(\lceil A\rceil / \lceil C\rceil)$. Thus $\delta(\lceil A\rceil \lceil B\rceil X) - \delta(\lceil B\rceil) \geq \delta(\lceil A\rceil) - \delta(\lceil C\rceil)$, which rearranges to $\delta(\lceil A\rceil \lceil B\rceil X) - \delta(\lceil A\rceil) \geq \delta(\lceil B\rceil) - \delta(\lceil C\rceil)$, wherefore $\delta(\lceil A\rceil \lceil B\rceil X / \lceil A\rceil) \geq \delta(\lceil B\rceil / \lceil C\rceil)$.

(6) Finite Character

Follows from the finite character built into both the predimension function and the definition of strongness.

(7) Invariance

Follows from the invariance of the predimension.

(8) Superstability

Let $\lambda$ be as in Def. 4.11. We will show that the strong closure of $A$ has at most cardinality $\lambda + |A|$.

By the local character of the predimension, it suffices to show this for finitely generated $A$. Let $A \to B$ be finitely generated such that $\delta(B) = \dim(A)$. Then $B$ is finitely generated and thus by $\lambda$-superstability of cardinality at most $\lambda$. However, $B$ contains the strong closure of $A$, which proves the statement. Therefore and again by $\lambda$-superstability, there are at most $|A| + \lambda$ finite types over the strong closure of $A$, and a fortiori over $A$ itself.

$\square$

This result allows us to characterise the SU-rank of regular amalgamation-by-predimension constructions. We will first prove a simple lemma on boundedness:

LEMMA 4.15. *Let $\mathbb{M}$ be the amalgamation limit of a regular amalgamagtion-by-predimension construction and $A \subset \mathbb{M}$. Then the strong hull of $A$ consists exactly of those elements whose type over $A$ is bounded.*

PROOF. $a \in \lceil A \rceil \Leftrightarrow a \perp_A a$ by the characterisation of freeness given in Theorem 4.14. This exactly coincides with boundedness by Lemmas 2.6 and 2.9 of [8]. $\square$

THEOREM 4.16. *Let $A \subseteq B \subset \mathbb{M}$ with* $\dim(B/A) < 1$. *Then* $\mathrm{SU}(B/A) = \mathfrak{l}(\lceil B \rceil / \lceil A \rceil)$ *wherever* $\mathfrak{l}(\lceil B \rceil / \lceil A \rceil)$ *is finite.*

PROOF. We will show the theorem by induction on $\mathfrak{l}(\lceil Aa \rceil / \lceil A \rceil)$:

$\mathfrak{l}(\lceil B \rceil / \lceil A \rceil) = 0$: This implies that $\lceil B \rceil = \lceil a \rceil$ and therefore $a \in \lceil A \rceil$. But by Lemma 4.15, strong closure and bounded closure coincide and thus $\mathrm{SU}(a/A) = 0$ by Prop. 1.29.

$\mathfrak{l}(\lceil B \rceil / \lceil A \rceil) = n + 1$: This implies that there is a $A' \subset \mathbb{M}$ such that $\lceil A \rceil \to A'$ is a strong extension of length $n$ and $A' \to \lceil B \rceil$ is a minimal strong extension. By the induction hypothesis and the pairs formula for SU-rank, it suffices to show that $\mathrm{SU}(\lceil B \rceil / A') = 1$.

$\mathrm{SU}(\lceil B \rceil / A') \geq 1$: By Lemma 4.15, strong extensions are unbounded and therefore have at least SU-rank 1.

$\mathrm{SU}(\lceil B \rceil / A') \leq 1$: We need to show that every forking extension $\mathrm{tp}(\lceil B \rceil / C)$ of $\mathrm{tp}(\lceil B \rceil / A')$ has SU-rank 0, which is to say that it is bounded or equivalently that $B$ is contained in the strong closure of $C$. So let $C$ be such that $\lceil B \rceil$ is not independent of $C$ over $A'$. Then either $\lceil B \rceil \cap \lceil C \rceil \supsetneq A'$ or $\dim(\lceil B \rceil \lceil C \rceil / \lceil C \rceil) < \dim(B/A') = 0$. But as $\lceil C \rceil$ is strong closed, $\dim(\lceil B \rceil \lceil C \rceil / \lceil C \rceil) \geq 0$, excluding the second case.

Therefore, $\lceil B \rceil \cap \lceil C \rceil \supsetneq A'$, and by minimality of the extension, this implies that $\lceil B \rceil = \lceil \lceil B \rceil \cap C \rceil$ and thus $B \subseteq \lceil C \rceil$. □

In categories with additional properties, we can complete the picture:

DEFINITION 4.17. Call a predimension category $C$ *strictly regular* if it is regular and also satisfies the following three clauses:

(1) If $A \to B$ is generated by a singleton, then $\delta(A \to B) \leq 1$.
(2) If $\lceil A \rceil \to B$ is an extension, $\mathrm{tp}(a/\lceil A \rceil) = \mathrm{tp}(b/\lceil A \rceil)$, $a \downfree_{\lceil A \rceil} B$ and $b \downfree_{\lceil A \rceil} B$, then $\mathrm{tp}(a/B) = \mathrm{tp}(b/B)$.
(3) For all $A \subset \mathbb{M}$ there are singletons $a \in \mathbb{M}$ with dimension 0 over $A$ whose strong hulls have arbitrary length over $\lceil A \rceil$, but who have dimension 1 over $\emptyset$.

LEMMA 4.18. *Let $\mathbb{M}$ be the amalgamation limit of a strictly regular predimension category. Let $a \in \mathbb{M}$ and $A \subset \mathbb{M}$ with $\dim(a/A) = 1$. Then $\mathrm{tp}(a/A)$ is the unique non-dividing extension of $\mathrm{tp}(a/\emptyset)$.*

PROOF. We need to prove $a \downfree_{\emptyset} A$ and we will use the characterisation provided in Thm. 4.14.

$\lceil a \rceil \cap \lceil A \rceil = \emptyset$: Assume not. Then there is a $b \in \lceil a \rceil$ with $\delta(b/A) < 0$. But then

$$\delta(ab/A) = \delta(b/A) + \delta(a/bA) \leq \delta(b/A) + 1 \leq 0$$

in contradiction to $\dim(a/A) = 1$.

$\dim(a/A) = \delta(\lceil a \rceil / \emptyset)$ is clear, as the latter cannot exceed 1 by strict regularity.

Uniqueness follows from the fact that free extensions are unique in strictly regular predimension categories. □

THEOREM 4.19. *Let* $\mathbb{M}$ *be the amalgamation limit of a strictly regular predimension category and let* $a \in \mathbb{M}$ *and* $A \subset \mathbb{M}$ *with* $\dim(a/A) = 1$. *Then* $\mathrm{SU}(a/A) = \aleph_0$.

PROOF. By Lemma 4.18 and Fact 1.28 it suffices to consider types over $\emptyset$. We see that $\mathrm{SU}(a/A) \leq \aleph_0$ as every dividing extension of the type has dimension smaller than 1 and therefore by Thm. 4.16 has finite SU-rank. By the additional condition in the theorem, there is a dividing extension of $\mathrm{tp}(a/A)$ of every finite SU-rank, and therefore $\mathrm{SU}(a/A) = \aleph_0$. $\square$

## 4.4. Dividing and Morley rank over Bounded Subsets

The $\Gamma$-fields investigated in [**7**], the main examples of amalgamation-by-predimension constructions considered so far, satisfy an even stronger amalgamation property than the one in clause g) of the definition above, called *asymmetric free amalgamation:*

CONDITION 4.20. A regular amalgamation-by-predimension construction *has asymmetric free amalgamation* if for any $A$,$B$,$C$ as below we can complete the diagram as indicated:

$$\begin{tikzcd}
& B \arrow[dr, dashed, "\text{not necessarily strong}"] & \\
A \arrow[ur, "\text{strong}"] \arrow[dr, "\text{not necessarily strong}"'] & & B \oplus_A C \\
& C \arrow[ur, dashed, "\text{strong}"'] &
\end{tikzcd}$$

This has proven to be a very important condition in axiomatising the amalgamation limit; here, we show that it is a sufficient condition under which the Morley rank over bounded subsets introduced earlier characterises dividing over strong extensions.

