Stacks in Derived Bornological Geometry

Rhiannon Savage

Balliol College

University of Oxford

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Abstract

In this thesis, we describe a higher categorical framework for discussing derived analytic and derived smooth geometry. Analogous to the Toën and Vezzosi model of derived algebraic geometry which uses simplicial commutative rings as its building blocks, in our model we use simplicial commutative complete bornological rings. This work builds upon foundational work of Ben-Bassat, Kelly, and Kremnizer.

This general framework allows us to prove several results about derived stacks, in particular we develop the obstruction theory of stacks and use it to prove a representability theorem. This theorem cements this new theory of derived bornological geometry as strong and versatile, and gives differential and analytic geometers a new perspective on their own representability problems.

In this thesis, we begin by studying a generalisation of the Koszul duality theory of Beilinson, Ginzburg, and Soergel to the setting of algebra objects in a bicomplete closed symmetric monoidal exact category E with enough flat projectives. Examples include the category CBorn_R of complete bornological spaces and the derived equivalent category $\mathrm{Ind}(\mathrm{Ban}_R)$ of formal filtered colimits of Banach modules over a Banach ring R.

We then define a general categorical context we call a *derived geometry context*. In these contexts we obtain our representability theorem. If a derived stack has a geometric truncation, is compatible with Postnikov towers, and has a well defined obstruction theory, our theorem shows that it is representable by a derived geometric stack.

Working relative to CBorn_R for an appropriately chosen Banach ring R, we can define suitable derived geometry contexts modelling derived complex analytic and derived smooth geometry. In the derived smooth geometry setting, we develop a theory of $\mathcal{C}^\infty$-bornological rings extending the theory of $\mathcal{C}^\infty$-rings. Finally, we show that the derived moduli stack of non-linear elliptic PDEs is representable by a derived $\mathcal{C}^\infty$-bornological affine scheme.

Notation

In this thesis, we use the following notation.

- We use normal font C and Mod_A to denote ordinary categories, and normal font F for ordinary stacks,
- We use mathcal font $\mathcal{C}$ and bold font $\mathbf{Mod}_A$ to denote $(\infty, 1)$-categories, and mathcal font $\mathcal{F}$ for higher stacks,
- We use the letters A, B for elements of an $(\infty, 1)$-category $\mathcal{C}$, and X, Y for elements of an $(\infty, 1)$-category $\mathcal{A}$ or $\mathcal{M}$. We tend to think of our category $\mathcal{A}$ as our relevant category of affines.

Glossary of Categories

- Ban_k denotes the category of Banach spaces over some valued field k,
- Fr_k denotes the category of Fréchet spaces over some valued field k,
- CBorn_R denotes the category of complete bornological spaces over a Banach ring R,
- Con denotes the category of convenient spaces over $\mathbb{R}$ and ConMfd the category of convenient manifolds,
- sC denotes the category of simplicial objects in a category C,
- Comm(C) denotes the category of commutative monoids (algebra objects) in a symmetric monoidal category C,
- Aff denotes the category of affine schemes,
- CRing denotes the category of commutative rings,
- Ind(C) and SInd(C) denote the free filtered and sifted cocompletions respectively of an ordinary category C. These are defined in Appendix C,
- $\mathcal{P}_\Sigma(\mathcal{C})$ denotes the free sifted cocompletion of an $(\infty, 1)$-category $\mathcal{C}$,
- For an $(\infty, 1)$-category $\mathcal{C}$ and an object $X \in \mathcal{C}$, we use the notation $\mathcal{C}_{/X}$ and ${}^{X/}\mathcal{C}$ to denote the over/slice and under/coslice categories respectively,
- For any $(\infty, 1)$-category $\mathcal{C}$, we denote by $h : \mathcal{C} \to \mathrm{PSh}(\mathcal{C})$ the Yoneda embedding.

Contents

Introduction

In *Récoltes et Semailles* [33], Grothendieck uses the analogy of the 'rising sea' to describe his theory-building approach towards Mathematics. Analogous to how a mass of land slowly and silently becomes engulfed by the rising sea, he describes how solving complex problems can be done gradually step-by-step until the resulting problem becomes almost trivial.

We use a similar approach in this thesis. The bulk of the thesis is devoted to describing a very general framework for doing derived geometry relative to complete bornological rings. This allows us to talk about derived algebraic, analytic, and smooth geometry essentially in the same breath. With this strong framework in place, we can prove a representability theorem for derived stacks in these contexts, which gives several natural conditions under which a derived stack can be built up from derived affines. A proof of representability of the derived moduli stack of solutions to non-linear elliptic partial differential equations then naturally arises as a corollary.

We hope that this thesis lays the groundwork for further study of the representability of derived stacks, for example the derived moduli stack of Galois representations. We also believe that the new theory of $\mathcal{C}^\infty$-bornological rings proposed in this thesis could provide us with a rich and interesting perspective on infinite dimensional manifolds.

A Relative Approach to Geometry

In this thesis, we take the perspective that, simplistically, geometry is the combination of algebra and topology. For example, schemes are glued together from algebraic objects, affine schemes, using the Zariski topology. In [84], Toën and Vaquié use this approach to define a theory of *algebraic geometry relative to a category.* In order to define an appropriate notion of an algebra object, they start off with the classical result that there is an equivalence of categories

$$\mathrm{Aff} \simeq \mathrm{CRing}^{op} \simeq \mathrm{Comm}(\mathrm{Mod}_{\mathbb{Z}})^{op}$$

where Comm($\mathrm{Mod}_{\mathbb{Z}}$) denotes the category of commutative monoids in $\mathrm{Mod}_{\mathbb{Z}}$.

The idea is that, to obtain geometry relative to some nice symmetric monoidal category C, we can just replace $\mathrm{Mod}_{\mathbb{Z}}$ with C and define the category of affines, Aff(C), to be the category $\mathrm{Comm}(\mathrm{C})^{op}$. Endowing this category with a suitably chosen *Grothendieck topology* allows us to build up analogues of geometric objects.

The real power of this relative approach comes in defining appropriate models for derived geometry. Derived algebraic geometry originated because of frustration around a lack of 'higher structure' in classical algebraic geometry. A main motivation for derived algebraic geometry was to study non-transversal intersections of subvarieties. The intersection can be realised as a *derived scheme*.

The relative approach to geometry described above admits a natural generalisation to derived geometry. We instead work with an appropriate symmetric monoidal $(\infty, 1)$-category $\mathcal{C}$. We can then consider some well-defined category of algebra objects in $\mathcal{C}$, and then define the category of derived affines to be the opposite category of algebra objects. In practice, we want to be able to pick out a full subcategory of geometrically interesting derived affines to work with, but have them be embedded in a larger category with stronger and more versatile properties. This is explained in more detail in Section 3.2.1.

$$\begin{tikzcd}
\text{Classical Geometry} \arrow[r, squiggly, "\textit{add homotopical information}"] & \text{Derived Geometry} \\
\text{1-categories} \arrow[r, squiggly] & (\infty,1)\text{-categories}
\end{tikzcd}$$

In [85] and [86], Toën and Vezzosi propose that derived algebraic geometry can be done relative to the $(\infty, 1)$-category $\mathbf{Ch}(\mathrm{Mod}_{\mathbb{Z}})$ of chain complexes in $\mathrm{Mod}_{\mathbb{Z}}$. The *category of (connective) derived affines*, $\mathbf{DAff}^{cn}$, is defined to be the $(\infty, 1)$-category

$$\mathbf{DAff}^{cn} := \mathbf{sCRing}^{op} \simeq \mathbf{L}^{H}(\mathrm{Comm}(\mathrm{sMod}_{\mathbb{Z}}))^{op}$$

where $\mathbf{L}^{H}(\mathrm{Comm}(\mathrm{sMod}_{\mathbb{Z}}))$ denotes the $(\infty, 1)$-category associated to the model category of simplicial commutative rings (also known as *animated rings*). They also provide definitions of Zariski open immersions, and flat, étale, and smooth morphisms of derived affines which all have explicit descriptions as 'strong' versions of the classical notions. This essentially means that on π_0 we obtain the classical notion, and then higher homotopy groups are well behaved with respect to each other (see Definition 5.1.2.8). In this thesis, we primarily work with $(\infty, 1)$-categories for their simplicity and versatility, but we note that much of the early literature in this area is in the language of model categories.

We should note that Lurie has another model of derived algebraic geometry using his notion of *structured spaces* [51]. This theory utilises $\mathbb{E}_\infty$-rings rather than simplicial commutative rings. These two approaches essentially correspond in characteristic zero, but have their own advantages depending on whether one is coming from an algebraic or topological viewpoint. Derived algebraic geometry serves as an interesting subject in its own right, but it has numerous uses in intersection theory, deformation theory, and moduli theory [83].

Derived Analytic Geometry

The development of derived analytic geometry is driven by motivations similar to those behind derived algebraic geometry. Lurie has an approach to derived complex analytic geometry, detailed in [54, Sections 11 and 12], built up using the pregeometry associated to the $(\infty, 1)$-category of finite dimensional Stein manifolds.

Porta and Yue Yu's formulation of derived complex analytic geometry [63] is based on this theory of Lurie's, but also includes an approach to derived non-archimedean analytic geometry [65]. There are several limitations to the Porta and Yue Yu theory, as detailed in [10]. In particular, it is hard to formulate a definition and work with quasi-coherent sheaves on their derived analytic spaces. Recent work of Ben-Bassat, Kelly, and Kremnizer circumnavigates these problems by working with a more relative approach, motivated by the work of Toën and Vezzosi [86] and Raksit [72].

In [10], they present a model of geometry relative to an $(\infty, 1)$-algebra context. This is essentially a locally presentable symmetric monoidal $(\infty, 1)$-category $\mathcal{C}$ equipped with a graded monad $\mathbf{D}$. Our algebras will then be algebras over this monad. In this thesis, we will only be interested in *derived algebraic contexts* in the sense of Raksit [72, Definition 4.2.1] where we use the **LSym** monad. The full formalism of $(\infty, 1)$-algebra contexts is useful if one wants to do spectral algebraic geometry.

Working with this relative approach requires you to choose a category with good properties for doing geometry, within which your classical 'affine objects' can be realised as a full subcategory of derived affines. In choosing an appropriate category to work with for derived analytic geometry, there are several natural candidates. In rigid analytic geometry, the algebraic building blocks are affinoid algebras. In complex analytic geometry, the algebras of interest are Stein algebras. Affinoid algebras and

Stein algebras are examples of commutative Fréchet algebras. Therefore, it would perhaps seem sensible to choose the category Fr_k of Fréchet spaces as your base symmetric monoidal category. However, this category has several limitations which restrict us from making certain geometric constructions. In particular, it is not closed symmetric monoidal and is neither complete nor cocomplete. Moreover, it does not have enough projectives and the tensor product is not derivable.

Therefore, we want to find a larger category which contains the category Fr_k as a full subcategory and which has good properties which we can exploit. In several papers ([4],[5],[7],[10],[11],[12],[41]) spanning almost a decade, Bambozzi, Ben-Bassat, Kelly, Kremnizer, and Mukherjee establish that we can consider derived analytic geometry as geometry relative to the $(\infty, 1)$-category of chain complexes of *complete bornological spaces.*

Derived Bornological Geometry

Analogous to how a topology is a collection of open sets, one can think of a *bornology* as a collection of bounded sets. The category CBorn_k of *complete bornological spaces over a valued field k* is equivalent to the full subcategory of essentially monomorphic objects in $\mathrm{Ind}(\mathrm{Ban}_k)$, the free filtered cocompletion of Ban_k. Moreover, there is a derived equivalence

$$\mathrm{D}(\mathrm{CBorn}_k) \simeq \mathrm{D}(\mathrm{Ind}(\mathrm{Ban}_k))$$

We note that CBorn_k is a concrete category, whereas $\mathrm{Ind}(\mathrm{Ban}_k)$ is not. However, we will often need to exploit the universal properties of $\mathrm{Ind}(\mathrm{Ban}_k)$ as described in Appendix C.1. In practice, we will work with both categories interchangeably.

We note that the categories Ban_k and Fr_k are full subcategories of CBorn_k and $\mathrm{Ind}(\mathrm{Ban}_k)$. Moreover, $\mathrm{Ind}(\mathrm{Ban}_k)$ and CBorn_k have several useful properties. In particular, they are bicomplete closed symmetric monoidal elementary exact categories with enough flat projectives.

The *category of (connective) derived affines in* $\mathbf{Ch}(\mathrm{Ind}(\mathrm{Ban}_k))$ is the $(\infty, 1)$-category

$$\mathbf{DAff}^{cn}(\mathbf{Ch}(\mathrm{Ind}(\mathrm{Ban}_k))) := \mathbf{L}^H(\mathrm{Comm}(\mathrm{sInd}(\mathrm{Ban}_k)))^{op}$$

There is an adjunction between the classical and derived affines

$$\iota : \mathrm{Aff}(\mathrm{Ind}(\mathrm{Ban}_k)) \leftrightarrows \mathbf{DAff}^{cn}(\mathbf{Ch}(\mathrm{Ind}(\mathrm{Ban}_k))) : t_0$$

Although this is a relatively new theory, there is already a wealth of new developments and applications. In particular, there is a version of the Hochschild-Kostant-Rosenberg Theorem [41], there are six functor formalisms for rigid analytic sheaves [80], there are derived blow ups [9], and there are developments in the theory of analytic geometry over $\mathbb{F}_1$ [6].

In this thesis, we further develop the foundational theory of derived bornological geometry, in particular the theory of derived stacks. We define notions of geometric stacks, cotangent complexes, obstruction theories, and also show that we have a representability theorem for certain derived stacks. We also broaden the scope of the theory by establishing the groundwork for a bornological theory of derived smooth geometry.

There are other models for derived smooth geometry. A Lurie-esque model using structured spaces appears in the work of Spivak [81] with his notion of derived manifolds. This was extended by Carchedi and Steffens [22]. For a less abstract perspective, there is also Joyce's approach to derived differential geometry [36] using d-manifolds.

Derived Moduli Stacks of Solutions to Elliptic PDEs

Geometric stacks are a class of stacks which are glued together from affines in a compatible way. Key examples are algebraic stacks and Deligne-Mumford stacks. A main motivation for defining geometric stacks is to endow certain moduli stacks with useful geometric structures. For example, in [53], Lurie shows that the derived moduli stack of elliptic curves is a 1-geometric derived Deligne-Mumford stack.

In recent years, there has been a great deal of interest in proving representability of the derived moduli stack of solutions to non-linear elliptic PDEs. Given a system of non-linear elliptic partial differential equations on a compact manifold, one can express the moduli space of solutions locally as a *Kuranishi chart* [48], i.e. the zero set of a smooth map $f : \mathbb{R}^n \to \mathbb{R}^m$. We can consider the concept of a Kuranishi atlas on such a moduli space, see work by Fukaya and Ono [29]. Traditionally, proving representability of the associated moduli functors as derived manifolds (or d-manifolds) involves gluing together Kuranishi atlases, which is in general quite a complicated procedure.

There have recently been a number of alternative simpler approaches. In [62], Pardon considers this problem in a constructive way using non-linear elliptic Fredholm analysis. His definition of derived smooth manifolds is obtained from the usual category of smooth manifolds by formally adjoining finite limits modulo preserving finite transverse limits [62, Definition 3.5]. In [82], Steffens takes a different approach to proving representability of the moduli stack as a *derived $\mathcal{C}^\infty$-scheme* which is locally of finite presentation and quasi-smooth. These objects can be considered to be glued together from Kuranishi spaces [82, Remark 1.0.1].

In the final chapter of this thesis, we show that representability of the derived moduli stack of solutions to non-linear elliptic PDEs is a simple corollary of the representability theorem I have proven in derived bornological geometry. The proof requires little analytic machinery. It instead requires the development of a theory of *$\mathcal{C}^\infty$-bornological rings*, an extension of the classical theory of $\mathcal{C}^\infty$-rings to include algebras of smooth functions on ℓ^1s.

Connections with Condensed Mathematics

Some differences and similarities between bornological and condensed mathematics are highlighted in [10]. We note that the category of analytic rings, in the sense of Clausen and Scholze [77], is not quite a full subcategory of commutative monoids in some symmetric monoidal category and therefore cannot be dealt with in a similar way. However, we expect that the theory of geometric stacks lines up in the analytic setting. There is yet to be a representability theorem in this setting.

In the *Future Research Directions* section of this thesis, we discuss how we can prove representability of the moduli stack of solutions to non-linear elliptic PDEs as what we could perhaps call an *ultrasolid $\mathcal{C}^\infty$-ring*, i.e. an object in the free sifted cocompletion of the category of $\mathcal{C}^\infty$-rings. However, we think that in practice the more concrete approach of $\mathcal{C}^\infty$-bornological rings would be more accessible and useful to differential geometers.

Layout of Chapters

Chapter One: Exact Categories

We first set the scene with a discussion of the exact categories of interest to us in this thesis. Exact categories are additive categories equipped with a notion of a short exact sequence. This allows us to define various homological algebra constructions, in particular derived categories. Moreover, by [34, Proposition 3.11, Theorem 3.12], for any weakly idempotent complete exact category E with kernels, there is a fully faithful exact functor $I : \mathrm{E} \to \mathrm{LH}(\mathrm{E})$, where $\mathrm{LH}(\mathrm{E})$ is an abelian category. Further, there is an equivalence

$$\mathrm{D}(I) : \mathrm{D}(\mathrm{E}) \to \mathrm{D}(\mathrm{LH}(\mathrm{E}))$$

on the level of derived categories.

The aforementioned categories CBorn_k and $\mathrm{Ind}(\mathrm{Ban}_k)$ are our key examples of exact categories. A key fact that we will use in this thesis is that there are equivalences of categories

$$\mathrm{LH}(\mathrm{CBorn}_k) \simeq \mathrm{LH}(\mathrm{Ind}(\mathrm{Ban}_k)) \simeq \mathrm{SInd}(\mathrm{Lin}_k)$$

where Lin_k is the category whose objects are of the form $\ell^1(\kappa)$ for κ a cardinal. The morphisms are bounded linear maps.

Chapter Two: Koszul Monoids in Exact Categories

In [8], Beilinson, Ginzburg, and Soergel show that for any *Koszul ring*, for example $\mathrm{Sym}(V)$ for V a finite dimensional vector space over a field k, there is an associated *Koszul complex*. Moreover, there is an equivalence of certain derived categories of graded modules over a Koszul ring and its Koszul dual ring. In Chapter 2, we generalise this result to algebra objects in a bicomplete closed symmetric monoidal exact category E with enough flat projectives, for example CBorn_k or $\mathrm{Ind}(\mathrm{Ban}_k)$. This chapter is a generalisation of results appearing in my paper *Koszul Monoids in Quasi-abelian Categories* [74] to the setting of exact categories.

A Koszul monoid in E is a positively graded algebra object $A = \bigoplus_{j \geq 0} A_j$ in E which, among other conditions, admits a projective resolution with each P^i generated by its degree i-component over A_0. Examples include the symmetric algebra over a nuclear Fréchet space, and its dual Koszul monoid, the exterior algebra over the dual space. If a Koszul monoid A has a dual $A^!$ in the sense of Definition 2.2.2.7, then A has a Koszul complex

$$\cdots \to A \otimes_{A_0} {}^*(A_2^!) \to A \otimes_{A_0} {}^*(A_1^!) \to A$$

Our main result is that there is an equivalence between the following subcategories of the derived categories of graded left modules.

Theorem. *(Theorem 2.5.0.1) Suppose that A is a left dualisable Koszul monoid with quadratic dual monoid $A^!$. Then, there is an equivalence of categories*

$$\mathrm{D}^{\downarrow}(\mathrm{gr}_A\mathrm{Mod}) \simeq \mathrm{D}^{\uparrow}(\mathrm{gr}_{A^!}\mathrm{Mod})$$

The proof of this theorem exploits the embedding of exact categories into their abelian left hearts in order to use the classical theory of spectral sequences.

Chapter Three: Derived Geometric Stacks

In [86], Toën and Vezzosi present their notion of a *homotopical algebraic geometry (HAG) context* which essentially consists of a category of derived affines along with a Grothendieck topology $\boldsymbol{\tau}$ on them, and a class of maps $\mathbf{P}$. In the derived algebraic geometry context we discussed previously, we could endow our class $\mathbf{DAff}^{cn}$ with the étale topology and take the class of smooth maps. These contexts provide enough structure to discuss a theory of *geometric stacks* in the derived setting, examples in the classical setting coming from algebraic and Deligne-Mumford stacks

In Chapter 3, we define notions of geometry tuples. These provide enough structure for our intended purposes, but avoid some of the more onerous conditions in Toën and Vezzosi's HAG contexts. An $(\infty,1)$-*geometry tuple* consists of a tuple $(\mathcal{M}, \boldsymbol{\tau}, \mathbf{P}, \mathcal{A})$ where, motivated by our previous remarks, we think of $\mathcal{M}$ as a large class of derived affines within which some distinguished category $\mathcal{A}$ is chosen. In such a context we can define n-geometric stacks on $\mathcal{A}$. These are inductively defined starting with the definition that a (-1)-geometric stack on $\mathcal{A}$ is one of the form $\mathrm{Map}_{\mathcal{M}}(-, X)$ for some $X \in \mathcal{M}$.

We have the following examples, which are explained in more detail in Chapters 5 and 6. We will not consider the non-Archimedean setting in this thesis but this is discussed further in [10],[11], and [42].

	Derived Algebraic Geometry	Derived Complex Analytic Geometry	Derived Smooth Geometry
$\mathcal{M}$	$\mathbf{DAff}^{cn}$	$\mathbf{DAff}^{cn}(\mathbf{Ch}(\mathrm{Ind}(\mathrm{Ban}_{\mathbb{C}})))$	$\mathbf{DAff}^{cn}(\mathbf{Ch}(\mathrm{Ind}(\mathrm{Ban}_{\mathbb{R}})))$
$\boldsymbol{\tau}$	étale topology	finite Stein homotopy monomorphism topology	finite $\mathcal{C}^{\infty}$-topology
$\mathbf{P}$	smooth maps	formally perfect maps	$\mathcal{C}^{\infty}$-open immersions
$\mathcal{A}$	$\mathbf{DAff}^{cn}$	derived Steins	derived $\mathcal{C}^{\infty}$-bornological affines

In this chapter, we also explore conditions under which n-geometric stacks are preserved under adjunctions. In particular, several conditions can be relaxed if $\mathcal{A}$ is closed under $\boldsymbol{\tau}$-descent, which in particular implies that any stack with an n-atlas is n-geometric.

Chapter Four: Derived Geometry Contexts

Recall that we defined $\mathbf{DAff}^{cn}(\mathbf{Ch}(\mathrm{Ind}(\mathrm{Ban}_k))) := \mathbf{L}^H(\mathrm{Comm}(\mathrm{sInd}(\mathrm{Ban}_k)))^{op}$. A key aim of this work is to build a very general and easily adaptable theory, and therefore we need to be able to define a general notion of derived affines $\mathcal{M}$ relative to a category. In Chapter 4, we utilise the notion of a *derived algebraic context* $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, \mathcal{C}^0)$, formalised by Raksit in [72], to construct categories $\mathbf{DAff}^{cn}(\mathcal{C})$ of (connective) derived affines. A derived algebraic context consists of a stable locally presentable symmetric monoidal $(\infty, 1)$-category $\mathcal{C}$ with a well-defined t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ generated by $\mathcal{C}^0$. In this thesis we will only consider the connective setting. There is some work in the non-connective setting in [9] and [10]. The contents of Chapters 4 and 5 largely appear in the preprint *A Representability Theorem for Stacks in Derived Geometry Contexts* [75].

If we incorporate our definition of relative $(\infty, 1)$-geometry tuples and define a suitable collection $\mathbf{M}$ of modules for each $X = \mathrm{Spec}(A) \in \mathcal{A} \subseteq \mathbf{DAff}^{cn}(\mathcal{C})$, we obtain what we call a *derived geometry context* $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, \mathcal{C}^0, \boldsymbol{\tau}, \mathbf{P}, \mathcal{A}, \mathbf{M})$. This provides us with enough structure to be able to define the cotangent complex $\mathbb{L}_{\mathcal{F}/\mathcal{G},x}$ of a morphism of presheaves $\mathcal{F} \to \mathcal{G}$ in $\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))$ at a point $x : X = \mathrm{Spec}(A) \to \mathcal{F}$.

We are interested in finding conditions under which derived stacks are geometric. A key part of the proof is lifting maps $X = \mathrm{Spec}(A) \to \mathcal{F}$ along small extensions $X \to X_d[\Omega M]$. This necessitates the use of *obstruction theories*, see Definition 4.3.1.1. Under some simple conditions on $\boldsymbol{\tau}$ and $\mathbf{P}$, which we call the *obstruction conditions*, we obtain the following result.

Theorem. *(c.f. Theorem 4.3.4.1) Suppose that $\boldsymbol{\tau}$ and $\mathbf{P}$ satisfy the obstruction conditions relative to $\mathcal{A}$. Then any n-geometric stack $\mathcal{F} \in \mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$ has an obstruction theory.*

In the final part of this chapter, we study obstruction theories in more detail. Suppose that $\boldsymbol{\tau}$ and $\mathbf{P}$ satisfy the obstruction conditions, finite coproducts of (-1)-geometric stacks on $\mathcal{A}$ are (-1)-geometric, and there is a specific infinitesimal criteria on morphisms in $\mathbf{P}$, see Assumption 4.3.6.4. Then, under these conditions we obtain liftings along small extensions $X \to X_d[\Omega M]$ as desired.

Chapter Five: A Representability Theorem for Stacks in Derived Geometry Contexts

Representability is an important concept in mathematics and has many different interpretations. In algebraic geometry, we may want to know whether a presheaf is the image under the Yoneda embedding of a nice geometric object, for example an affine scheme. We might further ask, is it equivalent to a geometric stack, for example an algebraic stack. In our setting, we want to find conditions under which derived stacks in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$ are n-geometric. Our ultimate motivation is to discuss conditions under which derived moduli stacks, for example moduli stacks of solutions to PDEs or of Galois representations, are geometric.

Our conditions on representability are not as simple as those of Lurie in [56, Theorem 3.2.1], and in future work it would be interesting to explore such conditions in our derived geometry contexts. Instead, we prove a version of this theorem sketched in Toën and Vezzosi [86, Appendix C]. For this we need to introduce the notion of a *representability context* $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, \mathcal{C}^0, \boldsymbol{\tau}, \mathbf{P}, \mathcal{A}, \mathbf{M}, \mathbf{S})$ which is a derived geometry context along with some extra conditions. In particular, we will need $\boldsymbol{\tau}$ and $\mathbf{P}$ to satisfy the obstruction conditions relative to $\mathcal{A}$. We will also need certain compatibility conditions between $\mathcal{A}$ and its classical truncation $\mathcal{A}^{\heartsuit}$. These conditions are stated in Definition 5.1.3.1. An example of a representability context for derived complex analytic geometry is described in the latter parts of this chapter.

We prove the following representability theorem by lifting an atlas of the truncated stack $t_0(\mathcal{F})$ to an atlas for $\mathcal{F}$.

Theorem. *(Theorem 5.1.6.1) Suppose that* $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, \mathcal{C}^0, \boldsymbol{\tau}, \mathbf{P}, \mathcal{A}, \mathbf{M}, \mathbf{S})$ *is a representability context and that* $\mathcal{F}$ *is a stack in* $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$*. The following conditions are equivalent.*

1. *$\mathcal{F}$ is an n-geometric stack relative to $\mathcal{A}$,*
2. *$\mathcal{F}$ satisfies the following three conditions:*
   - *(a) The truncation $t_0(\mathcal{F})$ is an n-geometric stack relative to $\mathcal{A}^{\heartsuit}$,*
   - *(b) $\mathcal{F}$ has an obstruction theory relative to $\mathcal{A}$,*
   - *(c) For every $X = \mathrm{Spec}(A) \in \mathcal{A}$, $\mathcal{F}(A) \simeq \varprojlim_k \mathcal{F}(A_{\leq k})$.*

We also explore conditions under which mapping stacks are n-geometric. In particular, the following result is a corollary of our representability theorem.

Corollary. *(Corollary 5.3.3.2) Suppose that $X = \mathrm{Spec}(A) \in \mathcal{A}^{\heartsuit}$. Suppose that $\mathcal{G}$ is in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})_{/X}$. Under the following conditions, $\underline{\mathrm{Map}}_{\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})_{/X}}(X, \mathcal{G})$ is an n-geometric stack relative to $\mathcal{A}$,*

1. *The stack $t_0(\underline{\mathrm{Map}}_{\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})_{/X}}(X, \mathcal{G}))$ is an n-geometric stack relative to $\mathcal{A}^{\heartsuit}$,*
2. *$\mathcal{G}$ is n-geometric and the cotangent complex of the morphism $\mathcal{G} \to X$ is in $\mathbf{Perf}(\mathcal{G})$.*

Chapter Six: $\mathcal{C}^\infty$-Bornological Rings

As mentioned earlier in the introduction, we want to extend the category of $\mathcal{C}^\infty$-rings to a category of *$\mathcal{C}^\infty$-bornological rings* which has more useful properties and allows us to discuss smooth functions on infinite dimensional manifolds. Recall that objects of the form $\ell^1(\kappa)$ generate $\mathrm{CBorn}_{\mathbb{R}}$. Therefore, if we consider the category $\mathcal{C}^\infty$ consisting of Banach spaces of the form $\ell^1(\kappa)$ along with smooth maps between them, then we can define the category $\mathcal{C}^\infty\mathrm{BornRing}$ to be the free sifted cocompletion of $\mathcal{C}^{\infty,op}$. We can consider the *free $\mathcal{C}^\infty$-bornological ring* on a complete bornological space using the following result.

Corollary. *(Corollary 6.1.3.5) There is an adjunction*

$$L : \mathrm{LH}(\mathrm{CBorn}_{\mathbb{R}}) \leftrightarrows \mathcal{C}^\infty\mathrm{BornRing} : R$$

We can obtain the category $\mathcal{C}^\infty\mathbf{DBornRing}$ of derived $\mathcal{C}^\infty$-bornological rings by considering the $(\infty,1)$-categorical free sifted cocompletion of $\mathcal{C}^{\infty,op}$. We show that we can obtain a representability context for derived smooth geometry by taking $\mathcal{A}$ to be the category $\mathcal{C}^\infty\mathbf{DBAff} := \mathcal{C}^\infty\mathbf{DBornRing}^{op}$. We note that the category of $\mathcal{C}^\infty$-rings is a full subcategory of the category of $\mathcal{C}^\infty$-bornological rings. We can similarly consider the category of derived $\mathcal{C}^\infty$-rings and take $\mathcal{C}^\infty\mathbf{Aff}$ to be its opposite category.

In the final parts of this chapter, we explain how we can describe a derived non-linear partial differential equation $\mathcal{G}$ on sections of a morphism $\mathcal{F} \to X = \mathrm{Spec}(A)$ in $\mathbf{Stk}(\mathcal{C}^\infty\mathbf{Aff}, \boldsymbol{\tau}_{\mathcal{C}^\infty})$ as a closed immersion of the jet stack associated to $\mathcal{F}$. Motivated by the theory of D-modules, we define *the derived moduli stack of solutions to $\mathcal{G}$*, denoted $\mathbf{Sol}_X(\mathcal{G})$, to be the mapping stack of sections

$$\mathbf{Sol}_X(\mathcal{G}) := \underline{\mathrm{Map}}_{\mathbf{Stk}(\mathcal{C}^\infty\mathbf{DAff}, \boldsymbol{\tau}_{\mathcal{C}^\infty})_{/X_{dR}}}(X_{dR}, \mathcal{G})$$

As a corollary of our representability theorem for mapping stacks, we can show representability of the solution stack under the following conditions.

Corollary. *(Corollary 6.2.3.5) Suppose that*

1. *$Y = \mathrm{Spec}(B)$ is a system of (-1)-representable non-linear elliptic partial differential equations on sections of $\mathcal{F}$ where B is a finitely presented $\mathcal{C}^\infty$-ring,*
2. *$X = \mathrm{Spec}(A)$ is such that $\mathcal{C}^\infty(A^{red})$ is strongly dualisable and reflexive.*

Then, $\mathbf{Sol}_X(Y)$ is a (-1)-geometric stack relative to $\mathcal{C}^\infty\mathbf{DBAff}$.

Chapter 1

Exact Categories

Quillen's exact categories [71] provide an appropriate setting to generalise homological algebra from abelian categories to categories where there might not be enough kernels and cokernels. In this section, we review the main properties of exact categories that we will exploit throughout this thesis. Our main reference is the survey of Bühler [19].

We are particularly interested in the notion of a *quasi-abelian category*. This is an additive category with all kernels and cokernels where the exact structure comes from the class of all kernel-cokernel pairs. These were first introduced by Schneiders [76] with a view to developing the cohomological theory of sheaves with values in categories such as that of filtered modules or of locally convex topological vector spaces.

Categories such as the category of Banach spaces and the category of Fréchet spaces are key examples of quasi-abelian categories that arise in functional analysis. In this chapter, we define our main quasi-abelian categories of interest in this thesis, namely the category of complete bornological spaces and the category of Ind-Banach spaces.

1.1 Homological Algebra in Exact Categories

1.1.1 Definition

Suppose that E is an additive category. A *kernel-cokernel pair* (i, p) is a pair of morphisms

$$X \xrightarrow{i} Y \xrightarrow{p} Z$$

such that i is a kernel of p and p is a cokernel of i. Suppose that we have a class Q of kernel-cokernel pairs and $(i, p) \in Q$. Then, we call i an *admissible monic* and p an *admissible epic* with respect to Q.

Definition 1.1.1.1. [19, Section 2] An *exact category* (E, Q) consists of an additive category E equipped with a class Q of kernel-cokernel pairs, which we will call *short exact sequences*, such that

1. Isomorphisms are both admissible monics and admissible epics,
2. The classes of admissible monics and admissible epics are closed under composition,
3. In a pullback square
$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$
If f is an admissible epic, then f' is an admissible epic.
4. In a pushout square
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$
If f is an admissible monic, then f' is an admissible monic.

In the case when we have a *weakly idempotent complete exact category* [19, Definition 7.2] where every coretraction has a cokernel and every retraction has a kernel, we have the following important result about admissible epics and monics.

Proposition 1.1.1.2. *[19, Propositions 2.16 and 7.6](Obscure Axiom) Suppose that* E *is a weakly idempotent complete exact category. Let*

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"] & & Z \end{tikzcd}$$

be a commutative diagram in E*. If h is an admissible epic, then g is an admissible epic. Dually, if h is an admissible monic, then f is an admissible monic.*

We note that any abelian category is exact if we equip it with the class of all short exact sequences. Relaxing the requirement of being abelian, we obtain the following weaker notion.

Definition 1.1.1.3. [76, c.f. Remark 1.1.11] An additive category E is *quasi-abelian* if E is closed under kernels and cokernels, and the class of all kernel-cokernel pairs defines an exact structure on E.

We will say that a morphism $f : X \to Y$ in an additive category is *strict* if the canonical morphism $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism. In a quasi-abelian category, the admissible epics are the strict epimorphisms and the admissible monics are the strict monomorphisms. In an abelian category all morphisms are strict.

Example 1.1.1.4. *The canonical example of a category which is quasi-abelian but not abelian is the category* Ban_k *of Banach spaces over a valued field, equipped with bounded linear maps. We recall that a Banach space is a complete normed vector space. Consider the inclusion of* $C[0,1]$*, the Banach space of continuous real-valued functions on* $[0,1]$*, into* $L^1[0,1]$*, the space of Lebesgue classes of integrable real-valued functions on* $[0,1]$*. Then, this morphism is not strict in* Ban_k *because the image is dense and non-closed.*

Definition 1.1.1.5. [19, Definition 5.1] Suppose that we have an additive functor $F : \mathrm{E} \to \mathrm{E}'$ between exact categories (E, Q) and $(\mathrm{E}', \mathrm{Q}')$. Then F is *exact* if $F(\mathrm{Q}) \subseteq \mathrm{Q}'$.

Definition 1.1.1.6. An object X in an exact category E is said to be

- *projective* if the functor $\mathrm{Hom}_{\mathrm{E}}(X, -) : \mathrm{E} \to \mathrm{Mod}_{\mathbb{Z}}$ is exact,
- *injective* if the functor $\mathrm{Hom}_{\mathrm{E}}(-, X) : \mathrm{E}^{op} \to \mathrm{Mod}_{\mathbb{Z}}$ is exact,

If, in addition, E is monoidal with monoidal product $\otimes : \mathrm{E} \times \mathrm{E} \to \mathrm{E}$, then X is

- *flat* if the functor $X \otimes - : \mathrm{E} \to \mathrm{E}$ is exact.

Lemma 1.1.1.7. *[19, Proposition 11.3] An object X of E is projective if and only if every admissible epic $U \to X$ splits.*

Corollary 1.1.1.8. *Suppose that we have a short exact sequence* $X \xrightarrow{i} Y \xrightarrow{p} Z$ *in* E*. Then, if Y and Z are projective, so is X.*

Proof. Since Z is projective, then, by the previous lemma, the admissible epic $Y \xrightarrow{p} Z$ splits. Hence, $Y \simeq X \oplus Z$. Since Y is also projective, it follows that X is projective by [19, Corollary 11.6]. □

Definition 1.1.1.9. We will say that a (monoidal) exact category has *enough (flat) projectives* if, for any object Y of E, there is an admissible epic $X \twoheadrightarrow Y$ where X is a (flat) projective object of E.

Lemma 1.1.1.10. *Suppose that* E *is a weakly idempotent complete closed symmetric monoidal exact category with enough flat projectives. Then,*

1. *Every projective object of* E *is flat,*

2. *A morphism $f : X \to Y$ in* E *is an admissible epic if and only if the associated morphism*

$$\mathrm{Hom}_{\mathrm{E}}(P, X) \to \mathrm{Hom}_{\mathrm{E}}(P, Y)$$

is surjective for all flat projective generators P,

3. $\underline{\mathrm{Hom}}_{\mathrm{E}}(-, I)$ *is exact if I is injective.*

Proof. The first result follows from [40, Proposition 2.118]. The remaining results follow in a similar way to the proofs of [74, Propositions 2.22 and 2.25] using Proposition 1.1.1.2. □

We will need some conditions under which exact categories have nice generators. We recall the definition of compact from Appendix B.

Definition 1.1.1.11. [40, c.f. Definitions 2.92 and 2.97] An exact category E

1. *has a compact projective generating set* if there is a set P^0 consisting of compact projective objects in E such that, for every object X of E, there is a small coproduct $\coprod Q_i$ of objects Q_i in P^0 along with an admissible epic $\coprod Q_i \twoheadrightarrow X$,
2. is *elementary* if it is cocomplete and has a small compact projective generating set.

1.1.2 The Derived Category

We remark that familiar notions from homological algebra, for example the definition of a cochain complex, hold in any additive category. Therefore we will not redefine these notions and will instead refer to the book of Weibel [88]. We will in general refer to $\mathrm{Ch}^\bullet(\mathrm{E})$ as the category of cochain complexes, and by $\mathrm{Ch}_\bullet(\mathrm{E})$ the category of chain complexes. In later sections, for ease of notation, we will just use the notation Ch(E) to mean the category of chain complexes

We note that the category $\mathrm{Ch}^\bullet(\mathrm{E})$ of cochain complexes in an exact category E is also an exact category [19, Lemma 9.1] equipped with the class of sequences $X^\bullet \to Y^\bullet \to Z^\bullet$ such that, for each $i \in \mathbb{Z}$, $X^i \to Y^i \to Z^i$ is a short exact sequence in E.

In an exact category we don't have a well defined notion of *cohomology*, even in the situation where we have kernels and cokernels. Indeed, for a null sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

possible candidates for the cohomology at Y could be $\mathrm{Coker}(\mathrm{Im}(f) \to \mathrm{Ker}(g))$ or $\mathrm{Im}(\mathrm{Ker}(g) \to \mathrm{Coker}(f))$. These objects are isomorphic in abelian categories but not necesssarily in all exact categories, for example consider the morphism in Example 1.1.1.4.

Definition 1.1.2.1. [19, Definition 10.1] A cochain complex $X^\bullet$ in $\mathrm{Ch}^\bullet(\mathrm{E})$ is *acyclic* if each differential factors in E as some $X^n \to Z^{n+1}(X^\bullet) \to X^{n+1}$ such that each sequence $Z^n(X^\bullet) \to X^n \to Z^{n+1}(X^\bullet)$ is a short exact sequence.

Definition 1.1.2.2. [19, Definition 10.16] A cochain map $f^\bullet : X^\bullet \to Y^\bullet$ is a *quasi-isomorphism* in the homotopy category $\mathrm{K}^\bullet(\mathrm{E})$ if its mapping cone, $cone(f^\bullet)^\bullet$, [88, Section 1.5.1], is homotopy equivalent to an acyclic complex. The *derived category* $\mathrm{D}(\mathrm{E})$ is the localisation of the homotopy category $\mathrm{K}^\bullet(\mathrm{E})$ at the class of quasi-isomorphisms.

The following theorem, see [19, Theorem A.1], allows us to, in some sense, 'do homological algebra' in any exact category by just working in an abelian category.

Theorem 1.1.2.3. *Suppose that we have a small exact category* (E, Q). *Then, there is an abelian category* Ab_E *and a fully faithful exact additive functor* $I : \mathrm{E} \to \mathrm{Ab}_\mathrm{E}$ *which reflects exactness.*

Proposition 1.1.2.4. *[40, c.f. Proposition 2.43] Suppose that we have an abelianisation* $I : \mathrm{E} \to \mathrm{Ab}_\mathrm{E}$ *of an exact category* E *and that* $f^\bullet : X^\bullet \to Y^\bullet$ *is a morphism of cochain complexes. Suppose that* E *has all cokernels, then* $f^\bullet$ *is a quasi-isomorphism if and only if* $I(f^\bullet)$ *is.*

We get a stronger abelianisation result when E is a weakly idempotent complete exact category with kernels. Indeed, we can define the *left t-structure* on the homotopy category $\mathrm{K}^\bullet(\mathrm{E})$ as in [34, Section 3.1] with

$$\mathrm{K}^{\geq 0}(\mathrm{E}) := \{X^\bullet \in \mathrm{K}^\bullet(\mathrm{E}) \mid \tau^L_{\geq 0} X^\bullet \to X^\bullet \text{ is a quasi-isomorphism}\}$$

where $\tau^L_{\geq 0} X^\bullet$ is the complex whose n^{th} entry is 0 for $n < -2$, is X^n for $n \geq -1$, and is $\mathrm{Ker}(d_1^X)$ for $n = -2$. By [34, Proposition 3.5], this defines a t-structure on the derived category. We let $\mathrm{LH}(\mathrm{E})$ be the heart $\mathrm{D}^\heartsuit(\mathrm{E}) := \mathrm{D}^{\leq 0}(\mathrm{E}) \cap D^{\geq 0}(\mathrm{E})$ of this t-structure. We note that this is an abelian category. We have the following result.

Theorem 1.1.2.5. *[34, Proposition 3.11, Theorem 3.12] There is a fully faithful exact functor* $I : \mathrm{E} \to \mathrm{LH}(\mathrm{E})$ *which reflects exactness and induces an equivalence of derived categories*

$$\mathrm{D}(I) : \mathrm{D}(\mathrm{E}) \to \mathrm{D}(\mathrm{LH}(\mathrm{E}))$$

1.2 Examples of Quasi-abelian Categories

In this section, we will introduce the main categories of interest to us in this thesis, namely the category of Ind-Banach spaces and its concrete subcategory, the category of complete bornological spaces.

1.2.1 Banach Modules

Definition 1.2.1.1. A *Banach ring* is a unital commutative ring R equipped with a function $|-|: R \to \mathbb{R}_{\geq 0}$ which is positive definite, satisfies the triangle inequality, and satisfies that $|rs| \leq |r| \cdot |s|$ for all $r, s \in R$. Further, we require that R is a complete metric space with respect to the metric defined by $|-|$.

We say that R is *non-Archimedean* if $|r+s| \leq max\{|r|, |s|\}$ and is *Archimedean* otherwise. An example of a non-Archimedean Banach ring is the ring $\mathbb{Z}_p$ equipped with the p-adic norm, and an example of an Archimedean Banach ring is the ring $\mathbb{R}$ equipped with the absolute value norm.

Definition 1.2.1.2. Suppose that R is a Banach ring. A *Banach R-module* is an R-module M together with a map $||-||: M \to \mathbb{R}_{\geq 0}$ which is positive definite, satisfies the triangle inequality, and satisfies that $||rm|| \leq |r| \cdot ||m||$ for all $r \in R$ and $m \in M$. Further, we require that M is complete with respect to the metric defined by $||-||$.

Definition 1.2.1.3. A homomorphism $f : M \to N$ between Banach R-modules is *bounded* if there exists a constant $C > 0$ such that, for all $m \in M$,

$$||f(m)||_N \leq C||m||_M$$

The category of Banach modules over an Archimedean (resp. non-Archimedean) Banach ring R will be denoted Ban_R^A (resp. Ban_R^{nA}). The morphisms are bounded R-linear maps.

Suppose that R is a Banach ring. The category of Archimedean Banach R-modules, Ban_R^A, is closed symmetric monoidal [4, Proposition 3.17]. The monoidal product is the *complete Archimedean projective tensor product*, denoted by $M\hat{\otimes}_R N$. This is the completion of the R-module $M \otimes_R N$ with respect to the semi-norm given by

$$|x|_{M\otimes_R N} = \inf\left\{ \sum_{i\in I} ||m_i||_M ||n_i||_N \;\middle|\; x = \sum_{i\in I} m_i \otimes n_i,\ |I| < \infty \right\}$$

The internal hom functor $\underline{\mathrm{Hom}}_{\mathrm{Ban}_R^A}(M,N)$, for $M,N \in \mathrm{Ban}_R^A$, is given by the R-module $\mathrm{Hom}_{\mathrm{Ban}_R^A}(M,N)$ equipped with the semi-norm given by

$$|f|_{\sup} = \sup_{m \in M\setminus\{0\}} \frac{|f(m)|_N}{|m|_M}$$

If R is a non-Archimedean Banach ring, we can consider the category Ban_R^{nA} of non-Archimedean R-modules. This category is also a closed symmetric monoidal category by [4, Proposition 3.18]. The monoidal structure is given by the *complete non-Archimedean projective tensor product* which is defined to be the completion of $M \otimes_R N$ with respect to the norm given by

$$|x|_{M\otimes_R N} = \inf\{\max_{i\in I} |m_i|_M |n_i|_N \quad | \quad x = \sum_{i\in I} m_i \otimes n_i,\ |I| < \infty\}$$

The internal hom is defined as in the Archimedean case.

Proposition 1.2.1.4. *[4, Proposition 3.15, 3.18] For R an Archimedean (resp. non-Archimedean) Banach ring, the category of Banach modules over R,* Ban_R^A *(resp.* Ban_R^{nA}*), is quasi-abelian.*

If there is no ambiguity or the theory is identical, we will just use the notation Ban_R to denote Banach R-modules over any Banach ring R.

1.2.2 Ind-Banach Modules

We note that Ban_R^A and Ban_R^{nA} do not have infinite limits and colimits, see [11, Lemma A.26], which will make it difficult to work with them in this thesis. We can remedy this by constructing the categories $\mathrm{Ind}(\mathrm{Ban}_R^A)$ and $\mathrm{Ind}(\mathrm{Ban}_R^{nA})$ of Ind-Objects as discussed in Appendix C.1. We note that these categories are bicomplete closed symmetric monoidal quasi-abelian categories by Propositions C.1.0.2 and C.1.0.4.

1.2.3 Complete Bornological Spaces

Bornological spaces are spaces which possess enough structure to consider questions of boundedness, and thus are an ideal setting for bringing together homological algebra and functional analysis. Our main references are [4] and [61].

Definition 1.2.3.1. Let X be a set. A *bornology* on X is a collection $\mathcal{B}$ of subsets of X such that

- $\mathcal{B}$ covers X, i.e. for every $x \in X$, there exists some $B \in \mathcal{B}$ such that $x \in \mathcal{B}$,

- $\mathcal{B}$ is stable under inclusions, i.e. for every inclusion $A \subset B \in \mathcal{B}$, we have $A \in \mathcal{B}$,
- $\mathcal{B}$ is stable under finite unions, i.e. for each $n \in \mathbb{N}$ and $B_1, \dots, B_n \in \mathcal{B}$, we have $\bigcup_{i=1}^n B_i \in \mathcal{B}$.

The pair $(X, \mathcal{B})$ is called a *bornological set*, and the elements of $\mathcal{B}$ are called bounded subsets of X. A family of subsets $\mathcal{A} \subset \mathcal{B}$ is called a *basis* for $\mathcal{B}$ if, for any $B \in \mathcal{B}$, there exist $A_1, \dots, A_n \in \mathcal{A}$ such that $B \subset A_1 \cup \dots \cup A_n$. A *morphism of bornological sets* is any map which sends bounded subsets to bounded subsets.

Suppose that we have a complete non-trivially valued field k. The case where k is a trivially valued field is addressed in [4, Section 6].

Definition 1.2.3.2. A *bornological vector space* over k is a k-vector space V together with a bornology on the underlying set of V such that the maps $(\lambda, v) \to \lambda v$ and $(v, w) \to v + w$ are bounded.

Example 1.2.3.3. *Suppose that V is a vector space,*

1. *The fine bornology on V is the smallest possible bornology on V. A subset $B \subseteq V$ belongs to this bornology if and only if there is a finite-dimensional subspace $W \subseteq V$ such that $B \subseteq W$ and B is bounded in W,*

Suppose that V is a locally convex topological vector space,

1. *The von Neumann bornology on V is the bornology consisting of von-Neumann bounded subsets, i.e. those which are absorbed by each neighbourhood of the origin in V,*
2. *The precompact bornology on V is the bornology consisting of those subsets $B \subseteq V$ such that their closure in the completion of V is compact.*

We will now detail certain categories of bornological vector spaces. We let $k^\circ = \{\lambda \in k \mid |\lambda| \leq 1\}$. Suppose that V is a k-vector space. Then, a subset W of V is *convex* if, for every $v, w \in W$ and $t \in [0, 1]$, we have that $(1-t)v + tw \in W$, and *balanced* if, for every $\lambda \in k^\circ$, $\lambda W \subset W$. We will say that W is *absolutely convex (or a disk)* if, for k Archimedean, W is convex and balanced, and if, for k non-Archimedean, W is a k°-submodule of V.

Definition 1.2.3.4. A bornological vector space is said to be of *convex type* if it has a basis made up of absolutely convex subsets. The category of bornological k-vector spaces of convex type, equipped with bounded linear maps, will be denoted by Born_k.

Definition 1.2.3.5. A bornological vector space over k is

- *separated* if its only bounded vector subspace is the trivial subspace $\{0\}$,

- *complete* if there exists a small filtered category I, a functor $I \to \mathrm{Ban}_k$, and an isomorphism $V \simeq \varinjlim_{i\in I} V_i$ for a filtered colimit of Banach spaces over k, for which the system morphisms are all injective and the colimit is calculated in Born_k.

We denote the full subcategories of Born_k consisting of separated and complete bornological k-vector spaces by SBorn_k and CBorn_k respectively.

We note that we have fully faithful inclusion functors

$$\mathrm{CBorn}_k \hookrightarrow \mathrm{SBorn}_k \hookrightarrow \mathrm{Born}_k$$

which have left adjoints given by the *separation functor* $\mathrm{sep} : \mathrm{Born}_k \to \mathrm{SBorn}_k$ sending any space V to $V/\overline{\{0\}}$, and the completion functor $\mathrm{comp} : \mathrm{SBorn}_k \to \mathrm{CBorn}_k$ whose construction is detailed in [4, Section 3.3].

Example 1.2.3.6. *1. Any vector space endowed with the fine bornology is a complete bornological space,*

2. The von-Neumann bornology and the precompact bornology on any locally convex topological vector space V is a convex bornology. It is separated if V is Hausdorff and complete if V is complete.

We note that the category Born_k is closed symmetric monoidal. Given $V, W \in \mathrm{Born}_k$, we can endow $V \otimes_k W$ with the *projective tensor product bornology* generated by the absolutely convex hulls of subsets of the form

$$X \otimes Y = \{x \otimes y \mid x \in X, y \in Y\}$$

for bounded disks X in V and Y in W. The space $\mathrm{Hom}(V, W)$ can be endowed with a bornology provided by the *equibounded subsets*, i.e. those subsets L consisting of linear maps $f : V \to W$ such that, for each B bounded in V, the set $\{f(v) \mid f \in L, v \in B\}$ is bounded in W.

The closed monoidal structure on CBorn_k is given by the completion $\mathrm{comp}(V \otimes W)$ of the projective tensor product. The internal hom is defined as in Born_k.

Proposition 1.2.3.7. *[4, Lemma 3.53]* CBorn_k *is a bicomplete closed symmetric monoidal quasi-abelian category.*

Proposition 1.2.3.8. *[4, Proposition 3.60] Suppose that k is a valued field. There is a functor, the dissection functor,*

$$\begin{aligned} \mathrm{diss} : \mathrm{CBorn}_k &\to \mathrm{Ind}(\mathrm{Ban}_k) \\ V &\to \text{“}\varinjlim_{i\in I}\text{”} V_i \end{aligned}$$

which defines an equivalence of CBorn_k *with the subcategory of essentially monomorphic objects of* $\mathrm{Ind}(\mathrm{Ban}_k)$*.*

Under this equivalence, CBorn_k is identified with a concrete full subcategory of $\mathrm{Ind}(\mathrm{Ban}_k)$. Motivated by this, we can make the following definition over a Banach ring R,

Definition 1.2.3.9. The category CBorn_R of *complete bornological R-modules* is defined to be the full subcategory of $\mathrm{Ind}(\mathrm{Ban}_R)$ consisting of essentially monomorphic objects.

We note that CBorn_R is also a bicomplete closed symmetric monoidal quasi-abelian category. We will use the notation CBorn_R^{nA} and CBorn_R^{A} when we want to distinguish between Archimedean and non-Archimedean Banach rings. For any Banach ring R, there is an adjunction

$$\varinjlim : \mathrm{Ind}(\mathrm{Ban}_R) \leftrightarrows \mathrm{CBorn}_R : \mathrm{diss}$$

which, by [7, c.f. Proposition 3.19], provides an equivalence of categories

$$\mathrm{LH}(\mathrm{CBorn}_R) \simeq \mathrm{LH}(\mathrm{Ind}(\mathrm{Ban}_R))$$

and an equivalence of derived categories

$$\mathrm{D}(\mathrm{CBorn}_R) \simeq \mathrm{D}(\mathrm{Ind}(\mathrm{Ban}_R))$$

which preserves the monoidal structure.

1.3 Projective Generating Sets

Suppose that k is a non-trivially valued field. For any set S, we will define the set $\ell^1(S) := \{f : S \to k \mid \sum_{s\in S} |f(s)| < \infty\}$. This defines a Banach space with norm given by $||f||_1 := \sum_{s\in S} |f(s)|$. The set $\ell^1(S)$ is determined by S up to isomorphism, and hence we will also denote it by $\ell^1(\kappa)$, for κ the cardinality of S.

Lemma 1.3.0.1. *[69, c.f. Proposition 3.2.2] The category* Ban_k *has enough projectives. For every Banach space V, the space $\ell^1(X_V)$, where $X_V := \{v \in V \mid ||v|| \leq 1\}$, is a projective Banach space and there is a strict epimorphism $\ell^1(X_V) \to V$ defined by sending $f \in \ell^1(X_V)$ to $\sum_{v\in X_V} f(v)v$.*

When the field has a trivial valuation, the proof of the above result does not work. However, it is still true that the category Ban_k has enough projectives. Indeed, more generally, it is true for Ban_R for any Banach ring R, as described in [4, Definition 3.24]. If R is a non-Archimedean Banach ring and $M \in \mathrm{Ban}_R^{nA}$, then we can define the projective Banach R-module

$$P^{nA}(M) = \{f : M^{\times} \to R \mid \varprojlim_{m \in M^{\times}} ||f(m)m|| = 0\}$$

where $M^{\times} := M \backslash \{0\}$. The norm is defined to be $||f|| = \sup_{m \in M^{\times}} ||f(m)m||$. There is a strict epimorphism $P^{nA}(M) \to M$ defined by sending $f \in P^{nA}(M)$ to $\sum_{m \in M^{\times}} f(m)m$. If R is Archimedean, then we can define the projective Banach R-module

$$P^{A}(M) = \{f : M^{\times} \to R \mid \sum_{m \in M^{\times}} ||f(m)m|| < \infty\}$$

with the norm defined by $||f|| = \sum_{m \in M^{\times}} ||f(m)m||$. There is a strict epimorphism $P^{nA}(M) \to M$ defined by sending $f \in P^{nA}(M)$ to $\sum_{m \in M^{\times}} f(m)m$.

Moreover, by [4, Lemma 3.26], these objects are flat, and hence Ban_R^{A} and Ban_R^{nA} have enough flat projectives. It follows that $\mathrm{Ind}(\mathrm{Ban}_R^{A})$ and $\mathrm{Ind}(\mathrm{Ban}_R^{nA})$ also have enough flat projectives by Proposition C.1.0.4.

By definition of the projective objects in $\mathrm{Ind}(\mathrm{Ban}_R^{A})$ and $\mathrm{Ind}(\mathrm{Ban}_R^{nA})$, we easily note that there are also enough flat projective objects in the subcategories CBorn_R^{A} and CBorn_R^{nA} of essentially monomorphic objects.

As explained in [41, Section 4.2.1], the projective generators of Ban_R^{nA} and Ban_R^{A} provide a set of compact projective generators for the Ind-completions. Hence, we see that $\mathrm{Ind}(\mathrm{Ban}_R^{nA})$ and $\mathrm{Ind}(\mathrm{Ban}_R^{A})$ are elementary quasi-abelian categories in the sense of Definition 1.1.1.11.

We want to study these compact projective generators in more detail. Suppose that E is a complete elementary exact category with enough projectives. Then, we can consider the *projective model structure* as described in [40, Theorem 4.65] where the weak equivalences are quasi-isomorphisms, and the fibrations are degree-wise admissible epics.

Theorem 1.3.0.2. *[40, Theorem 4.77] (Model Dold-Kan Correspondence) If we endow* $\mathrm{Ch}_{\geq 0}(\mathrm{E})$ *and the category of simplicial objects* sE *in* E *with their projective model structures, then there is a Quillen equivalence of model categories between* $\mathrm{Ch}_{\geq 0}(\mathrm{E})$ *and* sE.

We recall that a t-structure on a model category is a t-structure on the homotopy category. We define the left t-structure $(\mathrm{Ch}_{\geq 0}(\mathrm{E}), \mathrm{Ch}_{\leq 0}(\mathrm{E}))$ on $\mathrm{Ch}(\mathrm{E})$ similarly to how we defined the left t-structure on the homotopy category in Section 1.1.2, and denote by $\mathrm{LH}(\mathrm{E})$ its heart. We note that, by [41, Corollary 4.23], this corresponds to the projective t-structure on $\mathrm{Ch}(\mathrm{E})$.

Proposition 1.3.0.3. *Suppose that* E *is a complete elementary exact category. Let* P^0 *be a set of compact projective generators closed under finite coproducts. Then, there is an equivalence of categories*

$$\mathrm{SInd}(\mathrm{P}^0) \simeq \mathrm{LH}(\mathrm{E})$$

where $\mathrm{SInd}(-)$ *denotes the free sifted cocompletion, as described in Appendix C.2.*

Proof. By our assumptions, we see that E has enough projectives. We note that P^0 is a set of fibrant-cofibrant compact projective generators for the model category $\mathrm{Ch}_{\geq 0}(\mathrm{E})$. Therefore, by [10, Lemma A.1.1] and the Dold-Kan correspondence, there is a Quillen equivalence of categories

$$\mathrm{sSInd}(\mathrm{P}^0) \simeq \mathrm{Ch}_{\geq 0}(\mathrm{E})$$

where $\mathrm{sSind}(\mathrm{P}^0)$ denotes the simplicial objects in $\mathrm{SInd}(\mathrm{P}^0)$. Therefore, by taking the 0-truncation, we get our desired equivalence. □

Remark. We note that our categories CBorn_R and $\mathrm{Ind}(\mathrm{Ban}_R)$ satisfy the conditions of the previous proposition.

We recall from Lemma 1.3.0.1 that, when k is a non-trivially valued field, we can describe the projective generators of Ban_k as the collection of objects of the form $\ell^1(S)$ for some set S. If $V \in \mathrm{Ind}(\mathrm{Ban}_k)$, then V is a formal filtered colimit "$\varinjlim_{i\in I}$"V_i of Banach spaces V_i over k. Hence, there exists a chain of strict epimorphisms

$$\bigoplus_{i\in I} \ell^1(X_{V_i}) \to \bigoplus_{i\in I} V_i \to \text{“}\varinjlim_{i\in I}\text{”} V_i \simeq V$$

with the direct sum taken in $\mathrm{Ind}(\mathrm{Ban}_k)$.

Consider the set of cardinals κ such that there exists a Banach space V containing a bounded unit disk of cardinality κ. This set is non-empty and so has a supremum which is also a cardinal. We will denote this by $\aleph$. We consider the small subcategory $\mathrm{Lin}_k \subseteq \mathrm{Ind}(\mathrm{Ban}_k)$ whose objects are all of the form $\ell^1(\kappa)$ for $\kappa < \aleph$, and whose morphisms are bounded linear maps.

Lemma 1.3.0.4. *The underlying set of* Lin_k *is a set of compact projective generators for* $\mathrm{Ind}(\mathrm{Ban}_k)$ *closed under finite direct sums and tensor products.*

Proof. Indeed, by our above reasoning, we know that this provides a set of compact projective generators. It is clearly closed under direct sums. For cardinals κ, μ, we have that $\ell^1(\kappa)\hat{\otimes}\ell^1(\mu) \simeq \ell^1(\kappa \times \mu)$ as Banach spaces, and hence also as Ind-Banach spaces by definition of the monoidal product in $\mathrm{Ind}(\mathrm{Ban}_k)$. $\square$

The following result is clear from Proposition 1.3.0.3.

Corollary 1.3.0.5. *There are equivalences of categories*

$$\mathrm{LH}(\mathrm{CBorn}_k) \simeq \mathrm{LH}(\mathrm{Ind}(\mathrm{Ban}_k)) \simeq \mathrm{SInd}(\mathrm{Lin}_k)$$

Chapter 2

Koszul Monoids in Exact Categories

Suppose that we have a finite dimensional vector space V over a field k. Then, there exists a projective resolution of k

$$\cdots \to \mathrm{Sym}(V) \otimes_k \bigwedge^2 V \to \mathrm{Sym}(V) \otimes_k V \to \mathrm{Sym}(V) \twoheadrightarrow k$$

where $\mathrm{Sym}(V)$ denotes the symmetric algebra and $\bigwedge^n V$ denotes the n^{th} graded component of the exterior algebra. In 1978, Bernstein, Gelfand, and Gelfand [15] showed the following equivalence of bounded derived categories over the categories of graded $\mathrm{Sym}(V)$ and $\bigwedge V^*$ modules.

$$\mathrm{D}^b(\mathrm{gr}_{\mathrm{Sym}(V)}\mathrm{Mod}) \simeq \mathrm{D}^b(\mathrm{gr}_{\bigwedge V^*}\mathrm{Mod})$$

Beilinson, Ginzburg, and Soergel, in their seminal paper [8] of 1996, extend these constructions to a more general collection of graded rings known as Koszul rings, of which $\mathrm{Sym}(V)$ and $\bigwedge V$ are toy examples. In particular, these Koszul rings are examples of *quadratic rings*, rings generated by degree one elements with relations of degree 2.

In my publication, *Koszul Monoids in Quasi-abelian Categories* [74], I show that most of the results of Beilinson, Ginzburg, and Soergel generalise over to algebra objects in a bicomplete closed symmetric monoidal quasi-abelian category E with enough flat projectives, for example the categories CBorn_R and $\mathrm{Ind}(\mathrm{Ban}_R)$. My work provides an extension of their work to graded algebra objects in categories other than the category of rings with semisimple 0-part, and also extends their results to a wider class of graded rings called *pre-Koszul rings*.

In the following chapter, I will present an abridged and generalised version of my paper [74] with numerous results and proofs omitted and instead cited to make for a more concise exposition. Instead of quasi-abelian categories, we will work in the more general setting of exact categories. We will fix for the rest of this chapter a bicomplete closed symmetric monoidal exact category E with enough flat projectives. Several examples of such categories are given in the previous chapter.

2.1 Koszul Monoids

2.1.1 Graded Monoids

Suppose that A is a monoid in the monoidal category E, i.e. an object A equipped with multiplication and unit morphisms satisfying natural associativity and unit conditions. Then, by [40, Section 6.1.2], we see that there is a free-forgetful adjunction

$$A \otimes - : \mathrm{E} \leftrightarrows {}_A\mathrm{Mod} : \mathrm{Forget}$$

between E and the category ${}_A\mathrm{Mod}$ of left A-modules. We note that, since E is cocomplete, there is an induced closed symmetric monoidal structure on ${}_A\mathrm{Mod}$ as described in [74, Section 1.4]. Moreover, by [40, Section 2.4], there is an induced exact structure on ${}_A\mathrm{Mod}$ where the class of short exact sequences consists of those such that their image under the forgetful functor defines a short exact sequence in E. The category ${}_A\mathrm{Mod}$ is bicomplete, with limits and colimits computed in E. The following result follows in the same way as [74, Corollary 3.3].

Corollary 2.1.1.1. *P is a (flat) projective object in* E *if and only if $A \otimes P$ is a (flat) projective object in ${}_A\mathrm{Mod}$. Moreover, ${}_A\mathrm{Mod}$ has enough flat projectives.*

Definition 2.1.1.2.
1. A ($\mathbb{Z}$)-*graded monoid* A in E consists of a family of objects $\{A_i\}_{i\in\mathbb{Z}}$ in E, a unit morphism $\eta : I \to A_0$ and, for all $i, j \in \mathbb{Z}$, a multiplication morphism $\mu_{i,j} : A_i \otimes A_j \to A_{i+j}$. The multiplication and unit satisfy natural associativity and unit conditions,
2. A *graded left A-module* M over a graded monoid A in E consists of a family of objects $\{M_i\}_{i\in\mathbb{Z}}$ in E and, for all $i, j \in \mathbb{Z}$, morphisms $\lambda_{i,j} : A_i \otimes M_j \to M_{i+j}$. The action maps satisfy natural associativity and unit conditions.

We note that we can apply similar reasoning to before to deduce that there is an exact structure on the category $\mathrm{gr}_A\mathrm{Mod}$ of graded left A-modules with exact sequences defined in each grading. We also see that it is bicomplete closed symmetric monoidal by [74, Corollary 1.26] and has enough projectives using similar reasoning to [74, Lemma 3.10] and the following result.

Proposition 2.1.1.3. *Suppose that M is a graded left A-module over a positively graded monoid A. Then, there exists a map $A \otimes_{A_0} M \to M$ which is a graded admissible epic. Moreover, if M is a projective left A_0-module, then $A \otimes_{A_0} M$ is a projective left A-module.*

Proof. This follows in the same way as [74, Proposition 3.8, 3.9] using Proposition 1.1.1.2 and the extension of scalars adjunction from [40, Section 6.1.2].

□

2.1.2 Definition of Koszul Monoids

For a positively graded monoid A in E, we let $A_{>0} := \bigoplus_{i>0} A_i$. We note that $A_{>0}$ is isomorphic to the kernel of the projection map $\pi : A \to A_0$. We can consider A_0 as a graded left A-module with, for each j, the action $A_j \otimes A_0 \to A_0$ defined to be the composition

$$A_j \otimes A_0 \xrightarrow{\mu_{j,0}} A_j \hookrightarrow A \twoheadrightarrow \mathrm{Coker}(A_{>0} \xrightarrow{\iota} A) \simeq A_0$$

As graded left A-modules, we have a short exact sequence $A_{>0} \to A \to A_0$.

Definition 2.1.2.1. We will say that a graded left A-module M is

1. *generated by its degree i component over A_0* if the map $A \otimes_{A_0} M_i \twoheadrightarrow M$ is an admissible epic,
2. *pure of weight n* if it is concentrated in degree n, i.e. $M = M_n$.

For a graded left A-module M, we define the grading shifts by $(M\langle n\rangle)_i = M_{i-n}$. We can consider A_0 to be a pure graded left A-module concentrated in degree 0 by defining the action of A_i on A_0 to be 0 unless $i = 0$. We remark that, for any A-modules M and N pure of weights m, n respectively, $\mathrm{Hom}_{\mathrm{gr}_A\mathrm{Mod}}(M, N) = 0$ unless $m = n$.

Proposition 2.1.2.2. *If M is an A-module generated by its degree i component over A_0 and N is pure of weight n, then $\mathrm{Hom}_{\mathrm{gr}_A\mathrm{Mod}}(M, N) = 0$ unless $i = n$.*

Proof. We note that, since there is an admissible epic $A \otimes_{A_0} M_i \twoheadrightarrow M$, there is an injection

$$\mathrm{Hom}_{\mathrm{gr}_A\mathrm{Mod}}(M, N) \hookrightarrow \mathrm{Hom}_{\mathrm{gr}_A\mathrm{Mod}}(A \otimes_{A_0} M_i, N)$$

The result then follows since $\mathrm{Hom}_{\mathrm{gr}_A\mathrm{Mod}}(A \otimes_{A_0} M_i, N) \simeq \mathrm{Hom}_{\mathrm{gr}_{A_0}\mathrm{Mod}}(M_i, N_n) = 0$ if $i \neq n$. □

The following notion should not be confused with the definition of pre-Koszul given by Priddy [68, Section 2].

Definition 2.1.2.3. We will say that a positively graded monoid A is *pre-Koszul* if A_0 is injective as a module over itself, each A_i is projective as an A_0-module, and, for any graded A-module M living only in degrees $\geq i$, whenever

$$\mathrm{Hom}_{\mathrm{gr}_A\mathrm{Mod}}(M, A_0\langle n\rangle) = 0$$

unless $n = i$, then M is generated by its i^{th} component over A_0.

Example 2.1.2.4. *If we work in the category of rings, then if A_0 is a semisimple ring, then A is pre-Koszul by [74, Proposition 4.5]. However, if A is pre-Koszul in the category of rings, then A_0 is not necessarily semisimple. Indeed, not every module over A_0 is guaranteed to be projective. See [32, Example 4.4] for an example of a ring A such that A_0 is not semisimple but A is pre-Koszul.*

Definition 2.1.2.5. A *Koszul monoid* in E is a positively graded pre-Koszul monoid A such that A_0, considered as a graded left A-module, admits a graded projective resolution of A-modules.

$$\cdots \to P^2 \to P^1 \to P^0 \twoheadrightarrow A_0$$

with each P^i generated by its degree i component over A_0.

Remark. The condition on P^i says that the diagonal part of the resolution generates the rest.

Example 2.1.2.6. *We note that in the category of abelian groups, any Koszul ring in the sense of [8, Definition 1.1.2] is a Koszul monoid.*

Example 2.1.2.7. *Suppose that we have a Koszul ring R over a field k in the sense of [8, Definition 1.1.2]. We consider the fine bornology on its underlying vector space such that R becomes a complete bornological ring [61, Example 1.27]. Suppose further that k is self-injective in* CBorn_k*, e.g. $k = \mathbb{R}, \mathbb{C}$. Then, R is a Koszul monoid in* CBorn_k *since the functor* $\mathrm{Fin} : \mathrm{Vect}_k \to \mathrm{CBorn}_k$*, endowing any vector space with its fine bornology, is fully faithful, exact, and left adjoint to the forgetful functor [61, Example 1.27, 1.77].*

2.1.3 Projective Resolutions

We now discuss in what sense a Koszul monoid is 'as close to being semisimple' as a graded monoid can be. For the rest of this section, we fix a positively graded monoid A in E.

Proposition 2.1.3.1. *Let $M \in \mathrm{gr}_A\mathrm{Mod}$ be a projective left A_0-module living only in degrees $\geq n$. Then, M admits a graded projective resolution of A-modules*

$$\cdots \to P^2 \to P^1 \to P^0 \twoheadrightarrow M$$

such that P^i lives only in degrees $\geq n+i$. So, $P^i = \bigoplus_{j \geq n+i} P^i_j$.

Proof. We may assume, without any loss of generality, that $n = 0$. We consider the module $P^0 = A \otimes_{A_0} M$. This is a graded projective A-module by Proposition 2.1.1.3 and, moreover, it lives only in positive degree. There also exists a graded admissible epic $d^0 : P^0 \to M$. Therefore, there must be a short exact sequence of graded A-modules

$$0 \to K^0 \xrightarrow{\iota^0} P^0 \xrightarrow{d^0} M \to 0$$

with $K^0 = \mathrm{Ker}(d^0)$, a graded A_0-module, and $\iota^0 : K^0 \to P^0$ the inclusion map. Since $P^0_0 = A_0 \otimes_{A_0} M_0 \simeq M_0$ we see that, in degree 0, d^0 is a monomorphism. Hence, K^0 lives only in degree ≥ 1.

We now let $P^1 = A \otimes_{A_0} K^0$. We note that K^0 is a projective A_0-module by Corollary 1.1.1.8. Hence, P^1 is a projective A-module by Proposition 2.1.1.3. We note that P^1 is graded and lives only in degrees ≥ 1. We construct a map $d^1 : P^1 \to P^0$ as the composition of the map $P^1 \to K^0$ and the inclusion map $\iota^0 : K^0 \to P^0$. We have an exact sequence

$$P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} M \to 0$$

We can continue in this way to construct our desired projective resolution. Indeed, suppose we have an exact sequence of graded projective A-modules defined up to degree i

$$P^i \xrightarrow{d^i} \ldots \to P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} M \to 0$$

with each P^j living only in degree $\geq j$. Suppose also that the sequence is constructed such that, for each j, $P^j = A \otimes_{A_0} K^{j-1}$ with K^{j-1} the graded A_0-module $\mathrm{Ker}(d^{j-1})$ where $d^{j-1} : P^{j-1} \twoheadrightarrow \mathrm{Im}(d^{j-1}) \simeq K^{j-2}$. We let $K^{i+1} = \mathrm{Ker}(d^{i+1})$, which is a projective A_0-module, and define $P^{i+1} = A \otimes_{A_0} K^i$. This is a projective graded A-module living only in degrees $\geq i+1$. The differential $d^{i+1} : P^{i+1} \to P^i$ is defined to be the composition of the map $P^{i+1} = A \otimes_{A_0} K^i \to K^i$ and the inclusion $K^i \to P^i$. The extended sequence

$$P^{i+1} \xrightarrow{d^{i+1}} P^i \xrightarrow{d^i} \ldots \to P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} M \to 0$$

is exact.

□

Since $\mathrm{gr}_A\mathrm{Mod}$ has enough projectives, $\mathrm{Hom}_{\mathrm{gr}_A\mathrm{Mod}}(-, N)$ is right derivable by [19, Remark 12.11]. We call the right derived functor $\mathrm{Ext}_{\mathrm{gr}_A\mathrm{Mod}}(-, N)$.

