Isolated singular points and the Gauss-Manin connection

D.Phil. Thesis

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Abstract

This thesis studies the Gauss - Manin connection for the Milnor fibration of a complex hypersurface with an isolated singular point. Using the formalism developed by Brieskorn and Malgrange, we give an explicit basis, with respect to which the connection has a simple pole and its residue matrix has a manageable form. We calculate the residue for several examples. The method of calculation enables us to give some examples relevant to questions arising in Malgrange's work on the Bernstein polynomial. We also make several conjectures related to the monodromy theorem.

Errata

page	11	line 5	C-linear
	16	7	cuts S_ε, transversely for $\varepsilon' \leqslant \varepsilon$
	17	18	locally constant
	18	14	simple pole
	22	22	parallel
	23	15	of the equation y'=Ay on page 18
	27	14	They occur at most
	49	19	within

Introduction

Suppose $f = f(x_0,\ldots,x_n)$ is an analytic function defined in an open ball B about 0 in C^{n+1}. Assume that $f(0) = 0$ and that 0 is the only critical point of f in B. Milnor [Mi] proves the following theorem: if B is chosen small enough, there exists a small punctured disc T' about 0 in C such that

$$f: X' = f^{-1}(T') \cap B \to T'$$

is a C^∞ fibre bundle, whose fibre $f^{-1}(t_0)$ has the homotopy type of a bouquet of n-spheres. The fundamental group $\pi_1(T',t_0)$ then acts on $H_n(f^{-1}(t_0),Z)$, and by choosing a generator of $\pi_1(T',t_0)$, one determines an automorphism h_* of $H_n(f^{-1}(t_0),Z)$, called the monodromy of f. h_* does not determine the topology of the pair $(B, B\cap f^{-1}(t_0))$ completely, but it can be used to prove some striking results, e.g.[Mi], pp. 68,69.

There is another way of looking at the monodromy, which has its origins in the theory of the hypergeometric equation (see introduction of [M]): $H^n = \bigcup_{t\varepsilon T'} H^n(f^{-1}(t), C)$ is a flat complex vector bundle over T'. Therefore it carries a canonical integrable connection. Now $H^n \to T'$ is trivialisable as an analytic vector bundle. With respect to any such trivialisation, the connection determines a system of ordinary linear differential equations

$$y'(t) = A(t)y(t),$$

where A(t) is a matrix of holomorphic functions on T'. The solutions of this equation are essentially period integrals (cf.4.4) and its monodromy is just h_*. This connection is called the Gauss-Manin connection of f.

In [B], Brieskorn describes the connection more explicitly. He remarks that any analytic section of $H^n \to T'$ can be represented by a family of holomorphic n-forms

$$\omega(t) \in \Gamma(f^{-1}(t), \Omega^n_{f^{-1}(t)}) \qquad t \in T'$$

depending analytically on t. (Each fibre is an n-dimensional Stein manifold.) In fact as he proves, there exist μ n-forms

$$\omega_1, \ldots, \omega_\mu \in \Gamma(B, \Omega^n_B)$$

where $\mu = \dim H^n(f^{-1}(t), C)$, such that the sections

$$w_j(t) = \omega_j|_{f^{-1}(t)} \qquad t \in T', \quad j=1, \ldots, \mu$$

provide a trivialisation of $H^n \to T'$. The corresponding matrix A of the connection has a pole at 0, and one can show ([B], [G], or [M1]) that the system of equations $y' = Ay$ is regular singular, ie. there exists a meromorphic substitution $x = Qy$ such that the transformed equation has a simple pole at 0.

Brieskorn gives an algebraic formula for the Gauss-Manin connection applied to sections of the form $w(t) = \omega|_{f^{-1}(t)}$, $\omega \in \Gamma(B, \Omega^n_B)$, with which the complex monodromy can be computed in a finite number of steps ([B], p.148). However, his paper leaves one with considerable practical problems. One goal of this thesis is to resolve these difficulties to such an extent that one can calculate the connection for interesting functions in a reasonable length of time. In particular, theorem 26 gives a trivialisation of the form above such that the corresponding system of equations has a simple pole.

Recent work suggests that there are perhaps more important reasons for wanting to compute the Gauss-Manin connection:

Suppose f is a real analytic function. Using the existence of a resolution of the singularity $0 \in f^{-1}(0)$, one can show that

$$I(\tau) = \int_{R^{n+1}} e^{i\tau f} \varphi dx_0 \cdots dx_n \sim \sum_{\alpha,p,q} c_{\alpha pq}(\varphi) \tau^{\alpha-p} (\log \tau)^q \qquad \text{as } \tau \to \infty$$

where $\varphi \in C^\infty_c(R^{n+1})$, $p,q \in N$, $0 \leqslant q \leqslant n$, and $\alpha \in A$, a finite set of negative numbers and $c_{\alpha pq}$ is a distribution with support 0 ([M1],p.406)

In [M1] Malgrange shows that the numbers α are eigenvalues of the Gauss-Manin connection . Discussions with K. Saito and A. Varchenko suggest that one may be able to define a "residue" of the connection whose set of eigenvalues is precisely A.

A second application occurs in the study of deformations of f: as is shown in the appendix of [B],

$$\mu = \dim_{\mathbb{C}} \mathbb{C}\{x_0,\ldots,x_n\}/(\partial f/\partial x_0,\ldots,\partial f/\partial x_n).$$

Let $1,\varphi_1,\ldots,\varphi_{\mu-1}$ be a basis of $\mathbb{C}\{x_0,\ldots,x_n\}/(\partial f/\partial x_0,\ldots,\partial f/\partial x_n)$, and set

$$f(x,s_1,..,s_{\mu-1}) = f(x) + s_1\varphi_1(x) + \ldots + s_{\mu-1}\varphi_{\mu-1}(x) \quad x=(x_0,..,x_n)$$

Define $F: \mathbb{C}^{n+\mu} \to \mathbb{C}^{\mu}$ by $F(x,s) = (f(x,s),s)$ where $s=(s_0,..,s_{\mu-1})$. F is a versal deformation of f ie. any other deformation can be induced from it. Let C be the critical set of F and $\Delta=F(C)$ the discriminant. As above, there exists an open ball $S \subset \mathbb{C}^{\mu}$ about 0 and a ball B about 0 in $\mathbb{C}^{n+\mu}$ such that

$$F: Y' = F^{-1}(S') \cap B \to S' \quad , \quad S'=S\setminus\Delta$$

is a C^{∞} fibre bundle. This bundle restricted to $s=0$ is just the bundle defined by f. Again $H^n = \bigcup_{s\in S'} H^n(F^{-1}(s), \mathbb{C})$ is a flat vector bundle and the canonical connection on it is also called the Gauss-Manin connection. This can be computed in terms of the Gauss-Manin connection of f. It is very useful in studying the action of the fundamental group $\pi_1(S',s_0)$ on $H^n(F^{-1}(s_0))$. In recent work on the period mapping $P: S \to H^n(F^{-1}(s_0))$, K. Saito in fact made use of some of the calculations of this thesis.

Lastly, I must mention the relation between the Gauss-Manin connection and the Bernstein polynomial, explained by Malgrange in [M2]. Let $\mathcal{D}$ be the ring of linear differential operators in $\partial/\partial x_0,\ldots,\partial/\partial x_n$ with coefficients in $\mathbb{C}\{x_0,\ldots,x_n\}$. Bernstein, Björk, and Kashiwara

([K2], §3) have shown that given $f \in C\{x_0,\ldots,x_n\}$, there exists $P(s) \in \mathcal{D}[s]$ such that

$$P(s)f^s = B(s)f^{s-1} , \qquad B(s)\in C[s].$$

The polynomials B(s) which satisfy such an equation form an ideal in C[s]. So let b(s) be a monic generator of this ideal; b is called the Bernstein polynomial of f. By setting s=0 above, one sees that $b(s) = s\tilde{b}(s)$, $\tilde{b} \in C[s]$. Malgrange proves that if f has an isolated singularity in 0, then $\tilde{b}(-s)$ is the minimal polynomial of a residue of the Gauss-Manin connection (see 6.7 below). Surprisingly b(s) is not a topological invariant , but can jump within well-behaved families of functions (in the sense of theorem 29). The point of view taken here makes it perhaps clearer why such behavior can be expected.

The second goal of this thesis was a "purely algebraic" proof of the monodromy theorem:

"The eigenvalues of h_* are roots of unity and occur with multiplicity $\leq n+1$ in the minimal polynomial of h_* ."

An attractive conjecture in this direction is the following:

"If $f^{r+1} \in (\partial f/\partial x_0,\ldots,\partial f/\partial x_n)$, then the zeroes of the minimal polynomial of h_* have multiplicity $\leq r+1$."

(In [B-S] Briançon and Skoda show that for any $f \in C\{x_0,\ldots,x_n\}$, with $f(0) = 0$, $f^{n+1} \in (\partial f/\partial x_0,\ldots,\partial f/\partial x_n)$.) Some explanation will be given why this conjecture should hold true (see 5.10); but unfortunately I did not succeed in proving it.

I would like to thank my supervisor Prof. I.M. James for his encouragement and patience with my efforts. The problems discussed in this thesis are due to Prof. E.V. Brieskorn. His, too, is the underlying philosophy that one should try to study the invariants of an isolated hypersurface singularity using the structure of the

ring $C\{x_0,\ldots,x_n\}/(\partial f/\partial x_0,\ldots,\partial f/\partial x_n)$. I am grateful to him, and to many other members and guests of the Sonderforschungsbereich 40 in Bonn, for much help and advice. Lastly I wish to thank Prof. B. Malgrange for a very stimulating letter and conversation.

The thesis is organized as follows:

Section 1: A discussion of systems of ordinary linear differential equations in the punctured disc, and their monodromy; in particular systems of regular singular equations p.6

Section 2: Some basic facts about Stein manifolds p.14

Section 3: The definition of the Milnor fibration and the Gauss-Manin connection p.16

Section 4: Brieskorn's version of the Gauss-Manin connection, regularity, and the monodromy theorem p.20

Section 5: Calculating the residue of the Gauss-Manin connection; in particular, a basis which gives a matrix with a simple pole p.28

Section 6: An example calculated in detail and some counterexamples p.38

Section 7: Further examples p.52

T_{pqr}: $x^p+ y^q+ z^r+ axyz$, $a\neq 0$, $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$ p.52

$x^8+ y^8+ z^8+ x^2y^2z^2$ p.66

$x^{p+q}+ y^{p+q}+ x^qy^q$ $((x^p+ y^q)(x^q+ y^p))$ $p>q\geq 2$ p.72

$Q_{k,i}$: $x^3+ yz^2+ x^2y^k+ b_0y^{3k+i}+ b_1y^{3k+i+1}+ \cdots + b_{k-1}y^{4k+i-1}$
$k>1$, $i>0$, $b_0\neq 0$ p.75

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§1 Differential equations on the punctured disc

1.1 Let $T = \{t \in C \mid |t| < \varepsilon\}$, $\varepsilon > 0$ and set $T' = T \setminus 0$. Let $p: U \to T'$ be the universal covering space of T'. Suppose $A(t)$ is an $n \times n$ matrix of holomorphic functions on T'. In this section we shall consider the homogeneous system of linear differential equations

$$y' = Ay \quad (1)$$

On U we can solve this system of equations analytically. So let $B(u)$ be a fundamental matrix on U. (For example, choose a point $u_0 \in U$ and take $B(u) = \exp \int_{u_0}^{u} A(v)dv$.) For convenience we shall regard B as a matrix of many-valued functions on T' and write $B(t)$.

Since $B(te^{2\pi i})$ is also a fundamental matrix, we have

$$B(te^{2\pi i}) = B(t)M \qquad ([C\text{-}L], p.70)$$

for some $M \in Gl(n,C)$. M is called the monodromy of the system (1), with respect to the fundamental matrix B. If $C(t) = B(t)N$, $N \in GL(n,C)$ is another fundamental matrix, then

$$C(te^{2\pi i}) = B(t)MN = C(t)(N^{-1}MN) ;$$

so $N^{-1}MN$ is the monodromy with respect to C.

Now pick a constant matrix R such that $M = \exp(2\pi iR)$. Then we see (cf. [C-L],p.109) that

$$P(t) = B(t)t^{-R} = B(t)\exp(-R\log t)$$

is a matrix of single-valued holomorphic functions on T': for

$$P(te^{2\pi i}) = B(te^{2\pi i})e^{-2\pi iR}e^{-R\log t} = B(t)t^{-R} = P(t).$$

1.2 We shall be concerned only with a special class of equations, which are characterized in the theorem below:

Theorem 1: The following properties of a system of equations $y' = Ay$ are equivalent:

(i) Let $B(t)$ be any fundamental matrix; write

$$B(t) = P(t)t^{R}$$

as above. Then the entries of P have a pole at 0.

(ii) Suppose $y(t) = (y_1(t),\ldots,y_n(t))$ is a solution of $y' = Ay$. Then in any sector $t\varepsilon T'$, $a \leq \arg t \leq b$, $b < a+2\pi$, every branch of y satisfies

$$|y_j(t)| = o(|t|^{m}) \text{ , for some integer } m,\ 1 \leq j \leq n .$$

(iii) There exists an invertible matrix $Q(t)$ of holomorphic functions on T' with a pole at 0, such that the substitution $x = Qy$ transforms $y' = Ay$ into an equation with a simple pole at 0,

ie. $x' = Q'y + Qy' = (Q'Q^{-1} + QAQ^{-1})x$

and $\overline{A} = Q'Q^{-1} + QAQ^{-1}$ has a simple pole at 0.

(In particular A is meromorphic on T.)

Definition: A system of equations with these properties is called regular singular.

Proof: It is easy to see that (i) $\Leftrightarrow$ (ii) . (iii) $\Rightarrow$ (ii) is a theorem of Fuchs. For a proof due to Birkhoff see [C-L], p.112. (i) $\Rightarrow$ (iii) is also simple: just use the substitution $x = P^{-1}y$. Then t^{R} is a fundamental matrix of the transformed system, which therefore has the form

$$x' = t^{-1}Rx \quad . \tag{2}$$

(Of course any equation $y' = Ay$ can be brought into this form by the substitution $x = P^{-1}y$, but P will not always be meromorphic.)

1.3 The monodromy M determines the form of the solutions of the equation (1): Suppose R is in Jordan normal form. Then each entry of t^R is of the form

$$\frac{t^{\alpha}(\log t)^r}{r!}$$

where α is an eigenvalue of multiplicity $\geq r+1$ in the minimal polynomial of R. Therefore the entries of $B(t) = P(t)t^R$ will, for arbitrary R, have the form

$$\sum_{\alpha,r} c_{\alpha,r} t^{\alpha}(\log t)^r$$

where (i) $\exp 2\pi i\alpha$ is an eigenvalue of multiplicity $\geq r+1$ in the minimal polynomial of the monodromy

(ii) $\alpha \geq m$ for some integer m.

1.4 When we calculate the Gauss - Manin connection, we will not usually end up with an equation that looks like (2). The best we can expect seems to be one of the form

$$y' = t^{-1}A_{-1}y + A_o y \tag{3}$$

where A_o is holomorphic in T and A_{-1} is constant. A_{-1} is called the residue of the equation.

Unfortunately such a residue needn't be a logarithm of the monodromy, as the following example shows: Let

$$y_1' = t^{-1}y_1 + y_2$$

$$y_2' = 0$$

Thus the residue of this system of equations is $A_{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

They have the solutions

$$y_1 = at + btlogt$$

$$y_2 = b \qquad a,b \in C$$

Their monodromy, with respect to the basis $y_1 = t$, $y_2 = 0$ and $y_1 = tlogt$, $y_2 = 1$ is

$$M = \begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix}$$

So $\exp 2\pi i A_{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \neq M$. But if we make the substitution

$$x_1 = t^{-1}y_1 \quad x_2 = y_2$$

the equations become

$$x_1' = t^{-1}x_2$$

$$x_2' = 0$$

and we have $M = \exp 2\pi i R$ where $R = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

The following theorem explains when A_{-1} will be a logarithm of the monodromy.

Theorem 2: ([C-L], p.119) If the eigenvalues of A_{-1} do not differ by positive integers, then $M = \exp 2\pi i A_{-1}$.

If A_{-1} does not satisfy the condition of theorem 2, then one must make a substitution $x = Qy$, Q meromorphic , so that the residue of

the new equation is well-behaved. That this is possible is shown by the following lemma:

Lemma 3: ([C-L], p.120) Let α be an eigenvalue of A_{-1}. Suppose A_{-1} is in Jordan normal form and

$$A_{-1} = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}$$

where S_1 is an $m \times m$ matrix consisting of all the blocks of A_{-1} with eigenvalue α, $m \leq n$. Define a substitution $x=Qy$ by

$$Q = \begin{pmatrix} t^{-1}I_m & 0 \\ 0 & I_{n-m} \end{pmatrix}$$

Then if $x' = (t^{-1}\bar{A}_{-1} + \bar{A}_0)x$ is the transformed system, $\bar{A}_{-1}$ has the form

$$\bar{A}_{-1} = \begin{pmatrix} S_1 - I_m & A^* \\ 0 & S_2 \end{pmatrix}$$

for some $m \times (n-m)$ matrix A^*.

A sequence of such substitutions will transform (3) into an equation to which theorem 2 applies. Thus $\exp 2\pi i\alpha$ will be an eigenvalue of the monodromy M. However, as the example above shows, the blocks in the Jordan normal form of A_{-1} belonging to α may be smaller than the blocks in the Jordan normal form of M belonging to $\exp 2\pi i\alpha$.

1.5 It is convenient to formalize the theory above in the following way ([D], pp.41,52 or [G-L]). Let $k = C\{t\}[t^{-1}]$ be the quotient field of the ring of convergent power series $C\{t\}$. Let V be an n-dimensional vector space over k.

Definition: A connection on V is a C-linear mapping

$$\nabla: \quad V \to \Omega^1_{C,0} \otimes_{C\{t\}} V$$

such that $\nabla(av) = da \otimes v + a\nabla v$, $a(t) \in k$, $v \in V$.

For our purposes it is sufficient to consider only the covariant derivative $D = \nabla_{\frac{d}{dt}}$ along the vector field $\frac{d}{dt}$. Thus $D: V \to V$ is C-and satisfies

$$D(av) = a'v + aDv \qquad \text{(Leibniz's rule)}$$

In what follows, a mapping D with these properties will also be called a connection.

Let $v_1,\ldots,v_n$ be a k-basis of V. Suppose

$$Dv_k = \sum_{j=1}^{n} a_{jk}v_j \qquad a_{jk}(t) \in k \ .$$

Set $A(t) = -(a_{jk}(t))$. Then if $v = \sum_{j=1}^{n} y_j v_j$, $y_j \in k$

$$Dv = \sum_{j=1}^{n} y_j' v_j + \sum_{j,k} y_k a_{jk} v_j \ .$$

Choose a punctured disc T' about 0 in which A converges. Then one can associate the system of equations $y' = Ay$ on T' to the connection D. Notice that the system of equations corresponding to any other basis of V can be obtained from $y' = Ay$ by a meromorphic substitution. Conversely, take a system of n equations $y' = Ay$, where A is holomorphic in some punctured disc about 0, and has a pole in 0. Define

$$D: k^n \to k^n$$

by

$$Dy = y' - Ay \qquad y \in k^n \ .$$

It is easy to check that B is a connection. To avoid working with bases, we make the following definition.

Definition: A lattice $L \subset V$ is a finitely generated $C\{t\}$-submodule such that $k \cdot L = V$.

Thus a lattice is a free submodule of rank n. Furthermore, if M is a second lattice, one sees that for some $k \varepsilon Z$

$$t^k L \subset M \quad .$$

Definition: A connection D on a space V is called regular if there exists a lattice $L \subset V$ such that $(tD)L \subset L$.

If we pick a basis $v_1, \ldots, v_n$ of such an L, then the matrix A of D with respect to this basis will have a simple pole. Therefore the system of equations corresponding to any basis of V will be regular singular.

When we are discussing the Gauss-Manin connection, the following characterization of regularity will be very useful.

Theorem 4: (cf. 4.1 in [G-L]) Let $L \subset V$ be any lattice. Then a connection $D: V \to V$ is regular if and only if the $C\{t\}$ module

$$\tilde{L} = L + (tD)L + \ldots + (tD)^r L + \ldots \subset V$$

is a lattice.

Proof: (i) Suppose D is regular. Clearly

$$L_r = L + (tD)L + \ldots + (tD)^r L \qquad r \geq 0$$

is a lattice and $L = L_0 \subset L_1 \subset \ldots \subset L_r \subset \ldots \subset \tilde{L}$. We will show that for r large enough

$$L_r = L_{r+1} = \ldots . \quad .$$

By definition, there exists a lattice M with $(tD)M \subset M$. For some $k \varepsilon Z$, $t^k L \subset M$. Now $t^k L$ is a lattice and using Leibniz's rule it is easy to see that $(t^k L)_r = t^k L_r$. So we can assume that $L \subset M$. But then

$$L \subset L_1 \subset \ldots \subset L_r \subset \ldots \subset M \quad .$$

$C\{t\}$ is Noetherian; therefore for some r

$$L_r = L_{r+1} = \ldots = \tilde{L} \quad .$$

(ii) Suppose $\tilde{L}$ is a lattice. By construction $(tD)\tilde{L} \subset \tilde{L}$.

