

Divergences in Kaluza–Klein models and their string regularization

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Abstract. Effective field theories with (large) extra dimensions are studied within a physical regularization scheme provided by string theory. Explicit string calculations then allow us to consistently analyse the ultraviolet sensitivity of Kaluza–Klein theories in the presence or absence of low-energy supersymmetry.

Contents

1	Introduction	2
2	Field theory calculation of the scalar potential	3
2.1	The scalar potential	3
2.1.1	The limits $l \gg (\Lambda\mathcal{R})/2$ and $l \rightarrow \infty$ (‘Kaluza–Klein regularization’) . .	6
2.1.2	The minimum condition	9
2.2	The scalar potential: further analysis	9
2.2.1	Case 1	10
2.2.2	Case 2	12
2.3	Remarks on perturbative expansion and ‘Kaluza–Klein regularization’	14
2.4	The need for a string regularization scheme	17
2.5	String regularization of the vacuum energy	18

3	String calculation of the vacuum energy	21
3.1	Cosmological constant in models with $N = 4 \rightarrow N = 0$ breaking	21
3.1.1	Harvey model in five dimensions	21
3.1.2	Harvey model in four dimensions	23
3.2	Cosmological constant in models with $N = 1 \rightarrow N = 0$ breaking	24
3.2.1	Cosmological constant for Scherk–Schwarz $N = 4 \rightarrow N = 0$ breaking	25
3.2.2	Cosmological constant for Scherk–Schwarz $N = 2 \rightarrow N = 0$ breaking	26
3.2.3	Validity of perturbative string calculation and the Hagedorn transition .	28
3.3	Cosmological constant: heterotic versus type I	29
3.4	Wilson line dependence of the cosmological constant	31
3.5	One-loop corrections in heterotic Scherk–Schwarz models	32
3.5.1	One-loop gauge couplings in theories with broken supersymmetry . . .	34
3.5.2	One-loop Yukawa couplings in theories with broken supersymmetry . .	36
4	Conclusions	37
	Appendix A Kaluza–Klein sums	38
	Appendix B Asymptotic expansions for the potential	40
	Appendix C Kaluza–Klein integrals	41
	Appendix D Lattice functions	43
	Appendix E Modular functions for gauge thresholds	44
	Appendix F Partition functions for Wilson line dependence	45

1. Introduction

The main motivation to consider generalizations of the standard model (SM) of strong and electroweak interactions is the hierarchy problem: the instability of the weak scale $M_w \approx G_F^{-1/2} \approx 300$ GeV in the presence of higher scales, like $M_{Planck} \approx 10^{19}$ GeV connected to the existence of gravitational interactions. The hierarchy problem is linked to a power-law sensitivity of the low-energy effective field theory (EFT) to the ultraviolet (UV) energy region. In the SM such a sensitivity (for field-dependent quantities³) manifests itself in the quadratic divergence of the (mass)² of the scalar Higgs particle.

Supersymmetry has been suggested as a possible solution to this problem. In the supersymmetric extension of the SM with softly broken supersymmetry, no power-like divergences appear (with the exception of the cosmological constant) to all orders in perturbation theory. The Higgs (mass)² still shows a logarithmic sensitivity to the UV scale, but this is consistent with a stabilization of the weak scale.

More recently, with the thorough discussion of EFTs with (large) extra dimensions the hierarchy problem reappears again and it has been claimed in the literature that in specific

³ In addition there is a quartic divergence of the vacuum energy (cosmological constant).

cases the weak scale does not show a power-like UV behaviour even in the absence of low-energy supersymmetry. These claims are very surprising since the EFTs considered are non-renormalizable and therefore contain many more sources of power-like divergences. While in the SM (as a renormalizable $D = 4$ field theory) the $(\text{mass})^2$ of a scalar particle is the only source of a quadratic divergence (for field-dependent quantities), in higher-dimensional theories we might find power-law sensitivity for gauge couplings, Yukawa couplings as well as new higher-dimensional operators. It is very difficult to analyse these questions within a naive low-energy EFT approach (well suited for renormalizable theories). This is especially true if the discussion relies exclusively on the consideration of one-loop calculations since the (infinitely many) new counterterms might manifest themselves only in higher-loop contributions to the effective Higgs mass as a power-like divergence. A careful examination of this question is thus needed in the framework of these higher-dimensional theories, especially since the finiteness of one-loop results might be obtained by a specific choice of the regularization procedure.

In the present paper we provide a physical regularization scheme for such theories via the embedding in string theory. The motivation is twofold. First of all we believe that a meaningful description of these extra-dimensional (non-renormalizable) theories requires a more fundamental theory at the high scale (for which string theory is a candidate). Secondly, in a consistent (finite) string theory, perturbative calculations give a finite result and we do not have to deal with the often arguable choice of a field theory regularization scheme. String theory provides us with a field theory UV cut-off, the string tension $M_{\text{string}} = \alpha'^{-1/2}$ with α' the slope parameter. The UV sensitivity of a physical parameter (like the Higgs mass) can be read off from the dependence on M_{string} , taking into account the appropriate sensitivity of all quantities appearing in that expression (e.g. a power-like sensitivity of the coupling constants). Such a discussion is conceptually simpler and more powerful than a naive field theory consideration, as it is independent of specifically chosen regularization schemes. Of course, the low-energy field theory description can be obtained in the limit $M_{\text{string}} \rightarrow \infty$ ($\alpha' \rightarrow 0$, zero-slope limit) where the power-like sensitivity with respect to M_{string} manifests itself in a UV, power-like divergence.

The first part of the paper addresses the scalar potential in EFTs with an additional compact dimension and the need for a string regularization scheme. This is provided in section 2.5, where the link with string theory is made explicit. The second part of the paper provides details of the calculation of the vacuum energy, one-loop gauge and Yukawa couplings in a significant class of string models.

2. Field theory calculation of the scalar potential

2.1. The scalar potential

In this section we investigate the scalar potential in a class of models with Kaluza–Klein (KK) towers of states associated with one additional spatial dimension (for early research see e.g. [1]). This class of models [2]–[6] attracted renewed interest recently due to their very interesting phenomenological consequences. The introductory part of this analysis was already published in [7] and we only review it for continuity of our presentation.

The aforementioned models usually consider as a starting point a higher-dimensional theory (e.g. 5D) with Scherk–Schwarz [8] and/or orbifold mechanism for supersymmetry breaking. The presence of an additional (compact) dimension induces in the 4D (boundary) effective theory a tower of KK modes which, depending on the particular model, can be associated with various

states of the low-energy spectrum. These modes fall into $N = 2$ multiplets, as a consequence of the enhanced supersymmetry in the 5D ‘bulk’ being broken to $N = 0$ on the 4D boundary by an orbifold-like compactification (for example $S_1/(Z_2 \times Z'_2)$ [5]). Yukawa interactions are localized on the 4D ‘boundary’ and induce (together with the gauge part) corrections to the scalar potential of these models. These will be addressed in the following. Gauge corrections will be considered elsewhere [9]. For a detailed description of generic models, see for example [3, 5].

To begin with, it is instructive for our purposes to explicitly present the contribution of one KK bosonic or fermionic mode of mass m_k to the scalar potential, computed in a one-loop expansion of the latter and in all orders in perturbation theory. We denote this contribution as \mathcal{V}_k^o and we use a cut-off regularization method, with Λ the momentum cut-off of the EFT. For a review and applications of this regularization method to the scalar potential, see [10]. Every regularization scheme has its own shortcomings, and our choice is motivated on physical grounds. Unlike other regularization methods, it is appropriate for displaying some phenomenological implications of what is called [4] ‘KK regularization’ which cannot be ‘seen’ in dimensional regularization for example. Other, more elegant regularization schemes are possible, but the present choice is more suitable for understanding the need for and link with a full string calculation/regularization (see later). Thus Λ is the scale where new physics (string physics) comes in. We have

$$\begin{aligned}\mathcal{V}_k^o &= \frac{1}{(2\pi)^4} \int d^4p \ln(1 + m_k^2/p^2) \\ &= \frac{2\pi^2}{(2\pi)^4} \left\{ \frac{1}{4} \left[m_k^2 \Lambda^2 - m_k^4 \ln \left(1 + \frac{\Lambda^2}{m_k^2} \right) + \Lambda^4 \ln \left(1 + \frac{m_k^2}{\Lambda^2} \right) \right] \right\}.\end{aligned}\quad (2.1)$$

As is well known, the contribution to the scalar potential of one (KK) state contains quadratic and logarithmic terms, the first and second term in (2.1).

We consider now the overall contribution to the one-loop expansion of the scalar potential of towers of KK modes which, following [5], are associated with the top quark due to its larger Yukawa coupling. The scalar potential for the Higgs field ϕ has the following generic structure after taking into account individual bosonic and fermionic KK contributions (2.1):

$$\mathcal{V}(\phi) = \frac{1}{2} \text{Tr} \sum_{k=-l}^l \int \frac{d^4p}{(2\pi)^4} \ln \frac{p^2 + m_{B_k}^2(\phi)}{p^2 + m_{F_k}^2(\phi)} \quad (2.2)$$

with the trace taken over the top hypermultiplet of fixed k -level contributing a factor of $(4N_c)$ where N_c is the number of colours.

In this analysis we will sum over the whole KK tower of states, $l \rightarrow \infty$. For this purpose we first consider the case of an arbitrarily fixed number (l) of KK modes that we sum over and arbitrarily fixed momentum cut-off Λ . Thus there are two cut-offs corresponding to the compact and non-compact sectors. We then take the limit $l \gg (\Lambda\mathcal{R})/2$. Further, we consider the limit $l \rightarrow \infty$ and then $\Lambda \rightarrow \infty$ to recover the case of ‘KK regularization’. For a more suitable regularization for models with compact and non-compact sectors, see [11].

The field-dependent boson and fermion masses in (2.2) are usually given by

$$m_{F_k}(\phi) = \frac{2k}{\mathcal{R}} + m_t(\phi) = \frac{2}{\mathcal{R}}(k + \omega), \quad \omega = \frac{m_t(\phi)\mathcal{R}}{2} \quad (2.3)$$

and

$$m_{B_k}(\phi) = \frac{2k+1}{\mathcal{R}} + m_t(\phi) = \frac{2}{\mathcal{R}}(k + \omega'), \quad \omega' = \omega + \frac{1}{2} \quad (2.4)$$

where we follow [5] and consider any integer values of k in the range $-\infty < k < \infty$. These mass formulae are generic for the states of the KK tower and depend (via ω and ω') on the boundary conditions that one chooses for fermions/bosons. Other cases for the mass assignment exist in the literature [2, 3], which we review in section 2.2.2. There, the mass (squared) of KK states is in addition shifted by an equal amount for both bosons and fermions.

The mass splitting between fermions and bosons, following the breaking of supersymmetry via an orbifold or Scherk–Schwarz mechanism, is then encoded in the difference $\omega' - \omega$ which for this case is positive, equation (2.4). The sign of the scalar potential and also of its second derivative (at $\phi = 0$) will then depend on the sign of this difference (see equation (2.2)), with implications for the existence of the (radiatively induced) electroweak symmetry breaking mechanism. The existence of this mechanism is then traced back to the type of boundary conditions that one chooses for the fermions and for the bosons respectively.

From equations (2.2)–(2.4) we find

$$\mathcal{V}(\phi) = \frac{\eta}{\mathcal{R}^4} \sum_{k=-l}^l \int_0^{\bar{\Lambda}} d\rho \, 2\rho^3 \ln \frac{\rho^2 + \pi^2(k + \omega')^2}{\rho^2 + \pi^2(k + \omega)^2}, \quad \bar{\Lambda} = \frac{\pi \mathcal{R} \Lambda}{2}, \quad \eta = \frac{4N_c}{2\pi^6} \quad (2.5)$$

where Λ is the momentum cut-off of the loop integral in (2.2) and l the number of states in the tower. Intuitively, in an EFT (of fixed cut-off Λ) the number of KK states should be restricted (unless a symmetry prevents us from doing this—we return to this point in section 2.5) to those states whose mass is smaller than the momentum cut-off of the loop integral, a condition which would lead to

$$m_{B_k, F_k} \approx \frac{2k}{\mathcal{R}} \leq \Lambda \quad \Rightarrow \quad \bar{\Lambda} \approx \pi l. \quad (2.6)$$

Here l stands for the KK state of the largest mass. Performing the sum over the tower of KK states under the momentum integral equations (2.2), (2.5) gives (see appendix A)

$$\sum_{k=-l}^l \ln \frac{\rho^2 + \pi^2(k + \omega')^2}{\rho^2 + \pi^2(k + \omega)^2} = \mathcal{Z}_0 + \mathcal{Z}_1 + \mathcal{Z}_2 \quad (2.7)$$

with the notation

$$\mathcal{Z}_0 = \ln \frac{\cosh(2\rho) - \cos(2\pi\omega')}{\cosh(2\rho) - \cos(2\pi\omega)} \quad (2.8)$$

and

$$\mathcal{Z}_1 = \ln \frac{[\rho^2 + \pi^2(l \pm \omega')^2]_*}{[\rho^2 + \pi^2(l \pm \omega)^2]_*} \quad (2.9)$$

and finally

$$\mathcal{Z}_2 = \ln \frac{[\Gamma(l \pm \omega' \pm i\rho/\pi)]_*}{[\Gamma(l \pm \omega \pm i\rho/\pi)]_*}. \quad (2.10)$$

The symbol $[\rho^2 + \pi^2(l \pm \omega)^2]_*$ stands for a product of the quantity within the brackets with all possible combinations of plus and minus signs. Accordingly, the potential can be written (with the ϕ -dependence hidden in ω and ω') as

$$\mathcal{V}(\phi) = \mathcal{V}_0(\phi) + \mathcal{V}_1(\phi) + \mathcal{V}_2(\phi) \quad (2.11)$$

where we have

$$\mathcal{V}_0(\phi) = \frac{\eta}{\mathcal{R}^4} \int_0^{\bar{\Lambda}} d\rho \, 2\rho^3 \mathcal{Z}_0 = \frac{\eta}{\mathcal{R}^4} \int_0^{\bar{\Lambda}} d\rho \, 2\rho^3 \ln \frac{\cosh(2\rho) - \cos(2\pi\omega')}{\cosh(2\rho) - \cos(2\pi\omega)} \quad (2.12)$$

and

$$\mathcal{V}_1(\phi) = \frac{\eta}{\mathcal{R}^4} \int_0^{\bar{\Lambda}} d\rho \, 2\rho^3 \mathcal{Z}_1 = \frac{\eta}{\mathcal{R}^4} \int_0^{\bar{\Lambda}} d\rho \, 2\rho^3 \ln \frac{\rho^2 + \pi^2(l + \omega')^2}{\rho^2 + \pi^2(l + \omega)^2} + (l \rightarrow -l) \quad (2.13)$$

and finally

$$\mathcal{V}_2(\phi) = \frac{\eta}{\mathcal{R}^4} \int_0^{\bar{\Lambda}} d\rho \, 2\rho^3 \mathcal{Z}_2 = \frac{\eta}{\mathcal{R}^4} \int_0^{\bar{\Lambda}} d\rho \, 2\rho^3 \ln \frac{[\Gamma(l \pm \omega' \pm i\rho/\pi)]_*}{[\Gamma(l \pm \omega \pm i\rho/\pi)]_*}. \quad (2.14)$$

For later reference, it is useful to recall that the numerators (ω' -dependent) of the integrands of \mathcal{V}_0 , \mathcal{V}_1 , \mathcal{V}_2 correspond to bosonic degrees of freedom, while the denominators correspond to the contribution (ω -dependent) of the fermions.

In the following we analyse each of the three contributions to the scalar potential \mathcal{V} . \mathcal{V}_0 is part of the scalar potential corresponding to the result of [5] which is finite and UV insensitive, as we discuss below. The expression for \mathcal{V}_0 simply corresponds to summing over an infinite number of KK states in equation (2.2) and is therefore independent of the KK level. It is considered [2]–[5] that this is the KK regularized part of the scalar potential.

The contributions \mathcal{V}_1 and \mathcal{V}_2 to the scalar potential are each vanishing in the limit of an infinite number of KK states. Indeed

$$\mathcal{V}_1(l \rightarrow \infty) = 0 \quad (2.15)$$

and

$$\mathcal{V}_2(l \rightarrow \infty) = 0 \quad (2.16)$$

while $\bar{\Lambda}$ is kept fixed, as a consequence of the vanishing of the integrands of \mathcal{V}_1 and \mathcal{V}_2 respectively. In this limit only \mathcal{V}_0 will contribute to the scalar potential:

$$\mathcal{V}(l \rightarrow \infty) = \mathcal{V}_0 \quad (2.17)$$

to give the result of [5]. We emphasize that it is instructive for our purposes to explicitly compute \mathcal{V}_1 and \mathcal{V}_2 in the general case of finite l and then take the limit $l \rightarrow \infty$ ($\bar{\Lambda}$ fixed) to understand its physical implications. This procedure/limit will clarify the role of supersymmetry in the cancellation of individual quadratic and logarithmic terms of the potential. Note that this limit in equation (2.1) means that its first two terms (quadratic and logarithmic) are not manifest in the final result obtained for \mathcal{V} after summing infinitely many contributions to the scalar potential⁴. For clarity, it is perhaps worth anticipating some of our conclusions, that in the limits of either $l \gg (\Lambda\mathcal{R})/2$ or of summing the whole KK tower the quadratic and logarithmic divergences in \mathcal{V} are absent in the bosonic sector alone and the same mechanism applies for the fermionic sector, *without* the need for supersymmetry.

2.1.1. The limits $l \gg (\Lambda\mathcal{R})/2$ and $l \rightarrow \infty$ ('Kaluza–Klein regularization'). After performing the integral for \mathcal{V}_0 , one finds the following result ($\gamma(\alpha, x)$ is the incomplete Gamma function [12]).

$$\mathcal{V}_0(\phi) = -\frac{N_c}{2\pi^6\mathcal{R}^4} \left\{ \frac{1}{2} \sum_{n=1}^{\infty} \frac{e^{2i\pi n\omega'}}{n^5} \gamma(4, 2n\bar{\Lambda}) + (\omega' \rightarrow -\omega') \right\} - (\omega' \rightarrow \omega). \quad (2.18)$$

⁴ One can see that the limit of very large m_k (large KK level), $m_k \gg \Lambda$, in equation (2.1) means that the first and second terms cancel in the fermionic or bosonic part only, to leave Λ^4 terms (to be cancelled due to equal bosonic/fermionic degrees of freedom). We return to this observation later in the text.

For $\omega' = \omega + 1/2$ (equation (2.4)) the above relation reduces to

$$\mathcal{V}_0(\phi) = \frac{3\eta}{\mathcal{R}^4} \sum_{k=0}^{\infty} \frac{\cos(2\pi\omega(2k+1))}{(2k+1)^5} \left\{ 1 - e^{-2\bar{\Lambda}(2k+1)} \sum_{m=0}^3 \frac{(2\bar{\Lambda}(2k+1))^m}{m!} \right\} \quad (2.19)$$

where the factor within curly braces is equal to unity in the limit of large $\bar{\Lambda}$. (If we set $\omega = 0$, we find that the potential is equal to

$$\mathcal{V}_0(\omega = 0) = \frac{93(4N_c)}{64\pi^6\mathcal{R}^4} \zeta(5) \quad (2.20)$$

and is (up to trace factor of $(4N_c)$) the result of [13] in our notation/assignment conventions for the KK tower.) The sign of the potential (fixed by the choice of boundary conditions for the bosons/fermions) is appropriate for triggering the electroweak symmetry breaking. Further analysis of this potential and phenomenological implications have already been discussed [5].

The result of integrating $\mathcal{V}_1(\phi)$ of equation (2.13) is given, irrespective of the dependence $m_t(\phi)$, by the following expression:

$$\mathcal{V}_1 = \bar{\mathcal{V}}_1(\omega') - \bar{\mathcal{V}}_1(\omega). \quad (2.21)$$

With the exception of some Λ^4 terms cancelled in equation (2.21) between bosonic and fermionic contributions (due to the equal numbers of bosonic/fermionic degrees of freedom), $\bar{\mathcal{V}}_1(\omega')$ represents the sole contribution of the bosonic (ω' -dependent) sector, while $\bar{\mathcal{V}}_1(\omega)$ is that of the fermionic sector. We have

$$\begin{aligned} \bar{\mathcal{V}}_1(\omega') = \frac{\eta}{2\mathcal{R}^4} \left\{ \pi^2(l + \omega')^2 \bar{\Lambda}^2 - \pi^4(l + \omega')^4 \ln \left[1 + \frac{\bar{\Lambda}^2}{\pi^2(l + \omega')^2} \right] \right. \\ \left. + \bar{\Lambda}^4 \ln(\pi^2(l + \omega')^2 + \bar{\Lambda}^2) \right\} + (l \rightarrow -l). \end{aligned} \quad (2.22)$$

As expected, both $\bar{\mathcal{V}}_1(\omega')$ and $\bar{\mathcal{V}}_1(\omega)$ have quadratic and logarithmic divergences and these are not cancelled between bosons and fermions for any finite summation over the KK tower of states. The absence of such UV terms in the final result for \mathcal{V} in the limit of summing over an infinite tower of states is sometimes considered as due to the ‘soft nature’ of breaking supersymmetry by a Scherk–Schwarz mechanism. This would apparently ensure a cancellation between bosonic and fermionic contributions not only at the level of quartic divergences (Λ^4), but also at the level of quadratic (Λ^2) and logarithmic ($\log \Lambda$) terms. This is not necessarily true, for reasons that we discuss below.

We consider in the following the limit of large l and fixed Λ , $\pi l \gg \bar{\Lambda} \equiv \pi\Lambda\mathcal{R}/2$. This seems a strange limit to take in a 4D EFT, considering KK modes of mass $l/\mathcal{R} \gg \Lambda$. In this case the logarithm in the second term of the bosonic term $\bar{\mathcal{V}}_1(\omega')$ can be expanded in a (rapidly convergent) power series, with the first term in the expansion cancelling the quadratic divergence $\bar{\Lambda}^2$ of the first term in $\bar{\mathcal{V}}_1$; more explicitly, the second term in (2.22) denoted as $A(l, \omega')$ is given by

$$A(l, \omega') \approx \frac{1}{2} \frac{\eta}{\mathcal{R}^4} \left\{ -\pi^2(l + \omega')^2 \bar{\Lambda}^2 + \frac{1}{2} \bar{\Lambda}^4 - \frac{1}{3} \frac{\bar{\Lambda}^6}{\pi^2(l + \omega')^2} + \dots \right\} \quad (2.23)$$

and its quadratic term cancels the first term in $\bar{\mathcal{V}}_1(\omega')$ (the Λ^4 term in the expansion (2.23) is cancelled in \mathcal{V}_1 upon including both bosonic and fermionic contributions). This cancellation takes place separately for the bosons and for the fermions. The last term (and higher ones)

in (2.23) is suppressed when $l \gg (\Lambda\mathcal{R})/2$. The interpretation of this observation is that states of mass larger than the momentum cut-off scale Λ of the loop integral ($2l/\mathcal{R} \gg \Lambda$) are required to cancel individual quadratic divergences of KK states. This situation has an analogue in equation (2.1) where we would take the limit $m_k \gg \Lambda$, with m_k the mass of a KK bosonic or fermionic state. There the quadratic term in (2.1) would be cancelled by the first term in the (convergent) expansion of the logarithmic contribution.