Corollary 2.1.3.2. *Let $M, N \in \mathrm{gr}_A\mathrm{Mod}$ be pure of weights m, n, with M projective as a left module over A_0. Then, $\mathrm{Ext}^i_{\mathrm{gr}_A\mathrm{Mod}}(M, N) = 0$ for $i > n - m$.*

Proof. Without loss of generality we may assume $m = 0$. By Proposition 2.1.3.1, M admits a graded projective resolution of A-modules

$$\cdots \to P^2 \to P^1 \to P^0 \twoheadrightarrow M$$

such that each P^i lives only in degrees $\geq i$, i.e. $P^i = \bigoplus_{j \geq i} P^i_j$. We examine the complex $\mathrm{Hom}_{\mathrm{gr}_A\mathrm{Mod}}(P^\bullet, N)$ which has objects

$$\mathrm{Hom}_{\mathrm{gr}_A\mathrm{Mod}}(P^i, N) = \mathrm{Hom}_{\mathrm{gr}_A\mathrm{Mod}}\left(\bigoplus_{j \geq i} P^i_j, N\right)$$

Since N is pure of weight n, $\mathrm{Hom}_{\mathrm{gr}_A\mathrm{Mod}}(\bigoplus_{j \geq i} P^i_j, N) = 0$ for $i > n$. Therefore, we have that $\mathrm{Ext}^i_{\mathrm{gr}_A\mathrm{Mod}}(M, N) = 0$ for $i > n$. □

Proposition 2.1.3.3. *If A is a Koszul monoid then, for any pure left A-module M of weight n, we have $\mathrm{Ext}^i_{\mathrm{gr}_A\mathrm{Mod}}(A_0, M) = 0$ unless $i = n$.*

Proof. Suppose that A is a Koszul monoid. Then, by definition, A_0 admits a graded projective resolution of A-modules.

$$\cdots \to P^2 \to P^1 \to P^0 \twoheadrightarrow A_0$$

with each P^i generated by its degree i component over A_0. We note that the i^{th} cohomology object of the complex $\mathrm{Hom}_{\mathrm{gr}_A\mathrm{Mod}}(P^\bullet, M)$ is $\mathrm{Ext}^i_{\mathrm{gr}_A\mathrm{Mod}}(A_0, M)$. But, since P^i is generated by its degree i component and M is pure of weight n, so $M = M_n$, we see that all the terms in the complex $\mathrm{Hom}_{\mathrm{gr}_A\mathrm{Mod}}(P^\bullet, M)$ are zero other than $\mathrm{Hom}_{\mathrm{gr}_A\mathrm{Mod}}(P^n, M)$. Thus, $\mathrm{Ext}^i_{\mathrm{gr}_A\mathrm{Mod}}(A_0, M) = 0$ unless $i = n$. □

Lemma 2.1.3.4. *Suppose that there exists a graded projective exact sequence of A-modules*

$$P^i \xrightarrow{d^i} \ldots \to P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} M \to 0$$

with P^i generated by its degree i component over A_0. Then, if $K^i := \mathrm{Ker}(d^i)$,

$$\mathrm{Ext}^{i+1}_{\mathrm{gr}_A\mathrm{Mod}}(M, N) = \mathrm{Hom}_{\mathrm{gr}_A\mathrm{Mod}}(K^i, N)$$

for any pure $N \in \mathrm{gr}_A\mathrm{Mod}$.

Proof. By dimension shifting, we see that there is an exact sequence

$$\to \mathrm{Hom}_{\mathrm{gr}_A\mathrm{Mod}}(P^i, N) \to \mathrm{Hom}_{\mathrm{gr}_A\mathrm{Mod}}(K^i, N) \to \mathrm{Ext}^{i+1}_{\mathrm{gr}_A\mathrm{Mod}}(M, N) \to 0$$

Therefore,

$$\mathrm{Ext}^{i+1}_{\mathrm{gr}_A\mathrm{Mod}}(M, N) \simeq \mathrm{Coker}(\mathrm{Hom}_{\mathrm{gr}_A\mathrm{Mod}}(P^i, N) \to \mathrm{Hom}_{\mathrm{gr}_A\mathrm{Mod}}(K^i, N))$$

Now, since P^i is generated by its degree i component over A_0 and K^i lives only in degrees $\geq i+1$, we see that $\mathrm{Hom}_{\mathrm{gr}_A\mathrm{Mod}}(P^i, N) \to \mathrm{Hom}_{\mathrm{gr}_A\mathrm{Mod}}(K^i, N)$ is the zero map, and hence

$$\mathrm{Ext}^{i+1}_{\mathrm{gr}_A\mathrm{Mod}}(M, N) \simeq \mathrm{Hom}_{\mathrm{gr}_A\mathrm{Mod}}(K^i, N)$$

□

Proposition 2.1.3.5. *If A is a pre-Koszul monoid and $\mathrm{Ext}^i_{\mathrm{gr}_A\mathrm{Mod}}(A_0, A_0\langle n\rangle) = 0$ unless $i = n$, then A is Koszul.*

Proof. We will construct a projective resolution $P^\bullet$ for A_0 using a similar method to Proposition 2.1.3.1. For A to be Koszul, we want each of our modules P^i to be generated in degree i over A_0. As before, we can take $P^0 = A$. Suppose that the resolution in question is constructed up to degree i

$$P^i \xrightarrow{d^i} \ldots \to P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} A_0 \to 0$$

with each P^j a graded projective A-module generated by its degree j component over A_0 and with d^j a monomorphism in degree j. We consider $K^i := \mathrm{Ker}(d^i)$. This is a projective left A_0-module living only in degrees $\geq i+1$. Since P^i is generated in degree i and $A_0\langle n\rangle$ is pure of weight n, we see that, by Lemma 2.1.3.4, for each n,

$$\mathrm{Ext}^{i+1}_{\mathrm{gr}_A\mathrm{Mod}}(A_0, A_0\langle n\rangle) = \mathrm{Hom}_{\mathrm{gr}_A\mathrm{Mod}}(K^i, A_0\langle n\rangle)$$

which, by assumption, is zero for $i+1 \neq n$. Hence, since K^i lives only in degree $\geq i+1$, K^i is generated by its $(i+1)^{th}$-component because A is pre-Koszul. We take P^{i+1} to be the graded projective A-module $A \otimes_{A_0} K^i_{i+1}$ and note that it is clearly generated by its $(i+1)^{th}$-component.

□

Proposition 2.1.3.6. *If A is a pre-Koszul monoid, then it is a Koszul monoid if and only if* $\mathrm{Ext}^i_{\mathrm{gr}_A\mathrm{Mod}}(A_0, A_0\langle n\rangle) = 0$ *unless $i = n$.*

Proof. This is clear from the two propositions above. □

Definition 2.1.3.7. If A is a Koszul monoid, we define the *Koszul complex* to be the sequence

$$\cdots \to P^3 \to P^2 \to P^1$$

constructed in Proposition 2.1.3.5. This complex gives a resolution of A_0.

2.2 Quadratic Monoids

Quadratic algebras are algebras generated by degree one elements with relations of degree two. In this section, we generalise this idea to obtain what we call *quadratic monoids*.

2.2.1 The Tensor Monoid

Suppose that A is a monoid in E. Then, we recall the definition of the tensor monoid $T(A)$ from Appendix A. If A is a monoid and M is an (A, A)-bimodule, we can define the tensor module $T_A(M) = A \oplus M \oplus (M \otimes_A M) \oplus \ldots$.

Lemma 2.2.1.1. *If $T_{A_0}(A_1)$ is a pre-Koszul monoid, then it is Koszul.*

Proof. If $T_{A_0}(A_1)$ is pre-Koszul, then A_1 is projective over A_0, and hence A_0 admits a projective resolution $0 \to T_{A_0}(A_1) \otimes_{A_0} A_1 \to T_{A_0}(A_1) \to A_0 \to 0$ as a $T_{A_0}(A_1)$-module.

□

Example 2.2.1.2. *Suppose that R_0 is a semisimple Banach algebra over a non-trivially valued field k and suppose that R_0 is injective over itself as an element of* CBorn_k*, for example $R_0 = \mathbb{C}$ in* $\mathrm{CBorn}_{\mathbb{C}}$*. Suppose further that R_1 is a projective R_0-module, e.g. $R_1 = \ell^1(\mathbb{C}) := \{(c_i)_{i \in \mathbb{N}-\{0\}} \mid c_i \in \mathbb{C}, \sum_{i\in\mathbb{N}-\{0\}} ||c_i|| < \infty\}$. We note* *that R_1 is also projective as an R_0-module when considered in* CBorn_k*. Then, we can consider the tensor algebra $T_{R_0}(R_1)$ in* CBorn_k*. To show that $T_{R_0}(R_1)$ is pre-Koszul it suffices to prove the hom condition. Suppose that $M = \bigoplus_{k\geq i}$ "$\varinjlim_{j\in J}$"$(M_j)_k$ is a graded $T_{R_0}(R_1)$-module in* CBorn_k *living only in degrees $\geq i$ and that*

$$\mathrm{Hom}_{\mathrm{gr}_{T_{R_0}(R_1)}\mathrm{Mod}(\mathrm{CBorn}_k)}(M, R_0\langle n\rangle) = 0$$

unless $n = i$. Then, we see that, identifying CBorn_k *with the full subcategory of essentially monomorphic objects in* $\mathrm{Ind}(\mathrm{Ban}_k)$*,*

$$\bigoplus_{k\geq i} \varprojlim_{j\in J} \mathrm{Hom}_{T_{R_0}(R_1)\mathrm{Mod}(\mathrm{Ban}_k)}((M_j)_k, R_0\langle n\rangle) = 0$$

unless $n = i$. Hence, for each $k \geq i$,

$$\varprojlim_{j\in J} \mathrm{Hom}_{T_{R_0}(R_1)\mathrm{Mod}(\mathrm{Ban}_k)}((M_j)_k, R_0\langle n\rangle) = 0$$

unless $n = i$. Since $M \in$ CBorn_k*, each of the system morphisms in M are monomorphisms. Hence, the morphisms in the system* $(\mathrm{Hom}_{T_{R_0}(R_1)\mathrm{Mod}(\mathrm{Ban}_k)}((M_j)_k, R_0\langle n\rangle))_{j\in J}$ *must be epimorphisms. If the inverse limit of these abelian groups is zero, then*

$$\mathrm{Hom}_{T_{R_0}(R_1)\mathrm{Mod}(\mathrm{Ban}_k)}((M_j)_k, R_0\langle n\rangle) = 0$$

unless $n = i$. Since R_0 is semisimple, we can use a similar reasoning to the proof of [74, Proposition 4.5], noting that, by [11, Lemma A.29], a strict epimorphism of Banach spaces is equivalently a surjection, to show that, for all $j \in J$, there is a strict epimorphism

$$T_{R_0}(R_1) \otimes_{R_0} (M_j)_i \to M_j$$

in ${}_{T_{R_0}(R_1)}\mathrm{Mod}(\mathrm{Ban}_k)$*. Hence, by [4, Proposition 2.10], there is a strict epimorphism*

$$T_{R_0}(R_1) \otimes_{R_0} \text{“}\varinjlim_{j\in J}\text{”}(M_j)_i \simeq \text{“}\varinjlim_{j\in J}\text{”} T_{R_0}(R_1) \otimes_{R_0} (M_j)_i \to \text{“}\varinjlim_{j'\in J}\text{”} M_{j'}$$

in $\mathrm{gr}_{T_{R_0}(R_1)}\mathrm{Mod}(\mathrm{CBorn}_k)$*.*

If A is positively graded, we can consider A_1 as an (A_0, A_0)-bimodule. There is a canonical morphism $\pi : T_{A_0}(A_1) \to A$ formed by 'linearly' extending the multiplication $\mu_i : A_1^{\otimes_{A_0} i} \to A$ for all $i \in \mathbb{N}$. We make the following definition.

Definition 2.2.1.3. We say that A is a *quadratic monoid* with quadratic data (A_1, R) if A is pre-Koszul and there exists a graded admissible epic $\pi : T_{A_0}(A_1) \twoheadrightarrow A$ such that there exists an admissible epic

$$T_{A_0}(A_1) \otimes_{A_0} R \otimes_{A_0} T_{A_0}(A_1) \twoheadrightarrow \mathrm{Ker}(\pi)$$

with $R = K_2 := \mathrm{Ker}(A_1 \otimes_{A_0} A_1 \twoheadrightarrow A_2)$.

Remark. We see that $A \simeq \mathrm{Coker}(\mathrm{Ker}(\pi) \to T_{A_0}(A_1))$ with $\mathrm{Ker}(\pi)$ generated by R. By some abuse of notation, we denote this quadratic monoid by $A = T_{A_0}(A_1)/(R)$. We note that A is in some sense generated by A_1 over A_0 with relations of degree two.

Example 2.2.1.4. *If the tensor monoid $T_{A_0}(A_1)$ is pre-Koszul, then it is quadratic with $R = 0$.*

We fix for the rest of this section a positively graded pre-Koszul monoid A. We will say that A is a *quotient of* $T_{A_0}(A_1)$ if there exists an admissible epic $T_{A_0}(A_1) \twoheadrightarrow A$.

Lemma 2.2.1.5. *If A is a quotient of $T_{A_0}(A_1)$, then there exists an admissible epic $A \otimes_{A_0} A_1 \twoheadrightarrow A_{>0}$ given, in each degree $i > 0$, by the map $\lambda_{i-1,1} : A_{i-1} \otimes_{A_0} A_1 \to A_i$.*

Proof. It suffices to show that $\lambda_{i-1,1}$ is an admissible epic for each i. We see that the following diagram commutes

$$\begin{tikzcd} & A_{i-1} \otimes_{A_0} A_1 \arrow[dr, "\lambda_{i-1,1}"] & \\ A_1^{\otimes_{A_0} i} \arrow[ur] \arrow[rr, "\mu_i"] & & A_i \end{tikzcd}$$

Since A is a quotient of $T_{A_0}(A_1)$, we see that μ_i is an admissible epic. Therefore, $\lambda_{i-1,1}$ is also an admissible epic by Proposition 1.1.1.2. □

Proposition 2.2.1.6. *For any pure $M \in \mathrm{gr}_A\mathrm{Mod}$,*

$$\mathrm{Ext}^1_{\mathrm{gr}_A\mathrm{Mod}}(A_0, M) = \mathrm{Hom}_{\mathrm{gr}_A\mathrm{Mod}}(A_{>0}, M)$$

Proof. We consider the graded projective exact sequence of A-modules $A \to A_0 \to 0$. We note that A is generated by its degree 0 component over A_0 and that $A_{>0} = \mathrm{Ker}(A \to A_0)$. Therefore, by Lemma 2.1.3.4, $\mathrm{Ext}^1_{\mathrm{gr}_A\mathrm{Mod}}(A_0, M) = \mathrm{Hom}_{\mathrm{gr}_A\mathrm{Mod}}(A_{>0}, N)$. □

Corollary 2.2.1.7. *If $\mathrm{Ext}^1_{\mathrm{gr}_A\mathrm{Mod}}(A_0, A_0\langle n\rangle) = 0$ unless $n = 1$, then there exists an admissible epic $A \otimes_{A_0} A_1 \to A_{>0}$. Moreover, A is a quotient of $T_{A_0}(A_1)$.*

Proof. Consider $A_0\langle n\rangle$ as a pure graded A-module of weight n. By the previous proposition, we see that $\mathrm{Hom}_{\mathrm{gr}_A\mathrm{Mod}}(A_{>0}, A_0\langle n\rangle) = 0$ unless $n = 1$. Then, since A is pre-Koszul, $A_{>0}$ is generated by its component in degree 1, so there exists an admissible epic $A \otimes_{A_0} A_1 \twoheadrightarrow A_{>0}$. Hence, for each i, there exists an admissible epic $A_{i-1} \otimes_{A_0} A_1 \twoheadrightarrow A_i$. For each $i > 0$, we can construct a chain of admissible epics

$$A_1^{\otimes i} = (A_1 \otimes_{A_0} A_1) \otimes_{A_0} A_1^{\otimes i-2} \twoheadrightarrow A_2 \otimes A_1^{\otimes i-2} \twoheadrightarrow \ldots \twoheadrightarrow A_i$$

and hence there exists an admissible epic $T_{A_0}(A_1) \twoheadrightarrow A$. □

Proposition 2.2.1.8. *Let A be a quotient of $T_{A_0}(A_1)$. If $\mathrm{Ext}^2_{\mathrm{gr}_A\mathrm{Mod}}(A_0, A_0\langle n\rangle) = 0$ unless $n = 2$, then A is a quadratic monoid.*

Proof. It suffices to show that there is an admissible epic

$$T_{A_0}(A_1) \otimes_{A_0} R \otimes_{A_0} T_{A_0}(A_1) \twoheadrightarrow \mathrm{Ker}(\pi)$$

where $R = K_2$. Let $K = \mathrm{Ker}(\pi)$. We note that, since the 0^{th} and 1^{st} components of $T_{A_0}(A_1)$ are A_0 and A_1 respectively, then K only exists in degree ≥ 2. We have an exact sequence

$$0 \to K \xrightarrow{\iota} T_{A_0}(A_1)_{>0} \xrightarrow{\pi} A \to A_0 \to 0$$

where $T_{A_0}(A_1)$ is a projective A-module generated by its degree 1 component. By Lemma 2.1.3.4,

$$\mathrm{Ext}^2_{\mathrm{gr}_A\mathrm{Mod}}(A_0, A_0\langle n\rangle) = \mathrm{Hom}_{\mathrm{gr}_A\mathrm{Mod}}(K, A_0\langle n\rangle)$$

Hence, $\mathrm{Hom}_{\mathrm{gr}_A\mathrm{Mod}}(K, A_0\langle n\rangle) = 0$ unless $n = 2$. Since A is pre-Koszul, there exists an admissible epic $A \otimes_{A_0} K_2 \twoheadrightarrow K$. Composing with the admissible epic $T_{A_0}(A_1) \twoheadrightarrow A$, we obtain an admissible epic

$$T_{A_0}(A_1) \otimes_{A_0} K_2 \otimes_{A_0} T_{A_0}(A_1) \twoheadrightarrow A \otimes_{A_0} K_2 \otimes_{A_0} A \twoheadrightarrow A \otimes_{A_0} K_2 \twoheadrightarrow K$$

where the second step follows by Proposition 2.1.1.3. □

The key result of this section is the following.

Corollary 2.2.1.9. *Any Koszul monoid is quadratic.*

Proof. Suppose that A is a Koszul monoid. Then, by Proposition 2.1.3.6, we have that $\mathrm{Ext}^i_{\mathrm{gr}_A\mathrm{Mod}}(A_0, A_0\langle n\rangle) = 0$ unless $n = i$. Hence, $\mathrm{Ext}^1_{\mathrm{gr}_A\mathrm{Mod}}(A_0, A_0\langle n\rangle) = 0$ unless $n = 1$. Therefore, by Corollary 2.2.1.7, A is a quotient of $T_{A_0}(A_1)$. Moreover, since $\mathrm{Ext}^2_{\mathrm{gr}_A\mathrm{Mod}}(A_0, A_0\langle n\rangle) = 0$ unless $n = 2$, then by Proposition 2.2.1.8, A is a quadratic monoid. □

2.2.2 Dual Quadratic Monoids

We refer the reader to [74, Section 6] for a more detailed exposition on what dual objects should look like in ${}_{A_0}\mathrm{Mod}$.

Definition 2.2.2.1. Suppose that A is a positively graded monoid in E.

1. If M is a left A_0-module, the *left dual A_0-module* is $M^* := \underline{\mathrm{Hom}}_{{}_{A_0}\mathrm{Mod}}(M, A_0)$,

2. If M is a right A_0-module, the *right dual A_0-module* is ${}^*M := \underline{\mathrm{Hom}}_{\mathrm{Mod}_{A_0}}(M, A_0)$.

By the internal hom adjunction, for any left A_0-module M there is an isomorphism

$$\mathrm{Hom}_{{}_{A_0}\mathrm{Mod}}(M^* \otimes_{A_0} M, A_0) \simeq \mathrm{Hom}_{{}_{A_0}\mathrm{Mod}}(M^*, M^*)$$

Definition 2.2.2.2. We define the *evaluation morphism* $ev_M : M^* \otimes_{A_0} M \to A_0$ to be the image of id_{M^*} under the above isomorphism.

Definition 2.2.2.3. An (A_0, A_0)-bimodule M is *left dualisable* if ${}^*(M^*) \simeq M$ and there exists a coevalution morphism $coev_M : A_0 \to M \otimes_{A_0} M^*$ such that the compositions

$$M \xrightarrow{coev_M \otimes_{A_0} \mathrm{id}_M} (M \otimes_{A_0} M^*) \otimes_{A_0} M \to M \otimes_{A_0} (M^* \otimes_{A_0} M) \xrightarrow{\mathrm{id}_M \otimes_{A_0} ev_M} M$$

and

$$M^* \xrightarrow{\mathrm{id}_{M^*} \otimes_{A_0} coev_M} M^* \otimes_{A_0} (M \otimes_{A_0} M^*) \to (M^* \otimes_{A_0} M) \otimes_{A_0} M^* \xrightarrow{ev_M \otimes_{A_0} \mathrm{id}_{M^*}} M^*$$

are the identity morphisms.

Example 2.2.2.4. *We note that when $A_0 = k$ is a field, the dualisable modules are precisely the finite dimensional vector spaces. When A_0 is a semisimple ring, the dualisable modules are precisely the finitely generated ones.*

Proposition 2.2.2.5. *Suppose that M is a left dualisable (A_0, A_0)-bimodule, N_1 is an (A_0, A_0)-bimodule, and N_2 is a left A_0-module. Then, we have that*

$$\underline{\mathrm{Hom}}_{{}_{A_0}\mathrm{Mod}}(N_1 \otimes_{A_0} M, N_2) \simeq \underline{\mathrm{Hom}}_{{}_{A_0}\mathrm{Mod}}(N_1, M^* \otimes_{A_0} N_2)$$

Proof. This isomorphism is induced by the isomorphism

$$N_1 \simeq N_1 \otimes_{A_0} A_0 \xrightarrow{\mathrm{id}_{N_1} \otimes_{A_0} coev_M} N_1 \otimes_{A_0} M \otimes_{A_0} M^* \to N_2 \otimes_{A_0} M^* \xrightarrow{s_{N_2, M^*}} M^* \otimes_{A_0} N_2$$

□

Definition 2.2.2.6. We will say that a positively graded monoid A is *left dualisable* if each A_i is a left dualisable (A_0, A_0)-bimodule.

Suppose that A is a left dualisable quadratic monoid with quadratic data (A_1, R). We note that $R = K_2$ is a left dualisable (A_0, A_0)-bimodule by [74, Corollary 6.16]. If we dualise the monomorphism $\iota : R \hookrightarrow A_1 \otimes_{A_0} A_1$, we obtain a map

$$\iota^* : A_1^* \otimes_{A_0} A_1^* \simeq (A_1 \otimes_{A_0} A_1)^* \to R^*$$

We define the *left orthogonal A_0-submodule* $R^\perp$ to be the kernel of ι^*.

Definition 2.2.2.7. Suppose that A is a left dualisable quadratic monoid (A_1, R). We say that a positively graded pre-Koszul monoid $A^!$ is the *left dual quadratic monoid* of A if

$$A^! \simeq T_{A_0}(A_1^*)/(R^\perp)$$

where we use the notation of Definition 2.2.1.3.

Example 2.2.2.8. *If we once again consider the quadratic tensor monoid $T_{A_0}(A_1)$, we see that $R^\perp = \mathrm{Ker}(A_1^* \otimes_{A_0} A_1^* \to 0) \simeq A_1^* \otimes_{A_0} A_1^*$. Therefore,*

$$A^! = \mathrm{Coker}(T_{A_0}(A_1^*) \otimes_{A_0} R^\perp \otimes_{A_0} T_{A_0}(A_1^*) \hookrightarrow T_{A_0}(A_1^*)) \simeq A_0 \oplus A_1^*$$

If A is instead a *right dualisable* quadratic monoid, we can define the *right orthogonal A_0-submodule* ${}^\perp R$ to be the kernel of the map ${}^*\iota : {}^*A_1 \otimes_{A_0} {}^*A_1 \to {}^*R$. The *right dual quadratic monoid* is then a positively graded pre-Koszul monoid with presentation ${}^*A \simeq T_{A_0}({}^*A_1)/({}^\perp R)$.

Proposition 2.2.2.9. *[74, Propositions 6.23 and 6.27] Suppose that A is a left dualisable quadratic monoid with left dual quadratic monoid $A^!$. Then,*

1. ${}^!(A^!) \simeq A$,
2. ${}^*(A^!_i) \simeq {}^\perp(K^!_i)$ *where* $K^!_i := \mathrm{Ker}((A_1^*)^{\otimes i} \to A_i^!)$.

Proposition 2.2.2.10. *Define P_i to be the pullback of all the monomorphisms*

$$A_1^{\otimes j} \otimes_{A_0} R \otimes_{A_0} A_1^{\otimes(i-j-2)} \hookrightarrow A_1^{\otimes i}$$

where j ranges from 0 to $i-2$. Then, there is an isomorphism $P_i \simeq {}^\perp(K^!_i) \simeq {}^*(A^!_i)$ for each i.

Proof. There exists an admissible epic of right dualisable (A_0, A_0)-bimodules

$$\bigoplus_{j=0}^{i-2} (A_1^*)^{\otimes j} \otimes_{A_0} R^\perp \otimes_{A_0} (A_1^*)^{\otimes(i-j-2)} \twoheadrightarrow K^!_i$$

Dualising this map we obtain, by [74, Proposition 6.13], a monomorphism

$${}^*(K^!_i) \hookrightarrow \bigoplus_{j=0}^{i-2} A_1^{\otimes j} \otimes_{A_0} {}^*(R^\perp) \otimes_{A_0} A_1^{\otimes(i-j-2)}$$

We know that ${}^\perp(K^!_i) = \mathrm{Ker}(A_1^{\otimes i} \to {}^*(K^!_i))$. Hence,

$${}^\perp(K^!_i) = \mathrm{Ker}\left(A_1^{\otimes i} \to \bigoplus_{j=0}^{i-2} A_1^{\otimes j} \otimes_{A_0} {}^*(R^\perp) \otimes_{A_0} A_1^{\otimes(i-j-2)}\right)$$

To show that $P_i \simeq {}^{\perp}(K_i^!)$, it suffices to show that P_i is also the kernel of this map. Now, since ${}^*(R^{\perp}) \simeq \mathrm{Coker}(\iota : R \hookrightarrow A_1 \otimes_{A_0} A_1)$, we see that the map

$$P_i \to A_1^{\otimes i} \to \bigoplus_{j=0}^{i-2} A_1^{\otimes j} \otimes_{A_0} {}^*(R^{\perp}) \otimes_{A_0} A_1^{\otimes(i-j-2)}$$

is zero since, by definition of the pullback, it factors through $A_1^{\otimes j} \otimes_{A_0} R \otimes_{A_0} A_1^{\otimes(i-j-2)}$ for all $0 \leq j \leq i-2$.

Now, suppose that we have an object W such that the map

$$W \to A_1^{\otimes i} \to \bigoplus_{j=0}^{i-2} A_1^{\otimes j} \otimes_{A_0} {}^*(R^{\perp}) \otimes_{A_0} A_1^{\otimes(i-j-2)}$$

is zero. We see that, since the composite $R \hookrightarrow A_1 \otimes_{A_0} A_1 \to A_2$ is zero, there exists a map ${}^*(R^{\perp}) \to A_2$ such that the following diagram commutes

$$\begin{tikzcd} A_1 \otimes_{A_0} A_1 \arrow[r] \arrow[dr] & {}^*(R^{\perp}) \arrow[d] \\ & A_2 \end{tikzcd}$$

and hence the following diagram commutes

$$\begin{tikzcd} A_1^{\otimes i} \arrow[r] \arrow[dr] & A_1^{\otimes j} \otimes_{A_0} {}^*(R^{\perp}) \otimes_{A_0} A_1^{\otimes(i-j-2)} \arrow[d] \\ & A_1^{\otimes j} \otimes_{A_0} A_2 \otimes_{A_0} A_1^{\otimes(i-j-2)} \end{tikzcd}$$

The map $W \to A_1^{\otimes i} \to A_1^{\otimes j} \otimes_{A_0} A_2 \otimes_{A_0} A_1^{\otimes(i-j-2)}$ is therefore zero since it factors through the zero map $W \to A_1^{\otimes j} \otimes_{A_0} {}^*(R^{\perp}) \otimes_{A_0} A_1^{\otimes(i-j-2)}$. Since $R = K_2$, there exists a map $W \to A_1^{\otimes j} \otimes_{A_0} R \otimes_{A_0} A_1^{\otimes(i-j-2)}$ such that the following diagram commutes

$$\begin{tikzcd} A_1^{\otimes j} \otimes_{A_0} R \otimes_{A_0} A_1^{\otimes(i-j-2)} \arrow[r] & A_1^{\otimes i} \\ W \arrow[u] \arrow[ur] & \end{tikzcd}$$

Hence, since, for each $0 \leq j \leq i-2$, all the maps

$$W \to A_1^{\otimes j} \otimes_{A_0} R \otimes_{A_0} A_1^{\otimes(i-j-2)} \to A_1^{\otimes i}$$

are equal, then by definition of the pullback there exists a map $W \to P_i$ such that the following diagram commutes for all $0 \leq j \leq i-2$,

$$\begin{tikzcd} A_1^{\otimes j} \otimes_{A_0} R \otimes_{A_0} A_1^{\otimes(i-j-2)} \arrow[r] & A_1^{\otimes i} \arrow[r] & \bigoplus_{j=0}^{i-2} A_1^{\otimes j} \otimes_{A_0} {}^*(R^{\perp}) \otimes_{A_0} A_1^{\otimes(i-j-2)} \\ P_i \arrow[u] & W \arrow[l] \arrow[u] & \end{tikzcd}$$

Hence, P_i is a kernel of the map $A_1^{\otimes i} \to \bigoplus_{j=0}^{i-2} A_1^{\otimes j} \otimes_{A_0} {}^*(R^\perp) \otimes_{A_0} A_1^{\otimes(i-j-2)}$ so, by the uniqueness of the kernel, $P_i \simeq {}^\perp(K_i^!)$. $\square$

2.3 The Koszul Complex

Suppose that A is a left dualisable quadratic monoid (A_1, R). Suppose further that the dual quadratic monoid exists, and denote it by $A^!$. For each $i \geq 0$, we let $\mathcal{K}^i$ be the A-module

$$\mathcal{K}^i = A \otimes_{A_0} {}^*(A_i^!)$$

We note that this module lives only in degree $\geq i$.

Proposition 2.3.0.1. *$\mathcal{K}^i$ is a projective A-module.*

Proof. By Proposition 2.1.1.3, it suffices to show that ${}^*(A_i^!)$ is a projective A_0-module. Indeed, this follows using that $\mathrm{Hom}_{\mathrm{gr}_{A_0}\mathrm{Mod}}({}^*(A_i^!), -) \simeq \mathrm{Hom}_{\mathrm{gr}_{A_0}\mathrm{Mod}}(A_0, A_i^! \otimes_{A_0} -)$ and the statement that $A_i^! \otimes_{A_0} -$ preserves admissible epics.

$\square$

We consider the multiplication map $\mu_{i,1}^! : A_i^! \otimes_{A_0} A_1^! \to A_{i+1}^!$ on $A^!$. Dualising, we obtain a map ${}^*\mu_{i,1}^! : {}^*(A_{i+1}^!) \to {}^*(A_1^!) \otimes_{A_0} {}^*(A_i^!) \simeq A_1 \otimes_{A_0} {}^*(A_i^!)$ which, when tensored with A, gives a map

$$\begin{aligned} \mathcal{K}^{i+1} = A \otimes_{A_0} {}^*(A_{i+1}^!) &\xrightarrow{\mathrm{id}_A \otimes_{A_0} {}^*\mu_{i,1}^!} A \otimes_{A_0} A_1 \otimes_{A_0} {}^*(A_i^!) \\ &\xrightarrow{\mu \otimes_{A_0} \mathrm{id}_{{}^*(A_i^!)}} A \otimes_{A_0} {}^*(A_i^!) = \mathcal{K}^i \end{aligned} \tag{2.1}$$

which we will denote by $d^{i+1} : \mathcal{K}^{i+1} \to \mathcal{K}^i$. We let $Z^i = \mathrm{Ker}(d^i : \mathcal{K}^i \to \mathcal{K}^{i-1})$ and $B^i = \mathrm{Im}(d^{i+1} : \mathcal{K}^{i+1} \to \mathcal{K}^i)$.

Lemma 2.3.0.2. *We have $d^i \circ d^{i+1} = 0$.*

Proof. By Proposition 2.2.2.10, we see that ${}^*(A_i^!)$ is isomorphic to the pullback of all the monomorphisms

$$A_1^{\otimes j} \otimes_{A_0} R \otimes_{A_0} A_1^{\otimes(i-j-2)} \hookrightarrow A_1^{\otimes i}$$

where j ranges from 0 to $i-2$. In particular, we can identify the morphism ${}^*\mu_{i,1}^!$: ${}^*(A_{i+1}^!) \to A_1 \otimes_{A_0} {}^*(A_i^!)$ with the morphism $P_{i+1} \to A_1 \otimes_{A_0} P_i$ induced by the pullback. Similarly, the morphism $\mathrm{id}_{A_1} \otimes_{A_0} {}^*\mu_{i-1,1}^!$ can be identified with the morphism

$A_1 \otimes_{A_0} P_i \to A_1 \otimes_{A_0} A_1 \otimes_{A_0} P_{i-1}$. In particular, using the properties of the pullback, we note that we have the following commutative diagram.

$$\begin{tikzcd}
A \otimes_{A_0} P_{i+1} \arrow[r] \arrow[d, "\mathrm{id}_A \otimes_{A_0} {}^*\mu^!_{i,1}"] & A \otimes_{A_0} R \otimes_{A_0} A_1^{\otimes i-1} \arrow[dd] \\
A \otimes_{A_0} A_1 \otimes_{A_0} P_i \arrow[d, "\mathrm{id}_A \otimes_{A_0} \mathrm{id}_{A_1} \otimes_{A_0} {}^*\mu^!_{i-1,1}"] & \\
A \otimes_{A_0} A_1 \otimes_{A_0} A_1 \otimes_{A_0} P_{i-1} \arrow[r] \arrow[d, "\mathrm{id}_A \otimes_{A_0} \mu_2 \otimes_{A_0} \mathrm{id}_{*(A^!_{i-1})}"] & A \otimes_{A_0} A_1^{\otimes i+1} \arrow[d, "\mathrm{id}_A \otimes_{A_0} \mu_2 \otimes_{A_0} \mathrm{id}_{A_1}^{\otimes i-1}"] \\
A \otimes_{A_0} P_{i-1} \arrow[r] & A \otimes_{A_0} A_1^{\otimes i-1}
\end{tikzcd}$$

We see that the left vertical morphism is precisely the composition $d^i \circ d^{i+1}$. Since $R = \mathrm{Ker}(\mu_2)$, then the right vertical morphism is zero. Hence, since the bottom horizontal morphism is a monomorphism, being a pullback of monomorphisms, then $d^i \circ d^{i+1} = 0$. $\square$

Definition 2.3.0.3. The *Koszul complex* $\mathcal{K}^\bullet$ of A is the complex

$$\mathcal{K}^\bullet = \cdots \to A \otimes_{A_0} {}^*(A^!_2) \to A \otimes_{A_0} {}^*(A^!_1) \to A$$

with differentials $d^{i+1} : \mathcal{K}^{i+1} \to \mathcal{K}^i$ defined as in Equation (2.1).

We note that the objects of the Koszul complex are graded with components $\mathcal{K}^i_j = A_{j-i} \otimes_{A_0} {}^*(A^!_i)$. Each $\mathcal{K}^i$ is projective as an A-module and is generated by its degree i component over A_0 since $\mathcal{K}^i_i = A_0 \otimes_{A_0} {}^*(A^!_i) \simeq {}^*(A^!_i)$. We will show in this section that A is a Koszul monoid if and only if this complex provides a resolution of A_0.

Proposition 2.3.0.4. *The $(i+1)$-th component of the graded map $d^{i+1} : \mathcal{K}^{i+1} \to \mathcal{K}^i$ is a monomorphism. It follows that $B^i_{i+1} \simeq {}^*(A^!_{i+1})$.*

Proof. We note that $\mathcal{K}^{i+1}_{i+1} \simeq A_0 \otimes_{A_0} {}^*(A^!_{i+1}) \simeq {}^*(A^!_{i+1})$ and $\mathcal{K}^i_{i+1} \simeq A_1 \otimes_{A_0} {}^*(A^!_i)$. Hence, the map d^{i+1}_{i+1} is equivalent to the map ${}^*\mu^!_{i,1} : {}^*(A^!_{i+1}) \to A_1 \otimes_{A_0} {}^*(A^!_i)$ which, being the dual of the epimorphism $\mu^!_{i,1} : A^!_i \otimes_{A_0} A^!_1 \to A^!_{i+1}$, is a monomorphism. $\square$

Corollary 2.3.0.5. *For each $i \geq 0$, Z^i lives only in degrees $\geq i+1$.*

Proof. We note that $\mathcal{K}^i_j = 0$ for $j < i$, and hence $Z^i_j = 0$ for $j < i$. By Proposition 2.3.0.4, $Z^i_i = 0$, and our result follows. $\square$

Proposition 2.3.0.6. $Z^i_{i+1} \simeq {}^*(A^!_{i+1})$.

Proof. We note that $Z^i_{i+1} = \operatorname{Ker}(d^i_{i+1} : A_1 \otimes_{A_0} {}^*(A^!_i) \to A_2 \otimes_{A_0} {}^*(A^!_{i-1}))$. Then, since $d^i_{i+1} \circ d^{i+1}_{i+1} = 0$ by Proposition 2.3.0.2, there exists a map ${}^*(A^!_{i+1}) \to Z^i_{i+1}$ such that the following diagram commutes

$$\begin{tikzcd}
Z^i_{i+1} \arrow[r, hook] & A_1 \otimes_{A_0} {}^*(A^!_i) \\
{}^*(A^!_{i+1}) \arrow[u] \arrow[ur, hook, "d^{i+1}_{i+1}"] &
\end{tikzcd}$$

By Proposition 2.2.2.10, ${}^*(A^!_{i+1}) \simeq P_{i+1}$. We will show that there exists a map $Z^i_{i+1} \to P_{i+1}$. We consider the monomorphism ${}^*\mu^!_{i,1} : {}^*(A^!_i) \to A_1 \otimes_{A_0} {}^*(A^!_{i-1})$. The kernel of the map $A_1 \otimes_{A_0} A_1 \otimes_{A_0} {}^*(A^!_{i-1}) \to A_2 \otimes_{A_0} {}^*(A^!_{i-1})$ is exactly $R \otimes_{A_0} {}^*(A^!_{i-1})$. Since the map

$$Z^i_{i+1} \to A_1 \otimes_{A_0} A_1 \otimes_{A_0} {}^*(A^!_{i-1}) \to A_2 \otimes_{A_0} {}^*(A^!_{i-1})$$

is exactly the zero map $Z^i_{i+1} \to A_1 \otimes_{A_0} {}^*(A^!_i) \xrightarrow{d^i_{i+1}} A_2 \otimes_{A_0} {}^*(A^!_{i-1})$ then, by the universal property of the kernel, there exists a map $Z^i_{i+1} \to R \otimes_{A_0} {}^*(A^!_{i-1})$ such that the following diagram commutes

$$\begin{tikzcd}
Z^i_{i+1} \arrow[r] \arrow[drr] & A_1 \otimes_{A_0} {}^*(A^!_i) \arrow[r] \arrow[rr, bend left, "d^i_{i+1}"] & A_1 \otimes_{A_0} A_1 \otimes_{A_0} {}^*(A^!_{i-1}) \arrow[r] & A_2 \otimes_{A_0} {}^*(A^!_{i-1}) \\
 & & R \otimes_{A_0} {}^*(A^!_{i-1}) \arrow[u] &
\end{tikzcd}$$

Using the definition of $P_i \simeq {}^*(A^!_i)$ from Proposition 2.2.2.10 we see that, for all $0 \le j \le i-2$, the maps

$$Z^i_{i+1} \to A_1 \otimes_{A_0} {}^*(A^!_i) \to A_1 \otimes_{A_0} (A_1^{\otimes j} \otimes_{A_0} R \otimes_{A_0} A_1^{\otimes i-j-2}) \to A_1^{\otimes(i+1)}$$

are all equal. Further, using the definition of ${}^*(A^!_{i-1}) \simeq P_{i-1}$ and the commutative diagram, for each $0 \le j \le i-3$ the map

$$Z^i_{i+1} \to A_1 \otimes_{A_0} A_1 \otimes_{A_0} {}^*(A^!_{i-1}) \to A_1 \otimes_{A_0} A_1 \otimes_{A_0} (A_1^{\otimes j} \otimes_{A_0} R \otimes_{A_0} A_1^{\otimes i-j-3}) \to A_1^{\otimes(i+1)}$$

is equal to the map $Z^i_{i+1} \to R \otimes_{A_0} A_1^{\otimes(i-1)} \to A_1^{\otimes(i+1)}$. Hence, for all $0 \le j \le i+1$, the maps

$$Z^i_{i+1} \to A_1^{\otimes j} \otimes_{A_0} R \otimes_{A_0} A_1^{\otimes(i+j-1)} \to A_1^{\otimes(i+1)}$$

are equal, and thus there exists a map $Z^i_{i+1} \to P_{i+1} \simeq {}^*(A^!_{i+1})$. This clearly makes the following diagram commute

$$\begin{tikzcd} Z^i_{i+1} \arrow[r, hook] \arrow[d, shift left] & A_1 \otimes_{A_0} {}^*(A^!_i) \\ {}^*(A^!_{i+1}) \arrow[u, shift left] \arrow[ur, hook, "d^{i+1}_{i+1}"] & \end{tikzcd}$$

Therefore, we have an isomorphism $Z^i_{i+1} \simeq {}^*(A^!_{i+1})$.

□

Theorem 2.3.0.7. *Suppose that A is a left dualisable quadratic monoid (A_1, R) with dual quadratic monoid $A^!$. Then, A is a Koszul monoid if and only if its Koszul complex is a resolution of A_0.*

Proof. If the Koszul complex of A is a resolution of A_0, then clearly A is a Koszul monoid since its Koszul complex consists of projective A-modules $\mathcal{K}^i$ generated by their i^{th} components.

On the other hand, suppose that A is a Koszul monoid. We need to show that the complex $\mathcal{K}^\bullet \to A_0$ is exact. We prove this inductively. Indeed, the map $\mathcal{K}^0 = A \to A_0$ is the natural projection map which is an admissible epic in $\mathrm{gr}_A\mathrm{Mod}$. Its kernel is isomorphic to $A_{>0}$ which is isomorphic to the image of the map

$$\mathcal{K}^1 = A \otimes_{A_0} {}^*(A^!_1) \simeq A \otimes_{A_0} A_1 \to A = \mathcal{K}^0$$

by Lemma 2.2.1.5. Now, suppose that the Koszul complex is exact up to degree i. Then, by Lemma 2.1.3.4,

$$\mathrm{Ext}^{i+1}_{\mathrm{gr}_A\mathrm{Mod}}(A_0, A_0\langle n\rangle) = \mathrm{Hom}_{\mathrm{gr}_A\mathrm{Mod}}(Z^i, A_0\langle n\rangle)$$

By Proposition 2.1.3.6, if A is Koszul then $\mathrm{Ext}^{i+1}_{\mathrm{gr}_A\mathrm{Mod}}(A_0, A_0\langle n\rangle) = 0$ unless $i+1 = n$. Therefore, $\mathrm{Hom}_{\mathrm{gr}_A\mathrm{Mod}}(Z^i, A_0\langle n\rangle) = 0$ unless $i+1 = n$. Since A is pre-Koszul, Z^i is generated by its $(i+1)^{th}$ component. Therefore, there exists an admissible epic $A \otimes_{A_0} Z^i_{i+1} \twoheadrightarrow Z^i$. Hence, by Propositions 2.3.0.4 and 2.3.0.6, there is an admissible epic

$$\mathcal{K}^{i+1} = A \otimes_{A_0} {}^*(A^!_{i+1}) \simeq A \otimes_{A_0} Z^i_{i+1} \simeq A \otimes_{A_0} B^i_{i+1} \twoheadrightarrow Z^i$$

and the following diagram commutes

$$\begin{tikzcd} \mathcal{K}^{i+1} \arrow[rr, "d^{i+1}"] \arrow[dr, two heads] & & \mathcal{K}^i \\ & Z^i \arrow[ur, hook] & \end{tikzcd}$$

By the universal property of the image B^i, there exists a unique map $B^i \to Z^i$ such that the following diagram commutes

$$\begin{tikzcd}
\mathcal{K}^{i+1} \arrow[rr] \arrow[dr, two heads] \arrow[ddr, two heads] & & \mathcal{K}^i \\
& B^i \arrow[ur, hook] \arrow[d] & \\
& Z^i \arrow[uur, hook] &
\end{tikzcd}$$

By Proposition 1.1.1.2, the map $B^i \to Z^i$ is an admissible epic. Since $d^i \circ d^{i+1} = 0$, we also know that there is a unique monomorphism $B^i \to Z^i$ such that the following diagram commutes

$$\begin{tikzcd}
Z^i \arrow[r, hook] & \mathcal{K}^i \\
B^i \arrow[u, hook] \arrow[ur, hook] &
\end{tikzcd}$$

By uniqueness, the two maps $B^i \to Z^i$ must be the same. Since this map is both a monomorphism and an epimorphism, it must be an isomorphism. Therefore $B^i \simeq Z^i$ and we are done. $\square$

Proposition 2.3.0.8. *If A is a Koszul monoid, then $A^!$ is too.*

Proof. If A is Koszul, then the complex

$$\cdots \to A \otimes_{A_0} {}^*(A^!_2) \to A \otimes_{A_0} {}^*(A^!_1) \to A \to A_0 \to 0$$

is exact by Theorem 2.3.0.7. Since our differentials are graded, for each $j > 0$ the complex

$$0 \to {}^*(A^!_j) \to \cdots \to A_{j-2} \otimes_{A_0} {}^*(A^!_2) \to A_{j-1} \otimes_{A_0} {}^*(A^!_1) \to A_j \to 0$$

is exact. Taking the left dual of this complex we obtain the complex

$$0 \leftarrow {}^*(A^!_j)^* \leftarrow \cdots \leftarrow (A_{j-2} \otimes_{A_0} {}^*(A^!_2))^* \leftarrow (A_{j-1} \otimes_{A_0} {}^*(A^!_1))^* \leftarrow A^*_j \leftarrow 0$$

Equivalently, since $A^!$ is right dualisable, we have the complex

$$0 \leftarrow A^!_j \leftarrow \cdots \leftarrow A^!_2 \otimes_{A_0} A^*_{j-2} \leftarrow A^!_1 \otimes_{A_0} A^*_{j-1} \leftarrow A^*_j \leftarrow 0$$

which is exact since the dual functor is exact by Lemma 1.1.1.10. By [19, Proposition 2.9], we obtain an exact sequence

$$\cdots \to A^! \otimes_{A_0} A^*_2 \to A^! \otimes_{A_0} A^*_1 \to A^! \to A_0 \to 0$$

The objects of the complex are all projective $A^!$-modules and the differentials are $A^!$-module morphisms. This complex gives the Koszul complex, $A^! \otimes_{A_0} A^*_\bullet$, of $A^!$ and provides a resolution of A_0. Hence, $A^!$ is a Koszul monoid. $\square$

2.4 Symmetric and Exterior Monoids

For a monoid A in E, we refer to Appendix A for the definition of the symmetric monoid $\mathrm{Sym}(A)$ and exterior monoid $\bigwedge A$. If M is an (A,A)-bimodule, we can easily define the symmetric module $\mathrm{Sym}_A(M)$ by defining $\mathrm{Sym}^n_A(M)$ to be the coequaliser of all the maps $\sigma : T_A(M)_n \to T_A(M)_n$. We can similarly define the exterior module $\bigwedge_A M$.

Example 2.4.0.1. *Suppose that* E *is enriched over the category of* $\mathbb{Q}$*-vector spaces and that* $T_{A_0}(A_1)$ *is a pre-Koszul monoid, see Section 2.2.1 for examples. Then, we can easily show that there is a map* $q_n : \mathrm{Sym}^n_{A_0}(A_1) \to T_{A_0}(A_1)_n$ *which is a section of the coequaliser map* $\pi_n : T_{A_0}(A_1)_n \to \mathrm{Sym}^n_{A_0}(A_1)$*. Therefore, each* $\mathrm{Sym}^n_{A_0}(A_1)$ *is a projective* A_0*-module. The hom-condition in the definition of pre-Koszul follows from the isomorphism*

$$\mathrm{Hom}_{\mathrm{gr}_{\mathrm{Sym}_{A_0}(A_1)}\mathrm{Mod}}(M, A_0\langle n\rangle) \simeq \mathrm{Hom}_{\mathrm{gr}_{T_{A_0}(A_1)}\mathrm{Mod}}(T_{A_0}(A_1) \otimes_{\mathrm{Sym}_{A_0}(A_1)} M, A_0\langle n\rangle)$$

for every $\mathrm{Sym}_{A_0}(A_1)$*-module* M*. Similarly, we can also show that the exterior module* $\bigwedge_{A_0} A_1$ *is pre-Koszul. If* A_1 *is left dualisable, for example is a nuclear Fréchet space, then the quadratic dual of* $\mathrm{Sym}_{A_0}(A_1)$ *is* $\bigwedge_{A_0} A_1^*$*. We consider the Koszul complex*

$$\mathcal{K}^\bullet = \cdots \to \mathrm{Sym}_{A_0}(A_1) \otimes_{A_0} \bigwedge_{A_0}^{2} A_1 \xrightarrow{d^2} \mathrm{Sym}_{A_0}(A_1) \otimes_{A_0} A_1 \xrightarrow{d^1} \mathrm{Sym}_{A_0}(A_1) \twoheadrightarrow A_0 \to 0$$

where the differentials are $d^i : \mathrm{Sym}_{A_0}(A_1) \otimes_{A_0} \bigwedge^i_{A_0} A_1 \to \mathrm{Sym}_{A_0}(A_1) \otimes_{A_0} \bigwedge^{i-1}_{A_0} A_1$ *with graded part*

$$d^i_j : \mathrm{Sym}^{j-i}_{A_0}(A_1) \otimes_{A_0} \bigwedge_{A_0}^{i} A_1 \to \mathrm{Sym}^{j-i+1}_{A_0}(A_1) \otimes_{A_0} \bigwedge_{A_0}^{i-1} A_1$$

given by the composition

$$\mathrm{Sym}^{j-i}_{A_0}(A_1) \otimes_{A_0} \bigwedge_{A_0}^{i} A_1 \xrightarrow{\mathrm{id}_{\mathrm{Sym}^{j-i}_{A_0}(A_1)} \otimes_{A_0} {}^*\mu^!_{i-1,1}} \mathrm{Sym}^{j-i}_{A_0}(A_1) \otimes_{A_0} A_1 \otimes_{A_0} \bigwedge_{A_0}^{i-1} A_1$$
$$\xrightarrow{\mu_{j-i,1} \otimes_{A_0} \mathrm{id}_{\bigwedge^{i-1}_{A_0} A_1}} \mathrm{Sym}^{j-i+1}_{A_0}(A_1) \otimes_{A_0} \bigwedge_{A_0}^{i-1} A_1$$

We then define maps

$$h^i_j : \mathrm{Sym}^{j-i}_{A_0}(A_1) \otimes_{A_0} \bigwedge_{A_0}^{i} A_1 \to \mathrm{Sym}^{j-i-1}_{A_0}(A_1) \otimes_{A_0} \bigwedge_{A_0}^{i+1} A_1$$

for each i, j by

$$\mathrm{Sym}^{j-i}_{A_0}(A_1) \otimes_{A_0} \bigwedge^{i}_{A_0} A_1 \xrightarrow{{}^*\mu_{1,j-i-1} \otimes_{A_0} \mathrm{id}_{\bigwedge^i_{A_0} A_1}} \mathrm{Sym}^{j-i-1}_{A_0}(A_1) \otimes_{A_0} A_1 \otimes_{A_0} \bigwedge^{i}_{A_0} A_1$$
$$\xrightarrow{\mathrm{id}_{\mathrm{Sym}^{j-i-1}_{A_0}(A_1)} \otimes_{A_0} \mu^!_{1,i}} \mathrm{Sym}^{j-i-1}_{A_0}(A_1) \otimes_{A_0} \bigwedge^{i+1}_{A_0} A_1$$

and we can then show that

$$h^{i-1}_j \circ d^i_j + d^{i+1}_j \circ h^i_j = i \circ id + (j-i) \circ id = j \circ id$$

which, since we are working in characteristic 0*, defines a homotopy between the cochain maps $id, 0 : \mathcal{K}^\bullet_j \to \mathcal{K}^\bullet_j$. It then follows, by [40, Proposition 2.39], that $\mathcal{K}^\bullet_j$ is exact for each j, and hence $\mathcal{K}^\bullet$ is exact. Since* $\mathrm{Sym}_{A_0}(A_1)$ *has a Koszul complex which is a resolution of A_0, it is a Koszul monoid by Proposition 2.3.0.7. Similarly, we can show that $\bigwedge_{A_0} A_1$ is a Koszul monoid.*

2.5 Our Main Koszul Duality Result

Suppose that A is a left dualisable quadratic monoid, with quadratic dual monoid $A^!$. We denote by $\mathrm{Ch}^\bullet(\mathrm{gr}_A\mathrm{Mod})$ the category of cochain complexes $M^\bullet$ of graded A-modules $M^i = \bigoplus_j M^i_j$. We can consider $M^\bullet \in \mathrm{Ch}^\bullet(\mathrm{gr}_A\mathrm{Mod})$ and $N^\bullet \in \mathrm{Ch}^\bullet(\mathrm{gr}_{A^!}\mathrm{Mod})$ as modules $\bigoplus_i M^i$ and $\bigoplus_i N^i$ over A and $A^!$ respectively. Define the full subcategories $\mathrm{Ch}^\uparrow(\mathrm{gr}_A\mathrm{Mod})$ and $\mathrm{Ch}^\downarrow(\mathrm{gr}_A\mathrm{Mod})$ as follows.

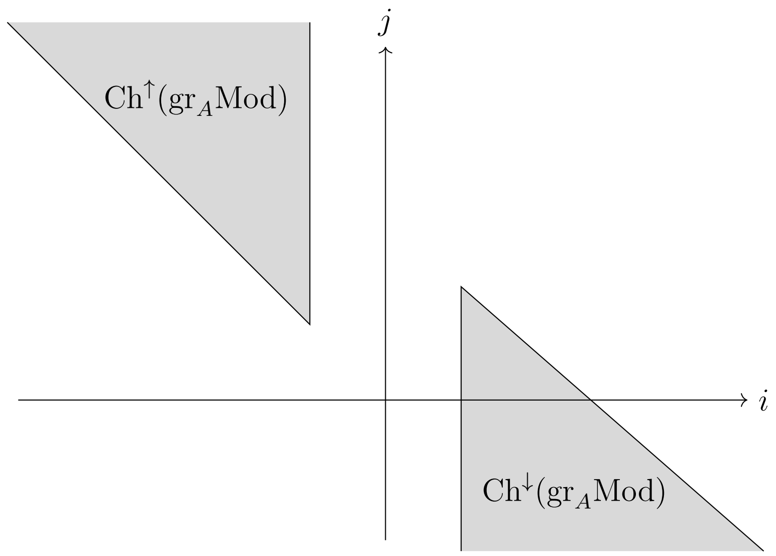
If $M^\bullet \in \mathrm{Ch}^\bullet(\mathrm{gr}_A\mathrm{Mod})$, with each $M^i = \bigoplus_j M^i_j$, then $M^\bullet \in \mathrm{Ch}^\uparrow(\mathrm{gr}_A\mathrm{Mod})$ if

$$M^i_j = 0 \text{ for } i \gg 0 \text{ or } i + j \ll 0$$

and $M^\bullet \in \mathrm{Ch}^\downarrow(\mathrm{gr}_A\mathrm{Mod})$ if

$$M^i_j = 0 \text{ for } i \ll 0 \text{ or } i + j \gg 0$$

We let $\mathrm{D}^\uparrow(\mathrm{gr}_A\mathrm{Mod})$ and $\mathrm{D}^\downarrow(\mathrm{gr}_A\mathrm{Mod})$ denote the localisations of $\mathrm{Ch}^\uparrow(\mathrm{gr}_A\mathrm{Mod})$ and $\mathrm{Ch}^\downarrow(\mathrm{gr}_A\mathrm{Mod})$ at quasi-isomorphisms. Note that these are triangulated subcategories of $\mathrm{Ch}^\bullet(\mathrm{gr}_A\mathrm{Mod})$. The following diagram illustrates where the non zero components M^i_j lie.



Motivated by the Tensor-Hom adjunction for modules, we define functors

$$\begin{aligned} F : \mathrm{gr}_A\mathrm{Mod} &\leftrightarrow \mathrm{gr}_{A^!}\mathrm{Mod} : G \\ M^\bullet &\to A^! \otimes_{A_0} M^\bullet \\ \underline{\mathrm{Hom}}_{\mathrm{gr}_{A_0}\mathrm{Mod}}(A, N^\bullet) &\leftarrow N^\bullet \end{aligned}$$

We will show that these functors descend to triangulated functors

$$\mathrm{D}F : \mathrm{D}^{\downarrow}(\mathrm{gr}_A\mathrm{Mod}) \to \mathrm{D}^{\uparrow}(\mathrm{gr}_{A^!}\mathrm{Mod}) \text{ and } \mathrm{D}G : \mathrm{D}^{\uparrow}(\mathrm{gr}_{A^!}\mathrm{Mod}) \to \mathrm{D}^{\downarrow}(\mathrm{gr}_A\mathrm{Mod})$$

inducing the following equivalence.

Theorem 2.5.0.1. *Suppose that A is a left dualisable Koszul monoid with quadratic dual monoid $A^!$. Then, there is an equivalence of categories*

$$\mathrm{D}^{\downarrow}(\mathrm{gr}_A\mathrm{Mod}) \simeq \mathrm{D}^{\uparrow}(\mathrm{gr}_{A^!}\mathrm{Mod})$$

We prove this theorem in several parts using similar methods to [8, Theorem 2.12.1].

Lemma 2.5.0.2. *There exists a functor* $\mathrm{C}F : \mathrm{Ch}^\bullet(\mathrm{gr}_A\mathrm{Mod}) \to \mathrm{Ch}^\bullet(\mathrm{gr}_{A^!}\mathrm{Mod})$.

Proof. Suppose that we have $(M^\bullet, d_M^\bullet) \in \mathrm{Ch}^\bullet(\mathrm{gr}_A\mathrm{Mod})$. We consider $A^!$ as a complex $((A^!)^\bullet, d_{A^!}^\bullet)$ in $\mathrm{Ch}^\bullet(\mathrm{gr}_{A^!}\mathrm{Mod})$ and define $F(M^\bullet) = A^! \otimes_{A_0} M^\bullet \in {}_{A^!}\mathrm{Mod}$. We construct the double complex of $A^!$ modules $F(M^\bullet)^{\bullet\bullet} = A^!_\bullet \otimes_{A_0} M^\bullet$ whose objects are defined by $F(M^\bullet)^{i,l} = A^!_l \otimes_{A_0} M^i$ and graded by $F(M^\bullet)^{i,l}_j = A^!_l \otimes_{A_0} M^i_j$.

The differentials are defined as follows. We first note that, since $A^!$ is right dualisable,

$$A^!_l \otimes_{A_0} M^i_j \simeq \underline{\mathrm{Hom}}_{A_0\mathrm{Mod}}({}^*(A^!_l), M^i_j) \tag{2.2}$$

Hence the j^{th} component $(d^{i,l}_v)_j : A^!_l \otimes_{A_0} M^i_j \to A^!_{l+1} \otimes_{A_0} M^i_{j+1}$ of the vertical differential is $(-1)^{i+j}(D^{i,l}_v)_j$, where $(D^{i,l}_v)_j$ can be defined using Equation (2.2) and the composition

$${}^*(A^!_{l+1}) \xrightarrow{{}^*\mu^!_{l,1}} A_1 \otimes_{A_0} {}^*(A^!_l) \to A_1 \otimes_{A_0} M^i_j \to M^i_{j+1}$$

We also remark that ${}^*\mu^!_{l,1}$ is one of the maps involved in the definition of the differential for the Koszul complex of A given in Section 2.3. The j^{th} component of the horizontal differential, $(d^{i,l}_h)_j : A^!_l \otimes_{A_0} M^i_j \to A^!_l \otimes_{A_0} M^{i+1}_j$, is defined by $(d^{i,l}_h)_j = \mathrm{id}_{A^!_l} \otimes_{A_0} (d^i_M)_j$. It is easy to check that we obtain a double complex. In degree k, we can visualise the double complex as follows

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\cdots \arrow[r] & A^!_2 \otimes_{A_0} M^i_{2-k} \arrow[r] \arrow[u] & A^!_2 \otimes_{A_0} M^{i+1}_{2-k} \arrow[r] \arrow[u] & A^!_2 \otimes_{A_0} M^{i+2}_{2-k} \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & A^!_1 \otimes_{A_0} M^i_{1-k} \arrow[r] \arrow[u] & A^!_1 \otimes_{A_0} M^{i+1}_{1-k} \arrow[r] \arrow[u] & A^!_1 \otimes_{A_0} M^{i+2}_{1-k} \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & M^i_{-k} \arrow[r] \arrow[u] & M^{i+1}_{-k} \arrow[r] \arrow[u] & M^{i+2}_{-k} \arrow[r] \arrow[u] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

We see that our double complex is bounded from below since $A^!$ is positively graded. We now consider a kind of 'twisted' total complex $FM^\bullet$ with

$$(FM)^n_k = \bigoplus_{i+j=n,k=l-j} A^!_l \otimes_{A_0} M^i_j$$

and with associated total differential $d^\bullet_F = d^{\bullet\bullet}_h + d^{\bullet\bullet}_v$. This complex is clearly a complex of $A^!$-modules. We therefore have a functor

$$CF : \mathrm{Ch}^\bullet(\mathrm{gr}_A\mathrm{Mod}) \to \mathrm{Ch}^\bullet(\mathrm{gr}_{A^!}\mathrm{Mod})$$
$$M^\bullet \to (FM)^\bullet$$

□

Proposition 2.5.0.3. $\mathrm{C}F$ *induces a functor* $\mathrm{D}F : \mathrm{D}^{\downarrow}(\mathrm{gr}_A\mathrm{Mod}) \to \mathrm{D}^{\uparrow}(\mathrm{gr}_{A^!}\mathrm{Mod})$.

Proof. Suppose that $M^\bullet \in \mathrm{Ch}^{\downarrow}(\mathrm{gr}_A\mathrm{Mod})$ and consider the functor

$$\begin{aligned} \mathrm{C}F : \mathrm{Ch}^\bullet(\mathrm{gr}_A\mathrm{Mod}) &\to \mathrm{Ch}^\bullet(\mathrm{gr}_{A^!}\mathrm{Mod}) \\ M^\bullet &\to (FM)^\bullet \end{aligned}$$

defined in the previous lemma. We now see why we specified the boundedness conditions on $M^\bullet$. If $n \gg 0$, then $i + j \gg 0$, and hence, since $M^\bullet \in \mathrm{Ch}^{\downarrow}(\mathrm{gr}_A\mathrm{Mod})$, we see that $M_j^i = 0$. Now, if $n + k \ll 0$, then $i + j + k \ll 0$. Since $A^!$ is positively graded, we know that $l = j + k \geq 0$, and hence $i \ll 0$. Therefore, $M_j^i = 0$ once again. We see that in both cases, $(FM)_k^n = 0$, and hence $(FM)^\bullet \in \mathrm{Ch}^{\uparrow}(\mathrm{gr}_{A^!}\mathrm{Mod})$. We can therefore restrict $\mathrm{C}F$ to a functor

$$\mathrm{C}F : \mathrm{Ch}^{\downarrow}(\mathrm{gr}_A\mathrm{Mod}) \to \mathrm{Ch}^{\uparrow}(\mathrm{gr}_{A^!}\mathrm{Mod})$$

To show that this functor induces the desired functor, it will suffice to show that this functor preserves exactness as it will then preserve all quasi-isomorphisms. Indeed, suppose that $M^\bullet$ is an exact complex. We note that, in each grading j, our double complex $F(M^\bullet)_j^{\bullet\bullet}$ can be considered as first quadrant since $M_j^i = 0$ for $i \ll 0$. This double complex has exact rows since $M^\bullet$ is exact and each $A_i^!$ is a flat A_0-module. Under the canonical embedding

$$I : \mathrm{gr}_{A^!}\mathrm{Mod} \to \mathrm{LH}(\mathrm{gr}_{A^!}\mathrm{Mod})$$

from Theorem 1.1.2.5, we note that $I(F(M^\bullet)_j^{\bullet\bullet})$ is a first quadrant double complex with exact rows.

Hence, we see that, in each degree j, associated to our first quadrant double complex $I(F(M^\bullet))_j^{\bullet\bullet}$, there exists a spectral sequence $\{{}^{II}E_{r,j}^{i,l}\}$ for $r \geq 0$ with first term

$${}^{II}E_{1,j}^{i,l} = H_h^i(I(F(M^\bullet))_j^{\bullet,l})$$

with differentials ${}^{II}d_{1,j}^{i,l} = H_h^i(I(d_h)_j^{\bullet,l})$. Furthermore, by comparing the total complex with respect to the grading (i, l) with the twisted total complex with respect to (i, j), we have the convergence

$$\begin{aligned} {}^{II}E_{2,j}^{i,l} = H_v^l(H_h^i(I(F(M^\bullet))_j^{\bullet\bullet})) &\Rightarrow H^{i+l}(\mathrm{Tot}(I(F(M^\bullet))_j^{\bullet\bullet})^\bullet) \\ &\simeq H^{n+k}(I(FM)_k^{\bullet-k}) \\ &\simeq H^n(I(FM)_k^\bullet) \end{aligned}$$

where $n = i + j, k = l - j$. Since $I(F(M^\bullet))_j^{\bullet\bullet}$ has exact rows, $H_h^i(I(F(M^\bullet))_j^{\bullet\bullet})$ is the zero complex. Hence, since I is additive, then

$$\bigoplus_{k=l-j} H^n(I(FM)_k^\bullet) = H^n\left(\bigoplus_{k=l-j} I(FM)_k^\bullet\right) = H^n(I(FM)^\bullet) = 0$$

Hence, $I(FM)^\bullet = I(\mathrm{C}F(M^\bullet))$ is an exact complex in $\mathrm{LH}(\mathrm{gr}_{A^!}\mathrm{Mod})$. It follows that $\mathrm{C}F(M^\bullet)$ is exact in $\mathrm{gr}_{A^!}\mathrm{Mod}$ and therefore $\mathrm{C}F$ induces a functor

$$\mathrm{D}F : \mathrm{D}^\downarrow(\mathrm{gr}_A\mathrm{Mod}) \to \mathrm{D}^\uparrow(\mathrm{gr}_{A^!}\mathrm{Mod})$$

□

Proposition 2.5.0.4. *There exists a functor* $\mathrm{C}G : \mathrm{Ch}^\bullet(\mathrm{gr}_{A^!}\mathrm{Mod}) \to \mathrm{Ch}^\bullet(\mathrm{gr}_A\mathrm{Mod})$ *which induces a functor* $\mathrm{D}G : \mathrm{D}^\uparrow(\mathrm{gr}_{A^!}\mathrm{Mod}) \to \mathrm{D}^\downarrow(\mathrm{gr}_A\mathrm{Mod})$.

Proof. Suppose that we have $(N^\bullet, d_N^\bullet) \in \mathrm{Ch}^\bullet(\mathrm{gr}_{A^!}\mathrm{Mod})$. We consider A as a complex $(A^\bullet, d_A^\bullet)$ in $\mathrm{Ch}^\bullet(\mathrm{gr}_A\mathrm{Mod})$ and consider $G(N^\bullet) = \underline{\mathrm{Hom}}_{A_0\mathrm{Mod}}(A, N^\bullet) \in {}_A\mathrm{Mod}$. We construct the double complex $G(N^\bullet)^{\bullet\bullet} = \underline{\mathrm{Hom}}_{A_0\mathrm{Mod}}(A_\bullet, N^\bullet)$ with objects defined by $G(N^\bullet)^{i,l} = \underline{\mathrm{Hom}}_{A_0\mathrm{Mod}}(A_{-l}, N^i)$ and graded by $G(N^\bullet)_j^{i,l} = \underline{\mathrm{Hom}}_{A_0\mathrm{Mod}}(A_{-l}, N_j^i)$. The differentials are defined as follows. We note that, since A is left dualisable

$$\underline{\mathrm{Hom}}_{A_0\mathrm{Mod}}(A_{-l}, N_j^i) \simeq A_{-l}^* \otimes_{A_0} N_j^i \tag{2.3}$$

Hence, the j^{th} component $(d_v^{i,l})_j : \underline{\mathrm{Hom}}_{A_0\mathrm{Mod}}(A_{-l}, N_j^i) \to \underline{\mathrm{Hom}}_{A_0\mathrm{Mod}}(A_{-(l+1)}, N_{j+1}^i)$ of the vertical differential is $(-1)^{i+j}(D_v^{i,l})_j$ where $(D_v^{i,l})_j$ can be defined, using Equation (2.3) and the composition

$$A_{-l}^* \otimes_{A_0} N_j^i \xrightarrow{\mu_{1,-(l+1)}^* \otimes_{A_0} \mathrm{id}_{N_j^i}} A_{-(l+1)}^* \otimes_{A_0} A_1 \otimes_{A_0} N_j^i \to A_{-(l+1)}^* \otimes_{A_0} N_{j+1}^i$$

The j^{th} component $(d_h^{i,l})_j : \underline{\mathrm{Hom}}_{A_0\mathrm{Mod}}(A_{-l}, N_j^i) \to \underline{\mathrm{Hom}}_{A_0\mathrm{Mod}}(A_{-l}, N_j^{i+1})$ of the horizontal differential is defined by the composition $A_{-l} \to N_j^i \xrightarrow{(d_N^i)_j} N_j^{i+1}$. The remainder of this proposition follows in a similar way to the previous two results, and more details can be found in [74, Lemma 8.2 and Proposition 8.3]. □

We note that, for modules $M \in {}_A\mathrm{Mod}$ and $N \in {}_{A^!}\mathrm{Mod}$, there is a chain of isomorphisms

$$\mathrm{Hom}_{A^!\mathrm{Mod}}(A^! \otimes_{A_0} M, N) \simeq \mathrm{Hom}_{A_0\mathrm{Mod}}(M, N) \simeq \mathrm{Hom}_{A\mathrm{Mod}}(M, \underline{\mathrm{Hom}}_{A_0\mathrm{Mod}}(A, N))$$

natural in M and N.

Lemma 2.5.0.5. *Suppose that $M^\bullet \in \mathrm{Ch}^\bullet(\mathrm{gr}_A\mathrm{Mod})$ and $N^\bullet \in \mathrm{Ch}^\bullet(\mathrm{gr}_{A^!}\mathrm{Mod})$. There is an isomorphism*

$$\mathrm{Hom}_{\mathrm{gr}_{A^!}\mathrm{Mod}}((FM)^\bullet, N^\bullet) \simeq \mathrm{Hom}_{\mathrm{gr}_A\mathrm{Mod}}(M^\bullet, (GN)^\bullet)$$

Proof. We note that we have an isomorphism

$$\mathrm{Hom}_{A^!\mathrm{Mod}}((FM)^\bullet, N^\bullet) \simeq \mathrm{Hom}_{A\mathrm{Mod}}(M^\bullet, (GN)^\bullet)$$

We will show that this isomorphism respects the grading. Since

$$\begin{aligned}\mathrm{Hom}_{A^!\mathrm{Mod}}((FM)^n_k, N^n_k) &= \mathrm{Hom}_{A^!\mathrm{Mod}}\left(\bigoplus_{i+j=n,k=l-j} A^!_l \otimes_{A_0} M^i_j, N^n_k\right) \\ &\simeq \bigoplus_{i+j=n} \mathrm{Hom}_{A_0\mathrm{Mod}}\left(M^i_j, \bigoplus_{k=l-j} \underline{\mathrm{Hom}}_{A^!\mathrm{Mod}}(A^!_l, N^n_k)\right)\end{aligned}$$

we see that a morphism $f^n : (FM)^n \to N^n$ respects the grading if and only if each morphism $M^i \to \underline{\mathrm{Hom}}_{A^!\mathrm{Mod}}(A^!, N^n) \simeq N^n$ sends pieces graded in degree j to pieces graded in degree $-j$ for all $i+j=n$. Similarly, since

$$\begin{aligned}\mathrm{Hom}_{A\mathrm{Mod}}(M^i_q, (GN)^i_q) &= \mathrm{Hom}_{A\mathrm{Mod}}\left(M^i_q, \bigoplus_{n-j=i,q=l+j} \underline{\mathrm{Hom}}_{A_0\mathrm{Mod}}(A_{-l}, N^n_{-j})\right) \\ &\simeq \bigoplus_{n-j=i} \mathrm{Hom}_{A_0\mathrm{Mod}}\left(\bigoplus_{q=l+j} A_{-l} \otimes_A M^i_q, N^n_{-j}\right)\end{aligned}$$

we see that a morphism $g^i : M^i \to (GN)^i$ respects the grading if and only if each morphism $M^i \simeq A \otimes_A M^i \to N^n$ sends pieces graded in degree j to pieces graded in degree $-j$ for all $n = i+j$. Therefore, we see that f^n preserves the grading for all n if and only if g^m preserves the grading for all m. □

Proposition 2.5.0.6. *The above isomorphisms induce an adjunction*

$$\mathrm{C}F : \mathrm{Ch}^\bullet(\mathrm{gr}_A\mathrm{Mod}) \leftrightarrows \mathrm{Ch}^\bullet(\mathrm{gr}_{A^!}\mathrm{Mod}) : \mathrm{C}G$$

Proof. By the previous lemma, it remains to show that the isomorphism

$$\mathrm{Hom}_{\mathrm{gr}_{A^!}\mathrm{Mod}}((FM)^\bullet, N^\bullet) \simeq \mathrm{Hom}_{\mathrm{gr}_A\mathrm{Mod}}(M^\bullet, (GN)^\bullet)$$

respects cochain maps for all $M^\bullet \in \mathrm{Ch}^\bullet(\mathrm{gr}_A\mathrm{Mod})$ and $N^\bullet \in \mathrm{Ch}^\bullet(\mathrm{gr}_{A^!}\mathrm{Mod})$. Indeed, suppose that there exists an element $f^\bullet \in \mathrm{Hom}_{\mathrm{Ch}^\bullet(\mathrm{gr}_{A^!}\mathrm{Mod})}((FM)^\bullet, N^\bullet)$. Let its image be $g^\bullet \in \mathrm{Hom}_{\mathrm{Ch}^\bullet(\mathrm{gr}_A\mathrm{Mod})}(M^\bullet, (GN)^\bullet)$. We need to show that $f^\bullet$ commutes with the differentials if and only if $g^\bullet$ does. We note that, by the previous Lemma, f_k^n induces morphisms $M_{-k}^{n+k} \to N_k^n$. We consider the following diagram

$$\begin{tikzcd}
M_{-k}^{n+k} \arrow[r, "(d_M^{n+k})_{-k}"] \arrow[d, hook] & M_{-k}^{n+k+1} \arrow[d, hook] \\
\bigoplus_{i+j=n,k=l-j} A_l^! \otimes_{A_0} M_j^i \arrow[r, "(d_F^n)_k"] \arrow[d, "f_k^n"] & \bigoplus_{i+j=n+1,k=l-j} A_l^! \otimes_{A_0} M_j^i \arrow[d, "f_k^{n+1}"] \\
N_k^n \arrow[r, "(d_N^n)_k"] & N_k^{n+1}
\end{tikzcd}$$

We easily see that the top square of the diagram commutes. Commutativity of the bottom square, when composed with the inclusion $M_{-k}^{n+k} \to \bigoplus_{i+j=n,k=l-j} A_l^! \otimes_{A_0} M_j^i$ is equivalent to the commutativity on M_{-k}^{n+k} of f_k^n with the differentials. Therefore, f_k^n commutes with the differentials for all n, k if and only if the outer square commutes. Similarly, we obtain the following diagram

$$\begin{tikzcd}
M_{-k}^{n+k} \arrow[r, "(d_M^{n+k})_{-k}"] \arrow[d, "g_{-k}^{n+k}"] & M_{-k}^{n+k+1} \arrow[d, "g_{-k}^{n+k+1}"] \\
\bigoplus_{i+j=n+k,-k=l-j} \underline{\mathrm{Hom}}_{A_0\mathrm{Mod}}(A_{-l}, N_j^i) \arrow[r, "(d_G^{n+k})_{-k}"] \arrow[d] & \bigoplus_{i+j=n+k+1,-k=l-j} \underline{\mathrm{Hom}}_{A_0\mathrm{Mod}}(A_{-l}, N_j^i) \arrow[d] \\
N_k^n \arrow[r, "(d_N^n)_k"] & N_k^{n+1}
\end{tikzcd}$$

We note that the bottom square of this diagram commutes. Commutativity of the top square when composed with the evaluation

$$\bigoplus_{i+j=n+k+1,-k=l-j} \underline{\mathrm{Hom}}_{A_0\mathrm{Mod}}(A_{-l}, N_j^i) \to N_k^{n+1}$$

is equivalent to the commutativity of g_{-k}^{n+k} with the differentials. Therefore, g_{-k}^{n+k} commutes with the differentials for all n, k if and only if the outer square commutes.