Corollary 5: D is regular if and only if, for any $v \varepsilon V$, the cyclic submodule generated by v, ie. the $C\{t\}$ module generated by $v, tDv, \ldots, (tD)^r v, \ldots$, is finitely generated.

Notice that $L_r = L + DtL + \ldots + (Dt)^r L = L + DtL + \ldots + D^r t^r L$.

Example: Let $V = k$, $L = C\{t\} \subset k$, and $Dy = y' - t^{-2}y$, $y \varepsilon k$. Then L_r is generated by t^{-r}. So $\tilde{L} = k$, which is not finitely generated over $C\{t\}$.

Definition: Suppose the connection D on V has a simple pole on the lattice L ie. $(tD)L \subset L$. Then the residue of D on L is the endomorphism $\overline{tD}$ of L/tL induced by tD.

If $v_1, \ldots, v_n$ is a basis of L and $\bar{v}_1, \ldots, \bar{v}_n$ is the induced basis of L/tL , then the matrix of $\overline{tD}$ with respect to $\bar{v}_1, \ldots, \bar{v}_n$ is the residue of the system of equations corresponding to $v_1, \ldots, v_n$.

§2 Some basic facts about Stein manifolds

Definition: (see [Gr-F], pp.55,135) A complex manifold X is a Stein manifold if

(i) given an infinite discrete set $\{x_m\}_{m=1}^{\infty} \subset X$, there exists $f \in \Gamma(X,\mathcal{O}_X)$ such that $\sup_m |f(x_m)| = \infty$.

(ii) given $x_0 \in X$, there exist $f_1,\ldots,f_l \in \Gamma(X,\mathcal{O}_X)$ so that x_0 is an isolated point in $\{f_1(x) = \ldots = f_l(x) = 0\}$

Any open convex set in C^n is a Stein manifold ([Gr-F], p.46)

Lemma 6: If X is a Stein manifold, then

(a) every closed submanifold is Stein,

(b) if $f \in \Gamma(X,\mathcal{O}_X)$ then $X \setminus f^{-1}(0)$ is Stein,

(c) suppose Y is a complex manifold, $Z \subset Y$ a polycylinder, and $f: X \to Y$ holomorphic; then $f^{-1}(Z)$ is also Stein.

Proof: (a) and (b) follow immediately from the definition. So does property (ii) for (c). To see property (i) for (c), consider

$$X \times_Y Z = \{(x,z) \in X \times Z \mid f(x) = z\} \simeq f^{-1}(Z) .$$

It is a closed subset of $X \times Z$, which satisfies (i).

The most important results on Stein manifolds are

Theorem A: Let S be a coherent $\mathcal{O}_X$ module. Then for all $x \in X$, $\Gamma(X,S)$ generates S_x as an $\mathcal{O}_{X,x}$ module.

Theorem B: If S is a coherent $\mathcal{O}_X$ module, then

$$H^q(X,S) = 0 \quad \text{for} \quad q > 0 \ .$$

see [Gr-F], p.182 .

Corollary 7: If Y is a complex manifold and $f: X \to Y$ an analytic map, then

$$R^q f_* S = 0 \quad \text{for} \quad q > 0 \ .$$

Proof: Let $Z \subset Y$ be a polycylinder. Then by lemma 6(c) and theorem B

$$H^q(f^{-1}(Z), S) = 0 \quad \text{for} \quad q > 0 \ .$$

A relative version of the following result will be needed later.

Theorem 8: $H^q(X, C) \simeq \ker d\ (\Gamma(\Omega_X^q) \to \Gamma(\Omega_X^{q+1}))\ /\ d\Gamma(\Omega_X^{q-1}) = H^q(\Gamma(\Omega_X^\cdot))$.

Proof: See appendix.

Lastly we recall a theorem of Grauert:

Theorem 9: ([Gr], p.268) Two analytic vector bundles on X which are topologically equivalent are also analytically equivalent.

§3 Milnor's fibration and the Gauss-Manin connection

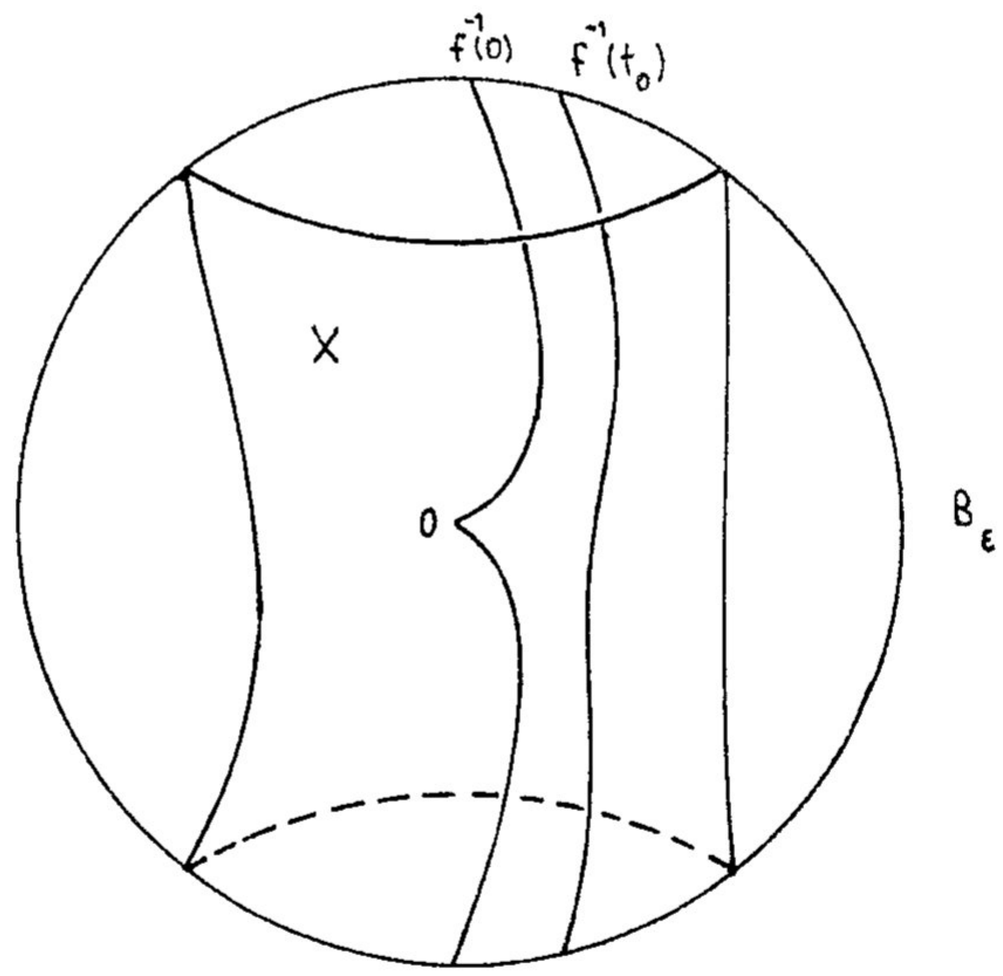
3.1 Suppose $f \in C\{x_0,\ldots,x_n\}$, $f(0) = 0$, and 0 is an isolated critical point of f. For $\varepsilon > 0$, let $B_\varepsilon = \{x \in C^{n+1} \mid |x| < \varepsilon\}$, $S_\varepsilon = \partial B_\varepsilon$. Choose ε small enough, that

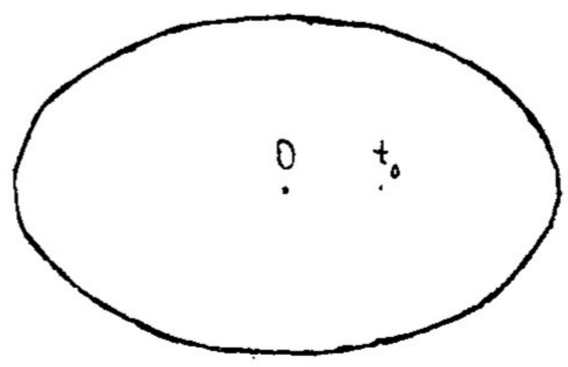
(i) f converges in a neighbourhood of $\overline{B}_\varepsilon$

(ii) 0 is the only critical point of f in $\overline{B}_\varepsilon$

(iii) $f^{-1}(0)$ cuts S_ε transversely. (This is possible by [Mi], p.17 .)

Then there exists $\eta > 0$ such that for all $t \in C$, $|t| < \eta$, $f^{-1}(t)$ cuts S_ε transversely. Let $T = \{t \in C \mid |t| < \eta\}$ and $X = f^{-1}(T) \cap B_\varepsilon$. By lemma 6(c), X is Stein.





Set $X^* = \overline{X} \setminus f^{-1}(0) = (f^{-1}(\overline{T}) \cap \overline{B}_\varepsilon) \setminus f^{-1}(0)$. The mapping $f: X^* \to \overline{T} \setminus 0$ has maximal rank and is proper. So according to Ehresmann's theorem (see[W]) $f: X^* \to \overline{T} \setminus 0$, is a C^∞ fibre bundle, whose

fibre is a 2n-dimensional manifold with boundary. (see also [L])

Let $X' = X \setminus f^{-1}(0)$, $T' = T \setminus 0$. Then $f: X' \to T'$ is also a C^∞ fibre bundle, whose fibre $f^{-1}(t_0) = X_{t_0}$ is an n-dimensional Stein manifold by lemma 6(a). Milnor shows in [Mi], Th.6.5 and 5.11 that X_{t_0} has the homotopy type of a bouquet of μ n-spheres.

The hypersurface X_0 determines the function f only up to a unit $u \in C\{x_0,\ldots,x_n\}$. Write $u = re^{i\varphi} + v$ where $v(0) = 0$. If we let $u_a = \dfrac{re^{ia\varphi}}{(1-a)r+a} + av$ and $f_a = u_a f$, for $0 \leq a \leq 1$, then f_a defines a fibre bundle over $T' \times [0,1]$. Since $f_0 = f$ and $f_1 = uf$, it follows that the Milnor fibrations belonging to f and uf are isomorphic. Therefore the bundle $X' \to T'$ above is an invariant of the hypersurface X_0 (or of the hypersurface germ represented by X_0).

3.2 Since T' is a punctured disc, the bundle $X' \to T'$ determines, up to isotopy, a diffeomorphism $h: X_{t_0} \to X_{t_0}$ called the geometric monodromy of f. Conversely, h determines the fibre bundle.

Let $H^n = \bigcup_{t \in T'} H^n(X_t, C)$. $H^n \to T'$ is a flat complex vector bundle It is flat because the transition functions are already defined on the integral cohomology and must therefore be constant. This bundle is determined up to isomorphism by h^*, the monodromy of f. h^* is independent of the choices made so far.

3.3 Because $H^n \to T'$ is flat, it carries a natural connection ∇: pick any $t \in T'$ and let $s_1,\ldots,s_\mu$ be μ linearly independent locally constant sections defined in a neighbourhood of t. Suppose $s = \sum_{j=1}^{\mu} c_j s_j$ is an arbitrary holomorphic section defined near t. Then $\nabla_{\frac{d}{dt}} s$ is the section given by

$$\nabla_{\frac{d}{dt}} s = \sum_{j=1}^{\mu} c_j' s_j \quad .$$

Because the transition functions of the bundle are constant, $\nabla_{\frac{d}{dt}} s$ does not depend on $s_1,\ldots,s_\mu$. ∇ is called the Gauss-Manin connection of f.

By theorem 9, $H^n \to T'$ is trivial as an analytic vector bundle. If we choose μ linearly independent analytic sections $w_1,\ldots,w_\mu$, then

$$(4) \qquad \nabla_{\frac{d}{dt}} w_k = -\sum_{j=1}^{\mu} a_{jk} w_j \qquad k = 1,\ldots,\mu$$

where $A = (a_{jk})$ is a matrix of holomorphic functions on T'. If we write the section s in the form

$$s = \sum_{j=1}^{\mu} y_j w_j \ ,$$

then

$$\nabla_{\frac{d}{dt}} s = \sum_j y_j'(t) w_j - \sum_{j,k} y_k a_{jk} w_j \ ,$$

and $\nabla_{\frac{d}{dt}} s = 0$ if and only if $y' = Ay$ where $y = \begin{pmatrix} y_1 \\ \vdots \\ y_\mu \end{pmatrix}$.

So the solutions of the equation $y' = Ay$ are the flat sections of $H^n \to T'$ and its monodromy is h^*. From the remark at the end of 1.2, we see that $w_1,\ldots,w_\mu$ can be chosen so that A has a simple. However, since we know nothing else about this basis, or the corresponding residue, it is of little use.

We would like to calculate the Gauss-Manin connection explicitly enough that we can compute the monodromy or asymptotic integrals from it. This was done by Brieskorn in [B] by describing the connection in terms of holomorphic differential forms.

3.4 But before going on, we need the following characterization of μ.

Theorem 10: ([B], p.153) $\mu = \dim_C C\{x_0,\ldots,x_n\} / (f_0,\ldots,f_n)$

where f_j denotes the derivative $\partial f/\partial x_j$, $j = 1,\ldots,n$.

Remark: $\dim_C C\{x_0,\ldots,x_n\} / (f_0,\ldots,f_n) < \infty$ if and only if, for some r, $m^{r+1} \subset (f_0,\ldots,f_n)$, m the maximal ideal of $C\{x_0,\ldots,x_n\}$. But by the Nullstellensatz, this is the case if and only if 0 is an isolated critical point of f.

Since $f \in m$, we have in particular that $f^{r+1} \in (f_0,\ldots,f_n)$. The following theorem of Briançon and Skoda asserts much more. Unfortunately their proof is analytic, and no algebraic proof seems to be known.

Theorem 11:([B-S]) If $f \in C\{x_0,\ldots,x_n\}$, $f(0) = 0$, then

$$f^{n+1} \in (f_0,\ldots,f_n) \quad .$$

§4 Brieskorn's description of the Gauss-Manin connection

4.1 Each fibre of the bundle $X' \to T'$ is a Stein manifold. So by theorem 8, any cohomology class in X_t can be represented by a holomorphic differential form. What one would like to do is to represent any analytic section of the bundle $H^n \to T'$ by a family of holomorphic n-forms

$$\omega_t \varepsilon\ \Gamma(\Omega^n_{X_t}) \quad , \quad t\ \varepsilon\ T' \ .$$

depending analytically on t. One way to do this is the following:

Definition: (EGA IV §16.6) The sheaf of relative p-forms $\Omega^p_{X/T}$ on X is defined by

$$\Omega^p_{X/T} = \Omega^p_X \ / \ df \wedge \Omega^{p-1}_X \qquad p \geq 0$$

and the relative exterior derivative

$$d: \Omega^p_{X/T} \to \Omega^{p+1}_{X/T}$$

is the homomorphism induced by $d: \Omega^p_X \to \Omega^{p+1}_X$.

Thus $\Omega^p_{X/T}|_{X'} = \Omega^p_{X'/T'}$ is the sheaf of p-forms "along the fibres of f". Denote by H^n the sheaf of analytic sections of $H^n \to T'$. What we need is now just a straightforward generalization of theorem 8:

Lemma 12: $H^n \simeq H^n(f_*\Omega_{X'/T'})$ The isomorphism is as follows: let ω be a local section of $f_*\Omega^n_{X'/T'}$ with $d\omega = 0$; then

$$t \to [\omega_t] \ \varepsilon\ H^n(X_t, C) \qquad t\ \varepsilon\ T'$$

is a local section of H^n.

Proof: See the appendix.

4.2 As was suggested in the introduction, there is a class of sections of H^n which are "well-behaved" as $t \to 0$, and which give a regular singular equation for the connection. Brieskorn introduces these by defining a number of coherent sheaves which extend H^n to the whole disc. Let

$$H = H^n(f_*\Omega^{\cdot}_{X/T})$$
$$H' = f_*\Omega^n_{X/T} \,/\, d(f_*\Omega^{n-1}_{X/T})$$
$$H'' = f_*\Omega^{n+1}_{X} \,/\, dt \wedge d(f_*\Omega^{n-1}_{X})$$

$H \subset H'$ and H and H' are both clearly extensions of H^n. Furthermore, the $\mathcal{O}_T$ homomorphism

$$dt \wedge \cdot : H' \to H''$$

is injective, and over T', it is an isomorphism. Thus H'' is also an extension of H^n. The basic facts about these sheaves are summarized below.

Theorem 13: (i) H, H', and H'' are coherent $\mathcal{O}_T$ modules ([B], p.123 and [G], pp.246,250)

(ii) As $\mathcal{O}_{T,0}$ modules

$$H_0 \simeq H^n(\Omega^{\cdot}_{X/T,0})$$
$$H'_0 \simeq \Omega^n_{X/T,0} \,/\, d\Omega^{n-1}_{X/T,0} \simeq \Omega^n_{X,0} \,/\{d\Omega^{n-1}_{X,0} + df \wedge \Omega^{n-1}_{X,0}\}$$
$$H''_0 \simeq \Omega^{n+1}_{X,0} \,/\, df \wedge d\Omega^{n-1}_{X,0}$$

([B], p.124 and [G], p.251)

(iii) H_0, H'_0, and H''_0 are free $\mathcal{O}_{T,0}$ modules of rank μ.

([M1], p.416)

From (ii) follows that

$$H''_0 \,/\, dt \wedge H'_0 \simeq \Omega^{n+1}_{X,0} \,/\, df \wedge \Omega^n_{X,0}$$
$$\simeq C\{x_0,\ldots,x_n\} \,/\, (f_0,\ldots,f_n) \quad .$$

Hence $\dim_C H''_0 / dt \wedge H'_0 = \mu$. We also have an isomorphism

$$d: H'_0 / H_0 \xrightarrow{\cong} H''_0 / dt \wedge H'_0 \quad ,$$

so that $\dim_C H'_0 / H_0 = \mu$ as well. If $f^{r+1} \varepsilon (f_0,\ldots,f_n)$, $r \geq 0$, then

$$f^{r+1}H'_0 \subset H_0 \quad \text{and} \quad f^{r+1}H''_0 \subset dt \wedge H'_0 \quad .$$

Thus $V = H_0 \otimes_{\mathcal{O}_{T,0}} k = H'_0 \otimes_{\mathcal{O}_{T,0}} k \simeq H''_0 \otimes_{\mathcal{O}_{T,0}} k$ is a μ-dimensicnal k vector space, where k is the quotient field of $\mathcal{O}_{T,0}$. H_0, H'_0, and H''_0 are lattices in V.

There is a very natural map

$$H''_{|T'} \to H^n$$

given by the following construction: suppose $U \subset T'$ and $\omega \varepsilon \Gamma(f^{-1}(U),\Omega_X^{n+1})$ represents $[\omega] \varepsilon \Gamma(U,H'')$; then

$$t \to [\text{res } \frac{\omega}{f - t}] \varepsilon H^n(X_t)$$

defines a section of H^n on U. Theorem 13 implies that there exist μ forms $\omega_1,\ldots,\omega_\mu \varepsilon \Gamma(X,\Omega_X^{n+1})$ so that the corresponding sections of H^n give a trivialization of the vector bundle $H^n \to T'$.

Before giving Brieskorn's formula for the Gauss-Manin connection on V I would like to recall a more classical description of it, in terms of period integrals.

4.3 Let $p: U \to T'$ be the universal covering of T'. Choose $u_0 \varepsilon U$ so that $p(u_0) = t_0$. For any $u \varepsilon U$, $H_n(X_{p(u)},Z)$ is naturally isomorphic to $H_n(X_{t_0},Z)$. So take any $\gamma_0 = \gamma(u_0) \varepsilon H_n(X_{t_0},Z)$ and define $\gamma(u) \varepsilon H_n(X_{p(u)},Z)$ by "parellel translation".

Suppose $\omega \varepsilon \Gamma(\Omega_X^n,)$; consider the function

$$I(u) = \int_{\gamma(u)} \omega$$

on U. For convenience, one usually writes

$$I(t) = \int_{\gamma(t)} \omega$$

and calls I a many-valued function on T'.

Lemma 14: ([B], p.127) I is a holomorphic function on T'.

