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Solving two-parameter eigenvalue problems using an alternating method

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ABSTRACT

We present a new approach to compute selected eigenvalues and eigenvectors of the two-parameter eigenvalue problem. Our method requires computing generalized eigenvalue problems of the same size as the matrices of the initial two-parameter eigenvalue problem. The method is applicable for right definite problems, possibly after performing an affine transformation. This includes a class of Helmholtz equations when separation of variables is applied. We provide a convergence proof for extremal eigenvalues and empirical evidence along with a local convergence proof for other eigenvalues.

1. Introduction

In this work we consider the *two-parameter eigenvalue problem*

$$\begin{aligned}(A_1 + \lambda B_1 + \mu C_1)u &= 0,\\ (A_2 + \lambda B_2 + \mu C_2)v &= 0\end{aligned} \tag{1}$$

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with matrices $A_1, B_1, C_1 \in \mathbb{R}^{n\times n}$ and $A_2, B_2, C_2 \in \mathbb{R}^{m\times m}$. A solution to this problem is given by possibly complex (λ, μ, u, v) if they fulfill (1) and $u, v \neq 0$. We call the pair (λ, μ) an *eigenvalue of the two-parameter eigenvalue problem* if $u, v \neq 0$ exist such that (λ, μ, u, v) satisfies (1) and we call the tensor product $u \otimes v$ an *eigenvector of the two parameter eigenvalue problem* (in the literature the pair (u, v) is often referred to as the eigenvector; the terminology here is simply for convenience).

Two-parameter eigenvalue problems have been extensively studied [2,26] and naturally arise in mathematical physics when separation of variables is applied. Consider for example the Helmholtz equation

$$\begin{aligned} \Delta u + \lambda u = 0 \quad &\text{in } \Omega, \\ u = 0 \quad &\text{on } \partial\Omega \end{aligned}$$

where $\Omega = \{(x, y) \in \mathbb{R}^2 : \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 < 1, y > 0\}$ is half of an open ellipse. Using elliptical coordinates, the problem can be reformulated into

$$\begin{aligned} v''(r) + (\lambda c^2 \sinh^2(r) + \mu)v(r) = 0, \quad v(0) = 0 = v(R) \\ w''(\varphi) + (\lambda c^2 \sin^2(\varphi) - \mu)w(\varphi) = 0, \quad w(0) = 0 = w(\pi) \end{aligned} \tag{2}$$

with $u(c\cosh(r)\cos(\varphi), c\sinh(r)\sin(\varphi)) = v(r)w(\varphi)$ and $\mu \in \mathbb{R}$ (this will be explained in Section 3). This is of the form (1) after discretization. There are also many other applications that lead to (1), including delay differential equations [14] and optimization [25].

There are several numerical methods for solving this problem. The traditional approach [2] solves an $nm \times nm$ generalized eigenvalue problem (4) to find all solutions. Another possibility are Jacobi-Davidson type methods discussed in [8,10]. These methods work well in finding eigenvalues close to a given target value. In [18] a Sylvester-Arnoldi type method is described that can be used to find a small subset of eigenvalues and uses that Sylvester equations can be solved efficiently. Another possibility is based on homotopy continuation, for example discussed in [6,21]. These aim to find all eigenvalues.

We present a new algorithm that can be seen as an *alternating method* as they are for example considered in [11] for tensors. A similar approach is used in [27]for two-parameter eigenvalue problems and in [3,17] for two-parameter Sturm-Liouville problems. Similar ideas from the 1970s can be found in [1]. In these methods they alternatingly solve the first eigenvalue problem for λ while fixing μ and the second eigenvalue problem for μ while fixing λ, and repeating the process. Another similar approach was taken in [23]. In there it is described how to transform a two-parameter eigenvalue problem with $n \gg m$ into a nonlinear eigenvalue problem and use techniques for nonlinear eigenvalue problems. By contrast, our method will choose either λ or μ and solve two coupled linear eigenvalue problems alternatingly. This aims to reduce the complexity of finding one solution to (1). The complexity is the one of solving a generalized eigenvalue problem with matrices of size $n \times n$ and $m \times m$. Hencefor $n = m$, the number of operations is

$O(n^3)$ for one solution. While we only establish global convergence for extremal eigenpairs, empirically our algorithm also finds all solutions, with $O(n^5)$ cost. It will turn out that this method can find eigenvalues based on their *index*. The index of an eigenvalue of multiparameter eigenvalue problems is a generalization of ordering real eigenvalues of a standard symmetric eigenvalue problems [26, Ch. 1]. We will need the following assumptions to hold:

A.1 All matrices are symmetric;
A.2 The matrices C_2 and $C_1 \otimes B_2 - B_1 \otimes C_2$ are positive definite and C_1 is negative definite.

Notice that in example (2) the second derivative is indeed a symmetric operator, and $\sinh^2(r) > 0$ and $\sin^2(\varphi) > 0$ almost everywhere. A discrete version of these equations hence leads to a problem of the type (1) that satisfies the assumptions **A.1** and **A.2** up to signs.

Remark 1. In standard literature it is usual to instead require $\Delta_0 = B_1 \otimes C_2 - C_1 \otimes B_2$ as positive definite, i.e., when interchanging the first and second equation. We chose these definiteness assumptions to make Algorithm 1 in the following section more natural.

Remark 2. We only require the matrices C_1, C_2, and $C_1 \otimes B_2 - B_1 \otimes C_2$ to be definite, however we chose these specific definiteness assumptions for convenience as to not have a list of all cases in the following theorems.

Remark 3. The assumptions are not restrictive for right definite two-parameter eigenvalue problems, that is for two-parameter problems with positive or negative definite $C_1 \otimes B_2 - B_1 \otimes C_2$. Indeed we can always find an affine transformation of (1) satisfying the definiteness assumptions. For a theoretical argument, see for example [26, Theorem 1.4.3.]. For two-parameter eigenvalue problems, we have the following explicit construction:

First we note, that at least one of the matrices B_1 and B_2 is definite [20, Lemma 2.1]: If both matrices are not definite, then there exist vectors u and v such that $u^\top B_1 u = 0 = v^\top B_2 v$ contradicting the definiteness of $C_1 \otimes B_2 - B_1 \otimes C_2$. We may therefore assume B_1 to be negative definite, the other cases can be derived analogously. Now let $u = \arg\min \frac{u^\top C_1 u}{u^\top B_1 u}$ be a minimizer. We can perform the affine transformation

$$C_1 \leftarrow C_1 - \left(\frac{u^\top C_1 u}{u^\top B_1 u} - \epsilon\right) B_1$$

$$C_2 \leftarrow C_2 - \left(\frac{u^\top C_1 u}{u^\top B_1 u} - \epsilon\right) B_2$$

for a sufficiently small $\epsilon > 0$. Now the positive definiteness of $C_1 \otimes B_2 - B_1 \otimes C_2$ and negative definiteness of B_1 implies that the new C_2 is positive definite and negative definiteness of B_1 and minimality of u implies negative definiteness of the new C_1.

Note that when replacing B_1, B_2 and C_1, C_2 by the linear combination of matrices

$$\tilde{B}_1 = b_1 B_1 + b_2 C_1, \quad \tilde{B}_2 = b_1 B_2 + b_2 C_2$$

and

$$\tilde{C}_1 = c_1 B_1 + c_2 C_1, \quad \tilde{C}_2 = c_1 B_2 + c_2 C_2$$

and after finding the eigenvalue $(\tilde{\lambda}, \tilde{\mu})$, we can recover the original eigenvalue (λ, μ) via

$$\lambda = b_1 \tilde{\lambda} + c_1 \tilde{\mu}, \quad \mu = b_2 \tilde{\lambda} + c_2 \tilde{\mu}.$$

This article is organized as follows: In Section 2, we first motivate our method and state useful results from multiparameter eigenvalue theory. Afterwards we discuss fixed point properties of the algorithm and prove convergence for extremal eigenvalues. Finally, we examine the time complexity of our method briefly. In Section 3, we show that a class of boundary value problems satisfies our assumptions if properly discretized. Finally, we exhibit results of numerical experiments in Section 4.

