Available online at www.sciencedirect.com
ScienceDirect
Annals of Pure and Applied Logic 145 (2007) 96–114
ANNALS OF PURE AND APPLIED LOGIC
www.elsevier.com/locate/apal

Interpreting structures of finite Morley Rank in strongly minimal sets

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Received 27 January 2004; received in revised form 24 January 2006; accepted 4 October 2006
Available online 9 November 2006
Communicated by A.J. Wilkie

Abstract

We show that any structure of finite Morley Rank having the definable multiplicity property (DMP) has a rank and multiplicity preserving interpretation in a strongly minimal set. In particular, every totally categorical theory admits such an interpretation. We also show that a slightly weaker version of the DMP is necessary for a structure of finite rank to have a strongly minimal expansion. We conclude by constructing an almost strongly minimal set which does not have the DMP in any rank preserving expansion, and ask whether this structure is interpretable in a strongly minimal set.

Keywords: Classification theory stability and related concepts; Categoricity and completeness of theories

0. Introduction

Given a complete theory T of finite Morley Rank, we are interested in the problem of finding a strongly minimal expansion of T, or equivalently, interpreting T in a strongly minimal theory. As a rule we will be interested in rank preserving interpretations, namely that if S is definable and of rank k in the original structure, then the corresponding set's rank in the interpretation is k as well. Such an interpretation does not always exist, as the following result from Chapter VI of [4] implies:

Definition 1. A theory T does not have the finite cover property (equivalently, T has NFCP) if for every formula $\varphi(\bar{x}, \bar{y})$ there is a natural number k_φ such that for any A and any φ-type q over A, q is k_φ-consistent iff q is consistent.

Lemma 1. *If a theory T has FCP then no expansion of T is uncountably categorical.*

The following will be useful:

Claim 1. *Let T be a stable theory.*

(1) *T has NFCP iff for every formula $E(\bar{x}, \bar{y}, \bar{z})$ with* $\text{length}(\bar{x}) = \text{length}(\bar{y})$*, there is a number n_E such that for every $\bar{c}$, if $E(\bar{x}, \bar{y}, \bar{c})$ is an equivalence relation then $E(\bar{x}, \bar{y}, \bar{c})$ has either less than n_E or infinitely many equivalence classes.*

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0168-0072/$ - see front matter
doi:10.1016/j.apal.2006.10.002

(2) *If T is $\aleph_0$-categorical then T has NFCP.*
(3) *If for every finite $L' \subseteq L(T)$, $T \restriction_{L'}$ is $\aleph_0$-categorical then T has NFCP.*

Proof. (1) is part of Shelah's FCP theorem (II.4.4 of [4]), (2) follows from (1), and (3) follows from (2) and the locality of FCP. □

In particular if T has FCP it cannot be interpreted in a strongly minimal structure. But NFCP is not enough. Consider the following:

Example 1. Let $L = \{E, E_1, E_2, E_3, \ldots\}$ and T be the theory stating that E is an equivalence relation with infinitely many classes which are all infinite and such that:

- Each E_i is an equivalence relation agreeing with E except for one E-class, which E_i divides into i infinite classes.
- If $i \neq j$ then E_i and E_j have different exceptional classes.

T is ω-stable of MR 2 ($\mathrm{MR}(T) = \mathrm{MR}(T \restriction_{\{E\}}) = 2$), and by the last part of Claim 1 T does not have FCP. However, there is no rank preserving interpretation of T in any almost strongly minimal (asm) set. To see this, assume that $M \models T$ is $\aleph_0$-saturated, and that there exists an asm expansion T', i.e. there exists a strongly minimal set D definable in T' (maybe over some set of parameters) and such that $M \subseteq \mathrm{acl}(D)$. A generic E-class in M would have, by the assumption on the rank, rank 1 and multiplicity m for some m. By Lemma 2 below, there must be a universal bound on the multiplicity of E-classes, that is, $\mathrm{mult}\{E(x, a) : a \in M\} \leq m'$ for some m'. But this is contradicted by the exceptional $E_{m'}$-class (as multiplicity can only increase in any rank preserving expansion).

The following is even stronger:

Example 2. Consider the following theory T:

(1) E is an equivalence relation with infinitely many classes, which are all infinite.
(2) For every prime p, E_p is a unary predicate which corresponds to exactly one E-class, and the E_p's are all disjoint.
(3) E carries uniformly on each equivalence class the structure of an Abelian group.
(4) For every prime $p : (\forall x)(E_p(x) \vee (\exists y)(py = x))$.
(5) For every prime p an axiom: $(\forall x \exists y, z)(E_p(x) \rightarrow z \neq 0 \wedge pz = 0 \wedge (py = x + z \vee py = x))$
(6) For every prime p an axiom: $(\forall x)\,(px = 0 \rightarrow (x = 0 \vee E_p(x)))$.
(7) For every prime p an axiom: $(\exists x \forall y)(py = 0 \rightarrow \bigvee\{y = nx : n < p\})$.
(8) For every prime p an axiom: $\forall x(p^2x = 0 \iff px = 0)$.

Thus the generic E-class is divisible by p for every prime p (since by axiom 4 every element of the generic class is p-divisible for all p). Moreover, by axioms 6 and 2, all torsion elements have prime order (since, if $nx = 0$ for some x, so for $p \mid n$ prime $p(\frac{n}{p})x = 0$, and by axiom 6 this implies that unless $\frac{n}{p}x = 0$, $x \in E_p$, and by axiom 2 all the E_p's are disjoint. Whence $x \in E_p$ and therefore has a prime power order, and by axiom 8 the order must in fact be prime). By axiom 6, a torsion element of order p is in E_p, so that the generic E-class is torsion free. By axiom 5, any E_p-class has torsion elements of order p and by axiom 7 the torsion elements of order p form a cyclic subgroup G_p of order p in E_p. By axiom 5 again, E_p/G_p is divisible by p, whence (axiom 4) divisible. Since all torsion element of E_p are of order p (by axioms 2 and 8), E_p/G_p is torsion free.

Now define, for a prime p, $E_p^0 = \{x : (\exists y)\,(py = x)\}$; then E_p^0 naturally divides E_p into p disjoint sets given by $E_p^n = E_p^0 + nx$ for $x \in G_p \setminus \{0\}$ and $0 \leq n < p$. It is now straightforward to check that $\mathrm{MR}(T) = 2$, that a generic E-class is strongly minimal, as is E_p/G_p.

We show that T has NFCP but T cannot be interpreted in any asm expansion (we do not require the interpretation to be rank preserving). To see the second part of this claim, assume that T' is asm and D is the universe of an interpretation of T in T'. Necessarily, D has finite MR; so infinitely many of the exceptional classes E_p must have the same rank in the interpretation, and for simplicity we may assume they all have the same rank, n. Since E_p/G_p is a stable, divisible torsion-free Abelian group, it must remain connected in D. But the natural projection $E_p^0 \rightarrow E_p/G_p$ is a definable bijection, implying that $\mathrm{mult}_{T'}\{E_p\} = p$. Thus the formula stating that $x \in D \wedge rk\{x : E(x, y)\} = n$ contradicts Lemma 2.

It remains to check that T has NFCP. As above, it is enough to check the reducts of T to any of the sub-languages $L_p = L \setminus \{E_q\}_{q>p}$. If any such reduct had FCP, it would not be interpretable in any uncountably categorical theory. We construct an uncountably categorical expansion of $T\restriction_{L_p}$. Let V be a vector space over $\mathbb{Q}$, and consider the 3-sorted structure M: V being the first sort, $\mathbb{Z}/(p!)\mathbb{Z}$ the second sort, and $V\times V\times\mathbb{Z}/(p!)\mathbb{Z}$ the third. Let $L(M)=\{E,E_1,\dots,E_p,+,\pi_i\}$, where π_i $(i=1,2,3)$ is the natural projection to the ith coordinate, E satisfying $E(x,y)\iff\pi_1(x)=\pi_1(y)$, where addition is defined naturally on each of the E classes $((a_1,a_2,a_3)+(a_1,b_2,b_3)=(a_1,a_2+b_2,a_3+b_3))$, and E_p is interpreted by $a_i\times V\times\mathbb{Z}/p!\mathbb{Z}$ for some $\mathbb{Q}$-independent $a_1,\dots,a_p\in V$. Let $U=\{x:\bigwedge(\pi_1(x)\neq a_i\to\pi_3(x)=0)\wedge(\pi_1(x)=a_i\to(p!/q_i)\mid\pi_3(x))\}$, and with the restrictions of $E,+,E_{q_1},\dots,E_{q_p}$ we have that U is an interpretation of $T\restriction_{L_p}$. It remains to check that any two models of $Th(M)$ of cardinality $\aleph_1$ are isomorphic. But this is now trivial, since the first sort of M is a vector space and therefore $\aleph_1$-categorical, and M is just a $p!$-cover of the Cartesian product of this vector space with itself, without any additional structure.

From now on, we will restrict ourselves to rank preserving interpretations. In this context consider the following:

Example 3. Let T be the theory of an equivalence relation E with infinitely many equivalence classes, which are all infinite. Let $\{E_n\}_{n=1}^{\infty}$ be equivalence relations, each refining a unique E class into infinitely many infinite classes, and such that no two of the E_n's refine the same E-class. To simplify things, we add to the language a new sort for T/E and constants c_n corresponding to the special classes refined by E_n. In this language, a simple back and forth argument shows that T eliminates quantifiers and T has MR 3 (because each special E-class has rank 2, and there are infinitely many special classes). T has NFCP (again by Lemma 1) but, MR is not definable in T: If $M\models T$ is a saturated model, then M/E is strongly minimal. But were MR definable in T, we would get a partition of M/E into two infinite definable sets (namely, those classes of rank 2 and those of rank 1). As MR is definable in any almost strongly minimal theory, this implies that no rank preserving expansion of T is asm (because the set of exceptional E-classes in T is countable in all models and therefore not definable in any expansion — see a detailed proof below). This means that even if the multiplicity of M/E increases in the expansion, there will be a set of multiplicity 1 containing infinitely many exceptional classes whose generic element is not exceptional, which is impossible.

These examples lead us to the following definition:

Definition 2. Assume T has finite, definable MR. T has the weak definable multiplicity property (wDMP) if for every definable (in T^{eq}) family $\{S_a : a\in\psi\}$ of definable sets, there is a natural number m such that $\mathrm{mult}\{S_a\}\leq m$ for all $a\in\psi$.

We show that as noted in the previous examples, wDMP is a necessary condition:

Lemma 2. *If T is almost strongly minimal, then T has wDMP.*

Proof. This is a special case of Proposition B.1 of [3], but the proof in the present case being much simpler, we give the details. Since T is almost strongly minimal, it has definable MR — see [1]. Let $M\models T$ be a saturated model. By asm there is a strongly minimal set D such that $\mathrm{acl}(D)=M$. Since wDMP is invariant under naming constants, we may assume that D is $\emptyset$-definable.

Assume, first, that $S_b=\varphi(\bar{x},\bar{b})$ is strongly minimal, then if $\bar{c}\in S_{\bar{b}}$ is any generic, there are by assumption $d_1,\dots,d_n\in D$ such that $\bar{c}\in\mathrm{acl}(D)$. If we choose n minimal, we get that $d_1,\dots,d_{n-1}$ are independent. Denote $\bar{e}=(d_1,\dots,d_{n-1})$ and $d=d_n$. There is a formula $\psi(\bar{x},\bar{b},\bar{e},d)$ such that $\bar{c}\models\psi(\bar{x},\bar{b},\bar{e},d)$ and $\models(\exists^{\leq m}x)\psi(\bar{x},\bar{b},\bar{e},d)$, moreover $\models(\exists^{\leq m}x)\psi(\bar{x},\bar{b}',\bar{e}',d')$ may be assumed for all $\bar{b}',\bar{e}',d'$ Because $\bar{c}$ was a generic in $S_{\bar{b}}$ and $S_{\bar{b}}$ is strongly minimal, we get that there is a definable set $F_{\bar{b}}$ of size, say, m' such that $\models(\forall\bar{x})(S_{\bar{b}}(\bar{x})\wedge\neg F_{\bar{b}}(x)\to(\exists y)\psi(\bar{x},\bar{b},\bar{e},y))$ and so, $\models(\exists\bar{u}\forall\bar{x})(S_{\bar{b}}(\bar{x})\wedge\neg F_{\bar{b}}(x)\to(\exists y)\psi(\bar{x},\bar{b},\bar{u},y)\wedge\left|F_{\bar{b}}\right|\leq m')$. Denote this last formula by $\theta(\bar{b})$ and assume that $\bar{b}'\models\theta$; then we claim that $S_{\bar{b}'}$ has multiplicity at most m. To see this, denote $R_{\bar{b}',\bar{e}'}=\{(\bar{x},y):\bar{x}\in S_{\bar{b}'},\ y\in D,\ \psi(\bar{x},\bar{b}',\bar{e}',y)\}$; then the projection $(x,y)\to y$ is, by the choice of ψ at most $m:1$ from $R_{\bar{b}',\bar{e}'}$ to D, and therefore $\mathrm{mult}\{R_{\bar{b}',\bar{e}'}\}\leq m$. On the other hand, the projection $(x,y)\to x$ is onto $S_{\bar{b}'}\setminus F_{\bar{b}}$, and so (as $F_{\bar{b}'}$ is finite) $S_{\bar{b}'}$ too is of multiplicity at most m.