Proof: Choose $t_1 \in T'$ and let $D \subset T'$ be an open disc about t_1. We have the Leray coboundary (or Gysin map)

$$\delta: H_n(X_{t_1}) \to H_{n+1}(X' \setminus f^{-1}(D)) \quad .$$

Set $\gamma' = \delta(\gamma(t_1))$. Then $\gamma' = \delta(\gamma(t))$ for all $t \in D$. Now apply the Leray residue theorem: for $t \in D$,

$$I(t) = \int_{\gamma(t)} \omega = \frac{1}{2\pi i} \int_{\gamma'} \frac{df \wedge \omega}{f - t} \quad .$$

The second integral is clearly an analytic function of t.

Notice that the I only depends on ω as a section of H^n. It follows then from the definition of the Gauss-Manin connection that

$$\frac{d}{dt} \int_{\gamma(t)} \omega = \int_{\gamma(t)} \nabla_{\frac{d}{dt}} \omega$$

4.4 The solutions of equation (4) can now be described in terms of such period integrals. According to lemma 12, there exist forms $\omega_j \in \Gamma(\Omega_{X'})$ such that

$$w_j(t) = [\omega_j|_{X_t}] \in H^n(X_t) \qquad t \in T' \ , \ 1 \leq j \leq \mu$$

Let $\gamma_1, \ldots, \gamma_\mu$ be a basis of $H_n(X_{t_0}, Z)$ and define $\gamma_k(t) \in H_n(X_t, Z)$ as above. Set

$$I_{jk}(t) = \int_{\gamma_k(t)} \omega_j \quad .$$

Then

$$\frac{d}{dt} I_{jk} = - \sum_{i=1}^{\mu} a_{ij} I_{ik} \quad .$$

So (I_{jk}) is a fundamental matrix of the system of equations dual to $y' = Ay$ (see [D], p.8) or equivalently $(I_{jk})^{-1}$ is a fundamental matrix of $y' = Ay$.

4.5 We need the following well-known lemma.

Lemma 15: The sequence

$$0 \to \mathcal{O}_{X,0} \overset{df\wedge\cdot}{\to} \Omega_{X,0} \overset{df\wedge\cdot}{\to} \cdots \overset{df\wedge\cdot}{\to} \Omega_{X,0}$$

is exact.

Proof: The above sequence is just the dual of the Koszul complex for $f_0,\ldots,f_n \in C\{x_0,\ldots,x_n\}$. Thus it is exact

if and only if $f_0,\ldots,f_n$ is a regular sequence in $C\{x_0,\ldots,x_n\}$

([Se], IV, proposition 3)

if and only if 0 is an isolated singularity of f.

Brieskorn's formula for the connection is the following:

Definition: Let $\varphi \in \Omega^n_{X,0}$ represent $[\varphi] \in H^n(\Omega^{\cdot}_{X/T,0}) \simeq H_0$. Thus $d\varphi = df \wedge \omega$, $\omega \in \Omega^n_{X,0}$; let

$$D[\varphi] = [\omega] \in \Omega^n_{X/T,0} / d\Omega^{n-1}_{X/T,0} \simeq H_0'$$

Lemma 15 ensures that $D: H_0 \to H_0'$ is well-defined. D is clearly C-linear and satisfies

$$D(a\varphi) = a'\varphi + aD\varphi$$

for $a \in \mathcal{O}_{T,0}$, $\varphi \in H_0$. Therefore one can extend D to the k vector space

V to get a connection (or covariant derivative) on V, in the sense of 1.5. A priori, D has a pole of order $\leq r+1$ if $f^{r+1}H_0' \subset H_0$.

One can equally well define $D: dt \wedge H_0' \to H_0''$ by

$$D[df \wedge \varphi] = [d\varphi] \qquad \varphi \in \Omega^n_{X,0}$$

Then the diagram

$$\begin{array}{ccc} H_0 & \xrightarrow{D} & H_0' \\ dt \wedge \cdot \downarrow & & \downarrow dt \wedge \cdot \\ dt \wedge H_0' & \xrightarrow{D} & H_0'' \end{array} \qquad \text{commutes.}$$

In terms of coordinates

$$D\left[df \wedge \sum_{j=0}^{n} (-1)^j g_j dx_0 \wedge \cdots \wedge \hat{dx_j} \wedge \cdots \wedge dx_n\right] = D\left[\sum_{j=0}^{n} g_j \frac{\partial f}{\partial x_j} dx_0 \wedge \cdots \wedge dx_n\right]$$

$$= \left[\sum_{j=0} \frac{\partial g_j}{\partial x_j} dx_0 \wedge \cdots \wedge dx_n\right]$$

where $g_j \in \mathcal{O}_{X,0}$, $j = 0,\ldots,n$. In practice it is easier to calculate with this second form of D.

Lemma 16: $D: H_0 \to H_0'$ and $D: H_0' \to H_0''$ are bijective.

The proof is straightforward. See also [M1], (3.5). We must check that D really is the Gauss-Manin connection:

Theorem 17: ([B], p.126) $$\frac{d}{dt}\int_{\gamma(t)}\omega = \int_{\gamma(t)} D\omega ,$$

where $\gamma(t)$ is a flat family of n-cycles as above, and $\omega \in \Gamma(X, H)$.

Proof: As in the proof of lemma 14, write

$$\int_{\gamma(t)}\omega = \frac{1}{2\pi i}\int_{\gamma'} \frac{df\wedge\omega}{f-t}$$

Then

$$\frac{d}{dt}\int_{\gamma(t)}\omega = \frac{1}{2\pi i}\int_{\gamma}\frac{df\wedge\omega}{(f-t)^2} = \frac{1}{2\pi i}\int_{\gamma}\left\{\frac{d\omega}{f-t} - d\left(\frac{\omega}{f-t}\right)\right\} = \frac{1}{2\pi i}\int_{\gamma}\frac{d\omega}{f-t}$$

$$= \frac{1}{2\pi i}\int_{\gamma}\frac{df\wedge D\omega}{f-t} = \int_{\gamma(t)}D\omega \ .$$

4.6 Theorem 18: D is regular.

There are many proofs. Most, like those of Katz [K1] and Brieskorn ([B], p.134ff.) make use of the existence of a resolution of the singularity $0 \in X_0 \subset C^{n+1}$. Malgrange ([M1], p.413ff.) gives a proof based on the following lemma.

Lemma 19: ([M1], 4.5) $\lim_{\substack{t\to 0\\ \arg t=0}} \int_{\gamma(t)}\omega = 0$ $\gamma(t)$, ω as above.

From 1.3 and 4.4, we can now see that

$$\int_{\gamma(t)}\omega = \sum_{\alpha,q} c_{\alpha,q}(\gamma,\omega)t^{\alpha}(\log t)^{q}$$

where $\alpha > 0$, $\exp 2\pi i\alpha$ is an eigenvalue of multiplicity $\geq q+1$ in the minimal polynomial of the monodromy, and $c_{\alpha,q}(\gamma,\omega) \in C$. This result is the starting point for Malgrange in his paper [M1] on asymptotic integrals.

The regularity theorem shows in what sense the sections of H^n in V are well-behaved in 0:

Corollary 20: $V = \{$germs of sections $\overset{\omega}{\wedge}$ of H^n over punctured neighbourhoods of $0 \mid \lim_{\substack{t\to 0\\ a\leq \arg t\leq b}} |\int_{\gamma(t)}\omega| = o(|t|^m)$ for all flat families $\gamma(t)$ of n-cycles, $m\in Z$, $a+2\pi > b\}$

Proof: (due to Brieskorn) It follows from theorem 18 that the space on

the right contains V. So choose $\omega_1,\ldots,\omega_\mu \in \Gamma(X,\Omega^n_{X,0})$ so that the corresponding sections of H^n are a trivialization. Let ω be from the right-hand space. Then $\omega = \sum_j a_j\omega_j$ where a_j, $j = 1,\ldots,\mu$ are holomorphic in a punctured neighbourhood of 0. Define flat families of cycles $\gamma_k(t)$, $k = 1,\ldots,\mu$, and integrals $I_{jk}(t)$ as in 4.4. As already remarked, the functions $I_{jk}(t)$ have the growth property. So do

$$y_k(t) = \int_{\gamma_k(t)}\omega \qquad\qquad k = 1,\ldots,\mu$$

But

$$y_k = \sum_j I_{jk}a_j$$

Therefore $a = (I)^{-1}y$, $I = (I_{jk})$, has the growth property. Hence the functions a_j have at most a pole at 0.

Theorem 21: (The monodromy theorem)

(i) The eigenvalues of the monodromy are roots of unity.

(ii) They with at most with multiplicity n+1 as roots of its minimal polynomial.

(i) is proved in [B],p.113 (see also [D], p.126). Another very attractive proof of (i) for "generic functions", using Newton polygons, is given in [V] and [E]. (i) and (ii) are proved in [K1] using the existence of a resolution of the singularity.

§5 Calculating the residue of the Gauss-Manin connection

For convenience $df \wedge H_0'$ and H_0'' will be denoted by H' and H" respectively.

5.1 The easiest functions to calculate with are the quasihomogeneous polynomials (cf.[B], pp.150).

Definition: Given positive integers $a_0,\ldots,a_n$ and d , a function $f \in C\{x_0,\ldots,x_n\}$ is called quasihomogeneous (or weighted homogeneous) of type $(a_0,\ldots,a_n)$ and degree d if

$$f(t^{a_0}x_0,\ldots,t^{a_n}x_n) = t^d f(x_0,\ldots,x_n)$$

for all $(x_0,\ldots,x_n) \in C^{n+1}$, $t \in C$.

Equivalently, one can require that f satisfy Euler's equation:

$$f = \sum_{j=0}^{n} \frac{a_j}{d} x_j f_j$$

In particular, $f \in (f_0,\ldots,f_n)$, and we can take $r = 0$ in 3.4. Saito showed that this property characterizes quasihomogeneous polynomials in the following sense:

Theorem 22: [S] $f \in C\{x_0,\ldots,x_n\}$ is quasihomogeneous, up to a change of coordinates, if and only if $f \in (f_0,\ldots,f_n)$.

Corollary 23: f is quasi$\hat{}$homogenous, up to a change of coordinates if and only if D has a simple pole on H" (or H').

Proof: $f \in (f_0,\ldots,f_n)$ if and only if $fH'' \subset H'$. But $D: H' \to H''$

is bijective by lemma 16. So f is quasihomogenous if and only if $DfH'' \subset H''$, or since $Df = fD + id.$, if and only if $fDH'' \subset H''$.

5.2 To compute a matrix for the Gauss-Manin connection, we first need a basis of H". Since $fH'' \subset H'$, the map

$$H''/fH'' \to H''/H'$$

is onto. But H"/fH" and H"/H' both have dimension μ. So in fact

$$H''/fH'' \simeq H''/H'$$

and any basis of $H''/H' \simeq C\{x_0,\ldots,x_n\}/(f_0,\ldots,f_n)$ can be lifted to a $C\{t\}$ basis of H". Let

$$\{ x_0^{m_0} \cdots x_n^{m_n} \mid (m_0,\ldots,m_n) \in J \subset Z^{n+1} \}$$

be a set of monomials which represent a basis of the vector space $C\{x_0,\ldots,x_n\}/(f_0,\ldots,f_n)$. Let $\omega = dx_0 \wedge \cdots \wedge dx_n \in \Omega^{n+1}_{C^{n+1},0}$. Thus

$$\{ [x_0^{m_0} \cdots x_n^{m_n}\omega] \mid (m_0,\ldots,m_n) \in J \}$$

is a $C\{t\}$ basis of H". Now Euler's equation implies that

$$f[x_0^{m_0} \cdots x_n^{m_n}\omega] = \sum_{j=0}^{n} [a_j f_j x_0^{m_0} \cdots x_j^{m_j+1} \cdots x_n^{m_n}\omega]$$

$$= [df \wedge \{ \sum_{j=o}^{n} (-1)^j a_j \, x_0^{m_0} \cdots x_j^{m_j+1} \cdots x_n^{m_n} \, dx_0 \wedge \cdots \wedge \hat{dx_j} \wedge \cdots \wedge dx_n \}]$$

Therefore

$$Df[x_0^{m_0} \cdots x_n^{m_n} \omega] = \sum_{j=0}^{n} a_j(m_j+1)[x_0^{m_0} \cdots x_n^{m_n} \omega]$$

and

$$D[x_0^{m_0} \cdots x_n^{m_n} \omega] = t^{-1}\{ \sum_{j=0}^{n} a_j(m_j+1) - 1\}[x_0^{m_0} \cdots x_n^{m_n} \omega]$$

Thus the equation for the Gauss-Manin connection with respect to the

basis chosen above (cf. construction in 4.2) is

$$\begin{pmatrix} y_1' \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ y_\mu' \end{pmatrix} = t^{-1} \begin{pmatrix} \cdot & & & & & \\ & \cdot & & & & \\ & & \cdot & & & \\ & & & -\sum_{j=0}^{n} \frac{a_j}{d^j}(m_j+1) + 1 & & \\ & & & & \cdot & \\ & & & & & \cdot \end{pmatrix} \begin{pmatrix} y_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ y_\mu \end{pmatrix}$$

and the monodromy of f is

$$\begin{pmatrix} \cdot & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \exp -\sum_{j=0}^{n} \frac{a_j}{d^j}(m_j+1) \\ & & & & \cdot \\ & & & & & \cdot \end{pmatrix}$$

In [M-O], formulas for μ and the characteristic polynomial of the monodromy are given.

5.3 Corollary 23 shows that D has a simple pole on H" if and only if $f \in (f_0,\ldots,f_n)$. This is not in general the case, as is clear from [A] or the examples below. So if we want to calculate D for arbitrary f, the first (and greatest problem) is to find a basis of the vector space V with respect to which D has a simple pole. Since the residue matrix tends to be large, one should also choose the basis so that the residue has a reasonable form.

One way to resolve the first difficulty is to look for a basis of $\tilde{H}''$ (for the definition see theorem 4). According to [M1], p.418 and [M2], p.113

$$fD: \tilde{H}' \to \tilde{H}'$$

is bijective. From lemma 16, it follows easily that

$$D: \tilde{H}' \to \tilde{H}''$$

is surjective (cf.[M2], p.113). Thus $f\tilde{H}'' = \tilde{H}'$ So to find a basis of $\tilde{H}''$ it is sufficient to find a subset of $\tilde{H}''$ which represents a basis of $\tilde{H}''/\tilde{H}' \simeq \tilde{H}''/f\tilde{H}''$ as a C vector space.

Remark: The residue of D on $\tilde{H}''$ is fD as an endomorphism of $\tilde{H}''/f\tilde{H}''$. As in the case where f is quasihomogeneous, it turns out in general to be more natural to compute Df on $\tilde{H}''/f\tilde{H}''$. Of course $fD = Df - id$. Since the diagram

$$\begin{array}{ccc} & Df & \\ \tilde{H}'' & \longrightarrow & \tilde{H}'' \\ f\downarrow\simeq & & f\downarrow\simeq \\ \tilde{H}' & \longrightarrow & \tilde{H}' \\ & fD & \end{array}$$

commutes trivially. we see that calculating Df on $\tilde{H}''/f\tilde{H}''$ is the same as calculating the residue of D on $\tilde{H}'$. Malgrange shows in [M2], p.116ff. that the eigenvalues of the residue of D on $\tilde{H}'$ or of Df on $\tilde{H}''/f\tilde{H}''$ are positive.

5.4 We now construct a basis of $\tilde{H}''/f\tilde{H}''$. First we need a "good" basis of H"/H': choose forms $\omega_1,\ldots,\omega_\nu \in H''$ which generate H"/H' as a C{t} module, and such that

$\omega_1,\ldots,\omega_\nu$ are a basis of $H''/(fH''+H')$

$\omega_1,\ldots,\omega_\nu,f\omega_1,\ldots,f\omega_{\nu_1}$ are a basis of $H''/(f^2H''+H')$ $\qquad \nu_1\leq\nu$

$\vdots$

$\omega_1,\ldots,\omega_\nu,f\omega_1,\ldots,f\omega_{\nu_1},\ldots\ldots\ldots,f^j\omega_1,\ldots,f^j\omega_{\nu_j}$ are a basis of $H''/(f^{j+1}H''+H')$, $\nu_j\leq\ldots\leq\nu_1\leq\nu$

$\vdots$

$\omega_1,\ldots,\omega_\nu,f\omega_1,\ldots,f\omega_{\nu_1},\ldots\ldots\ldots,f^r\omega_1,\ldots,f^r\omega_{\nu_r}$ are a basis of H"/H',

where $\nu_r\leq\ldots\leq\nu_1\leq\nu$, $f^{r+1} \in (f_0,\ldots,f_n)$, but $f^r \notin (f_0,\ldots,f_n)$. In practice, it seems easiest to choose monomials $m_1,\ldots,m_\nu \in C\{x_0,\ldots,x_n\}$ which are a minimal set of generators of $C\{x_0,\ldots,x_n\}/(f_0,\ldots,f_n)$ as a

$C\{t\}$ module, and are ordered as above, and then set $\omega_j = [m_j\omega] \in H''$.

5.5 A basis of $\tilde{H}''/f\tilde{H}''$

H'' has three filtrations: $\{D^{-k}H''\}_{k\geq 0}$, $\{f^kH''\}_{k\geq 0}$ and the quotient filtration from $\left\{m^k\Omega^{n+1}_{C^{n+1},0}\right\}_{k\geq 0}$ where $m = (x_0,\ldots,x_n)$. Malgrange uses lemma 19 to prove the following.

Lemma 24: (5.3 and 5.6 in [M1]) The three filtrations induce the same topology on H''.

Denote by $\hat{H}''$ the completion of H'' with respect to this topology.

Define $l(j) \geq 0$ for $1\leq j\leq \nu$ by

$$l(j) = \max\left\{l \mid D^l\omega_j \in \tilde{H}''\right\} .$$

For most j, one expects $l(j)=0$. However, the examples considered later show that this is not always so.

Lemma 25: $D^{l(1)}\omega_1,\ldots,D^{l(\nu)}\omega_\nu, D^{l(1)+1}f\omega_1,\ldots,D^{l(\nu_1)+1}f\omega_{\nu_1},\ldots\ldots\ldots,$
$D^{l(1)+r}f^r\omega_1,\ldots,D^{l(\nu_r)+r}f^r\omega_{\nu_r}$
are a C basis of $\tilde{H}''/f\tilde{H}''$ and a $C\{t\}$ basis of $\tilde{H}''$

Proof: (i) First of all, let us check that all the elements above lie in $\tilde{H}''$ and are not 0 in $\tilde{H}''/f\tilde{H}''$. Since $f\tilde{H}'' = \tilde{H}' = D^{-1}\tilde{H}''$, it's clear that $D^{l(j)}\omega_j \notin f\tilde{H}''$, $1\leq j\leq\nu$. We must show that

$D^{l(j)+k}f^k\omega_j \in \tilde{H}''$, but $D^{l(j)+k}f^k\omega_j \notin f\tilde{H}''$, for $1\leq j\leq\nu_k$, $1\leq k\leq r$

But the Leibniz rule implies that

$$(5) \qquad D^{l+k}f^k\omega_j = (Df+l)\cdots(Df+l+k-1)D^l\omega_j \qquad l\geq 0$$

So $D^{l+k}f^k\omega_j \in \tilde{H}''$ if $D^l\omega_j \in \tilde{H}''$. According to the remark

in 5.4, Df has positive eigenvalues on $\tilde{H}''/f\tilde{H}''$. Therefore $D^{l+k}f^k\omega_j \notin f\tilde{H}''$ if $D^l\omega_j \notin f\tilde{H}''$.

(ii) We now prove that these elements are in fact linearly independent: following Malgrange ([M1], p.422), define

$$\tau^{-1}\varphi = -D^{-1}\varphi \quad \text{for} \quad \varphi \in \hat{H}'' .$$

Then $\hat{H}''$ becomes a free $C[[\tau^{-1}]]$ module of rank μ, and $\hat{V} = \hat{H}'' \otimes_{C[[t]]} k$ becomes a vector space over $\ell = C[[\tau^{-1}]][\tau]$. Now $\dim_\ell \hat{V} = \mu$: for we have $\hat{H}'' \subset \tilde{\hat{H}}'' = \hat{\tilde{H}}'' \subset \hat{V}$ and by induction, we see that $\tau^k\hat{\tilde{H}}'' = f^{-k}\hat{\tilde{H}}''$. So $\ell\hat{\tilde{H}}'' = \hat{V}$. But $\hat{\tilde{H}}''$ is also of rank μ over $C[[\tau^{-1}]]$. Thus $\dim_\ell \hat{V} = \mu$. Since

$$\omega_1,\ldots,\omega_\nu, f\omega_1,\ldots,f\omega_{\nu_1},\ldots\ldots\ldots,f^r\omega_1,\ldots,f^r\omega_{\nu_r}$$

are representatives of a basis of $H''/H' \simeq \hat{H}''/\hat{H}' = \hat{H}''/\tau^{-1}\hat{H}''$, they are a $C[[\tau^{-1}]]$ basis of $\hat{H}''$ and hence an ℓ basis of $\hat{V}$. Therefore

$$D^{l(1)}\omega_1,\ldots,D^{l(\nu)}\omega_\nu, D^{l(1)+1}f\omega_1,\ldots,D^{l(\nu_1)+1}f\omega_{\nu_1},\ldots\ldots\ldots,$$
$$D^{l(1)+r}f^r\omega_1,\ldots,D^{l(\nu_r)+r}f^r\omega_{\nu_r}$$

form an ℓ basis of $\hat{V}$. It follows then from (i) that they are a $C[[\tau^{-1}]]$ basis of $\hat{\tilde{H}}''$ and a basis of $\hat{\tilde{H}}''/\tau^{-1}\hat{\tilde{H}}'' \simeq \tilde{H}''/\tilde{H}' \simeq \tilde{H}''/f\tilde{H}''$.