THEOREM 4.21. *Let* $\mathbb{M}$ *be the amalgamation limit of a strictly regular superstable amalgamation-by-predimension construction with asymmetric free amalgamation, let* $\vec{a} \in \mathbb{M}$ *and* $C \subseteq B \subset \mathbb{M}$*, where* $C \to B$ *is a strong extension in* $\mathbb{M}$*, i. e.* $\lceil C \rceil \cap B = C$*. Then* $\vec{a} \mathop{\smile\!\!\!\!\!\!|}_C B$ *if and only if* $\mathrm{RM_b}(\mathrm{tp}(\vec{a}/B)) = \mathrm{RM_b}(\mathrm{tp}(\vec{a}/C))$*.*

PROOF. "$\Leftarrow$": We will proceed by induction on the SU-rank of $C$. Assume that the fact is true for all $C' \to B'$ where $\mathrm{SU}(C') < \mathrm{SU}(C)$, and for the sake of contradiction assume that $\vec{a}$ is not free from $B$ over $C$. As the transcendental type is unique, there is a $B$-formula $\phi(\vec{x}, \vec{b})$ true for $\vec{a}$ which together with $\mathrm{tp}(\vec{a}/C)$ implies that $\vec{x}$ is not free from $B$ over $C$. By monotonicity of Morley rank we have $\mathrm{RM_b}(\mathrm{tp}(\vec{a}/B)) \leq \mathrm{RM_b}(\mathrm{tp}(\vec{a}/C) \cup \{\phi\})$. By free amalgamation, there are infinitely many mutually independent copies $B_i$ of the type of $B$ over $C$. By the induction hypothesis,

$$\mathrm{RM_b}(\mathrm{tp}(\vec{a}/C) \cup \{\phi(\vec{x}, b_i)\}) > \mathrm{RM_b}(\mathrm{tp}(\vec{a}/C) \cup \{\phi(\vec{x}, b_i)\} \cup \{\phi(\vec{x}, b_i)\}).$$

Thus the sequence

$$\{\phi(\vec{x}, \vec{b_1}), \phi(\vec{x}, \vec{b}_2) \cap \neg\phi(\vec{x}, \vec{b}_1), \phi(\vec{x}, \vec{b}_3) \cap \neg\phi(\vec{x}, \vec{b}_2) \cap \neg\phi(\vec{x}, \vec{b_1}), \ldots\}$$

defines an infinite sequence of disjoint definable subsets of $\mathrm{tp}(\vec{a}/C)$ of Morley rank $\mathrm{RM_b}(\mathrm{tp}(\vec{a}/C) \cup \{\phi\})$. As the additional clause in the inductive step of the definition of (b-)Morley rank is easily seen to be satisfied, this contradicts the fact that $\mathrm{RM_b}(\mathrm{tp}(\vec{a}/C)) = \mathrm{RM_b}(\mathrm{tp}(\vec{a}/B))$.

"$\Rightarrow$": When $\mathrm{tp}(\vec{a}/C)$ is bounded, its b-Morley-rank is 0 and since $\vec{a}$ is a witness for $\mathrm{tp}(\vec{a}/B)$, the second condition is automatically fulfilled. So assume that $\mathrm{tp}(\vec{a}/C)$ is unbounded. Let $\mathrm{RM_b}(\mathrm{tp}(\vec{a}/C)) \geq \alpha + 1$. We will show that $\mathrm{RM_b}(\mathrm{tp}(\vec{a}/B)) \geq \alpha + 1$. Let $D_i$ be parameter sets such that $\{\mathrm{tp}(\vec{a}/CD_i) | i \in I\}$ is an infinite disjoint union of complete types of Morley rank $\alpha$, obtained from extending the $\phi_i$ in the inductive definition of Morley rank to a complete type over its set of parameters (possible by the additional clause). In the following diagram, the top left arrow is strong because $B \to C$ is strong and $\vec{a} \mathop{\downarrow}_C B$.

$$\begin{tikzcd}
 & B\vec{a} \arrow[dr, dashed, "\text{not necessarily strong}"] & \\
C\vec{a} \arrow[ur, "\text{strong}"] \arrow[dr, "\text{not necessarily strong}"'] & & B\vec{a} \oplus_{C\vec{a}} CD_i\vec{a} \\
 & CD_i\vec{a} \arrow[ur, dashed, "\text{strong}"'] &
\end{tikzcd}$$

□

This implies that b-Morley-rank is bounded below by SU-rank, just as in the first-order case.

COROLLARY 4.22. *Let $\mathbb{M}$ be the amalgamation limit of a strictly regular superstable amalgamation-by-predimension construction with free amalgamation, let $\vec{a} \in \mathbb{M}$ and $C \subset \mathbb{M}$. Then*

$$\mathrm{RM_b}(\mathrm{tp}(\vec{a}/C)) \geq \mathrm{SU}(\mathrm{tp}(\vec{a}/C))$$

Proof. By induction on $\alpha \coloneqq \mathrm{SU}(\mathrm{tp}(\vec{a}/C))$:

The statement is clear for $\alpha = 0$ and in the limit case.

So assume it to be true for $\alpha$ and let $\mathrm{SU}(\mathrm{tp}(\vec{a}/C)) = \alpha+1$. This means there is a $C \subseteq B \subset \mathbb{M}$ such that the $\mathrm{tp}(\vec{a}/B)$ forks over $C$ and is of SU-rank $\alpha$. By Remark 2.1 of [**8**] and transitivity of forking, we can assume that $C \to B$ is a strong extension in $\mathbb{M}$. Then by the induction hypothesis, $\mathrm{RM_b}(\mathrm{tp}(\vec{a}/B)) \geq \mathrm{SU}(\mathrm{tp}(\vec{a}/B))$, and by the preceding theorem $\mathrm{RM_b}(\mathrm{tp}(\vec{a}/C)) > \mathrm{RM_b}(\mathrm{tp}(\vec{a}/B)) = \alpha$. □

## 4.5. Hrushovski's ab initio Construction

Before we close this chapter with an outlook on possible new applications, we will briefly compare our results to what is known for the prototype of all amalgamation-by-predimension constructions, the uncollapsed combinatorial structure that Hrushovski uses in [**17**] as the first step towards the new strongly minimal set. The exposition most useful to us is given by Martin Ziegler in Section 4 of [**41**]. There, he introduces the notion of a minimal extension just as we did in Definition 4.8 and uses an analysis similar to ours to relate the length of an extension to the rank of a type. Interestingly, he uses Morley Rank rather than U-Rank throughout. This has several consequences. On the one hand, he does not obtain the one-to-one correspondence of rank and length of our Theorem 4.14, since Morley Rank is generally larger than U-Rank. On the other hand, he is able to deduce in his Corollary 4.9 that the Morley Rank of any extension of predimension 0 is finite. We have not been able to reproduce this for our homogeneous setting, having to content ourselves with characterising dividing through a drop in b-Morley Rank (This latter fact is of course trivially true in the classical setting). The reason for our inability to move from the properties of U-Rank to those of Morley Rank lie in the comparatively low level of development of the theory of (b-)Morley Rank in the homogeneous context. In particular, we are unable to reproduce the Shelah-Erimbetov inequality, which is central to any induction argument that general (i. e. non-minimal) extensions of predimension 0 have finite Morley Rank.