2. An alternating algorithm for the two-parameter eigenvalue problem

In this section, we derive our algorithm and prove local convergence for all eigenvalues and global convergence for extremal eigenvalues. First, let us fix our notation. By $\otimes$ we denote the *tensor product*. For vectors this can be seen as the outer product, i.e., $u \otimes v$ corresponds to the rank-one matrix $uv^\top$. The tensor product of matrices then corresponds to a linear operator acting on matrices, i.e., the product $A \otimes B$ acts on the matrix X by $AXB^\top$. This means that $(A \otimes B)(u \otimes v) = (Au \otimes Bv)$. It can also be seen as the Kronecker product for matrices [7, Ch. 12]. Then the product of two vectors $u \otimes v$ can be reshaped into a rank-one matrix.

2.1. Derivation of the algorithm

The two-parameter eigenvalue problem (1) can be reduced to two generalized eigenvalue problems [2]. For this we define the operators acting on $n \times m$ matrices

$$\begin{aligned} \Delta_0 &= B_1 \otimes C_2 - C_1 \otimes B_2, \\ \Delta_1 &= A_1 \otimes C_2 - C_1 \otimes A_2, \\ \Delta_2 &= B_1 \otimes A_2 - A_1 \otimes B_2. \end{aligned} \tag{3}$$

A straightforward computation shows that a solution of (1) satisfies

$$\Delta_1(u \otimes v) + \lambda\Delta_0(u \otimes v) = 0,$$
$$\Delta_2(u \otimes v) + \mu\Delta_0(u \otimes v) = 0.$$

Hence, solutions of (1) are also rank-one solutions X of the system of generalized matrix eigenvalue problems

$$\begin{aligned} \Delta_1(X) + \lambda\Delta_0(X) = 0, \\ \Delta_2(X) + \mu\Delta_0(X) = 0, \end{aligned} \tag{4}$$

which are of size $nm \times nm$. Note that the eigenvectors are shared between the two generalized eigenvalue problems. Also under our assumptions solutions of (4) lead to solutions of (1). This is a well known consequence of classical theory [2,26]. We provide a self contained proof for completeness.

Lemma 4. *Under the assumptions **A.1** and **A.2**, the eigenvectors of the generalized eigenvalue problem $\Delta_1(X) + \lambda\Delta_0(X) = 0$ are either rank one or are a linear combination of rank-one eigenvectors. A rank-one eigenvector is then also an eigenvector of the corresponding two-parameter eigenvalue problem.*

Proof. The assumption **A.2** ensures that the eigenspaces of each subproblem in (4) are spanned by rank-one matrices. To see this, first note that the operator Δ_0 is negative definite, and Δ_1 and Δ_2 are symmetric. Therefore all eigenvalues of (4) are real. Second, we can without loss of generality assume $-C_1$ and C_2 to be identity matrices, else we just transform the matrices to $\tilde{A}_1 = (-C_1)^{-\frac{1}{2}} A_1 (-C_1)^{-\frac{1}{2}}$, $\tilde{B}_1 = (-C_1)^{-\frac{1}{2}} B_1 (-C_1)^{-\frac{1}{2}}$, etc., and $\tilde{u} = (-C_1)^{\frac{1}{2}} u$, $\tilde{v} = C_2^{\frac{1}{2}} v$. Then Δ_0 and Δ_1 are operators in the form of a *Sylvester equation*, i.e., they have the form

$$\Delta_0 = B_1 \otimes I_m + I_n \otimes B_2,$$
$$\Delta_1 = A_1 \otimes I_m + I_n \otimes A_2.$$

A solution to (4) then satisfies

$$\left((A_1 + \lambda B_1) \otimes I_m + I_n \otimes (A_2 + \lambda B_2)\right)(X) = 0,$$

i.e., X is a zero solution of a Sylvester equation. Such operators possess an orthonormal basis of rank-one eigenvectors, namely $u \otimes v$, where u is an eigenvector of $(A_1 + \lambda B_1)$ and v of $(A_2 + \lambda B_2)$ [12, Theorem 4.4.5], both of which are real symmetric and therefore possess an orthonormal basis of eigenvectors. It follows that X is a sum of rank-one eigenvectors corresponding to the eigenvalue 0. The dimension equals that of the null space of $\Delta_1(X) + \lambda\Delta_0(X)$, i.e., the geometric multiplicity of λ.

We next show that a rank-one solution to just one of the eigenvalue problems in (4) suffices to get a solution of the initial problem (1). Indeed, assume that $X = u \otimes v$ solves the second eigenvalue problem in (4). We then get

$$\begin{aligned} 0 =& \Delta_1(u \otimes v) + \lambda \Delta_0(u \otimes v) \\ =& A_1 u \otimes C_2 v - C_1 u \otimes A_2 v + \lambda B_1 u \otimes C_2 v - \lambda C_1 u \otimes B_2 v \\ =& (A_1 + \lambda B_1) u \otimes C_2 v - C_1 u \otimes (A_2 + \lambda B_2) v, \end{aligned}$$

which holds true if and only if there is a μ such that

$$\begin{aligned} -\mu C_1 u &= (A_1 + \lambda B_1) u, \\ -\mu C_2 v &= (A_2 + \lambda B_2) v. \end{aligned}$$

This implies (1). □

We can conclude that (1) has essentially nm solutions which can be obtained by computing the rank-one solutions of one of the eigenvalue problems in (4). In the following, we select the first of these eigenvalue problems.

The assumptions imply that the operators Δ_0 and Δ_1 are symmetric and Δ_0 is negative definite. Thus, the solution of

$$\Delta_1(X) + \lambda \Delta_0(X) = 0 \tag{5}$$

with *maximal* eigenvalue λ can be obtained by maximizing the *Rayleigh quotient*

$$\mathfrak{R}(X) = \frac{\langle X, \Delta_1(X) \rangle}{\langle X, -\Delta_0(X) \rangle}, \tag{6}$$

and since the solution is a rank-one matrix, we can just maximize $\mathfrak{R}(u \otimes v)$ over u and v. For convenience, define the functions

$$\begin{aligned} a_1(u) &= u^\top A_1 u, & b_1(u) &= u^\top B_1 u, & c_1(u) &= u^\top C_1 u, \\ a_2(v) &= v^\top A_2 v, & b_2(v) &= v^\top B_2 v, & c_2(v) &= v^\top C_2 v. \end{aligned}$$

The assumption **A.2** assures that $b_2(v)C_1 - c_2(v)B_1$ and $c_1(u)B_2 - b_1(u)C_2$ are positive definite. We can then write

$$\mathfrak{R}(u \otimes v) = -\frac{u^\top (c_2(v)A_1 - a_2(v)C_1)u}{u^\top (c_2(v)B_1 - b_2(v)C_1)u}.$$

For fixed v the matrix in the denominator is negative definite. Hence, the maximal value is given by the maximal eigenvalue of the generalized eigenvalue problem

$$(c_2(v)A_1 - a_2(v)C_1)\, u = \lambda(b_2(v)C_1 - c_2(v)B_1)u, \tag{7}$$

and respectively fixing u, the maximal value of

$$\mathfrak{R}(u \otimes v) = -\frac{v^\top(a_1(u)C_2 - c_1(u)A_2)v}{v^\top(b_1(u)C_2 - c_1(u)B_2)v}$$

is given by the maximal eigenvalue of

$$(a_1(u)C_2 - c_1(u)A_2)\, v = \lambda(c_1(u)B_2 - b_1(u)C_2)v. \tag{8}$$

These are generalized eigenvalue problems with matrices of size $n \times n$ and $m \times m$, while (5) is a generalized eigenvalue problem of size $nm \times nm$.

A similar alternating procedure is of course obtained when minimizing the Rayleigh quotient in (6), i.e., when aiming at the smallest eigenvalue of (5). More generally, nothing even prevents us from updating u and v with non-extremal eigenpairs of the subproblems (7) and (8) in the hope of finding non-extremal eigenpairs of (5). For instance, we can solve (7) for the i-th eigenvalue and (8) for the j-th eigenvalue, for some fixed i and j. This idea motivates Algorithm 1.

We will call a pair of eigenvectors u and v a *fixed point of Algorithm 1* if it simultaneously solves the eigenvalue problems (7) and (8). Then choosing the corresponding index (i, j) of the eigenvalue, i.e., u corresponds to the i-th smallest eigenvalue of (8) and v corresponds to the j-th smallest eigenvalue of (7), the algorithm will not change u and v anymore, provided eigenvalues are simple.

Algorithm 1: Alternating Algorithm for solving two-parameter eigenvalue problems.