Summing over an infinite tower of KK states corresponds to the limit $l \rightarrow \infty$, in which case one interchanges the infinite KK sum and the momentum integral. While this is possible (we discuss this later in the text), it includes modes of mass larger than the momentum cut-off of the loop integral, also intended to be the cut-off of our EFT. When $l \rightarrow \infty$, the quadratic and logarithmic dependence in the bosonic part $\bar{\mathcal{V}}_1(\omega')$ of (2.22) disappears (higher-order terms in (2.23) vanish) and a similar mechanism separately applies for the fermionic part. Indeed,

$$\lim_{l \rightarrow \infty} \frac{1}{2} \frac{\eta}{\mathcal{R}^4} \left\{ \pi^2 (l + \omega')^2 \bar{\Lambda}^2 - \pi^4 (l + \omega')^4 \ln \left[1 + \frac{\bar{\Lambda}^2}{\pi^2 (l + \omega')^2} \right] + (l \rightarrow -l) \right\} = \frac{1}{2} \frac{\eta}{\mathcal{R}^4} \bar{\Lambda}^4 \quad (2.24)$$

where the rhs $\bar{\Lambda}^4$ is cancelled between bosons and fermions due to their equal numbers, ensured by the initial (now broken) supersymmetry.

The expression for \mathcal{V}_2 , equation (2.14), can be integrated in the approximation of large modulus of $l + i\bar{\Lambda}$ which does not restrict the relative values of l , $\bar{\Lambda}$, and will thus not affect the result that we obtain for \mathcal{V}_2 in the limit $l \gg \bar{\Lambda}$. The approximation that we use for the integrand of \mathcal{V}_2 is detailed in appendix B. After some algebra, the result that we obtain for \mathcal{V}_2 , equation (2.14), may be written as

$$\begin{aligned} \mathcal{V}_2 \approx & \frac{\eta}{\mathcal{R}^4} \left\{ \pi^2 g(l, \omega') (l + \omega') \left[\bar{\Lambda}^2 - \pi^2 (l + \omega')^2 \ln \left(1 + \frac{\bar{\Lambda}^2}{\pi^2 (l + \omega')^2} \right) \right] + (\omega' \rightarrow -\omega') \right\} \\ & + \frac{\eta}{\mathcal{R}^4} \frac{\omega' \bar{\Lambda}^4}{2} \left\{ \ln \frac{\bar{\Lambda}^2 + \pi^2 (l + \omega')^2}{\bar{\Lambda}^2 + \pi^2 (l - \omega')^2} \right\} \\ & - \frac{\eta}{\mathcal{R}^4} \frac{4\bar{\Lambda}^5}{5\pi} \left\{ \arctan \frac{\bar{\Lambda}}{\pi(l - \omega')} + \arctan \frac{\bar{\Lambda}}{\pi(l + \omega')} \right\} \\ & - (\omega' \rightarrow \omega) + \frac{\eta}{\mathcal{R}^4} \frac{\bar{\Lambda}^4}{2} \left(l - \frac{1}{2} \right) \ln \left\{ \frac{[\bar{\Lambda}^2 + \pi^2 (l \pm \omega')^2]_*}{[\bar{\Lambda}^2 + \pi^2 (l \pm \omega)^2]_*} \right\} \end{aligned} \quad (2.25)$$

where the substitution ($\omega' \rightarrow \omega$) only applies to terms in front of it, to give the (separate, ω -dependent) fermionic contribution. The last term in (2.25) contains both the fermionic and the bosonic contributions. Also

$$g(l, \omega') = \frac{1}{60} \{ 10 - 15l + 6l^2 + 3(-5\omega' + 4l\omega' + 2\omega'^2) \}. \quad (2.26)$$

If we sum over a finite tower of KK states (finite l), we find quadratic and logarithmic contributions not cancelled in the sum $\mathcal{V}_1 + \mathcal{V}_2$. If in \mathcal{V}_2 we consider the limit $l \gg \bar{\Lambda}$ or $l \rightarrow \infty$, the logarithmic term in the first curly braces of (2.25) may be expanded in a power series and its first term in this expansion cancels the quadratic terms $\bar{\Lambda}^2$ due to states up to level l , similarly to the case for \mathcal{V}_1 .

To conclude, just as in the case of \mathcal{V}_1 , quadratic or logarithmic divergences are absent in the bosonic sector and the same applies to the fermionic sector in the limits of either $l \gg (\Lambda\mathcal{R})/2$ or of summing the whole KK tower. In this mechanism, the presence of KK modes of mass larger than the momentum cut-off of the loop integral was essential. The presence or need for

KK states beyond the momentum cut-off of the *effective* theory therefore requires a full string regularization/calculation of this model, where the inclusion of such states can be addressed consistently. A detailed string regularization/calculation will be provided in sections 2.5 and 3.

2.1.2. The minimum condition. We impose the minimum condition for the scalar potential with fixed number of KK states (represented by l), to find the compactification radius in terms of its value R_* derived from the minimum condition for \mathcal{V}_0 which is the limiting case when $l \rightarrow \infty$. The condition for the minimum of the potential \mathcal{V} leads to

$$\mathcal{R}^4 \approx R_*^4 \frac{\sin(\pi m_t(\phi)\mathcal{R})}{\sin(\pi m_t(\phi)R_*)} \left[1 - \frac{1}{6\pi} \frac{\mathcal{R}^4}{\eta} \frac{\partial(\mathcal{V}_1 + \mathcal{V}_2)}{\partial\omega} \right] \quad (2.27)$$

where $R_* \approx 1/(2m_t)(\sin(\pi m_t R_*) \approx 1)$ and where an explicit dependence of the top mass on the v.e.v. of Higgs is used (this dependence is that of [5] appropriate for our analysis, with $m_t(\phi) = 2/\pi\mathcal{R} \arctan(\pi\mathcal{R}\phi y_t/2)$, but other cases may be considered too). Tedious calculations give the following approximation for the derivative in equation (2.27)

$$\begin{aligned} \frac{\mathcal{R}^4}{\eta} \frac{\partial(\mathcal{V}_1 + \mathcal{V}_2)}{\partial\omega} \approx \frac{\pi^2}{2} \left\{ 4\bar{\Lambda}^2 \omega'(l+1) - \left[\pi^2(l+\omega')^2(l+\omega'+1)^2 \ln\left(1 + \frac{\bar{\Lambda}^2}{\pi^2(l+\omega')^2}\right) \right. \right. \\ \left. \left. - (\omega' \rightarrow -\omega') \right] \right\} + \frac{\bar{\Lambda}^4}{2} \ln \frac{\bar{\Lambda}^2 + \pi^2(l+\omega')^2}{\bar{\Lambda}^2 + \pi^2(l-\omega')^2} - (\omega' \rightarrow \omega). \end{aligned} \quad (2.28)$$

The last substitution ($\omega' \rightarrow \omega$) applies to all terms in front of it, including those within curly braces. In the limit $l \rightarrow \infty$ (fixed $\bar{\Lambda}$) the derivative of $\mathcal{V}_1 + \mathcal{V}_2$ vanishes (separately for ω - and ω' -dependent terms respectively) and we recover the case $\mathcal{R} = R_*$. We now assume there is a correlation between the momentum cut-off of the loop integrals (2.12)–(2.14) and the number of KK states in the tower. Therefore we assume the following correlation:

$$\bar{\Lambda} \approx \pi\xi l \quad (2.29)$$

which would be expected in 4D effective theory since $\Lambda\mathcal{R}$ approximates the number of KK states between the scales $1/\mathcal{R}$ and Λ . Using (2.29) in (2.28) and taking again the limit $l \rightarrow \infty$ of $\partial(\mathcal{V}_1 + \mathcal{V}_2)/\partial\omega$, we may obtain a finite result only if $\xi \rightarrow 0$. This means that no correlation between the momentum cut-off Λ and the (largest) mass in the KK tower (controlled by l) can be established, if the minimum condition is to be maintained. The existence of the latter is thus a direct consequence of infinitely many KK state contributions and its origin should eventually be investigated and traced back in the context of the original five-dimensional theory after supersymmetry breaking.

2.2. The scalar potential: further analysis

In this section we provide a different mathematical approach to the behaviour of the scalar potential. The analysis is closer to a string approach. In this case, while summing up infinitely many KK modes, the presence of KK states of mass larger than Λ is not manifest. For case 1 below, we again show that quartic and quintic divergences are cancelled due to the equal numbers of bosons and fermions (due to initial, now broken supersymmetry), and there are no quadratic and logarithmic divergences in either the bosonic or the fermionic sectors alone, if we sum over the whole KK tower. This result is in agreement with our findings so far (see also appendix C) and with those of [7, 11].

To compute the KK states' contributions to the potential, one may alternatively use an integral representation of their individual logarithmic contributions. This approach generalizes that using the Riemann representation of the zeta function in appendix C (computed for $\omega = 0$ only). Consider an individual mode m_n and its contribution \mathcal{V}_n to the scalar potential, either bosonic ($\mathcal{V}_n^b \equiv \mathcal{V}_n(m_n \rightarrow m_{B_n})$) or fermionic ($\mathcal{V}_n^f \equiv \mathcal{V}_n(m_n \rightarrow m_{F_n})$) part. This contribution has the structure (the limits below are taken in this order: first $x \rightarrow 0$, then $\Lambda \rightarrow \infty$)

$$\begin{aligned} \mathcal{V}_n &\equiv \frac{1}{(2\pi)^4} \int d^4k \ln(k^2 + m_n^2)/\Lambda^2 = \lim_{\epsilon \rightarrow 0} \frac{2\pi^2}{(2\pi)^4} \int_0^\Lambda dk k^3 \int_\epsilon^\infty \frac{dt}{t} [e^{-t} - e^{-t(k^2 + m_n^2)/\Lambda^2}] \\ &= \lim_{x \rightarrow 0} \frac{2\pi^2}{(2\pi)^4} \left\{ \frac{\Lambda^4}{4} \int_{x^2}^\infty \frac{dt}{t} e^{-t\Lambda^2} - \frac{1}{2} \int_{x^2}^\infty \frac{dt}{t^3} [e^{-tm_n^2} - e^{-t(\Lambda^2 + m_n^2)}] \right. \\ &\quad \left. + \frac{\Lambda^2}{2} \int_{x^2}^\infty \frac{dt}{t^2} e^{-t(m_n^2 + \Lambda^2)} \right\} \end{aligned} \quad (2.30)$$

where we defined $x^2 = \epsilon\Lambda^{-2}$, i.e. x has dimension $(\text{mass})^{-1}$. The first term in the curly braces is a quartic divergence which cancels between bosonic and fermionic degrees of freedom. The second term gives quadratic Λ^2 (also logarithmic $\ln \Lambda$) dependence, which can be seen by taking the limit $t \rightarrow 0$; these terms are the analogue of $m_n^2 \Lambda^2$ and $\ln(1 + \Lambda^2/m_k^2)$ in equation (2.1) which may be shown by explicitly performing the integrals in (2.30) and then taking the limit $x \rightarrow 0$. As long as one only performs a finite summation over the KK level n in equation (2.30) to obtain the scalar potential, the quadratic and logarithmic terms survive in the final result. If we perform an infinite summation over n , the (UV) behaviour in $t \rightarrow 0$ of the integrand may change.

From equation (2.30), using the definition

$$\Gamma(\alpha, x) = \int_x^\infty dt e^{-t} t^{\alpha-1}, \quad (2.31)$$

we find the following expression for the scalar potential (in which case $m_n \rightarrow m_{B_n}$ or $m_n \rightarrow m_{F_n}$ respectively):

$$\mathcal{V} = \mathcal{V}^b - \mathcal{V}^f \equiv \frac{1}{2}(4N_c) \sum_n (\mathcal{V}_n^b - \mathcal{V}_n^f) = - \lim_{x \rightarrow 0} \frac{2\pi^2}{(2\pi)^4} \frac{1}{2} (4N_c) \{ \Omega|_{m_n \rightarrow m_{B_n}} - \Omega|_{m_n \rightarrow m_{F_n}} \} \quad (2.32)$$

where only a Λ^4 term was cancelled (the first term in equation (2.30)) between the bosons and the fermions, while the remaining bosonic and fermionic contributions each have the structure

$$\Omega = \frac{1}{2} \sum_n \int_{x^2}^\infty \frac{dt}{t^3} \{ 1 - e^{-t\Lambda^2} - t\Lambda^2 e^{-t\Lambda^2} \} e^{-tm_n^2}. \quad (2.33)$$

We consider the case when the sum over n in (2.32), (2.33) is not restricted and is indeed taken over all positive/negative integers and address the implications of this procedure. In the following we consider for the mass m_n of the n th KK mode two generic cases:

2.2.1. Case 1. The mass is in this case assumed to be

$$m_n = \frac{2}{\mathcal{R}}(n + \sigma) \quad (2.34)$$

where $\sigma = \omega$ (fermions) or $\sigma = \omega'$ (bosons), which corresponds to the case of [5] also discussed in previous sections. In the limit of including all KK states of the tower, as can be noticed from the

KK bosonic/fermionic integrals (C.11) and (C.12) of appendix C (with $\sigma = 0, 1/2$), the infinite summation behaves like $x^{-5} = (\text{mass})^5$ and no logarithmic or quadratic divergences are present. The absence of the latter two is ensured not by supersymmetry, but by the infinite summation that we perform, as we show below. For this consider a slightly different (but equivalent) form of the Poisson-resummed expression of (2.33) which gives that

$$\begin{aligned}\Omega &= \frac{1}{2} \int_{x^2}^{\infty} \frac{dt}{t^3} (1 - e^{-t\Lambda^2} - t\Lambda^2 e^{-t\Lambda^2}) \sum_{n \in \mathbb{Z}} e^{-t(n+\sigma)^2/(\mathcal{R}/2)^2} \\ &= \frac{\sqrt{\pi}}{2} \frac{\mathcal{R}}{2} \int_{x^2}^{\infty} \frac{dt}{t^{7/2}} (1 - e^{-t\Lambda^2} - t\Lambda^2 e^{-t\Lambda^2}) \sum_{k \in \mathbb{Z}} e^{-\pi^2(\mathcal{R}/2)^2 k^2/t} e^{2i\pi k\sigma} \\ &\equiv \Omega_{k=0}(\Lambda) + \Omega_{k \neq 0}(\Lambda)\end{aligned}\quad (2.35)$$

where in the last step we separated the contribution due to the resummed index $k = 0$ (divergent) from the $\Omega_{k \neq 0}$ part. After some algebra we find

$$\Omega_{k=0}(\Lambda) = \frac{\sqrt{\pi}}{2} \frac{\mathcal{R}}{2} \frac{2}{5} \{2\sqrt{\pi}\Lambda^5(\text{erf}(\Lambda x) - 1) + x^{-5}(1 - e^{-\Lambda^2 x^2}(1 + \Lambda^2 x^2 - 2\Lambda^4 x^4))\} \quad (2.36)$$

which in the limit of $x \rightarrow 0$ has the following behaviour:

$$\Omega_{k=0}(\Lambda) \propto \frac{\sqrt{\pi}}{2} \frac{\mathcal{R}}{2} \frac{2}{5} \left\{ -2\sqrt{\pi}\Lambda^5 + \frac{5}{2}\Lambda^4 x^{-1} \right\} + \mathcal{O}(x). \quad (2.37)$$

Thus all divergences ($x^{-1}\Lambda^4$ and Λ^5) are σ -independent, and no quadratic or logarithmic terms are manifest any longer. To explain the absence of the latter two, note that even though every integrand in the first line in the rhs of (2.35) has a quadratic divergence, because we sum an infinite number of such contributions, the behaviour of the integral in $t = 0$ changes. This may be seen after a Poisson resummation needed to compute the leading behaviour in $t = 0$. Indeed, Poisson resummation transforms individual, infinitely many exponential contributions ($\exp(-t) \rightarrow 1, t \rightarrow 0$) in the rhs of (2.35) to suppressed ones, ($\exp(-1/t) \rightarrow 0, t \rightarrow 0$). Note also that in (2.36) one could exchange the order of the limits in x and Λ when evaluating the supertrace over bosonic and fermionic contributions, since its divergent terms cancel anyway due to the equal number of bosonic and fermionic degrees of freedom (due to the initial, underlying supersymmetry).

Further, using the notation

$$\mathcal{L}i_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}; \quad \gamma(1+m, x) = m! \left\{ 1 - e^{-x} \sum_{p=0}^m \frac{x^p}{p!} \right\}, \quad m = 0, 1, 2, \dots, \quad (2.38)$$

the contribution $\Omega_{k \neq 0}(\Lambda)$ can be integrated from $x \rightarrow 0$ to give

$$\begin{aligned}\Omega_{k \neq 0}(\Lambda) &= \frac{1}{8\pi^4(\mathcal{R}/2)^4} \{3\mathcal{L}i_5(e^{2i\pi\sigma}) - 3\mathcal{L}i_5(y) - 4\bar{\Lambda}^3 \mathcal{L}i_2(y) - 6\bar{\Lambda}^2 \mathcal{L}i_3(y) - 6\bar{\Lambda} \mathcal{L}i_4(y)\} \\ &\quad + (\sigma \rightarrow -\sigma) = \frac{1}{8\pi^4(\mathcal{R}/2)^4} \left\{ \frac{1}{2} \sum_{n=1}^{\infty} \frac{e^{2i\pi n\sigma}}{n^5} \gamma(4, 2n\bar{\Lambda}) + (\sigma \rightarrow -\sigma) \right\}\end{aligned}\quad (2.39)$$

where

$$y = \exp(-2\bar{\Lambda} + 2i\pi\sigma), \quad \bar{\Lambda} = \pi\Lambda \frac{\mathcal{R}}{2}. \quad (2.40)$$

Therefore $\Omega_{k \neq 0}$ is finite in the limit of large Λ and all divergences of Ω , of the type $x^{-1}\Lambda^4$ and Λ^5 , are present in $\Omega_{k=0}$. No quadratic or logarithmic divergences are present in the scalar

potential equation (2.32), which supports our previous findings where we used a summation over an arbitrary number (l) of KK states and then took the limit $l \rightarrow \infty$. It is the infinite number of states that is responsible for the absence of quadratic/logarithmic terms in either the bosonic or fermionic sectors alone [7]. The remaining divergences $x^{-1}\Lambda^4$ and Λ^5 have the same coefficient for bosonic and fermionic sectors. It is thus sufficient that 5D theory had equal numbers of bosonic and fermionic degrees of freedom for these to disappear too from the scalar potential. Thus it is not necessary that supersymmetry be present for these cancellations to take place. Finally, we find from (2.32), (2.35), (2.39) that

$$\begin{aligned} \mathcal{V}(\phi) &= -\frac{4N_c}{2} \frac{2\pi^2}{(2\pi)^4} (\Omega|_{\sigma \rightarrow \omega'} - \Omega|_{\sigma \rightarrow \omega}) = \frac{3}{2\pi^6} \frac{4N_c}{\mathcal{R}^4} \frac{1}{3!} \sum_{k=0}^{\infty} \frac{\cos(2(2k+1)\pi\omega(\phi))}{(2k+1)^5} \gamma(4, 2n\bar{\Lambda}) \\ &\rightarrow \frac{3}{2\pi^6} \frac{4N_c}{\mathcal{R}^4} \sum_{k=0}^{\infty} \frac{\cos(2(2k+1)\pi\omega(\phi))}{(2k+1)^5}. \end{aligned} \quad (2.41)$$

In the last step the limit $\bar{\Lambda} \equiv \pi\Lambda\mathcal{R}/2 \rightarrow \infty$ was taken to recover the result \mathcal{V}_0 of (2.19) and already investigated elsewhere [5]. This means either Λ or $\mathcal{R} \rightarrow \infty$, with the other fixed. Such limits seem unclear from the string theory point of view that we adopt. In a string embedding of this model, one would replace the cut-off Λ by the string scale M_{string} . Taking $\Lambda \rightarrow \infty$ cannot be considered consistent without turning on winding modes, not included here. Taking instead $\mathcal{R} \rightarrow \infty$ is again problematic in a string orbifold construction, as we discuss in section 2.3. Both limits asymptotically decouple the effects of winding modes, thus corresponding to a string calculation in a very particular/singular point in the moduli space, $T \equiv \mathcal{R}^2/\alpha' \rightarrow \infty$. A full (heterotic) string calculation which will recover the result (2.41) as a particular case is provided in section 3. However, the string result cannot be applied to values of $1/\mathcal{R}$ in the TeV region (as models corresponding to case 1 require for phenomenological purposes) due to perturbativity constraints on the 10D (heterotic) string coupling which forbid large-volume compactification.

2.2.2. *Case 2.* The mass of the KK states is

$$m_n^2 = \left[\frac{2}{\mathcal{R}} \right]^2 (n+q)^2 + M_\phi^2 \quad (2.42)$$

where $q = 1/2$ (0) for bosons and $q = 0$ ($1/2$) for fermions respectively. The exact choice for charge assignment (to obtain (string theory) insight into this assignment, see section 3, footnote 9) to fermions/bosons is crucial for the presence/absence of the electroweak symmetry breaking. The mass dependence (2.42) corresponds to models of the type of [2, 3] and bears some similarities to heterotic string models in the sense that the (mass)² of KK modes is shifted by $N = M_\phi^2$ where N is an excited heavy mode of the string (for this, see equation (3.37)).