## 4.6. Outlook on Applications

The first application we have in mind is to the algebraic Hrushovski constructions that heretofore have only been tackled with the tools of quasiminimal excellence; that means, only their finitely generated structures have been amalgamated and then sophisticated theory has been applied to lift the amalgamation limit to uncountable cardinalities. In the process, however, it becomes much more difficult to apply classical model-theoretic concepts such as dividing and rank, and the machinery of first-order stability in general becomes, for the most part, unavailable.

Constructing uncountable amalgamation limits directly in the way outlined here allows one to retain most of those techniques, as homogeneous model theory is very close to classical first-order stability. This has been worked out for the case of pseudo-exponentiation in the final chapter of this thesis, but should be possible for the whole array of structures discussed in [**7**].

In a completely different direction, Vahagn Aslanyan has been investigating when reducts of differentially closed fields (DCFs) are obtainable by an amalgamation-by-predimension construction. It has been very hard to obtain any characterisation using classical techniques, and even the main theorem of Aslanyan's thesis in this area, stating that a reduct in which the derivation is interpretable is not obtainable by a predimension construction, still requires the inductivity of its first-theory. We believe that the results given here, that characterise SU-rank and dividing in amalgamation-by-predimension construction independently of any axiomatisation constraints, might enable a direct comparison with U-rank (modulo the fixed field) in DCFs. This could possibly lead to a proof that a structure that "looks like a DCF" (i. e. interprets the derivation) can never be obtained as the limit of a predimension construction, regardless whether its theory is inductive (or indeed whether the prospective limit is even saturated).

CHAPTER 5

# Exponential Fields: General Notions and Quasiminimality

## 5.1. Introduction

We will now apply the results of the previous chapters to analyse the model theory of exponential fields. This is a rather modern branch of algebraic model theory that uses inherently non-first-order methods. The reason for this can readily be seen in $(\mathbb{C}, 0, 1, +, *, \exp)$, which is in many ways the prototype for studying exponential fields:

PROPOSITION 5.1. *The structure* $(\mathbb{C}, 0, 1, +, *, \exp)$ *interprets the full first-order theory of arithmetic.*

PROOF. $\mathbb{Z}$ is definable via $\forall x(\exp(x) = 1 \to \exp(ax) = 1)$. $\square$

It therefore took until 2002 for the visionary work of Prof Boris Zilber to obtain the first positive results in the model theory of exponential fields. In his seminal paper [**42**], he showed that there is a *quasiminimal pregeometry class*, uncountably categorical by default, of *strongly exponentially algebraically closed* exponential fields and conjectured that $(\mathbb{C}, 0, 1, +, *, \exp)$ is isomorphic the representative of this class in cardinality $2^{\aleph_0}$. This mirrors the situation in the model theory of fields: Algebraically closed fields are strongly minimal and therefore uncountably categorical. In this chapter we will investigate the scope and the limitations of this analogy.

Definition 5.2. A *partial exponential field* is a field $F$ of characteristic zero along with a homomorphism $\exp_F : (D(F), +) \to F^*$, where $D(F)$ is a $\mathbb{Q}$-linear subspace of $F$.

It is called an *(total) exponential field* if $D(F) = F^+$.

It is referred to as an *ELA-field* if it is total, algebraically closed as a pure field and $\exp_F$ is surjective.

Notation 5.3. From now on, we will refer to the graph of $\exp_F$ as $\mathrm{E}(F)$.

Applying classical first-order stability theory to exponential fields is hopeless, as the following generalisation of Prop. 5.1 shows:

Proposition 5.4. *Every ELA-field* $(F, 0, 1, +, *, \exp)$ *is first-order unstable.*

Proof. We claim that $R := \forall x(\exp(x) = 1 \to \exp(ax) = 1)$ defines an unstable infinite subring.

Firstly, $R$ is infinite because it includes $\mathbb{Z}$.

To see that $R$ is a subring, we observe that $\exp((a + b)x) = \exp(ax + bx) = \exp(ax) * \exp(bx) = 1 * 1 = 1$ and that $\exp(bx) = 1 \to \exp(abx) = 1$.

It is well known that every stable commutative ring is already a field, and therefore it suffices to demonstrate that $R$ cannot be a subfield of $F$. But every such subfield would have to extend $\mathbb{Q}$, contradicting the fact that $(\mathbb{Q}*\ker(\exp))/\ker(\exp) \cong \mu \neq 1$, where $\mu$ is the group of roots of unity in $F^*$. (This last fact is proven in Prop. 2.12 of [**26**]). $\square$

## 5.2. General exponential fields

The most lucid construction of the exponentially algebraically closed exponential fields can be found in [**7**], which we will follow. To formulate it, we will need some terminology; while everything can be found in the preprint, the exponential-field-versions of the notions can also be found in [**26**].

Definition 5.5. We define the following notions:

(1) A field extension $A \hookrightarrow B$ between partial exponential fields is called an *extension of partial exponential fields* if $\exp_A \subseteq \exp_B$.
(2) Let $B$ be a partial exponential field, $A \subseteq D(B)$ and $\vec{b} \in D(B)$. Then $\delta(\vec{b}/A) := \operatorname{trd}(\vec{b}, \exp(\vec{b})/A, \exp(A)) - \operatorname{ldim}_{\mathbb{Q}}(\vec{b}/A)$.
(3) Let $A \subseteq B$ be an extension of partial exponential fields. Then $B$ is called a *strong extension* of $A$, written $A \lhd B$, if
    (a) $\{x \in B^+ | \exp(x) = 1\} = \{x \in A^+ | \exp(x) = 1\}$.
    (b) For all $\vec{b} \in D(B)$, $\delta(\vec{b}/A) \geq 0$.
(4) Let $A \lhd B$ be a strong extension of partial exponential fields, $C$ an intermediate partial exponential field. Then the *strong hull of $C$ in $B$*, written $\lceil C \rceil$, is the set of all $x \in B$ such that $x \in C$ or there is a finite $X \subseteq B$ with $x \in X$ such that $\delta(X/C) < 0$.
(5) Let $A \lhd B$ be a strong extension of partial exponential fields, $C$ an intermediate partial exponential field. Then the *exponential closure of $C$ in $B$*, written $\operatorname{Ecl}^B(C)$, is the set of all $x \in B$ such that there is a finite $X \subseteq B$ with $x \in X$ and $\delta(X/\lceil C \rceil) = 0$.

(6) Let $(K_{\text{base}}, \exp)$ be the partial exponential field with underlying pure field $\mathbb{Q}^{\text{ab}}(\tau)$ for a transcendental $\tau$. For each $m \in \mathbb{N}_{>0}$, let $\omega_m$ be a primitive $m^{\text{th}}$ root of unity such that $(\omega_{mn})^n = \omega_m$ for all positive integers $m$ and $n$. Then let exp be a homomorphism from the $\mathbb{Q}$-linear span of $\tau$ to the roots of unity such that $\exp(\tau/m) = \omega_m$. □

The $\delta$ defined above is a well-behaved predimension function, which has been intensely studied and partly motivated the notions introduced for regular and strictly regular predimension categories earlier.

We collect some elementary properties here.