Input : Matrices A_i, B_i, C_i for $i = 1, 2$ satisfying **A.1** and **A.2** and index (i, j).
Output: Eigenvalue (λ, μ) of index (i, j) with corresponding eigenvector $u \otimes v$.

select random nonzero $u_0 \in \mathbb{R}^n$;
for *k=1,2,3,...* **do**
  $a_1 := a_1(u_{k-1}), \quad b_1 := b_1(u_{k-1}), \quad c_1 := c_1(u_{k-1})$;
  compute the eigenvector v_k corresponding to the j-th smallest eigenvalue of the symmetric right definite generalized eigenvalue problem

$$(a_1C_2 - c_1A_2)\, v = \lambda(c_1B_2 - b_1C_2)v;$$

  $a_2 := a_2(v_k); \quad b_2 := b_2(v_k); \quad c_2 := c_2(v_k)$;
  compute the eigenpair (u_k, λ_k) corresponding to the i-th smallest eigenvalue of the symmetric right definite generalized eigenvalue problem

$$(c_2A_1 - a_2C_1)\, u = \lambda(b_2C_1 - c_2B_1)u;$$

end
$\lambda := \lambda_k, \quad u := u_k, \quad v := v_k, \quad \mu := -\frac{a_2+\lambda b_2}{c_2}$;
return (λ, μ) *and* $u \otimes v$.

2.2. Fixed point properties

Our aim in this section is to show that in principle we can find all rank-one eigenpairs of the problem (5), and hence of (1), by finding fixed points of the subproblems (7) and (8) solved in Algorithm 1 for all possible input indices $(i, j) \in \{1, \dots, n\} \times \{1, \dots, m\}$. The first step in this direction is the following lemma, which shows that a fixed point of the algorithm indeed provides a solution to (5) and (1).

Lemma 5. *Let u and v simultaneously solve (7) and (8). Under the assumptions **A.1** and **A.2** both eigenvalues coincide. Moreover, setting $\mu := -\frac{a_2(v)+\lambda b_2(v)}{c_2(v)}$, $(u \otimes v, \lambda)$ is an eigenpair of (5) and (u, v, λ, μ) is a solution for the two-parameter eigenvalue problem (1).*

Proof. Denote the eigenvalue in (7) and (8) by $\hat{\lambda}$ and $\tilde{\lambda}$, respectively. Then multiplying (7) by $u^\top$ and (8) by $v^\top$, we get

$$c_2(v)a_1(u) - a_2(v)c_1(u) = -\hat{\lambda}(c_2(v)b_1(u) - b_2(v)c_1(u))$$

and

$$c_2(v)a_1(u) - a_2(v)c_1(u) = -\tilde{\lambda}(c_2(v)b_1(u) - b_2(v)c_1(u)).$$

Since **A.2** implies $c_2(v)b_1(u) - b_2(v)c_1(u) < 0$, we have $\hat{\lambda} = \tilde{\lambda} =: \lambda$. Collecting terms in (7) and (8) gives

$$\begin{aligned} c_2(v)A_1u + c_2\lambda B_1u - (a_2 + \lambda b_2(v))C_1u = 0, \\ -c_1(u)A_2v - c_1\lambda B_2v + (a_1 + \lambda b_1(u))C_2v = 0. \end{aligned}$$

Dividing the first equation by $c_2(v) > 0$ and the second by $-c_1(u) > 0$, we get (1) with $\mu = -\frac{a_1(u)+\lambda b_1(u)}{c_1(u)}$, if

$$\frac{a_2(v) + \lambda b_2(v)}{c_2(v)} = \frac{a_1(u) + \lambda b_1(u)}{c_1(u)}.$$

This equation is however just a consequence of

$$c_2(v)a_1(u) - a_2(v)c_1(u) = -\lambda(c_2(v)b_1(u) - b_2(v)c_1(u)).$$

By the considerations in Section 2.1 $(u \otimes v, \lambda)$ is also a solution of (5). □

The previous lemma does not yet let us conclude that all solutions of (1) occur as fixed points of Algorithm 1 when varying the input index (i, j). To show this, we need to introduce the notion of the *index of an eigenvalue* (λ, μ) of the two-parameter eigenvalue problem.

Definition 6. An eigenvalue (λ, μ) of the two-parameter eigenvalue problem has the index (i, j) if 0 is the i-th smallest eigenvalue of $A_1+\lambda B_1+\mu C_1$ and the j-th smallest eigenvalue of $A_2 + \lambda B_2 + \mu C_2$.

Remark 7. If the two-parameter eigenvalue problem comes from the discretization of two-parameter Sturm-Liouville eigenvalue problem, then the index corresponds to the number of internal zeros of the eigenfunctions, see e.g. [5, Ch. 8, Theorem 2.1].

If 0 is a multiple eigenvalue of $A_1+\lambda B_1+\mu C_1$ or $A_2+\lambda B_2+\mu C_2$, then the corresponding eigenvalue of the two-parameter eigenvalue problem has multiple indices as well. Under the assumption **A.1** every real valued eigenvalue of the two-parameter eigenvalue problem has an index since the matrices $A_1+\lambda B_1+\mu C_1$ and $A_2+\lambda B_2+\mu C_2$ are symmetric. More important for us is the following result, which immediately follows from [26, Theorem 1.4.1].

Theorem 8. *Under the assumptions **A.1** and **A.2** there is a unique eigenvalue (λ, μ) to every index $(i, j) \in \{1, \dots, n\} \times \{1, \dots, m\}$.*

The idea is now to show that the index in the sense of Definition 6 of the eigenvalue solution provided by a fixed point of Algorithm 1 coincides with the given input index (i, j). Together with Theorem 8 this then implies that all solutions can be obtained this way. For this we need the following version of Sylvester's law of inertia.

Lemma 9. *Let A, B be symmetric matrices and $I + B$ be positive definite. Then λ_i is the i-th largest eigenvalue of A if and only if it is the i-th largest eigenvalue of the generalized eigenvalue problem*

$$(A + \lambda_i B)u = \lambda(I + B)u.$$

Proof. Consider the matrices $\Delta_1 = A - \lambda_i I$ and $\Delta_2 = (I + B)^{-\frac{1}{2}}(A - \lambda_i I)(I + B)^{-\frac{1}{2}}$. The matrix Δ_2 is well defined since $I + B$ is positive definite and since $(I + B)^{-\frac{1}{2}} = ((I + B)^{-\frac{1}{2}})^T)$ is invertible, Δ_1 and Δ_2 are congruent and therefore have the same number of positive and respectively negative eigenvalues by Sylvester's law of inertia [13, Theorem 4.5.8].

We can rewrite Δ_2 in the following way:

$$\begin{aligned}\Delta_2 &= (I + B)^{-\frac{1}{2}}(A - \lambda_i I)(I + B)^{-\frac{1}{2}} \\ &= (I + B)^{-\frac{1}{2}}(A + \lambda_i B - \lambda_i B - \lambda_i I)(I + B)^{-\frac{1}{2}} \\ &= (I + B)^{-\frac{1}{2}}(A + \lambda_i B)(I + B)^{-\frac{1}{2}} - \lambda_i I.\end{aligned}$$

This implies that A and $(I + B)^{-\frac{1}{2}}(A + \lambda_i B)(I + B)^{-\frac{1}{2}}$ have the same number of eigenvalues that are smaller or respectively larger than λ_i. Since the eigenvalues of

$(I+B)^{-\frac{1}{2}}(A+\lambda_i B)(I+B)^{-\frac{1}{2}}$ are the same as the eigenvalues of the generalized eigenvalue problem

$$(A + \lambda_i B)u = \lambda(I + B)u$$

the claim is proven. □

We are now in a position to prove the main result.

Theorem 10. *Let (i, j) be the input index of Algorithm 1 and let (u, v, λ, μ) be a fixed point of Algorithm 1. Then the output (λ, μ) is the eigenvalue of problem (1) with index (i, j) in the sense of Definition 6 and $u \otimes v$ is the corresponding eigenvector.*

Proof. Let (u, v) be a fixed point of Algorithm 1 and let (λ, μ) be the corresponding eigenvalue, i.e., λ is the i-th smallest eigenvalue of the generalized eigenvalue problem

$$(c_2(v)A_1 - a_2(v)C_1)\, u = \lambda(b_2(v)C_1 - c_2(v)B_1)u.$$

Assumption **A.2** guarantees that $b_2(v)C_1 - c_2(v)B_1$ is positive definite, we may therefore apply Lemma 9. It follows that λ is the i-th smallest eigenvalue of the matrix

$$c_2(v)A_1 - a_2(v)C_1 - \lambda(b_2(v)C_1 - c_2(v)B_1) + \lambda I_n = c_2(v)(A_1 + \lambda B_1 + \mu C_1) + \lambda I_n,$$

where we substituted $a_2(v) = -\lambda b_2(v) - \mu c_2(v)$. This implies that 0 is the i-th smallest eigenvalue of $A_1 + \lambda B_1 + \mu C_1$ since $c_2(v) > 0$ is positive by Assumption **A.2**.