Hence, we see that commutativity of f_k^n and g_{-k}^{n+k} with the differentials is equivalent to commutativity of the following diagram

$$\begin{tikzcd}
M_{-k}^{n+k} \arrow[r, "(d_M^{n+k})_{-k}"] \arrow[d] & M_{-k}^{n+k+1} \arrow[d] \\
N_k^n \arrow[r, "(d_N^n)_k"] & N_k^{n+1}
\end{tikzcd}$$

Therefore, f_k^n commutes with the differentials for all n, k if and only if g_l^m does for all m, l.

□

Consider the counit $\varepsilon : \mathrm{C}F \circ \mathrm{C}G \to \mathrm{id}_{\mathrm{Ch}^\bullet(\mathrm{gr}_{A^!}\mathrm{Mod})}$ of the adjunction. Suppose that $N^\bullet \in \mathrm{Ch}^\bullet(\mathrm{gr}_{A^!}\mathrm{Mod})$. Then, we may consider the double complex $(F \circ \mathrm{C}G)(N^\bullet)^{\bullet\bullet}$ with

$$(F \circ \mathrm{C}G)(N^\bullet)^{i,l} = A^!_l \otimes_{A_0} \mathrm{C}G(N^\bullet)^i = A^!_l \otimes_{A_0} \bigoplus_{p+q=i} \underline{\mathrm{Hom}}_{A_0\mathrm{Mod}}(A, N^p_q)$$

and graded by

$$(F \circ \mathrm{C}G)(N^\bullet)^{i,l}_j = A^!_l \otimes_{A_0} \bigoplus_{p+q=i, j=r-q} \underline{\mathrm{Hom}}_{A_0\mathrm{Mod}}(A_{-r}, N^p_q)$$

Since $A^!_l$ is right dualisable, we see that

$$\begin{aligned} A^!_l \otimes_{A_0} \underline{\mathrm{Hom}}_{A_0\mathrm{Mod}}(A_{-r}, N^p_q) &\simeq \underline{\mathrm{Hom}}_{A_0\mathrm{Mod}}({}^*(A^!_l), \underline{\mathrm{Hom}}_{A_0\mathrm{Mod}}(A_{-r}, N^p_q)) \\ &\simeq \underline{\mathrm{Hom}}_{A_0\mathrm{Mod}}(A_{-r} \otimes_{A_0} {}^*(A^!_l), N^p_q) \end{aligned}$$

and hence

$$(F \circ \mathrm{C}G)(N^\bullet)^{i,l}_j \simeq \bigoplus_{p+q=i, j=r-q} \underline{\mathrm{Hom}}_{A_0\mathrm{Mod}}(A_{-r} \otimes_{A_0} {}^*(A^!_l), N^p_q)$$

The horizontal differential is given by

$$(d^{i,l}_h)_j = \mathrm{id}_{A^!_l} \otimes_{A_0} (d^i_G)_j$$

where $(d^i_G)_j$ is the total differential defined in Proposition 2.5.0.4. The vertical differential

$$\begin{aligned} (d^{i,l}_v)_j : &\bigoplus_{p+q=i, j=r-q} \underline{\mathrm{Hom}}_{A_0\mathrm{Mod}}(A_{-r} \otimes_{A_0} {}^*(A^!_l), N^p_q) \\ &\to \bigoplus_{p+q=i, j=r-q} \underline{\mathrm{Hom}}_{A_0\mathrm{Mod}}(A_{-(r+1)} \otimes_{A_0} {}^*(A^!_{l+1}), N^p_q) \end{aligned}$$

is given by $(-1)^i (D^{i,l}_v)_j$, where $(D^{i,l}_v)_j$ can be defined by taking the coproduct of the maps

$$\underline{\mathrm{Hom}}_{A_0\mathrm{Mod}}(A_{-r} \otimes_{A_0} {}^*(A^!_l), N^p_q) \to \underline{\mathrm{Hom}}_{A_0\mathrm{Mod}}(A_{-(r+1)} \otimes_{A_0} {}^*(A^!_{l+1}), N^p_q)$$

given by the composition

$$A_{-(r+1)} \otimes_{A_0} {}^*(A^!_{l+1}) \xrightarrow{d^{l+1}_{l-r}} A_{-r} \otimes_{A_0} {}^*(A^!_l) \to N^p_q$$

where $d^\bullet$ is the differential from the Koszul complex of A. Taking the total complex with respect to i and j we obtain

$$\begin{aligned} (\mathrm{C}F \circ \mathrm{C}G)(N^\bullet)^n_k &= \bigoplus_{i+j=n, k=l-j} A^!_l \otimes_{A_0} \bigoplus_{p+q=i, j=r-q} \underline{\mathrm{Hom}}_{A_0}(A_{-r}, N^p_q) \\ &= \bigoplus_{i+j=n, k=l-j} \bigoplus_{p+q=i, j=r-q} \underline{\mathrm{Hom}}_{A_0\mathrm{Mod}}(A_{-r} \otimes_{A_0} {}^*(A^!_l), N^p_q) \end{aligned}$$

with differential given by the total differential.

Lemma 2.5.0.7. *The counit ε is a split epimorphism.*

Proof. It is easy to see that the counit is the natural transformation with components $\varepsilon_{N^\bullet}$ given by

$$(\varepsilon^n_{N^\bullet})_k : (\mathrm{C}F \circ \mathrm{C}G)(N^\bullet)^n_k = \bigoplus_{i+j=n,k=l-j} A^!_l \otimes_{A_0} \bigoplus_{p+q=i,j=r-q} \underline{\mathrm{Hom}}_{A_0\mathrm{Mod}}(A_{-r}, N^p_q) \to N^n_k$$

given by the composition of the following maps

$$\begin{aligned} &\bigoplus_{i+j=n,k=l-j} A^!_l \otimes_{A_0} \bigoplus_{p+q=i,j=r-q} \underline{\mathrm{Hom}}_{A_0\mathrm{Mod}}(A_{-r}, N^p_q) \otimes_{A_0} A_0 \\ &\xrightarrow{\mathrm{id}_{A^!}\otimes_{A_0} ev} \bigoplus_{i+j=n,k=l-j} A^!_l \otimes_{A_0} N^{i+j}_{-j} \to N^n_{l-j} = N^n_k \end{aligned}$$

where the last map is from Proposition 2.1.1.3. We consider the map $\sigma_{N^\bullet} : N^\bullet \to (\mathrm{C}F \circ \mathrm{C}G)(N^\bullet)$ given in degree n by the inclusion map

$$\begin{aligned} N^n_k &\simeq A_0 \otimes_{A_0} \underline{\mathrm{Hom}}_{A_0\mathrm{Mod}}(A_0, N^n_k) \\ &\to \bigoplus_{i+j=n,k=l-j} A^!_l \otimes_{A_0} \bigoplus_{p+q=i,j=r-q} \underline{\mathrm{Hom}}_{A_0\mathrm{Mod}}(A_{-r}, N^p_q) \end{aligned}$$

We note that the map $((\varepsilon \circ \sigma)^n_{N^\bullet})_k$ is equivalent to the map

$$N^n_k \simeq A_0 \otimes_{A_0} \underline{\mathrm{Hom}}_{A_0\mathrm{Mod}}(A_0, N^n_k) \xrightarrow[ev]{\simeq} N^n_k$$

which is the identity map. Hence, $\varepsilon \circ \sigma = id$ and so ε is a split epimorphism. □

Proposition 2.5.0.8. *There is an equivalence* $\mathrm{D}F \circ \mathrm{D}G \to \mathrm{id}_{\mathrm{D}^\uparrow(\mathrm{gr}_{A^!}\mathrm{Mod})}$.

Proof. Since quasi-isomorphisms become isomorphisms in $\mathrm{D}^\uparrow(\mathrm{gr}_{A^!}\mathrm{Mod})$, it suffices to prove that, for $N^\bullet \in \mathrm{Ch}^\uparrow(\mathrm{gr}_{A^!}\mathrm{Mod})$, the counit $\epsilon_{N^\bullet} : (\mathrm{C}F \circ \mathrm{C}G)(N^\bullet) \to N^\bullet$ is a quasi-isomorphism. If we consider an abelianisation $I : \mathrm{gr}_{A^!}\mathrm{Mod} \to \mathrm{LH}(\mathrm{gr}_{A^!}\mathrm{Mod})$ then, by Lemma 1.1.2.4, we can just show that $I(\epsilon_{N^\bullet})$ is a quasi-isomorphism. However, since this map is also a split epimorphism, it suffices, by [74, Lemma 8.9], to show that the map $I(\sigma_{N^\bullet}) : I(N^\bullet) \to I((\mathrm{C}F \circ \mathrm{C}G)(N^\bullet))$ is a quasi-isomorphism. We want to show that, for each n, $I(\sigma^n_{N^\bullet})$ induces an isomorphism $H^n(I(N^\bullet)) \simeq H^n(I((\mathrm{C}F \circ \mathrm{C}G)(N^\bullet)))$ in cohomology.

The image of the first quadrant double complex $(F \circ \mathrm{C}G)(N^\bullet)^{\bullet\bullet}$ under the embedding is a first quadrant double complex $I((F \circ \mathrm{C}G)(N^\bullet))^{\bullet\bullet}$. We consider the spectral sequence $\{{}^I E^{i,l}_r\}$ for $r \geq 0$, with differentials ${}^I d^{i,l}_r$, such that

$${}^I E^{i,l}_0 = I((F \circ \mathrm{C}G)(N^\bullet)^{i,l})$$

and the maps ${}^Id_0^{i,l}$ are just the vertical differentials $I(d_v^{i,l})$. We also consider

$$ {}^IE_1^{i,l} = H_v^l(I((F \circ \mathrm{C}G)(N^\bullet))^{i,\bullet}) $$

with differentials ${}^Id_1^{i,l} = H_v^l(I(d_v)^{i,\bullet})$. We have that

$$ \begin{aligned} H_v^l(I((F \circ \mathrm{C}G)(N^\bullet))^{i,\bullet}) &= H_v^l\Bigg(I\Bigg(\bigoplus_{p+q=i} \underline{\mathrm{Hom}}_{A_0\mathrm{Mod}}(A \otimes_{A_0} {}^*(A^!_\bullet), N_q^p)\Bigg)\Bigg) \\ &\simeq \bigoplus_{p+q=i} H_v^l(I(\underline{\mathrm{Hom}}_{A_0\mathrm{Mod}}(A \otimes_{A_0} {}^*(A^!_\bullet), N_q^p))) \end{aligned} $$

Now, since A is Koszul, the complex $A \otimes_{A_0} {}^*(A^!_\bullet)$ is a projective resolution of A_0. Therefore,

$$ \begin{aligned} &\simeq \begin{cases} \bigoplus_{p+q=i} I(\underline{\mathrm{Ext}}^0_{A_0\mathrm{Mod}}(A_0, N_q^p)) & \text{if } l = 0 \\ 0 & \text{otherwise} \end{cases} \\ &\simeq \begin{cases} \bigoplus_{p+q=i} I(\underline{\mathrm{Hom}}_{A_0\mathrm{Mod}}(A_0, N_q^p)) & \text{if } l = 0 \\ 0 & \text{otherwise} \end{cases} \\ &\simeq \begin{cases} \bigoplus_{p+q=i} I(N_q^p) & \text{if } l = 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned} $$

The differentials are given by

$$ {}^Id_1^{i,l} = \begin{cases} \bigoplus_{p+q=i} I((d_N^p)_q) & \text{if } l = 0 \\ 0 & \text{otherwise} \end{cases} $$

We have the convergence

$$ \begin{aligned} {}^IE_2^{i,l} = H_h^i(H_v^l(I((F \circ \mathrm{C}G)(N^\bullet))^{i,\bullet})) &\Rightarrow H^{i+l}(\mathrm{Tot}(I((F \circ \mathrm{C}G)(N^\bullet))^{\bullet\bullet})^\bullet) \\ &\simeq H^n(I((\mathrm{C}F \circ \mathrm{C}G)(N^\bullet))^\bullet) \end{aligned} $$

where $n = i + l$.

We may view $N^\bullet$ as a double complex $N^{\bullet\bullet}$ with

$$ N^{i,l} = \begin{cases} \bigoplus_{p+q=i} N_q^p & \text{if } l = 0 \\ 0 & \text{otherwise} \end{cases} $$

The horizontal differentials are given by

$$ d^{i,l} = \begin{cases} \bigoplus_{p+q=i} (d_N^p)_q & \text{if } l = 0 \\ 0 & \text{otherwise} \end{cases} $$

and the vertical differentials are zero. The total complex is

$$\mathrm{Tot}(N^{\bullet\bullet})^n = \bigoplus_{i+l=n} N^{i,l} = \bigoplus_{p+q=n} N^p_q$$

We note that, since $N^\bullet \in \mathrm{Ch}^{\uparrow}(\mathrm{gr}_{A^!}\mathrm{Mod})$, then $N^{\bullet\bullet}$, and hence $I(N^{\bullet\bullet})$, is a first quadrant double complex. We construct a spectral sequence $\{{}^I\overline{E}_r^{i,l}\}$ for $r \geq 0$, with differentials ${}^I\overline{d}_r^{i,l}$, such that ${}^I\overline{E}_0^{i,l} = I(N)^{i,l}$ and the maps ${}^I\overline{d}_0^{i,l}$ are just the corresponding differentials of $N^{\bullet\bullet}$. We have

$${}^I\overline{E}_1^{i,l} = H^l_v(I(N^{i,\bullet})) = \begin{cases} \bigoplus_{p+q=i} I(N^p_q) & \text{if } l = 0 \\ 0 & \text{otherwise} \end{cases}$$

and we have the convergence ${}^I\overline{E}_2^{i,l} \Rightarrow H^{i+l}(\mathrm{Tot}(I(N)^{\bullet\bullet})^\bullet) = H^n(I(N^\bullet))$. The map σ induces a morphism of double complexes. Hence, we see that, since $\{{}^I\overline{E}_{r,j}^{i,l}\}$ and $\{{}^I E_{r,j}^{i,l}\}$ have the same first page and the differentials on the first page are the same, these spectral sequences must be equal for $r \geq 1$. Hence, we see that $H^n(I(N^\bullet)) = H^n(I((\mathrm{C}F \circ \mathrm{C}G)(N^\bullet)^\bullet))$ and therefore $I(\sigma_{N^\bullet})$ is a quasi-isomorphism. It follows that σ is a quasi-isomorphism, and hence so is ε.

$\square$

Proposition 2.5.0.9. *[74, Lemma 8.11, Proposition 8.12] The unit η of the adjunction is a split monomorphism and, moreover, there is an equivalence*

$$\mathrm{id}_{\mathrm{D}^{\downarrow}(\mathrm{gr}_A\mathrm{Mod})} \to \mathrm{D}G \circ \mathrm{D}F$$

We are now ready to prove our theorem.

Proof of Theorem 2.5.0.1 . We have seen, by Propositions 2.5.0.3 and 2.5.0.4, that there exist functors $\mathrm{C}F$ and $\mathrm{C}G$ inducing functors

$$\mathrm{D}F : \mathrm{D}^{\downarrow}(\mathrm{gr}_A\mathrm{Mod}) \to \mathrm{D}^{\uparrow}(\mathrm{gr}_{A^!}\mathrm{Mod}) \text{ and } \mathrm{D}G : \mathrm{D}^{\uparrow}(\mathrm{gr}_{A^!}\mathrm{Mod}) \to \mathrm{D}^{\downarrow}(\mathrm{gr}_A\mathrm{Mod})$$

The functors $\mathrm{C}F$ and $\mathrm{C}G$ are adjoint, and the unit and counit coming from this adjunction induce natural isomorphisms

$$\mathrm{D}F \circ \mathrm{D}G \to \mathrm{id}_{\mathrm{D}^{\uparrow}(\mathrm{gr}_{A^!}\mathrm{Mod})} \quad \text{and} \quad \mathrm{id}_{\mathrm{D}^{\downarrow}(\mathrm{gr}_A\mathrm{Mod})} \to \mathrm{D}G \circ \mathrm{D}F$$

Hence, there is an equivalence of categories

$$\mathrm{D}^{\downarrow}(\mathrm{gr}_A\mathrm{Mod}) \simeq \mathrm{D}^{\uparrow}(\mathrm{gr}_{A^!}\mathrm{Mod})$$

$\square$

Chapter 3

Derived Geometric Stacks

The notion of an n-geometric stack originates with Simpson [79]. They were introduced in order to provide a higher analogue of Artin's algebraic stacks [3] and are 'geometric' in the sense that locally they look like schemes. Toën and Vezzosi's notion of a homotopical algebraic geometry context [86] provides an appropriate setting to define n-geometric stacks on certain model categories. In this chapter we define an $(\infty, 1)$-categorical context, called a *relative* $(\infty, 1)$*-pre-geometry tuple*, within which we can define n-geometric ∞-stacks.

When there is no room for confusion we will drop the $(\infty, 1)$- prefix from the notions of limits, colimits, adjunctions etc. in $(\infty, 1)$-categories. We will not redefine all standard notions in the language of $(\infty, 1)$-categories, and will instead refer the reader to Lurie [52] and [57].

3.1 $(\infty, 1)$-Categories and ∞-Stacks

In this section, we will fix what we mean by an '∞-stack', a notion which can mean many different things in the literature. We will then prove certain functoriality results.

3.1.1 $(\infty, 1)$-Categories

There are various models for an $(\infty, 1)$-category, such as quasi-categories [52], Segal spaces [78], complete Segal spaces [73] or simplicial categories [85]. These specific models all have different advantages depending on what context you are working in, but are equivalent to one another in the sense that they can be connected by chains of Quillen equivalent model categories [14].

In [30], Gaitsgory and Rozenblyum work with the philosophy that 'one believes that the notion of an $(\infty, 1)$-category exists, and all one needs to know is how to use the words correctly'. We will use this philosophy in this thesis and the reader should feel free to use whichever model for an $(\infty,1)$-category they feel most comfortable with. However, it will at times be clearer and more useful to fix a model for an $(\infty,1)$-category, and in this case we will take our $(\infty,1)$-categories to be simplicial categories along with the model structure described by Bergner in [13, Theorem 1.1].

We note that any *relative category* M, i.e. a category with weak equivalences, presents an $(\infty,1)$-category $\mathbf{L}^H(\mathrm{M})$ known as its *hammock localisation* [25, Section 2]. If M is a simplicial model category, then the hammock localisation $\mathbf{L}^H(\mathrm{M})$ is connected by an equivalence of $(\infty,1)$-categories to the simplicial nerve $\mathbf{N}(\mathrm{M}^\circ)$ (see [52, Definition 1.1.5.5]) of the subcategory M° of M consisting of fibrant-cofibrant objects.

There is in fact an intimate connection between combinatorial simplicial model categories and certain $(\infty,1)$-categories known as locally presentable $(\infty,1)$-categories [52, Definition 5.5.0.1].

Proposition 3.1.1.1. *[52, c.f. Proposition A.3.7.6][26, c.f. Proposition 4.8] Suppose that $\mathcal{M}$ is an $(\infty,1)$-category. Then, the following two conditions are equivalent,*

1. *The $(\infty,1)$-category $\mathcal{M}$ is locally presentable,*
2. *There exists a combinatorial simplicial model category* M *and an equivalence $\mathcal{M}\simeq\mathbf{L}^H(\mathrm{M})$,*

We will call $\mathcal{M}$ the underlying $(\infty,1)$-category of M.

We define the category of ∞-groupoids, $\infty\mathbf{Grpd}$, to be $\mathbf{L}^H(\mathrm{sSet}_Q)$ where sSet_Q denotes the category of simplicial sets endowed with the standard Quillen model structure. There is a Quillen adjunction

$$i:\mathrm{Set}_Q\leftrightarrows\mathrm{sSet}_Q:\pi_0$$

which induces an adjunction of $(\infty,1)$ categories

$$i:\mathrm{Set}\leftrightarrows\infty\mathbf{Grpd}:\pi_0$$

We can use this definition of π_0 to define an associated ordinary category, the *homotopy category*, to any $(\infty,1)$-category.

Definition 3.1.1.2. [52, c.f. Definition 1.1.3.2] Suppose that $\mathcal{M}$ is an $(\infty,1)$-category. Then, the *homotopy category* $\mathrm{Ho}(\mathcal{M})$ has the same objects as $\mathcal{M}$ but, for $X,Y\in\mathcal{M}$,

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(X,Y)=\pi_0\mathrm{Map}_{\mathcal{M}}(X,Y)$$

Composition of morphisms in $\mathrm{Ho}(\mathcal{M})$ is induced from composition of morphisms in $\mathcal{M}$ by applying π_0.

3.1.2 Groupoid Objects

Suppose that we have an $(\infty, 1)$-category $\mathcal{M}$ with all $(\infty, 1)$-pullbacks. Compare the following definition with that of a *Segal groupoid object* in [86, Definition 1.3.1.6].

Definition 3.1.2.1. [52, c.f. Proposition 6.1.2.6 (4")] A *groupoid object in* $\mathcal{M}$ is a simplicial object $X_* : \Delta^{op} \to \mathcal{M}$ such that, for every $n \geq 0$ and every partition $[n] = S \cup S'$ such that $S \cap S'$ consists of a single element s, the diagram

$$\begin{array}{ccc} X_*([n]) & \longrightarrow & X_*(S) \\ \downarrow & & \downarrow \\ X_*(S') & \longrightarrow & X_*(\{s\}) \end{array}$$

is an $(\infty, 1)$-pullback square in $\mathcal{M}$.

Remark. For a simplicial object X_*, we denote $X_*([n])$ by X_n. We note that, for any groupoid object X_* in $\mathcal{M}$ and any $n \geq 0$, there are equivalences $X_n \to X_1 \times_{X_0} \cdots_{X_0} X_1$ where we are considering an n-fold fibre product of X_1 over X_0.

Definition 3.1.2.2. [52, c.f. Proposition 6.1.2.11] Suppose that we have a morphism $f : X \to Y$ in $\mathcal{M}$. An augmented simplicial object $\check{C}(f)_* \to Y$ is the *Čech nerve of* f if it is a groupoid object of $\mathcal{M}$ and there is an $(\infty, 1)$-pullback square in $\mathcal{M}$

$$\begin{array}{ccc} \check{C}(f)_1 & \longrightarrow & X \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

Suppose that we have a simplicial object X_*. If the colimit exists, then we denote it by $|X_*|$ and call it the *geometric realisation of* X_*.

Definition 3.1.2.3. [52, c.f. Corollary 6.2.3.5] In any $(\infty, 1)$-category $\mathcal{M}$ with $(\infty, 1)$-pullbacks and all geometric realisations of simplicial objects, a morphism $f : X \to Y$ is an *effective epimorphism* if and only if we have an equivalence $|\check{C}(f)_*| \to Y$.

3.1.3 ∞-Stacks

Suppose that $\mathcal{M}$ is an $(\infty, 1)$-category. The *category of* $(\infty, 1)$*-presheaves on* $\mathcal{M}$, denoted by $\mathbf{PSh}(\mathcal{M})$, is defined to be the $(\infty, 1)$-functor category $\mathbf{Fun}(\mathcal{M}^{op}, \infty\mathbf{Grpd})$. We note that there is a Yoneda embedding $h : \mathcal{M} \to \mathbf{PSh}(\mathcal{M})$ defined on objects $X \in \mathcal{M}$ by $\mathrm{Map}_{\mathcal{M}}(-, X)$.

In [52, Definition 6.2.2.1], Lurie defines the notion of a *Grothendieck topology* $\boldsymbol{\tau}$ on an $(\infty, 1)$-category $\mathcal{M}$ using the notion of a *sieve*. We note that, by [52, Proposition 6.2.2.5], for each $X \in \mathcal{M}$, each sieve on X corresponds to an equivalence class of monomorphisms $\{\mathcal{U}_i \to h(X)\}_{i\in I}$ in $\mathbf{PSh}(\mathcal{M})$. Let S denote a collection of representative monomorphisms corresponding to covering sieves in $\boldsymbol{\tau}$.

Classically, sheaves are presheaves whose values are determined by evaluating on covers. We can generalise this notion to the $(\infty, 1)$-categorical setting.

Definition 3.1.3.1. [52, c.f. Definition 6.2.2.6] A presheaf $\mathcal{F}$ in $\mathbf{PSh}(\mathcal{M})$ is *S-local* if, for every monomorphism $\mathcal{U} \to h(X)$ in S, we have an equivalence of ∞-groupoids

$$\mathcal{F}(X) \simeq \mathrm{Map}_{\mathbf{PSh}(\mathcal{M})}(h(X), \mathcal{F}) \to \mathrm{Map}_{\mathbf{PSh}(\mathcal{M})}(\mathcal{U}, \mathcal{F})$$

The *category of* $(\infty, 1)$*-sheaves*, denoted $\mathbf{Sh}(\mathcal{M}, \boldsymbol{\tau})$, is the full subcategory of $\mathbf{PSh}(\mathcal{M})$ spanned by S-local presheaves.

We note that the category $\mathbf{PSh}(\mathcal{M})$ is complete and cocomplete by [52, Corollary 5.1.2.4]. The category of sheaves $\mathbf{Sh}(\mathcal{M}, \boldsymbol{\tau})$ is an $(\infty, 1)$-topos by [52, Section 6.5.2], so in particular it is a reflective subcategory of $\mathbf{PSh}(\mathcal{M})$ and is also complete and cocomplete. Moreover, the sheafification functor $\mathbf{PSh}(\mathcal{M}) \to \mathbf{Sh}(\mathcal{M}, \boldsymbol{\tau})$ preserves colimits.

The definition of a Grothendieck topology on an $(\infty, 1)$-category $\mathcal{M}$ lines up with the usual notion of a Grothendieck topology on its homotopy category $\mathrm{Ho}(\mathcal{M})$ by [52, Remark 6.2.2.3]. We fix the following notion of a *Grothendieck pre-topology* on $\mathrm{Ho}(\mathcal{M})$.

Definition 3.1.3.2. A *Grothendieck pre-topology*, $\boldsymbol{\tau}$, on $\mathrm{Ho}(\mathcal{M})$ is a collection $\boldsymbol{\tau}$ of families of maps $\{f_i : U_i \to X\}_{i\in I}$ such that

- For any isomorphism f_i, we have that $\{f_i\}$ is in $\boldsymbol{\tau}$,
- If $\{U_i \to X\}_{i\in I}$ is in $\boldsymbol{\tau}$ and $\{V_{i,j} \to U_i\}_{j\in J}$ is in $\boldsymbol{\tau}$ for each i, then the composition $\{V_{i,j} \to X\}_{(i,j)\in I\times J}$ is in $\boldsymbol{\tau}$,
- If $\{U_i \to X\}_{i\in I}$ is in $\boldsymbol{\tau}$ and $V \to X$ is a morphism, then $U_i \times_X V$ exists and $\{U_i \times_X V \to V\}_{i\in I}$ is in $\boldsymbol{\tau}$.

We will call the covers in our Grothendieck pre-topology *$\boldsymbol{\tau}$-covering families* or *$\boldsymbol{\tau}$-covers*. We will call a category $\mathcal{M}$ equipped with a pre-topology $\boldsymbol{\tau}$ on $\mathcal{M}$ *an* $(\infty, 1)$*-site*.

Suppose that we have a Grothendieck pre-topology $\boldsymbol{\tau}$ on $\mathrm{Ho}(\mathcal{M})$. We note that $\boldsymbol{\tau}$ generates a Grothendieck topology on $\mathrm{Ho}(\mathcal{M})$ by defining the covering sieves to be those which contain $\boldsymbol{\tau}$-covering families. Conversely, when $\mathrm{Ho}(\mathcal{M})$ has pullbacks, any Grothendieck topology defines a pre-topology whose covering families are families of morphisms which generate covering sieves.

Definition 3.1.3.3. A presheaf $\mathcal{F}$ in $\mathbf{PSh}(\mathcal{M})$ satisfies *(Čech-)descent for* $\boldsymbol{\tau}$*-covers* if, whenever we have a $\boldsymbol{\tau}$-covering family $\{U_i \to X\}_{i\in I}$, we have an equivalence of ∞-groupoids

$$\mathcal{F}(X) \simeq \mathrm{Map}_{\mathbf{PSh}(\mathcal{M})}(h(X), \mathcal{F}) \to \mathrm{Map}_{\mathbf{PSh}(\mathcal{M})}(|\mathcal{U}_*|, \mathcal{F})$$

where $\mathcal{U}_* \to h(X)$ is the Čech nerve of the morphism $\mathcal{U} = \coprod_{i\in I} h(U_i) \to h(X)$ in $\mathbf{PSh}(\mathcal{M})$.

Proposition 3.1.3.4. *Let* $(\mathcal{M}, \boldsymbol{\tau})$ *be an* $(\infty,1)$*-site. The category of presheaves satisfying descent for* $\boldsymbol{\tau}$*-covers is precisely the category of sheaves* $\mathbf{Sh}(\mathcal{M}, \boldsymbol{\tau})$*.*

Proof. It suffices to show that $|\mathcal{U}_*|$ corresponds to the sieve generated by the covering family $\{U_i \to X\}_{i\in I}$. Indeed, this follows because $\mathcal{U}_*$ computes (-1)-truncations in $\mathbf{PSh}(\mathcal{M})_{/h(X)}$ by [52, Proposition 6.2.3.4] and therefore defines a monomorphism $|\mathcal{U}_*| \to h(X)$, and hence a sieve on X by [52, Proposition 6.2.2.5]. □

We note however that the category of $(\infty,1)$-sheaves is not hypercomplete in the sense of [52, Section 6.5.2], and therefore doesn't satisfy Whitehead's theorem. Given an $(\infty,1)$-site $(\mathcal{M}, \boldsymbol{\tau})$, we will define the category $\mathbf{Stk}(\mathcal{M}, \boldsymbol{\tau})$ of ∞-stacks to be the hypercompletion of $\mathbf{Sh}(\mathcal{M}, \boldsymbol{\tau})$. Equivalently, we can describe this category as follows.

We will say that a morphism $f : \coprod_{i\in I} h(U_i) \to \coprod_{j\in J} h(V_j)$ in $\mathbf{PSh}(\mathcal{M})$ is a *generalised* $\boldsymbol{\tau}$*-cover* if, for each $j \in J$, the family of morphisms $\{U_k \to V_j\}_{k\in K}$ corresponds to a $\boldsymbol{\tau}$-cover [24, c.f. p.13], where $k \in K$ if there is a map $h(U_k) \to h(V_j)$.

Definition 3.1.3.5. An augmented simplicial object $\mathcal{U}_* \to h(X)$ in $\mathbf{PSh}(\mathcal{M})$ is a *pseudo-representable* $\boldsymbol{\tau}$*-hypercover of* $h(X)$ if each $\mathcal{U}_n$ is a coproduct of representables and, for each $n \geq 0$, the map $\mathcal{U}_n \to (cosk_{n-1}\mathcal{U}_*)_n$ corresponds to a generalised $\boldsymbol{\tau}$-cover of $(cosk_{n-1}\mathcal{U}_*)_n$.

Remark. We note that the Čech nerve $\mathcal{U}_*$ of the morphism $\coprod_{i\in I} h(U_i) \to h(X)$ associated to a $\boldsymbol{\tau}$-cover $\{U_i \to X\}_{i\in I}$ is a $\boldsymbol{\tau}$-hypercover of height zero; the morphisms $\mathcal{U}_n \to (cosk_{n-1}\mathcal{U}_*)_n$ are all isomorphisms.

Definition 3.1.3.6. 1. A presheaf $\mathcal{F}$ satisfies *descent for $\boldsymbol{\tau}$-hypercovers* if, whenever we have a pseudo-representable $\boldsymbol{\tau}$-hypercover $\mathcal{U}_* \to h(X)$, we have an equivalence of ∞-groupoids

$$\mathcal{F}(X) \simeq \mathrm{Map}_{\mathbf{PSh}(\mathcal{M})}(|\mathcal{U}_*|, \mathcal{F})$$

2. The category of *∞-stacks* (or *hypercomplete* $(\infty,1)$*-sheaves*) is the full subcategory of $\mathbf{PSh}(\mathcal{M})$ consisting of presheaves satisfying descent for $\boldsymbol{\tau}$-hypercovers.

Remark. We note that, by [52, Proposition 6.5.2.14], if M is a small ordinary category equipped with a Grothendieck pre-topology $\boldsymbol{\tau}$ and A is the category of simplicial presheaves on M endowed with the local model structure, then $\mathcal{A} = \mathbf{L}^H(\mathrm{A})$ is equivalent to $\mathbf{Stk}(\mathcal{M}, \boldsymbol{\tau})$ where $\mathcal{M} = \mathbf{L}^H(\mathrm{M})$.

The category of ∞-stacks is an $(\infty,1)$-topos, and is moreover a reflective subcategory of $\mathbf{Sh}(\mathcal{M}, \boldsymbol{\tau})$ [52, c.f. Section 6.5.2]. Therefore, it is complete and cocomplete. Moreover, we note that the stackification functor $\mathbf{PSh}(\mathcal{M}) \to \mathbf{Stk}(\mathcal{M}, \boldsymbol{\tau})$ preserves colimits and finite limits.

3.1.4 Continuous and Cocontinuous Functors

Recall that there is an $(\infty,1)$-adjunction

$$i : \mathrm{Set} \leftrightarrows \infty\mathbf{Grpd} : \pi_0$$

Suppose that $(\mathcal{M}, \boldsymbol{\tau})$ is an $(\infty,1)$-site and that $\mathrm{F} : \mathrm{Ho}(\mathcal{M})^{op} \to \mathrm{Set}$ is an ordinary presheaf. We will denote the ordinary categories of presheaves and sheaves by $\mathrm{PSh}(\mathrm{Ho}(\mathcal{M}))$ and $\mathrm{Sh}(\mathrm{Ho}(\mathcal{M}), \boldsymbol{\tau})$ respectively. We let $i^{\boldsymbol{\tau}}(\mathrm{F})$ be the ∞-stackification of the $(\infty,1)$-presheaf

$$\begin{aligned} i : \mathcal{M}^{op} &\to \infty\mathbf{Grpd} \\ A &\to i(\mathrm{F}(A)) \end{aligned}$$

on the $(\infty,1)$-site $(\mathcal{M}, \boldsymbol{\tau})$.

Now, suppose that $\mathcal{F} : \mathcal{M}^{op} \to \infty\mathbf{Grpd}$ is an $(\infty,1)$-presheaf on $\mathcal{M}$. We let $\pi_0^{\boldsymbol{\tau}}(\mathcal{F})$ be the sheaf associated to the presheaf

$$\begin{aligned} \pi_0 : \mathrm{Ho}(\mathcal{M})^{op} &\to \mathrm{Set} \\ A &\to \pi_0(\mathcal{F}(A)) \end{aligned}$$

on the site $(\mathrm{Ho}(\mathcal{M}), \boldsymbol{\tau})$. If we denote by $\tau_{\leq 0}$ the 0-truncation functor defined in [52, Proposition 5.5.6.18] then we can easily see that $\pi_0^{\boldsymbol{\tau}}$ defines an equivalence of ordinary categories

$$\mathrm{Ho}(\tau_{\leq 0}(\mathbf{Stk}(\mathcal{M}, \boldsymbol{\tau}))) \simeq \mathrm{Sh}(\mathrm{Ho}(\mathcal{M}), \boldsymbol{\tau})$$

By [52, Proposition 7.2.1.14], a morphism in an $(\infty,1)$-topos $\mathcal{M}$ is an effective epimorphism precisely if its 0-truncation is an epimorphism in $\mathrm{Ho}(\tau_{\leq 0}\mathcal{M})$. This motivates the following definition.

Definition 3.1.4.1. A morphism of stacks $f:\mathcal{F}\to\mathcal{G}$ in $\mathbf{Stk}(\mathcal{M},\boldsymbol{\tau})$ is an *epimorphism* if the induced morphism $\pi_0^{\boldsymbol{\tau}}(f):\pi_0^{\boldsymbol{\tau}}(\mathcal{F})\to\pi_0^{\boldsymbol{\tau}}(\mathcal{G})$ is an epimorphism in $\mathrm{Sh}(\mathrm{Ho}(\mathrm{C}),\boldsymbol{\tau})$.

Using [52, Corollary 6.2.3.12] and [52, Proposition 6.2.3.14], we easily see that epimorphisms of stacks are stable by compositions, equivalences, and pullbacks.

Definition 3.1.4.2. Suppose that $(\mathcal{M},\boldsymbol{\tau})$ and $(\mathcal{N},\boldsymbol{\sigma})$ are $(\infty,1)$-sites, and that we have a functor $F:\mathcal{M}\to\mathcal{N}$. Then, F is a *continuous functor of* $(\infty,1)$*-sites* if

1. Whenever $\{U_i\to X\}_{i\in I}$ is a $\boldsymbol{\tau}$-cover, then $\{F(U_i)\to F(X)\}_{i\in I}$ is a $\boldsymbol{\sigma}$-cover,
2. For any morphism $X\to Y$ in $\mathcal{M}$ and every $\boldsymbol{\tau}$-cover $\{U_i\to Y\}_{i\in I}$, we have an isomorphism
$$F(X\times_Y U_i)\simeq F(X)\times_{F(Y)}F(U_i)$$

Suppose that $F:\mathcal{M}\to\mathcal{N}$ is a functor of $(\infty,1)$-categories and consider the precomposition functor $F^*:\mathbf{PSh}(\mathcal{N})\to\mathbf{PSh}(\mathcal{M})$. Then, by left and right $(\infty,1)$-Kan extension, we obtain a chain of adjunctions $F_!\dashv F^*\dashv F_*$ on the level of presheaves. Now, if we have a continuous functor $F:(\mathcal{M},\boldsymbol{\tau})\to(\mathcal{N},\boldsymbol{\sigma})$ of $(\infty,1)$-sites, then $\boldsymbol{\tau}$-covers get sent to $\boldsymbol{\sigma}$-covers, and we obtain a functor on stacks

$$F^*:\mathbf{Stk}(\mathcal{N},\boldsymbol{\sigma})\to\mathbf{Stk}(\mathcal{M},\boldsymbol{\tau})$$

with left adjoint
$$F_\#:\mathbf{Stk}(\mathcal{M},\boldsymbol{\tau})\to\mathbf{Stk}(\mathcal{N},\boldsymbol{\sigma})$$

given by composing the stackification functor with $F_!$. We remark that, for a representable stack $\mathcal{F}\simeq\mathrm{Map}_{\mathcal{M}}(-,X)$, $F_\#$ acts by $F_\#(\mathcal{F})=\mathrm{Map}_{\mathcal{N}}(-,F(X))$. We refer the reader to [64, Section 2] for full proofs of these statements.

Proposition 3.1.4.3. *Suppose that $F:(\mathcal{M},\boldsymbol{\tau})\to(\mathcal{N},\boldsymbol{\sigma})$ is a continuous functor of $(\infty,1)$-sites. Then, $F_\#$ preserves epimorphisms of ∞-stacks.*

Proof. Suppose that $f:\mathcal{F}\to\mathcal{G}$ is an epimorphism of stacks. It suffices to show that $\pi_0^{\boldsymbol{\sigma}}(F_\#(f)):\pi_0^{\boldsymbol{\sigma}}(F_\#(\mathcal{F}))\to\pi_0^{\boldsymbol{\sigma}}(F_\#(\mathcal{G}))$ is an epimorphism in the category of sheaves. We note that, if $\mathcal{H}$ is any stack over an $(\infty,1)$-site $(\mathcal{N},\boldsymbol{\sigma})$, then $\pi_0^{\boldsymbol{\sigma}}(\tilde{\mathcal{H}})\simeq\pi_0^{\boldsymbol{\sigma}}(\mathcal{H})$, where $\tilde{\mathcal{H}}$ is $\mathcal{H}$, considered as a presheaf. Hence, it suffices to show that

$$\pi_0^{\boldsymbol{\sigma}}(F_!(f)):\pi_0^{\boldsymbol{\sigma}}(F_!(\mathcal{F}))\to\pi_0^{\boldsymbol{\sigma}}(F_!(\mathcal{G}))$$

is an epimorphism of sheaves. Since both $\pi_0 : \infty\mathbf{Grpd} \to \mathrm{Set}$ and the sheafification functor commutes with left $(\infty,1)$-Kan extensions, we have an equivalence

$$\pi_0^{\boldsymbol{\sigma}}(F_!(\mathcal{F})) \simeq \mathrm{Ho}(F)_{\#}(\pi_0^{\boldsymbol{\tau}}(\mathcal{F}))$$

The result follows since $\mathrm{Ho}(F)_{\#}$, being a left adjoint, must preserve the epimorphism of sheaves given by $\pi_0^{\boldsymbol{\tau}}(f) : \pi_0^{\boldsymbol{\tau}}(\mathcal{F}) \to \pi_0^{\boldsymbol{\tau}}(\mathcal{G})$. $\square$

Definition 3.1.4.4. Suppose that $(\mathcal{M}, \boldsymbol{\tau})$ and $(\mathcal{N}, \boldsymbol{\sigma})$ are $(\infty,1)$-sites, and that we have a functor $F : \mathcal{M} \to \mathcal{N}$. Then, F is a *cocontinuous functor of* $(\infty,1)$*-sites* if, whenever $\{Y_j \to F(X)\}_{j\in J}$ is a $\boldsymbol{\sigma}$-cover, there exists a $\boldsymbol{\tau}$-cover $\{U_i \to X\}_{i\in I}$ such that the family of maps $\{F(U_i) \to F(X)\}_{i\in I}$ refines the cover $\{Y_j \to F(X)\}_{j\in J}$.

In the same setting as before, we see that if $F : (\mathcal{M}, \boldsymbol{\tau}) \to (\mathcal{N}, \boldsymbol{\sigma})$ is a cocontinuous functor of $(\infty,1)$-sites, then we obtain a functor $F_* : \mathbf{Stk}(\mathcal{M}, \boldsymbol{\tau}) \to \mathbf{Stk}(\mathcal{N}, \boldsymbol{\sigma})$ whose left adjoint is F^*. Similarly to Proposition 3.1.4.3, we have the following result.

Proposition 3.1.4.5. *Suppose that* $F : (\mathcal{M}, \boldsymbol{\tau}) \to (\mathcal{N}, \boldsymbol{\sigma})$ *is a cocontinuous functor of sites. Then,* F^* *preserves epimorphisms of* ∞*-stacks.*

Suppose that we have an $(\infty,1)$-site $(\mathcal{M}, \boldsymbol{\tau})$. We will say that an object X of $\mathcal{M}$ is *admissible* if $h(X) := \mathrm{Map}_{\mathcal{M}}(-, X)$ is a stack. By an application of [52, c.f. Corollary 5.1.5.8] along with the statement that stackification preserves colimits, we see that if every $X \in \mathcal{M}$ is admissible, then the Yoneda embedding generates $\mathbf{Stk}(\mathcal{M}, \boldsymbol{\tau})$ under small colimits.

Proposition 3.1.4.6. *Suppose that* $F : (\mathcal{M}, \boldsymbol{\tau}) \to (\mathcal{N}, \boldsymbol{\sigma})$ *is a fully faithful, continuous, and cocontinuous functor of sites, and that every* $X \in \mathcal{M}$ *is admissible. Then, the induced functor* $F_{\#} : \mathbf{Stk}(\mathcal{M}, \boldsymbol{\tau}) \to \mathbf{Stk}(\mathcal{N}, \boldsymbol{\sigma})$ *is fully faithful.*

Proof. Since $F_{\#}$ has a right adjoint F^*, it suffices to show that there is an equivalence of $(\infty,1)$-functors $F^* \circ F_{\#} \simeq \mathrm{id}_{\mathbf{Stk}(\mathcal{M},\boldsymbol{\tau})}$ by [52, Proposition 5.2.7.4]. Since $F_{\#}$ and F^* commute with colimits and every stack in $\mathbf{Stk}(\mathcal{M}, \boldsymbol{\tau})$ is a colimit of objects of the form $h(X)$ for $X \in \mathcal{M}$, it suffices to check on objects of the form $h(X)$. By fully faithfulness of F we have that

$$F^* \circ F_{\#}(h(X)) = F^*(\mathrm{Map}_{\mathcal{N}}(-, F(X))) \simeq \mathrm{Map}_{\mathcal{N}}(F(-), F(X)) \simeq h(X)$$

$\square$

3.2 Homotopical Algebraic Geometry

In this section, we will provide an analogue of Toën and Vezzosi's *HAG (homotopical algebraic geometry) contexts* [86] suitable for our applications.

3.2.1 Geometries

The following definition of an $(\infty,1)$-pre-geometry triple was first stated in [41]. As a guiding example, it may help to think of the classical algebraic geometry triple with $\mathcal{M}$ as the ordinary category of affines, $\boldsymbol{\tau}$ the étale topology, and $\mathbf{P}$ the class of smooth maps.

Definition 3.2.1.1. [41, Definition 6.3] An $(\infty,1)$-*pre-geometry triple* is a triple $(\mathcal{M},\boldsymbol{\tau},\mathbf{P})$ where $\mathcal{M}$ is an $(\infty,1)$-category, $\boldsymbol{\tau}$ is a Grothendieck pre-topology on $\mathrm{Ho}(\mathcal{M})$, and $\mathbf{P}$ is a class of maps in $\mathcal{M}$ such that

1. If $\{U_i \to X\}_{i\in I}$ is a $\boldsymbol{\tau}$-covering family, then each $U_i \to X$ is in $\mathbf{P}$,
2. $\mathbf{P}$ is *local for the topology* $\boldsymbol{\tau}$, in the sense that, whenever we have a morphism $f: Y \to X$ in $\mathcal{M}$ along with a $\boldsymbol{\tau}$-covering family $\{U_i \to Y\}_{i\in I}$ such that each induced morphism $U_i \to X$ is in $\mathbf{P}$, then $f \in \mathbf{P}$,
3. The class $\mathbf{P}$ is stable under equivalences, compositions, and pullbacks.

An $(\infty,1)$-pre-geometry triple is said to be an $(\infty,1)$-*geometry triple* if every object of $\mathcal{M}$ is admissible.

Often it is not clear in certain settings when a class of maps $\mathbf{P}$ is local for the topology. We can let $\mathbf{P}^{\boldsymbol{\tau}}$ be the class of maps $f: Y \to X$ in $\mathcal{M}$ such that there is a $\boldsymbol{\tau}$-cover $\{g_i : U_i \to Y\}_{i\in I}$ with $f \circ g_i \in \mathbf{P}$. Then, we see that $\mathbf{P}^{\boldsymbol{\tau}}$ is local and $\mathbf{P} \subseteq \mathbf{P}^{\boldsymbol{\tau}}$ with equality if and only if $\mathbf{P}$ is local for the $\boldsymbol{\tau}$-topology. By [10, Proposition 6.1.4], if morphisms in $\mathbf{P}$ are stable by equivalences, compositions, and pushouts, then so are morphisms in $\mathbf{P}^{\boldsymbol{\tau}}$.

A philosophy we use throughout this thesis is that our categories of interest, generally some subcategory of 'affines', should be embedded in larger categories with more useful properties. This leads naturally to talking about the notion of a relative geometry tuple as described in [41, Definition 6.4].

Definition 3.2.1.2.

1. A *relative* $(\infty,1)$-*pre-geometry tuple* consists of a tuple $(\mathcal{M},\boldsymbol{\tau},\mathbf{P},\mathcal{A})$ with $(\mathcal{M},\boldsymbol{\tau},\mathbf{P})$ an $(\infty,1)$-pre-geometry triple and $\mathcal{A}$ a full subcategory of $\mathcal{M}$ such that, if $f: Y \to X$ is a map in $\mathbf{P} \cap \mathcal{A}$ and $Z \to X$ is any map with Z in $\mathcal{A}$, then $Y \times_X Z$ exists and is in $\mathcal{A}$,

2. A relative $(\infty, 1)$-pre-geometry tuple $(\mathcal{M}, \boldsymbol{\tau}, \mathbf{P}, \mathcal{A})$ is *strong* if, whenever we have a cover $\{U_i \to X\}_{i \in I}$ in $\mathcal{M}$ and $Y \to X$ is a map with $Y \in \mathcal{A}$, then $\{U_i \times_X Y \to Y\}_{i \in I}$ is a cover in $\boldsymbol{\tau}|_{\mathcal{A}}$, where $\boldsymbol{\tau}|_{\mathcal{A}}$ denotes the restriction of $\boldsymbol{\tau}$ to $\mathcal{A}$.

Remark. We note that if we have a Grothendieck pre-topology defined on $\mathcal{A}$, then we can extend it to a Grothendieck pre-topology on $\mathcal{M}$ such that the resulting topology is strong relative to $\mathcal{A}$.

Definition 3.2.1.3. [41, Definition 6.6] Let $(\mathcal{M}, \boldsymbol{\tau}, \mathbf{P}, \mathcal{A})$ be a (strong) relative $(\infty, 1)$-pre-geometry tuple.

1. An object X of $\mathcal{M}$ is said to be $\mathcal{A}$-*admissible* if the restriction of $\mathrm{Map}_{\mathcal{M}}(-, X)$ to $\mathcal{A}$ is a stack in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$,
2. $(\mathcal{M}, \boldsymbol{\tau}, \mathbf{P}, \mathcal{A})$ is a *(strong) relative* $(\infty, 1)$-*geometry tuple* if each $X \in \mathcal{A}$ is $\mathcal{A}$-admissible.

We note that, if $(\mathcal{M}, \boldsymbol{\tau}, \mathbf{P}, \mathcal{A})$ is a relative $(\infty, 1)$-pre-geometry tuple, then if we restrict our topology $\boldsymbol{\tau}$ and our class of maps $\mathbf{P}$, we see that $(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}}, \mathbf{P}|_{\mathcal{A}})$ is an $(\infty, 1)$-pre-geometry triple. We can also construct strong tuples using the following lemma.

Lemma 3.2.1.4. *To any relative* $(\infty, 1)$*-pre-geometry tuple* $(\mathcal{M}, \boldsymbol{\tau}, \mathbf{P}, \mathcal{A})$ *we can define an associated strong relative* $(\infty, 1)$*-pre-geometry tuple* $(\mathcal{M}, \boldsymbol{\tau}_{\mathcal{A}}, \mathbf{P}_{\mathcal{A}}, \mathcal{A})$.

Proof. This follows by taking $\mathbf{P}_{\mathcal{A}} \subseteq \mathbf{P}$ to be the class of maps $f : Y \to X$ in $\mathbf{P}$ such that, whenever $Z \to X$ is a map with $Z \in \mathcal{A}$, then $Y \times_X Z$ is in $\mathcal{A}$. Our topology $\boldsymbol{\tau}_{\mathcal{A}}$ will be the class of covers $\{U_i \to X\}_{i \in I}$ in $\boldsymbol{\tau}$ such that, whenever there exists some map $Y \to X$ in $\mathcal{M}$, then $U_i \times_X Y$ is in $\mathcal{A}$ if and only if Y is in $\mathcal{A}$. $\square$

Example 3.2.1.5. *In the model category theoretic version of Toën and Vezzosi [86], a homotopical algebraic geometry (HAG) context is a tuple* $(\mathrm{C}, \mathrm{C}_0, \mathrm{A}, \boldsymbol{\tau}, \mathbf{P})$. *Here,* A *is taken to be a full subcategory of* $\mathrm{Comm}(\mathrm{C})$*, the category of commutative monoids in* C. *When we take* $\mathrm{C}_0 = \mathrm{C}$ *and consider the associated* $(\infty, 1)$*-categories, then* $(\mathbf{Aff}_{\mathbf{L}^H(\mathrm{C})}, \boldsymbol{\tau}, \mathbf{P}, \mathbf{L}^H(\mathrm{A})^{op})$ *is a relative* $(\infty, 1)$*-geometry tuple, where*

$$\mathbf{Aff}_{\mathbf{L}^H(\mathrm{C})} := \mathbf{L}^H(\mathrm{Comm}(\mathrm{C}))^{op}$$

Consider the inclusion functor $i : \mathbf{PSh}(\mathcal{A}) \to \mathbf{PSh}(\mathcal{M})$, which induces a functor on the level of stacks $i_\# : \mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}}) \to \mathbf{Stk}(\mathcal{M}, \boldsymbol{\tau})$. The following result shows that, in the case where the geometry tuple is strong, we can consider stacks on $\mathcal{A}$ by realising them as stacks on $\mathcal{M}$.

Proposition 3.2.1.6. *[41, Proposition 6.7] If $(\mathcal{M}, \boldsymbol{\tau}, \mathbf{P}, \mathcal{A})$ is a strong relative $(\infty, 1)$-geometry tuple, then $i_\#$ is fully faithful.*

3.2.2 Geometric Stacks

Fix a relative $(\infty, 1)$-pre-geometry tuple $(\mathcal{M}, \boldsymbol{\tau}, \mathbf{P}, \mathcal{A})$. We will say that $X \in \mathcal{A}$ is a *representable stack* if X is $\mathcal{A}$-admissible.

Definition 3.2.2.1.

1. A stack $\mathcal{F}$ in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_\mathcal{A})$ is (-1)-*geometric* if it is of the form $\mathcal{F} \simeq \mathrm{Map}_\mathcal{M}(-, X)$ for some $\mathcal{A}$-admissible $X \in \mathcal{M}$,
2. A morphism of stacks $f : \mathcal{F} \to \mathcal{G}$ in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_\mathcal{A})$ is (-1)-*representable* if, for any map $X \to \mathcal{G}$, with $X \in \mathcal{A}$ a representable stack, the pullback $\mathcal{F} \times_\mathcal{G} X$ is (-1)-geometric,
3. A morphism of stacks $f : \mathcal{F} \to \mathcal{G}$ in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_\mathcal{A})$ is *in* (-1)-$\mathbf{P}$ if it is (-1)-representable and, for any map $X \to \mathcal{G}$, with $X \in \mathcal{A}$ a representable stack, the induced map of (-1)-geometric stacks $\mathcal{F} \times_\mathcal{G} X \to X$ is represented by a morphism in $\mathbf{P}$.

Remark. We will often, when it is clear from context, denote a (-1)-geometric stack by the object $X \in \mathcal{M}$ that represents it.

Now, for $n \geq 0$, we can inductively build up notions of higher geometric stacks by glueing together representables as follows.

Definition 3.2.2.2.

1. Let $\mathcal{F}$ be a stack in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_\mathcal{A})$. An *$n$-atlas* for $\mathcal{F}$ is a set of morphisms $\{U_i \to \mathcal{F}\}_{i \in I}$ such that each U_i is (-1)-geometric, each map $U_i \to \mathcal{F}$ is in $(n-1)$-$\mathbf{P}$, and there is an epimorphism of stacks
$$\coprod_{i \in I} U_i \to \mathcal{F}$$
in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_\mathcal{A})$,
2. A stack $\mathcal{F}$ in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_\mathcal{A})$ is *n-geometric* if the diagonal morphism $\mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is $(n-1)$-representable and $\mathcal{F}$ admits an n-atlas,
3. A morphism of stacks $f : \mathcal{F} \to \mathcal{G}$ in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_\mathcal{A})$ is *n-representable* if, for any map $X \to \mathcal{G}$ with $X \in \mathcal{A}$ a representable stack, the pullback $\mathcal{F} \times_\mathcal{G} X$ is n-geometric,
4. A morphism of stacks $f : \mathcal{F} \to \mathcal{G}$ in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_\mathcal{A})$ is in n-$\mathbf{P}$ if it is n-representable and, for any map $X \to \mathcal{G}$ with $X \in \mathcal{A}$ a representable stack, there exists an n-atlas of the form $\{U_i \to \mathcal{F} \times_\mathcal{G} X\}_{i \in I}$ such that each map $U_i \to X$ is in $\mathbf{P}$.

We easily see that the collection, $\mathbf{Stk}_n(\mathcal{M}, \boldsymbol{\tau}, \mathbf{P}, \mathcal{A})$, of n-geometric stacks in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$ is a subset of the collection of n-geometric stacks $\mathbf{Stk}_n(\mathcal{M}, \boldsymbol{\tau}, \mathbf{P}, \mathcal{M})$ in $\mathbf{Stk}(\mathcal{M}, \boldsymbol{\tau})$ when $(\mathcal{M}, \boldsymbol{\tau}, \mathbf{P}, \mathcal{M})$ is considered as a relative $(\infty, 1)$-pre-geometry tuple. Moreover, if $(\mathcal{M}, \boldsymbol{\tau}, \mathbf{P})$ is an $(\infty, 1)$-geometry tuple and $\mathcal{A}$ has all finite limits which are preserved by the inclusion, then there is an induced functor

$$i_{\#} : \mathbf{Stk}_n(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}}, \mathbf{P}|_{\mathcal{A}}, \mathcal{A}) \to \mathbf{Stk}_n(\mathcal{M}, \boldsymbol{\tau}, \mathbf{P}, \mathcal{A})$$

which is fully faithful if $(\mathcal{M}, \boldsymbol{\tau}, \mathbf{P}, \mathcal{A})$ is a strong tuple [41, Corollary 6.13].

Notation 3.2.2.3. We will want to specify when we are working with n-geometric stacks within the $(\infty, 1)$-geometry tuple $(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}}, \mathbf{P}|_{\mathcal{A}})$. Hence, we will abbreviate the category of n-geometric stacks $\mathbf{Stk}_n(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}}, \mathbf{P}|_{\mathcal{A}}, \mathcal{A})$ to $\mathbf{Stk}_n(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}}, \mathbf{P}|_{\mathcal{A}})$. We will refer to n-representable morphisms with respect to this geometry tuple by the notation n-representable$|_{\mathcal{A}}$. The notation n-$\mathbf{P}|_{\mathcal{A}}$ in this context should be clear.

Example 3.2.2.4. *Suppose that k is a commutative ring.*

1. *When $\mathcal{M} = \mathcal{A} = \mathrm{Aff}_k$, the category of affine schemes, $\boldsymbol{\tau}$ is the étale topology, and $\mathbf{P}$ is the class of smooth maps, n-geometric stacks correspond to algebraic n-stacks [86],*
2. *When $\mathcal{M} = \mathcal{A} = \mathbf{DAff}_k^{cn} := \mathbf{L}^H(\mathrm{Comm}(\mathrm{sMod}_k))^{op}$, $\boldsymbol{\tau}$ is the étale topology, and $\mathbf{P}$ is the class of smooth morphisms (see Section 5.2.2) we obtain the derived algebraic geometry context of Toën and Vezzosi. Several examples of n-geometric stacks appear in this context such as the 1-geometric stack of rank n-vector bundles $\mathbf{Vect}_n$ [86, Section 2.2.6.1],*
3. *We will explore examples where $\mathcal{M}$ is not necessarily equal to $\mathcal{A}$ in Section 5.2.*

In a similar way to [86, Proposition 1.3.3.3], we can prove the following statements about n-geometric stacks.

Proposition 3.2.2.5.

1. *$\mathcal{F}$ is an n-geometric stack if and only if the map $\mathcal{F} \to *$ is n-representable,*
2. *Any $(n-1)$-geometric stack is n-geometric,*
3. *Any $(n-1)$-representable morphism is n-representable,*
4. *Any $(n-1)$-$\mathbf{P}$-morphism is an n-$\mathbf{P}$-morphism,*
5. *n-representable morphisms are stable by isomorphisms, pullbacks and compositions,*
6. *n-$\mathbf{P}$-morphisms are stable by isomorphisms, pullbacks and compositions.*

In the course of proving that n-representable morphisms are stable by composition we obtain the following result.

Corollary 3.2.2.6. *Suppose that $f : \mathcal{F} \to \mathcal{G}$ is an n-representable morphism of stacks. If $\mathcal{G}$ is n-geometric, then so is $\mathcal{F}$.*

Proposition 3.2.2.7. *Suppose that we have morphisms $\mathcal{F} \to \mathcal{G}$ and $\mathcal{H} \to \mathcal{G}$ with $\mathcal{F}, \mathcal{H}$ n-geometric stacks and suppose that the diagonal morphism $\mathcal{G} \to \mathcal{G} \times \mathcal{G}$ is n-representable for some $n \geq -1$. Then, $\mathcal{F} \times_{\mathcal{G}} \mathcal{H}$ is an n-geometric stack.*

Proof. We will show that the morphism $\mathcal{F} \times_{\mathcal{G}} \mathcal{H} \to *$ is n-representable. We have the following pullback diagram,

$$\begin{tikzcd} \mathcal{F} \times_{\mathcal{G}} \mathcal{H} \arrow[r] \arrow[d] & \mathcal{G} \arrow[d] \\ \mathcal{F} \times \mathcal{H} \arrow[r] & \mathcal{G} \times \mathcal{G} \end{tikzcd}$$

By stability of n-representable maps under pullback, the map $\mathcal{F} \times_{\mathcal{G}} \mathcal{H} \to \mathcal{F} \times \mathcal{H}$ is n-representable. Moreover, if we consider the pullback square,

$$\begin{tikzcd} \mathcal{F} \times \mathcal{H} \arrow[r] \arrow[d] & \mathcal{H} \arrow[d] \\ \mathcal{F} \arrow[r] & * \end{tikzcd}$$

then we see that, since $\mathcal{F}$ is n-geometric, the morphism $\mathcal{F} \times \mathcal{H} \to \mathcal{H}$ is n-representable. Hence, since the composition of n-representable maps is n-representable, the morphism $\mathcal{F} \times_{\mathcal{G}} \mathcal{H} \to *$ is n-representable. $\square$

Corollary 3.2.2.8. *The full subcategory of n-geometric stacks in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$ is stable under $(\infty, 1)$-pullbacks for $n \geq 0$.*

3.2.3 Closure under τ-descent

Fix a relative $(\infty, 1)$-geometry tuple $(\mathcal{M}, \mathbf{P}, \boldsymbol{\tau}, \mathcal{A})$. Suppose that we have a $\boldsymbol{\tau}$-cover $\{U_i \to X\}_{i \in I}$ in $\mathcal{M}$. Then, this defines an epimorphism of stacks $\coprod_{i \in I} U_i \to X$ in the sense of Definition 3.1.4.1. However, it does not follow that every epimorphism of representable stacks defines a $\boldsymbol{\tau}$-cover.

We will say that a stack $\mathcal{F}$ is *coverable* if there is an epimorphism of stacks $\coprod_{i \in I} U_i \to \mathcal{F}$ with each U_i a (-1)-geometric stack on $\mathcal{A}$. In this case, we will say that $\mathcal{F}$ is *covered by* $\{U_i\}_{i \in I}$.

Proposition 3.2.3.1. *Suppose that $\mathcal{F}$ is a coverable stack in* $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$*, covered by $\{U_i\}_{i\in I}$, and that $X \in \mathcal{A}$ is a representable stack along with a morphism $X \to \mathcal{F}$. Then, there exists a $\boldsymbol{\tau}$-cover $\{V_j \to X\}_{j\in J}$ in $\mathcal{A}$, a morphism $u : J \to I$ and, for all j, a commutative diagram in* $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$,

$$\begin{tikzcd} V_j \arrow[r] \arrow[d] & U_{u(j)} \arrow[d] \\ X \arrow[r] & \mathcal{F} \end{tikzcd}$$

Proof. Since $\mathcal{F}$ is coverable, there is an epimorphism of stacks $\coprod_{i\in I} U_i \to \mathcal{F}$ corresponding to an epimorphism of sheaves $\coprod_{i\in I} \pi_0^{\boldsymbol{\tau}}(U_i) \to \pi_0^{\boldsymbol{\tau}}(\mathcal{F})$ in $\mathrm{Sh}(\mathrm{Ho}(\mathcal{A}), \boldsymbol{\tau}|_{\mathcal{A}})$. It is known (see [59, Corollary III.7.5]) that a morphism of sheaves is an epimorphism if and only if it satisfies a local surjectivity property. Given the map $\pi_0^{\boldsymbol{\tau}}(X) \to \pi_0^{\boldsymbol{\tau}}(\mathcal{F})$, this local surjectivity property asserts that there exists a $\boldsymbol{\tau}$-cover $\{V_j \to X\}_{j\in J}$ and a morphism $u : J \to I$ such that the following diagram commutes in $\mathrm{Sh}(\mathrm{Ho}(\mathcal{A}), \boldsymbol{\tau}|_{\mathcal{A}})$,

$$\begin{tikzcd} V_j \arrow[r] \arrow[d] & \pi_0^{\boldsymbol{\tau}}(U_{u(j)}) \arrow[d] \\ \pi_0^{\boldsymbol{\tau}}(X) \arrow[r] & \pi_0^{\boldsymbol{\tau}}(\mathcal{F}) \end{tikzcd}$$

Under the inclusion map $i^{\boldsymbol{\tau}} : \mathrm{Sh}(\mathrm{Ho}(\mathcal{A}), \boldsymbol{\tau}|_{\mathcal{A}}) \to \mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$, we note that representable sheaves get sent to representable stacks since every element of $\mathcal{A}$ is $\mathcal{A}$-admissible. We have the following commutative diagram

$$\begin{tikzcd} V_j \arrow[r] \arrow[d] & U_{u(j)} \arrow[d] \\ X \arrow[r] & i^{\boldsymbol{\tau}} \circ \pi_0^{\boldsymbol{\tau}}(\mathcal{F}) \end{tikzcd}$$

Using the counit of the adjunction $i \circ \pi_0 \to \mathrm{id}$, we obtain a map $i^{\boldsymbol{\tau}} \circ \pi_0^{\boldsymbol{\tau}}(\mathcal{F}) \to \mathcal{F}$ such that the required diagram commutes. □

Definition 3.2.3.2. [65, c.f. Definition 8.3] We say that $\mathcal{A}$ is *closed under $\boldsymbol{\tau}$-descent relative to $\mathcal{M}$* if, for any stack $\mathcal{F}$ in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$, any morphism $\mathcal{F} \to X$ with X a (-1)-geometric stack, and any $\boldsymbol{\tau}$-cover $\{U_i \to X\}_{i\in I}$ of $\mathcal{A}$-admissible objects, we have that, if $\mathcal{F} \times_X U_i$ is a (-1)-geometric stack for every $i \in I$, then so is $\mathcal{F}$.

When $\mathcal{A}$ is closed under $\boldsymbol{\tau}$-descent we get several useful criteria for a stack to be n-geometric, including the following local condition for checking geometricity.

Lemma 3.2.3.3. *Suppose that $\mathcal{A}$ is closed under $\boldsymbol{\tau}$-descent. Let $f : \mathcal{F} \to \mathcal{G}$ be any morphism of stacks in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$ with $\mathcal{G}$ an n-geometric stack. Suppose that there exists an n-atlas $\{U_i \to \mathcal{G}\}_{i \in I}$ of $\mathcal{G}$ such that each stack $\mathcal{F} \times_{\mathcal{G}} U_i$ is n-geometric. Then, $\mathcal{F}$ is n-geometric.*

Furthermore, if every map $\mathcal{F} \times_{\mathcal{G}} U_i \to U_i$ is in n-$\mathbf{P}$, then so is f.

Proof. We proceed by induction on n. Indeed, when $n = -1$ the result follows since $\mathcal{A}$ is closed under $\boldsymbol{\tau}$-descent. If $n \geq 0$, then it suffices to show that f is n-representable by Corollary 3.2.2.6. Suppose that X is a representable stack along with a morphism $X \to \mathcal{G}$. It remains to show that $\mathcal{F} \times_{\mathcal{G}} X$ is n-geometric for all representable stacks X. Suppose that $\{U_i \to \mathcal{G}\}_{i \in I}$ is an n-atlas for $\mathcal{G}$. By Proposition 3.2.3.1, we can construct a $\boldsymbol{\tau}$-covering family $\{V_j \to X\}_{j \in J}$ for X such that the morphism $V_j \to \mathcal{G}$ factors through $U_{u(j)}$ for some morphism $u : J \to I$. Since the induced map

$$\mathcal{F} \times_{\mathcal{G}} V_j \to \mathcal{F} \times_{\mathcal{G}} U_{u(j)}$$

is n-representable for all $n \geq -1$ and $\mathcal{F} \times_{\mathcal{G}} U_{u(j)}$ is n-geometric by assumption then, by Corollary 3.2.2.6, it follows that each $\mathcal{F} \times_{\mathcal{G}} V_j \simeq (\mathcal{F} \times_{\mathcal{G}} U_{u(j)}) \times_{U_{u(j)}} V_j$ is n-geometric. Furthermore, if each map $\mathcal{F} \times_{\mathcal{G}} U_i \to U_i$ is in n-$\mathbf{P}$, then each map $\mathcal{F} \times_{\mathcal{G}} V_j \to V_j$ is also in n-$\mathbf{P}$ since n-$\mathbf{P}$ morphisms are stable under pullback. Therefore, it suffices to reduce to the situation where $\mathcal{G}$ is a (-1)-geometric stack. The proof then follows identically to [86, Proposition 1.3.3.4]. □

Corollary 3.2.3.4. *[86, c.f. Corollary 1.3.3.5] Suppose that $\mathcal{A}$ is closed under $\boldsymbol{\tau}$-descent relative to $\mathcal{M}$. The full subcategory of n-geometric stacks in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$ is closed under disjoint unions for $n \geq 0$.*

When $\mathcal{A}$ is closed under $\boldsymbol{\tau}$-descent relative to $\mathcal{M}$, the condition that the diagonal morphism $\mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is $(n-1)$-representable for an n-geometric stack $\mathcal{F}$ is superfluous.

Proposition 3.2.3.5. *[65, c.f. Corollary 8.6] Suppose that $\mathcal{A}$ is closed under $\boldsymbol{\tau}$-descent relative to $\mathcal{M}$. Suppose that $\mathcal{F}$ is a stack in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$ with an n-atlas $\{U_i \to \mathcal{F}\}_{i \in I}$ for some $n \geq 0$. Then, $\mathcal{F}$ is an n-geometric stack.*

We now want to explore some conditions under which $\mathcal{A}$ is closed under $\boldsymbol{\tau}$-descent relative to $\mathcal{M}$. Suppose that $\mathcal{F}$ is an n-geometric stack in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$ with n-atlas $\{U_i \to \mathcal{F}\}_{i \in I}$. Consider the Čech nerve, which we will denote by $\mathcal{U}_* \to \mathcal{F}$, of the epimorphism $\coprod_{i \in I} U_i \to \mathcal{F}$. We notice that

$$\mathcal{U}_m = \coprod_{\underline{i} \in I^{m+1}} U_{\underline{i}}$$

with $\underline{i} = (i_0, \ldots, i_m) \in I^{m+1}$ and $U_{\underline{i}} := U_{i_0} \times_{\mathcal{F}} U_{i_1} \times_{\mathcal{F}} \times \cdots \times_{\mathcal{F}} U_{i_m}$ in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$. Each $U_{\underline{i}}$ is an $(n-1)$-geometric stack since each morphism $U_{i_j} \to \mathcal{F}$ is $(n-1)$-representable and $(n-1)$-representable morphisms are stable under pullback and composition.

By Proposition 3.1.2.3, since an epimorphism of stacks is an effective epimorphism in the $(\infty, 1)$-topos $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$, we have the following result.

Proposition 3.2.3.6. *The natural morphism $|\mathcal{U}_*| := \varinjlim_{[m] \in \Delta} \mathcal{U}_m \to \mathcal{F}$ is an equivalence of stacks.*

Corollary 3.2.3.7. *Suppose that $\mathcal{M}$ is closed under finite coproducts and that coproducts are disjoint. If, for any finite family of $\mathcal{A}$-admissible objects $\{U_i\}_{i \in I}$ in $\mathcal{M}$, the family of morphisms*

$$\{U_i \to \coprod_{j \in I} U_j\}_{i \in I}$$

is a $\boldsymbol{\tau}$-covering family of $\coprod_{j \in I} U_j$, then the natural morphism

$$\coprod_{i \in I} h(U_i) \to h(\coprod_{i \in I} U_i)$$

is an equivalence of stacks in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$.

Proof. The case when I is empty follows easily as the empty family covers the initial object. Now, if I is non-empty it suffices, by induction, to consider the case where the finite family consists of two objects X, Y, say. We then see, by our assumption, that $\{X \to X \coprod Y, Y \to X \coprod Y\}$ is a $\boldsymbol{\tau}$-covering family, and hence there is an epimorphism of stacks

$$h(X) \coprod h(Y) \to h(X \coprod Y)$$

Therefore, $h(X \coprod Y)$ is equivalent to the geometric realisation of the Čech nerve of the epimorphism by Proposition 3.2.3.6. Hence, to show our desired equivalence it suffices to show that

$$h(X \times_{X \coprod Y} Y) \simeq h(X) \times_{h(X \coprod Y)} h(Y) \simeq \emptyset$$

which follows since coproducts are disjoint in $\mathcal{M}$. □

We consider the category $s\mathcal{M}$ of simplicial objects in $\mathcal{M}$. Suppose that $\mathcal{M}$ is closed under geometric realisations of simplicial objects. There exists an $(\infty, 1)$-adjunction

$$|-| : s\mathcal{M} \leftrightarrows \mathcal{M} : \mathrm{const}$$

where const denotes the constant functor taking an object in $\mathcal{M}$ to the constant simplicial object, and $|-|$ is the geometric realisation functor, taking $Y_* \in s\mathcal{M}$ to its geometric realisation $|Y_*| = \varinjlim_{[m]\in\Delta} Y_m$ in $\mathcal{M}$. Given an augmented simplicial object $U_* \to X \in \mathcal{M}$, this defines an adjunction on the over-categories

$$\int : s\mathcal{M}_{/U_*} \leftrightarrows \mathcal{M}_{/X} : - \times_X U_*$$

by [52, Lemma 5.2.5.2], where $\int$ takes an object $Y_* \to U_*$ of $s\mathcal{M}_{/U_*}$ to the object $|Y_*| \to |U_*| \to X$ in $\mathcal{M}_{/X}$. Motivated by the 'descent conditions' defined in [86] we make the following definitions.

Definition 3.2.3.8. 1. We say that a morphism of simplicial objects $Y_* \to U_*$ in $s\mathcal{M}$ is *co-cartesian* if, for each $[m] \to [n]$ in Δ, $Y_n \times_{U_n} U_m$ exists and the natural morphism $Y_m \to Y_n \times_{U_n} U_m$ is an isomorphism.

2. We say that an augmented simplicial object $U_* \to X$ in $s\mathcal{M}$ satisfies *the co-cartesian descent condition* if the unit of the above adjunction is an isomorphism when restricted to the full subcategory of $s\mathcal{M}_{/U_*}$ consisting of co-cartesian morphisms $Y_* \to U_*$.

Remark. We note that, if $U_* \in s\mathcal{M}$ satisfies the co-cartesian descent condition, then for all objects $Y_* \to U_*$ which are co-cartesian, $Y \times_X U_* \simeq Y_*$ where $Y = |Y_*|$.

Proposition 3.2.3.9. *Suppose that*

1. *$\mathcal{M}$ is closed under geometric realisations,*
2. *Any $\boldsymbol{\tau}$-cover has a finite subcover,*
3. *For any finite collection $\{U_i\}_{i\in I}$ of $\mathcal{A}$-admissible objects, the map $\coprod_{i\in I} h(U_i) \to h(\coprod_{i\in I} U_i)$ is an equivalence in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$,*
4. *The Čech nerve of any $\boldsymbol{\tau}$-cover of $\mathcal{A}$-admissible objects in $\mathcal{M}$ satisfies the co-cartesian descent condition.*

Then, $\mathcal{A}$ is closed under $\boldsymbol{\tau}$-descent relative to $\mathcal{M}$ in the sense of Definition 3.2.3.2.

Proof. Suppose that we have a stack $\mathcal{F}$ in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$ and a morphism $\mathcal{F} \to h(X)$ for some (-1)-geometric X. Suppose that we have a $\boldsymbol{\tau}$-cover $\{U_i \to X\}_{i\in I}$ such that $\mathcal{F} \times_{h(X)} h(U_i)$ is a (-1)-geometric stack for every $i \in I$. By Assumption (2), we may assume that I is a finite set. We want to show that $\mathcal{F}$ is a (-1)-geometric stack.