The assumption of summing over an infinite number of KK states allows one to perform a Poisson resummation to find

$$\begin{aligned} \Omega &= \frac{1}{2} \int_{x^2}^{\infty} \frac{dt}{t^3} (1 - e^{-t\Lambda^2} - t\Lambda^2 e^{-t\Lambda^2}) e^{-tM_\phi^2} \sum_{n \in \mathbf{Z}} e^{-t(n+q)^2/(\mathcal{R}/2)^2} \\ &= \frac{\sqrt{\pi}}{2} \frac{\mathcal{R}}{2} \int_{x^2}^{\infty} \frac{dt}{t^{7/2}} (1 - e^{-t\Lambda^2} - t\Lambda^2 e^{-t\Lambda^2}) e^{-tM_\phi^2} \sum_{k \in \mathbf{Z}} e^{-\pi^2(\mathcal{R}/2)^2 k^2/t} e^{2i\pi k q} \\ &\equiv \Omega_{k=0}(\Lambda) + \Omega_{k \neq 0}(\Lambda) \end{aligned} \quad (2.43)$$

where in the last step we again separated the contribution due to the resummed KK index $k = 0$ (divergent) from the $\Omega_{k \neq 0}$ part. It is then clear that the $\Omega_{k=0}$ part is independent of q

which distinguishes between bosons and fermions. Its divergent behaviour is cancelled between bosons and fermions as a consequence of the equal numbers of bosonic and fermionic degrees of freedom *and*, equally important, of the summation over infinitely many KK states. Also

$$\Omega_{k \neq 0}(\Lambda) = \Omega_1(z) - \Omega_2(\tilde{z}) - \Omega_3(\tilde{z}) \quad (2.44)$$

where

$$\begin{aligned} \Omega_1(z) &= \frac{1}{8\pi^4(\mathcal{R}/2)^4} \{3\mathcal{L}i_5(z) + 6\overline{M}_\phi \mathcal{L}i_4(z) + 4\overline{M}_\phi^2 \mathcal{L}i_3(z) + (q \rightarrow -q)\} \\ \Omega_2(\tilde{z}) &= \Omega_1(z \rightarrow \tilde{z})|_{M_\phi \rightarrow (M_\phi^2 + \Lambda^2)^{1/2}} \rightarrow 0 \quad \text{if } \Lambda \rightarrow \infty \\ \Omega_3(\tilde{z}) &= \frac{\Lambda^2}{4\pi^2(\mathcal{R}/2)^2} \{\mathcal{L}i_3(\tilde{z}) + 2\pi(\mathcal{R}/2)(M_\phi^2 + \Lambda^2)^{1/2} \mathcal{L}i_2(\tilde{z}) \\ &\quad + (q \rightarrow -q)\} \rightarrow 0 \quad \text{if } \Lambda \rightarrow \infty \end{aligned} \quad (2.45)$$

and with

$$\overline{M}_\phi = \pi M_\phi \frac{\mathcal{R}}{2}, \quad z = \exp(-2\overline{M}_\phi + 2i\pi q), \quad \tilde{z} = \exp(-2\pi(\mathcal{R}/2)(M_\phi^2 + \Lambda^2)^{1/2} + 2i\pi q). \quad (2.46)$$

In the case $M_\phi \rightarrow 0$ (Λ fixed) we can recover Ω and the potential of the previous case, equation (2.41). As in the previous case, models with ‘KK regularization’ take the limit $\Lambda \rightarrow \infty$ which at string embedding level most probably ‘avoids’ any winding mode contribution. Taking this limit we find

$$\Omega_{k \neq 0}(\Lambda \rightarrow \infty) = \Omega_1(z) \quad (2.47)$$

which gives for the scalar potential (using (2.32), (2.43), (2.47))

$$\begin{aligned} \mathcal{V}(\phi) &= -\frac{4N_c}{2} \frac{2\pi^2}{(2\pi)^4} (\Omega_1|_{q \rightarrow 1/2} - \Omega_1|_{q \rightarrow 0}) \\ &= -\frac{N_c}{2\pi^6 \mathcal{R}^4} \{[3\mathcal{L}i_5(z) + 6\overline{M}_\phi \mathcal{L}i_4(z) + 4\overline{M}_\phi^2 \mathcal{L}i_3(z) + \text{h.c.}]_{q=1/2} - (q \rightarrow 0)\}, \end{aligned} \quad (2.48)$$

a result computed in a different approach in [3]. For $M_\phi = 0$ we recover case 1, equation (2.41) with $\omega = 0$. Both cases are particular results of the string calculation, as discussed later in the text (see again equation (3.37) and text the thereafter).

While summing over an infinite number of KK states in (2.33), as was done in cases 1, 2 above, we interchanged the sum and the integral in equations (2.35), (2.43). This is possible if the integrand in (2.33) is exponentially suppressed in $x \rightarrow 0$. Since this is not the case, one should introduce a $(1 - \exp(-\rho t))$, $\rho \rightarrow \infty$, regulator and see whether any ρ -presence is manifest in the final result. Such a regulator already exists $(1 - \exp(-t\Lambda^2))$ for part of the integrand, provided by the 4D cut-off, and the would-be divergent dependence on it (present in $\Omega_{k \neq 0}$) cancels in the bosonic–fermionic difference. For the last term in curly braces in (2.33), one can again multiply it by a regulator and show that upon taking $\rho \rightarrow \infty$ such dependence vanishes in the potential, thus enabling one to interchange the sum and the integral (see also section 3.2.3 for a discussion at string level).

Although not fully manifest as in section 2.1.1, the use of KK states of mass larger than the field theory cut-off Λ may be traced in that we performed a Poisson resummation (which requires infinitely many states of mass l/R , l unbound) and which necessarily includes states

of mass larger than any EFT cut-off. Additionally, in (2.33) we integrated from the (deep-UV region) $t = x^2 = 0$, with Λ fixed, which means that we probe energies beyond any fixed cut-off $1/t > \Lambda^2$. However, if we insist on considering only the (EFT) region $1/t \ll \Lambda^2$, we find from equations (2.32), (2.33)

$$\mathcal{V} = \frac{2\pi}{(2\pi)^4} \frac{4N_c}{4} \sum_n \int_{x^2}^{\infty} \frac{dt}{t^3} \{e^{-tm_{F_n}^2} - e^{-tm_{B_n}^2}\} \quad (2.49)$$

where the sum in front of the integral must be restricted to integers giving KK masses of values smaller than the cut-off Λ (cut-off regularization of the potential gives the same result as proper time regularization; see (2.49)). This equation will be useful when establishing the link of the EFT (with truncated KK tower) with string results for the vacuum energy (section 2.5).

2.3. Remarks on perturbative expansion and ‘Kaluza–Klein regularization’

The KK models usually overlook the issue of validity of the perturbation theory that they employ and implicitly assume it to hold. The presence of KK states (due to the extra dimension) which have Yukawa and gauge interactions is manifest as a divergence of the associated couplings, which may thus lead to divergences for other quantities depending on them such as the Higgs mass.

As an example, consider the threshold corrections to the gauge couplings in the 4D theory with a tower of $N = 2$ KK states (associated with an extra dimension) in the loop contributing to the running coupling. Adding up the KK states’ contributions below a cut-off $n \leq \Lambda\mathcal{R}$ gives

$$\alpha^{-1}|_{n \leq \Lambda\mathcal{R}} \propto \sum_n \ln \frac{\Lambda}{m_n} = \sum_n \ln(\Lambda\mathcal{R}) - \sum_n \ln n = \Lambda\mathcal{R} - \frac{1}{2} \ln \Lambda\mathcal{R} + \dots \quad (2.50)$$

Thus a linearly divergent behaviour [14] with the cut-off emerges. This behaviour is further supported by heterotic string calculations of the gauge thresholds [15, 16] due to $N = 2$ sectors (that KK states are part of) in two-torus compactifications. In this case the behaviour is indeed power-like (quadratic). If one dimension of the two torus is fixed to a value equal to the string length, the string result is then linearly dependent on the (string) scale, in agreement with (2.50) above for the case of one additional compact dimension. This linear behaviour also exists in type I models in the limit of one large compact co-dimension [17]. Such a behaviour of the gauge coupling above the $1/\mathcal{R}$ scale can then lead to a non-perturbative regime, the more so when $1/\mathcal{R}$ has a small value (TeV). Thus the perturbative analysis of the model considered may become unreliable. This situation may eventually be avoided in some (heterotic) models with $N = 2$ subsectors using the Scherk–Schwarz mechanism (defined in section 3.2) for the breaking $N = 2 \rightarrow N = 0$ supersymmetry. This will be examined in section 3.5.1. This case preserves the possibility of having only logarithmic string thresholds to gauge couplings. In general, dependence on additional moduli exists in string models which can spoil this behaviour.

The case of Yukawa couplings obtained in (5D) models with KK states due to the extra dimension depends on the superpotential W assumed. Yukawa coupling running above the scale $1/\mathcal{R}$ is induced by wavefunction renormalization (due to KK states) and has two contributions. One is a gauge contribution (independent of W) which may be linear with the scale, thus giving a linear behaviour of Yukawa coupling on the high scale. A second contribution to Yukawa couplings due to wavefunction renormalization is controlled by W alone. This part can be quadratic with the scale if a ‘delta-function’ 5D ‘localized’ W at some fixed points of the

orbifold is present. In 4D language this means we sum over two towers of KK states (of different levels, since 5D momentum is not conserved) in the one-loop correction (induced by W) to wavefunction renormalization, and thus to the Yukawa coupling. In this way Yukawa coupling takes on a strong (UV) sensitivity to the scale/cut-off. Examples of this type were provided in [5] and [18]. The behaviour may eventually be changed in some (heterotic) string models where string corrections to Yukawa couplings are related to gauge coupling thresholds (see section 3.5.2).

In calculations, Yukawa couplings (gauge as well) become non-perturbative well before reaching the cut-off due to the large number of KK states present in the beta functions. For this reason the loop expansion of the effective potential which is after all a perturbative calculation involving Yukawa and gauge couplings may break down when summing up the effects of an infinite tower of KK states. In particular the *one-loop improved* [10] scalar potential, in which one-loop corrected couplings are used instead of their tree level values, is expected to differ significantly from its one-loop expansion, due to the divergent behaviour of the couplings. Alternatively, the (soft) Higgs (mass)² derived from \mathcal{V} is proportional at one-loop level to Yukawa coupling [5] which in turn has a one-loop correction (two-loop correction to the Higgs mass²) which is large due to the Yukawa coupling varying quadratically/linearly [5, 18] with the scale. Thus two-loop Higgs (mass)² will receive an induced sensitivity to the high scale [7] via the one-loop Yukawa couplings. An explicit example was provided in [18] where two-loop Higgs (mass)² was shown to be linearly sensitive to the UV cut-off (via the top coupling), although the conclusion was that Higgs mass is finite. This conclusion seems to imply that the Higgs mass is (two-loop) finite if expressed in terms of (rescaled/renormalized) Yukawa coupling (this is the non-renormalization theorem for the superpotential at one-loop level in the model considered), but the latter is however linearly divergent. Therefore the behaviour of the Higgs mass is worse than in the MSSM, where only logarithmic dependence on the high scale exists. Further high-scale sensitivity may be brought in by the fact that the models are non-renormalizable. For the model [5], a quadratic divergence also exists [7, 9].

Perturbative constraints on the couplings, and thus on the loop expansion of the effective potential, are not easy to avoid. One possibility is that the compactification scale $1/\mathcal{R}$ at which KK states set in is very close to the cut-off of the model, and thus rather high, ruling out models with large compactification radius. In a (heterotic) string embedding case, the same situation requires a compactification scale close to the string scale, for the 10D string coupling must be smaller than unity. Such a condition would mean that EFT (requiring $1/\mathcal{R} \ll M_{string}$) may be unreliable and a full string calculation is needed. Phenomenological constraints usually require $1/\mathcal{R} \approx \text{TeV}$ (Higgs (mass)² proportional to $1/\mathcal{R}^2$), so a type I embedding is then necessary. Thus the question of a mechanism for fixing the value of $1/\mathcal{R}$ must be addressed, since this is simply ‘fixed’ *ad hoc* to the TeV region by phenomenology. This is important since a (one-loop) Higgs (mass)² proportional to $1/\mathcal{R}^2$ (where supersymmetry is broken) with $1/\mathcal{R}$ large just restores the problem of quadratic divergences in the SM formulation.

Various issues related to ‘KK regularization’ exist in the literature, and recent analyses [19, 20] were performed to investigate this procedure. It has been shown [7] (reviewed in section 2.1) that quadratic/logarithmic divergences are absent in the bosonic (also in the fermionic) parts of the potential, due to summing over an infinite number of states in the KK tower (‘KK regularization’ limit). In this mechanism, KK states beyond the cut-off of the model played an important role. These remarks prompted the conclusion of [7], which raised concerns regarding the physical meaning of ‘KK regularization’ in an EFT (of fixed cut-off) approach and stressed the need

for a string understanding of this problem (for this, see sections 2.5 and 3). Other works [20] investigated the ‘KK regularization’, observing that it corresponds to an extremely anisotropic distribution of the momenta; more explicitly, p_4 and $p_5 = k/\mathcal{R}$ are highly correlated (the sum of their squared values should be smaller than the 5D momentum cut-off, M_5). Thus the ‘KK regularization’ of summing infinitely many KK states corresponds to $p_4^2 \ll M_5^2 \equiv l/\mathcal{R}$, i.e. a very anisotropic compactification. This is actually manifest in the models examined, if one observes that after summing the effects of infinitely many KK states, the contribution of the 4D momentum in the loop integrals of the Higgs mass is actually constrained to very low values, as observed in [5].

To justify an EFT approach with KK modes of mass larger than the momentum cut-off, one could consider the inclusion of some form factors ($\exp(-n^2/(\mathcal{R}\Lambda)^2)$) [19] to exponentially suppress the couplings of these KK states, i.e. to ‘smoothly’ decouple them. It was shown that *a priori* suitable decoupling form factors (exponentially suppressed couplings) may still give an infinite result for the Higgs mass, although the conclusion in [19] seems somewhat different. The results are dependent on the decoupling procedure of these heavy KK modes, i.e. this procedure is not clearly defined on field theory grounds and does not lead to unique results.

A cut-off regularization of the potential (2.2) is certainly not the best procedure, but is intuitive and suitable to prove the requirement of states of masses larger than the momentum cut-off, i.e. the need for a full string calculation where such a situation can be consistently addressed. This regularization will enable us to match the (field theory limit of the) string result to (that of) the EFT with a truncated tower of KK states. Such momentum cut-off regularization was previously used for the scalar potential (for a review, see [10]). Adopting a dimensional regularization for the 4D momentum integral only is not a better procedure, because we actually need a simultaneous treatment of the UV effects due to *all five* dimensions. Currently there is no regularization scheme able to treat on *equal footing* the (momentum integral over the) four non-compact dimensions and the (KK sum due to the) extra compact dimension. This should be the case, as both are sources of UV effects and one would expect a ‘unified’ treatment of their effects (see [11] with applications in [9] and [23]).

Finite-temperature arguments are sometimes used (see for example [2, 22]) to justify a soft UV behaviour of the vacuum energy (or Higgs mass) in 4D effective models with one additional compact dimension (the 5D supersymmetric model or $N = 2$ in 4D). The inverse radius of the compact dimension corresponds to the temperature. Supersymmetry in the initial 5D model then ensures a vanishing of the potential in the $\mathcal{R} \rightarrow \infty$ (or $T = 0$) limit. Finite values of \mathcal{R} would then trigger only finite corrections to the scalar potential, by analogy with finite-temperature calculations of the free energy, since in going from $T = 0$ (finite) to $T \neq 0$ only finite corrections could appear. It seems to us that it is hard to reconcile this mechanism with the phenomenological requirement of obtaining a chiral spectrum in 4D, while smoothly varying the radius from $\mathcal{R} \rightarrow \infty$ for the $N = 2$ supersymmetric limit in 4D to a broken supersymmetry phase. An orbifold-like compactification is then necessary, but then the limit $\mathcal{R} \rightarrow \infty$ on the orbifold does not restore the full initial supersymmetry at the fixed points. In other words, finite-temperature arguments may apply to a circle compactification of the extra dimension, but not necessarily to a manifold with fixed points (orbifold compactifications).

We consider that one necessarily needs a departure from the field theory approach to models which are ultimately string models and for which naive field theory methods may not necessarily apply.

2.4. The need for a string regularization scheme

One of the compelling arguments for a string regularization scheme for field theory models with KK states comes, somewhat surprisingly, from the field theory itself. The presence in the ‘KK regularization’ limit of states of masses larger than the momentum cut-off immediately raises the question of whether this is a legitimate procedure in an EFT approach. There are two possibilities. The first is that the 5D theory is finite and has no fundamental length/cut-off at all, but this seems highly unlikely, given the absence of a description of quantum gravity in such a case. The second possibility is that it has a cut-off and correspondingly the 4D theory will have one as well, above which the inclusion/presence of KK states may not be easy to justify in EFT of fixed cut-off. A symmetry argument may however be used for motivating the summation over an infinite tower of states in field theory, and the nature of this symmetry and its relationship to modular invariance will be briefly addressed in section 2.5 and carefully investigated in [23]. This issue is relevant, since in some cases the effect of states of mass larger than the cut-off is to ensure the absence of quadratic/logarithmic divergences in the bosonic and in the fermionic sector respectively, even in the absence of supersymmetry. This prompted us to investigate KK models in the larger context of string theory.

Topologically charged excitations associated with dimensional reduction from five to four dimensions (winding modes in string theory) which also have mass larger than the EFT cut-off also play an important UV role. To conclude that the UV behaviour of a model with KK states only is that found in the field theory limit is not necessarily correct, since the interplay between KK and winding states cannot necessarily be described in models with KK states only, because modular invariance interchanges these states. Summing infinitely many KK states may actually correspond, at string level, to a very special/singular point in the moduli space where winding modes are decoupled, but the question of whether this decoupling limit is well defined remains to be answered (a symmetry argument may however be used to avoid this situation; see [23]). In fact winding modes are infinitely massive (thus asymptotically decoupled) only in the limit $\mathcal{R} \rightarrow \infty$, which cannot be considered in string theory. In fact their density equals $1/\mathcal{R}$ [24] and only vanishes when $\mathcal{R} \rightarrow \infty$. Hence the effect of the winding modes must not be neglected and one may expect, in a string embedding of models with additional extra dimensions, the presence of some additional (UV and/or regularization) effects due to them. Since the effects of winding modes cannot be dealt with in a field theory framework, it may be appropriate to truncate the tower of KK states to a finite number, at a mass scale smaller than that (of the order of the string scale) at which winding states are turned on. In the next section we explore the link with string theory when one sums over either a truncated or the whole, infinite KK tower.

Truncating the KK tower in an EFT (of fixed cut-off) approach may be required by the fact that in the supersymmetric limit $\mathcal{R} \rightarrow \infty$ when the vacuum energy vanishes, we have all KK states mass degenerate. Since the mass of KK states is $\propto l/\mathcal{R}$, for the limit $\lim_{\mathcal{R} \rightarrow \infty} l/\mathcal{R}$ the highest KK level l must indeed be finite in the EFT theory approach. As discussed, one may try to avoid truncating the KK tower by exponentially/smoothly decoupling heavy KK states situated above the cut-off, and still keep infinitely many KK states in the EFT approach. However, it seems that this procedure is not uniquely defined in field theory; therefore a full string approach is necessary.

One example where string theory regularizes an EFT result obtained in the presence of (a truncated tower of) KK states is the case of the running of the effective gauge couplings. KK states charged under the SM gauge group affect their running to give a power-law dependence

with the scale. The EFT result is thus divergent (cut-off dependent); equation (2.50). There exist however full heterotic string calculations which compute such effects due to (two-torus) compactifications which preserve an $N = 2$ supersymmetric sector [15, 16]. Provided that the cut-off of the effective theory is smaller than the string scale, one can indeed approximate the string threshold corrections to the gauge couplings by an EFT result obtained using a truncated tower of KK states [25]. The reason why the string can regularize the (otherwise divergent) field theory result is in this case due to the geometry of the string (two-torus geometry in the heterotic case) which manifests itself at field theory level as form factors whose effect is small for scales smaller than the string scale. Winding modes also play an essential role in the finite character of the string result.

We therefore need a phenomenologically viable string model whose low-energy limit is the class of KK models investigated so far. If such a model existed, one could explicitly see the regularization of the vacuum energy by the string, along the lines discussed for the gauge thresholds.

2.5. String regularization of the vacuum energy

Before proceeding with detailed string calculations of the vacuum energy (denoted by V), we anticipate the full string result of the following sections (see section 3), to show how the (heterotic) string regularizes the corresponding EFT result denoted by \mathcal{V} (for a related discussion, see [26]). The string result has as its starting point the expression below (where $D = 4$ non-compact dimensions, R is the radius used in string calculations expressed in units of α' ; however, in this section only, we show explicitly the dependence on $\alpha' = 1$).

$$V(R) = \alpha'^{-D/2} \int_{\mathcal{F}_2} \frac{d^2\tau}{\tau_2^{1+D/2}} \sum_{m,n \in \mathbf{Z}} (-1)^m e^{2\pi i \tau_1 m n} e^{-\pi \tau_2 (m^2 \alpha' / R^2 + n^2 R^2 / \alpha')} C_{(1,\theta)}(q, \bar{q}). \quad (2.51)$$

The structure of the function C above is not important for the qualitative discussion of this section. In particular, it encodes massive string states with masses equal to or above the string scale $\alpha'^{-1/2}$. \mathcal{F}_2 is the extended fundamental domain of integration. The effect of KK states (labelled by m) alone on the vacuum energy is obtained by discarding the winding modes:

$$V(R)_{KK \text{ only}} = \alpha'^{-D/2} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^{1+D/2}} \sum_{k \in \mathbf{Z}} \{e^{-\pi \tau_2 (k^2 \alpha' / (R/2)^2)} - e^{-\pi \tau_2 ((k+1/2)^2 \alpha' / (R/2)^2)}\} C_{(1,\theta)}(q, \bar{q}). \quad (2.52)$$

This relation has some similarities with the ($D = 4$) EFT result, equations (2.32), (2.33), where the limit $\Lambda \rightarrow \infty$ is formally taken:

$$\mathcal{V}(\mathcal{R}) = \frac{2\pi}{(2\pi)^4} \frac{4N_c}{4} \sum_n \int_{x^2}^{\infty} \frac{dt}{t^3} \{e^{-tm_{F_n}^2} - e^{-tm_{B_n}^2}\} \quad (2.53)$$

with the replacements $m_{F_n}^2 = n^2 / (\mathcal{R}/2)^2$ and $m_{B_n}^2 = (n + 1/2)^2 / (\mathcal{R}/2)^2$ and the sum extended over the whole KK tower. However, the integral (2.52) is not over the region (x^2, ∞) ($x \rightarrow 0$) (as in field theory (2.53)) which would make it diverge in $t = 0$, but over the extended fundamental domain \mathcal{F} . Given a different integration region in (2.52) from the EFT case, one should expect a different string theory value for the KK effects alone compared to that of EFT in the limit of summing over *all* KK states in the tower ('KK regularization'). The reason is that the (infinite) KK tower alone integrated over \mathcal{F} cannot be equal to the same tower integrated over $(0, \infty)$ only

(to give the EFT result), because it is the sum over winding modes which plays the essential role in changing the integration region \mathcal{F} into $(0, \infty)$. This is discussed below.