Similarly, λ is the j-th smallest eigenvalue of the generalized eigenvalue problem

$$(a_1(u)C_2 - c_1(u)A_2)\, v = \lambda(c_1(u)B_2 - b_1(u)C_2)v.$$

We can again use Lemma 9 to conclude that λ is the j-th smallest eigenvalue of the matrix

$$a_1(u)C_2 - c_1(u)A_2 - \lambda(c_1(u)B_2 - b_1(u)C_2) + \lambda I_m = -c_1(u)(A_2 + \lambda B_2 + \mu C_2) + \lambda I_m,$$

where we substituted $a_1(u) = -\lambda b_1(u) - \mu c_1(u)$. Note that the values for μ coincide in a fixed point, as was shown in Lemma 5. The Assumption **A.2** implies that $c_1(u) < 0$ and therefore 0 is the j-th smallest eigenvalue of $A_2 + \lambda B_2 + \mu C_2$. □

Remark 11. This result made use of the definiteness of C_1 and C_2. If instead B_1 and B_2 are definite one obtains a similar correspondence of input indices and indices of the eigenvalue when considering an alternating method resulting from the eigenvalue problem $(\Delta_2 + \mu\Delta_0)(X) = 0$ instead of $(\Delta_1 + \lambda\Delta_0)(X) = 0$.

Theorem 10 implies that if Algorithm 1 converges, then it computes a solution to the two-parameter eigenvalue problem (1).

Corollary 12. *Let (u_k, v_k, λ_k) be a sequence generated by Algorithm 1. If u_k and v_k converge in the projective sense to u and v, i.e., convergence is up to sign flip if we choose normalized u_k and v_k in each step, then λ_k converge to λ and u, v, λ are fixed by Algorithm 1. Therefore the output (λ, μ) is the eigenvalue of problem (1) with index (i, j) in the sense of Definition 6 and $u \otimes v$ is the corresponding eigenvector.*

Proof. Let $X_{2k} = u_k \otimes v_k$ and $X_{2k+1} = u_{k+1} \otimes v_k$. Since u_k and v_k converge to u and v respectively X_k converges to $u \otimes v$. Define $\hat{\lambda}_k = \mathfrak{R}(X_{2k+1})$. Notice that $\lambda_k = \mathfrak{R}(X_{2k})$ is the eigenvalue corresponding to (7) and $\hat{\lambda}_k$ is the eigenvalue corresponding to (8). By continuity of the Rayleigh quotient $\mathfrak{R}$, we get $\lim_{k\to\infty} \lambda_k = \lim_{k\to\infty} \hat{\lambda}_k = \lambda$. By continuity of the functions $a_1, b_1, c_1, a_2, b_2, c_2$ and continuity of eigenvalues of a nonsingular generalized eigenvalue problem, we get that u and v are eigenvectors of the eigenvalue problems (7) and (8) with eigenvalue λ and continuity also ensures that the eigenvalue is still the i-th or respectively j-th largest. Hence, (u, v, λ, μ) satisfies the conditions of Theorem 10. □

2.3. Geometric interpretation and rate of convergence

Algorithm 1 can be interpreted geometrically. For a given input index (i, j) we look for the intersection of the curves

$$\gamma_i = \{(\lambda, \mu) : \ 0 \text{ is the } i\text{-th largest eigenvalue of } A_1 + \lambda B_1 + \mu C_1\},$$
$$\zeta_j = \{(\lambda, \mu) : \ 0 \text{ is the } j\text{-th largest eigenvalue of } A_2 + \lambda B_2 + \mu C_2\}.$$

As a consequence of [15, Theorem II.6.1] and definiteness of C_1 and C_2, these curves are continuous and piecewise analytic, and the corresponding eigenvectors are also piecewise analytic. In a fashion similar to [4, Lemma 2.2] we can derive the tangent line of γ_i and ζ_j at an analytic point (λ, μ) with the corresponding eigenvectors. We can choose λ as a local variable and obtain

$$(A_1 + \lambda B_1 + \mu(\lambda) C_1) u(\lambda) = 0$$

which implies

$$(A_1 + \lambda B_1 + \mu(\lambda) C_1) u'(\lambda) = -(B_1 + \mu'(\lambda) C_1) u(\lambda).$$

Multiplying both equations with $u(\lambda)^\top$ on the left results in

$$\begin{aligned} a_1(u(\lambda)) + \lambda b_1(u(\lambda)) + \mu(\lambda) c_1(u(\lambda)) &= 0, \\ b_1(u(\lambda)) + \mu'(\lambda) c_1(u(\lambda) &= 0. \end{aligned} \tag{9}$$

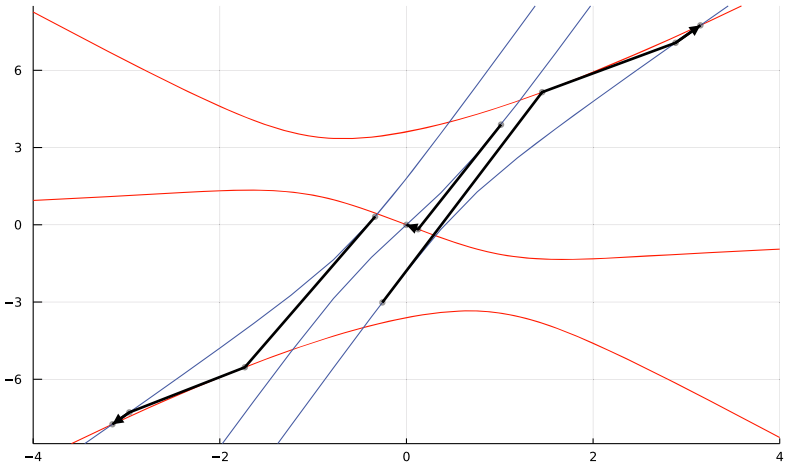


Fig. 1. Iterates in the (λ, μ) plane for input indices $(1,1), (2,2), (3,3)$. The next iterate is given by the intersection of the tangent line at the current iterate to one curve and the other curve.

The tangent line T_γ of γ_i at a point $(\lambda_0, \mu(\lambda_0))$ is therefore given by

$$T_\gamma = \{(\lambda, \mu) : \mu c_1(u(\lambda_0)) = -\big(a_1(u(\lambda_0)) + \lambda b_1(u(\lambda_0))\big)\}$$

and similarly the tangent line T_ζ of ζ_j at a point $(\lambda_0, \mu(\lambda_0))$ is given by

$$T_\zeta = \{(\lambda, \mu) : \mu c_2(v(\lambda_0)) = -\big(a_2(v(\lambda_0)) + \lambda b_2(v(\lambda_0))\big)\},$$

where $v(\lambda_0)$ is a vector in the nullspace of $A_2 + \lambda_0 B_2 + \mu(\lambda_0) C_2$. Examining Algorithm 1, it can therefore be interpreted as follows:

1. Start at a point (λ, μ) of γ_i and compute it's tangent line T_γ;
2. compute the intersection point (λ, μ) of T_γ and ζ_j and compute the tangent line T_ζ;
3. compute the intersection point (λ, μ) of T_ζ and γ_i and compute the tangent line T_γ;
4. go to step 2.

For a visualization of these steps, see Fig. 1. We can use this interpretation to estimate the local rate of convergence. Notably, this interpretation is related to Newton's method for computing an intersection point of two functions f and g. While Newton's method would compute a new iterate via

$$f(x_k) - g(x_k) + (f'(x_k) - g'(x_k))(x_{k+1} - x_k) = 0,$$

our method performs two steps via

$$\begin{aligned} &f(x_k) + f'(x_k)(x_{k+1} - x_k) = g(x_{k+1}) \quad \text{and} \\ &g(x_{k+1}) + g'(x_{k+1})(x_{k+2} - x_{k+1}) = f(x_{k+2}). \end{aligned} \tag{10}$$

This allows us to prove local quadratic convergence in a similar fashion to a standard Newton's method.