FACT 5.6. *Let $(F, \exp)$ be an exponential field, $G \subseteq F$ an exponential subfield. Then*

*(1) $\delta(G/G) = 0$.*

*(2) Let $G \subseteq F$ be an exponential subfield. If $b \in F$ is a finite tuple, then there is a finitely generated subfield $G_0$ of $G$ such that for any intermediate exponential field $G_0 \subseteq G' \subseteq G$ we have $\delta(b/G) = \delta(b/G')$.*

*(3) Let $X, Y$ be exponential subfields of $F$ with $X$ finitely generated over $X \cap Y$. Then*

$$\delta(XY/Y) \leq \delta(X/X \cap Y).$$

*(4) Let $X, Y$ be exponential subfields of $F$ with $X \subseteq Y$. Then*

$$\delta(Y/A) = \delta(Y/X) + \delta(X/A).$$

PROOF. Number 1 is clear from the definition, numbers 2-4 are Lemma 4.2 of **[7]**. □

The notions of being finitely generated as a partial or as an ELA-field do not coincide; we thus give both here.

DEFINITION 5.7. Let $F$ be a partial E-Field and $X \subseteq \mathrm{E}(F)$. Then the *partial E-field generated by $X$ in $F$* is the intersection of all partial E-subfields of $F$ with the same kernel as $F$ the graph of whose exponential function contains $X$. If $F$ is an ELA-field, then the *ELA-field generated by $X$ in $F$* is the intersection of all ELA-subfields of $F$ containing $X$, again with the same kernel as $F$. If $F$ is generated as a partial E-field (resp. as an ELA-field) by a finite $X \subseteq \mathrm{E}(F)$ in itself, then $F$ is called *finitely generated* as a partial E-field (resp. as an ELA-field). As usual, one can define being generated *over an $A \subseteq F$* and can thus talk about *finitely generated partial E-field* and *ELA-field extensions.* Furthermore, a partial E-field is called *essentially finitary* if it is finitely generated as a partial E-field or if it is a finitely generated partial E-field extension of a countable ELA-field.

The classification of strong extensions of exponential fields by the exponential-algebraic type of a generating set is crucial to the study of exponential fields and thus we will need the terminology to formulate it here.

DEFINITION 5.8. Let $A$ be a partial E-field and $B$ a finitely generated partial E-field extension of $A$. Then the linear dimension of $\mathrm{E}(B)$ over $\mathrm{E}(A)$ is finite. Thus we can find a *basis* for the extension, by which we mean a tuple $\vec{b} = (b_1, \ldots, b_n) \in \mathrm{E}(B)^n$ of minimal length $n$ such that$\vec{b} \cup \mathrm{E}(A)$ generates $\mathrm{E}(B)$. We denote by $\mathrm{loc}(\vec{b}/A)$ of $\vec{b}$ the *locus of $\vec{b}$ over $A$*, that is, the smallest Zariski-closed subset of $F \times F^{\times}$ defined over $A$ and containing $\vec{b}$. A basis $\vec{b} \in \mathrm{E}(B)^n$ for a finitely generated extension $A \subseteq B$ of partial E-fields is *good* if the isomorphism type of the extension is determined up to isomorphism by the $\mathrm{loc}(\vec{b}/A)$. That is, whenever $B'$ is another extension of $A$ which is generated by a basis $\vec{b'}$ such that $\mathrm{loc}(\vec{b'}/A) = \mathrm{loc}(\vec{b}/A)$ then there is an isomorphism of partial E-fields $B \xrightarrow{\sim} B'$ fixing $A$ pointwise which takes $\vec{b}$ to $\vec{b'}$.

A main result of the exponential-algebraic analysis of the essentially finitary case is the following

FACT 5.9. *Let $A$ be an essentially finitary partial E-field, and let $B$ be a finitely generated partial E-field extension of $A$ with the same kernel as $A$. Then there is a good basis of that extension.*

We will now give another structure result on strong extensions that will be needed for constructions below.

FACT 5.10. *Let $F$ be an ELA-field. If $A \lhd F$ and $a \in F \setminus \mathrm{Ecl}(A)$, then there is an $a' \in D(F)$ such that $a'$ is interalgebraic with $a$ over $A$ and $\delta(a'/A) = 1$, and the partial E-subfield of $F$ generated by $A$ and $(a', \exp(a'))$ is strongly embedded in $F$. We can choose $a'$ such that $a$ is rational over $A(a')$, and, if $A$ is essentially finitary, also such that $(a', \exp(a'))$ is a good basis. Furthermore, the locus $\mathrm{loc}((a', \exp(a')), a/A)$ can be taken to be a particular algebraic curve defined over $\mathbb{Q}$, not depending on $A$ or $a$.*

The predimension gives rise to a pregeometry and thus to a dimension function.

FACT 5.11. *Let $F$ be an ELA-field. Then the* Ecl*-closed subsets of $F$ are the closed sets of a pregeometry on $F$. Furthermore, suppose that $A \lhd F$ and $B$ is a finitely generated partial E-field extension of $A$ in $F$. Then:*

*(1)* $\mathrm{Ecldim}(B/A) = \min\{\delta(C/A) | B \subseteq C \subseteq F\}$.

*(2)* $B \lhd F$ *if and only if* $\mathrm{Ecldim}(B/A) = \delta(B/A)$.

In the work of Kirby and Zilber, the model theory of exponential fields has been studied from the point of view of quasiminimality. In this framework, one uses the pregeometry coming from the predimension to construct uncountable models in which the pregeometric closure of a countable set is suppressed to countability itself; i.e. the preogemetry from the predimension is forced to coincide with the pregeometry of countable closure. The advantage of this framework, as we will see below, is that one can "artificially" obtain uncountable categoricity of the structure built in this way. However, finer-grained model theoretic study (such as dividing and U-rank) becomes more difficult as all non-transcendental types are bounded. Therefore, in the next chapter, we propose homogeneous model theory as an alternative framework for studying exponential fields.

## 5.3. Quasiminimal Exponential Fields

Definition 5.12. An $\mathcal{L}$-structure $M$ is called *quasiminimal* if every definable subset of $M$ is either countable or cocountable.

Just like minimality, quasiminimality gives rise to a pregeometry: A formula $\phi(\vec{x})$, which could have parameters, is considered small iff the set of its realisations in $M$ is countable. If this notion is well-behaved, it can be used to derive categoricity, just as in the strongly minimal case.

Definition 5.13. Let $L$ be a countable language, $M$ an $\mathcal{L}$-structure and let cl be a pregeometry on $M$. Then $(M, \mathrm{cl})$ is a *quasiminimal pregeometry structure* if

(1) If $\mathrm{qftp}(a, \vec{b}) = \mathrm{qftp}(a', \vec{b'})$, then $a \in \mathrm{cl}(\vec{b}) \leftrightarrow a' \in \mathrm{cl}(\vec{b'})$.

(2) $M$ is infinite-dimensional with respect to cl.

(3) If $A \subseteq M$ is finite then $\mathrm{cl}(A)$ is countable.

(4) Assume that $C, C' \subseteq M$ are countable closed subsets, enumerated such that $\mathrm{qftp}(C) = \mathrm{qftp}(C')$. If $a \in M \backslash C$ and $a' \in M \backslash C'$, then $\mathrm{qftp}(a, C) = \mathrm{qftp}(a', C')$.