Consider the Čech nerve $\mathcal{U}_*$ of the epimorphism of stacks $\coprod_{i\in I} h(U_i) \to h(X)$. We note that, by our assumptions, $\mathcal{U}_m = h((U)_m)$ where $(U)_*$ is the Čech nerve of the morphism $\coprod_{i\in I} U_i \to X$ in $\mathcal{M}$. Moreover, by Proposition 3.2.3.6, there is an equivalence

$$|\mathcal{U}_*| := \varinjlim_{[m]\in\Delta} \mathcal{U}_m \to h(X)$$

We let $\mathcal{F}_m$ be the pullback $\mathcal{F} \times_{h(X)} \mathcal{U}_m$ which we know, by our assumptions, is representable for every m, say $\mathcal{F}_m = h(Y_m)$ for some $\mathcal{A}$-admissible $Y_m \in \mathcal{M}$. Consider the induced simplicial objects Y_* and $\mathcal{F}_*$ as objects of s$\mathcal{M}$ and s$\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$ respectively. There is a fully faithful functor $\underline{h}$: s$\mathcal{M} \to$ s$\mathbf{Stk}(\mathcal{M}, \boldsymbol{\tau}|_{\mathcal{A}})$ induced by the Yoneda embedding and we easily see that $\mathcal{F}_* = \underline{h}(Y_*)$. By construction, $\mathcal{F}_*$ is the Čech nerve of the morphism $\mathcal{F} \times_{h(X)} \coprod_{i\in I} h(U_i) \to \mathcal{F}$, and hence, by Proposition 3.2.3.6, the natural morphism

$$|\mathcal{F}_*| := \varinjlim_{[m]\in\Delta} \mathcal{F}_m \to \mathcal{F}$$

is an equivalence in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$.

Let $Y := \varinjlim_{[m]\in\Delta} Y_m$ which is in $\mathcal{M}$ since $\mathcal{M}$ is closed under geometric realisations. By fully faithfulness of $\underline{h}$, there is a morphism $Y_* \to (U)_*$ corresponding to the morphism $\mathcal{F}_* = \underline{h}(Y_*) \to \underline{h}((U)_*) = \mathcal{U}_*$. We note that, for any morphism $[m] \to [n]$ in Δ, we have, by construction, the following isomorphism in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$,

$$\mathcal{F}_n \times_{\mathcal{U}_n} \mathcal{U}_m \simeq (\mathcal{F} \times_{h(X)} \mathcal{U}_n) \times_{\mathcal{U}_n} \mathcal{U}_m \simeq \mathcal{F} \times_{h(X)} \mathcal{U}_m \simeq \mathcal{F}_m$$

and hence we have an isomorphism $Y_n \times_{(U)_n} (U)_m \to Y_m$ in $\mathcal{M}$. Since the Čech nerve $(U)_* \to X$ satisfies the co-cartesian descent condition, there is an isomorphism $Y \times_X (U)_* \to Y_*$ in $\mathbf{s}\mathcal{M}$, and hence an isomorphism $h(Y) \times_{h(X)} \underline{h}((U)_*) \to \underline{h}(Y_*)$ in $s\mathbf{PSh}(\mathcal{M})$. It then follows that

$$|\mathcal{F}_*| \simeq |\underline{h}(Y_*)| \simeq h(Y) \times_{h(X)} |\underline{h}((U)_*)| \simeq h(Y) \times_{h(X)} h(X) \simeq h(Y)$$

Therefore $|\mathcal{F}_*|$, and hence $\mathcal{F}$, is a (-1)-geometric stack in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$.

□

3.2.4 Preservation of Geometric Stacks

In this subsection, we will fix two relative $(\infty, 1)$-geometry tuples $(\mathcal{M}, \boldsymbol{\tau}, \mathbf{P}, \mathcal{A})$ and $(\mathcal{N}, \boldsymbol{\sigma}, \mathbf{Q}, \mathcal{B})$ along with a colimit-preserving functor $L : \mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}}) \to \mathbf{Stk}(\mathcal{B}, \boldsymbol{\sigma}|_{\mathcal{B}})$. Then, by [52, Corollary 5.5.2.9], there is an induced adjunction

$$L : \mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}}) \leftrightarrows \mathbf{Stk}(\mathcal{B}, \boldsymbol{\sigma}|_{\mathcal{B}}) : R$$

Proposition 3.2.4.1. *Suppose that L preserves pullbacks of (-1)-geometric stacks along m-$\mathbf{P}$-morphisms for all $m \geq -1$. Then, for each $n \geq -1$, L preserves pullbacks of n-geometric stacks along m-$\mathbf{P}$-morphisms for all $m \geq -1$.*

Proof. We prove this claim by induction on n, with the case $n = -1$ true by assumption. Suppose that we have an m-$\mathbf{P}$-morphism $\mathcal{F} \to \mathcal{G}$ of n-geometric stacks for some $m \geq -1$ and a morphism of stacks $\mathcal{H} \to \mathcal{G}$, with $\mathcal{H}$ n-geometric. We note that, by Proposition 3.2.3.6, if $\{U_i \to \mathcal{G}\}_{i \in I}$ is an n-atlas for $\mathcal{G}$, then we can write $\mathcal{G} = |\mathcal{U}_*|$. Let

$$\mathcal{F}_* = \mathcal{F} \times_{\mathcal{G}} \mathcal{U}_* \quad \text{and} \quad \mathcal{H}_* = \mathcal{H} \times_{\mathcal{G}} \mathcal{U}_*$$

We note that $\mathcal{F} \simeq |\mathcal{F}_*|$ and $\mathcal{H} \simeq |\mathcal{H}_*|$ and $\mathcal{F} \times_{\mathcal{G}} \mathcal{H} \simeq |\mathcal{F}_* \times_{\mathcal{U}_*} \mathcal{H}_*|$. Since L commutes with colimits,

$$\begin{aligned} L(\mathcal{F} \times_{\mathcal{G}} \mathcal{H}) &= L(\varinjlim_{[l] \in \Delta} (\mathcal{F}_l \times_{\mathcal{U}_l} \mathcal{H}_l)) \\ &\simeq \varinjlim_{[l] \in \Delta} L(\mathcal{F}_l \times_{\mathcal{U}_l} \mathcal{H}_l) \\ &\simeq \varinjlim_{[l] \in \Delta} \coprod_{\underline{i} \in I^{l+1}} L((\mathcal{F} \times_{\mathcal{G}} U_{\underline{i}}) \times_{U_{\underline{i}}} (\mathcal{H} \times_{\mathcal{G}} U_{\underline{i}})) \end{aligned}$$

Suppose that $[l] \in \Delta$. For each $\underline{i}$ in I^{l+1}, we note that $\mathcal{F} \times_{\mathcal{G}} U_{\underline{i}}$, $U_{\underline{i}}$ and $\mathcal{H} \times_{\mathcal{G}} U_{\underline{i}}$ are all $(n-1)$-geometric stacks. Therefore, by the inductive hypothesis, since the morphism $\mathcal{F} \times_{\mathcal{G}} U_{\underline{i}} \to U_{\underline{i}}$ is in m-$\mathbf{P}$ for some m, then we have that

$$\begin{aligned} L(\mathcal{F} \times_{\mathcal{G}} \mathcal{H}) &\simeq \varinjlim_{[l] \in \Delta} \coprod_{i \in I^{l+1}} L(\mathcal{F} \times_{\mathcal{G}} U_{\underline{i}}) \times_{L(U_{\underline{i}})} L(\mathcal{H} \times_{\mathcal{G}} U_{\underline{i}}) \\ &\simeq \varinjlim_{l \in \Delta} L(\mathcal{F}_l) \times_{L(\mathcal{U}_l)} L(\mathcal{H}_l) \\ &\simeq L(\mathcal{F}) \times_{L(\mathcal{G})} L(\mathcal{H}) \end{aligned}$$

□

Suppose for the rest of the section that L preserves epimorphisms of stacks, and that L sends (-1)-geometric stacks on $\mathcal{A}$ to (-1)-geometric stacks on $\mathcal{B}$.

Example 3.2.4.2. *Suppose that we have a functor of sites $F : (\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}}) \to (\mathcal{B}, \boldsymbol{\sigma}|_{\mathcal{B}})$. We recall from Section 3.1.4 that, if F is continuous, we have an $(\infty, 1)$-adjunction*

$$F_\# : \mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}}) \leftrightarrows \mathbf{Stk}(\mathcal{B}, \boldsymbol{\sigma}|_{\mathcal{B}}) : F^*$$

We showed that $F_\#$ preserves epimorphisms and sends (-1)-geometric stacks to (-1)-geometric stacks. We note that all the results hold in this section with L replaced with $F_\#$. Now, if F is instead cocontinuous, then we have an $(\infty, 1)$-adjunction

$$F^* : \mathbf{Stk}(\mathcal{B}, \boldsymbol{\tau}|_{\mathcal{B}}) \leftrightarrows \mathbf{Stk}(\mathcal{A}, \boldsymbol{\sigma}|_{\mathcal{A}}) : F_*$$

We also showed that F^* preserves epimorphisms and sends (-1)-geometric stacks to (-1)-geometric stacks. Therefore, all the results in this section also hold with L replaced with F^* and with our tuples $(\mathcal{M}, \boldsymbol{\tau}, \mathbf{P}, \mathcal{A})$ and $(\mathcal{N}, \boldsymbol{\sigma}, \mathbf{Q}, \mathcal{B})$ swapped.

Proposition 3.2.4.3. *Suppose that*

1. *L preserves pullbacks of n-geometric stacks for all $n \geq 0$,*
2. *L sends $\mathbf{P}$-morphisms between (-1)-geometric stacks to $\mathbf{Q}$-morphisms between (-1)-geometric stacks,*
3. *Every representable stack on $\mathcal{B}$ is the image under L of a representable on $\mathcal{A}$.*

Then, for all $n \geq -1$, L sends n-geometric stacks on $\mathcal{A}$ to n-geometric stacks on $\mathcal{B}$. Moreover, L sends n-$\mathbf{P}$-morphisms to n-$\mathbf{Q}$-morphisms, not necessarily between geometric stacks.

Proof. We prove this by induction on n. By assumption, L preserves (-1)-geometric stacks. Now, suppose that we have a (-1)-$\mathbf{P}$-morphism $\mathcal{F} \to \mathcal{G}$ between stacks. Suppose further that we have a representable stack Y on $\mathcal{B}$ and a map $Y \to L(\mathcal{G})$. We note that, by Assumption 3, there exists some $X \in \mathcal{A}$ such that $L(X) \simeq Y$. Hence, since the map $\mathcal{F} \to \mathcal{G}$ is in (-1)-$\mathbf{P}$, then

$$L(\mathcal{F}) \times_{L(\mathcal{G})} Y \simeq L(\mathcal{F}) \times_{L(\mathcal{G})} L(X) \simeq L(\mathcal{F} \times_{\mathcal{G}} X)$$

is a (-1)-geometric stack and the induced map $\mathcal{F} \times_{\mathcal{G}} X \to X$ is in $\mathbf{P}$. Therefore, we see that $L(\mathcal{F}) \to L(\mathcal{G})$ is in (-1)-$\mathbf{Q}$.

Now, suppose that the statement holds for all m such that $-1 \leq m < n$. Suppose that $\mathcal{F} \in \mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$ is an n-geometric stack with n-atlas $\{U_i \to \mathcal{F}\}_{i \in I}$. Then, using our assumptions, we can show that $L(\mathcal{F})$ has an n-atlas. Moreover, we easily note that the diagonal morphism $L(\mathcal{F}) \to L(\mathcal{F} \times \mathcal{F}) \simeq L(\mathcal{F} \times \mathcal{F})$ is $(n-1)$-representable using Assumption (3).

To conclude, suppose that $f : \mathcal{F} \to \mathcal{G}$ is an n-$\mathbf{P}$ morphism of stacks in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$ and that there is a map $Y \to L(\mathcal{G})$ for some representable stack Y on $\mathcal{B}$. We note that there exists a representable stack X on $\mathcal{A}$ such that $L(X) \simeq Y$, and we can easily see that $L(\mathcal{F}) \times_{L(\mathcal{G})} Y$ is n-geometric. Moreover, we know that there exists an n-atlas

$\{U_i \to \mathcal{F} \times_{\mathcal{G}} X\}_{i \in I}$ such that the induced morphism $U_i \to X$ is in $\mathbf{P}$. Hence, the n-atlas $\{L(U_i) \to L(\mathcal{F}) \times_{L(\mathcal{G})} Y\}_{i \in I}$ is such that the induced morphism $L(U_i) \to Y$ is in $\mathbf{Q}$.

□

If we now suppose that $\mathcal{B}$ is closed under $\boldsymbol{\tau}$-descent, then we can relax some of our conditions and obtain the following result.

Proposition 3.2.4.4. *Suppose that*

1. *L preserves pullbacks of n-geometric stacks along $(n-1)$-$\mathbf{P}$-morphisms for all $n \geq 0$,*
2. *L sends (-1)-$\mathbf{P}$-morphisms between (-1)-geometric stacks on $\mathcal{A}$ to (-1)-$\mathbf{Q}$-morphisms between (-1)-geometric stacks on $\mathcal{B}$.*

Then, for all $n \geq -1$, L sends n-geometric stacks on $\mathcal{A}$ to n-geometric stacks on $\mathcal{B}$. Moreover, it sends n-$\mathbf{P}$-morphisms between n-geometric stacks to n-$\mathbf{Q}$-morphisms between n-geometric stacks.

Proof. We prove this by induction on n. By assumption, the case $n = -1$ is satisfied. Now, suppose that the statement holds for all m such that $-1 \leq m < n$. Suppose that $\mathcal{F} \in \mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$ is an n-geometric stack with n-atlas $\{U_i \to \mathcal{F}\}_{i \in I}$. Then, using our assumptions, we can show that $L(\mathcal{F})$ has an n-atlas and therefore $L(\mathcal{F})$ is n-geometric by Proposition 3.2.3.5.

Now, suppose that $f : \mathcal{F} \to \mathcal{G}$ is an n-$\mathbf{P}$-morphism of geometric stacks in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$. Then, since $\mathcal{G}$ is n-geometric, it has an n-atlas $\{U_i \to \mathcal{G}\}_{i \in I}$ and, as before, $\{L(U_i) \to L(\mathcal{G})\}_{i \in I}$ is an n-atlas for $L(\mathcal{G})$. By Lemma 3.2.3.3, to show that $L(f)$ is in n-$\mathbf{Q}$, it suffices to show that each map $L(\mathcal{F}) \times_{L(\mathcal{G})} L(U_i) \to L(U_i)$ is in n-$\mathbf{Q}$. Since L preserves pullbacks of n-geometric stacks along $(n-1)$-$\mathbf{P}$-morphisms,

$$L(\mathcal{F}) \times_{L(\mathcal{G})} L(U_i) \simeq L(\mathcal{F} \times_{\mathcal{G}} U_i)$$

Hence, since $\mathcal{F} \times_{\mathcal{G}} U_i \to U_i$ is an n-$\mathbf{P}$-morphism of geometric stacks in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$ because n-$\mathbf{P}$-morphisms are stable by pullbacks, we can reduce to the case when $\mathcal{G}$ is a (-1)-geometric stack X.

So, suppose we have an n-$\mathbf{P}$ morphism $f : \mathcal{F} \to X$ with $\mathcal{F}$ n-geometric. In this case, we consider an n-atlas $\{V_i \to \mathcal{F}\}_{i \in I}$ for $\mathcal{F}$, which we know induces an n-atlas $\{L(V_i) \to L(\mathcal{F})\}_{i \in I}$ for $L(\mathcal{F})$. The induced morphism $V_i \to X$ is in $\mathbf{P}$ since f is in n-$\mathbf{P}$. Now, since L sends morphisms in $\mathbf{P}$ between (-1)-geometric stacks to morphisms in $\mathbf{Q}$, we see that the image of the map $V_i \to X$ under L is in $\mathbf{Q}$. Therefore, $L(f)$ is in n-$\mathbf{Q}$. □

Chapter 4

Derived Geometry Contexts

In order to be able to discuss cotangent complexes and obstruction theories for derived stacks, we need to introduce more structure. This is facilitated by what we call a *derived geometry context*. The notion of a *derived algebraic context*, formalised by Raksit [72], has provided us with a versatile framework in which to explore both connective and non-connective geometric settings. Applications of this framework can be seen in [9] and [41]. In [10], Ben-Bassat, Kelly, and Kremnizer introduce the notion of a *spectral algebraic context* which both encapsulates these derived algebraic contexts and provides an appropriate context in which to do spectral algebraic geometry in the sense of Lurie [58].

In this chapter we define the notion of a *derived geometry context* by identifying a particular category $\mathcal{A}$ of affines along with a well-behaved system $\mathbf{M}$ of categories of modules on $\mathcal{A}$. Within such a context, we obtain several important results about liftings of maps of stacks along first order infinitesimal deformations.

4.1 Derived Algebraic Contexts

4.1.1 Derived Algebraic Contexts

Suppose that $\mathcal{C}$ is a stable locally presentable symmetric monoidal $(\infty, 1)$-category with unit I and monoidal functor $\otimes$. Suppose that $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ is a t-structure on $\mathcal{C}$, i.e. a t-structure on the triangulated homotopy category $\mathrm{Ho}(\mathcal{C})$. We recall that the heart $\mathcal{C}^{\heartsuit}$ of a t-structure on a stable $(\infty, 1)$-category is an abelian category.

Definition 4.1.1.1. [72, Definition 3.3.1] The t-structure is *compatible* if

1. $\mathcal{C}_{\leq 0}$ is closed under filtered colimits in $\mathcal{C}$,
2. The unit object I of $\mathcal{C}$ lies in $\mathcal{C}_{\geq 0}$,
3. If $A, B \in \mathcal{C}_{\geq 0}$, then $A \otimes B \in \mathcal{C}_{\geq 0}$.

Definition 4.1.1.2. [72, Definition 4.2.1] A *derived algebraic context* is a tuple $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, \mathcal{C}^0)$ where

1. $\mathcal{C}$ is a stable locally presentable symmetric monoidal $(\infty, 1)$-category,
2. $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ is a complete t-structure on $\mathcal{C}$ which is compatible with the monoidal structure,
3. $\mathcal{C}^0$ is a small full subcategory of $\mathcal{C}^\heartsuit$ which is
   - a symmetric monoidal subcategory of $\mathcal{C}$,
   - a generating set of compact projectives for $\mathcal{C}_{\geq 0}$,
   - closed under the formation of $\mathcal{C}^\heartsuit$-symmetric powers, i.e. for any $A \in \mathcal{C}^0$ and $n \geq 0$, $\mathrm{Sym}^n_{\mathcal{C}^\heartsuit}(A) \in \mathcal{C}^0$,
   - closed under the formation of finite coproducts in $\mathcal{C}$.

We note that, by [52, c.f. Proposition 5.5.8.22], there is a symmetric monoidal equivalence of $(\infty, 1)$-categories

$$\mathcal{C}_{\geq 0} \simeq \mathcal{P}_\Sigma(\mathcal{C}^0) := \mathbf{Fun}^\times(\mathcal{C}^{0,op}, \infty\mathbf{Grpd})$$

The category $\mathcal{P}_\Sigma(\mathcal{C}^0)$ inherits a symmetric monoidal structure from $\mathcal{C}^0$ [57, Corollary 4.8.1.12].

Within such a derived algebraic context we can define the notion of a *derived commutative algebra object.* Indeed, consider the functor $\mathrm{Sym}_{\mathcal{C}^\heartsuit} : \mathcal{C}^0 \to \mathcal{C}_{\geq 0}$ defined to be the composition of the functor $\mathcal{C}^\heartsuit \to \mathcal{C}_{\geq 0}$ with the usual symmetric algebra functor on $\mathcal{C}^\heartsuit$, restricted to $\mathcal{C}^0$. By Proposition C.2.0.4, this functor extends to a functor

$$\mathbf{LSym}_{\mathcal{C}_{\geq 0}} : \mathcal{C}_{\geq 0} \to \mathcal{C}_{\geq 0}$$

This is the *derived symmetric algebra functor on* $\mathcal{C}_{\geq 0}$ and is a monadic functor. By [72, Construction 4.2.20], we can extend this functor to a monad on $\mathcal{C}$ which we will call the *derived symmetric algebra monad on* $\mathcal{C}$ and denote by $\mathbf{LSym}_{\mathcal{C}}$.

Definition 4.1.1.3. [72, Definition 4.2.22] A *derived commutative algebra object of* $\mathcal{C}$ is a module over the derived symmetric algebra monad on $\mathcal{C}$.

We denote by $\mathbf{DAlg}(\mathcal{C})$ the $(\infty, 1)$-category of derived commutative algebra objects. This category is locally presentable, in particular it admits all small limits and colimits. We note that there is an adjunction

$$\mathbf{LSym}_{\mathcal{C}} : \mathcal{C} \leftrightarrows \mathbf{DAlg}(\mathcal{C}) : U_{\mathcal{C}}$$

where $U_{\mathcal{C}}$ is the forgetful functor.

The t-structure on $\mathcal{C}$ allows us to define the following categories

$$\begin{aligned}\mathbf{DAlg}^{\geq n}(\mathcal{C}) &:= \mathbf{DAlg}(\mathcal{C}) \times_{\mathcal{C}} \mathcal{C}_{\geq n}\\ \mathbf{DAlg}^{\heartsuit}(\mathcal{C}) &:= \mathbf{DAlg}(\mathcal{C}) \times_{\mathcal{C}} \mathcal{C}^{\heartsuit}\\ \mathbf{DAlg}^{\leq n}(\mathcal{C}) &:= \mathbf{DAlg}(\mathcal{C}) \times_{\mathcal{C}} \mathcal{C}_{\leq n}\end{aligned}$$

In particular, we will denote by $\mathbf{DAlg}^{cn}(\mathcal{C})$ the $(\infty,1)$-*category of connective derived commutative algebra objects*, i.e. the category $\mathbf{DAlg}^{\geq 0}(\mathcal{C})$. We note that, as we only work in the connective setting in this thesis, we don't necessarily need the full machinery of derived algebraic contexts, but we use it in this work for consistency with the existing literature.

Example 4.1.1.4. *Let k be a commutative ring. Let Mod_k^{fgf} be the ordinary category of finitely generated free k-modules and let $\mathbf{Mod}_{k,\geq 0}$ be $\mathcal{P}_\Sigma(\mathrm{Mod}_k^{fgf})$. Define $\mathbf{Mod}_k := \mathbf{Stab}(\mathbf{Mod}_{k,\geq 0})$. Then, $(\mathbf{Mod}_k, \mathbf{Mod}_{k,\geq 0}, \mathbf{Mod}_{k,\leq 0}, \mathrm{Mod}_k^{fgf})$ is a derived algebraic context and $\mathbf{DAlg}^{cn}(\mathbf{Mod}_k)$ is equivalent to the $(\infty,1)$-category of simplicial commutative k-algebras as defined in [51, Definition 4.1.1].*

We note that there is a functor from the $(\infty,1)$-category of simplicial commutative k-algebras to the $(\infty,1)$-category of $\mathbb{E}_\infty$-k-algebras but that this functor is not necessarily an equivalence unless k contains the field $\mathbb{Q}$ of rational numbers [57, Warning 7.1.4.21]. More generally, by [72, Proposition 4.2.27], there is a functor

$$\Theta : \mathbf{DAlg}(\mathcal{C}) \to \mathbf{CAlg}(\mathcal{C})$$

which preserves small limits and colimits.

Definition 4.1.1.5. Suppose that $A \in \mathbf{DAlg}(\mathcal{C})$. Then,

1. We denote by $\mathbf{Mod}_A$ the $(\infty,1)$-category $\mathbf{Mod}_{\Theta(A)}(\mathcal{C})$,
2. We denote by $\mathbf{DAlg}_A(\mathcal{C})$ the under category ${}^{A/}\mathbf{DAlg}(\mathcal{C})$.

We note that, by [72, Notation 4.2.28], Θ induces a monadic adjunction

$$\mathbf{LSym}_A : \mathbf{Mod}_A \leftrightarrows \mathbf{DAlg}_A(\mathcal{C}) : U$$

By [50, Theorem 3.4.2 and Proposition 3.6.6] and [57, c.f. Remark 7.3.4.16], $\mathbf{Mod}_A$ is a stable locally presentable symmetric monoidal $(\infty,1)$-category. We denote by $\otimes_A^{\mathbb{L}}$ the coproduct on $\mathbf{Mod}_A$, which is given by the monoidal product $\otimes_{\Theta(A)}^{\mathbb{L}}$. Since $\mathbf{DAlg}_A(\mathcal{C})$ is co-cartesian monoidal and the functor U is monoidal we see that, for any two objects $B, B' \in \mathbf{DAlg}_A(\mathcal{C})$, there is an equivalence between $B \coprod_A B$ and

$B \otimes^{\mathbb{L}}_A B$ where $B \coprod_A B$ denotes the coproduct in $\mathbf{DAlg}_A(\mathcal{C})$, considered as an object of $\mathbf{Mod}_A$ under the forgetful functor.

We will denote by $\mathbf{Mod}_A^{\geq n}$ and $\mathbf{Mod}_A^{\leq n}$ the $(\infty,1)$-categories defined by the fibre products

$$\mathbf{Mod}_A^{\geq n} := \mathbf{Mod}_A \times_{\mathcal{C}} \mathcal{C}_{\geq n}$$
$$\mathbf{Mod}_A^{\heartsuit} := \mathbf{Mod}_A \times_{\mathcal{C}} \mathcal{C}^{\heartsuit}$$
$$\mathbf{Mod}_A^{\leq n} := \mathbf{Mod}_A \times_{\mathcal{C}} \mathcal{C}_{\leq n}$$

Denote by $\mathbf{Mod}_A^{cn}$ the category $\mathbf{Mod}_A^{\geq 0}$ of *connective A-modules*.

For each $n \in \mathbb{Z}$, there are inclusion functors $\iota_{\geq n} : \mathcal{C}_{\geq n} \to \mathcal{C}$ and $\iota_{\leq n} : \mathcal{C}_{\leq n} \to \mathcal{C}$. The functor $\iota_{\geq n}$ has a right adjoint $\tau_{\geq n} : \mathcal{C} \to \mathcal{C}_{\geq n}$ and the functor $\iota_{\leq n}$ has a left adjoint $\tau_{\leq n} : \mathcal{C} \to \mathcal{C}_{\leq n}$. We denote by $\pi_n : \mathcal{C} \to \mathcal{C}^{\heartsuit}$ the functors $\pi_n = \tau_{\leq 0} \circ \tau_{\geq 0} \circ [n]$ where $[n]$ denotes the n^{th} power of the suspension functor Σ^n on $\mathcal{C}$.

Remark. We note that these functors depend on the t-structure and don't necessarily correlate with the truncation functors defined in [52, Proposition 5.5.6.18].

Suppose that $A \in \mathbf{DAlg}^{cn}(\mathcal{C})$. Then, $(\mathbf{Mod}_A^{\geq 0}, \mathbf{Mod}_A^{\leq 0})$ is a t-structure on $\mathbf{Mod}_A$ by [10, Lemma 2.3.96]. Therefore, we get a well defined notion of truncation on $\mathbf{Mod}_A$ which lines up with the truncation functors on $\mathcal{C}$. We want to similarly be able to define good notions of truncation functors on the category $\mathbf{DAlg}^{cn}(\mathcal{C})$.

Definition 4.1.1.6. We say that $\mathbf{DAlg}^{cn}(\mathcal{C})$ is *compatible with the t-structure on $\mathcal{C}$* if, for all $n \geq 0$, there exist adjunctions

$$\tau_{\leq n} : \mathbf{DAlg}^{cn}(\mathcal{C}) \leftrightarrows \mathbf{DAlg}^{cn,\leq n}(\mathcal{C}) : \iota_{\leq n}$$
$$\iota_{\geq n} : \mathbf{DAlg}^{cn,\geq n}(\mathcal{C}) \leftrightarrows \mathbf{DAlg}^{cn}(\mathcal{C}) : \tau_{\geq n}$$

such that $\tau_{\leq n} \circ U_{\mathcal{C}} \simeq U_{\mathcal{C}} \circ \tau_{\leq n}$ and $\tau_{\geq n} \circ U_{\mathcal{C}} \simeq U_{\mathcal{C}} \circ \tau_{\geq n}$. In this situation, we will define the functor $\pi_n : \mathbf{DAlg}^{cn}(\mathcal{C}) \to \mathbf{DAlg}^{\heartsuit}(\mathcal{C})$ to be $\tau_{\leq 0} \circ \tau_{\geq 0} \circ [n]$.

Remark. In many of the cases we are considering, particularly the categories of simplicial objects in an exact category, this condition will hold.

4.1.2 Model Derived Algebraic Contexts

Many of the derived algebraic contexts we will use in this thesis will be presented by *model derived algebraic contexts* $(\mathrm{C}, \mathrm{C}_{\geq 0}, \mathrm{C}_{\leq 0}, \mathrm{C}^0)$. A model derived algebraic context has a similar definition to Definition 4.1.1.2 so we will not reproduce it here (see [41, Definition 3.14] for details). In particular, any model derived algebraic context presents a derived algebraic context.

Proposition 4.1.2.1. *[41, Proposition 3.15] Suppose that* $(\mathrm{C}, \mathrm{C}_{\geq 0}, \mathrm{C}_{\leq 0}, \mathrm{C}^0)$ *is a model derived algebraic context. Then,* $(\mathbf{L}^H(\mathrm{C}), \mathbf{L}^H(\mathrm{C}_{\geq 0}), \mathbf{L}^H(\mathrm{C}_{\leq 0}), \mathbf{L}^H(\mathrm{C}^0))$ *is a derived algebraic context.*

Recall that, in derived algebraic geometry, our derived affines correspond to simplicial commutative rings. By Example 4.1.1.4, the ∞-category of simplicial commutative rings can be realised as the category of connective algebras for some derived algebraic context. We obtain a similar result for simplicial algebra objects in certain exact categories.

Theorem 4.1.2.2. *[10, c.f. Theorems 3.1.41, 3.1.42] Suppose that* E *is a complete closed symmetric monoidal elementary exact category such that*

1. E *has symmetric projectives, i.e. for any projective* P *and any* $n \in \mathbb{N}$, $\mathrm{Sym}^n_{\mathrm{E}}(P)$ *is projective,*
2. *The tensor product of two projectives is projective,*
3. *The monoidal unit* I *is projective,*
4. *Projectives are flat.*

Let P^0 *be a set of compact projective generators closed under finite direct sums, tensor products, and the formation of symmetric powers. Then,*

1. *The category of simplicial objects in* E, sE, *forms a homotopical algebraic (HA) context* $(\mathrm{sE}, \mathrm{sE}, \mathrm{Comm}(\mathrm{sE}))$ *in the sense of [86, Definition 1.1.0.11],*
2. *When* $\mathrm{Ch}(\mathrm{E})$ *is equipped with the projective model structure and the left t-structure as described in Section 1.3,* $(\mathrm{Ch}(\mathrm{E}), \mathrm{Ch}_{\geq 0}(\mathrm{E}), \mathrm{Ch}_{\leq 0}(\mathrm{E}), \mathrm{P}^0)$ *is a model derived algebraic context.*

Example 4.1.2.3. *An example of an exact category satisfying the conditions in Theorem 4.1.2.2 is the category* Mod_k*, for* k *a commutative ring. This is exactly the situation in Example 4.1.1.4.*

Corollary 4.1.2.4. $(\mathbf{Ch}(\mathrm{E}), \mathbf{Ch}_{\geq 0}(\mathrm{E}), \mathbf{Ch}_{\leq 0}(\mathrm{E}), \mathbf{L}^H(P^0))$ *is a derived algebraic context where* $\mathbf{Ch}(\mathrm{E}) := \mathbf{L}^H(\mathrm{Ch}(\mathrm{E}))$.

Theorem 4.1.2.5. *[10, Proposition 3.1.66] There is an equivalence of* ∞*-categories*

$$\mathbf{DAlg}^{cn}(\mathbf{Ch}(\mathrm{E})) \simeq \mathbf{L}^H(\mathrm{Comm}(\mathrm{sE}))$$

Example 4.1.2.6. *Suppose that* R *is a Banach ring. We note, by [41, Section 4.6], that* $\mathrm{Ind}(\mathrm{Ban}_R)$ *is an example of a category satisfying the conditions of Theorem 4.1.2.2 with* P^0 *the set of compact projective generators described in Section*

1.3. For R a Banach ring, we note that CBorn_R is not necessarily elementary, but instead **AdMon***-elementary in the sense of [41, Definition 4.5], and so we cannot directly apply our theorem. However, by [41, Remark 4.38], we have a Quillen equivalence $\mathrm{Ch}(\mathrm{CBorn}_R) \simeq \mathrm{Ch}(\mathrm{Ind}(\mathrm{Ban}_R))$ and we can transfer the t-structure on $\mathbf{Ch}(\mathrm{Ind}(\mathrm{Ban}_R))$ to one on $\mathbf{Ch}(\mathrm{CBorn}_R)$ such that the positive parts agree. However, the transferred non-positive part $\widetilde{\mathbf{Ch}}_{\leq 0}(\mathrm{CBorn}_R)$ does not correspond to the category $\mathbf{Ch}_{\leq 0}(\mathrm{CBorn}_R)$. We obtain equivalent derived algebraic contexts*

$$(\mathbf{Ch}(\mathrm{Ind}(\mathrm{Ban}_R)), \mathbf{Ch}_{\geq 0}(\mathrm{Ind}(\mathrm{Ban}_R)), \mathbf{Ch}_{\leq 0}(\mathrm{Ind}(\mathrm{Ban}_R)), \mathbf{L}^H(P^0))$$

and

$$(\mathbf{Ch}(\mathrm{CBorn}_R), \mathbf{Ch}_{\geq 0}(\mathrm{CBorn}_R), \widetilde{\mathbf{Ch}}_{\leq 0}(\mathrm{CBorn}_R), \mathbf{L}^H(P^0))$$

In particular, the categories of connective derived algebras will be equivalent in both contexts.

4.2 The Cotangent Complex

The cotangent complex is a fundamental concept in derived algebraic geometry. It controls the *deformation theory* of geometric objects. In this section, we define cotangent complexes for derived affine objects in our derived algebraic contexts, and then generalise to cotangent complexes of derived stacks. For the purposes of this thesis we will only focus on the connective setting, the non-connective setting can be dealt with using the theory of [10].

The definitions we make in this section are analogous to the ones made in derived algebraic geometry, see [86].

4.2.1 The Cotangent Complex

Suppose that $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, \mathcal{C}^0)$ is a derived algebraic context. We denote by Σ and Ω the suspension and loop functors respectively on $\mathcal{C}$. We recall that the *stabilisation* $\mathbf{Stab}(\mathcal{D})$ of any $(\infty, 1)$-category $\mathcal{D}$ with finite limits exists by [57, Proposition 1.4.2.17] and that the suspension and loop functors on $\mathcal{D}$ induce functors $\Omega^\infty : \mathbf{Stab}(\mathcal{D}) \to \mathcal{D}$ and $\Sigma^\infty : \mathcal{D} \to \mathbf{Stab}(\mathcal{D})$.

Classically (see [70, Section 4]), for any commutative ring A, there is an equivalence between the category of A-modules and the category of *Beck modules*, i.e. abelian group objects in $\mathrm{CRing}_{/A}$. By [10, Remark 2.4.117] (see also [75, Theorem 3.3.3]), we can obtain a similar result in our derived algebraic contexts.

Suppose that $\mathcal{D}$ is an $(\infty, 1)$-category with finite products. The $(\infty, 1)$*-category of abelian group objects in* $\mathcal{D}$, denoted $\mathbf{Ab}(\mathcal{D})$, is defined to be

$$\mathbf{Ab}(\mathcal{D}) := \mathbf{Fun}^{\times}(\mathrm{Mod}_{\mathbb{Z}}^{fgf,op}, \mathcal{D})$$

Theorem 4.2.1.1. *Suppose that* $A \in \mathbf{DAlg}^{cn}(\mathcal{C})$*. Then, there is an equivalence of* $(\infty, 1)$*-categories*

$$\mathbf{Mod}_A \simeq \mathbf{Stab}(\mathbf{Ab}(\mathbf{DAlg}^{cn}(\mathcal{C})_{/A}))$$

This allows us to define the square-zero extension in a similar way to Lurie [57, Remark 7.3.4.16]. We denote the functor from $\mathbf{Stab}(\mathbf{Ab}(\mathbf{DAlg}^{cn}(\mathcal{C})_{/A}))$ to $\mathbf{Ab}(\mathbf{DAlg}^{cn}(\mathcal{C})_{/A})$ by $\Omega^{\infty}_{\mathbf{Ab}}$ and the natural forgetful functor from $\mathbf{Ab}(\mathbf{DAlg}^{cn}(\mathcal{C})_{/A})$ to $\mathbf{DAlg}^{cn}(\mathcal{C})_{/A}$ given by evaluating at the unit, by F.

Definition 4.2.1.2. Suppose that $A \in \mathbf{DAlg}^{cn}(\mathcal{C})$. The *square-zero extension functor* is defined to be the functor

$$\mathbf{sqz}_A : \mathbf{Mod}_A \xrightarrow{\simeq} \mathbf{Stab}(\mathbf{Ab}(\mathbf{DAlg}^{cn}(\mathcal{C})_{/A})) \xrightarrow{\Omega^{\infty}_{\mathbf{Ab}}} \mathbf{Ab}(\mathbf{DAlg}^{cn}(\mathcal{C})_{/A}) \xrightarrow{F} \mathbf{DAlg}^{cn}(\mathcal{C})_{/A}$$

We denote the image of $M \in \mathbf{Mod}_A$ under $\mathbf{sqz}_A$ by $A \oplus M$ and call it the *square-zero extension of* A *by* M.

Remark. We note that, using a similar reasoning to [57, Remark 7.3.4.16], this does in fact define something we can consider as the square-zero extension.

Suppose that $A \in \mathbf{DAlg}^{cn}(\mathcal{C})$. Consider the functor $\Theta : \mathbf{DAlg}(\mathcal{C}) \to \mathbf{CAlg}(\mathcal{C})$ and denote by Θ_A the induced functor on $\mathbf{DAlg}(\mathcal{C})_{/A}$. We note that $\Theta_A(A \oplus M)$ is equivalent to the square-zero extension, in the sense of [57, Remark 7.3.4.16], of $\Theta(A) \in \mathbf{CAlg}^{cn}(\mathcal{C})$ by M.

Example 4.2.1.3. *Consider the* $(\infty, 1)$*-category of simplicial commutative rings as a derived algebraic context as in Example 4.1.1.4. Consider the functor which takes an object* $A \simeq \mathbf{LSym}_{\mathbf{Mod}_{\mathbb{Z}}}(P)$ *for some* $P \in \mathrm{Mod}_{\mathbb{Z}}^{fgf}$ *and a module* M *of the form* $A^{\oplus n}$ *for some* $n \geq 0$*, and then computes the usual* 1*-categorical square-zero extension. In [58, Construction 25.3.1.1], Lurie defines square-zero extensions for simplicial commutative rings by left Kan extending this functor.*

By [58, Remark 25.3.1.2], for $A \in \mathbf{DAlg}^{cn}(\mathbf{Mod}_{\mathbb{Z}})$ *and* $M \in \mathbf{Mod}_A^{cn}$*, the underlying* $\mathbb{E}_{\infty}$*-algebra of this square-zero extension corresponds to* $\Theta_A(A \oplus M)$*. In particular, by [72, Remark 4.4.12], this corresponds to the underlying* $\mathbb{E}_{\infty}$*-algebra of the square-zero extension of Raksit defined in [72, Construction 4.4.5]. Hence, in* $\mathbf{CAlg}^{cn}(\mathbf{Mod}_{\mathbb{Z}})_{/A}$*, all the definitions of square-zero extension correspond.*

Using the definition of the square-zero extension, we can easily write down the following definition.

Definition 4.2.1.4. For a morphism $f : A \to B$ in $\mathbf{DAlg}^{cn}(\mathcal{C})$ and $M \in \mathbf{Mod}_B$, we define the ∞*-groupoid of A-derivations from B to M* to be

$$\mathbf{Der}_A(B, M) := \mathrm{Map}_{\mathbf{DAlg}_A^{cn}(\mathcal{C})_{/B}}(B, B \oplus M)$$

If $A = I$, then we denote the ∞-groupoid of derivations from B to M by $\mathbf{Der}(B, M)$.

Lemma 4.2.1.5. *There is an adjunction $L_A : \mathbf{DAlg}^{cn}(\mathcal{C})_{/A} \leftrightarrows \mathbf{Mod}_A : \mathbf{sqz}_A$.*

Proof. We note that $\mathbf{Mod}_A$ and $\mathbf{DAlg}^{cn}(\mathcal{C})_{/A}$ are presentable. The functor $\mathbf{sqz}_A$ preserves limits and is accessible since $\Omega^\infty_{\mathbf{Ab}}$ is a right adjoint by [57, Proposition 1.4.4.4] and the functor F is an accessible limit-preserving functor. Therefore, $\mathbf{sqz}_A$ has a left adjoint by [52, Corollary 5.5.2.9]. $\square$

For any morphism $f : A \to B$ in $\mathbf{DAlg}^{cn}(\mathcal{C})$, there is also an induced adjunction on the slice categories,

$$L_{B/A} : \mathbf{DAlg}_A^{cn}(\mathcal{C})_{/B} \leftrightarrows \mathbf{Mod}_B : \mathbf{sqz}_{B/A}$$

By setting $\mathbb{L}_{B/A} := L_{B/A}(B)$, we obtain the following result.

Corollary 4.2.1.6. *Suppose that we have a morphism $f : A \to B$ in $\mathbf{DAlg}^{cn}(\mathcal{C})$. Then, there exists a B-module $\mathbb{L}_{B/A}$ such that there is an equivalence of ∞-groupoids*

$$\mathbf{Der}_A(B, M) \simeq \mathrm{Map}_{\mathbf{Mod}_B}(\mathbb{L}_{B/A}, M)$$

for all $M \in \mathbf{Mod}_B$.

Definition 4.2.1.7. Suppose that we have a morphism $f : A \to B$ in $\mathbf{DAlg}^{cn}(\mathcal{C})$. The *relative cotangent complex of f* is the unique B-module $\mathbb{L}_{B/A}$ from Corollary 4.2.1.6. If $A = I$, then we denote the cotangent complex of the morphism $I \to B$ by $\mathbb{L}_B$.

Suppose that $f : A \to B$ is a morphism in $\mathbf{DAlg}(\mathcal{C})$. Then, there is a forgetful functor $f^* : \mathbf{Mod}_B \to \mathbf{Mod}_A$ which, by [50, c.f. Theorem 3.6.7], has a left adjoint given by $- \otimes_A^{\mathbb{L}} B = f_! : \mathbf{Mod}_A \to \mathbf{Mod}_B$. We have the following results about the cotangent complex.

Lemma 4.2.1.8. *Suppose that we have a morphism $f : A \to B$ in $\mathbf{DAlg}^{cn}(\mathcal{C})$. Then, there is a cofibre sequence in $\mathbf{Mod}_B$*

$$\mathbb{L}_A \otimes^{\mathbb{L}}_A B \to \mathbb{L}_B \to \mathbb{L}_{B/A}$$

Proof. It suffices to check, since the Yoneda embedding reflects limits, that for all $M \in \mathbf{Mod}_B$, $\mathrm{Map}_{\mathbf{Mod}_B}(\mathbb{L}_{B/A}, M)$ is the fibre of the map

$$\mathrm{Map}_{\mathbf{Mod}_B}(\mathbb{L}_B, M) \to \mathrm{Map}_{\mathbf{Mod}_B}(\mathbb{L}_A \otimes^{\mathbb{L}}_A B, M)$$

For any $M \in \mathbf{Mod}_B$, we have a chain of equivalences

$$\mathrm{Map}_{\mathbf{Mod}_B}(\mathbb{L}_A \otimes^{\mathbb{L}}_A B, M) \simeq \mathrm{Map}_{\mathbf{Mod}_A}(\mathbb{L}_A, M) \simeq \mathrm{Map}_{\mathbf{DAlg}^{cn}(\mathcal{C})_{/A}}(A, A \oplus M)$$

and therefore we are reduced to evaluating the fibre of the map

$$\mathrm{Map}_{\mathbf{DAlg}^{cn}(\mathcal{C})_{/B}}(B, B \oplus M) \to \mathrm{Map}_{\mathbf{DAlg}^{cn}(\mathcal{C})_{/A}}(A, A \oplus M)$$

which is equivalent to $\mathrm{Map}_{\mathbf{DAlg}^{cn}_A(\mathcal{C})_{/B}}(B, B \oplus M) \simeq \mathrm{Map}_{\mathbf{Mod}_B}(\mathbb{L}_{B/A}, M)$, i.e. the collection of derivations $B \to M$ which act as the identity on A. $\square$

The following results follow using Lemma 4.2.1.8 and using similar reasoning to [57, Corollary 7.3.3.6, Proposition 7.3.3.7].

Corollary 4.2.1.9. 1. *Suppose that $A \to B$ and $B \to C$ are maps in $\mathbf{DAlg}^{cn}(\mathcal{C})$. Then, we have a cofibre sequence*

$$\mathbb{L}_{B/A} \otimes^{\mathbb{L}}_B C \to \mathbb{L}_{C/A} \to \mathbb{L}_{C/B}$$

in $\mathbf{Mod}_C$,

2. *Suppose that we have a pushout square in $\mathbf{DAlg}^{cn}(\mathcal{C})$*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ A' \arrow[r] & B' \end{tikzcd}$$

Then, there is an equivalence $\mathbb{L}_{B/A} \otimes^{\mathbb{L}}_B B' \simeq \mathbb{L}_{B'/A'}$ in $\mathbf{Mod}_{B'}$. Moreover, there is a pushout square in $\mathbf{Mod}_{B'}$

$$\begin{tikzcd} \mathbb{L}_A \otimes^{\mathbb{L}}_A B' \arrow[r] \arrow[d] & \mathbb{L}_B \otimes^{\mathbb{L}}_B B' \arrow[d] \\ \mathbb{L}_{A'} \otimes^{\mathbb{L}}_{A'} B' \arrow[r] & \mathbb{L}_{B'} \end{tikzcd}$$

4.2.2 Infinitesimal Extensions

Suppose that $f : A \to B$ is a morphism in $\mathbf{DAlg}^{cn}(\mathcal{C})$ and suppose that $M \in \mathbf{Mod}_B$. If we consider the image of M under $\mathbf{sqz}_{B/A}$, then this defines a canonical morphism $i : B \oplus M \to B$. There is an inclusion map $s : B \to B \oplus M$ which comes from considering the image of the zero morphism $\mathbb{L}_{B/A} \to M$ under the equivalence

$$\pi_0(\mathrm{Map}_{\mathbf{Mod}_B}(\mathbb{L}_{B/A}, M)) \simeq \pi_0(\mathbf{Der}_A(B, M))$$

induced by Corollary 4.2.1.6. We can consider the map $i : B \to B \oplus M$ as a section of the map $s : B \oplus M \to B$ in $\mathbf{DAlg}^{cn}_A(\mathcal{C})_{/B}$.

We denote by $\Omega_A : \mathbf{Mod}_A \to \mathbf{Mod}_A$ the loop functor and the suspension functor by $\Sigma_A : \mathbf{Mod}_A \to \mathbf{Mod}_A$ on $\mathbf{Mod}_A$. For $M \in \mathbf{Mod}_A$, we will occasionally use the notation $M[n]$ to denote the object $\Sigma^n(M)$. If $A \in \mathbf{DAlg}^{cn}(\mathcal{C})$ and $M \in \mathbf{Mod}_A$ then, since $\mathbf{sqz}_A$ commutes with limits, there is a pullback square

$$\begin{tikzcd} A \oplus \Omega_A(M) \arrow[r] \arrow[d] & A \arrow[d, "s"] \\ A \arrow[r, "s"] & A \oplus M \end{tikzcd}$$

in $\mathbf{DAlg}^{cn}(\mathcal{C})$. This motivates the following definition.

Definition 4.2.2.1. Suppose that we have a morphism $f : A \to B$ in $\mathbf{DAlg}^{cn}(\mathcal{C})$, $M \in \mathbf{Mod}_B$, and a derivation $d \in \pi_0(\mathbf{Der}_A(B, M))$. Then, the *square zero extension associated to* d, denoted $B \oplus_d \Omega M$, is the pullback in $\mathbf{DAlg}^{cn}_A(\mathcal{C})_{/B}$ of the diagram

$$\begin{tikzcd} B \oplus_d \Omega M \arrow[r] \arrow[d] & B \arrow[d, "d"] \\ B \arrow[r, "s"] & B \oplus M \end{tikzcd}$$

Lemma 4.2.2.2. *Suppose that $A \in \mathbf{DAlg}^{cn}(\mathcal{C})$, $M \in \mathbf{Mod}_A$, and $d \in \pi_0(\mathbf{Der}(A, M))$. Then, the following commutative diagram is both a pullback and pushout square in $\mathbf{Mod}_A$.*

$$\begin{tikzcd} A \oplus_d \Omega M \arrow[r] \arrow[d] & A \arrow[d, "d"] \\ A \arrow[r, "s"] & A \oplus M \end{tikzcd}$$

Moreover, there is a fibre-cofibre sequence in $\mathbf{Mod}_A$, $A \oplus_d \Omega M \to A \to M$.

Proof. Consider the square as a pullback square in $\mathbf{Mod}_A$ under the forgetful functor $\mathbf{DAlg}_A^{cn}(\mathcal{C}) \to \mathbf{Mod}_A$. Since $\mathbf{Mod}_A$ is stable, the square is also a pushout square in $\mathbf{Mod}_A$. The cofibres of the two horizontal morphisms in $\mathbf{Mod}_A$ are equivalent to $M \in \mathbf{Mod}_A$, and hence we can conclude. $\square$

Remark. We note that, as we are working with connective objects, we will often have to restrict to modules M in $\mathbf{Mod}_A^{\geq 1}$ so that $\Omega_A(M) \in \mathbf{Mod}_A^{cn}$. We note that, if $M \in \mathbf{Mod}_A^{\geq 1}$, then the above sequence is also a fibre-cofibre sequence in $\mathbf{Mod}_A^{cn}$.

Corollary 4.2.2.3. *Suppose $A \in \mathbf{DAlg}^{cn}(\mathcal{C})$, $M \in \mathbf{Mod}_A^{\geq 1}$, and $N \in \mathbf{Mod}_{A\oplus_d \Omega M}^{cn}$. Then, there is an equivalence in $\mathbf{DAlg}^{cn}(\mathcal{C})$.*

$$N \simeq (N \otimes^{\mathbb{L}}_{A\oplus_d\Omega M} A) \times_{N\otimes^{\mathbb{L}}_{A\oplus_d\Omega M}(A\oplus M)} (N \otimes^{\mathbb{L}}_{A\oplus_d\Omega M} A)$$

Proof. By the previous lemma, we have the following commutative diagram in $\mathbf{Mod}_A$ where both squares are pushout diagrams.

$$\begin{tikzcd}
N \otimes^{\mathbb{L}}_{A\oplus_d\Omega M} A \arrow[r] & A \oplus M \arrow[r] & N \otimes^{\mathbb{L}}_{A\oplus_d\Omega M} (A \oplus M) \\
N \arrow[u] \arrow[r] & A \arrow[u] \arrow[r] & N \otimes^{\mathbb{L}}_{A\oplus_d\Omega M} A \arrow[u]
\end{tikzcd}$$

Hence, the outer rectangle is a pushout square in $\mathbf{Mod}_A$, equivalently a pullback square in $\mathbf{Mod}_A$. Since the forgetful functor $\mathbf{DAlg}_A(\mathcal{C}) \to \mathbf{Mod}_A$ is conservative, we obtain the desired equivalence.

$\square$

Suppose that we have a morphism $f : A \to B$ in $\mathbf{DAlg}^{cn}(\mathcal{C})$. Consider the induced map of B-modules $\eta : \mathbb{L}_B \to \mathbb{L}_{B/A}$. We note that this induces a derivation $d_\eta : B \to B \oplus \mathbb{L}_{B/A}$. Consider the associated square-zero extension $B \oplus_{d_\eta} \Omega\mathbb{L}_{B/A}$. Since the restriction of η to $\mathbb{L}_A$ is nullhomotopic, the map f factors as a composition

$$A \xrightarrow{f'} B \oplus_{d_\eta} \Omega\mathbb{L}_{B/A} \xrightarrow{f''} B$$

In particular, there is a map of A-modules $\mathrm{cofib}(f) \to \mathrm{cofib}(f'')$ which, using Lemma 4.2.2.2, induces a map of B-modules

$$\epsilon_f : B \otimes_A \mathrm{cofib}(f) \to \mathrm{cofib}(f'') \simeq \mathbb{L}_{B/A}$$

Lemma 4.2.2.4. *Suppose that $f : A \to B$ is a morphism in $\mathbf{DAlg}^{cn}(\mathcal{C})$ and that f induces an equivalence $\tau_{\leq n}(A) \to \tau_{\leq n}(B)$ for some $n \in \mathbb{N}$. Then $\tau_{\leq n}(\mathbb{L}_{B/A}) \simeq 0$.*

Proof. This follows in a similar way to [57, Lemma 7.4.3.17] using Lemma 4.2.1.8. □

By applying this lemma to the morphism $f : I \to A$, we see that if $A \in \mathbf{DAlg}^{cn}(\mathcal{C})$, then $\mathbb{L}_A$ is connective.

Now, suppose that for all $m \geq 2$, if $A \in \mathcal{C}$ is n-connective, then $\mathbf{LSym}^m(A)$ is $(n+2)$-connective. This condition holds, for example, for $\mathbb{E}_\infty$-rings and simplicial commutative rings [58, Proposition 25.2.4.1]. In this situation, we have the following.

Theorem 4.2.2.5. *[57, c.f. Theorem 7.4.3.12, Corollary 7.4.3.2] Suppose that $f : A \to B$ is a morphism in $\mathbf{DAlg}^{cn}(\mathcal{C})$ and that* $\mathrm{cofib}(f)$ *is n-connective for some $n \geq 1$. Then,*

1. $\mathrm{fib}(\epsilon_f)$ *is $(n+2)$-connective,*
2. *The induced morphism $\mathbb{L}_A \to \mathbb{L}_B$ has n-connective cofibre. The converse is true if the morphism $\pi_0(A) \to \pi_0(B)$ is an isomorphism.*

Corollary 4.2.2.6. *Suppose that $f : A \to B$ is a morphism in $\mathbf{DAlg}^{cn}(\mathcal{C})$. Then, as a $\pi_0(B)$-module, $\pi_0(\mathbb{L}_{B/A})$ is isomorphic to $\pi_0(\mathbb{L}_{\pi_0(B)/\pi_0(A)})$.*

Proof. By the previous result applied to the morphism $A \to \pi_0(A)$, we can deduce that the canonical map $\pi_0(\mathbb{L}_A) \to \pi_0(\mathbb{L}_{\pi_0(A)})$ is an isomorphism. The result then follows using the following morphism of long exact sequences.

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_0(\mathbb{L}_A \otimes_A B) \arrow[r] \arrow[d] & \pi_0(\mathbb{L}_B) \arrow[r] \arrow[d] & \pi_0(\mathbb{L}_{B/A}) \arrow[d] \\
\cdots \arrow[r] & \mathbb{L}_{\pi_0(A)} \otimes_{\pi_0(A)} \pi_0(B) \arrow[r] & \pi_0(\mathbb{L}_{\pi_0(B)}) \arrow[r] & \pi_0(\mathbb{L}_{\pi_0(B)/\pi_0(A)})
\end{tikzcd}$$

□

4.2.3 Smooth, Étale, and Perfect

We now use our definition of the cotangent complex to make the following definitions, motivated by the classical definitions. The definitions of projective and perfect in this context can be found in Appendix B.

Definition 4.2.3.1. [86, c.f. Definitions 1.2.6.1 and 1.2.7.1] Suppose that $f : A \to B$ is a morphism in $\mathbf{DAlg}^{cn}(\mathcal{C})$. Then,

- f is *formally étale* if the natural morphism $\mathbb{L}_A \otimes^{\mathbb{L}}_A B \to \mathbb{L}_B$ is an equivalence in $\mathbf{Mod}_B$,
- f is *formally unramified* if $\mathbb{L}_{B/A} \simeq 0$ in $\mathbf{Mod}_B$,

- f is *formally perfect* if the B-module $\mathbb{L}_{B/A}$ is perfect.

We can easily show that formally étale and formally unramified morphisms are stable by equivalences, compositions, and $(\infty, 1)$-pushouts using Corollary 4.2.1.9.

Lemma 4.2.3.2. *Formally perfect morphisms in* $\mathbf{DAlg}^{cn}(\mathcal{C})$ *are stable under equivalences, compositions, and pushouts.*

Proof. It is easy to see that equivalences are formally perfect. If we have formally perfect maps $A \to B$ and $B \to C$, then we have a cofibre sequence

$$\mathbb{L}_{B/A} \otimes_B^{\mathbb{L}} C \to \mathbb{L}_{C/A} \to \mathbb{L}_{C/B}$$

by Corollary 4.2.1.9. This sequence splits by [57, Proposition 7.2.2.6] since perfect C-modules are projective. Hence, by [20, Lemma 1.6.11],

$$\mathbb{L}_{C/A} \simeq (\mathbb{L}_{B/A} \otimes_B^{\mathbb{L}} C) \coprod \mathbb{L}_{C/B}$$

We note that perfect objects are stable under base-change by [10, Section 2.1.5]. Hence, since $\mathbb{L}_{C/A}$ is the coproduct of two perfect modules, it is perfect. Now, suppose that we have a formally perfect map $A \to B$ along with a map $A \to C$. Then, we see that, by Corollary 4.2.1.9,

$$\mathbb{L}_{B \otimes_A^{\mathbb{L}} C/C} \simeq \mathbb{L}_{B/A} \otimes_A^{\mathbb{L}} C$$

which is the base change of a perfect module and is therefore perfect. □

Definition 4.2.3.3. Suppose that $A \in \mathbf{DAlg}^{cn}(\mathcal{C})$ and that $P \in \mathcal{C}_{\geq 0}$ is projective. Then, an A-module M is *P-projective* if M is a retract in $\mathbf{Mod}_A$ of $A \otimes P$.

Definition 4.2.3.4. [10, Definition 2.5.125] Let $P \in \mathcal{C}_{\geq 0}$ be projective. Then, a map $f : A \to B$ in $\mathbf{DAlg}^{cn}(\mathcal{C})$ is *formally P-smooth* if

1. $\mathbb{L}_{B/A}$ is P-projective,
2. The morphism $\mathbb{L}_A \otimes_A^{\mathbb{L}} B \to \mathbb{L}_B$ has a retraction in $\mathbf{Mod}_B^{cn}$.

We say that f is *formally smooth* if it is formally P-smooth for some P.

We note that formally smooth morphisms in $\mathbf{DAlg}^{cn}(\mathcal{C})$ are stable under equivalences, compositions, and pushouts by [10, Lemma 2.5.129].

Proposition 4.2.3.5. *Suppose that* $f : A \to B$ *is a morphism in* $\mathbf{DAlg}^{cn}(\mathcal{C})$*. Then,*

1. *f is formally étale if and only if it is formally unramified,*
2. *If f is formally étale, then it is formally perfect and formally smooth.*

Proof. For the first result, we consider the cofibre sequence in $\mathbf{Mod}_B$

$$\mathbb{L}_A \otimes^{\mathbb{L}}_A B \to \mathbb{L}_B \to \mathbb{L}_{B/A}$$

Since $\mathbf{Mod}_B$ is a stable $(\infty, 1)$-category, the sequence is also a fibre sequence, from which the first result easily follows. If $f : A \to B$ is formally étale, then the natural morphism $\mathbb{L}_A \otimes^{\mathbb{L}}_A B \to \mathbb{L}_B$ is an equivalence and $\mathbb{L}_{B/A} \simeq 0$, which is perfect and projective. Hence, the second result follows. □

Lemma 4.2.3.6. *Suppose that $A \in \mathbf{DAlg}^{cn}(\mathcal{C})$ and that $P \in \mathbf{Mod}_A$ is projective. Then,*

1. *If M is an object of $\mathbf{Mod}^{cn}_A$, then*

$$\mathrm{Hom}_{\mathbf{Mod}_{\pi_0(A)}}(\pi_0(P), \pi_0(M)) \simeq \pi_0(\mathrm{Map}_{\mathbf{Mod}_A}(P, M))$$

2. *For any $M \in \mathbf{Mod}^{\geq 1}_A$, $\pi_0(\mathrm{Map}_{\mathbf{Mod}_A}(P, M)) = 0$.*

Proof. This follows immediately using [10, Lemma 2.3.75]. □

Proposition 4.2.3.7. *Formally perfect morphisms in $\mathbf{DAlg}^{cn}(\mathcal{C})$ are formally smooth.*

Proof. Indeed, we note that if $f : A \to B$ is formally perfect, then the cotangent complex $\mathbb{L}_{B/A}$ is perfect, and hence projective. Now, if we consider the shifted fibre sequence

$$\mathbb{L}_B \to \mathbb{L}_{B/A} \to (\mathbb{L}_A \otimes^{\mathbb{L}}_A B)[1]$$

and, if we denote by $[-, -]$ the set $\pi_0(\mathrm{Map}_{\mathbf{Mod}_B}(-, -))$, then we obtain a long exact sequence in homotopy

$$\cdots \to [\mathbb{L}_{B/A}, \mathbb{L}_B] \to [\mathbb{L}_{B/A}, \mathbb{L}_{B/A}] \to [\mathbb{L}_{B/A}, (\mathbb{L}_A \otimes^{\mathbb{L}}_A B)[1]] \to 0$$

then, since $[\mathbb{L}_{B/A}, (\mathbb{L}_A \otimes^{\mathbb{L}}_A B)[1]] \simeq 0$ by Lemma 4.2.3.6, there is a surjection

$$[\mathbb{L}_{B/A}, \mathbb{L}_B] \to [\mathbb{L}_{B/A}, \mathbb{L}_{B/A}]$$

and hence the morphism $\mathbb{L}_B \to \mathbb{L}_{B/A}$ has a section. Therefore, the fibre sequence $\mathbb{L}_A \otimes^{\mathbb{L}}_A B \to \mathbb{L}_B \to \mathbb{L}_{B/A}$ splits, and so $\mathbb{L}_A \otimes^{\mathbb{L}}_A B \to \mathbb{L}_B$ has a retraction. □

4.2.4 Derived Geometry Contexts

Suppose that we have a derived algebraic context $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, \mathcal{C}^0)$. We define the categories of affines and connective affines by

$$\mathbf{DAff}(\mathcal{C}) := \mathbf{DAlg}(\mathcal{C})^{op} \quad \text{and} \quad \mathbf{DAff}^{cn}(\mathcal{C}) := \mathbf{DAlg}^{cn}(\mathcal{C})^{op}$$

For $A \in \mathbf{DAlg}(\mathcal{C})$, we will denote by $\mathrm{Spec}(A)$ its image in $\mathbf{DAff}(\mathcal{C})$. Suppose that $\mathcal{A}$ is some full subcategory of $\mathbf{DAff}^{cn}(\mathcal{C})$. Since we are not considering all affines, we need to restrict our categories of modules to the following *good systems* in order to be able to define the relevant notions.

For a collection $\mathbf{M}_A$ of $\mathbf{A}$-modules, we will denote by $\mathbf{M}_{A,n}$ the fibre product $\mathbf{M}_A \times_{\mathbf{Mod}_A} \mathbf{Mod}_A^{\geq n}$. In particular, $\mathbf{M}_A^{cn}$ is the fibre product $\mathbf{M}_A \times_{\mathbf{Mod}_A} \mathbf{Mod}_A^{cn}$ and $\mathbf{M}_A^{\heartsuit}$ is the fibre product $\mathbf{M}_A \times_{\mathbf{Mod}_A} \mathbf{Mod}_A^{\heartsuit}$.

Definition 4.2.4.1. A *good system of categories of modules* on $\mathcal{A}$, denoted $\mathbf{M}$, is an assignment, to each $A \in \mathcal{A}^{op} \subseteq \mathbf{DAlg}^{cn}(\mathcal{C})$, of a full subcategory $\mathbf{M}_A$ of $\mathbf{Mod}_A$ satisfying the following properties

1. $A \in \mathbf{M}_A$,
2. $\mathbb{L}_A \in \mathbf{M}_A$,
3. $\pi_n(A) \in \mathbf{M}_A$ for all $n \geq 0$,
4. Whenever $M \in \mathbf{M}_A$ and $f : A \to B$ is a morphism in $\mathcal{A}^{op}$, $M \otimes_A^{\mathbb{L}} B \in \mathbf{M}_B$,
5. For any $M \in \mathbf{M}_{A,1}$ and any derivation $d \in \pi_0(\mathbf{Der}(A, M))$, $\mathrm{Spec}(A \oplus_d \Omega M)$ is $\mathcal{A}$-admissible,
6. If $X = \mathrm{Spec}(A) \in \mathcal{A}^{\leq k} := \mathcal{A} \times_{\mathbf{DAff}^{cn}(\mathcal{C})} \mathbf{DAff}^{\leq k}(\mathcal{C})$ for some $k > 0$, then for any $M \in \mathbf{M}_A^{\heartsuit}$ and any derivation $d \in \pi_0(\mathbf{Der}(A, M))$, $\mathrm{Spec}(A \oplus_d \Omega M[k+1]) \in \mathcal{A}$,
7. If $M \in \mathbf{M}_A$, then $\Omega_A(M) \in \mathbf{M}_A$ and $\Sigma_A(M) \in \mathbf{M}_A$,
8. $\mathbf{M}_A$ is closed under equivalences, retracts, and finite colimits.

Combining the definitions of derived algebraic contexts with our notion of a geometry tuple from Section 3.2.1, we can define a *derived geometry context*.

Definition 4.2.4.2. We say that $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, \mathcal{C}^0, \boldsymbol{\tau}, \mathbf{P}, \mathcal{A}, \mathbf{M})$ is a *derived geometry context* if

1. $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, \mathcal{C}^0)$ is a derived algebraic context,
2. $\mathcal{A}$ is a full subcategory of $\mathbf{DAff}^{cn}(\mathcal{C})$,
3. $(\mathbf{DAff}^{cn}(\mathcal{C}), \boldsymbol{\tau}, \mathbf{P}, \mathcal{A})$ is a strong relative $(\infty, 1)$-geometry tuple,
4. $\mathbf{M}$ is a good system of categories of modules on $\mathcal{A}$.

4.2.5 Cotangent Complexes of Presheaves

Fix a derived geometry context $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, \mathcal{C}^0, \boldsymbol{\tau}, \mathbf{P}, \mathcal{A}, \mathbf{M})$. We recall that there is a fully faithful functor $i : \mathbf{PSh}(\mathcal{A}) \to \mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))$ which, since we are assuming our tuple is strong, induces a fully faithful functor on stacks.

Suppose that $X = \mathrm{Spec}(A) \in \mathcal{A}$ and $M \in \mathbf{M}_A^{cn}$. Define $X[M] := \mathrm{Spec}(A \oplus M)$, which is a stack on $\mathcal{A}$ by our assumptions. If we consider the map $A \oplus M \to A$ specified by the square-zero extension construction, then we obtain an induced morphism of affines $X \to X[M]$. Now, suppose that we have some morphism of presheaves $f : \mathcal{F} \to \mathcal{G}$ in $\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))$ and a morphism $x : X \to \mathcal{F}$.

Definition 4.2.5.1. The ∞*-groupoid of derived* $\mathcal{F}/\mathcal{G}$ *derivations from* $\mathcal{F}$ *to* M *at* x is defined to be

$$\mathbf{Der}_{\mathcal{F}/\mathcal{G}}(X, M) := \mathrm{Map}_{X/\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))_{/\mathcal{G}}}(X[M], \mathcal{F})$$

When $\mathcal{G} = *$, we denote the ∞-groupoid of derivations by $\mathbf{Der}_{\mathcal{F}}(X, M)$.

Remark. We see that, by [52, c.f. Lemma 5.5.5.12], $\mathbf{Der}_{\mathcal{F}/\mathcal{G}}(X, M)$ is the fibre of the map $\mathbf{Der}_{\mathcal{F}}(X, M) \to \mathbf{Der}_{\mathcal{G}}(X, M)$ at x.

Lemma 4.2.5.2. *Let* $X = \mathrm{Spec}(A) \in \mathcal{A}$ *and* $M \in \mathbf{M}_A^{cn}$. *Suppose that we have a derivation* $d \in \pi_0(\mathbf{Der}(A, M))$ *and let* p *be the map* $\mathcal{F}(A) \times_{\mathcal{F}(A \oplus M)} \mathcal{F}(A) \to \mathcal{F}(A)$ *induced by the map* $A \oplus M \to A$. *Then, there is an equivalence of* $(\infty, 1)$*-groupoids*

$$\Omega_{d(x),0} \mathbf{Der}_{\mathcal{F}}(X, M) \simeq \mathrm{fib}(\mathcal{F}(A) \times_{\mathcal{F}(A \oplus M)} \mathcal{F}(A) \xrightarrow{p} \mathcal{F}(A))$$

where $\Omega_{d(x),0}(-)$ *is the path space and the fibre on the right is taken at* $x : X \to \mathcal{F}$.

Proof. We construct the following commutative diagram

$$\begin{tikzcd}
\mathrm{fib}(p) \arrow[r] \arrow[d] & * \arrow[d, "x"] & \\
\mathcal{F}(A) \times_{\mathcal{F}(A \oplus M)} \mathcal{F}(A) \arrow[r, "p"] \arrow[d] & \mathcal{F}(A) \arrow[d, "d"] & \\
* \arrow[r, "0"] \arrow[d] & \mathbf{Der}_{\mathcal{F}}(X, M) \arrow[r] \arrow[d] & * \arrow[d] \\
\mathcal{F}(A) \arrow[r] & \mathcal{F}(A \oplus M) \arrow[r] & \mathcal{F}(A)
\end{tikzcd}$$

We note that the bottom right square is a pullback square by the above remarks and we can easily deduce that all other squares are pullback squares. Hence, we obtain the desired result. □

Definition 4.2.5.3. We say that f has a *relative cotangent complex at* x if there is an integer n, a module $\mathbb{L}_{\mathcal{F}/\mathcal{G},x}$ in $\mathbf{M}_{A,-n}$, and an equivalence

$$\mathbf{Der}_{\mathcal{F}/\mathcal{G}}(X,-) \simeq \mathrm{Map}_{\mathbf{Mod}_A}(\mathbb{L}_{\mathcal{F}/\mathcal{G},x},-)$$

in $\mathbf{PSh}((\mathbf{M}_A^{cn})^{op})$. If the cotangent complex exists, then we say that $\mathbb{L}_{\mathcal{F}/\mathcal{G},x}$ is the *relative cotangent complex of* f *at* x. If $\mathcal{G} = *$, then we denote the cotangent complex of the morphism $\mathcal{F} \to *$ at $x : X \to \mathcal{F}$ by $\mathbb{L}_{\mathcal{F},x}$.

Proposition 4.2.5.4. *Any representable presheaf* X *in* $\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))$ *has a cotangent complex for any map* $y : Y \to X$ *where* $Y \in \mathcal{A}$. *Furthermore, this cotangent complex is connective.*

Proof. Suppose that $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$ are representable presheaves and that there is a point $y : Y \to X$ corresponding to a morphism $f : A \to B$. Then, we can easily show that, for any $M \in \mathbf{M}_B^{cn}$, there is an equivalence

$$\mathbf{Der}_X(Y,M) := \mathrm{Map}_{Y/\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))}(Y[M],X) \simeq \mathrm{Map}_{\mathbf{Mod}_B}(\mathbb{L}_A \otimes_A^{\mathbb{L}} B, M)$$

Hence, let $\mathbb{L}_{X,y} := \mathbb{L}_A \otimes_A^{\mathbb{L}} B$ which we know is connective and, moreover, lies in $\mathbf{M}_B$ by our assumptions. $\square$

The proof of the following statement is similar to [86, Proposition 1.2.11.3] and is detailed in [75, Lemma 4.1.7].

Lemma 4.2.5.5. *Suppose that* $A \in \mathcal{A}^{op}$. *The functor*

$$\begin{aligned} D : \mathbf{Mod}_A &\to \mathbf{PSh}((\mathbf{M}_A^{cn})^{op}) \\ M &\to \mathrm{Map}_{\mathbf{Mod}_A}(M,-) \end{aligned}$$

is fully faithful when restricted to the subcategory $\mathbf{M}_{A,-n}$.

Now, consider the morphism $f : \mathcal{F} \to \mathcal{G}$ of presheaves in $\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))$ and suppose that we have morphisms $x : X \to \mathcal{F}$ and $y : Y \to \mathcal{F}$. Then, there is an induced morphism

$$\mathbf{Der}_{\mathcal{F}/\mathcal{G}}(Y,-) \to \mathbf{Der}_{\mathcal{F}/\mathcal{G}}(X,-)$$

which, if f has relative cotangent complexes at x and y, induces a morphism

$$\mathrm{Map}_{\mathbf{Mod}_B}(\mathbb{L}_{\mathcal{F}/\mathcal{G},y},-) \to \mathrm{Map}_{\mathbf{Mod}_A}(\mathbb{L}_{\mathcal{F}/\mathcal{G},x},-) \simeq \mathrm{Map}_{\mathbf{Mod}_B}(\mathbb{L}_{\mathcal{F}/\mathcal{G},x} \otimes_A^{\mathbb{L}} B,-)$$

and hence, by Lemma 4.2.5.5, there is a morphism

$$\mathbb{L}_{\mathcal{F}/\mathcal{G},x} \otimes_A^{\mathbb{L}} B \to \mathbb{L}_{\mathcal{F}/\mathcal{G},y}$$

Definition 4.2.5.6. We say that a morphism $f : \mathcal{F} \to \mathcal{G}$ in $\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))$ has a *relative global cotangent complex (relative to $\mathcal{A}$)* if it satisfies the following two conditions.

1. For any $X \in \mathcal{A}$ and any point $x : X \to \mathcal{F}$, the morphism f has a cotangent complex $\mathbb{L}_{\mathcal{F}/\mathcal{G},x}$,
2. For any morphism $g : A \to B$ in $\mathcal{A}^{op}$ and any morphism

$$\begin{tikzcd} Y \arrow[rr, "g"] \arrow[dr, "y"'] & & X \arrow[dl, "x"] \\ & \mathcal{F} & \end{tikzcd}$$

in $\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))$, the induced morphism $\mathbb{L}_{\mathcal{F}/\mathcal{G},x} \otimes^{\mathbb{L}}_{A} B \to \mathbb{L}_{\mathcal{F}/\mathcal{G},y}$ is an equivalence of modules in $\mathbf{M}_B$.

It is clear, using Proposition 4.2.5.4, that any representable in $\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))$ has a global cotangent complex. We collect together the following results about the global cotangent complex. The proofs follow in the same way as in [86, Lemma 1.4.1.16].

Lemma 4.2.5.7. *Suppose that $f : \mathcal{F} \to \mathcal{G}$ is a morphism in $\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))$.*

1. *If both $\mathcal{F}$ and $\mathcal{G}$ have global cotangent complexes, then f has a relative global cotangent complex. Furthermore, for any $X = \mathrm{Spec}(A) \in \mathcal{A}$ and any morphism of presheaves $x : X \to \mathcal{F}$, there is a natural fibre-cofibre sequence of modules in $\mathbf{M}_A$,*
$$\mathbb{L}_{\mathcal{G},x} \to \mathbb{L}_{\mathcal{F},x} \to \mathbb{L}_{\mathcal{F}/\mathcal{G},x}$$
2. *If the morphism f has a relative global cotangent complex, then for any presheaf $\mathcal{H} \in \mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))$ and any morphism $\mathcal{H} \to \mathcal{G}$, the morphism $\mathcal{F} \times_{\mathcal{G}} \mathcal{H} \to \mathcal{H}$ has a relative global cotangent complex, and furthermore we have*
$$\mathbb{L}_{\mathcal{F}/\mathcal{G},x} \simeq \mathbb{L}_{\mathcal{F}\times_{\mathcal{G}}\mathcal{H}/\mathcal{H},x}$$
for any $X \in \mathcal{A}$ and any morphism of presheaves $x : X \to \mathcal{F} \times_{\mathcal{G}} \mathcal{H}$,
3. *If, for any $X = \mathrm{Spec}(A) \in \mathcal{A}$ and any morphism of presheaves $x : X \to \mathcal{F}$, the morphism $\mathcal{F} \times_{\mathcal{G}} X \to X$ has a relative global cotangent complex, then the morphism f has a relative global cotangent complex. Furthermore, we have*
$$\mathbb{L}_{\mathcal{F}/\mathcal{G},x} \simeq \mathbb{L}_{\mathcal{F}\times_{\mathcal{G}}X/X,x^{\cdot}}$$
where $x^{\cdot} : X \to \mathcal{F} \times_{\mathcal{G}} X$ denotes the lift of the point x.