Theorem 26: $D^{l(1)}\omega_1,\ldots,D^{l(\nu)}\omega_\nu, (Df)D^{l(1)}\omega_1,\ldots,(Df)D^{l(\nu_1)}\omega_{\nu_1},\ldots\ldots\ldots,$
$(Df)^rD^{l(1)}\omega_1,\ldots,(Df)^rD^{l(\nu_r)}\omega_{\nu_r}$ are a basis of $\tilde{H}''/f\tilde{H}''$.

Proof: Use equation (5) and the previous lemma.

5.6 The following result, although vague, is useful to have.

Theorem 27: There exists a basis of $\tilde{H}''/f\tilde{H}''$ consisting of elements of the form $(Df)^q\omega_j$, for suitable q, and suitable $j\varepsilon\ 1,..\nu$

Proof: Since D is regular, we have $\tilde{H}'' = H'' + \ldots + (Df)^pH''$ for large p. So if $\varphi_1,\ldots,\varphi_\mu \varepsilon H''$ represent a basis of H"/H' as a complex vector space, then

$$\varphi_1,\ldots,\varphi_\mu,Df\varphi_1,\ldots,Df\varphi_\mu,\ldots\ldots\ldots,(Df)^p\varphi_1,\ldots,(Df)^p\varphi_\mu$$

generate $\tilde{H}''/\tilde{H}'$. If we take as our basis, the basis from 5.4, we see that $(Df)^qf^k\omega_j \varepsilon f\tilde{H}''$ for $k\geq 1$, $q\geq 0$. Therefore

$$\omega_1,\ldots,\omega_\nu,Df\omega_1,\ldots,Df\omega_\nu,\ldots\ldots\ldots,(Df)^p\omega_1,\ldots,(Df)^p\omega_\nu$$

generate $\tilde{H}''/\tilde{H}'$. If $l(j)>0$, then $\omega_j\varepsilon\tilde{H}'$; so $(Df)^q\omega_j\varepsilon\ \tilde{H}'$ for all $q\geq 0$ and these elements too are unnecessary. From the remaining elements we can then pick a basis.

5.7 In this and the following section, we shall explain how to go about computing a matrix for Df with respect to the basis of theorem 26 (and 27). First, define $r(j)\geq 0$ for $1\leq j\leq\nu$ by the condition

$$f^{r(j)+1}\omega_j \varepsilon H' \quad \text{but} \quad f^{r(j)}\omega_j \notin H' .$$

Thus $r\geq r(1)\geq\cdots\geq r(\nu)\geq 0$. The basis for H"/H' can now be written

$$\begin{array}{cccc}
\omega_1, & \ldots\ldots, & \omega_j, & \ldots\ldots, \omega_\nu \\
f\omega_1, & \ldots\ldots, & f\omega_j, & \ldots\ldots, f\omega_\nu \\
\vdots & & \vdots & \\
 & & f^{r(j)}\omega_j & \\
\vdots & & & \\
f^{r(1)}\omega_1 & & & f^{r(\nu)}\omega_\nu
\end{array}$$

As was seen in the proof of lemma 24, these elements also form an ℓ basis of $\hat{V}$. So using the definition of D we can compute $Df^{r(j)+1}\omega_j$ in terms of

this basis; likewise $(Df)^2 f^{r(j)-1}\omega_j, \ldots$, and finally $(Df)^{r(j)+1}\omega_j$, $1 \leqslant j \leqslant \nu$ The equation for an element $(Df)^{r(j)+1}\omega_j$ may contain terms of the form $(Df)^q D^l \omega_k$, for some $k \neq j$. By computing $(Df)^{r(j)+s}\omega_j$, $s=2,\ldots$. we can then eliminate all but one such term, thus determining $l(k)$. The same calculation gives us a basis as in theorem 27.

5.8 We would like to have a matrix for Df which is in a "roughly" triangular shape, ie. of the form

$$\begin{pmatrix} A_1 & & \\ \vdots & \ddots & \\ & A_j & \\ & \vdots & \ddots \\ & & A_\nu \end{pmatrix}$$

where the block $A_j = \begin{pmatrix} 0 & & a_0 \\ 1 & \ddots & \vdots \\ & \ddots & 0 \vdots \\ & & 1\ a_{r(j)} \end{pmatrix}$ $a_0, \ldots, a_{r(j)} \in C$

comes from the basis elements $D^{l(j)}\omega_j, \ldots, (Df)^{r(j)} D^{l(j)}\omega_j$, and the dots below the "diagonal" indicate possible non-zero entries below the last column of each block. The advantage of this form is of course that we then get the characteristic polynomial of Df immediately. If we can compute the eigenvalues, it also gives us the Jordan normal form.

It is only possible to achieve such a matrix if we order the cyclic submodules C_j (cf. corollary 5) generated by the elements $D^{l(j)}\omega_j$ appropriately. The dots below the diagonal then represent coefficients of terms $(Df)^q D^{l(k)}\omega_k$, for some q, where $C_k \subset C_j$, and do not have to be explicitly computed.

5.9 Here is a summary of our method for calculating a residue of D.

I. Compute a good basis of $C\{x_0,\ldots,x_n\}/(f_0,\ldots,f_n)$ and of H''/H'.
(5.1 and 5.4)

II. Compute equations for the remaining relations $f^{r(j)+1}\omega_j \in H'$.

III. Starting with $[\omega]=[dx_0\wedge\cdots\wedge dx_n]$, calculate $Df^{r(j)+1}\omega_j, D^2f^{r(j)+1}\omega_j,\ldots$
$D^{r(j)+1}f^{r(j)+1}\omega_j$. (5.2 and 5.7)

IV. While calculating, keep an eye open for numbers $l(j)>0$. Knowing $(Df)^{r(j)+1}\omega_j$, one then gets $(Df)^{r(j)+1}D^{l(j)}\omega_j$ from equation (5).

V. Compute the $(Df)^{r(j)+1}\omega_j$ in such an order as to give a matrix in a "roughly" triangular form. (5.8)

VI. The resulting matrix for Df on $\tilde{H}''/f\tilde{H}''$ is the residue of D on $\tilde{H}'$, and has positive eigenvalues. (5.2 and 5.3)

In §6, these general principles will be applied to a concrete example.

5.10 Following 5.7 the basis of theorem 26 can be written

$$
\begin{array}{cccc}
D^{l(1)}\omega_1 & \cdots\cdots & D^{l(j)}\omega_j & \cdots\cdots\ D^{l(\nu)}\omega_\nu \\
(Df)D^{l(1)}\omega_1 & \cdots\cdots & (Df)D^{l(j)}\omega_j & \cdots\cdots\ (Df)D^{l(\nu)}\omega_\nu \\
\vdots & & \vdots & \vdots \\
\vdots & & \vdots & (Df)^{r(\nu)}D^{l(\nu)}\omega_\nu \\
\vdots & & (Df)^{r(j)}D^{l(j)}\omega_j & \\
(Df)^{r(1)}D^{l(1)}\omega_1 & &
\end{array}
$$

The form of this basis encourages one to sharpen the conjecture made in the introduction.

Conjecture: On the cyclic subspaces $C_j/fC_j \subset \tilde{H}''/f\tilde{H}''$ the zeroes of the minimal polynomial of Df have multiplicity $\leq r(j)+1$, $1\leq j\leq\nu$

In the examples calculated in §6 and §7 this is in fact the case.

A useful way of looking at the operator Df is the following(cf.[K3] and [M2]): Suppose $f^{r+1} = \sum_j a_j f_j$, $a_0,\ldots,a_n \in C\{x_0,\ldots,x_n\}$. Let $\xi = \sum_j a_j \partial/\partial x_j$.

Lemma 28: The Lie derivative $L(f^{-r}\xi)$ is well-defined on V and

$$Df = L(f^{-r}\xi) - r \ .$$

Proof: On $\Omega^{n+1}_{C^{n+1},0}$, $L(\xi)(g\omega) = \sum_j \frac{\partial}{\partial x_j}(a_j g)\,\omega$, $g \in C\{x_0,\ldots,x_n\}$.

Remark: ξ is only determined up to vector fields η such that $\eta f = 0$. Let $\eta_{jk} = f_k \frac{\partial}{\partial x_j} - f_j \frac{\partial}{\partial x_k}$, $0 \leq j,k \leq n$. Then [Se], IV,prop.3 implies that $\eta = \sum_{j,k} g_{jk}\eta_{jk}$ for some $g_{jk} \in C\{x_0,\ldots,x_n\}$. One can check that on H", $L(\eta_{jk}) = 0$.

For r=0 it can be shown directly that ξ and $L(\xi)$ have positive rational eigenvalues ([Sch-W], 3.5). The same should be possible for r>0

§6 An Example

6.1 Let us calculate the residue of D for the function

$$f(x,y) = ax^5 + y^6 + x^4y \qquad a\varepsilon C$$

Thus

$$f_x' = 5ax^4 + 4x^3y \qquad f_y = 6y^5 + x^4$$

and

$$(A) \qquad f = \frac{5}{24}xf_x + \frac{1}{6}yf_y - \frac{a}{24}x^5 \ .$$

I. A good basis of $C\{x,y\}/(f_x,f_y)$.

If a=0, then (A) is Euler's equation, and f is quasihomogeneous of type (5,4) and degree 24. In this case a good basis is

$$1,x,y,x^2,xy,y^2,x^3,x^2y,xy^2,y^3,x^4,x^2y^2,xy^3,y^4,x^2y^3,xy^4,x^2y^4,x^5,x^6$$

So μ=19. We can calculate a matrix for Df on H"/H' as explained in 5.2 . The result is shown on page 48 .

From here on, we assume that $a\neq 0$ (although most equations also hold for a=0).

Claim: $x^7 \varepsilon (f_x,f_y)$

Proof: We have

$$x^3f_y = 6x^3y^5 + x^7$$

$$-\frac{3}{2}y^4f_x = -\frac{15}{2}ax^4y^4 - 6x^3y^5$$

$$\frac{15}{8}axy^3f_x = \frac{75}{8}a^2x^5y^3 + \frac{15}{2}ax^4y^4$$

$$-\frac{75}{32}a^2x^2y^2f_x = -\frac{375}{32}a^3x^6y^2 - \frac{75}{8}a^2x^5y^3$$

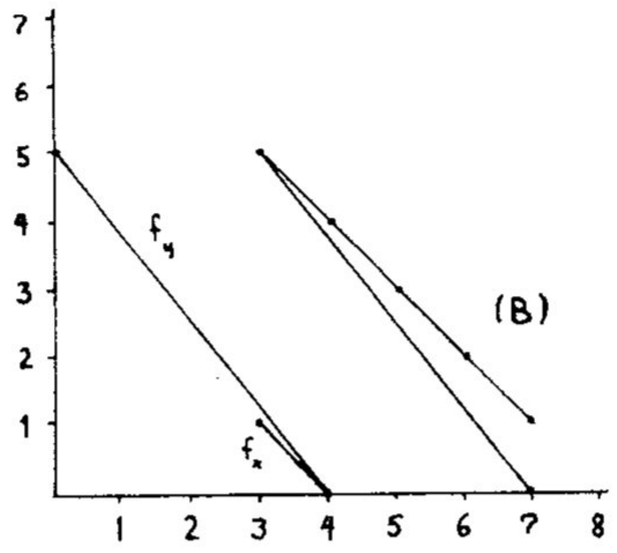
$$\frac{375}{128}a^3x^3yf_x = \frac{1875}{128}a^4x^7y + \frac{375}{32}a^3x^6y^2$$

Adding these equations gives

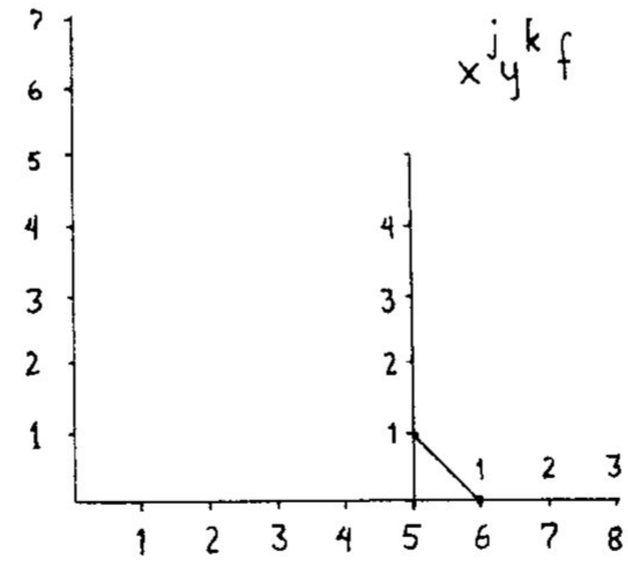
$$(B) \qquad x^3f_y + \frac{3}{2}\left(-y^4 + \frac{5}{4}axy^3 - \frac{25}{16}a^2x^2y^2 + \frac{125}{64}a^3x^3y\right)f_x = x^7\left(1 + \frac{1875}{128}a^4y\right)$$

The easiest way to find equation (B) is to use the Newton diagrams of

f_x and f_y (cf.[A], §15). One starts with the point (7,0) , representing x^7 , and the builds a "chain" using as "links" the diagrams of f_x and f_y appropriately translated. Such a chain represents an expression in (f_x,f_y). It is constructed to end above and to the right of x^7, so that the expression represented by the chain is equal to (unit)$\cdot x^7$.



Newton diagrams can also help to find a relation between xf and yf in $C\{x,y\}/(f_x,f_y)$: Thanks to equation (A), we can represent x^jy^kf by the point (j,k) in a coordinate system whose origin is at (5,0). We then see that a chain consisting of a single link, x^2f_x , joins xf and yf.



Here are the corresponding expressions with the right coefficients:

$$yf = \frac{5}{24}\,xyf_x + \frac{1}{6}\,y^2f_y - \frac{1}{24}\,ax^5y \qquad yA$$

$$\frac{1}{96}\,ax^2f_x = \frac{5}{96}\,a^2x^6 + \frac{1}{24}\,ax^5y$$

$$\frac{5}{4}\,axf = \frac{25}{96}\,ax^2f_x + \frac{5}{24}\,axyf_y - \frac{5}{96}\,a^2x^6 \qquad \frac{5}{4}\,axA$$

Thus

$$(C) \qquad yf + \frac{5}{4}\,axf = (\frac{5}{24}\,xy + \frac{1}{4}\,ax^2)f_x + (\frac{1}{6}\,y^2 + \frac{5}{24}\,axy)f_y$$

It follows from (A),(B), and (C) that

$$x^2f,\ xyf,\ y^2f\ \varepsilon\ (f_x,f_y)$$

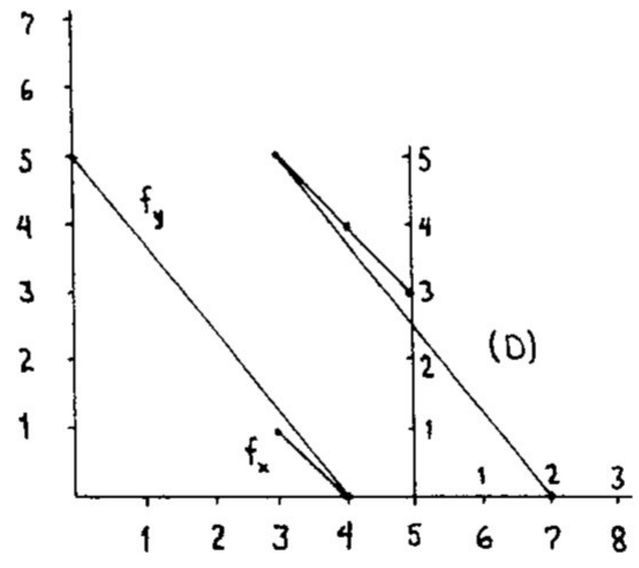
One can then check that

$$1,x,y,x^2,xy,y^2,x^3,x^2y,xy^2,y^3,x^4,x^2y^2,xy^3,y^4,x^2y^3,xy^4,x^2y^4,f(=-\frac{1}{24}\,ax^5),$$

$$xf(=-\frac{1}{24}\,ax^6)$$

are a good basis of $C\{x,y\}/(f_x,f_y)$. So $\mu=19$ for $a\neq 0$ as well.

6.2 II. From theorem 11 we know that $f^2\varepsilon\ (f_x,f_y)$. We must find an explicit equation. First we derive a relation for x^2f and y^3f. This can be done using a Newton diagram just as with equation (C).



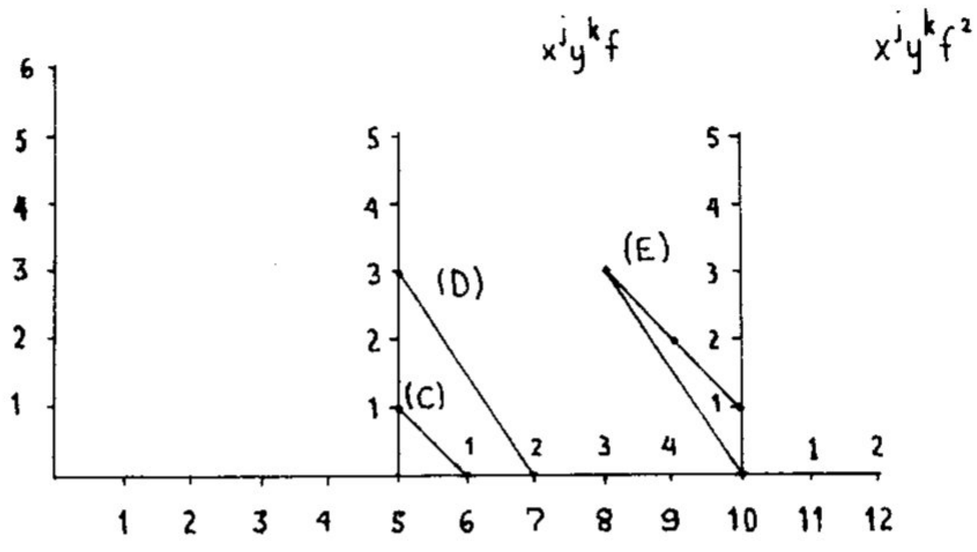
The corresponding equation is

$$(D) \qquad x^2f + \frac{75}{8}\,a^2y^3f = (\frac{5}{24}\,x^3 + \frac{1}{16}\,ay^4 + \frac{15}{8}\,a^2xy^3)f_x + (\frac{1}{6}\,x^2y - \frac{1}{24}\,ax^3 + \frac{25}{16}\,a^2y^4)f_y$$

(Equations (C) and (D) seem to generate the module of non-trivial

relations between f, f_x, and f_y).

We can now find an equation for f^2: Think of equations (C) and (D) as links in the second coordinate system joining yf to xf and x^2f to y^3f respectively. Introduce a third coordinate system with origin (10,0) (=(5,0) in the second system) to represent expressions $x^jy^kf^2$. Then starting at f^2, build a chain using (C) and (D), which ends above and to the right of f^2.



