On the projective normality of an embedding of the moduli space of stable marked rational curves into projective space

by

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Abstract. We study an embedding of the fine moduli space of stable rational curves with n marked points into projective space for each $n \geqslant 5$. This embedding is developed from work of A. Gibney and D. Maclagan. It is related to the embedding given by the log canonical system, which is shown to be projectively normal in a paper by S. Keel and J. Tevelev using methods in sheaf cohomology. In this thesis, we prove the projective normality of our embedding for $n = 5$ and $n = 6$ using combinatorial methods. From this we can give, in a new way, a description of equations defining $\overline{M}_{0,6}$ as a projective variety. We begin work on extending these ideas to obtain analogous results for each $n \geqslant 7$, as well as give conjectures on equations defining $\overline{M}_{0,n}$ as a projective variety in this general case.

1 Introduction

The theory of moduli spaces provides a valuable setting in which one can study a given class of objects in algebraic geometry. For our purposes, a moduli space parametrising a given class of objects will be a scheme over $\mathbb{C}$ whose closed points correspond one-to-one with (equivalence classes of) these objects. All of our schemes will be over $\mathbb{C}$, although many of the arguments we use may be generalised to any algebraically closed field with characteristic 0.

One of the most fundamental examples of a moduli space is the projective n-space, $\mathbb{P}^n = \mathbb{P}^n_{\mathbb{C}}$, defined for each (integer) $n \geqslant 1$. A more complex example that begins to show how moduli theory deserves study as a topic of its own is the moduli space, $M_{0,n}$, for a given integer $n \geqslant 3$. The scheme $M_{0,n}$ parametrises smooth, complete, rational curves with n distinct labelled (closed) points. For $n \geqslant 4$, the scheme $M_{0,n}$ is smooth but not complete (the corresponding analytic space is not compact [Ser56]). In this case, it is of interest to mathematicians to find a suitable completion, $\overline{M}_{0,n}$. Complete moduli spaces allow one to use tools from intersection theory and deformation theory to study the geometry of the objects being parametrised.

We start by defining fine moduli spaces, which arise from moduli functors, and we give $\mathbb{P}^n$ as a detailed example. We then give a thorough introduction to $M_{0,n}$ and its usual completion $\overline{M}_{0,n}$, which is also a smooth fine moduli space. The boundary, $\partial M_{0,n} = \overline{M}_{0,n} \backslash M_{0,n}$, is studied both structurally and combinatorially in §2.4. We will later see that this boundary is the union of prime divisors whose equivalence classes generate the divisor class group of $\overline{M}_{0,n}$.

The log canonical bundle on $\overline{M}_{0,n}$ is the line bundle associated with the divisor class, κ, that is the sum of the canonical class and all boundary divisors. In [KT09], it is shown that κ is very ample and that the associated complete linear system gives rise

to a closed embedding of $\overline{M}_{0,n}$ into $\mathbb{P}^{(n-2)!-1}$. In the same paper, it is proven that this embedding is projectively normal. This means that for each $l \geqslant 0$, the induced map on global sections,

$$H^0(\mathbb{P}^{(n-2)!-1}, \mathcal{O}_{\mathbb{P}^{(n-2)!-1}}(l)) \longrightarrow H^0(\overline{M}_{0,n}, \mathcal{O}_{\overline{M}_{0,n}}(l\kappa)),$$

is surjective. In this thesis, we will work towards an alternative proof of this fact using combinatorial methods. This takes inspiration from a method used in [HMSV09], where the monomials of a polynomial ring are represented by graphs. A benefit to this approach is that one can hope to go on to give a combinatorial description of equations defining $\overline{M}_{0,n}$ as a projective variety.

The first step is to embed $\overline{M}_{0,n}$ as a closed subscheme of a smooth toric variety, X_Δ. The relevant toric theory is outlined in Chapter 3. In Chapter 4, we describe $M_{0,n}$ and $\overline{M}_{0,n}$ as quotients of a Grassmannian by the action of an algebraic group and use this to give equations defining $\overline{M}_{0,n}$ as a subscheme of X_Δ in the Cox ring, S, of X_Δ. We also show that the homomorphism, $\mathrm{Cl}(X_\Delta) \to \mathrm{Cl}(\overline{M}_{0,n})$, induced by the inclusion, is an isomorphism. Since $\overline{M}_{0,n}$ and X_Δ are both smooth, the Picard group and divisor class group coincide for each scheme.

In Chapter 5 we compute the divisor class κ in an explicit basis of $\mathrm{Cl}(\overline{M}_{0,n})$ as a subgroup of a free abelian group. If $\kappa' \in \mathrm{Cl}(X_\Delta)$ is the unique pre-image of κ, then this calculation also gives κ' in an explicit basis. For each $l \geqslant 0$ we get an induced homomorphism of global sections,

$$H^0(X_\Delta, \mathcal{O}_{X_\Delta}(l\kappa')) \longrightarrow H^0(\overline{M}_{0,n}, \mathcal{O}_{\overline{M}_{0,n}}(l\kappa)).$$

These homomorphisms are the graded pieces of a $\mathbb{C}$-algebra homomorphism from a subring, $S^{\kappa'}$ of S, to a subring of the total coordinate ring of $\overline{M}_{0,n}$. In §5.2 we will give a conjecture that the quotient of $S^{\kappa'}$ by a homogeneous ideal $I^{\kappa'}$ gives a homogeneous

coordinate ring for $\overline{M}_{0,n}$. We will prove this conjecture for $n = 5$ and $n = 6$. An explanation of how one may go on to prove this conjecture is given in Chapter 6. Assuming this conjecture is true, then for each $n \geqslant 5$ we obtain an embedding of $\overline{M}_{0,n}$ into $\mathbb{P}^N$ for some $N > 0$.

Proving that this embedding of $\overline{M}_{0,n}$ into $\mathbb{P}^N$ is projectively normal is equivalent to showing that $(S/I)^\kappa$ is generated as a $\mathbb{C}$-algebra by degree one elements. A basis for the graded piece $(S^\kappa)_l$, for each $l \geqslant 0$, is more intuitively realised by a collection of graphs. In Chapter 6 this correspondence is defined. We use this correspondence in Chapter 7 to prove the projective normality of the studied embedding in the cases $n = 5$ and $n = 6$.

In Chapter 8 we focus on the $n = 6$ case. By hand, we find a basis for the space $(S^\kappa)_1$. Using the combinatorial methods established in §6.2, we find a set of 24 monomials in $(S/I)^\kappa_1$ that generate $(S/I)^\kappa$ as a $\mathbb{C}$-algebra. This implies that $\overline{M}_{0,6}$ is a closed subscheme of $\mathbb{P}^{23}$. In Chapter 9 we begin to explore properties of the ideal of equations defining $\overline{M}_{0,6}$ in $\mathbb{P}^{23}$. In particular, we make progress on showing that $\overline{M}_{0,6}$ in $\mathbb{P}^{23}$ is cut out by quadrics, something also proven in [KT09]. Unlike this paper however, our method would automatically provide us with a combinatorial description of a finite set of quadric equations that cut out $\overline{M}_{0,6}$. We also note that $23 = (6-2)! - 1$ is the dimension of the embedding space in [KT09], raising questions on how our embedding may be related to the one associated with the complete linear system of κ.

Chapter 10 comments on obstacles that arise when trying to extend our method of proof for the projective normality in the $n = 5$ and $n = 6$ cases to the general case. In contrast, we also give reasons for optimism in why one could hope to extend this graphical method to obtain an inductive proof for all $n \geqslant 5$. We conclude the thesis by combining our work with results in the literature to suggest a finite set of degree two equations that may define $\overline{M}_{0,n}$ as a closed subscheme of projective space for general $n \geqslant 5$.

Basics and notation

Let Sch be the category of separated schemes of finite type over $\mathbb{C}$. By a scheme, we will always mean an object in Sch, unless stated otherwise. We assume knowledge of the material in [Har97, §II-III]. We give a brief introduction, more for notation purposes, to some of this below.

A scheme is affine if it is isomorphic to $\operatorname{Spec} R$ for some finitely generated $\mathbb{C}$-algebra, R. A general scheme has an open cover by affine schemes. Any morphism of affine schemes, $f : \operatorname{Spec} R' \to \operatorname{Spec} R$, is determined by a unique $\mathbb{C}$-algebra homomorphism, $R \to R'$, which is denoted by f^*. In this way, there is an equivalence of categories between affine schemes and finitely generated $\mathbb{C}$-algebras.

Definition 1.1. A finitely generated $\mathbb{C}$-algebra, S, is graded by an abelian group, A (we may also say that S is A-graded), if there is a direct sum decomposition,

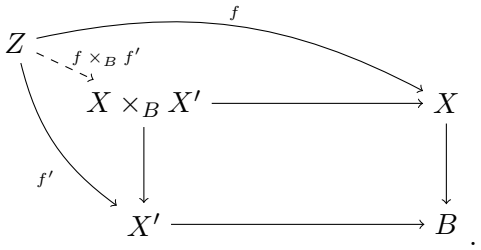
$$S = \bigoplus_{a \in A} S_a,$$

so that for each pair a, a' in A, whenever $s \in S_a$ and $s' \in S_{a'}$, we have $s \cdot s' \in S_{a+a'}$.

Let S be a finitely generated, $\mathbb{Z}$-graded $\mathbb{C}$-algebra with $S_a = 0$ for each $a < 0$, then we can form the scheme $\operatorname{Proj} S$. If $S = \mathbb{C}[x_0, \ldots, x_n]$ with its usual $\mathbb{Z}$-grading, then $\operatorname{Proj} S$ is projective n-space over $\mathbb{C}$. If S' is another finitely generated, $\mathbb{Z}$-graded $\mathbb{C}$-algebra, then a homomorphism of $\mathbb{Z}$-graded $\mathbb{C}$-algebras, $S \to S'$, induces a rational map $\operatorname{Proj} S' \dashrightarrow \operatorname{Proj} S$. Note that there may be rational maps between these two schemes that are not induced by any $\mathbb{C}$-algebra homomorphism between their homogeneous coordinate rings.

If X and X' are schemes over a scheme B (called the base), then we can form the fibre product, $X \times_B X'$, of X and X' over B. If $B = \operatorname{Spec} \mathbb{C}$ we will just write $X \times X'$ instead of $X \times_{\operatorname{Spec} \mathbb{C}} X'$. Fibre products are defined by their universal property which we give now as a proposition.

Proposition 1.2. Let X and X' be schemes over a base B. Let Z be a scheme and suppose we are given morphisms, $f : Z \to X$ and $f' : Z \to X'$, that agree over B. Then there is a unique morphism, which we denote by $f \times_B f'$, from Z to $X \times_B X'$, so that the diagram below commutes.



□

If X is a scheme, then by a point of X we will mean a closed point. Furthermore, if $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules on X, then $H^0(X, \mathcal{F})$ is the $\mathbb{C}$-vector space of global sections of $\mathcal{F}$ on X. If D is a Cartier divisor on X, whose class is $\alpha \in \mathrm{Pic}(X)$, then both $\mathcal{O}_X(D)$ and $\mathcal{O}_X(\alpha)$ refer to the corresponding invertible sheaf on X.

Let X be a projective variety (that is, a closed subscheme of $\mathbb{P}^N$, for some $N \geqslant 1$). Then one can define the Hilbert polynomial of X. It is well-known that if $\pi : X \to B$ is a flat family of schemes, then the Hilbert polynomials of each fibre of π are equal. In this setting, we write $\mathrm{Hilb}(\mathbb{P}^N)$ for the Hilbert scheme of $\mathbb{P}^N$. This scheme has connected components stratified by the Hilbert polynomials of closed subschemes of $\mathbb{P}^N$. Details and results on these can be found in [Nak99]. We will only require them in §4.2 to define Hilbert quotients.

2 The fine moduli spaces $M_{0,n}$ and $\overline{M}_{0,n}$

In this chapter we will introduce perhaps the nicest type of moduli space: the fine moduli space. In general, we will want to study a given class, $\mathcal{T}$, of objects that can be described as objects in Sch, possibly with some added data that will be formalised in context. A moduli space for this class will be a scheme, $M = M_{\mathcal{T}}$, whose points are in bijection with objects in $\mathcal{T}$ up to some notion of equivalence.

We want the structure of such a scheme M to reflect the geometry between objects in $\mathcal{T}$. With fine moduli spaces, this happens in the most desirable way - for example, a curve in M will correspond to a continuous transformation of objects in $\mathcal{T}$ parametrised by one variable. A precise notion of this concept is given by the language of families. We will see that the existence of a fine moduli space is equivalent to the existence of a universal family.

We illustrate these definitions with a running example of $\mathbb{P}^n$ as a fine moduli space parametrising lines through the origin in $\mathbb{A}^{n+1}$. This is usually expressed in terms of locally free subsheaves, but I have opted to use the definitions directly instead.

With these definitions in place, we briefly introduce complete, nonsingular, rational curves. From this, we define the class, $\mathcal{T}$, of complete, nonsingular, n-labelled, rational curves. We can then introduce $M_{0,n}$ and show that it is a fine moduli space for this class. The final two parts of this chapter define $\overline{M}_{0,n}$ and explore the boundary $\partial M_{0,n}$.

The content here follows the notes in [Cav16] from the start up to around page 27. We will only speak in the language of schemes however, and we will provide more details in results that are only sketched or left as exercises in [Cav16].

2.1 Fine moduli spaces

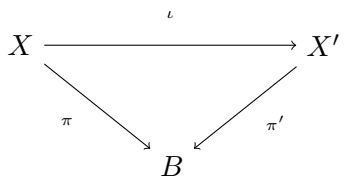
Let $\mathcal{T}$ be a class of objects, with some notion of equivalence, that we want to parametrise by a scheme $M = M_{\mathcal{T}}$. An object in $\mathcal{T}$ will be a scheme with some extra data that will always be made clear in context. We will call an object in $\mathcal{T}$ a $\mathcal{T}$-scheme. Intuitively, a family of $\mathcal{T}$-schemes over a base scheme B is a collection of $\mathcal{T}$-schemes parametrised by points in B.

Definition 2.1.1. Let B be a scheme. A family of $\mathcal{T}$-schemes over B (or simply a $\mathcal{T}$-family over B) consists of the following data:

- A surjective morphism of schemes, $\pi : X \to B$;
- Some extra data that varies algebraically over the base B, which restricts naturally to the fibres over points of B, making each of these fibres a $\mathcal{T}$-scheme.

For ease, we will often refer to a $\mathcal{T}$-family just by its morphism π as long as it is clear what the extra data of this $\mathcal{T}$-family is. Note that a $\mathcal{T}$-scheme is really just a $\mathcal{T}$-family over $\operatorname{Spec}\mathbb{C}$. Given the definition of a $\mathcal{T}$-scheme, there may be multiple ways of defining $\mathcal{T}$-families. Defining $\mathcal{T}$-families however, also defines $\mathcal{T}$-schemes at the same time. Extra conditions we often place on $\mathcal{T}$-families are common 'niceness' properties of morphisms of schemes, such as flatness.

The notion of equivalence on $\mathcal{T}$-schemes can be extended to $\mathcal{T}$-families. We say that two $\mathcal{T}$-families, $\pi : X \to B$ and $\pi' : X' \to B$, are equivalent if there is an isomorphism, $\iota : X \to X'$, making the diagram below commute and additionally, the restriction of this diagram to each fibre over a point gives an equivalence of $\mathcal{T}$-schemes.



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In the text above, we've been a bit vague in the use of the phrase 'extra data' and its restriction to fibres. The reason for this is that there are many appropriate definitions of data on a class of schemes depending on what one wants to study. The definition above can apply to a vast range of geometric situations. The best way to give clarity is to give an example.

Example 2.1.2. Let $\mathcal{T}$ be the class of closed subschemes, X, of $\mathbb{A}^{n+1} = \mathbb{A}^{n+1}_{\mathbb{C}}$ that correspond to lines through the origin in $\mathbb{C}^{n+1}$. Here, the extra data is that $\mathcal{T}$-schemes are embedded in $\mathbb{A}^{n+1}$. Two objects in $\mathcal{T}$ are equivalent only if they are equal as subschemes of a fixed copy of $\mathbb{A}^{n+1}$. A $\mathcal{T}$-family over a base scheme, B, can be described as follows: We must have a surjective morphism of schemes $\pi : X \to B$. The extra data is that X must be a closed sub-line bundle of $\mathbb{A}^{n+1} \times B$, so that π is just a restriction of the projection to B. For each point b in B, the fibre X_b is closed in the fibre $(\mathbb{A}^{n+1} \times B)_b \cong \mathbb{A}^{n+1}$, and must correspond to a line through the origin in this $\mathbb{A}^{n+1}$. We will see that, unsurprisingly, the scheme, $M = M_{\mathcal{T}}$, parametrising this class $\mathcal{T}$ will be the projective space $\mathbb{P}^n = \mathbb{P}^n_{\mathbb{C}}$. $\Diamond$

We will continue to extend the example above as we introduce more definitions. Back in the case of a general class $\mathcal{T}$, a scheme $M = M_{\mathcal{T}}$ that is a fine moduli space for $\mathcal{T}$ will have the following property: If $\pi : X \to B$ is a $\mathcal{T}$-family, then the natural map of sets from points of B to points of M sending a point b in B to the point representing the class of the fibre X_b will be part of a morphism of schemes from B to M. This correspondence will be bijective in the sense that any morphism of schemes $B \to M$ arises from a unique $\mathcal{T}$-family, $\pi : X \to B$, up to equivalence.

This leads us to moduli functors. For this to work, we will need another structure within our class $\mathcal{T}$. We need the extra data on $\mathcal{T}$-families to be compatible with base change. In particular, if $\pi : X \to B$ is a $\mathcal{T}$-family and $\phi : B' \to B$ is a morphism of schemes then we should get a unique and well-defined pull-back family, $\phi^*(\pi) : X \times_B B' \to B'$. In the

example above with closed lines in $\mathbb{A}^{n+1}$, $\mathcal{T}$-families have this property. Indeed, closed immersions are preserved under base extension [Hart, §II, Ex 3.11], so if X is closed in $\mathbb{A}^{n+1} \times B$, then $X \times_B B'$ is closed in $(\mathbb{A}^{n+1} \times B) \times_B B' = \mathbb{A}^{n+1} \times B'$.

For the rest of this section (excluding the continuation of Example 2.1.2), our class $\mathcal{T}$ will be fixed and will be compatible with fibre products. Before we define the moduli functor associated to $\mathcal{T}$, we need to introduce some category theory.

Let *Set* be the category of sets and fix a scheme X. The functor of points of X is the contravariant functor, $h_X = \mathrm{Hom}(-, X)$, from *Sch* to *Set*. This functor sends any scheme, Y, to the set of morphisms, $\mathrm{Hom}(Y, X)$, from Y to X. If $\phi : Y' \to Y$ is any morphism of schemes then h_X sends ϕ to the map of sets

$$\mathrm{Hom}(Y, X) \longrightarrow \mathrm{Hom}(Y', X), \quad f \longmapsto f \circ \phi.$$

Definition 2.1.3. A contravariant functor $F : Sch \to Set$ is called representable if there is a scheme, X, such that $h_X \cong F$. In this case we say that the scheme X represents F.

Recall that two functors, F and G, from a category, $\mathcal{C}$, to a category, $\mathcal{D}$, are isomorphic if there is a natural transformation, ϵ, from F to G, such that for each object C of $\mathcal{C}$, the map ϵ_C in $\mathrm{Hom}_{\mathcal{C}}(F(C), G(C))$ is an isomorphism.

We can recover a scheme, X, uniquely from its functor of points, h_X. This makes sense intuitively - for suppose we have $h_X \cong h_{X'}$ for two schemes X and X'. Then $h_X(\mathrm{Spec}\,\mathbb{C}) \cong h_{X'}(\mathrm{Spec}\,\mathbb{C})$ gives a bijection between the points of X and the points of X'. More generally, if U is a scheme admitting morphisms to X then the isomorphism between $h_X(U)$ and $h_{X'}(U)$ will give us a bijection between the ways in which we can map U into X and the ways we can map U into X'. Thinking of affine opens and their overlaps we can see the following result is expected. The statement itself makes up part of Yoneda's lemma, a fundamental result in category theory.

Lemma 2.1.4. Suppose that X and X' are two schemes (so objects in Sch). Then $h_X \cong h_{X'}$ if and only if $X \cong X'$. □

We are now ready to define the moduli functor associated to $\mathcal{T}$. A fine moduli space for $\mathcal{T}$ will be a scheme that represents this functor.

Definition 2.1.5. Define a functor, $F = F_{\mathcal{T}}$, called the moduli functor of $\mathcal{T}$, from the category Sch to the category Set, as follows. A scheme, B, is sent to the set of equivalence classes of $\mathcal{T}$-families over B. If $\phi : B' \to B$ is a morphism of schemes, then $F(\phi)$ sends a $\mathcal{T}$-family, $\pi : X \to B$, to its pull-back family $\phi^*(\pi) : X \times_B B' \to B'$.

Note that a moduli functor is contravariant. Indeed, if $\pi : X \to B$ is a $\mathcal{T}$-family, and $\phi : B' \to B$ and $\phi' : B'' \to B'$ are morphisms of schemes, then we have

$$(\phi \circ \phi')^*(\pi) = \phi'^* \left(\phi^*(\pi)\right).$$

Defintion 2.1.6. A fine moduli space for $\mathcal{T}$ is a scheme, $M = M_{\mathcal{T}}$, that represents the moduli functor $F_{\mathcal{T}}$.

Lemma 2.1.4 implies that a fine moduli space, if it exists, is unique up to isomorphism. We will now continue Example 2.1.2, attempting to parametrise lines through the origin in $\mathbb{A}^{n+1}$.

Example 2.1.2 (continued)**.** Let $\mathcal{T}$ be the class of closed subschemes of $\mathbb{A}^{n+1}$ that are lines through the origin, as in the previous part. We will construct an isomorphism of functors between $h_{\mathbb{P}^n}$ and $F = F_{\mathcal{T}}$, showing that $\mathbb{P}^n$ is a fine moduli space for $\mathcal{T}$. For each scheme B, define $\epsilon_B : h_{\mathbb{P}^n}(B) \to F(B)$ as follows. If $f \in h_{\mathbb{P}^n}$, i.e. $f : B \to \mathbb{P}^n$ is a morphism, then consider also the usual morphism $q : \mathbb{A}^{n+1}\backslash\{0\} \to \mathbb{P}^n$. Using f and q we can construct the fibre product $U = \mathbb{A}^{n+1}\backslash\{0\} \times_{\mathbb{P}^n} B$. The projection of the first factor of U followed by the inclusion into $\mathbb{A}^{n+1}$, along with the projection of U to the

second factor gives a natural map from U to $\mathbb{A}^{n+1} \times B$ by Proposition 1.2. This realises U as a subscheme of $\mathbb{A}^{n+1} \times B$.

Let X be the Zariski closure of U in $\mathbb{A}^{n+1} \times B$. We have a morphism, $\pi : X \to B$, which is just the restriction of the second projection. Furthermore, π is surjective since it is surjective on U (which follows from the fact that q is surjective). Consider a point b in B. The fibre U_b is a line in $(\mathbb{A}^{n+1})_b \cong \mathbb{A}^{n+1}$ with its origin removed. Namely, it is the line corresponding to $f(b) \in \mathbb{P}^n$. The Zariski closure of this fibre is obtained simply by adding the origin back into this line. The fibre X_b is thus a line through the origin in $(\mathbb{A}^{n+1})_b$. Varying over all points in B, each fibre is a line through the origin in $(\mathbb{A}^{n+1} \times B)_b \cong \mathbb{A}^{n+1}$, thus the morphism π defines a $\mathcal{T}$-family. We define $\epsilon_B(f)$ to be this $\mathcal{T}$-family π.

These maps are compatible with base change, thus we get a well-defined natural transformation, $\epsilon : h_{\mathbb{P}^n} \to F$, from these ϵ_B as B ranges over all schemes. It is easy to check that each ϵ_B is injective. For surjectivity, fix a scheme, B, and let $\pi : X \to B$ define a $\mathcal{T}$-family over B. By definition, X is a closed sub-line bundle of $\mathbb{A}^{n+1} \times B$. Let $\operatorname{Spec} R = V$ be an affine open of B on which X restricts to a trivial line bundle, so that $\pi^{-1}(V) \cong \mathbb{A}^1 \times V$. Then we get a composition,

$$V \hookrightarrow \mathbb{A}^1 \backslash \{0\} \times V \to \mathbb{A}^{n+1} \backslash \{0\} \times V \to \mathbb{A}^{n+1} \backslash \{0\} \to \mathbb{P}^n.$$

Here, the first map is induced by the homomorphism of coordinate rings from $\mathbb{C}[t, t^{-1}] \otimes R$ to R defined by sending $t \otimes r$ to r (remember that R is a $\mathbb{C}$-algebra). The second map is induced by the restriction of π, identifying $\mathbb{A}^1 \times V$ with its isomorphic subscheme $\pi^{-1}(V)$ of X. The final two maps are the projection and q respectively. Covering B with open affines V_i so that each $\pi^{-1}(V_i)$ is trivial, these maps from V_i to $\mathbb{P}^n$ glue to form a morphism from B to $\mathbb{P}^n$. This proves surjectivity of ϵ_B, and thus completes the proof that $\mathbb{P}^n$ is a fine moduli space for this $\mathcal{T}$. $\Diamond$

The example above shows that proving that two functors are isomorphic can be quite cumbersome - even in the case of an elementary example of a fine moduli space. For this reason, instead of trying to just find a scheme, $M_{\mathcal{T}}$, which represents a moduli functor, $F_{\mathcal{T}}$, it is often easier to try and find a universal family, $\pi : U_{\mathcal{T}} \to M_{\mathcal{T}}$.

Definition 2.1.7. A $\mathcal{T}$-family, $\pi : U \to M$, is universal if any other $\mathcal{T}$-family is a pull-back of π. More precisely, if $\pi' : X \to B$ is any other $\mathcal{T}$-family, then there is a unique morphism, $\varphi : B \to M$, such that $X = U \times_M B$ and $\pi' = \varphi^*(\pi)$

Theorem 2.1.8. Let $F = F_{\mathcal{T}}$ be our moduli functor. Then there is a fine moduli space, $M = M_{\mathcal{T}}$, representing this functor if and only if there is a universal family, $\pi : U \to M$, over M.

Proof. Suppose M is a fine moduli space for $\mathcal{T}$, so $h_M \cong F$. Then, in particular, there is an isomorphism $h_M(M) \cong F(M)$. Let $\pi : U \to M$ be the $\mathcal{T}$-family over M corresponding to the identity morphism in $h_M(M)$ under this isomorphism. Now let $\pi' : X \to B$ be any $\mathcal{T}$-family over any scheme B. This is an element of $F(B)$, so by the isomorphism between functors, must correspond to a unique element $\varphi \in h_M(B)$, i.e. a unique morphism $\varphi : B \to M$. The morphism $h_M(\varphi) : h_M(M) \to h_M(B)$ sends any given morphism, $f \in h_M(M)$, to $f \circ \varphi$.

So now we have an isomorphism $h_M \cong F$, where π corresponds to id_M, and π' corresponds to φ. Furthermore, the morphism $h_M(\varphi)$ sends id_M to φ. Hence, in F rather than h_M, we must have that $F(\varphi)$ sends $\pi : U \to M$ to $\pi' : X \to B$. But by definition, $F(\varphi)$ is the pull-back family $\varphi^*(\pi) : U \times_M B \to B$. This proves one direction of the theorem.

The converse is straightforward; if $\pi : U \to M$ is universal, then our natural transformation from h_M to F can be defined for each scheme, B, by sending any φ in $h_M(B)$ to the pullback family $\varphi^*(\pi)$ in $F(B)$. This natural functor is an isomorphism of functors.

□

Example 2.1.2 (continued)**.** We return one last time to the class $\mathcal{T}$ of closed subschemes of $\mathbb{A}^{n+1}$ that are lines through the origin. Instead of constructing an isomorphism of functors, we now seek to find a universal $\mathcal{T}$-family, $\pi : U \to \mathbb{P}^n$. Let $y_0, \ldots, y_n$ be affine coordinates on $\mathbb{A}^{n+1}$ and let $X_0, \ldots, X_n$ be homogeneous coordinates on $\mathbb{P}^n$. Let U be the closed subscheme of $\mathbb{A}^{n+1} \times \mathbb{P}^n$ defined by the vanishing of the 2×2 minors of the matrix

$$\begin{pmatrix} y_0 & \ldots & y_n \\ X_0 & \ldots & X_n \end{pmatrix}.$$

We claim that the restriction of $\mathbb{A}^{n+1} \times \mathbb{P}^n \to \mathbb{P}^n$ to U defines a universal $\mathcal{T}$-family $\pi : U \to \mathbb{P}^n$. If $P = [P_0 : \cdots : P_n]$ is a point in $\mathbb{P}^n$, the fibre U_P is the line in $\mathbb{A}^{n+1} \times \{P\} \cong \mathbb{A}^{n+1}$ corresponding to $P \in \mathbb{P}^n$ in the usual way. Thus π defines a $\mathcal{T}$-family.

Now let $\psi : X \to B$ be any $\mathcal{T}$-family. By definition, X is a closed sub-line bundle of $\mathbb{A}^{n+1} \times B$. If we let p_1, p_2 be the usual projections from $\mathbb{A}^{n+1} \times B$ then we also know that ψ is the restriction of p_2 to X. Using the earlier part of this example, let $\varphi : B \to \mathbb{P}^n$ be the composition

$$B \to \mathbb{A}^{n+1} \backslash \{0\} \times B \to \mathbb{A}^{n+1} \backslash \{0\} \to \mathbb{P}^n.$$

Put $\rho = p_1 \times (\varphi \circ \psi)$. As defined so far, ρ is a morphism from X to $\mathbb{A}^{n+1} \times \mathbb{P}^n$. By construction, the image of ρ lies in U and thus we consider ρ as a morphism from X to U. By Proposition 1.2 we get a morphism $\rho \times_{\mathbb{P}^n} \psi : X \to U \times_{\mathbb{P}^n} B$. Our construction extends further to show that $\rho \times_{\mathbb{P}^n} \psi$ is an isomorphism inducing an equivalence of $\mathcal{T}$-families between ψ and the pull-back of π along φ. This implies that π is a universal $\mathcal{T}$-family.

Note that this is consistent with the proof of Theorem 2.1.8. Indeed, if ϵ is the natural

transformation as defined in the previous part of this example, then we find $\epsilon_{\mathbb{P}^n}(\mathrm{id}_{\mathbb{P}^n})$ as follows. We let U be the fibre product of $\mathbb{A}^{n+1}\backslash\{0\}$ and $\mathbb{P}^n$ over $\mathbb{P}^n$, embedded inside the usual product $\mathbb{A}^{n+1} \times \mathbb{P}^n$. For each point $P \in \mathbb{P}^n$, the fibre U_P is the line in $\mathbb{A}^{n+1} \times \{P\}$ corresponding to P with its origin removed. The Zariski closure of such a fibre is the fibre over P of the universal $\mathcal{T}$-family defined above. Since the Zariski closure of U contains the union of the Zariski closures of each U_P, we see that we obtain the universal $\mathcal{T}$-family π as expected. This concludes our example of $\mathbb{P}^n$ as a fine moduli space. $\Diamond$

For later on, we need the following proposition. This says that a universal $\mathcal{T}$-family over a base M, if it exists, is unique up to equivalence.

Proposition 2.1.9. Suppose that $\pi : U \to M$ and $\pi' : U' \to M$ are universal $\mathcal{T}$-families over a base M. Then π and π' define equivalent $\mathcal{T}$-families.

Proof. By the universality of π, there is a unique morphism, $\varphi : M \to M$, so that $\pi' = \varphi^*(\pi)$, making $U' = U \times_M M$. Similarly, as π' is universal, there is a unique morphism, $\varphi' : M \to M$, with $\pi = \varphi'^*(\pi')$, so that $U = U' \times_M M$. Then,

$$\pi = \varphi'^*(\pi') = \varphi'^* \circ \varphi^*(\pi) = (\varphi \circ \varphi')^*(\pi).$$

Applying universality of π to the $\mathcal{T}$-family π itself, π is a pullback of itself by a unique morphism, which must be the identity, thus the above equality implies that $\varphi \circ \varphi' = \mathrm{id}_M$. Similarly, $\varphi' \circ \varphi = \mathrm{id}_M$. As such, φ and φ' are automorphisms of M, and the claim follows since the pull-back of a scheme over a base, B, by an automorphism of B is an isomorphism. $\square$

We conclude this section by justifying that points of a fine moduli space, $M = M_{\mathcal{T}}$, are in bijection with equivalence classes of $\mathcal{T}$-schemes. Indeed, let $\pi : U \to M$ be a universal family. Then a $\mathcal{T}$-scheme, $X \to \operatorname{Spec} \mathbb{C}$, corresponds to a morphism, $\operatorname{Spec} \mathbb{C} \to M$, which in turn determines a point in M. Conversely, a point m in M corresponds to the class of

the fibre U_m in the universal family. It is straightforward to check this gives a bijection. This argument also extends to describe the morphism associated with a given $\mathcal{T}$-family. Namely, if $\pi : X \to B$ is a $\mathcal{T}$-family, then the corresponding morphism, $\varphi : B \to M$, sends a point b in B to the point m in M corresponding the class of the fibre ($\mathcal{T}$-scheme) X_b.

2.2 Nonsingular rational curves and $M_{0,n}$

In this section, we will introduce the fine moduli space, $M_{0,n}$, parametrising complete, nonsingular, rational curves with n distinct labelled points up to automorphism. We give more details than in [Cav16, §1.2]. Here and throughout the rest of this chapter, a curve will mean a complete, one-dimensional $\mathbb{C}$-scheme.