Now, given a definable family $\{\varphi(\bar{x},\bar{b}):\bar{b}\models\theta\}$ for some θ, we have to show wDMP. By the definability of MR, we may assume wlog that $\bar{b}\models\theta\Rightarrow rk\{\varphi(\bar{x},\bar{b})\}=k$ for some natural number k. We now proceed by induction on $(rk\{\theta\},\mathrm{mult}\{\theta\})$ and on $(rk\{\varphi(\bar{x},\bar{b})\},\mathrm{mult}\{\varphi(\bar{x},\bar{b})\})$: take any generic $\bar{b}\in\theta$, if $\varphi(\bar{x},\bar{b})$ is strongly minimal, then by the above argument there exists a formula $\theta_{\bar{b}}$ (over $\emptyset$) such that for all $\bar{b}\models\theta_{\bar{b}}$, $\mathrm{mult}\{\varphi(\bar{x},\bar{b})\}\leq m_b$, and as

$(rk\{\theta \wedge \neg\theta_{\bar{b}}\}, \text{mult}\{\theta \wedge \neg\theta_{\bar{b}}\})$, we can proceed by induction. If $\varphi(\bar{x}, \bar{b})$ is of rank 1, and multiplicity $l > 1$, we can divide $\varphi(\bar{x}, \bar{b})$ into l disjoint strongly minimal sets and proceed by induction. So the remaining case is for $\varphi(\bar{x}, \bar{b})$ of rank $r > 1$; but this follows easily by induction if we divide the variables in $\bar{x}$ into two sets, say, $x_1, \ldots, x_{n-1}$ and x_n such that for a generic $\bar{a} \models \varphi(\bar{x}, \bar{b})$, $rk(a_n/a_1, \ldots, a_{n-1}, \bar{b}) = 1$. □

Corollary 3. *If T has definable MR but does not have wDMP, then there is no rank preserving asm expansion of T.*

Proof. Multiplicity can only go up in a rank-preserving expansion. □

We can summarize the above examples in the following:

Lemma 4. *Let T be a theory of finite MR. If there exists a rank preserving interpretation of T in a strongly minimal theory T', then T has NFCP and the following weak version of wDMP: for any $\varphi(x, y)$ there exists $n \in \mathbb{N}$ such that* $\text{mult}\{\varphi(x, b')\} \leq n$ *whenever* $rk\{\varphi(x, b')\} = rk\{\varphi(x, b)\}$.

Proof. The fact that T has NFCT is given in Lemma 1. As for the rest of the claim, it follows from the fact that if T' is a strongly minimal interpretation of T, then T' has wDMP. If there were no such n, we could find $b_1, b_2, \ldots$ with $rk\{\varphi(x, b_i)\} = rk\{\varphi(x, b)\}$ and $\text{mult}\{\varphi(x, b_i)\} \geq i$. Since the interpretatation is rank preserving, this is impossible. □

At this stage, we do not know whether these assumptions are sufficient to assure the existence of rank preserving strongly minimal expansions for theories of finite MR. What we can prove requires a somewhat stronger condition. We recall the following from [2]:

Definition 3. Let T be of finite and definable MR. T has the definable multiplicity property (DMP) if whenever $\varphi(\bar{x}, \bar{a})$ (in T^{eq}) defines a set of rank k and multiplicity m, there exists some $\psi(\bar{y}) \in tp(\bar{a})$ such that $\bar{a}' \models \psi(\bar{y}) \Rightarrow rk\{\varphi(\bar{x}, \bar{a}')\} = k \wedge \text{mult}\{\varphi(\bar{x}, \bar{a}')\} = m$.

It is worth noting, as has been pointed out to me by E. Hrushovski, that already the definability of Morley rank (and in particular DMP) is not a necessary condition for the existence of rank preserving interpretations. In fact, it is an easy exercise to find reducts of the theory of pure equality in which the Morley rank is not definable.

As will be discussed in more detail in the third part of the paper, our results apply also if T only has a rank preserving expansion with DMP. At present, there are no known examples of strongly minimal theories which do not admit a rank preserving DMP expansion. This not only suggests that DMP is a relatively weak requirement (in the context of the present question), but also stresses that the main result of this paper covers all known examples of theories of finite MR which actually have a rank preserving interpretation in a strongly minimal set.

1. Main results

We can now state our main claim:

Theorem 5. *Let T_0 be a theory of finite MR in a countable language such that MR is definable and T_0 has DMP. There exists a theory T in $L(T_0) \cup \{R, D\}$ for D, a new sort, and R, a new binary predicate, giving T the structure of a bipartite graph satisfying:*

- *If $(M, D) \models T$, then $M \models T_0$, D is strongly minimal and $M \subseteq dcl(D)$.*
- *The natural interpretation of T_0 in T is rank and multiplicity preserving.*
- *T has DMP.*

Proof. The proof is a variation on the fusion technique of [2]: given a saturated structure $M \models T_0$, we will follow Hrushovski's steps in constructing a structure N equipped with a strongly minimal set D such that $N = dcl(D)$ and M is interpretable in N. This technique differs from ours mainly in the definition of the fusion and the corresponding change in the definition of the fusion-code (2-code in [2]).

We may assume that T_0 has elimination of quantifiers and that the language of T_0 has no function (or constant) symbols. For simplicity, we will also assume that T_0 has elimination of imaginaries. Although we can probably do without the EOI, it is worthwhile to note that this assumption does not harm the generality of the proof, as the statement still holds if T_0 has finite but unbounded MR (i.e. has infinitely many sorts, each of finite MR but with no uniform bound).

Let $M \models T_0$ be a countable saturated model. Define $\mathcal{B} = \{B \models T_0^{\forall} : B \text{ finite}\}$ (this is not a trivial class, as the language is assumed to have no function symbols. Thus any finite subset of M has a representative in $\mathcal{B}$). For $\emptyset \neq B \in \mathcal{B}$, define $d_1(B) = \mathrm{MR}(tp(f(B)/\emptyset))$, where $f : B \hookrightarrow M$ is an embedding (this is well defined because of the QE assumption) and $d_1(\emptyset) = 0$. Let $\mathcal{D} = [I]^{<\omega}$ for some infinite set I, and consider the family of "2-sorted" structures $\mathcal{C} = \{(B, D, R) : B \in \mathcal{B}, D \in \mathcal{D}, R \subseteq B \times D\}$. Of course, there may be many sorts in T_0, so "2-sorted" only stresses the division of our new structures into the "old sorts" of T_0 and the "new sort" of $\mathcal{D}$. In fact, to be precise we have, for each sort $s \in \mathcal{S}$ (the set of sorts of T_0), to add a binary relation R_s, and we actually consider the family $\mathcal{C} = \{(B, D, R_s)_{s\in\mathcal{S}} : B \in \mathcal{B}, D \in \mathcal{D}, R_s \subseteq B_s \times D\}$; but to make the notation simpler, and as no ambiguity can arise, we will keep the simpler notation.

For $C = (C_{\mathcal{B}}, C_{\mathcal{D}}, C_R) \in \mathcal{C}$ define $d_0(C) = d_1(C_{\mathcal{B}}) + |C_{\mathcal{D}}| - |C_R|$, and for $C, C' \in \mathcal{C}$, let $C \subseteq C'$ denote that C is a substructure of C' (i.e. $C_{\mathcal{B}} \subseteq (C')_{\mathcal{B}}, C_{\mathcal{D}} \subseteq (C')_{\mathcal{D}}, C_R \subseteq (C')_R$). We will also write $d_1(X/Y) = \mathrm{MR}(X_{\mathcal{B}}/Y_{\mathcal{B}})$; $(X/Y)_{\mathcal{D}} = (XY)_{\mathcal{D}} \setminus Y_{\mathcal{D}}$; $(X/Y)_R := (XY)_R \setminus Y_R$, and if all of the above are finite we define

$$d_0(X/Y) := d_1(X/Y) + |(X/Y)_{\mathcal{D}}| - |(X/Y)_R|.$$

Definition 4. (1) $C \in \mathcal{C}$ is self-sufficient in $A \in \mathcal{C}$, denoted $C \le A$, if
  (a) $C \subseteq A$.
  (b) For all $C \subseteq B \subseteq A$, $d_0(C) \le d_0(B)$.
(2) Let $\mathcal{C}_0 = \{C \in \mathcal{C} : \emptyset \le C\}$.

Claim 2. *If $C_1 \le C_2$ and $X \subseteq C_2$, then $d_0(X \cap C_1) \le d_0(X)$.*

Proof. This is a simple calculation. Denote $Y = (X \setminus C_1)$; we have to show is that $d_0(Y/X \cap C_1) \ge 0$. Because $C_1 \le C_2$, we have that $d_0(Y/C_1) \ge 0$, and therefore it is enough to show that $d_0(Y/X \cap C_1) \ge d_0(Y/C_1)$, which is an easy counting argument: let $(Y \to Z)_R = \{(y, x) : y \in Y, x \in Z, C_2 \models R(y, x)\}$; then

$$\begin{aligned} d_0(Y/X \cap C_1) &= d_0(Y \cup (X \cap C_1)) - d_0(X \cap C_1) \\ &= d_1((Y \cup X \cap C_1)_{\mathcal{B}}) + |(Y \cup X \cap C_1)_{\mathcal{D}}| - |(Y \cup X \cap C_1)_R| \\ &\quad - (d_1((X \cap C_1)_{\mathcal{B}}) + |(X \cap C_1)_{\mathcal{D}}| - |(X \cap C_1)_R|) \\ &\ge \mathrm{MR}(Y/X \cap C_1) + |Y_{\mathcal{D}}| - |Y_R| - |(Y \to X \cap C_1)_R| \\ &\ge \mathrm{MR}(Y/C_1) + |Y_{\mathcal{D}}| - |Y_R| - |(Y \to C_1)_R| \\ &= d_0(Y/C_1) \ge 0 \end{aligned}$$

as required. □

Corollary 6. *The relation $\le$ is transitive.*

Proof. Standard. □

Definition 5. (1) For $A \subseteq B \in \mathcal{C}_0$, define $d^B(A) = \min\{d_0(A') : A \subseteq A' \subseteq B\}$ (we will usually write simply $d(A)$ when the context is clear).
(2) Let N be a structure for $L = \{L(T_0), R, D\}$, $\emptyset \le N \models T_0^{\forall}$ and $A \subseteq N$ finite; then $d^N(A)$ is defined as in (1) above.
(3) Let N be as in (2) above $A, B \subseteq N$, A finite; then $d^N(A/B) = \min\{d^N(A \cup B') - d^N(B') : B' \subseteq B \text{ finite}\}$.
(4) Let N be as in (2) above. Then $A \subseteq N$ is self-sufficient in N, denoted $A \le N$, if for all finite $B \le A$, $B \le N$.
(5) For $A \subseteq N$, with N as above denote $cl_0^N(A) = \bigcap\{A' \le N : A \subseteq A'\}$.

Remark 1. $cl_0^N(A) \le N$.

Proof. If A is finite, then by Claim 2 for all A_1, A_2 with $A \le A_i$, $A \le A_1 \cap A_2$ as well. From this, we obtain that if A is finite then by the assumption that $\emptyset \le N$, there exists a finite $A' \supseteq A$ with $A' \le N$, so $cl_0^N(A) = \bigcap\{A'' \le A' : A \subseteq A'\}$. This is a finite intersection, so that by the remark at the head of the proof it follows that $cl_0^N(A) \le A'$, and by transitivity $cl_0^N(A) \le N$. If A is infinite, then note that by definition $\bigcup\{cl_0^N(A') : A' \subseteq A, \text{ finite}\} \le N$, and that $\bigcup\{cl_0^N(A') : A' \subseteq A, \text{ finite}\} \subseteq cl_0^N(A)$, so that we have equality. □

To obtain a structure of finite MR with D strongly minimal, we propose to use Hrushovski's μ functions. Toward that goal we recall the following definitions:

Definition 6. Let S be a structure of finite MR. A normal code for S consists of the following information:

(1) An integer m and a formula $\psi(\bar{y}_1, \dots, \bar{y}_m)$;
(2) A definable function $f(\bar{y}_1, \dots, \bar{y}_m)$;
(3) A formula $\varphi(\bar{x}, \bar{y})$;
(4) A formula $\theta(\bar{u})$ such that whenever $\models \theta(\bar{b})$:
  (a) $C = C(\bar{b}) =_{\text{def}} \{\bar{x} : \varphi(\bar{x}, \bar{b})\}$ has rank k multiplicity 1. If $\bar{x}, \bar{y} \in C$ and $\bar{x}_i = \bar{x}_j$ then $\bar{y}_i = \bar{y}_j$.
  (b) ψ is true of any m independent realizations of C.
  (c) f takes constant value $\bar{b}$ on m-tuples of realizations of C satisfying ψ.
  (d) Let $\bar{x} = \bar{x}^1{}^\frown\bar{x}^2$ be any partition of $\bar{x}$ into two sets. Then for any $\bar{b}$ such that $\theta(\bar{b})$, if for a generic $\bar{a} \in C(\bar{b})$, $\bar{a} = \bar{a}^1{}^\frown\bar{a}^2$, the set $\{\bar{x}^2 : (\bar{a}^1, \bar{x}^2) \in C(\bar{b})\}$ has rank j, then for all $\bar{c}^1$, the set $\{\bar{x}^2 : (\bar{c}^1, \bar{x}^2) \in C(\bar{b})\}$ has rank $\leq j$;
  (e) If $\bar{e}_1, \dots, \bar{e}_m \in C$ are such that $\models \psi(\bar{e}_1, \dots, \bar{e}_m)$ and $\bar{e} \in C$ is generic over $\bar{e}_1, \dots, \bar{e}_m$ then $\models \psi(\bar{e}, \bar{e}_1, \dots, \bar{e}_{m-1})$;
(5) ψ is symmetric in its arguments.
(6) ψ, f, φ, θ have no parameters.
(7) If c is a normal code, we write $m_c, \varphi_c, f_c, \theta_c, n_c$, and k_c, where n_c is the arity of $\bar{x}$, and k_c the rank of $C(\bar{b})$ for an element $\bar{b} \models \theta_c(\bar{u})$. If $\models \theta_c(\bar{b})$ we say that $(c, \bar{b})$ is a normal code for $(C, \bar{b})$.

Definition 7. Let S be a structure of finite MR. A set Q of normal codes for S is standard if:

(1) If $c_1, c_2 \in Q, n_{c_1} = n_{c_2}, k(c_1) = k(c_2)$, and $\mathrm{MR}(\{x : \varphi_{c_1}(\bar{x}, \bar{a}_1)\} \triangle \{x : \varphi_{c_2}(\bar{x}, \bar{a}_2)\}) < k$ for some $\bar{a}_1, \bar{a}_2 \models \theta_{c_i}(\bar{u})$, then $c_1 = c_2$ (and $\bar{a}_1 = \bar{a}_2$).
(2) For any definable $C \subseteq S^n$ of multiplicity 1, there exists $C' \subseteq S^n$ with $\mathrm{MR}(C \triangle C') < \mathrm{MR}(C)$ such that C' has a normal code (c, b) with $c \in Q$.
(3) If $c \in Q$, and c' is obtained from c by a permutation of the variables in $\bar{x}$, then $c' \in Q$.