The full string result, equation (2.51), is finite in the deep-UV region ($t = 0$) (for details, see section 3, equations (3.8), (3.18), (3.27) and (3.32)). First a Poisson resummation over the KK mode m is performed in (2.51) (resummed KK index l). Further, a technical argument used in the following sections (this follows from the arguments which precede equation (3.8)) ‘unfolds’ the fundamental domain \mathcal{F}_2 of integration in (2.51) into the half-strip and in this process winding modes n in (2.51) play a ‘regularization’ role. The ‘original’ winding number is written as $n = pc$, while the resummed KK number l is written as $l = pd$, with $(c, d) = 1$ (prime integers). The sums over n and l are thus replaced by a sum over the (prime integer) pairs (c, d) and a sum over p . The former sum transforms \mathcal{F}_2 into the half-strip to give $\tau_2 \in (0, \infty)$ as in (regularized) EFT. We are thus left with a sum over the integer p . Winding modes $n \neq 0$ (also resummed KK states, l) are manifest in the definition of the integer p that we sum over in the full string result. This result takes the following *generic* form⁵ (with model dependence only present in the coefficients γ_N):

$$V(R) = \alpha'^{-(1+D)/2} R \int_0^\infty \frac{dt}{t^{3/2+D/2}} \sum_{p>0} \sum_{N \geq 0} \{1 - (-1)^p\} e^{-(\pi(R/2)^2/\alpha' t)p^2} e^{-4\pi t N \alpha'} \gamma_N \quad (2.54)$$

$$= 4^{D/2} \int_{\epsilon^2 \alpha'}^\infty \frac{dz}{z^{1+D/2}} \sum_{s=-\infty}^\infty \sum_{N \geq 0} \{e^{-(\pi z/R^2)s^2} - e^{-(\pi z/R^2)(s+1/2)^2}\} e^{-\pi z N} \gamma_N. \quad (2.55)$$

This full string result thus includes the contribution of the winding modes. In the last step we made the change $4t\alpha' \rightarrow z$; we performed a Poisson resummation over p (after including the $p = 0$ case) to write the finite string result in a form close to that of EFT calculations. Since each of the two terms in (2.55) is divergent in $z = 0$ (see appendix C for $N = 0$, $D = 4$), we must introduce a UV cut-off ϵ^2 to avoid their divergence of type ϵ^{-5} which cancels between them (at field theory level this is ensured by the equal numbers of bosonic/fermionic degrees of freedom).

Note that in the deep-UV region $t \rightarrow 0$, most contributions are exponentially suppressed in (2.54), except those of small radius, $R^2/\alpha' \ll 1$, which may still contribute. The latter may be interpreted as corresponding to KK masses of value $k/R > M_{string}$, which agrees with previous findings at EFT level that KK states of mass larger than the momentum cut-off make a significant contribution. Their presence in the string framework is however justified, unlike in the case of (effective-) field theory.

The cut-off at $\epsilon^2 \alpha'$ in (2.55) thus excludes the deep-UV ($\tau_2 \rightarrow 0$) momentum region, retaining under the integral over z only the momentum range $0 \leq 1/z < 1/(\epsilon^2 \alpha')$. (That the lower integration range of the t -integral in (2.54) has to be changed from $t = 0$ to some finite cut-off value, when one wants to interpret (2.54) as a field theory result, may also be understood as follows: in the limit $t \equiv \tau_2 \rightarrow 0$ the integrand accounts for pure winding modes: after applying an S -transformation $\tau \rightarrow -1/\tau$ ($(c, d) = (1, 0)$) to the integrand, which changes this integration region to the region $\tau_2 \rightarrow \infty$, the number p is converted to pure windings $n = p$, $l = 0$, which clearly should not enter in the field theoretical limit of (2.51) that we are discussing.) This procedure therefore excludes from the string modes (that we sum over) those of deep-UV momentum which would otherwise contribute to $V(R)$. In (2.55) these string modes are represented by a mixture of Poisson-resummed winding and usual KK modes. This means that the summation over the index ‘ s ’ in (2.55) is

⁵ See again (3.8); also N is a heavy string mode.

replaced by two sums over k and w (with ‘ k ’: KK number; ‘ w ’: ‘resummed’ winding mode number) such that

$$|s| \equiv |k + wR^2\alpha'^{-1}| \leq \frac{R\alpha'^{-1/2}}{\epsilon}, \quad \epsilon > 0, \quad k \neq 0. \quad (2.56)$$

Therefore the KK contribution alone has to respect the same upper bound:

$$|k| \leq \frac{M_{string}R}{\epsilon} < \infty, \quad \epsilon > 0. \quad (2.57)$$

We conclude that the effect of the KK states alone ($w = 0$) may be written in the form

$$V(R)_{KK \text{ only}} = 4^{D/2} \int_{\epsilon^2\alpha'}^{\infty} \frac{dz}{z^{1+D/2}} \sum_{|k| \leq M_{string}R/\epsilon} \sum_{N \geq 0} \{e^{-(\pi z/R^2)k^2} - e^{-(\pi z/R^2)(k+1/2)^2}\} e^{-\pi z N} \gamma_N. \quad (2.58)$$

This result due to KK states only may be compared to EFT results with KK towers. According to (2.58) the right procedure in the EFT seems to be that one should sum only a finite number of KK states, $k < \infty$ (since $\epsilon > 0$), a condition imposed to avoid the deep-UV region. The result in string theory is finite and well defined in this region, equation (2.54). However, only when establishing the link with field theory is a regulator $\epsilon > 0$ required, which truncates the momentum integration as well as the summation over the KK modes alone.

Equation (2.58) may be mapped (up to an overall factor, model dependent) onto the EFT result of equation (2.49) with $m_n^2 \propto N + (k+q)^2/R^2$ where $N = 0$ as in section 2.2.1 or $N = M_\phi^2$ as in section 2.2.2. The limit $1/t \ll \Lambda^2 \leq \alpha'^{-1}$ of the ‘truncated KK’ tower of equation (2.49) is here $1/z \ll \alpha'^{-1}$ and the KK states of (2.58) which give a significant contribution (not exponentially suppressed) are those for which $\pi z k^2/R^2 \approx \mathcal{O}(1)$, i.e. $k^2 \approx R^2/z \ll (RM_{string})^2$; thus $k/R \ll M_{string}$. This is in agreement with equation (2.49) in including only KK states of mass below the cut-off of the order of M_{string} . In this way the string provides a physical regularization of an EFT result which includes (a truncated tower of) KK states only.

From (2.58) it is obvious that for field theory calculations in the ‘KK regularization’ limit, in which one ‘removes’ the restriction of summing over a finite KK number, one ‘recovers’ a full string result, equations (2.54), (2.55). However, this result actually includes ‘more’ than that of field theory does, such as the effects of winding modes as well as the modular invariance symmetry which play a crucial role in ensuring a finite-string result. Winding modes however do not have a counterpart at field theory level. Therefore, if the procedure of summing the whole KK tower is justified in field theory, a remnant of the modular invariance symmetry present in the string calculations should be present at the field theory level and be respected/implicitly assumed by ‘KK regularization’. This then explains why the field theory results obtained while summing over infinitely many KK states are compatible with a *full* string calculation (including momentum *and* winding states) and not with the string contribution due to (infinitely many) KK states alone. A possible source of such a remnant symmetry of modular invariance may be related to the discrete shift-symmetry equation (3.21) for zero-windings: $m \rightarrow m + p$ (which thus allows one to sum over the whole KK tower in field theory). This ‘shift symmetry’⁶ will be carefully examined in [23] and exploited in [11] in building a truly consistent (5D) field theory

⁶ We thank S Groot Nibbelink for discussions on this issue.

regularization scheme. We hope that this sheds some light on the meaning of taking the limit of ‘KK regularization’ from the point of view of string theory.

The safe procedure to follow in phenomenological studies is to apply a field theory calculation of the scalar potential up to the compactification scale and simply add to this the full string result as a ‘threshold correction’. This procedure is similar to that of adding string threshold corrections for the gauge couplings to the 4D field theory value of the gauge couplings, obtained from calculations below the compactification scale.

3. String calculation of the vacuum energy

In this section we want to address the question of UV finiteness of the one-loop cosmological constant in string theories with broken supersymmetry. Throughout the whole section, the compactification radius R is measured in units of $\alpha'^{1/2}$. We consider string theories in D space-time dimensions ($D \leq 9$) with one internal dimension of size $M_{comp} = 1/R$. The remaining $9 - D$ dimensions are supposed to be of string scale size M_{string} . As we will explain in section 3.3, the cosmological constant in heterotic string theories has a much better UV behaviour⁷ in contrast to the type I case. This explains why in this section we mainly focus on heterotic string vacua.

3.1. Cosmological constant in models with $N = 4 \rightarrow N = 0$ breaking

3.1.1. Harvey model in five dimensions. As an example, we want to calculate the one-loop cosmological constant of the model in [27]. This is a heterotic string model in five dimensions. Four dimensions are compactified on a torus of string size and one dimension is compactified on a circle of radius R . With respect to this circle an orbifold twist is introduced, which breaks supersymmetry from $N = 4$ to 0. At the same time the gauge group is broken from $E_8 \times E_8$ to a single E_8 of level two. The full calculation uses a combination of methods developed in [28] and [29]. The starting point is the world-sheet string one-loop integral:

$$V(R) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \frac{1}{\tau_2^{5/2}} \sum_{(h,g)} Z_{(h,g)}(q, \bar{q}) = \int_{\mathcal{F}_2} \frac{d^2\tau}{\tau_2} \frac{1}{\tau_2^{5/2}} Z_{(1,\theta)}(q, \bar{q}). \quad (3.1)$$

The sector $(h, g) = (1, 1)$ from the untwisted sector does not contribute, since it has $N = 4$ supersymmetry, i.e. $Z_{(1,1)}(q, \bar{q}) = 0$. The region \mathcal{F}_2 is the (extended) fundamental region $\{1, S, ST\}\mathcal{F}$ of the modular subgroup $\Gamma_0(2)$ [29]. The partition function in the $(1, \theta)$ sector is

$$Z_{(1,\theta)}(q, \bar{q}) = \mathcal{N}_{(1,\theta)}^{SO(1,1)}(q, \bar{q}) \mathcal{C}_{(1,\theta)}(q, \bar{q}) = \mathcal{N}_{(1,\theta)}^{SO(1,1)}(q, \bar{q}) f_{(1,\theta)}(\bar{q}) g_{(1,\theta)}(q) \quad (3.2)$$

with the three building blocks ($q = e^{2\pi i\tau}$, $\bar{q} = e^{-2\pi i\bar{\tau}}$):

$$\begin{aligned} f_{(1,\theta)}(\bar{q}) &= \frac{\theta_4(\bar{q}^2)^4}{\eta(\bar{q})^{12}} [\theta_3^4(\bar{q}) - \theta_4^4(\bar{q}) + \theta_2^4(\bar{q})] = \sum_{N \geq 0} c(N) \bar{q}^N = 32(1 + 8\bar{q} + 40\bar{q}^2 + 160\bar{q}^3 + \dots) \\ g_{(1,\theta)}(q) &= \frac{E_4(q^2)}{\eta(q^2)^{12}} = \sum_{M \geq -1} d(M) q^M = q^{-1} + 252q + 5130q^3 + 54760q^5 + \dots, \quad (3.3) \\ \mathcal{N}_{(1,\theta)}^{SO(1,1)}(q, \bar{q}) &= \sum_{(p_L, p_R)} (-1)^m q^{(1/2)|p_L|^2} \bar{q}^{(1/2)|p_R|^2} = \sum_{m, n \in \mathbf{Z}} (-1)^m e^{2\pi i \tau_1 m n} e^{-\pi \tau_2 ([m^2/R^2] + R^2 n^2)} \end{aligned}$$

⁷ Only harmless IR divergences due to massless states may show up. These effects can be regularized by different methods.

and the Narain momenta:

$$p_{R/L} = \frac{1}{\sqrt{2}} \left(\frac{m}{R} \mp Rn \right). \quad (3.4)$$

It should be kept in mind that the functions $f_{(1,\theta)}(\bar{q})$ and $g_{(1,\theta)}(q)$ transform automorphically under the modular subgroup $\Gamma_0(2)$ [29]. Poisson resummation on m in $\mathcal{N}_{(1,\theta)}^{SO(1,1)}(q, \bar{q})$ yields

$$\mathcal{N}_{(1,\theta)}^{SO(1,1)}(q, \bar{q}) = \frac{R}{\tau_2^{1/2}} \sum_{n,l \in \mathbf{Z}} e^{-(\pi R^2/\tau_2)|l+(1/2)+\tau n|^2}. \quad (3.5)$$

We rewrite the lattice sum (3.5) in the integrand of equation (3.1):

$$\sum_{n,l \in \mathbf{Z}} e^{-(\pi R^2/\tau_2)|l+(1/2)+\tau n|^2} = \frac{1}{2} \sum_{(n,l) \neq (0,0)} [1 - (-1)^l] e^{-(\pi R^2/4\tau_2)|l+2n\tau|^2}. \quad (3.6)$$

Now we shall see how the orbit decomposition of [28] can be applied (the convergence of this integral depends on the value of the radius R . Below the critical radius $R_H = \frac{1}{2}\sqrt{2}$, a tachyon appears in the spectrum [27], which spoils the convergence. We shall be more precise about this issue in section 3.2.3) to

$$V(R) = \frac{1}{2} R \int_{\mathcal{F}_2} \frac{d^2\tau}{\tau_2} \frac{1}{\tau_2^3} \sum_{(n,l) \neq (0,0)} [1 - (-1)^l] e^{-(\pi R^2/4\tau_2)|l+2n\tau|^2} \mathcal{C}_{(1,\theta)}(q, \bar{q}). \quad (3.7)$$

As in [28] we define $n = pc$, $l = pd$ with $(c, d) = 1$, and the exponential in (3.7) becomes now $\exp(-(\pi R^2/4\tau_2)p^2|d + 2c\tau|^2)$, which can be obtained from $\exp(-(\pi R^2/4\tau_2)p^2)$ after the modular transformation

$$\begin{pmatrix} a & b/2 \\ 2c & d \end{pmatrix} \in \Gamma_0(2)$$

(with $ad - bc = 1$) acting on τ . Taking all those $\Gamma_0(2)$ elements in (3.7), i.e. all $(c, d) = 1$ with the above conditions, which imply $(-1)^l = (-1)^{pd} = (-1)^p$, since a, d are always odd integers, we unfold the fundamental region \mathcal{F}_2 in (3.7) to the half-strip $\mathcal{H} = \{\tau | -1/2 \leq \tau_1 \leq 1/2; 0 \leq \tau_2 < \infty\}$. Note that these manipulations are possible, since the modular function $\mathcal{C}_{(1,\theta)}(q, \bar{q})$ is automorphic under the modular subgroup $\Gamma_0(2)$ [29]. Finally we arrive at

$$\begin{aligned} V(R) &= \frac{1}{2} R \int_{\mathcal{H}} \frac{d^2\tau}{\tau_2^4} \sum_{p>0} [1 - (-1)^p] e^{-(\pi R^2/4\tau_2)p^2} \mathcal{C}_{(1,\theta)}(q, \bar{q}) \\ &= \frac{1}{2} R \int_0^\infty \frac{dt}{t^4} \sum_{p>0} \sum_N [1 - (-1)^p] e^{-(\pi R^2/4t)p^2} e^{-4\pi t N} \gamma_N. \end{aligned} \quad (3.8)$$

In the last step we performed the τ_1 -integration

$$g(\tau_2) := \int_{-1/2}^{1/2} d\tau_1 \mathcal{C}_{(1,\theta)}(q, \bar{q}) = \sum_N \gamma_N e^{-4\pi\tau_2 N}, \quad (3.9)$$

which projects all states to equal-mass levels $M = N$. For $\tau_2 \rightarrow 0$, the function $g(\tau_2)$ counts the physical states (of the sector $(1, \theta)$) [24]. We defined the numbers $\gamma_N = c(N)d(N) \equiv d_F(N) - d_B(N)$, which correspond to the bosonic–fermionic degeneracy at the mass level N :

$$\sum_{N \geq 1} \gamma_N \bar{q}^N q^N = 32(2016 \bar{q}q + 820\,800 \bar{q}^3 q^3 + 93\,749\,120 \bar{q}^5 q^5 + \dots). \quad (3.10)$$

For the lowest level $N = 0$, the last expression in equation (3.8) may be compared with [13]

$$\frac{1}{2}\gamma_0 R \int_0^\infty \frac{dt}{t^4} \sum_{p>0} [1 - (-1)^p] e^{-(\pi R^2/4t)p^2} = \frac{128}{\pi^3} \frac{\gamma_0}{R^5} \sum_{\substack{p>0 \\ p \text{ odd}}} \frac{1}{p^6} = \frac{2\pi^3}{15} \frac{\gamma_0}{R^5}, \quad (3.11)$$

which looks similar to the result of [13]. Since (3.11) comes from string theory, the dangerous state $p = 0$ does not occur. However, according to (3.10), in the particular model of [27], we have $\gamma_0 = 0$, which means boson–fermion degeneracy at the massless level. All contributions to the cosmological constant come from the massive sector $N \neq 0$ and thus give an exponential contribution. Note also how the tachyonic contribution, encoded in the q^{-1} -power of (3.3), is projected out in (3.10). To perform the integral (3.8) we use equation (3.471.12) of [12]

$$\int_0^\infty \frac{dx}{x^{1-\nu}} e^{-x-(\mu^2/4x)} = 2\left(\frac{\mu}{2}\right)^\nu K_\nu(\mu), \quad (3.12)$$

for $|\arg(\mu)| < \frac{\pi}{2}$ and $\text{Re}(\mu^2) > 0$ and arrive at

$$V(R) = \frac{128}{R^2} \sum_{\substack{p>0, \text{ odd} \\ N \geq 1}} \gamma_N \frac{N^{3/2}}{p^3} K_3[2\pi R p \sqrt{N}]. \quad (3.13)$$

One may wish to further manipulate the result (3.13). With the asymptotic expansion [30]

$$K_3(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \left(1 + \frac{35}{8} \frac{1}{z} + \frac{945}{128} \frac{1}{z^2} + \frac{3465}{1024} \frac{1}{z^3} + \dots\right) \quad (3.14)$$

for large arguments $|z|, |\arg(z)| < \frac{3}{2}\pi$, equation (3.13) can be cast into the form

$$V(R) = \frac{64}{R^{5/2}} \sum_{\substack{p>0, \text{ odd} \\ N \geq 1}} \gamma_N \frac{N^{5/4}}{p^{7/2}} e^{-2\pi R p \sqrt{N}} \left(1 + \frac{35}{16\pi R p \sqrt{N}} + \frac{945}{512\pi^2 R^2 p^2 N} + \dots\right). \quad (3.15)$$

The first order gives precisely the $V(R) \sim R^{-5/2} e^{-R\sqrt{N}}$ behaviour of the cosmological constant, which has been determined in [27].

3.1.2. Harvey model in four dimensions. To obtain a model in four dimensions with $N = 4 \rightarrow N = 0$ breaking, we may compactify one more dimension on a radius of string size. Only a few details will change in the partition function (3.3). Essentially, everywhere in (3.3) one has to insert the lattice partition function $\tau_2^{1/2} \mathcal{N}_{R=1}(q, \bar{q})$ accounting for the additional compactified dimension. Here,

$$\mathcal{N}_{R=1}(q, \bar{q}) = \theta_3(q^2)\theta_3(\bar{q}^2) + \theta_2(q^2)\theta_2(\bar{q}^2) \quad (3.16)$$

is the partition function for the KK momenta and windings w.r.t. a string-size ($R = 1$) circle compactification. Thus we have

$$V(R) = \int_{\mathcal{F}_2} \frac{d^2\tau}{\tau_2} \frac{1}{\tau_2^2} \mathcal{N}_{(1,\theta)}^{SO(1,1)}(q, \bar{q}) \mathcal{N}_{R=1}(q, \bar{q}) \frac{E_4(q^2)}{\eta(q^2)^{12}} \frac{\theta_4(\bar{q}^2)^4}{\eta(\bar{q})^{12}} [\theta_3^4(\bar{q}) - \theta_4^4(\bar{q}) + \theta_2^4(\bar{q})]. \quad (3.17)$$

We can proceed as in the previous subsection to arrive at

$$V(R) = \frac{1}{2} R \int_0^\infty \frac{dt}{t^{7/2}} \sum_{p>0} \sum_N [1 - (-1)^p] e^{-(\pi R^2/4t)p^2} e^{-4\pi t N} \gamma_N. \quad (3.18)$$

Again, only the relative number of boson and fermions γ_N :

$$\sum_{N \geq 1} \gamma_N \bar{q}^N q^N = 32(2 + 2520 q \bar{q} + 8096 q^{5/4} \bar{q}^{5/4} + 28\,224 q^2 \bar{q}^2 + 1231\,680 q^3 \bar{q}^3 + \dots) \quad (3.19)$$

contributes in (3.18). We use (3.12) to evaluate

$$V(R) = \frac{93\zeta(5)\gamma_0}{4\pi^2 R^4} + \frac{64}{R^{3/2}} \sum_{\substack{p > 0, \text{ odd} \\ N \geq 1}} \gamma_N \frac{N^{5/4}}{p^{5/2}} K_{5/2}[2\pi R p \sqrt{N}]. \quad (3.20)$$

When one uses the explicit representation for $K_{5/2}$, one can re-express (3.20) in a form closer to that of field theory results. We refer the reader to section 3.2.2.

One might be surprised to see that the Harvey model compactified to four dimensions has a non-vanishing γ_0 rather than five dimensions (3.10). However, we compactify on an additional circle at the self-dual radius, where two additional gauge bosons become massless. Therefore, the relative number of bosons and fermions changes by 2 as we go down from five to four dimensions. This can be verified from (3.19). The factor 32 is the usual ground-state degeneracy.

3.2. Cosmological constant in models with $N = 1 \rightarrow N = 0$ breaking

In the following we start with heterotic orbifold compactifications with $N = 1$ supersymmetry in four dimensions. To break supersymmetry with a Scherk–Schwarz mechanism (also known as coordinate-dependent compactification), one needs one unrotated compactified dimension, i.e. our $N = 1$ orbifold has to possess $N = 2$ subsectors⁸, whose twist leaves invariant one two-dimensional subplane [31]. This means that the breaking takes place in an $N = 2$ subsector of the full $N = 1$ orbifold. One does not introduce mass shifts in the twisted sector. Technically, this means that one breaks an $N = 2$ subsector of the full orbifold to $N = 0$ and leaves the $N = 1$ sector untouched [31]. In addition, one has the completely untwisted sector, the so-called $N = 4$ sector, which is broken to $N = 0$ by the Scherk–Schwarz mechanism.