Corollary 4.2.5.8. *Suppose that we have a commutative diagram in* $\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))$

$$\begin{tikzcd} X \arrow[r, "x"] & \mathcal{F} \arrow[r] & \mathcal{G} \\ Y \arrow[u] \arrow[r, "y"] & \mathcal{H} \arrow[u] \arrow[ur] & \end{tikzcd}$$

with $X = \mathrm{Spec}(A)$ *and* $Y = \mathrm{Spec}(B)$ *in* $\mathcal{A}$*. If* $\mathcal{F}, \mathcal{G}$*, and* $\mathcal{H}$ *have global cotangent complexes, then there exists a fibre-cofibre sequence of modules in* $\mathbf{M}_B$*,*

$$\mathbb{L}_{\mathcal{F}/\mathcal{G},x} \otimes^{\mathbb{L}}_A B \to \mathbb{L}_{\mathcal{H}/\mathcal{G},y} \to \mathbb{L}_{\mathcal{H}/\mathcal{F},y}$$

Proof. Using the previous lemma, we obtain the following commutative diagram of modules in $\mathbf{M}_B$,

$$\begin{tikzcd} \mathbb{L}_{\mathcal{G},y} \arrow[r] \arrow[d] & \mathbb{L}_{\mathcal{F},y} \arrow[r] \arrow[d] & \mathbb{L}_{\mathcal{H},y} \arrow[d] \\ * \arrow[r] & \mathbb{L}_{\mathcal{F}/\mathcal{G},y} \arrow[r] \arrow[d] & \mathbb{L}_{\mathcal{H}/\mathcal{G},y} \arrow[d] \\ & * \arrow[r] & \mathbb{L}_{\mathcal{H}/\mathcal{F},y} \end{tikzcd}$$

and hence we obtain a fibre-cofibre sequence of modules in $\mathbf{M}_B$

$$\mathbb{L}_{\mathcal{F}/\mathcal{G},y} \to \mathbb{L}_{\mathcal{H}/\mathcal{G},y} \to \mathbb{L}_{\mathcal{H}/\mathcal{F},y}$$

Since the morphism $f : \mathcal{F} \to \mathcal{G}$ has a global cotangent complex, we obtain the following fibre-cofibre sequence in $\mathbf{M}_B$

$$\mathbb{L}_{\mathcal{F}/\mathcal{G},x} \otimes^{\mathbb{L}}_A B \to \mathbb{L}_{\mathcal{H}/\mathcal{G},y} \to \mathbb{L}_{\mathcal{H}/\mathcal{F},y}$$

□

4.3 Obstruction Theories

In this section, we consider the problem of when a presheaf has an *obstruction theory.* These obstruction theories control lifting problems along first-order infinitesimal deformations.

4.3.1 Obstruction Theory of Presheaves

Suppose that we have some derived geometry context $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, \mathcal{C}^0, \boldsymbol{\tau}, \mathbf{P}, \mathcal{A}, \mathbf{M})$. Suppose that $X = \mathrm{Spec}(A) \in \mathcal{A}$, $M \in \mathbf{M}^{cn}_A$, and $d \in \pi_0(\mathbf{Der}(A, M))$. We define the object

$$X_d[\Omega M] := \mathrm{Spec}(A \oplus_d \Omega M)$$

If $M \in \mathbf{M}_{A,1}$, then $X_d[\Omega M]$ is $\mathcal{A}$-admissible by our assumptions on the good system of categories of modules $\mathbf{M}$.

Definition 4.3.1.1. Suppose that $f : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves in $\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))$. Then,

1. f is *infinitesimally cartesian relative to* $\mathcal{A}$ if, for any $X = \operatorname{Spec}(A) \in \mathcal{A}$, any A-module $M \in \mathbf{M}_{A,1}$, and any derivation $d \in \pi_0(\mathbf{Der}(A, M))$, corresponding to a morphism $d : A \to A \oplus M$, the square

$$\begin{array}{ccc} \mathcal{F}(A \oplus_d \Omega M) & \longrightarrow & \mathcal{G}(A \oplus_d \Omega M) \\ \downarrow & & \downarrow \\ \mathcal{F}(A) \times_{\mathcal{F}(A \oplus M)} \mathcal{F}(A) & \longrightarrow & \mathcal{G}(A) \times_{\mathcal{G}(A \oplus M)} \mathcal{G}(A) \end{array}$$

   is a pullback square in $\infty\mathbf{Grpd}$,
2. f *has an obstruction theory relative to* $\mathcal{A}$ if it has a global cotangent complex relative to $\mathcal{A}$ and is infinitesimally cartesian relative to $\mathcal{A}$.

We say that a presheaf $\mathcal{F}$ in $\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))$ *has an obstruction theory* if the morphism $\mathcal{F} \to *$ does.

Proposition 4.3.1.2. *Any representable presheaf X in $\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))$ has an obstruction theory relative to $\mathcal{A}$.*

Proof. We know that any representable presheaf has a global cotangent complex. It remains to show that it is infinitesimally cartesian relative to $\mathcal{A}$. Indeed, suppose that $X = \operatorname{Spec}(A)$ and that $Y = \operatorname{Spec}(B) \in \mathcal{A}$. Suppose that M is in $\mathbf{M}_{B,1}$ and that $d \in \pi_0(\mathbf{Der}(B, M))$. Then,

$$\begin{aligned} &\operatorname{Map}_{\mathbf{DAlg}^{cn}(\mathcal{C})}(A, B \oplus_d \Omega M) \\ &\simeq \operatorname{Map}_{\mathbf{DAlg}^{cn}(\mathcal{C})}(A, B \times_{B \oplus M} B) \\ &\simeq \operatorname{Map}_{\mathbf{DAlg}^{cn}(\mathcal{C})}(A, B) \times_{\operatorname{Map}_{\mathbf{DAlg}^{cn}(\mathcal{C})}(A, B \oplus M)} \operatorname{Map}_{\mathbf{DAlg}^{cn}(\mathcal{C})}(A, B) \end{aligned}$$

from which the result follows. □

Proposition 4.3.1.3. *Suppose that $\mathcal{F}$ is a stack in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$ whose diagonal morphism $\mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is $(n-1)$-representable for some $n \geq 0$. Then, $\mathcal{F}$ has an obstruction theory if and only if it is infinitesimally cartesian.*

Proof. Follows in a similar way to [86, c.f. Proposition 1.4.2.7]. □

Corollary 4.3.1.4. *Any n-geometric stack has an obstruction theory if and only if it is infinitesimally cartesian.*

We collect together the following results. The proofs follow in the same way as in [86, c.f. Lemma 1.4.2.3] using the results in Lemma 4.2.5.7.

Lemma 4.3.1.5. *Suppose that $f : \mathcal{F} \to \mathcal{G}$ is a morphism in $\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))$. Then,*

1. *If both presheaves $\mathcal{F}$ and $\mathcal{G}$ have an obstruction theory, then the morphism f has an obstruction theory,*
2. *If the morphism f has an obstruction theory, then for any presheaf $\mathcal{H}$ and any morphism $\mathcal{H} \to \mathcal{G}$, the morphism $\mathcal{F} \times_{\mathcal{G}} \mathcal{H} \to \mathcal{H}$ has a relative obstruction theory,*
3. *If, for any $X \in \mathcal{A}$ and any morphism of presheaves $x : X \to \mathcal{G}$, the presheaf $\mathcal{F} \times_{\mathcal{G}} X$ has an obstruction theory, then the morphism f has a relative obstruction theory.*

Corollary 4.3.1.6. *Any (-1)-representable morphism of stacks in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$ has an obstruction theory.*

Proof. Follows using Proposition 4.3.1.2 and item (3) of the previous lemma. □

We state the following two propositions motivating the definition of a presheaf 'having an obstruction theory'. Their proofs follow in a similar way to [86, Propositions 1.4.2.5 and 1.4.2.6].

Proposition 4.3.1.7. *Suppose that $\mathcal{F}$ is a presheaf in $\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))$ which has an obstruction theory relative to $\mathcal{A}$. Let $X = \mathrm{Spec}(A) \in \mathcal{A}$, $M \in \mathbf{M}_{A,1}$, and take some $d \in \pi_0(\mathbf{Der}(A, M))$. Suppose that $x : X = \mathrm{Spec}(A) \to \mathcal{F}$ is a morphism. Then, there exists a natural obstruction*

$$\alpha(x) \in \pi_0(\mathrm{Map}_{\mathbf{Mod}_A}(\mathbb{L}_{\mathcal{F},x}, M))$$

vanishing if and only if x extends to a morphism x' in ${}^{X/}\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))$

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "x"'] & & X_d[\Omega M] \arrow[dl, dotted, "x'"] \\ & \mathcal{F} & \end{tikzcd}$$

Proposition 4.3.1.8. *Suppose that $f : \mathcal{F} \to \mathcal{G}$ is a map in $\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))$ which has an obstruction theory relative to $\mathcal{A}$. Let $X = \mathrm{Spec}(A) \in \mathcal{A}$, $M \in \mathbf{M}_{A,1}$, and $d \in \pi_0(\mathbf{Der}(A, M))$. Suppose that x is a point in $\mathcal{F}(A) \times_{\mathcal{G}(A)} \mathcal{G}(A \oplus_d \Omega M)$ with projection $y \in \mathcal{F}(A)$ and let $L(x)$ be the homotopy fibre, taken at x, of the morphism*

$$\mathcal{F}(A \oplus_d \Omega M) \to \mathcal{F}(A) \times_{\mathcal{G}(A)} \mathcal{G}(A \oplus_d \Omega M)$$

Then, there exists a natural point $\alpha(x)$ in $\pi_0(\mathrm{Map}_{\mathbf{Mod}_A}(\mathbb{L}_{\mathcal{F}/\mathcal{G},y}, M))$ and a natural equivalence of ∞-groupoids

$$L(x) \simeq \Omega_{\alpha(x),0}\mathrm{Map}_{\mathbf{Mod}_A}(\mathbb{L}_{\mathcal{F}/\mathcal{G},y}, M)$$

4.3.2 Infinitesimally Smooth

We will need the following notion of *infinitesimal smoothness* which is somewhat weaker than the notion of smoothness. For a more detailed description in homotopical algebraic contexts see [86].

Definition 4.3.2.1. Suppose that $f : A \to B$ is a morphism in $\mathbf{DAlg}^{cn}(\mathcal{C})$. Then, f is *formally infinitesimally smooth (or formally i-smooth) relative to* $\mathcal{A}$ if, for any $C \in \mathcal{A}^{op}$, any morphism $A \to C$, any $M \in \mathbf{M}_{C,1}$ and any $d \in \pi_0(\mathbf{Der}_A(C, M))$, the natural morphism

$$\pi_0(\mathrm{Map}_{\mathbf{DAlg}_A^{cn}(\mathcal{C})}(B, C \oplus_d \Omega M)) \to \pi_0(\mathrm{Map}_{\mathbf{DAlg}_A^{cn}(\mathcal{C})}(B, C))$$

is surjective.

We note that formally i-smooth morphisms are stable by equivalences, compositions, and pushouts. Moreover,

Proposition 4.3.2.2. *[86, c.f. Proposition 1.2.8.3] A morphism $f : A \to B$ in $\mathbf{DAlg}^{cn}(\mathcal{C})$ is formally i-smooth relative to $\mathcal{A}$ if and only if, for any morphism $B \to C$ with $C \in \mathcal{A}^{op}$ and any $M \in \mathbf{M}_{C,1}$, the natural morphism*

$$\pi_0(\mathrm{Map}_{\mathbf{Mod}_C}(\mathbb{L}_{C/A}, M)) \to \pi_0(\mathrm{Map}_{\mathbf{Mod}_B}(\mathbb{L}_{B/A}, M))$$

is zero.

Corollary 4.3.2.3. *Any formally étale or formally unramified morphism $f : A \to B$ in $\mathbf{DAlg}^{cn}(\mathcal{C})$ is formally i-smooth relative to $\mathcal{A}$.*

Proof. This follows from Proposition 4.2.3.5 and the previous result. □

Corollary 4.3.2.4. *Any formally smooth or formally perfect morphism in $\mathbf{DAlg}^{cn}(\mathcal{C})$ is formally i-smooth.*

Proof. Indeed, suppose that $f : A \to B$ is a formally smooth (or perfect) morphism in $\mathbf{DAlg}^{cn}(\mathcal{C})$ and that $C \in \mathbf{DAlg}^{cn}(\mathcal{C})$ along with a morphism $B \to C$. Then, we have a fibre sequence

$$\mathbb{L}_{B/A} \otimes_B^{\mathbb{L}} C \to \mathbb{L}_{C/A} \to \mathbb{L}_{C/B}$$

in $\mathbf{Mod}_C$. Consider some $M \in \mathbf{M}_{C,1}$ and a derivation $d \in \pi_0(\mathbf{Der}_A(C, M))$. Then, since $\mathbb{L}_{B/A}$ is projective, we see that $\pi_0(\mathrm{Map}_{\mathbf{Mod}_B}(\mathbb{L}_{B/A}, M)) = 0$ by Lemma 4.2.3.6, and hence the morphism

$$\pi_0(\mathrm{Map}_{\mathbf{Mod}_C}(\mathbb{L}_{C/A}, M)) \to \pi_0(\mathrm{Map}_{\mathbf{Mod}_B}(\mathbb{L}_{B/A}, M))$$

is zero. The result then follows by Proposition 4.3.2.2. □

4.3.3 Obstruction Conditions

Suppose that we have some derived geometry context $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, \mathcal{C}^0, \boldsymbol{\tau}, \mathbf{P}, \mathcal{A}, \mathbf{M})$. We will now impose some extra conditions on $\boldsymbol{\tau}$ and $\mathbf{P}$ and define a suitable class $\mathbf{S}$ of formally smooth morphisms in $\mathcal{A}$ such that we obtain obstruction theories for geometric stacks.

We will call a family $\{U_i \to X\}_{i \in I}$ of morphisms of $\mathcal{A}$-admissible objects in $\mathbf{DAff}^{cn}(\mathcal{C})$ *an* $\mathbf{S}$*-covering family* if each morphism $U_i \to X$ is in $\mathbf{S}$ and there is an epimorphism of stacks $\coprod_{i \in I} U_i \to X$ in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$. The induced epimorphism $\coprod_{i \in I} U_i \to X$ will be called an $\mathbf{S}$*-cover*.

Lemma 4.3.3.1. *Suppose that we have a formally smooth morphism $A \to B$ in $\mathbf{DAlg}^{cn}(\mathcal{C})$, $M \in \mathbf{Mod}_A$, and $d \in \pi_0(\mathbf{Der}(A, M))$. Then, d lifts to a derivation d' which lies in $\pi_0(\mathbf{Der}(B, M \otimes^{\mathbb{L}}_A B))$.*

Proof. Indeed, we note that since $A \to B$ is formally smooth, we have a morphism $\mathbb{L}_B \to \mathbb{L}_A \otimes^{\mathbb{L}}_A B$ in $\mathbf{Mod}_B$. We define d' to be the image of d under the following map.

$$\begin{aligned}\pi_0(\mathbf{Der}(A, M)) = \pi_0(\mathrm{Map}_{\mathbf{Mod}_A}(\mathbb{L}_A, M)) &\to \pi_0(\mathrm{Map}_{\mathbf{Mod}_B}(\mathbb{L}_A \otimes^{\mathbb{L}}_A B, M \otimes^{\mathbb{L}}_A B)) \\ &\to \pi_0(\mathrm{Map}_{\mathbf{Mod}_B}(\mathbb{L}_B, M \otimes^{\mathbb{L}}_A B)) \\ &= \pi_0(\mathbf{Der}(B, M \otimes^{\mathbb{L}}_A B))\end{aligned}$$

□

Definition 4.3.3.2. We say that $\boldsymbol{\tau}$ and $\mathbf{P}$ satisfy *the obstruction conditions relative to* $\mathcal{A}$ if there exists a class $\mathbf{S}$ of morphisms in $\mathcal{A}$ such that

1. Any morphism in $\mathbf{P}$ is formally i-smooth,
2. The class $\mathbf{S}$ consists of formally smooth morphisms and is stable under equivalences, compositions, and pullbacks,
3. Suppose that $X = \mathrm{Spec}(A) \in \mathcal{A}$, $M \in \mathbf{M}_{A,1}$, and that $d \in \pi_0(\mathbf{Der}(A, M))$. Consider the natural morphism

$$X = \mathrm{Spec}(A) \to \mathrm{Spec}(A \oplus_d \Omega M) = X_d[\Omega M]$$

in $\mathbf{DAff}^{cn}(\mathcal{C})$. Suppose that $\{V_j \to X\}_{j \in J}$ is a $\boldsymbol{\tau}|_{\mathcal{A}}$-covering family of X. Then, there exists a finite $\mathbf{S}$-covering family $\{V'_k = \mathrm{Spec}(A_k) \to X\}_{k \in K}$ in $\mathcal{A}$ and a morphism $v : K \to J$ such that, for each $k \in K$, there is a commutative diagram

$$\begin{tikzcd} & V_{v(k)} \arrow[d] \\ V'_k \arrow[ur] \arrow[r] & X \end{tikzcd}$$

and such that $\{W_k \to X_d[\Omega M]\}_{k\in K}$ is an $\mathbf{S}$-covering family of $X_d[\Omega M]$ in $\mathcal{A}$, where $W_k = \mathrm{Spec}(A_k \oplus_{d'_k} \Omega M'_k)$ with d'_k the derivation induced by Lemma 4.3.3.1 and $M'_k = M \otimes^{\mathbb{L}}_A A_k$.

Remark. We note that this provides an extension of the analogous results in [75, Sections 4.5 and 4.6] since we work with a class of formally smooth rather than formally étale morphisms. Moreover, we note that our obstruction conditions differ from the Artin conditions of [86, Definition 1.4.3.1] because we are not assuming quasi-compactness of our topology. In particular, Condition (3) is a weaker condition.

Lemma 4.3.3.3. *Suppose that $X = \mathrm{Spec}(A) \in \mathcal{A}$, $M \in \mathbf{M}_{A,1}$, and that $d \in \pi_0(\mathbf{Der}(A, M))$. Suppose that we have a morphism $A \to A_j$ in $\mathcal{A}^{op}$. Then, there is an equivalence*

$$B_j \otimes^{\mathbb{L}}_{A\oplus_d \Omega M} A \to A_j$$

in $\mathbf{DAlg}^{cn}(\mathcal{C})$ where $B_j = A_j \oplus_{d'_j} \Omega M'_j$, d'_j is the derivation induced by Lemma 4.3.3.1, and $M'_j = M \otimes^{\mathbb{L}}_A A_j$.

Proof. We consider the morphism $A \oplus_d \Omega M \to B_j$ induced by the composition of morphisms $A \to A_j \to A_j \oplus M'_j$. By Lemma 4.2.2.2, there are fibre-cofibre sequences

$$A \oplus_d \Omega M \to A \to M$$
$$B_j \to A_j \to M'_j$$

in $\mathbf{Mod}^{cn}_A$ and $\mathbf{Mod}^{cn}_{A_j}$ respectively, which induce the following commutative diagram where the horizontal maps form fibre-cofibre sequences in $\mathbf{Mod}_A$

$$\begin{tikzcd}
B_j \otimes^{\mathbb{L}}_{A\oplus_d\Omega M} A \arrow[r] \arrow[d] & A_j \otimes^{\mathbb{L}}_{A\oplus_d\Omega M} A \arrow[r] \arrow[d] & M'_j \otimes^{\mathbb{L}}_{A\oplus_d\Omega M} A \arrow[d] \\
A_j \otimes_{A\oplus_d\Omega M} A \oplus_d \Omega M \arrow[r] & A_j \otimes^{\mathbb{L}}_{A\oplus_d\Omega M} A \arrow[r] & A_j \otimes^{\mathbb{L}}_{A\oplus_d\Omega M} M
\end{tikzcd}$$

Since the two right hand side vertical maps are equivalences, it follows that the first vertical map is an equivalence, and hence we have an equivalence $B_j \otimes^{\mathbb{L}}_{A\oplus_d\Omega M} A \simeq A_j$ in $\mathbf{Mod}^{cn}_A$, and hence also in $\mathbf{DAlg}^{cn}(\mathcal{C})$. $\square$

4.3.4 Obstruction Theory of Geometric Stacks

Suppose that we have a derived geometry context $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, \mathcal{C}^0, \boldsymbol{\tau}, \mathbf{P}, \mathcal{A}, \mathbf{M})$. Suppose also that, for any finite collection $\{U_i\}_{i\in I}$ of $\mathcal{A}$-admissible objects in $\mathcal{M}$, the map $\coprod_{i\in I} h(U_i) \to h(\coprod_{i\in I} U_i)$ is an equivalence in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$. Compare the following proof with [86, c.f. Theorem 1.4.3.2].

Theorem 4.3.4.1. *Suppose that* $\boldsymbol{\tau}$ *and* $\mathbf{P}$ *satisfy the obstruction conditions relative to* $\mathcal{A}$.

1. *If* $f : \mathcal{F} \to \mathcal{G}$ *is an* n*-representable morphism of stacks in* $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$*, then it has an obstruction theory,*
2. *If* $\mathcal{F}$ *is an* n*-geometric stack in* $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$ *and* $g : U \to \mathcal{F}$ *is an* n-$\mathbf{P}$*-morphism with* U *a* (-1)*-geometric stack then, for any* $X \in \mathcal{A}$ *and* $x : X = \mathrm{Spec}(A) \to U$*, there exists an* $\mathbf{S}$*-cover*
$$x' : X' = \mathrm{Spec}(A') \to X$$
such that, for any $M \in \mathbf{M}_{A',1}$*, the natural map*
$$\pi_0(\mathrm{Map}_{\mathbf{Mod}_{A'}}(\mathbb{L}_{X'/U}, M)) \to \pi_0(\mathrm{Map}_{\mathbf{Mod}_A}(\mathbb{L}_{U/\mathcal{F},x}, M))$$
is zero.

Proof. We prove this by induction on n. Indeed, when $n = -1$, Statement (1) follows from Corollary 4.3.1.6.

To prove Statement (2) at level $n = -1$, suppose that $\mathcal{F}$ is a (-1)-geometric stack, say $\mathcal{F} = Y = \mathrm{Spec}(B)$ for some $\mathcal{A}$-admissible $B \in \mathcal{M}$. Suppose that there is a (-1)-$\mathbf{P}$-morphism $g : U = \mathrm{Spec}(C) \to Y$ in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$. By Assumption (1), this morphism is formally i-smooth. Therefore, we see that for any morphism $x : X = \mathrm{Spec}(A) \to U$, and taking the trivial $\mathbf{S}$-cover of X, our result follows immediately using Proposition 4.3.2.2.

Now, suppose that $f : \mathcal{F} \to \mathcal{G}$ is an n-representable morphism of stacks and that the theorem holds for k-representable morphisms for $k < n$. Suppose that $X \in \mathcal{A}$ and that there is a morphism $x : X \to \mathcal{G}$. We note that, by Lemma 4.3.1.5, it suffices to show that the n-geometric stack $\mathcal{F} \times_{\mathcal{G}} X$ has an obstruction theory. This leads to the following lemma.

Lemma 4.3.4.2. *Any* n*-geometric stack has an obstruction theory.*

Proof. Suppose that $\mathcal{F}$ is an n-geometric stack. By Corollary 4.3.1.4, it suffices to show that $\mathcal{F}$ is infinitesimally cartesian. Indeed, suppose that $X = \mathrm{Spec}(A) \in \mathcal{A}$, $M \in \mathbf{M}_{A,1}$ and that there is a derivation $d \in \pi_0(\mathbf{Der}(A, M))$. We want to prove that the morphism
$$\mathcal{F}(A \oplus_d \Omega M) \to \mathcal{F}(A) \times_{\mathcal{F}(A \oplus M)} \mathcal{F}(A)$$
is an equivalence in $\infty\mathbf{Grpd}$. Suppose that $x \in \pi_0(\mathcal{F}(A) \times_{\mathcal{F}(A \oplus M)} \mathcal{F}(A))$ with projection $x_1 \in \pi_0(\mathcal{F}(A))$. By adapting [52, Lemma 2.1.3.4] to the case of ∞-groupoids, it suffices to prove that the fibre of the above map at x is contractible.

We define a functor $\mathcal{S} : (\mathbf{DAff}^{cn}(\mathcal{C})_{/X_d[\Omega M]})^{op} \to \infty\mathbf{Grpd}$ which takes a map $A \oplus_d \Omega M \to B$ to the fibre of the morphism

$$\mathcal{F}(B) \to \mathcal{F}(B \otimes^{\mathbb{L}}_{A\oplus_d \Omega M} A) \times_{\mathcal{F}(B\otimes^{\mathbb{L}}_{A\oplus_d\Omega M} A\oplus M)} \mathcal{F}(B \otimes^{\mathbb{L}}_{A\oplus_d\Omega M} A)$$

taken at the image of x. Since $\mathcal{F}$ is a stack on $\mathcal{A}$, we see that $\mathcal{S}$ defines a stack on $\mathbf{DAff}^{cn}(\mathcal{C})_{/X_d[\Omega M]}$. Suppose that we have some epimorphism of representable stacks $\coprod_{k\in K} \mathrm{Spec}(B_k) \to X_d[\Omega M]$ in $\mathbf{Stk}(\mathbf{DAff}^{cn}(\mathcal{C})_{/X_d[\Omega M]}, \boldsymbol{\tau})$. To show that $\mathcal{S}(A \oplus_d \Omega M)$ is contractible, it suffices to show that $\prod_{k\in K} \mathcal{S}(B_k)$ is contractible.

Suppose that we have an $(n-1)$-atlas $\{U_i \to \mathcal{F}\}_{i\in I}$ of $\mathcal{F}$. Then, by Proposition 3.2.3.1, we may find a $\boldsymbol{\tau}$-covering family $\{V_j \to X\}_{j\in J}$ of X such that the morphism $V_j \to \mathcal{F}$ factors through some $U_{u(j)}$. We may assume, by Assumption (3), that this cover refines to a finite $\mathbf{S}$-covering family $\{V'_k = \mathrm{Spec}(A'_k) \to X\}_{k\in K}$ and defines an $\mathbf{S}$-covering family $\{W_k = \mathrm{Spec}(B_k) \to X_d[\Omega M]\}_{k\in K}$ of $X_d[\Omega M]$, where $B_k = A_k \oplus_{d'_k} \Omega M'_k$ for d'_k the derivation induced by Lemma 4.3.3.1 and $M'_k = M \otimes^{\mathbb{L}}_A A_k$.

Since $B_k \otimes^{\mathbb{L}}_{A\oplus_d\Omega M} (A \oplus M) \simeq A_k \oplus M'_k$ and $B_k \otimes^{\mathbb{L}}_{A\oplus_d\Omega M} A \simeq A_k$ by Lemma 4.3.3.3 then, to show that $\prod_{k\in K} \mathcal{S}(B_k)$ is contractible, we just need to check that

$$\prod_{k\in K} \mathcal{F}(B_k) \to \prod_{k\in K} \mathcal{F}(A_k) \times_{\mathcal{F}(A_k\oplus M'_k)} \mathcal{F}(A_k)$$

is an equivalence of ∞-groupoids. Therefore, for each $k \in K$, we can just replace $X = \mathrm{Spec}(A)$ by $V'_k = \mathrm{Spec}(A'_k)$, d by d'_k, and M by M'_k. Hence, the point $x_1 \in \pi_0(\mathcal{F}(A))$, which is the image of $x \in \pi_0(\mathcal{F}(A) \times_{\mathcal{F}(A\oplus M)} \mathcal{F}(A))$, lifts to a point $y_1 \in \pi_0(U_{u(k)}(A))$ for some $k \in K$. Let $U := U_{u(k)} \in \mathcal{A}$.

Sublemma 4.3.4.3. *The point $x \in \pi_0(\mathcal{F}(A) \times_{\mathcal{F}(A\oplus M)} \mathcal{F}(A))$ lifts to a point y in $\pi_0(U(A) \times_{U(A\oplus M)} U(A))$*

Proof. We note that there is a commutative square of ∞-groupoids

$$\begin{tikzcd}
U(A) \times_{U(A\oplus M)} U(A) \arrow[r, "f"] \arrow[d, "p"] & \mathcal{F}(A) \times_{\mathcal{F}(A\oplus M)} \mathcal{F}(A) \arrow[d, "q"] \\
U(A) \arrow[r] & \mathcal{F}(A)
\end{tikzcd}$$

induced by the map $A \oplus M \to A$. We let $\mathrm{fib}(p)$ and $\mathrm{fib}(q)$ be the fibres of the morphisms p and q taken at y_1 and x_1 respectively and consider the induced morphism $h : \mathrm{fib}(p) \to \mathrm{fib}(q)$. We note that there is an induced morphism from $\mathrm{fib}(h)$ taken at

x to $\mathrm{fib}(f)$ at x. It therefore suffices to show that the fibre of h at x is non-empty. By Lemma 4.2.5.2, we see that h is equivalent to the morphism

$$\Omega_{d(y_1),0}\mathbf{Der}_U(X, M) \to \Omega_{d(x_1),0}\mathbf{Der}_{\mathcal{F}}(X, M)$$

Hence, we see that the fibre of h at x is equivalent to

$$\Omega_{d(y_1),0}\mathbf{Der}_{U/\mathcal{F}}(X, M) \simeq \Omega_{d(y_1),0}\mathrm{Map}_{\mathbf{Mod}_A}(\mathbb{L}_{U/\mathcal{F},y_1}, M)$$

Since $\mathcal{F}$ is an n-geometric stack and $g : U \to \mathcal{F}$ is an $(n-1)$-$\mathbf{P}$ morphism, then by Statement (2) of the theorem at level $n-1$, we note that there is an $\mathbf{S}$-cover $x' : X' = \mathrm{Spec}(A') \to X$ such that, for any $M \in \mathbf{M}_{A',1}$, the natural map

$$\pi_0(\mathrm{Map}_{\mathbf{Mod}_{A'}}(\mathbb{L}_{X'/U}, M)) \to \pi_0(\mathrm{Map}_{\mathbf{Mod}_A}(\mathbb{L}_{U/\mathcal{F},y_1}, M)) \tag{4.1}$$

is zero. By Corollary 4.2.5.8, we have a fibre-cofibre sequence in $\mathbf{Mod}_A$

$$\mathbb{L}_{U/\mathcal{F},y_1} \to \mathbb{L}_{X'/\mathcal{F},x'} \to \mathbb{L}_{X'/U}$$

which induces the long exact sequence

$$\cdots \to \pi_0(\mathrm{Map}(\mathbb{L}_{X'/\mathcal{F},x'}, M)) \to \pi_0(\mathrm{Map}(\mathbb{L}_{U/\mathcal{F},y_1}, M)) \to \pi_{-1}(\mathrm{Map}(\mathbb{L}_{X'/U}, M)) \to \dots$$

where the first map is zero by Equation (4.1). We note that the image of $d(y_1)$ under the second map is zero, and hence we see that $d(y_1)$ must lie in the image of the first map. Therefore, 0 and $d(y_1)$ lie in the same connected component of $\mathrm{Map}_{\mathbf{Mod}_A}(\mathbb{L}_{U/\mathcal{F},y_1}, M)$, and hence $\mathrm{fib}(h)$ is non-empty. $\square$

Now, since the morphism $g : U \to \mathcal{F}$ is $(n-1)$-representable, by our inductive hypothesis it is infinitesimally cartesian and we have a pullback square

$$\begin{tikzcd}
U(A \oplus_d \Omega M) \arrow[r] \arrow[d, "p'"] & \mathcal{F}(A \oplus_d \Omega M) \arrow[d, "q'"] \\
U(A) \times_{U(A \oplus M)} U(A) \arrow[r] & \mathcal{F}(A) \times_{\mathcal{F}(A \oplus M)} \mathcal{F}(A)
\end{tikzcd}$$

We note that, since U is (-1)-geometric, it is infinitesimally cartesian, and hence p' is an equivalence. Therefore, the fibre of q' at x is either contractible or empty by [52, c.f. Proposition 1.2.12.9]. But, x lifts to a point in $\pi_0(U(A) \times_{U(A\oplus M)} U(A))$ by the sublemma, and hence the fibre must be non empty. Therefore, $\mathcal{F}$ is infinitesimally cartesian. $\square$

With the proof of Statement (1) complete, it remains to show Statement (2) holds at level n. Suppose that we have an n-geometric stack $\mathcal{F}$ and an n-$\mathbf{P}$ morphism $g : U \to \mathcal{F}$. Take an n-atlas $\{U_i \to \mathcal{F}\}_{i \in I}$ for $\mathcal{F}$, and consider the induced n-atlas $\{V_j = \mathrm{Spec}(B_j) \to U \times_{\mathcal{F}} U_i\}_{j \in J}$ for the n-geometric stacks $U \times_{\mathcal{F}} U_i$. We have an induced morphism $v_j : X \times_U V_j \to V_j$ of (-1)-geometric stacks. By Lemma 4.2.5.7, it suffices to show that there exists an $\mathbf{S}$-cover $X' = \mathrm{Spec}(A') \to X$ such that, for any $M \in \mathbf{M}_{A',1}$, the natural map

$$\pi_0(\mathrm{Map}_{\mathbf{Mod}_{A'}}(\mathbb{L}_{X' \times_U V_j / V_j}, M)) \to \pi_0(\mathrm{Map}_{\mathbf{Mod}_A}(\mathbb{L}_{U \times_{\mathcal{F}} U_i / U_i, x^{\cdot}}, M))$$

is zero. This map factors through the morphism

$$\pi_0(\mathrm{Map}_{\mathbf{Mod}_{B_j}}(\mathbb{L}_{V_j / U \times_{\mathcal{F}} U_i, v_j}, M)) \to \pi_0(\mathrm{Map}_{\mathbf{Mod}_A}(\mathbb{L}_{U \times_{\mathcal{F}} U_i / U_i, x^{\cdot}}, M)) \tag{4.2}$$

and hence it suffices to show, up to replacing V_j with an $\mathbf{S}$-cover, that this latter morphism is zero.

We note that, since the morphism $U_i \to \mathcal{F}$ is in $(n-1)$-$\mathbf{P}$, then the induced morphism $V_j \to U_i$ is in $\mathbf{P}$. Hence, we see, by Proposition 4.3.2.2 and Condition (1), that $\pi_0(\mathrm{Map}_{\mathbf{Mod}_{B_j}}(\mathbb{L}_{V_j/U_i}, M)) = 0$. Hence, since the morphism in Equation (4.2) factors through $\pi_0(\mathrm{Map}_{\mathbf{Mod}_{B_j}}(\mathbb{L}_{V_j/U_i}, M)) = 0$ by Corollary 4.2.5.8, then we see that our morphism is zero as desired.

□

4.3.5 Postnikov Towers

A Postnikov tower of a topological space gives you a way to 'build up' the homotopy groups of the space. We will generalise this idea to our derived geometry contexts as follows.

Suppose that $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, \mathcal{C}^0)$ is a derived algebraic context and that $\mathbf{DAlg}^{cn}(\mathcal{C})$ is compatible with the t-structure on $\mathcal{C}$ in the sense of Definition 4.1.1.6. For $n \geq 0$, we will denote by $A_{\leq n}$ the object $\tau_{\leq n}(A) \in \mathbf{DAlg}^{cn}(\mathcal{C})$. There is an induced morphism $A_{\leq n} \to A_{\leq n-1}$ for each n.

Definition 4.3.5.1. Suppose that $A \in \mathbf{DAlg}^{cn}(\mathcal{C})$. The *Postnikov tower* of A is

$$A \to \ldots \to A_{\leq n} \to A_{\leq n-1} \to \ldots \to A_{\leq 0} = \pi_0(A)$$

We will say that Postnikov towers in $\mathbf{DAlg}^{cn}(\mathcal{C})$ *converge* if A is the limit of this sequence in $\mathbf{DAlg}^{cn}(\mathcal{C})$.

Definition 4.3.5.2. A derived geometry context $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, \mathcal{C}^0, \boldsymbol{\tau}, \mathbf{P}, \mathcal{A}, \mathbf{M})$ will be *Postnikov compatible* if

1. The t-structure on $\mathcal{C}$ is left complete,
2. $\mathbf{DAlg}^{cn}(\mathcal{C})$ is compatible with the t-structure on $\mathcal{C}$,
3. Given a Postnikov tower in $\mathbf{DAlg}^{cn}(\mathcal{C})$, then $A_{\leq n} \in \mathcal{A}^{op}$ for all $n \geq 0$ if and only if $A \in \mathcal{A}^{op}$,
4. For all $m \geq 2$, if $A \in \mathcal{C}$ is n-connective, then $\mathbf{LSym}^m(A)$ is $(n+2)$-connective.

Remark. We remark that the last condition is dealt with in more generality in [10, Definition 2.1.8], and is included here in its present form so that Theorem 4.2.2.5 holds.

For the rest of the section, suppose that we have a Postnikov compatible derived geometry context $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, \mathcal{C}^0, \boldsymbol{\tau}, \mathbf{P}, \mathcal{A}, \mathbf{M})$. We note that, since $\mathcal{C}$ is left complete, Postnikov towers in $\mathcal{C}$ converge. Hence, since $\mathbf{DAlg}^{cn}(\mathcal{C})$ is closed under limits and using our assumptions, Postnikov towers in $\mathbf{DAlg}^{cn}(\mathcal{C})$ also converge. In particular, our assumptions imply that, if $X = \mathrm{Spec}(A)$ is in the full subcategory $\mathcal{A}$ of $\mathbf{DAff}^{cn}(\mathcal{C})$, then X is the colimit in $\mathcal{A}$ of the sequence

$$X \leftarrow \ldots \leftarrow X_{\leq n} \leftarrow X_{\leq n-1} \leftarrow \ldots \leftarrow X_{\leq 0}$$

In particular, we can use this to show that every (-1)-geometric stack on $\mathcal{A}$ is nilcomplete with respect to $\mathcal{A}$ in the following sense.

Definition 4.3.5.3. Suppose that $f : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves in $\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))$. Then, f is said to be *nilcomplete with respect to* $\mathcal{A}$ if, for every $X = \mathrm{Spec}(A) \in \mathcal{A}$, the square

$$\begin{array}{ccc} \mathcal{F}(A) & \longrightarrow & \varprojlim_k \mathcal{F}(A_{\leq k}) \\ \downarrow & & \downarrow \\ \mathcal{G}(A) & \longrightarrow & \varprojlim_k \mathcal{G}(A_{\leq k}) \end{array}$$

is a pullback square in $\infty\mathbf{Grpd}$. We say that $\mathcal{F}$ is *nilcomplete* with respect to $\mathcal{A}$ if the morphism $\mathcal{F} \to *$ is nilcomplete.

Definition 4.3.5.4. Suppose that $f : \mathcal{F} \to \mathcal{G}$ is an n-representable morphism in $\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))$. Then, f satisfies the *Postnikov lifting property with respect to* $\mathcal{A}$

if, for every $X \in \mathcal{A}$ and every $k > 0$, the following lifting problem has at least one solution $X_{\leq k} \to \mathcal{F}$

$$\begin{tikzcd} X_{\leq k-1} \arrow[r] \arrow[d] & \mathcal{F} \arrow[d, "f"] \\ X_{\leq k} \arrow[r] \arrow[ur, dotted] & \mathcal{G} \end{tikzcd}$$

Proposition 4.3.5.5. *Suppose that*

1. *$\boldsymbol{\tau}$ and $\mathbf{P}$ satisfy the obstruction conditions relative to $\mathcal{A}$ for a suitable class of morphisms $\mathbf{S}$,*
2. *For any finite collection $\{U_i\}_{i\in I}$ of $\mathcal{A}$-admissible objects, the map $\coprod_{i\in I} h(U_i) \to h(\coprod_{i\in I} U_i)$ is an equivalence in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$,*
3. *Every m-$\mathbf{P}$-representable map of stacks in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$ satisfies the Postnikov lifting property for $m \geq -1$.*

Let $f : \mathcal{F} \to \mathcal{G}$ be an n-representable morphism of stacks in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$. Then, f is nilcomplete for all $n \geq -1$.

Proof. We want to prove that, for any $X = \mathrm{Spec}(A) \in \mathcal{A}$, the map

$$\mathcal{F}(A) \to \mathcal{G}(A) \times_{\varprojlim_k \mathcal{G}(A_{\leq k})} \varprojlim_k \mathcal{F}(A_{\leq k})$$

is an equivalence in $\infty\mathbf{Grpd}$. It suffices to show that the fibre of this morphism taken at any vertex is contractible. Indeed, take $x \in \pi_0(\mathcal{G}(A) \times_{\varprojlim_k \mathcal{G}(A_{\leq k})} \varprojlim_k \mathcal{F}(A_{\leq k}))$ and denote by x' the projection of x into $\pi_0(\mathcal{G}(A))$. The map $\mathcal{F} \times_{\mathcal{G}} X \to \mathcal{F}$ is n-representable since n-representable maps are stable under pullback. Since nilcompleteness of the map $\mathcal{F} \times_{\mathcal{G}} X \to X$ at A implies nilcompleteness of f at A, we can just replace $\mathcal{F}$ by $\mathcal{F} \times_{\mathcal{G}} X$ and $\mathcal{G}$ by X in our statement. Hence, we are reduced to proving the statement in the case when we have an n-representable map $f : \mathcal{F} \to Y$, with $\mathcal{F}$ an n-geometric stack and $Y = \mathrm{Spec}(B)$ a (-1)-geometric stack on $\mathcal{A}$. We prove the statement by induction on n.

We note that the (-1)-geometric stack Y is nilcomplete with respect to $\mathcal{A}$. It therefore suffices to show that the map

$$\mathcal{F}(A) \to \varprojlim_k \mathcal{F}(A_{\leq k})$$

is an equivalence for every $A \in \mathcal{A}^{op}$. The point x is now a point in $\pi_0(\varprojlim_k \mathcal{F}(A_{\leq k}))$. We will let $x_k : X_{\leq k} \to \mathcal{F}$ denote the morphism of stacks corresponding to the projection of x into $\pi_0(\mathcal{F}(A_{\leq k}))$.

When $n = -1$, we note that $\mathcal{F}$ is a (-1)-geometric stack and the result easily follows. Now suppose that $n \geq 0$ and consider the morphism $x_0 : X_{\leq 0} \to \mathcal{F}$. Since $\mathcal{F}$ is n-geometric, it has an n-atlas $\{U_i \to \mathcal{F}\}_{i \in I}$. By Proposition 3.2.3.1, there exists a $\boldsymbol{\tau}$-covering family $\{V_j \to X_{\leq 0}\}_{j \in J}$ of (-1)-geometric stacks on $\mathcal{A}$ along with a morphism $u : J \to I$ such that the induced morphism $V_j \to \mathcal{F}$ factors through $U_{u(j)}$. We note that, by Condition (3) of Definition 4.3.3.2, we may refine this to a finite $\mathbf{S}$-covering family $\{V'_k \to X_{\leq 0}\}_{k \in K}$. We let $V = \coprod_{k \in K} V'_k$, and consider this as a representable stack $\mathrm{Spec}(A')$ on $\mathcal{A}$.

We define a functor $\mathcal{S} : (\mathcal{A}_{/X_{\leq 0}})^{op} \to \infty\mathbf{Grpd}$ which takes a map $\pi_0(A) \to A'$ to the fibre of the morphism

$$\mathcal{F}(A') \to \varprojlim_k \mathcal{F}(A'_{\leq k})$$

at x_0. We see that this defines a stack on $\mathcal{A}_{/X_{\leq 0}}$. To show that $\mathcal{S}(A)$ is contractible, it suffices to show that $\mathcal{S}(A')$ is contractible for any epimorphism $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ of stacks. Therefore, we can replace $X_{\leq 0}$ with its $\mathbf{S}$-cover V and there will exist a corresponding $U := U_i$ for some $i \in I$ such that we have a commutative diagram

$$\begin{tikzcd} & U \arrow[d] \\ X_{\leq 0} \arrow[ur, "y_0"] \arrow[r, "x_0"] & \mathcal{F} \end{tikzcd}$$

with the morphism $U \to \mathcal{F}$ in $(n-1)$-$\mathbf{P}$. We claim that there is a point $y \in \pi_0(\varprojlim_k U(A_{\leq k}))$ whose image in $\pi_0(\varprojlim_k \mathcal{F}(A_{\leq k}))$ is x. We proceed by induction on $k \geq 0$ to construct a sequence of compatible maps $y_k : X_{\leq k} \to U$. Indeed, when $k = 0$, we can consider the induced map $y_0 : X_{\leq 0} \to U$ coming from the above commutative diagram. Now suppose that $k > 0$ and that we have a map $y_{k-1} : X_{\leq k-1} \to U$. Using our Postnikov lifting property for the $(n-1)$-$\mathbf{P}$ map $U \to \mathcal{F}$, there exists a map $y_k : X_{\leq k} \to U$ such that the subdiagrams in the following commutative diagram commute

$$\begin{tikzcd} X_{\leq k-1} \arrow[r, "y_{k-1}"] \arrow[d] & U \arrow[d] \\ X_{\leq k} \arrow[ur, dotted, "y_k"] \arrow[r, "x_k"] & \mathcal{F} \end{tikzcd}$$

Given such a sequence (y_k), then $y = \varprojlim_k y_k \in \pi_0(\varprojlim_k U(A_{\leq k}))$ has image x in $\pi_0(\varprojlim_k \mathcal{F}(A_{\leq k}))$.

We examine the following commutative diagram

$$\begin{tikzcd} U(A) \arrow[r] \arrow[d] & \varprojlim_k U(A_{\leq k}) \arrow[d] \\ \mathcal{F}(A) \arrow[r] & \varprojlim_k \mathcal{F}(A_{\leq k}) \end{tikzcd}$$

We note that, since $U \to \mathcal{F}$ is $(n-1)$-representable, then by our inductive hypothesis this is a pullback square. Since each $x \in \pi_0(\varprojlim_k \mathcal{F}(A_{\leq k}))$ corresponds to some point $y \in \pi_0(\varprojlim_k U(A_{\leq k}))$, the fibre of the morphism $\mathcal{F}(A) \to \varprojlim_k \mathcal{F}(A_{\leq k})$ at x is equivalent to the fibre of the morphism $U(A) \to \varprojlim_k U(A_{\leq k})$ at y. Now, since $U(A) \to \varprojlim_k U(A_{\leq k})$ is an equivalence by the case $n = -1$, its fibre at y is contractible, and hence the fibre of $\mathcal{F}(A) \to \varprojlim_k \mathcal{F}(A_{\leq k})$ at x is also contractible. □

Remark. As an immediate corollary we see that in this situation any n-geometric stack is nilcomplete.

4.3.6 Climbing the Postnikov Tower

Suppose that $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, \mathcal{C}^0, \boldsymbol{\tau}, \mathbf{P}, \mathcal{A}, \mathbf{M})$ is a Postnikov compatible derived geometry context and that $k \geq 1$.

Lemma 4.3.6.1. *There exists a derivation $d_k \in \pi_0(\mathbf{Der}(A_{\leq k-1}, \pi_k(A)[k+1]))$ such that $A_{\leq k-1} \oplus_{d_k} \pi_k(A)[k]$ is isomorphic to $A_{\leq k}$.*

Proof. This follows from [10, c.f. Theorem 2.1.35] using Theorem 4.2.2.5 and Assumption (4) of Postnikov compatibility. □

Lemma 4.3.6.2. *There exist natural equivalences*

$$\begin{aligned} \pi_{k+2}(\mathbb{L}_{A_{\leq k-1}/A_{\leq k}}) &\simeq 0 \\ \pi_{k+1}(\mathbb{L}_{A_{\leq k-1}/A_{\leq k}}) &\simeq \pi_k(A) \\ \pi_i(\mathbb{L}_{A_{\leq k-1}/A_{\leq k}}) &= 0 \quad \textit{for } i \leq k \end{aligned}$$

Proof. Denote the morphism $A_{\leq k} \to A_{\leq k-1}$ by f. Since $\mathrm{cofib}(f)$ is k-connective, by Theorem 4.2.2.5 and Assumption (4) of Postnikov compatibility, the fibre of the induced map $A_{\leq k-1} \otimes^{\mathbb{L}}_{A_{\leq k}} \mathrm{cofib}(f) \to \mathbb{L}_{A_{\leq k-1}/A_{\leq k}}$ is $(k+2)$-connective. It follows, by considering the induced long exact sequence, that

$$\pi_i(\mathbb{L}_{A_{\leq k-1}/A_{\leq k}}) \simeq \pi_i(A_{\leq k-1} \otimes^{\mathbb{L}}_{A_{\leq k}} \mathrm{cofib}(f))$$

for $i \leq k+1$ and there is a surjection $\pi_{k+2}(A_{\leq k-1} \otimes_{A_{\leq k}} \mathrm{cofib}(f)) \to \pi_{k+2}(\mathbb{L}_{A_{\leq k-1}/A_{\leq k}})$. Hence, we just need to calculate $\pi_i(A_{\leq k-1} \otimes^{\mathbb{L}}_{A_{\leq k}} \mathrm{cofib}(f))$.

Let $K = \pi_k(A)[k+1] = \mathrm{cofib}(f)$. We easily note that there is a fibre-cofibre sequence in $\mathbf{Mod}_{A_{\leq k-1}}$

$$K \otimes^{\mathbb{L}}_{A_{\leq k}} K \xrightarrow{m} K \simeq A_{\leq k} \otimes^{\mathbb{L}}_{A_{\leq k}} K \to A_{\leq k-1} \otimes^{\mathbb{L}}_{A_{\leq k}} \mathrm{cofib}(f)$$

where m is the multiplication map. Therefore, it suffices to calculate $\pi_i(\mathrm{cofib}(m))$. We note that the multiplication morphism $m : K \otimes^{\mathbb{L}}_{A_{\leq k}} K \to K$ is nullhomotopic, using the previous lemma and similar reasoning to [57, Proposition 7.4.1.14]. Therefore, we have a splitting

$$\mathrm{cofib}(m) \simeq K \coprod (K \otimes^{\mathbb{L}}_{A_{\leq k}} K)[1]$$

from which our result easily follows.

□

Suppose we want to show that an n-representable map in $\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))$ satisfies the Postnikov lifting property with respect to $\mathcal{A}$ as in Definition 4.3.5.4. By Lemma 4.3.6.1, the morphism $A_{\leq k} \to A_{\leq k-1}$ is given by a square zero extension. Hence, it suffices to show the following condition.

Definition 4.3.6.3. Suppose that $f : \mathcal{F} \to \mathcal{G}$ is an n-representable morphism of presheaves in $\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))$. Then, f satisfies the *square-zero lifting property with respect to* $\mathcal{A}$ if, for every $X = \mathrm{Spec}(A) \in \mathcal{A}$, $M \in \mathbf{M}_{A,1}$, and $d \in \pi_0(\mathbf{Der}(A, M))$, the following lifting problem has at least one solution $X_d[\Omega M] \to \mathcal{F}$

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathcal{F} \arrow[d, "f"] \\ X_d[\Omega M] \arrow[r] \arrow[ur, dotted] & \mathcal{G} \end{tikzcd}$$

Recall the definition of a morphism of stacks being in n-$\mathbf{P}|_{\mathcal{A}}$ from Notation 3.2.2.3. We will make the following assumption on such morphisms.

Assumption 4.3.6.4. If a morphism $f : \mathcal{F} \to \mathcal{G}$ of stacks in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$ is in n-$\mathbf{P}|_{\mathcal{A}}$ then, for any $X = \mathrm{Spec}(A) \in \mathcal{A}$, any $x : X \to \mathcal{F}$ and any $M \in \mathbf{M}_{A,1}$, we have that $\pi_0(\mathrm{Map}_{\mathbf{Mod}_A}(\mathbb{L}_{\mathcal{F}/\mathcal{G},x}, M)) = 0$.

Proposition 4.3.6.5. *Suppose that*

1. $\boldsymbol{\tau}$ *and* $\mathbf{P}$ *satisfy the obstruction conditions relative to* $\mathcal{A}$ *for a suitable class of morphisms* $\mathbf{S}$*,*

2. *For any finite collection $\{U_i\}_{i\in I}$ of $\mathcal{A}$-admissible objects, the map $\coprod_{i\in I} h(U_i) \to h(\coprod_{i\in I} U_i)$ is an equivalence in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$,*
3. *Assumption 4.3.6.4 is satisfied.*

Then, every n-$\mathbf{P}|_{\mathcal{A}}$-representable morphism of stacks in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$ satisfies the square-zero lifting property, and hence the Postnikov lifting property with respect to $\mathcal{A}$.

Proof. Indeed, suppose that we have an n-$\mathbf{P}|_{\mathcal{A}}$-representable map $f : \mathcal{F} \to \mathcal{G}$ in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$ and a lifting problem

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathcal{F} \arrow[d, "f"] \\ X_d[\Omega M] \arrow[r] \arrow[ur, dotted] & \mathcal{G} \end{tikzcd}$$

where $X = \operatorname{Spec}(A)$, $M \in \mathbf{M}_{A,1}$, and $d \in \pi_0(\mathbf{Der}(A, M))$. Consider the n-geometric stack $\mathcal{H} := \mathcal{F} \times_{\mathcal{G}} X_d[\Omega M]$ and consider the map $x : X \to \mathcal{F} \times_{\mathcal{G}} X_d[\Omega M] = \mathcal{H}$ induced by the pullback. We see that it suffices to find a solution to the following lifting problem

$$\begin{tikzcd} X \arrow[r, "x"] \arrow[d] & \mathcal{H} \arrow[d] \\ X_d[\Omega M] \arrow[r, "id"] \arrow[ur, dotted] & X_d[\Omega M] \end{tikzcd}$$

Now, since $\boldsymbol{\tau}$ and $\mathbf{P}$ satisfy the obstruction conditions, then by Theorem 4.3.4.1, $\mathcal{H}$ has an obstruction theory. Hence, by Proposition 4.3.1.8, the obstruction to lifting the map $X \to X_d[\Omega M]$ to a map $X_d[\Omega M] \to \mathcal{H}$ lies in

$$\pi_0(\mathrm{Map}_{\mathbf{Mod}_A}(\mathbb{L}_{\mathcal{H}/X_d[\Omega M],x}, M))$$

which is zero by our assumption, since the map $\mathcal{H} \to X_d[\Omega M]$ is in n-$\mathbf{P}|_{\mathcal{A}}$.

$\square$

Remark. It follows that, in the above context, n-representable$|_{\mathcal{A}}$ morphisms of stacks in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$ are nilcomplete.

Chapter 5

A Representability Theorem for Stacks in Derived Geometry Contexts

Representability is one of the most fundamental concepts in algebraic geometry and algebraic topology. Classically, a representability theorem aims to give conditions under which certain functors are representable, i.e. in the image of the Yoneda embedding. If a functor is representable by an object X then we can study (i.e. do geometry or topology on) the functor by just studying X. We can interpret this idea in several different settings, and also provide higher and derived analogues.

In derived algebraic geometry, the Artin-Lurie representability theorem [56, Theorem 3.2.1] provides natural conditions under which derived stacks are n-geometric. The main motivation is the study of moduli stacks. In particular, the representability theorem was instrumental in constructing the derived moduli stack of elliptic curves and in Lurie's reproof of the Goerss-Hopkins-Miller Theorem [53].

A version of the Artin-Lurie representability theorem is proved by Toën and Vezzosi in [86, Theorem C.0.9] that holds for their model of derived algebraic geometry. In this chapter, we vastly extend this result so that it holds in certain derived geometry contexts, our *representability contexts*. In particular, our models of derived analytic and derived smooth geometry will provide representability contexts. Our motivation is to be able to study moduli stacks arising in these contexts in a new way.

5.1 Representability Contexts

In this section, we introduce the notion of a representability context and prove the representability theorem. In the following section, we give several examples of such contexts.

5.1.1 Affines in the Heart

Suppose that $(\mathcal{C},\mathcal{C}_{\geq 0},\mathcal{C}_{\leq 0},\mathcal{C}^0,\boldsymbol{\tau},\mathbf{P},\mathcal{A},\mathbf{M})$ is a Postnikov compatible derived geometry context. We recall that this consists of a derived algebraic context $(\mathcal{C},\mathcal{C}_{\geq 0},\mathcal{C}_{\leq 0},\mathcal{C}^0)$, a collection of connective affines $\mathcal{A}\subseteq\mathbf{DAff}^{cn}(\mathcal{C})$, and a certain collection $\mathbf{M}$ of *good modules*. Since we are assuming that our context is Postnikov compatible, there is an adjunction

$$\pi_0=\tau_{\leq 0}|_{\mathbf{DAlg}^{cn}(\mathcal{C})}:\mathbf{DAlg}^{cn}(\mathcal{C})\leftrightarrows\mathbf{DAlg}^{\heartsuit}(\mathcal{C}):\iota_{\leq 0}|_{\mathbf{DAlg}^{\heartsuit}(\mathcal{C})}$$

We will denote the induced adjunction on the opposite categories by

$$\iota:\mathbf{DAff}^{\heartsuit}(\mathcal{C})\leftrightarrows\mathbf{DAff}^{cn}(\mathcal{C}):t_0$$

Definition 5.1.1.1. Define the full subcategory $\mathcal{A}^{\heartsuit}\subseteq\mathcal{A}\subseteq\mathbf{DAff}^{\heartsuit}(\mathcal{C})$ to consist of objects X in $\mathbf{DAff}^{\heartsuit}(\mathcal{C})$ such that $X=t_0(Y)$ for some $Y\in\mathcal{A}$.

Definition 5.1.1.2. We define a class $\mathbf{P}^{\heartsuit}$ of maps in $\mathcal{A}^{\heartsuit}$ and a collection $\boldsymbol{\tau}^{\heartsuit}$ of covering families in $\mathrm{Ho}(\mathcal{A}^{\heartsuit})$ as follows

1. The collection $\mathbf{P}^{\heartsuit}$ is defined to be the collection of morphisms f such that $f\in\mathbf{P}\cap\mathcal{A}^{\heartsuit}$,
2. The collection of $\boldsymbol{\tau}^{\heartsuit}$-covers is the collection $\{t_0(U_i)\rightarrow t_0(X)\}_{i\in I}$ such that $\{U_i\rightarrow X\}_{i\in I}$ is a $\boldsymbol{\tau}$-cover in $\mathcal{A}$.

It is clear from our definition that the collection of $\boldsymbol{\tau}^{\heartsuit}$-covering families in $\mathrm{Ho}(\mathcal{A}^{\heartsuit})$ defines a Grothendieck pre-topology on $\mathcal{A}^{\heartsuit}$. If $\iota|_{\mathcal{A}^{\heartsuit}}:(\mathcal{A}^{\heartsuit},\boldsymbol{\tau}^{\heartsuit})\rightarrow(\mathcal{A},\boldsymbol{\tau}|_{\mathcal{A}})$ is a continuous functor of $(\infty,1)$-sites, then by our remarks in Section 3.1.4, $\iota|_{\mathcal{A}^{\heartsuit}}$ induces the following functors on the corresponding categories of stacks.

Definition 5.1.1.3. 1. The *truncation functor* t_0 is defined to be the functor

$$t_0:=(\iota|_{\mathcal{A}^{\heartsuit}})^*:\mathbf{Stk}(\mathcal{A},\boldsymbol{\tau}|_{\mathcal{A}})\rightarrow\mathbf{Stk}(\mathcal{A}^{\heartsuit},\boldsymbol{\tau}^{\heartsuit})$$

2. The *extension functor* i is defined to be its left adjoint

$$i:=(\iota|_{\mathcal{A}^{\heartsuit}})_{\#}:\mathbf{Stk}(\mathcal{A}^{\heartsuit},\boldsymbol{\tau}^{\heartsuit})\rightarrow\mathbf{Stk}(\mathcal{A},\boldsymbol{\tau}|_{\mathcal{A}})$$

Recall from Section 3.2.2, that we use the notation $\mathbf{Stk}_n(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}}, \mathbf{P}|_{\mathcal{A}})$ to denote the collection of n-geometric stacks defined relative to the strong $(\infty, 1)$-geometry tuple $(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}}, \mathbf{P}|_{\mathcal{A}}, \mathcal{A})$. In particular, (-1)-geometric stacks are those represented by objects in $\mathcal{A}$. We also recall that there is a fully faithful functor

$$\mathbf{Stk}_n(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}}, \mathbf{P}|_{\mathcal{A}}) \to \mathbf{Stk}_n(\mathbf{DAff}^{cn}(\mathcal{C}), \boldsymbol{\tau}, \mathbf{P}, \mathcal{A})$$

5.1.2 Transverse, Flat, and Derived Strong Modules

We recall that the t-structure on $\mathbf{Mod}_A$ is induced from the t-structure on $\mathcal{C}$, and hence, by our Postnikov compatibility condition, lines up with the t-structure on $\mathbf{DAlg}^{cn}(\mathcal{C})$. In general the homotopy groups π_i don't commute with base-change. Under some conditions, such as derived strongness or with some flatness assumptions, they do. We note that these conditions are explored more in [10, Section 2.3.2].

Lemma 5.1.2.1. *Suppose that $C \in \mathbf{DAlg}^{cn}(\mathcal{C})$ and $A, B \in \mathbf{DAlg}^{\heartsuit}(\mathcal{C})$ along with morphisms $\iota(A) \to \iota(B)$ and $\iota(A) \to C$. Then, $\pi_i(C \otimes^{\mathbb{L}}_{\iota(A)} \iota(B)) \simeq \pi_i(C) \otimes^{\mathbb{L}}_A B$ in $\mathbf{Mod}^{\heartsuit}_B(\mathcal{C})$ for each i.*

Proof. We first note that, since the tensor product of objects in the image of the forgetful functor $U : \mathbf{DAlg}^{cn}_B(\mathcal{C}) \to \mathbf{Mod}_B$ is given by the image of the coproduct in $\mathbf{DAlg}^{cn}_B(\mathcal{C})$, there is an equivalence

$$\pi_0(C \otimes^{\mathbb{L}}_{\iota(A)} \iota(B)) \simeq \pi_0(C) \otimes^{\mathbb{L}}_A B$$

in $\mathbf{Mod}^{\heartsuit}_B$, since π_0 is a left adjoint. By shifting, the result holds in the situation when C has non-vanishing homotopy group only in degree k for some $k \geq 0$. Now, consider $C \simeq \varinjlim_k C_{\leq k}$ in $\mathbf{DAlg}^{cn}(\mathcal{C})$. For $i \leq k$, we note that $\pi_i(C) \simeq \pi_i(C_{\leq k})$ and $\pi_i(C \otimes^{\mathbb{L}}_{\iota(A)} \iota(B)) \simeq \pi_i(C_{\leq k} \otimes^{\mathbb{L}}_{\iota(A)} \iota(B))$. Hence, it suffices to show that we have an equivalence $\pi_i(C_{\leq k} \otimes^{\mathbb{L}}_{\iota(A)} \iota(B)) \simeq \pi_i(C_{\leq k}) \otimes^{\mathbb{L}}_A B$ for each k.

When $k = 0$, the result holds since $C_{\leq 0}$ is concentrated in degree 0. Suppose that $k > 0$ and that the result holds for all $n < k$. We define K_k to be the fibre in $\mathbf{DAlg}^{cn}_{\iota(A)}(\mathcal{C})$ of the morphism $C_{\leq k} \to C_{\leq k-1}$. Base-changing by the morphism $\iota(A) \to \iota(B)$, we obtain the following fibre sequence

$$K_k \otimes^{\mathbb{L}}_{\iota(A)} \iota(B) \to C_{\leq k} \otimes^{\mathbb{L}}_{\iota(A)} \iota(B) \to C_{\leq k-1} \otimes^{\mathbb{L}}_{\iota(A)} \iota(B)$$

and hence obtain the following long exact sequence in homotopy

$$\cdots \to \pi_i(K_k \otimes^{\mathbb{L}}_{\iota(A)} \iota(B)) \to \pi_i(C_{\leq k} \otimes^{\mathbb{L}}_{\iota(A)} \iota(B)) \to \pi_i(C_{\leq k-1} \otimes^{\mathbb{L}}_{\iota(A)} \iota(B)) \to \dots$$

We also have a long exact sequence in $\mathbf{Mod}_B^{\heartsuit}$

$$\cdots \to \pi_i(K_k) \otimes_A^{\mathbb{L}} B \to \pi_i(C_{\leq k}) \otimes_A^{\mathbb{L}} B \to \pi_i(C_{\leq k-1}) \otimes_A^{\mathbb{L}} B \to \dots$$

We note that K_k is concentrated in degree k for each $k \geq 1$. Therefore, we see that $\pi_i(K_k \otimes^{\mathbb{L}}_{\iota(A)} \iota(B)) \simeq \pi_i(K_k) \otimes^{\mathbb{L}}_A B$. By induction, $\pi_i(C_{\leq k-1} \otimes^{\mathbb{L}}_{\iota(A)} \iota(B)) \simeq \pi_i(C_{\leq k-1}) \otimes^{\mathbb{L}}_A B$, and hence it follows that $\pi_i(C_{\leq k} \otimes^{\mathbb{L}}_{\iota(A)} \iota(B)) \simeq \pi_i(C_{\leq k}) \otimes^{\mathbb{L}}_A B$.

□

If we impose a flatness condition on our generators, then we can obtain spectral sequences in our derived algebraic context $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, \mathcal{C}^0)$ in a similar way to [57, Proposition 7.2.1.17]. Suppose that $A \in \mathcal{C}$ and consider $\pi_*(A) = \bigoplus_i \pi_i(A)$ as an object of $\mathcal{C}^{\heartsuit}$.

Definition 5.1.2.2. [10, Definition 2.3.77] An object P of $\mathcal{C}_{\geq 0}$ is *homotopy flat* if, for any object A of $\mathcal{C}$, the natural map of graded objects

$$\pi_*(A) \otimes \pi_*(P) \to \pi_*(A \otimes^{\mathbb{L}} P)$$

is an equivalence. A derived algebraic context $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, \mathcal{C}^0)$ is *flat* if every object of $\mathcal{C}^0$ is homotopy flat. A derived geometry context is *flat* if the underlying derived algebraic context is flat.

Example 5.1.2.3. *In the setting of Theorem 4.1.2.2, the resulting derived algebraic context will be flat by [10, Proposition 3.1.69].*

For the rest of this section we fix a flat derived algebraic context $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, \mathcal{C}^0)$.

Proposition 5.1.2.4. *[10, Lemma 2.3.79] There is a spectral sequence*

$$\mathrm{Tor}^p_{\pi_*(A)}(\pi_*(M), \pi_*(N))_q \Rightarrow \pi_{p+q}(M \otimes^{\mathbb{L}}_A N)$$

for $M, N \in \mathbf{Mod}^{cn}_A$.

Corollary 5.1.2.5. *Suppose that $A \in \mathbf{DAlg}^{cn}(\mathcal{C})$ and that M is a connective A-module such that $M \otimes^{\mathbb{L}}_A \pi_0(A) \simeq 0$. Then, $M \simeq 0$.*

Proof. Indeed, consider the spectral sequence

$$\mathrm{Tor}^p_{\pi_*(A)}(\pi_*(M), \pi_0(A))_q \Rightarrow \pi_{p+q}(M \otimes^{\mathbb{L}}_A \pi_0(A)) \simeq 0$$

and proceed by induction on $i \geq 0$ to show that $\pi_i(M) \simeq 0$. □

Definition 5.1.2.6. Suppose that $A \in \mathbf{DAlg}^{cn}(\mathcal{C})$, $N \in \mathbf{Mod}_A^{\heartsuit}$, and $M \in \mathbf{Mod}_A^{cn}$. Then,

- N is *transverse to* $M \in \mathbf{Mod}_A^{cn}$ if $M \otimes_A^{\mathbb{L}} N$ is in $\mathbf{Mod}_A^{\heartsuit}$,
- M is *flat* if every $N \in \mathbf{Mod}_A^{\heartsuit}$ is transverse to it.

Proposition 5.1.2.7. *[10, c.f. Lemma 2.3.82] Suppose that $A \in \mathbf{DAlg}^{cn}(\mathcal{C})$. Let M be a connective A-module and N a connective A-module. Suppose that, for each $i \leq n$, $\pi_i(N)$ is transverse to M. Then, there is an equivalence*

$$\pi_i(M \otimes_A^{\mathbb{L}} N) \simeq \pi_0(M) \otimes_{\pi_0(A)}^{\mathbb{L}} \pi_i(N)$$

for all $i \leq n$.

Definition 5.1.2.8. Suppose that $A \in \mathbf{DAlg}^{cn}(\mathcal{C})$ and $M \in \mathbf{Mod}_A^{cn}$. Then, M is *derived strong* if the map

$$\pi_*(A) \otimes_{\pi_0(A)}^{\mathbb{L}} \pi_0(M) \to \pi_*(M)$$

is an equivalence. A morphism $f : A \to B$ is *derived strong* if B is derived strong as an A-module.

We note that equivalences are derived strong maps, and compositions of derived strong maps are derived strong. Using the spectral sequence, we get more general results about homotopy groups of tensor products under some strongness assumptions.

Proposition 5.1.2.9. *Suppose that $A \in \mathbf{DAlg}^{cn}(\mathcal{C})$. Suppose that M is a derived strong A-module and that N is an A-module such that $\pi_*(N)$ is transverse to $\pi_0(M)$ over $\pi_0(A)$. Then, there is an isomorphism*

$$\pi_*(M \otimes_A^{\mathbb{L}} N) \simeq \pi_0(M) \otimes_{\pi_0(A)}^{\mathbb{L}} \pi_*(N)$$

Proof. Indeed, if we consider the spectral sequence

$$\mathrm{Tor}^p_{\pi_*(A)}(\pi_*(M), \pi_*(N))_q \Rightarrow \pi_{p+q}(M \otimes_A^{\mathbb{L}} N)$$

Then, since M is a derived strong A-module, we see that

$$\pi_*(M) \otimes_{\pi_*(A)}^{\mathbb{L}} \pi_*(N) \simeq \pi_0(M) \otimes_{\pi_0(A)}^{\mathbb{L}} \pi_*(N)$$

and hence, using our transversality assumption, the spectral sequence degenerates on the first page, and we obtain our result. $\square$

Lemma 5.1.2.10. *Suppose that $f : A \to B$ is a derived strong map in $\mathbf{DAlg}^{cn}(\mathcal{C})$, then $\pi_0(B) \simeq B \otimes^{\mathbb{L}}_A \pi_0(A)$.*

Proof. Indeed, if we consider the spectral sequence

$$\mathrm{Tor}^p_{\pi_*(A)}(\pi_*(B), \pi_0(A))_q \Rightarrow \pi_{p+q}(B \otimes^{\mathbb{L}}_A \pi_0(A))$$

Then, since f is derived strong, we see that

$$\pi_*(B) \otimes^{\mathbb{L}}_{\pi_*(A)} \pi_0(A) \simeq (\pi_*(A) \otimes^{\mathbb{L}}_{\pi_0(A)} \pi_0(B)) \otimes^{\mathbb{L}}_{\pi_*(A)} \pi_0(A) \simeq \pi_0(B)$$

and hence the spectral sequence degenerates on the first page, and we obtain our result. $\square$

Corollary 5.1.2.11. *If $f : A \to B$ in $\mathbf{DAlg}^{cn}(\mathcal{C})$ is derived strong, then*

$$\mathbb{L}_{B/A} \otimes^{\mathbb{L}}_B \pi_0(B) \simeq \mathbb{L}_{\pi_0(B)/\pi_0(A)}$$

Proof. By the previous lemma, there is a pushout square in $\mathbf{DAlg}^{cn}(\mathcal{C})$

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ \pi_0(A) & \longrightarrow & \pi_0(B) \end{array}$$

Hence, the result follows by Corollary 4.2.1.9. $\square$

Suppose that we have some derived algebraic context $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, \mathcal{C}^0)$, $\mathcal{A}$ is a full subcategory of $\mathbf{DAff}^{cn}(\mathcal{C})$, and that we have a class $\mathbf{P}$ of maps in $\mathbf{DAff}^{cn}(\mathcal{C})$. In order for morphisms in $\mathbf{P}$ to be well behaved with respect to their truncations, we will impose the following strongness condition.

Definition 5.1.2.12. Suppose that $f : Y = \mathrm{Spec}(B) \to X = \mathrm{Spec}(A)$ is a morphism in $\mathcal{A}$. Then, f is *derived strong relative to* $\mathbf{P}|_{\mathcal{A}}$ if

1. The corresponding morphism $f : A \to B$ is derived strong,
2. The morphism $t_0(f) : t_0(Y) \to t_0(X)$ is in $\mathbf{P}^{\heartsuit}$.

Remark. In the derived algebraic context corresponding to simplicial commutative rings, maps are smooth if and only if they are strongly smooth, and étale if and only if they are strongly étale.

Recall from Section 3.2.1, that if $\boldsymbol{\tau}$ is a topology on $\mathrm{Ho}(\mathbf{DAff}^{cn}(\mathcal{C}))$, then we can extend a class of maps $\mathbf{P}$ to be local for a topology $\boldsymbol{\tau}$ by defining the class $\mathbf{P}^{\boldsymbol{\tau}}$. Suppose that, whenever a morphism is derived strong relative to $\mathbf{P}|_{\mathcal{A}}$, then it is in $\mathbf{P}|_{\mathcal{A}}$. Then, we note that if a morphism is derived strong relative to $\mathbf{P}^{\boldsymbol{\tau}}|_{\mathcal{A}}$ and morphisms in $\boldsymbol{\tau}$ are derived strong, then it is in $\mathbf{P}^{\boldsymbol{\tau}}|_{\mathcal{A}}$.

5.1.3 Representability Contexts

In our proof of the representability theorem, we will assume representability of the truncated stack and then lift certain constructions to the derived stack. We will need to put suitable conditions on our derived geometry contexts such that they are well-behaved with respect to their truncation and such that we have the necessary obstruction theories. Recall the notation n-representable$|_{\mathcal{A}}$ from Notation 3.2.2.3.

Definition 5.1.3.1. $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, \mathcal{C}^0, \boldsymbol{\tau}, \mathbf{P}, \mathcal{A}, \mathbf{M}, \mathbf{S})$ is a *representability context* if

- $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, \mathcal{C}^0, \boldsymbol{\tau}, \mathbf{P}, \mathcal{A}, \mathbf{M})$ is a flat Postnikov compatible derived geometry context,
- $\mathbf{S}$ is a class of morphisms in $\mathcal{A}$,

such that

1. $(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}}, \mathbf{P}|_{\mathcal{A}}, \mathcal{A}^{\heartsuit})$ is a strong relative $(\infty, 1)$-geometry tuple,
2. $\iota|_{\mathcal{A}^{\heartsuit}} : (\mathcal{A}^{\heartsuit}, \boldsymbol{\tau}^{\heartsuit}) \to (\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$ is a continuous functor of $(\infty, 1)$-sites,
3. For any finite collection $\{U_i\}_{i \in I}$ of $\mathcal{A}$-admissible objects, the map $\coprod_{i \in I} h(U_i) \to h(\coprod_{i \in I} U_i)$ is an equivalence in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$,
4. A morphism is in $\mathbf{P}|_{\mathcal{A}}$ if it is derived strong relative to $\mathbf{P}|_{\mathcal{A}}$,
5. An n-representable$|_{\mathcal{A}}$ morphism of stacks $f : \mathcal{F} \to \mathcal{G}$ in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$ is in n-$\mathbf{P}|_{\mathcal{A}}$ if f satisfies that, for any $x : X = \mathrm{Spec}(A) \to \mathcal{F}$ and any $M \in \mathbf{M}_{A,1}$,
$$\pi_0(\mathrm{Map}_{\mathbf{Mod}_A}(\mathbb{L}_{\mathcal{F}/\mathcal{G},x}, M)) = 0$$
The converse holds if $t_0(\mathcal{F}) \to t_0(\mathcal{G})$ is in n-$\mathbf{P}^{\heartsuit}$,
6. $\boldsymbol{\tau}$ and $\mathbf{P}$ satisfy the obstruction conditions of Definition 4.3.3.2 relative to $\mathcal{A}$, for the class $\mathbf{S}$ of morphisms.