To answer what a rational curve is, we will introduce two notions of genus on a given curve. Each of these is a birational invariant. In this section, curves will also be nonsingular unless stated otherwise.

Since we are working over $\mathbb{C}$, all complete, nonsingular curves are smooth, compact, complex manifolds [Ser56]. In this setting, our algebraic definitions of genus will coincide with the topological genus (see [HKP$^+$03, §22]). In particular, they are non-negative integers. For example, any curve of genus 0 will be homeomorphic to the sphere, S^2, whereas a curve of genus 1 is homeomorphic to the topological torus, $S^1 \times S^1$.

Definition 2.2.1. Let X be a curve. The arithmetic genus, $\rho_a(X)$, of X is defined to be $1-\chi(X, \mathcal{O}_X)$, where $\chi(X, \mathcal{O}_X)$ is the Euler characteristic of the structure sheaf of X. If $X \subset \mathbb{P}^n$, then $\chi(X, \mathcal{O}_X) = h(0)$, where h is the Hilbert polynomial of the embedding of X in $\mathbb{P}^n$. If $\rho_a(X) = 0$ then we say that the curve X is rational.

The arithmetic genus is constant over a flat family of schemes since Hilbert polynomials are. We will see that this allows one to compute the arithmetic genus of many singular curves over $\mathbb{C}$. In our class $\mathcal{T}$ that will be parametrised by $M_{0,n}$, the $\mathcal{T}$-families will

necessarily be flat for this reason.

There is also a geometric genus, $\rho_g(X)$, defined for any curve X. This is equal to the arithmetic genus of the normalisation, $\tilde{X}$, of X. This is not constant over flat families. To see this, consider a curve, X, with $\rho_a(X) = 1$, so that X is homeomorphic to the topological torus. Consider a flat family over $\mathbb{A}^1$ whose fibre over the point 1 is X, and whose fibre over the point 0 is the singular curve obtained by contracting a cycle whose class generates a rank one summand of the first homology group, $H_1(X;\mathbb{Z})$, to a point. This is illustrated in the diagram below by the first arrow. Normalising this singular curve removes the singularity, effectively pulling it apart. This is illustrated in the second arrow below. The degeneration at 0 has geometric genus 0.

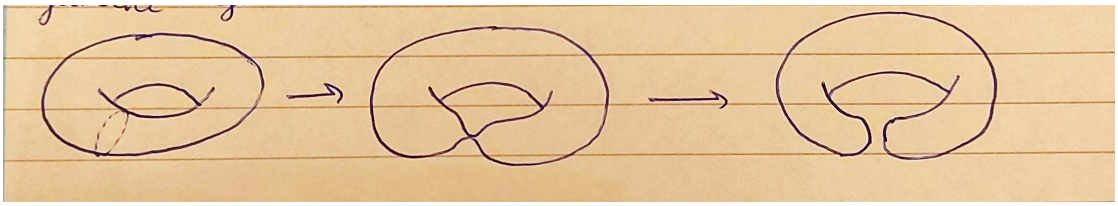


Figure 1: Degeneration of a genus 1 curve, followed by normalisation

We will discuss further the difference between the two types of genus, as well as how to compute them for curves, in §2.3. In the nonsingular case, the notions coincide and we are free to use the term genus unambiguously.

Ideally, for each (integer) $g \geqslant 0$, there would be a fine moduli space, M_g, parametrising curves of genus g up to isomorphism. A moduli functor for this class should send a scheme, B, to the set of flat and proper families of curves of genus g over B.

That this moduli functor is not representable is well-known. The reason for this is that for each $g \geqslant 0$, there is a curve of genus g admitting non-trivial automorphisms. This allows one to construct a non-trivial family by gluing together trivial families. In the

$g = 0$ case for instance, any nonsingular, complete, genus 0 curve is isomorphic to $\mathbb{P}^1$ and thus M_0 should be a point if it exists. If M_0 is a point, then any pull-back of the universal family will be trivial, but we know that non-trivial $\mathbb{P}^1$-bundles exist.

For $g \geqslant 2$, it is well-known that the automorphism group of a genus g curve may be non-trivial but finite. This means that no fine moduli space exists, but there is a weaker construction called a coarse moduli space. For $g = 0$ and $g = 1$, not even a coarse moduli space exists as our automorphism groups are infinite. We only study fine moduli spaces, but [HKP$^+$03, §23] and [KLP18, §14] are references for the moduli spaces M_g with $g \geqslant 2$.

The intuition is that to make our functor representable we need extra data on curves that will restrict the automorphism group to the trivial group. We turn our focus purely to rational curves, i.e. those of genus 0. Recall that we assume all our curves are complete and nonsingular. A basic fact is that any rational curve is isomorphic to $\mathbb{P}^1$. The automorphism group, $\mathrm{Aut}(\mathbb{P}^1)$, of $\mathbb{P}^1$ is well understood.

Any automorphism of $\mathbb{P}^1$ is induced by an automorphism of its homogeneous coordinate ring, $\mathbb{C}[x_0, x_1]$. Each of these is given by an element of the projective linear group $\mathrm{PGL}_2(\mathbb{C})$ via

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto (x \mapsto ax + by, y \mapsto cx + dy), \quad ad - bc \neq 0.$$

It is worth thinking of the closed points of $\mathbb{P}^1$ as elements of the extended Riemann sphere, $\tilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, where addition and multiplication are partially defined. Explicitly, we have

$$\mathbb{C} \equiv \{[z : 1] : z \in \mathbb{C}\} \subset \mathbb{P}^1, \quad \infty = [1 : 0] \in \mathbb{P}^1.$$

Each element of $\mathrm{PGL}_2(\mathbb{C})$ corresponds to an automorphism of $\tilde{\mathbb{C}}$ (known as a Möbius

transformation):

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \left(z \mapsto \frac{az+b}{cz+d} \right), \quad ad - bc \neq 0, z \in \tilde{\mathbb{C}}.$$

It is well known that given any three distinct points, p_1, p_2, p_3, in $\mathbb{P}^1$, that there is a unique automorphism of $\mathbb{P}^1$ sending the ordered triple (p_1, p_2, p_3) to $(0, 1, \infty)$. The corresponding Möbius transformation of $\tilde{\mathbb{C}}$ is given by

$$\Phi_{cr} : z \longmapsto \frac{p_1 - z}{p_3 - z} \cdot \frac{p_3 - p_2}{p_1 - p_2}.$$

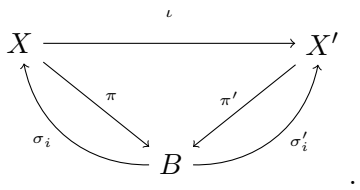
If $p_i = \infty$ for some $i \in \{1, 2, 3\}$ then we remove the two factors in Φ_{cr} above containing p_i. Geometrically, Φ_{cr} sends any point, $p \in \mathbb{P}^1$, to the cross-ratio of the four ordered points (p_1, p_2, p_3, p). An important fact about the cross-ratio of four points is that it is invariant under applying an automorphism of $\mathbb{P}^1$. By composing with inverses it is clear that given any two triples of distinct points, (p_1, p_2, p_3) and (q_1, q_2, q_3), in $\mathbb{P}^1$, there is a unique automorphism of $\mathbb{P}^1$ mapping p_i to q_i for each $i \in \{1, 2, 3\}$.

Let $\mathcal{T}$ be the class of nonsingular, rational curves with n labelled points, where n is always an integer with $n \geqslant 3$. By n labelled points we will always mean ordered, distinct points. An equivalence of $\mathcal{T}$-schemes is an isomorphism of rational curves inducing an order preserving bijection on the labelled points. The previous discussion implies that the automorphism group of a $\mathcal{T}$-scheme is trivial, thus we can think about constructing a fine moduli space, $M_{0,n}$, for $\mathcal{T}$. We will often use the notation $[n]$ to denote the set $\{1, \ldots, n\}$ from now on.

We define our moduli functor (or equivalently, define a $\mathcal{T}$-family), $F = F_{\mathcal{T}}$, as follows. Given any scheme, B, a $\mathcal{T}$-family over B is a scheme, X, with a surjective, flat and proper morphism, $\pi : X \to B$, so that each fibre is a rational curve. The extra data

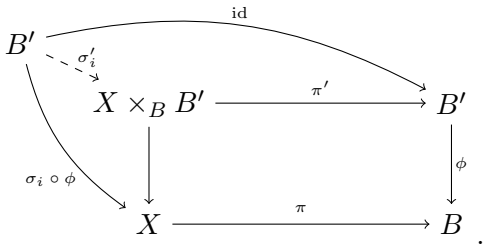
are n disjoint sections, $\sigma_i : B \to X$, for each $i \in [n]$, of π. That is, each σ_i satisfies $\pi \circ \sigma_i = \mathrm{id}_B$, and for each point b in B, we have $\sigma_i(b) = \sigma_j(b)$ if and only if $i = j$. These sections play the role of labelling the points on each fibre, thus a $\mathcal{T}$-family over $\operatorname{Spec} \mathbb{C}$ is compatible with what we defined a $\mathcal{T}$-scheme to be.

Two $\mathcal{T}$-families, $\pi : X \to B$ and $\pi' : X' \to B$, the first having sections $\sigma_1, \ldots, \sigma_n$ and the second having sections $\sigma'_1, \ldots, \sigma'_n$, are equivalent if there is an isomorphism, $\iota : X \to X'$, so that for each $i \in [n]$, the following diagram commutes:



.

We need to check that our $\mathcal{T}$-families are compatible with fibre products before we can be sure F is a well-defined moduli functor. Let $\pi : X \to B$ be a $\mathcal{T}$-family with sections $\sigma_1, \ldots, \sigma_n$. Suppose we have a morphism of schemes, $\phi : B' \to B$. Then, for each $i \in [n]$, we get the pull-back diagram below.



.

Proposition 1.2 gives us n sections $\sigma'_1, \ldots \sigma'_n$ for $\pi' = \phi^*(\pi) : X \times_B B' \to B'$. Note that if b' is a point of B' with $\sigma'_i(b') = \sigma'_j(b')$ for some $i, j \in [n]$, then the diagrams above imply that that $\sigma_i(\phi(b')) = \sigma_j(\phi(b'))$, hence $i = j$. Thus our sections are disjoint and we have well-defined moduli functor.

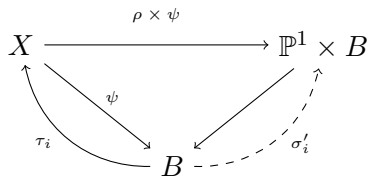
Let's look at the case $n = 3$ first. A $\mathcal{T}$-scheme is a nonsingular rational curve, so isomorphic to $\mathbb{P}^1$, with three labelled points. There is then a unique automorphism of $\mathbb{P}^1$ mapping these three points, in order, to the triple $(0, 1, \infty)$. In particular, there is only one $\mathcal{T}$-scheme up to equivalence. If the fine moduli space $M_{0,3}$ exists then it must be a point, i.e. $M_{0,3} = \operatorname{Spec} \mathbb{C}$.

Let $\pi : \mathbb{P}^1 \to \operatorname{Spec} \mathbb{C}$ be the terminal morphism. Define sections, σ_1, σ_2 and σ_3, from $\operatorname{Spec} \mathbb{C}$ to $\mathbb{P}^1$ by sending the unique point to $0, 1$ and ∞ respectively. This defines a $\mathcal{T}$-family over $\operatorname{Spec} \mathbb{C}$. We want to show that this $\mathcal{T}$-family is universal.

Suppose that $\psi : X \to B$, with sections τ_1, τ_2 and τ_3, is another $\mathcal{T}$-family. If $x \in X$, then let $b = \psi(x)$. If x is closed then so is b and $x \in X_b \cong \mathbb{P}^1$. Additionally, let $x_i = \tau_i(b) \in X_b$ for each $i \in [3]$. Define a Möbius transformation, ρ, of X_b (identified by an isomorphism with $\mathbb{P}^1$) that is the cross-ratio of these four points, i.e.

$$\rho(x) = \frac{x_1 - x}{x_3 - x} \cdot \frac{x_3 - x_2}{x_1 - x_2} \in \mathbb{P}^1.$$

This is well-defined since the cross-ratio is independent of the isomorphism with $\mathbb{P}^1$ chosen. From each of these Möbius transformations we can recover the element of $\mathrm{PGL}_2(\mathbb{C})$ defining the corresponding isomorphism of X_b and $\mathbb{P}^1$ as schemes. The matrix entries depend on the points x_1, x_2, x_3 and x, and hence this determines a unique morphism, $\rho : X \to \mathbb{P}^1$, since τ_1, τ_2, τ_3 and ψ are morphisms. From ρ and ψ we can form the morphism $\rho \times \psi : X \to \mathbb{P}^1 \times B$ which fits into the diagram drawn below.



There are unique sections σ_1', σ_2' and σ_3' that make this diagram commute. Namely, for

each $i \in [3]$ we have $\sigma'_i = (\rho \times \psi) \circ \tau_i$. By construction, for each point b in B we have

$$\sigma'_1(b) = (0, b), \quad \sigma'_2(b) = (1, b), \quad \sigma'_3(b) = (\infty, b).$$

From this, we see that the projection, $\mathbb{P}^1 \times B \to B$, defines a $\mathcal{T}$-family which is the pull-back of the terminal $\mathcal{T}$-family π, by the terminal morphism $B \to \operatorname{Spec} \mathbb{C}$. Furthermore, $\rho \times \psi$ is an isomorphism of schemes. To see this, note that it is a collection of automorphisms of the fibres parametrised by the base B, varying algebraically over this base. Since matrix inversion in $\mathrm{PGL}_2(\mathbb{C})$ is an automorphism of $\mathrm{PGL}_2(\mathbb{C})$, we can invert $\rho \times \psi$ on each fibre to obtain an inverse isomorphism. This implies that $\pi : \mathbb{P}^1 \to \operatorname{Spec} \mathbb{C}$ is universal, so that $M_{0,3} = \operatorname{Spec} \mathbb{C}$ is a fine moduli space.

Now, suppose that $n \geqslant 4$. Consider the scheme

$$M_{0,n} = \left(\mathbb{P}^1 \backslash \{0, 1, \infty\}\right)^{n-3} \backslash \bigcup_{4 \leqslant i < j \leqslant n} \{z_i = z_j\},$$

where z_i is the coordinate of the $(i-3)^{\text{th}}$ copy of $\mathbb{P}^1 \backslash \{0, 1, \infty\}$. Let $\pi : \mathbb{P}^1 \times M_{0,n} \to M_{0,n}$ be the usual projection. Let $i \in [n]$, then we need to define a section, σ_i, of π.

Note that $M_{0,n}$ is an open subscheme of $\mathbb{A}^{n-3}$, which has coordinate ring $\mathbb{C}[z_4, \ldots, z_n]$. If y_0, y_1 are homogeneous coordinates on $\mathbb{P}^1$, then let J_i be the ideal in

$$\mathbb{C}[y_0, y_1] \bigotimes \mathbb{C}[z_4, \ldots, z_n] \cong \mathbb{C}[y_0, y_1, z_4, \ldots, z_n]$$

generated by $y_0 - z_i y_1$ if $i \neq 3$. Here we put $z_1 = 0$ and $z_2 = 1$. Let J_3 be the ideal generated by y_1. Since each J_i is homogeneous in the variables y_0, y_1, the ring homomorphism

$$\mathbb{C}[y_0, y_1, z_4, \ldots, z_n] \longrightarrow \mathbb{C}[y_0, y_1, z_4, \ldots, z_n]/J_i$$

defines a morphism from $\mathbb{A}^{n-3}$ to $\mathbb{P}^1 \times \mathbb{A}^{n-3}$. This restricts to a section $M_{0,n} \to \mathbb{P}^1 \times M_{0,n}$,

which we set to be σ_i.

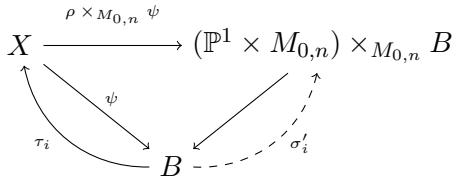
Let $\psi : X \to B$, with sections τ_i for each $i \in [n]$, be any $\mathcal{T}$-family. We first define a morphism, φ, from B to $M_{0,n}$, and then define a morphism, ρ, from X to $\mathbb{P}^1 \times M_{0,n}$.

For each $b \in B$, write $x_i = \tau_i(b)$ for each $i \in [n]$. Let b be a point in B, so that $X_b \cong \mathbb{P}^1$. Define a Möbius transformation of X_b (in any isomorphism with $\mathbb{P}^1$), $\Phi_{b,cr}$, by

$$\Phi_{b,cr} : z \longmapsto \frac{x_1 - z}{x_3 - z} \cdot \frac{x_3 - x_2}{x_1 - x_2}.$$

This determines an element of $\mathrm{PGL}_2(\mathbb{C})$. The entries of this matrix depend on the morphisms τ_1, τ_2 and τ_3 so these morphisms from X_b to $\mathbb{P}^1$ fit together to form a morphism $\Phi : X \to \mathbb{P}^1$. For each $i \in [n]\backslash[3]$, let $\varphi_i : B \to \mathbb{P}^1\backslash\{0, 1, \infty\}$ be given by $\varphi_i = \Phi \circ \tau_i$. Then put $\varphi = \prod_{i\in[n]\backslash[3]} \varphi_i$. Since the τ_i are disjoint, φ is a morphism from B to $M_{0,n}$. Now define $\rho : X \to \mathbb{P}^1 \times M_{0,n}$ to be the product $\Phi \times (\varphi \circ \psi)$.

Using Proposition 1.2, we get a morphism $\rho \times_{M_{0,n}} \psi$ from X to the fibre product $(\mathbb{P}^1 \times M_{0,n}) \times_{M_{0,n}} B$. This fits into the diagram below.



.

As in the case where $n = 3$, the map $\rho \times_{M_{0,n}} \psi$ is an isomorphism. Furthermore, the projection that is the arrow on the right of the above diagram is the pull-back of π by φ. This implies that $\pi : \mathbb{P}^1 \times M_{0,n} \to M_{0,n}$ is a universal $\mathcal{T}$-family and we have a fine moduli space $M_{0,n} = M_{\mathcal{T}}$ for each $n \geqslant 3$.

We see that the dimension of $M_{0,n}$ is $n - 3$. Another thing we notice is that it is not complete for $n \geqslant 4$. In general, one desires moduli spaces that are complete. One reason

for this is so that we can utilise intersection theory as a way to study the objects being parametrised. We would also like to be able to talk about the limits of flat families. For example, in $M_{0,n}$, we can ask what happens when we let one labelled point tend towards another. Our next section introduces a moduli space of labelled curves that are allowed to be singular. We will see that this is a complete moduli space containing $M_{0,n}$ as a dense open subset. In other words, $\overline{M}_{0,n}$ is a completion of $M_{0,n}$.

2.3 A modular completion of $M_{0,n}$

At the end of the preceding section, we noted that $M_{0,n}$ is not compact when $n \geqslant 4$ and explained our desire to find a completion. Of course, there are many complete schemes containing $M_{0,n}$ as a dense open subscheme - the most obvious one being $(\mathbb{P}^1)^{n-3}$. For moduli theory, we would like a completion that is itself a moduli space parametrising a broader class, $\overline{\mathcal{T}}$. In this way, each point in $M_{0,n} \subset \overline{M}_{0,n}$ will still correspond to an n-labelled rational curve. The points on the boundary, $\partial M_{0,n} = \overline{M}_{0,n} \backslash M_{0,n}$, will correspond to $\overline{\mathcal{T}}$-schemes that are not $\mathcal{T}$-schemes.

The direction that this section takes includes the rhetoric of [Cav16, §1.3] along with the exploration of some exercises in [Cav16, §1.4]. Again, we give much more detail than is usually seen in the literature.

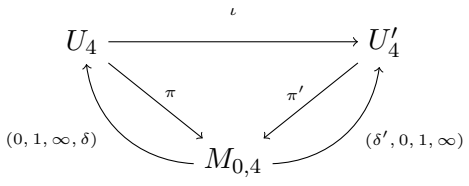
The new objects in $\overline{\mathcal{T}}$ should not be too dissimilar from n-labelled, nonsingular, rational curves. A straightforward suggestion is to allow labelled points to coincide. Then our $\overline{\mathcal{T}}$-schemes will be nonsingular, rational curves with an n-tuple of points (up to equivalence). However, in this case, we lose the representability of our moduli functor. Indeed, $\mathbb{P}^1$ with the n-tuple $(0, \ldots, 0)$ would be such an object. This is effectively $\mathbb{P}^1$ with one labelled point, and thus it has a non-trivial (and in fact infinite) automorphism group.

We get a better intuition on how to complete $M_{0,n}$ if we also try to complete $U_n = \mathbb{P}^1 \times M_{0,n}$, where $\pi : U_n \to M_{0,n}$ is the universal $\mathcal{T}$-family with its sections $\sigma_1, \ldots, \sigma_n$ as

defined in §2.2. We start by considering the simplest case of when $n = 4$.

Let $n = 4$. Then $U_4 = \mathbb{P}^1 \times \mathbb{P}^1 \backslash \{0, 1, \infty\}$ and π is the projection to the second factor. The first three sections have image that is constant when projected to the first factor, and to reflect this we denote these sections by $0 = \sigma_1, 1 = \sigma_2$ and $\infty = \sigma_3$ respectively. The fourth section is the diagonal section, $\delta = \sigma_4 : M_{0,4} \to U_4$. The universality of this $\mathcal{T}$-family was realised from an arbitrary $\mathcal{T}$-family, $\psi : X \to B$, with sections $\tau_1, \ldots, \tau_4$, by applying isomorphisms of each fibre X_b, where b is closed, with a copy of $\mathbb{P}^1$ so that the images of τ_1, τ_2 and τ_3 were sent to $0, 1$ and ∞ respectively. The image of $\tau_4(b)$ is sent to the cross-ratio of the ordered points $\tau_1(b), \ldots, \tau_4(b)$.

Instead, consider a universal family, $\pi' : U_4' \to M_{0,4}$, characterised by sending the images of the sections τ_2, τ_3 and τ_4 to $0, 1$ and ∞ respectively. For each point b in B, the image of $\tau_1(b)$ must be sent to the cross-ratio of the four ordered points $\tau_2(b), \tau_3(b), \tau_4(b), \tau_1(b)$. Note that these are the same four points as before, but the change in order changes the cross-ratio. Let δ' denote the first section of π', then the last three sections of π' are $0, 1$ and ∞ in that order. By Proposition 2.1.9, there must be an isomorphism $\iota : U_4 \to U_4'$ making the diagram below commute (we have abused notation here by labelling the outside arrows by the ordered quadruples of sections, whereas there are really four diagrams).



Let b be a point in $M_{0,4}$ and consider the fibres $(U_4)_b$ and $(U_4')_b$. Let ι_b be the restriction of ι to these fibres. Then ι_b must be an equivalence of $\mathcal{T}$-schemes, thus an isomorphism of the underlying schemes which are each a $\mathbb{P}^1$. Thus ι_b is given by a unique element of

$\mathrm{PGL}_2(\mathbb{C})$. For the diagram above to commute we must have that

$$\iota_b \equiv \begin{bmatrix} 1 & -1 \\ 1 & -b \end{bmatrix}.$$

If we complete both U_4 and $M_{0,n}$ to obtain a universal $\overline{\mathcal{T}}$-family, $\overline{\pi} : \overline{U}_4 \to \overline{M}_{0,n}$ (where $\overline{\mathcal{T}}$ can be defined by describing the added fibres), then applying the same construction to U_4' and π', the isomorphism ι should extend to an isomorphism between $\overline{U}_4$ and $\overline{U}_4'$ that induces an equivalence of $\overline{\mathcal{T}}$-families.

Let $\overline{M}_{0,4} = \mathbb{P}^1$ and put $\hat{U}_4 = \hat{U}_4' = \mathbb{P}^1 \times \mathbb{P}^1$, closing up the holes in the obvious way. Then $\hat{\pi} : \hat{U}_4 \to \overline{M}_{0,4}$, obtained by extending π continuously, is still projection to the second factor. The sections $0, 1, \infty$ and δ all extend continuously to sections of $\hat{\pi}$. We can similarly define $\hat{\pi}' : \hat{U}_4' \to \overline{M}_{0,4}$ with extended sections $\delta', 0, 1$ and ∞.

The morphism ι defines a rational map from $\hat{U}_4$ to $\hat{U}_4'$. We can see from the matrix defining each ι_b, given above, this rational map can be extended everywhere except at the points $(1,1)$ and (∞, ∞). The restriction $\iota_1 : (\hat{U}_4)_1 \backslash \{(1,1)\} \to (\hat{U}_4')_1$ has image a single point (which corresponds to the matrix above being singular for $b = 1$). We have a similar situation in the fibre over ∞.

In general, by choosing different sections to trivialise to the constant sections $0, 1$ and ∞, we obtain different universal families over $M_{0,4}$, and thus different rational maps from $\mathbb{P}^1 \times \mathbb{P}^1$ to itself. The only points at which we cannot always extend this rational map to a morphism will be among $(0,0), (1,1)$ and (∞, ∞). To try and rectify this, let $\overline{U}_4$ be the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at these three points. Let $\overline{\pi} : \overline{U}_4 \to \overline{M}_{0,4}$ be the composition of the blow-up morphism with $\hat{\pi}$. We can extend the four sections of π continuously to sections of $\bar{\pi}$. For their images in the exceptional divisors, E_0, E_1 and E_∞ (each isomorphic to $\mathbb{P}^1$) over $(0,0), (1,1)$ and (∞, ∞) respectively, the constant sections have direction $0 = [0:1]$, whereas the diagonal section δ has direction $1 = [1:1]$. Notice that

blowing up preserves the disjointness of our sections.

Each of the rational maps from $\mathbb{P}^1 \times \mathbb{P}^1$, obtained as in the previous paragraph, now extend to isomorphisms. Where ι_1 mapped $(\hat{U}_4)_1 \backslash \{(1,1)\}$ to a point, it now maps $(\overline{U}_4)_1 \backslash E_1$ isomorphically onto the exceptional divisor E_1' over $(1,1)$ in $\overline{U}_4'$ (minus the point at infinity). The extended morphism ι maps E_1 isomorphically onto a copy of $\mathbb{P}^1$ whose image in $\mathbb{P}^1 \times \mathbb{P}^1$ is $\mathbb{P}^1 \times \{1\}$. The isomorphism $\iota : \overline{U}_4 \to \overline{U}_4'$ is also compatible with the ordered sections. This makes $\overline{\pi}$ a good candidate for a universal family for some class $\overline{\mathcal{T}}$ that will give a completion of $M_{0,4}$.

If $\overline{\pi} : \overline{U}_4 \to \overline{M}_{0,4}$ were a universal $\overline{\mathcal{T}}$-family, then we have three equivalence classes of $\overline{\mathcal{T}}$-schemes that are not $\mathcal{T}$-schemes. Namely, these are the classes of the fibres of $\overline{\pi}$ over $0, 1$ and ∞. Each of these is the union of two irreducible components, each isomorphic to $\mathbb{P}^1$, intersecting in a singular point that is a node (see below). Furthermore, each of these irreducible components contain the image of exactly two out of the four sections of $\overline{\pi}$. The way in which these four points are shared between the two irreducible components determines the equivalence class. Each of these fibres is a so-called stable rational curve with four labelled points.

We now spend some time defining and classifying stable, n-labelled, rational curves. These objects will form our new class $\overline{\mathcal{T}}$ so that $\overline{M}_{0,n} = M_{\overline{\mathcal{T}}}$ is a complete, fine moduli space containing $M_{0,n}$ as a dense open subscheme. For $n = 4$, we will obtain that $\overline{M}_{0,n} = \mathbb{P}^1$ with universal family $\overline{\pi}$ as defined above.

Definition 2.3.1. A stable, n-labelled, rational curve is a connected curve, X, with the following properties:

- It is nodal, i.e. the only the singularities in X are locally isomorphic to a neighbourhood of the origin in $\{xy = 0\} \subset \mathbb{A}^2$.
- It has finitely many irreducible components $X_1, \ldots, X_t$.

- The arithmetic genus of X, $\rho_a(X)$, is equal to zero (i.e. X is rational).
- There are n distinct, labelled, smooth points in X with the condition that each irreducible component of X contains at least three points that are either a labelled point or a node.

In the definition above, the second bullet is actually implied by the other three, but we have incorporated it into our definition for simplicity. Let $\overline{\mathcal{T}}$ be the class of stable, n-labelled, rational curves. We clearly have $\mathcal{T} \subset \overline{\mathcal{T}}$, with $\mathcal{T}$ consisting of the nonsingular $\overline{\mathcal{T}}$-schemes. To understand the singular objects making up $\overline{\mathcal{T}} \backslash \mathcal{T}$, we need to discuss how one can find the arithmetic genus of a connected, nodal curve with finitely many irreducible components. For the rest of this section, by a curve we will mean a connected, nodal curve with finitely many irreducible components.

Over the complex numbers, one can understand these curves using topology. To this end, let X be a curve with irreducible components $X_1, \ldots, X_t$, where $t \geqslant 1$. Then X is a compact, complex manifold and for each $i \in [t]$, X_i is homeomorphic to a compact, orientable surface. In this setting we can 'smooth' X to a nonsingular curve, X', by replacing each node in X with a loop as illustrated in the figure below.

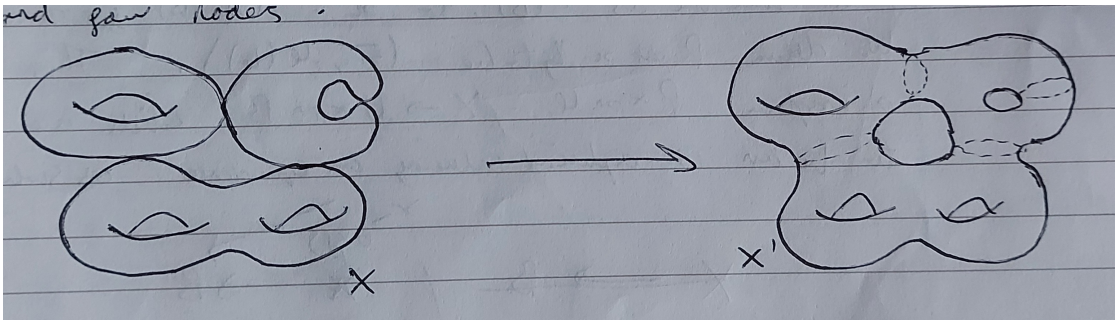


Figure 2: Smoothing a curve X to find its arithmetic genus

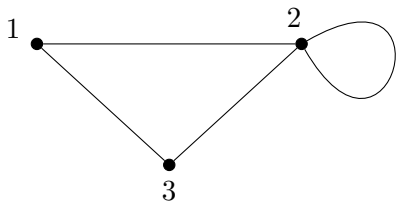
Now, starting with X', we can define a flat family of schemes contracting each of these

loops in X' back to nodes, recovering X. Since arithmetic genus is constant over flat families, we must have $\rho_a(X) = \rho_a(X')$. Since X' is nonsingular, its arithmetic genus is its genus as a topological space as discussed in §2.2. In the diagram above for example, we have $\rho_a(X) = \rho_a(X') = 5$. In practice, this allows us to compute the arithmetic genus of any complex curve X.

We want to classify the curves with arithmetic genus 0. For this we introduce a so-called dual graph associated to each curve.

Definition 2.3.2. Let X be a curve with irreducible components $X_1, \ldots, X_t$. The dual graph of X, denoted by $\Gamma(X)$, is described as follows. Its vertex set is just $[t]$. There is one edge between two vertices i and j in $[t]$ for each node in $X_i \cap X_j$. Note that $\Gamma(X)$ may contain loops, and may have more than one edge between any pair of vertices.

Example 2.3.4. Consider the curve X with arithmetic genus 5 as in Figure 2 above. It has three components and four nodes. Its dual graph therefore has three vertices and four edges, and it is drawn below.



$\Diamond$

In general, properties of the dual graph of a curve can give us information on the arithmetic genus.

Proposition 2.3.5. Let X be a curve with components $X_1, \ldots, X_t$. If its dual graph contains a cycle (including a loop or an instance of more than one edge between some pair of vertices), then $\rho_a(X) > 0$.

Sketch Proof. Let $i_0, \ldots, i_l = i_0 \in [t]$ be a cycle of shortest length in $\Gamma(X)$, where $l = 0$ if $\Gamma(X)$ has a loop. If $l > 0$, let $e_{j,j+1}$, for each $j = 0, \ldots, l-1$, be the edge between i_j and i_{j+1} in this cycle. If $l = 0$ then let e be the corresponding loop. Now let p_i be an arbitrary smooth point in X_i for each $i \in [t]$. We know that each X_i is path connected, so if $l > 0$ then for each $j = 0, \ldots, l-1$, there is a path from p_{i_j} to $p_{i_{j+1}}$, whose image lies entirely in $X_{i_j} \cup X_{i_{j+1}}$, passing through the node corresponding to the edge $e_{j,j+1}$ exactly once and passing through no other node. If $l = 0$, then there is a path from p_{i_0} to itself, whose image lies entirely in X_{i_0}, passing through the node corresponding to the edge e exactly once and passing through no other node.