To be more explicit, let us consider the $\mathbf{Z}_2 \times \mathbf{Z}_2$ orbifold with twists $\theta = \frac{1}{2}(-1, -1, 2)$ and $\omega = \frac{1}{2}(-1, 2, -1)$. The $N = 2$ sectors are $\mathcal{T}_\theta = \{(1, \theta), (\theta, 1), (\theta, \theta)\}$, $\mathcal{T}_\omega = \{(1, \omega), (\omega, 1), (\omega, \omega)\}$, and $\mathcal{T}_{\theta\omega} = \{(1, \theta\omega), (\theta\omega, 1), (\theta\omega, \theta\omega)\}$. We shall introduce a Scherk–Schwarz mechanism w.r.t. one coordinate of the third torus. Therefore only the $N = 2$ sector \mathcal{T}_θ and the completely untwisted sector $\mathcal{T}_0 = (1, 1)$ are appropriate for this procedure. (In the following, whenever we talk about the $N = 2$ sector we have in mind this special sector.) As a concrete example for the Scherk–Schwarz breaking w.r.t. the latter dimension (third torus subplane) we choose the following shifts on the internal charges [31]:

$$n \rightarrow n, \quad m \rightarrow m + p - \frac{1}{2}n, \quad p \rightarrow p - n. \quad (3.21)$$

⁸ A large internal dimension may lead to unwanted large gauge couplings. It was argued in [31] that this can be avoided in the *supersymmetric case*, when one discusses models with vanishing $N = 2$ beta-function coefficients $b_a^{N=2}$. As a side remark, let us point out that this is not the right condition to meet. Even for $b_a^{N=2} = 0$ there are always additional universal gauge threshold corrections, which become huge for large extra dimensions [32]. The correct condition for avoiding large threshold corrections is $b_a^{N=2} = 12$ [32, 33]. In [33] examples of $N = 1$ orbifolds were presented, which have $N = 2$ sectors and meet this condition. However, there may be further contributions to the gauge coupling, which go like $\ln(T_2), 1/T_2, 1/T_2^2, \dots$ from the D -density, accounting for IR effects, anomalies, or higher loops. Their size is much smaller than T_2 , but should also be cancelled in addition and impose further string constraints on the models.

Here n, m are the winding and KK quantum numbers and p is the internal $U(1)$ fermion charge. For more details see [31].

Introducing a Scherk–Schwarz mechanism w.r.t. an untwisted plane of an heterotic $N = 1$ orbifold which has $N = 4$, $N = 2$, and $N = 1$ subsectors will break these subsectors to $N = 0$, $N = 0$, and $N = 1$ subsectors, respectively. Thus in the following two subsections we shall discuss separately the breaking $N = 4 \rightarrow N = 0$ in the $N = 4$ sector \mathcal{T}_0 and the breaking $N = 2 \rightarrow N = 0$ in the $N = 2$ sector \mathcal{T}_θ .

3.2.1. Cosmological constant for Scherk–Schwarz $N = 4 \rightarrow N = 0$ breaking. The breaking from $N = 4$ to 0 represents one of the basic supersymmetry breaking patterns triggered by a Scherk–Schwarz mechanism [34]. Let us calculate the cosmological constant for this case. We consider the heterotic string compactified on five string-size circles and one circle of dimension R . The shifts (3.21) refer to this latter circle. In the $N = 4$ partition function these shifts imply a fermion charge-dependent coupling of the circle partition function $\mathcal{N}^{SO(1,1)}(q, \bar{q})$ to the fermionic sectors

$$\sum_{(\alpha, \beta)} (-1)^{\alpha + \beta} \bar{\theta} \left[\begin{matrix} \alpha \\ \beta \end{matrix} \right]^4$$

whose effect can be written as

$$\mathcal{N}^{SO(1,1)}(q, \bar{q}) \sum_{(\alpha, \beta)} (-1)^{\alpha + \beta} \bar{\theta} \left[\begin{matrix} \alpha \\ \beta \end{matrix} \right]^4 \xrightarrow{\text{equation (3.21)}} \sum_{(\alpha, \beta)} (-1)^{\alpha + \beta} \mathcal{N}_{(\alpha, \beta)}^{SO(1,1)}(q, \bar{q}) \bar{\theta} \left[\begin{matrix} \alpha \\ \beta \end{matrix} \right]^4, \quad (3.22)$$

with [31]

$$\mathcal{N}_{(\alpha, \beta)}^{SO(1,1)}(q, \bar{q}) = \sum_{(\tilde{p}_L, \tilde{p}_R)} (-1)^{n\beta} q^{(1/2)|\tilde{p}_L|^2} \bar{q}^{(1/2)|\tilde{p}_R|^2}, \quad (3.23)$$

and

$$\tilde{p}_{R/L} = \frac{1}{\sqrt{2}} \left(\frac{1}{R} \left[m + \frac{1}{2}(\alpha + n) \right] \mp Rn \right). \quad (3.24)$$

With the lattice functions (D.1) defined in appendix D, we may express the rhs of (3.22) as

$$\sum_{(\alpha, \beta)} (-1)^{\alpha + \beta} \mathcal{N}_{(\alpha, \beta)}^{SO(1,1)}(q, \bar{q}) \bar{\theta} \left[\begin{matrix} \alpha \\ \beta \end{matrix} \right]^4 = -\bar{\theta}_3^4 (\mathcal{O}_0 - \mathcal{O}_{1/2}) + \bar{\theta}_2^4 (\mathcal{E}_0 - \mathcal{E}_{1/2}) + \bar{\theta}_4^4 (\mathcal{O}_0 + \mathcal{O}_{1/2}). \quad (3.25)$$

Thus, the total partition function can be written⁹ as

$$Z_{\mathcal{T}_0}(q, \bar{q}) = \tau_2^{-2} \mathcal{C}(q, \bar{q}) [-\bar{\theta}_3^4 (\mathcal{O}_0 - \mathcal{O}_{1/2}) + \bar{\theta}_2^4 (\mathcal{E}_0 - \mathcal{E}_{1/2}) + \bar{\theta}_4^4 (\mathcal{O}_0 + \mathcal{O}_{1/2})], \quad (3.26)$$

with the modular function $\mathcal{C}(q, \bar{q}) = \bar{\eta}^{-12} \eta^{-24} E_4^2 \mathcal{N}_{SO(5,5)}(q, \bar{q})$. The cosmological constant

$$V_{N=4 \rightarrow N=0}(R) = \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2} Z_{\mathcal{T}_0}(q, \bar{q}) = \int_{\mathcal{F}_2} \frac{d^2 \tau}{\tau_2} \frac{1}{\tau_2^2} (\mathcal{E}_0 - \mathcal{E}_{1/2}) \bar{\theta}_2^4 \mathcal{C}(q, \bar{q}) \quad (3.27)$$

⁹ Whereas this form is convenient for issues like modular invariance, it may be not obvious that in the field theory limit (zero-windings), integer KK momenta correspond to bosons and half-integer momenta to fermions. However, after explicitly working out the lhs of (3.25) and dropping the odd winding functions $\mathcal{O}_0, \mathcal{O}_{1/2}$, one realizes that the (NS, NS) sector comes with \mathcal{E}_0 while the (R, NS) sector appears with $\mathcal{E}_{1/2}$. Thus in the field theory limit of (3.25), \mathcal{E}_0 and $\mathcal{E}_{1/2}$ are to be associated with bosons and fermionic states, respectively.

may be evaluated in a way similar to that of the previous subsection on noting that $\mathcal{E}_0(R/2) - \mathcal{E}_{1/2}(R/2) = \mathcal{N}_{(1,\theta)}^{SO(1,1)}$. We arrive at

$$V_{N=4 \rightarrow N=0}(R) = \frac{93\zeta(5)\gamma_0^{\mathcal{T}_0}}{64\pi^2 R^4} + \frac{16\sqrt{2}}{R^{3/2}} \sum_{\substack{p>0, \text{ odd} \\ N \geq 1}} \gamma_N^{\mathcal{T}_0} \frac{N^{5/4}}{p^{5/2}} K_{5/2}[4\pi R p \sqrt{N}]. \quad (3.28)$$

Again, only the relative number of boson and fermions $\gamma_N^{\mathcal{T}_0}$ contributes:

$$\sum_{N \geq 1} \gamma_N^{\mathcal{T}_0} q^N \bar{q}^N = 32(257 + 5120 q^{1/4} \bar{q}^{1/4} + 40\,800 q^{1/2} \bar{q}^{1/2} + 162\,560 q^{3/4} \bar{q}^{3/4} + \dots). \quad (3.29)$$

We assumed the five-dimensional string-size lattice $SO(5,5)$ to be orthogonal, i.e. $\mathcal{N}_{SO(5,5)}(q, \bar{q}) = [\mathcal{N}_{R=1}(q, \bar{q})]^5$. Let us point out that after imposing the shifts (3.21), the $R \rightarrow 1/R$ duality is no longer a symmetry of the theory. This may be easily checked from expression (3.26).

3.2.2. Cosmological constant for Scherk–Schwarz $N = 2 \rightarrow N = 0$ breaking. An $N = 2$ sector of an $N = 1$ heterotic toroidal orbifold represents an orbifold limit of a heterotic compactification on $K3 \times T^2$. This means that two of the four complex fermions $\bar{\theta}[\beta]^\alpha$ in (3.25) are twisted by the orbifold group. Furthermore, the bosonic oscillators and gauge bosons are twisted. Embedding the twist $\theta = \frac{1}{2}(1, 1, -2)$ in the partition function (3.26) gives us the relevant partition function for the $N = 2$ sector \mathcal{T}_θ :

$$\begin{aligned} Z_{\mathcal{T}_\theta}(q, \bar{q}) = & (\mathcal{O}_0 - \mathcal{O}_{1/2}) \left(- \left| \frac{\theta_3^2 \theta_4^2}{\theta_2^2} \right|^2 g_{(1,\theta)} + \left| \frac{\theta_2^2 \theta_3^2}{\theta_4^2} \right|^2 g_{(\theta,1)} \right) \\ & + (\mathcal{E}_0 - \mathcal{E}_{1/2}) \left(- \left| \frac{\theta_2^2 \theta_3^2}{\theta_4^2} \right|^2 g_{(\theta,1)} - \left| \frac{\theta_2^2 \theta_4^2}{\theta_3^2} \right|^2 g_{(\theta,\theta)} \right) \\ & + (\mathcal{O}_0 + \mathcal{O}_{1/2}) \left(\left| \frac{\theta_3^2 \theta_4^2}{\theta_2^2} \right|^2 g_{(1,\theta)} + \left| \frac{\theta_2^2 \theta_4^2}{\theta_3^2} \right|^2 g_{(\theta,\theta)} \right), \end{aligned} \quad (3.30)$$

with the functions

$$\begin{aligned} g_{(1,\theta)} &= -\tau_2^{-2} \frac{E_4}{\eta^{18} \bar{\eta}^6} (\theta_3^4 + \theta_4^4) \mathcal{N}_{R=1}(q, \bar{q}), \\ g_{(\theta,1)} &= \tau_2^{-2} \frac{E_4}{\eta^{18} \bar{\eta}^6} (\theta_3^4 + \theta_2^4) \mathcal{N}_{R=1}(q, \bar{q}), \\ g_{(\theta,\theta)} &= \tau_2^{-2} \frac{E_4}{\eta^{18} \bar{\eta}^6} (-\theta_4^4 + \theta_2^4) \mathcal{N}_{R=1}(q, \bar{q}), \end{aligned} \quad (3.31)$$

which take into account the (untwisted) $E_7 \times E_8$ gauge degrees of freedom, the other unrotated string-size internal dimension, and the untwisted oscillators and zero-modes. As in the Harvey model, we may conveniently express the integral for the cosmological constant as an integral over the extended fundamental region \mathcal{F}_2 , which accounts for the three different $N = 2$ subsectors $(1, \theta)$, (θ, θ) , and $(\theta, 1)$:

$$V_{N=2 \rightarrow N=0}(R) = \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2} Z_{\mathcal{T}_\theta}(q, \bar{q}) = \int_{\mathcal{F}_2} \frac{d^2 \tau}{\tau_2} \frac{1}{\tau_2^2} (\mathcal{E}_0 - \mathcal{E}_{1/2}) \left(- \left| \frac{\theta_2^2 \theta_3^2}{\theta_4^2} \right|^2 g_{(\theta,1)} - \left| \frac{\theta_2^2 \theta_4^2}{\theta_3^2} \right|^2 g_{(\theta,\theta)} \right). \quad (3.32)$$

Noting that $\mathcal{E}_0(R/2) - \mathcal{E}_{1/2}(R/2) = \mathcal{N}_{(1,\theta)}^{SO(1,1)}$, we may proceed similarly to in the previous sections to obtain for (3.32)

$$V_{N=2 \rightarrow N=0}(R) = \frac{93\zeta(5)\gamma_0^{\mathcal{T}_\theta}}{64\pi^2 R^4} + \frac{16\sqrt{2}}{R^{3/2}} \sum_{\substack{p>0, \text{ odd} \\ N \geq 1}} \gamma_N^{\mathcal{T}_\theta} \frac{N^{5/4}}{p^{5/2}} K_{5/2}[4\pi R p \sqrt{N}], \quad (3.33)$$

with the degeneracy coefficients

$$\begin{aligned} \sum_{N \geq 0} \gamma_N^{\mathcal{T}_\theta} \bar{q}^N q^N = & -512(2 + 8q^{1/4}\bar{q}^{1/4} + 255q^{1/2}\bar{q}^{1/2} + 1016q^{3/4}\bar{q}^{3/4} \\ & + 25\,032q\bar{q} + 94\,720q^{5/4}\bar{q}^{5/4} + 844\,822q^{3/2} \\ & + \bar{q}^{3/2}3031\,480q^{7/4}\bar{q}^{7/4} + 22\,155\,392q^2\bar{q}^2 + \dots). \end{aligned} \quad (3.34)$$

We see that in contrast to the Harvey model for $D = 5$ we now have $\gamma_0^{\mathcal{T}_\theta} = -1024 \neq 0$, i.e. boson–fermion non-degeneracy already, at the massless level.

Ultimately, we are interested in the cosmological constant for our $\mathbf{Z}_2 \times \mathbf{Z}_2$ orbifold model, whose $N = 1$ SUSY is broken to $N = 0$ by the Scherk–Schwarz mechanism (3.21). The cosmological constant of the untwisted sector (3.28) and that of the $N = 2$ sector (3.33) are the only R -dependent contributions to the complete cosmological constant. Thus we may write

$$V_{N=1 \rightarrow N=0}(R) = \frac{1}{4}V_{N=4 \rightarrow N=0}(R) + \frac{1}{4}V_{N=2 \rightarrow N=0}(R). \quad (3.35)$$

Using the expression equation (8.451.6) of [12]:

$$K_{5/2}(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \left(1 + \frac{3}{z} + \frac{3}{z^2} \right) \quad (3.36)$$

we eventually rewrite (3.33) and also (3.28) in terms of polylogarithms ($x = e^{-4\pi R}$), as is done in field theory calculations [3]. Thus, equation (3.35) finally takes the form

$$\begin{aligned} V(R) = & \frac{93\zeta(5)\gamma_0}{64\pi^2 R^4} + \frac{4}{R^2} \sum_{N>0} \gamma_N N \left[\mathcal{L}i_3(x^{\sqrt{N}}) + \frac{3}{4\pi R \sqrt{N}} \mathcal{L}i_4(x^{\sqrt{N}}) + \frac{3}{16\pi^2 R^2 N} \mathcal{L}i_5(x^{\sqrt{N}}) \right. \\ & \left. - \mathcal{L}i_3(-x^{\sqrt{N}}) - \frac{3}{4\pi R \sqrt{N}} \mathcal{L}i_4(-x^{\sqrt{N}}) - \frac{3}{16\pi^2 R^2 N} \mathcal{L}i_5(-x^{\sqrt{N}}) \right], \end{aligned} \quad (3.37)$$

with $\gamma_N = (1/4)(\gamma_N^{\mathcal{T}_0} + \gamma_N^{\mathcal{T}_\theta})$. It is interesting that equation (3.37) has (up to an overall model-dependent factor, represented by γ_N) the same analytic form as the scalar potential has in (‘KK regularized’) field theory, but the latter cannot explain/account for the contribution of winding mode effects, which however it includes. For comparison, see sections 2.2.1, and 2.2.2, equations (2.39), (2.48), which are just particular cases of the above result for $N = 0$, $N = M_\phi^2$ respectively. The origin of this similarity was discussed in section 2.5 and is traced back to equation (3.18), which bears some similarity in form to the scalar potential equations (2.35), (2.43) (with the formal limit $\Lambda \rightarrow \infty$ in the expression for the potential). See [6] and references therein for further (field theoretical) details.

Although we have calculated the result (3.35) for a $\mathbf{Z}_2 \times \mathbf{Z}_2$ orbifold model (with Scherk–Schwarz breaking), up to trivial factors this result should also hold for other orbifold vacua. Similar generalizations are known for $N = 2$ orbifolds [33]. In particular, the modular functions appearing in (3.30) should serve as generic building blocks for $N = 2$ orbifolds with Scherk–Schwarz breaking.

3.2.3. Validity of perturbative string calculation and the Hagedorn transition. String theories in D space-time dimensions with supersymmetry broken by a Scherk–Schwarz mechanism w.r.t. an internal dimension of radius R behave like $(D + 1)$ -dimensional supersymmetric string theories at finite temperature $T = 1/(2\pi R)$. It is a notorious problem that in such theories a Hagedorn temperature exists, at which various physical quantities develop singularities. See [35] for an interesting recent account on this topic and further references. In this section we have to ask what such a critical radius means for our cosmological constant calculations in the previous sections.

We have seen that in perturbative heterotic string theories in D space-time dimensions with one internal dimension of size R (in string units) and $9 - D$ dimensions of string size, the cosmological constant always takes the generic form

$$V(R) = \int_{\mathcal{F}} \frac{d\tau}{\tau_2^{D/2+1}} [F_{(1,\theta)}(\mathcal{E}_0 - \mathcal{E}_{1/2}) + F_{(\theta,1)}(\mathcal{O}_0 + \mathcal{O}_{1/2}) + F_{(\theta,\theta)}(\mathcal{O}_0 - \mathcal{O}_{1/2})] \quad (3.38)$$

if supersymmetry is broken by some discrete \mathbf{Z}_2 -action θ . Here, the $F_{(h,g)}$ are functions depending on the compactification and supersymmetry breaking details, and the \mathcal{E} and \mathcal{O} are R -dependent partition functions encoding the information about the KK and winding numbers (see appendix D for further details). One may worry about whether the integrand is absolutely convergent for generic R , which is a necessary condition for being able to exchange the order of the integration and the summation. Usually the partition functions are expanded as a series in q and \bar{q} , which represents a good expansion as long as $\tau_2 > 1$ and the mass squared is positive. Of course, all the lattice functions \mathcal{E} and \mathcal{O} share this property. However, the functions $F_{(h,g)}$ may have negative powers of q and \bar{q} , which in the integrand lead to an IR singularity at $\tau_2 \rightarrow \infty$. In practice it proved to be very useful to rewrite (3.38):

$$V(R) = \int_{\mathcal{F}_2} \frac{d\tau}{\tau_2^{D/2+1}} F_{(1,\theta)}(\mathcal{E}_0 - \mathcal{E}_{1/2}). \quad (3.39)$$

Then, due to $F_{(\theta,1)}(\mathcal{O}_0 + \mathcal{O}_{1/2})(-1/\tau) = |\tau|^{2-D} F_{(1,\theta)}(\mathcal{E}_0 - \mathcal{E}_{1/2})(\tau)$, negative powers of q, \bar{q} in $F_{(\theta,1)}(\tau)$ manifest themselves as the UV singularity $\tau_2 \rightarrow 0$ in the integrand $F_{(1,\theta)}(\mathcal{E}_0 - \mathcal{E}_{1/2})(\tau)$. The question is to what extent these negative powers may be cancelled by positive powers coming from the lattice functions \mathcal{E}, \mathcal{O} . Thus the lattice functions play the role of a regulator in the integrand. As these lattice functions depend on the radius R , this role may be lost when R reaches a critical value R_H , and the integrand becomes divergent.

For concreteness, let us investigate this issue for the calculations in (3.27) and (3.32). In the integral (3.27), the functions appear as follows:

$$\begin{aligned} -\theta_3^4(\bar{q})\mathcal{C}(q, \bar{q})(\mathcal{O}_0 - \mathcal{O}_{1/2}) &= \left(-\frac{8}{q} - \frac{160}{q^{1/2}} - \frac{512}{\bar{q}^{1/2}} - \frac{1}{q\bar{q}^{1/2}} - \frac{20}{q^{3/4}\bar{q}^{1/4}} - \dots\right)(\mathcal{O}_0 - \mathcal{O}_{1/2}), \\ \theta_4^4(\bar{q})\mathcal{C}(q, \bar{q})(\mathcal{O}_0 + \mathcal{O}_{1/2}) &= \left(-\frac{8}{q} + \frac{160}{q^{1/2}} + \frac{512}{\bar{q}^{1/2}} + \frac{1}{q\bar{q}^{1/2}} + \frac{20}{q^{3/4}\bar{q}^{1/4}} + \dots\right)(\mathcal{O}_0 + \mathcal{O}_{1/2}), \\ \theta_2^4(\bar{q})\mathcal{C}(q, \bar{q})(\mathcal{E}_0 - \mathcal{E}_{1/2}) &= \frac{16}{q}(\mathcal{E}_0 - \mathcal{E}_{1/2}) + \dots \end{aligned} \quad (3.40)$$

The dots stand for terms with at least one positive power (of q or \bar{q}). First, one may wonder whether the $(16/q)$ -pole in the last function gives rise to a tachyon in the spectrum for *any* value of R . This may happen when we take the zero-winding and zero-momentum state $(m, n) = (0, 0)$ from \mathcal{E}_0 . However, after Poisson resummation (cf appendix D), the state $(m, n) = (0, 0)$ is removed in the combination $\mathcal{E}_0 - \mathcal{E}_{1/2}$. Moreover, since $m_R^2 = 0, m_L^2 = -1$, it does not even fulfil level

matching. There is also a physical argument that the $(16/q)$ -pole may not lead to a tachyon: the θ_2 refers to the Ramond sector, where tachyons never appear due to the vanishing zero-point energy in this sector. Since there is no $(m, n) = (0, 0)$ state in the function \mathcal{O}_0 , the pole $-8/q$ in the first two functions of (3.40) does not give rise to a tachyon, either. On the other hand, the negative-power combinations $q^{-1}\bar{q}^{-1/2}$ may combine with the states $(\pm 1/2, \pm 1)$ from $\mathcal{O}_{1/2}$, to give rise to tachyons (with $N_L = 0 = N_R$), whose mass becomes zero at $R_H = 1 + (1/2)\sqrt{2}$, but negative below this critical radius R_H . (Let us recall that one of the main differences between gauge bosons and tachyons becoming massless at a critical radius is that the mass of the gauge boson is positive for $R \neq R_{cr}$, whereas the tachyon mass is positive only for $R > R_H$ and becomes negative for $R < R_H$. Of course, the latter fact is just the reason that a particle which becomes massless at R_H is called a tachyon. For example, the power $\bar{q}^{-1/2}$ is responsible for the gauge boson $(m, n) = \pm(1/2, -1)$ becoming massless at $R_{cr} = (1/2)\sqrt{2}$ (with $N_L = 1, N_R = 0$.) Note that a Poisson resummation removes this state in the combination $\mathcal{O}_0 - \mathcal{O}_{1/2}$, but not in $\mathcal{O}_0 + \mathcal{O}_{1/2}$.