Proposition 13. *Let $f, g : U \subset \mathbb{R} \to \mathbb{R}$ be differentiable with Lipschitz continuous derivatives f' and g' and corresponding constants α and β. Furthermore, let $|f'(x) - g'(x)| \geq \gamma > 0$ for all x. Then a sequence x_k generated by (10) satisfies*

$$|\Delta x_k|(1 - \frac{\beta}{2}|\Delta x_k|) \leq \frac{\alpha}{2\gamma}|\Delta x_{k-1}|^2,$$
$$|\Delta x_{k+1}|(1 - \frac{\alpha}{2}|\Delta x_{k+1}|) \leq \frac{\beta}{2\gamma}|\Delta x_k|^2,$$

where $\Delta x_k = x_{k+1} - x_k$.

Proof. We start with (10) and with a Taylor expansion for g we get

$$f(x_k) + f'(x_k)\Delta x_k = g(x_{k+1}) = g(x_k) + g'(x_k)\Delta x_k + \int_0^1 \big(g'(x_k + s\Delta x_k) - g'(x_k)\big)\Delta x_k ds.$$

A Taylor expansion of f and $g(x_k) = f(x_{k-1}) + f'(x_{k-1})\Delta x_{k-1}$ leads to

$$f(x_k) = g(x_k) + \int_0^1 \big(f'(x_{k-1} + s\Delta x_{k-1}) - f'(x_{k-1})\big)\Delta x_{k-1} ds.$$

Combining both equations, we get

$$\big(f'(x_k) - g'(x_k)\big)\Delta x_k = \int_0^1 \big(g'(x_k + s\Delta x_k) - g'(x_k)\big)\Delta x_k - \big(f'(x_{k-1} + s\Delta x_{k-1}) - f'(x_{k-1})\big)\Delta x_{k-1} ds.$$

Using Lipschitz continuity and $|f'(x) - g'(x)| > \gamma$, we arrive at

$$\gamma|\Delta x_k| \leq \frac{\beta}{2}|\Delta x_k|^2 + \frac{\alpha}{2}|\Delta x_{k-1}|^2.$$

The first inequality follows directly, and the second one can be derived analogously. □

We can apply this result, when the intersection point of γ_i and ζ_j is an analytic point of both curves. Then we can describe both locally via smooth functions $\mu = f(\lambda)$ for γ_i and $\mu = g(\lambda)$ for ζ_j. The difference $|f' - g'|$ can be bounded with (9). We get $|f' - g'| \geq |\frac{b_1(u)}{c_1(u)} - \frac{b_2(v)}{c_2(v)}| > 0$ by the definiteness assumption **A.2**.

2.4. Global convergence for extremal eigenpairs

So far, we have shown that if the algorithm converges, it returns the eigenvalue of the two-parameter eigenvalue problem with the desired index. For extremal indices, we can actually prove global convergence.

Theorem 14. *The sequences u_k and v_k generated by Algorithm 1 converge (up to normalization and sign flip) to a solution of (1) if the input index is either $(1,1)$ or (n,m) and under the assumption that the respective eigenvalue is simple.*

We use the following lemma for proving convergence when each step is at least as good as a line search.

Lemma 15. *Let $f : D \to \mathbb{R}$ be continuously differentiable, where $D \subset \mathbb{R}^n$ is open. Let $\{x_k\}_{k\in\mathbb{N}} \subset K \subset D$ and $g_k = \nabla f(x_k)$ satisfy*

$$f(x_{k+1}) \leq f(x_k - \sigma g_k)$$

for every $\sigma \in \mathbb{R}$ such that $x_k - \sigma g_k \in D$, where K is compact. Then g_k converges to zero as $k \to \infty$.

Proof of Lemma 15. First note that $f(x_{k+1}) \leq f(x_k - \sigma g_k)$ for all σ implies that $f(x_k)$ is monotonically nonincreasing and $\{x_k\}_{k\in\mathbb{N}} \subset K$ implies that $f(x_k)$ is bounded from below. Hence, $f(x_k)$ converges to some value $\tilde{f}$ and since x_k lies in a compact set, the gradients g_k are bounded as well.

We have that either g_k converges to zero or there exists a subsequence $g_{k'}$ such that $\|g_{k'}\| \geq 2\epsilon > 0$. We therefore assume towards a contradiction that $\|g_k\| \geq 2\epsilon > 0$ for some $\epsilon > 0$ and infinite many k. Since g_k are bounded, there exists $\delta_1 > 0$ such that $x_k - \sigma g_k \in \tilde{K} \subset D$ for every $\sigma \in [0, \delta_1]$, where $\tilde{K}$ is also compact. Since f is continuously differentiable, ∇f is uniformly continuous on $\tilde{K}$. Therefore, there is $\delta_2 \in (0, \delta_1]$ such that

$$\|\nabla f(x_k - \xi g_k) - g_k\| < \epsilon,$$

for all $\xi \in [0, \delta_2]$. By the mean value theorem, Cauchy-Schwarz inequality, and the assumption $\|g_k\| \geq 2\epsilon$, we have

$$\begin{aligned} f(x_k) - f(x_k - \delta_2 g_k) &= \delta_2 \nabla f(x_k - \xi g_k)^\top g_k \\ &\geq \delta_2 \|g_k\|^2 - \delta_2 \|\nabla f(x_k - \xi g_k) - g_k\| \|g_k\| \\ &\geq \delta_2 \|g_k\|^2 - \delta_2 \|g_k\| \epsilon \\ &\geq \frac{\delta_2}{2} \|g_k\|^2 \geq 2\delta_2 \epsilon^2 \end{aligned}$$

for infinite many k. This yields the contradiction

$$\infty > f(x_0) - \tilde{f} = \sum_{k=0}^{\infty} f(x_k) - f(x_{k+1}) \geq \sum_{k=0}^{\infty} f(x_k) - f(x_k - \delta_2 g_k) \geq \sum_{k'=0}^{\infty} 2\delta_2 \epsilon^2 = \infty,$$

therefore g_k converges to zero. □

Proof of Theorem 14. Without loss of generality, assume that $C_1 = -I_n$ and $C_2 = I_m$. Finding the vectors u and v for the indices $(1,1)$ and (n,m) corresponds to either minimizing or maximizing the Rayleigh quotient

$$\mathfrak{R}(X) = -\frac{\langle X, \Delta_1(X)\rangle}{\langle X, \Delta_0(X)\rangle}.$$

Its gradient is given by

$$\nabla\mathfrak{R}(X) = -\frac{2}{\langle X, \Delta_0(X)\rangle}\left(\Delta_1(X) + \mathfrak{R}(X)\Delta_0(X)\right)$$

and if $X = u \otimes v$, then since $C_1 = -I_n$ and $C_2 = I_m$, we have

$$\begin{aligned}\nabla\mathfrak{R}(u\otimes v) = &-\frac{2}{\langle u\otimes v, \Delta_0(u\otimes v)\rangle}\big(u \otimes (A_2 + \mathfrak{R}(u\otimes v)B_2)v \\ &+ (A_1 + \mathfrak{R}(u\otimes v)B_1)u \otimes v\big).\end{aligned}$$

This shows that the gradient is an element in the tangent space of the rank-one matrix manifold at $u \otimes v$, which is given by

$$T_{u\otimes v}\mathcal{M}_1 = \{x \otimes v + u \otimes y : x \in \mathbb{R}^n, y \in \mathbb{R}^m\}.$$