Concatenating these paths, we obtain a loop in X. When we smooth X to X' to compute the arithmetic genus we don't affect the path-connectedness between intersecting components, hence we can lift this to a loop in X'. This loop is not contractible in X' for it goes around the 'hole' in X' between the components $X_{i_0}, \ldots, X_{i_{l-1}}$ exactly once (since the cycle is of shortest length). This implies that the fundamental group of X' is non-trivial, so X' cannot be homeomorphic to the sphere, and thus $\rho_a(X') > 0$. □

The proposition implies that the dual graph of a stable, n-labelled, rational curve must be a tree (note that $\Gamma(X)$ is connected since X is). Conversely, let X be a curve with components $X_1, \ldots, X_t$ and write $\rho_a(X_i) = g_i \geqslant 0$ for each $i \in [t]$. Suppose further that there are m nodes in X, so m is a nonnegative integer. An induction argument [HKP$^+$03, Ex 22.2.1] on m shows that

$$\rho_a(X) = 1 + m - t + \Sigma_{i=1}^{t} g_i. \tag{2.1}$$

Now m is just the number of edges in $\Gamma(X)$. If $\Gamma(X)$ is a tree then we have $m = t - 1$ necessarily and hence in this case $\rho_a(X) = \Sigma_{i=1}^{t} g_i$. Clearly, the only way that we can have $\rho_a(X) = 0$ is if $g_i = 0$ for each $i \in [t]$. Since each irreducible component of X is nonsingular we have proven the following:

Proposition 2.3.6. A stable, n-labelled, rational curve, X, is a connected nodal curve with irreducible components $X_1, \ldots, X_t$, all isomorphic to $\mathbb{P}^1$. Furthermore, the dual graph of X is a tree and each component contains at least three points that are either labelled points or a node. $\square$

We could have defined our $\overline{\mathcal{T}}$-schemes by the equivalent description given in the proposition to start with. However, one can also study the moduli spaces of stable, n-labelled curves with arithmetic genus equal to some $g > 0$. For larger g there will be many more 'dual graph types' of curves with arithmetic genus g. For $g = 0$ we have proven that the only dual graph types are trees.

Returning to the case where $g = 0$, fix an integer $n \geqslant 3$. We have an understanding of the class $\overline{\mathcal{T}}$ of stable, n-labelled, rational curves. An equivalence of $\overline{\mathcal{T}}$-schemes is an isomorphism of curves inducing an order preserving bijection on the labelled points. One can use the stability condition that each irreducible component of a $\overline{\mathcal{T}}$-scheme contains at least three points that are either labelled or a node to prove that the automorphism group of a $\overline{\mathcal{T}}$-scheme is trivial. The only issue is dealing with nodal points that are not ordered in any way. To deal with this, if a $\overline{\mathcal{T}}$-scheme is reducible, then choose an irreducible component whose vertex in the dual graph is a leaf. It is easy to argue that this component must be fixed under any automorphism. One can then use induction on the number of irreducible components.

We define our moduli functor $F = F_{\overline{\mathcal{T}}}$ as follows. Given any scheme, B, a $\overline{\mathcal{T}}$-family over B is a scheme, X, with a flat and proper morphism, $\pi : X \to B$, along with n disjoint sections, $\sigma_i : B \to X$, $i \in [n]$, of π making each fibre a $\overline{\mathcal{T}}$-scheme. Note that the sections must have their images contained in the smooth locus of each fibre. The notion of equivalence between $\overline{\mathcal{T}}$-schemes, as well as the compatibility with fibre products, use the same diagrams as in §2.2 for $\mathcal{T}$-families.

If $n = 3$, then there is only one $\overline{\mathcal{T}}$-scheme up to equivalence. Indeed, if X is a $\overline{\mathcal{T}}$-

scheme with more than one irreducible component, then $\Gamma(X)$ is a tree with at least two vertices. As such, $\Gamma(X)$ has at least two leaves, which correspond to irreducible components containing exactly one node. For stability, both of these components must have at least two labelled points, but there are only three in total. Thus, in this case, we have $\overline{\mathcal{T}} = \mathcal{T}$ and $\bar{M}_{0,3} = M_{0,3}$ is a fine moduli space representing $F_{\overline{\mathcal{T}}}$. This is unsurprising as $M_{0,3}$ is already complete.

The argument above using leaves of dual graphs with more than one vertex implies that, if $n = 4$, then the only singular $\overline{\mathcal{T}}$-schemes are those curves with two irreducible components, each isomorphic to $\mathbb{P}^1$, intersecting in a node, and each containing two of the four labelled points.

Suppose that $n = 4$ and X is a singular $\overline{\mathcal{T}}$-scheme with irreducible components X_1 and X_2, each isomorphic to $\mathbb{P}^1$, and labelled points p_1, p_2, p_3 and p_4. We can assume that $p_1 \in X_1$.

Suppose that $p_4 \in X_1$ as well, so p_2 and p_3 must lie in X_2. Denote by p_* the singular point of X, which is precisely $X_1 \cap X_2$. Then X_1 and X_2 are each 3-labelled, nonsingular, rational curves as in §2.2. Indeed, X_1 is labelled by p_1, p_4 and p_* and X_2 is labelled by p_2, p_3 and p_*. There is a unique isomorphism of X with the fibre $(\overline{U}_4)_0$ of $\overline{\pi}$ inducing an equivalence of $\overline{\mathcal{T}}$-schemes. Indeed, there is a unique isomorphism of X_1 with E_0 sending p_1 to $\sigma_1(0), p_2$ to $\sigma_2(0)$ and p_* to the singular point of the fibre, since both X_1 and E_0 are isomorphic to $\mathbb{P}^1$. Similarly, there is a unique isomorphism of X_2 with the remaining irreducible component of $(\overline{U}_4)_0$ sending p_2 to $\sigma_3(0)$, p_3 to $\sigma_4(0)$ and p_* to the correct point. This defines an isomorphism from X to $(\overline{U}_4)_0$ since they agree on $X_1 \cap X_2 = p_*$. In the next section we dive deeper into this occurence of n'-labelled curves, with $3 \leqslant n' < n$, being glued together to form the boundary curves.

If instead, p_2 shares an irreducible component with p_4, then X is equivalent as a $\overline{\mathcal{T}}$-scheme to the fibre $(\overline{U}_4)_1$. Finally, if p_3 and p_4 share an irreducible component, then

X is equivalent to the $\overline{\mathcal{T}}$-scheme $(\overline{U}_4)_\infty$. Since any automorphism of a curve with two irreducible components must restrict to a pair of isomorphisms, these three equivalence classes of $\overline{\mathcal{T}}$-schemes are distinct (simply put, applying an isomorphism can't change how the four points are shared between the two components).

We briefly explain how to show that $\overline{\pi}$ is a universal $\overline{\mathcal{T}}$-family. Let $\psi : X \to B$, with sections $\tau_1, \ldots, \tau_4$, be any $\overline{\mathcal{T}}$-family. Similarly to in §2.2, we want to define morphisms, $\varphi : B \to \overline{M}_{0,4}$ and $\rho : X \to \overline{U}_4$, so that Proposition 1.2 gives us a morphism $\rho \times_{\overline{M}_{0,4}} \psi$ that is an isomorphism realising an equivalence of $\overline{\mathcal{T}}$-families between ψ and the pull-back of $\overline{\pi}$ along φ.

On the open loci of X and B defined by those $\overline{\mathcal{T}}$-schemes in the family that are non-singular, these morphisms will be exactly the ones defined in §2.2. The classification of singular $\overline{\mathcal{T}}$-schemes given above, along with the flatness of $\overline{\pi}$ and ψ (which can be thought of as ensuring that the limiting schemes in families are well-defined and unique) imply that φ and ρ extend uniquely to all of B and X respectively in a way so that the product has the desired properties.

For general $n \geqslant 4$, let $U_n = \mathbb{P}^1 \times (\mathbb{P}^1 \backslash \{0, 1, \infty\})^{n-3}$ be so that $\pi : U_n \to M_{0,n}$, with sections $\sigma_1, \ldots, \sigma_n$, is the universal $\mathcal{T}$-family as defined in §2.2. Inductively, we obtain the complete universal scheme $\overline{U}_n$ from $\overline{U}_{n-1}$ by blowing up $\overline{U}_{n-1} \times \mathbb{P}^1$ along subschemes corresponding to when the final, newly added, section may come together with the original $n-1$ sections. We explain an instance of this with the example below.

Example 2.3.7. Let $n = 5$ and consider $\hat{U}_5 = \overline{U}_4 \times \mathbb{P}^1$. Let z and b be the coordinates on $\mathbb{P}^1 \times \mathbb{P}^1$ in the blow-up $\overline{U}_4 \to \mathbb{P}^1 \times \mathbb{P}^1$. Let c be the coordinate of the second factor in $\hat{U}_5$. Then $\overline{U}_5$ is obtained by blowing up $\hat{U}_5$ along the subschemes defined by

$$\{z = c = 0\}, \quad \{z = c = 1\}, \quad \{z = c = \infty\}, \quad \{b = c\}.$$

The first three subschemes above correspond to when the fifth section comes together with the first, second and third sections respectively. The fourth subscheme corresponds to when the fifth section comes together with the fourth. The fibres of $\overline{U}_5$ lying over exactly one of these subschemes have two irreducible components - one labelled by two points and the other labelled by three points.

The fibres of $\overline{U}_5$ lying over the intersection of two of the subschemes above have three irreducible components. The dual graph of any such curve must be a path. The irreducible components corresponding to the leaf vertices must have two labelled points. The irreducible component corresponding to the non-leaf vertex (which has degree two) must contain the other labelled point. We can then compute that there are $5 \cdot \frac{1}{2}\binom{4}{2} = 15$ equivalence classes of $\overline{\mathcal{T}}$-schemes with three irreducible components. $\Diamond$

In this way, for each $n \geqslant 4$, we get a morphism, $\overline{\pi} : \overline{U}_n \to \overline{U}_{n-1}$, which corresponds to 'forgetting' the last section, and contracting any irreducible components that are left with less than three points that are either labelled or a node (see [Cav16, §2.2]). This morphism defines a universal $\overline{\mathcal{T}}$-family, so $\overline{M}_{0,n} = \overline{U}_{n-1}$ is a fine moduli space. This description also shows that $\overline{M}_{0,n}$ is smooth for all $n \geqslant 3$, which will be a fact we use often when talking about divisors.

A more detailed description of $\overline{U}_n$ as a successive blow-up over codimension two subschemes is given by [Kee92]. The construction of the universal family from contraction (corresponding to forgetting the final labelled point) and stabilisation (adding a new labelled point) was originally given in [KM77].

2.4 The boundary of $M_{0,n}$ in $\overline{M}_{0,n}$

The previous two sections describe in detail the moduli spaces $M_{0,n}$ and $\overline{M}_{0,n}$ for each $n \geqslant 3$. The $\overline{\mathcal{T}}$-schemes that make up the boundary, $\partial M_{0,n}$, of $M_{0,n}$ in $\overline{M}_{0,n}$ are the singular $\overline{\mathcal{T}}$-schemes (of course, when we say that a $\overline{\mathcal{T}}$-scheme is singular we really mean

that the underlying scheme is singular). We want to describe the structure and the combinatorics of this boundary. We will see that it has codimension one, and its irreducible components are gluings of families of n'-labelled, stable, rational curves for some n' (possibly varying) with $3 \leqslant n' < n$. In §2.3 this was already illustrated in the case $n = 4$. Indeed, $\partial M_{0,4}$ consisted of three points, each corresponding to a pair of 3-labelled, stable, rational curves glued along their third labelled points.

The labelled dual graphs we define in this section are equivalent to those given in [Cav16, §2.1]. Our modified definition will reflect the different topological types of the schemes underlying objects in $\overline{\mathcal{T}}$.

We first extend our idea of the dual graph of a curve (with the same conventions as the preceding section), X. It is clear that the underlying schemes of two $\overline{\mathcal{T}}$-schemes are isomorphic if and only if their dual graphs are isomorphic. Adding some more data to these dual graphs allows us to stratify the boundary $\partial M_{0,n}$.

Definition 2.4.1. Let X be an n-labelled, stable, rational curve with irreducible components $X_1, \ldots, X_t$ (where $t \geqslant 1$). The labelled dual graph, $\Gamma^+(X)$, of X is its dual graph $\Gamma(X)$, along with the following extra data: For each i in the vertex set $[t]$, we associate the subset I_i of $[n]$ defined by the labelled points lying in X_i.

Let X be a curve and let $\Gamma^+(X)$ be its dual graph. The three properties below must hold. Furthermore, any tree with these extra data on vertices satisfying these three properties is a labelled dual graph of some curve in $\overline{M}_{0,n}$. We thus call such graphs labelled dual graphs.

- For each $i \in [t]$, we have $|I_i| + \deg(i) \geqslant 3$, where $\deg(i)$ is the degree of the vertex i in $\Gamma(X)$.
- Whenever $i \neq j$ in $[t]$, we have $I_i \cap I_j = \emptyset$.
- We have $\bigcup_{i=1}^{t} I_i = [n]$.

We say that two labelled dual graphs, $\Gamma^+(X)$ and $\Gamma^+(X')$, associated to curves, X and X' in $\overline{M}_{0,n}$, are isomorphic if there is an isomorphism of graphs, $f : \Gamma^+(X) \to \Gamma^+(X')$, so that $I_i = I_{f(i)}$ for each $i \in [t]$. Let $\mathcal{G}$ be the set of isomorphism classes of labelled dual graphs. For each $\gamma \in \mathcal{G}$, let $\overline{M}^{\gamma}_{0,n}$ be the subscheme of $\overline{M}_{0,n}$ that is the closure of the subset corresponding to those curves whose labelled dual graph is in the class γ.

For each $\gamma \in \mathcal{G}$, the subscheme $\overline{M}^{\gamma}_{0,n}$ can be viewed by taking the product of $M_{0,|I_i|+\deg(i)}$ as i ranges over $[t]$, and then gluing along the labelled points corresponding to edges of the dual graph.

We define an operation on labelled dual graphs as follows: If X is a curve in $\overline{M}_{0,n}$, then given any two vertices i and j in $\Gamma^+(X)$ connected by an edge, we can contract this edge to a new vertex, w, obtaining a new tree, T, with vertex set $[t]\backslash\{i,j\} \cup \{w\}$. We label the vertices of T by subsets of $[n]$ by putting $I_w = I_i \cup I_j$ and labelling the other vertices by the same subset as in $\Gamma^+(X)$. Any T obtained in this way is itself a labelled dual graph.

The above operation defines a partial order on the set $\mathcal{G}$, where we put $\gamma \prec \gamma'$ in $\mathcal{G}$ if we can obtain γ' from γ from a sequence of these operations. The class of the labelled dual graph with one vertex is the unique maximal element of $\mathcal{G}$ under this partial order. This labelled dual graph corresponds to those curves in the open and dense $M_{0,n}$, i.e. those curves not on the boundary.

Proposition 2.4.2 [Cav16, §2.1]**.** Let $\gamma, \gamma' \in \mathcal{G}$ with $\gamma \prec \gamma'$. Then $\overline{M}^{\gamma}_{0,n}$ is contained in the Zariski closure of $\overline{M}^{\gamma'}_{0,n}$ in $\overline{M}_{0,n}$. □

We are nearly ready to describe the boundary $\partial M_{0,n}$. We first define a set that will feature regularly throughout this document.

Definition 2.4.3. Let $\mathcal{I}$ be the set of unordered pairs, $\{I, I^c\}$, where $I \subset [n]$, and the complement is taken in $[n]$, such that both $|I| \geqslant 2$ and $|I^c| \geqslant 2$. We will write I, where

$I \subset [n]$ and $2 \leqslant |I| \leqslant n-2$, for a given element of $\mathcal{I}$, understanding that $I \equiv I^c$.

For each $I \in \mathcal{I}$, we define $|I| = \min\{|I|, |I^c|\}$. Note that $2 \leqslant |I| \leqslant \frac{n}{2}$. If we are in a situation where the sizes of both I and I^c appear in the same equation we will write $\#I$ and $\#I^c$ in these equations for clarity.

We have seen that $\overline{M}_{0,n}$ is the disjoint union of $\overline{M}^{\gamma}_{0,n}$ as γ ranges over $\mathcal{G}$. Consider the elements of $\mathcal{G}$ whose graphs contain just two vertices. These classes are in bijection with the set $\mathcal{I}$ defined above. Indeed, elements of $\mathcal{I}$ correspond to the different ways of labelling the two vertices in the labelled dual graphs.

Let γ be a class in $\mathcal{G}$ corresponding to some $I \in \mathcal{I}$ as above. Then there is an isomorphism from the product of $M_{0,\#I+1}$ and $M_{0,\#I^c+1}$ to $\overline{M}^{\gamma}_{0,n}$ which identifies the final marked points in each entry of the product. As such, by §2.2, we have

$$\dim(\overline{M}^{\gamma}_{0,n}) = (\#I + 1 - 3) + (\#I^c + 1 - 3) = n - 4.$$

We also see from this construction that $\overline{M}^{\gamma}_{0,n}$ is irreducible. Combining this with Proposition 2.4.2 we see that the irreducible, closed, codimension one subschemes of $\overline{M}_{0,n}$ contained in the boundary $\partial M_{0,n}$ are in one-to-one correspondence with elements of $\mathcal{I}$. Indeed, this correspondence is obtained by taking the closure of the corresponding $\overline{M}^{\gamma}_{0,n}$.

Let $I_1, I_2 \in \mathcal{I}$. Let γ_1 and γ_2 be the corresponding elements of $\mathcal{G}$. Then $\overline{M}^{\gamma_1}_{0,n} \cap \overline{M}^{\gamma_2}_{0,n}$ admits a nice description as the union of those $\overline{M}^{\gamma'}_{0,n}$, with $\gamma' \in \mathcal{G}$, such that $\gamma' \prec \gamma_1$ and $\gamma' \prec \gamma_2$. Inductively, the codimension of $\overline{M}^{\gamma}_{0,n}$, where $\gamma \in \mathcal{G}$, is one less than the number of vertices in those graphs in γ.

This concludes our discussion on $\partial\overline{M}_{0,n}$. As we will see later, the divisor class group of $\overline{M}_{0,n}$ is generated by those prime divisors contained in the boundary.

3 Toric Geometry

This document concerns an embedding of $\overline{M}_{0,n}$ into projective space for $n \geqslant 5$. To do this we will show that $\overline{M}_{0,n}$ is a closed subscheme of some smooth toric variety, which is the purpose of this chapter.

In this chapter we introduce notation and basic results on toric theory that we will use. The usual reference for this is [Ful93]. We adopt the same conventions for toric varieties as in this book except that we will not require them to be normal. For this reason, [Cox95] is a good second reference, with an appendix on toric varieties that are not normal. Detailed proofs of key results in [Ful93, §3.1-2] are found in [OM78, §4-5].

We will also briefly introduce some tropical geometry, which is a vital tool for embedding $\overline{M}_{0,n}$ into a toric variety. A good introduction to this topic can be found in [MS15].

3.1 Algebraic groups and actions

Before defining toric varieties, we need to introduce algebraic groups and what we mean by an action of an algebraic group on a scheme. All definitions and statements here can be found in the opening sections of [MFK94].

Definition 3.1.1. A scheme, G, is an algebraic group if there are morphisms,

$$e : \operatorname{Spec} \mathbb{C} \to G, \quad m : G \times G \to G, \quad i : G \to G,$$

that give the points of G the structure of a group. Here, $e(\langle 0 \rangle)$ plays the role of the identity, and the maps m and i define multiplication and inversion respectively.

We can talk about usual properties of schemes for algebraic groups. For example, an affine algebraic group is an algebraic group whose underlying scheme is affine. A homomorphism of algebraic groups is a morphism of the underlying schemes that induces a group homomorphism between the points. Note that a closed subgroup of an algebraic

group can be given the structure of an algebraic group by restricting the defining morphisms in Definition 3.3.1. The product (over $\operatorname{Spec}\mathbb{C}$) of any two algebraic groups is itself an algebraic group (just take products of the defining morphisms).

Example 3.1.2. One of the simplest examples of an algebraic group are the algebraic tori. We define the one-dimensional torus first. Let $T = \operatorname{Spec}\mathbb{C}[t, t^{-1}]$. Since T is affine, we can define the morphisms e, m and i by their corresponding homomorphisms of coordinate rings. Note that

$$\mathbb{C}[T \times T] \cong \mathbb{C}[t_1, t_1^{-1}, t_2, t_2^{-1}].$$

The ring homomorphisms are described below.

$$\begin{aligned} e^* : \mathbb{C}[T] &\longrightarrow \mathbb{C}, & f(t) &\mapsto f(1) \\ m^* : \mathbb{C}[T] &\longrightarrow \mathbb{C}[T \times T], & f(t) &\to f(t_1 t_2) \\ i^* : \mathbb{C}[T] &\longrightarrow \mathbb{C}[T], & f(t) &\to f(t^{-1}) \end{aligned}$$

One can check that these give the points of T the structure of a group isomorphic to $\mathbb{C}^*$, the multiplicative group obtained from the field $\mathbb{C}$ by removing 0.

Definition 3.1.3. For each (integer) $m \geqslant 1$, the m-dimensional algebraic torus is the algebraic group, T^m, whose group structure is defined by the product of the group structures on each copy of T. A given algebraic group, T_N, is an algebraic torus if there exists an $m \geqslant 1$ so that $T_N \cong T^m$ as algebraic groups.

Of course, the underlying group of any algebraic torus is isomorphic to the group $(\mathbb{C}^*)^m$ for some $m \geqslant 1$, where m is the dimension of the underlying scheme.

Let X be a scheme and let G be an algebraic group. An action of G on X is a morphism, $a : G \times X \to X$, that satisfies the usual properties making the group G act on the set of

points of X. Note that for any algebraic group, G, the multiplication morphism defines an action of G on itself.

When a group acts on a set, we have orbits of this action. In this algebraic setting, one wonders when a nice scheme exists whose points are in bijection with orbits. This is the motivation behind geometric invariant theory, of which we give some of the most basic definitions.

Definition 3.1.4. Let $a : G \times X \to X$ be an action of an algebraic group, G, on a scheme, X. A scheme, Y, along with a morphism, $\pi : X \to Y$, is a categorical quotient if the following diagram commutes:

$$\begin{array}{ccc} G \times X & \xrightarrow{\quad a \quad} & X \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{\quad \pi \quad} & Y \end{array} .$$

Here, the left vertical arrow is the usual projection. Furthermore, (Y, π) is universal in the sense that if Z and $\psi : X \to Z$ is any other pair making the diagram above commute, then ψ factors uniquely through π. In this case we often write X/G for the scheme Y.

Definition 3.1.5. Let $a : G \times X \to X$ be as in Definition 3.1.4 and suppose that (Y, π) is a categorical quotient of X by G. Then (Y, π) is said to be a geometric quotient if the properties below also hold.

- For each point y in Y, the points of the fibre of π over y is an orbit of the group G acting on the set of points of X.
- A subset, $U \subset Y$, is open if and only if $\pi^{-1}(U)$ is open in X, i.e. π is submersive.
- For each open $U \subset Y$, the k-algebra $\mathcal{O}_Y(U)$ is isomorphic to the G-invariant sections in $\pi_* \mathcal{O}_X(U)$ via the map induced by π.

Geometric quotients satisfy ideal properties, but they do not exist in general, even if a categorical quotient does exist. A general theme is that one may need to remove some points from X in order to form a geometric quotient. For example, if $m \geqslant 1$, then the torus T acts algebraically on affine m-space, $\mathbb{A}^m$. This is the usual scalar multiplication of $\mathbb{C}$ on $\mathbb{C}^m$. A geometric quotient $\mathbb{A}^m/T$ doesn't exist. This is because the origin is a unique fixed point of this action. Removing the origin, we can find a geometric quotient $\left(\mathbb{A}^m \backslash \{0\}\right)/T$. This can be given by the usual projective m-space, $\mathbb{P}^m$.

Definition 3.1.6. Let G be an algebraic group. Let H be its maximal, normal (in the group theoretical sense), solvable subgroup. Let H_0 be the connected component of H containing the identity. We say that G is reductive if H is an algebraic torus.

We now state a key result from geometric invariant theory. For this, let X be a closed subscheme of $\mathbb{P}^m$, for some $m \geqslant 1$. Suppose that G is a reductive algebraic group acting on $\mathbb{P}^m$. We say that this action is linear if the induced action of G on the homogeneous coordinate ring of $\mathbb{P}^m$ preserves its grading. We say that this action is equivariant with respect to the closed subscheme X if the action of G on $\mathbb{P}^m$ restricts to an action of G on X.

Theorem 3.1.7. Let X be a closed subscheme of $\mathbb{P}^m$. Suppose that G is a reductive algebraic group acting linearly on $\mathbb{P}^m$ in a way that is equivariant with respect to X. Suppose that U is an open subscheme of X upon which G acts freely (meaning that there are no fixed points of this action). Then a geometric quotient of U by G exists. $\square$

In the setting of Theorem 3.1.7, finding a geometric quotient, X/G, amounts to finding a good completion of U/G. We will see later a case with $U/G \cong M_{0,n}$, and the completion X/G will be a copy of $\overline{M}_{0,n}$ in a smooth toric variety.

3.2 Toric varieties

With the definitions of the preceding section, we are ready to define toric varieties and state relevant, well-known results.

Definition 3.2.1. A toric variety, X, is an irreducible, separated scheme of finite type (over $\mathbb{C}$) that contains a dense, open subscheme, T_N, which is an algebraic torus. Furthermore, the natural action of T_N on itself extends to an algebraic action of T_N on X.

We can make the collection of all toric varieties into a category. A morphism of toric varieties is a morphism of schemes that restricts to a homomorphism of algebraic groups between the dense open tori.

To any m-dimensional algebraic torus, T_N, let N be its lattice of one-parameter subgroups, i.e. the abelian group $\mathrm{Hom}_{alg}(T, T_N)$, where T is just the one-dimensional torus. Let M be the dual lattice to N, which can be identified with the character lattice of T_N, which is the abelian group $\mathrm{Hom}_{alg}(T_N, T)$. Both N and M are isomorphic to $\mathbb{Z}^m$. We have the associated real vector spaces $N_{\mathbb{R}} = N \bigotimes \mathbb{R}$ and $M_{\mathbb{R}} = M \bigotimes \mathbb{R}$.

By a cone in $N_{\mathbb{R}}$ we will always mean a strongly convex, rational, polyhedral cone as defined in [Ful93, §1]. If σ is a cone in $N_{\mathbb{R}}$ then we write $\tau \prec \sigma$ whenever τ is a face of σ. A fan, Δ, of cones in $N_{\mathbb{R}}$ is a (non-empty) collection of cones satisfying the following two properties:

- If $\sigma \in \Delta$ and $\tau \prec \sigma$, then $\tau \in \Delta$.
- If $\sigma, \sigma' \in \Delta$, then we have $\sigma \cap \sigma' \prec \sigma$.

Let Δ be a fan in $N_{\mathbb{R}}$ as above. We briefly explain how this corresponds to a normal toric variety X_Δ. The toric variety X_Δ can covered by affine open pieces in bijection with cones in Δ. If $\sigma \in \Delta$, then its dual cone $\sigma^\vee$ lies in $M_{\mathbb{R}}$. The set $S_\sigma = \sigma^\vee \cap M$

has the structure of an abelian semigroup with a zero element, and thus we can form the ring $A_\sigma = \mathbb{C}[S_\sigma]$. The ring A_σ is integrally closed. If U_σ is the corresponding affine scheme, then this is the affine open of X_Δ corresponding to the cone σ in Δ. If $\tau \prec \sigma$ in Δ, then in the dual cone we have $\sigma^\vee \prec \tau^\vee$. This gives us an inclusion of coordinate rings $A_\sigma \hookrightarrow A_\tau$. These maps of rings induce open embeddings $U_\tau \hookrightarrow U_\sigma$, and these morphisms show how the affine opens U_σ, with $\sigma \in \Delta$, are glued together to form X_Δ. Finally, since each cone in Δ is strongly convex, $\{0\} \in \Delta$. The inclusion $U_{\{0\}} \hookrightarrow X_\Delta$ is the embedding of a torus, with dimension $\dim_{\mathbb{R}} N_{\mathbb{R}}$, that makes X_Δ a toric variety.

Now, let N and N' be a pair of lattices. Let Δ be a fan in $N_{\mathbb{R}}$ and let Δ' be a fan in $N'_{\mathbb{R}}$. A morphism of fans from Δ to Δ' is a homomorphism, $\varphi : N \to N'$, of lattices so that for each cone σ in Δ, we have $\varphi(\sigma) \subset \sigma'$ for some cone σ' in Δ'. In this way, we can form the category of fans in lattices.

Theorem 3.2.2 [OM78, Theorem 4.1]**.** There is an equivalence of categories between the category of fans in lattices and the category of normal toric varieties. This equivalence sends a fan Δ in a lattice N to the normal toric variety X_Δ constructed above. □

We can now introduce a normal toric variety as X_Δ, where Δ is a fan in a lattice N (that can be taken to be the character lattice).

That toric varieties which are not normal may not correspond to a fan in any lattice is a result of Sumihiro's theorem, originally seen in [Sum74], possibly failing to be true. See [CLS11, §3.A] for an example of this.

If X is a toric variety that is not normal, then let $\tilde{X} \to X$ be its normalisation. If T_N is the dense open torus in X, then since T_N is normal, it is also an open subscheme of $\tilde{X}$. That T_N is also dense in $\tilde{X}$ follows from the affine case, where normalisation is induced by an injection of rings. That the action of T_N on itself extends to $\tilde{X}$ can also be checked by considering the affine case first. Then $\tilde{X}$ is a normal toric variety, hence

is equal to X_Σ for some fan Σ in some lattice N.

Theorem 3.2.2 assigns to each normal toric variety a fan that can be described combinatorially. For these varieties, many common properties of schemes can be identified just from the corresponding fan (and lattice). For example, the dimension of a toric variety, X_Δ, is just the dimension of the lattice N. The examples given in the rest of this section can be found in [Ful93, §2 - 3.2]. We start with how to check whether a toric variety X_Δ is smooth just using Δ.

Proposition 3.2.3. Let X_Δ be a normal toric variety. Then X_Δ is nonsingular if and only if each cone $\sigma \in \Delta$ has a set of generators that form part of a $\mathbb{Z}$-basis of the lattice N. □

This inspires the use of terminology for fans that we would usually use for schemes. We say that a fan is simplicial if each cone in the fan has a generating set that is linearly independent. We say that a fan is nonsingular if each cone in the fan has a generating set that forms part of a $\mathbb{Z}$-basis of N.

Let Δ be a fan in a lattice N. A fan Δ' in N is a refinement of Δ if each cone in Δ is a union of cones in Δ', and the supports of Δ and Δ' (the set of points in $N_\mathbb{R}$ contained in these fans) are equal. If Δ is singular, then we can always find a refinement, Δ', of Δ so that Δ' is nonsingular. The obvious map of fans $(\Delta', N) \to (\Delta, N)$ induces a birational morphism of toric varieties, $X'_\Delta \to X_\Delta$. This map is a resolution of singularities of X_Δ.

Let X_Δ be a normal toric variety with dense open torus T_N. One can ask what are the orbits of the action of T_N on X_Δ. Again, it turns out that we can describe our answer just using Δ. Namely, there is one T_N-orbit of X_Δ for each cone σ in Δ. Let $\sigma \in \Delta$ and let O_σ be the corresponding orbit. Then O_σ is itself an algebraic torus . The dimension of O_σ is $\dim N - \dim \sigma$. In particular, $O_{\{0\}} = T_N$.

Let $\tau \in \Delta$. Consider the closure, $V(\tau)$, of the orbit O_τ in X_Δ. This is the disjoint union

of orbits corresponding to cones of Δ that contain τ as a face. In other words,

$$V(\tau) = \bigsqcup_{\tau \prec \sigma} O_\sigma.$$

The subscheme $V(\tau)$ has codimension $\dim(\tau)$ and is a toric variety in its own right, with dense open torus O_τ. These $V(\tau)$ give a complete classification of all irreducible, T_N-invariant, closed subschemes of X_Δ.

In particular, the irreducible, T_N-invariant, closed subschemes of codimension one are in bijection with minimal generators (often called rays) of the cones in Δ. In the next section, we will use this fact to show how one can compute the Weil divisor class group and the Picard group of normal toric varieties. For general schemes, this is often difficult.

3.3 Divisors and the Cox ring

We will recap the definitions of both Weil divisors and Cartier divisors on any normal scheme. In general, any Cartier divisor defines a Weil divisor, but not vice versa. On a smooth scheme however, all Weil divisors are Cartier. We then define the Weil divisor class group, $\mathrm{Cl}(X)$, and the Picard group, $\mathrm{Pic}(X)$, for any normal scheme, X. These can be found in [Har97, §II.6]. We then establish how one can compute these groups for a normal toric variety, X_Δ, following [Ful93, §3.3-3.4].

After this we introduce the Cox ring of a normal toric variety, X_Δ. This is a polynomial ring whose variables are indexed by rays of the fan Δ. This ring is graded by $\mathrm{Cl}(X_\Delta)$, and the variables play the role that homogeneous coordinates do in $\mathbb{P}^m$ [Cox95]

Let X be a normal scheme. Recall that a prime divisor on X is a closed, irreducible subscheme of codimension one. A Weil divisor on X is an element of the free abelian group generated by all prime divisors on X. We write $\mathrm{WDiv}(X)$ for this abelian group. A general Weil divisor, $D \in \mathrm{WDiv}(X)$, is written as a finite formal sum, $\Sigma_i n_i Y_i$, where

each n_i is an integer and each Y_i is a prime divisor on X.

Each non-zero rational function, $f \in \mathbb{C}(X)^*$, on X defines an element $(f) \in \mathrm{WDiv}(X)$. This is known as the divisor of zeroes of f.