Here are the corresponding equations:

$$f^2 = \frac{5}{24}\,xff_x + \frac{1}{6}\,yff_y - \frac{a}{24}\,x^5f \qquad fA$$

$$-\frac{1}{24}\,ax^5f - \frac{25}{64}\,a^3x^3y^3f = (-\frac{5}{576}\,ax^6 - \frac{1}{384}\,a^2x^3y^4 - \frac{5}{64}\,a^3x^4y^3)f_x + (-\frac{1}{144}\,ax^5y + \frac{1}{576}\,a^2x^6 - \frac{25}{384}\,a^3x^3y^4)f_y \qquad -\frac{a}{24}\,x^3D$$

$$\frac{25}{64}\,a^3x^3y^3f + \frac{125}{256}\,a^4x^4y^2f = (\frac{125}{1536}\,a^3x^4y^3 + \frac{25}{256}\,a^4x^5y^2)f_x + (\frac{25}{384}\,a^3x^3y^4 + \frac{125}{1536}\,a^4x^4y^3)f_y \qquad \frac{25}{64}\,a^3x^3y^2C$$

$$-\frac{125}{256}\,a^4x^4y^2f - \frac{625}{1024}\,a^5x^5yf = (-\frac{625}{6144}\,a^4x^5y^2 - \frac{125}{1024}\,a^5x^6y)f_x + (-\frac{125}{1536}\,a^4x^4y^3 - \frac{625}{6144}\,a^5x^5y^2)f_y \qquad -\frac{125}{256}a^4x^4yC$$

$$\frac{3750}{256}\,a^4yf^2 = \frac{3125}{1024}\,a^4xyff_x + \frac{625}{256}\,a^4y^2ff_y - \frac{625}{1024}\,a^5x^5yf \qquad \frac{3750}{256}a^4yfA$$

Add these equations up:

$$(E)\qquad f^2+\frac{3750}{256}a^4yf^2=(\frac{5}{24}xf-\frac{5a}{576}x^6-\frac{1}{384}a^2x^3y^4+\frac{5}{1536}a^3x^4y^3$$
$$-\frac{25}{6144}a^4x^5y^2-\frac{125}{1024}a^5x^6y+\frac{3125}{1024}a^4xyf)f_x$$
$$+(\frac{1}{6}yf-\frac{a}{144}x^5y+\frac{1}{576}a^2x^6-\frac{625}{6144}a^5x^5y^2$$
$$+\frac{625}{256}a^4y^2f)f_y$$

6.3 III. For convenience we shall from now on use the same symbol $x^jy^k\omega$, $j,k\geqslant 0$, to denote the objects in $\Omega^2_{C^2,0}$, H", $\tilde{H}$", H"/H', $\tilde{H}$"/f$\tilde{H}$",etc.

We start by calculating $(Df)^2\omega$. Applying D to equation (E), we get

$$Df^2\omega=(\frac{9}{24}f\omega+\frac{5}{24}xf_x\omega+\frac{1}{6}yf_y\omega-\frac{17}{288}ax^5\omega-\frac{1}{128}a^2x^2y^4\omega)+\frac{5}{384}a^3x^3y^3\omega$$
$$-\frac{125}{6144}a^4x^4y^2\omega-\frac{2875}{3072}a^5x^5y\omega+\frac{8125}{1024}a^4yf\omega+\frac{3125}{1024}a^4xyf_x\omega$$
$$+\frac{625}{256}a^4y^2f_y\omega-\frac{3750}{256}a^4Df^2y\omega$$

As was suggested in 5.8 (V.) we must separate the relevant from the irrelevant terms in the above equation. To do this, denote by $C_{j,k}\subset\tilde{H}$" the cyclic submodule generated by $x^jy^k\omega$, $j,k\geqslant 0$.

Claim: All terms outside the brackets are in $D^{-1}C_{0,1}\subset D^{-1}\tilde{H}"=\tilde{H}'=f\tilde{H}"$.

Proof:
$$4x^3y^3=y^2f_x-5ax^4y^2$$
$$4x^4y^2=xyf_x-5ax^5y$$
$$\frac{a}{24}x^5y=-yf+\frac{5}{24}xyf_x+\frac{1}{6}y^2f_y\ .\qquad y\ A$$

So we need only check that $y^2f_x\omega$, $xyf_x\omega$, $yf\omega$, $y^2f_y\omega$, $Df^2\omega\ \varepsilon\ D^{-1}C_{0,1}$.
But $D(y^2f_x\omega)=0$, $D(xyf_x\omega)=y\omega$, $D(y^2f_y\omega)=2y\omega$, $D(Df^2y\omega)=(Df)^2y\omega+Dfy\omega$.

Now from (A) we get

$$-\frac{17}{288}ax^5=\frac{17}{12}f-\frac{85}{288}xf_y-\frac{17}{72}yf_y\ .$$

Thus

$$Df^2\omega = \frac{43}{24} f\omega - \frac{25}{288} xf_x\omega - \frac{5}{72} yf_y\omega - \frac{1}{128} a^2x^2y^4\omega + \text{terms in } D^{-1}C_{0,1} .$$

and therefore

$$(Df)^2\omega = D^2f^2\omega - Df\omega = \frac{19}{24} Df\omega - \frac{5}{32} \omega - \frac{1}{128} a^2Dx^2y^4\omega + \text{terms in } C_{0,1} \qquad (F)$$

Remark: This implies that $Dx^2y^4\omega \in \tilde{H}''$ (cf. IV). Since $x^2y^4\omega \notin H'+ fH''$ we have $Dx^2y^4\omega \notin H''+ DfH''$. We shall see that $C_{0,1} \subset H''+ DfH''$. Therefore $(Df)^2\omega \notin H''+ DfH''$.

In view of V., we shall compute $(Df)Dx^2y^4\omega$ next. For $j,k \geq 0$,

$$x^jy^{k+1}f + \frac{5}{4} ax^{j+1}y^kf = (\frac{5}{24} x^{j+1}y^{k+1} + \frac{a}{4} x^{j+2}y^k)f_x + (\frac{1}{6} x^jy^{k+2} + \frac{5}{24} ax^{j+1}y^{k+1})f_y \qquad x^jy^kC$$

and applying D,

$$Dfx^jy^{k+1}\omega + \frac{5}{4} aDfx^{j+1}y^k\omega = \frac{5j+4k+13}{24} x^jy^{k+1}\omega + \frac{6j+5k+17}{24} ax^{j+1}y^k\omega \qquad (G)$$

Thus

$$fDx^2y^4\omega = Dfx^2y^4\omega - x^2y^4\omega = \frac{11}{24} x^2y^4\omega + \frac{11}{6} ax^3y^3\omega - \frac{5}{4} aDfx^3y^3\omega .$$

But as was shown above, $x^3y^3\omega$, $Dfx^3y^3\omega \in D^{-1}C_{0,1}$. Therefore

$$(Df)Dx^2y^4\omega = \frac{11}{24} Dx^2y^4\omega + \text{terms in } C_{0,1} .$$

If we want a basis of the type described in theorem 27, we can eliminate $Dx^2y^4\omega$: apply D to equation (F) to obtain

$$(Df)^3\omega = \frac{19}{24} (Df)^2\omega - \frac{5}{32} Df\omega - \frac{1}{128} a^2(Df)Dx^2y^4\omega + \text{terms in } C_{0,1}$$

$$= \frac{19}{24} (Df)^2\omega - \frac{5}{32} Df\omega - \frac{11}{3072} a^2Dx^2y^4\omega + \text{terms in } C_{0,1}$$

Use equation (F) to eliminate $Dx^2y^4\omega$. The result is

$$(Df)^3\omega = \frac{5}{4} (Df)^2\omega - \frac{299}{576} Df\omega + \frac{55}{768} \omega + \text{terms in } C_{0,1} .$$

In particular $(Df)^3\omega \in (Df)^2H''+ DfH''+ H''$.

6.4 Keeping V. in mind, we now compute $Dfy\omega$. By setting j=k=0 in equation (G), we get

$$Dfy\omega = \frac{13}{24}\, y\omega + \frac{17}{24}\, ax\omega - \frac{5}{4}\, aDfx\omega$$

or

$$Dfy\omega = \frac{13}{24}\, y\omega + \text{terms in } C_{1,0} .$$

Since $fx \notin (f_x, f_y)$, we shall compute $(Df)^2 x\omega$ next:

$$xf^2 + \frac{3750}{256}\, a^4 xyf^2 = (\frac{5}{24}\, x^2 f - \frac{5a}{576}\, x^7 - \frac{1}{384}\, a^2 x^4 y^4 + \frac{5}{1536}\, a^3 x^5 y^3 - \frac{25}{6144}\, a^4 x^6 y^2$$
$$\frac{125}{1024}\, a^5 x^7 y + \frac{3125}{1024}\, a^4 x^2 yf) f_x$$
$$+(\frac{1}{6}\, xyf - \frac{a}{144}\, x^6 y + \frac{1}{576}\, a^2 x^7 - \frac{625}{6144}\, a^5 x^6 y^2 + \frac{625}{256}\, a^4 xy^2 f) f_y$$
$$xE$$

$$Df^2 x\omega = (\frac{7}{12}\, xf\omega + \frac{5}{24}\, x^2 f_x\omega + \frac{1}{6}\, xyf_y\omega - \frac{13}{192}\, ax^6\omega) - \frac{1}{96}\, a^2 x^3 y^4 \omega$$
$$+ \frac{25}{1536}\, a^3 x^4 y^3 \omega - \frac{25}{1024}\, a^4 x^5 y^2 \omega - \frac{1625}{1536}\, a^5 x^6 y\omega + \frac{5625}{512}\, a^4 xyf\omega$$
$$+ \frac{3125}{1024}\, a^4 x^2 yf_x\omega + \frac{625}{256}\, a^4 xy^2 f_y\omega - \frac{3750}{256}\, a^4 Df^2 xy\omega$$

Just as in 6.3, we can check that all the terms outside the brackets belong to $D^{-1}C_{0,2}$ or $D^{-1}C_{1,1}$. As we shall see shortly, $C_{1,1} \subset C_{0,2}$. Furthermore

$$-\frac{13}{192}\, ax^6 = \frac{13}{8}\, xf - \frac{65}{192}\, x^2 f_x - \frac{13}{48} xyf_y \qquad \frac{13}{8}\, xA$$

Therefore

$$Df^2 x\omega = \frac{53}{24}\, xf\omega - \frac{25}{192}\, x^2 f_x\omega - \frac{5}{48}\, xyf_y\omega + \text{terms in } D^{-1}C_{0,2}$$

and

$$(Df)^2 x\omega = D^2 f^2 x\omega - Dfx\omega = \frac{29}{24}\, Dfx\omega - \frac{35}{96}\, x\omega + \text{terms in } C_{0,2} .$$

6.5 To complete the calculation of Df, we need one more equation: for $j \geq 0$,

$$x^{j+2} f + \frac{75}{8}\, a^2 x^j y^3 f = (\frac{5}{24}\, x^{j+3} + \frac{a}{16}\, x^j y^4 + \frac{15}{8}\, a^2 x^{j+1} y^3) f_x +$$

$$+ \left(\frac{1}{6} x^{j+2}y - \frac{a}{24} x^{j+3} + \frac{25}{16} a^2 x^j y^4\right) f_y \qquad x^j D$$

So

$$Dfx^{j+2}\omega = \frac{5j+19}{24} x^{j+2}\omega + \frac{ja}{16} x^{j-1}y^4\omega + \frac{30j+130}{16} a^2 x^j y^3 \omega - \frac{75}{8} a^2 Dfx^j y^3 \omega \quad (H)$$

Equation (G) shows that

$$Dfx^j y^{k+1}\omega = \frac{5j+4k+13}{24} x^j y^{k+1}\omega + \text{terms in } C_{j+1,k} \qquad j,k \geq 0$$

From equation (H), it follows that

$$Dfx^2\omega = \frac{19}{24} x^2\omega + \text{terms in } C_{0,3}$$

$$Dfx^3\omega = x^3\omega + \text{terms in } C_{0,4}$$

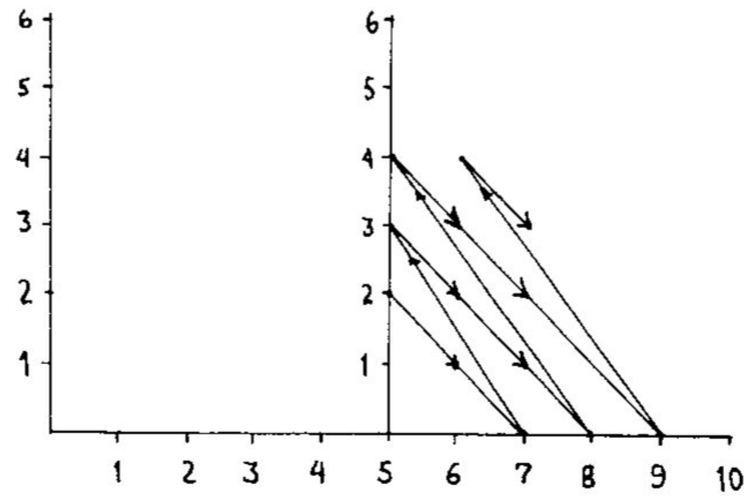
$$Dfx^4\omega = \frac{29}{24} x^4\omega + \text{terms in } C_{1,4}$$

From these equations, we see how the cyclic modules $C_{j,k}$, $j+k \geq 2$, are ordered:

$$C_{0,2} \supset C_{1,1} \supset C_{2,0} \supset C_{0,3} \supset C_{1,2} \supset C_{2,1} \supset C_{3,0} \supset C_{0,4} \supset C_{1,3} \supset$$

$$C_{2,2} \supset C_{4,0} \supset C_{1,4} \supset C_{2,3}$$

Equations (G) and (H) simply reflect the relations (C) and (D) between f_x, f_y, and f; so we could have predicted all the inclusions above from a Newton diagram by constructing a long chain, using (C) and (D), which started at the point (0,2) in the second coordinate system.



$C_{2,3}/fC_{2,3} \subset \tilde{H}''/f\tilde{H}''$ is 1-dimensional. For from (G) we get

$$Dfx^2y^3\omega = \frac{31}{24}\, x^2y^3\omega + \text{terms in } C_{3,2}$$

and as in 6.3, we see that $x^3y^2\omega \in H' + fH'' \subset f\tilde{H}''$. So we are finished. The results of the calculation are summed up in the matrix below (cf.VI) The non-zero entries below the diagonal, indicated by "•", in general depend on the parameter a and are therefore not rational. The characteristic polynomials of the two 2×2 blocks are $(s - 3/8)(s - 5/12)$ and $(s - 7/12)(s - 5/8)$, so the matrix is diagonalisable. No two eigenvalues are equal or differ by a positive integer. The characteristic polynomial of the monodromy is

$$\frac{(s^{24} - 1)(s - 1)}{(s^6 - 1)} \quad .$$

6.6 Matrix of Df on $\tilde{H}''/f\tilde{H}''$ for $f(x,y) = ax^5 + y^6 + x^4y$ $\quad 0 \neq a \in C$

	ω	$Df\omega$	$Dx^2y^4\omega$	$y\omega$	$x\omega$	$Dfx\omega$	$y^2\omega$	$xy\omega$	$x^2\omega$	$y^3\omega$	$xy^2\omega$	$x^2y\omega$	$x^3\omega$	$y^4\omega$	$xy^3\omega$	$x^2y^2\omega$	$x^4\omega$	$xy^4\omega$	$x^2y^3\omega$
ω	0	$-\frac{5}{32}$																	
$Df\omega$	1	$\frac{19}{24}$																	
$Dx^2y^4\omega$		.	$\frac{11}{24}$																
$y\omega$		.	.	$\frac{13}{24}$															
$x\omega$		.	.	.	0	$\frac{35}{96}$													
$Dfx\omega$		.	.	.	1	$\frac{29}{24}$													
$y^2\omega$		.	.			.	$\frac{17}{24}$												
$xy\omega$		.	.			.	.	$\frac{3}{4}$											
$x^2\omega$		.	.			.	.	.	$\frac{19}{24}$										
$y^3\omega$		.	.			.	.	.	.	$\frac{7}{8}$									
$xy^2\omega$		.	.			.	.	.	.	.	$\frac{11}{12}$								
$x^2y\omega$		.	.			.	.	.	.	.	.	$\frac{23}{24}$							
$x^3\omega$		.	.			.	.	.	.	.	.	.	1						
$y^4\omega$		.	.			.	.	.	.	.	.	.	.	$\frac{25}{24}$					
$xy^3\omega$		.	.			.	.	.	.	.	.	.	.	.	$\frac{13}{12}$				
$x^2y^2\omega$		.	.			.	.	.	.	.	.	.	.	.	.	$\frac{9}{8}$			
$x^4\omega$		.	.			.	.	.	.	.	.	.	.	.	.	.	$\frac{29}{24}$		
$xy^4\omega$		.	.			.	.	.	.	.	.	.	.	.	.	.	.	$\frac{5}{4}$	
$x^2y^3\omega$		.	.			.	.	.	.	.	.	.	.	.	.	.	.	.	$\frac{31}{24}$

Matrix of Df on H''/H' for $f(x,y) = y^6 + x^4y$

	ω	$x^5\omega$	$x^2y^4\omega$	$y\omega$	$x\omega$	$x^6\omega$	$y^2\omega$	$xy\omega$	$x^2\omega$	$y^3\omega$	$xy^2\omega$	$x^2y\omega$	$x^3\omega$	$y^4\omega$	$xy^3\omega$	$x^2y^2\omega$	$x^4\omega$	$xy^4\omega$	$x^2y^3\omega$
ω	$\frac{3}{8}$																		
$x^5\omega$		$\frac{17}{12}$																	
$x^2y^4\omega$			$\frac{35}{24}$																
$y\omega$				$\frac{13}{24}$															
$x\omega$					$\frac{7}{12}$														
$x^6\omega$						$\frac{13}{8}$													
$y^2\omega$							$\frac{17}{24}$												
$xy\omega$								$\frac{3}{4}$											
$x^2\omega$									$\frac{19}{24}$										
$y^3\omega$									$\frac{21}{24}$										
$xy^2\omega$										$\frac{11}{12}$									
$x^2y\omega$											$\frac{23}{24}$								
$x^3\omega$												1							
$y^4\omega$													$\frac{25}{24}$						
$xy^3\omega$														$\frac{13}{12}$					
$x^2y^2\omega$															$\frac{9}{8}$				
$x^4\omega$																$\frac{29}{24}$			
$xy^4\omega$																	$\frac{5}{4}$		
$x^2y^3\omega$																		$\frac{31}{24}$	

6.7 There are two reasons why this function was used as an example. First, it was the simplest polynomial I could find where

$$\tilde{H}'' = H'' + DfH'' + (Df)^2H'' \ , \ \tilde{H}'' \neq H'' + DfH'' \qquad \text{(cf.6.3,6.4,6.5):}$$

I asked Malgrange whether he thought it likely, that if

$$f^{r+1} \varepsilon (f_0,\ldots,f_n) = I_f$$

then

$$(Df)^{r+1}H'' \subset H'' + \cdots + (Df)^rH'' \ .$$

He replied that if

$$(Df)^2H'' \subset DfH'' + H''$$

then

$$f^2 \varepsilon fI_f + I_f^2 \ ,$$

but that Briancon had found a polynomial, namely $f(x,y) = x^6 + y^7 + x^5y$, which did not satisfy this integrality condition. Of course, by theorem 11, $f^2 \varepsilon I_f$.

As was remarked in 6.3, for $f = ax^5 + y^6 + x^4y$, $a \neq 0$, $(Df)^2\omega \notin H'' + DfH''$ although $f^2 \varepsilon I_f$. So this function also settles the above question.

Claim: $f^2 \varepsilon fI_f + I_f^2$

Proof: Consider equation (E). The left-hand side is of the form $(\text{unit})f^2$. We must check that all the terms mithin the brackets on the right-hand side lie in $(f)+I_f$. We have

$$4x^3y^4 = y^3f_x - 5ax^4y^3$$

$$4x^4y^3 = xy^2f_x - 5ax^5y^2$$

$$\frac{a}{24}\, x^5y^2 = -\, y^2f + \frac{5}{24}\, xy^2f_x + \frac{1}{6}\, y^3f_y$$

Thus x^4y^3, x^3y^4, $x^5y^2 \varepsilon (f)+I_f$. To see that x^6, x^6y, $x^5y \varepsilon (f) + I_f$, look at xA, xyA, yA.

Thus the converse to Malgrange's result does not hold.

In [T],0.5 it is shown that any function $f\varepsilon\, C\{x_0,\ldots,x_n\}$ with $f(0)=0$ is integral over the ideal I_f. If we take

$$f = ax^{k+5} + y^{k+6} + x^{k+4}y \qquad a\neq 0\ ,\ k\geqslant 0$$

then by using the analogs of equations (C), (D), and (E), it is not hard to see that

$$f^{k+2} \varepsilon f^{k+1}I_f + \cdots + I_f^{k+2}\ .$$

These equations, together with the fact that f_x, f_y is a regular sequence in $C\{x,y\}$, show that f is not integral over I_f of degree k+1. Thus there is no result like theorem 11 for integrality over I_f. Similarly one checks that

$$(Df)^{k+3}H'' \subset H'' + \cdots + (Df)^{k+2}H''\ .$$

Using theorem 26, one can verify that this too is best possible.

One would conjecture of course that Malgrange's result can be generalized:

$$\text{If}\quad (Df)^{r+1}H'' \subset H'' + \cdots + (Df)^{r}H'',$$

$$\text{then f is integral of degree } r+1 \text{ over } I_f.$$

Here is a summary of the various properties that have come up:

$$f^{r+1} \varepsilon I_f \quad \overset{\text{conjecture true for } r=0}{\Rightarrow \not\Leftarrow} \quad \text{Df has eigenvalues of multiplicity} \leqslant r+1$$

$$\Uparrow \not\Downarrow$$

$$f^{r+1} \varepsilon fI_f^r + \cdots + I_f^{r+1} \quad \not\Rightarrow \underset{\text{conjecture true for } r=0,1}{\Leftarrow} \quad \tilde{H}'' = (Df)^rH'' + \cdots + H''$$

There is a second reason why $ax^5 + y^6 + x^4y$ is interesting: it shows how the Gauss-Manin connection, together with the module $\tilde{H}''$, is very much an analytic invariant rather than a topological one. To make this plain,

we need the following result of Lê and Ramanujam [L-R], Th. 2.1 .