Lemma 7. *If S is a structure of finite, definable MR and S has DMP, then S has a standard set of normal codes.*

Proof. The proof of the corresponding lemma in [2] translates verbally to the present context. □

Our next step is to find the corresponding analogue to the 2-codes of [2]:

Definition 8. Let T be a theory as above, with D a new sort not appearing in $L(T)$, and R a new binary relation symbol defined on couples (t, d) of elements, where t is in one of the sorts of T and $d \in D$. An f-code c in $L(T) \cup \{R, D\}$ contains the following information:

(1) A standard code $\hat{c}$ in $L(T)$;
(2) A natural number $n_c \geq n_{\hat{c}}$;
(3) An R-formula χ_c of the form:

$$\chi_c(\bar{x}, \bar{y}, \bar{w}) = \bigwedge_{i,j} R(x_i, y_j)^{\epsilon_{i,j}} \wedge \bigwedge_{i,j} R(x_i, w_j)^{\epsilon_{i,j}} \wedge \bigwedge_{i,j} R(w_j, y_i)^{\epsilon_{i,j}}$$

where $\mathrm{length}(\bar{x}) = n_{\hat{c}}$, $\mathrm{length}(\bar{y}) = n_c - n_{\hat{c}}$, $\epsilon_{i,j} \in \{-1, 1\}$, $R(x, y)^1 := R(x, y)$, $R(x, y)^{-1} := \neg R(x, y)$, and in each case (i, j) ranges over all the possibilities;
(4) A formula $\eta_c(\bar{x}, \bar{y})$ stating that no two of the variables of $\bar{x}, \bar{y}$ are equal;
(5) We require that:
  - The variables $\bar{x}$ are of the sorts of T, the variables $\bar{y}$ are from D, and the variables of $\bar{w}$ are from both D and from the sorts of T.
  - Let s denote the number of positive literals in χ_c; then $k_{\hat{c}} + (n_c - n_{\hat{c}}) = s$.
  - For each $z \in \bar{w}_{\mathcal{D}}$, denote by s_z the number of positive literals in χ_c in which z appears. Then $s_z \geq 1$.

- For every $\bar{b} \models \theta_{\hat{c}}(\bar{u})$ and for every partition pr of the variables of $\bar{x} = \bar{x}^1{}^\frown\bar{x}^2$ and $\bar{y} = \bar{y}^1{}^\frown\bar{y}^2$, let s_{pr} denote the number of positive literals in χ_c in the variables $\bar{x}^2$, $\bar{y}^2$ and $\bar{w}$. Then $rk\{\bar{x}^2 : \varphi_{\hat{c}}(\bar{x}^1{}^\frown\bar{x}^2, \bar{b})\} < s_{\text{pr}} - \text{length}(\bar{y}^2)$ for every $\bar{x}^1$.

For later use, it will be convenient to denote by h_c the maximum of the ranks (in T_0) of the sorts appearing in $\bar{x}$. The triple $(c, \bar{b}, \bar{d})$ will be called an f-code for $C = \{(\bar{x}, \bar{y}) : \varphi_c(\bar{x}, \bar{b}) \wedge \chi_c(\bar{x}, \bar{y}, \bar{d}) \wedge \eta_c(\bar{x}, \bar{y})\}$.

Remark 2. Note that if for an f-code c, $\varphi_{\hat{c}}(x, b)$ is algebraic for some $b \models \theta_{\hat{c}}$ then $\varphi_{\hat{c}}(x, b)$ is algebraic for all $b \models \theta_{\hat{c}}$. Notice, moreover, that in that case χ_c is trivial, and the whole f-code trivializes. To avoid notational problems, we will exclude algebraic codes form the definition.

Note that the last condition in the above definition is well-defined, thanks to condition 4(d) of the definition of a normal code.

To simplify the notation, when $\text{length}(\bar{z}_{\mathcal{B}}) = n_c$ and $\text{length}(\bar{z}_{\mathcal{D}}) = n_{\hat{c}} - n_c$, we may write $\chi_c(\bar{z}, \bar{w})$ instead of $\chi_c(\bar{z}_{\mathcal{B}}, \bar{z}_{\mathcal{D}}, \bar{w})$.

The following claim shows that the definition of f-code is indeed what we were looking for:

Claim 3. *Let c be an f-code for $L(T_0) \cup \{R, D\}$, $B \subseteq A \in \mathcal{C}_0$, $\bar{b} \in B_{\mathcal{B}}$, $\bar{d} \in B$ and $\bar{a} \in A$ such that:*

- $\models \theta_{\hat{c}}(\bar{b})$;
- $\models \chi_c(\bar{a}, \bar{d}) \wedge \varphi_c(\bar{a}_{\mathcal{B}}, \bar{b}) \wedge \eta(\bar{a}_{\mathcal{B}}, \bar{a}_{\mathcal{D}})$.

Then

(1) $d_0(\bar{a}/B) \leq 0$;
(2) *If* $d_0(\bar{a}/B) = 0$, *then either* $\bar{a} \in B$ *or* $\bar{a} \cap B = \emptyset$.
(3) *If* $d_0(\bar{a}/B) = 0$ *and* $\bar{a}' \subseteq \bar{a}$ *with* $d_0(\bar{a}'/B) = 0$, *then either* $\bar{a}' = a$ *or* $\bar{a}' \subseteq B$.

Proof. (1) $d_0(\bar{a}/B) = d_0(\bar{a}B) - d_0(B) = d_1(\bar{a}_{\mathcal{B}}/B_{\mathcal{B}}) + |(\bar{a}B)_{\mathcal{D}}| - |B_{\mathcal{D}}| - |(\bar{a}B)_R| + |B_R|$. Now $d_1(\bar{a}_{\mathcal{B}}/B_{\mathcal{B}}) \leq k_{\hat{c}}$ by definition of $\varphi_{\hat{c}}$ and η_c. Next, $|(\bar{a}B)_{\mathcal{D}}| - |B_{\mathcal{D}}| \leq \left|(\bar{a}\bar{d})_{\mathcal{D}}\right| - \left|\bar{d}_{\mathcal{D}}\right|$ but $\left|(\bar{a}\bar{d})_{\mathcal{D}}\right| - \left|\bar{d}_{\mathcal{D}}\right| = n_c - n_{\hat{c}}$. Finally, if $\bar{a} \cap B = \emptyset$ we know that $|B_R| - |(\bar{a}B)_R| \leq \left|(\bar{b}\bar{d})_R\right| - \left|(\bar{a}\bar{b}\bar{d})_R\right|$ because if $b' \in B \setminus (\bar{b}\bar{d})$ we have $\left|(\bar{b}\bar{d}b')_R\right| - \left|(\bar{a}\bar{b}\bar{d}b')_R\right| \leq \left|(\bar{b}\bar{d})_R\right| - \left|(\bar{a}\bar{b}\bar{d})_R\right|$. But $\left|(\bar{b}\bar{d})_R\right| - \left|(\bar{a}\bar{b}\bar{d})_R\right| \leq -s$, so we get that $d_0(\bar{a}/B) \leq k_{\hat{c}} + (n_c - n_{\hat{c}}) - s \leq 0$, by the definition of f-codes. If $\bar{a} \cap B \neq \emptyset$, repeat the same argument with $\bar{a}' = \bar{a} \setminus B$, and use the last part of (5) in the definition of f-codes to get the same conclusion.

(2) Assume for contradiction that both $\bar{a} \cap B \neq \emptyset$ and $\bar{a} \setminus B \neq \emptyset$, and consider the partition induced on the variables $\bar{x}$ and $\bar{y}$ of $\varphi_{\hat{c}}$ and χ_c. Then, by the last condition in the definition of f-codes (and an argument as in (1) above), we get that $d_0(a/B) = d_0((\bar{a} \setminus B)/B) < 0$, contradicting the assumption.

(3) Apply (1) and (2) above to $B \cup \bar{a}'$: because $\bar{a}' \subseteq \bar{a}$ and $d_0(\bar{a}'/B) \leq 0$, by assumption, it cannot be that $d_0(\bar{a}/B \cup \bar{a}') < 0$ (because this will imply $d_0(\bar{a}/B) < 0$). Thus $d_0(\bar{a}/B \cup \bar{a}') = 0$, and therefore $\bar{a} \in B \cup \bar{a}'$, which means that either $\bar{a} = \bar{a}'$ or $\bar{a}$ (and therefore also $\bar{a}'$) is in B. □

The next claim shows that not only are f-codes well behaved, but also that they capture all the information we need:

Claim 4. *Let $N := (N_1, D)$ be an L-structure such that $N_1 \models T_0^{\forall}$ and $\emptyset \leq N$. Let $B \subseteq A' \subseteq N$, and denote $A = A' \setminus B$. Assume that $d_0(A/B) = 0$, and that there is no proper non-empty $\tilde{A} \subseteq A$ with $d_0(\tilde{A}/B) \leq 0$. Then there exists a unique f-code c, and tuples $\bar{b} \in \text{acl}_{T_0}(B_{\mathcal{B}})$, $\bar{d} \in B_{\mathcal{D}}$, such that $\models \theta_{\hat{c}}(\bar{b})$ and $\models \chi_c(\bar{a}, \bar{b}, \bar{d}) \wedge \varphi_c(\bar{a}_{\mathcal{B}}, \bar{b})$, where $\bar{a}$ is any given enumeration of A.*

Proof. Embed $(A')_{\mathcal{B}}$ into M (the saturated model of T_0 we started with). $f(A_{\mathcal{B}})$ is a generic element of a unique type q, with $\text{MR}(q) = d_1(\bar{a}_{\mathcal{B}}/B_{\mathcal{B}})$ and multiplicity 1. So there are a unique standard code $\hat{c}$ and $\bar{b} \in \text{acl}(B_{\mathcal{B}})$ such that $M \models \theta_{\hat{c}}(\bar{b})$ and $M \models \varphi_{\hat{c}}(\bar{a}_{\mathcal{B}}, \bar{b})$. Now for each $b \in B$, denote $s_b = \{b' \in A : \ \models R(b, b')\}$ and let $\bar{d} = \{b \in B : s_b \neq \emptyset\}$. As $\bar{d}$ is finite, we may take χ_c to be the formula describing all the R-relations holding in $\bar{a}_R \cup (\bar{a} \to (\bar{b}\bar{d}))_R$ (as defined in the proof of Claim 2). Finally, let $n_c = \text{length}(\bar{a})$. We now show that the resulting code c is an f-code:

(1) By our choice, $\hat{c}$ is a standard code and $\models \theta_{\hat{c}}(\bar{b})$, so by definition we have that $rk\{\bar{x} : \varphi_{\hat{c}}(x, b)\} = d_1(\bar{a}_{\mathcal{B}}/\bar{b}) = k_{\hat{c}}$.

(2) Because χ_c describes all the R-relations in $\bar{a}_R \cup (\bar{a} \to (\bar{b}\bar{d}))_R$, the number s of positive literals in χ_c is exactly equal to the number of relations in $\bar{a}_R \cup (\bar{a} \to \bar{b}\bar{d})_R$.
(3) By the assumption that $d_0(\bar{a}/B) = 0$, we get, using (1) and (2) above, that $k_{\hat{c}} + (n_c - n_{\hat{c}}) = s$, as required.
(4) By definition of $\bar{d}$ for each $z \in \bar{d}$ we have $s_z \geq 1$.
(5) Any non-trivial partition of the variables of χ corresponds to a proper non-empty subset $\tilde{A} \subseteq A$. Let $\tilde{k} = d_1(\tilde{A}_{\mathcal{B}}/B)$; then $d_0(\tilde{A}/B) = \tilde{k} + |\tilde{A}_D| - (s - s_{pr}) > 0$, by the assumption on A. Using condition 4(d) of the definition of normal codes, we get that

$$rk\{\bar{x}^2 :\models \varphi_{\hat{c}}(\tilde{a}\hat{\ }x^2, \bar{b})\} + \left|A_{\mathcal{D}} \setminus \tilde{A}_{\mathcal{D}}\right| - s_{pr} < 0$$

which is exactly what we needed.

As χ_c is uniquely determined by $\bar{a}_{\mathcal{B}}, \bar{a}_{\mathcal{D}}$ and $\bar{d}$, it only remains to check the uniqueness of $\bar{d}$, which is obvious if we want (3) and (4) above to hold. □

From now on, when we say that an $L_0 \cup \{R, D\}$-structure $N \models T_0^{\forall}$, we will mean, as in the statement of the last claim, that $N = (N_1, D)$ and $N_1 \models T_0^{\forall}$.

Claim 5. *Let $B \subseteq A' \subseteq N$ be as above, and c an f-code, $\bar{b} \in \mathrm{acl}(B_{\mathcal{B}})$, and $\bar{d} \in B$ as provided by the previous claim. Then there is a natural number $m_c \geq m_{\hat{c}}$ and a $\emptyset$-definable (in $L(T_0) \cup R$) m_c-ary function g such that for any disjoint $\bar{a}_1, \ldots, \bar{a}_{m_c}$ satisfying $\varphi_{\hat{c}}((\bar{a}_i)_{\mathcal{B}}, \bar{b}) \wedge \chi_c(\bar{a}_i, \bar{d})$ for all i, if $\psi_{\hat{c}}((\bar{a}_1)_{\mathcal{B}}, \ldots, (\bar{a}_{m_c})_{\mathcal{B}})$, then $g(\bar{a}_1, \ldots, \bar{a}_{m_c}) = \bar{d}$.*

Proof. If $\bar{d} = \emptyset$, there is nothing do to; take any $d \in \bar{d}$; by definition of $\bar{d}$ there exists an element $a \in A' \setminus B$ with $R(a, d)$, so this holds for any $\bar{a}_i \models \chi_c(\bar{x}, \bar{d})$, and thus $d_0(d/\bar{a}_i, \bar{b}) \leq d_0(d) - 1$. By the disjointedness of the $\bar{a}_i$'s, $d_0(d/\bar{a}_0, \ldots, \bar{a}_n, \bar{b}) \leq d_0(d) - n$. By the assumption on $A = A' \setminus B$, $d_0(\bar{a}_i/\bar{b}, \bar{d}) = 0$ giving: $d_0(\bar{a}_0, \ldots, \bar{a}_n, \bar{b}, \bar{d}) \leq d_0(\bar{b}, \bar{d}) \leq d_0(\bar{b}) + d_0(\bar{d})$.