Explicitly, any prime divisor, Y, on X has a generic point, ζ, in X. This means that the closure of ζ in X is Y. The local ring $\mathcal{O}_{X,\zeta}$ is a discrete valuation ring with quotient field equal to $\mathbb{C}(X)$. Let val_Y denote the corresponding valuation of $\mathbb{C}(X)$. Then the divisor of zeroes of $f \in \mathbb{C}(X)^*$ is the Weil divisor $(f) = \Sigma_Y \, \mathrm{val}_Y(f)Y$, where Y ranges over all prime divisors on X.

The map $f \mapsto (f)$ defines a group homomorphism from $\mathbb{C}(X)^*$ to $\mathrm{WDiv}(X)$. We call divisors in the image of this homomorphism principal divisors.

Definition 3.3.1. Let X be a normal scheme. The divisor class group, $\mathrm{Cl}(X)$, is the quotient of $\mathrm{WDiv}(X)$ by the subgroup of principal divisors.

Recall that a Cartier divisor on X consists of the following data:

- A covering of X by a collection of affine opens, $\{U_\alpha\}_{\alpha \in I}$, where I is some indexing set.
- For each $\alpha \in I$, a rational function $f_\alpha \in \mathbb{C}(X)^*$, so that for each pair, α and α' in I, the rational function $\frac{f_\alpha}{f_{\alpha'}}$ restricted to $U_\alpha \cap U_{\alpha'}$ is a non-vanishing regular function.

The f_α in the data above are called the local equations of the Cartier divisor. If $D = \{U_\alpha, f_\alpha\}_{\alpha \in I}$ is a Cartier divisor, then we define the invertible sheaf, $\mathcal{O}_X(D)$, to be the subsheaf of the constant sheaf of rational functions on X generated on U_α by $\frac{1}{f_\alpha}$.

Any Cartier divisor, D, on X defines a Weil divisor on X which we will also call D. If $D = \{U_\alpha, f_\alpha\}_{\alpha \in I}$ and $D' = \{U_{\alpha'}, f_{\alpha'}\}_{\alpha' \in I'}$ are two Cartier divisors on X, then the sum $D + D'$ is defined to be the Cartier divisor on X with local equation $f_\alpha f_{\alpha'}$ on the affine

open $U_\alpha \cap U_{\alpha'}$ for each $\alpha \in I, \alpha' \in I'$. Note that each such $U_\alpha \cap U_{\alpha'}$ is affine since we have assumed that X is separated. This makes the set of Cartier divisors a subgroup of $\mathrm{WDiv}(X)$, which we call $\mathrm{CDiv}(X)$. Note that any principal divisor is Cartier via $(f) \mapsto \{X, f\}$.

The Picard group of X, $\mathrm{Pic}(X)$, is the abelian group of invertible sheaves on X up to isomorphism. The map $D \mapsto \mathcal{O}_X(D)$ is a homomorphism from $\mathrm{CDiv}(X)$ to $\mathrm{Pic}(X)$. Its kernel is precisely the subgroup of principal divisors. Therefore $\mathrm{Pic}(X) \subset \mathrm{Cl}(X)$. If X is smooth then every Weil divisor is Cartier and this inclusion is an equality.

The groups $\mathrm{Cl}(X)$ and $\mathrm{Pic}(X)$ are useful invariants of a normal scheme, X. In practice however, they are often difficult to compute. For normal toric varieties, the computation is much easier. The collection of all divisors on a scheme X is generally very large. With normal toric varieties, we only need to look at divisors that are invariant under the torus action.

Definition 3.3.2. Let X_Δ be a normal toric variety with dense open torus T_N. Let $D = \Sigma_i n_i Y_i$, where $n_i \neq 0$ for each i, be a Weil divisor on X_Δ. We say that D is T_N-invariant if each Y_i is invariant under the action of T_N on X_Δ. Let $\mathrm{WDiv}_T(X_\Delta)$ and $\mathrm{CDiv}_T(X_\Delta)$ denote the abelian groups of T_N-invariant Weil divisors and T_N-invariant Cartier divisors on X_Δ respectively.

Let X_Δ be a normal toric variety. In the previous section we characterised all T_N-invariant prime divisors on X_Δ. These are the closures of the T_N-orbits of X_Δ corresponding to rays of Δ. Let $\tau_1, \ldots, \tau_s$ be the rays of Δ and for each $i \in [s]$, let $v_i \in N$ be the minimal rational generator of τ_i and let $D_i = V(\tau_i)$. We need to describe $\mathrm{CDiv}_T(X_\Delta)$ as a subgroup of $\mathrm{WDiv}_T(X_\Delta)$.

Recall that N is the character lattice of T_N, and M is its dual. We denote by $\langle -, - \rangle$ the dual pairing from $M \times N$ to $\mathbb{Z}$.

Proposition 3.3.3. Let X_Δ be a normal toric variety as above. The T_N-invariant Weil divisor $\Sigma_{i=1}^s a_i D_i$ on X is a T_N-invariant Cartier divisor on X if and only if for each (maximal) cone σ in Δ, there exists some u_σ in M so that $\langle u_\sigma, v_i \rangle = -a_i$ for each v_i in σ. □

Any T_N-invariant principal divisor is determined by a fixed $u \in M$. If $u \in M$ then the corresponding divisor is $\Sigma_{i=1}^s \langle u, v_i \rangle D_i$. This defines a group homomorphism from M to $\mathrm{CDiv}_T(X) \subset \mathrm{WDiv}_T(X)$.

Proposition 3.3.4. Let X_Δ be a normal toric variety as above. Suppose that Δ is not contained in any proper linear subspace of $N_\mathbb{R}$. Then there is a commutative diagram, whose rows are exact, as drawn below.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & \mathrm{CDiv}_T(X_\Delta) & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & \mathrm{WDiv}_T(X_\Delta) & \longrightarrow & \mathrm{Cl}(X_\Delta) & \longrightarrow & 0 \end{array}.$$

The vertical arrows here are inclusions. The first non-trivial map in each row is the homomorphism described above and the second are restrictions of the quotient by the subgroup of principal divisors. Furthermore, if X_Δ is smooth, then $\mathrm{Cl}(X_\Delta)$ has no torsion.

Proof. The diagram can be found in [Ful93, §3.4]. The last statement follows from Proposition 3.2.3 and Proposition 3.3.3. □

Associated to the normal toric variety $X = X_\Delta$ is a coordinate ring S that is graded by $\mathrm{Cl}(X_\Delta)$. This acts like the homogeneous coordinate ring $\mathbb{C}[x_0, \ldots, x_n]$ of $\mathbb{P}^n$. Write $\Delta(1)$ for the set of rays in Δ. For each ρ in $\Delta(1)$, let v_ρ be the first lattice point of N lying on ρ. Let D_ρ be the corresponding T_N-invariant Weil Divisor. The Cox ring, S, of

X is

$$S = \mathbb{C}[x_\rho : \rho \in \Delta(1)].$$

The $\mathrm{Cl}(X_\Delta)$-grading of S is given by $\deg(x_\rho) = [D_\rho] \in \mathrm{Cl}(X_\Delta)$ for each ρ in $\Delta(1)$. The theory and three propositions below are initially given in [Cox95].

Proposition 3.3.5. Let $X = X_\Delta$ be a normal toric variety. Let $\alpha = [D] \in \mathrm{Cl}(X_\Delta)$, and denote by S_α the corresponding graded piece of S. Then S_α is a $\mathbb{C}$-vector space that is isomorphic to the space of global sections of the coherent sheaf, $\mathcal{O}_X(D)$, on X. In other words, we have

$$S_\alpha \cong H^0(X, \mathcal{O}_X(D)).$$

□

We can see that S and its $\mathrm{Cl}(X_\Delta)$-grading depend only on $\Delta(1)$, thus we don't have enough information to recover X from its graded Cox ring. For each cone σ in Δ we define a monomial, x_σ in S, by

$$x_\sigma = \prod_{\rho \not< \sigma} x_\rho.$$

We can then define a monomial ideal, B of S, generated by these x_σ as σ ranges over all the cones in Δ. A minimal generating set of monomials for B is given by those x_σ indexed by maximal cones of the fan. In the case of projective space, B is the usual irrelevant ideal.

Associated to S is the affine scheme $\operatorname{Spec} S$, which we denote by $\mathbb{A}^{\Delta(1)}$. Given an ideal, $I \subset S$, we can define the closed subset $V(I)$ of $\mathbb{A}^{\Delta(1)}$ as usual. Namely, $V(I)$ is the set of prime ideals containing I. In particular we have a closed subset $Z = V(B)$ in $\mathbb{A}^{\Delta(1)}$. Assume further that Δ is smooth, and let G be the abelian group $\mathrm{Hom}_{gps}(\mathrm{Cl}(X), \mathbb{C}^*)$. This is isomorphic to a torus since $\mathrm{Cl}(X)$ is free abelian by Proposition 3.3.4. Applying the exact functor $\mathrm{Hom}_{gps}(-, \mathbb{C}^*)$ to the bottom exact sequence in Proposition 3.3.4, we

see that G is a subgroup of $(\mathbb{C}^*)^{\Delta(1)} \equiv T^{\Delta(1)}$, and hence G acts naturally on $\mathbb{A}^{\Delta(1)}$.

Proposition 3.3.6. Let $X = X_\Delta$ be a smooth toric variety, and let S, B and Z be as defined above. Then X is naturally isomorphic to the geometric quotient of $\mathbb{A}^{\Delta(1)} \backslash Z$ by the action of G. □

It is well known that any closed subscheme of $\mathbb{P}^n$ arises from a homogeneous ideal of $\mathbb{C}[x_0, \ldots, x_n]$. Furthermore, two ideals in $\mathbb{C}[x_0, \ldots, x_n]$ define the same closed subscheme of $\mathbb{P}^n$ if and only if their saturations with respect to the irrelevant ideal are equal. Recall that if B and I are ideals of a ring R then the saturation of I in R with respect to B is the ideal

$$(I : B^\infty) = \{f \in R : B^m f \subset I \text{ for some } m \geqslant 0\}.$$

Saturation with respect to the irrelevant ideal gives the largest ideal defining the corresponding closed subscheme of $\mathbb{P}^n$. We get a similar picture for ideals in the Cox ring of a smooth toric variety:

Proposition 3.3.7. Let X be a smooth toric variety as above. Any closed subscheme of X arises from a homogeneous (with respect to the $\mathrm{Cl}(X)$-grading of S) ideal in S. Furthermore, two homogeneous ideals, I and J of S, define the same closed subscheme if and only if $(I : B^\infty) = (J : B^\infty)$. Saturating an ideal with respect to B gives the largest ideal defining the corresponding subscheme. □

The correspondence in this proposition is a generalisation of the correspondence between homogeneous ideals in $\mathbb{C}[x_0, \ldots, x_m]$ and closed subschemes of $\mathbb{P}^m$. Indeed, to any graded S-module, F, one can define a quasi-coherent sheaf, $\tilde{F}$, on X. We have $\tilde{S} = \mathcal{O}_X$. If I is a $\mathrm{Cl}(X)$-homogeneous ideal in S, then $\tilde{I}$ is an ideal sheaf on X. Then I determines the closed subscheme, M of X, with $\mathcal{O}_M = \mathcal{O}_X/\tilde{I}$. Note that since I is $\mathrm{Cl}(X)$-homogeneous, the quotient S/I is also $\mathrm{Cl}(X)$-graded.

Let X be a normal toric variety with Cox ring S. Suppose that M is a normal, projective

scheme that is a closed subscheme of X (which may itself not be projective) defined by a $\mathrm{Cl}(X)$-homogeneous ideal I in S. Suppose we are given a very ample divisor class, κ, in $\mathrm{Cl}(M)$. Suppose that $\kappa' \in \mathrm{Cl}(X)$ maps to κ in the restriction homomorphism of divisor classes induced by the inclusion. This inclusion induces a map of vector spaces,

$$H^0(X, \mathcal{O}_X(\kappa')) \longrightarrow H^0(M, \mathcal{O}_M(\kappa)).$$

In fact, this works when replacing κ by $l\kappa$ for each $l \geqslant 0$. We thus get a map

$$\bigoplus_{l \geqslant 0} H^0(X, \mathcal{O}_X(l\kappa')) \longrightarrow \bigoplus_{l \geqslant 0} H^0(M, \mathcal{O}_M(l\kappa)). \tag{3.1}$$

Taking into account the usual maps

$$\mathcal{O}_X(l_1\kappa') \bigotimes \mathcal{O}_X(l_2\kappa') \to \mathcal{O}_X((l_1 + l_2)\kappa'), \quad \mathcal{O}_M(l_1\kappa) \bigotimes \mathcal{O}_M(l_2\kappa) \to \mathcal{O}_M((l_1 + l_2)\kappa),$$

we actually have a homomorphism of $\mathbb{Z}$-graded $\mathbb{C}$-algebras. The left hand side of (3.1) is the subalgebra, $S^{\kappa'} = \bigoplus_{l \geqslant 0} S_{l\kappa'}$, of S by Proposition 3.3.5. In general, this homomorphism may not be surjective. Since κ is very ample, we do know that the right hand side is a projective coordinate ring for the embedding of M into projective space corresponding to the complete linear system of κ.

Later on, we will be in a case where the class groups $\mathrm{Cl}(X)$ and $\mathrm{Cl}(M)$ are isomorphic and κ' will be globally generated. This will allow us to show that $R^\kappa = (S/I)^{\kappa'} = S^{\kappa'}/I^{\kappa'}$ is a projective coordinate ring for M. More explicitly we will have $\operatorname{Proj} R^\kappa \cong M$.

3.4 Tropical geometry applied to toric varieties

We end our chapter on the relevant theory of toric varieties with a method of completing closed subschemes of a torus. Suppose that M is a closed subscheme of a torus, $T_N \cong T^m$, where $m \geqslant 1$. Suppose that X_Σ is a complete normal toric variety containing T_N as its

dense torus. The Zariski closure, $\bar{M}$, of M in X_Σ is complete.

Tropical geometry gives us a way of finding which T_N-orbits of X_Σ intersect $\bar{M}$. In the situation later where X_Σ is too large to do computations with, this allows us to find a simpler toric subvariety of X_Σ that still contains $\bar{M}$ as a closed subscheme.

We first pass from $\mathbb{C}$ to an algebraically closed extension, L, that has a non-trivial valuation, $\text{val} : L^* \to \mathbb{R}$, that is trivial when restricted to $\mathbb{C}^*$. We also assume that the image of L^* under this valuation is dense in $\mathbb{R}$. We can take L to be the field of Puiseux series, $\mathbb{C}\{\{t\}\}$. The elements of this field are formal power series

$$f(t) = \sum_{i \geqslant 1} c_i t^{a_i},$$

with $c_i \in \mathbb{C}$ and $a_i \in \mathbb{Q}$ subject to the condition that there exists an $N \in \mathbb{Z}$ so that $Na_i \in \mathbb{Z}$ for all $i \geqslant 1$. If $f(t) \in L^*$, then its valuation is set to be the lowest exponent of t appearing in the series. This field extension has the properties we desire - this is proved in Theorem 2.1.5 in [MS15].

Since M is a closed subscheme of the m-dimensional torus T_N, it is defined by an ideal, J, in the coordinate ring $\mathbb{C}[t_1, t_1^{-1}, \ldots, t_m, t_m^{-1}]$. Consider the subvariety, $V({}_L J)$, of the m-dimensional torus over L defined by the ideal generated by J considered as a subset of $L[t_1, t_1^{-1}, \ldots, t_m, t_m^{-1}]$. The tropical variety of M, $\text{Trop}(M)$, is defined to be the closure in $\mathbb{R}^m$ of the set

$$\{(\text{val}(u_1), \ldots, \text{val}(u_m)) \in \mathbb{R}^m : (u_1, \ldots, u_m) \in V({}_L J)\}.$$

To see that this is equivalent to other definitions of $\text{Trop}(M)$ the reader may be familiar with, see [MS15, §3]. A key fact we will use is the following:

Proposition 3.4.1. Let M be a closed subscheme of an m-dimensional torus T_N (over

$\mathbb{C}$) and let $\mathbb{C} \subset L$ and $\text{val} : L^* \to \mathbb{R}$ be as above. Then there is a fan, Δ, in $\mathbb{R}^m$ so that the support of Δ is equal to $\text{Trop}(M)$.

Proof. Let $I \subset \mathbb{C}[T_N]$ be the ideal defined by M. Then $\text{Trop}(M)$ is the intersection of all $\text{Trop}(V(f))$ with $f \in I$ by [MS15, Theorem 3.2.3]. Each such $\text{Trop}(V(f))$ is the support of a polyhedral complex by [MS15, Proposition 3.1.6]. Since all of the equations defining M have their coefficients in $\mathbb{C}$, on which val is trivial, these polyhedral complexes are fans, and the result follows. □

Recall that M is a closed subscheme of T_N, which is in turn a dense, open subscheme of a toric variety X_Σ.

Proposition 3.4.2 [MS15, Theorem 6.3.4]**.** Let $M \subset T_N$ as above, and let $\bar{M}$ be the Zariski closure of M in X_Σ. For each cone σ in Σ, let O_σ denote the corresponding T_N-orbit in X_Σ. Then O_σ intersects $\bar{M}$ if and only if $\text{Trop}(M)$ intersects the relative interior of σ. □

This concludes our short section on tropical geometry. This method is a valuable source of finding useful completions of closed subschemes of tori.

4 Embedding $\overline{M}_{0,n}$ into a smooth toric variety

Fix $n \geqslant 5$. The purpose of this chapter is to apply the theory in Chapter 3 to realise $\overline{M}_{0,n}$ as a closed subscheme of a smooth, normal toric variety, X_Δ. We also give the equations in the Cox ring, S, of X_Δ that defines this embedding.

We start by showing that $M_{0,n}$ is a geometric quotient of an open subscheme of the Grassmannian parametrising lines in $\mathbb{P}^{n-1}$, denoted as usual by $G(2,n)$, by an action of the torus T^{n-1}. In §4.2, we define Hilbert quotients. The completion, $\overline{M}_{0,n}$, is then the Hilbert quotient of $G(2,n)$ by the aforementioned torus action.

At this point, we will see that $\overline{M}_{0,n}$ is a closed subscheme of a toric variety X_Σ. The toric variety X_Σ is too complex for manageable computations, so we use tropical geometry as in §3.4 to find a smooth, normal toric subvariety, X_Δ, of X_Σ still containing $\overline{M}_{0,n}$ as a closed subscheme.

We conclude this chapter by proving that $\mathrm{Cl}(X_\Delta) \cong \mathrm{Cl}(\overline{M}_{0,n})$, as well as giving the equations in S defining $\overline{M}_{0,n}$ in X_Δ.

The material in this chapter is a collection of examples and results in [GM11].

4.1 The moduli space $M_{0,n}$ as a geometric quotient

Consider the affine space, $\mathbb{A}^{\binom{n}{2}}$, with coordinates $\{x_{ij}\}_{1\leqslant i<j\leqslant n}$. Consider also the n-dimensional torus, $T^n = \operatorname{Spec}\mathbb{C}\left[t_i, t_i^{-1} : i \in [n]\right]$. The ring homomorphism

$$\mathbb{C}[x_{ij} : 1 \leqslant i < j \leqslant n] \longrightarrow \mathbb{C}\left[t_i, t_i^{-1} : i \in [n]\right] \bigotimes \mathbb{C}[x_{ij} : 1 \leqslant i < j \leqslant n],$$

sending x_{ij} to $t_i t_j \otimes x_{ij}$ for each pair $\{i,j\} \subset [n]$ defines an action of the algebraic group T^n on $\mathbb{A}^{\binom{n}{2}}$. Recall that $\mathbb{P}^{n-1}$ is a geometric quotient of $\mathbb{A}^n$ by T and $\mathbb{P}^{\binom{n}{2}-1}$ is the geometric quotient of $\mathbb{A}^{\binom{n}{2}}$ by T, where in both cases, T acts by scalar multiplication on

the coordinates. Note also that $T^n/T \subset \mathbb{P}^{n-1}$ is a dense, open, $(n-1)$-dimensional torus in $\mathbb{P}^{n-1}$. For each pair $\{i,j\} \subset [n]$, the polynomial $t_i t_j \otimes x_{ij}$ has homogeneous degree two in the variables $t_1, \ldots, t_n$ and homogeneous degree one in the variables $\{x_{ij}\}_{1 \leqslant i < j \leqslant n}$. Thus the above action defines an action of the torus T^n/T on $\mathbb{P}^{\binom{n}{2}-1}$. From now we identify T^n/T with T^{n-1}.

The Grassmannian, $G(2,n)$, is a fine moduli space parametrising 2-dimensional linear subspaces of a fixed copy of $\mathbb{A}^n$. A closed point in $G(2,n)$, corresponding to a linear subspace $V \subset \mathbb{A}^n$, can be specified by a rank two matrix,

$$\begin{pmatrix} x_1 & \ldots & x_n \\ y_1 & \ldots & y_n \end{pmatrix}, \tag{4.1}$$

whose rows form a basis of the vector space V. For each $\{i,j\} \subset [n]$, with $i < j$, the Plücker coordinate, x_{ij}, of V is the 2×2 minor of the matrix above obtained from the i^{th} and j^{th} columns. It is well known that this is well-defined and embeds $G(2,n)$ as a closed subscheme of $\mathbb{P}^{\binom{n}{2}-1}$ defined by the ideal

$$I = \langle x_{ij}x_{kl} - x_{ik}x_{jl} + x_{il}x_{jk} : 1 \leqslant i < j < k < l \leqslant n \rangle.$$

Note that I is invariant under the action of T^{n-1} defined above. This implies that the action of T^{n-1} on $\mathbb{P}^{\binom{n}{2}-1}$ is equivariant with respect to $G(2,n)$.

Proposition 4.1.1. Let U be the open subscheme of $G(2,n) \subset \mathbb{P}^{\binom{n}{2}-1}$ defined by those 2-dimensional linear subspaces of $\mathbb{A}^n$ whose Plücker coordinates are all nonzero. Then a geometric quotient of U by T^{n-1} exists and is isomorphic to the fine moduli space $M_{0,n}$.

Proof. The action of T^{n-1} on $G(2,n)$ restricts to an action on U. This action of T^{n-1} on U is free, and thus the geometric quotient U/T^{n-1} exists by Theorem 3.1.7.

Let V be a 2-dimensional linear subspace of $\mathbb{A}^n$ corresponding to a point in U. Then

it is given by a $2 \times n$ matrix as in (4.1). That V corresponds to a point in U means that all 2×2 minors of this matrix are non-zero, thus the points $[x_1 : y_1], \ldots, [x_n : y_n]$ are all distinct points of $\mathbb{P}^1$. Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$. Any other $2 \times n$ matrix whose row space is V has its first row equal to $a\mathbf{x} + b\mathbf{y}$ and its bottom row equal to $c\mathbf{x} + d\mathbf{y}$ for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$. The n distinct points, $[ax_1 + by_1 : cx_1 + dy_1], \ldots, [ax_n + by_n : cx_n + dy_n]$ of $\mathbb{P}^1$ obtained from this matrix are exactly the images of the original n points under the automorphism of $\mathbb{P}^1$ corresponding to the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PGL}_2(\mathbb{C})$. This implies that we have a well-defined morphism from U to $M_{0,n}$.

We need to argue that the preimage of a point of m in $M_{0,n}$ in $U \to M_{0,n}$ is the T^{n-1}-orbit of a point in U. For this, let m be a point in $M_{0,n}$ corresponding to n labelled points $[x_1 : y_1], \ldots, [x_n : y_n]$ on $\mathbb{P}^1$, and let V be the $2 \times n$ matrix whose rows are $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$. For each $t = [t_1 : \cdots : t_n] \in T^{n-1}$, the $2 \times n$ matrix,

$$\begin{pmatrix} t_1 x_1 & \ldots & t_n x_n \\ t_1 y_1 & \ldots & t_n y_n \end{pmatrix},$$

defines the same element of $M_{0,n}$. For each $\{i, j\} \subset [n]$ with $i < j$, the 2×2 minor of this matrix is $t_i t_j$ times the corresponding 2×2 minor of the original matrix corresponding to V. This implies that the morphism $U \to M_{0,n}$ sends the T^{n-1}-orbit of V to m. It follows from checking the definitions that $M_{0,n}$ is the geometric quotient of U by T^{n-1}. $\square$

To give a good quotient for all of $G(2, n)$ by T^{n-1}, we need a good completion of $U/T^{n-1} \equiv M_{0,n}$ from Proposition 4.1.1. For our purpose, we would like this completion to be isomorphic to the moduli space $\overline{M}_{0,n}$. We will see that this happens when we take the so-called Hilbert quotient of $G(2, n)$ by T^{n-1}.

4.2 Hilbert quotients and $\overline{M}_{0,n}$

Fix integers d and m satisfying $1 \leqslant d \leqslant m$. Identify T^d with the torus $T^{d+1}/T \subset \mathbb{P}^d$ and T^m with the torus T^{m+1}/T, as we did for T^{n-1} at the start of the previous section. Suppose that T^d acts on $\mathbb{P}^m$ so that the action restricted to T^m is just the natural action induced by a homomorphism of algebraic groups from T^d to T^m. Assume that this action is faithful, that is, the only point of T^d that fixes all of $\mathbb{P}^m$ is the identity.

Now let X be a closed subscheme of $\mathbb{P}^m$, and suppose that the action of T^d on $\mathbb{P}^m$ restricts to an action of T^d on X. Suppose further that U is an open subscheme of X on which T^d acts freely. Then the geometric quotient U/T^d exists by Theorem 3.1.7.

Note that in §4.1, we have the situation above with $d = n-1$, $m = \binom{n}{2} - 1$, $X = G(2,n)$ and $U \subset X$ defined by those points with non-vanishing Plücker coordinates or equivalently, $U = X \cap T^m$.

The closure of each T^d-orbit of T^m has the same Hilbert polynomial. We can therefore define a map from T^d-orbits of T^m to points in the same connected component of the Hilbert scheme, $\mathrm{Hilb}(\mathbb{P}^m)$.

Definition 4.2.1. The Hilbert quotient of $\mathbb{P}^m$ by T^d, denoted by $\mathbb{P}^m//^H T^d$, is the closure of the image of the map to $\mathrm{Hilb}(\mathbb{P}^m)$ defined above.

The Hilbert quotient $\mathbb{P}^m//^H T^d$ is a complete toric variety that is not normal in general [KSZ92]. Its normalisation is a normal toric variety, thus it is equal to X_Σ for some fan Σ in a lattice N by the material in Chapter 3.

Definition 4.2.2. Let $X \subset \mathbb{P}^m$ be as above. Since T^d acts on X and acts freely on T^m, it acts freely on $X \cap T^m$. The Hilbert quotient of X by T^d, denoted by $X//^H T^d$, is the closure of the image of $(X \cap T^m)/T^d$ in $\mathrm{Hilb}(\mathbb{P}^m)$.

Since X is a closed subscheme of $\mathbb{P}^m$, the Hilbert quotient $X//^H T^d$ is a closed subscheme

of $\mathbb{P}^m//^H T^d$.

In the setting of §4.1, the Hilbert quotient $G(2,n)//^H T^{n-1}$ is the closure of $M_{0,n}$ in the complete toric variety $\mathbb{P}^{\binom{n}{2}-1}//^H T^{n-1}$. This closure is isomorphic to the completion $\overline{M}_{0,n}$ by [Kap92, Theorem 4.1.8]. Let X_Σ be the normalisation of $\mathbb{P}^{\binom{n}{2}-1}//^H T^{n-1}$. Since $\overline{M}_{0,n}$ is smooth and irreducible, the inverse image of $G(2,n)//^H T^{n-1}$ in X_Σ is also isomorphic to $\overline{M}_{0,n}$.

Let A_n be the $n \times \binom{n}{2}$ matrix, with columns indexed by pairs in $[n]$, whose ij^{th} column has ones in the i^{th} and j^{th} rows and has zeroes everywhere else. Consider the toric ideal, $I_{A_n} \subset k[x_{ij} : 1 \leqslant i < j \leqslant n]$, generated by binomials,

$$\prod_{1 \leqslant i < j \leqslant n} x_{ij}^{u_{ij}} - \prod_{1 \leqslant i < j \leqslant n} x_{ij}^{v_{ij}},$$

where $u = (u_{ij})_{ij}$ and $v = (v_{ij})_{ij}$ satisfy $A_n \cdot u = A_n \cdot v$. Then I_{A_n} defines the toric variety that is the closure of the T^{n-1}-orbit of $[1 : \cdots : 1]$ (see [Stu96, §4], [GKZ94, §5]). Then Σ is the Gröbner fan of I_{A_n}, details of which are found in [MR88].

We have given $\overline{M}_{0,n}$ as a closed subscheme of a complete normal toric variety. The fan structure of Σ is too complex to be able to do useful computations. We apply the tropical geometry theory outlined in §3.4 to find a manageable subvariety of X_Σ that also contains $\overline{M}_{0,n}$ as a closed subscheme.

4.3 A simpler toric variety containing $\overline{M}_{0,n}$

We first define a fan, Δ, in the lattice $\mathbb{Z}^{\binom{n}{2}-n}$. Recall the set $\mathcal{I}$ given in Definition 2.4.3 and conventions for the size of elements of $\mathcal{I}$ immediately after. For each $I \in \mathcal{I}$, let $\gamma_I \in \mathcal{G}$ be the corresponding equivalence class of labelled dual graphs. Recall that these $\overline{M}_{0,n}^{\gamma_I}$ make up the boundary $\partial M_{0,n}$.

Define a simplicial complex, $\tilde{\Delta}$, with vertex set equal to $\mathcal{I}$, where a subset $\tilde{\sigma}$ of $\mathcal{I}$ is

a simplex in $\tilde{\Delta}$ if for each pair I, J in $\tilde{\Delta}$, the boundary pieces $\overline{M}_{0,n}^{\gamma_I}$ and $\overline{M}_{0,n}^{\gamma_J}$ have non-empty intersection. From the remarks at the end of §2.4, one can check that this occurs if and only if one of the following four conditions hold:

$$I \subset J, \quad J \subset I, \quad I \subset J^c, \quad J \subset I^c.$$

Now let $\mathcal{P} = \{ij : 2 \leqslant i < j \leqslant n, ij \neq 23\}$ be an indexing set for a basis of $\mathbb{R}^{\binom{n}{2}-n}$. We temporarily assume that each $I \in \mathcal{I}$ contains 1, since we can always identify I with I^c. At the start of Chapter 5 we remark that breaking symmetry in this fashion is not natural, and we won't make this assumption later in the document. For each $I \in \mathcal{I}$, define a vector $r_I = (R_{ij \in \mathcal{P}}) \in \mathbb{Z}^{\binom{n}{2}-n}$ as follows.

$R_{ij,I} = 1$	if either 2 or 3 is in I and $\{i, j\} \subset I^c$
$R_{ij,I} = -1$	if either i or j is in I and $\{2, 3\} \subset I^c$
$R_{ij,I} = 0$	otherwise

From this we can define an $\left(\binom{n}{2} - n\right) \times |\mathcal{I}|$ matrix, R, with columns indexed by $\mathcal{I}$, with I^{th} column equal to r_I.

Definition 4.3.1. For each $\tilde{\sigma}$ in $\tilde{\Delta}$, let σ be the cone in $\mathbb{R}^{\binom{n}{2}-n}$ that is generated by all of the r_I with $I \in \tilde{\sigma}$. Let Δ be the collection of all cones σ obtained in this way from the simplicial complex $\tilde{\Delta}$.

The collection of cones Δ defined above form a fan. In fact, this fan is homeomorphic to the space of phylogenetic trees as seen in [BHV01]. For the following result, recall that Σ is the fan associated with the normalisation of the Hilbert quotient $\mathbb{P}^{\binom{n}{2}-1} /\!/^H T^{n-1}$.

Proposition 4.3.2. The fan Δ is a smooth subfan of Σ. In particular, X_Δ is a smooth toric subvariety of X_Σ.

Proof. That Δ is a subfan of Σ is [GM11, Proposition 5.4]. Smoothness is proven in [GM11, Proposition 5.2]. The final statement of this proposition follows from Theorem 3.2.2 and Proposition 3.2.3. $\square$

In fact, [GM11, Proposition 5.2] goes further than just proving the smoothness of Δ. It shows that the support of the fan Δ is precisely the tropical variety $\mathrm{Trop}(M_{0,n})$, as in Proposition 3.4.1. Applying Proposition 3.4.2, X_Δ is precisely the union of those T^{n-1}-orbits of X_Σ that intersect $\overline{M}_{0,n}$. As such, $\overline{M}_{0,n} \subset X_\Sigma$ is actually contained in X_Δ as a closed subscheme.

Let S be the Cox ring of X_Δ. Note that $S \cong \mathbb{C}[x_I : I \in \mathcal{I}]$. Let D_I denote the T^{n-1}-invariant Weil divisor associated to the ray generated by r_I for each $I \in \mathcal{I}$. In §3.3 we saw that S is naturally graded by $\mathrm{Cl}(X_\Delta)$ with $\deg(x_I) = [D_I]$ for each $I \in \mathcal{I}$.