Theorem 29: Let $f(a,x_0,\ldots,x_n)$ be a polynomial in $x_0,\ldots,x_n$ whose coefficients are smooth complex-valued functions of $a\varepsilon[0,1]$. Suppose that $f(a,0)=0$, and that $f(a,\cdot)$ has an isolated singularity in 0, for all $a\varepsilon[0,1]$. Assume that μ_a is independent of a.

Then the Milnor fibrations of f_0 and f_1 have the same fibre homotopy type. If $n\neq 2$, they are even isomorphic as C^∞ fibre bundles and the topological types of the singularities are the same.

$ax^5+y^6+x^4y$ is just such a family of polynomials (see 6.1). So y^6+x^4y and $ax^5+y^6+x^4y$, $a\neq 0$ have the same topological invariants, eg. Seifert form or monodromy. However the structure of $\tilde{H}''$ changes suddenly as $a\to 0$: for y^6+x^4y is quasihomogeneous and therefore has $\tilde{H}''=H''$. This behavior is the result of a corresponding jump in the structure of the ring $C\{x,y\}/(f_x,f_y)$ and is reflected in the behavior of the characteristic polynomial of Df:

$$s-\frac{5}{12}\to s-\frac{17}{12}\ ,\ s-\frac{11}{24}\to s-\frac{35}{24}\ ,\ s-\frac{5}{8}\to s-\frac{13}{8}$$

In [M2] Malgrange proves that the reduced Bernstein polynomial $\tilde{b}$ (see introduction) is the minimal polynomial of the negative of the residue of D on $\tilde{H}''$. (In the appendix a shorter proof of this result is given). For $ax^5+y^6+x^4y$, the minimal polynomial is the characteristic polynomial; so $\tilde{b}$ too will jump as $a\to 0$. I suspect that such discontinuous behavior of the ring $C\{x_0,\ldots,x_n\}/(f_0,\ldots,f_n)$ will generally have this effect on the Bernstein polynomial.

§7 Further Examples

In this section, the Gauss-Manin connection is calculated for several other interesting functions. They are all generic in the sense of [E] and [V]. Our results for the characteristic polynomial of the monodromy agree with what one gets using their formula.

At present this formula is the best way of calculating the characteristic polynomial of the monodromy. However the functions to which it applies do not include for example curves, any of whose branches at the singular point has more than one Puiseux pair. Sumners and Woods [Su-W] give an algorithm for computing the characteristic polynomial for any curve. They can also say something about the minimal polynomial. G.L. Gordon is currently studying how to compute the Jordan normal form from an arbitrary resolution of the singularity. F. Ehlers is working on the same question using the Newton polygon of the function.

There is an unpublished thesis in Japanese which calculates the Bernstein polynomial for many functions, including the family T_{pqr} and $x^{p+q}+ y^{p+q}+ x^q y^q$ considered here.

T_{pqr}: $f(x,y,z) = x^p+ y^q+ z^r+ axyz$ $\quad 0\neq a\in C$ $\quad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$

The singularities T_{pqr} are one of the three types of unimodular singularities (see [A]). They are the first in Arnold's classification of functions which have no quasihomogeneous representatives.

$f = \frac{1}{p} xf_x+ \frac{1}{q} yf_y+ \frac{1}{r} zf_z+ asxyz$ $\quad s = 1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r}$

$f_x = px^{p-1}+ ayz$ $\quad f_y = qy^{q-1}+ axz$ $\quad f_z = rz^{r-1}+ axy$

Case 1: $p,q,r\geq 3$

$x^2y(1 + \frac{pqr}{a^3} x^{p-3}y^{q-3}z^{r-3}) = \frac{qr}{a^3} y^{q-2}z^{r-3}f_x - \frac{r}{a^2}z^{r-2}f_y + \frac{1}{a} xf_z$

By permuting x,y,z (and p,q,r) one finds that

$$x^2y, x^2z, xy^2, y^2z\ xz^2, yz^2 \in (f_x, f_y, f_z)$$

Basis for $C\{x,y,z\}/(f_x,f_y,f_z)$: $1, x, \ldots, x^{p-1}, y, \ldots, y^{q-1}, z, \ldots, z^{r-1}, f$

$$\mu = p+q+r-1$$

The equations needed for calculating the Gauss-Manin connection are:

$$\text{(A)}\quad xf + \frac{pqr}{a^3} x^{p-2}y^{q-3}z^{r-3}f = (\frac{1}{p} x^2 + \frac{qrs}{a^2} y^{q-2}z^{r-2} + \frac{qr}{a^3} x^{p-1}y^{q-3}z^{r-3})f_x$$
$$+ (\frac{1}{q} xy - \frac{rs}{a} z^{r-1} + \frac{pr}{a^3} x^{p-2}y^{q-2}z^{r-3})f_y$$
$$+ (\frac{1}{r} xz + sxz + \frac{pq}{a^3} x^{p-2}y^{q-3}z^{r-2})f_z$$

and similar equations obtained by permuting x,y,z, and

$$\text{(B)}\quad f^2 + \frac{pqr}{a^3} x^{p-3}y^{q-3}z^{r-3}f^2 = (\frac{1}{p} xf + \frac{as}{p} x^2yz + \frac{qr}{a} s^2y^{q-1}z^{r-1} +$$
$$+ \frac{qrs}{a^2} x^{p-1}y^{q-2}z^{r-2} + \frac{qr}{a^3} x^{p-2}y^{q-3}z^{r-3}f)f_x$$
$$+ (\frac{1}{q} yf + \frac{as}{q} xy^2z - rs^2yz^r + \frac{prs}{a^2} x^{p-2}y^{q-1}z^{r-2}$$
$$+ \frac{pr}{a^3} x^{p-3}y^{q-2}z^{r-3}f)f_y$$
$$+ (\frac{1}{r} zf + \frac{as}{r} xyz^2 + as^2xyz^2 + \frac{pqs}{a^2} x^{p-2}y^{q-2}z^{r-1}$$
$$+ \frac{pq}{a^3} x^{p-3}y^{q-3}z^{r-2}f)f_z \quad .$$

(B) is obtained from (A). By differentiating (B), one can compute $(Df)^2\omega$. Differentiating $x^{j-1}\wedge$(A) gives

$$\text{(C)}\quad Dfx^j\omega + \frac{prq}{a^3} Dfx^{j+p-3}y^{q-3}z^{r-3}\omega = \frac{j+p}{p}x^j\omega + \frac{qrs}{a^2}(j-1)x^{j-2}y^{q-2}z^{r-2}\omega$$
$$+ \frac{pqr}{a^3}(2s+1+\frac{j}{p})x^{j+p-3}y^{q-3}z^{r-3}\omega \qquad j\geq 1$$

From (C) and similar equations for $Dfy^k\omega$, $Dfz^l\omega$, $k,l\geq 1$, one can compute the matrix below. The characteristic polynomial of the monodromy is $(t^p-1)(t^q-1)(t^r-1)/(t-1)$. The Jordan normal form has one 2×2 block.

	ω	$Df\omega$	$z\omega$	$z^2\omega$	$z^3\omega$	$z^3\omega$	$x\omega$	$y^2\omega$	$y\omega$	$x^2\omega$	
ω	0	-1									
$Df\omega$	1	2									
$z\omega$		.	$\frac{5}{4}$								
$z^2\omega$		.	.	$\frac{3}{2}$							p=3
$z^3\omega$		.	.	.	$\frac{7}{4}$						q=3
$x\omega$						$\frac{4}{3}$					r=4
$y^2\omega$						.	$\frac{5}{3}$				
$y\omega$								$\frac{4}{3}$			
$x^2\omega$								.	$\frac{5}{3}$		

	ω	$Df\omega$	$z^2\omega$	$z\omega$	$x\omega$	$x^2\omega$	$x^3\omega$	$y\omega$	$y^2\omega$	$y^3\omega$	
ω	0	-1									
$Df\omega$	1	2									
$z^2\omega$		.	$\frac{5}{3}$								
$z\omega$				$\frac{4}{3}$							p=4
$x\omega$					$\frac{5}{4}$						q=4
$x^2\omega$						$\frac{3}{2}$					r=3
$x^3\omega$							$\frac{7}{4}$				
$y\omega$								$\frac{5}{4}$			
$y^2\omega$									$\frac{3}{2}$		
$y^3\omega$										$\frac{7}{4}$	

	ω	$Df\omega$	$x\omega \cdots x^{p-1}\omega$	$y\omega \cdots y^{q-1}\omega$	$z\omega \cdots z^{r-1}\omega$
ω	0	-1			
$Df\omega$	1	2			
$x\omega$			$\frac{p+1}{p}$		
$\vdots$					
$x^{p-1}\omega$			$\frac{2p-1}{p}$		
$y\omega$				$\frac{q+1}{q}$	
$\vdots$					
$y^{q-1}\omega$				$\frac{2q-1}{q}$	
$z\omega$				$\frac{r+1}{r}$	
$\vdots$					
$z^{r-1}\omega$					$\frac{2r-1}{r}$

$p,q,r \geq 4$

Case 2: $p,q\geq 4$, $r=2$

$$y^2z(1 - \frac{4pq}{a^4} x^{p-4}y^{q-4}) = \frac{1}{a} yf_x + \frac{2p}{a^3} x^{p-3}f_y - (\frac{p}{a^2}x^{p-2} + \frac{2pq}{a^4} x^{p-4}y^{q-2})f_z$$

Interchanging x and y gives a similar equation for x^2z. So

$$x^2z, y^2z, z^3, xf, yf, zf \in (f_x, f_y, f_z)$$

Basis for $C\{x,y,z\}/(f_x,f_y,f_z)$: $1,x,\ldots,x^{p-1},y,\ldots,y^{q-1},z,f$ $\mu = p+q+1$

The equations needed for computing the Gauss-Manin connection are:

$$(A)\quad zf + \frac{2pq}{a^3} x^{p-3}y^{q-3}f = (\frac{1}{p} xz + sxz + \frac{2q}{a^3} x^{p-2}y^{q-3})f_x$$
$$+ (\frac{1}{q} yz - \frac{ps}{a} x^{p-1} + \frac{2p}{a^3} x^{p-3}y^{q-2})f_y$$
$$+ (\frac{1}{2} z^2 + \frac{pqs}{a^2} x^{p-2}y^{q-2} + \frac{pq}{a^3} x^{p-3}y^{q-3}z)f_z$$

$$(B)\quad yf - \frac{4pq}{a^4} x^{p-4}y^{q-3}f = (\frac{1}{p} xy + sxy - \frac{4q}{a^4} x^{p-3}y^{q-3})f_x$$
$$+ (\frac{1}{q} y^2 + \frac{2ps}{a^2} x^{p-2} - \frac{4p}{a^4} x^{p-4}y^{q-2})f_y$$
$$+ (\frac{yz}{2} - \frac{ps}{a} x^{p-1} - \frac{2pqs}{a^3} x^{p-3}y^{q-2} - \frac{2pq}{a^4} x^{p-4}y^{q-3}z)f_z$$

and an equation (C) obtained from (B) by interchanging x and y .

Differentiating $\overset{y^{k-1}}{\wedge}$(B) gives

$$(D)\quad Dfy^k\omega - \frac{4pq}{a^4} Dfx^{p-4}y^{q+k-4}\omega = \frac{k+q}{q} y^k\omega + \frac{2ps}{a^2} (k-1)x^{p-2}y^{k-2}\omega$$
$$+ (12q-4p(k-3)-10pq) \frac{1}{a^4} x^{p-4}y^{k+q-4}\omega \quad k\geq 1$$

Differentiating (A) and $\overset{x^{j-1}}{\wedge}$(C) gives similar equations for $Dfz\omega$ and $Dfx^j\omega$, $j\geq 1$. From (A) one can also derive an equation for f^2 from which $(Df)^2\omega$ can be calculated. The resulting matrices for Df are shown below. The characteristic and minimal polynomials of the monodromy are the same as in case 1, with r=2.

	ω	$Df\omega$	$x\omega$	$x^2\omega$	$y^2\omega$	$x^3\omega$	$x^4\omega$	$y\omega$	$z\omega$	$y^3\omega$
ω	0	-1								
$Df\omega$	1	2								
$x\omega$		.	$\frac{6}{5}$							
$x^2\omega$		.	.	$\frac{7}{5}$						
$y^2\omega$		.	.	.	$\frac{3}{2}$					
$x^3\omega$		.	.	.	.	$\frac{8}{5}$				
$x^4\omega$		.	.	.	.	.	$\frac{9}{5}$			
$y\omega$								$\frac{5}{4}$		
$z\omega$								.	$\frac{3}{2}$	
$y^3\omega$								.	.	$\frac{7}{4}$

p=5
q=4
r=2

	ω	$Df\omega$	$z\omega$	$x\omega$	$y^4\omega$	$y\omega$	$x^4\omega$	$x^2\omega$	$y^3\omega$	$y^2\omega$	$x^3\omega$
ω	0	-1									
$Df\omega$	1	2									
$z\omega$		.	$\frac{3}{2}$								
$x\omega$				$\frac{6}{5}$							
$y^4\omega$				.	$\frac{9}{5}$						
$y\omega$						$\frac{6}{5}$					
$x^4\omega$						.	$\frac{9}{5}$				
$x^2\omega$								$\frac{7}{5}$			
$y^3\omega$								.	$\frac{8}{5}$		
$y^2\omega$										$\frac{7}{5}$	
$x^3\omega$										.	$\frac{8}{5}$

p=5
q=5
r=2

	ω	$Df\omega$	$y^4\omega$	$x^2\omega$	$y^3\omega$	$y^2\omega$	$x^4\omega$	$x\omega$	$x^3\omega$	$x^5\omega$	$y\omega$	$z\omega$	
ω	0	-1											
$Df\omega$	1	2											
$y^4\omega$		.	$\frac{9}{5}$										
$x^2\omega$				$\frac{4}{3}$									p=6
$y^3\omega$				.	$\frac{8}{5}$								q=5
$y^2\omega$						$\frac{7}{5}$							r=2
$x^4\omega$					.		$\frac{5}{3}$						
$x\omega$								$\frac{7}{6}$					
$x^3\omega$									$\frac{3}{2}$				
$x^5\omega$										$\frac{11}{6}$			
$y\omega$											$\frac{6}{5}$		
$z\omega$												$\frac{3}{2}$	

	ω	$Df\omega$	$x^2\omega$	$y^2\omega$	$x^4\omega$	$x\omega$	$x^3\omega$	$x^5\omega$	$y\omega$	$y^3\omega$	$z\omega$	
ω	0	-1										
$Df\omega$	1	2										
$x^2\omega$		.	$\frac{4}{3}$									
$y^2\omega$		.	.	$\frac{3}{2}$								p=6
$x^4\omega$		.	,	.	$\frac{5}{3}$							q=4
$x\omega$						$\frac{7}{6}$						r=2
$x^3\omega$						.	$\frac{3}{2}$					
$x^5\omega$						.	.	$\frac{11}{6}$				
$y\omega$									$\frac{5}{4}$			
$y^3\omega$									.	$\frac{7}{4}$		
$z\omega$											$\frac{3}{2}$	

	ω	$Df\omega$	$x^2{}_\omega$	$y^{q-2}{}_\omega$	$y^2{}_\omega$	$x^{p-2}{}_\omega$	$x\omega$	$x^3{}_\omega$	$\cdots$	$x^{p-3}{}_\omega$	$x^{p-1}{}_\omega$	$y\omega$	$y^3{}_\omega$	$\cdots$	$y^{q-3}{}_\omega$	$y^{q-1}{}_\omega$	$z\omega$
ω	0	-1															
$Df\omega$	1	2															
$x^2{}_\omega$			$\frac{p+2}{p}$														
$y^{q-2}{}_\omega$			.	$\frac{2q-2}{q}$													
$y^2{}_\omega$					$\frac{q+2}{q}$												
$x^{p-2}{}_\omega$					.	$\frac{2p-2}{p}$											
$x\omega$							$\frac{p+1}{p}$										
$x^3{}_\omega$								$\frac{p+3}{p}$	.	.							
$\vdots$																	
$x^{p-3}{}_\omega$										$\frac{2p-3}{p}$							
$x^{p-1}{}_\omega$											$\frac{2p-1}{p}$						
$y\omega$											$\frac{q+1}{q}$						
$y^3{}_\omega$												$\frac{q+3}{q}$	.	.			
$\vdots$																	
$y^{q-3}{}_\omega$															$\frac{2q-3}{q}$		
$y^{q-1}{}_\omega$																$\frac{2q-1}{q}$	
$z\omega$																	$\frac{3}{2}$

$p \geq 6$, $q \geq 6$

Case 3: $p>6$, $q=3$, $r=2$

$$xz^2(1 - \frac{72p}{a^6} x^{p-6}) = - \frac{3}{a^2} yf_x + (\frac{z}{a} - \frac{6p}{a^4} x^{p-3})f_y + (\frac{3p}{a^3} x^{p-2} + \frac{18p}{a^5} x^{p-4}y - \frac{36p}{a^6} x^{p-5}z)f_z$$

$$yz^2(1 - \frac{72p}{a^6} x^{p-6}) = \frac{z}{a} f_x + (- \frac{p}{a^2} x^{p-2} + \frac{12p}{a^5} x^{p-5}z)f_y + (\frac{3p}{a^3} x^{p-3}y - \frac{6p}{a^4} x^{p-4}z - \frac{36p}{a^6} x^{p-6}yz)f_z$$

Thus

$$xz^2, yz^2, z^3, xf, yf, zf \in (f_x, f_y, f_z)$$

Basis for $C\{x,y,z\}/(f_x,f_y,f_z)$: $1, x, \ldots, x^{p-1}, y, y^2, z, f$ $\qquad \mu = p+4$

The equations needed for computing the Gauss-Manin connection are:

$$(A) \quad xf - \frac{72p}{a^6} x^{p-5}f = (\frac{1}{p} x^2 + \frac{6s}{a^2} y - \frac{72}{a^6} x^{p-4})f_x + (\frac{1}{3} xy - \frac{2s}{a} z + \frac{12ps}{a^4} x^{p-3} - \frac{24p}{a^6} x^{p-5}y)f_y + (\frac{1}{2} xz + sxz - \frac{6ps}{a^3} x^{p-2} - \frac{36ps}{a^5} x^{p-4}y - \frac{36p}{a^6} x^{p-5}z)f_z$$

$$(B) \quad yf - \frac{72p}{a^6} x^{p-6}yf = (\frac{xy}{p} - \frac{2s}{a} z - \frac{72}{a^6} x^{p-5}y)f_x + (\frac{1}{3} y^2 + \frac{2ps}{a^2} x^{p-2} - \frac{24ps}{a^5} x^{p-5}z - \frac{24p}{a^6} x^{p-6}y^2)f_y + (\frac{yz}{2} + syz - \frac{6ps}{a^3} x^{p-3}y + \frac{12ps}{a^4} x^{p-4}z - \frac{36p}{a^6} x^{p-6}yz)f_z$$

$$(C) \quad zf - \frac{72p}{a^6} x^{p-6}zf = (\frac{xz}{p} + sxz - \frac{72}{a^6} x^{p-5}z)f_x + (\frac{yz}{3} - \frac{ps}{a} x^{p-1} + \frac{12ps}{a^4} x^{p-4}z - \frac{24p}{a^6} x^{p-6}yz)f_y + (\frac{1}{2} z^2 + \frac{3ps}{a^2} x^{p-2}y - \frac{6ps}{a^3} x^{p-3}z - \frac{36ps}{a^5} x^{p-3}yz - \frac{36p}{a^6} x^{p-6}z^2)f_z$$

Differentiating $\overset{x^{j-1}}{\wedge}$(A) and $\overset{y^{k-1}}{\wedge}$(B) gives

$$(D)\quad Dfx^j{}_\omega - \frac{72p}{a^6} Dfx^{j+p-6}{}_\omega = \frac{j+p}{p} x^j{}_\omega + \frac{6s}{a^2}(j-1)x^{j-2}y_\omega$$
$$- \left(\frac{72}{a^6}(j-5) + \frac{132p}{a^6}\right)x^{j+p-6}{}_\omega \qquad j\geq 1$$

$$(E)\quad Dfy^k{}_\omega - \frac{72p}{a^6} Dfx^{p-6}y^k{}_\omega = \frac{k+3}{3} y^k{}_\omega + \frac{2ps}{a^2}(k-1)x^{p-2}y^{k-2}{}_\omega$$
$$+ \frac{12ps}{a^4} x^{p-4}y^{k-1}{}_\omega - \frac{24ps}{a^5}(k-1)x^{p-5}y^{k-2}z_\omega$$
$$- \left(\frac{132p}{a^6} + \frac{24pk}{a^6} - \frac{360}{a^6}\right)x^{p-6}y^k{}_\omega \qquad k=1,2$$

Differentiating (C) gives a similar equation for Dfz_ω. From (C) one also obtains an equation for f^2, from which $(Df)^2{}_\omega$ can be computed.