Now assume every iterate of u and v are unit vectors. Then $-c_1(u) = 1 = c_2(v)$ and the Rayleigh quotient reads

$$\mathfrak{R}(u\otimes v) = \frac{a_1(u) + a_2(v)}{b_1(u) + b_2(v)}.$$

Now let (λ, v) be an eigenpair of

$$\left(a_1(u)I_m + A_2\right)v = -\lambda(b_1(u)I_m + B_2)v,$$

which is one step of Algorithm 1. It follows that $\lambda = \mathfrak{R}(u \otimes v)$ and

$$-\left(A_2 + \lambda B_2\right)v = (a_1(u) + \lambda b_1(u))v.$$

The gradient is then

$$\nabla\mathfrak{R}(u\otimes v) = \frac{2}{\langle u\otimes v, \Delta_0(u\otimes v)\rangle}\left(-u \otimes (a_1(u) + \lambda b_1(u))v + (A_1 + \lambda B_1)u \otimes v\right),$$

and is orthogonal to the subspace $\{u \otimes y : y \in \mathbb{R}^m\} \subset T_{u\otimes v}\mathcal{M}_1$ as

$$\begin{aligned}\langle \nabla\mathfrak{R}(u\otimes v), u\otimes y\rangle &= \frac{2}{\langle u\otimes v, \Delta_0(u\otimes v)\rangle}\big(-(a_1(u)+\lambda b_1(u))v^\top y + (a_1(u)+\lambda b_1(u))v^\top y\big)\\ &= 0.\end{aligned}$$

This was to be expected, as in each step we optimize the Rayleigh quotient with respect to the subspace $\{u \otimes y : y \in \mathbb{R}^m\}$ or respectively $\{x \otimes v : x \in \mathbb{R}^n\}$, which makes the gradient orthogonal to that respective space. This, together with $\nabla\mathfrak{R}(u\otimes v) \in T_{u\otimes v}\mathcal{M}_1$, implies that the gradient lies in $\{u \otimes y : y \in \mathbb{R}^m\}$ or respectively $\{x \otimes v : x \in \mathbb{R}^n\}$ after each half step.

We now only consider the case of minimizing the Rayleigh quotient, as maximizing can be done analogously. As we can choose our iterates u_k and v_k to be normalized, the iterates $X_k = u_k \otimes v_k$ lie in a compact set. As

$$v_{k+1} = \operatorname*{arg\,min}_{\|v\|=1} \mathfrak{R}(u_k \otimes v),$$

we have $\mathfrak{R}(X_{k+1}) \leq \mathfrak{R}(X_k - \sigma\nabla\mathfrak{R}(X_k))$ since $X_k - \sigma\nabla\mathfrak{R}(X_k)$ is of the form $u_k \otimes v$, and the $\mathfrak{R}$ is scaling invariant. We can thus use Lemma 15 and therefore $\nabla\mathfrak{R}(X_k)$ converges to zero, implying that any convergent subsequence of X_k converges to an eigenvector of $\Delta_1(X) + \lambda\Delta_0(X) = 0$. Corollary 12 implies that this eigenvector corresponds to the eigenvalue of the correct index. Hence, since this eigenvalue is simple, the sequence X_k only has one accumulation point up to sign flip. □

2.5. Complexity

We discuss time complexity of Algorithm 1. We assume $m = n$. Computing the eigendecomposition of a generalized eigenvalue problem needs $O(n^3)$ operations. This implies that the complexity for computing one eigenvalue of the two-parameter eigenvalue problem is $O(n^3k)$, where k is the number of iterations in Algorithm 1 until a sufficient accuracy is achieved (empirically we observe $k = O(1)$). When computing all n^2 eigenvalues of the two-parameter eigenvalue problem, we therefore get a complexity of $O(n^5k)$ (and with $k = O(1)$ we get a complexity of $O(n^5)$). However, a full eigenvalue decomposition is not always necessary. For extremal eigenvalues, we only need to compute the eigenvector corresponding to the largest or smallest eigenvalue, which is typically possible in $O(n^2)$ operations, for example using Lanczos or LOBPCG [16]. Similarly, the i-th largest eigenvalue can often be computed in $O(n^2i)$ operations. In summary, computing an eigenvalue with index (i, j) needs $O(n^2k\max(\min(i, n-i+1), \min(j, n-j+1))$ operations, where k is the number of iterations necessary for the desired accuracy. If the matrices allow for fast matrix vector multiplication, for example if they are sparse with $O(n)$ nonzero entries, the complexity can be reduced further up to $O(nk\max(\min(i, n-i+1), \min(j, n-j+1))$ operations.

3. A class of PDE eigenvalue problems

We now present a class of PDE eigenvalue problems, which can be separated into an appropriate two-parameter eigenvalue problem. Let $\phi : (a,b) \times (c,d) \to U \subset \mathbb{R}^2$ be a diffeomorphism, with

$$g(x,y) := (D\phi(x,y))^\top D\phi(x,y) = \begin{pmatrix} g_1(x) + g_2(y) & 0 \\ 0 & g_1(x) + g_2(y) \end{pmatrix},$$

and choose $g_1, g_2 > 0$. This is always possible for such a function g as $g(x,y)$ is positive definite and therefore $g_1(x) + g_2(y) > 0$. Now let $\Omega = \phi((a,b) \times (c,d))$. We consider the Helmholtz equation

$$\begin{aligned} \Delta u + \lambda u &= 0 \quad \text{on } \Omega, \\ u &= 0 \quad \text{at } \partial\Omega. \end{aligned}$$

We can now write Δu in coordinates given by ϕ. Then

$$\begin{aligned} (\Delta u)(\phi(x,y)) =& \frac{1}{\sqrt{\det(g(x,y))}} \nabla \cdot \left(\sqrt{\det(g(x,y))} g^{-1}(x,y) \nabla (u(\phi(x,y)) \right) \\ =& \frac{1}{g_1(x) + g_2(y)} \Delta (u(\phi(x,y))). \end{aligned}$$

Making an ansatz $u(x,y) = v(x)w(y)$, we get

$$\begin{aligned} 0 &= v''(x)w(y) + v(x)w''(y) + \lambda g_1(x) v(x) w(y) + \lambda g_2(y) v(x) w(y) \\ &= v(x)\big(w''(y) + \lambda g_1(x) w(y) - \mu w(y)\big) + w(y)\big(v''(x) + \lambda g_2(y) v(x) + \mu v(x)\big). \end{aligned}$$

Now let v and w satisfy

$$v''(x) + \lambda g_1(x) v(x) + \mu v(x) = 0 \tag{11}$$

$$w''(y) + \lambda g_2(y) w(y) - \mu w(y) = 0, \tag{12}$$

and $0 = v(a) = v(b) = w(c) = w(d)$. Then $u(x,y) = v(x)w(y)$ solves the original eigenvalue problem, and on a given discretization (11) satisfies **A.1** and **A.2**.

Example 16. The complex function $\phi : \mathbb{C} \to \mathbb{C}, z \mapsto z^2$. Then $D\phi(z) = 2z$. Therefore with $z = x + iy$, we have

$$g(x,y) = \begin{pmatrix} 2x & 2y \\ -2y & 2x \end{pmatrix} \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix} = \begin{pmatrix} 4x^2 + 4y^2 & 0 \\ 0 & 4x^2 + 4y^2 \end{pmatrix}.$$

Example 17. The complex function $\phi : \mathbb{C} \to \mathbb{C}, z \mapsto e^z$. Then $D\phi(z) = e^z$. Therefore with $z = x + iy$, we have

$$g(x,y) = \begin{pmatrix} e^x\cos(y) & e^x\sin(y) \\ -e^x\sin(y) & e^x\cos(y) \end{pmatrix} \begin{pmatrix} e^x\cos(y) & -e^x\sin(y) \\ e^x\sin(y) & e^x\cos(y) \end{pmatrix} = \begin{pmatrix} e^{2x} & 0 \\ 0 & e^{2x} \end{pmatrix}.$$

These coordinates describe an annulus.

Example 18. The complex function $\phi : \mathbb{C} \to \mathbb{C}, z \mapsto \cosh(z)$. Then $D\phi(z) = \sinh(z)$. Therefore with $z = x + iy$, we have

$$\begin{aligned} g(x,y) &= \begin{pmatrix} \sinh(x)\cos(y) & \cosh(x)\sin(y) \\ -\cosh(x)\sin(y) & \sinh(x)\cos(y) \end{pmatrix} \begin{pmatrix} \sinh(x)\cos(y) & -\cosh(x)\sin(y) \\ \cosh(x)\sin(y) & \sinh(x)\cos(y) \end{pmatrix} \\ &= \begin{pmatrix} \sinh^2(x) + \sin^2(y) & 0 \\ 0 & \sinh^2(x) + \sin^2(y) \end{pmatrix}. \end{aligned}$$

These are elliptical coordinates.