(5) Let $C, C' \subseteq M$ be countable closed subsets or empty, enumerated such that $\mathrm{qftp}(C) = \mathrm{qftp}(C')$, let $\vec{b}, \vec{b'} \in M$ such that $\mathrm{qftp}(\vec{b}, C) = \mathrm{qftp}(\vec{b'}, C')$ and let $a \in \mathrm{cl}(C, \vec{b})$. Then there is an $a' \in M$ such that $\mathrm{qftp}(a, \vec{b}, C) = \mathrm{qftp}(a', \vec{b'}, C')$.

The theory of these quasiminimal pregeometry structures has been developed in [**6**] and culminates in the following

FACT 5.14. *Let $M$ be a quasiminimal pregeometry structure and let $\mathcal{K}(M)$ be the smallest class of $\mathcal{L}$-structures which contains $M$ and its closed substructures and is closed under taking unions of directed systems of closed embeddings as well as under isomorphisms (We will call such a class a* quasiminimal pregeometry class*). Then $\mathcal{K}(M)$ is $\mathcal{L}_{\omega_1,\omega}(Q)$-axiomatisable, every structure in $\mathcal{K}(M)$ satisfies clauses 1., 3., 4. and 5. and up to isomorphism there is exactly on structure in $\mathcal{K}(M)$ of each cardinal dimension. In particular, $\mathcal{K}(M)$ is uncountably categorical.* $\square$

We will build a countable-dimensional quasiminimal pregeometry structure using the Amalgamation Theorem 2.22 (with the generalisation contained in the remark following), our prospective amalgamation categories being defined as follows:

Definition 5.15. For every partial exponential field $K$, let $\mathfrak{C}(K)$ be the category of partial exponential fields strongly extending $K$, with strong extensions as morphisms.

For every partial exponential field $K$, let $\mathfrak{C}^{\text{full}}(K)$ be the category of ELA-fields strongly extending $K$, with strong extensions as morphisms.□

Thm. 5.9 of [**7**] now gives us the amalgamation result we require:

Theorem 5.16. *The categories $\mathfrak{C}_{\leq\aleph_0}(K_{\text{base}})$ and $\mathfrak{C}^{\text{full}}_{\leq\aleph_0}(K_{\text{base}})$ are $\aleph_0$-amalgamation categories with the same amalgamation limit.* □

Bays and Kirby [**7**] now go on to prove that the amalgamation limit is a quasiminimal pregeometry class in an appropriate language. In fact, their Theorem 6.9 proves more than stated.

Definition 5.17. Let $\mathcal{L}^{\text{QE}}$ be the language extending the field language by

(1) A binary relation symbol $E$ (to code the exponential).
(2) Parameters for every element in $K_{\text{base}}$.
(3) Relation symbols for formulas of the form $\phi_W(\vec{x},\vec{y})$ and $\psi_W(\vec{y})$, which are defined as follows:
    (a) Let $C$ be the algebraic closure of $K_{\text{base}}$ as a pure field, $W$ be any subvariety of $(C^n \times (C^*)^n) \times C^r$ defined over $K_{\text{base}}$. Then let $\phi_W(y)$ be the formula defining the subset
    "$(\vec{x},\vec{y}) \in W$ and $x \in E^n$ and $x$ is $\mathbb{Q}$-linearly independent over $E_{|K_{\text{base}}}$".
    (b) For every $W$ as above, let $\psi_W(\vec{y})$ stand for $\exists_x(\phi_W(\vec{x},\vec{y}))$. □

THEOREM 5.18. *Let $K$ be either a finitely generated partial exponential field or a countable ELA-field. Let $\mathfrak{C}'$ be an $\aleph_0$-amalgamation category which is a full subcategory of $\mathfrak{C}(K)$ and closed under considering strong partial exponential subfields strongly extending $K$. Then the amalgamation limit of $\mathfrak{C}'_{\leq\aleph_0}$, viewed as an $\mathcal{L}^{\mathrm{QE}}$-structure in the natural way and equipped with the exponential closure* Ecl*, is a quasiminimal pregeometry structure if it is infinite-dimensional with respect to* Ecl.

PROOF. The proof of the clauses QM1', QM4, QM5a and QM5b in the terminology of Proposition 6.5 of [**7**] go through word-for-word as in the proof of Thm. 6.9 of [**7**]; QM2 (infinite-dimensionality) is true by default. □

The next step to studying these quasiminimal exponential fields is giving an adequate axiomatisation for them, bearing in mind that Fact 5.14 guarantees an $\mathcal{L}_{\omega_1,\omega}(Q)$-axiomatisation.

This has already been done in the original [**42**], the formulation adopted here given by [**7**] in Thm. 9.1.

THEOREM 5.19. *The class* $\mathrm{ECF}_{\mathrm{SK,CCP}}$ *of exponential fields* $(F, +, *, \exp)$ *satisfying the following list of axioms is uncountably categorical and quasiminimal. Considered in a language $\mathcal{L}^{QE}$ expanded by first-order definitions,* $\mathrm{ECF}_{\mathrm{SK,CCP}}$ *is a quasiminimal pregeometry class.*

*(1)* $(F, +, *, \exp)$ *is an ELA-field.*
*(2) The kernel of* exp *is an infinite cyclic group generated by a transcendental element $\tau$.*
*(3) For all tuples $\vec{x}$ from $F$,* $\mathrm{trd}(\vec{x}, \exp(\vec{x})) \geq \mathrm{ldim}_{\mathbb{Q}}(\vec{x})$.

(4) *If $V$ is a rotund, additively and multiplicatively free subvariety of* $(F^{+})^{n} \times (F^{*})^{n}$ *defined over $F$ and of dimension $n$ and $\vec{a} \in F$, there is $\vec{x} \in F$ such that $(\vec{x}, \exp(\vec{x})) \in V$ and $\vec{x}$ is $\mathbb{Q}$-linearly independent over $\vec{a}$ (*strong exponential algebraic closedness*).*

(5) *For each finite subset $X$ of $F$, the exponential algebraic closure* $\mathrm{ecl}^{F}(X)$ *of $X$ in $F$ is countable.* □

The terminology used in these axioms is explained in most papers dealing with this area, [**7**] again being a complete reference. The name rotundity has been chosen because it suggests largeness in every direction; it means that if $(\vec{x_i}, \exp(\vec{x_i}))$ are in the variety, this does not force any linear combination of the $x_i$ to violate Condition 3.

Theorem 5.19 seems to fit the analogy to pure fields very well, $\mathrm{ECF}_{\mathrm{SK,CCP}}$ corresponding to ACF and quasiminimal pregeometry structures to strongly minimal fields. However, in the latter case, ACF is indeed the *only* strongly minimal theory of pure fields. The analogy would therefore suggest that $\mathrm{ECF}_{\mathrm{SK,CCP}}$ might be the only quasiminimal pregeometry class of ELA-fields. This, however, is not the case, as [**7**] find significantly weaker requirements for being a quasiminimal pregeometry class.

Definition 5.20. Let $(F, +, *, \exp)$ be an exponential field and $E$ the graph of exp. Then $F$ is called *almost exponentially algebraically closed* if for all but countably many rotund, additively and multiplicatively free subvariety of $(F^{+})^{n} \times (F^{*})^{n}$ defined over $F$ and of dimension $n$, $E(F)^{n} \cap V(F)$ is Zariski-dense on $V$.

THEOREM 5.21 (Theorem 11.6 of [**7**]). *If an exponential field satisfies numbers 1. and 5. of the axioms for* $\mathrm{ECF}_{\mathrm{SK,CCP}}$ *and is generically exponentially algebraically closed, it is a quasiminimal pregeometry structure.*

## 5.4. A New Quasiminally Excellent Exponential Field

In Remark 11.10 at the end of their preprint, [**7**] ask whether this significantly relaxed closure property is now enough to characterise quasiminimality. We will use a construction similar in spirit to the one from Chapter 3 to show that this is not the case, as we will construct an example of an uncountable quasiminimal exponential field $F$ (which even is a quasiminimal pregeometry structure) and give a definable family $(V_y)_{y\in F}$ of rotund and free varieties such that for only countably many $y$ there is an $(x, e^x) \in V_y(F)$.