By Proposition 3.3.7, the closed subscheme $\overline{M}_{0,n}$ in X_Δ is defined by some ideal $I_{\overline{M}_{0,n}}$ of S that is homogeneous with respect to the $\mathrm{Cl}(X_\Delta)$-grading. It is [GM11, Theorem 3.2] that explains how to find the largest such ideal. Indeed, $X_\Delta \subset X_\Sigma$ satisfies the hypotheses of this theorem with respect to the subscheme $\overline{M}_{0,n}$. The calculations in our case are also done for us in [GM11, Theorem 6.3]:

Theorem 4.3.3. The largest homogeneous ideal of S defining $\overline{M}_{0,n}$ in X_Δ is given by

$$I_{\overline{M}_{0,n}} = \left(\left\langle \prod_{i,j \in I, k,l \in I^c} x_I - \prod_{i,k \in I, j,l \in I^c} x_I + \prod_{i,l \in I, j,k \in I^c} x_I \right\rangle : \left(\prod_I x_I \right)^\infty \right).$$

These equations are obtained by homogenising the Plücker relations with respect to the $\mathrm{Cl}(X_\Delta)$-grading of S, and saturating with respect to the product of all variables in S. $\square$

4.4 Divisors on X_Δ and $\overline{M}_{0,n}$

The final piece of the algebraic puzzle is to show that the groups $\mathrm{Cl}(X_\Delta)$ and $\mathrm{Cl}(\overline{M}_{0,n})$ are isomorphic. Note that both X_Δ and $\overline{M}_{0,n}$ are smooth, thus their respective Picard groups are equal to their Weil divisor class groups. A description of $\mathrm{Cl}(\overline{M}_{0,n})$ is known:

Proposition 4.4.1 [Kee92, Theorem 1]**.** Let $\{e_I\}_{I\in\mathcal{I}}$ be a basis for $\mathbb{Z}^{|\mathcal{I}|}$. For each quadruple $1 \leqslant i < j < k < l \leqslant n$, define a vector,

$$w_{ijkl} = \sum_{i,j\in I;k,l\notin\mathcal{I}} e_I - \sum_{i,j\notin I;k,l\in\mathcal{I}} e_I \in \mathbb{Z}^{|\mathcal{I}|}.$$

Let W be the sublattice of $\mathbb{Z}^{|\mathcal{I}|}$ generated by all of these w_{ijkl}. Then $\mathrm{Cl}(\overline{M}_{0,n})$ is isomorphic to $\mathbb{Z}^{|\mathcal{I}|}/W$. Furthermore, for each $I \in \mathcal{I}$, the image of the I^{th} standard basis vector of $\mathbb{Z}^{|\mathcal{I}|}$ in $\mathrm{Cl}(\overline{M}_{0,n})$ is the equivalence class of the boundary divisor $\overline{M}_{0,n}^{\gamma_I}$. □

Now $\mathbb{Z}^{|\mathcal{I}|}$ is equal to the middle term of the bottom row of the exact sequence in Proposition 3.3.4. The I^{th} component of the map $M \to \mathbb{Z}^{|\mathcal{I}|}$ in this sequence is given by $\langle -, r_I\rangle$, where $\langle -,-\rangle$ is the dual lattice product of M and N. Thus the map $M \to \mathbb{Z}^{|\mathcal{I}|}$ is defined by the $|\mathcal{I}|\times\left(\binom{n}{2} - n\right)$ matrix R^T. By Proposition 3.3.4, showing that $\mathrm{Cl}(\overline{M}_{0,n}) \cong \mathrm{Cl}(X_\Delta)$ is equivalent to showing that $\operatorname{im} R^T = W$.

We want to give the definition of the map $\mathbb{Z}^{|\mathcal{I}|} \to \mathrm{Cl}(X_\Delta)$ in this exact sequence. Since X_Δ is smooth, the fan Δ is smooth. This means that $\mathrm{Cl}(X_\Delta)$ has no torsion, so is free abelian of rank $|\mathcal{I}| - \left(\binom{n}{2} - n\right)$. As such, the map $\mathbb{Z}^{|\mathcal{I}|} \to \mathrm{Cl}(X_\Delta)$ is defined by an $\left(|\mathcal{I}| - \left(\binom{n}{2} - n\right)\right) \times |\mathcal{I}|$ matrix.

We define a matrix G with these dimensions. Let the first $\binom{n}{2}$ columns of G be indexed by those $I \in \mathcal{I}$ with $|I| = 2$, so that $I = ij$ for some pair $\{i, j\} \subset [n]$. The remaining $|\mathcal{I}| - \binom{n}{2}$ columns are indexed by those $I \in \mathcal{I}$ with $|I| \geqslant 3$. Let the first n rows of G be indexed by $[n]$, and let the bottom $|\mathcal{I}| - \binom{n}{2}$ rows of G be indexed by those $I \in \mathcal{I}$ with

$|I| \geqslant 3$. This indexing of the rows and columns of G allows us to define G in block form with four blocks.

The top left block of G is the $n \times \binom{n}{2}$ matrix A_n as defined in §4.2. Recall that if $ij \in \mathcal{I}$, then the ij^{th} column of A_n is $e_i + e_j$. Note that this is the matrix defining the action of T^{n-1} on $\mathbb{P}^{\binom{n}{2}-1}$.

The bottom left block of G is the $\left(|\mathcal{I}| - \binom{n}{2}\right) \times \binom{n}{2}$ matrix, C, where for a given $I \in \mathcal{I}$, with $|I| \geqslant 3$, and a given $ij \in \mathcal{I}$, the $(I, ij)^{\text{th}}$ entry of C is equal to -1 if $\{i, j\} \subset I$, and is equal to 0 otherwise.

The top right block of G is the $n \times \left(|\mathcal{I}| - \binom{n}{2}\right)$ zero matrix. The bottom right block of G is the $\left(|\mathcal{I}| - \binom{n}{2}\right) \times \left(|\mathcal{I}| - \binom{n}{2}\right)$ identity matrix.

Proposition 4.4.2. In the bottom exact sequence of Proposition 3.3.4 applied to X_Δ, the map $\mathbb{Z}^{|\mathcal{I}|} \to \mathrm{Cl}(X_\Delta) \cong \mathbb{Z}^{|\mathcal{I}|-\left(\binom{n}{2}-n\right)}$ is given by the matrix G. Furthermore, the kernel of G is precisely the lattice W defined in Proposition 4.4.1. In particular, $\mathrm{Cl}(X_\Delta) \cong \mathrm{Cl}(\overline{M}_{0,n})$.

Proof. The first two parts of this lemma are given by [GM11, Lemma 6.2]. The bottom exact row of Proposition 3.3.4 in our situation is the composition of R^T with G. By exactness we have $\ker G = \operatorname{im} R^T$, thus the final statement follows. □

Proposition 4.4.1 combined with Proposition 4.4.2 tells us that the isomorphism of $\mathrm{Cl}(X_\Delta)$ with $\mathrm{Cl}(\overline{M}_{0,n})$ sends the class of the T^{n-1}-invariant Weil divisor D_I to the class of the boundary divisor $\overline{M}_{0,n}^{\gamma_I}$.

5 An ample bundle on $\overline{M}_{0,n}$

In this chapter we explore a particular divisor class, κ, in $\mathrm{Cl}(\overline{M}_{0,n})$. This will be the sum of the canonical divisor plus the sum of all the boundary divisors. This is known in the literature as the log canonical divisor. It is often natural to study log canonical divisors since, given any scheme, this divisor class is an invariant. As commented in the introduction, we already know that κ is a very ample divisor class.

We will use the facts that $\overline{M}_{0,n}$ is a closed subscheme of X_Δ and that $\mathrm{Cl}(\overline{M}_{0,n}) \cong \mathrm{Cl}(X_\Delta)$ to describe a projective coordinate ring for the embedding of $\overline{M}_{0,n}$ into projective space associated with κ. This will continue from the discussion at the end of §3.3 with $X = X_\Delta$ and $M = \overline{M}_{0,n}$.

5.1 The definition of κ

The results of the previous chapter give an isomorphism of $\mathrm{Cl}(\overline{M}_{0,n})$ with a subgroup of $\mathbb{Z}^d$, where $d = |\mathcal{I}| - \left(\binom{n}{2} - n\right)$, sending the class of the boundary divisor $\overline{M}_{0,n}^{\gamma_I}$, where $I \in \mathcal{I}$, to the vector g_I. The matrix G, whose columns are the g_I, and the matrix R (defined in §4.3) defined the short exact sequence used to calculate $\mathrm{Cl}(X_\Delta)$ and hence also $\mathrm{Cl}(\overline{M}_{0,n})$ by isomorphism. The columns of R were indexed by $\mathcal{I}$ and were defined with the assumption that $1 \in I$ for each $I \in \mathcal{I}$. This assumption breaks a natural symmetry that we will exploit in our combinatorial method of Chapter 6. We thus return to the convention that each $I \equiv I^c$ in $\mathcal{I}$ is not assumed to contain 1. Recall that if $I \in \mathcal{I}$, then by saying $i, j \in \mathcal{I}$ we really mean that either $\{i, j\} \subset I$ or $\{i, j\} \subset I^c$.

For each $I \in \mathcal{I}$, we will define a new vector in $\mathbb{Z}^d$. We still give $\mathbb{Z}^d$ the standard basis

indexed by $[n] \sqcup \{I \in \mathcal{I} : |I| \geqslant 3\}$. If $I \in \mathcal{I}$ then put

$$v_I = e_i + e_j - \sum_{J \in \mathcal{I}; |J| \geqslant 3; i,j \in J} e_J \qquad \text{if } I \equiv ij$$

$$v_I = 2e_I \qquad \text{if } |I| \geqslant 3$$

Proposition 5.1.1. The abelian group $\mathrm{Cl}(\overline{M}_{0,n})$, and hence also $\mathrm{Cl}(X_\Delta)$, embeds as a subgroup of $\mathbb{Z}^d$ via the map sending the class of the boundary divisor $\overline{M}_{0,n}^{\gamma_I}$, where $I \in \mathcal{I}$, to v_I.

Proof. One can verify that the v_I satisfy the relations of Proposition 4.4.1. Alternatively, one can obtain the v_I by a change of basis from the g_I of §4.4. For more details, see [GM17, §3]. □

Using this proposition, we now identify $\mathrm{Cl}(\overline{M}_{0,n})$ and $\mathrm{Cl}(X_\Delta)$ with the subgroup of $\mathbb{Z}^d$ generated by the v_I. We need to compute κ in $\mathbb{Z}^d$ explicitly. The preimage, $\kappa' \in \mathrm{Cl}(X_\Delta)$, of κ will of course correspond to the same vector in $\mathbb{Z}^d$. For this, we require a formula for the log canonical divisor of $\overline{M}_{0,n}$.

Lemma 5.1.2. For each integer s with $2 \leqslant s \leqslant \frac{n}{2}$, define $\mathcal{D}_s = \sum_{I \in \mathcal{I}: |I| = s} v_I \in \mathbb{Z}^d$. Then κ is given in $\mathbb{Z}^d$ under the isomorphism of Proposition 5.1.1 by

$$\kappa = \sum_{2 \leqslant s \leqslant \frac{n}{2}} \left(\frac{s(n-s)}{n-1} - 1 \right) \mathcal{D}_s = \sum_{2 \leqslant s \leqslant \frac{n}{2}} \frac{s(n-s) - (n-1)}{n-1} \mathcal{D}_s.$$

Proof. The formula for the canonical divisor of $\overline{M}_{0,n}$ in terms of the boundary divisors is given in Proposition 1 in [Pan97]. The sum in the lemma is precisely this canonical divisor plus the sum of all boundary divisors and hence is the log canonical class. □

We can use this lemma to compute κ explicitly. Before we do this, we will explain the method we will use to make this computation manageable.

Recall that the first n components of elements of $\mathbb{Z}^d$ are indexed by the integers from 1 to n, which can be thought of as one element subsets of $[n]$. The remaining components of elements of $\mathbb{Z}^d$ are indexed by those $I \in \mathcal{I}$ with $|I| \geqslant 3$, which can be taken to be subsets of $[n]$ of size at least 3 and at most $\frac{n}{2}$. We can stratify these components by the size, r, of the indexing subset of $[n]$. Then either $r = 1$ or $3 \leqslant r \leqslant \frac{n}{2}$. In what follows, r will always obey these bounds.

For each r, we define a counting function $c_r : \mathbb{Z}^d \to \mathbb{Z}$. This takes a vector $v \in \mathbb{Z}^d$ to the sum of those coordinates of v indexed by subsets of $[n]$ of size r. For example, c_1 takes the sum of the first n components. Clearly, each c_r is linear, and we can exploit this to compute each $c_r(\kappa)$.

Once we have computed $c_r(\kappa)$ for each r, we can use the fact that all of the definitions we have used are symmetric with respect to permutations on the set $[n]$. This implies that each of the coordinates of κ indexed by subsets of $[n]$ of a fixed size r will be equal. Explicitly, the first n coordinates of κ will be equal to $\frac{c_1(\kappa)}{n}$. The coordinates of κ indexed by $I \in \mathcal{I}$ with $3 \leqslant |I| = r < \frac{n}{2}$ will be $\frac{c_r(\kappa)}{\binom{n}{r}}$. If n is even and $r = \frac{n}{2}$, then we divide $c_r(\kappa)$ by $\frac{1}{2}\binom{n}{r}$ instead (since complements of these subsets are of the same size and identified).

Let us compute $c_1(\kappa)$. By the definition of the v_I, we have $c_1(\mathcal{D}_s) = 0$ for each $s \geqslant 3$. For each $\{i, j\} \subset [n]$ we have $c_1(v_{ij}) = 2$. This implies that $c_1(\mathcal{D}_2) = 2\binom{n}{2} = n(n-1)$. By linearity and Lemma 5.1.2, we conclude that

$$c_1(\kappa) = \frac{2(n-2) - (n-1)}{n-1} n(n-1) = n^2 - 3n.$$

As discussed above, this means that each of the first n coordinates of κ are equal to $n - 3$.

Now let us compute $c_r(\kappa)$ for $3 \leqslant r < \frac{n}{2}$. We have $c_r(\mathcal{D}_s) = 0$ except when s is equal to either 2 or r. Clearly $c_r(\mathcal{D}_r) = 2\binom{n}{r}$. Computing $c_r(\mathcal{D}_2)$ is a case of computing $c_r(v_{ij})$

for an arbitrary pair $\{i,j\} \subset [n]$ and multiplying this number by $\binom{n}{2}$. We claim that if $\{i,j\} \subset [n]$, then

$$c_r(v_{ij}) = -\binom{n-2}{r-2} - \binom{n-2}{r}. \tag{5.1}$$

The first binomial coefficient here corresponds to those $I \in \mathcal{I}$ with $|I| = r$ and $\{i,j\} \subset I$, since the remaining $r-2$ elements of I are free to be picked. The second binomial coefficient is when $\{i,j\} \subset I^c$ and I is formed from any r elements of the remaining $n-2$ numbers. Using Lemma 5.1.2 we can now write

$$c_r(\kappa) = \frac{r(n-r)-(n-1)}{n-1} 2\binom{n}{r} + \frac{2(n-2)-(n-1)}{n-1}\binom{n}{2}\left(-\binom{n-2}{r-2} - \binom{n-2}{r}\right).$$

Dividing this expression by $\binom{n}{r}$ and simplifying, we find that the coordinates of κ indexed by those $I \in \mathcal{I}$ of size r are each equal to

$$-\frac{1}{2}n^2 + \frac{3}{2}n + rn - r^2 - 2. \tag{5.2}$$

If n is even and $r = \frac{n}{2}$, then $c_r(\mathcal{D}_r) = 2\frac{1}{2}\binom{n}{r} = \binom{n}{r}$. For $c_r(v_{ij})$, where $\{i,j\} \subset [n]$, we only have one of the binomial coefficients in (5.1) since we can assume that i is contained in I and not I^c. Note that for this r, both of these binomial coefficients are equal. This means that the expressions in n and r of $c_r(\mathcal{D}_2)$ and $c_r(\mathcal{D}_r)$ for $r = \frac{n}{2}$ are half of the same expressions for $3 \leqslant r < \frac{n}{2}$. We also divide $c_r(\kappa)$ by $\frac{1}{2}\binom{n}{r}$ rather than $\binom{n}{r}$ to compute the coordinates. The extra half factors cancel hence the expression (5.2) also holds for $r = \frac{n}{2}$.

We have established a formula for κ for each $n \geqslant 5$. For clarity we will state this as a proposition:

Proposition 5.1.3. For each $i \in [n]$, the coordinate of e_i in κ is given by $n-3$. If $I \in \mathcal{I}$, with $|I| = r \geqslant 3$, then the coordinate of e_I in κ is given by the expression (5.2). □

The expression (5.2) will not be as random as it first seems. To get an idea, if n is even and $r = \frac{n}{2}$ then this expression becomes $-(r-1)(r-2)$, which is -2 times the $(r-2)^{\text{th}}$ triangle number. When we translate this algebraic setting into graph theory, we will get a much nicer intuition for these coefficients in terms of triangle numbers.

Example 5.1.4. Let $n = 5$, then $\mathcal{I}$ is just the set of pairs in $[5]$. In this case $\kappa = (2,2,2,2,2)$. Going up one to $n = 6$, the set $\mathcal{I}$ consists of the 15 pairs $\{i,j\} \subset [n]$ and 10 pairs of complementary subsets of $[n]$ of size 3. The formula above implies that

$$\kappa = 3\sum_{i=1}^{6} e_i - 2\sum_{I\in\mathcal{I},|I|=3} e_I.$$

$\Diamond$

5.2 A homogeneous coordinate ring for $\overline{M}_{0,n}$

In Theorem 4.3.3, we gave the $\mathrm{Cl}(X_\Delta)$-homogeneous ideal $I_{\overline{M}_{0,n}}$, in the Cox ring S of X_Δ, defining $\overline{M}_{0,n}$ as a closed subscheme as in Proposition 3.3.7. With the added information that $\mathrm{Cl}(X_\Delta) \cong \mathrm{Cl}(\overline{M}_{0,n})$ from Proposition 4.4.2, we continue our discussion at the end of §3.3 in the case that $X = X_\Delta$ and $M = \overline{M}_{0,n}$. We will work towards proving that $\mathrm{Proj}(S/I_{\overline{M}_{0,n}})^{\kappa'} \cong \overline{M}_{0,n}$. The results and notation of this section are from [GM11, §4].

Definition 5.2.1. Recall that $\{D_I : I \in \mathcal{I}\}$ is the set of torus invariant divisors on X_Δ. Define $\mathcal{G}(X_\Delta)$ to be the cone in $\mathrm{Cl}(X_\Delta)\bigotimes\mathbb{R}$ generated by the globally generated divisor classes. Explicitly, we can write

$$\mathcal{G}(X_\Delta) = \bigcap_{\sigma\in\Delta}\left\{\sum_{I\notin\sigma} x_I[D_I] : x_I \geqslant 0\right\} \subset \mathrm{Cl}(X_\Delta)\bigotimes\mathbb{R}.$$

If we embed $\mathrm{Cl}(X_\Delta)$ into $\mathbb{Z}^d$ as in the previous section then we can understand $\mathcal{G}(X_\Delta)$

as a cone in $\mathbb{R}^d$. Indeed, for each maximal cone σ, we have a cone in $\mathbb{R}^d$ generated by those v_I for which the ray corresponding to I is not contained in σ. Then $\mathcal{G}(X_\Delta)$ is the intersection of these cones (as σ ranges over maximal cones in Δ).

By §4 in [GM11] if we can show that $\kappa' \in \text{Relint}\,(\mathcal{G}(X_\Delta))$, where $\text{Relint}(\sigma)$ denotes the relative interior of a cone σ, then we can conclude that $\text{Proj}(S/I)^{\kappa'} \cong \overline{M}_{0,n}$. To show that κ' is in $\mathcal{G}(X_\Delta)$ we need to show that it is in the positive span of the v_I indexed by those I whose rays are not contained in σ for each maximal cone σ. We show this by hand for $n = 5$ below. For general n the question of whether κ' lies in $\mathcal{G}(X_\Delta)$ can be reformulated as a simple combinatorics question after the correspondence of Chapter 6.

For $n = 5$, any subset of $\mathcal{I}$ whose rays generate a maximal cone in Δ is of the form $\{ij, kp\} \subset \mathcal{I}$, where $i, j, k, p \in [5]$ are distinct. If we let a be the remaining element of $[5]$ then the equality

$$\kappa' = (2, 2, 2, 2, 2) = v_{ik} + v_{kj} + v_{jp} + v_{pa} + v_{ai}$$

implies that $\kappa' \in \text{Relint}\,(\mathcal{G}(X_\Delta))$.

Conjecture 5.2.2. Let $n \geqslant 5$. Then κ' is contained in $\text{Relint}\,(\mathcal{G}(X_\Delta))$. In particular, if we write R^κ for $(S/I)^{\kappa'}$, then $\text{Proj}\, R^\kappa = \overline{M}_{0,n}$.

For the rest of this thesis we will assume that this conjecture is true, even though we will only have proven it for $n = 5$ and $n = 6$. We have then obtained a manageable homogeneous coordinate ring for $\overline{M}_{0,n}$. This coordinate ring is generated by the images of degree l monomials in $S^{\kappa'}$. In the next chapter we will find a combinatorial description of all of these monomials.

6 Translation to the language of graph theory

We now make little reference to the geometry involved and focus purely on the algebra. As such, we will identify $\mathrm{Cl}(X_\Delta)$ and $\mathrm{Cl}(\overline{M}_{0,n})$ with $\mathbb{Z}^d$, where $d = |\mathcal{I}| - \left(\binom{n}{2} - n\right)$, via the isomorphisms in Proposition 5.1.1. We index the first n components of $\mathbb{Z}^d$ by $[n]$ and the rest by $\{I \in \mathcal{I} : |I| \geqslant 3\}$.

Then S is the $\mathbb{Z}^d$-graded polynomial ring $k[x_I : I \in \mathcal{I}]$. The grading is given by $\deg(x_I) = v_I$, where v_I is defined by:

$$v_I = \begin{cases} e_i + e_j - \sum_{i,j \in J} e_J & \text{if } I = ij \\ 2e_I & \text{if } |I| \geqslant 3 \end{cases}$$

As usual, in the sum defining v_I when $I = ij$, by $i, j \in I$ we really mean that either $\{i, j\} \subset I$ or $\{i, j\} \subset I^c$.

We have the $\mathbb{Z}^d$-homogeneous ideal, $I_{\overline{M}_{0,n}}$, in S, found in Theorem 4.3.3. Namely,

$$I_{\overline{M}_{0,n}} = \left(\left\langle \prod_{i,j \in I, k,l \in I^c} x_I - \prod_{i,k \in I, j,l \in I^c} x_I + \prod_{i,l \in I, j,k \in I^c} x_I \right\rangle : \left(\prod_I x_I \right)^\infty \right).$$

Then R is the $\mathbb{Z}^d$-graded ring $S/I_{\overline{M}_{0,n}}$. Let κ be the element of $\mathbb{Z}^d$ computed in §5.1 (identifying κ and κ' in $\mathbb{Z}^d$).

The vector κ gives us two section rings which are sub-algebras of S and R respectively. They are each graded by $\mathbb{Z}$. Namely,

$$S^\kappa = \bigoplus_{l \geqslant 0} S_{l\kappa}, \quad R^\kappa = \bigoplus_{l \geqslant 0} R_{l\kappa}.$$

The main result of this thesis will be showing that R^κ is generated in degree one.

This chapter will give a correspondence between monomials in S and graphs on n vertices.

For the rest of this document, a graph on n vertices will mean a labelled multigraph with no loops.

Definition 6.0.1. Let G and G' be two graphs on n vertices. Their graph product, $G \cdot G'$, is also a graph on n vertices. The number of edges between a pair of vertices, $i, j \in [n]$, in $G \cdot G'$ is the sum of the number of edges between i and j in G and the number of edges between i and j in G'.

This notion of product will allow us to view S^κ as a 'graph algebra', i.e. a k-algebra generated by graphs with formal addition and multiplication defined from the graph product. The quotient ring R^κ will be a graph algebra too, generated by the same graphs. Of course, there will be extra relations between graphs in R^κ corresponding to equations in $I_{\overline{M}_{0,n}}$. We will see that many of these relations, which involve many variables in S, can be written in a more simple way in the graph setting.

The inspiration for this idea is from [HMSV09], where a simpler ring in the case $n = 6$ is considered. We will outline this example in §6.1 before defining our own correspondence in the $n = 6$ case.

In §6.2 we generalise the graph correspondence to the case of all $n \geqslant 5$.

6.1 Section rings as 'graph algebras' : n = 6

Details of the following example can be found in [HMSV09]. It will be easier to define our own correspondence after seeing a similar, though more simple, example.

Let $S' = k[x_{ij} : 1 \leqslant i < j \leqslant 6]$. This ring is $\mathbb{Z}^6$-graded with $\deg x_{ij} = e_i + e_j$ for each pair $\{i, j\} \subset [6]$. We have an ideal, I_M, in S that is generated by the Plücker relations, that is the relations

$$x_{ij}x_{kp} - x_{ik}x_{jp} + x_{ip}x_{jk}$$

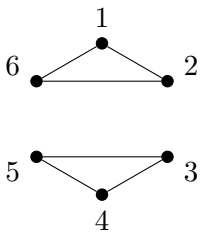
for each $1 \leqslant i < j < k < p \leqslant 6$. Let $\lambda = (1, 1, 1, 1, 1, 1)$, and let S'^λ be the subring

$S'^{\lambda} = \bigoplus_{l \geqslant 0} S'_{l\lambda}$ of S'.

To each monomial, X, in S' we associate a graph on 6 vertices. The number of edges between a pair of vertices i and j in the graph corresponding to X is set to be the highest power of x_{ij} dividing X. It is clear by construction that this correspondence is injective, and multiplication of monomials in S' is compatible with the graph product of the corresponding graphs. As such, S', and its sub-algebra S'^{λ}, can be written as graph algebras. If $v \in \mathbb{Z}^6$, then we can talk about degree v monomials or degree v graphs in S'.

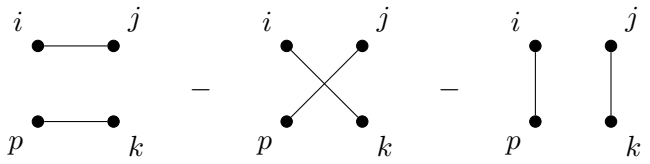
Let $l \geqslant 0$. Degree $l\lambda$ graphs are precisely l-regular graphs. Recall that a graph is said to be l-regular if each of its vertices is adjacent to l edges. A k-basis of the graded piece $S'_{l\lambda}$ is given by the collection of all l-regular graphs.

Monomials of degree λ correspond to 1-regular graphs on 6 vertices, i.e. perfect pairings of the six vertices. In general, there are l-regular graphs, with $l > 1$, that cannot be written as the product of 1-regular graphs. The simplest example, with $l = 2$, is drawn below.



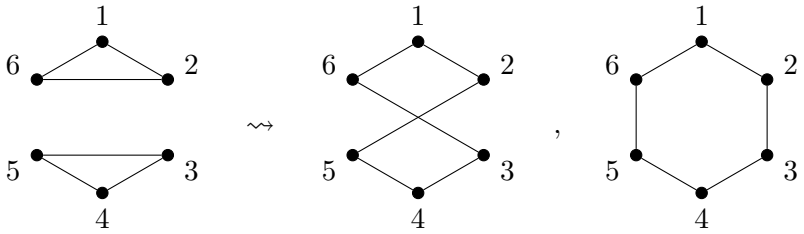
This shows that S'^{λ} is not generated as a k-algebra by monomials of degree λ. That is, S'^{λ} with its natural $\mathbb{Z}$-grading is not generated in degree one. Consider the quotient, $R' = S'/I_M$. Since the generators of I_M are homogeneous with respect to the $\mathbb{Z}^6$-grading of S, we can write R', and thus also R'^{λ}, as a graph algebra. The graph algebra R'^{λ} is also generated by all l-regular graphs for all $l \geqslant 0$. There are extra relations in this algebra though, corresponding to the Plücker relations generating I_M. Explicitly, the

Plücker equation $x_{ij}x_{kp} - x_{ik}x_{jp} + x_{ip}x_{jk} = 0$, where $1 \leqslant i < j < k < p \leqslant 6$, holding in R' corresponds to the graph algebra equation



(6.1)

One can use these graph equations to write a given l-regular graph as a signed sum of two new l-regular graphs. We call this applying a Plücker operation to the l-regular graph. We don't care about the signs of the newly obtained graphs, just the graphs themselves. Take the example of a 2-regular graph above that cannot be written as the product of two 1-regular graphs. Applying a Plücker operation between the edges 26 and 35 replaces this graph with two new ones as drawn below:



Both of these newly obtained 2-regular graphs can be written as the (graph) product of two 1-regular graphs. This means that the degree 2λ monomial corresponding to the graph on the left hand side can be written as a linear combination of products of monomials of degree λ in R'.

In fact, showing that R'^{λ} is generated in degree one becomes a combinatorics problem - indeed, it is equivalent to show that given any l-regular graph, one can apply a sequence

of Plücker operations to obtain a collection of graphs that are each a product of 1-regular graphs. This is proven to be possible in [HMSV09, Theorem 2.3], using Hall's Marriage Theorem in the proof.

Let's return to our own case with $S, I_{\overline{M}_{0,n}}, R$ and κ. For $n = 5$, the set $\mathcal{I}$ consists of pairs in [5] and the graph situation is very similar to the example above. For $n = 6$, the extra complexities we will face start to show, hence we will do this correspondence first to get a taste for the general situation.

Fix $n = 6$. By Example 5.1.4, we have

$$\kappa = 3\sum_{i=1}^{6} e_i - 2\sum_{I\in\mathcal{I},|I|\geqslant 3} e_I.$$

Each monomial, X, in S can be written as a product, X_pX_q, where X_p is a product of variables indexed by pairs and X_q is a product of variables indexed by those I in $\mathcal{I}$ with $|I| = 3$. Associate $X = X_pX_q$ with the graph on 6 vertices where the number of edges between a pair of vertices, $i, j \in [6]$, is equal to the largest power of x_{ij} dividing X_p. This correspondence is not injective on S since it depends only on X_p. We claim that for a monomial $X = X_pX_q$ of degree $l\kappa$, for some $l \geqslant 0$, the product X_q is uniquely determined by X_p. This will imply that the graph correspondence is injective on monomials in S^κ.

This claim is quite straightforward. Let $l \geqslant 0$ and let $X = X_pX_q$ be a monomial of degree $l\kappa$. Note that, by the definition of the v_I given in the introductory segment of this chapter, the only variables of S contributing to the first 6 components of the degree are those indexed by pairs. That $\deg(X) = l\kappa$ must then imply that the first 6 coordinates of $\deg(X_p)$ are all equal to $3l$. Now let $I \in \mathcal{I}$ with $|I| = 3$. Let a_I be the coefficient of e_I in $\deg(X_p)$. The largest power of x_I dividing X_q must be $-l - \frac{a_I}{2}$, and thus X_q is uniquely determined.

We need to characterise all graphs arising from a degree $l\kappa$ monomial for each $l \geqslant 0$.

Suppose that $l \geqslant 0$ and let G be a graph obtained from a degree $l\kappa$ monomial, $X = X_p X_q$. For each $I \in \mathcal{I}$ with $|I| = 3$, let a_I be the coefficient of e_I in $\deg(X_p)$. For a suitable X_q to exist, the previous paragraph implies, for each $I \in \mathcal{I}$ with $|I| = 3$, that a_I must be even and satisfy $-a_I \geqslant 2l$. We claim that $-a_I$ is equal to twice the number of edges in the subgraph of G induced by those vertices in I.

To verify this claim, it is an easy to prove fact of regular graphs on 6 vertices that if $I \in \mathcal{I}$ has $|I| = 3$, then the number of edges in the subgraph of G induced by those vertices in I is equal to the number of edges in the subgraph of G induced by vertices of I^c. Each edge in the subgraphs induced by I or I^c corresponds to a variable x_{ij} which contributes -1 to the I^{th} component of the degree. The claim follows immediately and this argument already tells us that a_I is even. The numerical condition that $-a_I \geqslant 2l$ translates to the condition that any subgraph of G induced by three vertices contains at least l edges. We will say that any triangle in G contains at least l edges.

In practice, to check that each triangle in a $3l$-regular graph on 6 vertices containes at least l edges, we only need to check those triangles containing the vertex 1 (or indeed any chosen vertex). This is because in such a graph the number of edges in a triangle is equal to the number of edges in its complimentary triangle.

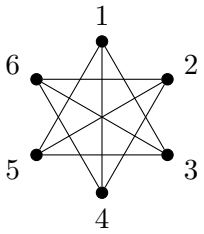
We have now proved the following:

Proposition 6.1.1. Let $l \geqslant 0$. Then there is a one-to-one correspondence between monomials in S of degree $l\kappa$, and $3l$-regular graphs on 6 vertices containing at least l edges in each triangle. Furthermore, this correspondence is compatible with products, and allows us to write S^κ as a graph algebra generated by these graphs. □

If G is a graph fitting the description in Proposition 6.1.1 then we say that G has degree l. This is consistent with the natural $\mathbb{Z}$-gradings of S^κ and R^κ.

We can use this correspondence to prove Conjecture 5.2.2 for $n = 6$. The subsets of

$\mathcal{I}$ whose corresponding rays in Δ generate a maximal cone in Δ, up to symmetry, are $J_1 = \{12, 34, 56\}$ and $J_2 = \{12, 34, 126\}$. We thus need to show that κ lies in the cone, τ_i in R^d, generated by $\{v_I : I \notin J_t\}$ for $t = 1$ and $t = 2$. This is equivalent to finding a degree κ monomial in S that divides by no variable indexed by an $I \in J_t$ for $t = 1$ and $t = 2$. For J_1 this corresponds to a a degree one graph where the pairs $12, 34, 56$ are empty. For J_2 this corresponds to a degree one graph where the pairs $12, 34$ are empty and the triangle 126 contains exactly one edge. The degree one graph below is an example for both of these conditions. The corresponding linear combinations of v_I lie in the relative interiors of τ_1 and τ_2 respectively and this implies that κ is in the relative interior of $\mathcal{G}(X_\Delta)$ as required. In particular we can write Proj $R^\kappa \cong \overline{M}_{0,6}$ with certainty.