4. Numerical experiments

In this section, we present results from numerical experiments to test the performance of Algorithm 1. For a measure of the error we use Definition 6. For a given approximate eigenvalue (λ, μ) we compute the i-th smallest eigenvalue of $A_1 + \lambda B_1 + \mu C_1$ and the j-th smallest eigenvalue of $A_2 + \lambda B_2 + \mu C_2$ and take the sum of the respective absolute values. This quantity is zero if and only if (λ, μ) is an eigenvalue with index (i, j). This is also the sum of the corresponding residual norms with the corresponding normalized eigenvectors.

In a first experiment we generated matrices satisfying Assumptions **A.1** and **A.2** randomly. For the matrices A_1 and A_2 we generated $n \times n$ and $m \times m$ matrices with independent standard Gaussian distributed entries and took the symmetric part. For the matrices B_1, B_2, C_1, C_2 we generated matrices S_1 and S_2 with Gaussian distributed entries and an n dimensional array b_1 with values uniformly distributed between -0.5 and 0.5 and an m dimensional array b_2 with values uniformly distributed between -1.5 and -0.5. We then chose the matrices

$$B_1 = S_1 \operatorname{Diag}(b_1) S_1^\top, \quad B_2 = S_2 \operatorname{Diag}(b_2) S_2^\top, \quad C_1 = -S_1 S_1^\top, \quad C_2 = S_2 S_2^\top,$$

where $\operatorname{Diag}(b_i)$ is the matrix with the entries of b_i on its diagonal.

As a second example, we discretized the Helmholtz equation on half of an ellipse as in the example in Section 1, i.e.,

$$\begin{aligned} v''(r) + (\lambda c^2 \sinh^2(r) + \mu) v(r) = 0, \quad v(0) = 0 = v(1) \\ w''(\varphi) + (\lambda c^2 \sin^2(\varphi) - \mu) w(\varphi) = 0, \quad w(0) = 0 = w(\pi). \end{aligned} \tag{13}$$

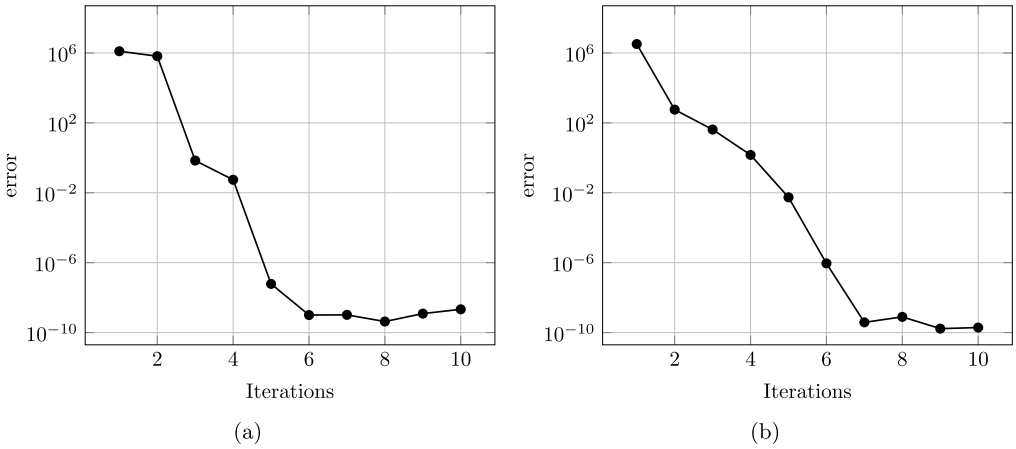


Fig. 2. Testing Algorithm 1 with input index $(1, 1)$. Figure (a) depicts results for randomly generated 1000×1000 matrices fulfilling Assumptions **A.1** and **A.2** and Figure (b) depicts results for a discretization of (2) on a 1000×1000 grid.

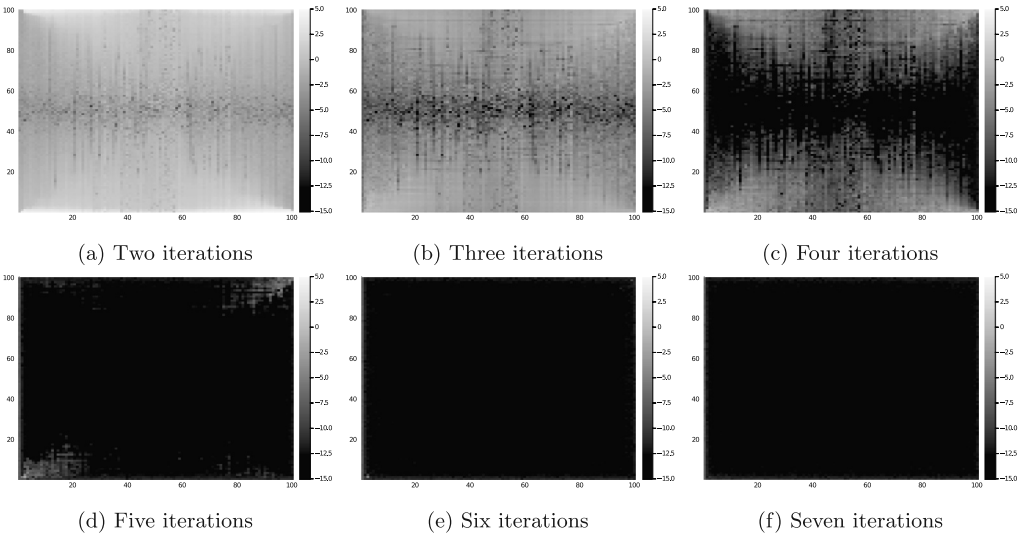
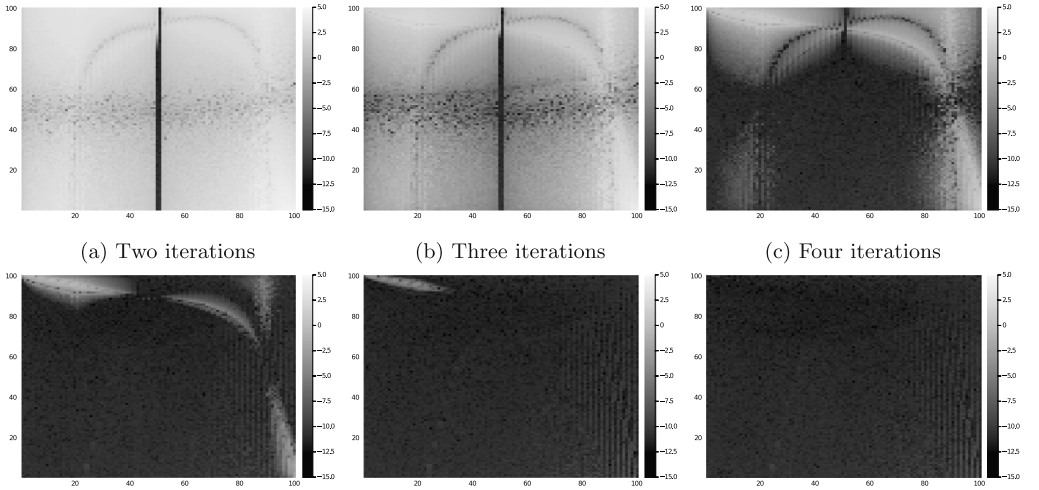


Fig. 3. Testing Algorithm 1 with random 100×100 matrices fulfilling the assumptions. Every picture is taken after solving one eigenvalue problem in line 4 or 6 of the algorithm. The axis label the indices of the computed eigenvalue and the colorscale shows the sum of the absolute values of the i-th and j-th eigenvalue in (1) on a logarithmic scale.

For Fig. 2, we generated 1000×1000 matrices randomly in (a) and used a discretization on a 1000×1000 grid in (b) as described above and used Algorithm 1 to find an eigenvalue with minimal λ, i.e., with input index $(1, 1)$. After solving 6 and 7 eigenvalue problems respectively as described in lines 4 and 6 of Algorithm 1, we found an eigenvalue with an error of approximately 10^{-9} and 10^{-10} respectively, confirming the result of Theorem 14.



(a) Two iterations (b) Three iterations (c) Four iterations

(d) Five iterations (e) Six iterations (f) Seven iterations

Fig. 4. Testing Algorithm 1 with a discretization of (2) on a 100×100 grid. Every picture is taken after solving one eigenvalue problem in line 4 or 6 of the algorithm. The axis label the indices of the computed eigenvalue and the colorscale shows the sum of the absolute values of the i-th and j-th eigenvalue in (1) on a logarithmic scale.