If $K$ is an ELA-field, let $\mathfrak{C}_{\mathrm{tr},(*)}(K)$ be the class of all strong transcendental partial exponential field extensions of $K$ with the following property $(*)$:

$$\forall_{x,y}(x + e^x + y + e^y = 0 \to y \in K).$$

From now on, fix $K \coloneqq \mathrm{cl}_{\mathbb{B}}(\emptyset)$, where $\mathbb{B}$ is the unique infinite-dimensional countable pseudo-exponential field.

The example we are constructing will itself have $(*)$. This will immediately give us:

PROPOSITION 5.22. *Let $F$ have property $(*)$ and let $V_y(x, z) \coloneqq \{x+z+y+e^y = 0\}$. Then $\{x \in F|(x, e^x) \in V_y(x, z)\}$ is empty for all but countably many $y$.*

PROOF. $K$ is countable. □ □

As the $V_y$ are of a very simple form and only in one variable and its exponential, rotundity and freeness can be read off the definition immediately.

Our goal is therefore to find an uncountable quasiminimal $F \in \mathfrak{C}^{\text{full}}_{\text{tr},(*)}(K)$.

We will show that $\mathfrak{C}_{\text{tr},(*)}(K)$ and $\mathfrak{C}^{\text{full}}_{\text{tr},(*)}(K)$ are $\aleph_0$-amalgamation categories and investigate their common amalgamation limit. We cover the main points in two lemmas:

Lemma 5.23. *The following equalities hold:*

$$\mathfrak{C}_{\text{tr},(*),<\aleph_0}(K) = \mathfrak{C}_{\text{tr},(*)}(K) \cap \mathfrak{C}_{\text{tr},<\aleph_0}(K)$$

*and*

$$\mathfrak{C}^{\text{full}}_{\text{tr},(*),<\aleph_0}(K) = \mathfrak{C}^{\text{full}}_{\text{tr},(*)}(K) \cap \mathfrak{C}^{\text{full},}_{\text{tr},<\aleph_0}(K).$$

.

Proof. It is clear that if $L \in \mathfrak{C}_{\text{tr},(*)}(K)$ is $\aleph_0$-small as a structure in $\mathfrak{C}_{\text{tr}}(K)$, it is also $\aleph_0$-small as a structure in $\mathfrak{C}_{\text{tr},(*)}(K)$. The converse follows as every non-factoring strong embedding into a limit of arbitrary partial exponential fields can be converted to a non-factoring embedding into a limit of partial exponential fields with $(*)$ by intersecting with the image of $L$ under the embedding. The same holds with the "full" categories throughout. □ □

Lemma 5.24. *Let $P$ be a partial E-field extending $K$ and satisfying $(*)$. Then $P^{\text{full}}$ satisfies $(*)$.*

Proof. We will show that any $x, y, x', y' \in P^{\text{full}}$ such that $x' = e^x$, $y' = e^y$ and $x + x' + y + y' = 0$ already have to lie in $P$. We turn to the construction of

$P^{\text{full}}$ given in Th. 4.17 of [**7**]. $P^{\text{full}}$ is obtained there by embedding $P$ in a large algebraically closed field and then iterating two different kinds of extensions: In the first kind, a new element $e^a$ is added for an $a$ that is not yet in the domain of the exponential function, along with a division system for $e^a$. The resulting partial exponential function is then extended in the obvious way to the $\mathbb{Q}$-linear subspace of this domain. This $e^a$ is generic over those elements that have already been constructed. Interlaced with those are steps of the second kind, where a non-zero element that is not yet in the range of the exponential function is complemented by a preimage and its division system in the same manner.

We assume for the sake of contradiction that not all of $x$, $y$, $x'$ and $y'$ lie in $P$ and cover two cases:

Case 1: $y'$ was added last and not simultaneously with $x$, $y$ or $x'$ . Then $y'$ is generic and therefore field-theoretically transcendental over $x$, $x'$ and $y$, contradicting $x + x' + y + y' = 0$.

Case 2: $y'$ was added last and simultaneously with $x'$. Then $x$ and $y$ are part of a division system for an $a \in P^{\text{full}}$ and $x'$ and $y'$ are part of a division system for a $b \in P^{\text{full}}$ generic over $a$. This genericity, however, contradicts $x + x' + y + y' = 0$ as above.

This distinction covers all cases in which $y'$ was added last as if it were added simultaneously with $x$ then $x'$ would have to be added afterwards and, as no exponential relations are added between members of a division system, it could not be appended together with $y$.

The case of $x'$ being added last is entirely symmetric and the same is true for $x$ and $y$ being added last in an interlaced step. $\square$ $\square$

THEOREM 5.25. $\mathfrak{C}^{\mathrm{full}}_{\mathrm{tr},(*),\leq\aleph_0}(K)$ *is an* $\aleph_0$*-amalgamation-category.*

PROOF. AC1: Obvious.

AC2: Clear as limits of $\omega$-chains are ELA-fields and $(*)$ is a local property.

AC3, AC4: As $\mathfrak{C}^{\mathrm{full}}_{\mathrm{tr},(*)}(K)$ is a subcategory of $\mathfrak{C}^{\mathrm{full}}_{\mathrm{tr}}(K)$ where AC3 and AC4 hold by Th. 5.9 of [**7**], this follows from Lemma 5.23.

AC5: Following the proof to Prop. 5.7 in [**7**] and using Thm. 5.21 there, we merely have to show that the $A$ constructed there has $(*)$ if $A_L$ and $A_R$ do. By the second lemma above, it suffices to show that $A_1$ has $(*)$. However, for $x =: x_L + x_R$ and $y =: y_L + y_R$ the equation $x + x' + y + y' = 0$ is equivalent to $x_L + x_R + y_L + y_R + x_L x_R + y_L y_R = 0$ which implies $x_L, y_L \in A$ or $x_R, y_R \in A$ by algebraic independence of $A_L$ and $A_R$ over $A$. As $A_L$ and $A_R$ have $(*)$ by assumption, so therefore does $A_1$.

AC6: Follows from AC5 as all $F \in \mathfrak{C}^{\mathrm{full}}_{\mathrm{tr},(*)}(K)$ extend $K$ strongly. □ □

COROLLARY 5.26. $\mathfrak{C}^{\mathrm{full}}_{\mathrm{tr},(*),\leq\aleph_0}(K)$ *contains a* $\mathfrak{C}^{\mathrm{full}}_{\mathrm{tr},(*),\leq\aleph_0}(K)$*-universal and* $\mathfrak{C}^{\mathrm{full}}_{\mathrm{tr},(*),<\aleph_0}(K)$*-saturated object* $F$*.*

PROOF. Theorem 2.22. □

THEOREM 5.27. $\mathfrak{C}_{\mathrm{tr},(*),\leq\aleph_0}(K)$ *is an amalgamation category.*

PROOF. AC1-4 and 6 are again clear.