In any case, if the radius reaches the Hagedorn radius R_H (in string units)

$$R_H = 1 + \frac{1}{2}\sqrt{2}, \quad (3.41)$$

tachyons appear in the spectrum and the convergence of the integrand in (3.27) has to be questioned. In the previous calculations we assumed

$$R > R_H. \quad (3.42)$$

For that case our cosmological constant calculation managed to project out these potential tachyonic states. This is why we did not encounter any singularity in the final result. Nonetheless, these states are part of the spectrum and become tachyonic as soon as R is smaller than R_H .

The temperature $T_H = 1/(2\pi R_H)$ associated with the radius (3.41) is known as the perturbative Hagedorn temperature in heterotic string theory. Such a behaviour is quite generic in heterotic string models with supersymmetry broken by a Scherk–Schwarz mechanism. The functions appearing in (3.32) take an expansion similar to (3.40). Thus the above discussion may be repeated to verify the limiting radius (3.41) also for the integrand in (3.32). This behaviour is quite different from the model discussed in [36], which interpolates between a heterotic string with gauge group $O(16) \times O(16)$ at large radii and $E_8 \times E_8$ at small radii, and no Hagedorn behaviour is met.

Equation (3.42) may be in conflict with the requirement that we should have a small radius to keep the gauge couplings small. The gauge couplings increase with the radius due to threshold effects. In effect they may take the gauge couplings into the strong-coupling regime, where our cosmological constant calculations are clearly no longer valid, as new non-perturbative states (e.g. the NS 5-brane) may give rise to additional contributions. However, there are models where such growth of the gauge couplings is avoided (footnote 8).

3.3. Cosmological constant: heterotic versus type I

One of the main lessons of the previous section is that for heterotic string vacua with supersymmetry broken by a Scherk–Schwarz mechanism w.r.t. an internal dimension of radius R , the string calculation always reduces to the generic form (cf equation (3.8)):

$$V(R) = R \int_0^\infty \frac{dt}{t^{D/2+3/2}} \sum_{p>0} \Pi(p) \sum_N \gamma_N e^{-(\pi R^2/4t)p^2} e^{-4\pi t N}. \quad (3.43)$$

Here, $\Pi(p)$ is a projector. Without supersymmetry breaking, $\Pi(p) = 0$; in the simplest models with supersymmetry breaking, we have $\Pi(p) = \frac{1}{2}[1 - (-1)^p]$.

One of the intriguing features of expression (3.43) is that it equally describes the result of a (regulated) open-string one-loop calculation of type I with supersymmetry broken by a Scherk–Schwarz mechanism. For example, the closed-string result (3.27) has its type I counterpart as open descendants:

$$V(R) = \int_0^\infty \frac{dt}{t^3} \theta_4(it/R^2) \Theta(it) + \text{regularization}, \quad (3.44)$$

with the modular function [37]

$$\Theta(it) = \left[(d_E + d_O) \frac{\theta_2(it/2)^4}{\eta(it/2)^{12}} + (d_E - d_O) \frac{\theta_2(it/2 + 1/2)^4}{\eta(it/2 + 1/2)^{12}} \right] \theta_3(it)^5 = \sum_{N \geq 0} \tilde{\gamma}_N e^{-\pi t N} \quad (3.45)$$

encoding the open-string spectrum and θ_4 accounting for the KK states of the open string. The two contributions to the cosmological constant originate from the annulus and Möbius amplitude, respectively. The Klein bottle contribution vanishes. For a recent review and a detailed list of references on supersymmetry breaking in type I string we refer the reader to [38]. The coefficients d_E, d_O are Chan–Paton factors. We shall comment on the second term, which comes from a regularization, in a moment. After a Poisson resummation in $\theta_4(it/R^2)$, we obtain

$$V(R) = \frac{1}{2} R \int_0^\infty \frac{dt}{t^{7/2}} \sum_{p > 0} [1 - (-1)^p] e^{-(\pi R^2/4t)p^2} \sum_{N \geq 0} \tilde{\gamma}_N e^{-\pi t N} + \text{regularization}, \quad (3.46)$$

which should be compared/contrasted with (3.43). However, there is one important issue to consider regarding the second term. The type I expression (3.44) receives an UV divergence $t \rightarrow 0$ for $N = 0$ from the zero-momentum state $m = 0$ in $\theta_4(it/R^2)$. (It has been already pointed out in [37] that this type I model also has a critical radius R_H , below which string excitations N become less massive than the momentum excitations, and the cosmological constant (3.44) diverges. In this case the regularization method by which the integral (3.44) is defined is insufficient to control that divergence.) Usually this state is present in any one-loop type I calculation and has to be regularized, which gives rise to this additional term. This is a notorious problem from field theory with KK modes (cf section 2). Since an open string does not have windings, a one-loop open-string calculation always reduces to a (naive) field theoretical calculation with a tower of massive string states (encoded in Θ) and one should expect potential UV problems.

It is argued in [39] that the state $m = 0$ is not present if (3.44) is uniformly regularized in the transverse closed-string channel and the one-particle reducible diagrams are subtracted [39]. Another argument for the absence of the divergence is that when summing over all one-loop open-string diagrams, the divergence drops out if certain conditions on the spectrum, which guarantee tadpole cancellation, hold. In practice this means that one performs a Poisson resummation on $\theta_4(t)$ in (3.44), which converts the open-channel KK momenta to the closed-channel windings and subtracts the contribution of the zero-winding state (the ‘Poisson-resummation prescription’). This situation reminds us of that of [13] and of the discussion in section 2.

Nevertheless it is remarkable that the result of a one-loop heterotic string calculation with both KK and winding states looks like a (carefully regularized) open-string one-loop calculation with only KK modes running in the loop. In particular, the appearance of windings and/or modular invariance tells us that this correspondence follows to a lesser extent from the common underlying four-dimensional EFT (this fact just explains why the cosmological constant is given

by some sort of Schwinger one-loop integral with massive states running in the loop), but may be a consequence of a similar underlying regularization procedure using windings (and implying some sort of modular invariance) in both theories. Open strings with a non-vanishing NS B -field, whose KK modes look like windings, may shed more light on this connection [40].

The analogy between the heterotic (3.43) and type I result (3.46) may sound familiar from heterotic type I dualities, where one-loop heterotic calculations are mapped to type I one-loop results. However, this duality is somewhat more subtle, as e.g. in nine (or eight) dimensions an exact one-loop heterotic result comprises tree to three-loop results on the type I side [41, 42]. Moreover, non-perturbative corrections below nine dimensions complicate this map. But it is remarkable (and related to our discussion) that e.g. the leading¹⁰ part of the one-loop heterotic threshold of the $\text{Tr } F^4$ coupling in nine dimensions gives (3.43) with $\Pi(p) = 1$ (and $N = 0$). Heterotic type I duality maps this correction, which is finite, to a one-loop open-string result, written in the closed-string channel with the zero-winding dropped: $\int_0^\infty (dt/t^2) \sum_{n \neq 0} e^{-2\pi R^2 n^2/t}$. In fact, the heterotic zero-winding and KK contribution maps to a tree-level type I.

3.4. Wilson line dependence of the cosmological constant

In this subsection we want to investigate how our previous results for the cosmological constant change under the inclusion of one Wilson line w.r.t. the gauge group. The mass of the heavy particles running in the loop depends on this additional modulus and thus gives rise to a Wilson line dependence of the cosmological constant.

We introduce the (continuous) Wilson line:

$$A_i^I = a(1, -1, 0^6) \quad (3.47)$$

(written in an E_8 orthonormal basis) in the second E_8 gauge group. This choice makes sure that an $SU(2)$ subgroup of the E_8 is broken at generic values of a . In fact, equation (3.47) reads $a_i^I = a(1; 0^7)$ w.r.t. an $SU(2) \times E_7$ root lattice basis. With this choice (3.47), the Narain momenta (3.24) change as usual (see also [43]):

$$\tilde{p}_{R/L} = \frac{1}{\sqrt{2}} \left(\frac{1}{R} \left[m + \frac{1}{2}(\alpha + n) + \frac{1}{2}A^2 n + AN \right] \mp Rn \right) \quad (3.48)$$

with the E_8 gauge charges N_I . These momenta enter the lattice functions (D.1), which account for the different spin structures. However, now they are also coupled with the gauge group:

$$\mathcal{E}_{\delta/2} = \sum_{\substack{n, m \\ N \in E_8}} e^{-2\pi i \tau_1 (m + (\delta/2) + (1/2)A^2 n + N)} n e^{-(\pi \tau_2 / R^2) [(m + (\delta/2) + (1/2)A^2 n + AN)^2 + n^2 R^4]} q^{(1/2)(N + An)^2} \quad (3.49)$$

with $\delta = 0, 1$ and similarly for $\mathcal{O}_{\delta/2}$ with odd windings n . After Poisson resummation on m they become

$$\mathcal{E}_{\delta/2} = \frac{R}{\tau_2^{1/2}} \sum_{n, l \in \mathbf{Z}} (-1)^{l\delta} e^{2\pi i ((1/2)lnA^2 + lAN)} e^{-(\pi R^2 / \tau_2) |l + \tau n|^2} q^{(1/2)(K + An)^2}. \quad (3.50)$$

Due to the special choice (3.47), which breaks only an $SU(2)$ subgroup of the E_8 gauge group, it is possible to disentangle the remaining E_7 partition function in the lattice sums (3.49) and (3.50).

¹⁰ That part of the full heterotic one-loop calculation which is mapped to a one-loop correction on the type I side. As stated before, the remaining parts are mapped to tree and higher-than-one-loop parts for type I.

As explained in appendix A of [33], this is accomplished by splitting the E_8 character function into a product of E_7 and $SU(2)$ characters:

$$E_4(q) = \sum_{N \in E_8} q^{(1/2)N^2} \rightarrow \sum_{K \text{ even}} q^{(1/4)K^2} E_{4,1}^{\text{even}}(q) + \sum_{K \text{ odd}} q^{1/4 K^2} E_{4,1}^{\text{odd}}(q) \quad (3.51)$$

and analogously

$$\sum_{N \in E_8} q^{(1/2)(N+An)^2} \rightarrow \sum_{K \text{ even}} q^{(1/4)(K+2an)^2} E_{4,1}^{\text{even}}(q) + \sum_{K \text{ odd}} q^{(1/4)(K+2an)^2} E_{4,1}^{\text{odd}}(q), \quad (3.52)$$

with the E_7 character functions

$$\begin{aligned} E_{4,1}^{\text{even}}(q) &= \theta_3(q^2)^7 + 7\theta_3(q^2)^3\theta_2(q^2)^4 = 1 + 126q + 756q^2 + \dots, \\ E_{4,1}^{\text{odd}}(q) &= \theta_2(q^2)^7 + 7\theta_2(q^2)^3\theta_3(q^2)^4 = 56q^{3/4} + 576q^{7/4} + 1512q^{11/4} + \dots. \end{aligned} \quad (3.53)$$

Thus, the lattice partition function (3.50) can be written as

$$\begin{aligned} \mathcal{E}_{\delta/2} &= \frac{R}{\tau_2^{1/2}} \sum_{n,l} (-1)^{l\delta} e^{-(\pi R^2/\tau_2)|l+\tau n|^2} \left\{ \sum_{K \text{ even}} e^{2\pi i(lna^2+alK)} q^{(1/4)(K+2an)^2} E_{4,1}^{\text{even}}(q) \right. \\ &\quad \left. + \sum_{K \text{ odd}} e^{2\pi i(lna^2+alK)} q^{(1/4)(K+2an)^2} E_{4,1}^{\text{odd}}(q) \right\}. \end{aligned} \quad (3.54)$$

With this in mind, we introduce the Wilson line (3.47) in the partition functions (3.26) and (3.30): the lattice functions $\mathcal{E}_{\delta/2}$ and $\mathcal{O}_{\delta/2}$ have to be replaced by (3.54) and one $E_4(q)$ from the gauge partition function has to be dropped, because it is already contained in (3.54). For the τ -integration we can make use of the same techniques as before, i.e. only the sector with the combination $\mathcal{E}_0 - \mathcal{E}_{1/2}$ contributes at the cost of extending the integration from \mathcal{F} to \mathcal{F}_2 . Furthermore, the orbit decomposition allows us to stick to the case $n = 0, l = p$ by extending the integration \mathcal{F}_2 to the full strip \mathcal{H} . Thus, we end up with integrals

$$\Lambda(R, a) = R \int_0^\infty \frac{dt}{t^{7/2}} \sum_{p>0} \sum_{N \geq 0} \sum_{K \in \mathbb{Z}} [1 - (-1)^p] e^{-(\pi R^2/t)p^2} e^{2\pi iapK} e^{-4\pi tN} \gamma\left(N - \frac{1}{4}K^2, N\right), \quad (3.55)$$

which following the previous sections integrates to

$$\begin{aligned} \Lambda(R, a) &= \frac{3}{2\pi^2 R^4} \sum_{p>0, \text{ odd}} \sum_{K=\{0, \pm 1, \pm 2\}} \frac{e^{2\pi iapK}}{p^5} \gamma\left(-\frac{1}{4}K^2, 0\right) \\ &\quad + \frac{2^{9/2}}{R^{3/2}} \sum_{p>0, \text{ odd}} \sum_{\substack{N \geq 0 \\ K \in \mathbb{Z}}} e^{2\pi iapK} \gamma\left(N - \frac{1}{4}K^2, N\right) \frac{N^{5/4}}{p^{5/2}} K_{5/2}[4\pi R p \sqrt{N}]. \end{aligned} \quad (3.56)$$

The expansion coefficients $\gamma(M, N)$ are given in appendix F for the model discussed in section 3.3. Again, only massless states, classified w.r.t. their $SU(2)$ charge K contribute to the first term of (3.56). Thus, essentially, a Wilson line gives rise to an additional ‘projector’ $e^{2\pi iapK}$. Its effect can be easily seen for discrete values of a .

3.5. One-loop corrections in heterotic Scherk–Schwarz models

In this section we want to investigate the general structure of one-loop corrections in heterotic string theories in D space-time dimensions with one large extra dimension of size R and

supersymmetry broken by a Scherk–Schwarz mechanism (3.21). The analysis will enable us to give the explicit form for the one-loop gauge threshold corrections in such models. From the previous subsections it is obvious that the generic expression reads

$$I_{D,n}(R) = \int_{\mathcal{F}_2} \frac{d^2\tau}{\tau_2^{D/2-n+1}} [\mathcal{E}_0(R) - \mathcal{E}_{1/2}(R)] \mathcal{C}(q, \bar{q}). \quad (3.57)$$

We assume that the condition $D/2 \geq n$ holds. The integer n depends on what kind of amplitude we are calculating. For example, $n = 0$ for a cosmological constant calculation and $n = 2$ for gauge thresholds. Furthermore, the function $\mathcal{C}(q, \bar{q})$ is a weighted partition function (elliptic genus) encoding the massive string states. It has modular weight $n + 1/2 - D/2$ under modular transformations of both τ and $\bar{\tau}$. On using the procedure described in section 3.1, the integral $I_{D,n}(R)$ takes the generic Schwinger-integral form (cf equation (3.8)):

$$I_{D,n}(R) = R \int_0^\infty \frac{dt}{t^{D/2+3/2-n}} \sum_{N \geq 0} \gamma_N \sum_{p > 0} [1 - (-1)^p] e^{-(\pi R^2 p^2/t)} e^{-4\pi N t}. \quad (3.58)$$

We suppose that the analogue of the τ_1 -integration (3.9) cancels any negative N -powers in (3.58). We point out that the integrand is only convergent for the case (3.42) and the limit $R \rightarrow \infty$ has to be taken with care. In this limit, supersymmetry is restored and the projection onto odd numbers p is suspended. For completeness, we may present the analogue of (3.57) for the supersymmetric case. Then we have an unshifted Narain lattice for the bosonic zero-modes of the internal dimensions (cf (3.3)) and integrate only over the fundamental domain \mathcal{F} :

$$\begin{aligned} I_{D,n}^{susy}(R) &= \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^{D/2-n+1}} \sum_{(p_L, p_R)} q^{(1/2)|p_L|^2} \bar{q}^{(1/2)|p_R|^2} \mathcal{C}(q, \bar{q}) \\ &= R \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^{D/2-n+3/2}} \sum_{(n, l)} e^{-(\pi R^2/\tau_2)|l+\tau n|^2} \mathcal{C}(q, \bar{q}). \end{aligned} \quad (3.59)$$

As usual, we performed a Poisson resummation on the momentum m (cf (3.5)). Apart from $\mathcal{C}(q, \bar{q})$, which depends on the number of supersymmetries and may even be zero for certain couplings, the main difference from (3.6) is that the sum also includes the state $(n, l) = (0, 0)$. It gives rise to a linear R -dependence in $I_{D,n}^{susy}(R)$:

$$R\gamma := R \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^{D/2-n+3/2}} \mathcal{C}(q, \bar{q}),$$

in contrast to the case for $I_{D,n}(R)$. At present, we do not have a general expression for γ , except for $D/2 - n = 1/2$. In that case, $\gamma = \frac{\pi}{3} \lim_{\tau_2 \rightarrow 0} g(\tau_2)$, provided that the (in that case) modular invariant function $\mathcal{C}(q, \bar{q})$ meets certain conditions [24]. For a definition of g , see (3.9). Altogether, we obtain for $I_{D,n}^{susy}(R)$

$$\begin{aligned} I_{D,n}^{susy}(R) &= R\gamma + 2R \sum_{N > 0} \sum_{p > 0} \left(\frac{2N^{1/2}}{Rp} \right)^{1/2+D/2-n} \gamma_N K_{1/2+D/2-n}(4\pi R p \sqrt{N}) \\ &\quad + \begin{cases} \gamma_0 \pi^{-1/2-D/2+n} R^{2n-D} \Gamma(1/2 + D/2 - n) \zeta(1 + D - 2n), & D \neq 2n \\ \gamma_0 (\frac{1}{2} \gamma_E - \ln R - \frac{1}{2} \ln 4\pi), & D = 2n. \end{cases} \end{aligned} \quad (3.60)$$

There will then also be a contribution from the so-called zero-orbit $p = 0$ (given by the first term of (3.60)). Thus in the following, we shall assume $R \neq \infty$.

The contributions $N \neq 0$ and $N = 0$ are evaluated, separately. For $N \neq 0$, equation (3.58) gives

$$4R \sum_{N>0} \sum_{p>0, \text{ odd}} \left(\frac{2N^{1/2}}{Rp} \right)^{1/2+D/2-n} \gamma_N K_{1/2+D/2-n}(4\pi R p \sqrt{N}) \quad (3.61)$$

and for the $N = 0$ term (this contribution is reminiscent from a one-loop field theory calculation with KK modes running in the loop (cut-off sent to infinity)) we obtain

$$2R\gamma_0 \int_0^\infty \frac{dt}{t^{D/2+3/2-n+\epsilon}} \sum_{p>0, \text{ odd}} e^{-(\pi R^2 p^2/t)}. \quad (3.62)$$

For $N = 0$ the parameter ϵ has to be introduced for the case $D/2 = n$, to regularize the integral (3.62). This corresponds to some sort of dimensional regularization. See also appendix C for its evaluation using a different procedure and [42] for a similar situation and more details. To further simplify (3.61), we restrict to even space-time dimensions D . In that case, the explicit representation for $K_{1/2+D/2-n}$ in equation (8.468) of [12] can be used to proceed further. After some algebra, equations (3.61) and (3.62) lead to

$$\begin{aligned} I_{D,n}(R) = & \frac{2^{D/2-n}}{R^{D/2-n}} \sum_{N>0} \gamma_N N^{D/4-n/2} \sum_{k=0}^{D/2-n} \frac{(D/2-n+k)!}{4^k k! (D/2-n-k)!} \frac{1}{(2\pi R \sqrt{N})^k} \\ & \times \mathcal{L}i_{k+1+D/2-n}(x^{\sqrt{N}}) - (x \rightarrow -x) \\ & + \begin{cases} \gamma_0 (2 - 2^{2n-D}) \pi^{-1/2-D/2+n} R^{2n-D} \\ \quad \times \Gamma(1/2 + D/2 - n) \zeta(1 + D - 2n), & D \neq 2n \\ \gamma_0 (\frac{1}{2} \gamma_E - \ln R - \frac{1}{2} \ln \pi), & D = 2n, \end{cases} \end{aligned} \quad (3.63)$$

with $x = e^{-4\pi R}$. Though we restricted to even dimensions D , the last term in (3.63), which comes from the integral (3.62), is valid for any integer D . On the other hand, this term also shows us that the structure of the one-loop threshold correction (3.57) is quite different for even and odd dimensions D . For odd dimensions the Riemann ζ -function becomes the elementary $\zeta(2s) = \sum_{n=1}^\infty (1/n^{2s}) = (-1)^{s+1} ((2\pi)^{2s}/2(2s)!) B_{2s}$, $s \geq 1$, with no need for a ζ -function regularization. This is one of the reasons that field theories in odd dimensions have a better UV behaviour.