Assume toward a contradiction that for all m, there were $B \in \mathcal{C}_0$, elements $\bar{a}_1, \ldots, \bar{a}_m, \bar{b}, \bar{d}_1, \bar{d}_2$ such that $\models \bigwedge_{i,j} \varphi_{\hat{c}}((\bar{a}_i)_{\mathcal{B}}, \bar{b}) \wedge \chi_c(\bar{a}_i, \bar{d}_j)$ and $\bar{d}_1 \neq \bar{d}_2$, then

$$d_0(\bar{a}_0, \ldots, \bar{a}_m, \bar{b}, \bar{d}_1, \bar{d}_2) \leq d_0(\bar{b}) + d_0(\bar{d}_1) + m d_0(\bar{a}_i/\bar{d}_1, \bar{b}) + d_0(\bar{d}_2/\bar{a}_0, \ldots, \bar{a}_m).$$

Since $\bar{d}_1 \neq \bar{d}_2$, there exists some $d' \in \bar{d}_2 \setminus \bar{d}_1$, and because $\bar{d}_2 \models \chi_c(\bar{a}_i, \bar{d}_j)$, it follows from (5) in the definition of f-codes that $d_0(d'/\bar{d}_1\bar{a}_0, \ldots, \bar{a}_m) \leq d_0(d') - m$. By what we have already calculated, this gives:

$$d_0(\bar{a}_0, \ldots, \bar{a}_m, \bar{b}, \bar{d}_1, \bar{d}_2) \leq d_0(\bar{b}) + 2d_0(\bar{d}) - m\left|\bar{d}_2 \setminus \bar{d}_1\right|$$

which, for large enough m, gives negative d_0, a contradiction.

To make $m_c \geq m_{\hat{c}}$, just take m_c provided by the argument above large enough. □

Consider the special case of a formula of the form $R(a, d)$ for a generic (in its sort) $a \in M_{\mathcal{B}}$ and $d \in \mathcal{D}$. Clearly $d_0(d/a) = 0$, and $R(x, y)$ is an f-code (setting the standard code $\theta_{\hat{c}}(y) := y = y$ – in the appropriate sort – $\varphi_{\hat{c}}$ any sentence in T, $n_{\hat{c}} = 0$ and $n_c = 1$, $\chi_c = R(x, y)$). The above proof shows that for this specific code, we have $m_c > 2\mathrm{MR}(a)$.

Using the above claim, we can now safely strengthen the definition of f-codes as follows:

Definition 9. Let T, D, R be as in Definition 8. From now on we will assume that an f-code for $L(T) \cup \{R, D\}$ contains, in addition to the information required in Definition 8, a natural number m_c and a $\emptyset$-definable m_c-placed function g, as provided in the previous lemma.

At this stage we can already introduce Hrushovski's μ-functions:

Let μ^* be a finite-to-one integer valued function defined on f-codes satisfying:

- $\mu^*(c) \geq m_c - 1$.
- $\mu^*(c) = \mu^*(c')$ if c differs from c' only by permutation of the variables $x_1, \ldots, x_{n_{\hat{c}}}$, and $y_1, \ldots, y_{n_c - n_{\hat{c}}}$.

Now let $\mu(c) = h_c m_c n_c + \mu^*(c)$.

Given an f-code c and an integer $m \geq m_c$, let $\Theta_c'(\bar{u}_1, \bar{u}_2, \bar{y}_1, \ldots, \bar{y}_m)$ be the conjunction of the following conditions:

- The $\bar{y}_i$'s are pairwise disjoint.
- $\psi_{\hat{c}}$ is true of each $m_{\hat{c}}$-tuple of the $(\bar{y}_i)_{\mathcal{B}}$'s.
- $f_{\hat{c}}((\bar{y}_1)_{\mathcal{B}}, \ldots, (\bar{y}_{m_{\hat{c}}})_{\mathcal{B}}) = \bar{u}_1$.
- $g_c(\bar{y}_1, \ldots, \bar{y}_{m_c}) = \bar{u}_2$.
- $\varphi_{\hat{c}}((\bar{y}_i)_{\mathcal{B}}, \bar{u}_1)$ for all $1 \le i \le m$.
- $\chi_c(\bar{y}_i, \bar{u}_2)$ for all $1 \le i \le m$.

Now let

$$\Theta_c(\bar{y}_1, \ldots, \bar{y}_m) = \Theta'_c(f_{\hat{c}}((\bar{y}_1)_{\mathcal{B}}, \ldots, (\bar{y}_{m_{\hat{c}}})_{\mathcal{B}}), g_c(\bar{y}_1, \ldots, \bar{y}_{m_c}), \bar{y}_1, \ldots, \bar{y}_m).$$

We are now ready to formulate the axioms of the first order theory T promised in the statement of the theorem:

Universal axioms. (1) $T_0^{\forall}$.
(2) $\Theta_c(\bar{y}_1, \ldots, \bar{y}_m)$ has no solution if $m \ge \mu(c)$.
(3) If $N \models T$, then $\emptyset \le N$ (which is first order by the definability of MR in T_0).

AE axioms. (1) Axioms stating that if $M \models T$, then $M|_{L(T_0)}$ is algebraically closed.
(2) For each f-code c and integer l, an axiom stating that for every set W of ln_c-tuples and every tuples $\bar{b}, \bar{d}$, the code instance $c(\bar{b}, \bar{d})$ has a solution outside W unless W contains a "maximal" set of solutions, or for some other f-code, c_0, adding a generic solution would create too many solutions to c_0. Formally — for all sets W of ln_c-tuples and all $\bar{u}, \bar{w}$ such that $\theta_{\hat{c}}(\bar{u}) \wedge (\exists \bar{x})\chi_c(\bar{x}, \bar{w})$ one of the following holds:
(a) $(\exists \bar{x})(\bar{x} \not\subseteq W$, and $\varphi_{\hat{c}}(\bar{x}_{\mathcal{B}}, \bar{u}) \wedge \chi_c(\bar{x}, \bar{u}, \bar{w}))$.
(b) $(\exists \bar{y}_1, \ldots, \bar{y}_r \in W)(\Theta_c(\bar{u}, \bar{w}, \bar{y}_1, \ldots, \bar{y}_r))$ and $r = \mu(c)$.
(c) For some f-code c_0 and some choice of variables $\bar{y}$ (as explained below): $(\exists \bar{y} \setminus \bar{x})(\forall^* \bar{x}$ s.t. $\varphi_{\hat{c}}(\bar{x}_{\mathcal{B}}, \bar{u}) \wedge \chi_c(\bar{x}, \bar{w}))\,\Theta_{c_0}(\bar{y}_0, \ldots, \bar{y}_{\mu(c_0)})$, where:
$\bar{y}_v = (y_{v,1}, \ldots, y_{v,n_{c_0}})$;
if $v \ge h_{c_0} m_{c_0} n_{c_0}$, then $y_{v,l} = x_j$ for some j (and x_j a variable in $\bar{x}$).
if $v < h_{c_0} m_{c_0} n_{c_0}$, then $y_{v,l} = x_j$ for some j (and x_j a variable in $\bar{x}$), or $y_{v,l}$ is a new variable.
$(\exists \bar{y} \setminus \bar{x})$ quantifies only over those variables in $\bar{y}$ which are not in $\bar{x}$.
$(\forall^* x \ldots)$ means "for generic x such that $\ldots$" (in the present case, it means that $rk\{\bar{x} : \varphi_{\hat{c}}(\bar{x}_{\mathcal{B}}, \bar{u})\} = k_{c_0}$ and $\chi_c(\bar{x}, \bar{w})$).

Note that all the variables $y_{v,l}$ may be assumed to be distinct (because otherwise θ_{c_0} will not hold). This can be achieved (for $v \ge h_{c_0} m_{c_0} n_{c_0}$) only if $(\mu(c_0) + 1 - h_{c_0} m_{c_0} n_{c_0}) n_{c_0} \le n_c$, but by the definition of μ, we get that $\mu^*(c_0) \le n_c / n_{c_0}$, and because μ^* is finite to one, only a finite number of f-codes should be considered. Therefore the axiom is indeed first order.

EA axioms. Axioms stating that in a saturated model M of T, there exists an infinite d-independent set I in $\mathcal{D}^M$.

We proceed to show that T has the desired properties. First, we have to show that T is consistent:

Lemma 8. *Suppose $M \models T^{\forall}$ and:*

(1) *There exists an infinite d-independent set in M.*
(2) *Whenever $M \le N$, $N \models T^{\forall}$, $\varphi(\bar{x}, \bar{y})$ is a quantifier free formula and $\bar{b} \in M$ such that $N \models (\exists \bar{x})\varphi(\bar{x}, \bar{b})$, then $M \models (\exists \bar{x})\varphi(\bar{x}, \bar{b})$.*

Then $M \models T$.

Proof. The following set of claims will show that M satisfies each of the axioms of T:

Claim 6. *Let $N = M \cup \{a\}$ be a structure for $L(T_0) \cup \{R, D\}$ such that $N \models T_0^{\forall}$. Assume that $a \in \mathrm{acl}(M) \setminus M$, in the sense of T_0 and $N \models (\forall x)\neg R(a, x)$; then $N \models T^{\forall}$.*

Proof. By the assumptions on a, we get that $d_0(a/M) = 0$ and $\emptyset \le N$. Now suppose $N \models \theta_c(\bar{a}_1, \ldots, \bar{a}_m)$ with $m > \mu(c)$. By the definition of θ_c, the $\bar{a}_i$ are disjoint, and by the assumption on M, it follows that $a \in \bar{a}_i$ for exactly one of the $\bar{a}_i$, say $\bar{a}_m$. Then, as $d_0(\bar{a}_m/M) = d_0(a/M)$, we get by Claim 3 that either $\bar{a}_m \subseteq M$, contradicting the assumption that $a \in \bar{a}_m$, or $\bar{a}_m \cap M = \emptyset$, which implies that $\bar{a}_m = a$. This implies that $\bar{a}_1, \ldots, \bar{a}_m$ all satisfy the same algebraic code, but algebraic codes are excluded from our set of codes, a contradiction. □

Claim 7. *M is algebraically closed in the sense of T_0.*

Proof. Suppose not. Then let $a \in \mathrm{acl}_{T_0}(M) \setminus M$, and let $N = M \cup \{a\}$ be a model of $T_0^\forall$. Make N a model for R by making a R-related to no element in N. By the previous claim, $N \models T^\forall$. Let $\varphi(\bar{x}, \bar{y})$ be such that $T_0 \models (\forall \bar{y} \forall x_1, \ldots, x_m)(\bigwedge_{i=1}^m \varphi(x_i, \bar{y}) \to \bigvee_{i \neq j}(x_i = x_j))$, and $N \models \varphi(a, \bar{b})$ for some $\bar{b} \in M$. By QE, we may assume that $\varphi(x, \bar{y})$ is quantifier free, and so if we let $S = \varphi(x, \bar{b})^M$, then a is a solution in N of $\varphi(\bar{x}, \bar{b}) \wedge (x \notin S)$, which does not have a solution in M, a contradiction. □

Claim 8. *Let $N = M \cup \{a_1, \ldots, a_n\}$, $N \models T_0^\forall$ in $L \cup \{R, D\}$, $d(\bar{a}/M) \geq 0$. Let c be an f-code, $\bar{b}, \bar{d} \in M$ and $N \models \varphi_{\hat{c}}(\bar{a}, \bar{b}) \wedge \chi_c(\bar{a}, \bar{d})$; then either*

(1) $N \models T^\forall$, or
(2) There are $\bar{a}_0, \ldots, \bar{a}_{r-1} \in M$ such that $N \models \Theta'_c(\bar{b}, \bar{d}, \bar{a}_0, \ldots, \bar{a}_{r-1})$ and $r = \mu(c)$, or
(3) There exists an f-code c_0, $r_1 < h_{c_0} m_{c_0} n_{c_0}$, tuples $\bar{a}_0, \ldots, \bar{a}_{r_1 - 1} \in N$ with $\bar{a}_{r_1}, \ldots, \bar{a}_r$ from $\{a_1, \ldots, a_n\}$ and $r = \mu(c_0)$, such that $N \models \Theta_{c_0}(\bar{a}_1, \ldots, \bar{a}_r)$.

Proof. Suppose $N \not\models T^\forall$. By assumption, $N \models T_0^\forall$ and from $d(a_{i_1}, \ldots, a_{i_j}/M) \geq 0$ it follows that $\emptyset \leq N$ so that there must be some f-code c_0 such that $\Theta_{c_0}(\bar{y}_0, \ldots, \bar{y}_r)$ has a solution in N for $r \geq \mu(c)$. Let $\bar{a}_0, \ldots, \bar{a}_r$ be such a solution. As Θ_{c_0} is symmetric in its arguments, we may assume that for some $0 \leq r_0 \leq r_1 \leq r$, we have that $\bar{a}_0, \ldots, \bar{a}_{r_0 - 1} \subseteq M$, and $\bar{a}_{r_0}, \ldots, \bar{a}_{r_1 - 1}$ are those of the $\bar{a}_i$ whose intersection with M is not empty. Note that by our assumption on M, it cannot be that all the $\bar{a}_i$ are in M, so necessarily $r_0 < r$. Now let

$$k(i) = d_0(\bar{a}_i / M\bar{a}_0, \ldots, \bar{a}_{i-1});$$

then by Claim 3 $k(i) \leq 0$ if $i \geq m_{c_0}$ (because the canonical parameters $\bar{b}_{c_0}, \bar{d}_{c_0}$ corresponding to the given set of realizations are definable over $\bar{a}_0, \ldots, \bar{a}_{m_{c_0}}$). Moreover, in that case if $\bar{a}_i \cap M \neq \emptyset$ and $\bar{a}_i \not\subseteq M$, then $k(i) < 0$.

Case A. $r_0 \geq m_{c_0}$.