Our proof of AC 5 closely follows the proof of Corollary 5.8 of [**7**]. By Lemma 5.24 we can form $A_L^{\mathrm{full}}$, $A_R^{\mathrm{full}}$ and $A_0^{\mathrm{full}}$ and the embeddings between them just as there. We now invoke Theorem 5.25 above rather than Thm. 5.7 of [**7**] to finish the proof. □

Corollary 5.28. *$\mathfrak{C}_{\mathrm{tr},(*),\leq\aleph_0}(K)$ contains a $\mathfrak{C}_{\mathrm{tr},(*),\leq\aleph_0}(K)$-universal and $\mathfrak{C}_{\mathrm{tr},(*),<\aleph_0}(K)$-saturated object $F$.*

Proof. Theorem 2.22. □

We can conclude:

Proposition 5.29. *$\mathfrak{C}_{\mathrm{tr},(*),\leq\aleph_0}(K)$ and $\mathfrak{C}^{\mathrm{full}}_{\mathrm{tr},(*),\leq\aleph_0}(K)$ have the same $\aleph_0$-amalgamation limit $F$, which is a quasiminmal pregeometry structure.*

Proof. Just as in the proof of Thm. 5.9 of **[7]** and again invoking Lemma 5.24 wherever necessary, we can see that the amalgamation limits coincide. That it is a quasiminimal pregeometry structure now follows immediately from Theorem 5.18 above. □ □

The counterexample now rests on the simple observation that $(*)$ is preserved under directed colimits.

Lemma 5.30. *$(*)$ is preserved under directed colimits.* □

Proof. By directedness, every quadruple that contradicts $(*)$ would lie in one of the limitands and, since $(*)$ holds there, lie in $K$. □ □

As preservation under closed substructure and isomorphism is clear, this implies

Proposition 5.31. *Every structure in $\mathcal{K}(F)$ satisfies $(*)$.*

By Fact 5.14 we know that there is an uncountable structure $F_{\aleph_1}$ in $\mathcal{K}(F)$, thereby satisfying $(*)$, and Prop. 5.22 ensures that this $F_{\aleph_1}$ is not almost exponentially closed.

Overall, we have seen that sufficiently exponentially-algebraically closed ELA-fields are quasiminimal, but that the quasiminimal ELA-fields are far more diverse than the strongly minimal pure fields. We will close our discussion of quasiminimal exponential fields by noting that this does not result merely from the fact that quasiminimality is just generally too weak to characterise structures, but from a richness in the class of exponential fields.

FACT 5.32. *Every quasiminimal field of cardinality greater than* $\aleph_1$ *is algebraically closed.*

PROOF. See [**13**]. □

CHAPTER 6

# Homogeneous Exponential Fields

We are primarily interested, however, in exploring exponential fields as homogeneous structures. The existence of large superstable homogeneous exponential fields has first been discovered by Kirby and Zilber [**27**], and we can apply the theory we developed in the preceding chapters of this thesis to provide a more lucid account of their construction and to give a much finer analysis of their model theory. As we have developed all the necessary theoretical results in the preceding chapters, this will just be a brief and sketchy description of how (and why) they specialise to that particular case.

NOTATION 6.1. For any (partial) exponential field $F$, let $Z(F) := \{a \in F | \forall x(\exp(x) = 1 \to \exp(ax) = 1)\}$.

In the quasiminimal excellent exponential fields with standard kernel, that is, those intended to mimic the complex numbers with respect to exponentiation, $Z(F)$ is always isomorphic to the ring of integers. However, number-theoretic problems around division points and the existence of good bases mean that we will follow [**27**] in considering exponential fields with more saturated kernels. More precisely, we demand $Z(F)$ to be *algebraically compact*. Referring to [**27**] for the precise definition, it suffices for us to know from there:

FACT 6.2. *Let $M$ be an $\aleph_0$-saturated model of the theory of arithmetic* $\mathrm{Th}(\mathbb{Z},+)$. *Then $M$ is algebraically compact.*

DEFINITION 6.3. Let $K$ be an ELA-Field such that $Z(K)$ is an algebraically compact model of $\mathrm{Th}(\mathbb{Z},+)$. Then $\mathfrak{C}'_K$ is the category of all strong kernel-preserving extensions of $K$, the $K$-subscript sometimes being omitted to avoid overcrowding of the notation.

THEOREM 6.4. $\mathfrak{C}'_K$ *is a $\lambda$-superstable strictly regular predimension category for any $\lambda \geq |K|$.*

PROOF. It is clear that it is a hull category, and that $\mathrm{id} : K \to K$ is an intial object. The hull of a subset is just the free ELA-closure, existing by Proposition 3.13 of [**27**]

The properties of $\delta$, including clause (e), follow from Fact 5.6 above.

We will show that the hull operator satisfies the exchange principle. Let $b \in (A \cup \{c\})^{\mathrm{hull}} \backslash A^{\mathrm{hull}}$. Then enumerate $(A \cup \{c\})^{\mathrm{hull}}$ in such a way that for each $\alpha < |(A \cup \{c\})^{\mathrm{hull}}|$, either

- $s_{\alpha+1}$ is algebraic over $\{s_\beta | \beta \leq \alpha\}$.
- $s_{\alpha+1} = \exp(s_\beta)$ for a $\beta \leq \alpha$.
- $\exp(s_{\alpha+1}) = s_\beta$ for a $\beta \leq \alpha$.

Among those enumerations, choose one in which the index of $b$ is minimal possible. Let $m$, $n$ be the indices of $c$ and $b$ respectively, and note that they will both be finite.

We will prove the statement by induction on $m - n$.

So let the statement be true for all $b, c$ with indices less than $m-n$ apart. In particular, the statement holds for $b$ and $s_{n+1}$. By minimality of $m$, $b \notin (A \cup \{c\} - \{s_{n+1}\})^{\text{hull}}$. Thus, $s_{n+1} \in (A \cup \{b\})^{\text{hull}}$. But then we can divide into the three cases above and conclude by using the exchange principle for the algebraic closure and that the other two cases are symmetrical. ($\square$)

Free (even asymmetrical) amalgamation follows from Lemma 3.18 of [**27**], $\lambda$-superstability form the fact that the isomorphism types of strong extensions are determined by the types of a good basis. It is clear from the general theory on exponential fields that they also satisfy the additional clauses stipulated for strict regularity. $\square$

The theory presented in the chapter on amalgamation constructions now gives us the following existence results and characterisation of independence:

THEOREM 6.5. *Let $K$ be an ELA-Field whose kernel is algebraically compact. Then for every regular $\lambda \geq |K|$ there is a simply superstable homogeneous ELA-field $\mathbb{F}$ with $\ker \mathbb{F} = \ker K$ (and where the language has been augmented by constants for the elements of $K$). $\mathbb{F}$ is $\mathfrak{C}'_{\leq\lambda}$-universal and $\mathfrak{C}'_{<\lambda}$-homogeneous.*

PROOF. Follows from Prop. 4.12. $\square$

We can also apply the results that characterise dividing and U-rank in $\mathbb{F}$:

THEOREM 6.6. *For any $A, B, C \subset F$, $A \downarrow_C B$ if and only if*

$$\lceil (AC)^{ELA} \rceil \cap \lceil (BC)^{ELA} \rceil = \lceil C^{ELA} \rceil$$

*and*

$$\delta(\lceil (AC)^{ELA} \rceil \,/\, \lceil C^{ELA} \rceil) = \dim(\lceil (AC)^{ELA} \rceil \, \lceil (BC)^{ELA} \rceil \,/\, \lceil (BC)^{ELA} \rceil).$$

PROOF. Follows from Theorem 4.14. □

THEOREM 6.7. *Let and* $A \subseteq B \subset \mathbb{F}$ *with* $\dim(B/A) < 1$. *Then* $\mathrm{SU}(B/A) = \mathfrak{l}(\lceil B \rceil \,/\, \lceil A \rceil)$ *whenever* $\mathfrak{l}(\lceil B \rceil \,/\, \lceil A \rceil)$ *is finite.*

PROOF. Theorem 4.16. □

By Theorem 4.19, the SU-rank of the transcendental type and thus of $\mathbb{F}$ itself is $\aleph_0$.