We will give several motivating examples of these representability contexts in Section 5.2. In particular, we will describe how derived algebraic geometry in the sense of Toën and Vezzosi [86], where we work with simplicial commutative rings, can be reframed as a representability context in the above sense.

5.1.4 Truncation and Extension

Fix a representability context $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, \mathcal{C}^0, \boldsymbol{\tau}, \mathbf{P}, \mathcal{A}, \mathbf{M}, \mathbf{S})$. By Assumption (2), the functor $\iota|_{\mathcal{A}^{\heartsuit}} : (\mathcal{A}^{\heartsuit}, \boldsymbol{\tau}^{\heartsuit}) \to (\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$ induces an adjunction

$$i : \mathbf{Stk}(\mathcal{A}^{\heartsuit}, \boldsymbol{\tau}^{\heartsuit}) \leftrightarrows \mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}}) : t_0$$

Lemma 5.1.4.1. *$\iota|_{\mathcal{A}^{\heartsuit}}$ is a cocontinuous functor.*

Proof. Consider a $\boldsymbol{\tau}$-cover $\{U_j \to \iota(X)\}_{j\in J}$ for $U_j \in \mathcal{A}$ and $X \in \mathcal{A}^\heartsuit$. We consider the collection $\{t_0(U_j) \to t_0 \circ \iota(X) \simeq X\}_{j\in J}$, which is a $\boldsymbol{\tau}^\heartsuit$-covering family in $\mathrm{Ho}(\mathcal{A}^\heartsuit)$ by definition. We note that the family $\{\iota \circ t_0(U_j) \to \iota(X)\}_{j\in J}$ refines the cover $\{U_j \to \iota(X)\}_{j\in J}$ since each morphism $\iota \circ t_0(U_j) \to \iota(X)$ factors through $U_j \to \iota(X)$ via the counit. □

Lemma 5.1.4.2. *The extension functor i is fully faithful.*

Proof. This follows from Proposition 3.1.4.6 since every object of $\mathcal{A}^\heartsuit$ is admissible by Assumption (1) and the functor ι is fully faithful and both continuous and cocontinuous. □

Lemma 5.1.4.3. *The extension functor i sends (-1)-$\mathbf{P}^\heartsuit$-morphisms between representable stacks on $\mathcal{A}^\heartsuit$ to (-1)-$\mathbf{P}|_{\mathcal{A}}$-morphisms between representable stacks on $\mathcal{A}$.*

Proof. Suppose that $f : Y \to X$ is a (-1)-$\mathbf{P}^\heartsuit$-morphism between representable stacks on $\mathcal{A}^\heartsuit$. Consider the induced morphism $i(f) : i(Y) \to i(X)$ of stacks on $\mathcal{A}$ and suppose that there exists some representable stack Z on $\mathcal{A}$ along with a morphism $Z \to i(X)$. We note that the pullback $i(Y) \times_{i(X)} Z$ is representable, and hence, by Assumption (4), it suffices to show that the induced morphism $i(Y) \times_{i(X)} Z \to Z$ is derived strong relative to $\mathbf{P}|_{\mathcal{A}}$. Since $t_0(i(Y) \times_{i(X)} Z) \simeq Y \times_X t_0(Z)$ and f is in (-1)-$\mathbf{P}^\heartsuit$, we know that $t_0(i(Y) \times_{i(X)} Z) \to t_0(Z)$ is in $\mathbf{P}^\heartsuit$.

Suppose that $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$ for $A, B \in \mathcal{A}^\heartsuit$, and that $Z = \mathrm{Spec}(C)$ for $C \in \mathcal{A}$. We conclude that the map is derived strong by noticing that, by Proposition 5.1.2.1 and our assumptions on our representability context, there is an equivalence

$$\pi_i(C) \otimes^{\mathbb{L}}_{\pi_0(C)} \pi_0(i(B) \otimes^{\mathbb{L}}_{i(A)} C) \simeq B \otimes^{\mathbb{L}}_A \pi_i(C) \simeq \pi_i(i(B) \otimes^{\mathbb{L}}_{i(A)} C)$$

for each $i \geq 0$. □

Proposition 5.1.4.4. *If $\mathcal{A}$ is closed under $\boldsymbol{\tau}$-descent relative to $\mathcal{A}$, then the extension functor i satisfies the following properties:*

1. *i preserves pullbacks of stacks in $\mathbf{Stk}_n(\mathcal{A}^\heartsuit, \boldsymbol{\tau}^\heartsuit, \mathbf{P}^\heartsuit)$ along m-$\mathbf{P}|_{\mathcal{A}}$-morphisms for all $m \geq -1$,*
2. *i sends stacks in $\mathbf{Stk}_n(\mathcal{A}^\heartsuit, \boldsymbol{\tau}^\heartsuit, \mathbf{P}^\heartsuit)$ to stacks in $\mathbf{Stk}_n(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}}, \mathbf{P}|_{\mathcal{A}})$ and sends n-$\mathbf{P}^\heartsuit$ morphisms between stacks in $\mathbf{Stk}_n(\mathcal{A}^\heartsuit, \boldsymbol{\tau}^\heartsuit, \mathbf{P}^\heartsuit)$ to n-$\mathbf{P}$ morphisms between stacks in $\mathbf{Stk}_n(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}}, \mathbf{P}|_{\mathcal{A}})$.*

Proof. Indeed, we note that i preserves pullbacks of representable stacks because $i(\mathrm{Spec}(A)) = \mathrm{Spec}(A)$ and every object of $\mathcal{A}$ is $\mathcal{A}$-admissible. Therefore, Statement (1) follows by Proposition 3.2.4.1. To show Statement (2), we use Proposition 3.2.4.4 along with Statement (1), the assumption that $\mathcal{A}$ is closed under $\boldsymbol{\tau}$-descent relative to $\mathcal{A}$, and that i sends (-1)-$\mathbf{P}^{\heartsuit}$-morphisms between representable stacks to (-1)-$\mathbf{P}|_{\mathcal{A}}$-morphisms between representable stacks by Lemma 5.1.4.3. $\square$

Proposition 5.1.4.5. *If $\mathcal{A}$ is closed under $\boldsymbol{\tau}$-descent relative to $\mathcal{A}$, then $\mathcal{A}^{\heartsuit}$ is closed under $\boldsymbol{\tau}^{\heartsuit}$-descent relative to $\mathcal{A}^{\heartsuit}$.*

Proof. Indeed, suppose that $\mathcal{F}$ is a stack in $\mathbf{Stk}(\mathcal{A}^{\heartsuit}, \boldsymbol{\tau}^{\heartsuit})$ and that we have a morphism $\mathcal{F} \to X$ for some $X \in \mathcal{A}^{\heartsuit}$. Suppose that we have a $\boldsymbol{\tau}^{\heartsuit}$-covering family $\{U_i \to X\}_{i \in I}$ such that $\mathcal{F} \times_X U_i$ is a representable stack for every i. We note that, since ι is continuous, $\{\iota(U_i) \to \iota(X)\}_{i \in I}$ is a $\boldsymbol{\tau}$-cover, and, moreover, since each morphism $\iota(U_i) \to \iota(X)$ is in $\mathbf{P}|_{\mathcal{A}}$, then we see that $i(\mathcal{F} \times_X U_i) \simeq i(\mathcal{F}) \times_{i(X)} i(U_i)$ is representable for each i by Proposition 5.1.4.4. Therefore, since $\mathcal{A}$ is closed under $\boldsymbol{\tau}$-descent relative to $\mathcal{A}$, it follows that $i(\mathcal{F})$ is representable, say $i(\mathcal{F}) = Y$ for some $Y \in \mathcal{A}$. Since i is fully faithful, $\mathcal{F} \simeq t_0(i(\mathcal{F})) = t_0(Y)$ is a representable stack on $\mathcal{A}^{\heartsuit}$. $\square$

Proposition 5.1.4.6. *The truncation functor t_0 satisfies the following properties:*

1. *t_0 preserves epimorphisms of stacks,*
2. *t_0 sends stacks in $\mathbf{Stk}_n(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}}, \mathbf{P}|_{\mathcal{A}})$ to stacks in $\mathbf{Stk}_n(\mathcal{A}^{\heartsuit}, \boldsymbol{\tau}^{\heartsuit}, \mathbf{P}^{\heartsuit})$ and sends n-$\mathbf{P}|_{\mathcal{A}}$ morphisms between stacks in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$ to n-$\mathbf{P}^{\heartsuit}$ morphisms between stacks in $\mathbf{Stk}(\mathcal{A}^{\heartsuit}, \boldsymbol{\tau}^{\heartsuit})$.*

Proof. Property number (1) follows by Proposition 3.1.4.5 because ι is a cocontinuous functor by Lemma 5.1.4.1. Now we use the setting of Example 3.2.4.2 to show Property number (2). We note that t_0 preserves finite limits because it is a right adjoint. It sends morphisms in $\mathbf{P}|_{\mathcal{A}}$ between representable stacks to morphisms in $\mathbf{P}^{\heartsuit}$ between representable stacks. Moreover, we note that every representable stack X in $\mathbf{Stk}(\mathcal{A}^{\heartsuit}, \boldsymbol{\tau}^{\heartsuit})$ is equivalent to $t_0(\iota(X))$. Hence, we can conclude using Proposition 3.2.4.3. $\square$

5.1.5 The Relative Cotangent Complex of the Truncation

In this section, we will explore connectivity of the relative cotangent complex of the natural morphism $t_0(\mathcal{F}) \to \mathcal{F}$ of stacks in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$. The following result follows in a similar way to [66, Lemma 7.15]. Compare with [75, Lemma 5.5.1].

Lemma 5.1.5.1. *Suppose that $A \in \mathcal{A}^{\heartsuit,op}$. If $M_1, M_2 \in \mathbf{M}_{A,-n}$ for some n and*

$$\mathrm{Map}_{\mathbf{Mod}_A}(M_1, N[m]) \simeq \mathrm{Map}_{\mathbf{Mod}_A}(M_2, N[m])$$

for every $N \in \mathbf{M}_A^{\heartsuit}$, then in $\mathbf{Mod}_A$

$$\tau_{\leq m}(M_1) \simeq \tau_{\leq m}(M_2)$$

Lemma 5.1.5.2. *Suppose that $\mathcal{F}$ is a stack in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$ such that the morphism $t_0(\mathcal{F}) \to \mathcal{F}$ has an obstruction theory. Suppose that $u_0 : U_0 = \mathrm{Spec}(A_0) \to t_0(\mathcal{F})$ is a morphism in $\mathbf{P}^{\heartsuit}$ such that $U_0 \in \mathcal{A}^{\heartsuit}$. Then, $\pi_i(\mathbb{L}_{t_0(\mathcal{F})/\mathcal{F},u_0}) \simeq 0$ for all $i \leq 1$.*

Proof. We consider the fibre-cofibre sequence $\mathbb{L}_{\mathcal{F},u_0} \to \mathbb{L}_{t_0(\mathcal{F}),u_0} \to \mathbb{L}_{t_0(\mathcal{F})/\mathcal{F},u_0}$ from Lemma 4.2.5.7. There is an induced long exact sequence in homotopy

$$\dots \pi_1(\mathbb{L}_{t_0(\mathcal{F})/\mathcal{F},u_0}) \to \pi_0(\mathbb{L}_{\mathcal{F},u_0}) \to \pi_0(\mathbb{L}_{t_0(\mathcal{F}),u_0}) \to \pi_0(\mathbb{L}_{t_0(\mathcal{F})/\mathcal{F},u_0}) \to \dots \tag{5.1}$$

We note that, for any $M \in \mathbf{M}_{A_0}^{\heartsuit}$, the vertical fibres in the following diagram

$$\begin{array}{ccc} t_0(\mathcal{F})(A_0 \oplus M) & \longrightarrow & \mathcal{F}(A_0 \oplus M) \\ \downarrow & & \downarrow \\ t_0(\mathcal{F})(A_0) & \longrightarrow & \mathcal{F}(A_0) \end{array}$$

induce a map

$$\mathrm{Map}_{\mathbf{Mod}_{A_0}}(\mathbb{L}_{t_0(\mathcal{F}),u_0}, M) \to \mathrm{Map}_{\mathbf{Mod}_{A_0}}(\mathbb{L}_{\mathcal{F},u_0}, M) \tag{5.2}$$

We note that $A_0 \oplus M \in \mathbf{DAlg}^{\heartsuit}(\mathcal{C})$. Therefore, the horizontal morphisms in the diagram are equivalences and hence the map in Equation (5.2) is an equivalence. By Lemma 5.1.5.1, we see that $\tau_{\leq 0}(\mathbb{L}_{t_0(\mathcal{F}),u_0}) \simeq \tau_{\leq 0}(\mathbb{L}_{\mathcal{F},u_0})$. Hence, by examining the long exact sequence in Equation (5.1), we see that $\pi_i(\mathbb{L}_{t_0(\mathcal{F})/\mathcal{F},u_0}) = 0$ for $i \leq 0$.

Now, define $M := \pi_1(\mathbb{L}_{t_0(\mathcal{F})/\mathcal{F},u_0})$ and suppose that it is non-zero. We have a long exact sequence in homotopy

$$\dots \pi_2(\mathbb{L}_{t_0(\mathcal{F})/\mathcal{F},u_0}) \to \pi_1(\mathbb{L}_{\mathcal{F},u_0}) \to \pi_1(\mathbb{L}_{t_0(\mathcal{F}),u_0}) \to \pi_1(\mathbb{L}_{t_0(\mathcal{F})/\mathcal{F},u_0}) \to 0$$

and hence the morphism $\pi_1(\mathbb{L}_{t_0(\mathcal{F}),u_0}) \to \pi_1(\mathbb{L}_{t_0(\mathcal{F})/\mathcal{F},u_0})$ must be non-zero. Since

$$\begin{aligned} \mathrm{Map}_{\mathbf{Mod}_{A_0}}(\mathbb{L}_{t_0(\mathcal{F}),u_0}, M[1]) &\simeq \mathrm{Map}_{\mathbf{Mod}_{A_0}^{\leq 1}}(\tau_{\leq 1}(\mathbb{L}_{t_0(\mathcal{F}),u_0}), M[1]) \\ &\simeq \mathrm{Map}_{\mathbf{Mod}_{A_0}^{\leq 1}}(\pi_1(\mathbb{L}_{t_0(\mathcal{F}),u_0})[1], \pi_1(\mathbb{L}_{t_0(\mathcal{F})/\mathcal{F},u_0})[1]) \end{aligned}$$

we see that there exists a non-zero map $\mathbb{L}_{t_0(\mathcal{F}),u_0} \to M[1]$. Since the morphism $U_0 \to t_0(\mathcal{F})$ is in $\mathbf{P}^{\heartsuit}$, we see that

$$\pi_0(\mathrm{Map}_{\mathbf{Mod}_{A_0}}(\mathbb{L}_{U_0/t_0(\mathcal{F}),u_0}, M[1])) = 0$$

by Assumption (5). Hence, by considering the fibre sequence of cotangent complexes $\mathbb{L}_{t_0(\mathcal{F}),u_0} \to \mathbb{L}_{U_0,u_0} \to \mathbb{L}_{U_0/t_0(\mathcal{F}),u_0}$, we see that there must exist a non-zero map $\mathbb{L}_{U_0,u_0} \to M[1]$. This defines a non-zero derivation $d : A_0 \to A_0 \oplus M[1]$.

Since the morphism $t_0(\mathcal{F}) \to \mathcal{F}$ has an obstruction theory, then by Proposition 4.3.1.8, the fibre of the morphism

$$t_0(\mathcal{F})(A_0 \oplus_d \Omega M[1]) \to t_0(\mathcal{F})(A_0) \times_{\mathcal{F}(A_0)} \mathcal{F}(A_0 \oplus_d \Omega M[1]) \tag{5.3}$$

at the point $x \in \pi_0(t_0(\mathcal{F})(A_0) \times_{\mathcal{F}(A_0)} \mathcal{F}(A_0 \oplus_d \Omega M[1]))$ defined by the following commutative diagram

$$\begin{tikzcd} U_0 \arrow[r, "u_0"] \arrow[d] & t_0(\mathcal{F}) \arrow[d, "j"] \\ (U_0)_d[\Omega M[1]] \arrow[r] & \mathcal{F} \end{tikzcd}$$

is isomorphic, for some $\alpha(x) \in \pi_0(\mathrm{Map}_{\mathbf{Mod}_{A_0}}(\mathbb{L}_{t_0(\mathcal{F})/\mathcal{F},u_0}, M[1]))$, to the path space

$$\Omega_{\alpha(x),0}\mathrm{Map}_{\mathbf{Mod}_{A_0}}(\mathbb{L}_{t_0(\mathcal{F})/\mathcal{F},u_0}, M[1])$$

The morphism in Equation (5.3) is an equivalence because $A_0 \oplus_d \Omega M[1]$ and A_0 are in $\mathbf{DAlg}^{\heartsuit}(\mathcal{C})$. Hence, the path space is zero. However, since d is non-zero and provides a path between $\alpha(x)$ and 0, we get a contradiction. Therefore, $\pi_1(\mathbb{L}_{t_0(\mathcal{F})/\mathcal{F},u_0}) \simeq 0$. $\square$

5.1.6 The Representability Theorem

The following representability theorem bears many similarities to the one stated in [86, Theorem C.0.9] and will be sufficient for our purposes.

Theorem 5.1.6.1. *Suppose that $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, \mathcal{C}^0, \boldsymbol{\tau}, \mathbf{P}, \mathcal{A}, \mathbf{M}, \mathbf{S})$ is a representability context and that $\mathcal{F}$ is a stack in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$. The following conditions are equivalent.*

1. *$\mathcal{F}$ is an n-geometric stack in $\mathbf{Stk}_n(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}}, \mathbf{P}|_{\mathcal{A}})$,*
2. *$\mathcal{F}$ satisfies the following three conditions:*
   (a) The truncation $t_0(\mathcal{F})$ is an n-geometric stack in $\mathbf{Stk}_n(\mathcal{A}^{\heartsuit}, \boldsymbol{\tau}^{\heartsuit}, \mathbf{P}^{\heartsuit})$,
   (b) $\mathcal{F}$ has an obstruction theory relative to $\mathcal{A}$,
   (c) $\mathcal{F}$ is nilcomplete with respect to $\mathcal{A}$.

Proof. We first suppose that $\mathcal{F}$ is an n-geometric stack. Then, (a) is satisfied by Proposition 5.1.4.6. Condition (b) follows from Theorem 4.3.4.1 and Assumptions (3) and (6). Finally, Condition (c) follows from Proposition 4.3.5.5 along with Proposition 4.3.6.5 and Assumptions (3), (5) and (6).

To prove the converse, suppose that $\mathcal{F}$ is a stack in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$ satisfying conditions (a) to (c). The proof goes by induction on n. Suppose that $n = -1$. We prove the following lemma.

Lemma 5.1.6.2. *Suppose that $U_0 \in \mathcal{A}^{\heartsuit}$ and that we have a morphism $U_0 \to t_0(\mathcal{F})$ in n-$\mathbf{P}^{\heartsuit}$ such that, for any base point $x : X \to U_0$, we have that $\mathbb{L}_{U_0/t_0(\mathcal{F}),x} \simeq 0$. Then, there exists a representable stack $U = \mathrm{Spec}(A) \in \mathcal{A}$ in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$ and a morphism $u : U \to \mathcal{F}$ such that $t_0(U) \simeq U_0$ and $\mathbb{L}_{U/\mathcal{F},u} \simeq 0$.*

Proof. We will construct by induction a sequence of representable stacks

$$U_0 \to U_1 \to \ldots \to U_k \to U_{k+1} \to \ldots \to \mathcal{F}$$

satisfying the following properties:

- We have $U_k = \mathrm{Spec}(A_k)$ with $A_k \in \mathcal{A}^{op}$ k-truncated,
- The corresponding morphism $A_{k+1} \to A_k$ induces an equivalence on the k-th truncation,
- The morphisms $u_k : U_k \to \mathcal{F}$ are such that $\pi_i(\mathbb{L}_{U_k/\mathcal{F},u_k}) = 0$ for all $i \leq k+1$.

Let $A = \varprojlim_k A_k$ and let $U = \mathrm{Spec}(A)$. By construction $A_k = \tau_{\leq k}(A)$, and we note that $A \in \mathcal{A}^{op}$ by Assumption (3) on our Postnikov compatible derived geometry context. Therefore, U is a representable stack on $\mathcal{A}$. The points u_k define a well defined point in $\pi_0(\varprojlim_k \mathcal{F}(A_k))$ which we know, by condition (c), defines a point in $\pi_0(\mathcal{F}(A))$. Therefore, there is a well-defined morphism of stacks $u : U \to \mathcal{F}$ which, by Lemma 4.3.1.5, has a global cotangent complex. If $M \in \mathbf{M}_{A,1}$, then

$$\begin{aligned}
\mathrm{Map}_{\mathbf{Mod}_A}(\mathbb{L}_{U/\mathcal{F},u}, M) &\simeq \mathrm{Map}_{U/\mathbf{Stk}(\mathcal{A},\boldsymbol{\tau}|_{\mathcal{A}})/\mathcal{F}}(U[M], \mathcal{F}) \\
&\simeq \mathrm{Map}_{U/\mathbf{Stk}(\mathcal{A},\boldsymbol{\tau}|_{\mathcal{A}})/\mathcal{F}}(\varinjlim_k U_k[M_{\leq k}], \mathcal{F}) \\
&\simeq \varprojlim_k \mathrm{Map}_{U_k/\mathbf{Stk}(\mathcal{A},\boldsymbol{\tau}|_{\mathcal{A}})/\mathcal{F}}(U_k[M_{\leq k}], \mathcal{F}) \\
&\simeq \varprojlim_k \mathrm{Map}_{\mathbf{Mod}_{A_k}}(\mathbb{L}_{U_k/\mathcal{F},u_k}, M_{\leq k})
\end{aligned}$$

By our last assumption on the U_k, $\mathrm{Map}_{\mathbf{Mod}_A}(\mathbb{L}_{U/\mathcal{F},u}, M) \simeq 0$. Hence, by Lemma 4.2.5.5, $\mathbb{L}_{U/\mathcal{F},u} \simeq 0$.

It now remains to construct such a sequence. Indeed, when $k = 0$, since U_0 is a representable stack on $\mathcal{A}^{\heartsuit}$, $U_0 = \mathrm{Spec}(A_0)$ for some $A_0 \in \mathcal{A}^{\heartsuit,op}$. We let $u_0 : U_0 \to \mathcal{F}$ denote the morphism induced by $u_0 : U_0 \to t_0(\mathcal{F})$ and the morphism $j : t_0(\mathcal{F}) \to \mathcal{F}$.

If we consider the fibre sequence of A_0-modules from Corollary 4.2.5.8,

$$\mathbb{L}_{t_0(\mathcal{F})/\mathcal{F},u_0} \to \mathbb{L}_{U_0/\mathcal{F},u_0} \to \mathbb{L}_{U_0/t_0(\mathcal{F}),u_0}$$

then we obtain the following long exact sequence

$$\dots \to \pi_1(\mathbb{L}_{U_0/t_0(\mathcal{F}),u_0}) \to \pi_0(\mathbb{L}_{t_0(\mathcal{F})/\mathcal{F},u_0}) \to \pi_0(\mathbb{L}_{U_0/\mathcal{F},u_0}) \to \pi_0(\mathbb{L}_{U_0/t_0(\mathcal{F}),u_0}) \to \dots$$

By assumption, $\mathbb{L}_{U_0/t_0(\mathcal{F}),u_0} \simeq 0$. By Lemma 5.1.5.2, since $t_0(\mathcal{F})$ and $\mathcal{F}$ have obstruction theories, we see that $\pi_i(\mathbb{L}_{t_0(\mathcal{F})/\mathcal{F},u_0}) = 0$ for all $i \leq 1$. Therefore, $\pi_i(\mathbb{L}_{U_0/\mathcal{F},u_0}) = 0$ for all $i \leq 1$.

Now, suppose that all the U_i for $i \leq k$ have been constructed for some $k \geq 0$. Consider $u_k : U_k \to \mathcal{F}$ and the natural morphism

$$d_k : \mathbb{L}_{U_k} \to \mathbb{L}_{U_k/\mathcal{F},u_k} \xrightarrow{\tau_{\leq k+2}} (\mathbb{L}_{U_k/\mathcal{F},u_k})_{\leq k+2} \simeq \pi_{k+2}(\mathbb{L}_{U_k/\mathcal{F},u_k})[k+2]$$

We note that this defines an element of

$$\mathbf{Der}_{U_k}(U_k, \pi_{k+2}(\mathbb{L}_{U_k/\mathcal{F},u_k})[k+2]) \simeq \mathrm{Map}_{U_k/\mathbf{Stk}}(\mathrm{Spec}(A_k \oplus \pi_{k+2}(\mathbb{L}_{U_k/\mathcal{F},u_k})[k+2]), U_k)$$

We let $A_{k+1} = A_k \oplus_{d_k} \pi_{k+2}(\mathbb{L}_{U_k/\mathcal{F},u_k})[k+1]$ and $U_{k+1} = \mathrm{Spec}(A_{k+1})$, which, by our assumptions on $\mathbf{M}$, lies in $\mathcal{A}$. Since the obstruction $\alpha(u_k) : \mathbb{L}_{\mathcal{F},u_k} \to \pi_{k+2}(\mathbb{L}_{U_k/\mathcal{F},u_k})[k+2]$ from Proposition 4.3.1.7 is induced by d_k, we see that it is nullhomotopic. It follows that the morphism $u_k : U_k \to \mathcal{F}$ extends to a morphism $u_{k+1} : U_{k+1} \to \mathcal{F}$

$$\begin{tikzcd} U_k \arrow[rr] \arrow[dr, "u_k"'] & & U_{k+1} \arrow[dl, "u_{k+1}"] \\ & \mathcal{F} & \end{tikzcd}$$

in ${}^{U_k/}\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})_{/\mathcal{F}}$. We note that U_{k+1} satisfies the first two conditions by construction. It remains to prove the final condition. Consider the fibre sequence in $\mathbf{Mod}_{A_k}$ associated to the above triangle

$$\mathbb{L}_{U_{k+1}/\mathcal{F},u_{k+1}} \otimes^{\mathbb{L}}_{A_{k+1}} A_k \to \mathbb{L}_{U_k/\mathcal{F},u_k} \to \mathbb{L}_{U_k/U_{k+1}}$$

and consider the corresponding long exact sequence

$$.. \to \pi_{i+1}(\mathbb{L}_{U_k/U_{k+1}}) \to \pi_i(\mathbb{L}_{U_{k+1}/\mathcal{F},u_{k+1}} \otimes^{\mathbb{L}}_{A_{k+1}} A_k) \to \pi_i(\mathbb{L}_{U_k/\mathcal{F},u_k}) \to \pi_i(\mathbb{L}_{U_k/U_{k+1}}) \to ..$$

We note that $\pi_i(\mathbb{L}_{U_k/\mathcal{F},u_k}) = 0$ for all $i \leq k+1$. Moreover, by Lemma 4.3.6.2, we have that $\pi_i(\mathbb{L}_{U_k/U_{k+1}}) = 0$ for $i \leq k+1$ and $i = k+3$ and

$$\pi_{k+2}(\mathbb{L}_{U_k/U_{k+1}}) \simeq \pi_{k+1}(A_{k+1}) \simeq \pi_{k+1}(A_k \oplus_{d_k} \pi_{k+2}(\mathbb{L}_{U_k/\mathcal{F},u_k})[k+1]) \simeq \pi_{k+2}(\mathbb{L}_{U_k/\mathcal{F},u_k})$$

Therefore, using the long exact sequence, we see that $\pi_i(\mathbb{L}_{U_{k+1}/\mathcal{F},u_{k+1}} \otimes^{\mathbb{L}}_{A_{k+1}} A_k) = 0$ for $i \leq k+2$.

It remains to show that $\pi_i(\mathbb{L}_{U_{k+1}/\mathcal{F},u_{k+1}}) = 0$ for all $i \leq k+2$. We denote by K_{k+1} the fibre of the morphism $A_{k+1} \to A_k$. We note that this is concentrated in degree $k+1$. By considering the following long exact sequence of homotopy groups

$$..\to \pi_i(\mathbb{L}_{U_{k+1}/\mathcal{F},u_{k+1}} \otimes^{\mathbb{L}}_{A_{k+1}} K_{k+1}) \to \pi_i(\mathbb{L}_{U_{k+1}/\mathcal{F},u_{k+1}}) \to \pi_i(\mathbb{L}_{U_{k+1}/\mathcal{F},u_{k+1}} \otimes^{\mathbb{L}}_{A_{k+1}} A_k) \to \dots$$

we see that it suffices to show that $\pi_i(\mathbb{L}_{U_{k+1}/\mathcal{F},u_{k+1}} \otimes^{\mathbb{L}}_{A_{k+1}} K_{k+1}) \simeq 0$ for all $i \leq k+2$. Indeed, using the Tor-spectral sequence

$$\mathrm{Tor}^p_{\pi_*(A_{k+1})}(\pi_*(\mathbb{L}_{U_{k+1}/\mathcal{F},u_{k+1}}), \pi_*(K_{k+1}))_q \Rightarrow \pi_{p+q}(\mathbb{L}_{U_{k+1}/\mathcal{F},u_{k+1}} \otimes^{\mathbb{L}}_{A_{k+1}} K_{k+1})$$

we easily see that the bottom k-rows of the second page are zero. Therefore, we can see that $\pi_i(\mathbb{L}_{U_{k+1}/\mathcal{F},u_{k+1}} \otimes^{\mathbb{L}}_{A_{k+1}} K_{k+1}) = 0$ for $i \leq k$. Since $k \geq 1$, this implies, in particular, that $\pi_0(\mathbb{L}_{U_{k+1}/\mathcal{F},u_{k+1}})$ and $\pi_1(\mathbb{L}_{U_{k+1}/\mathcal{F},u_{k+1}})$ are zero. We then see that $\pi_i(\mathbb{L}_{U_{k+1}/\mathcal{F},u_{k+1}} \otimes^{\mathbb{L}}_{A_{k+1}} K_{k+1}) = 0$ for $i = k+1$ and $k+2$. $\square$

Now, coming back to the original theorem. Suppose that $n = -1$. Then, we note that $t_0(\mathcal{F})$ is a representable stack. Therefore, by Lemma 5.1.6.2 applied to the identity morphism $t_0(\mathcal{F}) \to t_0(\mathcal{F})$, we see that there exists a representable stack U in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$ and a morphism $u : U \to \mathcal{F}$ with $\mathbb{L}_{U/\mathcal{F},u} \simeq 0$ and $t_0(U) \simeq t_0(\mathcal{F})$.

We will prove by induction on $k \geq 0$ that, for every $A \in \mathcal{A}^{op}$, $U(\tau_{\leq k} A) \simeq \mathcal{F}(\tau_{\leq k} A)$. Then, using Condition (c), we see that $U(A) \simeq \mathcal{F}(A)$ for every $A \in \mathcal{A}^{op}$, and hence $\mathcal{F} \simeq U$ is representable. When $k = 0$, we note that $t_0(U) \simeq t_0(\mathcal{F})$, and hence we are done. Now, suppose that $k > 0$ and that $U(\tau_{\leq m} A) \simeq \mathcal{F}(\tau_{\leq m} A)$ for all $m < k$ and all $X = \mathrm{Spec}(A) \in \mathcal{A}$. By Lemma 4.3.6.1, the map $A_{\leq k} \to A_{\leq k-1}$ is isomorphic in $\mathcal{A}^{op}_{/A_{\leq k-1}}$ to the morphism

$$A_{\leq k-1} \oplus_{d_k} \pi_k(A)[k] \to A_{\leq k-1}$$

for some derivation $d_k \in \pi_0(\mathbf{Der}(A_{\leq k-1}, \pi_k(A)[k+1]))$. Since $\mathbb{L}_{U/\mathcal{F},u} \simeq 0$, then $\mathbb{L}_U \simeq \mathbb{L}_{\mathcal{F},u}$. Hence, since $\mathcal{F}$ has a global cotangent complex, we see that

$$\mathrm{Map}_{X_{\leq k-1}/\mathbf{Stk}}(X_{\leq k-1}[\pi_k(A)[k+1]], U) \simeq \mathrm{Map}_{X_{\leq k-1}/\mathbf{Stk}}(X_{\leq k-1}[\pi_k(A)[k+1]], \mathcal{F})$$

Now, since U and $\mathcal{F}$ are infinitesimally cartesian, then, using our inductive hypothesis,

$$\begin{aligned} U(A_{\leq k-1} \oplus_{d_k} \pi_k(A)[k]) &\simeq U(A_{\leq k-1}) \times_{U(A_{\leq k-1} \oplus \pi_k(A)[k+1])} U(A_{\leq k-1}) \\ &\simeq \mathcal{F}(A_{\leq k-1}) \times_{\mathcal{F}(A_{\leq k-1} \oplus \pi_k(A)[k+1])} \mathcal{F}(A_{\leq k-1}) \\ &\simeq \mathcal{F}(A_{\leq k-1} \oplus_{d_k} \pi_k(A)[k]) \end{aligned}$$

from which it follows that $U(A_{\leq k}) \simeq \mathcal{F}(A_{\leq k})$.

Now, suppose that $n > -1$ and that the theorem holds for $m < n$. Suppose that $\mathcal{F}$ is a stack in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$ satisfying conditions (a) to (c) of the theorem. We will first show that the diagonal morphism $\mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is $(n-1)$-representable. Suppose that we have a map $X \to \mathcal{F} \times \mathcal{F}$ where X is a (-1)-geometric stack. It suffices to show that $\mathcal{F} \times_{\mathcal{F}\times\mathcal{F}} X$ is $(n-1)$-geometric. We note that, since $t_0(\mathcal{F})$ is an n-geometric stack, then the diagonal morphism $t_0(\mathcal{F}) \to t_0(\mathcal{F}) \times t_0(\mathcal{F})$ is $(n-1)$-representable, and hence

$$t_0(\mathcal{F} \times_{\mathcal{F}\times\mathcal{F}} X) \simeq t_0(\mathcal{F}) \times_{t_0(\mathcal{F})\times t_0(\mathcal{F})} t_0(X)$$

is an $(n-1)$-geometric stack. Since $\mathcal{F}$ has an obstruction theory, it follows that $\mathcal{F} \times_{\mathcal{F}\times\mathcal{F}} X$ has an obstruction theory by Lemma 4.3.1.5. Moreover, it is easy to see that

$$(\mathcal{F} \times_{\mathcal{F}\times\mathcal{F}} X)(A) = \varprojlim_k (\mathcal{F} \times_{\mathcal{F}\times\mathcal{F}} X)(\tau_{\leq k} A)$$

for any $A \in \mathcal{A}^{op}$. Therefore, since $\mathcal{F} \times_X (\mathcal{F} \times \mathcal{F})$ satisfies Conditions (a) to (c) of the theorem at level $n-1$, we can conclude by the induction hypothesis.

It remains to show that $\mathcal{F}$ has an n-atlas. Suppose that $t_0(\mathcal{F})$ has an n-atlas $\{V_j \to t_0(\mathcal{F})\}_{j\in J}$ in $\mathbf{Stk}(\mathcal{A}^{\heartsuit}, \boldsymbol{\tau}^{\heartsuit})$. We will lift this to an atlas of $\mathcal{F}$. Consider any morphism $V_j \to t_0(\mathcal{F})$, which we note is in $(n-1)$-$\mathbf{P}^{\heartsuit}$, and let $U_0 := V_j$. Similarly to before, we will construct by induction a sequence of representable stacks

$$U_0 \to U_1 \to \ldots \to U_k \to U_{k+1} \to \ldots \to \mathcal{F}$$

in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$ satisfying the following properties

- We have $U_k = \mathrm{Spec}(A_k)$ with $A_k \in \mathcal{A}^{op}$ k-truncated,
- The corresponding morphism $A_{k+1} \to A_k$ induces an equivalence on the k-th truncation,
- The morphisms $u_k : U_k \to \mathcal{F}$ are such that, for any $M \in \mathbf{M}_{A_k,1}$ with $\pi_i(M) = 0$ for $i > k+1$, one has $[\mathbb{L}_{U_k/\mathcal{F},u_k}, M] := \pi_0(\mathrm{Map}_{\mathbf{Mod}_{A_k}}(\mathbb{L}_{U_k/\mathcal{F},u_k}, M)) \simeq 0$.

We construct the sequence inductively. Indeed, when $k = 0$, we let $u_0 : U_0 \to \mathcal{F}$ denote the morphism induced by $u_0 : U_0 \to t_0(\mathcal{F})$ and the morphism $j : t_0(\mathcal{F}) \to \mathcal{F}$. Suppose that $M \in \mathbf{M}_{A_0}$ is concentrated in degree 1 and consider the long exact sequence

$$\cdots \to [\mathbb{L}_{U_0/t_0(\mathcal{F}),u_0}, M] \to [\mathbb{L}_{U_0/\mathcal{F},u_0}, M] \to [\mathbb{L}_{t_0(\mathcal{F})/\mathcal{F},u_0}, M] \to 0$$

Since $\mathbb{L}_{t_0(\mathcal{F})/\mathcal{F},u_0}$ is 1-connective by Lemma 5.1.5.2 and the morphism $U_0 \to t_0(\mathcal{F})$ is in $(n-1)$-$\mathbf{P}$, then it follows using Assumption (5), that

$$\pi_0(\mathrm{Map}_{\mathbf{Mod}_{A_0}}(\mathbb{L}_{U_0/\mathcal{F},u_0}, M)) = 0$$

Now, suppose that all the U_i for $i \leq k$ have been constructed. We may construct the next term in the sequence in a similar way to Lemma 5.1.6.2. Indeed, consider $u_k : U_k \to \mathcal{F}$ and the natural morphism

$$d_k : \mathbb{L}_{U_k} \to \mathbb{L}_{U_k/\mathcal{F},u_k} \xrightarrow{\tau_{\leq k+2}} (\mathbb{L}_{U_k/\mathcal{F},u_k})_{\leq k+2} \xrightarrow{\pi_{k+2}} \pi_{k+2}(\mathbb{L}_{U_k/\mathcal{F},u_k})$$

Then, as before, we let $A_{k+1} = A_k \oplus_{d_k} \pi_{k+2}(\mathbb{L}_{U_k/\mathcal{F}})[k+1]$ and $U_{k+1} = \mathrm{Spec}(A_{k+1})$. As before, there is an induced morphism $u_{k+1} : U_{k+1} \to \mathcal{F}$. The only thing we need to check is the third condition. Suppose we have a $(k+2)$-truncated $M \in \mathbf{M}_{A_{k+1},1}$. From the sequence $U_k \to U_{k+1} \to \mathcal{F}$, we obtain a long exact sequence

$$\cdots \to [\mathbb{L}_{U_k/U_{k+1}}, M_{\leq k+1}] \to [\mathbb{L}_{U_k/\mathcal{F},u_k}, M_{\leq k+1}] \to [\mathbb{L}_{U_{k+1}/\mathcal{F},u_{k+1}} \otimes^{\mathbb{L}}_{A_{k+1}} A_k, M_{\leq k+1}] \to 0$$

Since $M_{\leq k+1}$ is a $(k+1)$-truncated module in $\mathbf{M}_{A_k,1}$, then, by our assumptions on our sequence, $\pi_0(\mathrm{Map}_{\mathbf{Mod}_{A_k}}(\mathbb{L}_{U_k/\mathcal{F},u_k}, M_{\leq k+1})) \simeq 0$, and hence

$$\pi_0(\mathrm{Map}_{\mathbf{Mod}_{A_k}}(\mathbb{L}_{U_{k+1}/\mathcal{F},u_{k+1}} \otimes^{\mathbb{L}}_{A_{k+1}} A_k, M_{\leq k+1})) \simeq 0$$

Using similar reasoning to Lemma 5.1.6.2 and using Lemma 4.3.6.2, we can see that $\pi_{k+2}(\mathbb{L}_{U_{k+1}/\mathcal{F},u_{k+1}} \otimes^{\mathbb{L}}_{A_{k+1}} A_k) = 0$. Therefore, since $\pi_i(M) = 0$ for $i > k+2$,

$$\pi_0(\mathrm{Map}_{\mathbf{Mod}_{A_{k+1}}}(\mathbb{L}_{U_{k+1}/\mathcal{F},u_{k+1}}, M)) \simeq \pi_0(\mathrm{Map}_{\mathbf{Mod}_{A_k}}(\mathbb{L}_{U_{k+1}/\mathcal{F},u_{k+1}} \otimes^{\mathbb{L}}_{A_{k+1}} A_k, M)) \simeq 0$$

as required.

Suppose that the sequence in question has been constructed. We let $A = \varprojlim_k A_k$ and let $U = \mathrm{Spec}(A)$. Since the diagonal morphism $\mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is $(n-1)$-representable, we can see that the morphism $u : U \to \mathcal{F}$ is $(n-1)$-representable using Proposition 3.2.2.7.

Suppose that $M \in \mathbf{M}_{A,1}$. Then, for each $k \geq 0$, there is a fibre sequence

$$\mathbb{L}_{U/\mathcal{F},u} \otimes^{\mathbb{L}}_{A} A_k \to \mathbb{L}_{U_k/\mathcal{F},u_k} \to \mathbb{L}_{U_k/U,u_k}$$

and hence we have a long exact sequence

$$\cdots \to [\mathbb{L}_{U_k/U,u_k}, M_{\leq k+1}] \to [\mathbb{L}_{U_k/\mathcal{F},u_k}, M_{\leq k+1}] \to [\mathbb{L}_{U/\mathcal{F},u} \otimes^{\mathbb{L}}_{A} A_k, M_{\leq k+1}] \to 0$$

Since $[\mathbb{L}_{U_k/\mathcal{F},u_k}, M_{\leq k+1}] \simeq 0$ by construction, then

$$\pi_0(\mathrm{Map}_{\mathbf{Mod}_A}(\mathbb{L}_{U/\mathcal{F},u}, M_{\leq k+1})) \simeq \pi_0(\mathrm{Map}_{\mathbf{Mod}_{A_k}}(\mathbb{L}_{U/\mathcal{F},u} \otimes^{\mathbb{L}}_A A_k, M_{\leq k+1})) \simeq 0$$

Therefore, since this holds for all $k \geq 0$, then

$$\pi_0(\mathrm{Map}_{\mathbf{Mod}_A}(\mathbb{L}_{U/\mathcal{F},u}, M)) = 0$$

Since the morphism $U_0 = V_j \to t_0(\mathcal{F})$ is in $(n-1)$-$\mathbf{P}^{\heartsuit}$, we see that $U \to \mathcal{F}$ is in $(n-1)$-$\mathbf{P}$ by Assumption (5).

Let $U_{(j)}$ denote the U corresponding to taking $U_0 = V_j$. Now, since the total morphism $\coprod_j V_j \to t_0(\mathcal{F})$ is an epimorphism of stacks, it follows that the total morphism $\coprod_{j \in J} U_{(j)} \to \mathcal{F}$ is an epimorphism of stacks since $\pi_0^{\tau}(\mathcal{F}) \simeq \pi_0^{\tau}(t_0(\mathcal{F}))$. Hence, we see that we have constructed an n-atlas $\{U_{(j)} \to \mathcal{F}\}_{j \in J}$ for $\mathcal{F}$.

$\square$

5.2 Derived Analytic Geometry Contexts

Suppose that R is a Banach ring. Recall from Example 4.1.2.6, that we obtain a flat derived algebraic context

$$(\mathbf{Ch}(\mathrm{Ind}(\mathrm{Ban}_R)), \mathbf{Ch}_{\geq 0}(\mathrm{Ind}(\mathrm{Ban}_R)), \mathbf{Ch}_{\leq 0}(\mathrm{Ind}(\mathrm{Ban}_R)), \mathbf{L}^H(P^0))$$

where P^0 is the collection of compact projective generators described in Section 1.3, differing depending on whether we are working over a non-Archimedean or Archimedean Banach ring. Moreover, we have an equivalence of categories

$$\mathbf{DAlg}^{cn}(\mathbf{Ch}(\mathrm{Ind}(\mathrm{Ban}_R))) \simeq \mathbf{L}^H(\mathrm{Comm}(\mathrm{sInd}(\mathrm{Ban}_R)))$$

As explained in the Introduction, this is an appropriate setting to develop theories of derived analytic and derived smooth geometry.

In this section, we will endow a general derived algebraic context $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, \mathcal{C}^0)$ with some extra structures such that we obtain representability contexts of the form

$$(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, \mathcal{C}^0, \boldsymbol{\tau}, \mathbf{P}, \mathcal{A}, \mathbf{M}, \mathbf{S})$$

We will then give a specific example of a derived complex analytic geometry context. A context for derived smooth geometry is presented in Chapter 6. We note that, for brevity, we have omitted several details about the formally étale topology. These can be found in [75].

5.2.1 Formal Covering Families

Suppose that $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, \mathcal{C}^0)$ is a flat Postnikov compatible derived algebraic context.

Definition 5.2.1.1. A family of morphisms $f_i : U_i = \mathrm{Spec}(B_i) \to \mathrm{Spec}(A) = X$ in $\mathrm{Ho}(\mathbf{DAff}^{cn}(\mathcal{C}))$ is a *formal covering family* if the family of functors

$$\{(f_i)_! : \mathbf{Mod}_A^{cn} \to \mathbf{Mod}_{B_i}^{cn}\}_{i \in I}$$

is conservative, i.e. for every morphism $u \in \mathbf{Mod}_A^{cn}$, u is an equivalence if and only if all the $(f_i)_!(u)$ are equivalences.

It is easy to see that equivalences define formal covering families, and that compositions and pullbacks of formal covering families also define formal covering families. Moreover, the truncation of any formal covering family is a formal covering family.

Lemma 5.2.1.2. *If $\{f_i : U_i = \mathrm{Spec}(B_i) \to \mathrm{Spec}(A) = X\}_{i \in I}$ is a formal covering family, then so is $\{t_0(f_i) : t_0(U_i) = \mathrm{Spec}(\pi_0(B_i)) \to \mathrm{Spec}(\pi_0(A)) = t_0(X)\}_{i \in I}$.*

Proof. We just need to show that the family $\{t_0(f_i)_! : \mathbf{Mod}_{\pi_0(A)}^{cn} \to \mathbf{Mod}_{\pi_0(B_i)}^{cn}\}_{i \in I}$ is conservative. Suppose that we have a morphism $u : M \to N$ in $\mathbf{Mod}_{\pi_0(A)}^{cn}$. Then, since

$$M \otimes_{\pi_0(A)}^{\mathbb{L}} \pi_0(B_i) \simeq \pi_0(M \otimes_A^{\mathbb{L}} B_i)$$

it is clear that u is an equivalence if and only if $t_0(f_i)_!(u)$ is an equivalence for each $i \in I$. ☐

Recall from Section 3.2.1, that we can extend a class of maps $\mathbf{P}$ to be local for a topology $\boldsymbol{\tau}$ by defining the category $\mathbf{P}^{\boldsymbol{\tau}}$.

Lemma 5.2.1.3. *Suppose that $\mathbf{P}$ is a class of maps which are formally étale, and $\boldsymbol{\tau}$ is a Grothendieck topology whose covers are formal covering families. Then, $\mathbf{P}^{\boldsymbol{\tau}}$ consists of formally étale morphisms.*

Proof. Suppose that we have a morphism $Y = \mathrm{Spec}(B) \to \mathrm{Spec}(A) = X$ in $\mathbf{P}^{\boldsymbol{\tau}}$. Then, we note that there is a $\boldsymbol{\tau}$-cover $\{U_i = \mathrm{Spec}(C_i) \to \mathrm{Spec}(B) = Y\}_{i \in I}$ such that each induced morphism $A \to C_i$ is in $\mathbf{P}$. There is a fibre-cofibre sequence

$$\mathbb{L}_{B/A} \otimes_B^{\mathbb{L}} C_i \to \mathbb{L}_{C_i/A} \to \mathbb{L}_{C_i/B}$$

in $\mathbf{Mod}_{C_i}$. Therefore, since the morphisms $A \to C_i$ and $B \to C_i$ are formally étale, then they are formally unramified and we see that, for each $i \in I$, $\mathbb{L}_{B/A} \otimes_B^{\mathbb{L}} C_i \simeq 0$. Since our covering family is formal, it follows that $\mathbb{L}_{B/A} \simeq 0$. Therefore, $Y \to X$ is formally étale.

☐

Lemma 5.2.1.4. *Suppose that* $\mathbf{P}$ *is a class of maps which are derived strong, and that* $\boldsymbol{\tau}$ *is a Grothendieck topology whose covers are formal covering families and consist of derived strong morphisms. Then,* $\mathbf{P}^{\boldsymbol{\tau}}$ *consists of derived strong morphisms.*

Proof. Indeed, suppose that we have a morphism $f : Y = \mathrm{Spec}(B) \to \mathrm{Spec}(A) = X$ in $\mathbf{P}^{\boldsymbol{\tau}}$. Then, there exists a cover $\{U_i = \mathrm{Spec}(C_i) \to Y\}_{i \in I}$ in $\boldsymbol{\tau}$ such that the induced morphism $A \to C_i$ is in $\mathbf{P}$. We can then see, using Lemma 5.1.2.10, that for each $i \in I$,

$$(\pi_*(A) \otimes^{\mathbb{L}}_{\pi_0(A)} \pi_0(B)) \otimes^{\mathbb{L}}_B C_i \simeq \pi_*(C_i) \simeq \pi_*(B) \otimes^{\mathbb{L}}_{\pi_0(B)} \pi_0(C_i) \simeq \pi_*(B) \otimes^{\mathbb{L}}_B C_i$$

and therefore, since $\{U_i \to Y\}_{i \in I}$ is a formal covering family, we see that the morphism $A \to B$ is derived strong. □

Lemma 5.2.1.5. *Suppose that* $X = \mathrm{Spec}(A) \in \mathbf{DAff}^{cn}(\mathcal{C})$*,* $M \in \mathbf{Mod}_A^{\geq 1}$*, and that* $d \in \pi_0(\mathbf{Der}(A, M))$ *is a derivation. Suppose that* $\{V_i = \mathrm{Spec}(A_i) \to X\}_{i \in I}$ *is a formal covering family with each morphism* $V_i \to X$ *formally smooth. Then,* $\{W_i \to X_d[\Omega M]\}_{i \in I}$ *is a formal covering family where* W_i *is defined to be* $\mathrm{Spec}(A_i \oplus_{d_i'} \Omega M_i')$ *with* d_i' *the derivation induced by Lemma 4.3.3.1 and* $M_i' = M \otimes^{\mathbb{L}}_A A_i$*.*

Proof. Indeed, suppose that $W_i = \mathrm{Spec}(B_i)$ with $B_i = A_i \oplus_{d_i'} \Omega M_i'$. Consider the family $\{g_i : W_i \to X_d[\Omega M]\}_{i \in I}$. Suppose that we have a morphism $u : N \to N'$ in $\mathbf{Mod}^{cn}_{A \oplus_d \Omega M}$. It suffices to show that if, for each $i \in I$, we have an equivalence $(g_i)_!(u) : N \otimes^{\mathbb{L}}_{A \oplus_d \Omega M} B_i \to N' \otimes^{\mathbb{L}}_{A \oplus_d \Omega M} B_i$, then u is also an equivalence. We note that, by Lemma 4.3.3.3, there is an equivalence

$$N \otimes^{\mathbb{L}}_A A_i \simeq N \otimes^{\mathbb{L}}_{A \oplus_d \Omega M} B_i \simeq N' \otimes^{\mathbb{L}}_{A \oplus_d \Omega M} B_i \simeq N' \otimes^{\mathbb{L}}_A A_i$$

for all $i \in I$. Hence, $u : N \to N'$ is an equivalence because $\{V_i \to X\}_{i \in I}$ is a conservative subfamily.

□

5.2.2 Derived Algebraic Geometry

We recall from Example 4.1.1.4 that, for k a commutative ring, there is an equivalence of $(\infty, 1)$-categories

$$\mathbf{DAlg}^{cn}(\mathbf{Mod}_k) \simeq \mathbf{L}^H(\mathrm{Comm}(\mathrm{sMod}_k))$$

Definition 5.2.2.1. A morphism $f : A \to B$ in $\mathbf{DAlg}^{cn}(\mathbf{Mod}_k)$ is

- *finitely presented* if, for any filtered diagram of objects $\{A \to C_i\}_{i\in I}$, there is an equivalence of ∞-groupoids

$$\varinjlim_{i\in I} \mathrm{Map}_{\mathbf{DAlg}^{cn}_A(\mathbf{Mod}_k)}(B, C_i) \to \mathrm{Map}_{\mathbf{DAlg}^{cn}_A(\mathbf{Mod}_k)}(B, \varinjlim_{i\in I} C_i)$$

- *étale* (resp. *smooth*) if it is formally étale (resp. formally smooth) and finitely presented.

Definition 5.2.2.2. The *étale topology* on $\mathrm{Ho}(\mathbf{DAff}^{cn}(\mathbf{Mod}_k))$ has covers of the form $\{U_j \to X\}_{j\in J}$ with $U_j = \mathrm{Spec}(B_j)$ and $X = \mathrm{Spec}(A)$, satisfying the following properties.

1. Each morphism $A \to B_j$ is étale,
2. There is a finite subset $K \subseteq J$ such that the family $\{U_k \to X\}_{k\in K}$ is a formal covering family.

We denote this topology by $\boldsymbol{\acute{e}t}$.

Consider the category $\mathbf{DAff}^{cn}(\mathbf{Mod}_k)$ endowed with the étale topology, $\boldsymbol{\acute{e}t}$, and the class of smooth maps, $\mathbf{Sm}$. Let $\mathbf{M}$ denote the full categories of modules $\mathbf{Mod}_A$ for each $A \in \mathbf{DAlg}^{cn}(\mathbf{Mod}_k)$. Let $\mathbf{E}$ denote the class of étale morphisms.

Proposition 5.2.2.3. *The tuple*

$$(\mathbf{Mod}_k, \mathbf{Mod}_{k,\geq 0}, \mathbf{Mod}_{k,\leq 0}, \mathrm{Mod}_k^{fgf}, \boldsymbol{\acute{e}t}, \mathbf{Sm}, \mathbf{DAff}^{cn}(\mathbf{Mod}_k), \mathbf{M}, \mathbf{E})$$

is a representability context and is moreover closed under $\boldsymbol{\acute{e}t}$*-descent.*

Proof. We see that $(\mathbf{Mod}_k, \mathbf{Mod}_{k,\geq 0}, \mathbf{Mod}_{k,\leq 0}, \mathrm{Mod}_k^{fgf}, \boldsymbol{\acute{e}t}, \mathbf{Sm}, \mathbf{DAff}^{cn}(\mathbf{Mod}_k), \mathbf{M})$ is a flat Postnikov compatible derived geometry context using results from [86]. We note that Conditions (1) and (2) for being a representability context are clearly satisfied. Furthermore, $\mathbf{DAff}^{cn}(\mathbf{Mod}_k)$ satisfies the descent condition in Definition 3.2.3.8 for hypercovers by [86, Lemma 2.2.2.13], and hence can easily be shown to be closed under $\boldsymbol{\acute{e}t}$-descent by Proposition 3.2.3.9. Condition (3) follows by [86, Lemma 1.3.2.3]. Conditions (4) and (5) follow from [86, Theorem 2.2.2.6] and [86, Corollary 2.2.5.3] respectively. Finally, Condition (6) follows from [86, Proposition 2.2.3.2]. □

This is the correct representability context to work in to do derived algebraic geometry in the sense of Toën and Vezzosi [86]. In [86, Section 2.2.6], they apply their version of the representability theorem to prove conditions under which certain mapping stacks are n-geometric.

5.2.3 Derived Strongly Formally Perfect Morphisms

Suppose that $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, \mathcal{C}^0)$ is a flat Postnikov compatible derived algebraic context. Recall from Proposition 4.2.3.7, that formally perfect morphisms are formally smooth. In the derived algebraic geometry context from the previous section, a finitely presented morphism is formally perfect if and only if it is formally smooth. In the rest of this section, we will work with formally perfect morphisms rather than formally smooth morphisms because we obtain stronger results. Several results have analogues for formally smooth morphisms as described in [10] and also [75].

Definition 5.2.3.1. A morphism $f : A \to B$ in $\mathbf{DAlg}^{cn}(\mathcal{C})$ is

1. *flat* if, whenever M is an A-module in $\mathbf{DAlg}^{\heartsuit}(\mathcal{C})$, then $M \otimes^{\mathbb{L}}_A B$ is in $\mathbf{DAlg}^{\heartsuit}(\mathcal{C})$,
2. *derived strongly flat* if it is derived strong and $\pi_0(f)$ is flat,
3. *flat formally perfect* if it is flat and formally perfect.

Lemma 5.2.3.2. *[10, c.f. Corollary 2.3.86] Given a morphism $f : A \to B$ in $\mathbf{DAlg}^{cn}(\mathcal{C})$, the following are equivalent,*

1. *f is a flat morphism,*
2. *f is a derived strongly flat morphism,*
3. *$B \otimes^{\mathbb{L}}_A - : \mathbf{Mod}^{cn}_A \to \mathbf{Mod}^{cn}_B$ commutes with finite limits.*

In the definition of a representability context, we specified that our class of maps should consist of maps which are derived strong.

Definition 5.2.3.3. Suppose that we have a morphism $f : A \to B$ in $\mathbf{DAlg}^{cn}(\mathcal{C})$. Then, f is *derived strongly (flat) formally perfect* if it is derived strong and $\pi_0(f)$ is (flat) formally perfect.

Corollary 5.2.3.4. *Suppose that $f : A \to B$ is a morphism in $\mathbf{DAlg}^{cn}(\mathcal{C})$. Then,*

1. *If f is derived strongly (flat) formally perfect, then it is (flat) formally perfect,*
2. *If f is formally perfect and $\pi_0(f)$ is formally perfect with $\pi_n(A)$ transverse to $\pi_0(B)$ as $\pi_0(A)$-modules for all n, then f is derived strong.*

Proof. For (1), suppose that f is derived strongly formally perfect. By Corollary 5.1.2.11, there is an equivalence

$$\mathbb{L}_{B/A} \otimes^{\mathbb{L}}_B \pi_0(B) \simeq \mathbb{L}_{\pi_0(A)/\pi_0(B)}$$

Since $\mathbb{L}_{\pi_0(B)/\pi_0(A)}$ is a perfect $\pi_0(B)$-module, it is a retract of some finite colimit of objects of the form $\coprod_{E_i} \pi_0(B)$ where E_i is a finite set, i.e. there exist maps

$r : \varinjlim_i \coprod_{E_i} \pi_0(B) \to \mathbb{L}_{\pi_0(B)/\pi_0(A)}$ and $i : \mathbb{L}_{\pi_0(B)/\pi_0(A)} \to \varinjlim_i \coprod_{E_i} \pi_0(B)$ such that $r \circ i$ is the identity. We note that, by Lemma 4.2.3.6,

$$\pi_0(\mathrm{Map}_{\mathbf{Mod}_B}(\varinjlim_i \coprod_{E_i} B, \varinjlim_i \coprod_{E_i} B)) \simeq \mathrm{Hom}_{\mathbf{Mod}_{\pi_0(B)}}(\varinjlim_i \coprod_{E_i} \pi_0(B), \varinjlim_i \coprod_{E_i} \pi_0(B))$$

and hence there is a map $i' : \varinjlim_i \coprod_{E_i} B \to \varinjlim_i \coprod_{E_i} B$ such that $i' \circ i' = i'$ and $\pi_0(i') = i \circ r$. Similarly, there is a map $r' : \varinjlim_i \coprod_{E_i} B \to \mathbb{L}_{B/A}$ such that $\pi_0(r') = r$. Since $\mathbf{Mod}_B$ is idempotent complete by [52, Corollary 5.4.3.6], i' induces a split fibre sequence of B-modules

$$K \to \varinjlim_i \coprod_{E_i} B \to C$$

where K is the fibre of i' and C is the retract of i'. We note that r' induces a map $r'' : C \to \mathbb{L}_{B/A}$ which is an equivalence on π_0. If we can show that $C \simeq \mathbb{L}_{B/A}$, then since C is a retract of $\varinjlim_i \coprod_{E_i} B$, it follows that $\mathbb{L}_{B/A}$ is perfect, as desired.

We note that C is derived strong as an A-module since it is a retract of a free derived strong A-module. Therefore, by Lemma 5.1.2.10, we have an equivalence $C \otimes^{\mathbb{L}}_B \pi_0(B) \simeq C \otimes^{\mathbb{L}}_A \pi_0(A) \simeq \pi_0(C)$. If we let K' be the fibre of the morphism $C \to \mathbb{L}_{B/A}$, then we note that $K' \otimes^{\mathbb{L}}_B \pi_0(B) \simeq 0$. Therefore, by Lemma 5.1.2.5, $K' \simeq 0$, and hence $\mathbb{L}_{B/A} \simeq C$ is perfect. The result about flatness follows from Lemma 5.2.3.2.

Now, for (2), suppose that f is a formally perfect morphism and that $\pi_0(f)$ is formally perfect with $\pi_n(A)$ transverse to $\pi_0(B)$ as $\pi_0(A)$-modules for all n. Then, we note that f and $\pi_0(f)$ are formally smooth by Lemma 4.2.3.7, and hence we can conclude by [10, Proposition 2.6.160]. $\square$

5.2.4 Quasicoherent Sheaves

Suppose that we have an $(\infty,1)$-site $(\mathbf{DAff}^{cn}(\mathcal{C}), \boldsymbol{\tau})$. For $X = \mathrm{Spec}(A) \in \mathbf{DAff}^{cn}(\mathcal{C})$, we define $\mathbf{QCoh}(X) := \mathbf{Mod}_A$ and for any morphism $f : Y \to X$, $\mathbf{QCoh}(f)$ is defined to be the colimit-preserving functor $B \otimes^{\mathbb{L}}_A - : \mathbf{Mod}_A \to \mathbf{Mod}_B$. This defines a $\mathbf{Cat}$-valued presheaf

$$\mathbf{QCoh} : \mathbf{DAff}^{cn}(\mathcal{C})^{op} \to \mathbf{Pr}^{\mathbb{L},\otimes}$$

where $\mathbf{Pr}^{\mathbb{L},\otimes}$ is the $(\infty,1)$-category whose objects are locally presentable monoidal $(\infty,1)$-categories and morphisms are left adjoint functors. If we restrict to perfect modules, we obtain a $\mathbf{Pr}^{\mathbb{L},\otimes}$-valued presheaf which we will denote by $\mathbf{Perf}$. If we restrict to strongly dualisable modules, as in Appendix B, we obtain a presheaf which we will denote by $\mathbf{Dls}$. We note that we can define similar notions of (Čech-)descent and hyperdescent for $\boldsymbol{\tau}$-covers for $\mathbf{Pr}^{\mathbb{L},\otimes}$-valued presheaves as in Section 3.1.3. See also [10, Section 7.1.3].

Lemma 5.2.4.1. *[10, c.f. Lemma 7.2.35] Suppose that $\mathcal{N} \subseteq \mathbf{QCoh}$ is such that*

1. *$\mathcal{N}$ satisfies descent for $\boldsymbol{\tau}$-covers,*
2. *$\mathcal{N}(\mathrm{Spec}(A))$ contains A,*
3. *$\mathcal{N}(\mathrm{Spec}(A)) \subseteq \mathbf{Mod}_A$ is closed under the tensor product and internal hom.*

Then, **Dls** *is a local sub-presheaf of $\mathcal{N}$. In particular it satisfies (Čech-)descent for $\boldsymbol{\tau}$-covers, and also satisfies hyperdescent for $\boldsymbol{\tau}$-covers whenever $\mathcal{N}$ does.*

Lemma 5.2.4.2. *Suppose that*

1. *$\mathcal{A}$ is closed under geometric realisations,*
2. *Any $\boldsymbol{\tau}$-cover has a finite subcover,*
3. *For any finite collection $\{U_i\}_{i\in I}$ of $U_i \in \mathcal{A}$, the map $\coprod_{i\in I} h(U_i) \to h(\coprod_{i\in I} U_i)$ is an equivalence in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$,*
4. ***QCoh** satisfies (Čech)-descent for $\boldsymbol{\tau}$-covers.*

Then,

1. *Any Čech nerve of a $\boldsymbol{\tau}$-cover of objects in $\mathcal{A}$ satisfies the co-cartesian descent condition of Definition 3.2.3.8,*
2. *$\mathcal{A}$ is closed under $\boldsymbol{\tau}$-descent relative to $\mathcal{A}$ in the sense of Definition 3.2.3.2.*

Proof. We note that (2) follows from (1) and Proposition 3.2.3.9.

To prove (1), suppose that $\{U_i = \mathrm{Spec}(A_i) \to \mathrm{Spec}(A) = X\}_{i\in I}$ is a $\boldsymbol{\tau}$-cover of objects in $\mathcal{A}$. By the second assumption, we can assume it is finite. Consider the Čech nerve $\mathcal{U}_*$ of the epimorphism of stacks $\coprod_{i\in I} h(U_i) \to h(X)$. By our assumptions, $\mathcal{U}_m = h((U)_m)$ where $(U)_*$ is the Čech nerve of the morphism $\coprod_{i\in I} U_i \to X$ in $\mathcal{A}$. Further, $(U)_m = \mathrm{Spec}(B_m)$ for some $B_m \in \mathcal{A}$. Let B_* be the associated simplicial object.

By an extension of [86, Theorem B.0.7], we can identify $\varprojlim_{[n]\in\Delta} \mathbf{Mod}_{B_n}$ with the category $\mathbf{csMod}_{B_*,cart}$ of cosimplicial cartesian B_*-modules, i.e. cosimplicial B_*-modules M_* such that, for each $[n] \to [m]$ in Δ, $M_n \otimes^{\mathbb{L}}_{B_n} B_m \to M_m$ is an equivalence. By descent for quasi-coherent sheaves with respect to $\boldsymbol{\tau}$-covers, there is therefore an equivalence of categories $B_* \otimes^{\mathbb{L}}_A - : \mathbf{Mod}_A \to \mathbf{csMod}_{B_*,cart}$. Hence, if we have a co-cartesian morphism $Y_* = \mathrm{Spec}(C_*) \to (U)_*$ in the sense of Definition 3.2.3.8, then C_* is a cosimplicial cartesian B_*-module. Hence, $C_* \simeq B_* \otimes^{\mathbb{L}}_A C$, where $C := \varprojlim_{[n]\in\Delta} C_n$. Therefore, $U_* \times_X Y \simeq Y_*$ as required. □

Remark. In practice, many of our representability contexts will satisfy that $\mathcal{A}$ is closed under $\boldsymbol{\tau}$-descent relative to $\mathbf{DAff}^{cn}(\mathcal{C})$, by a similar proof to the above, but not necessarily with respect to $\mathcal{A}$. For example, the category $\mathbf{DSt}^{op}$ of derived Steins is not closed under geometric realisations and so the conditions of the above proof will not hold.

5.2.5 Infinitesimal Criteria for Formally Perfect Morphisms

Suppose that $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, \mathcal{C}^0)$ is a flat Postnikov compatible derived algebraic context. We will denote by $\mathbf{fP}$ the class of formally perfect morphisms in $\mathbf{DAff}^{cn}(\mathcal{C})$. Suppose that $\mathbf{P} \subseteq \mathbf{fP}$ is a subclass of formally perfect morphisms satisfying the following conditions:

1. The class $\mathbf{P}$ is stable under equivalences, compositions, and pullbacks,
2. If a morphism $f : Y = \mathrm{Spec}(B) \to \mathrm{Spec}(A) = X$ in $\mathcal{A}$ is formally perfect and $t_0(f) \in \mathbf{P}^{\heartsuit}$, then f is in $\mathbf{P}$.

Suppose that there exists a topology $\boldsymbol{\tau}$ such that $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, \mathcal{C}^0, \boldsymbol{\tau}, \mathbf{P}^{\boldsymbol{\tau}}, \mathcal{A}, \mathbf{M})$ is a flat Postnikov compatible derived geometry context. Suppose that $\boldsymbol{\tau}$ and $\mathbf{P}^{\boldsymbol{\tau}}$ satisfy the obstruction conditions for an appropriate class $\mathbf{S}$ of morphisms and that $\iota|_{\mathcal{A}^{\heartsuit}} : (\mathcal{A}^{\heartsuit}, \boldsymbol{\tau}^{\heartsuit}) \to (\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$ is a continuous functor of $(\infty, 1)$-sites.

Lemma 5.2.5.1. *Suppose that $\mathbf{Dls}|_{\mathcal{A}^{op}}$ satisfies descent for $\boldsymbol{\tau}$-covers. Suppose that $f : \mathcal{F} \to Y$ is an n-$\mathbf{P}^{\boldsymbol{\tau}}|_{\mathcal{A}}$-morphism from an n-geometric stack in $\mathbf{Stk}_n(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}}, \mathbf{P}^{\boldsymbol{\tau}}|_{\mathcal{A}})$ to a representable stack $Y = \mathrm{Spec}(B) \in \mathcal{A}$. Then, for any $x : X = \mathrm{Spec}(A) \to \mathcal{F}$ with $A \in \mathcal{A}^{op}$, we have that $\mathbb{L}_{\mathcal{F}/Y,x}$ is strongly dualisable and its dual $\mathbb{L}^{\vee}_{\mathcal{F}/Y,x}$ is 0-connective.*

Proof. We proceed by induction on n. Indeed, when $n = -1$ and $\mathcal{F}$ is representable, say by $Z = \mathrm{Spec}(D) \in \mathcal{A}$, then f is in $\mathbf{P}^{\boldsymbol{\tau}}$. Hence, there exists a $\boldsymbol{\tau}$-cover of Z, $\{U_i = \mathrm{Spec}(C_i) \to Z\}_{i \in I}$, such that the induced morphism $U_i \to Y$ is in $\mathbf{P}$. By considering the fibre sequence

$$\mathbb{L}_{Z/Y} \otimes^{\mathbb{L}}_D C_i \to \mathbb{L}_{U_i/Y} \to \mathbb{L}_{U_i/Z}$$

we see that, since $\mathbb{L}_{U_i/Z}$ is perfect and $\mathbb{L}_{U_i/Y}$ is perfect, then $\mathbb{L}_{Z/Y} \otimes^{\mathbb{L}}_D C_i$ is strongly dualisable by Lemma B.0.0.2. Since $\mathbf{Dls}|_{\mathcal{A}^{op}}$ satisfies descent for $\boldsymbol{\tau}$-covers, we see that $\mathbb{L}_{Z/Y}$ is a strongly dualisable 0-connective D-module. Moreover, $\mathbb{L}^{\vee}_{Z/Y}$ is 0-connective.

Now, when $n > -1$, since $\mathbf{Dls}|_{\mathcal{A}^{op}}$ satisfies descent for $\boldsymbol{\tau}$-covers and $\mathcal{F}$ has a global cotangent complex by Theorem 4.3.4.1, we see that the properties we need to show are local for the $\boldsymbol{\tau}$-topology. We can therefore assume, since $\mathcal{F}$ is n-geometric, that the point x lifts to a point of an n-atlas for $\mathcal{F}$ by Proposition 3.2.3.1. So, suppose that there is a representable stack $U \in \mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$, and an $(n-1)$-$\mathbf{P}^{\boldsymbol{\tau}}|_{\mathcal{A}}$-morphism $U \to \mathcal{F}$ such that $x \in \pi_0(\mathcal{F}(A))$ is the image of a point $u \in \pi_0(U(A))$. We consider the induced fibre sequence of A-modules

$$\mathbb{L}_{\mathcal{F}/Y,x} \to \mathbb{L}_{U/Y,u} \to \mathbb{L}_{U/\mathcal{F},u}$$

Up to replacing U by an appropriate cover, we can assume that the induced morphism $U \to Y$ is in $\mathbf{P}|_{\mathcal{A}}$. Hence, $\mathbb{L}_{U/Y,u}$ is perfect and its dual is 0-connective. Since the morphism $U \to \mathcal{F}$ is in $(n-1)$-$\mathbf{P}^{\boldsymbol{\tau}}|_{\mathcal{A}}$, by our inductive hypothesis $\mathbb{L}_{U/\mathcal{F},u}$ is strongly dualisable and its dual is 0-connective. We can then easily see that $\mathbb{L}_{\mathcal{F}/Y,x}$ is strongly dualisable and its dual is 0-connective using Lemma B.0.0.2 and the induced long exact sequence of homotopy groups. $\square$

Proposition 5.2.5.2. *Suppose that $\mathbf{Dls}|_{\mathcal{A}^{op}}$ satisfies descent for $\boldsymbol{\tau}$-covers. Suppose that $f : \mathcal{F} \to \mathcal{G}$ is an n-representable$|_{\mathcal{A}}$ morphism of stacks in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$ with $t_0(f) : t_0(\mathcal{F}) \to t_0(\mathcal{G})$ in n-$\mathbf{P}^{\boldsymbol{\tau},\heartsuit}$. Then the following statements are equivalent*

1. *f is in n-$\mathbf{P}^{\boldsymbol{\tau}}|_{\mathcal{A}}$,*
2. *For any $x : X = \mathrm{Spec}(A) \to \mathcal{F}$, $\mathbb{L}_{\mathcal{F}/\mathcal{G},x}$ is strongly dualisable and $\mathbb{L}^{\vee}_{\mathcal{F},\mathcal{G},x}$ is 0-connective,*
3. *For any $x : X = \mathrm{Spec}(A) \to \mathcal{F}$ and $M \in \mathbf{M}_{A,1}$, $\pi_0(\mathrm{Map}_{\mathbf{Mod}_A}(\mathbb{L}_{\mathcal{F}/\mathcal{G},x}, M)) = 0$.*

Proof. To show that (1) implies (2), suppose that f is in n-$\mathbf{P}^{\boldsymbol{\tau}}|_{\mathcal{A}}$ and that there is a morphism $x : X = \mathrm{Spec}(A) \to \mathcal{F}$ with $X \in \mathcal{A}$. Then, we note that $\mathcal{F} \times_{\mathcal{G}} X$ is n-geometric and $\mathcal{F} \times_{\mathcal{G}} X \to X$ is in n-$\mathbf{P}^{\boldsymbol{\tau}}|_{\mathcal{A}}$. Moreover, by Lemma 4.2.5.7, we have that $\mathbb{L}_{\mathcal{F}/\mathcal{G},x} \simeq \mathbb{L}_{\mathcal{F}\times_{\mathcal{G}}X/X,x'}$, and hence we just need to show that (2) holds for any n-$\mathbf{P}^{\boldsymbol{\tau}}|_{\mathcal{A}}$-morphism from an n-geometric stack to a representable stack on $\mathcal{A}$. This is the content of Lemma 5.2.5.1.

To show that (2) implies (3), since $\mathbb{L}_{\mathcal{F}/\mathcal{G},x}$ is strongly dualisable, we have that

$$\pi_0(\mathrm{Map}_{\mathbf{Mod}_A}(\mathbb{L}_{\mathcal{F}/\mathcal{G},x}, M)) \simeq \pi_0(\mathrm{Map}_{\mathbf{Mod}_A}(A, \mathbb{L}^{\vee}_{\mathcal{F}/\mathcal{G},x} \otimes^{\mathbb{L}}_A M))$$

Since $M \in \mathbf{M}_{A,1}$, it follows that $\mathbb{L}^{\vee}_{\mathcal{F}/\mathcal{G},x} \otimes^{\mathbb{L}}_A M$ is 1-connective. Hence, the left hand side must be zero by Lemma 4.2.3.6.

Finally, to prove that (3) implies (1), we suppose that $f : \mathcal{F} \to \mathcal{G}$ is an n-representable$|_{\mathcal{A}}$ morphism of stacks with $t_0(f) : t_0(\mathcal{F}) \to t_0(\mathcal{G})$ in n-$\mathbf{P}^{\boldsymbol{\tau},\heartsuit}$. Since f is n-representable we can just suppose, using familiar reasoning and Lemma 4.2.5.7, that $\mathcal{F}$ is n-geometric and that $\mathcal{G}$ is a representable stack X on $\mathcal{A}$. It also suffices to show that, whenever we have a representable stack Y along with an $(n-1)$-$\mathbf{P}^{\boldsymbol{\tau}}|_{\mathcal{A}}$ morphism $Y \to \mathcal{F}$, the composite morphism $Y \to X$ is in $\mathbf{P}^{\boldsymbol{\tau}}$.