As in cases 1 and 2, the characteristic polynomial of the monodromy is $(t^p-1)(t^3-1)(t^2-1)/(t-1)$ and the minimal polynomial is divisible by $(t-1)^2$.

	ω	$Df\omega$	$x\omega$	$x^2\omega$	$y\omega$	$x^3\omega$	$z\omega$	$x^4\omega$	$y^2\omega$	$x^5\omega$	$x^6\omega$	
ω	0	-1										
$Df\omega$	1	2										
$x\omega$		.	$\frac{8}{7}$									p=7
$x^2\omega$		.	.	$\frac{9}{7}$								q=3
$y\omega$		.	.	.	$\frac{4}{3}$							r=2
$x^3\omega$		.	.	.	.	$\frac{10}{7}$						
$z\omega$		.	.	.	.	.	$\frac{3}{2}$					
$x^4\omega$		.	.	.	.	.	.	$\frac{11}{7}$				
$y^2\omega$		.	.	.	.	.	.	.	$\frac{5}{3}$			
$x^5\omega$		.	.	.	.	.	.	.	.	$\frac{12}{7}$		
$x^6\omega$		.	.	.	.	.	.	.	.	.	$\frac{13}{7}$	

	ω	$Df\omega$	$x^2\omega$	$y\omega$	$x^4\omega$	$y^2\omega$	$x^6\omega$	$x\omega$	$x^3\omega$	$z\omega$	$x^5\omega$	$x^7\omega$	
ω	0	-1											
$Df\omega$	1	2											
$x^2\omega$		.	$\frac{5}{4}$										
$y\omega$		.	.	$\frac{4}{3}$									
$x^4\omega$		.	.	.	$\frac{3}{2}$								p=8
$y^2\omega$		.	.	.	.	$\frac{5}{3}$							q=3
$x^6\omega$		.	.	.	.	.	$\frac{7}{4}$						r=2
$x\omega$							.	$\frac{9}{8}$					
$x^3\omega$							.	.	$\frac{11}{8}$				
$z\omega$							.	.	.	$\frac{3}{2}$			
$x^5\omega$							.	.	.	.	$\frac{13}{8}$		
$x^7\omega$							.	.	.	.	.	$\frac{15}{8}$	

	ω	$Df\omega$	$x^3\omega$	$z\omega$	$x^6\omega$	$x\omega$	$x^4\omega$	$y^2\omega$	$x^7\omega$	$x^2\omega$	$y\omega$	$x^5\omega$	$x^8\omega$
ω	0	-1											
$Df\omega$	1	2											
$x^3\omega$		.	$\frac{4}{3}$										
$z\omega$		.	.	$\frac{3}{2}$									
$x^6\omega$		.	.	.	$\frac{5}{3}$								
$x\omega$						$\frac{10}{9}$							
$x^4\omega$						.	$\frac{13}{9}$						
$y^2\omega$						.	.	$\frac{5}{3}$					
$x^7\omega$						.	.	.	$\frac{16}{9}$				
$x^2\omega$										$\frac{11}{9}$			
$y\omega$										.	$\frac{4}{3}$		
$x^5\omega$										.	.	$\frac{14}{9}$	
$x^8\omega$										.	.	.	$\frac{17}{9}$

$p=9,\ q=3,\ r=2$

For $p \geq 10$, the matrix of Df is made up of the following blocks:

	ω	$Df\omega$	$x^{p-6}\omega$	....	$x^{j(p-6)}\omega$	....
ω	0	-1				
$Df\omega$	1	2				
$x^{p-6}\omega$		.	$\frac{2p-6}{p}$			
$x^{j(p-6)}\omega$		.	.		$\frac{p+j(p-6)}{p}$	
		.	.		.	

	$x^2\omega$	$y\omega$	$x^{p-4}\omega$	....	$x^{p-4+j(p-6)}\omega$	....
$x^2\omega$	$\frac{p+2}{p}$					
$y\omega$	.	$\frac{4}{3}$				
$x^{p-4}\omega$	.	.	$\frac{2p-4}{p}$			
$x^{p-4+j(p-6)}\omega$	.	.	.		$\frac{2p-4+j(p-6)}{p}$	
	.	.	.		.	

	$x^3\omega$	$z\omega$	$x^{p-3}\omega$	....	$x^{p-3+j(p-6)}\omega$	....
$x^3\omega$	$\frac{p+3}{p}$					
$z\omega$	.	$\frac{3}{2}$				
$x^{p-3}\omega$	.	.	$\frac{2p-3}{p}$			
$x^{p-3+j(p-6)}\omega$	.	.	.		$\frac{2p-3+j(p-6)}{p}$	
	.	.	.		.	

	$x^4{}_\omega$	$y^2{}_\omega$	$x^{p-2}{}_\omega$	$x^{p-2+j(p-6)}{}_\omega$ ....
$x^4{}_\omega$	$\frac{p+4}{p}$			
$y^2{}_\omega$	.	$\frac{5}{3}$		
$x^{p-2}{}_\omega$	.	.	$\frac{2p-2}{p}$ . .	
$x^{p-2+j(p-6)}{}_\omega$	.	.	.	$\frac{2p-2+j(p-6)}{p}$

	$x^k{}_\omega$	$x^{k+j(p-6)}{}_\omega$	
$x^k{}_\omega$	$\frac{p+k}{p}$		
$x^{k+j(p-6)}{}_\omega$	.	$\frac{p+k+j(p-6)}{p}$	$k\neq 2,3,4$

In [M3] Malgrange gives an example

$$f(x_1,\ldots,x_n) = x_1^{2n+2} + \cdots + x_n^{2n+2} + (x_1\cdots x_n)^2$$

whose monodromy has an n×n block when in Jordan normal form. This he demonstrates by proving that

$$\int_{\gamma(t)} x_1 dx_2\wedge\cdots\wedge dx_n = O(\sqrt{t}(\log t)^{n-1})$$

where $\gamma(t) = \{x\varepsilon R^n \mid f(x)=t\}$, $t>0$ (see 4.5). Thus the upper bound on the size of the blocks in the Jordan normal form, contained in theorem 21, is best possible. We compute the Gauss-Manin connection for n=3.

$$f(x,y,z) = x^8 + y^8 + z^8 + x^2y^2z^2$$

$$f_x = 8x^7 + 2xy^2z^2 \qquad f_y = 8y^7 + 2x^2yz^2 \qquad f_z = 8z^7 + 2x^2y^2z$$

$$f = \frac{1}{8}xf_x + \frac{1}{8}yf_y + \frac{1}{8}zf_z + \frac{1}{4}x^2y^2z^2$$

$$x^5y^3z(1 + 64x^2y^2z^2) = 8y^5z^3f_x - 2xz^5f_y + \frac{1}{2}x^3yf_z$$

$$xy^4z^4 = -16x^5f + (2x^6 + \frac{1}{2}y^2z^2)f_x + 2x^5yf_y + 2x^5zf_z$$

Similar equations are obtained by permuting x,y, and z.

Basis for $C\{x,y,z\}/(f_x,f_y,f_z)$: $\mu=215$

$1, f, f^2$

x, xf

x^2, x^2f, y^6z^6 xy, xyf

x^3, x^3f x^2y, x^2yf $xyz, xyzf$

x^4, x^4f x^2y^2, x^2y^2f, z^6 x^2yz, x^2yzf x^3y

x^5, x^5f x^2y^2z, x^2y^2zf x^4y x^3y^2 x^3yz

x^5y x^4y^2 x^4yz x^3y^3 x^3y^2z

x^6y x^5y^2 x^5yz x^4y^3 x^4y^2z x^3y^3z

x^6y^2 x^6yz x^5y^3 x^5y^2z x^4y^4 x^4y^3z
x^6y^3 x^6y^2z x^5y^4
x^6y^4 x^5y^5
x^6y^5

and all polynomials obtained from these by permuting x,y, and z.

The equations used for computing the Gauss-Manin connection are:

$$(A)\quad x^3yf + 64x^5y^3z^2f = (\tfrac{1}{8}x^4y + 2y^5z^4 + 8x^6y^3z^2)f_x + (\tfrac{1}{8}x^3y^2 - \tfrac{1}{2}xz^6 + 8x^5y^4z^2)f_y + (\tfrac{1}{4}x^3yz + 8x^5y^3z^3)f_z$$

$$(B)\quad (8y^6 + 2x^2z^2)f = (xy^6 + \tfrac{1}{4}x^3z^2)f_x + (y^7 + \tfrac{1}{2}x^2yz^2)f_y + (y^6z + \tfrac{1}{4}x^2z^3)f_z$$

$$(C)\quad xf^2 + 64x^3y^2z^2f^2 = (\tfrac{1}{8}x^2f + 8x^4y^2z^2f + \tfrac{1}{32}x^4y^2z^2 + \tfrac{1}{2}y^6z^6 + 2x^6y^4z^4)f_x + (\tfrac{1}{8}xyf + 8x^3y^3z^2f + \tfrac{1}{32}x^3y^3z^2 - \tfrac{1}{8}xyz^8 + 2x^5y^5z^4)f_y + (\tfrac{1}{8}xzf + 8x^3y^2z^3f + \tfrac{1}{16}x^3y^2z^3 + 2x^5y^4z^5)f_z$$

$$(D)\quad f^3 + 64x^2y^2z^2f^3 = (\tfrac{1}{8}xf^2 + 8x^3y^2z^2f^2 + \tfrac{1}{32}x^3y^2z^2f + 2x^5y^4z^4f + \tfrac{1}{128}x^5y^4z^4 + \tfrac{1}{8}xy^8z^8 + \tfrac{1}{2}x^7y^6z^6)f_x + (\tfrac{1}{8}yf^2 + 8x^2y^3z^2f^2 + \tfrac{1}{32}x^2y^3z^2f + 2x^4y^5z^4f + \tfrac{1}{128}x^4y^5z^4 - \tfrac{1}{32}x^2y^3z^{10} + \tfrac{1}{2}x^6y^7z^6)f_y + (\tfrac{1}{8}zf^2 + 8x^2y^2z^3f^2 + \tfrac{1}{32}x^2y^2z^3f + 2x^4y^4z^5f + \tfrac{1}{64}x^4y^4z^5 + \tfrac{1}{2}x^6y^6z^7)f_z$$

and similar expressions obtained by permuting x,y, and z. (C) is derived from (A) and (D) from (C). From (A) follows

$$(E)\quad Df x^j y^k z^l{}_\omega = \frac{j+k+2l+4}{8}x^j y^k z^l{}_\omega + 2(j-3)x^{j-4}y^{k+4}z^{l+4}{}_\omega - \frac{k-1}{2}x^{j-2}y^{k-2}z^{l+6}{}_\omega \mod f\tilde{H}'' \qquad \text{for } j\geq 3,\ k\geq 1$$

From (C) follows

$$(F)\quad Df^2x^jy^kz^l\omega = \frac{j+k+l+3}{8}x^jy^kz^lf\omega + \frac{j+k+2l+12}{32}x^{j+2}y^{k+2}z^{l+2}\omega +$$
$$\frac{j-1}{2}x^{j-2}y^{k+6}z^{l+6}\omega - \frac{k+1}{8}x^jy^kz^{l+8}\omega + \frac{1}{8}x^{j+1}y^kz^lf_x\omega +$$
$$\frac{1}{8}x^jy^{k+1}z^lf_y\omega + \frac{1}{8}x^jy^kz^{l+1}f_z\omega \quad \text{mod } f^2\tilde{H}'' \qquad \text{for } j\geq 1$$

$(Df)^3\omega$ can be calculated then from (D), $(Df)^3x^2\omega$ and $(Df)^3x^2y^2\omega$ from (B) and (F), and the rest from (E) or (F). The matrix of Df consists of the following blocks:

	ω	$Df\omega$	$(Df)^2\omega$	Characteristic polynomial
ω	0	0	$\frac{1}{8}$	
$Df\omega$	1	0	$-\frac{3}{4}$	$(s-\frac{1}{2})^3$
$(Df)^2\omega$	0	1	$\frac{3}{2}$	

	$x\omega$	$Dfx\omega$	
$x\omega$	0	$-\frac{25}{64}$	$(s-\frac{5}{8})^2$
$Dfx\omega$	1	$\frac{5}{4}$	

	$x^2\omega$	$Dfx^2\omega$	$(Df)^2x^2\omega$	
$x^2\omega$	0	0	$\frac{9}{16}$	
$Dfx^2\omega$	1	0	$-\frac{33}{16}$	$(s-\frac{3}{4})^2(s-1)$
$(Df)^2x^2\omega$	0	1	$\frac{5}{2}$	

	$xy\omega$	$Dfxy\omega$	Characteristic polynomial
$xy\omega$	0	$-\frac{21}{32}$	$(s-\frac{3}{4})(s-\frac{7}{8})$
$Dfxy\omega$	1	$\frac{13}{8}$	

	$x^3\omega$	$Dfx^3\omega$	
$x^3\omega$	0	$-\frac{49}{64}$	$(s-\frac{7}{8})^2$
$Dfx^3\omega$	1	$\frac{7}{4}$	

	$x^2y\omega$	$Dfx^2y\omega$	
$x^2y\omega$	0	$-\frac{63}{64}$	$(s-\frac{7}{8})(s-\frac{9}{8})$
$Dfx^2y\omega$	1	2	

	$xyz\omega$	$Dfxyz\omega$	
$xyz\omega$	0	-1	$(s-1)^2$
$Dfxyz\omega$	1	2	

	$x^4\omega$	$Dfx^4\omega$	
$x^4\omega$	0	-1	$(s-1)^2$
$Dfx^4\omega$	1	2	

	$x^2y^2\omega$	$Dfx^2y^2\omega$	$(Df)^2x^2y^2\omega$	
$x^2y^2\omega$	0	0	$\frac{25}{16}$	
$Dfx^2y^2\omega$	1	0	$-\frac{65}{16}$	$(s-\frac{5}{4})^2(s-1)$
$(Df)^2x^2y^2\omega$	0	1	$\frac{7}{2}$	

Characteristic polynomial

	$x^2yz\omega$	$Dfx^2yz\omega$	
$x^2yz\omega$	0	$-\frac{45}{32}$	$(s-\frac{9}{8})(s-\frac{5}{4})$
$Dfx^2yz\omega$	1	$\frac{19}{8}$	

$$Dfx^3y\omega = x^3y\omega$$

	$x^5\omega$	$Dfx^5\omega$	
$x^5\omega$	0	$-\frac{81}{64}$	$(s-\frac{9}{8})^2$
$Dfx^5\omega$	1	$\frac{9}{4}$	

	$x^2y^2z\omega$	$Dfx^2y^2z\omega$	
$x^2y^2z\omega$	0	$-\frac{121}{64}$	$(s-\frac{11}{8})^2$
$Dfx^2y^2z\omega$	1	$\frac{11}{4}$	

	$x^4y\omega$	$y^5z^4\omega$	$x^3y^2\omega$	$xz^6\omega$	$x^3yz\omega$	$x^5y\omega$
$x^4y\omega$	$\frac{9}{8}$					
$y^5z^4\omega$	2	$\frac{13}{8}$				
$x^3y^2\omega$			$\frac{9}{8}$			
$xz^6\omega$			$-\frac{1}{2}$	$\frac{11}{8}$		
$x^3yz\omega$					$\frac{5}{4}$	
$x^5y\omega$						$\frac{5}{4}$

	$x^4y^2\omega$	$x^2z^6\omega$	$y^6z^4\omega$	$x^4yz\omega$	$y^5z^5\omega$	$x^3y^3\omega$	$xyz^6\omega$	$x^3y^2z\omega$	$x^5y^2\omega$	$x^3z^6\omega$
$x^4y^2\omega$	$\frac{5}{4}$									
$x^2z^6\omega$	$-\frac{1}{2}$	$\frac{3}{2}$								
$y^6z^4\omega$	2	$-\frac{1}{2}$	$\frac{7}{4}$							
$x^4yz\omega$				$\frac{11}{8}$						
$y^5z^5\omega$				2	$\frac{7}{4}$					
$x^3y^3\omega$						$\frac{5}{4}$				
$xyz^6\omega$						-1	$\frac{13}{8}$			
$x^3y^2z\omega$								$\frac{3}{2}$		
$x^5y^2\omega$									$\frac{11}{8}$	
$x^3z^6\omega$									$\frac{1}{2}$	$\frac{13}{8}$

	$x^5yz\omega$	$x^4y^3\omega$	$x^2yz^6\omega$	$x^4y^2z\omega$	$y^6z^5\omega$	$x^3y^3z\omega$	$x^5y^3\omega$	$x^5y^2z\omega$	$x^4y^4\omega$	$x^4y^3z\omega$
$x^5yz\omega$	$\frac{3}{2}$									
$x^4y^3\omega$		$\frac{11}{8}$								
$x^2yz^6\omega$		$-\frac{3}{2}$	$\frac{15}{8}$							
$x^4y^2z\omega$				$\frac{13}{8}$						
$y^6z^5\omega$		2	$\frac{15}{8}$							
$x^3y^3z\omega$				$\frac{7}{4}$						
$x^5y^3\omega$					$\frac{3}{2}$					
$x^5y^2z\omega$						$\frac{7}{4}$				
$x^4y^4\omega$							$\frac{3}{2}$			
$x^4y^3z\omega$								2		

The characteristic polynomial of the monodromy is $(s^8-1)^{27}/(s-1)$. The Jordan normal form of the monodromy has 1 3×3 block and 22 2×2 blocks.

$$f(x,y) = x^{p+q} + y^{p+q} + x^q y^q \qquad p>q\geq 2$$

Let $f_a(x,y) = x^{p+q} + y^{p+q} + x^q y^q + ax^p y^p$, $a\varepsilon C$. By calculating dim $C\{x,y\}/(f_x,f_y)$ or by using [V], one sees that μ_a is constant. Therefore by theorem 29, $f=f_0$ is topologically equivalent to

$$f_1 = (x^p + y^q)(x^q + y^p) .$$

$x^6 + y^6 + x^2y^2$ is the two-variable case of Malgrange's example [M3].

$$f_x = (p+q)x^{p+q-1} + qx^{q-1}y^q \qquad f_y = (p+q)y^{p+q-1} + qx^q y^{q-1}$$

$$f = \frac{x}{p+q} f_x + \frac{y}{p+q} f_y + \frac{p-q}{p+q} x^q y^q$$

$$x^{2q-1}y^{q-1}\left(1 - \frac{(p+q)^2}{q^2} x^{p-q}y^{p-q}\right) = \frac{1}{q} x^{q-1} f_y - \frac{p+q}{q^2} y^{p-1} f_x$$

A similar equation holds with x and y switched.