For Fig. 3 we generated 100×100 matrices satisfying Assumptions **A.1** and **A.2** as described above and use Algorithm 1 to find all eigenvalue, i.e., for every input index $(i, j) \in \{1, \ldots, 100\} \times \{1, \ldots, 100\}$. The axis describes the input index of Algorithm 1 and the greyscale describes the base ten logarithm of the error described above. An iteration is computing the solution of one eigenvalue problem in lines 4 and 6 of Algorithm 1. After 7 iterations the highest error was $4 \cdot 10^{-8}$ and the smallest errors were in the range of machine precision.

In Fig. 4 we used a discretization of the Helmholtz equation (13) on a 100×100 grid and repeated the experiment of Fig. 3 for the resulting matrices. Again, after 7 iterations the highest error was $3 \cdot 10^{-8}$ and the smallest errors were in range of machine precision. These experiments suggest that Algorithm 1 can indeed be used to find every eigenvalue of right definite two-parameter eigenvalue problems.

In Fig. 5(a) we compared the time of both Algorithm 1 and the algorithm `twopareig` [22] for computing every eigenvalue of a two-parameter eigenvalue problem of varying sizes $n = m$. We generated matrices satisfying the assumptions as above. Notice that Algorithm 1 can be run in parallel for different input indices to further improve efficiency. The experiment was run on a Intel Xeon Gold 6144 at 3.5 GHz with 384 GB RAM. We used 8 cores with 16 threads. We chose to solve 10 small eigenvalue problems in Algorithm 1, corresponding to $k = 1, \ldots, 5$ in line 2. For larger n our method was indeed faster, and the asymptotic slope in Fig. 5(a) on a loglog scale is smaller, indicating that the asymptotic computational cost is lower. In Fig. 5(b) we measure the sum of the n^2 errors as described above. We observe that our method computed eigenvalues with higher accuracy.

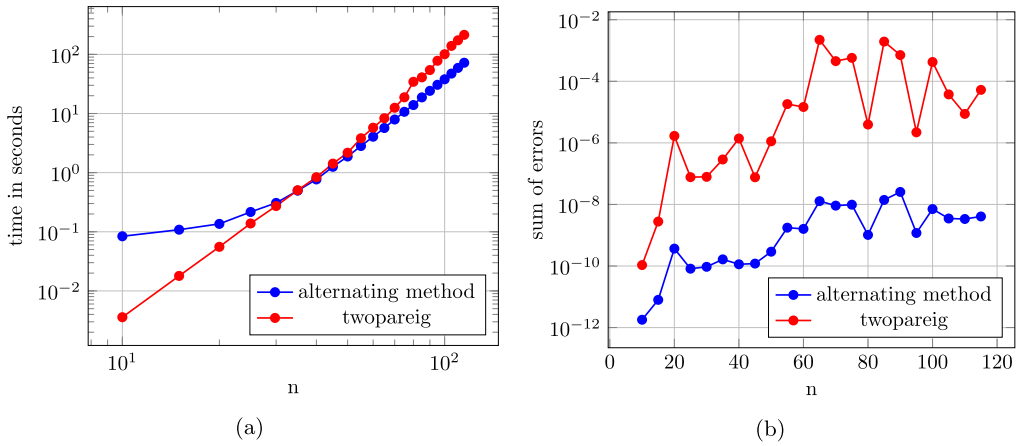


Fig. 5. Comparing the computational time (a) and precision (b) for randomly generated examples with matrices of size $n \times n$ satisfying the assumption **A.1** and **A.2** for our method and `twopareig` from [22].

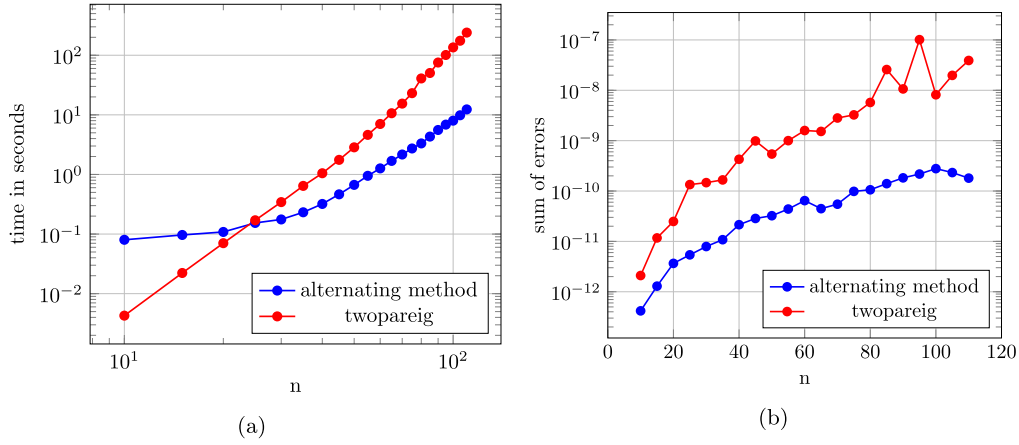


Fig. 6. Comparing the computational time (a) and precision (b) for randomly generated examples with matrices of size $n \times n$ satisfying the assumption **A.1** and **A.2** for our method and `twopareig` from [22]. For this example the matrices B_1, B_2, C_1, C_2 are diagonal matrices.

Finally, we repeated the last experiment, but we chose $S_1 = I_n = S_2$. This made the matrices B_1, B_2, C_1 and C_2 diagonal, which effectively transformed the generalized eigenvalue problems in Algorithm 1 into ordinary eigenvalue problems to further improve efficiency. When solving for every eigenvalue of the two-parameter problem, we can perform the left and right actions of $(-C_1)^{-\frac{1}{2}}$ and $C_2^{-\frac{1}{2}}$ respectively and afterwards diagonalize B_1 and B_2. This justifies this experiment. The results are depicted in Fig. 6. In Fig. 6(a) we see that the alternating method is faster than the method `twopareig` [22] for even smaller n. Again our method is more accurate.

Remark 19. In principle Algorithm 1 can be made more efficient by a fitting choice of u_0. If the two-parameter eigenvalue problem comes from the discretization of a two-parameter Sturm-Liouville eigenvalue problem, then the factor u_{ij} of an eigenvector $u_{ij} \otimes v_{ij}$ corresponding to index (i, j) has the same number of zeros as the factor u_{ik} of $u_{ik} \otimes v_{ik}$ corresponding to the index (i, k). Then u_{ik} can be very similar to u_{ij} and is possibly a good initial guess. We observed that such a choice for u_0 usually reduces the number of iterations needed until the algorithm converges by one. This saves around 10% to 25% computing time.

5. Conclusion and outlook

We presented a new method for computing eigenvalues of the two-parameter eigenvalue problem. Our approach only requires solving generalized eigenvalue problems of the size of the matrices of the two-parameter problem and can therefore reduce the complexity compared to conventional methods. Our method also uses a search for the eigenvalues by index, which makes it possible to find successive eigenvalues of the two-parameter eigenvalue problem without deflation.

So far the technique of our proof only established global convergence for extremal eigenvalues and under definiteness assumptions. The numerical experiments however indicated convergence for every eigenvalue. Proposition 13 gives insight into the local convergence, but a global convergence proof remains an open problem. Although there are many classes of two-parameter eigenvalue problems that satisfy the assumptions (eventually after performing an affine transformation), many other interesting applications do not. Therefore it would be important to investigate if a generalization to non-singular problems, i.e., the operator Δ_0 is invertible, as in [8] or even to the general case as in [9] is possible.

This paper relies heavily on the assumptions **A.1** and **A.2**. A natural question is to relax these conditions, in particular the definiteness condition **A.2**. Indeed under weaker assumptions (such as when the matrices are *almost* definite, with a few eigenvalues of the opposite sign), using inertia laws [19] one can show that the generalized eigenvalue problem (4), and hence also (1), has many real eigenvalues. It would be of interest to investigate the applicability of the results here in such situations.

Finally, another interesting generalization that could be considered is to multiparameter eigenvalue problems with more than 2 parameters. The eigenvectors then form rank-one tensors similar to (4) [2]. Again an alternating approach as in [11] can be used, however our proof technique will not work and in practice the generalization will not easily assure convergence even for extremal indices. A similar approach using the Tensor-Train format is used in [24].

Declaration of competing interest

The authors declare no competing interests.

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