As $r_0 < r \bar{a}_{r_0}$ exists and $d_0(\bar{a}_{r_0}/M) = k(r_0) \leq 0$. Again by Claim 3, and by the assumption that $d(\bar{a}/M) \geq 0$, we get that $\bar{a}_{r_0} = a^\sigma$, where $\sigma \in \mathrm{Sym}(\{1, \ldots, n\})$. So by the definition of standard codes and by the uniqueness of f-codes, it follows that $c_0 = c^\sigma$, and we obtain (2) of the claim.

Case B. $r_0 < m_{c_0}$.

By definition, $\Sigma_{i<r_1} k(i) = d_0(\bar{a}_0, \ldots, \bar{a}_{r_1 - 1}/M) \geq 0$ (by the assumption $d(\bar{a}/M) \geq 0$). Now, $k(i) \leq h_{c_0} n_{c_0} - 1$ for $i < \min\{m_{c_0}, r_1\}$ (because a_i has at least one coordinate in M). For $m_{c_0} \leq i < r_1$, we have that $k(i) < 0$, and so $\Sigma_{i<r_1} k(i) \leq m_{c_0}(h_{c_0} n_{c_0} - 1) - (r_1 - m_{c_0})$, and we get that $r_1 \leq h_{c_0} m_{c_0} n_{c_0}$. □

Claim 9. *M is a model of the axioms of AE(2).*

Proof. Let c be an f-code, $\bar{b}, \bar{d} \in M$, $M \models \theta_{\hat{c}}(\bar{b}) \wedge (\exists \bar{x}) \chi_c(\bar{x}, \bar{d})$. Let W be a finite set of n_c-tuples from M. Let $N = M \cup \{a_1, \ldots, a_n\}$, where $\{a_1, \ldots, a_n\}$ are new elements. Make N into an $L(T_0)$ -extension of M in such a way that $N \models \varphi_{\hat{c}}(\bar{a}_{\mathcal{B}}, \bar{b})$, and into an R-extension of M in such a way that $N \models \chi_c(\bar{a}, \bar{d})$. By Claim 3 $d(\bar{a}/M) = 0$, so the previous claim is true:

If (1) of the claim is true, then in N there is a solution to $\varphi_c(\bar{x}_{\mathcal{B}}, \bar{b}) \wedge \chi_c(\bar{x}, \bar{d})$ outside W, and therefore there is such a solution in M too.

If (2) of the claim holds, then either one of the $\bar{a}_i$ is outside W and again we are fine, or they are all in W, and then (2) of the corresponding axiom in (2) is true.

If (3) of the claim is true, then for $\nu \geq h_{c_0} m_{c_0} n_{c_0}$, take $y_{\nu,l} = x_j$ where a_j is the lth coordinate of $\bar{a}_\nu$, and for $\nu < h_{c_0} m_{c_0} n_{c_0}$ take $y_{\nu,l} = x_j$ if $a_{\nu,j} \in \bar{a}$, and a new variable otherwise. Then (3) of the corresponding axiom is satisfied. □

This finishes the proof of the lemma. □

Corollary 9. *T is consistent.*

Proof. Let M be a saturated model of T_0, and let J be an infinite independent set in M. Let I be an infinite set with no structure. Make an $L(T_0) \cup \{R, D\}$ structure M_0 of $J \cup I$ by assigning all the elements of I a new sort, such that no R-relations hold between I and J. Clearly, $M_0 \models T^{\forall}$, and so any existentially closed (in the sense of $\leq$, not of elementary embeddings) extension $M_0 \leq M$ is a model of T. □

The next step is to show that the sort $\mathcal{D}$, when interpreted in T, is strongly minimal, and that $N \models T$ in every model $N = dcl(N_{\mathcal{D}})$:

Claim 10. *Let M be as in Lemma 8, and let $a \in M_{\mathcal{B}}$. Denote by r the MR (in T_0) of the sort of a. Then there are at least $2r + 1$ elements $d_i \in M_{\mathcal{D}}$ such that $\models R(a, d_i)$.*

Proof. Suppose not. Let $N = M \cup \{d\}$, where $N_{\mathcal{D}} = M_{\mathcal{D}} \cup \{d\}$, and $N \models R(x, d) \iff x = a$. Then obviously $N \models T_0^{\forall}$, and $d_0(d/M) = 0$, so $\emptyset \leq N$. Now suppose $N \models \theta_c(\bar{a}_1, \ldots, \bar{a}_m)$ with $m > \mu(c)$ for some f-code c. As in the proof of Lemma 8, we get that $d = \bar{a}_m$, and therefore $R(a, a_i)$ is true for all $1 \leq i \leq m$ (because $R(a, d)$ is the unique formula satisfied by d/M, and therefore must appear in c). But by the discussion following the proof of Claim 5 and by our choice of μ, we have that $\mu(c) > 2r + 1$, which contradicts the assumption that $m > \mu(c)$. So $N \models T^{\forall}$. Let $S = R(a, x)^M$; then by our assumption, S is finite and the formula $R(a, x) \wedge (x \notin S)$ has a solution in N, but not in M — a direct contradiction to assumption 2 of Lemma 8. □

Corollary 10. *Let $N \models T$ then $dcl(N_{\mathcal{D}}) = N$.*

Proof. Assume $a \in N \setminus N_{\mathcal{D}}$, and $r = \mathrm{MR}(x = x)$ in the type of a in T_0. By the last claim and the AE axioms there are at least $2r + 1$ elements $\{d_i\}_{i=1}^{2r+1} \in D$ with $N \models R(d_i, a)$ for all i. Now assume that a' is of the same sort as a and that $N \models \bigwedge_{i=1}^{2r+1} R(d_i, a')$; then $d_0(a, a', d_1, \ldots, d_{2r+1}) = d_1(a) + d_1(a'/a) + 2r + 1 - 2(2r + 1) \leq -1$, contradicting the axiom asserting that any finite subset of N has positive d_0. □

The following is easy and resembles the previous argument:

Lemma 11. (1) *Let M be a model of $T^{\forall}$ which is algebraically closed (in the sense of $L(T_0)$). Let $\bar{x}$ be a set of variables, $\varphi(\bar{x}_{\mathcal{B}}, \bar{a})$ an $L(T_0)$-formula, and $\chi(\bar{x}, \bar{d})$ an R-formula over M such that $\varphi(\bar{x}_{\mathcal{B}}, \bar{a})$ has rank k (in the sense of T_0) and χ has s positive literals. If $k + s + (\mathrm{length}(\bar{x}) - \mathrm{length}(\bar{x}_{\mathcal{B}})) \leq 0$, then there is a finite number of disjoint solutions of $\varphi(\bar{x}, \bar{a}) \wedge \chi(\bar{x}, \bar{d})$ in M.*

(2) *Let M be as above, let c be an f-code, and $\bar{a}, \bar{d} \in M$ with $M \models \theta_{\hat{c}}(\bar{a}) \wedge (\exists \bar{x})\chi_c(\bar{x}, \bar{d})$; then there is a finite number of solutions in M to $\varphi_{\hat{c}}(\bar{x}_{\mathcal{B}}, \bar{a}) \wedge \chi_c(\bar{x}, \bar{d})$.*

Proof. By induction on $n = \mathrm{length}(\bar{x})$. Assume for contradiction that $J \subseteq M^n$ is an infinite set of disjoint solutions of $\varphi(\bar{x}, \bar{a}) \wedge \chi(\bar{x}, \bar{d})$. We may replace J by a subset such that J forms a Morley sequence over a finite set $A \supseteq \bar{a}, \bar{d}$. Denote $k' = d_1(\bar{c}_{\mathcal{B}}/A)$ for $\bar{c} \in J$; then $k' \leq k$. Let $J_m \subseteq J$ be of size m; then $d_1((J_m)_{\mathcal{B}}/A) \leq mk'$ so $d_0(J_m A) \leq mk' + mn' - ms + d_0(A)$, but as this should be non-negative for all m, we get that $k' = k$ and $k + n' - s = 0$, so $d_0(\bar{c}/A) = 0$.

(1) If there is $\bar{c}' \subseteq \bar{c}$ with $d_0(\bar{c}'/A) = 0$, then the claim follows by induction. Otherwise there is, by Claim 4, a unique f-code c, and elements $\bar{a}'\bar{d}' \in \mathrm{acl}(A) \subseteq M$, such that $M \models \theta_{\hat{c}}(\bar{x}_{\mathcal{B}}, \bar{a}') \wedge \chi_c(\bar{c}, \bar{d}') \wedge \varphi_{\hat{c}}(\bar{c}, \bar{a})$. Now J being infinite will contradict the assumption that $M \models T^{\forall}$.

(2) Easy, using the first part of the lemma and Claim 3. □

Lemma 12. *Let B_1, B_2 be substructures of $M_1, M_2 \models T$ such that $B_i \leq M_i$ and $d(M_i/B_i) = 0$. Let $f : B_1 \to B_2$ be a bijection preserving the atomic relations of $L(T_0)$ and R. Then f extends to an isomorphism $M_1 \to M_2$.*

Proof. It is enough to show that given any element $a \in M_1 \setminus B_1$, f can be extended to another map meeting the same conditions whose domain is $B_1 a$. Note that wlog we may assume that $B_i = \mathrm{acl}_{T_0}(B_i)$: the assumption that $B_i \leq M_i$ implies that no $a' \in \mathrm{acl}_{T_0}(B_i)$ has any R-relations with any element in B_i, and therefore any $L(T_0)$-isomorphism between $\mathrm{acl}_{T_0}(B_1)$ and $\mathrm{acl}_{T_0}(B_2)$ extending f will do the job. Moreover, we have easily that $\mathrm{acl}_{L(T_0)}(B_i) \leq M_i$.

By changing names, we may also assume that $B_1 = B_2 = B$ and that $f = id$. Let $a_0 \in M_1 \setminus B$; then by the assumption that $d(M_i/B) = 0$, we get that $d(a_0/B) = 0$ and therefore $d(a_0/B_0) = 0$ for some finite $B_0 \subseteq B$. So there is a finite set $a_0 \in A = \{a_i\}_{i=0}^{n} \subseteq M_1 \setminus B$ with $d_0(A/B_0) = 0$. If we choose A of minimal size and an enumeration

$\bar{a}$ thereof, then by Claim 4 there is an f-code c and $\bar{b}, \bar{d} \in B_0$ (wlog) such that $M_1 \models \theta_{\hat{c}}(\bar{b}) \wedge \varphi_{\hat{c}}(\bar{a}_{\mathcal{B}}, \bar{b}) \wedge \chi_c(\bar{a}, \bar{d})$. By Lemma 11 the set $W = \{\bar{x} \in B^n : \models \varphi_{\hat{c}}(\bar{x}_{\mathcal{B}}, \bar{b}) \wedge \chi_c(\bar{x}, \bar{d})\}$ is finite. Because $M_i \models T$, one of the following cases must occur:

(1) There exists $\bar{a}' \in M_2^n \setminus W$ such that $M_2 \models \varphi_{\hat{c}}(\bar{a}_{\mathcal{B}}, \bar{b}) \wedge \chi_c(\bar{a}, \bar{d})$.

In that case, extend f to $B \cup \{\bar{a}\}$ by $\bar{a} \to \bar{a}'$. $\bar{a}'$ does not lie entirely in B, since $\bar{a}' \notin W$. Because $B \leq M_2$ too, $d_0(\bar{a}'/B) \geq 0$, and so by Claim 3 $d_0(\bar{a}'/B) = 0$ and $\bar{a}' \cap B = \emptyset$. From this and the fact that $d_0(\bar{a}'/B) = 0$ it follows that $d_1((\bar{a}')_{\mathcal{B}}/B) = d_1(\bar{a}_{\mathcal{B}}/B)$ (because length$(\bar{a})$ = length$(\bar{a}')$, length$(\bar{a}_{\mathcal{B}})$ = length$((\bar{a}')_{\mathcal{B}})$, $M_1 \models \chi_c(\bar{a}, \bar{d})$, $M_2 \models \chi_c(\bar{a}', \bar{d})$). Now, $\varphi_{\hat{c}}(\bar{x}_{\mathcal{B}}, \bar{b})$ determines a unique $L(T_0)$-type in $S(B)$ of rank $k_{\hat{c}}$, so the $L(T_0)$ type of $\bar{a}'/B$ is that of $\bar{a}/B$, and so f is indeed an $L(T_0)$-embedding. Because $M_1 \models \chi_c(\bar{a}, \bar{d})$, $M_2 \models \chi_c(\bar{a}', \bar{d})$ and no other R-relations hold between $\bar{a}, \bar{a}'$ and B, it is also an R-embedding. The fact that $d(M_i/B\bar{a}) = d(M_i/B\bar{a}') = 0$ follows from monotonicity.

(2) There are $\bar{a}_1, \ldots, \bar{a}_r$ in W, $r = \mu(c)$, $\Theta'_c(\bar{b}, \bar{d}, \bar{a}_1, \ldots, \bar{a}_r)$.

By definition of Θ'_c, we will also have that $\models \Theta'_c(\bar{b}, \bar{d}, \bar{a}_1, \ldots, \bar{a}_r, \bar{a})$ (because $\bar{a}_{\mathcal{B}} \perp_{B_0} (\bar{a}_1)_{\mathcal{B}}, \ldots, (\bar{a}_r)_{\mathcal{B}}$ implies – by the definition of normal codes – that $\psi_{\hat{c}}(\bar{a}_1, \ldots, \hat{a}_i, \ldots, \bar{a}_r, \bar{a})$ for all i, and therefore $\Theta'_c(\bar{b}, \bar{d}, \bar{a}_1, \ldots, \bar{a}_r, \bar{a})$ by definition). This contradicts $M_1 \models T^{\forall}$, and therefore this case cannot occur.

(3) Let $\bar{a}' \models tp(\bar{a}/\bar{b}\bar{d})|_{M_2}$ (i.e. $\bar{a}'_{\mathcal{B}}$ has the same $L(T_0)$-type over $(\bar{b}\bar{d})_{\mathcal{B}}$ as $\bar{a}_{\mathcal{B}}$ and satisfies exactly the same R-relations with B, as does $\bar{a}$ – in particular $\models \chi_c(\bar{a}', \bar{d})$ – and $(\bar{a}')_{\mathcal{B}} \perp_{(\bar{b}\bar{d})_{\mathcal{B}}} (M_2)_{\mathcal{B}}$). Moreover, we may choose $\bar{a}'$ such that it has no R-relations, save those already explicit in χ_c. This can be done because if $d \in M_{\mathcal{D}}$ and $M_2 \models R(a'', d)$ for some $a'' \in \bar{a}'$ then $d_0(d/\bar{a}') \leq 0$, and therefore there are only a finite number of such elements in M_2, and by compactness we can find $\bar{a}'$, as required. By our assumption $\Theta_c(\bar{a}_0, \ldots, \bar{a}_{\mu(c_0)})$ is true of some f-code c_0 and $\bar{a}_0, \ldots, \bar{a}_{\mu(c_0)} \subseteq M_2\bar{a}'$. Moreover, $\bar{a}_i \subseteq (\bar{a}')^{n_{c_0}}$ for $i \geq h_{c_0} m_{c_0} n_{c_0}$.