We note that $t_0(Y) \to t_0(\mathcal{F})$ is in $(n-1)$-$\mathbf{P}^{\boldsymbol{\tau},\heartsuit}$ by Proposition 5.1.4.6, and hence the composite $t_0(Y) \to t_0(X)$ is in $\mathbf{P}^{\boldsymbol{\tau},\heartsuit}$. Therefore, we see that there exists a $\boldsymbol{\tau}^{\heartsuit}$-cover $\{U_i \to t_0(Y)\}_{i\in I}$ such that the induced morphisms $U_i \to t_0(X)$ are in $\mathbf{P}^{\heartsuit}$.

By definition, the $\boldsymbol{\tau}^{\heartsuit}$-cover is the truncation of a $\boldsymbol{\tau}$-cover $\{U_i' = \mathrm{Spec}(C_i) \to Y\}_{i\in I}$. We need to show that the induced morphisms $U_i' \to X$ are in $\mathbf{P}$. Since the morphism $U_i \to t_0(X)$ is in $\mathbf{P}^{\heartsuit}$, we see that $\mathbb{L}_{\pi_0(C_i)/\pi_0(A)}$ is a perfect $\pi_0(C_i)$-module and there is a retract

$$\varinjlim_{j\in J} \coprod_{E_j} \pi_0(C_i) \to \mathbb{L}_{\pi_0(C_i)/\pi_0(A)}$$

for finite sets J and E_j. By Corollary 4.2.2.6 and Lemma 4.2.3.6, this lifts to a map $\varinjlim_{j\in J}\coprod_{E_j} C_i \to \mathbb{L}_{C_i/A}$. Let K be the fibre of this morphism. If we consider the fibre sequence

$$\mathrm{Map}_{\mathbf{Mod}_{C_i}}\left(\mathbb{L}_{C_i/A}, \varinjlim_{j\in J}\coprod_{E_j} C_i\right) \to \mathrm{Map}_{\mathbf{Mod}_{C_i}}(\mathbb{L}_{C_i/A}, \mathbb{L}_{C_i/A}) \to \mathrm{Map}_{\mathbf{Mod}_{C_i}}(\mathbb{L}_{C_i/A}, K[1])$$

and the induced long exact sequence in homotopy then, by our assumption, since $K[1] \in \mathbf{M}_{C_i,1}$, there is a surjection

$$\pi_0\left(\mathrm{Map}_{\mathbf{Mod}_{C_i}}\left(\mathbb{L}_{C_i/A}, \varinjlim_{j\in J}\coprod_{E_j} C_i\right)\right) \to \pi_0(\mathrm{Map}_{\mathbf{Mod}_{C_i}}(\mathbb{L}_{C_i/A}, \mathbb{L}_{C_i/A}))$$

and hence we see that $\mathbb{L}_{C_i/A}$ is perfect, being a retract of $\varinjlim_{j\in J}\coprod_{E_j} C_i$. Therefore, the morphism $U_i' \to X$ is formally perfect and its truncation lies in $\mathbf{P}^{\heartsuit}$. By our assumptions on $\mathbf{P}$, it must therefore lie in $\mathbf{P}$.

□

5.2.6 The Finite Homotopy Monomorphism Topology

Suppose that $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, \mathcal{C}^0)$ is a flat Postnikov compatible derived algebraic context.

Definition 5.2.6.1. Suppose that $f : A \to B$ is a morphism in $\mathbf{DAlg}^{cn}(\mathcal{C})$. Then, f is a *homotopy epimorphism* if the map $B \otimes_A^{\mathbb{L}} B \to B$ is an equivalence. The corresponding morphism in $\mathbf{DAff}^{cn}(\mathcal{C})$ is called a *homotopy monomorphism.*

Lemma 5.2.6.2. *[10, c.f. Lemma 2.1.42] If $f : A \to B$ in $\mathbf{DAlg}^{cn}(\mathcal{C})$ is a homotopy epimorphism, then f is formally étale, and hence formally perfect.*

Definition 5.2.6.3. The *finite homotopy monomorphism topology* on $\mathrm{Ho}(\mathbf{DAff}^{cn}(\mathcal{C}))$ has finite covers $\{U_j \to X\}_{j \in J}$ such that

1. Each morphism $U_j \to X$ is a homotopy monomorphism,
2. The family $\{U_j \to X\}_{j \in J}$ is a formal covering family.

It is easy to see that this defines a Grothendieck topology. We will denote the topology by $\boldsymbol{hm}^{fin}$.

Lemma 5.2.6.4. **QCoh** *satisfies (Čech-)descent for* $\boldsymbol{hm}^{fin}$*-covers.*

Proof. Suppose that we have a $\boldsymbol{hm}^{fin}$-cover $\{U_j = \mathrm{Spec}(B_j) \to \mathrm{Spec}(A) = X\}_{j \in J}$. Let $B = \prod_{j \in J} B_j$ and consider the cobar resolution $\mathbf{CB}^*(B)$ defined in each degree by $\mathbf{CB}^n(B) = B^{\otimes^{\mathbb{L}}_A n+1}$. We note that, since each morphism $A \to B_j$ is a homotopy epimorphism, the limit of this resolution is equivalent to B as a B-module. Hence, by conservativity of our cover, we see that $A \to \mathbf{CB}^*(B)$ is a limit diagram. Therefore, by [60, Proposition 3.20], $A \to B$ is descendable in the sense of [60, Definition 3.18].

Consider the Čech nerve $\mathcal{U}_*$ of the morphism $\coprod_{j \in J} h(U_j) \to h(X)$. We note that $h(\mathrm{Spec}(\mathbf{CB}^n(B)))$ is equivalent to $\mathcal{U}_n$. Therefore, by [60, Proposition 3.22], there is an equivalence of categories

$$\mathbf{QCoh}(X) = \mathbf{Mod}_A \to \varprojlim_n \mathbf{Mod}_{B^{\otimes^{\mathbb{L}}_A n+1}} \simeq \varprojlim_{n \in \Delta} \mathbf{Mod}_{\mathbf{CB}^n(B)} = \varprojlim_{n \in \Delta} \mathbf{QCoh}(\mathcal{U}_n)$$

and therefore we have descent in the sense of Definition 3.1.3.3. $\square$

Lemma 5.2.6.5. *[10, Proposition 2.6.165] Suppose that $f : A \to B$ is a morphism in* $\mathbf{DAlg}^{cn}(\mathcal{C})$*.*

1. *If f is a homotopy epimorphism, then $\pi_0(f) : \pi_0(A) \to \pi_0(B)$ is an epimorphism,*
2. *If f is a derived strong morphism such that $\pi_0(f) : \pi_0(A) \to \pi_0(B)$ is an epimorphism, then f is a homotopy epimorphism,*
3. *If f is a homotopy epimorphism such that $\pi_0(f)$ is a homotopy epimorphism, and each $\pi_n(A)$ is transverse to $\pi_0(B)$ over $\pi_0(A)$, then f is derived strong.*

Suppose that there exists a category $\mathcal{A}$ such that $(\mathbf{DAff}^{cn}(\mathcal{C}), \boldsymbol{hm}^{fin}, \mathbf{fP}^{\boldsymbol{hm}^{fin}}, \mathcal{A})$ is a relative $(\infty, 1)$-geometry tuple.

Lemma 5.2.6.6. *For any finite collection $\{U_i\}_{i \in I}$ of $\mathcal{A}$-admissible $U_i \in \mathbf{DAff}^{cn}(\mathcal{C})$, the map $\coprod_{i \in I} h(U_i) \to h(\coprod_{i \in I} U_i)$ is an equivalence in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{hm}^{fin}|_{\mathcal{A}})$.*

Proof. We note that the morphism $\prod_{j\in I} A_j \to A_i$ is a homotopy epimorphism by [10, Proposition 2.5.134] and Lemma 5.2.6.5. Moreover, it is clear that the family $\{U_i \to \coprod_{j\in I} U_j\}_{i\in I}$ is a formal covering family. Hence, we can conclude by Corollary 3.2.3.7. $\square$

In this situation, we obtain the following representability context.

Corollary 5.2.6.7. *The tuple* $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, \mathcal{C}^0, \boldsymbol{hm}^{fin}, \mathbf{fP}^{\boldsymbol{hm}^{fin}}, \mathcal{A}, \mathbf{M}, \mathbf{S})$ *is a representability context in the sense of Definition 5.1.3.1 if*

$$(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, \mathcal{C}^0, \boldsymbol{hm}^{fin}, \mathbf{fP}^{\boldsymbol{hm}^{fin}}, \mathcal{A}, \mathbf{M})$$

is a flat Postnikov compatible derived geometry context, such that

1. $(\mathcal{A}, \boldsymbol{hm}^{fin}|_{\mathcal{A}}, \mathbf{fP}^{\boldsymbol{hm}^{fin}}|_{\mathcal{A}}, \mathcal{A}^{\heartsuit})$ *is a strong relative* $(\infty,1)$*-geometry tuple,*
2. $\iota|_{\mathcal{A}^{\heartsuit}} : (\mathcal{A}^{\heartsuit}, \boldsymbol{\tau}^{\heartsuit}) \to (\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$ *is a continuous functor of* $(\infty,1)$*-sites,*
3. *A morphism is in* $\mathbf{fP}^{\boldsymbol{hm}^{fin}}|_{\mathcal{A}}$ *if it is derived strong relative to* $\mathbf{fP}^{\boldsymbol{hm}^{fin}}|_{\mathcal{A}}$*,*
4. $\boldsymbol{hm}^{fin}$ *and* $\mathbf{fP}^{\boldsymbol{hm}^{fin}}$ *satisfy the obstruction conditions of Definition 4.3.3.2 relative to* $\mathcal{A}$ *for* $\mathbf{S}$*.*

Proof. We note that Conditions (1), (2), (4), and (6) are satisfied by assumption. Condition (3) is satisfied by Lemma 5.2.6.6. Condition (5) follows from the proof of Proposition 5.2.5.2 and the statement that **Dls** satisfies descent for $\boldsymbol{hm}^{fin}$-covers by Lemma 5.2.4.1 and Lemma 5.2.6.4.

$\square$

Remark. We note that if covers in $\boldsymbol{hm}^{fin}|_{\mathcal{A}}$ consist of derived strong morphisms, then Condition (3) is satisfied using Lemma 5.2.6.5.

5.2.7 Derived Complex Analytic Geometry

We will not give a full introduction to complex analytic geometry, a good reference is [31]. The basic algebraic building blocks of complex analytic geometry are Stein algebras. These are objects in the image of the functor

$$\mathcal{O} : \text{Stein Spaces} \to \text{Comm}(\text{Fr}_{\mathbb{C}})^{op}$$

taking a Stein space X, for example $X = \mathbb{C}^n$, to the algebra of holomorphic functions on X.

We will say that a Stein algebra A is *finitely presented* if it is of the form $\mathcal{O}(\mathbb{C}^n)/I$ for some $n \geq 0$ and some finitely generated ideal I. We note that we can write A

as a *Fréchet-Stein algebra* in the sense of [10, Definition 5.2.52] in the following way. Write $\mathcal{O}(\mathbb{C}^n)$ as a limit $\varprojlim_k \mathcal{O}(D_k)$ where $\mathcal{O}(D_k)$ is the Banach space of holomorphic functions on the disk D_k of radius k in $\mathbb{C}^n$. Then, we can define I_k to be the closure of the ideal generated by the image of I in $\mathcal{O}(D_k)$. We then easily see that $A = \varprojlim_k A_k$ where $A_k = \mathcal{O}(D_k)/I_k$.

Definition 5.2.7.1. [10, c.f. Definition 5.2.56] A module M over a finitely presented Stein algebra A is *coadmissible* if it can be written as a limit $\varprojlim_k M_k$ such that

1. Each M_k is finitely generated over A_k,
2. The natural morphism $A_k \hat{\otimes}_{A_{k+1}} M_{k+1} \to M_k$ is an equivalence.

Remark. We think of coadmissible modules as being the analogues for complex analytic geometry of coherent modules in algebraic geometry.

Consider the derived algebraic context

$$(\mathbf{Ch}(\mathrm{Ind}(\mathrm{Ban}_{\mathbb{C}})), \mathbf{Ch}_{\geq 0}(\mathrm{Ind}(\mathrm{Ban}_{\mathbb{C}})), \mathbf{Ch}_{\leq 0}(\mathrm{Ind}(\mathrm{Ban}_{\mathbb{C}})), \mathbf{L}^H(P^0))$$

We remark that the following definition of a derived Stein is due to Kelly and we refer to [10] for more details.

Definition 5.2.7.2. $A \in \mathbf{DAlg}^{cn}(\mathbf{Ch}(\mathrm{Ind}(\mathrm{Ban}_{\mathbb{C}})))$ is a *derived Stein algebra over* $\mathbb{C}$ if

1. $\pi_0(A)$ is a finitely presented Stein algebra,
2. $\pi_n(A)$ is a coadmissible $\pi_0(A)$-module.

Remark. It is shown in [10, Section 9.2.1.1] that the complex analytic theory of Porta and Yue Yu, as described in [63], built up from their notion of derived Stein algebras, embeds fully faithfully into our theory.

Let $\mathbf{DSt}$ denote the full subcategory of $\mathbf{DAlg}^{cn}(\mathbf{Ch}(\mathrm{Ind}(\mathrm{Ban}_{\mathbb{C}})))$ consisting of derived Stein algebras, and St the ordinary category of Stein algebras. We note that $\mathbf{DSt}$ is closed under finite products by [10, Proposition 5.4.115] and [31, Theorem V.1.1]. We recall that $\mathbf{fP}^{\boldsymbol{hm}^{fin}}$ denotes the expansion of our class of formally perfect maps to those which are local for the $\boldsymbol{hm}^{fin}$-topology. We note that covers of Steins are often not flat, but we can impose a transversality condition, as described in the following definition.

Definition 5.2.7.3. We make the following definitions.

- The class $\mathbf{fP}^{\boldsymbol{hm}}_{\mathbf{DSt}}$ is the subclass of maps $f : Y = \mathrm{Spec}(B) \to \mathrm{Spec}(A) = X$ in $\mathbf{fP}^{\boldsymbol{hm}^{fin}}$ such that

1. Whenever $A \to C$ is a map with $C \in \mathrm{St}$, then $B \otimes_A^{\mathbb{L}} C$ is in St,
2. Any coadmissible $\pi_0(A)$-module M is transverse to $\pi_0(B)$.

- The *finite Stein homotopy monomorphism topology*, which we will denote by $\boldsymbol{hm}_{\mathbf{DSt}}^{fin}$, consists of covers $\{U_j = \mathrm{Spec}(B_j) \to \mathrm{Spec}(A) = X\}_{j \in J}$ in $\boldsymbol{hm}^{fin}$ such that

1. Whenever there is some map $A \to C$, then $C \in \mathrm{St}$ if and only if $B_j \otimes_A^{\mathbb{L}} C$ is in St for each $j \in J$,
2. Any coadmissible $\pi_0(A)$-module M is transverse to $\pi_0(B_j)$.

Remark. We note that an epimorphism $A \to B$ of Stein algebras such that $\pi_n(A)$ is transverse to $\pi_0(B)$ as a $\pi_0(A)$-module is equivalent to a homotopy epimorphism of Stein algebras. This follows easily using the Tor spectral sequence from Proposition 5.1.2.4.

It is easy to see that $\mathbf{fP}_{\mathbf{DSt}}^{\boldsymbol{hm}}$ is closed under equivalences, compositions, and pullbacks. We can also see that the finite Stein homotopy monomorphism topology defines a Grothendieck pre-topology on $\mathrm{Ho}(\mathbf{DAff}^{cn}(\mathbf{Ch}(\mathrm{Ind}(\mathrm{Ban}_{\mathbb{C}}))))$. Using Lemma 5.2.6.4, we see that **QCoh** satisfies descent for $\boldsymbol{hm}_{\mathbf{DSt}}^{fin}$-covers. It follows by Lemma 5.2.4.1, that **Dls** satisfies descent for $\boldsymbol{hm}_{\mathbf{DSt}}^{fin}$-covers.

Lemma 5.2.7.4. *Morphisms in* $\mathbf{fP}_{\mathbf{DSt}}^{\boldsymbol{hm}}$ *are derived strong.*

Proof. This is a consequence of Lemma 5.2.1.4. Indeed, if $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a formally perfect morphism or a homotopy monomorphism with each $\pi_n(A)$ transverse to $\pi_0(B)$ as a $\pi_0(A)$-module, then the morphism $A \to B$ is derived strong by Corollary 5.2.3.4 and Lemma 5.2.6.5. $\square$

Lemma 5.2.7.5. *Suppose that we have a morphism* $Y = \mathrm{Spec}(B) \to \mathrm{Spec}(A) = X$ *in* $\mathbf{fP}_{\mathbf{DSt}}^{\boldsymbol{hm}}$ *and a map* $A \to C$ *with* $C \in \mathbf{DSt}$. *Then,* $B \otimes_A^{\mathbb{L}} C$ *is in* $\mathbf{DSt}$.

Proof. We note that, since $C \in \mathbf{DSt}$, $B \otimes_A^{\mathbb{L}} \pi_0(C) \simeq \pi_0(B \otimes_A^{\mathbb{L}} C)$ is a Stein algebra. The map $C \to B \otimes_A^{\mathbb{L}} C$ is derived strong by Lemma 5.2.7.4 since morphisms in $\mathbf{fP}_{\mathbf{DSt}}^{\boldsymbol{hm}}$ are closed under pullback. Therefore,

$$\pi_i(C) \otimes_{\pi_0(C)}^{\mathbb{L}} \pi_0(B \otimes_A^{\mathbb{L}} C) \simeq \pi_i(B \otimes_A^{\mathbb{L}} C)$$

and, since $\pi_i(C)$ is coadmissible as a $\pi_0(C)$-module, we see that $\pi_i(B \otimes_A^{\mathbb{L}} C)$ is coadmissible as a $\pi_0(B \otimes_A^{\mathbb{L}} C)$-module. $\square$

Lemma 5.2.7.6. *Suppose that* $\{U_j = \mathrm{Spec}(B_j) \to \mathrm{Spec}(A) = X\}_{j\in J}$ *is a* $\boldsymbol{hm}^{fin}_{\mathbf{DSt}}$*-cover and we have some map* $A \to C$*. Then, if* $B_j \otimes^{\mathbb{L}}_A C \in \mathbf{DSt}$ *for each* j*, then* $C \in \mathbf{DSt}$*.*

Proof. It is clear that $\pi_0(C) \in \mathrm{St}$ by our conditions on our topology. Since we have descent for coadmissible modules on Steins by [10, Theorem 9.2.14], then we can conclude that $\pi_n(C)$ is a coadmissible $\pi_0(C)$-module. □

Lemma 5.2.7.7. *A morphism* $f : Y = \mathrm{Spec}(B) \to \mathrm{Spec}(A) = X$ *is in* $\mathbf{fP}^{\boldsymbol{hm}}_{\mathbf{DSt}}|_{\mathbf{DSt}^{op}}$ *if it is derived strong and* $t_0(f)$ *is in* $\mathbf{fP}^{\boldsymbol{hm},\heartsuit}_{\mathrm{St}}$*.*

Proof. Suppose that there is a morphism $Z = \mathrm{Spec}(C) \to X$ with $Z \in \mathrm{St}^{op}$. Then, we note that, since $t_0(f) \in \mathbf{fP}^{\boldsymbol{hm},\heartsuit}_{\mathrm{St}}$, $\pi_0(B \otimes^{\mathbb{L}}_A C) \simeq \pi_0(B) \otimes^{\mathbb{L}}_{\pi_0(A)} C \in \mathrm{St}$. Since f is derived strong, $\pi_n(B \otimes^{\mathbb{L}}_A C) \simeq \pi_0(B) \otimes^{\mathbb{L}}_{\pi_0(A)} \pi_n(C) \simeq 0$ for $n > 0$, by Proposition 5.1.2.9. Now, we note that, since covers in $\boldsymbol{hm}^{fin}$ are derived strong by Lemma 5.2.6.5, we can conclude that the morphism $f : Y \to X$ is in $\mathbf{fP}^{\boldsymbol{hm}^{fin}}$ using Corollary 5.2.3.4. The condition on transversality of coadmissible modules follows from the statement that $t_0(f)$ is in $\mathbf{fP}^{\boldsymbol{hm},\heartsuit}_{\mathrm{St}}$ and that $X, Y \in \mathbf{DSt}^{op}$. □

Lemma 5.2.7.8. *Every object of* $\mathbf{DAff}^{cn}(\mathbf{Ch}(\mathrm{Ind}(\mathrm{Ban}_{\mathbb{C}})))$ *is* $\mathbf{DSt}^{op}$*-admissible with respect to the finite Stein homotopy monomorphism topology.*

Proof. We adapt the proof of [86, Lemma 2.2.2.13] to our setting. Suppose that we have a $\boldsymbol{hm}^{fin}_{\mathbf{DSt}}$-hypercover $\mathcal{U}_* \to h(X)$ in $\mathbf{PSh}(\mathbf{DSt}^{op})$. We note that each $\mathcal{U}_n$ is equivalent to a coproduct $\coprod_{i_n \in I_n} h(U_{i_n})$ of representable stacks such that we have a corresponding $\boldsymbol{hm}^{fin}_{\mathbf{DSt}}$-cover $\{U_{i_n} \to (\mathrm{cosk}_{n-1}\mathcal{U}_*)_n\}_{i_n \in I_n}$. Let $U_{i_n} = \mathrm{Spec}(B_{i_n})$ and $X = \mathrm{Spec}(A)$ for some $A, B_{i_n} \in \mathbf{DSt}$. Let $U_n = \coprod_{i_n \in I_n} U_{i_n}$ and consider the augmented simplicial object $\mathrm{Spec}(B_*) = U_* \to X$.

We will show that $A \simeq \varprojlim_n B_n$ as then $X \simeq |U_*|$. It then follows easily that any $Y \in \mathbf{DAff}^{cn}(\mathbf{Ch}(\mathrm{Ind}(\mathrm{Ban}_{\mathbb{C}})))$ satisfies descent for $\boldsymbol{hm}^{fin}_{\mathbf{DSt}}$-hypercovers as required. Indeed, consider the spectral sequence

$$\pi_p(\mathrm{Tot}(\pi_q(B_*))) \Rightarrow \pi_{q-p}\left(\varprojlim_n \prod_{i_n \in I_n} B_{i_n}\right)$$

We note that, by definition of our topology, each $\pi_q(B_{i_n})$ is transverse to $\pi_0(B_{i_n})$ over $\pi_0(A)$. Using the definition of derived Stein algebras, we therefore see that the $\pi_q(B_{i_n})$ glue together to define an object of $\mathrm{Coad}(t_0(U_*))$. Since we have hyperdescent for coadmissible sheaves on St^{op} with respect to $\boldsymbol{hm}^{fin,\heartsuit}$ by [10, Theorem 9.2.14], we see that $\pi_p(\mathrm{Tot}(\pi_q(B_*))) \simeq 0$ for $p \neq 0$. Therefore, the spectral sequence degenerates and we have that

$$\pi_p\left(\varprojlim_n \prod_{i_n \in I_n} B_{i_n}\right) \simeq \mathrm{Ker}\left(\pi_p\left(\prod_{i_0 \in I_0} B_{i_0}\right) \to \pi_p\left(\prod_{i_1 \in I_1} B_{i_1}\right)\right)$$

This implies that $\pi_p\left(\varprojlim_n \prod_{i_n \in I_n} B_{i_n}\right)$ is the $\pi_0(A)$-module obtained by descent from $\pi_p(B_{i_*})$. Therefore, it follows that the natural morphism

$$\pi_p\left(\varprojlim_n \prod_{i_n \in I_n} B_{i_n}\right) \otimes^{\mathbb{L}}_{\pi_0(A)} \pi_0(B_{i_0}) \to \pi_p(B_{i_0})$$

is an isomorphism for all $p > 0$ and $i_0 \in I_0$. We note that the morphism $A \to B_{i_0}$ is derived strong by Lemma 5.2.7.4. Hence, we can use Lemma 5.1.2.10 to show that there is an equivalence

$$B_{i_0} \to \left(\varinjlim_n \prod_{i_n \in I_n} B_{i_n}\right) \otimes^{\mathbb{L}}_A B_{i_0}$$

Since B_{i_0} was arbitrary, and the family $\{A \to B_{i_0}\}_{i_0 \in I_0}$ corresponds to a formal covering family, we see that $|B_*| \simeq \varprojlim_n \prod_{i \in I_n} B_{i_n} \simeq A$ as required. □

Corollary 5.2.7.9. *Suppose that $f : \mathcal{F} \to \mathcal{G}$ is an n-representable$|_{\mathbf{DSt}^{op}}$ morphism of stacks in $\mathbf{Stk}(\mathbf{DSt}^{op}, \boldsymbol{hm}^{fin}_{\mathbf{DSt}})$. Then, f is in n-$\mathbf{fP}^{\boldsymbol{hm}}_{\mathbf{DSt}}|_{\mathbf{DSt}^{op}}$ if f satisfies that, for any $x : X = \mathrm{Spec}(A) \to \mathcal{F}$ and any $M \in \mathbf{M}_{A,1}$,*

$$\pi_0(\mathrm{Map}_{\mathbf{Mod}_A}(\mathbb{L}_{\mathcal{F}/\mathcal{G},x}, M)) = 0$$

The converse holds if $t_0(\mathcal{F}) \to t_0(\mathcal{G})$ is in n-$\mathbf{fP}^{\boldsymbol{hm},\heartsuit}_{\mathrm{St}}$.

Proof. Indeed, we note that, by Lemma 5.2.7.8 and Lemma 5.2.4.1, $\mathbf{Dls}$ satisfies descent for $\boldsymbol{hm}^{fin}_{\mathbf{DSt}}$-covers. Therefore, the forwards direction follows from Proposition 5.2.5.2 since $\mathbf{fP}^{hm}_{\mathbf{DSt}} \subseteq \mathbf{fP}^{\boldsymbol{hm}^{fin}_{\mathbf{DSt}}}$. To prove the converse, we note that, by the proof of Proposition 5.2.5.2, it suffices to show that if we have a morphism $f : Y \to X$ in $\mathbf{DSt}^{op}$ which is in $\mathbf{fP}^{\boldsymbol{hm}^{fin}_{\mathbf{DSt}}}$ and whose truncation lies in $\mathbf{fP}^{\boldsymbol{hm},\heartsuit}_{\mathrm{St}}$, then f lies in $\mathbf{fP}^{\boldsymbol{hm}}_{\mathbf{DSt}}$. Condition (1) of Definition 5.2.7.3 follows by definition of $\mathbf{hm}^{fin}_{\mathbf{DSt}}$ and Condition (2) follows since $t_0(f)$ is in $\mathbf{fP}^{\boldsymbol{hm},\heartsuit}_{\mathrm{St}}$. □

Corollary 5.2.7.10. $(\mathbf{DAff}^{cn}(\mathbf{Ch}(\mathrm{Ind}(\mathrm{Ban}_{\mathbb{C}}))), \boldsymbol{hm}^{fin}_{\mathbf{DSt}}, \mathbf{fP}^{hm}_{\mathbf{DSt}}, \mathbf{DSt})$ *is a strong relative $(\infty, 1)$-geometry tuple.*

Proof. We note that, by Lemmas 5.2.7.5 and 5.2.7.6, together with descent for transversality, we can show that it is a relative $(\infty, 1)$-pre-geometry tuple. Further, by Lemma 3.2.1.4 it is strong. By Lemma 5.2.7.8, every object of $\mathbf{DAff}^{cn}(\mathbf{Ch}(\mathrm{Ind}(\mathrm{Ban}_{\mathbb{C}})))$ is $\mathbf{DSt}^{op}$-admissible. □

5.2.8 A Representability Context for Derived Complex Analytic Geometry

We now want to check that the geometry constructed in the previous section defines a representability context.

Lemma 5.2.8.1. $(\mathbf{DSt}^{op}, \boldsymbol{hm}^{fin}_{\mathbf{DSt}}|_{\mathbf{DSt}^{op}}, \mathbf{fP}^{\boldsymbol{hm}}_{\mathbf{DSt}}|_{\mathbf{DSt}^{op}}, \mathrm{St}^{op})$ *is a strong relative* $(\infty,1)$*-geometry tuple.*

Proof. We easily see, using Corollary 5.2.7.10 and our definition of $\mathbf{fP}^{\boldsymbol{hm}}_{\mathbf{DSt}}$, that the tuple is a relative $(\infty,1)$-geometry tuple. Furthermore, every object of $\mathbf{DSt}^{op}$ is St^{op}-admissible since Stein algebras satisfy descent for the $\boldsymbol{hm}^{fin,\heartsuit}_{\mathrm{St}}$-topology. It is strong by our conditions on morphisms in $\mathbf{fP}^{\boldsymbol{hm}}_{\mathbf{DSt}}$. $\square$

Lemma 5.2.8.2. $\iota|_{\mathrm{St}^{op}} : (\mathrm{St}^{op}, \boldsymbol{hm}^{fin,\heartsuit}_{\mathrm{St}}) \to (\mathbf{DSt}^{op}, \boldsymbol{hm}^{fin}_{\mathbf{DSt}})$ *is a continuous functor of* $(\infty,1)$*-sites.*

Proof. Suppose that $\{t_0(U_j) = \mathrm{Spec}(\pi_0(B_j)) \to \mathrm{Spec}(\pi_0(A)) = t_0(X)\}_{j\in J}$ is a cover in $\boldsymbol{hm}^{fin,\heartsuit}_{\mathrm{St}}$ corresponding to the truncation of a cover $\{U_j \to X\}_{j\in J}$ in $\boldsymbol{hm}^{fin}_{\mathbf{DSt}}$. Then, we note that $\pi_0(A) \to \pi_0(B_j)$ is an epimorphism of Stein algebras such that $\pi_n(A)$ is transverse to $\pi_0(B_j)$ as a $\pi_0(A)$-module. Hence, it is a homotopy epimorphism. Moreover, we know that $U_j \to X$ is strong, and hence we can show the remaining condition on $\boldsymbol{hm}^{fin}_{\mathbf{DSt}}$ using Lemma 5.1.2.10. We know that the truncation of a formal covering family is a formal covering family by Lemma 5.2.1.2. $\square$

Lemma 5.2.8.3. *For any finite collection* $\{U_i\}_{i\in I}$ *of* $U_i \in \mathbf{DAff}^{cn}(\mathbf{Ch}(\mathrm{Ind}(\mathrm{Ban}_{\mathbb{C}})))$*, the map* $\coprod_{i\in I} h(U_i) \to h(\coprod_{i\in I} U_i)$ *is an equivalence in* $\mathbf{Stk}(\mathbf{DSt}^{op}, \boldsymbol{hm}^{fin}_{\mathbf{DSt}})$*,*

Proof. By Lemma 5.2.6.6, we note that the family $\{U_i \to \coprod_{j\in I} U_j\}_{i\in I}$ is a $\boldsymbol{hm}^{fin}$-covering family. Moreover, we can easily check that each morphism $U_i \to \coprod_{j\in I} U_j$ lies in $\mathbf{fP}^{\boldsymbol{hm}}_{\mathbf{DSt}}$. Hence, we can conclude by Corollary 3.2.3.7. $\square$

Suppose that $A \in \mathbf{DSt}$. Let $\mathbf{Mod}^{coad,cn}_A$ denote the subcategory of connective A-modules M such that $\pi_n(M)$ is coadmissible over $\pi_0(A)$ for all $n \geq 0$. Consider the system $\mathbf{Mod}^{coad,cn}$ defined by these categories.

Lemma 5.2.8.4. $\mathbf{Mod}^{coad,cn}$ *is a good system of categories of modules on* $\mathbf{DSt}^{op}$ *in the sense of Definition 4.2.4.1.*

Proof. Indeed, we note that Conditions (1), (3), and (7) follow easily from our definition of derived Stein algebras. We note that Condition (2) follows using [10, Lemma 5.4.121], (4) follows by [10, Theorem 5.4.118] and (8) follows by [10, Proposition 5.4.116]. Condition (5) follows by Lemma 5.2.7.8. To show that Condition (6) holds, suppose that $A \in \mathbf{DSt}^{\leq k}$ for some k, $M \in \mathbf{Mod}_A^{coad,cn,\heartsuit}$, and $d \in \pi_0(\mathbf{Der}(A, M))$, and consider the following fibre sequence

$$A \oplus_d \Omega M[k+1] \to A \to M[k+1]$$

from Lemma 4.2.2.2. Then, for $n < k$ we note that, by considering the long exact sequence in homotopy, $\pi_n(A \oplus_d \Omega M[k+1]) \simeq \pi_n(A)$, which is a finitely presented Stein algebra for $n = 0$ and is coadmissible as a $\pi_0(A)$-module for $0 < n < k$. When $n = k$, we see that we have an exact sequence

$$0 \to M \to \pi_k(A \oplus_d \Omega M[k+1]) \to \pi_k(A) \to 0$$

and it follows that $\pi_k(A \oplus_d \Omega M[k+1])$ is a coadmissible $\pi_0(A)$-module since the category of coadmissible $\pi_0(A)$-modules is abelian by [10, Proposition 5.2.58]. For $n > k$, $\pi_n(A \oplus_d \Omega M[k+1]) \simeq 0$. $\square$

Lemma 5.2.8.5. *$\boldsymbol{hm}^{fin}_{\mathbf{DSt}}$ and $\mathbf{fP}^{\boldsymbol{hm}}_{\mathbf{DSt}}$ satisfy the obstruction conditions relative to $\mathbf{DSt}^{op}$ for the class* $\mathbf{hm}$ *of homotopy monomorphisms.*

Proof. Indeed, we note that morphisms in $\mathbf{fP}^{\boldsymbol{hm}}_{\mathbf{DSt}}$ are formally étale by Lemma 5.2.1.3, and hence are formally i-smooth. Therefore, Condition (1) is satisfied. Condition (2) is also clearly satisfied. To show that Condition (3) is satisfied, we suppose that we have a $\boldsymbol{hm}^{fin}_{\mathbf{DSt}}$-covering family $\{V_j = \mathrm{Spec}(A_j) \to \mathrm{Spec}(A) = X\}_{j \in J}$ and some $M \in \mathbf{Mod}_{A,1}^{coad,cn}$. Now, define $W_j = \mathrm{Spec}(B_j) = \mathrm{Spec}(A_j \oplus_{d'_j} \Omega M'_j) \in \mathbf{DSt}^{op}$ with d'_j the derivation induced by Lemma 4.3.3.1 and $M'_j = M \otimes^{\mathbb{L}}_A A_j$. This can trivially be refined to a $\boldsymbol{hm}^{fin}$-covering family. The collection $\{W_j \to X_d[\Omega M]\}_{j \in J}$ is a formal covering family by Lemma 5.2.1.5. By Lemma 4.3.3.3, and using that the morphism $V_j \to X$ is a homotopy monomorphism, we have an equivalence

$$B_j \otimes^{\mathbb{L}}_{A \oplus_d \Omega M} A \simeq A_j \simeq A_j \otimes^{\mathbb{L}}_A A_j \simeq (B_j \otimes^{\mathbb{L}}_{A \oplus_d \Omega M} B_j) \otimes^{\mathbb{L}}_{A \oplus_d \Omega M} A$$

Hence, by Corollary 4.2.2.3, we have an equivalence $B_j \simeq B_j \otimes^{\mathbb{L}}_{A \oplus_d \Omega M} B_j$. Therefore, $\{W_j \to X_d[\Omega M]\}_{j \in J}$ is also a $\boldsymbol{hm}^{fin}$-covering family.

By Lemma 5.2.6.4, since we have $\mathbf{QCoh}$-descent for $\boldsymbol{hm}^{fin}$-covers and since W_j and $X_d[\Omega M]$ are $\mathbf{DSt}^{op}$-admissible, we see that this defines an epimorphism of stacks $\coprod_{j \in J} W_j \to X_d[\Omega M]$ in $\mathbf{Stk}(\mathbf{DSt}^{op}, \boldsymbol{hm}^{fin}_{\mathbf{DSt}})$. $\square$

Denote by $\underline{\mathbf{Ch}}(\mathrm{Ind}(\mathrm{Ban}_{\mathbb{C}}))$ the derived algebraic context

$$(\mathbf{Ch}(\mathrm{Ind}(\mathrm{Ban}_{\mathbb{C}})), \mathbf{Ch}_{\geq 0}(\mathrm{Ind}(\mathrm{Ban}_{\mathbb{C}})), \mathbf{Ch}_{\leq 0}(\mathrm{Ind}(\mathrm{Ban}_{\mathbb{C}})), \mathbf{L}^{H}(P^{0}))$$

Corollary 5.2.8.6. *The tuple* $(\underline{\mathbf{Ch}}(\mathrm{Ind}(\mathrm{Ban}_{\mathbb{C}})), \boldsymbol{hm}^{fin}_{\mathbf{DSt}}, \mathbf{fP}^{\boldsymbol{hm}}_{\mathbf{DSt}}, \mathbf{DSt}^{op}, \mathbf{Mod}^{coad,cn})$ *is a flat Postnikov compatible derived geometry context.*

Proof. Indeed, we know that it is a derived geometry context by Corollary 5.2.7.10 and Lemma 5.2.8.4. To see that it is Postnikov compatible, we note that the first two conditions follow easily from standard results. We note that Assumption (3) follows from the definition of derived Stein algebras and, since we are working over $\mathbb{C}$, Assumption (4) is satisfied using a similar proof to [58, Proposition 25.2.4.1]. □

Corollary 5.2.8.7. *The tuple*

$$(\underline{\mathbf{Ch}}(\mathrm{Ind}(\mathrm{Ban}_{\mathbb{C}})), \boldsymbol{hm}^{fin}_{\mathbf{DSt}}, \mathbf{fP}^{\boldsymbol{hm}}_{\mathbf{DSt}}, \mathbf{DSt}^{op}, \mathbf{Mod}^{coad,cn}, \mathbf{hm})$$

is a representability context in the sense of Definition 5.1.3.1

Proof. By Corollary 5.2.8.6, our context is a flat Postnikov compatible derived geometry context. Condition (1) is satisfied by Lemma 5.2.8.1. Condition (2) follows by Lemma 5.2.8.2. Condition (3) follows by Lemma 5.2.8.3. Condition (4) follows by Lemma 5.2.7.7 and Condition (5) by Corollary 5.2.7.9. Finally, Condition (6) follows by Lemma 5.2.8.5.

□

5.3 Representability of Mapping Stacks

Many moduli stacks naturally appear as mapping stacks when families of geometric objects over some base object naturally correspond to maps from the base into a suitable *classifying stack*. For example, the moduli stack of principal G-bundles on some geometric object X (e.g. a scheme, or a stack), where G is some group object, can be expressed as the mapping stack from X to $\mathbf{BG}$.

In this section, we will apply our representability theorem to find conditions under which mapping stacks in our representability contexts are representable.

5.3.1 The Mapping Stack

Suppose that we have a representability context $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, \mathcal{C}^0, \boldsymbol{\tau}, \mathbf{P}, \mathcal{A}, \mathbf{M}, \mathbf{S})$. We note that the category of presheaves $\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))$ is locally cartesian closed, and hence, for any $X \in \mathcal{A}$, the slice category $\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))_{/X}$ is cartesian closed. Suppose that $\mathcal{F}$ and $\mathcal{G}$ are in $\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))_{/X}$. By the Yoneda Lemma, the presheaf $\underline{\mathrm{Map}}_{\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))_{/X}}(\mathcal{F}, \mathcal{G})$ acts by

$$\underline{\mathrm{Map}}_{\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))_{/X}}(\mathcal{F}, \mathcal{G})(B) = \mathrm{Map}_{\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))_{/X}}(\mathcal{F} \times_X Y, \mathcal{G})$$

where $Y = \mathrm{Spec}(B)$.

If $\mathcal{F}$ and $\mathcal{G}$ are stacks in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})_{/X}$, then we obtain similar results and the mapping stack $\underline{\mathrm{Map}}_{\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})_{/X}}(\mathcal{F}, \mathcal{G})$ is obtained by stackifying $\underline{\mathrm{Map}}_{\mathbf{PSh}(\mathcal{A})_{/X}}(\mathcal{F}, \mathcal{G})$.

Theorem 5.3.1.1. *Suppose that $X = \mathrm{Spec}(A) \in \mathcal{A}^{\heartsuit}$. Suppose that $\mathcal{F}$ and $\mathcal{G}$ are stacks in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})_{/X}$. Under the following conditions, $\underline{\mathrm{Map}}_{\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})_{/X}}(\mathcal{F}, \mathcal{G})$ is an n-geometric stack in $\mathbf{Stk}_n(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}}, \mathbf{P}|_{\mathcal{A}})$,*

1. *The truncation $t_0(\underline{\mathrm{Map}}_{\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})_{/X}}(\mathcal{F}, \mathcal{G}))$ is in $\mathbf{Stk}_n(\mathcal{A}^{\heartsuit}, \boldsymbol{\tau}^{\heartsuit}, \mathbf{P}^{\heartsuit})$,*
2. *The stack $\underline{\mathrm{Map}}_{\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})_{/X}}(\mathcal{F}, \mathcal{G})$ has a global cotangent complex relative to $\mathcal{A}$,*
3. *$\mathcal{G}$ is n-geometric,*
4. *The stack $\mathcal{F}$ can be written as a colimit of representable stacks $\varinjlim_i U_i$ where $U_i = \mathrm{Spec}(C_i) \in \mathcal{A}^{\heartsuit}$ and each morphism $U_i \to X$ is flat and in $\mathbf{P}$.*

Proof. We just need to check that the conditions of Theorem 5.1.6.1 hold. Indeed, Condition (a) immediately holds. Condition (b) is satisfied if we can show that the mapping stack is infinitesimally cartesian relative to $\mathcal{A}$. Indeed, since $\mathcal{F}$ can be written as a colimit $\varinjlim_i U_i$ where $U_i = \mathrm{Spec}(C_i) \in \mathcal{A}^{\heartsuit}$ with C_i flat as an A-module, we see that

$$\underline{\mathrm{Map}}_{\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})_{/X}}(\mathcal{F}, \mathcal{G}) \simeq \varprojlim_i \underline{\mathrm{Map}}_{\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})_{/X}}(U_i, \mathcal{G})$$

Therefore, since the property of being infinitesimally cartesian is stable by limits, we can assume that $\mathcal{F}$ is representable by some $U = \mathrm{Spec}(C) \in \mathcal{A}^{\heartsuit}$. Suppose that $Y = \mathrm{Spec}(B)$ is in $\mathcal{A}$, $M \in \mathbf{M}_{B,1}$, and that we have a derivation $d \in \pi_0(\mathbf{Der}(B, M))$ corresponding to a morphism $d : B \to B \oplus M$. Then, using the Yoneda Lemma, we note that the commutative square

$$\begin{array}{ccc}
\underline{\mathrm{Map}}_{\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})_{/X}}(U, \mathcal{G})(B \oplus_d \Omega M) & \longrightarrow & \underline{\mathrm{Map}}_{\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})_{/X}}(U, \mathcal{G})(B) \\
\downarrow & & \downarrow \\
\underline{\mathrm{Map}}_{\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})_{/X}}(U, \mathcal{G})(B) & \longrightarrow & \underline{\mathrm{Map}}_{\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})_{/X}}(U, \mathcal{G})(B \oplus M)
\end{array}$$

is equivalent to the commutative square

$$\begin{array}{ccc}
\mathcal{G}((B \oplus_d \Omega M) \otimes^{\mathbb{L}}_A C) & \longrightarrow & \mathcal{G}(B \otimes^{\mathbb{L}}_A C) \\
\big\downarrow & & \big\downarrow \\
\mathcal{G}(B \otimes^{\mathbb{L}}_A C) & \longrightarrow & \mathcal{G}((B \oplus M) \otimes^{\mathbb{L}}_A C)
\end{array}$$

Using the flatness of C, we can see that $(B \oplus_d \Omega M) \otimes^{\mathbb{L}}_A C \simeq (B \otimes^{\mathbb{L}}_A C) \oplus_{d'} \Omega M'$ where $M' = M \otimes^{\mathbb{L}}_A C$ and d' is the induced derivation. Now, since $\mathcal{G}$ is n-geometric, it is infinitesimally cartesian relative to $\mathcal{A}$ by Theorem 4.3.4.2. Therefore, since $B \otimes^{\mathbb{L}}_A C$ is in $\mathcal{A}^{op}$ since each morphism $A \to C$ is in $\mathbf{P}$, we see that the above square is a pullback square, and hence $\underline{\mathrm{Map}}_{\mathbf{Stk}(\mathcal{A},\boldsymbol{\tau}|_{\mathcal{A}})_{/X}}(\mathcal{F}, \mathcal{G})$ is infinitesimally cartesian.

Finally, we need to show that $\underline{\mathrm{Map}}_{\mathbf{Stk}(\mathcal{A},\boldsymbol{\tau}|_{\mathcal{A}})_{/X}}(\mathcal{F}, \mathcal{G})$ is nilcomplete. Once again it suffices, using our fourth condition, to prove the statement when $\mathcal{F}$ is a representable stack $U = \mathrm{Spec}(C) \in \mathcal{A}^{\heartsuit}$ such that the morphism $U \to X$ is flat and in $\mathbf{P}$. Indeed, suppose that $Y = \mathrm{Spec}(B) \in \mathcal{A}$. Then,

$$\underline{\mathrm{Map}}_{\mathbf{Stk}(\mathcal{A},\boldsymbol{\tau}|_{\mathcal{A}})_{/X}}(U, \mathcal{G})(B) \simeq \mathcal{G}(B \otimes^{\mathbb{L}}_A C)$$

By Lemma 5.1.2.1, we see that $(B \otimes^{\mathbb{L}}_A C)_{\leq k} \simeq B_{\leq k} \otimes^{\mathbb{L}}_A C$. Therefore, our result follows since $\mathcal{G}$ is n-geometric, and hence nilcomplete. $\square$

5.3.2 Perfect Quasi-coherent Sheaves and Base-Change

Recall that we have a functor $\mathbf{QCoh} : \mathbf{DAff}^{cn}(\mathcal{C})^{op} \to \mathbf{Pr}^{\mathbb{L},\otimes}$. We left Kan extend this functor to a functor defined on $\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))^{op}$. For any $\mathcal{F} \in \mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))$, we denote by $\mathcal{O}_{\mathcal{F}}$ the monoidal unit in $\mathbf{QCoh}(\mathcal{F})$. We define $\mathbf{Perf}(\mathcal{F})$ to be the subcategory of perfect objects in $\mathbf{QCoh}(\mathcal{F})$ in the sense of Appendix B, i.e. retracts of finite colimits of $\coprod_E \mathcal{O}_{\mathcal{F}}$ for some finite set E.

Suppose that we have a morphism $f : \mathcal{F} \to \mathcal{G}$ of presheaves in $\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))$. Then, we obtain a natural functor $f^* : \mathbf{QCoh}(\mathcal{G}) \to \mathbf{QCoh}(\mathcal{F})$ which has a right adjoint given by right Kan extension $f_* : \mathbf{QCoh}(\mathcal{F}) \to \mathbf{QCoh}(\mathcal{G})$.

We note that objects in $\mathbf{Perf}(\mathcal{F})$ are strongly dualisable in $\mathbf{QCoh}(\mathcal{F})$. If we have a morphism $f : \mathcal{F} \to \mathcal{G}$ of presheaves in $\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))$, then since f^* is symmetric monoidal and preserves colimits and retracts, $f^*\mathbf{Perf}(\mathcal{G}) \subseteq \mathbf{Perf}(\mathcal{F})$. If we have a morphism of representables $f : Y = \mathrm{Spec}(B) \to \mathrm{Spec}(A) = X$ in $\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))$, then we note that the morphism f_* corresponds to the morphism $f_* : \mathbf{Mod}_B \to \mathbf{Mod}_A$ which is exact. Hence, in particular, f_* preserves finite colimits and retracts, and therefore $f_*\mathbf{Perf}(Y) \subseteq \mathbf{Perf}(X)$.

Definition 5.3.2.1. Suppose that we have a morphism $f : \mathcal{F} \to \mathcal{G}$ of presheaves in $\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))$. Then, we say that f *satisfies the perfect projection formula* if, for any $\mathcal{M} \in \mathbf{Perf}(\mathcal{F})$ and $\mathcal{N} \in \mathbf{QCoh}(\mathcal{G})$, there is an equivalence

$$f_*(\mathcal{M} \otimes_{\mathbf{QCoh}(\mathcal{F})} f^*(\mathcal{N})) \simeq f_*(\mathcal{M}) \otimes_{\mathbf{QCoh}(\mathcal{G})} \mathcal{N}$$

Remark. If $f : Y \to X$ is a morphism of representables, then f satisfies the perfect projection formula.

Suppose that $\mathcal{F} \in \mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))$. For any $\mathcal{M} \in \mathbf{QCoh}(\mathcal{F})$, we consider the dual object $\mathcal{M}^\vee := \underline{\mathrm{Map}}_{\mathbf{QCoh}(\mathcal{F})}(\mathcal{M}, \mathcal{O}_{\mathcal{F}})$ and denote the associated functor by $(-)^\vee_{\mathcal{F}} : \mathbf{QCoh}(\mathcal{F}) \to \mathbf{QCoh}(\mathcal{F})$. Suppose that we have a map $f : \mathcal{F} \to \mathcal{G}$ in $\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))$. Then, we define the plus pushforward $f_+ : \mathbf{QCoh}(\mathcal{F}) \to \mathbf{QCoh}(\mathcal{G})$ by

$$f_+ := (-)^\vee_{\mathcal{G}} \circ f_* \circ (-)^\vee_{\mathcal{F}}$$

Lemma 5.3.2.2. *Suppose that $f : \mathcal{F} \to \mathcal{G}$ is a map such that $f_*\mathbf{Perf}(\mathcal{F}) \subseteq \mathbf{Perf}(\mathcal{G})$ and which satisfies the perfect projection formula. Then, for any $\mathcal{M} \in \mathbf{Perf}(\mathcal{F})$ and $\mathcal{N} \in \mathbf{QCoh}(\mathcal{G})$, there is an equivalence, natural in $\mathcal{M}$ and $\mathcal{N}$,*

$$\mathrm{Map}_{\mathbf{QCoh}(\mathcal{G})}(f_+(\mathcal{M}), \mathcal{N}) \simeq \mathrm{Map}_{\mathbf{QCoh}(\mathcal{F})}(\mathcal{M}, f^*(\mathcal{N}))$$

Proof. Using the perfect projection formula and our conditions, the result easily follows from the following chain of equivalences.

$$\begin{aligned} f_*(\underline{\mathrm{Map}}_{\mathbf{QCoh}(\mathcal{F})}(\mathcal{M}, f^*(\mathcal{N}))) &\simeq f_*(\mathcal{M}^\vee \otimes_{\mathbf{QCoh}(\mathcal{F})} f^*(\mathcal{N})) \\ &\simeq f_*(\mathcal{M}^\vee) \otimes_{\mathbf{QCoh}(\mathcal{G})} \mathcal{N} \\ &\simeq \underline{\mathrm{Map}}_{\mathbf{QCoh}(\mathcal{G})}(f_*(\mathcal{M}^\vee)^\vee, \mathcal{N}) \\ &\simeq \underline{\mathrm{Map}}_{\mathbf{QCoh}(\mathcal{G})}(f_+(\mathcal{M}), \mathcal{N}) \end{aligned}$$

□

Definition 5.3.2.3. Suppose that $f : \mathcal{F} \to \mathcal{G}$ and $g : \mathcal{H} \to \mathcal{G}$ are morphisms of presheaves in $\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))$. Then, (f, g) *satisfies perfect base-change* if, given the pullback diagram

$$\begin{tikzcd} \mathcal{F} \times_{\mathcal{G}} \mathcal{H} \arrow[r, "g'"] \arrow[d, "f'"] & \mathcal{F} \arrow[d, "f"] \\ \mathcal{H} \arrow[r, "g"] & \mathcal{G} \end{tikzcd}$$

the natural map $g^* f_* \mathcal{M} \to f'_*(g')^* \mathcal{M}$ is an equivalence whenever $\mathcal{M} \in \mathbf{Perf}(\mathcal{F})$.

Proposition 5.3.2.4. *Suppose that we have a pullback diagram*

$$\begin{tikzcd} \mathcal{F} \times_{\mathcal{G}} \mathcal{H} \arrow[r, "g'"] \arrow[d, "f'"] & \mathcal{F} \arrow[d, "f"] \\ \mathcal{H} \arrow[r, "g"] & \mathcal{G} \end{tikzcd}$$

Suppose that (f, g) satisfies perfect base change and that $f_*\mathbf{Perf}(\mathcal{F}) \subseteq \mathbf{Perf}(\mathcal{G})$. Suppose that $\mathcal{M} \in \mathbf{Perf}(\mathcal{F})$. Then, there is a natural equivalence

$$(f')_+(g')^*\mathcal{M} \to g^* f_+\mathcal{M}$$

Proof. Using our assumptions, we have the following chain of equivalences.

$$\begin{aligned} (f')_+(g')^*\mathcal{M} &\simeq \underline{\mathrm{Map}}_{\mathbf{QCoh}(\mathcal{H})}((f')_* \circ (-)^\vee_{\mathcal{F}'} \circ (g')^*\mathcal{M}, \mathcal{O}_{\mathcal{H}}) \\ &\simeq \underline{\mathrm{Map}}_{\mathbf{QCoh}(\mathcal{H})}((f')_*(g')^*\mathcal{M}^\vee, \mathcal{O}_{\mathcal{H}}) \\ &\simeq \underline{\mathrm{Map}}_{\mathbf{QCoh}(\mathcal{H})}(g^* f_*\mathcal{M}^\vee, \mathcal{O}_{\mathcal{H}}) \\ &\simeq \underline{\mathrm{Map}}_{\mathbf{QCoh}(\mathcal{H})}(g^* \circ (-)^\vee_{\mathcal{G}} \circ f_+\mathcal{M}, \mathcal{O}_{\mathcal{H}}) \\ &\simeq g^*\underline{\mathrm{Map}}_{\mathbf{QCoh}(\mathcal{G})}((-)^\vee_{\mathcal{G}} \circ f_+\mathcal{M}, \mathcal{O}_{\mathcal{G}}) \\ &\simeq g^* f_+\mathcal{M} \end{aligned}$$

□

5.3.3 The Cotangent Complex of the Mapping Presheaf

We note that, if a morphism $f : \mathcal{F} \to \mathcal{G}$ of presheaves in $\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))$ has a global cotangent complex $\mathbb{L}_{\mathcal{F}/\mathcal{G},u}$ for every point $u : U \to \mathcal{F}$, then this determines an object $\mathbb{L}_{\mathcal{F}/\mathcal{G}} \in \mathbf{QCoh}(\mathcal{F})$ such that $u^*(\mathbb{L}_{\mathcal{F}/\mathcal{G}}) \simeq \mathbb{L}_{\mathcal{F}/\mathcal{G},u}$.

Suppose that $X = \mathrm{Spec}(A) \in \mathcal{A}^\heartsuit$ and that $\mathcal{F}$ and $\mathcal{G}$ are in $\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))_{/X}$. Suppose that we have a point $u : U = \mathrm{Spec}(C) \to \underline{\mathrm{Map}}_{\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))_{/X}}(\mathcal{F}, \mathcal{G})$ and consider the following pullback diagram in $\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))$.

$$\begin{tikzcd} \mathcal{F} \times_X U \arrow[r, "u'"] \arrow[d, "\pi_u"] & \mathcal{F} \times_X \underline{\mathrm{Map}}_{\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))_{/X}}(\mathcal{F}, \mathcal{G}) \arrow[d, "\pi"] \\ U \arrow[r, "u"] & \underline{\mathrm{Map}}_{\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))_{/X}}(\mathcal{F}, \mathcal{G}) \end{tikzcd} \tag{5.4}$$

Proposition 5.3.3.1. *Suppose that the following conditions are satisfied.*

1. *The morphism $\mathcal{G} \to X$ has a global cotangent complex and this is an object in $\mathbf{Perf}(\mathcal{G})$,*

2. *The morphism $\mathcal{F} \to X$ is (-1)-representable,*
3. *(π, u) satisfies perfect base change for any morphism u and π_* preserves perfect objects.*

Then, the morphism $\underline{\mathrm{Map}}_{\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))_{/X}}(\mathcal{F}, \mathcal{G}) \to X$ has a global cotangent complex given by

$$\mathbb{L}_{\underline{\mathrm{Map}}_{\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))_{/X}}(\mathcal{F},\mathcal{G})/X} = \pi_+ \circ ev^*(\mathbb{L}_{\mathcal{G}/X})$$

where ev is the evaluation morphism $ev : \mathcal{F} \times_X \underline{\mathrm{Map}}_{\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))_{/X}}(\mathcal{F}, \mathcal{G}) \to \mathcal{G}$

Proof. Denote by f_u the following composition of maps defining $\mathcal{F} \times_X U$ as a point of $\mathcal{G}$,

$$\mathcal{F} \times_X U \xrightarrow{u'} \mathcal{F} \times_X \underline{\mathrm{Map}}_{\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))_{/X}}(\mathcal{F}, \mathcal{G}) \xrightarrow{ev} \mathcal{G}$$

By our assumptions, $\mathcal{F} \times_X U$ is representable. Suppose that $M \in \mathbf{M}_C^{cn}$. In the following, we will abbreviate $\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))_{/X}$ to $\mathbf{PSh}_{/X}$. Then,

$$\begin{aligned}
\mathbf{Der}_{\underline{\mathrm{Map}}_{\mathbf{PSh}_{/X}}(\mathcal{F},\mathcal{G})}(U, M) &:= \mathrm{Map}_{U/\mathbf{PSh}_{/X}}(U[M], \underline{\mathrm{Map}}_{\mathbf{PSh}_{/X}}(\mathcal{F}, \mathcal{G})) \\
&\simeq \mathrm{Map}_{\mathcal{F}\times_X U/\mathbf{PSh}_{/X}}(\mathcal{F} \times_X U[M], \mathcal{G}) \\
&\simeq \mathrm{Map}_{\mathcal{F}\times_X U/\mathbf{PSh}_{/X}}((\mathcal{F} \times_X U)[(\pi_u)^*(M)], \mathcal{G}) \\
&\simeq \mathrm{Map}_{\mathbf{QCoh}(\mathcal{F}\times_X U)}(f_u^*(\mathbb{L}_{\mathcal{G}/X}), (\pi_u)^*(M))
\end{aligned}$$

Now, since π_u satisfies the perfect projection formula, $(\pi_u)_*$ preserves perfect objects, and $f_u^*(\mathbb{L}_{\mathcal{G}/X}) \in \mathbf{Perf}(\mathcal{F} \times_X U)$, we can apply Lemma 5.3.2.2,

$$\begin{aligned}
&\simeq \mathrm{Map}_{\mathbf{QCoh}(U)}((\pi_u)_+ \circ f_u^*(\mathbb{L}_{\mathcal{G}/X}), M) \\
&\simeq \mathrm{Map}_{\mathbf{QCoh}(U)}((\pi_u)_+ \circ (u')^* \circ ev^*(\mathbb{L}_{\mathcal{G}/X}), M)
\end{aligned}$$

Now, since (π, u) satisfies perfect base-change and π_* preserves perfect objects then, since $ev^*(\mathbb{L}_{\mathcal{G}/X})$ is perfect, we can apply Proposition 5.3.2.4 to conclude that

$$\simeq \mathrm{Map}_{\mathbf{QCoh}(U)}(u^* \circ \pi_+ \circ ev^*(\mathbb{L}_{\mathcal{G}/X}), M)$$

By its definition, we can easily see that $\mathbb{L}_{\underline{\mathrm{Map}}_{\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))_{/X}}(\mathcal{F},\mathcal{G})/X}$ is global. □

Remark. We can then define $\mathbb{L}_{\underline{\mathrm{Map}}_{\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))_{/X}}(\mathcal{F},\mathcal{G})}$ to be the fibre of the morphism $\mathbb{L}_{\underline{\mathrm{Map}}_{\mathbf{PSh}(\mathbf{DAff}^{cn}(\mathcal{C}))_{/X}}(\mathcal{F},\mathcal{G})/X} \to \mathbb{L}_X[1]$.

Corollary 5.3.3.2. *Suppose that $X = \mathrm{Spec}(A) \in \mathcal{A}^{\heartsuit}$ and that $\mathcal{G}$ is in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})_{/X}$. Under the following conditions, $\underline{\mathrm{Map}}_{\mathbf{Stk}(\mathcal{A},\boldsymbol{\tau}|_{\mathcal{A}})_{/X}}(X, \mathcal{G})$ is an n-geometric stack in $\mathbf{Stk}_n(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}}, \mathbf{P}|_{\mathcal{A}})$.*

1. *The truncation* $t_0(\underline{\mathrm{Map}}_{\mathbf{Stk}(\mathcal{A},\boldsymbol{\tau}|_{\mathcal{A}})/X}(X,\mathcal{G}))$ *is in* $\mathbf{Stk}_n(\mathcal{A}^\heartsuit,\boldsymbol{\tau}^\heartsuit,\mathbf{P}^\heartsuit)$,
2. $\mathcal{G}$ *is* n*-geometric and the cotangent complex of the morphism* $\mathcal{G}\to X$ *is in* $\mathbf{Perf}(\mathcal{G})$.

Proof. This follows by Theorem 5.3.1.1 using Proposition 5.3.3.1. $\square$

Chapter 6

$\mathcal{C}^\infty$-Bornological Rings

Suppose that X is a finite-dimensional manifold. Then, the $\mathbb{R}$-algebra $\mathcal{C}^\infty(X)$ of smooth functions $X \to \mathbb{R}$ has the structure of a $\mathcal{C}^\infty$*-ring*. A $\mathcal{C}^\infty$-ring is an algebra A over $\mathbb{R}$ such that every smooth map $\mathbb{R}^n \to \mathbb{R}^m$ lifts to a corresponding map $A^n \to A^m$ on the algebra. In this chapter, we present an extension of these ideas to the situation where we are working with infinite dimensional manifolds. This leads us to define the notion of a $\mathcal{C}^\infty$*-bornological ring*. This new collection of objects contains $\mathcal{C}^\infty$-rings. We can easily extend these ideas to consider derived versions of these objects.

These derived $\mathcal{C}^\infty$-bornological rings give us a new perspective on current problems in differential geometry. In particular, under suitable conditions, we can easily apply the representability theorem to show that the derived moduli stack of solutions to non-linear elliptic PDEs is representable by a derived $\mathcal{C}^\infty$-bornological affine scheme.

We remark that there are several other approaches to derived smooth geometry, but they all start with some kind of notion of a derived $\mathcal{C}^\infty$-ring. There is also a theory of derived $\mathcal{C}^\infty$-superalgebras due to Carchedi [21] which also allows us to work with infinite dimensional manifolds. One can often obtain good theories of derived manifolds by restricting to finitely presented derived $\mathcal{C}^\infty$-rings. One approach by Carchedi and Steffens appears in [22], extending earlier work by Spivak [81].

6.1 $\mathcal{C}^\infty$-Bornological Rings

6.1.1 Convenient Manifolds

One approach to defining infinite dimensional manifolds comes from manifolds modelled on the notion of a *convenient space*. We remark that the category of smooth manifolds is not closed monoidal but the category of convenient manifolds is. Our main reference for this section is the book *The Convenient Setting of Global Analysis* by Kriegl and Michor [46].

Definition 6.1.1.1. [46, Section 1.2] Suppose that X is a locally convex topological vector space and that $c : \mathbb{R} \to X$ is a curve.

1. c is *differentiable* if the derivative $c'(t) := \lim_{s \to 0} \frac{1}{s}(c(t+s) - c(t))$ at t exists for all t,
2. c is *smooth* if all iterated derivatives exist,
3. The c^∞*-topology* on a locally convex space X is the final topology with respect to all smooth curves $\mathbb{R} \to X$. Its open sets will be called c^∞*-open*.

Recall that any locally convex topological vector space can be endowed with the von Neumann bornology as described in Example 1.2.3.3.

Lemma 6.1.1.2. *[46, Corollary 2.11] A linear map $f : X \to Y$ between locally convex topological vector spaces is bounded if and only if it maps smooth curves in X to smooth curves in Y.*

In the following, for X a locally convex topological vector space, X^* denotes the locally convex space of continuous linear functionals on X.

Definition 6.1.1.3. Suppose that X is a locally convex topological vector space. Then, X is *convenient* if it satisfies the equivalent conditions of [46, Theorem 2.14]. In particular, X is convenient if, whenever $c : \mathbb{R} \to X$ is a curve such that $f \circ c : \mathbb{R} \to \mathbb{R}$ is smooth for all $f \in X^*$, then c is smooth.

We denote by Con the category of convenient spaces equipped with the von Neumann bornology with morphisms being bounded linear maps. We note that the category Con is a full subcategory of $\text{CBorn}_{\mathbb{R}}$. The category Con is a cartesian closed category. As described in [16], the category Con can be described as the 'fixed points' of the adjunction between separated bornological spaces of convex type and locally convex topological vector spaces.

Definition 6.1.1.4. [46, Section 3.11] Suppose that we have a morphism $f : X \supset U \to Y$ defined on a c^∞-open subset U of X where X and Y are convenient spaces. Then, f is called *smooth* if it maps smooth curves in U to smooth curves in Y.

Definition 6.1.1.5. [46, Section 27.1] Suppose that we have a set M.

1. A *chart* (U, u) on M is a bijection $u \to u(U) \subseteq X_U$ from a subset $U \subseteq M$ onto a c^∞-open subset of a convenient vector space X_U,

2. M is a *convenient manifold* if it has an atlas consisting of charts of this form,
3. A mapping $f : M \to N$ between convenient manifolds is *smooth* if, for each $x \in M$ and each chart (V, v) on N with $f(x) \in V$, there is a chart (U, u) on M with $x \in U$ and $f(U) \subseteq V$, satisfying that $v \circ f \circ u^{-1}$ is smooth.

Example 6.1.1.6. *We can give any convenient space V the structure of a convenient manifold by defining the atlas to be $\{(V, \mathrm{id}_V)\}$. In particular, $\mathbb{R}$ can be considered as a convenient manifold.*

Definition 6.1.1.7. Given a convenient manifold M, we define $\mathcal{C}^\infty(M)$ to be the space $\mathcal{C}^\infty(M) := \mathrm{Hom}_{\mathrm{Smooth}}(M, \mathbb{R})$.

We denote the category of convenient manifolds equipped with smooth mappings by ConMfd. This category is cartesian closed. By [46, Section 27.17], $\mathcal{C}^\infty(M)$ is a convenient space. Hence, we see that there is a functor $\mathcal{C}^\infty : \mathrm{ConMfd} \to \mathrm{Con}^{op}$.

Lemma 6.1.1.8. *$\mathcal{C}^\infty(M)$ has a convenient algebra structure with multiplication determined by the multiplication map $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$.*

Proof. We first note that $\mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \simeq \mathrm{Hom}_{\mathrm{Smooth}}(M, \mathbb{R} \times \mathbb{R})$. Moreover, using the exponential law given in [46, Section 27.17], we see that

$$\mathrm{Hom}_{\mathrm{Smooth}}(\mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M), \mathcal{C}^\infty(M)) \simeq \mathrm{Hom}_{\mathrm{Smooth}}(\mathrm{Hom}_{\mathrm{Smooth}}(M, \mathbb{R} \times \mathbb{R}) \times M, \mathbb{R})$$

Composing the evaluation map $\mathrm{Hom}_{\mathrm{Smooth}}(M, \mathbb{R} \times \mathbb{R}) \times M \to \mathbb{R} \times \mathbb{R}$, which we can see is smooth by a simple application of the exponential law, with the smooth multiplication map $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$, we obtain a smooth map

$$\mathrm{Hom}_{\mathrm{Smooth}}(M, \mathbb{R} \times \mathbb{R}) \times M \to \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

Hence, we obtain a smooth map $\mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ of convenient spaces. Since this morphism is also linear it is bounded by Lemma 6.1.1.2. □

The following important result tells us that any smooth mapping of convenient spaces factors through a bounded linear morphism.

Theorem 6.1.1.9. *[46, c.f. Theorem 23.6] Suppose that we have a convenient vector space X. Then, there exists a convenient vector space λX along with a smooth mapping $\delta_X : X \to \lambda X$ such that, for every smooth mapping $f : X \to Y$ with values in a convenient vector space Y, there exists a unique linear bounded map $\tilde{f} : \lambda X \to Y$ such that $\tilde{f} \circ \delta_X = f$. Moreover, there is an isomorphism of convenient vector spaces*

$$\mathrm{Hom}_{\mathrm{Con}}(\lambda X, Y) \simeq \mathrm{Hom}_{\mathrm{Smooth}}(X, Y)$$

6.1.2 $\mathcal{C}^\infty$-Bornological Rings

Recall from Corollary 1.3.0.5 that there is an equivalence of categories

$$\mathrm{LH}(\mathrm{CBorn}_{\mathbb{R}}) \simeq \mathrm{SInd}(\mathrm{Lin}_{\mathbb{R}})$$

where the category on the right is the free sifted cocompletion of the category $\mathrm{Lin}_{\mathbb{R}}$ consisting of objects which are of the form $\ell^1(\kappa)$ for κ less than a cardinal $\aleph$, and whose morphisms are bounded linear maps. The space $\ell^1(\kappa)$ is a Banach space, and hence can naturally be considered as a convenient space, and moreover a convenient manifold. For ease of notation, in this section we will abbreviate $\mathrm{Lin}_{\mathbb{R}}$ to Lin.

Definition 6.1.2.1. We define the category $\mathcal{C}^\infty$ to be the full subcategory of ConMfd consisting of convenient manifolds of the form $\ell^1(\kappa)$ for some $\ell^1(\kappa) \in \mathrm{Lin}$, and whose morphisms are defined to be the smooth morphisms between objects.

Remark. We note that Lin and $\mathcal{C}^\infty$ are closed under finite coproducts with

$$\ell^1(\kappa) \coprod \ell^1(\mu) \simeq \ell^1(\kappa \coprod \mu)$$

The category $\mathcal{C}^\infty\mathrm{Ring}$ of $\mathcal{C}^\infty$-rings can be described as the free sifted cocompletion of the opposite category of the category CartSp consisting of manifolds of the form $\mathbb{R}^n$, for some $n \in \mathbb{N}$, along with smooth morphisms between them. Since $\mathbb{R}^n$ is equivalent to $\ell^1(n)$, we note that CartSp is a full subcategory of $\mathcal{C}^\infty$. Hence, we can extend the notion of a $\mathcal{C}^\infty$-ring by adding in all the infinite dimensional manifolds in $\mathcal{C}^\infty$.

Definition 6.1.2.2. The *category of $\mathcal{C}^\infty$-bornological rings*, denoted by $\mathcal{C}^\infty\mathrm{BornRing}$, is the free sifted cocompletion of $\mathcal{C}^{\infty,op}$, i.e.

$$\mathcal{C}^\infty\mathrm{BornRing} := \mathrm{SInd}(\mathcal{C}^{\infty,op}) = \mathrm{Fun}^\times(\mathcal{C}^\infty, \mathrm{Set})$$

Remark. We also note that $\mathcal{C}^\infty\mathrm{BornRing}$ can be described as the algebras for the multisorted Lawvere theory defined by $\mathcal{C}^\infty$.

Lemma 6.1.2.3. *There is a fully faithful embedding $i : \mathcal{C}^\infty\mathrm{Ring} \to \mathcal{C}^\infty\mathrm{BornRing}$.*

Proof. CartSp is a full subcategory of $\mathcal{C}^\infty$. Hence we obtain a fully faithful functor on the level of free sifted cocompletions by Lemma C.2.0.2. ☐

We note that, by Lemma 6.1.1.8, there is a functor $\mathcal{C}^\infty : \mathcal{C}^{\infty,op} \to \mathrm{Comm}(\mathrm{CBorn}_{\mathbb{R}})$. We can extend this, using Lemma C.2.0.3, to a sifted colimit-preserving functor

$$\mathcal{C}^\infty : \mathcal{C}^\infty\mathrm{BornRing} \to \mathrm{Comm}(\mathrm{CBorn}_{\mathbb{R}})$$

It is not clear whether this functor is fully faithful since $\mathcal{C}^\infty(\ell^1(\kappa))$ is not necessarily compact in $\mathrm{Comm}(\mathrm{CBorn}_{\mathbb{R}})$. We can get some partial results by considering the notion of a $\mathcal{C}^\infty_{lfcs}$-ring. In the following, a *locally finite sum* of smooth functions $f_i : M \to \mathbb{R}$ is one such that, for any point $x \in M$, there exists an open subset U of M such that only finitely many f_i are non-zero on U.

Definition 6.1.2.4. [1, Section 5] For a convenient space X, define $\mathcal{C}^\infty_{lfcs}(M)$ to be the smallest subalgebra of $\mathcal{C}^\infty(M)$ which contains $X^\vee := \underline{\mathrm{Hom}}_{\mathrm{Con}}(X, \mathbb{R})$, the space of bounded linear functionals on X, and which is closed under locally finite sums.

Remark. We note that $X^* \subseteq X^\vee$.

Example 6.1.2.5. *For any finite dimensional convenient space X we have that $\mathcal{C}^\infty_{lfcs}(X) \simeq \mathcal{C}^\infty(X)$.*

We can easily see that we also obtain a sifted colimit-preserving functor

$$\mathcal{C}^\infty_{lfcs} : \mathcal{C}^\infty\mathrm{BornRing} \to \mathrm{Comm}(\mathrm{CBorn}_{\mathbb{R}}) \tag{6.1}$$

Definition 6.1.2.6. [1, Theorem 5.1] A convenient space X is *weakly realcompact* if, for any $\phi : \mathcal{C}^\infty_{lfcs}(X) \to \mathbb{R}$, there exists some unique $x \in X$ such that, for any $f \in \mathcal{C}^\infty_{lfcs}(X)$, $\phi \circ f = f(x)$.

Example 6.1.2.7. *Suppose that κ is a non-measurable cardinal. Then, $\ell^1(\kappa)$ is weakly realcompact by [27, p. 575].*

Lemma 6.1.2.8. *Suppose that $\ell^1(\mu)$ and $\ell^1(\kappa)$ are in $\mathcal{C}^\infty$. Then, the map*

$$\mathrm{Hom}_{\mathrm{Smooth}}(\ell^1(\mu), \ell^1(\kappa)) \to \mathrm{Hom}_{\mathrm{Comm}(\mathrm{CBorn}_{\mathbb{R}})}(\mathcal{C}^\infty_{lfcs}(\ell^1(\kappa)), \mathcal{C}^\infty_{lfcs}(\ell^1(\mu)))$$

is a bijection.

Proof. Suppose that $x \in \ell^1(\mu)$ and $\phi \in \mathrm{Hom}_{\mathrm{Comm}(\mathrm{CBorn}_{\mathbb{R}})}(\mathcal{C}^\infty_{lfcs}(\ell^1(\kappa)), \mathcal{C}^\infty_{lfcs}(\ell^1(\mu)))$. Then, we note that $ev_x \circ \phi$ is a map in $\mathrm{Hom}_{\mathrm{Comm}(\mathrm{CBorn}_{\mathbb{R}})}(\mathcal{C}^\infty_{lfcs}(\ell^1(\kappa)), \mathbb{R})$. Consider the restriction of $ev_x \circ \phi$ to $\mathrm{Sym}(\ell^\infty(\kappa)) = \mathrm{Sym}(\ell^1(\kappa)^\vee) \subseteq \mathcal{C}^\infty_{lfcs}(\ell^1(\kappa))$. Then, since $\ell^1(\kappa)$ is weakly realcompact, we see that there exists some unique $y \in \ell^1(\kappa)$ such that, for all $f \in \mathrm{Sym}(\ell^\infty(\kappa))$, $(ev_x \circ \phi)(f) = f(y)$.