Basis for $C\{x,y\}/(f_x,f_y)$: $x^k y^l$ $0\leq k,l\leq q-2$

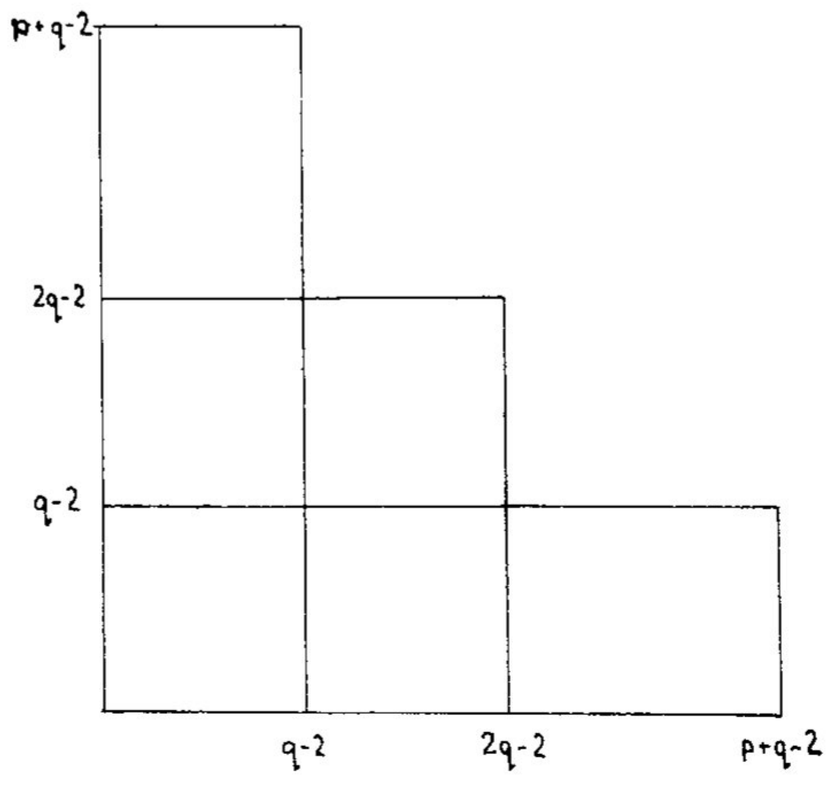
$fx^k y^l = \frac{p-q}{p+q} x^{k+q} y^{l+q}$ $0\leq k,l\leq q-2$

$x^{q-1}y^{q-1}$, $x^k y^{q-1}$, $x^{q-1}y^l$ $q\leq k,l\leq 2q-2$

$x^k y^l$ $q-1\leq k\leq p+q-2$ $0\leq l\leq q-2$

$q-1\leq l\leq p+q-2$ $0\leq k\leq q-2$

$\mu = 2q(p+q)-2(p+q)+1$



The equations needed for computing the Gauss-Manin connection are:

$$(A)\quad x^{q-1}f - \frac{(p+q)^2}{q^2}x^{p-1}y^{p-q}f = (\frac{1}{p+q}x^q - \frac{p-q}{q^2}y^p - \frac{p+q}{q^2}x^py^{p-q})f_x +$$

$$(\frac{p}{q(p+q)}x^{q-1}y - \frac{p+q}{q^2}x^{p-1}y^{p-q+1})f_y$$

and a similar equation with x and y switched, and

$$(B)\quad f^2 - \frac{(p+q)^2}{q^2}x^{p-q}y^{p-q}f^2 = (\frac{x}{p+q}f + \frac{p-q}{(p+q)^2}x^{q+1}y^q - \frac{(p-q)^2}{q^2(p+q)}xy^{p+q}$$

$$- \frac{p+q}{q^2}x^{p-q+1}y^{p-q}f - \frac{p-q}{q^2}x^{p+1}y^p)f_x +$$

$$(\frac{y}{p+q}f + \frac{(p-q)p}{(p+q)^2q}x^qy^{q+1} - \frac{p-q}{q^2}x^py^{p+1}$$

$$- \frac{p+q}{q^2}x^{p-q}y^{p-q+1}f)f_y$$

(B) is derived from (A). From (A) also follows, for $k \geq q-1$,

$$(C)\quad Dfx^ky^l\omega = \frac{q(k+1)+p(l+1)}{q(p+q)}x^ky^l\omega + \frac{(p+q)^2}{q^2}Dfx^{k+p-q}y^{l+p-q}\omega$$

$$- \frac{p+q}{q^2}(k+l+2p-2q+2)x^{k+p-q}y^{l+p-q}\omega - \frac{p-q}{q^2}(k-q+1)x^{k-q}y^{p+l}\omega$$

There is a similar equation for $l \geq q-1$.

Matrix of Df on the cyclic subspace generated by $x^ky^l\omega$, $k \geq q-1$:

	$x^ky^l\omega$	$x^{k-q}y^{l+p}\omega$	$x^{k+p-q}y^{l+p-q}\omega$	$x^{k+p-2q}y^{l+2p-q}\omega$	...
$x^ky^l\omega$	$e(k,l)$				
$x^{k-q}y^{l+p}\omega$	.	$e(k-q,l+p)$			
$x^{k+p-q}y^{l+p-q}\omega$	.	.	$e(k+p-q,l+p-q)$		
$x^{k+p-2q}y^{l+2p-q}\omega$	.	.	.	$e(k+p-2q,l+2p-q)$	
⋮	⋮	⋮	⋮	⋮	. .

where $e(k,l) = \frac{q(k+1)+p(l+1)}{q(p+q)}$

For $0 \leqslant l \leqslant k \leqslant q-2$, one obtains from (B)

	$x^k y^l \omega$	$Dfx^k y^l \omega$	$x^{k+p-q} y^{l+p-q} \omega$	$Dfx^{k+p-q} y^{l+p-q} \omega$	... $x^{k+m(p-q)} y^{l+m(p-q)} \omega$ ... as above
$x^k y^l \omega$	0	$-e(k,l)e(l,k)$			
$Dfx^k y^l \omega$	1	$\frac{k+l+2}{q}$			
$x^{k+p-q} y^{l+p-q} \omega$		.	0	$-e(k+p-q,l+p-q)e(l+p-q,k+p-q)$	
$Dfx^{k+p-q} y^{l+p-q} \omega$		.	1	$\frac{k+l+2p-2q+2}{q}$	
⋮		⋮		⋮	
$x^{k+m(p-q)} y^{l+m(p-q)} \omega$		.		.	$e(k+m(p-q),l+m(p-q))$
⋮		⋮		⋮	⋮
as above					

where $k+m(p-q) \geqslant q-1 > k+(m-1)(p-q)$. There is a similar matrix for $k \leqslant l$.

The characteristic polynomial of the monodromy is $(s^{\frac{q(p+q)}{r}}-1)^{2r}(s-1)/(s^{p+q}-1)^2$ where $r=(p,q)$. The Jordan normal form of the monodromy has q-1 2×2 blocks, belonging to the eigenvalues $\frac{1}{q},\ldots,\frac{q-1}{q}$.

$$f(x,y,z) = x^3 + yz^2 + x^2y^k + by^{3k+i} \qquad k>1\ ,\ i>0$$

$$\text{where}\quad b = b(y) = b_0 + b_1y + \cdots + b_{k-1}y^{k-1}\ ,\ b_0\neq 0$$

This family is $Q_{k,i}$ in Arnold's classification. It is presented here as a typical example of the functions which occur in Arnold's paper [A].

$$f_x = 3x^2 + 2xy^k \qquad f_y = z^2 + kx^2y^{k-1} + (3k+i)by^{3k+i-1} + b'y^{3k+i} \qquad f_z = 2yz$$

$$yf_y = \frac{z}{2} f_z + kx^2y^k + dy^{3k+i}\ ,\ \text{where}\quad d(y) = 3kb + ib + yb'\ ,\ d(0)=(3k+i)b_0\neq 0$$

$$f = \frac{x}{3} f_x + \frac{y}{3k} f_y + \frac{3k-1}{6k} zf_z - \frac{c}{3k} y^{3k+i}\ ,\ c(y) = ib + yb'\ ,\ c(0)=ib_0\neq 0$$

$$x^2y^{2k}(1 + \frac{9}{4k} dy^i) = (\frac{1}{2} xy^k + \frac{3}{4k} dy^{2k+i})f_x - \frac{3}{2k} xyf_y + \frac{3}{4k} xzf_z$$

It follows that

$$x^2y^{2k},\ xy^{3k},\ y^{4k+i},\ xf,\ y^kf\ \varepsilon\ (f_x,f_y,f_z)$$

Basis of $C\{x,y,z\}/(f_x,f_y,f_z)$: $1,y,\ldots,y^{3k+i-1},yf,\ldots,y^{k-1}f$

$x,xy,\ldots,xy^{2k-1}$

z,xz

$\mu = 6k+i+2$

The equations needed for computing the Gauss-Manin connection are:

$$(A)\quad xf - \frac{3}{2k} dy^{k+i}f = (\frac{1}{3} x^2 - \frac{c}{6k} y^{2k+i} - \frac{d}{2k} xy^{k+i})f_x + (\frac{xy}{3k} - \frac{3b}{2k} y^{k+i+1})f_y + (\frac{3k-1}{6k} xz - \frac{3(3k-1)b}{4k} y^{k+i}z - \frac{3c}{4k} y^{k+i}z)f_z$$

$$(B)\quad y^kf + \frac{3}{2} xf = (\frac{d-kb}{2d} xy^k + \frac{1}{2} x^2)f_x + (\frac{b}{d} y^{k+1} - \frac{3b}{2d} xy)f_y + (\frac{d-b}{2d} y^kz + \frac{3(d-b)}{4kb} xz)f_z$$

$$(C)\quad zf = \frac{xz}{3} f_x + \frac{yz}{3k} f_y + (\frac{3k-1}{6k} z^2 - \frac{c}{6k} y^{3k+i-1})f_z$$

$$(D)\quad f^2+\frac{9}{4k}\,dy^i f^2 = (\frac{x}{3}\,f - \frac{c(d-kb)}{6kd}\,xy^{3k+i} - \frac{c^2}{12k^2}y^{4k+2i} - \frac{cd}{4k^2}xy^{3k+2i}$$

$$+ \frac{3d}{4k}\,xy^i f)f_x +$$

$$(\frac{y}{3k}\,f - \frac{bc}{3kd}\,y^{3k+i+1} + \frac{c^2}{6k^2d}\,xy^{2k+i+1} - \frac{3bc}{4k^2}\,y^{3k+2i+1}$$

$$+ \frac{3d}{4k^2}y^{i+1}f)f_y +$$

$$(\frac{3k-1}{6k}\,zf - \frac{c(d-b)}{6kd}\,y^{3k+i}z + (\frac{c(3k-4)}{12k^2} + \frac{bc}{4k^2d})xy^{2k+i}z$$

$$- \frac{3c}{8k^2}(\frac{d-b)}{2}\,y^{3k+2i}z + \frac{3(3k-1)d}{8k^2}\,y^i zf)f_z$$

(D) is derived from (A) and (B). From (D) follows

$$(E)\quad Df^2y^j\omega = \frac{5k+2j+1}{6k}\,y^j f\omega + \frac{1}{3}\,xy^j f_x\omega + \frac{1}{3k}\,y^{j+1}f_y\omega + \frac{3k-1}{6k}\,y^j zf_z\omega$$

$$+ \frac{11k+4i+2j+1}{2(3k+i)}\,y^{3k+i+j}\omega + \text{terms in } D^{-1}C_{0,j+1,0} \qquad j\geq 0$$

From (B) follows

$$(F)\quad Dfy^j\omega = \frac{5k+2j+2i+1}{2(3k+i)}\,y^j\omega + \text{terms in } C_{1,j-k,0} \qquad \text{for } j\geq k$$

From (A) follows

$$(G)\quad Dfxy^j\omega = \frac{7k+2j+1}{6k}\,xy^j\omega + \text{terms in } C_{0,k+i+j,0} \qquad j\geq 0$$

where C_{jlm} is the cyclic submodule generated by $x^jy^lz^m\omega$. One can compute $Dfz\omega$ and $Dfxz\omega$ from (C). The characteristic polynomial of the monodromy is

$$(s^{3k}-1)(s^{3k+i}-1)(s^2+s+1) \qquad \text{for } k \text{ odd}$$

$$(s^{3k}+1)(s^{3k+i}+1)(s^2+s+1) \qquad \text{for } k \text{ even}$$

	$\cdots\ y^{j}_{\omega}$	Dfy^{j}_{ω}	$\cdots\ y^{k+j}_{\omega}$	xy^{j}_{ω}	$\cdots\ y^{k+j+i}_{\omega}$	xy^{j+i}_{ω}	$\cdots\ y^{3k-1}_{\omega}$	y^{3k}_{ω}	$\cdots\ y^{3k+i-1}_{\omega}$	$z\omega$	$xz\omega$
$\vdots$	$\ddots$										
y^{j}_{ω}	0	$-a(j)e(j)$									
Dfy^{j}_{ω}	1	$a(j)+e(j)$									
$\vdots$		$\vdots$	$\ddots$								
y^{k+j}_{ω}		.	$a(k+j)$								
xy^{j}_{ω}		.	.	$e(j+k)$							
$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$						
y^{k+j+i}_{ω}		.	.	.	$a(k+j+i)$						
xy^{j+i}_{ω}		.	.	.	.	$e(k+j+i)$					
$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$				
y^{3k-1}_{ω}		.	.	.	.	.	$a(3k-1)$				
y^{3k}_{ω}		.	.	.	.	.	.	$a(3k)$			
$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$			
y^{3k+i-1}_{ω}		.	.	.	.	.	.	.	$a(3k+i-1)$		
$z\omega$										$\frac{4}{3}$	
$xz\omega$											$\frac{5}{3}$

where $a(j) = \frac{5k+2i+2j+1}{2(3k+i)}$ and $e(j) = \frac{5k+2j+1}{6k}$

$a(j)=e(j)$ if and only if $k=2j+1$. Therefore the Jordan normal form of the monodromy has one 2×2 block if k is odd, and is a diagonal matrix if k is even.

Appendix

Here are proofs of theorem 8 and lemma 12. Although theorem 8 is well known, we give a proof here because the proof of lemma 12 is a simple generalization of it.

Proof of theorem 8: According to the holomorphic Poincaré lemma, the complex

$$\mathcal{O}_X \to \Omega_X^1 \to \cdots \to \Omega_X^n$$

is a resolution of the constant sheaf C_X. Therefore in the second spectral sequence in hypercohomology (EGA 0 §12.4),

$$''E_2^{pq} = H^p(X, H^q(\Omega_X^\cdot)) \Rightarrow \mathbb{H}^\cdot(X, \Omega_X^\cdot)$$

we have

$$''E_2^{pq} = 0 \quad \text{for } q > 0$$

and

$$H^p(X,C) = ''E_2^{p,0} = ''E_\infty^{p,0} = \mathbb{H}^p(X,\Omega_X^\cdot)$$

(Usually one simply resolves the complex above with the double complex Ω_X^{pq}, and then $\mathbb{H}^\cdot(X,\Omega_X^\cdot)$ is by definition the de Rham cohomology of X.)

Now look at the first spectral sequence in hypercohomology:

$$'E_2^{pq} = H^p(H^q(X,\Omega_X^\cdot)) \Rightarrow \mathbb{H}^\cdot(X,\Omega_X^\cdot) \simeq H^\cdot(X,C).$$

Since X is Stein and Ω_X^p is coherent, it follows from Cartan's theorem B that

$$H^q(X,\Omega_X^p) = 0 \qquad q>0,\ p\geq 0$$

So

$$'E_2^{pq} = 0 \quad \text{for} \quad q>0$$

and

$$H^p(\Gamma(\Omega_X^\cdot)) = 'E_2^{p,0} = 'E_\infty^{p,0} \simeq H^p(X,C).$$

Proof of lemma 12: The relative Poincaré lemma ([B], p.117) shows that the complex

$$\mathcal{O}_{X'} \to \Omega^1_{X'/T'} \to \cdots \to \Omega^n_{X'/T'}$$

of $f^*\mathcal{O}_{T'}$ modules is a resolution of $f^*\mathcal{O}_{T'}$. Therefore in the second spectral sequence in hypercohomology for the functor f_*

$$''E_2^{pq} = R^p f_*(H^q(\Omega^\cdot_{X'/T'})) \Rightarrow \mathbb{R}\cdot f_*\Omega^\cdot_{X'/T'} ,$$

we have

$$''E_2^{pq} = 0 \quad \text{for} \quad q>0$$

and

$$R^p f_*(f^*\mathcal{O}_{T'}) = ''E_2^{p,0} \simeq ''E_\infty^{p,0} \simeq \mathbb{R}^p f_*\Omega^\cdot_{X'/T'} .$$

Now consider the first spectral sequence:

$$'E_2^{pq} = H^p(R^q f_*\Omega^\cdot_{X'/T'}) \Rightarrow \mathbb{R}\cdot f_*\Omega^\cdot_{X'/T'} \simeq R\cdot f_*(f^*\mathcal{O}_{T'})$$

Since X' is Stein (lemma 6(b)) and $\Omega^p_{X'/T'}$ is coherent, corollary 7 implies that

$$R^q f_*\Omega^p_{X'/T'} = 0 \qquad q>0 \ , \ p\geqslant 0 .$$

So

$$'E_2^{pq} = 0 \quad \text{for} \quad q>0$$

and

$$H^p(f_*\Omega^\cdot_{X'/T'}) = 'E_2^{p,0} \simeq 'E_\infty^{p,0} \simeq R^p f_*(f^*\mathcal{O}_{T'}) .$$

We must check that

$$R^n f_*(f^*\mathcal{O}_{T'}) \simeq H^n \simeq R^n f_* C_{X'} \otimes_C \mathcal{O}_{T'} .$$

The inclusion $C_{X'} \to f^*\mathcal{O}_{T'}$ induces a homomorphism

$$R^p f_* C_{X'} \otimes_C \mathcal{O}_{T'} \to R^p f_*(f^*\mathcal{O}_{T'}) .$$

Suppose $t\varepsilon T'$ and $U\subset T'$ is a small disc about t, such that $f^{-1}U \simeq U\times X_t$. Let $p: U\times X_t \to U$ be the projection. Then using the Künneth formula, we have

$$\begin{array}{ccccc} R^n f_*(f^* \mathcal{O}_{T,t})(U) & \simeq & H^n(f^{-1}U, f^* \mathcal{O}_{T,t}) & \simeq & H^n(X_t, C) \otimes \mathcal{O}_{T,t}(U) \\ \uparrow & & \uparrow & & \uparrow \simeq \\ R^n f_* C_{X,t}(U) \otimes \mathcal{O}_{T,t}(U) & \simeq & H^n(f^{-1}U, C) \otimes \mathcal{O}_{T,t}(U) & = & H^n(U \times X_t, C) \otimes \mathcal{O}_{T,t}(U) \end{array}$$

and the lemma is proved.

We also give a proof of the main theorem of Malgrange's paper [M2] which avoids the lengthy calculations in §4 of his paper. The arguments used instead all come from [M2], 5.6 .

Let $\mathcal{D}$ be the ring of linear differential operators in $\partial/\partial x_0, \ldots, \partial/\partial x_n$ with coefficients in $\mathcal{O}=C\{x_0, \ldots, x_n\}$. We make $\Omega^{n+1} = \Omega^{n+1}_{C^{n+1},0}$ into a right $\mathcal{D}$ module in the following way. Let

$$\varphi g = g\varphi \ , \ \varphi\xi = - L(\xi)\varphi$$

for $\varphi \in \Omega^{n+1}$, $g \in \mathcal{O}$ and ξ a vector field over $\mathcal{O}$. These operations extend uniquely to all of $\mathcal{D}$. Let $\mathcal{O}[f^{-1},s]f^s$ be a free $\mathcal{O}[f^{-1},s]$ module with basis f^s. Let $\mathcal{D}$ act on $\mathcal{O}[f^{-1},s]f^s$ by

$$\frac{\partial}{\partial x_j}(hf^s) = \frac{\partial h}{\partial x_j} f^s + shf^{-1}f_j f^s \qquad 0 \leq j \leq n$$

for $h \in \mathcal{O}[f^{-1},s]$.

<u>Theorem</u>: ([M2],3.7) $\Omega^{n+1} \otimes_{\mathcal{D}} \mathcal{O}[f^{-1},s]f^s \simeq V$

where the isomorphism is given by

$$\varphi \otimes s^k f^s \to (-Df)^k[\varphi]$$

for $\varphi \in \Omega^{n+1}[f^{-1}]$, $k \geq 0$

Let $M = \mathcal{D}[s]f^s \subset \mathcal{O}[f^{-1},s]f^s$. It follows from the definition of the Bernstein polynomial b (see introduction) that b is the minimal polynomial of the operator s+1 on $\mathcal{D}[s]f^s/\mathcal{D}[s]f^{s+1}$. Our simplification lies in the proof of the following result.

Theorem: ([M2], 5.3) $\quad \Omega^{n+1}\otimes_{\mathcal{D}} M \simeq \tilde{H}''$

Proof: Let $M_k = s^k \mathcal{D}f^s + \cdots + \mathcal{D}f^s \subset M$. It's easy to see (cf.[M2], 5.6) that

$$\Omega^{n+1}\otimes_{\mathcal{D}} M_0 = \Omega^{n+1}\otimes_{\mathcal{D}} \mathcal{D}f^s \simeq H''.$$

Therefore

$$\Omega^{n+1}\otimes_{\mathcal{D}} M_k \simeq (Df)^k H'' + \cdots + H'' .$$

Now as Malgrange shows in 5.6, M_{k+1}/M_k is a $\mathcal{D}$ module of support 0. Therefore $\Omega^{n+1}\otimes_{\mathcal{D}} M_{k+1}/\Omega^{n+1}\otimes_{\mathcal{D}} M_k \simeq \Omega^{n+1}\otimes_{\mathcal{D}}(M_{k+1}/M_k) = 0$ if and only if $M_{k+1}/M_k = 0$ ([M2], 2.8, 2.9). But the regularity theorem says that for k sufficiently large,

$$(Df)^{k+1} H'' \subset (Df)^k H'' + \cdots + H''$$

Therefore for large k,

$$M_k = M_{k+1} = \cdots = M$$

and

$$\Omega^{n+1}\otimes_{\mathcal{D}} M = \Omega^{n+1}\otimes_{\mathcal{D}} M_k = (Df)^k H'' + \cdots + H'' = \tilde{H}'' .$$

Theorem: ([M2], 5.4) $\tilde{b}$ is the minimal polynomial of $-fD$ on $\tilde{H}''/f\tilde{H}''$.

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