As in the case with S' and λ, the ring S^κ is not generated in degree one. Passing to the quotient R^κ, the generating equations of $I_{\overline{M}_{0,6}}$ give relations on the graph algebra. The corresponding operation forming two new graphs from an original graph will be identical to the operation defined by the graph equation (6.1). There is an added degree of complexity in when we can apply such an operation, which we shall explain now.

Remember that each degree l graph in R^κ must have at least l edges in each triangle. Consider the operation defined by equation (6.1). Compare the number of edges in each triangle in the first term of (6.1) compared with the other two. The only triangles that contain fewer edges in the final two terms are ija, ijb, kpa and kpb, where a and b are the two vertices not involved in the operation. In order to check that the Plücker operation is valid, we only need to check that there are more than l edges in the triangles ija and

ijb (then regularity covers the complements kpa and kpb).

That R^κ is generated in degree one, and thus that our embedding of $\overline{M}_{0,6}$ obtained from the complete linear system for the divisor class κ is projectively normal, is now implied by the following theorem:

Theorem 6.1.2. Let $l \geqslant 0$, and let G be a degree l graph. Using valid Plücker operations, one can write G as a linear combination of degree l graphs that are each a product of degree 1 graphs.

Technically, Theorem 6.1.2 is stronger than projective normality of the embedding. This is because we haven't used any extra generators of $I_{\overline{M}_{0,n}}$ that may arise from the saturation. We will prove this theorem as one case of a conjecture we will give in Chapter 7.

Having now treat the first case where $\mathcal{I}$ contains subsets other than just pairs, we are ready to treat the case of general $n \geqslant 5$.

6.2 Section rings as graph algebras : $n \geqslant 5$

In this section we fix n, with $n \geqslant 5$, and generalise Proposition 6.1.1 and Theorem 6.1.2. The set of variables of the polynomial ring S is the disjoint union of those indexed by pairs and those indexed by I in $\mathcal{I}$ with $|I| \geqslant 3$. To each monomial in S we will associate a graph on n vertices. This won't give an injective correspondence on the monomials in S for $n \geqslant 6$, but it will when restricted to monomials in S^κ.

Notation. Let $l \geqslant 0$. If X is a monomial of S with degree $l\kappa$, then we call X a degree l monomial, consistent with the natural $\mathbb{Z}$-grading of S^κ. Let G be the graph corresponding to X. We call G a degree l graph. If G is any graph on n vertices, then if i, j is a pair of vertices, the multiplicity of ij is the number of edges between i and j in G. We may say that ij is empty in G if there are no edges between them.

Let X be a monomial of degree l in S^κ. Write $X = X_pX_q$, where X_p is a product of those variables indexed by pairs and X_q is a product of those variables indexed by those I in $\mathcal{I}$ with $|I| \geqslant 3$. We associate X with the graph, G, on n vertices where for each pair, $\{i, j\} \subset [n]$, the multiplicity of ij in G is equal to the highest power of x_{ij} dividing X_p (this justifies use of the word multiplicity in this context). To see that this correspondence is injective on monomials in S^κ, we recall that for each I in $\mathcal{I}$ with $|I| \geqslant 3$, the degree of x_I has a two in the I^{th} component and zeroes elsewhere. As such, these are all linearly independent. Thus, if $X = X_pX_q$ has degree l in S^κ, then X_q is uniquely determined by X_p. This means that the correspondence from degree l monomials to degree l graphs is injective.

We need to describe all degree l graphs for each $l \geqslant 0$. Let $X = X_pX_q$ be a degree l monomial in S^κ and let G be its corresponding graph. Recall the calculation of κ from §5.1. That the first n coordinates of κ are each equal to $n - 3$ implies that G must be $(n-3)l$-regular. This is because the variables in the product X_q do not contribute to the first n components of the degree. Let $I \in \mathcal{I}$ have $|I| \geqslant 3$. For a pair, $\{i, j\} \subset [n]$, the variable x_{ij} in S contributes -1 to the I^{th} component of the degree if either $\{i, j\} \subset I$ or $\{i, j\} \subset I^c$, otherwise it contributes 0 to this component. Thus the I^{th} coordinate of $\deg(X_p)$ is minus the number of edges in G lying completely inside the vertices in I or completely inside the vertices in I^c.

Notation. If $I \in \mathcal{I}$ has $|I| = r \geqslant 3$, then we call the subgraph of a graph, G, induced by the vertices in I the r-gon in G corresponding to I. We will often call this r-gon I as well. We will often say triangle instead of 3-gon, quadrilateral instead of 4-gon and so on. If I is an r-gon then I^c is its complementary $(n - r)$-gon.

For each I in $\mathcal{I}$ with $|I| \geqslant 3$, let a_I be the I^{th} coordinate of $\deg(X_p)$. The highest power of x_I that divides X_q must be $\frac{1}{2}(lb_I - a_I)$, where b_I is the I^{th} coordinate of κ, as given by (5.2). In order for a suitable X_q to exist, this number needs to be a non-negative

integer for each such I.

Lemma 6.2.1. Let $l \geqslant 0$ and let $X = X_p X_q$ be a degree l monomial in S^κ. Let G be the corresponding graph on n vertices. Let $I \in \mathcal{I}$ have $|I| = r \geqslant 3$ and let a_I and b_I be the I^{th} coordinates of $\deg(X_p)$ and κ respectively. The condition that $\frac{1}{2}(lb_I - a_I)$ is a non-negative integer is equivalent to the condition that the r-gon in G associated to I contains at least $l\binom{r-1}{2}$ edges.

Proof. The key thing here is that G is $l(n-3)$-regular. Let $f = l(n-3)$. If there are $t \geqslant 0$ edges in the r-gon I, then this accounts for $2t$ in the sum, rf, of the degrees of the vertices in I. The remaining $(fr - 2t)$ edges adjacent to at least one vertex in I pass from I to I^c. The remaining

$$\frac{1}{2}nf - t - (fr - 2t) = \left(\frac{1}{2}n - r\right) f + t$$

edges of G must lie completely in I^c. This means that there are $(\frac{1}{2}n - r)f + 2t$ edges lying completely in I or completely in I^c, which is exactly the number $-a_I$. Case analysis with the possible values of r and n modulo 2 shows that lb_I and a_I are the same modulo 2. The condition that $-a_I \geqslant -lb_I$ becomes

$$\left(\frac{1}{2}n - r\right) f + 2t \geqslant l\left(\frac{1}{2}n^2 - \frac{3}{2}n - rn + r^2 + 2\right).$$

This inequality simplifies to $t \geqslant l\binom{r-1}{2}$. Since t is just the number of edges in the r-gon I, this proves the lemma. □

We can now characterise all the degree l graphs for each $l \geqslant 0$. Note that this fits with the description given by Proposition 6.1.1, since $\binom{3-1}{2} = 1$.

Proposition 6.2.2. Let $l \geqslant 0$. Then the degree l graphs are precisely those $l(n-3)$-regular graphs with at least $l\binom{r-1}{2}$ edges contained in each r-gon for each r with $3 \leqslant$

$r \leqslant \frac{n}{2}$. Furthermore, this correspondence is compatible with products, and allows us to write S^κ as a graph algebra, generated by these graphs.

Proof. This follows from Lemma 6.2.1 and the paragraph preceding it. □

Like for $n = 6$, we would like to use this correspondence to prove Conjecture 5.2.2. To this end, fix $n \geqslant 7$ and suppose $J \subset \mathcal{I}$ corresponds to a maximal cone in Δ. Showing that κ is contained in the cone, τ, in $\mathbb{R}^d$ generated by those v_I with $I \notin J$ is equivalent to finding a degree one graph G on n vertices satisfying the following two conditions:

- If ij is a pair in J, then ij is empty in G.
- If $I \in J$ with $|I| = r \geqslant 3$, then the r-gon in G corresponding to I contains exactly $\binom{r-1}{2}$ edges.

To prove Conjecture 5.2.2 in generality, one must check that it is possible to find such a graph for each $J \subset \mathcal{I}$ corresponding to a maximal cone in Δ. If the corresponding positive linear combinations of the v_I lie in the relative interiors of each τ then we would be done. I suspect that this method could easily prove Conjecture 5.2.2 for other small n. We will instead continue under the assumption that Conjecture 5.2.2 is true, allowing us to write $\operatorname{Proj} R^\kappa \cong \overline{M}_{0,n}$.

The ring S^κ is not generated in degree one in general. The quotient R^κ can also be represented as a graph algebra, and is similarly generated by the graphs characterised in the above proposition. It has extra relations given by the equations in $I_{\overline{M}_{0,n}}$. These include the Plücker relations. Let r satisfy $3 \leqslant r \leqslant \frac{n}{2}$. From (6.1), the only r-gons containing less edges in either of the two new graphs obtained by a Plücker operation between two pairs of vertices, ij and kp, are those r-gons containing ij and neither k or p, or those r-gons containing kp and neither i or j. For this Plücker operation to be valid in R^κ, we therefore require that these r-gons contain more than $l\binom{r-1}{2}$ edges.

Let r satisfy $3 \leqslant r \leqslant \frac{n}{2}$ and let I be an r-gon in a degree l graph G. The proof of

Lemma 6.2.1 implies that I contains $l\binom{r-1}{2}+t$ edges if and only if the $(n-r)$-gon I^c contains $l\binom{n-r-1}{2}+t$ edges. When checking if we can apply a Plücker operation between pairs of vertices, ij and kp, we can use this fact to only need to check those r-gons, with $3 \leqslant r \leqslant n-3$, containing ij and not containing k or p. This is useful in our inductive method introduced in Chapter 7.

Our algebraic problem on the projective normality of the embedding of $\overline{M}_{0,n}$ into projective space associated with κ is now translated into a graph theory formulation. Again, the following theorem is technically stronger as it hasn't accounted for extra generators that may arise from the saturation in $I_{\overline{M}_{0,n}}$. Note that the $n=6$ case is just Theorem 6.1.2.

Conjecture 6.2.3. Let $l \geqslant 0$ and let G be a degree l graph on n vertices. Then, using Plücker operations, one can write G as a linear combination of degree l graphs that are each a product of degree 1 graphs.

7 Projective normality of the embeddings of $\bar{M}_{0,5}$ and $\bar{M}_{0,6}$

If Chapter 5 converted our geometric problem to an algebraic one, then Chapter 6 took the algebraic problem and formulated it as a combinatorics problem, namely Conjecture 6.2.3. In this chapter, we prove Conjecture 6.2.3 for $n = 5$ and $n = 6$ (thus proving Theorem 6.1.2).

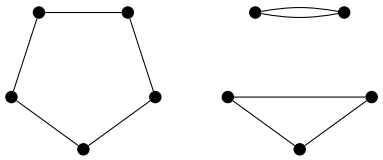
For $n = 6$ we will give a set of degree one generators of R^κ that is much smaller than the set of degree one monomials. It may be helpful in some cases to consider the co-graphs of these generators. Recall that the co-graph of a graph (with no multiple edges allowed), G, is the graph, $\bar{G}$, whose vertex set is the same as G, and a pair of vertices is adjacent in $\bar{G}$ if and only if they are not adjacent in G. Of course, for graphs with multiple edges allowed, it is unclear what a co-graph is. We will use Plücker relations to restrict to a set of generators (not necessarily in degree one) of R^κ on which we can define a working generalisation of co-graphs. This will make it much easier to describe equations defining $\overline{M}_{0,6}$.

7.1 When $n = 5$

In this section, we prove Conjecture 6.2.3 for $n = 5$. This is quite simple, and we structure our proof in the way we wish to continue with larger n. The proof of Conjecture 6.2.3 for $n = 6$ will partly use induction, having obtained the result for $n = 5$.

If $n = 5$ then $\mathcal{I}$ is just the set of pairs contained in [5]. Since $\kappa = (2, 2, 2, 2, 2)$, degree l graphs are just $2l$-regular graphs for each $l \geqslant 0$. It is easy to show that all degree one

graphs are labellings of one of the following two unlabelled graphs:



Degree one generation of R^κ amounts to saying that monomials corresponding to the labellings of the above two unlabelled graphs generate R^κ. We want to make this simpler and restrict to just the labellings of the graph on the left. To do this, we show that using Plücker operations, we can restrict to considering graphs that have no pairs with multiplicity greater than their degree.

Note that for $n = 5$, we can apply a Plücker operation between any two disjoint, non-empty pairs ij and kp since there are no conditions on the number of edges contained in any r-gon.

Lemma 7.1.1. Let G be a degree l graph for some $l \geqslant 0$. One can apply a sequence of Plücker operations to write G as a linear combination of degree l graphs, each of which contain no pair with multiplicity greater than l.

Proof. If G has no edges of multiplicity greater than l we are already done so assume, without loss of generality, that there are $l + t$ edges between 1 and 2, where $t > 0$. Assume further that $l + t$ is the maximal edge multiplicity in G.

Choose a non-empty pair in the triangle 345 in G (a $2l$-regularity argument implies this exists). Without loss of generality, this is the pair 34. Applying a Plücker operation between 12 and 34 produces two new graphs where 12 has multiplicity $l + t - 1$. There are $l + t > l$ edges between 1 and 2 in G. Since G is $2l$-regular, the multiplicities of $13, 14, 23$ and 24 in G are all less than l. Thus, applying this Plücker operation removes an edge with maximal multiplicity without creating a new one in either of the two new

graphs. Doing this for all maximal pairs, we can reduce the maximal edge multiplicity to $l + t - 1$.

Proceeding inductively, we can keep applying Plücker operations to obtain a collection of graphs, each of whose maximal edge multiplicity is at most l. This proves the lemma. □

The algebraic version of this lemma is the following:

Lemma 7.1.2. The section ring R^κ is generated as a k-algebra by those degree l monomials, X, where l ranges over all $l \geqslant 1$, so that x_{ij}^{l+1} does not divide X for any pair $\{i, j\} \subset [5]$. □

The consequence of Lemma 7.1.1 is that if R^κ is generated in degree one then it is generated by those graphs only containing single edges, i.e. labelled 5-cycles. We have reduced Conjecture 6.2.3 in the case that $n = 5$ to the following lemma.

Lemma 7.1.3. Let $l \geqslant 1$ and let G be a degree l graph with maximal edge multiplicity at most l. Then G has a subgraph that is a 5-cycle. In particular, G can be written as the product of a degree one graph and a degree $l - 1$ graph.

Before proving this lemma, we comment on how it implies Conjecture 6.2.3 in the case $n = 5$. If G is any degree l graph, then Lemma 7.1.1 allows us to write this as a linear combination of degree l graphs that have no pairs with multiplicity greater than l. Using the preceding lemma, each of these new degree l graphs can be written as the product of a degree one graph and a degree $l - 1$ graph. One can then work inductively in l.

Proof of Lemma 7.1.3. Fix $l \geqslant 1$ and let G be a degree l graph with maximal pair multiplicity at most l. Firstly, we need to argue that G has a subgraph that is a cycle. If G does not contain a cycle then let T be the disjoint union of trees on 5 vertices obtained from G by turning all non-empty edges into just single edges. Then T has a leaf. This leaf would have to correspond to a pair of multiplicity $2l$ in G, which is contrary to the

maximality assumption on G.

Let C be a cycle of maximal length in G. Suppose that C is a 3-cycle. Without loss of generality, this is 123. Now if 4 is adjacent to any two vertices from $1, 2$ or 3 in G, this creates a 4-cycle. This means that 4 must only be adjacent to 5 and at most one vertex from $1, 2$ and 3. The assumptions on G imply that the multiplicity of 45 is l and we can assume the multiplicity of 14 is also l. A similar argument with 5 instead of 4 implies that 5 is adjacent to exactly one of $1, 2$ and 3 with multiplicity l. We can't have 14 and 15 having multiplicity l, since 1 is also adjacent to both 2 and 3 and G is $2l$-regular. So we can assume 25 has multiplicity l. But then 14523 is a 5-cycle in G which contradicts the maximality of C.

Now assume that C is a 4-cycle, say 1234. If 5 is adjacent to any two consecutive vertices in C, then we can add 5 at this point to find a 5-cycle in G. Since this can't happen by assumption, we can assume that 5 is only adjacent to 1 and 3, and clearly both of these must have multiplicity l. The 4-paths 1532 and 1534 are both subgraphs of G. If an edge between 2 and 4 exists then we get a 5-cycle in G, so 24 must be empty. But then $24, 25$ and 45 are all empty, thus each of $2, 4$ and 5 must have all of their edges going to 1 and 3. This clearly isn't possible when G is $2l$-regular. This contradiction implies that G contains a 5-cycle, proving the lemma. □

We conclude this section with comments on finding a smallest generating set of R^κ. All of the above implies that R^κ is generated as an algebra by labelled 5-cycles. These give a generating set of size $\frac{4!}{2} = 12$. One can check that it is possible to apply Plücker operations to such a cycle containing the edge 15 to replace it with 5-cycles that do not contain the edge 15. This gives a set of generators of size 6. This means that there is a projectively normal embedding of $\bar{M}_{0,5}$ into $\mathbb{P}^5$ obtained from the complete linear system associated to κ.

For larger n, it is clear that n-cycles are not degree one graphs, for they are not $(n-3)l$-

regular. The general picture is that the co-graphs of n-cycles are degree one graphs. Of course, the co-graph of a 5-cycle is itself a 5-cycle, so we have seen the first example of this pattern.

7.2 When $n = 6$

Moving from $n = 5$ to $n = 6$, we have to start thinking about the validity of Plücker operations, checking how many edges are contained in each triangle. We will prove lemmas that are analogous to Lemma 7.1.1 and Lemma 7.1.3.

Fix $n = 6$. If $l \geqslant 0$, then recall that a degree l graph, G, is $3l$-regular with each triangle containing at least l edges. To apply a Plücker operation between two disjoint, non-empty pairs of vertices, ij and kp, we need to check that the triangles ija and ijb contain more than l edges, where a and b are the remaining two vertices in G.

Lemma 7.2.1. Let G be a degree l graph for some $l \geqslant 1$. One can apply a sequence of Plücker operations to write G as a linear combination of degree l graphs, each of which contain no pair with multiplicity greater than l.

Proof. If G has no edges of multiplicity greater than l we are already done. Assume that 12 has multiplicity $l + t$, with $t > 0$. Assume further that $l + t$ is the maximal edge multiplicity in G. Note that we can always apply a Plücker operation between 12 and any other disjoint, adjacent pair of vertices. Indeed, that there are more than l edges between 1 and 2 already implies there are more than l edges in the relevant triangles.

Suppose that i and j are adjacent in G, and that they are disjoint from 12. If the multiplicities of $1i, 1j, 2i$ and $2j$ are at most $l + t - 2$ then we can apply a Plücker operation between 12 and ij without creating a new maximal pair. If we can do this, we will do it recursively until it is no longer possible or until we satisfy the conditions in the lemma. As such, we assume this does not happen, and that there is an edge of multiplicity at least $l + t - 1$ between a vertex in 12 and a vertex not in 12. We can

assume that 23 has multiplicity at least $l + t - 1$.

The triangle 456 must contain an edge, so we can assume that 4 and 5 are adjacent. Again, our only problem may be if one of the pairs $14, 15, 24, 25$ have multiplicity at least $l + t - 1$. We have already accounted for $2l + 2t - 1 > 2l$ edges adjacent to 2 (between 12 and 23), so this pair cannot be 24 or 25. We thus assume that 14 has multiplicity at least $l + t - 1$. Now there are $3l + 3t - 2 > 3l$ edges in the quadrilateral 1234, thus there is an edge between 5 and 6 by $3l$-regularity. By considering the degrees of the vertices, each of the pairs $15, 16, 25$ and 26 have multiplicity less than $l + t - 1$. We can thus apply a Plücker operation between 12 and 56 without creating a new maximal pair (with respect to multiplicity).

We have shown that whenever there is a pair with multiplicity greater than l, we can apply a Plücker operation reducing the multiplicity of a maximal pair without creating a new maximal pair in either of the two new graphs. Doing this recursively we will achieve the conditions set out in the lemma. □

As in the $n = 5$ case, the above lemma has its algebraic version:

Lemma 7.2.2. The section ring R^κ is generated as a k-algebra by those degree l monomials, X, where l ranges over all $l \geqslant 1$, so that x_{ij}^{l+1} does not divide X for any pair $\{i, j\} \subset [6]$. □

The consequence of Lemma 7.2.1 is that if R^κ is generated in degree one then it is generated by those graphs only containing single edges. One can describe these easily by their co-graphs. Let G be a degree one graph with no pairs of vertices having multiplicity greater than one. Then the co-graph, $\bar{G}$, is 2-regular and each triangle contains at most two edges. It is easy to check that this characterises 6-cycles. We will extend this idea of considering the co-graphs of degree one generators further. Before doing so, we treat those degree l graphs with maximal edge multiplicity at most l that have no empty

edges.

Definition 7.2.3. Let G be a degree l graph for some $l \geqslant 0$. We will call G spread if the maximum edge multiplicity of the pairs of vertices in G is at most l.

Lemma 7.2.1 now says that R^κ is generated by all spread graphs of degree l as l ranges over all $l \geqslant 1$. We want to show that each spread, degree l graph is contained in the sub-algebra of R^κ generated by spread, degree one graphs. This will in turn prove that spread, degree one graphs generate R^κ.

If $l \geqslant 1$, then we split the set of spread, degree l graphs into two disjoint subsets. The first subset consists of those graphs that are 'complete', that is to say that every pair of vertices contains an edge. The second subset consists of those graphs that are not complete, so there is at least one empty pair of vertices.

Lemma 7.2.4. Let $l \geqslant 1$ and let G be a complete, spread, degree l graph. Then G can be written as a product of a spread, degree one graph with a degree $l-1$ graph.

Before proving Lemma 7.2.4, we introduce a useful vector for any degree l graph.

Definition 7.2.5. Let $l \geqslant 0$ and let G be any degree l graph. Index the components of $\mathbb{Z}^{10}$ by those $I \in \mathcal{I}$ with $|I| = 3$. The triangle edge vector of G, denoted by $\mathrm{te}(G)$, is the element of $\mathbb{Z}^{10}$ whose I^{th} coordinate is equal to the number of edges in the triangle in G corresponding to I.

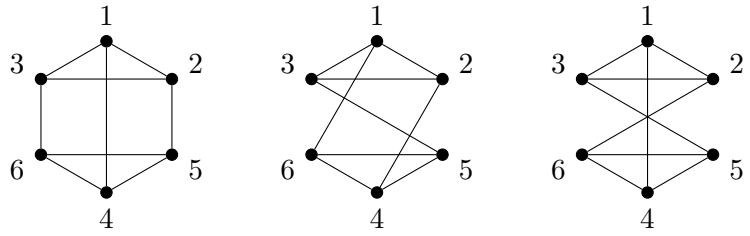
Note that a degree l graph is a $3l$-regular graph whose triangle edge vector has each coordinate at least l. Motivated by this fact, we write $\mathrm{te}(G) \geqslant v \in \mathbb{Z}^{10}$ if each coordinate of $\mathrm{te}(G)$ is at least the corresponding coordinate of v. If we write $\mathbf{1}_{10}$ for the vector of ones, a degree l graph satisfies $\mathrm{te}(G) \geqslant l \cdot \mathbf{1}_{10}$.

Dividing a degree l graph, G, by a degree one graph, H, consists of checking two things. Firstly we need to check that H is a subgraph of G. Secondly, one needs to check that

$\mathrm{te}(G) - \mathrm{te}(H) \geqslant (l-1) \cdot \mathbf{1}_{10}$. In Lemma 7.2.4, the first part is done for any spread degree one subgraph. We just need to show that there is a spread degree one subgraph satisfying the second condition on the triangle edge vectors.

Proof of Lemma 7.2.4. Since G is complete, we have that any spread, degree one graph is a subgraph of G. We need to check that there is a spread, degree one graph, H, so that $\mathrm{te}(G) - \mathrm{te}(H) \geqslant (l-1) \cdot \mathbf{1}_{10}$. Note that since G is complete we must have that $l \geqslant 2$.

There are $\frac{1}{2} \cdot 6 \cdot 3l = 9l$ edges in G. Each edge belongs to 4 triangles, and there are $\binom{6}{3} = 20$ triangles in G. We deduce that the sum of the coordinates of $2\,\mathrm{te}(G)$ is equal to $4 \cdot 9l = 36l$, thus the mean value of the coordinates of $\mathrm{te}(G)$ is $1.8l$. By the pigeon hole principle, and using that $l \geqslant 2$, there must be a coordinate of $\mathrm{te}(G)$ that is at least $l+2$. Without loss of generality, this is the coordinate indexed by $123 \equiv 456$. Consider the following three spread, degree one graphs in R^{κ}.



Name these three graphs H_1, H_2 and H_3 from left to right. Suppose that H_1 does not divide G. Then there must be a coordinate of $\mathrm{te}(G) - \mathrm{te}(H_1)$ that is less than $l-1$, say that this is I. Now $\mathrm{te}(G)_{123} - \mathrm{te}(H_1)_{123} \geqslant (l+2) - 3 = l-1$, so $I \neq 123$. If $\mathrm{te}(H_1)_I = 1$, then $\mathrm{te}(G)_I - \mathrm{te}(H)_I \geqslant l-1$ which is contrary to the choice of I. This means that I must be one of the triangles in H_1 containing two edges, which is one of $124, 125, 134, 136, 145$ and 146. These cases are symmetrical and thus we can assume that $I = 125$. That $\mathrm{te}(G)_{125} - \mathrm{te}(H_1)_{125} = \mathrm{te}(G)_{125} - 2 < l-1$ implies that $\mathrm{te}(G)_{125} = l$.

We now claim that both H_2 and H_3 divide G in R^κ. Since G is complete, and $\mathrm{te}(G)_{125} = l$, there are less than l edges between the pairs 12 and 15 combined. The $3l$-regularity of G implies that there are more than $2l$ edges between any two of the pairs $13, 14$ and 16 combined. Since G is spread, this implies that $\mathrm{te}(G)_{134}, \mathrm{te}(G)_{136}$ and $\mathrm{te}(G)_{146}$ are all greater than l.

Similarly, replacing the vertex 1 in the previous paragraph with 2, there are less than l edges between 12 and 25 combined. This implies that $\mathrm{te}(G)_{234}, \mathrm{te}(G)_{236}$ and $\mathrm{te}(G)_{246}$, which are the same as $\mathrm{te}(G)_{156}, \mathrm{te}(G)_{145}$ and $\mathrm{te}(G)_{135}$ respectively, are all greater than l. Arguing again with 5 instead of 1 or 2, we find that both $\mathrm{te}(G)_{126}$ and $\mathrm{te}(G)_{124}$ are greater than l.

Out of the ten coordinates of $\mathrm{te}(G)$, we have $\mathrm{te}(G)_{123} \geqslant l+2, \mathrm{te}(G)_{125} = l$, and each of the other eight coordinates are at least $l+1$. For $i = 2, 3$, we can see in the diagrams above that $\mathrm{te}(H_i)_{123} = 3, \mathrm{te}(H_i)_{125} = 1$ and the remaining eight coordinates of $\mathrm{te}(H_i)$ are ones or twos. This implies that $\mathrm{te}(G) - \mathrm{te}(H_i) \geqslant (l-1) \cdot \mathbf{1}_{10}$ for $i = 2, 3$. Therefore both H_2 and H_3 divide G in R^κ. □

We now turn to non-complete, spread, degree l graphs. Here we can make the situation easier by defining a generalisation of co-graphs.

Definition 7.2.6. Let G be a spread, degree l graph. The co-graph, $\bar{G}$, of G is the graph on 6 vertices defined as follows. For each pair of vertices, ij, in G, let m_{ij} be its multiplicity in G. Then the multiplicity of ij in $\bar{G}$ is $l - m_{ij}$.

Note that this generalises the notion of co-graphs where multiple edges are not allowed in the degree one case. If G, G' are spread, degree l and spread, degree l' graphs respectively, then $(G \cdot G')^-$ is the graph product of $\bar{G}$ and $\bar{G}'$.

If G is a spread, degree l graph, then we call $\bar{G}$ a degree l co-graph. By Proposition 6.1.1, the degree l co-graphs are those $2l$-regular graphs on 6 vertices with at most $3l - l = 2l$

edges in each triangle. It is easy to check that degree one co-graphs are 6-cycles.

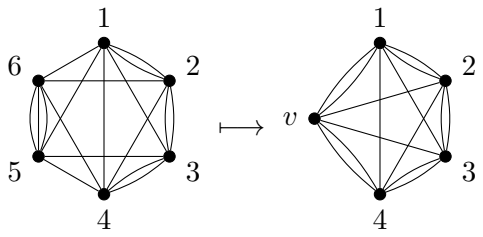
Lemma 7.2.7. Let $l \geqslant 0$ and suppose that G is a non-complete, spread, degree l graph. Then G can be written as the product of a spread, degree one graph with a degree $(l-1)$-graph.

The proof of Lemma 7.2.7 uses an induction method that we hope to extend to larger n. In order to spell out its details, we give it a section of its own.

7.3 Induction from $n = 5$ to $n = 6$

In this section we prove Lemma 7.2.7. We do this via an inductive method where we can then use the $n = 5$ case. There are reasons why we can hope this induction can be used for larger n as well, which we will explain in Chapter 10.

Let G be a non-complete, spread, degree l graph for some $l \geqslant 2$. Then, without loss of generality, the pair 56 is empty in G. Replace G with its co-graph, $\bar{G}$. Then 56 has multiplicity l in $\bar{G}$. Obtain the graph, $\bar{G}^-$, on 5 vertices by replacing the vertices 5 and 6 in $\bar{G}$ by a new vertex, v. In $\bar{G}^-$, the edges in the quadrilateral 1234 are the same as those in $\bar{G}$. For each $i \in 1234$, we put one edge between i and v in $\bar{G}^-$ for each edge in $i5$ or $i6$ in $\bar{G}$. The following diagram is an example of the operation $\bar{G} \mapsto \bar{G}^-$ with $l = 3$.

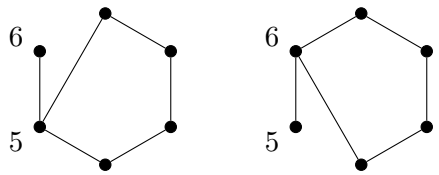


Since $\bar{G}$ is $2l$-regular, it is easy to check that $\bar{G}^-$ is as well. One can define co-graphs of spread, degree l graphs in the $n = 5$ case in the same way as for $n = 6$. Then $\bar{G}^-$ is a degree l co-graph for $n = 5$. The co-graph version of Lemma 7.1.3 implies that $\bar{G}^-$ can

be written as a product of 5-cycles. As such, write $\bar{G}^- = C_1 \dots C_l$, where C_i is a 5-cycle for each $i \in [l]$.

Consider C_1, and look at the edges adjacent to v in C_1. Say that these are iv and jv, with $i, j \in [4]$. If $i5$ and $j6$ are non-empty in $\bar{G}$, or if $i6$ and $j5$ are non-empty in $\bar{G}$, then we can find a 6-cycle, C_1^+, obtained by expanding v to an edge 56, that is a subgraph of $\bar{G}$. We then check if we can do this for each of $C_2, \dots, C_l$. Relabelling if necessary, we can write $\bar{G}$ as a graph product, $\bar{G} = C_1^+ \dots C_t^+ \cdot H$, where H is a $2(l-t)$-regular graph. Note that at the moment we cannot say whether this is a product in R^κ. The multiplicity of 56 in H is $l - t$, and we can form H^- in the same way, and it must be equal to the product $C_{t+1} \dots C_l$. We can assume that none of the C_i, with $t+1 \leqslant i \leqslant l$, can be expanded in v to form a 6-cycle that is a subgraph of H.

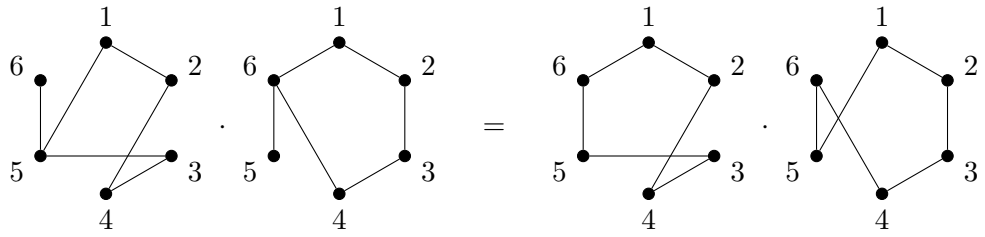
Since C_{t+1} cannot be expanded in v to form a 6-cycle contained in H, that C_{t+1} divides H^- implies that a labelling of one of the two following graphs is a subgraph of H.