3.5.1. One-loop gauge couplings in theories with broken supersymmetry. Let us apply these results for the gauge threshold corrections in four space-time dimensions $D = 4$, $n = 2$, in particular for the model discussed in section 3.2. We remind the reader that the model has an $N = 4$ subsector \mathcal{T}_0 and an $N = 2$ subsector \mathcal{T}_θ , which are broken to $N = 0$ by the Scherk–Schwarz mechanism. Furthermore, it has an $N = 1$ subsector, which is not affected by this mechanism. The former two subsectors \mathcal{T}_0 , \mathcal{T}_θ will give rise to radius-dependent contributions to the gauge couplings, while the $N = 1$ sector will not. For gauge thresholds, the function $\mathcal{C}(q, \bar{q})$ has to account for the gauge and fermion charge insertion: $\text{Tr } Q_a^2 F^2 (-1)^F$ [15]. Technically, this means that the (light-cone) space-time fermion partition function $(\theta_2(\bar{q})/\eta(\bar{q}))$ from the RNS sector in (3.26) and (3.30) has to be replaced by $\theta_2(\bar{q}) \rightarrow \frac{1}{24}[\theta_3(\bar{q})^4 + \theta_4(\bar{q})^4 + \hat{E}_2(\bar{q})]\theta_2(\bar{q})$ and similarly in the other sectors. Furthermore, the E'_8 gauge partition function $E_4(q)$ has to be changed to $\frac{1}{3}[\hat{E}_2(q)E_4(q) - E_6(q)]$ if we are interested in thresholds w.r.t. this gauge group $G_a = E_8$. Since $\hat{E}_2(q) = E_2(q) - 3/\pi\tau_2$, we encounter additional $1/\tau_2$ powers stemming from

pinching the positions of the two gauge vertex operators. They bring back the original power of $1/\tau_2^3$ of the cosmological constant calculation in (3.57), i.e. $I_{4,0}(R) = V(R)$, which is the subject of the preceding subsections. In supersymmetric vacua this term is cancelled by the spin structure sum [15]. In total, the gauge thresholds assemble from three contributions:

$$\Delta_a(R) = \frac{9}{\pi^2} V(R) - \frac{3}{\pi} I_{4,1}(R) + I_{4,2}(R) \quad (3.64)$$

which are classified by their power of $1/\tau_2$ (besides their functions $\mathcal{C}(q, \bar{q})$). The relevant functions $\mathcal{C}(q, \bar{q})$ for the model of section 3.2 will be detailed in appendix E. The second expression summarizes all other universal corrections to the gauge couplings, whereas the last term $I_{4,2}$ comprises their group-dependent part. See [32] for a detailed discussion on these terms in toroidal orbifolds. The contribution $I_{4,2}$ has only a total $1/\tau_2$ power in the integrand of (3.57), which gives rise to the logarithmic contributions of the gauge threshold corrections in four space-time dimensions. In that case the integrand has to be regularized, which results in additional $\ln(R)$ terms and some scheme-dependent constants (cf the last line of equation (3.63)).

It may be a surprise that the cosmological constant appears in (3.64). However, the following argument explains why the latter has to appear in the gauge threshold result for non-supersymmetric vacua: if these corrections are calculated in a background gauge field method, gravity has to be switched on to fulfil the Einstein equation. The latter are supplemented by a cosmological constant term and should therefore modify the gauge threshold result. This is a novel effect in contrast to the threshold corrections in supersymmetric string vacua discussed in [15].

Using the ingredients from appendix E and the result (3.63), the one-loop gauge couplings (3.64) for the model of section 3.2 take the form

$$\begin{aligned} \Delta_a(R) = & \frac{9}{\pi^2} \frac{93\zeta(5)\gamma_0}{64\pi^2 R^4} - \frac{3}{\pi} \frac{7\zeta(3)\beta_0}{8\pi R^2} + \alpha_0 \left(\frac{1}{2} \gamma_E - \ln R - \frac{1}{2} \ln \pi \right) \\ & + \frac{9}{\pi^2} \frac{4}{R^2} \sum_{N>0} \gamma_N N \sum_{k=0}^2 \frac{(2+k)!}{4^k k! (2-k)!} \frac{1}{(2\pi R \sqrt{N})^k} \mathcal{L}i_{k+3}(x^{\sqrt{N}}) \\ & - \frac{3}{\pi} \frac{2}{R} \sum_{N>0} \beta_N N^{1/2} \sum_{k=0}^1 \frac{(1+k)!}{4^k k! (1-k)!} \frac{1}{(2\pi R \sqrt{N})^k} \mathcal{L}i_{k+2}(x^{\sqrt{N}}) \\ & + \sum_{N>0} \alpha_N \mathcal{L}i_1(x^{\sqrt{N}}) - (x \rightarrow -x). \end{aligned} \quad (3.65)$$

The first line gives the contributions from the massless sector $N = 0$, whereas the polylogarithms account for the massive string states with mass level N . They are exponentially suppressed with the radius R . The second line is nothing else than (3.37). The coefficients $\alpha_N = (1/4)(\alpha_N^{\mathcal{T}_0} + \alpha_N^{\mathcal{T}_\theta})$ and $\beta_N = (1/4)(\beta_N^{\mathcal{T}_0} + \beta_N^{\mathcal{T}_\theta})$ can be picked up from the appendix, whereas γ_N is introduced in (3.37).

As we have pointed out above: in the limit $R \rightarrow \infty$, supersymmetry is restored. In that case the state $p = 0$, i.e. $(l, n) = (0, 0)$ in (3.58), gives a term linear in R in agreement with gauge threshold results for supersymmetric vacua with four or eight supercharges (cf (3.60)) (for sixteen and more supercharges, $\mathcal{C}(q, \bar{q}) = 0$). One interesting observation to make regarding (3.65) is that the gauge thresholds do not have a linear power dependence on R , which means that the gauge couplings do not blow up in the decompactification limit. It is clear that the $N = 4$ sector (entering (3.65) through the coefficients $\alpha_N^{\mathcal{T}_0}$ and $\beta_N^{\mathcal{T}_0}$) should show such a behaviour, because $N = 4$ supersymmetry is restored in the decompactification limit. This behaviour has also been demonstrated for heterotic models with $N = 4 \rightarrow N = 2$ Scherk–Schwarz

supersymmetry breaking [44]. However, we point out that the $N = 2$ sector (encoded in the coefficients $\alpha_N^{T_\theta}$ and $\beta_N^{T_\theta}$), which is broken to $N = 0$, also shows such a behaviour. Again, like the cosmological constant (cf the previous section), equation (3.65) takes the same analytic form as the corresponding *regularized* type I result [17].

3.5.2. One-loop Yukawa couplings in theories with broken supersymmetry. After having investigated the one-loop cosmological constant and the gauge threshold corrections in heterotic string vacua with supersymmetry broken by a Scherk–Schwarz mechanism, in this section we shall discuss the one-loop Yukawa couplings.

In four-dimensional effective supersymmetric string theories with canonical renormalized Einstein terms, the Yukawa interactions between massless matter fields appear in the form

$$\mathcal{L} \sim g_{string} e^{(1/2)G^{(0)}} W_{ijk} \psi^i \psi^j \phi^k + \text{h.c.} \quad (3.66)$$

The function W_{ijk} depends holomorphically on the moduli fields and $G^{(0)}$ is the matter- and dilaton-independent part of the Kähler potential. Thus for the simplest case of one modulus R in a string compactification from five to four dimensions, we have: $G^{(0)} = -2 \ln R$. Effective superstring theories are best described using conformal supergravity with a linear multiplet accounting for the dilaton superfield [45]. The scalar of the compensator is fixed (once) at tree level to get a canonical Einstein term. The first two additional factors $g_{string} e^{(1/2)G^{(0)}}$ in (3.66) result from this fixing. Thus they refer to tree-level quantities. Moreover in this set-up, the F -density encoding (3.66) may not depend on the linear multiplet, which governs the string-loop expansion. This has the consequence that there is no possibility that (3.66) may receive any perturbative corrections beyond one-loop level. Any possible renormalization of the Yukawa couplings has to arise from the D -density. More precisely, the physical Yukawa couplings λ_{ijk} are modified by corrections to the matter field Kähler metric $Z_{i\bar{i}} = Z_{i\bar{i}}^{(0)} + g_{string}^2 Z_{i\bar{i}}^{(1)} + \dots$:

$$\begin{aligned} \lambda_{ijk} &= g_{string} e^{(1/2)G^{(0)}} W_{ijk} (Z_{i\bar{i}} Z_{j\bar{j}} Z_{k\bar{k}})^{-1/2} \\ &\sim \lambda_{ijk}^{tree} \left[1 + g_{string}^2 \left(\frac{Z_{i\bar{i}}^{(1)}}{Z_{i\bar{i}}^{(0)}} + \frac{Z_{j\bar{j}}^{(1)}}{Z_{j\bar{j}}^{(0)}} + \frac{Z_{k\bar{k}}^{(1)}}{Z_{k\bar{k}}^{(0)}} \right) \right]^{-1/2}. \end{aligned} \quad (3.67)$$

The second line expresses the one-loop approximation. In particular, to obtain the fermion masses $m_{ij} = \lambda_{ijk} \langle h^k \rangle$, one of the metrics $Z_{j\bar{j}}$ must be related to the Higgs field h .

The question is how much of the above discussion can be taken over to non-supersymmetric effective string theories as they arise e.g. from the model discussed in section 3.2. The particular feature of toroidal orbifolds with supersymmetry broken by a Scherk–Schwarz mechanism is the survival of an $N = 1$ subsector, which is untouched by the supersymmetry breaking pattern. Thus its contribution to the effective action should still be organized by $N = 1$ supersymmetry, in particular by the linear multiplet formalism reviewed above. Thus Yukawa couplings which only involve matter fields from such an $N = 1$ sector should appear in the effective action in the form (3.66). Therefore, by the same arguments as above, the physical Yukawa couplings λ_{ijk} between twisted matter fields should only be corrected by their wavefunction renormalization (3.67).

Let us now come to string theory. There, the most canonical way to calculate possible one-loop corrections to Yukawa couplings is to consider the S -matrix $\langle V_\psi(z_1) V_\psi(z_2) V_\phi(z_3) \rangle$ of two fermions and one scalar. Depending on the region of integration z_i , this amplitude comprises both one-loop vertex corrections $|z_{12}|, |z_{13}|, |z_{23}| > \epsilon$ and one-loop propagator

corrections (to the external legs) from pinching, e.g. $z_1 \rightarrow z_2$. In a supersymmetric string vacuum, only the latter contributions give a non-vanishing result, whereas the one-loop vertex correction vanishes as a simple result of applying Riemann identities [46]. This has been explicitly demonstrated for heterotic orbifolds in [47]. This means that, in agreement with the above field theoretical arguments, Yukawa couplings are not renormalized at one-loop level in supersymmetric string theories (in fact, the whole holomorphic superpotential W is perturbatively not renormalized in supersymmetric string vacua [48]), although there are one-loop corrections $Z_{i\bar{i}}$ to the wavefunction, modifying the physical Yukawa couplings (3.67). The moduli-dependent corrections $Y_i := Z_{i\bar{i}}^{(1)} / Z_{i\bar{i}}^{(0)}$ appearing in equation (3.67) have been calculated for untwisted matter fields in the supersymmetric $N = 1$ orbifold models [50]. (The physical Yukawa couplings λ_{ijk}^{tree} between three untwisted or one untwisted and two twisted matter fields are constants. See [49] for further details on tree-level orbifold couplings.) There, the functions Y_i turned out to be proportional to the gauge threshold corrections [16].

On the other hand, in generic non-supersymmetric string vacua, the Yukawa string S -matrix may provide not only non-vanishing propagator corrections $Z_{i\bar{i}}^{(1)}$ but also non-zero vertex corrections δ_{ijk} , due to the lack of relevant Riemann identities. Therefore, in general the relation (3.67) is modified in non-supersymmetric vacua:

$$\lambda_{ijk} = \lambda_{ijk}^{tree} [1 - \frac{1}{2} g_{string}^2 (Y_i + Y_j + Y_k)] + g_{string}^2 \delta_{ijk}. \quad (3.68)$$

As in the case of gauge threshold corrections (section 3.5.1), the quantities Y_i receive radius-dependent contributions from the sectors \mathcal{T}_0 and \mathcal{T}_θ only. Like in the supersymmetric case, they take the same radius dependence as the one-loop gauge thresholds (3.65). It is a major task to determine δ_{ijk} for non-supersymmetric string vacua, as one has to deal with spin fields arising in the fermionic vertex V_ψ besides string world-sheet instanton corrections at one-loop level [51]. Though very important, a calculation of δ_{ijk} is beyond the scope of the present article and we leave this for an interesting future project. However, from the experience gained in section 3.5, it is obvious that the calculation of δ_{ijk} must result in an integral of the form (3.57), which always gives rise to power suppression in the radius R (cf equation (3.63)).

4. Conclusions

The paper investigated the properties of the scalar potential/vacuum energy in EFT models with one additional compact dimension, as well as in the heterotic and type I string compactifications.

In the EFT approach there are additional KK states associated with the extra dimension which significantly affect the scalar potential. The limit of summing the whole KK tower was investigated to show the absence at one-loop level of quadratic and logarithmic divergences in either the bosonic or the fermionic sector, respectively. In this mechanism, the presence of states of mass larger than the cut-off of the 4D effective theory played an important role [7]. This prompted us to investigate this problem in the context of string theory.

While the scalar potential/Higgs mass receives a finite (Yukawa) contribution in the ‘KK regularization’ limit at one-loop level, the dependence of the perturbative expansion on the Yukawa or gauge couplings may make this result UV sensitive at two-loop level and beyond. This dependence is due to the fact that the ‘running’ gauge and Yukawa couplings are UV sensitive in one-loop and higher orders. Gauge corrections can bring in a quadratic divergence even at one-loop level [9]. The non-renormalizable character of KK models may further enhance this UV sensitivity.

In section 3 we derived formulae for the one-loop cosmological constant (3.43) and gauge couplings (3.65) for generic $D = 4$ string compactifications with supersymmetry broken by a Scherk–Schwarz mechanism w.r.t. one internal circle of radius R . In particular, the one-loop gauge couplings show a logarithmic dependence (3.65) on the compactification scale R in contrast to the linear R -behaviour in the corresponding supersymmetric case (3.60). Our results, also relevant for $D = 5$ supersymmetric string theories at finite temperature $T = 1/(2\pi R)$, can be nicely rewritten in terms of polylogarithms appropriate for a comparison with field theory (cf (3.37), (3.63) and (2.41), (2.48)). We compared our heterotic string calculations, which are UV finite due to the power of world-sheet modular invariance or/and the interplay between KK and winding states, with corresponding type I calculations. In general, the latter need to be regularized like ordinary field theories. However, their finite part assumes the same analytic form (3.46) as the heterotic results. This correspondence at one-loop level may be a remnant of heterotic type I duality in nine dimensions.

Arguments in favour of a full string calculation were presented in the framework of general KK (orbifold) models. It has been shown that the limit of summing the *whole* KK tower provides a result compatible with a *full* (heterotic) string result which actually includes not only the effects of KK states, but also the effects due to winding modes as well, whose presence cannot be explained and is not accounted for by current field theory calculations or ‘KK regularization’. An explanation of why these two results are equal has been suggested in section 2.5: due to a discrete ‘shift symmetry’ in the absence of winding modes (see [23]), as a remnant of modular invariance in (heterotic) string theory. Further, the comparison of the effects of KK states alone in the EFT and string theory shows that they have different structures, and that the field theory limit of the string result requires a truncation of the KK tower to match it onto an EFT result with KK states of mass below some cut-off of the order of M_{string} . In this way the heterotic string can provide a physical regularization scheme for the 4D EFT result with a truncated tower of KK states.

From this analysis we conclude that radiative corrections generically tend to raise the value of the relevant physical quantities (as e.g. Higgs mass) towards the string scale M_{string} . Unless protected by a symmetry like low-energy softly broken supersymmetry, the UV sensitivity of physical quantities will be manifest, even in the case of a large extra dimension $1/R \ll M_{string}$.

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Appendix A. Kaluza–Klein sums

We have the following mathematical identities:

$$\sum_{k=-\infty}^{\infty} \ln \frac{\rho^2 + \pi^2(k + \omega')^2}{\rho^2 + \pi^2(k + \omega)^2} = \ln \frac{\cosh(2\rho) - \cos(2\pi\omega')}{\cosh(2\rho) - \cos(2\pi\omega)}. \quad (\text{A.1})$$

This relation can easily be proved using the following mathematical identity [12]:

$$\cosh(2\rho) - \cos(2\pi\omega') = 2\sin^2(\pi\omega') \prod_{k=-\infty}^{\infty} \left(1 + \frac{\rho^2}{\pi^2(k + \omega')^2}\right). \quad (\text{A.2})$$

We take the logarithm of this last equation which we then write for both ω and ω' and subtract the two results to reconstruct the left-hand side of equation (A.1). Further, we make use of the following identity [12]:

$$\frac{\sin \pi\omega'}{\sin \pi\omega} = \prod_{k=-\infty}^{\infty} \left(1 + \frac{\omega' - \omega}{k + \omega}\right). \quad (\text{A.3})$$

We take the logarithm of this expression and after some algebra we can easily recover from the last two equations the result of equation (A.1).

In the text we also made use of the following sum of the effects of a truncated tower of KK states, which is proved below:

$$\sum_{k=-l}^l \ln \frac{\rho^2 + \pi^2(k + \omega')^2}{\rho^2 + \pi^2(k + \omega)^2} = \mathcal{Z}_0 + \mathcal{Z}_1 + \mathcal{Z}_2 \quad (\text{A.4})$$

with the notation

$$\mathcal{Z}_0 = \ln \frac{\cosh(2\rho) - \cos(2\pi\omega')}{\cosh(2\rho) - \cos(2\pi\omega)} \quad (\text{A.5})$$

and

$$\mathcal{Z}_1 = \ln \frac{[\rho^2 + \pi^2(l \pm \omega')^2]_*}{[\rho^2 + \pi^2(l \pm \omega)^2]_*} \quad (\text{A.6})$$

and finally

$$\mathcal{Z}_2 = \ln \frac{[\Gamma(l \pm \omega' \pm i\rho/\pi)]_*}{[\Gamma(l \pm \omega \pm i\rho/\pi)]_*}. \quad (\text{A.7})$$

In these equations the symbol $[\rho^2 + \pi^2(l \pm \omega)^2]_*$ means that a product of the expression within $[\dots]_*$ with all combinations of plus and minus signs is considered (and similarly for $[\Gamma(l \pm \omega' \pm i\rho/\pi)]_*$). We observe that taking the limit $l \rightarrow \infty$ should recover the result of equation (A.1). Since the second term (\mathcal{Z}_1) in (2.7) has vanishing limit ($l \rightarrow \infty$, for fixed ρ), we conclude that the last term of (2.7) (\mathcal{Z}_2) has in such a case a vanishing limit as well. The integral of \mathcal{Z}_0 , \mathcal{Z}_1 , and \mathcal{Z}_2 with respect to ρ leads to the potentials \mathcal{V}_0 , \mathcal{V}_1 , and \mathcal{V}_2 respectively, equations (2.12)–(2.14) in the text. For the aforementioned reasons, in the limit of large l and fixed $\bar{\Lambda}$, the potentials \mathcal{V}_1 and \mathcal{V}_2 vanish.

To prove equation (A.4) we start with the following mathematical identity:

$$\begin{aligned} \sum_{k=-l}^l \ln(a + \pi(k + \omega')) \\ = -\ln(a + \pi\omega') + \ln \Gamma(1 + l - \omega' - a/\pi) + \ln \Gamma(1 + l + \omega' + a/\pi) \\ - \ln \Gamma(\omega' + a/\pi) - \ln \Gamma(-\omega' - a/\pi). \end{aligned} \quad (\text{A.8})$$

We write this equation with the replacement $a \rightarrow \pm i\rho$; then we rewrite the result obtained for the case with $\omega' \rightarrow \omega$; we then subtract the two results to reconstruct the truncated sum of the

left-hand side of (A.4). We find that

$$\begin{aligned} \sum_{k=-l}^l \ln \frac{\rho^2 + \pi^2(k + \omega')^2}{\rho^2 + \pi^2(k + \omega)^2} &= -\ln \frac{\rho^2 + \pi^2\omega'^2}{\rho^2 + \pi^2\omega^2} + \ln \frac{[\rho^2 + \pi^2(l \pm \omega')^2]_*}{[\rho^2 + \pi^2(l \pm \omega)^2]_*} \\ &\quad - \ln \frac{[\Gamma(\pm\omega' \pm i\rho/\pi)]_*}{[\Gamma(\pm\omega \pm i\rho/\pi)]_*} + \ln \frac{[\Gamma(l \pm \omega' \pm i\rho/\pi)]_*}{[\Gamma(l \pm \omega \pm i\rho/\pi)]_*} \end{aligned} \quad (\text{A.9})$$

where for the terms $\Gamma(1 + l \pm \omega' \pm i\rho/\pi)$ we made use of the relation

$$\Gamma[1 + x] = x\Gamma[x]. \quad (\text{A.10})$$

In equation (A.9) we apply the identity written out below [12]:

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \gamma)\Gamma(\beta - \gamma)} = \prod_{k=0}^{\infty} \left(1 + \frac{\gamma}{k + \alpha}\right) \left(1 - \frac{\gamma}{k + \beta}\right). \quad (\text{A.11})$$

More explicitly we consider the logarithm of this equality for the case $(\alpha, \beta \rightarrow \omega', \gamma \rightarrow i\rho/\pi)$ and also for the case $(\alpha, \beta \rightarrow -\omega', \gamma \rightarrow i\rho/\pi)$. This will provide us with an expression for the third term in (A.9), $\ln[\Gamma(\pm\omega' \pm i\rho/\pi)]_*$. We further replace the products of Γ functions of arguments of opposite signs, $\Gamma(-\omega')\Gamma(\omega')$, by the identity [12]

$$\Gamma[\omega']\Gamma[-\omega'] = -\frac{\pi}{\omega' \sin \pi\omega'}. \quad (\text{A.12})$$

Finally, in the result for the truncated sum (A.9) we replace the infinite series of (A.11) with the left-hand side of equation (A.2) to obtain the desired result, equation (A.4).

Appendix B. Asymptotic expansions for the potential

In this section we study the asymptotic behaviour of the product of Gamma functions

$$\ln \frac{[\Gamma(l \pm \omega' \pm i\rho/\pi)]_*}{[\Gamma(l \pm \omega \pm i\rho/\pi)]_*} \quad (\text{B.1})$$

when their arguments have a large modulus, i.e. when $|l + i\rho/\pi|$ is very large, without specifying the relationship between l and ρ . For this we use the following expansion, valid for large $|z|$ [12]:

$$\ln \Gamma(z) \approx z \ln z - z - \frac{1}{2} \ln z + \frac{1}{2} \ln(2\pi) + \frac{1}{12z} + \mathcal{O}(z^{-3}). \quad (\text{B.2})$$

We therefore apply this equation by replacing z with the various arguments of Gamma functions present in (B.1). After some algebra we find that

$$\begin{aligned} \ln \frac{[\Gamma(l \pm \omega' \pm i\rho/\pi)]_*}{[\Gamma(l \pm \omega \pm i\rho/\pi)]_*} &\approx \left\{ l - \frac{1}{2} \right\} \ln \frac{[\rho^2 + \pi^2(l \pm \omega')^2]_*}{[\rho^2 + \pi^2(l \pm \omega)^2]_*} \\ &\quad + \left\{ \frac{i\rho}{\pi} \ln \frac{[\pi(l \pm \omega') + i\rho]_*}{[\pi(l \pm \omega') - i\rho]_*} + \omega' \ln \frac{\pi^2(l + \omega')^2 + \rho^2}{\pi^2(l - \omega')^2 + \rho^2} - (\omega' \rightarrow \omega) \right\} \\ &\quad + \frac{\pi^2}{6} \left\{ \frac{l - \omega'}{\rho^2 + \pi^2(l - \omega')^2} + \frac{l + \omega'}{\rho^2 + \pi^2(l + \omega')^2} - (\omega' \rightarrow \omega) \right\}. \end{aligned} \quad (\text{B.3})$$

This expression is then integrated piecewise to provide an asymptotic behaviour of $\mathcal{V}_2(\phi)$ for large value of the absolute value of the complex number $l + i\bar{\Lambda}$, without a restriction on the relative behaviour of l and $\bar{\Lambda}$. The first term in (B.3), when integrated, gives a contribution proportional to \mathcal{V}_1 , so a cancellation of quadratic and logarithmic divergences as noticed for \mathcal{V}_1 applies in this case, too. For each of the other two terms, a similar cancellation takes place.