In order to analyse stability-preserving substructures, analogues of the non-exponentially-closed quasiminimal exponential fields of the preceding chapter, we have to find types that satisfy Assumption 3.1.

We will first show that the simplest strongly minimal sets have trivial pregeometry.

PROPOSITION 6.8. *Let $f$ be a polynomial in one variable over an $F \subset \mathbb{F}$ s. t. $f(0) \neq 1$. Then $\mathcal{V}_f$, the set of all solutions to $e^x = f(x)$ in $\mathbb{F}$, is a strongly minimal set, and for all $X \subseteq \mathcal{V}_f$ it holds that* $\mathrm{scl}(X) \cap \mathcal{V}_f = X$.

PROOF. We first prove the following *claim*: □

Let $S$ be a $\mathbb{Q}$-linearly independent finite subset of $\mathcal{V}_f$. Then for all $X \subseteq S$, $\mathrm{scl}(X) \cap S = X$.

*Proof* of the claim:

$$\delta(S) = \mathrm{trd}(S, \exp(S)) - \mathrm{ld}(S) = \mathrm{trd}(S, f(S)) - |S| = \mathrm{trd}(S) - |S| \geq 0$$

by the Schanuel property for $\mathbb{B}$ and therefore $\mathrm{trd}(S) = |S|$ and $S$ is algebraically independent. As for all $x \in S$, $\exp(X)$ is algebraic over $x$, $S$ is even $ELA$-independent.

Therefore, for any $x \in S \backslash X$, $\delta(x/X) \geq 0$ implies that $x \notin \mathrm{scl}(X)$. ($\square$)

We now show by induction on $n$ that every subset of size $n$ of $\mathcal{V}_f$ is linearly and therefore by the claim also algebraically independent.

$n = 1$: Follows from $f(0) \neq 1$.

Assume the result to hold for all subsets of size $\leq n$ and let $S =: \{a_1, \ldots, a_{n+1}\}$. We need to show that $S$ is linearly independent. Assume not. Then w.l.o.g. there are $k_1, \ldots, k_n, l_1, \ldots, l_n \in \mathbb{Z}$ s. t. $a_{n+1} = \sum_{i=1}^{n} \frac{k_i}{l_i} a_i$. This implies, however, that

$$f(\sum_{i=1}^{n} \frac{k_i}{l_i} a_i) = f(a_{n+1}) = \exp(a_{n+1}) = \exp(\sum_{i=1}^{n} \frac{k_i}{l_i} a_i) = \prod_{i=1}^{n} \exp(\frac{k_i}{l_i} a_i).$$

This in turn leads to

$$f(\sum_{i=1}^{n} \frac{k_i}{l_i} a_i)^{\prod l_i} = \prod_{i=1}^{n} \exp(k_i(\prod_{j \neq i} l_j) a_i) = \prod_{i=1}^{n} (\exp(a_i)^{k_i(\prod_{j \neq i} l_j)}) = \prod_{i=1}^{n} f(a_i)^{k_i(\prod_{j \neq i} l_j)},$$

an algebraic relation on $\{a_1, \ldots, a_n\}$. That however contradicts the induction hypothesis and the initial claim. $\square$

However, since we have not shown in Section 3 that Assumption 3.1 follows from triviality for homogeneous models, we will instead give a direct characterisation of orthogonality and almost-orthogonality.

Proposition 6.9. *Let $\mathcal{V}_f$ be as above, defined over an $F \subseteq \mathbb{F}$, and let $p$ be the generic type of $\mathcal{V}_f$. Let $q$ be any complete type over $F$. Then the following are equivalent:*

*(1) $p$ is orthogonal to $q$.*

*(2) $p$ is almost orthogonal to $q$.*

*(3) For any $\vec{x} \vdash q$, $\lceil F(\vec{x})^{ELA} \rceil \cap p(\mathbb{M}) = \emptyset$.*

Proof. 1. implies 2. by definition.

2. implies 3.: If $y \in p(\mathbb{M})$, then $\mathrm{tp}(y/F)$ is unbounded; however, by the definition of almost-orthogonality, $y$ is free from $\vec{x}$ over $F$ and thus $\mathrm{tp}(y/F(\vec{x}))$ is also unbounded, or in other words, $y \notin \lceil F(\vec{x})^{ELA} \rceil$.

So we only need to show that 3. implies 1.

Let $\lceil F(\vec{x})^{ELA} \rceil \cap p(\mathbb{M}) = \emptyset$ for $\vec{x} \vdash q$. Let $F \subseteq D$ and let $D$ be free from $\vec{x}$ over $F$. Let $p'$ be a non-dividing extension of $p$ to $D$. Then $p'(\mathbb{M})$ is a strongly minimal set. We need to show that $\lceil D(\vec{x})^{ELA} \rceil \cap p'(\mathbb{M}) = \emptyset$. By the second clause of Theorem 6.6,

$$\delta(\lceil (F(\vec{x})^{ELA} \rceil / \lceil F^{ELA} \rceil) = \dim(\lceil (D)^{ELA} \rceil \lceil F(\vec{x})^{ELA} \rceil / \lceil D^{ELA} \rceil).$$

Assume for contradiction that there is a $y \in \lceil D(\vec{x})^{ELA} \rceil \cap p'(\mathbb{M})$. Then we can conclude as in the proof to the proposition above that such a $y$ would have to be in the linear span of $D(\vec{x})$. However, this would imply that, for some finite $D' \subseteq D$, $\delta(D'(\vec{x})) < \delta(D') + \delta(\vec{x})$ in contradiction to the equation above. $\square$

Corollary 6.10. *Let $\mathcal{V}_f$ be as above, defined over an $F \subseteq \mathbb{F}$, and let $p$ be the generic type of $\mathcal{V}_f$. Then $p$ satisfies Assumption 3.1.*

Proof. Follows immediately from Proposition 3.2. $\square$

Thus, the theory from Chapter 3 is applicable to this situation, and while we refrain from spelling out the entire body of results again, we do give the analogue of Theorem 3.4.

THEOREM 6.11. *Let $\mathbb{F}$ be of cardinality $\lambda$ as constructed in Theorem 6.5, $\lambda$ a regular cardinal, $C \subset \mathbb{F}$ of cardinality less than $\lambda$ and $P$ a family of types defined by $\mathcal{V}_f$ over $C$. Then the category $\mathfrak{C}^{\mathbb{M}}_{(P)}$ of all substructures $A$ of $\mathbb{F}$ extending $C$ of cardinality $\leq \lambda$ orthogonal to every $p \in P$ over $C$ is a $\lambda$-stability category, and thus has a uniques amalgamation limit in cardinality $\lambda$. This limit does not realise any of the $\mathcal{V}_f$ in $P$ and has all the properties described in Chapter 3.*

REMARK 6.12. Also, although we have formulated the argument above only for the simple strongly minimal sets mentioned there, this line of proof should be able to prove analogues of the above for various trivial strongly minimal sets in $\mathbb{F}$.

Overall, we hope to have demonstrated in this chapter that homogeneous model theory is an appropriate framework within which to ask (and answer!) several interesting and a priori non-trivial questions about exponential fields. It significantly enriches the quasiminimal approach taken in previous papers by giving us an array of tools from classical stability theory that are not available otherwise.

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