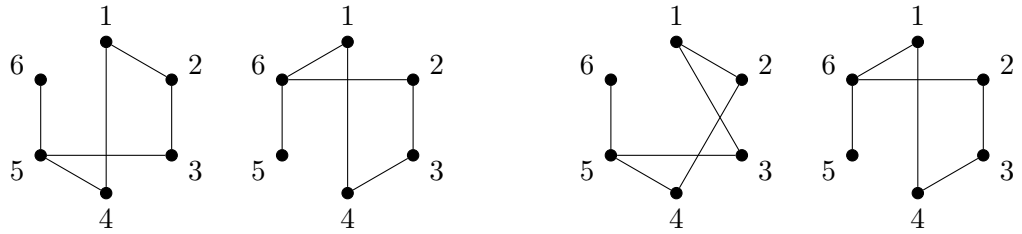
Choose one of these subgraphs and call it C'_{t+1}, noting that $C'^-_{t+1} = C_{t+1}$. Say that C'_{t+1} is of type 5 if it is a labelling of the graph on the left (characterised by the degree of the vertex 5 being equal to 3), and say that it is of type 6 if it is a labelling of the graph on the right. Then let H_1 be so that $H = C'_{t+1} \cdot H_1$. We now do the same with C_{t+2}, which is a subgraph of H_1^-. We can find C'_{t+2}, which is a type 5 or type 6 graph as above with $C'^-_{t+2} = C_{t+2}$, and H_2 on 6 vertices so that $H = C'_{t+1}C'_{t+2} \cdot H_2$. Continuing in this way, we can write $H = C'_{t+1} \dots C'_l$. For each $i \in [l] \backslash [t]$, we have $C'^-_i = C_i$, and the graph C'_i is of type 5 or type 6. That H is regular implies that these types occur

with equal frequency. Relabelling if necessary, we can assume that successive graphs are of opposite type. In other words, $C'_{t+1} \cdot C'_{t+2}, \ldots, C'_{l-1} \cdot C'_l$ are each 4-regular.

Ideally, we want to show that these 4-regular graphs can all be written as the product of two 6-cycles. Indeed, in this case, H, and therefore $\bar{G}$, would be a product of 6-cycles and we would be done. To this end, let C' and D' be one of these pairs making a 4-regular graph product. Assume that C' is of type 5 and D' is of type 6. If a vertex, $i \in [4]$, is adjacent to 5 in C' and 6 in D', then we can swap the edge $i5$ in C' with the edge $i6$ in D' to form two 6-cycles with the same graph product. The example drawn below illustrates this.

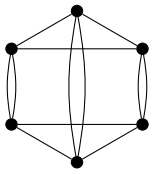


The only case we are left considering is when no vertex adjacent to 5 in C' is adjacent to 6 in D'. If we assume that 5 is adjacent to 3 and 4 in C', and 6 is adjacent to 1 and 2 in D', then there are two distinct possibilities (up to symmetry) for the pairs C' and D'. Both of these are illustrated below.

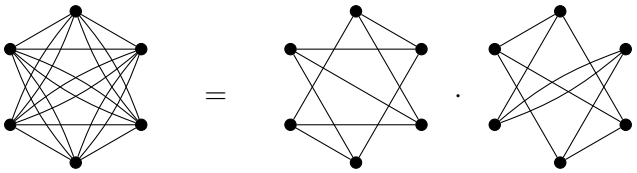


One can check that the graph product of the two graphs on the right is equal to the graph product of the two 6-cycles 126534 and 132456. The graph product of the two graphs on the left gives us our only exceptional case. That is, the graph drawn below

(with some labelling) is our only degree two co-graph containing at least one double edge that cannot be written as the product of two 6-cycles.



This degree two co-graph corresponds to the following degree two graph and product of degree one graphs:



Note that it is the product of a spread, degree one graph with a degree one graph that is not spread. The statement of Lemma 7.2.7 allowed for this possibility.

We have written $\bar{G}$ as a product of degree one co-graphs and some labellings of the exceptional graph given above. Since the co-graph of this exceptional graph is the product of two degree one graphs, we can see that G is the product of degree one graphs, some of which may not be spread. This completes the proof of Lemma 7.2.7.

With this, we have shown that Conjecture 6.2.3 is true for $n = 6$. Our proof also shows that R^κ is generated by spread, degree one graphs, which are precisely the co-graphs of labelled 6-cycles. There are 60 labelled 6-cycles, so this gives us a projectively normal embedding of $\bar{M}_{0,6}$ into $\mathbb{P}^{59}$. In the next chapter we will see that we can reduce this set of 60 generators to a subset of 24 generators. Over the next two chapters we will build on the degree one generation of R^κ proved above and describe equations defining

$\overline{M}_{0,6}$ as a projective variety. Whilst one can do this with $\bar{M}_{0,5}$ in $\mathbb{P}^5$, this moduli space is a surface, thus relatively simple and well understood. Discussion on applying the induction argument used in this section for larger n is given in Chapter 10.

8 Equations defining $\bar{M}_{0,6}$ as a projective variety

For this Chapter, fix $n = 6$. Recall that $S = k[x_I : I \in \mathcal{I}]$ is the $\mathrm{Cl}(X_\Delta)$-graded Cox ring of the toric variety X_Δ, and $\kappa \in \mathrm{Cl}(X_\Delta)$ is the very ample log canonical class. Let I_6 be the $\mathrm{Cl}(X_\Delta)$-homogeneous ideal of S given by

$$I_6 = \left\langle \prod_{i,j\in I,k,p\in I^c} x_I - \prod_{i,k\in I,j,p\in I^c} x_I + \prod_{i,p\in I,j,k\in I^c} x_I \right\rangle.$$

This is just the ideal $I_{\overline{M}_{0,6}}$ in Theorem 4.3.3 except without the saturation. The ideal I_6 also defines the closed subscheme $\overline{M}_{0,6}$ in X_Δ (see Proposition 3.3.7 and [GM11, §6]). Since the proof in Chapter 7 made no use of any extra generators that may arise from saturating I_6 to obtain $I_{\overline{M}_{0,6}}$, we can replace $R^\kappa = (S/I_{\overline{M}_{0,6}})^\kappa$ with $(S/I_6)^\kappa$ without changing $\operatorname{Proj} R^\kappa$ or the fact that R^κ is generated in degree one. From now on by the ring R we mean S/I_6.

We know from §5.2 and the remark following Proposition 6.1.1 that $\overline{M}_{0,6}$ can be given as the scheme $\operatorname{Proj} R^\kappa$. Since the dimension of R_κ is finite dimensional we deduce that $R^\kappa \cong k[X_1, \ldots, X_t]/J$. Here, for each $1 \leqslant i \leqslant t$, the variable X_i is a monomial of degree one in R^κ. The ideal J is a $\mathbb{Z}$-homogeneous ideal in the homogeneous coordinate ring $k[X_1, \ldots, X_t]$ of $\mathbb{P}^{t-1}$. This defines $\overline{M}_{0,6}$ as a closed subscheme of $\mathbb{P}^{t-1}$.

In this chapter we aim to find the smallest possible value for t. Making t small enough will allow us to perform combinatorics by hand to describe properties of the ideal J. We will describe combinatorially some equations lying in J. Our end goal would be to show that J is generated by quadrics by our combinatorial method. Using a different method of proof, [KT09, Theorem 1.2] tells us that $\overline{M}_{0,n}$ is cut out by quadrics in general.

8.1 A basis of the degree one piece of S^κ

In §6.2, we described the subring S^κ of S as a graph algebra. Namely, a degree l monomial corresponds to a $3l$-regular graph on 6 vertices with the condition that each triangle contains at least l edges. Addition in this graph algebra is formal and multiplication is given by the graph product. It is useful to formalise what we mean by saying that S^κ is a graph algebra. Before we can do this we need to introduce some terminology that will feature often in the rest of this thesis.

Definition 8.1.1. Suppose that T is the polynomial ring on a set of variables indexed by a set $\mathcal{B}$, i.e. $T = k[X_b : b \in \mathcal{B}]$. A binomial equation, or simply a binomial, in T is a polynomial in T with exactly two terms, so a polynomial of the form $\prod_{b\in\mathcal{B}} X_b^{n_b} - \prod_{b\in\mathcal{B}} X_b^{m_b}$ where only finitely many n_b and m_b are non-zero positive integers. If J is an ideal in T and $T' = T/J$ is the quotient, then we may call a binomial equation contained in J a binomial relation in T'. Note that the binomial $\prod_{b\in\mathcal{B}} X_b^{n_b} - \prod_{b\in\mathcal{B}} X_b^{m_b}$ is contained in J if and only if $\prod_{b\in\mathcal{B}} X_b^{n_b} = \prod_{b\in\mathcal{B}} X_b^{m_b}$ in T'.

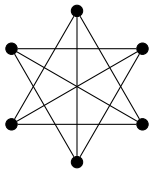
We now formalise the construction of S^κ as a graph algebra. Let S' be the polynomial ring whose variables are indexed by the set, $\mathcal{B}$, of all degree l graphs as l ranges over all $l \geqslant 0$. We give S' the obvious $\mathbb{Z}$-grading, so that the degree of a degree l graph is l. Then S^κ is the quotient of S' by its homogeneous ideal generated by all binomial relations corresponding to collections of graphs with the same graph product. More explicitly, if $(G_i)_{i=1}^t$ and $(H_i)_{i=1}^s$ are two collections of graphs in $\mathcal{B}$, then the binomial equation $\prod_{i=1}^t X_{G_i} - \prod_{i=1}^s X_{H_i}$ in S' is a binomial relation in S^κ if and only if $G_1 \cdot \ldots \cdot G_t = H_1 \cdot \ldots \cdot H_s$.

We now explain why this interpretation is useful for R^κ. From Chapter 7, we know that R^κ is generated in degree one. If $\mathcal{A}$ is a set of degree one graphs generating R^κ, then $R^\kappa \cong k[X_a : a \in \mathcal{A}]/J$, where J is a homogeneous ideal in $k[X_a : a \in \mathcal{A}]$. Since $R^\kappa = S^\kappa/I_6^\kappa$, and we have a complete grasp of equations in I_6 and relations in S^κ, we can

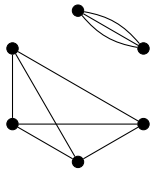
conclude that J is generated by just two types of equations - those binomial relations in S^κ those equations in I_6^κ (remember that each X_a, with $a \in \mathcal{A}$, represents a monomial in S).

Recall that if $\lambda \in \mathrm{Cl}(X_\Delta)$, then by S_λ we mean the homogeneous part of S of degree λ. By S^λ we mean the subring $S^\lambda = \bigoplus_{l \geqslant 0} S_{l\lambda}$. We can extend this to I_6, so that $(I_6)_\lambda$ is the degree λ piece of I_6 and I_6^λ is the ideal $\bigoplus_{l \geqslant 0}(I_6)_{l\kappa}$ of S^κ (which is not an ideal of S).

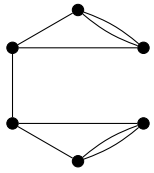
We want to give a basis for S_κ, i.e. a complete list of all degree one graphs. For larger n this becomes unreasonable, but $n = 6$ is small enough for us to do this by hand. Let G be an arbitrary degree one graph. If G only has single edges then we have already discussed that G is the complement of a labelled 6-cycle. There are 60 such graphs (up to isomorphism), each corresponding to some labelling of the graph drawn below.



Now let's consider the case that G contains edges of multiplicity greater than one. Since G is 3-regular, the most extreme case is that G contains a triple edge. Without loss of generality, say the pair 12 has multiplicity three in G. There can be no more edges adjacent to either 1 or 2. The triangles $134, 135, 136, 145, 146, 156$ (and their complements) all need an edge and we deduce that each of $34, 35, 36, 45, 46, 56$ have multiplicity one. Since G was determined uniquely by the choice of the triple edge, there are $\binom{6}{2} = 15$ degree one graphs containing a triple edge, corresponding to labellings of the following graph.



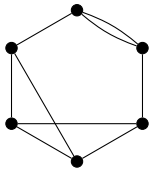
We are left only to find those degree one graphs G with no triple edges, but at least one double edge. Suppose that G has two double edges. Since G is 3-regular these cannot be adjacent, so we can assume the double edges are 12 and 34. There must be one edge from a vertex in 1234 to a vertex in 56 (otherwise 56 would have to be a triple edge), so we can assume that 16 is non-empty. Then 126 contains at least three edges and thus so does its complement 345. Again, since 34 is not a triple edge we can assume that 45 is non-empty. Now we have accounted for all edges adjacent to 1 and 4, yet we still need an edge in the triangles 135 and 246. The edges 35 and 26 must exist and we can complete the remaining edge to find that G is a labelling of the following graph.



There are 90 labellings, up to isomorphism, of the degree one graph above. To see this, there are $\frac{1}{2}\binom{6}{2}\binom{4}{2} = 45$ choices for the pair of double edges. Then, if one of these double edges is ij, there are two choices for the remaining vertex adjacent to both i and j and this choice fully determines the graph.

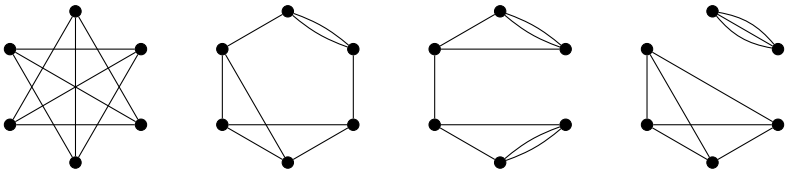
The final case is those degree one graphs G with just one double edge, which we can put to be the pair 12. All other pairs have multiplicity zero or one. Now if 1 and 2 are each adjacent to a third vertex i, then $12i$ has four edges and thus so must its complementary triangle. The pigeon hole principle would then imply this complementary triangle has a double edge which is contrary to our assumption. We can thus assume that the

remaining vertex adjacent to 1 is 6 and the remaining vertex adjacent to 2 is 3. Now we have accounted for all edges adjacent to 1 and 2 but we still need an edge in the triangles $134, 135, 145, 246, 256$. This uniquely gives us a labelling of the graph below.



There are 180 labellings of the graph above up to isomorphism. Indeed, there are 15 choices for the double edge. If this double edge is the pair ij, there are then 4 choices for the remaining vertex adjacent to i and 3 choices for the remaining vertex adjacent to j. These choices fully determine the graph.

We have argued that there are precisely four isomorphism classes (ignoring labelling) of degree one graphs in S^κ. These are each drawn below. From left to right, we will call labellings of the four graphs below Type 1 graphs, Type 2 graphs, Type 3 graphs and Type 4 graphs.

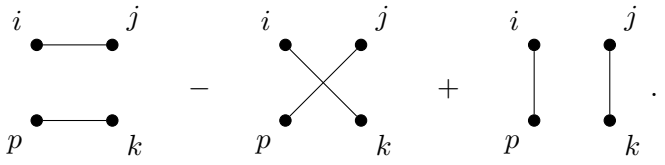


We have also found the number of non-isomorphic labellings of each of these graphs. Overall there are $60 + 180 + 90 + 15 = 345$ degree one graphs. This statement has the following interpretation in the toric geometry.

Proposition 8.1.2. Let X_Δ be the toric variety of §4.3, with $n = 6$. If κ' is the

preimage of the log canonical divisor of $\overline{M}_{0,6}$ under the isomorphism of $\mathrm{Cl}(\overline{M}_{0,6})$ with $\mathrm{Cl}(X_\Delta)$ given in Proposition 4.4.2, then the dimension of $H^0(X_\Delta, \mathcal{O}_{X_\Delta}(\kappa'))$ is 345. $\square$

These 345 graphs give all degree one monomials in both S^κ and R^κ. Since R^κ is generated in degree one and $\overline{M}_{0,6} \cong \operatorname{Proj} R^\kappa$, we can now describe $\overline{M}_{0,6}$ as a projective variety. Indeed, we have $R^\kappa = k[X_1, \ldots, X_{345}]/J$, where the X_i correspond to the degree one graphs described above. The ideal J is generated by just two sets of equations - those binomial relations corresponding to graph products and those equations contained in the the ideal I_6^κ defining $\overline{M}_{0,6}$ in X_Δ. We can describe a large set of generators for $I_6^\kappa \subset S^\kappa$. Consider the graph expression below, for some $1 \leqslant i < j < k < p \leqslant 6$, which corresponds to an arbitrary generator of I_6.



(8.1)

Suppose that H is a graph on 6 labelled vertices with $\deg(i) = \deg(j) = \deg(k) = \deg(p) = 3l - 1$, and the remaining two vertices have degree $3l$. Suppose further that the triangles ijk, ijp, ikp, jkp contain at least $l - 1$ edges, and all other triangles in H contain at least l edges. Then the graph product of H with the graph expression above is contained in I_6^κ (it would be a homogeneous polynomial with three terms of degree l in S^κ). We conclude that I_6^κ is generated by all such equations as we range over all possible $1 \leqslant i < j < k < p \leqslant 6$. This gives us an infinite number of generators, with generators in each degree, so whilst we do have a description of equations defining $\overline{M}_{0,6}$ in $\mathbb{P}^{344}$, we would like to improve on this.

The first thing to realise is that $\overline{M}_{0,6}$ actually lies in a linear subspace of $\mathbb{P}^{344}$. Indeed,

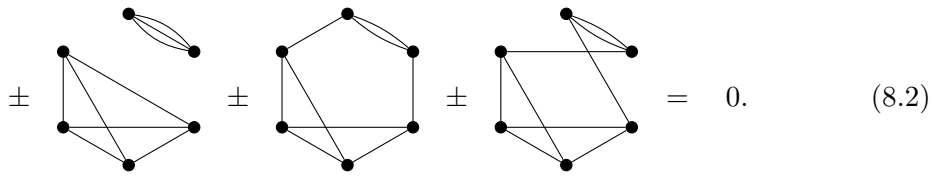
in Chapter 7 we proved that the 60 Type 1 degree one graphs generate R^κ, thus $\overline{M}_{0,6}$ lies in a 59-dimensional linear subspace of $\mathbb{P}^{344}$. In the next section, we will show that R^κ is generated by a size 24 subset of the Type 1 graphs, and thus $\overline{M}_{0,6}$ lies in a 23-dimensional subspace of $\mathbb{P}^{344}$. In Chapter 9 we will work towards giving a finite set of quadric equations defining $\overline{M}_{0,6}$ in $\mathbb{P}^{23}$.

8.2 A set of 24 generators for R^κ

Showing that the image of $\overline{M}_{0,6}$ inside $\mathbb{P}^{344}$ lies in a linear subspace amounts to finding linear equations satisfied by the 345 generators of R^κ. These linear equations (in the generators) lie in $(I_6)_\kappa$. We will describe a basis of $(I_6)_\kappa$. To do this we will look at each type of degree one graph and investigate the Plücker operations that are possible. We will start with Plücker operations on Type 4 graphs and end with Plücker operations on Type 1 graphs.

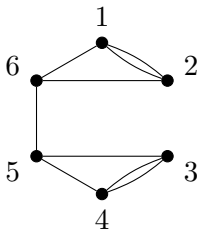
Recall that if G is a degree one graph, then to apply a Plücker operation between pairs of vertices ij and kp, these edges must exist and if a and b are the remaining two vertices in G, then the triangles ija and ijb must contain more than one edge.

Let G be a Type 4 graph, which is the disjoint union of a triple edge and a complete graph on four vertices. For ease, we can assume that the triple edge is 12. We cannot apply a Plücker operation between a disjoint pair of edges in the complete part of G. For example, 34 and 56 is such a pair but the triangle 134 contains exactly one edge. Thus the only Plücker operation one can apply up to symmetry (i.e. up to isomorphism if we forget labelling) is between the triple edge and some single edge. The corresponding graph equation is drawn below (the signs depend on labelling).

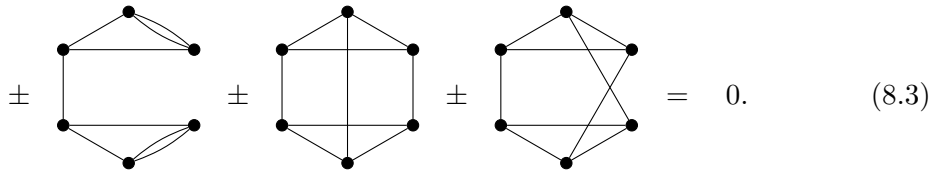


(8.2)

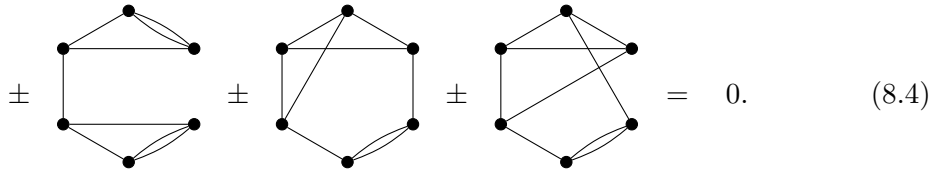
Now let's look at the Plücker operations that can be applied to Type 3 graphs. Consider the Type 3 graph, G, given below:



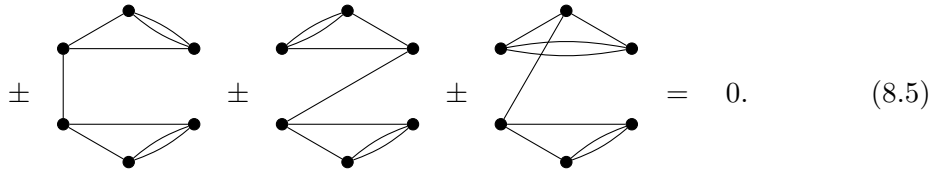
Suppose we were to try to apply a Plücker operation on G not involving a double edge. Using a symmetry argument we can assume we are trying the edges 16 and 45. But the triangle 136 contains exactly one edge, so this is not possible. We conclude that any Plücker operation on G must include a pair of multiplicity of two. Again, up to symmetry there are really only three cases. Firstly, we can apply a Plücker operation between the pairs 12 and 34 in G (so the two pairs of multiplicity two). We can also apply a Plücker operation between 12 and 35. Finally, we can apply a Plücker operation between 12 and 56. Forgetting the labellings, these three possibilities, in the order we have given them, give the three unlabelled graph equations below.



(8.3)

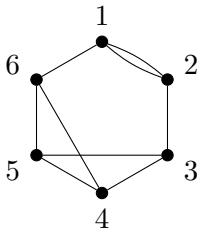


(8.4)



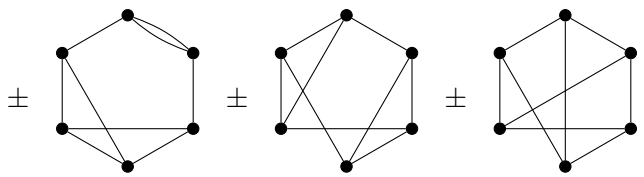
(8.5)

Next we want to look at the Plücker operations that can be applied to Type 2 graphs. Consider the following Type 2 graph, G:

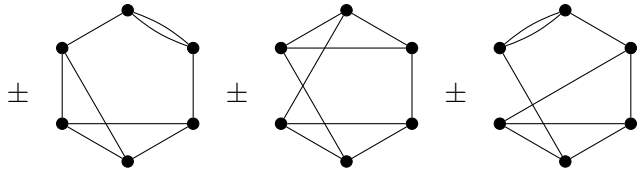


We can always apply a Plücker operation between the double edge 12 and any disjoint non-empty pair. Up to symmetry, in G there are two Plücker operations involving the pair 12 to consider - applying an operation between 12 and 45, or applying an operation

between 12 and 56. Now let us try to find Plücker operations on G that can be applied between two single edges. Since the triangles $134, 135, 145, 246, 256$ contain exactly one edge, there is no valid operation between a disjoint pair of edges in the quadrilateral 3456. As such, we can assume that 16 is involved in the operation. Since 136 contains exactly one edge, the second pair involved in the operation includes the vertex 3. Up to symmetry, the only choices for the second pair are 23 and 34. However, if we apply a Plücker operation between 16 and 23 we get a graph equation of the form (8.2) and if we apply a Plücker operation between 16 and 34 we get the same graph equation as in (8.4). This was expected as we saw that these two graph equations contained Type 2 terms. The only new (unlabelled) graph equations are those two obtained from the Plücker operations on Type 2 graphs involving the double edge:



(8.6)



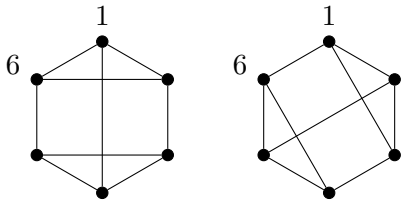
(8.7)

The only graphs we haven't yet considered Plücker operations on are Type 1 graphs. It is easy to check that applying any valid Plücker operation to a Type 1 graph will result in two new graphs, at least one of which will contain a double edge. But in this case the corresponding graph equation will already be covered by the six unlabelled graph

equations drawn above. By looking at the instances of Type 1 graphs amongst these six unlabelled graph equations, there are three essentially different Plücker operations that can be applied to Type 1 graphs. These will yield equations (8.3), (8.6) and (8.7).

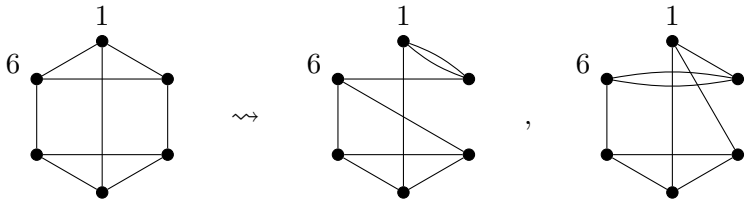
By ranging through all possible labellings of the six graph equations given in this section we have given all linear relations satisfied by the 345 degree one monomials in R^κ. These six equations can also be used to show that if R^κ is generated in degree one then it is generated by the 60 Type 1 graphs. Indeed, equation (8.6) can be used to write any Type 2 graph as a signed sum of two Type 1 graphs, equation (8.3) can be used to write any Type 3 graph as a signed sum of two Type 1 graphs. Finally, equation (8.2) can be used to write any Type 4 graph as a signed sum of two Type 2 graphs, both of which can be written as a signed sum of Type 1 graphs.

We can use the six unlabelled graph equations corresponding to Plücker operations to reduce this degree one generating set of R^κ from 60 monomials to just 24. Recall that Type 1 graphs are the complements of labelled 6-cycles. There are 24 labelled 6-cycles that contain the edge 16. Indeed, these naturally correspond to labelled 6-paths from 1 to 6. We claim that the 24 Type 1 graphs that do not contain the edge 16, i.e. the complements of 6-cycles that do contain the edge 16, generate R^κ. It is sufficient to show that given any Type 1 graph containing the edge 16, one can apply a sequence of Plücker operations to obtain a set of Type 1 graphs, none of which contain the edge 16. Let G be a Type 1 graph containing the edge 16, so that $\bar{G}$ is a 6-cycle not containing the edge 16. Then the distance (length of shortest path) from 1 to 6 in $\bar{G}$ is equal to 2 or 3. The graph drawn below on the left corresponds to G (with some labelling of the other 4 vertices) in the distance 2 case and the graph on the right corresponds to the distance 3 case.



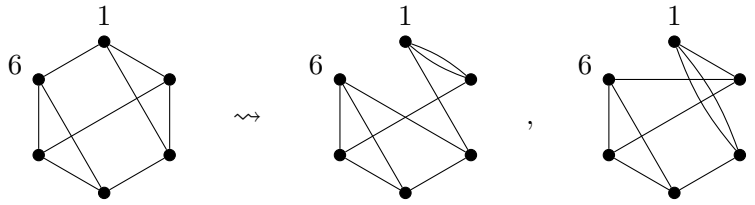
Another way to think of these two cases is that a Type 1 graph is the disjoint union of two complete triangles along with a perfect pairing of the vertices in one of these triangles with the vertices in the other. If we assume the edge 16 exists then it is either contained in a complete triangle or it is one of the edges in the pairing between the triangles. We can see that the two graphs drawn above cover these two possibilities.

Let's start with the graph on the left, where 16 is part of a complete triangle. For convenience, label the remaining vertices in this graph from 2 to 5 so the numbers increase going clockwise as usual. First, we apply a Plücker operation between 16 and 23 to obtain two Type 2 graphs - this will give a labelling of equation (8.7).

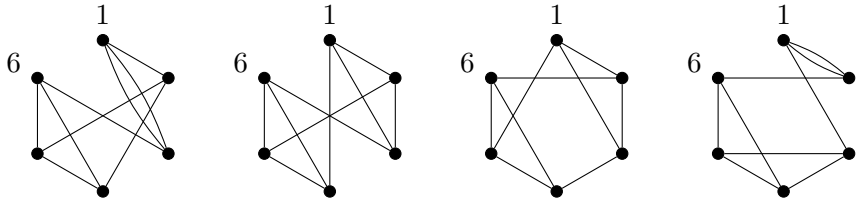


In the first of these Type 2 graphs we can apply a Plücker operation between 12 and 35 and in the second of these Type 2 graphs we can apply a Plücker operation between 26 and 34. These are both examples of equation (8.6) and we obtain a set of four Type 1 graphs, none of which contain the edge 16.

Now consider the second case, where 16 is not part of a complete triangle in a Type 1 graph. We first apply the following Plücker operation to obtain two Type 2 graphs.



In the first of these Type 2 graphs, one can apply a Plücker operation between 12 and 34. In the second of these Type 2 graphs, we apply a Plücker operation between 13 and 25. We are left with the following four graphs:



Both Type 1 graphs here do not contain the edge 16. Furthermore, the two Type 2 graphs here can be written as a signed sum of Type 1 graphs not containing the edge 16 using a graph equation of the form (8.6). But this means we have used Plücker operations to write our Type 1 graph containing the edge 16 as a signed sum of six Type 1 graphs that do not contain the edge 16. We have proven that the 24 Type 1 graphs not containing the edge 16 generate R^κ. Of course, the choice of the pair 16 is arbitrary and we can say that the set of Type 1 graphs not containing any chosen fixed edge generates R^κ. We now state the geometric consequence of this fact.

Proposition 8.2.1. The fine moduli space $\overline{M}_{0,6}$ is a closed subscheme of $\mathbb{P}^{23}$. If we give $\mathbb{P}^{23}$ the set of homogeneous coordinates corresponding to Type 1 graphs not containing the edge 16, then the homogeneous ideal of equations defining $\overline{M}_{0,6} \subset \mathbb{P}^{23}$ is generated by binomials corresponding to finite collections of the 24 graphs with equal graph products, and the collection of homogeneous equations in these 24 variables whose corresponding equations in S lie in $(I_6)^{\kappa}$. □

Reducing from 345 generators to just 24 makes it plausible to describe all of the binomial relations between these generators. Even having reduced to just the 60 generators that have no edges of multiplicity more than one makes the problem of finding binomial relations much easier. Indeed, in this case we can think about co-graphs as in Definition 7.2.6. Switching between graphs and co-graphs is compatible with the graph product. In other words if we are given two l-tuples, $(G_i)_{i=1}^{l}$ and $(H_i)_{i=1}^{l}$, of Type 1 graphs then we have

$$\prod_{i=1}^{l} G_i = \prod_{i=1}^{l} H_i \iff \prod_{i=1}^{l} \bar{G}_i = \prod_{i=1}^{l} \bar{H}_i.$$

Finding binomial relations amounts to finding l-tuples of labelled 6-cycles with the same graph product. If we are just looking at the 24 cycles containing the edge 16 then binomial relations between these amount to which labelled 6-paths from 1 to 6 have the same graph product (since the pair 16 will have multiplicity l in all such products). It is a much harder combinatorial problem to describe the equations between these 24 monomials taken in S that lie in I_6. Through six unlabelled equations we were able to describe $(I_6)_{\kappa}$, but we don't have information on $(I_6)_{l\kappa}$ for general $l \geqslant 0$. In the next chapter we begin to explore both types of relations between the 24 monomials.

9 Work on equations defining $\overline{M}_{0,6}$ in $\mathbb{P}^{23}$

As in §8.2, we now view R^κ as a $\mathbb{Z}$-graded quotient of a polynomial ring whose variables correspond to the 24 labelled 6-cycles necessarily containing the edge 16. Let $\mathcal{A}$ denote the set of such 6-cycles. We can identify $\mathcal{A}$ with the set of permutations on $\{2,3,4,5\}$. For example, the permutation $3425 \in \mathcal{A}$ corresponds to the labelled 6-cycle 134256. If $\sigma = abcd \in \mathcal{A}$, so that $\{a,b,c,d\} = \{2,3,4,5\}$, then we write $G_\sigma = G_{abcd}$ for the corresponding labelled 6-cycle (whose edges are $1a, ab, bc, cd, d6$ and 16). We write $X_\sigma = X_{abcd}$ for the corresponding degree one monomial in R^κ.

With these identifications we have $R^\kappa \cong k[X_\sigma : \sigma \in \mathcal{A}]/J$, where J is generated by binomials corresponding to unordered l-tuples of the G_σ with the same graph product and by equations corresponding to graph equations in the G_σ lying in $(I_6)^\kappa$. In the first section of this chapter, we will use linear algebra to obtain a good grasp of all of the binomial equations. In the second section we discuss how one may continue to try and prove that J is generated by equations in degree two.

9.1 Binomial relations between the 24 generators

In this section we will find a description of all binomial relations between our 24 generators of R^κ. For each $\sigma \in \mathcal{A}$, we define a vector, $v_\sigma \in \mathbb{Z}^{10}$. We index the components of $\mathbb{Z}^{10}$ by pairs ij with $1 \leqslant i < j \leqslant 5$ and put the ij component of v_σ equal to the multiplicity of the pair ij in the corresponding cycle (which will equal to zero or one). If G is a product of $l \geqslant 0$ cycles containing the edge 16 then we define similarly the vector $v_G \in \mathbb{Z}^{10}$, where the ij component of v_G is equal to the number of edges between i and j in G. We note two things. Firstly, if $G = \prod_{\sigma\in\mathcal{A}} G_\sigma^{n_\sigma}$, where the n_σ are non-negative integers, then $v_G = \sum_{\sigma\in\mathcal{A}} n_\sigma v_\sigma$. Secondly, we can determine G uniquely from v_G. Indeed the multiplicities of all pairs in G not containing the vertex 6 can be read from v_G. The multiplicities of the 5 pairs containing 6 can be deduced using the $2l$-regularity of

G.