We will show that this case cannot occur either. Denote $r = \mu(c_0)$, $r_1 = r - m_{c_0} + 1$. Because $\mu^*(c_0) \geq m_{c_0} - 1$ we get that $r_1 \geq h_{c_0} m_{c_0} n_{c_0}$. Let $\bar{b}' = f_{\hat{c}_0}((\bar{a}_1)_{\mathcal{B}}, \ldots, (\bar{a}_{m_{\hat{c}}})_{\mathcal{B}})$, $\bar{d}' = g_{c_0}(\bar{a}_1, \ldots, \bar{a}_{m_{c_0}})$ then $\models \varphi_{\hat{c}_0}((\bar{a}_i)_{\mathcal{B}}, \bar{b}') \wedge \chi_{c_0}(\bar{a}_i, \bar{d}')$ for all $0 \leq i \leq \mu(c_0)$. Note that $\bar{b}' \in dcl(B\bar{a}')$ (since $\bar{a}_{r_1}, \ldots, \bar{a}_r \subseteq \bar{a}'$ and $(\bar{a}')_{\mathcal{B}} \perp_{(\bar{b}\bar{d})_{\mathcal{B}}} (M_2)_{\mathcal{B}}$) and $\bar{d}' \in B\bar{a}'$ as well (again, because $\bar{a}_{r_1}, \ldots, \bar{a}_r \subseteq \bar{a}'$, and $\bar{a}'$ has no R-relations outside $B\bar{a}'$). Now it will be enough to show that $\bar{a}_j \in B\bar{a}'$, as this will imply that there are more than $\mu(c_0)$ realizations of a type coded by c_0 in M_1 (they will lie in $B\bar{a}$), contradicting the assumption that $M_1 \models T^{\forall}$.

Fix j and assume $\bar{c}_j = \bar{a}_j \setminus B\bar{a}' \neq \emptyset$. Then, by the choice of $\bar{a}'$, we have $(\bar{c}_j)_{\mathcal{B}} \perp_{B_{\mathcal{B}}} \bar{a}'_{\mathcal{B}}$ in the sense of $L(T_0)$. $B \leq M_i$ implies that $d_0(\bar{c}_j/B) \geq 0$. As $\models \chi_c(\bar{a}', \bar{d})$ and $\bar{a}'$ does not satisfy any other R-relations (and $\bar{d} \in B$), it follows that $d_0(\bar{c}_j/B\bar{a}') \geq 0$, and so $d_0(\bar{a}_j/B\bar{a}') \geq 0$. Because $\bar{b}', \bar{d} \in B\bar{a}'$ Claim 3 implies that $d_0(\bar{a}_j/B\bar{a}') = 0$ and $\bar{c}_j = \bar{a}_j$. In particular, we get that $\bar{d}' \in B$: we have already seen that $d' \in B\bar{a}'$, but $\bar{a}_j \cap B\bar{a}' = \emptyset$, and therefore, by the choice of $\bar{a}'$, has no R-relations with $\bar{a}'$. On the other hand, $\bar{c}_j = \bar{a}_j$ implies that $(\bar{a}_j)_{\mathcal{B}}$ is a solution of $\varphi_{\hat{c}_0}(\bar{x}_{\mathcal{B}}, \bar{b}')$ generic over $B\bar{a}'$. This means that $\bar{b}' \in \mathrm{acl}_{L(T_0)}(B) = B$ (since $\bar{b}'$ is the canonical base of $tp((\bar{a}_j)_{\mathcal{B}}/B\bar{a}')$, and $\bar{a}_j \perp_B \bar{a}'$). This means that $d_0(\bar{a}_j/B) = 0$ and once again, applying Claim 3, we get that $\bar{a}_j = \bar{a}'$. By the uniqueness of f-codes, this implies that $c_0 = c^{\sigma}$ and $\mu(c_0) = \mu(c)$. Moreover, for all $i \neq j$, $\bar{a}_i \subseteq M_2$, so either $\bar{a}_i \subseteq B$ for all $i \neq j$, contradicting the assumption that $M_1 \models T^{\forall}$, or $\bar{a}_i \not\subseteq B$ for some $i \neq j$, in which case $\bar{a}_i \cap B = \emptyset$ (because $B \leq M_2$ and $\bar{b}'\bar{d}' \subseteq \mathrm{acl}(B) = B$ by Claims 3 and 4 respectively), contradicting the assumption that there were no such $\bar{a}_i$ in $M_2 \setminus B$.

This completes the proof of the lemma. □

Lemma 13. *T is complete and almost strongly minimal.*

We will use the following claim:

Claim 11. *If $\mathcal{U} \models T$, then for any $a \in \mathcal{U}$ and any small sets $A \subseteq B \subseteq \mathcal{U}$, the relation $\Gamma(a, A, B) =_{\mathrm{def}}$ "$d(a/B) = d(a/A)$" is an independence relation equivalent to non-forking.*

Proof. (1) Clearly, Γ is invariant under automorphisms of $\mathcal{U}$, and by definition of $d(X/Y)$ is of finite character (i.e. $(a, A, B) \in \Gamma$ iff for any finite tuple $b \subseteq B$ $(a, A, A \cup \{b\}) \in \Gamma$).

(2) Again, by the definition of $d(X/Y)$, we get that for any a and B, there exists a finite $A \subseteq B$ such that $(a, A, B) \in \Gamma$.

(3) For any a, A and $B \supseteq A$, there exists an a' with $tp(a/A) = tp(a'/A)$, such that $(a', A, B) \in \Gamma$. If $d(a/A) = 0$ there is nothing to do. Otherwise there are two possibilities: if $a \in \mathcal{U}_{\mathcal{D}}$ then a has no R- relations with any element of A, and all we have to do is find a' which has no R-relations with B, which is easy by the AE-axioms; otherwise take a non-forking extension of $tp(a/A_{\mathcal{B}})$ to $B_{\mathcal{B}}$ which has no other R-relations with B save those it has with A, which is again easy — take a set D of elements in $\mathcal{U}_{\mathcal{D}}$ such that $cl(DB) = cl(B) \cup D$; then add R-relations between a and D to make $a/cl(BD)$ an instance of an f-code, and it is clear that it must be realized in $\mathcal{U}$.
(4) $(a, A, Ab) \in \Gamma \iff (b, A, Aa) \in \Gamma$, we know that $d(ab/A) = d(a/Ab) + d(b/A) = d(b/Aa) + d(a/A)$, so if $d(a/Ab) = d(a/A)$ the claim follows.
(5) If $A \subseteq B \subseteq C$, then $(a, B, C) \in \Gamma \wedge (a, A, B) \in \Gamma$ iff $(a, A, C) \in \Gamma$: the $\Rightarrow$ is trivial; the other direction follows directly from the definition.

The next step is to show that if $\bar{a} \subseteq \mathcal{U}$ and $A = \mathrm{acl}(A)$, then for any set $B \supseteq A$, there is a unique Γ-non-forking extension of $p = tp(\bar{a}/A)$ to B. Because enlarging B only makes our task harder, we may assume that $B \leq \mathcal{U}$. Denote $\bar{a}' := cl(A\bar{a}) \setminus A$. Denote also $q = tp(\bar{a}'/A)$, and choose $\phi(x, y, a) \in q \restriction_{L_0}$ stationary of minimal MR. Set $\chi(\bar{x}, \bar{y}, \bar{b}) := \bigwedge\{qftp_R(\bar{a}'/A))\}$, where in both χ and ϕ the variable $\bar{x}$ stands for variables corresponding to $\bar{a}$, and $\bar{y}$ stands for variables form $\bar{a}' \setminus \bar{a}$. Hence

$$\varphi(\bar{x}) := (\exists \bar{y})(\phi(\bar{x}, \bar{y}, \bar{a}) \wedge \chi(\bar{x}, \bar{y}, \bar{b})) \in tp(\bar{a}/A).$$

Clearly, $d(\bar{e}/B) \leq d(\bar{a}/A)$ for any $\bar{e} \models \varphi(\bar{x})$, and equality holds iff $\bar{e}$ is Γ-non-forking with B over A, and $d(\bar{e}/A) = d(\bar{a}/A)$. Choose such an $\bar{e}$ and $\bar{e}'$ containing the existential quantifier in the definition of φ. By the choice of $\bar{e}$, it follows $cl(B\bar{e}) = B\bar{e}\bar{e}'$, i.e. $B\bar{e}\bar{e}' \leq \mathcal{U}$. By the choice of ϕ, for any $\bar{e}_1, \bar{e}_2 \models \varphi\Gamma$-generic over B, there is a partial isomorphism $\bar{e}_1 \mapsto \bar{e}_2$, extending the identity on B and extendable to $cl(B\bar{e}_1)$. But since $cl(B\bar{e}_i) \leq \mathcal{U}$, this partial isomorphism extends to an automorphism of $\mathcal{U}$. Because Γ-non-forking is described by a unique (a priori, partial) type, the desired conclusion follows.

This implies that T is stable and Γ-non-forking is equivalent to non-forking. Moreover, this shows that T is ω-stable (because every 1-type has d-rank, and is isolated in its rank) and the d-rank is MR, and in particular $d(a/B) = 0$ iff $a \in \mathrm{acl}(B)$ (one direction was proved in Lemma 11, and the other direction follows from the uniqueness of non-forking extensions over cl-closed sets). □

Thus, for a generic $d \in \mathcal{D}$, $d(d) = 1$, and by what we have just shown, this implies that MR(d) = 1. For an arbitrary $M \models T$, take a d-base consisting only of elements of $\mathcal{D}$ – which can be done by Corollary 10 – then by Lemma 12, any two elements of this base have the same type. In particular, we get that $\mathcal{D}$ is strongly minimal, and by Corollary 10 T is almost strongly minimal.

Lemma 14. *Let $N \models T$, and denote N_0 the union of all the L_0-sorts in N, then $N_0 \models T_0$.*

Proof. We may assume that N is saturated. Let M be a big saturated model of T_0. As $N_0 \models T_0^{\forall}$, we may assume, by changing names, that N_0 is a substructure of M. We show that $N_0 \prec M$. Let $\varphi(x, \bar{y})$ be any formula in $L(T_0)$ (so, wlog, quantifier free), $\bar{b} \in N_0$, and assume that $M \models \varphi(a, \bar{b})$ for some $a \in M$. If $a \in N_0$ we are done, otherwise $a \notin \mathrm{acl}_{L_0}(\bar{b})$, and we may take a to be a generic realization of $\varphi(x, \bar{b})$. By possibly strengthening $\varphi(x, \bar{b})$ (and since $N_0 = \mathrm{acl}_{L_0}(N_0)$), we may assume that $\varphi(x, \bar{b})$ is stationary. Let $n = \mathrm{MR}_{L_0}(a/\bar{b})$. Choose $d_1, \ldots, d_n \in \mathcal{D}^N$ generic over $\bar{b}$, and consider the formula $\varphi(x, \bar{b}) \wedge \bigwedge_{i=1}^n R(x, d_i)$. Let $N_1 = N \cup \{a'\}$ be such that $N_1 \models \varphi(a', \bar{b}) \wedge \bigwedge_{i=1}^n R(a', d_i)$, and such that a' realizes the generic L_0-type of $\varphi(x, \bar{b})$, and does not realize any R relations with elements of N except those explicitly specified above. Then $d_0(a'/N) = 0$; so by Lemma 4 a' realizes some f-code $c(\bar{b}'\bar{d}')$ for some $\bar{b}', \bar{d}' \in N$. However since necessarily $\bar{d} = \bar{d}'$, it must be that $\varphi_{\hat{c}}(x, \bar{b}') \sim \varphi(x, \bar{b})$, so any generic realization of $c(\bar{b}'\bar{d}')$ also realizes $\varphi(x, \bar{b})$, and vice versa. Finally, since x is a singleton, it must be that $N \models (\exists x)\, c(\bar{b}'\bar{d}')$ (because clause (b) of Axiom AE(2) cannot be true). By what we have just shown, if $a \in (c(\bar{b}'\bar{d}') \wedge \neg\varphi(x, b))^N$, then $d_0(a/\bar{b}'\bar{d}'\bar{b}) < 0$. Making sure that $d_1, \ldots, d_n \in N$ were chosen to be independent generics (in the sense of T) over $\bar{b}', \bar{b}$, we get that $d_0(a/\bar{b}'\bar{d}'\bar{b}) = 0$, and therefore $N \models (\exists x)\varphi(x, \bar{b})$, implying that $N_0 \models (\exists x)\, \varphi(x, \bar{b})$, which proves $N_0 \prec M$, as required. □

To conclude the proof of the main part of the theorem, we now take $T_{\mathcal{D}} = Th(\mathcal{D})$, to obtain a strongly minimal theory interpreting T_0. □

Corollary 15. *Every $\aleph_0$-categorical ω-stable theory of finite rank is interpretable in a strongly minimal theory.*

Proof. Definability of the Morley rank and degree comes from $\aleph_0$-categoricity. □

Lemma 16. *T has DMP.*

Proof. From [2], we know that it is enough to prove DMP for strongly minimal subsets. So let $\varphi(\bar{x}, \bar{b})$ be strongly minimal, and let $\bar{a} \models \varphi(\bar{x}, \bar{b})$ be a generic realization. Since $\varphi(\bar{x}, \bar{b})$ is stationary, it has a unique non-forking extension to $cl(\bar{b})$, and as $cl(\bar{b})$ is finite, we may as well assume that $\bar{b} = cl(\bar{b})$. Moreover, if $a \in \bar{a}$ and $b' \in \mathrm{acl}(\bar{b})$ is such that $\models R(a, b')$, then $b' \in dcl(\bar{b})$ (by stationarity of $\varphi(\bar{x}, \bar{b})$), so we may also assume that $\models \neg R(a, b')$ for all $a \in \bar{a}$ and $b' \in \mathrm{acl}(\bar{b}) \setminus \bar{b}$. Denote $A = cl(\bar{a}\bar{b}) \setminus \bar{b}$ and $p = qftp(A/\bar{b})$ (the quantifier free type of A/b). Let $p_0(\bar{x}, \bar{y}, \bar{b}) = p \restriction_{L_0}$ and $p_R(\bar{x}, \bar{y}, \bar{b}) = p \restriction_R$, where $\bar{y}$ are variables corresponding to $A \setminus \bar{a}$, and let $\psi_0 \in p_0$ be of minimal rank and degree, and $\psi_R = \bigwedge_{\varphi \in p_R} \varphi$ Choose $\theta_0(\bar{u})$ such that for all $\bar{b}' \models \theta_0$:

- $rk\{(\bar{x}, \bar{y}) : \psi_0(\bar{x}, \bar{y}, \bar{b}')\} = rk\{(\bar{x}, \bar{y}) : \psi_0(\bar{x}, \bar{y}, \bar{b})\}$.
- For generic $\bar{x}, \bar{y} \models \psi_0(\bar{x}, \bar{y}, \bar{b}')$, $rk\{\bar{y}' : \psi_0(\bar{x}, \bar{y}', \bar{b}')\} = rk\{\bar{y}' : \psi_0(\bar{x}, \bar{y}', \bar{b})\}$.
- $rk\{\bar{x} : (\exists \bar{y})\psi_0(\bar{x}, \bar{y}, \bar{b}')\} = rk\{\bar{x} : (\exists \bar{y})\psi_0(\bar{x}, \bar{y}, \bar{b})\}$.
- The same for multiplicities in L_0.