Define a function $\psi : \ell^1(\mu) \to \ell^1(\kappa)$ by $\psi(x) = y$. Then, since

$$\mathrm{Hom}_{\mathrm{Comm}(\mathrm{CBorn}_{\mathbb{R}})}(\mathrm{Sym}(\ell^\infty(\kappa)), \mathcal{C}^\infty_{lfcs}(\ell^1(\mu))) \simeq \mathrm{Hom}_{\mathrm{CBorn}_{\mathbb{R}}}(\ell^\infty(\kappa), \mathcal{C}^\infty_{lfcs}(\ell^1(\mu)))$$

we see that, for all $f \in \ell^1(\kappa)^* \subseteq \ell^\infty(\kappa)$, the image $\bar{\phi}$ in $\mathrm{CBorn}_{\mathbb{R}}$ of ϕ restricted to $\mathrm{Sym}(\ell^\infty(\kappa))$ satisfies that $\bar{\phi}(f) = f \circ \psi$. Hence, since $\bar{\phi}(f)$ is bounded linear and we are working with convenient vector spaces, ψ is smooth by Definition 6.1.1.3. $\square$

Therefore, we see that $\mathcal{C}^\infty$BornRing can equivalently be described as the sifted cocompletion of the category $\mathcal{C}^\infty_{lfcs}$ consisting of complete bornological algebras of the form $\mathcal{C}^\infty_{lfcs}(\ell^1(\kappa))$ with morphisms being bounded algebra homomorphisms.

6.1.3 The Free $\mathcal{C}^\infty$-Bornological Ring

The categories Lin and $\mathcal{C}^\infty$ are closed under finite products. By Lemma 6.1.1.2, the bounded linear maps in Lin are smooth maps, and hence there is a natural functor $\iota : \mathrm{Lin} \to \mathcal{C}^\infty$.

Lemma 6.1.3.1. $\iota : \mathrm{Lin} \to \mathcal{C}^\infty$ *preserves finite products.*

Proof. Suppose that we have a product $\ell^1(\mu) \times \ell^1(\kappa)$ in Lin and an object $\ell^1(\nu) \in \mathcal{C}^\infty$ along with smooth morphisms $\ell^1(\nu) \to \ell^1(\mu)$ and $\ell^1(\nu) \to \ell^1(\kappa)$. Then, by Theorem 6.1.1.9, there exists a convenient space $\lambda(\ell^1(\nu))$ such that the smooth mappings $\ell^1(\nu) \to \ell^1(\mu)$ and $\ell^1(\nu) \to \ell^1(\kappa)$ define unique bounded linear maps $\lambda(\ell^1(\nu)) \to \ell^1(\mu)$ and $\lambda(\ell^1(\nu)) \to \ell^1(\kappa)$.

Therefore, by the universal property of the product in Lin, there exists a unique bounded linear mapping $\lambda(\ell^1(\nu)) \to \ell^1(\mu) \times \ell^1(\kappa)$ compatible with the projections into $\ell^1(\mu)$ and $\ell^1(\kappa)$. By composing with the smooth mapping $\delta_{\ell^1(\nu)} : \ell^1(\nu) \to \lambda(\ell^1(\nu))$ from Theorem 6.1.1.9, there exists a unique smooth mapping $\ell^1(\nu) \to \ell^1(\mu) \times \ell^1(\kappa)$ compatible with the projections. Therefore, $\ell^1(\mu) \times \ell^1(\kappa)$ is the product in $\mathcal{C}^\infty$. $\square$

Lemma 6.1.3.2. $\mathcal{C}^\infty$ *has finite direct sums.*

Proof. Consider the product $\ell^1(\mu) \times \ell^1(\kappa)$ of two objects in Lin, which, by the previous lemma, defines the product in $\mathcal{C}^\infty$. We want to show that this defines a coproduct in $\mathcal{C}^\infty$. Indeed, suppose that $\ell^1(\nu) \in \mathcal{C}^\infty$ and suppose that we have smooth morphisms $\ell^1(\mu) \to \ell^1(\nu)$ and $\ell^1(\kappa) \to \ell^1(\nu)$. Then, we note that there are induced bounded linear morphisms $\lambda(\ell^1(\mu)) \to \ell^1(\nu)$ and $\lambda(\ell^1(\kappa)) \to \ell^1(\nu)$.

By definition of the coproduct and product in Lin, we note that there is an induced bounded linear morphism $\lambda(\ell^1(\mu)) \times \lambda(\ell^1(\kappa)) \to \ell^1(\nu)$ and therefore, by the universal property of the tensor product in $\mathrm{Ban}_{\mathbb{R}}$, a bounded linear morphism $\lambda(\ell^1(\mu)) \hat{\otimes} \lambda(\ell^1(\kappa)) \to \ell^1(\nu)$. By [28, Proposition 5.2.4], there is an equivalence

$$\lambda(\ell^1(\mu)) \hat{\otimes} \lambda(\ell^1(\kappa)) \simeq \lambda(\ell^1(\mu) \times \ell^1(\kappa))$$

Therefore, by Theorem 6.1.1.9, we can lift the map $\lambda(\ell^1(\mu)) \hat{\otimes} \lambda(\ell^1(\kappa)) \to \ell^1(\nu)$ to a smooth map $\ell^1(\mu) \times \ell^1(\kappa) \to \ell^1(\nu)$ which defines the coproduct. $\square$

Lemma 6.1.3.3. $\mathcal{C}^\infty$BornRing *has all small limits and colimits.*

Proof. Indeed, we easily see, since limits and colimits in $\mathrm{Fun}(\mathcal{C}^\infty, \mathrm{Set})$ are computed pointwise, that $\mathcal{C}^\infty$BornRing has all small limits and sifted colimits, since these commute with finite products. We note that the coproduct in $\mathcal{C}^\infty$BornRing is induced using the definition of the direct sum in $\mathcal{C}^\infty$, and then extended by sifted colimits to $\mathcal{C}^\infty$BornRing. Therefore, since $\mathcal{C}^\infty$BornRing has all sifted colimits and finite coproducts, it has all small colimits. $\square$

By Lemma 6.1.3.1, ι^{op} preserves finite coproducts. Therefore, we can use Lemma C.2.0.3 to extend ι^{op} to a colimit-preserving functor

$$\tilde{L} : \mathrm{SInd}(\mathrm{Lin}^{op}) \to \mathrm{SInd}(\mathcal{C}^{\infty,op}) \simeq \mathcal{C}^\infty\mathrm{BornRing}$$

We note that there is a product preserving functor $\mathrm{Lin} \to \mathrm{SInd}(\mathrm{Lin}^{op})^{op}$. Since sifted colimits commute with finite products, we can left Kan extend to a finite product preserving functor $\mathrm{SInd}(\mathrm{Lin}) \to \mathrm{SInd}(\mathrm{Lin}^{op})^{op}$. Hence, taking the opposite functor and composing it with $\tilde{L}$ we obtain a finite coproduct preserving functor

$$\mathrm{LH}(\mathrm{CBorn}_{\mathbb{R}})^{op} \simeq \mathrm{SInd}(\mathrm{Lin})^{op} \to \mathcal{C}^\infty\mathrm{BornRing} \tag{6.2}$$

Theorem 6.1.3.4. *There is a colimit-preserving functor*

$$L : \mathrm{LH}(\mathrm{CBorn}_{\mathbb{R}}) \to \mathcal{C}^\infty\mathrm{BornRing}$$

which acts on elements of Lin *by* $L(\ell^1(\kappa)) = \ell^1(\kappa)^\vee := \underline{\mathrm{Hom}}_{\mathrm{CBorn}_{\mathbb{R}}}(\ell^1(\kappa), \mathbb{R})$.

Proof. We note that, since $\mathrm{CBorn}_{\mathbb{R}}$ is closed, there is a colimit-preserving functor $\mathrm{Lin} \xrightarrow{(-)^\vee} \mathrm{CBorn}_{\mathbb{R}}^{op}$ obtained by considering each element of Lin as a complete bornological space and then taking the dual in $\mathrm{CBorn}_{\mathbb{R}}$. Therefore, since there is a finite coproduct preserving functor $\mathrm{CBorn}_{\mathbb{R}}^{op} \hookrightarrow \mathrm{LH}(\mathrm{CBorn}_{\mathbb{R}})^{op}$ by Proposition 1.1.2.5, we can compose with the functor from Equation (6.2) to obtain a finite coproduct preserving functor

$$\mathrm{Lin} \to \mathrm{CBorn}_{\mathbb{R}}^{op} \to \mathrm{LH}(\mathrm{CBorn}_{\mathbb{R}})^{op} \to \mathrm{SInd}(\mathrm{Lin}^{op}) \to \mathcal{C}^\infty\mathrm{BornRing}$$

By Lemma C.2.0.3, we can extend to a colimit-preserving functor $L : \mathrm{LH}(\mathrm{CBorn}_{\mathbb{R}}) \to \mathcal{C}^\infty\mathrm{BornRing}$ $\square$

By the adjoint functor theorem, we obtain the following result. This shows that we can construct a *free $\mathcal{C}^\infty$-bornological ring* from any complete bornological space.

Corollary 6.1.3.5. *There is an adjunction* $L : \mathrm{LH}(\mathrm{CBorn}_{\mathbb{R}}) \leftrightarrows \mathcal{C}^{\infty}\mathrm{BornRing} : R$*.*

By the adjoint functor theorem, R is defined by

$$\begin{aligned} R : \mathcal{C}^{\infty}\mathrm{BornRing} &\to \mathrm{LH}(\mathrm{CBorn}_{\mathbb{R}}) \simeq \mathrm{SInd}(\mathrm{Lin}) \\ A &\to \mathrm{Hom}_{\mathcal{C}^{\infty}\mathrm{BornRing}}(L(-), A) \end{aligned}$$

Proposition 6.1.3.6. *R preserves the monoidal structure when restricted to* $\mathcal{C}^{\infty}$Ring*.*

Proof. Suppose that we have some element $\ell^1(\mu) \in \mathrm{Lin}$ and consider the dual object $\ell^1(\mu)^{\vee} \in \mathrm{CBorn}_{\mathbb{R}}^{op}$. Since every object in $\mathrm{CBorn}_{\mathbb{R}}$ can be written as a filtered colimit of objects in Lin, we can identify $\ell^1(\mu)^{\vee}$ with a cofiltered limit $\varprojlim_i \ell^1(\mu_i)$ in $\mathrm{CBorn}_{\mathbb{R}}^{op}$ and identify with a formal sifted colimit "$\varinjlim_i$"$\ell^1(\mu_i)$ in $\mathcal{C}^{\infty}$BornRing using Appendix C.2. Since $A \in \mathcal{C}^{\infty}\mathrm{Ring}$, A can be written as a formal sifted colimit "$\varinjlim_j$"$\mathbb{R}^{n_j}$. Then, we note that

$$\begin{aligned} R(A)(\ell^1(\mu)) &\simeq \mathrm{Hom}_{\mathcal{C}^{\infty}\mathrm{BornRing}}(L(\ell^1(\mu)), A) \\ &\simeq \mathrm{Hom}_{\mathcal{C}^{\infty}\mathrm{BornRing}}(\text{“}\varinjlim_i\text{”}\ell^1(\mu_i), \text{“}\varinjlim_j\text{”}\mathbb{R}^{n_j}) \\ &\simeq \varprojlim_i \varinjlim_j \mathrm{Hom}_{\mathrm{Smooth}}(\mathbb{R}^{n_j}, \ell^1(\mu_i)) \end{aligned}$$

Using Theorem 6.1.1.9,

$$\simeq \varprojlim_i \varinjlim_j \mathrm{Hom}_{\mathrm{Con}}(\lambda\mathbb{R}^{n_j}, \ell^1(\mu_i))$$

Since $\lambda\mathbb{R}^{n_j}$ is equivalent to $\mathcal{C}^{\infty}(\mathbb{R}^{n_j})^{\vee} = \underline{\mathrm{Hom}}_{\mathrm{Con}}(\mathcal{C}^{\infty}(\mathbb{R}^{n_j}), \mathbb{R})$ by [46, Corollary 23.11],

$$\simeq \varprojlim_i \varinjlim_j \mathrm{Hom}_{\mathrm{Con}}(\mathcal{C}^{\infty}(\mathbb{R}^{n_j})^{\vee}, \ell^1(\mu_i))$$

Now, since $\mathcal{C}^{\infty}(\mathbb{R}^{n_j})$ is a reflexive nuclear convenient space by [46, Result 6.5.7] and [35, Theorem 5.3.2],

$$\simeq \varprojlim_i \varinjlim_j \mathcal{C}^{\infty}(\mathbb{R}^{n_j})\hat{\otimes}\ell^1(\mu_i)$$

where we are considering the underlying set of the space. Since our limit is cofiltered, it commutes with the tensor product in Con,

$$\simeq (\varinjlim_j \mathcal{C}^{\infty}(\mathbb{R}^{n_j}))\hat{\otimes} \varprojlim_i \ell^1(\mu_i)$$

And, by definition of the functor $\mathcal{C}^\infty : \mathcal{C}^\infty\mathrm{BornRing} \to \mathrm{Comm}(\mathrm{CBorn}_\mathbb{R})$,

$$\simeq \mathcal{C}^\infty(A)\hat{\otimes}\ell^1(\mu)^\vee$$

Suppose that $B \in \mathcal{C}^\infty\mathrm{Ring}$. Since $\mathcal{C}^\infty(A \otimes B) \simeq \mathcal{C}^\infty(A)\hat{\otimes}\,\mathcal{C}^\infty(B)$ for $\mathcal{C}^\infty$-rings by [87, Theorem 51.6], we can use our above characterisation of the right adjoint to show that $R(A \otimes B) \simeq R(A)\hat{\otimes}R(B)$. □

6.1.4 Derived $\mathcal{C}^\infty$-Bornological Rings

We propose the following category of derived affines via a natural extension of our definition of $\mathcal{C}^\infty$-bornological rings.

Definition 6.1.4.1. The $(\infty,1)$-*category of derived $\mathcal{C}^\infty$-bornological rings*, which we denote by $\mathcal{C}^\infty\mathbf{DBornRing}$, is the $(\infty,1)$-category

$$\mathcal{C}^\infty\mathbf{DBornRing} := \mathcal{P}_\Sigma(\mathcal{C}^{\infty,op}) = \mathbf{Fun}^\times(\mathcal{C}^\infty, \infty\mathbf{Grpd})$$

We define the $(\infty,1)$-category of *derived $\mathcal{C}^\infty$-bornological affines* by

$$\mathcal{C}^\infty\mathbf{DBAff} := \mathcal{C}^\infty\mathbf{DBornRing}^{op}$$

The *ordinary category of $\mathcal{C}^\infty$-bornological affines* is $\mathcal{C}^\infty\mathrm{BAff} := \mathcal{C}^\infty\mathrm{BornRing}^{op}$.

We can similarly describe the $(\infty,1)$-*category of derived $\mathcal{C}^\infty$-rings* to be the $(\infty,1)$-category $\mathcal{C}^\infty\mathbf{DRing} := \mathcal{P}_\Sigma(\mathrm{CartSp}^{op})$. Similarly, the $(\infty,1)$-category of *derived $\mathcal{C}^\infty$-affines* is defined by $\mathcal{C}^\infty\mathbf{DAff} := \mathcal{C}^\infty\mathbf{DRing}^{op}$. By Lemma 6.1.2.3, together with [52, Proposition 5.5.8.22] and Proposition C.2.0.4, there exists a fully faithful inclusion functor $\mathcal{C}^\infty\mathbf{DAff} \hookrightarrow \mathcal{C}^\infty\mathbf{DBAff}$.

Remark. We note that derived manifolds in the sense of Carchedi and Steffens [22, Corollary 1.1] are a full subcategory of $\mathcal{C}^\infty\mathbf{DAff}$, and hence of $\mathcal{C}^\infty\mathbf{DBAff}$.

By [10, Lemma 5.3.91], there is a fully faithful functor

$$\mathcal{C}^\infty : \mathcal{C}^\infty\mathbf{DRing} \to \mathbf{DAlg}^{cn}(\mathbf{Ch}(\mathrm{Ind}(\mathrm{Ban}_\mathbb{R})))$$

We expect there to be a fully faithful functor

$$\widetilde{\mathcal{C}^\infty} : \mathcal{C}^\infty\mathbf{DBornRing} \to \mathbf{DAlg}^{cn}(\mathbf{Ch}(\mathrm{Ind}(\mathrm{Ban}_\mathbb{R})))$$

which restricts to $\mathcal{C}^\infty$ on $\mathcal{C}^\infty\mathbf{DRing}$. We make the following conjecture.

Conjecture 6.1.4.2. *There exists a functor* $\widetilde{\mathcal{C}^\infty} : \mathcal{C}^{\infty,op} \to \mathrm{Comm}(\mathrm{CBorn}_{\mathbb{R}})$ *such that*

1. *For each* $\ell^1(\kappa)$ *in* $\mathcal{C}^\infty$*, we have a chain of subalgebra inclusions*
$$\mathrm{Sym}(\ell^\infty(\kappa)) \subseteq \widetilde{\mathcal{C}^\infty}(\ell^1(\kappa)) \subseteq \mathcal{C}^\infty_{lfcs}(\ell^1(\kappa))$$
2. *For each* $\ell^1(\kappa)$ *in* $\mathcal{C}^\infty$*, the morphism* $\mathrm{Sym}(\ell^\infty(\kappa)) \to \widetilde{\mathcal{C}^\infty}(\ell^1(\kappa))$ *is a homotopy epimorphism,*
3. *For each* $n \in \mathbb{N}$*,* $\widetilde{\mathcal{C}^\infty}(\mathbb{R}^n) \simeq \mathcal{C}^\infty(\mathbb{R}^n)$*.*

Remark. We make some conjectures about what kind of object $\widetilde{\mathcal{C}^\infty}$ could be in the Further Research section of this thesis.

Assuming that such a functor $\widetilde{\mathcal{C}^\infty} : \mathcal{C}^{\infty,op} \to \mathrm{Comm}(\mathrm{CBorn}_{\mathbb{R}})$ exists, it can then be extended, using Lemma C.2.0.4, to a sifted colimit-preserving functor
$$\widetilde{\mathcal{C}^\infty} : \mathcal{C}^\infty\mathbf{DBornRing} \to \mathbf{DAlg}^{cn}(\mathbf{Ch}(\mathrm{Ind}(\mathrm{Ban}_{\mathbb{R}})))$$

Remark. We remark that ideas used in the following proof are similar to those used in [10, Section 4.3.2], as well as in unpublished work by Arun Soor in the affinoid setting.

Corollary 6.1.4.3. *Assuming Conjecture 6.1.4.2 is true, there is a fully faithful inclusion functor*
$$\widetilde{\mathcal{C}^\infty} : \mathcal{C}^\infty\mathbf{DBornRing} \hookrightarrow \mathbf{DAlg}^{cn}(\mathbf{Ch}(\mathrm{Ind}(\mathrm{Ban}_{\mathbb{R}})))$$

Proof. Suppose that $A, B \in \mathcal{C}^\infty\mathbf{DBornRing}$. We want to show that the canonical map
$$\mathrm{Map}_{\mathcal{C}^\infty\mathbf{DBornRing}}(A, B) \to \mathrm{Map}_{\mathbf{DAlg}^{cn}(\mathbf{Ch}(\mathrm{Ind}(\mathrm{Ban}_{\mathbb{R}})))}(\widetilde{\mathcal{C}^\infty}(A), \widetilde{\mathcal{C}^\infty}(B))$$
is an equivalence. We can immediately reduce to the case when A is of the form $\ell^1(\kappa)$ for some cardinal κ. Then, writing B as a formal sifted $(\infty,1)$-colimit "$\varinjlim_i$"$\ell^1(\mu_i)$, we consider the composite
$$\begin{aligned}&\mathrm{Map}_{\mathcal{C}^\infty\mathbf{DBornRing}}(\ell^1(\kappa), \text{“}\varinjlim_i\text{”}\ell^1(\mu_i))\\ &\to \mathrm{Map}_{\mathbf{DAlg}^{cn}(\mathbf{Ch}(\mathrm{Ind}(\mathrm{Ban}_{\mathbb{R}})))}(\widetilde{\mathcal{C}^\infty}(\ell^1(\kappa)), \varinjlim_i \widetilde{\mathcal{C}^\infty}(\ell^1(\mu_i)))\\ &\to \mathrm{Map}_{\mathbf{DAlg}^{cn}(\mathbf{Ch}(\mathrm{Ind}(\mathrm{Ban}_{\mathbb{R}})))}(\mathrm{Sym}(\ell^\infty(\kappa)), \varinjlim_i \widetilde{\mathcal{C}^\infty}(\ell^1(\mu_i)))\end{aligned} \tag{6.3}$$

We prove that this composition of maps is an equivalence. Since $\mathrm{Sym}(\ell^\infty(\kappa))$ is compact projective in $\mathrm{Comm}(\mathrm{Ind}(\mathrm{Ban}_{\mathbb{R}}))$, it suffices to show that, for each i, the map
$$\mathrm{Hom}_{Smooth}(\ell^1(\mu_i), \ell^1(\kappa)) \to \mathrm{Hom}_{\mathrm{Comm}(\mathrm{Ind}(\mathrm{Ban}_{\mathbb{R}}))}(\mathrm{Sym}(\ell^\infty(\kappa)), \widetilde{\mathcal{C}^\infty}(\ell^1(\mu_i)))$$

is an equivalence. This easily follows using our reasoning in Lemma 6.1.2.8.

Now, since the morphism $\mathrm{Sym}(\ell^\infty(\kappa)) \to \widetilde{\mathcal{C}^\infty}(\ell^1(\kappa))$ is a homotopy epimorphism, the morphism

$$\mathrm{Map}(\widetilde{\mathcal{C}^\infty}(\ell^1(\kappa)), \varinjlim_i \widetilde{\mathcal{C}^\infty}(\ell^1(\mu_i))) \to \mathrm{Map}(\mathrm{Sym}(\ell^\infty(\kappa)), \varinjlim_i \widetilde{\mathcal{C}^\infty}(\ell^1(\mu_i))) \tag{6.4}$$

is a monomorphism on π_0 and an equivalence on π_n for any n. Since the composition in Equation (6.3) is an equivalence, we note that the map in Equation (6.4) is also an epimorphism on π_0, and hence is an equivalence. We then easily deduce that the first map in Equation (6.3) is an equivalence, as desired. $\square$

For the rest of this chapter, we will assume that the conjecture and its corollary is true. We want to define a suitable representability context using derived $\mathcal{C}^\infty$-bornological rings.

Definition 6.1.4.4. Suppose that $f : Y = \mathrm{Spec}(B) \to \mathrm{Spec}(A) = X$ is a morphism in $\mathbf{DAff}^{cn}(\mathbf{Ch}(\mathrm{Ind}(\mathrm{Ban}_{\mathbb{R}})))$. Then, f is a *$\mathcal{C}^\infty$-localisation* if it is a flat homotopy monomorphism. We denote the class of $\mathcal{C}^\infty$-localisations by **loc**.

We define the following topology which we can think of as some kind of 'flat homotopy Zariski topology' on derived $\mathcal{C}^\infty$-bornological affines.

Definition 6.1.4.5. The *finite $\mathcal{C}^\infty$-localisation topology* on the homotopy category $\mathrm{Ho}(\mathbf{DAff}^{cn}(\mathbf{Ch}(\mathrm{Ind}(\mathrm{Ban}_{\mathbb{C}}))))$ has finite covers $\{U_j = \mathrm{Spec}(B_j) \to \mathrm{Spec}(A) = X\}_{j\in J}$, satisfying the following properties

1. Each $A \to B_j$ is a $\mathcal{C}^\infty$-localisation,
2. The family $\{U_j \to X\}_{j\in J}$ is a formal covering family.

We will denote this topology by $\boldsymbol{\tau}_{\mathbf{loc}}$.

Definition 6.1.4.6. We make the following definitions. In the following, **Alg** will denote each of the categories $\mathcal{C}^\infty$**DBornRing**, $\mathcal{C}^\infty$BornRing, $\mathcal{C}^\infty$**DRing**, and $\mathcal{C}^\infty$Ring.

- The class $\mathbf{open}_{\mathcal{C}^\infty}$ of *open immersions* is the subclass of $\mathbf{loc}^{\boldsymbol{\tau}_{\mathbf{loc}}}$ such that, whenever $A \to C$ is a map with $C \in \mathbf{Alg}$, then $B \otimes^{\mathbb{L}}_A C \in \mathbf{Alg}$,
- The *finite $\mathcal{C}^\infty$-topology*, which we will denote by $\boldsymbol{\tau}_{\mathcal{C}^\infty}$, consists of finite covers $\{U_j = \mathrm{Spec}(B_j) \to \mathrm{Spec}(A) = X\}_{j\in J}$ in $\boldsymbol{\tau}_{\mathbf{loc}}$ such that, whenever there is some map $A \to C$, then $C \in \mathbf{Alg}$ if and only if $B_j \otimes^{\mathbb{L}}_A C \in \mathbf{Alg}$.

We note that $\mathcal{C}^\infty$-localisations and $\mathcal{C}^\infty$-open immersions are closed under equivalences, compositions, and pushouts.

Corollary 6.1.4.7. *Suppose that $f : Y \to X$ is a map in $\mathbf{DAff}^{cn}(\mathbf{Ch}(\mathrm{Ind}(\mathrm{Ban}_{\mathbb{R}})))$. Then,*

1. *f is a $\mathcal{C}^\infty$-localisation if and only if it is derived strong and $t_0(f)$ is a $\mathcal{C}^\infty$-localisation,*
2. *f is an open immersion if it is derived strong and $t_0(f)$ is an open immersion.*

Proof. If a morphism $f : Y \to X$ is a $\mathcal{C}^\infty$-localisation then it is derived strong by Lemma 5.2.3.2. We can then use flatness to easily show that $t_0(f)$ is a $\mathcal{C}^\infty$-localisation. The converse is true using Lemmas 5.2.3.2 and 5.2.6.5. The result in the case of open immersions follows using Lemmas 5.1.2.10, 5.2.3.2, and 5.2.6.5, along with the statement that covers in $\boldsymbol{\tau}_{\mathcal{C}^\infty}$ are derived strong. □

Since covers in $\boldsymbol{\tau}_{\mathcal{C}^\infty}$ are covers in the faithfully flat topology, we can easily show that **QCoh** satisfies hyperdescent using a similar proof to Lemma 5.2.7.8 (see also [86, c.f. Lemma 2.2.2.13]). Therefore, by Lemma 5.2.4.1, **Dls** satisfies descent for $\boldsymbol{\tau}_{\mathcal{C}^\infty}$-covers.

Corollary 6.1.4.8. *Suppose that we have an n-representable$|_{\mathcal{C}^\infty\mathbf{DBAff}}$ morphism $f : \mathcal{F} \to \mathcal{G}$ of stacks in $\mathbf{Stk}(\mathcal{C}^\infty\mathbf{DBAff}, \boldsymbol{\tau}_{\mathcal{C}^\infty}|_{\mathcal{C}^\infty\mathbf{DBAff}})$. Then, f is in n-$\mathbf{open}_{\mathcal{C}^\infty}|_{\mathcal{C}^\infty\mathbf{DBAff}}$ if f satisfies that, for any $x : X = \mathrm{Spec}(A) \to \mathcal{F}$ and any $M \in \mathbf{M}_{A,1}$,*

$$\pi_0(\mathrm{Map}_{\mathbf{Mod}_A}(\mathbb{L}_{\mathcal{F}/\mathcal{G},x}, M)) = 0$$

The converse holds if $t_0(\mathcal{F}) \to t_0(\mathcal{G})$ is in n-$\mathbf{open}^{\heartsuit}_{\mathcal{C}^\infty}$.

Proof. Indeed, we note that, since **Dls** satisfies descent for $\boldsymbol{\tau}_{\mathcal{C}^\infty}$-covers, the forwards direction follows from Proposition 5.2.5.2 since $\mathbf{open}_{\mathcal{C}^\infty} \subseteq \mathbf{loc}^{\boldsymbol{\tau}_{\mathcal{C}^\infty}}$ and $\mathcal{C}^\infty$-localisations are formally perfect by Lemma 5.2.6.2. To prove the converse, we note that, by the proof of Proposition 5.2.5.2, it suffices to show that if we have a morphism $f : Y = \mathrm{Spec}(B) \to \mathrm{Spec}(A) = X$ in $\mathcal{C}^\infty\mathbf{DBAff}$ which is in $\mathbf{fP}^{\boldsymbol{\tau}_{\mathcal{C}^\infty}}$ and satisfies that its truncation lies in $\mathbf{open}^{\heartsuit}_{\mathcal{C}^\infty}$, then f lies in $\mathbf{open}_{\mathcal{C}^\infty}$. We can also assume that the chosen cover $\{U_i \to Y\}_{i \in I}$ in $\boldsymbol{\tau}_{\mathcal{C}^\infty}$ satisfies that $t_0(U_i) \to t_0(X)$ is in $\mathbf{loc}^{\heartsuit}$. Therefore, we see that, since $t_0(U_i) \to t_0(X)$ is flat and formally perfect, the formally perfect morphism $U_i \to X$ is derived strong by Lemma 5.2.3.4. Since covers in $\boldsymbol{\tau}_{\mathcal{C}^\infty}$ are formal covering families and derived strong by Corollary 6.1.4.7, we see that the morphism $f : Y \to X$ is derived strong by Lemma 5.2.1.4. Hence, we conclude that f is in $\mathbf{open}_{\mathcal{C}^\infty}$ by Corollary 6.1.4.7. □

By our definitions of $\mathbf{open}_{\mathcal{C}^\infty}$ and $\boldsymbol{\tau}_{\mathcal{C}^\infty}$, the following result easily follows.

Lemma 6.1.4.9. $(\mathbf{DAff}^{cn}(\mathbf{Ch}(\mathrm{Ind}(\mathrm{Ban}_{\mathbb{R}}))), \boldsymbol{\tau}_{\mathcal{C}^\infty}, \mathbf{open}_{\mathcal{C}^\infty}, \mathcal{C}^\infty\mathbf{DBAff})$ *is a strong relative* $(\infty,1)$*-geometry tuple.*

The following result easily follows from our definition of $\mathcal{C}^\infty$-open immersions.

Lemma 6.1.4.10. $(\mathcal{C}^\infty\mathbf{DBAff}, \boldsymbol{\tau}_{\mathcal{C}^\infty}|_{\mathcal{C}^\infty\mathbf{DBAff}}, \mathbf{open}_{\mathcal{C}^\infty}|_{\mathcal{C}^\infty\mathbf{DBAff}}, \mathcal{C}^\infty\mathbf{DAff})$ *is a strong relative* $(\infty,1)$*-geometry tuple.*

Therefore, by Proposition 3.2.1.6, we have a chain of fully faithful inclusions

$$\mathbf{Stk}(\mathcal{C}^\infty\mathbf{DAff}, \boldsymbol{\tau}_{\mathcal{C}^\infty}) \to \mathbf{Stk}(\mathcal{C}^\infty\mathbf{DBAff}, \boldsymbol{\tau}_{\mathcal{C}^\infty}) \to \mathbf{Stk}(\mathbf{DAff}^{cn}(\mathbf{Ch}(\mathrm{Ind}(\mathrm{Ban}_{\mathbb{R}}))), \boldsymbol{\tau}_{\mathcal{C}^\infty})$$

where we have dropped the restriction notation for ease of notation. Moreover, there is a chain of full subcategory inclusions on the level of n-geometric stacks.

6.1.5 A Representability Context for Derived Smooth Geometry

We now show that the geometry defined in the previous section defines a representability context. Indeed, the following is clear.

Lemma 6.1.5.1. $(\mathcal{C}^\infty\mathbf{DBAff}, \boldsymbol{\tau}_{\mathcal{C}^\infty}|_{\mathcal{C}^\infty\mathbf{DBAff}}, \mathbf{open}_{\mathcal{C}^\infty}|_{\mathcal{C}^\infty\mathbf{DBAff}}, \mathcal{C}^\infty\mathrm{BAff})$ *is a strong relative* $(\infty,1)$*-geometry tuple.*

Lemma 6.1.5.2. $\iota|_{\mathcal{C}^\infty\mathrm{BAff}} : (\mathcal{C}^\infty\mathrm{BAff}, \boldsymbol{\tau}^{\heartsuit}_{\mathcal{C}^\infty}) \to (\mathcal{C}^\infty\mathbf{DBAff}, \boldsymbol{\tau}_{\mathcal{C}^\infty}|_{\mathcal{C}^\infty\mathbf{DBAff}})$ *is a continuous functor of* $(\infty,1)$*-sites.*

Proof. Suppose that $\{t_0(U_j) = \mathrm{Spec}(\pi_0(B_j)) \to \mathrm{Spec}(\pi_0(A)) = t_0(X)\}_{j\in J}$ is a cover in $\boldsymbol{\tau}^{\heartsuit}_{\mathcal{C}^\infty}$ corresponding to the truncation of a cover $\{U_j \to X\}_{j\in J}$ in $\boldsymbol{\tau}_{\mathcal{C}^\infty}|_{\mathcal{C}^\infty\mathbf{DBAff}}$. The cover is a formal covering family by Lemma 5.2.1.2. We note that $\pi_0(A) \to \pi_0(B_j)$ is a $\mathcal{C}^\infty$-localisation and the morphism $A \to B_j$ is derived strong by Corollary 6.1.4.7. Therefore, using Lemma 5.1.2.10, we can show that $\pi_0(A) \to \pi_0(B_j)$ satisfies the conditions to be a cover in the finite $\mathcal{C}^\infty$-topology. □

Lemma 6.1.5.3. *For any finite collection* $\{U_i\}_{i\in I}$ *in* $\mathbf{DAff}^{cn}(\mathbf{Ch}(\mathrm{Ind}(\mathrm{Ban}_{\mathbb{R}})))$*, the map* $\coprod_{i\in I} h(U_i) \to h(\coprod_{i\in I} U_i)$ *is an equivalence in* $\mathbf{Stk}(\mathcal{C}^\infty\mathbf{DBAff}, \boldsymbol{\tau}_{\mathcal{C}^\infty}|_{\mathcal{C}^\infty\mathbf{DBAff}})$*,*

Proof. Suppose that $U_j = \mathrm{Spec}(A_j)$ and consider the morphism $U_i \to \coprod_{j\in I} U_j$. We note that, by Lemma 5.2.6.6, this is a homotopy monomorphism and, moreover, is flat. It is clear that the family $\{U_i \to \coprod_{j\in I} U_j\}_{i\in I}$ is a formal covering family. Moreover, for any morphism $\prod_{j\in I} A_j \to C$, $A_i \otimes^{\mathbb{L}}_{\prod_{j\in I} A_j} C \simeq C$, and hence we can deduce that the family $\{U_i \to \coprod_{j\in I} U_j\}_{i\in I}$ is a cover in $\boldsymbol{\tau}_{\mathcal{C}^\infty}$. We can then conclude by Corollary 3.2.3.7. □

Corollary 6.1.5.4. $\mathcal{C}^\infty\mathbf{DBAff}$ *is closed under* $\boldsymbol{\tau}_{\mathcal{C}^\infty}$*-descent relative to* $\mathcal{C}^\infty\mathbf{DBAff}$*.*

Proof. This follows from checking the conditions of Lemma 5.2.4.2. □

Let **Mod** denote the collection of, for each $X = \mathrm{Spec}(A) \in \mathcal{C}^\infty\mathbf{DBAff}$, the categories $\mathbf{Mod}_A$ of A-modules.

Lemma 6.1.5.5. $(\underline{\mathbf{Ch}}(\mathrm{Ind}(\mathrm{Ban}_\mathbb{R})), \boldsymbol{\tau}_{\mathcal{C}^\infty}, \mathbf{open}_{\mathcal{C}^\infty}, \mathcal{C}^\infty\mathbf{DBAff}, \mathbf{Mod})$ *is a flat Postnikov compatible derived geometry context.*

Proof. Indeed, we note that it is a flat derived geometry context using Lemma 6.1.4.9. The conditions for Postnikov compatibility also easily follow. In particular, since we are working over $\mathbb{R}$, Condition (4) follows using a similar proof to [58, Proposition 25.2.4.1]. It is easy to check that **Mod** is a good system of categories of modules. □

Lemma 6.1.5.6. $\boldsymbol{\tau}_{\mathcal{C}^\infty}$ *and* $\mathbf{open}_{\mathcal{C}^\infty}$ *satisfy the obstruction conditions relative to* $\mathcal{C}^\infty\mathbf{DBAff}$ *for the class* **hm** *of homotopy monomorphisms.*

Proof. Indeed, Conditions (1) and (2) follow easily by definition of **loc**. Using similar reasoning to Lemma 5.2.8.5, we can prove that Condition (3) holds. □

Corollary 6.1.5.7. *The tuple* $(\underline{\mathbf{Ch}}(\mathrm{Ind}(\mathrm{Ban}_\mathbb{R})), \boldsymbol{\tau}_{\mathcal{C}^\infty}, \mathbf{open}_{\mathcal{C}^\infty}, \mathcal{C}^\infty\mathbf{DBAff}, \mathbf{Mod}, \mathbf{hm})$ *is a representability context in the sense of Definition 5.1.3.1.*

Proof. By Lemma 6.1.5.5, our context is a flat Postnikov compatible derived geometry context. Condition (1) follows from Lemma 6.1.5.1. Condition (2) follows by Lemma 6.1.5.2. Condition (3) follows by 6.1.5.3. Condition (4) follows by Corollary 6.1.4.7 and Condition (5) by Corollary 6.1.4.8. Finally, Condition (6) follows by Lemma 6.1.5.6. □

6.2 The Derived Moduli Stack of Solutions to Partial Differential Equations

As explained in the Introduction, a main motivation for the development of this theory has been to apply it to the study of derived moduli stacks of solutions to PDEs. In particular, a higher categorical approach to this theory seems to necessitate the embedding of a suitable extended category of $\mathcal{C}^\infty$-rings, facilitated by our $\mathcal{C}^\infty$-bornological rings, into a closed monoidal category satisfying several nice properties, in our situation $\mathrm{CBorn}_\mathbb{R}$.

In this chapter, we describe a higher categorical approach to defining derived moduli stacks of solutions to non-linear elliptic PDEs. We then show that representability of such stacks as a derived $\mathcal{C}^\infty$-bornological affine scheme is a direct consequence of the representability theorem for mapping stacks described in Section 5.3.

6.2.1 Motivation

A geometric approach to the study of algebraic equations starts with the definition of an *algebraic variety.* Work by Vinogradov [43], amongst others, gives an analogous approach to the study of partial differential equations. A survey can be found in [44] and this will be our main reference in this section.

Suppose that we have a function $u : \mathbb{R}^2 \to \mathbb{R}$ and consider the following partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} = 0 \tag{6.5}$$

Then, consider the function $\pi : (\mathbb{R}^2) \times \mathbb{R} \to \mathbb{R}^2$ defined by sending $(x, y, u(x, y))$ to (x, y). We note that a section of π is u. Define the space $\mathrm{Jets}^2(\pi)$ to have coordinates $x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}$. We can interpret the partial differential equation from Equation (6.5) as a submanifold E of $\mathrm{Jets}^2(\pi)$ defined by

$$E = \{(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) \in \mathrm{Jets}^2(\pi) \mid u_{xx} + u_{xy} = 0\}$$

We can generalise this idea. Indeed, suppose that we have a locally trivial smooth vector bundle $\pi : E \to M$ over a smooth manifold M, where the dimension of M is n and the dimension of E is $m + n$. We will think of M as the space corresponding to the independent variables, and E as the space parameterising both independent and dependent variables. We denote the space of sections of π by $\Gamma(\pi)$. We can define an equivalence relation on $\Gamma(\pi)$ by defining two sections $s, s' \in \Gamma(\pi)$ to be k-equivalent at a point $x \in M$ if their graphs are tangent to each other with order k at $s(x) = s'(x) \in E$. The *k^{th}-order jet space of sections of* π, $\mathrm{Jets}^k(\pi)$, is the set of equivalence classes of sections at points $x \in M$, equipped with the natural structure of a smooth manifold. The *jet space of sections of* π, denoted $\mathrm{Jets}^\infty(\pi)$ is the limit of the chain

$$\ldots \mathrm{Jets}^{k+1}(\pi) \to \mathrm{Jets}^k(\pi) \ldots \mathrm{Jets}^1(\pi) \to \mathrm{Jets}^0 = E \xrightarrow{\pi} M$$

Definition 6.2.1.1. A *system of k^{th}-order partial differential equations on sections of* π is a closed submanifold $E_0 \subseteq \mathrm{Jets}^k(\pi)$. A *solution of a PDE* is an n-dimensional submanifold N such that $N^k \subseteq E_0$.

Suppose that we have some system $E_0 \subseteq \mathrm{Jets}^k(\pi)$ of PDEs, and define some coordinates $\{x_1, \ldots, x_n, u_I^1, \ldots, u_I^m\}$ for $\mathrm{Jets}^k(\pi)$ where I is all tuples of length k containing elements $(x_1, \ldots, x_n)$. We can interpret the u_I^j as corresponding to all possible partial derivatives up to order k with respect to $x_1, \ldots, x_n$. Then, this system of partial differential equation corresponds to a system of r equations

$$F_i(x_1, \ldots, x_n, u_I^1, \ldots, u_I^m) = 0, \quad i = 1, \ldots, r$$

which is what we would classically think of as a system of PDEs.

Algebraically, suppose that we have a system of partial differential equations E_0 of order k defined using the sheaf of differential operators associated to a smooth algebraic variety X. Then, we can define a *D-module* $\mathcal{M}_{E_0}$ encoding the system, see [23, Chapter 6], and then the space of solutions is given by

$$\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}_{E_0}, \mathcal{O}_X)$$

where $\mathcal{D}_X$ is the sheaf of differential operators on X. There is an equivalence between $\mathcal{D}_X$-modules and quasicoherent sheaves on the de Rham space X_{dR} associated to X.

In the next section, motivated by work of Kryczka, Sheshmani, and Yau [47], we will build an analogous theory in the differential setting, using the definition of the de Rham space for $\mathcal{C}^\infty$-rings provided by Borisov and Kremnizer [18]. We will then extend the theory of partial differential equations to the derived setting.

6.2.2 Derived Partial Differential Equations

Given a scheme X, the associated *de Rham space*, X_{dR}, is the presheaf on affine schemes defined by sending $\mathrm{Spec}(R)$ to $X(\mathrm{Spec}(R/I))$ where R/I is the quotient of R by the nilradical, the ideal of nilpotent elements. A similar idea can be used in the context of $\mathcal{C}^\infty$-schemes, as described in [18]. Recall that there exists a fully faithful sifted colimit preserving functor $\mathcal{C}^\infty : \mathcal{C}^\infty\mathrm{Ring} \to \mathrm{CRing} \subseteq \mathrm{Comm}(\mathrm{Ind}(\mathrm{Ban}_{\mathbb{R}}))$. Using [18, Definition 1], we can define a functor

$$\begin{aligned} R_{nil} : \mathcal{C}^\infty\mathrm{Ring} &\to \mathrm{Comm}(\mathrm{Ind}(\mathrm{Ban}_{\mathbb{R}})) \\ A &\to \mathcal{C}^\infty(A)/\sqrt[nil]{0} \end{aligned}$$

where, for any ideal $I \leq \mathcal{C}^\infty(A)$, $\sqrt[nil]{I} := \{f \in \mathcal{C}^\infty(A) \mid \exists k \in \mathbb{Z}_{>0} \text{ such that } f^k \in I\}$. We note that, since $R_{nil}(A)$ is a sifted colimit of an object in the essential image of $\mathcal{C}^\infty$, then it lies in the essential image of $\mathcal{C}^\infty$. For $A \in \mathcal{C}^\infty\mathrm{Ring}$, we define the *reduced $\mathcal{C}^\infty$-ring*, A^{red}, to be the object in $\mathcal{C}^\infty\mathrm{Ring}$ corresponding to $R_{nil}(A)$.

Definition 6.2.2.1. Suppose that $X = \mathrm{Spec}(A) \in \mathcal{C}^\infty\mathbf{DAff}$.

1. The *reduced derived* $\mathcal{C}^\infty$*-ring*, denoted A^{red}, is defined to be the object of $\mathcal{C}^\infty\mathrm{Ring}$ defined by $A^{red} = \pi_0(A)^{red}$,
2. The *de Rham space* is the object $X_{dR} = \mathrm{Spec}(A^{red})$.

Fix some $X = \mathrm{Spec}(A) \in \mathcal{C}^\infty\mathbf{DAff}$ and consider the induced morphism of representable stacks $p_{dR} : X \to X_{dR}$ in $\mathbf{Stk}(\mathcal{C}^\infty\mathbf{DAff}, \boldsymbol{\tau}_{\mathcal{C}^\infty})$. There is an induced functor

$$p_{dR,*} : \mathbf{Stk}(\mathcal{C}^\infty\mathbf{DAff}, \boldsymbol{\tau}_{\mathcal{C}^\infty})_{/X} \to \mathbf{Stk}(\mathcal{C}^\infty\mathbf{DAff}, \boldsymbol{\tau}_{\mathcal{C}^\infty})_{/X_{dR}}$$

which has a right adjoint given by

$$p_{dR}^* : \mathbf{Stk}(\mathcal{C}^\infty\mathbf{DAff}, \boldsymbol{\tau}_{\mathcal{C}^\infty})_{/X_{dR}} \to \mathbf{Stk}(\mathcal{C}^\infty\mathbf{DAff}, \boldsymbol{\tau}_{\mathcal{C}^\infty})_{/X}$$

The following definitions and their connections with classical notions is discussed in [47, Section 3.5].

Definition 6.2.2.2. Suppose that $\mathcal{F} \in \mathbf{Stk}(\mathcal{C}^\infty\mathbf{DAff}, \boldsymbol{\tau}_{\mathcal{C}^\infty})_{/X}$. Then,

1. The associated *de Rham jet stack* is defined to be $\mathbf{Jets}^\infty_{X_{dR}}(\mathcal{F}) := p_{dR,*}(\mathcal{F})$,
2. The associated *jet stack* is defined to be $\mathbf{Jets}^\infty_X(\mathcal{F}) := p_{dR}^* \circ p_{dR,*}(\mathcal{F})$

By definition of the base-change map, we see that we have a pullback square

$$\begin{tikzcd} \mathbf{Jets}^\infty_X(\mathcal{F}) \arrow[r] \arrow[d] & \mathbf{Jets}^\infty_{X_{dR}}(\mathcal{F}) \arrow[d] \\ X \arrow[r, "p_{dR}"] & X_{dR} \end{tikzcd}$$

in $\mathbf{Stk}(\mathcal{C}^\infty\mathbf{DAff}, \boldsymbol{\tau}_{\mathcal{C}^\infty})$.

Definition 6.2.2.3. [86, c.f. Definition 2.2.3.5] Suppose that we have a morphism $f : \mathcal{F} \to \mathcal{G}$ of stacks in $\mathbf{Stk}(\mathcal{C}^\infty\mathbf{DAff}, \boldsymbol{\tau}_{\mathcal{C}^\infty})$. Then, f is a *closed immersion* if it is (-1)-representable and if, for any $X = \mathrm{Spec}(A) \in \mathcal{C}^\infty\mathbf{DAff}$, the induced morphism

$$\mathcal{F} \times_{\mathcal{G}} X := \mathrm{Spec}(B) \to \mathrm{Spec}(A)$$

induces an epimorphism $\pi_0(A) \to \pi_0(B)$ in $\mathrm{Comm}(\mathrm{Ind}(\mathrm{Ban}_{\mathbb{R}}))$.

In particular, if we have a closed immersion $\mathcal{G} \to \mathbf{Jets}^\infty_{X_{dR}}(\mathcal{F})$, then this also defines a closed immersion $\mathcal{G} \times_{\mathbf{Jets}^\infty_{X_{dR}}(\mathcal{F})} \mathbf{Jets}^\infty_X(\mathcal{F}) \to \mathbf{Jets}^\infty_X(\mathcal{F})$.

Definition 6.2.2.4. Suppose that $\mathcal{F} \in \mathbf{Stk}(\mathcal{C}^\infty\mathbf{DAff}, \boldsymbol{\tau}_{\mathcal{C}^\infty})_{/X}$. Then, a stack $\mathcal{G}$ is a *system of derived non-linear partial differential equations on sections* of $\mathcal{F}$ if there is a closed immersion $\mathcal{G} \to \mathbf{Jets}^\infty_{X_{dR}}(\mathcal{F})$ in $\mathbf{Stk}(\mathcal{C}^\infty\mathbf{DAff}, \boldsymbol{\tau}_{\mathcal{C}^\infty})_{/X_{dR}}$.

We can then naturally make the following definition motivated by the classical theory.

Definition 6.2.2.5. Suppose that $X \in \mathcal{C}^\infty\mathbf{DAff}$ and that $\mathcal{F} \in \mathbf{Stk}(\mathcal{C}^\infty\mathbf{DAff}, \boldsymbol{\tau}_{\mathcal{C}^\infty})_{/X}$. Suppose that $\mathcal{G}$ is a system of derived non-linear partial differential equations on sections of $\mathcal{F}$. Then, *the derived moduli stack of solutions to* $\mathcal{G}$, denoted $\mathbf{Sol}_X(\mathcal{G})$, is the mapping stack of sections

$$\mathbf{Sol}_X(\mathcal{G}) := \underline{\mathrm{Map}}_{\mathbf{Stk}(\mathcal{C}^\infty\mathbf{DAff}, \boldsymbol{\tau}_{\mathcal{C}^\infty})_{/X_{dR}}}(X_{dR}, \mathcal{G})$$

Remark. By [47, Theorem 3.57], if X is a scheme we recover the derived version of the D-module solution space discussed briefly in Section 6.2.1.

6.2.3 Representability of the Derived Moduli Stack of Solutions

We recall that we developed a substantial amount of machinery for proving representability of mapping stacks in Section 5.3. By Theorem 6.1.5.7, there is a representability context

$$(\underline{\mathbf{Ch}}(\mathrm{Ind}(\mathrm{Ban}_{\mathbb{R}})), \boldsymbol{\tau}_{\mathcal{C}^\infty}, \mathbf{open}_{\mathcal{C}^\infty}, \mathcal{C}^\infty\mathbf{DBAff}, \mathbf{Mod}, \mathbf{hm})$$

We will suppose for the rest of this section that $X \in \mathcal{C}^\infty\mathbf{DAff}^\heartsuit \simeq \mathcal{C}^\infty\mathrm{Aff}$ and that $\mathcal{F} \in \mathbf{Stk}(\mathcal{C}^\infty\mathbf{DAff}, \boldsymbol{\tau}_{\mathcal{C}^\infty})_{/X}$. Suppose that $Y = \mathrm{Spec}(B)$ is a (-1)-representable system of non-linear partial differential equations on sections of $\mathcal{F}$. Consider $\mathbf{Sol}_X(Y)$ as a stack in $\mathbf{Stk}(\mathcal{C}^\infty\mathbf{DAff}, \boldsymbol{\tau}_{\mathcal{C}^\infty}) \subseteq \mathbf{Stk}(\mathcal{C}^\infty\mathbf{DBAff}, \boldsymbol{\tau}_{\mathcal{C}^\infty})$. We will show that it is geometric as a derived $\mathcal{C}^\infty$-bornological stack.

Recall from Corollary 6.1.3.5 that there is an adjunction

$$L : \mathrm{LH}(\mathrm{CBorn}_{\mathbb{R}}) \leftrightarrows \mathcal{C}^\infty\mathrm{BornRing} : R$$

Definition 6.2.3.1. We will say that $B \in \mathcal{C}^\infty\mathrm{Ring}$ is *of special presentation* if it is in the essential image of the functor $L : \mathrm{LH}(\mathrm{CBorn}_{\mathbb{R}}) \to \mathcal{C}^\infty\mathrm{BornRing}$.

Example 6.2.3.2. $\mathbb{R}^n \in \mathcal{C}^\infty\mathrm{BornRing}$ *for each* $n \in \mathbb{N}$ *is an object of special presentation, as well as any colimit of objects of this form. In particular, any* $\mathcal{C}^\infty$*-ring of finite presentation will be of special presentation.*

Suppose that $A \in \mathcal{C}^\infty\mathrm{Ring}$ and that $\mathcal{C}^\infty(A)$ is strongly dualisable and reflexive as a convenient space in the sense of Definition B.0.0.1. For example, if A is a finitely presented $\mathcal{C}^\infty$-ring, using our notation, then $\mathcal{C}^\infty(A)$ is a strongly dualisable and reflexive convenient space by [46, Result 6.5.7] and [35, Theorem 5.3.2]. Using our characterisation of $R(A)$ from Lemma 6.1.3.6, we can see that $R(A)$ is also strongly dualisable and reflexive.

Lemma 6.2.3.3. *Suppose that $Y = \mathrm{Spec}(B)$ is a (-1)-representable system of partial differential equations on sections of $\mathcal{F}$, with $B \in \mathcal{C}^\infty\mathrm{Ring}$ of special presentation. Suppose that $X = \mathrm{Spec}(A)$ is such that $\mathcal{C}^\infty(A^{red})$ is strongly dualisable and reflexive as a convenient space. Then, $t_0(\mathbf{Sol}_X(Y))$ is (-1)-representable by an object of $\mathcal{C}^\infty\mathrm{BAff}$.*

Proof. Since B is of special presentation it can be written as $L(B')$ where $B' \in \mathrm{LH}(\mathrm{CBorn}_{\mathbb{R}})$. Suppose that $Z = \mathrm{Spec}(C) \in \mathcal{C}^\infty\mathrm{Aff}$. Then,

$$\begin{aligned} t_0(\mathbf{Sol}_X(Y))(C) &= t_0(\underline{\mathrm{Map}}_{\mathbf{Stk}(\mathcal{C}^\infty\mathbf{DAff},\boldsymbol{\tau}_{\mathcal{C}^\infty})_{/X_{dR}}}(X_{dR}, Y))(C) \\ &\simeq \mathrm{Hom}_{\mathcal{C}^\infty\mathrm{BAff}_{/X_{dR}}}(X_{dR} \times Z, Y) \\ &\simeq \mathrm{Hom}_{A^{red}/\mathcal{C}^\infty\mathrm{BornRing}}(L(B'), A^{red} \otimes C) \end{aligned}$$

Now, using the adjunction from Corollary 6.1.3.5,

$$\simeq \mathrm{Hom}_{R(A^{red})/\mathrm{LH}(\mathrm{CBorn}_{\mathbb{R}})}(B', R(A^{red} \otimes C))$$

Now, by Proposition 6.1.3.6,

$$\simeq \mathrm{Hom}_{R(A^{red})/\mathrm{LH}(\mathrm{CBorn}_{\mathbb{R}})}(B', R(A^{red})\hat{\otimes}R(C))$$

Since $R(A^{red})$ is strongly dualisable and reflexive,

$$\begin{aligned} &\simeq \mathrm{Hom}_{R(A^{red})/\mathrm{LH}(\mathrm{CBorn}_{\mathbb{R}})}(B'\hat{\otimes}R(A^{red})^\vee, R(C)) \\ &\simeq \mathrm{Hom}_{A^{red}/\mathcal{C}^\infty\mathrm{BornRing}}(L(B'\hat{\otimes}R(A^{red})^\vee), C) \\ &\simeq \mathrm{Hom}_{\mathcal{C}^\infty\mathrm{BAff}_{/X_{dR}}}(Z, \mathrm{Spec}(L(B'\hat{\otimes}R(A^{red})^\vee))) \end{aligned}$$

Therefore, we see that $t_0(\mathbf{Sol}_X(Y))$ is equivalent, as a stack in $\mathbf{Stk}(\mathcal{C}^\infty\mathrm{Aff}, \boldsymbol{\tau}^{\heartsuit}_{\mathcal{C}^\infty})$, to the stack represented by $\mathrm{Spec}(L(B'\hat{\otimes}R(A^{red})^\vee)) \in \mathcal{C}^\infty\mathrm{BAff}$. $\square$

We will restrict our attention to elliptic PDEs, i.e. ones with perfect cotangent complex.

Definition 6.2.3.4. A system of derived non-linear partial differential equations $\mathcal{G}$ on sections of $\mathcal{F} \in \mathbf{Stk}(\mathcal{C}^\infty\mathbf{DAff}, \boldsymbol{\tau}_{\mathcal{C}^\infty})_{/X}$ is a *system of derived non-linear elliptic partial differential equations* if it is an n-geometric stack and the global cotangent complex $\mathbb{L}_{\mathcal{G}/X_{dR}}$ is an object of $\mathbf{Perf}(\mathcal{G})$.

Corollary 6.2.3.5. *Suppose that*

1. $Y = \mathrm{Spec}(B)$ *is a* (-1)*-representable system of non-linear elliptic partial differential equations on sections of* $\mathcal{F}$*, where* B *is of special presentation (for example, a finitely presented* $\mathcal{C}^\infty\mathrm{Ring}$*),*
2. $X = \mathrm{Spec}(A)$ *is such that* $\mathcal{C}^\infty(A^{red})$ *is strongly dualisable and reflexive as a convenient space.*

Then, $\mathbf{Sol}_X(Y)$ *is a* (-1)*-geometric stack in* $\mathbf{Stk}_n(\mathcal{C}^\infty\mathbf{DBAff}, \tau_{\mathcal{C}^\infty}, \mathbf{open}_{\mathcal{C}^\infty})$*.*

Proof. This follows from Corollary 5.3.3.2 together with Lemma 6.2.3.3. □

Future Research Directions

In this thesis, we have developed several useful tools for derived bornological geometry based on the foundational work by Kremnizer et al. [10]. In particular, we now have a strong representability theorem which holds in many different contexts and provides a new perspective on the study of moduli stacks of solutions to non-linear elliptic partial differential equations. The scope of potential future research is wide and is sketched in this section.

Smooth Functions on Contracting Coproducts

A major next step in my research is to prove Conjecture 6.1.4.2 as this will mean that we have a good representability context for derived $\mathcal{C}^\infty$-bornological geometry.

The category $\mathrm{Ban}_{\mathbb{R}}^{\leq 1}$ is the category whose objects are Banach spaces but morphisms are contracting linear maps, i.e. those of norm less than or equal to 1. The coproduct (resp. product) in this category is the *contracting coproduct (resp. contracting product)* $\coprod^{\leq 1}$ (resp. $\prod^{\leq 1}$). We can consider any $\ell^1(\kappa) \in \mathrm{Lin}$ as a contracting coproduct of $\mathbb{R}$, and then consider the resulting object in $\mathrm{Ban}_{\mathbb{R}}$.

We note that, by [45, Proposition 2.11], for any collection $X_i \in \mathrm{CBorn}_{\mathbb{R}}$ indexed by $i \in I$, there is an equivalence $(\coprod_{i\in I}^{\leq 1} X_i)^\vee \simeq \prod_{i\in I}^{\leq 1} X_i^\vee$. We hope that there is a similar relationship when we consider smooth functionals, i.e. some relation between $\mathcal{C}^\infty(\coprod_{i\in I}^{\leq 1} \mathbb{R})$ and $\prod_{i\in I}^{\leq 1} \mathcal{C}^\infty(\mathbb{R})$. Then, since any $\ell^1(\kappa)$ can be written as a contracting coproduct of $\mathbb{R}$s, we would be able to characterise smooth functions on them.

If $\ell^1(\kappa) \simeq \coprod_{i\in I}^{\leq 1} \mathbb{R}$, we propose that the appropriate algebra to take in Conjecture 6.1.4.2 is $\widetilde{\mathcal{C}^\infty}(\ell^1(\kappa)) := \prod^{\leq 1} \mathcal{C}^\infty(\mathbb{R})$. We can then use the statement that polynomials are bornologically dense in $\mathcal{C}^\infty(\mathbb{R})$ along with an appropriately defined Koszul complex to show that the morphism $\mathrm{Sym}(\ell^\infty(\kappa)) \to \widetilde{C^\infty}(\ell^1(\kappa))$ is a homotopy epimorphism.

Further, in the proof of Lemma 6.1.3.6, we assumed that our object A was a $\mathcal{C}^\infty$-ring. A main feature of the proof was that $\lambda\mathbb{R}^n \simeq \mathcal{C}^\infty(\mathbb{R}^n)^\vee$. With new results about contracting coproducts, we could perhaps characterise $\lambda\ell^1(\kappa)$ as the dual of some nuclear space and extend our result to $\mathcal{C}^\infty\mathrm{BornRing}$.

Derived $\mathcal{C}^\infty$-Bornological Geometry

In order to develop derived $\mathcal{C}^\infty$-bornological geometry further, there are a number of important constructions to make. In particular, we could define the category of derived bornological manifolds to be the opposite category of finitely presented derived $\mathcal{C}^\infty$-bornological rings. We would then like this category to satisfy the property that there exists a fully faithful functor into it from an appropriate closed monoidal subcategory of convenient manifolds which, in particular, preserves transverse pullbacks and the final object, and that it should be universal with respect to this property.

An alternative approach to defining $\mathcal{C}^\infty$-bornological rings, which is perhaps more in keeping with the condensed mathematics framework of Clausen and Scholze [77] would be to define some category of *ultrasolid $\mathcal{C}^\infty$-rings*. This would be the sifted pro-completion

$$\mathrm{SPro}(\mathcal{C}^\infty\mathrm{Ring}) := \mathrm{SInd}(\mathcal{C}^\infty\mathrm{Ring}^{op})^{op}$$

of the category of $\mathcal{C}^\infty$-rings. The functor $\mathrm{Cpt} : \mathcal{C}^\infty\mathrm{Ring} \to \mathrm{Comm}(\mathrm{CBorn}_{\mathbb{R}})$ from [17, Corollary 7], which considers any $\mathcal{C}^\infty$-ring as a Fréchet space and then endows it with the precompact bornology, can be extended to a limit-preserving monoidal functor $\widetilde{\mathrm{Cpt}}$. Hence, we can use the resulting adjunction to prove representability of the truncated stack as in Lemma 6.2.3.3.

Deformation Theory

Suppose that we have a derived algebraic context $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, \mathcal{C}^0)$. There should be a certain suitable class $\mathbf{DAlg}^{sm}(\mathcal{C})$ of 'small' objects in $\mathbf{DAlg}(\mathcal{C})$, for example reflexive algebras, such that we can obtain an appropriate definition of a formal moduli problem $X : \mathbf{DAlg}^{sm}(\mathcal{C}) \to \infty\mathbf{Grpd}$ as in [55, Definition 0.0.8]. In these contexts we hope to obtain certain equivalences between the categories of moduli problems and categories of differential graded bornological Lie algebras. In the derived smooth geometry context, as discussed in Chapter 6, we should be able to recover the classical Kodaira-Spencer maps.

Six-Functor Formalisms for Geometric Stacks

In [49], Liu and Zheng use the notion of a category of correspondences to define six-functor formalisms for higher Artin stacks. The six-functors are defined on the derived categories of étale sheaves on the higher Artin stacks and are instrumental in proving results about étale cohomology of Artin stacks. We would hope to be able to prove a generalisation for geometric stacks defined on a suitable derived geometry context. There is currently work proving other types of six-functor formalisms in these new theories of derived analytic geometry, namely in [80] for quasi-coherent sheaves on dagger analytic varieties and in [39] for rigid sheaves.

Representability of Mapping Stacks

In Section 5.3, we studied conditions under which mapping stacks are representable and obtained Corollary 5.3.3.2. There should be several variations of this result as the result in Proposition 5.3.3.1 is satisfied under very general conditions. In particular, we need to explore when the diagram in Equation (5.4) satisfies perfect base change for any such morphism u.

In future work, we hope to define suitable representability contexts for doing derived non-archimedean analytic geometry. It is not currently clear whether the theory of Porta and Yue Yu [65] embeds into the bornological framework. However, in [67, Theorem 6.8], they show that in their model of derived non-archimedean geometry, under conditions such as that π is proper and has finite coherent cohomological dimension, (π, u) satisfies perfect base-change. We expect to find similar conditions for our derived geometry contexts.

Derived Moduli Stacks of Galois Representations

Recall that we have the following derived complex analytic geometry context

$$(\underline{\mathbf{Ch}}(\mathrm{Ind}(\mathrm{Ban}_{\mathbb{C}})), \boldsymbol{hm}^{fin}_{\mathbf{DSt}}, \mathbf{fP}^{hm}_{\mathbf{DSt}}, \mathbf{DSt}^{op}, \mathbf{Mod}^{coad,cn}, \mathbf{hm})$$

from Corollary 5.2.8.7. In this setting, as well as perhaps in appropriately defined derived non-archimedean analytic geometry settings, we hope to study the moduli stack of Galois representations.

Suppose that we have a representability context $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}, \mathcal{C}^0, \boldsymbol{\tau}, \mathbf{P}, \mathcal{A}, \mathbf{M}, \mathbf{S})$ and that $\mathbf{G}$ is a (Galois) group object in $\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})$. Then, we can define the *derived moduli stack of n-dimensional (Galois) representations of* $\mathbf{G}$ by

$$\mathbf{Rep}_n(\mathbf{G}) := \underline{\mathrm{Map}}_{\mathbf{Stk}(\mathcal{A}, \boldsymbol{\tau}|_{\mathcal{A}})}(\mathbf{BG}, \mathbf{BGL}_n)$$

where $\mathbf{BG}$ is the quotient stack $[*/\mathbf{G}]$, defined by $\varinjlim_{n \in \Delta^{op}} \mathbf{G}^n \times *$. In our derived complex analytic geometry setting, a simple application of our representability theorem should provide conditions under which this is represented by a derived complex analytic space.

Appendix A

Symmetric Algebra Objects

Suppose that C is a symmetric monoidal category with countable coproducts. If we consider the category Mon(C) of (associative) monoids in C, then the forgetful functor $\mathrm{Mon(C)} \to \mathrm{C}$ has a left adjoint $T : \mathrm{C} \to \mathrm{Mon(C)}$. For a monoid $A \in \mathrm{E}$, the *tensor monoid*, denoted by $T(A)$, is the graded monoid

$$T(A) = \bigoplus_{n=0}^{\infty} T_n(A) := \bigoplus_{n=0}^{\infty} A^{\otimes n}$$

with multiplication $\mu : T(A) \otimes T(A) \to T(A)$ determined by the canonical isomorphism

$$\mu_{i,j} : T_i(A) \otimes T_j(A) \to T_{i+j}(A)$$

The unit is the inclusion $\eta : T_0(V) \to T(V)$.

Suppose that, in addition, C has finite coequalisers. If we consider the category Comm(C) of commutative monoids in C, then the forgetful functor $\mathrm{Comm(C)} \to \mathrm{C}$ has a left adjoint $\mathrm{Sym} : \mathrm{C} \to \mathrm{Comm(C)}$. For a monoid $A \in \mathrm{E}$, the symmetric group Σ_n acts on $T_n(A) = A^{\otimes n}$ by permutation. Let $\mathrm{Sym}^n(A)$ be the coequaliser of all the maps σ, for $\sigma \in \Sigma_n$. Then, the *symmetric monoid on* A, $\mathrm{Sym}(A)$, can be defined as

$$\mathrm{Sym}(A) = \bigoplus_{n \geq 0} \mathrm{Sym}^n(A)$$

Let $\bigwedge^n A$ be the coequaliser of all the maps $sgn(\sigma)\sigma$ for $\sigma \in \Sigma_n$. Then, the *exterior monoid* on A, $\bigwedge A$, is defined to be

$$\bigwedge A = \bigoplus_{n \geq 0} \bigwedge^n A$$

Appendix B

Perfect, Compact, Dualisable Objects

Let $\mathcal{C}$ be an additive closed symmetric monoidal $(\infty, 1)$-category with monoidal product $\otimes^{\mathbb{L}}$ and unit I. Denote the internal mapping space by $\underline{\mathrm{Map}}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$. We fix the following definitions.

Definition B.0.0.1. Suppose that $A \in \mathcal{C}$. Then its *dual* object is $A^{\vee} := \underline{\mathrm{Map}}_{\mathcal{C}}(A, I)$. An object $A \in \mathcal{C}$ is

1. *compact* if $\mathrm{Map}_{\mathcal{C}}(A, -)$ commutes with filtered colimits,
2. *projective* if $\mathrm{Map}_{\mathcal{C}}(A, -)$ commutes with geometric realisations,
3. *perfect* if it is a retract of a finite colimit of objects of the form $\coprod_E I$ for some finite set E,
4. *dualisable* if the map $A^{\vee} \otimes^{\mathbb{L}} A \to \underline{\mathrm{Map}}_{\mathcal{C}}(A, A)$ is an equivalence,
5. *strongly dualisable* if the map $A^{\vee} \otimes^{\mathbb{L}} B \to \underline{\mathrm{Map}}_{\mathcal{C}}(A, B)$ is an equivalence for any $B \in \mathcal{C}$,
6. *reflexive* if $(A^{\vee})^{\vee} \simeq A$.

We note that perfect objects are strongly dualisable and projective. We will frequently use the following result.

Lemma B.0.0.2. *If $\mathcal{C}$ is additionally a stable $(\infty, 1)$-category and $A \to B \to C$ is a fibre sequence in $\mathcal{C}$ with two of the objects strongly dualisable, then the third object is strongly dualisable.*

Proof. Suppose that $D \in \mathcal{C}$. The result follows from considering the morphism of fibre-cofibre sequences induced by $(-)^{\vee} \otimes^{\mathbb{L}} D$ and $\underline{\mathrm{Map}}_{\mathcal{C}}(-, D)$ and then applying the five lemma. $\square$

Appendix C

Sifted and Filtered Free Cocompletions

C.1 Ind-Objects

Suppose that C is a small category with finite colimits. Then,

Definition C.1.0.1. The *free filtered cocompletion of* C, denoted Ind(C), is defined to be the subcategory of presheaves consisting of those which preserve small limits

$$\mathrm{Ind}(\mathrm{C}) := \mathrm{Fun}^{lex}(\mathrm{C}^{op}, \mathrm{Set})$$

We will occasionally refer to this category as *the category of Ind-objects in* C.

Consider the category $\widetilde{\mathrm{Ind}}(\mathrm{C})$ whose objects are functors $X : I \to \mathrm{C}$ where I is a small filtered category and morphisms are defined by

$$\mathrm{Hom}_{\widetilde{\mathrm{Ind}}(\mathrm{C})}(X, Y) := \varprojlim_{i \in I} \varinjlim_{j \in J} \mathrm{Hom}_{\mathrm{C}}(X(i), Y(j))$$

There is a functor

$$\begin{aligned} \widetilde{\mathrm{Ind}}(\mathrm{C}) &\to \mathrm{Ind}(\mathrm{C}) \\ X &\to \varinjlim_{i \in I} h_{X(i)} \end{aligned}$$

By [37, Proposition 6.1.7], this functor defines an equivalence of categories. For ease of notation, we will define the *formal filtered colimit* of objects in C to be

$$\text{“}\varinjlim_{i \in I}\text{”} X_i := \varinjlim_{i \in I} h_{X(i)}$$

Remark. In this thesis, we will frequently switch between using these different types of notation.

Proposition C.1.0.2. *The category* Ind(C) *satisfies several useful properties*

1. Ind(C) *has small filtered colimits and the functor* $\mathrm{Ind(C)} \to \mathrm{PSh(C)}$ *preserves them,*
2. *The inclusion functor* $\iota_{\mathrm{C}} : \mathrm{C} \to \mathrm{Ind(C)}$ *is right exact,*
3. *If* C *has small limits, then so does* Ind(C) *and moreover the natural functor* $\mathrm{C} \to \mathrm{Ind(C)}$ *preserves small limits.*

Proof. Follows from Corollary 6.1.6, Theorem 6.1.8, and Corollary 6.1.17 in [37]. □

In particular, we have the following.

Proposition C.1.0.3. *[37, Proposition 6.1.9] Suppose that we have a functor* $F : \mathrm{C} \to \mathrm{D}$*. Then, there exists a unique functor* $IF : \mathrm{Ind(C)} \to \mathrm{Ind(D)}$ *such that the restriction to* C *is* F *and such that* F *commutes with small filtered colimits.*

Proposition C.1.0.4. *Suppose we have a category* C*,*

1. *If* C *is quasi-abelian, then so is* Ind(C)*,*
2. *If* C *is closed symmetric monoidal with enough (flat) projectives, then so is* Ind(C)*.*

Proof. Follows from [74, Example 2.10] and [76, Proposition 2.1.19]. The closed structure is induced by the closed structure on C as follows,

$$\text{“}\varinjlim_{i\in I}\text{”} X_i \otimes \text{“}\varinjlim_{j\in J}\text{”} Y_j := \text{“}\varinjlim\text{”}_{(i,j)\in I\times J} X_i \otimes Y_j$$

and

$$\underline{\mathrm{Hom}}_{\mathrm{Ind(C)}}(\text{“}\varinjlim_{i\in I}\text{”} X_i, \text{“}\varinjlim_{j\in J}\text{”} Y_j) := \varprojlim_{i\in I}\varinjlim_{j\in J}\underline{\mathrm{Hom}}_{\mathrm{C}}(X_i, Y_j)$$

□

The category Ind(C) is in general quite hard to work with as it is not necessarily concrete, i.e. it doesn't come equipped with a faithful functor $\mathrm{Ind(C)} \to \mathrm{Set}$.

Definition C.1.0.5. An object $X \in \mathrm{Ind(C)}$ is *essentially monomorphic* if it is isomorphic to an object in Ind(C) of the form $X' : I \to \mathrm{C}$ where, for every morphism $i \to j$ in I, the corresponding morphism $X'(i) \to X'(j)$ in C is a monomorphism.

We note that, if a category C is concrete and the functor $\mathrm{C} \to \mathrm{Set}$ preserves and reflects monomorphisms, then by [4, Proposition 3.31], Ind(C) is a concrete category.

C.2 SInd-Objects

Suppose that C is a category with finite coproducts. We recall that a sifted colimit is a colimit which commutes with finite products.

Definition C.2.0.1. The *free sifted cocompletion of* C, denoted SInd(C), is defined to be the subcategory of presheaves consisting of those which preserve small products,

$$\mathrm{SInd}(\mathrm{C}) := \mathrm{Fun}^{\times}(\mathrm{C}^{op}, \mathrm{Set})$$

Lemma C.2.0.2. *Suppose that we have a fully faithful functor $i : \mathrm{C} \to \mathrm{D}$ for C, D with finite coproducts. Then, there is a fully faithful functor $i : \mathrm{SInd}(\mathrm{C}) \to \mathrm{SInd}(\mathrm{D})$.*

Proof. This follows from [38, Theorem 4.99]. □

Lemma C.2.0.3. *Suppose that we have a functor $F : \mathrm{C} \to \mathrm{D}$ between two categories* C *and* D*. Suppose that* C *has finite coproducts.*

1. *If* D *has sifted colimits, then F extends to a sifted colimit-preserving functor $\tilde{F} : \mathrm{SInd}(\mathrm{C}) \to \mathrm{D}$,*
2. *If* D *has all colimits and* F *preserves finite coproducts, then F extends to a colimit-preserving functor $\tilde{F} : \mathrm{SInd}(\mathrm{C}) \to \mathrm{D}$.*

Proof. These are standard results and can be found in [2]. □

As in the previous section, we can use [2, Corollary 2.7] to argue that SInd(C) is equivalent to the category whose objects are functors $X : I \to \mathrm{C}$, where I is a small sifted category and morphisms are defined appropriately. Hence, we can write the objects of SInd(C) as *formal sifted colimits of objects* in C as

$$\text{“}\varinjlim_{i \in I}\text{”} X_i := \varinjlim_{i \in I} h_{X(i)}$$

We note that, in the context of $(\infty, 1)$-categories, it is customary to use the notation $\mathcal{P}_{\Sigma}(\mathcal{C})$ to denote the free sifted cocompletion. By definition, this is the following category of product preserving $(\infty, 1)$-functors

$$\mathcal{P}_{\Sigma}(\mathcal{C}) := \mathbf{Fun}^{\times}(\mathcal{C}^{op}, \infty\mathbf{Grpd})$$

We have the following important results analogous to the above.

Proposition C.2.0.4. *[52, Proposition 5.5.8.15] Let $\mathcal{C}$ be a small $(\infty, 1)$-category with finite coproducts and let $\mathcal{D}$ be an $(\infty, 1)$-category with filtered colimits and geometric realisations. Then, there is an equivalence of categories*

$$\mathrm{Fun}_{\Sigma}(\mathcal{P}_{\Sigma}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

where $\mathrm{Fun}_{\Sigma}(\mathcal{P}_{\Sigma}(\mathcal{C}), \mathcal{D})$ denotes the subcategory of functors preserving filtered colimits and geometric realisations. Moreover, any functor in $\mathrm{Fun}_{\Sigma}(\mathcal{P}_{\Sigma}(\mathcal{C}), \mathcal{D})$ preserves sifted colimits. If $\mathcal{D}$ has finite coproducts then any finite coproduct preserving functor $\mathcal{C} \to \mathcal{D}$ can be extended to a small colimit-preserving functor $\mathcal{P}_{\Sigma}(\mathcal{C}) \to \mathcal{D}$.

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