Appendix C. Kaluza–Klein integrals

In this section we provide useful mathematical formulae needed to evaluate various KK integrals. Their asymptotic behaviour will also be carefully investigated.

We would like to investigate the behaviour of the following KK integrals, for $\alpha = -2$, which were encountered in the text while evaluating the vacuum energy:

$$\mathcal{I}(\epsilon^2, \alpha) = \frac{1}{2} \int_{\epsilon^2}^{\infty} dt t^{\alpha-1} \sum_{k=-\infty}^{\infty} e^{-k^2 \pi t} \quad (\text{C.1})$$

and

$$\mathcal{J}(\epsilon^2, \alpha) = \frac{1}{2} \int_{\epsilon^2}^{\infty} dt t^{\alpha-1} \sum_{k=-\infty}^{\infty} e^{-(k+1/2)^2 \pi t}. \quad (\text{C.2})$$

As a first step, we provide a useful integral representation of the Riemann zeta function [12] valid for *all* complex/real values of its argument, α :

$$\zeta(2\alpha) = \frac{\pi^\alpha}{\Gamma(\alpha)} \left\{ \frac{1}{2\alpha(2\alpha-1)} + \mathcal{K} \right\}, \quad \mathcal{K} = \int_1^{\infty} dt (t^{-1/2-\alpha} + t^{\alpha-1}) \sum_{k=1}^{\infty} e^{-k^2 \pi t}. \quad (\text{C.3})$$

It is useful for our purposes to transform the integral of \mathcal{K} into a KK integral, with the lower limit of integration equal to $\epsilon^2 \rightarrow 0$:

$$\begin{aligned} \mathcal{K}(\alpha) &= \int_1^{\infty} dt t^{-1/2-\alpha} \sum_{k=1}^{\infty} e^{-k^2 \pi t} + \int_1^{\infty} dt t^{\alpha-1} \sum_{k=1}^{\infty} e^{-k^2 \pi t} \\ &= \int_{\epsilon^2}^1 dt t^{-3/2+\alpha} \sum_{k=1}^{\infty} e^{-k^2 \pi/t} + \int_1^{\infty} dt t^{\alpha-1} \sum_{k=1}^{\infty} e^{-k^2 \pi t} \\ &= \frac{1}{2} \int_{\epsilon^2}^1 dt t^{-3/2+\alpha} \left[\sum_{k=-\infty}^{\infty} e^{-k^2 \pi/t} - 1 \right] + \int_1^{\infty} dt t^{\alpha-1} \sum_{k=1}^{\infty} e^{-k^2 \pi t} \\ &= \frac{1}{2} \int_{\epsilon^2}^1 dt t^{-3/2+\alpha} \left[t^{1/2} \sum_{k=-\infty}^{\infty} e^{-k^2 \pi t} - 1 \right] + \frac{1}{2} \int_1^{\infty} dt t^{\alpha-1} \left[\sum_{k=-\infty}^{\infty} e^{-k^2 \pi t} - 1 \right] \\ &= \mathcal{I}(\epsilon^2, \alpha) - \frac{1}{2} \int_{\epsilon^2}^1 dt t^{-3/2+\alpha} - \frac{1}{2} \int_1^{\infty} dt t^{\alpha-1} \end{aligned} \quad (\text{C.4})$$

where in the last step a Poisson resummation over k has been performed under the first integral. We therefore find the following behaviour of $\mathcal{I}(\epsilon^2, \alpha = -2)$:

$$\mathcal{I}(\epsilon^2 \rightarrow 0, \alpha = -2) \equiv \frac{1}{2} \int_{\epsilon^2}^{\infty} \frac{dt}{t^3} \sum_{k=-\infty}^{\infty} e^{-k^2 \pi t} = \frac{3}{4\pi^2} \zeta(5) + \frac{1}{5} \epsilon^{-5}. \quad (\text{C.5})$$

To evaluate $\mathcal{J}(\epsilon^2, \alpha)$, we split the KK sum into sums over odd and even k and find that

$$\begin{aligned} \mathcal{I}(\epsilon^2, \alpha) &= \frac{1}{2} \int_{\epsilon^2}^{\infty} dt t^{\alpha-1} \sum_{k=-\infty}^{\infty} e^{-k^2 \pi t} \\ &= \frac{1}{2} \int_{\epsilon^2}^{\infty} dt t^{\alpha-1} \sum_{n=-\infty}^{\infty} e^{-4n^2 \pi t} + \frac{1}{2} \int_{\epsilon^2}^{\infty} dt t^{\alpha-1} \sum_{n=-\infty}^{\infty} e^{-4(n+1/2)^2 \pi t} \end{aligned} \quad (\text{C.6})$$

which after a rescaling of the integration variable t , $t \rightarrow t' = 4t$, gives

$$\mathcal{I}(\epsilon^2, \alpha) = 4^{-\alpha} \mathcal{I}(4\epsilon^2, \alpha) + 4^{-\alpha} \mathcal{J}(4\epsilon^2, \alpha). \quad (\text{C.7})$$

Using the expression for $\mathcal{I}(\epsilon^2, \alpha = -2)$ we find that

$$\mathcal{J}(\epsilon^2 \rightarrow 0, \alpha = -2) \equiv \frac{1}{2} \int_{\epsilon^2}^{\infty} \frac{dt}{t^3} \sum_{k=-\infty}^{\infty} e^{-(k+1/2)^2 \pi t} = \frac{-15}{16} \frac{3}{4\pi^2} \zeta(5) + \frac{1}{5} \epsilon^{-5}. \quad (\text{C.8})$$

We therefore find the following results for fermionic and bosonic KK integrals used in the literature:

$$\frac{1}{2} \int_{\epsilon^2 R^2}^{\infty} \frac{dt}{t^3} \sum_{k=-\infty}^{\infty} e^{-k^2 \pi t / R^2} = R^{-4} \mathcal{I}(\epsilon^2, \alpha = -2) = \frac{1}{R^4} \left\{ \frac{3}{4\pi^2} \zeta(5) + \frac{1}{5} \epsilon^{-5} \right\} \quad (\text{C.9})$$

and also

$$\frac{1}{2} \int_{\epsilon^2 R^2}^{\infty} \frac{dt}{t^3} \sum_{k=-\infty}^{\infty} e^{-(k+1/2)^2 \pi t / R^2} = R^{-4} \mathcal{J}(\epsilon^2, \alpha = -2) = \frac{1}{R^4} \left\{ \frac{-15}{16} \frac{3}{4\pi^2} \zeta(5) + \frac{1}{5} \epsilon^{-5} \right\} \quad (\text{C.10})$$

which shows that the two integrals above have divergences of the same type, in $\epsilon^2 = 0$, with equal coefficients multiplying it and cancelling in the difference. The origin of this divergence for $(\alpha = -2)$ can be traced back to the $k = 0$ KK mode added and subtracted under the first integral in the third line of equation (C.4). In terms of a dimensionful lower limit of integration, $\epsilon^2 = \epsilon^2 R^2$, $\epsilon \propto (\text{mass})^{-1}$, we find

$$\frac{1}{2} \int_{\epsilon^2}^{\infty} \frac{dt}{t^3} \sum_{k=-\infty}^{\infty} e^{-k^2 \pi t / R^2} = R^{-4} \mathcal{I}(\epsilon^2 / R^2, \alpha = -2) = \frac{1}{R^4} \frac{3}{4\pi^2} \zeta(5) + \frac{1}{5} \epsilon^{-5} R \quad (\text{C.11})$$

and also

$$\frac{1}{2} \int_{\epsilon^2}^{\infty} \frac{dt}{t^3} \sum_{k=-\infty}^{\infty} e^{-(k+1/2)^2 \pi t / R^2} = R^{-4} \mathcal{J}(\epsilon^2 / R^2, \alpha = -2) = \frac{1}{R^4} \frac{-15}{16} \frac{3}{4\pi^2} \zeta(5) + \frac{1}{5} \epsilon^{-5} R \quad (\text{C.12})$$

or

$$\frac{1}{2} \int_{\epsilon^2}^{\infty} \frac{dt}{t^3} \sum_{k=-\infty}^{\infty} [e^{-k^2 \pi t / R^2} - e^{-(k+1/2)^2 \pi t / R^2}] = \frac{1}{R^4} \frac{31}{16} \frac{3}{4\pi^2} \zeta(5) \quad (\text{C.13})$$

encountered in the text. The conclusion is that neither $\mathcal{I}(\epsilon^2 / R^2, \alpha = -2)$ nor $\mathcal{J}(\epsilon^2 / R^2, \alpha = -2)$ has any quadratic or logarithmic divergence after summing an infinite tower of states.

Using the same procedure, we also find the following result for the integrals used in section 3.5, equation (3.58), for $n = 2$, $D = 4$, $N = 0$:

$$\mathcal{I}(\epsilon^2, \alpha = 0, \Lambda) \equiv \frac{1}{2} \int_{\epsilon^2}^{\Lambda} \frac{dt}{t} \sum_{k=-\infty}^{\infty} e^{-k^2 \pi t} = \frac{1}{2} \gamma_E - \frac{1}{2} \ln(4\pi) + \epsilon^{-1} + \frac{1}{2} \ln \Lambda, \quad \Lambda \rightarrow \infty \quad (\text{C.14})$$

and

$$\mathcal{J}(\epsilon^2, \alpha = 0, \Lambda) \equiv \frac{1}{2} \int_{\epsilon^2}^{\Lambda} \frac{dt}{t} \sum_{k=-\infty}^{\infty} e^{-(k+1/2)^2 \pi t} = \epsilon^{-1} - \ln 2. \quad (\text{C.15})$$

Therefore the divergence ϵ^{-1} will cancel in the difference of the two integrals, which is equal to (for R finite)

$$\mathcal{P} \equiv \frac{1}{2} \int_{\epsilon^2 R^2 \rightarrow 0}^{\Lambda \rightarrow \infty} \frac{dt}{t} \sum_{k=-\infty}^{\infty} [e^{-k^2 \pi t / R^2} - e^{-(k+1/2)^2 \pi t / R^2}] = \frac{1}{2} (\gamma_E - \ln \pi + \ln \Lambda - \ln R^2). \quad (\text{C.16})$$

Note that \mathcal{P} is the Poisson-resummed form of equation (3.58). Indeed,

$$\mathcal{P} = R \int_{\epsilon^2 R^2}^{\infty} \frac{dt}{t^{3/2}} \sum_{n \geq 0} [1 - (-1)^n] e^{-n^2 \pi R^2 / t}. \quad (\text{C.17})$$

Note that the divergent part $\ln \Lambda$ is removed from the expression of \mathcal{P} , equation (C.16), by the massless modes accounted for by the ‘−1’ term subtracted from the partition function under the integral over the fundamental domain, which gives the general formula for the gauge thresholds [16]. For this reason $\ln \Lambda$ is not present in the final result, the last line of equation (3.63).

Appendix D. Lattice functions

It is convenient to introduce the lattice functions [37]

$$\begin{aligned} \mathcal{E}_0 &= \sum_{\substack{m \in \mathbf{Z} \\ n \in 2\mathbf{Z}}} e^{2\pi i \tau_1 m n} e^{-\pi \tau_2 [(m^2/R^2) + n^2 R^2]}, \\ \mathcal{E}_{1/2} &= \sum_{\substack{m \in \mathbf{Z} \\ n \in 2\mathbf{Z}}} e^{2\pi i \tau_1 (m+1/2)n} e^{-\pi \tau_2 [(m+1/2)^2/R^2 + n^2 R^2]}, \\ \mathcal{O}_0 &= \sum_{\substack{m \in \mathbf{Z} \\ n \in 2\mathbf{Z}}} e^{2\pi i \tau_1 m(n+1)} e^{-\pi \tau_2 [(m^2/R^2) + (n+1)^2 R^2]}, \\ \mathcal{O}_{1/2} &= \sum_{\substack{m \in \mathbf{Z} \\ n \in 2\mathbf{Z}}} e^{2\pi i \tau_1 (m+1/2)(n+1)} e^{-\pi \tau_2 [(m+1/2)^2/R^2 + (n+1)^2 R^2]}. \end{aligned} \quad (\text{D.1})$$

Poisson resummation on (D.1) leads to

$$\begin{aligned} \mathcal{E}_0 &= \frac{R}{\tau_2^{1/2}} \sum_{n, l \in \mathbf{Z}} [1 + (-1)^n] e^{-(\pi R^2/\tau_2)|l+n\tau|^2}, \\ \mathcal{E}_{1/2} &= \frac{R}{\tau_2^{1/2}} \sum_{n, l \in \mathbf{Z}} [1 + (-1)^n] (-1)^l e^{-(\pi R^2/\tau_2)|l+n\tau|^2}, \\ \mathcal{O}_0 &= \frac{R}{\tau_2^{1/2}} \sum_{n, l \in \mathbf{Z}} [1 - (-1)^n] e^{-(\pi R^2/\tau_2)|l+n\tau|^2}, \\ \mathcal{O}_{1/2} &= \frac{R}{\tau_2^{1/2}} \sum_{n, l \in \mathbf{Z}} [1 - (-1)^n] (-1)^l e^{-(\pi R^2/\tau_2)|l+n\tau|^2}. \end{aligned} \quad (\text{D.2})$$

The modular properties for these functions can be easily deduced:

$$\begin{aligned} \mathcal{E}_0(-1/\tau) &= \frac{1}{2} |\tau| (\mathcal{E}_0 + \mathcal{E}_{1/2} + \mathcal{O}_0 + \mathcal{O}_{1/2}), \\ \mathcal{E}_{1/2}(-1/\tau) &= \frac{1}{2} |\tau| (\mathcal{E}_0 + \mathcal{E}_{1/2} - \mathcal{O}_0 - \mathcal{O}_{1/2}), \\ \mathcal{O}_0(-1/\tau) &= \frac{1}{2} |\tau| (\mathcal{E}_0 - \mathcal{E}_{1/2} + \mathcal{O}_0 - \mathcal{O}_{1/2}), \\ \mathcal{O}_{1/2}(-1/\tau) &= \frac{1}{2} |\tau| (\mathcal{E}_0 - \mathcal{E}_{1/2} - \mathcal{O}_0 + \mathcal{O}_{1/2}), \end{aligned} \quad (\text{D.3})$$

and $\mathcal{E}_0(\tau + 1) = \mathcal{E}_0$, $\mathcal{E}_{1/2}(\tau + 1) = \mathcal{E}_{1/2}$, $\mathcal{O}_0(\tau + 1) = \mathcal{O}_0$, and $\mathcal{O}_{1/2}(\tau + 1) = -\mathcal{O}_{1/2}$.

Appendix E. Modular functions for gauge thresholds

In this appendix we present the modular functions needed for the gauge threshold calculation in section 3.5 of the model in section 3.2. For the sector \mathcal{T}_0 we have (the arrow symbolizes projection onto equal left/right masses, which appears after the τ_1 -integration, see also (3.9))

$$\begin{aligned}
 \mathcal{A}(q, \bar{q}) &= \frac{1}{72} \frac{E_4(q)}{\eta(\bar{q})^{12} \eta(q)^{24}} [\theta_3(\bar{q})^4 + \theta_4(\bar{q})^4 + E_2(\bar{q})] [E_4(q) E_2(q) - E_6(q)] [\mathcal{N}_{R=1}(q, \bar{q})]^5 \\
 &\rightarrow \sum_{N>0} \alpha_N^{\mathcal{T}_0} q^N \bar{q}^N, \\
 &= 480 + 9600 q^{1/4} \bar{q}^{1/4} + 76\,800 q^{1/2} \bar{q}^{1/2} + 307\,200 q^{3/4} \bar{q}^{3/4} + \dots, \\
 \mathcal{B}(q, \bar{q}) &= \frac{1}{72} \frac{E_4(q)}{\eta(\bar{q})^{12} \eta(q)^{24}} \{ [\theta_3(\bar{q})^4 + \theta_4(\bar{q})^4 + E_2(\bar{q})] E_4(q) + E_4(q) E_2(q) \\
 &\quad - E_6(q) \} [\mathcal{N}_{R=1}(q, \bar{q})]^5 \rightarrow \sum_{N>0} \beta_N^{\mathcal{T}_0} q^N \bar{q}^N \\
 &= \frac{1}{3} (1508 + 30\,080 q^{1/4} \bar{q}^{1/4} + 240\,000 q^{1/2} \bar{q}^{1/2} + 957\,440 q^{3/4} \bar{q}^{3/4} + \dots). \quad (\text{E.1})
 \end{aligned}$$

Of course, before the supersymmetry breaking, we have a vanishing β -function coefficient in the $N = 4$ sector:

$$b_{E'_8}^{\mathcal{T}_0} = 0 \quad (\text{E.2})$$

and upon introducing the Scherk–Schwarz shifts (3.21) we obtain

$$b_{E'_8}^{\mathcal{T}_0} = 480 \equiv \alpha_0^{\mathcal{T}_0}. \quad (\text{E.3})$$

For the sector \mathcal{T}_θ , we have the modular functions

$$\begin{aligned}
 \mathcal{A}(q, \bar{q}) &= -\frac{1}{72} \frac{1}{\eta(\bar{q})^6 \eta(q)^{18}} [\theta_3(\bar{q})^4 + \theta_4(\bar{q})^4 + E_2(\bar{q})] \mathcal{N}_{R=1}(q, \bar{q}) \\
 &\quad \times [E_4(q) E_2(q) - E_6(q)] \left\{ \left| \frac{\theta_2^2 \theta_3^2}{\theta_4^2} \right|^2 [\theta_3(q)^4 + \theta_2(q)^4] + \left| \frac{\theta_2^2 \theta_4^2}{\theta_3^2} \right|^2 [\theta_2(q)^4 - \theta_4(q)^4] \right\} \\
 &\rightarrow \sum_{N>0} \alpha_N^{\mathcal{T}_\theta} q^N \bar{q}^N, \\
 &= -7680 q^{1/2} \bar{q}^{1/2} - 30\,720 q^{3/4} \bar{q}^{3/4} - 1536\,000 q \bar{q} - 5898\,240 q^{5/4} \bar{q}^{5/4} \dots \quad (\text{E.4})
 \end{aligned}$$

Now in the $N = 2$ sector alone, before supersymmetry breaking we encounter the usual E'_8 β -function coefficient for standard gauge embedding (see e.g. [33]):

$$b_{E'_8}^{\mathcal{T}_\theta} = 60 \quad (\text{E.5})$$

and after introducing the Scherk–Schwarz shifts (3.21) we determine

$$b_{E'_8}^{\mathcal{T}_\theta} = -30 \quad (\text{E.6})$$

$$\begin{aligned}
 \mathcal{B}(q, \bar{q}) &= -\frac{1}{72} \frac{1}{\eta(\bar{q})^6 \eta(q)^{18}} \left\{ \left| \frac{\theta_2^2 \theta_3^2}{\theta_4^2} \right|^2 [\theta_3(q)^4 + \theta_2(q)^4] + \left| \frac{\theta_2^2 \theta_4^2}{\theta_3^2} \right|^2 [\theta_2(q)^4 - \theta_4(q)^4] \right\} \\
 &\quad \times \{ E_4(q) E_2(q) - E_6(q) + [\theta_3(\bar{q})^4 + \theta_4(\bar{q})^4 + E_2(\bar{q})] E_4(q) \} \mathcal{N}_{R=1}(q, \bar{q})
 \end{aligned}$$

$$\begin{aligned}
& \rightarrow \sum_{N>0} \beta_N^{\mathcal{T}_\theta} q^N \bar{q}^N \\
& = -\frac{1}{3} (128 + 512 q^{1/4} \bar{q}^{1/4} + 24\,000 q^{1/2} \bar{q}^{1/2} \\
& \quad + 95\,744 q^{3/4} \bar{q}^{3/4} + 3197\,440 q \bar{q} + \dots).
\end{aligned} \tag{E.7}$$

Appendix F. Partition functions for Wilson line dependence

In this section we want to present the partition functions needed in section 3.4 for the model of 3.2 when the Wilson line (3.47) is introduced.

Following the explanations in section 3.4, the partition function (3.26) of the $N = 4$ sector \mathcal{T}_0 is modified to

$$\sum_{\substack{M>-1 \\ N \geq 0}} \gamma^{\mathcal{T}_0}(M, N) q^M \bar{q}^N = \eta^{-12}(\bar{q}) \eta^{-24}(q) \theta_2(\bar{q})^4 E_4(q) [E_{4,1}^{\text{even}}(q) + E_{4,1}^{\text{odd}}(q)] \mathcal{N}_{SO(5,5)}(q, \bar{q}). \tag{F.1}$$

Furthermore, for the $N = 2$ sector \mathcal{T}_θ , we obtain

$$\sum_{\substack{M>-1/2 \\ N \geq 0}} \gamma^{\mathcal{T}_\theta}(M, N) q^M \bar{q}^N = -\frac{E_{4,1}^{\text{even}}(q) + E_{4,1}^{\text{odd}}(q)}{\eta^{18} \bar{\eta}^6} \left[\left| \frac{\theta_2^2 \theta_3^2}{\theta_4^2} \right|^2 (\theta_3^4 + \theta_2^4) + \left| \frac{\theta_2^2 \theta_4^2}{\theta_3^2} \right|^2 (-\theta_4^4 + \theta_2^4) \right]. \tag{F.2}$$

The combination of the two coefficients $\gamma^{\mathcal{T}_0}(M, N)$, $\gamma^{\mathcal{T}_\theta}(M, N)$ gives the total contribution, $\gamma(M, N) = \frac{1}{4} \gamma^{\mathcal{T}_0}(M, N) + \frac{1}{4} \gamma^{\mathcal{T}_\theta}(M, N)$, needed in (3.56).

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