We will finish if we show:

Claim. *Denote $\theta(\bar{u}) := \theta_0(\bar{u}) \wedge (\exists \bar{x}, \bar{y})(\psi_0(\bar{x}, \bar{y}, \bar{u}) \wedge \psi_R(\bar{x}, \bar{y}, \bar{u}))$, then for all $\bar{b} \models \theta_0$, $\varphi(\bar{x}, \bar{b}')$ is strongly minimal.*

Proof. Let $\bar{b}'$ be as above; it is clear that $rk\{\varphi(\bar{x}, \bar{b}')\} \leq 1$, and thus our first goal is to show that $rk\{\varphi(\bar{x}, \bar{b}')\} \geq 1$. Consider the following structure $C \in \mathcal{C}_0$: $C = C_1 \cup C_2 \cup C_3$, where the C_i's are disjoint, satisfying: $C_1 \cong \bar{b}'$, $C_2 \cup C_1 \cong cl^M(\bar{b}')$, $C \models \psi_0(C_3, C_1) \wedge \psi_R(C_3, C_1)$, $C_3 \mathop{\smile\!\!\!\!\!\!|}\,_{C_1}^{L_0} C_1 \cup C_2$, and there are no R-relations between C_3 and C_2 (since $\bar{b}' \models \theta_0$ there exists C_3 as required). Then $C_1 \cup C_2 \leq C$, and there is an embedding $f : C \hookrightarrow M$ such that $f(C_1 \cup C_2) = cl(\bar{b}')$ and $f(C) \leq M$. Since $d_0(C_3/C_1 \cup C_2) = 1$, we get that $d_0(f(C_3)/cl(\bar{b}')) = 1$, and if we take $\bar{a}' \subseteq C_3$ as the part corresponding to the variable $\bar{x}$, we get $rk\{\varphi(\bar{x}, \bar{b}')\} = 1$.

It remains to show that $\varphi(\bar{x}, \bar{b}')$ is stationary. Let $\bar{a}_1, \bar{a}_2 \models \varphi(\bar{x}, \bar{b}')$ be generic; by our choice of θ_0, $(\exists \bar{y})\ \psi_0(\bar{x}, \bar{y}, \bar{b}')$ is stationary in L_0, and by the genericity of $\bar{a}_i$ we get that $tp_{L_0}(\bar{a}_1/\bar{b}') = tp_{L_0}(\bar{a}_2/\bar{b}')$. By stationarity $stp_{L_0}(\bar{a}_1/\bar{b}') = stp_{L_0}(\bar{a}_2/\bar{b}')$, and again by genericity (in $L_0 \cup \{R\}$) this implies $tp_{L_0}(\bar{a}_1/\mathrm{acl}(\bar{b}')) = tp_{L_0}(\bar{a}_2/\mathrm{acl}(\bar{b}'))$ (taking acl in L not in L_0 this time). On the other hand, $\bar{a}_i$ cannot have any R-relations with elements in $\mathrm{acl}(\bar{a}_i\bar{b}')$ other than those explicitly appearing in $\psi_R(\bar{x}, \bar{y}, \bar{b}')$, since such relations will make $\bar{a}_i$ algebraic over $\bar{b}'$, contradicting the genericity of $\bar{a}_i$. Thus a map f extending the identity on $\mathrm{acl}(\bar{b}')$ and taking $\bar{a}_1$ to $\bar{a}_2$ is elementary (being elementary in L_0 and $\{R\}$). It remains to check that such a mapping exists, and that such f can be extended to $cl(\bar{a}_i \cup \mathrm{acl}(\bar{b}'))$, since then it would be extendable to an automorphism of M. We already know that such an L_0-elementary map exists, and we have to make sure that this map can be taken to be R-elementary as well. Recall that by our assumption, if $a \in \bar{a}_i$ and $b \in \mathrm{acl}(\bar{b}')$ is such that $\models R(a, b)$, then $b \in \bar{b}'$, so there is in fact an L-elementary map taking $\bar{a}_1$ to $\bar{a}_2$ over $\mathrm{acl}(\bar{b}')$. Now $C = cl(\bar{a}_i \cup \mathrm{acl}(\bar{b}')) = \bar{a}_i \cup \mathrm{acl}(\bar{b}')$, since $d_0(\bar{a}_i/\mathrm{acl}(b')) = d(\bar{a}_i/\mathrm{acl}(b'))$ by genericity of a_i (otherwise we would get that $rk\{\varphi(\bar{x}, \bar{b}')\} = 0$, contradicting what we have already proved. This follows from the fact that θ was modeled after $qftp(A/\bar{b})$, where $A = cl(\bar{a}\bar{b}) \setminus \bar{a}$, and therefore $d_0(A/\bar{b}) = d(A/b) = 1$). Hence, $C = \mathrm{acl}(b') \cup \{a_i\}$, and the claim is proved. □

This concludes the proof of the lemma. □

2. Generalizations and open questions

As a first corollary we show the following:

Corollary 17. *Let T_1, T_2 be theories of finite MR, such that both have DMP and definable MR; then there is a strongly minimal theory T interpreting both T_1 and T_2.*

Proof. Use the proposition to find strongly minimal theories T_1', T_2' interpreting T_1 and T_2 respectively. By the last lemma, we can find such T_i' with DMP, so T_1', T_2' can be fused to obtain a strongly minimal theory T interpreting both. □

Now we note that by practically the same proof we can obtain the following:

Theorem 18. *Let D_1, D_2 be strongly minimal structures with DMP; then for all natural numbers $n, m > 0$, there exists an almost strongly minimal structure D such that:*

(1) *There is a rank preserving interpretation of both D_1, D_2 in D.*
(2) *$D_1 \cup D_2 = D$ and $D_1 \cap D_2 = \emptyset$.*
(3) *There exists a zero definable correspondence $f \subseteq D_1 \times D_2$.*
(4) *f is $n : m$.*

Proof. First we may assume without loss of generality that $L(D_1) \cap L(D_2) = \emptyset$, $T(D_i)$ have QE and EOI, as in the main theorem. Now let $M \models T_i$ be countable saturated models. Define $\mathcal{B}_i = \{B \models T_0^\forall : B \text{ finite}\}$ (this is not a trivial class, as the language is assumed to have no function symbols. Thus any finite subset of M_i has a representative in $\mathcal{B}_i$). For $B \in \mathcal{B}_i$, define $d_i(B)$ as before. Consider the family of 2-sorted structures $\mathcal{C} = \{(B_1, B_2, R) : B_i \in \mathcal{B}_i, R \subseteq B_1 \times B_2\}$, and for $C = (C_{\mathcal{B}_1}, C_{\mathcal{B}_2}, C_R) \in \mathcal{C}$, define $d_0(C) = d_1(C_{\mathcal{B}_1}) + d_2(C_{\mathcal{B}_2}) - |C_R|$.

For $C, C' \in \mathcal{C}$, let $C \subseteq C'$ denote that C is a substructure of C' (i.e. $C_{\mathcal{B}_i} \subseteq (C')_{\mathcal{B}_i}$, $C_R \subseteq (C')_R$). Now we define $d(\bar{a}/B)$, $C_1 \leq C_2$, $\mathcal{C}_0$, cl_0 etc. exactly as above, and everything goes unaltered up to the definition of f-codes, where we need the following adjustments:

Definition 10. Let T_1, T_2 be theories as above, and let R be a new binary relation symbol defined on couples (t_1, t_2) of elements, where t_i is in one of the sorts of T_i. A 2-f-code c in $L(T_1) \cup L(T_2) \cup \{R\}$ contains the following information:

(1) A pair of standard codes $\hat{c}_i$ in $L(T_i)$;
(2) A natural number $n_c = n_{\hat{c}_1} + n_{\hat{c}_2}$;
(3) An R-formula χ of the form:
$$\chi(\bar{x}_1, \bar{x}_2, \bar{w}) = \bigwedge_{i,j} R(x_{1,i}, x_{2,j})^{\epsilon_{i,j}} \bigwedge_{i,j} R(x_{1,i}, w_j)^{\epsilon_{i,j}} \bigwedge_{i,j} R(x_{2,i}, w_j)^{\epsilon_{i,j}}$$
where in each case (i, j) ranges over all the possibilities;
(4) Formulae $\eta_i(\bar{x}_i)$ stating that no two of the variables of $\bar{x}_i$ are equal;
(5) and such that:
- The variables $\bar{x}_i$ are of the sorts of T_i, and the variables of $\bar{w}$ are from either sort.
- Let s denote the number of positive literals in χ; then $k_{\hat{c}_1} + k_{\hat{c}_2} = s$.
- For each $z \in \bar{w}$, denote by s_z the number of positive literals in χ in which z appears. Then $s_z \geq 1$.
- For $\bar{b}_i \models \theta_{\hat{c}_i}(\bar{u}_i)$ $(i = 1, 2)$, and for every partition pr of the variables of $\bar{x}_1 = \bar{x}_1^1{}^\frown\bar{x}_1^2$ and $\bar{x}_2 = \bar{x}_2^1{}^\frown\bar{x}_2^2$, let s_{pr} denote the number of positive literals in χ in the variables $\bar{x}_1^2$, $\bar{x}_2^2$ and $\bar{w}$. Then $rk_{L_1}\{\bar{x}_1^2 : \varphi_{\hat{c}}(\bar{x}_1^1{}^\frown\bar{x}_1^2, \bar{b}_1)\} + rk_{L_2}\{\bar{x}_2^2 : \varphi_{\hat{c}}(\bar{x}_2^1{}^\frown\bar{x}_2^2, \bar{b}_2)\} < s_{pr}$.

$(c, \bar{b}_1, b_2, \bar{d})$ will be called a 2-f-code for $C = \{(\bar{x}_1, \bar{x}_2) : \varphi_1(\bar{x}_1, \bar{b}_1) \wedge \varphi_2(\bar{x}_2, \bar{b}_2) \wedge \chi(\bar{x}_1, \bar{x}_2, \bar{d})\}$.

Now everything goes through trivially (modulo the amendments implied by the definition of the codes) up to the point where we introduce the theory T (for example, in the first part of the equivalent of Claim 3, we will have $d_0(\bar{a}/B) = d_0(\bar{a}B) - d_0(B) = d_1(\bar{a}/B) + d_2(\bar{a}/B) - |(\bar{a}B)_R| + |B_R|$, and from this point on the proof goes unaltered. Note that we have abused the notation and written $d_1(\bar{B})$ where it should have been $d_1(B_{\mathcal{B}_1})$, but there is no ambiguity). Now we define the theory T exactly as in the main theorem save that :

(1) We add to $T^\forall$ the axiom T_R, stating that R is (at most) $n : m$.
(2) We replace AE-axiom (2) by an axiom stating that R is exactly $n : m$.
(3) In AE-axiom (2), we have to add a condition stating that given $(\bar{u}_1, \bar{u}_2, \bar{w})$ there is no realization of $(c, u_1, u_2, \bar{w})$ if such a realization contradicts T_R (as T_R is a single axiom this new condition we add can be written in a single formula as well).

This implies a slight change in the statement of Claim 8 — we have to add a fourth option, namely that given $\bar{a}, N, M$ as in the claim, N may not be a model of $T^\forall$ if $N \not\models T_R$. But this happens iff there are $a'_1, \ldots, a'_k$ not all in M contradicting T_R, which is exactly the new condition we added to AE-axiom (2), and the proof of Lemma 8 goes through easily once we show that indeed R is $n : m$, which by now should be routine (if not, take $a \in M$, which does not have mR-images, and look at the structure $N = M \cup \{b\}$ where b is a generic (over M) of $T(D_2)$, $N \models R(b, a)$, but b does not satisfy any other R relations. By assumption, $N \models T_R$, and exactly as in the main theorem, it follows that $N \models T^\forall$ and $N \geq M$. Take $\{b_1, \ldots, b_{m'}\} = R(a, y)^M$ and $m' < m$ then $R(a, y) \wedge \bigwedge_{i \leq m'} y \neq b_i$ is satisfied in N, but not in M — a contradiction).