In particular, if G and H are each products of $l \geqslant 0$ cycles containing the edge 16 then $v_G = v_H$ if and only if $G = H$. We get the equivalence

$$\prod_{\sigma \in \mathcal{A}} G_\sigma^{n_\sigma} = G = \prod_{\sigma \in \mathcal{A}} G_\sigma^{m_\sigma} \iff \sum_{\sigma \in \mathcal{A}} n_\sigma v_\sigma = \sum_{\sigma \in \mathcal{A}} m_\sigma v_\sigma,$$

where the n_σ and m_σ are non-negative integers. From this, any binomial relation $\prod_{\sigma \in \mathcal{A}} X_\sigma^{n_\sigma} = \prod_{\sigma \in \mathcal{A}} X_\sigma^{m_\sigma}$ not dividing by any monomial corresponds to a linear dependency between the v_σ. If we let V be the 10×24 matrix whose columns are the v_σ, as σ ranges over $\mathcal{A}$, then V defines a linear map from $\mathbb{Z}^{24}$ to $\mathbb{Z}^{10}$. Binomial relations not dividing by any monomial correspond one-to-one with vectors in the kernel of this map. We would like to find a $\mathbb{Z}$-basis for the kernel of V.

We order the permutations in $\mathcal{A}$ in the following way. Given $abcd \neq a'b'c'd'$ in $\mathcal{A}$, we say that the larger of these two elements is the one in which 2 appears further left. If in each permutation 2 is in the same position we then compare where 3 lies, and then if necessary where 4 lies. This is the usual way to order these permutations, i.e. $2345, 2354, \ldots, 5432$. With this ordering on the columns of V, and the rows of V ordered in the usual way as well, the matrix V is given as follows:

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1
\end{pmatrix},$$

The first question to ask about the kernel of this matrix is its dimension (if we view the linear map over $\mathbb{Q}$). For this, we claim that the row rank of V is 9, so that the kernel of V has dimension $24 - 9 = 15$. If we let r_i, with $1 \leqslant i \leqslant 10$, be the i^{th} row of V as given above then it is straightforward to check that any dependency relation between the rows of V is a scalar multiple of

$$3r_1 + 3r_2 + 3r_3 + 3r_4 - r_5 - r_6 - r_7 - r_8 - r_9 - r_{10} = 0.$$

We are thus looking for a finite set (of at least 15 elements) that generate the kernel of V as a group. Of course, one can ask a computer to find a $\mathbb{Z}$-basis, but this won't necessarily give us a useful basis to work with. To clarify what we mean by this, we would like our set to include as many elements corresponding to degree two binomial relations as possible. A degree two binomial relation has the form $X_{abcd}X_{efgh} = X_{a'b'c'd'}X_{e'f'g'h'}$, corresponding to an equation $v_{abcd} + v_{a'b'c'd'} - v_{efgh} - v_{e'f'g'h'} = 0$. In other words, the corresponding element in $\ker V \subset \mathbb{Z}^{24}$ has two coordinates equal to 1, two coordinates

equal to -1, and the rest equal to 0. For this, we wish to find all degree two binomial equations, which we can do by hand.

Suppose that $X_{abcd}X_{efgh} = X_{a'b'c'd'}X_{e'f'g'h'}$, where the subscripts represent permutations of $\{2,3,4,5\}$ as above. Without loss of generality, we can assume that $abcd = 2345$. Let G be the degree two co-graph corresponding to both $X_{2345}X_{efgh}$ and $X_{a'b'c'd'}X_{e'f'g'h'}$. Since $G = G_{2345} \cdot G_{efgh}$, the vertex 1 in G is adjacent to 2 and e (if $e = 2$ then there is a double edge between 1 and 2). Similarly, the vertex 6 is adjacent to both 5 and h in G. Since $G = G_{a'b'c'd'} \cdot G_{e'f'g'h'}$ also, the vertex 1 is adjacent to a' and e' and the vertex 6 is adjacent to d' and h' in G. Swapping $a'b'c'd'$ and $e'f'g'h'$ if necessary we can assume that $a' = 2$ and $e' = e$. There are then two cases, either $d' = 5$ and $h' = h$, or $d' = h, h' = 5$ with $d' \neq 5$, $h' \neq h$.

In the first case we are dealing with the graph equation $G = G_{2345} \cdot G_{efgh} = G_{2b'c'5} \cdot G_{ef'g'h}$. For this not to be a trivial relation (i.e. something not of the form $xy = xy$) we must have that $2b'c'5 \neq 2345$, which forces $2b'c'5 = 2435$. Similarly we must have $ef'g'h \neq efgh$, which forces $ef'g'h = egfh$. Our graph equation becomes $G = G_{2345} \cdot G_{efgh} = G_{2435} \cdot G_{egfh}$. The edges 23 and 45 exist in G_{2345} but not in G_{2435}, so for this graph equation to hold the edges 23 and 45 must exist in G_{egfh}. This leaves the following eight possibilities for $egfh$:

$$2345, 2354, 3245, 3254, 4523, 4532, 5423, 5432. \tag{9.1}$$

Conversely, the edges 24 and 35 exist in G_{2435} but not in G_{2345}, so again, for the graph equation to hold, these edges must exist in G_{efgh}. This gives us eight possibilities for $efgh$:

$$2435, 2453, 3524, 3542, 4235, 4253, 5324, 5432. \tag{9.2}$$

We get $egfh$ from $efgh$ by transposing the middle two elements of the permutation.

There are only four possibilities out of the eight given for $egfh$ in (9.1) compatible with the eight possibilities given for $efgh$ in (9.2). We get that $egfh$ must be one of

$$2345, 3254, 4523, 5432.$$

The choice $egfh = 2345$ gives us a trivial relation (of the form $xy = yx$). The remaining three choices give us the following three binomial equations:

$$X_{2345}X_{3524} = X_{2435}X_{3254}, \quad X_{2345}X_{4253} = X_{2435}X_{4523}, \quad X_{2345}X_{5342} = X_{2435}X_{5432}.$$

Now we move onto the second case, where $d' = h, h' = 5$ with $d' \neq 5$, $h' \neq h$. We are then looking to find all graph equations of the form $G_{2345} \cdot G_{efgh} = G_{2b'c'h} \cdot G_{ef'g'5}$ with $h \neq 5$.

Now the edge 23 exists in G_{2345} thus must exist in either $G_{2b'c'h}$ or $G_{ef'g'5}$. If 23 exists in $G_{2b'c'h}$ then $b' = 3$ and $h \neq 5$ implies that $2b'c'h = 2354$ and we have $G_{2345} \cdot G_{efgh} = G_{2354} \cdot G_{ef'g'5}$. Now G_{2345} contains the edge 34 but G_{2354} does not, and so $G_{ef'g'5}$ must contain the edge 34. We are left with the following four possibilities for $ef'g'5$:

$$2345, 2435, 3425, 4325.$$

The first of these gives a trivial relation. In the second possibility, the edge 35 exists in both G_{2354} and G_{2435}, so must be a double edge in the graph product, however 35 is empty in G_{2345}, so cannot have a double edge in the same product. This contradiction implies we don't get a relation with $ef'g'5 = 2435$. The third possibility gives us the binomial relation $X_{2345}X_{3524} = X_{2354}X_{3425}$. The fourth possibility gives us $e = 4 = h$ which contradicts $efgh$ being a permutation. We have obtained just one new relation in taking 23 to exist in $G_{2b'c'h}$.

Let's now consider the case that 23 does not exist in $G_{2b'c'h}$, and thus 23 must exist in $G_{ef'g'5}$. The graph equation we are looking at is $G_{2345} \cdot G_{efgh} = G_{2b'c'h} \cdot G_{ef'g'5}$ with $h \neq 5$ and $b' \neq 3$. Since 23 exists in $G_{ef'g'5}$ we have the following four possibilities for $ef'g'5$:

$$2345, 3245, 4235, 4325.$$

The first of these gives us a trivial relation. If $ef'g'5 = 3245$ then 13 must be present in G_{efgh} which forces $e = 3$. Now $h \neq 5$ and h cannot be 2 or 3 since $efgh = 3fgh$ and $2b'c'h$ are permutations, so $h = 4$. The edge 24 exists in G_{3245} but not G_{2345}, so 24 exists in G_{3fg4}, meaning that $3fg4 = 3524$. We obtain the binomial relation $X_{2345}X_{3524} = X_{2534}X_{3245}$.

The third possibility, that $ef'g'5 = 4235$, implies that 14 and 24 exist in G_{efgh} (since they do not in G_{2345}). This implies that $G_{efgh} = G_{42gh}$. Along with the assumption that $h \neq 5$ we deduce that $efgh = 4253$ and we obtain the relation $X_{2345}X_{4253} = X_{2543}X_{4235}$.

The remaining possibility is that $ef'g'5 = 4325$. Then 25 must be an edge in $G_{efgh} = G_{4fgh}$. Furthermore, $b' \neq 3$ by assumption and 23 is an edge in G_{2345}, so 23 cannot be an edge in G_{4fgh}. Then since $h \neq 2$, as $2b'c'h$ is a permutation, we must have $4fgh = 4253$ and we get the relation $X_{2345}X_{4253} = X_{2453}X_{4325}$.

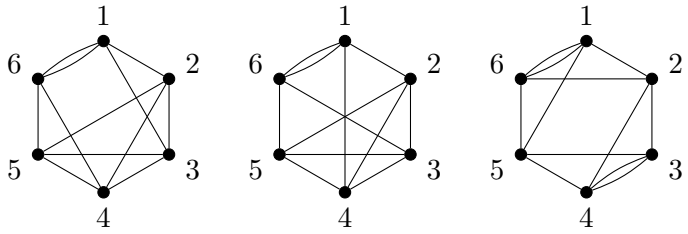
We have exhausted all possible degree two binomial relations where the corresponding degree two co-graph contains the cycle 123456 as a sub-graph. In fact, we see that there are only three degree two co-graphs giving rise to such relations. Indeed, combining the above results we have three sets of equalities:

$$X_{2345}X_{3524} = X_{2435}X_{3254} = X_{2354}X_{3425} = X_{2534}X_{3245},$$

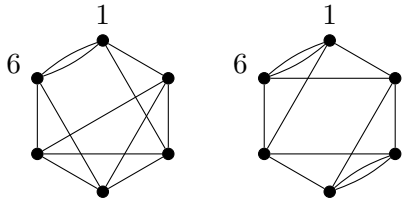
$$X_{2345}X_{4253} = X_{2435}X_{4523} = X_{2543}X_{4235} = X_{2453}X_{4325}, \tag{9.3}$$

$$X_{2345}X_{5342} = X_{2435}X_{5432}.$$

The three degree two co-graphs corresponding to these equalities are drawn below. The graph on the left represents the four monomials in the first row, the middle graph the four monomials in the middle row and the final graph the two monomials in the bottom row.



If we ignore the labelling of the points in $\{2, 3, 4, 5\}$, then the first two graphs given here are isomorphic. To see this, just swap the roles of 3 and 4. This means that up to labelling of the points in $\{2, 3, 4, 5\}$, there are just two degree two co-graphs that give rise to binomial relations, and these are drawn below.



There are six non-isomorphic labellings of both of these graphs. To see this, note that a labelling of the graph on the left is determined by which two vertices from $\{2, 3, 4, 5\}$ are adjacent to 1, and a labelling of graph on the right is determined by which two vertices from $\{2, 3, 4, 5\}$ have a double edge between them. From the equations (9.3), we see that

each labelling of the graph on the left can be written as a product of two degree one co-graphs in four different ways. This amounts to obtaining three linearly independent vectors in $\ker V$ for each of the six labellings. For example, the top row of equalities given in (9.3) give us the three kernel vectors

$$v_{2345} + v_{3524} - v_{2435} - v_{3254}, \quad v_{2345} + v_{3524} - v_{2354} - v_{3425}, \quad v_{2345} + v_{3524} - v_{2534} - v_{3245}.$$

Each labelling of the degree two co-graph on the right can be written as a product of degree one co-graphs in just two ways, giving us just one element in the kernel of V for each of the six labellings. Overall, we have found $3 \times 6 + 1 \times 6 = 24$ elements in $\ker V$ corresponding to degree two binomial relations. One can encode these 24 equations into a 24×24 matrix, A, whose columns are indexed by permutations of $\{2, 3, 4, 5\}$, ordered as before. For example, the vector $v_{2345} + v_{3524} - v_{2435} - v_{3254} \in \ker V$ will correspond to a row

$$(1, 0, -1, 0, 0, 0, 0, -1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$$

in A. We give the matrix A below. The first three rows correspond the three linearly independent elements of the kernel obtained from the top row in (9.3). The next 15 rows can also be taken in blocks of three rows each, one block for each non-isomorphic labelling of the degree two co-graph admitting four decompositions as a product of degree one co-graphs. The bottom six rows of A are the elements in $\ker V$ that correspond to the relations obtained from the different labellings of the degree two co-graph given on the right above.

$$\begin{pmatrix}
1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\
0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}$$

The Smith normal form of this matrix has its first 14 diagonal entries equal to 1 and all

other entries equal to 0. This means that the rows of this matrix generate a saturated lattice of rank 14 contained in the kernel of V. We know that the rank of $\ker V$ is 15, so we need to find one more vector in $\ker V$ that is linearly independent to the 24 rows of A. We know that this new vector won't correspond to a degree two binomial relation, but rather a binomial relation of larger degree. One can check that the graph equation $G_{2534} \cdot G_{5243} \cdot G_{5432} = G_{2543} \cdot G_{5234} \cdot G_{5342}$ holds, and thus we get a degree three binomial relation corresponding to the vector

$$v_{2534} + v_{5243} + v_{5432} - v_{2543} - v_{5234} - v_{5342} \in \ker V.$$

Let A' be the 25×24 matrix obtained from A by adding the row corresponding to this element of the kernel at the bottom. The Smith normal form of A' has its first 15 diagonal entries equal to 1 and all other entries equal to 0. This means that the rank of A' is 15, and the 25 vectors in $\ker V$ given by the rows of A' span $\ker V$ as a $\mathbb{Z}$-module (note that the Smith normal form having ones on its diagonal is necessary for this last statement, otherwise our 25 vectors would generate a proper subgroup of $\ker V$ of finite index).

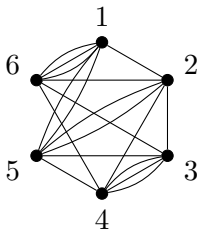
We have found 25 binomial relations, 24 of which are in degree two, that allow us to describe all binomial relations. Indeed, if $\prod_{\sigma \in \mathcal{A}} X^{n_\sigma} - \prod_{\sigma \in \mathcal{A}} X^{m_\sigma}$ is a binomial relation, then we can assume that this expression doesn't divide by any X_σ. This expression must correspond to a unique element in $\ker V$, and thus a (non-unique) linear combination of rows in A'. This means that the saturation of the ideal generated by the 25 binomial relations corresponding to rows of A' with respect to the ideal $\langle X_\sigma : \sigma \in \mathcal{A} \rangle$ contains every binomial relation between the 24 generators of R^κ.

9.2 Towards $\overline{M}_{0,6}$ being cut out by quadrics

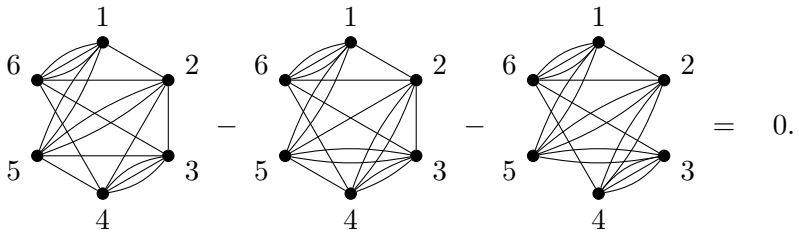
We have an ideal $J \subset k[X_\sigma : \sigma \in \mathcal{A}]$ defining the projective scheme $\overline{M}_{0,6}$ in $\mathbb{P}^{23}$. We know that this ideal is generated by binomials and equations in $(I_6)^\kappa$. We would like to show that J is generated by its degree two piece J_2. The first equation we will deal with is the degree three binomial equation that we added to the 24 degree two binomial equations in the previous section. Explicitly, this is

$$f = X_{2534}X_{5243}X_{5432} - X_{2543}X_{5234}X_{5342} \in J.$$

We will show that f lies in the ideal generated by J_2. For this, we draw the degree 3 co-graph, G, corresponding to the binomial equation f:



In this degree 3 co-graph G, the pairs 24 and 35 have multiplicity less than 3 and the edges $23, 34, 45, 25$ exist. Furthermore, the triangles 246 and 241 contain fewer than $3 \cdot 2 = 6$ edges. These facts imply that in the corresponding degree 3 graph $\bar{G}$, a Plücker operation can be applied between the edges 24 and 35. The corresponding degree 3 co-graph equation is



One can check that this graph equation can be factorised into the degree 3 equation

$$X_{2534}g = 0, \quad g = X_{5243}X_{5432} - X_{5342}X_{5423} - X_{5342}X_{5243}.$$

It can then be checked that g corresponds to a Plücker operation on degree two graphs, and thus $g \in J_2$. Now we calculate

$$f = X_{2534}g + X_{5342}h, \quad h = X_{2534}X_{5423} + X_{2534}X_{5243} - X_{2543}X_{5234}.$$

Once again, h corresponds to a Plücker operation on degree two graphs, and so $h \in J_2$. But then we have shown that f is contained in the ideal generated by J_2.

In the previous section, we could not quite conclude that all binomials lying in J were contained in the ideal generated by the 25 binomial relations corresponding to rows of A'. Let's assume for now that all binomial equations lying in J are contained in the ideal generated by the 25 binomials given in A'. We have just proven that the only equation of these 25 in a degree larger than 2 is generated by equations in J_2, and thus under this assumption all binomial equations are generated by equations in J_2. To establish that J is generated by J_2, and hence that $\overline{M}_{0,6}$ is cut out by quadrics, we would only need to show that any polynomial in $k[X_\sigma : \sigma \in \mathcal{A}]$ that lies in $(I_6)^\kappa$ is contained in the ideal generated by J_2.

10 Towards projective normality when $n \geqslant 7$

Having covered the $n = 6$ case in detail over the previous two chapters, we start this chapter from the end of Chapter 7, working on the general $n \geqslant 7$ case. We will continue with our assumption that Conjecture 5.2.2 is true, appealing to our discussion following Proposition 6.2.2. Our first step would then be to prove the projective normality of the embedding of $\overline{M}_{0,n}$ associated to κ by proving Conjecture 6.2.3. We will attempt to generalise much of the method used in Chapter 7. Whilst some parts generalise very well, we will encounter problems that need solving as well.

Fix $n \geqslant 7$ and recall that R^κ is a graph algebra generated by all degree l-graphs (as described in Proposition 6.2.2) for all $l \geqslant 0$. We can define spread degree l graphs in the same way that we did for $n = 6$. Namely, we say that a degree l graph on n vertices is spread if its maximal pair multiplicity is at most l. To each spread degree l graph we can define its degree l co-graph in the same way as in Definition 7.2.6. If G is a spread degree l graph on n vertices, then using Proposition 6.2.2, its co-graph $\bar{G}$ is $2l$-regular and, for any $I \in \mathcal{I}$ with $|I| = r \geqslant 3$, the corresponding r-gon in $\bar{G}$ contains at most $l\binom{r}{2} - l\binom{r-1}{2} = l(r-1)$ edges. For $l = 1$, it is easy to check that this characterises labelled n-cycles.

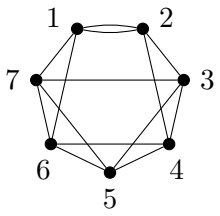
If G is a spread degree l graph on n vertices, then a Plücker operation between two disjoint pairs of vertices, ij and kl in G, is valid if and only if these two pairs do not have multiplicity l in $\bar{G}$ and any r-gon in $\bar{G}$, with $3 \leqslant r \leqslant n-3$, containing one of these pairs and no vertex from the other pair has less than $l(r-1)$ edges.

For $n = 6$, we used Lemma 7.2.1 to reduce the generating set of R^κ from all degree l graphs for all $l \geqslant 0$ to just those spread graphs. We generalise this idea for our larger n.

Idea 10.1. Let G be a degree l graph on n vertices for some $l \geqslant 1$. One can apply a sequence of Plücker operations to write G as a linear combination of degree l graphs,

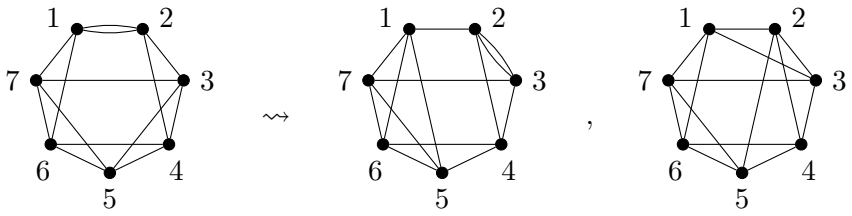
each of which contain no pair with multiplicity greater than l.

This looks to be the hardest part of our method for $n = 6$ to generalise. For $n = 5$ and $n = 6$ we showed that we can always apply a single Plücker operation to reduce the multiplicity of a maximal pair without creating a new one. This isn't always possible for larger n. Even when $n = 7$ and $l = 1$, the following degree one graph poses a problem:



The issue in this example is that there is only one vertex not adjacent to at least one of the vertices 1 or 2. Applying a Plücker operation between 12 and any other disjoint, non-empty pair will always create a new double edge.

This doesn't mean that this idea is not true. One may need to apply more than one Plücker operation to remove an edge of maximal multiplicity without having created a new one in any of the new graphs. In the $n = 7$, $l = 1$ example drawn above, we can first apply a Plücker operation between the pairs 12 and 35:



The second of the two new graphs obtained is spread. In the first of the two new graphs obtained, one can apply a Plücker operation between 23 and 56 to arrive at two spread

degree one graphs. This proves that our degree one graph on 7 vertices can be written as a signed sum of three spread degree one graphs.

As n increases, we would expect the number of Plücker operations needed to write a 'worst case' degree l graph as a signed sum of spread degree l graphs to grow without bound. It is this reason that we are some distance away from proving that Idea 10.1 holds. It is, however, worth continuing under the assumption that Idea 10.1 is true. The rest of the method used in Chapter 7 will translate remarkably well to general $n \geqslant 7$. This means that if one were to deal with Idea 10.1, we would be most of the way towards a proof of Conjecture 6.2.3.

Let us assume that Idea 10.1 is true, so we are reducing the generating set of R^κ to just those degree l graphs that are spread. As in the $n = 6$ case, we can split these graphs into two types - those that are complete and those that are not.

Idea 10.2. Let $l \geqslant 1$ and let G be a complete, spread degree l graph on n vertices. Then G can be written as the product of a spread degree one graph with a degree $l-1$ graph.

This is a generalisation of Lemma 7.2.4, which we proved by associating the triangle edge vector to any degree l graph on 6 vertices. For larger n, we still associate a vector to each degree l graph, indexed by those I in $\mathcal{I}$ with $|I| \geqslant 3$, recording the number of edges in each r-gon for each $3 \leqslant r \leqslant \frac{n}{2}$. We will still denote this vector, for a graph G, by $\mathrm{te}(G)$.

If G is a complete, spread degree l graph then let H be any spread degree one graph. We know that H is a subgraph of G. That H divides G in R^κ is equivalent to, for each $I \in \mathcal{I}$ with $|I| \geqslant 3$, the number $\mathrm{te}(G)_I - \mathrm{te}(H)_I$ being at least $(l-1) \cdot \binom{r-1}{2}$. Which vectors are equal to $\mathrm{te}(H)$ for some spread degree one graph H can be characterised using the fact that any such H is the co-graph of an n-cycle. For larger n we have to think about

many different r-gons, not just triangles. For this reason, Idea 10.2 is harder to prove than Lemma 7.2.4.

Idea 10.3. Let $l \geqslant 0$ and suppose that G is a non-complete, spread degree l graph. Then G is contained in the subalgebra of R^κ generated by degree one graphs.

This is a generalisation of Lemma 7.2.7 and a large amount of the induction step there is compatible with any n.

Suppose that G is a non-complete, spread degree l graph on n vertices. We can then assume that $n-1;n$ has multiplicity l in $\bar{G}$. By contracting this pair to a vertex v, as in §7.3, we get a graph, $\bar{G}^-$, on $n-1$ vertices. Note that $\bar{G}^-$ is $2l$-regular since $\bar{G}$ is.

Let I be an r-gon in $\bar{G}^-$ with $3 \leqslant r \leqslant \frac{n-1}{2}$. If $v \notin I$, then I is an r-gon in $\bar{G}$ and thus contains at most $l(r-1)$ edges. Of course, if $v \in I$ then we can just consider the complementary $(n-r)$-gon. This implies that $\bar{G}^-$ is a degree l co-graph on $n-1$ vertices.

By the inductive hypotheses, we can apply Plücker operations (in the co-graph form) to $\bar{G}^-$ to find it as a linear combination of degree l co-graphs that can each be written as a product of $(n-1)$-cycles. We next argue that a Plücker operation in $\bar{G}^-$ can be lifted to $\bar{G}$.

Suppose that we can apply a Plücker operation between two pairs in $\bar{G}^-$ that do not contain v. We can assume that these pairs are 12 and 34. Recall that this means that these pairs have multiplicity less than l in $\bar{G}^-$, and that there are less than $l(r-1)$ edges in any r-gon, with $3 \leqslant r \leqslant n-4$, containing 12 but not containing 3 or 4. It is easy to check that one can then apply a Plücker operation between 12 and 34 in $\bar{G}$.

The more interesting case is when we can apply a Plücker operation between two pairs of vertices in $\bar{G}^-$, one of which contains v. For this, suppose that we can apply a Plücker operation between 12 and $n-2;v$ in $\bar{G}^-$. Since $n-2;v$ has multiplicity less than l in $\bar{G}^-$, the pair $n-2;n-1$ has multiplicity less than l in $\bar{G}$.

Let I be an r-gon in $\bar{G}$, with $3 \leqslant r \leqslant n-3$, containing 12 but not containing $n-2$ or $n-1$. If $n \notin I$, then I is an r-gon in $\bar{G}^-$ and that we can apply a Plücker operation between 12 and $n-2; v$ in $\bar{G}^-$ means that I contains less than $l(r-1)$ edges. If $n \in I$, then consider the $(r-1)$-gon $I \backslash n$. This contains less than $l(r-2)$ edges since our operation is valid in $\bar{G}^-$. The number of edges from n to vertices in $I \backslash n$ is at most l, since there are l edges between $n-1$ and n in $\bar{G}$ and $\bar{G}$ is $2l$-regular. This implies there are less than $l(r-2)+l = l(r-1)$ edges in I. This means that we can apply a Plücker operation between the edges 12 and $n-2; n-1$ in $\bar{G}$, and this lifts the Plücker operation between 12 and $n-2; v$ in $\bar{G}^-$.

We have proven that any Plücker operation in $\bar{G}^-$ lifts to a Plücker operation in $\bar{G}$. This allows us to sidestep any exceptional cases for the $n-1$ case, like the one we encountered for $n = 6$ in §7.3. By induction, we can apply Plücker operations to write $\bar{G}^-$ as a linear combination of degree l co-graphs on $n-1$ vertices which are each a product of $(n-1)$-cycles. Lifting these Plücker operations to $\bar{G}$, we can assume that $\bar{G}^-$ can be written as a product of $(n-1)$-cycles.

We can now follow the same idea of §7.3. If $\bar{G}^- = C_1 \dots C_l$ is a product of $(n-1)$-cycles, we take out as many n-cycles, obtained by expanding these $(n-1)$-cycles in v, as possible. We are left with $(n-1)$-cycles that expand to subgraphs of $\bar{G}$ that are $2l$-regular everywhere except in the vertices $n-1$ and n which have degrees one and three in some order. These two possibilities occur in equal frequency, and we can pair them off.

Now we just need to check that all possible products of these pairs correspond to a product of two degree one graphs. In the $n = 6$ case, there were two possibilities up to symmetry, one of which could be written as a product of two 6-cycles, and one exceptional case. The number of exceptional cases grows with n, though if one can cover these exceptional cases then Idea 10.3 holds for n if Conjecture 6.2.3 holds for $n-1$.

11 Final Words

In this thesis we have described an embedding of $\overline{M}_{0,6}$ into projective space and shown that this embedding is projectively normal. We did this by developing a graph theoretic formulation inspired by [HMSV09]. We built upon our method further to give a combinatorial description of some equations contained in the homogeneous ideal defined by this embedding. Whilst we could not quite arrive at a combinatorial description of quadrics defining $\overline{M}_{0,6}$ as a projective subscheme, we are able to discuss the potential our method has.

For general $n \geqslant 7$, we made good progress in finding an embedding of $\overline{M}_{0,n}$ into projective space, generalising the method used for $n = 6$. One must still prove Conjecture 5.2.2, but the discussion following Proposition 6.2.2 gives us hope that this is not too difficult, particularly for other small n. Due to the obstacles we encountered in trying to prove Idea 10.1 for larger n, as well as the increasing number of exceptional cases one will encounter in proving Idea 10.3, it is unlikely that one will be able to give a complete description of equations defining $\overline{M}_{0,n}$ using just our combinatorial method except just for smaller n. Despite this, we have made enough progress to compare our findings to results in the literature.

Fix $n \geqslant 5$. In [KT09], it is proven that the very ample class κ embeds $\overline{M}_{0,n}$ into $\mathbb{P}^1 \times \ldots \times \mathbb{P}^{n-3}$. By the Segre embedding, $\mathbb{P}^1 \times \ldots \times \mathbb{P}^{n-3}$ lies in $\mathbb{P}^{(n-2)!-1}$ and so $\overline{M}_{0,n}$ is a closed subscheme of $\mathbb{P}^{(n-2)!-1}$. This paper also proves that this embedding is projectively normal and cut out by quadric equations. This means that $\bigoplus_{l \geqslant 0} H^0(\overline{M}_{0,n}, \mathcal{O}_{\overline{M}_{0,n}}(l\kappa))$ is generated in degree one by a set of 24 generators. Note that this is consistent with our findings for $n = 6$ since $(6-2)! - 1 = 23$ was the dimension of the projective space we embedded $\overline{M}_{0,6}$ into in Chapter 9.

We can compare this with the ring R^κ that we conjectured to be a homogeneous coor-

dinate ring for $\overline{M}_{0,n}$. For $n = 6$ we were able to show that R^κ can be viewed as a graph algebra generated in degree one by the 24 labelled 6-cycles containing the edge 16. In Chapter 10 we argued that, in general, labelled n-cycles are degree one graphs in R^κ. If we were to prove the ideas of Chapter 10 then we would have that the set of labelled n-cycles gives rise to a set of degree one generators of the coordinate ring. If one could mirror the $n = 6$ case and show that just those labelled n-cycles containing the edge $1n$ generate R^κ then this would be a set of $(n-2)!$ generators. As we have remarked above, this would be consistent with the dimension of the embedding space in [KT09]. This suggests that our embedding could be the embedding of $\overline{M}_{0,n}$ corresponding to the complete linear system of κ, which would answer Question 7.4(3) in [GM11] for those divisors equal to a positive multiple of κ.

Let us assume that R^κ is generated by the $(n-2)!$ labelled n-cycles containing the edge $1n$ (this is a large assumption based just on our work and the consistency with results in the literature). Let $\mathcal{A}$ denote the set of labelled n-cycles containing the edge $1n$, which can be identified with permutations of $\{2, \ldots, n-1\}$. Under our assumptions we have that $\overline{M}_{0,n}$, embedded in $\mathbb{P}^{(n-2)!-1}$, has homogeneous coordinate ring $k[X_\sigma : \sigma \in \mathcal{A}]/J$, where J is a homogeneous ideal in $k[X_\sigma : \sigma \in \mathcal{A}]$.

There are two types of equations generating the ideal J. The first of these are binomial relations arising from equal graph products. The second of these are equations in $k[X_\sigma : \sigma \in \mathcal{A}]$ that lie in $I_{\overline{M}_{0,n}}$ when taken in S. Applying Theorem 1.2 in [KT09], J is generated by its degree two part, thus we can restrict to equations in degree two. We give this discussion as a conjecture:

Conjecture 11.1. Let $\mathbb{P}^{(n-2)!-1}$ have homogeneous coordinates indexed by the set $\mathcal{A}$ of labelled n-cycles containing the edge $1n$. Let E be the set of degree two binomials in $k[X_\sigma : \sigma \in \mathcal{A}]$ corresponding to pairs of graphs in $\mathcal{A}$ with the same graph product. Let F be the set of degree two equations in $k[X_\sigma : \sigma \in \mathcal{A}]$ that lie in $(I_{\overline{M}_{0,n}})_{2\kappa}$ when

considered as their respective equations in S. Then $E \cup F$ is a finite set of quadrics defining $\overline{M}_{0,n}$ as a closed subscheme of $\mathbb{P}^{(n-2)!-1}$.

Finding equations defining $\overline{M}_{0,n}$ is still an active area of research. In [MR17] the authors follow on from the methods used in [KT09] to give their own conjecture on equations defining $\overline{M}_{0,n}$ and verify this conjecture for smaller n using Macaulay2. As recently as September 2022, [GGL22] was submitted to arXiv claiming to prove this conjecture. It would be interesting to investigate how our equations we have found for $n = 6$, and conjectured above in the general case, compare to the equations given in these papers.

Although our graphical method has allowed us to suggest a good candidate for quadric equations defining $\overline{M}_{0,n}$, it is unlikely that one will be able to prove this conjecture for all $n \geqslant 5$ just using the combinatorial methods of this thesis. One will probably need to combine ideas in this thesis with other geometric methods to obtain a proof.

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