We skip Corollary 10, which is irrelevant to the present situation and as above, modulo the trivial modifications, Lemma 11 goes through easily as well: e.g. the formulation of the lemma in the present situation will be

Lemma 19. (1) *Let M be a model of $T^\forall$ which is algebraically closed in the sense of $L(D_1)$ and $L(D_2)$. Let $\varphi_i(\bar{x}_i, \bar{b}_i)$ be formulae in $L_i(M)$, and let $\chi(\bar{x}_1, x_2, \bar{d})$ be an R-formula over M such that $\varphi_i(\bar{x}_i, \bar{a}_i)$ has rank k_i (in the sense of T_i) and χ has s positive literals. If $k_1 + k_2 + s \leq 0$, then there are only a finite number of disjoint solutions of $\varphi_1(\bar{x}_1, \bar{a}) \wedge \varphi_2(\bar{x}_2, \bar{a}) \wedge \chi(\bar{x}_1, \bar{x}_2, \bar{d})$ in M.*

(2) *Let M be as above, c a 2-f-code, and $\bar{a}_i, \bar{d} \in M$ with $M \models \theta_{\hat{c}_i}(\bar{a}_i) \wedge (\exists \bar{x})\chi_c(\bar{x}, \bar{d})$; then there are only a finite number of solutions in M to $\varphi_{\hat{c}_1}(\bar{x}_1, \bar{a}_1) \wedge \varphi_{\hat{c}_2}(\bar{x}_2, \bar{a}_2) \wedge \chi_c(\bar{x}_1, \bar{x}_2, \bar{d})$.*

And the modifications to the proof are clear, and go through smoothly. The proof of Lemma 12 requires, besides the by now standard modifications, another remark: recall that the proof of Lemma 12 divides into three cases according to the cases occurring in AE-axiom (2), so having added a fourth case to that axiom, a fourth case has to be dealt with here as well — but this fourth case obviously cannot occur (because given a generic solution $\bar{a}$ to a 2-f-code $(c, \bar{b}_1, \bar{b}_2)$ a has no R-relations outside $\bar{a} \cup \bar{b}_1 \cup \bar{b}_2$).

To show almost strong minimality, it will be enough to show that, say, D_1 is strongly minimal. Towards this end, note that the relation $d(a/B) = 0$ defines a dependence relation (which by strong minimality of D_i – in L_i – is symmetric) corresponding to algebraic dependence. Now the result follows from Lemma 12 (adapted to the present situation as described above) and the following general lemma, which is the last claim in the very last corollary of [2]:

Lemma 20. *Let T be a theory such that:*

(1) *T has no finite models.*
(2) *Algebraic closure is a dependence relation on every model of T.*
(3) *Any bijection between transcendence bases of models of T extends to an isomorphism of models.*

Then T is complete and strongly minimal.

This completes the proof of Theorem 18. □

Remark 3. The proof of Theorem 18 goes through unaltered if D_1 and D_2 are of rank k, have DMP and definable MR.

As we have stated in the introduction, we do not know whether wDMP and the definability of MR are sufficient conditions for a theory of finite rank to be interpretable in a strongly minimal theory. We conclude with a few observations concerning this question.

At first we remark that DMP is certainly not a necessary condition, for a trivial reason:

Example 4. Consider the theory T of a single infinite equivalence relation E with infinite classes, countably many of which are split into two infinite sets by relations $\{R_i\}_{i=1}^{\infty}$. Then T does not have DMP (as shown by $x = x$), but has a DMP (rank preserving) expansion (add a new relation R which splits each E class into two, and axioms stating that R corresponds with R_i for all i). Thus to interpret T in a strongly minimal theory, we apply our main theorem to the DMP expansion and then reduce back to the original language.

So, obviously, reducts of theories with DMP can always be fused. In order to find a theory which has wDMP but is not a reduct of a DMP theory, we note that the above argument has a simple generalization. Suppose that for every standard code c in a theory of finite MR with wDMP and def. MR such that $\max\{\text{mult}\{\varphi_c(x, b)\} : b \models \psi_c\} = l_c$, we can add to the language $L(T)$ a new equivalence relation E_c with $l!$ infinite classes such that:

- The domain of E is $\bigcup\{\varphi_c(x, b)^2 : b \models \psi_c\}$.
- If $b \models \psi$ and $\chi(x, b)$ splits $\varphi_c(x, b)$ into $r \leq l$ parts, then each of these is the union of $\frac{l!}{r}$ distinct E_c classes.

Then in the enriched language, T will have DMP as in the above example. However, our next example shows that this strategy cannot always work:

Example 5. Consider the following 2-sorted theory T_g:

(1) D is an Abelian divisible torsion-free group with distinguished elements $\{d_i, d_i'\}_{i<\omega}$.
(2) The d_i and d_i' are independent generic elements of D.
(3) $f : C \to D$ is an atomic function, with all fibers infinite.
(4) For each i, there exists a ternary relation $+_i$ which endows $f^{-1}(d_i')$ with a structure of an Abelian divisible torsion-free group.
(5) For each i, there exists an infinite unary relation R_i which splits $f^{-1}(d_i)$ into two infinite sets (and R_i holds only of elements in $f^{-1}(d_i)$).

Let $M \models T_g$ be a saturated model, and let $A_i \leq M$ $(i = 1, 2)$ be small substructures; then any elementary map $g : A_1 \to A_2$ extends to an automorphism of M. Towards this end, complete $A_i \cap D^M$ to a base B_i for D as a vector space over $\mathbb{Q}$. Clearly, if $\bar{g} : (B_1 \setminus A_1) \to (B_2 \setminus A_2)$ is any bijection, then $\bar{g} \cup g$ is elementary, and can be extended linearly to the whole of D^M, and we may therefore assume that $A_i \cap D^M = D^M$. Now assume that f has been extended to substructures B_1, B_2, and we want to extend it to some $b \in M \setminus B_1$. If $f(b) = d_i'$ for some i, then $b \notin \operatorname{span}_{\mathbb{Q}}(B_1 \cap f^{-1}(d_i'))$. Since the B_i are small, there is an element $b' \in f^{-1}(d_i') \setminus B_2$, and we extend g by defining $g(b) = b'$. Extending g linearly to $\operatorname{span}(bB_1 \cap f^{-1}(d_i'))$, we get back to the induction hypothesis, and may proceed by back-and-forth. The possibility that $f(b) = d_i$ for some i, or that $f(b) \notin \{d_i, d_i'\}_{i=1}^{\infty}$, are treated in the same way.

Consequently we get that $\mathrm{MR}(D) = 1$ (choose $A_1 = \operatorname{span}_{\mathbb{Q}}\{d_i, d_i', b\}_{i=1}^{\infty}$ and $A_1 = \operatorname{span}_{\mathbb{Q}}\{d_i, d_i', b'\}_{i=1}^{\infty}$ with $g : A_1 \to A_2\mathbb{Q}$-linear satisfying $f(b) = b'$), and so are all the f-fibers, except those over $\{d_i\}_{i=1}^{\infty}$, which are all of multiplicity 2.

A closer inspection of the above argument will show that we have also proved:

Claim 12. *Let $L_0 = L \setminus \{R_i\}_{i=1}^{\infty}$, $T_g' = T_g \restriction_{L_0}$, and $M \models T_g'$. For all $A \subseteq M$, $\operatorname{acl}_{T_{g'}}(A) = dcl_{T_g'}(A)$.*

Claim 13. *T_g has definable MR.*

Proof. First we note that it will be enough to show that formulae of the form $\varphi(x, \bar{y})$ have definable MR, where x is a singleton, and $\bar{y}$ of arbitrary length. To see this, consider $\varphi(x_1, \ldots, x_n, \bar{b})$. Denote $\hat{x} = (x_2, \ldots, x_n)$, and for $\bar{a} \models \varphi(\bar{x}, \bar{b})$ denote $\hat{a}$ accordingly. Take $\bar{a}$ generic in $\varphi(\bar{x}, \bar{b})$. By assumption, there is a formula $\psi(\hat{x}, y)$ defining MR for $\varphi(x_1, \hat{a}, b)$. By induction there is a formula $\chi(y)$ defining MR for $\psi(\hat{x}, b)$. Note that $\hat{a}$ is a generic of $\psi(\hat{x}, \bar{b})$ (for if $\hat{e} \models \psi(\hat{x}, \bar{b})$ and $\operatorname{rank}(\hat{e}/\bar{b}) > \operatorname{rank}(\hat{a}/\bar{b})$; then $\operatorname{rank}\{\varphi(x_1, \hat{e}, b)\} = \operatorname{rank}\{\varphi(x_1, \hat{a}, b)\}$ – as both parameters satisfy ψ – contradicting the choice of $\bar{a}$ as generic of $\varphi(\bar{x}, \bar{b})$). Thus, χ is a definition of MR for φ.

But now the claim follows trivially, for if $\varphi(x, \bar{b})$ is a formula as above, with the unique free variable x, there are 3 options:

- $\varphi(x, \bar{b})$ is algebraic — and the requirement follows easily from the previous claim (the multiplicity of $\varphi(x, \bar{b})$ is bounded by 2 for all $\bar{b}$).
- $\operatorname{rank}\{\varphi(x, \bar{b})\} = 2$, which implies that x is a generic of C, independent from $\bar{b}$. So φ asserts that x has no relations with $\bar{b}$, and therefore the formula $\bar{y} = \bar{y}$ is a definition of MR for φ (because each of the $\varphi_{i,j}$ appears with a negative sign, and has rank at most 1).
- $\operatorname{rank}\{\varphi(x, \bar{b})\} = 1$, which means that either x is a generic of D (in which case, as above, the formula $\bar{y} = \bar{y}$ has the desired property), or x is a generic of a fiber of f, algebraic over $\bar{b}$. In the latter case, if x is a generic of a generic fiber, again we have nothing to do. Otherwise, it is in a fiber algebraic over the special points of D, and the formula showing this will be enough. □

Corollary 21. *T_g has wDMP.*

Proof. For each $i \in \mathbb{N}$, let $E_i(x, y) := f(x) = f(y) \wedge (R_i(x) \leftrightarrow R_i(y))$. Expand T to T' by adding imaginary elements for E_i-classes. Using the same methods as in the previous claims, it is easy to check that for every $\varphi(\bar{x}, \bar{y})$ and every $\bar{b}$, $\operatorname{mult}\{\varphi(\bar{x}, \bar{b})\} \leq 2^n$ for $n = \operatorname{length}(\bar{x})$, since T has definable MR so does T', and to show wDMP it will be enough to check that T' has weak elimination of imaginaries. Towards this end, it will suffice to show that if $B \supseteq A = \operatorname{acl}(A)$ (where acl is taken in T, not in T^{eq}, of course) and $p \in S(A)$, then p has a unique non-forking extension to B, which by now should already be trivial. □

Claim 14. *T_g has no MR preserving DMP expansion.*

Proof. Assume for contradiction that $T' \supseteq T$ is an expansion with DMP, and consider the formula $\varphi(x, y) =$ "$y = f(x)$". Because D is a divisible group, D is strongly minimal in T', and therefore has a unique generic type. We now consider two cases:

(1) For a generic $b \in D$, $\text{mult}\{\varphi(x, b)\} = 1$. Then by DMP there is a formula $\psi \in tp(b)$ such that for all $b' \models \psi$, $\text{mult}\{\varphi(x, b')) = 1$. In particular, ψ holds of all generic elements of D, and therefore is infinite. On the other hand, because $\text{mult}\{\varphi(x, d_i)\} \geq 2$ for all i, clearly $\neg\psi$ is infinite, contradicting the strong minimality of D.
(2) For a generic $b \in D$, $\text{mult}\{\varphi(x, b)\} = m > 1$. The exact same argument holds, taking d'_i instead of d_i (using the fact that $\text{mult}\{\varphi(x, d'_i)\} = 1$ for all i). □

Problem 1. Does the theory T_g have a rank preserving interpretation in a strongly minimal theory?

The following gives a partial answer:

Claim 15. *The theory T_g of the above example has an interpretation in an almost strongly minimal theory.*

Proof. We add a new function $g : C \to D$ to the language such that:

(1) g is $2:1$ on every fiber of f.
(2) $g(x_1) = g(x_2) = d_i \to \neg R_i(x_1, x_2)$.
(3) The resulting theory T_g is almost strongly minimal.

To do this, we first add the function g for $T_0 =_{\text{def}} T \restriction_{L \setminus \{R_i\}}$. T_0 has DMP and therefore we can, very much like in the proof of Theorem 18, add a new binary relation symbol $R(x, y)$, which admits elements from C in the first argument and elements of D in the second. We add the axiom T_R:

$$(\forall x_1, x_2, x_3, y) \left(\bigwedge_{i \neq j} f(x_i) = f(x_j) \wedge \bigwedge_i R(x_i, y) \to \bigvee_{i \neq j} x_i = x_j \right).$$

For a finite structure $A \models T_0^{\forall} \cup T_R$ we define $d_0(A) = \text{MR}_{T_0}(A) - |R^A|$. It is now routine to check that the construction of Theorem 18 goes through, and that the resulting theory, T_1, is almost strongly minimal. Now expand T_1 by adding new binary relations $\{R_i\}_{i=1}^{\infty}$ such that:

- $R_i(x_1, x_2) \to f(x_1) = f(x_2) = d_i$.
- R_i is an equivalence relation with two infinite classes.
- For all j, an axiom: $\bigwedge_{i=1}^{2} R(x_i, y) \wedge f(x_i) = d_j \to \neg R_j(x_1, x_2)$.

And it is straightforward to check that the resulting theory is almost strongly minimal, or more precisely that the induced structure on D remains strongly minimal. To see this, denote L the restricted language – i.e. without the R_i – and assume that $\models R(a_1, b) \wedge R(a_2, b) \wedge f(a_1) = d_i$; then for all A such that $R^{-1}(A) \cap \{a_1, a_2\} = \emptyset$, $tp_L(a_1/A) = tp_L(a_2/A)$. Thus for any set of parameters A, any $a_1, a_2 \in D$ that are generic over A are still conjugate over A. To see this, construct by back-and-forth an L-automorphism $\sigma \in \text{Aut}(M/A)$ with $\sigma(a_1) = a_2$. By the last observation, σ can be modified into an automorphism in the expanded language. □

So we ask:

Problem 2. Does any almost strongly minimal theory have a rank preserving interpretation in a strongly minimal theory?

And a possibly easier question:

Problem 3. Is there a strongly minimal theory which does not have a rank preserving DMP expansion?

Acknowledgements

The results presented here are part of my Ph.D. thesis. I am very grateful to my supervisor, Ehud Hrushovski, for his invaluable help, comments and suggestions, and more particularly for many ideas of his that have made their way into this paper.

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