Zeta Functions and Modularity of Calabi-Yau Manifolds

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To my parents,
for their unfailing support and encouragement

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Abstract

We study the Dwork operator $\mathcal{U}(\lambda)$ on a four dimensional subspace of the Dwork cohomology in the large structure limit of a one parameter family of quintic threefolds. The Dwork cohomology becomes singular in this limit, but we recover it by introducing formal logarithmic states which remain well defined at $\lambda = 0$. In a suitable basis, $\mathcal{U}(0)$ is known to have a single nonzero off-diagonal component at the large structure point, whose value is a rational multiple of $\zeta_p(3)$. We compute the value of this off-diagonal component via a direct computation in our "logarithmic Dwork cohomology" at $\lambda = 0$, and obtain an expansion for $\mathcal{U}(\lambda)$ around $\lambda = 0$.

This expansion provides an efficient means to compute a quartic factor $R(T)$ of the local zeta function, not just for the quintic but for a large number of one parameter families of Calabi-Yau manifolds. We exploit the miraculous fact that $\mathcal{U}(\lambda)$ becomes a rational function mod p^4 to avoid evaluating the series for $\mathcal{U}(\lambda)$ directly. This is vital to the performance of the computation for large primes.

The behaviour of the zeta function at singular values of the parameter is particularly interesting. The Weil conjectures ensure rationality, but say nothing about the form of the zeta function when the variety is singular. What generally happens is that one or more eigenvalues of $\mathcal{U}(\lambda)$ go to zero and the degree of $R(T)$ falls. Typically a quadratic factor occurs in $R(T)$ with which a modular form can be associated. For a number of one parameter families, we evaluate $R(T)$ at the singular points, for $7 \leq p \leq 97$, and seek to identify the associated modular form. We reproduce various known results with higher certainty by checking up to $p = 97$. We identify modular forms in many new cases, including Hilbert modular forms of parallel weight 4 at irrational conifold singularities.

Contents

Chapter 1

Introduction

The principal object of interest in the study of arithmetic varieties is the zeta function. Given a prime p, consider a projective arithmetic variety $\mathcal{V}$ defined as the vanishing locus in $\mathbb{F}_p\mathbb{P}^{n+1}$ of a homogeneous polynomial $P(x)$. For $r \in \mathbb{N}$, the number of $\mathbb{F}_{p^r}$-points of $\mathcal{V}$ is

$$N_r(\mathcal{V}) = |\{x \in \mathbb{F}_{p^r}\mathbb{P}^n \colon P(x) = 0 \mod p\}|. \tag{1.1}$$

The zeta function is the generating function of the numbers $N_r(\mathcal{V})$,

$$\zeta_{\mathcal{V}}(T) = \exp\left(\sum_{r=1}^{\infty} N_r(\mathcal{V})\frac{T^r}{r}\right). \tag{1.2}$$

By virtue of the Weil conjectures, now proven [1, 2], we know that $\zeta_{\mathcal{V}}(T)$ is a rational function which, for $\mathcal{V}$ smooth over $\mathbb{F}_p$, takes the form

$$\zeta_{\mathcal{V}}(T) = \frac{P_1(T)P_3(T)\cdots P_{2n-1}(T)}{(1-T)P_2(T)\cdots P_{2n-2}(T)(1-p^nT)},$$

for polynomials $P_i(T)$ whose degrees are given by the Betti numbers b_i of the complex variety defined by the vanishing of $P(x)$ in $\mathbb{P}^{n+1}$. The rationality was proved by Dwork [1, 3, 4], who showed that $\zeta_{\mathcal{V}}(T)$ could be computed as the determinant of a certain operator (the *Dwork operator*) on a suitable (infinite dimensional) space of power series. This determinant reduces to a determinant over a finite dimensional differential complex of power series (the *Dwork cohomology*), whence it becomes clear that $\zeta_{\mathcal{V}}(T)$ is a rational function.

Perhaps the most elementary, and certainly the most studied, one-parameter family of Calabi-Yau manifolds is the quintic threefold $\mathcal{M}_\varphi$. Defined over $\mathbb{F}_p$ and parametrized by $\varphi \in \mathbb{F}_p$, it is given by the vanishing locus in $\mathbb{F}_p\mathbb{P}^4$ of the polynomial

$$P_\varphi(x) = \varphi \sum_{i=1}^{5} x_i^5 - \prod_{i=1}^{5} x_i. \tag{1.3}$$

The alternative parameter $\lambda = \varphi^5$ will prove useful. Defining the logarithmic derivative $\vartheta = \lambda \frac{\mathrm{d}}{\mathrm{d}\lambda}$, we can write down the *Picard-Fuchs* equation satisfied by its periods[1]

$$\mathcal{P}\varpi_k = \left(\vartheta^4 - 5\lambda \prod_{i=1}^{4} (5\vartheta + i)\right) \varpi_k = 0, \quad 0 \le k \le 3. \tag{1.4}$$

First computed by Dwork, the zeta function of the quintic has since been studied in some detail. Candelas, de la Ossa and Rodriguez Villegas [5, 6] expressed $N_r(\mathcal{M}_\varphi)$ directly in terms of the periods, but computing $\zeta_{\mathcal{M}_\varphi}(T)$ by substituting these numbers into equation (1.2) is only practical for small p. Samol and van Straten [7, 8] computed the zeta function directly using the unit root method, but this approach also scales badly with the prime p. In fact, Dwork's approach is the most amenable to computation. Candelas and de la Ossa [9] gave an account of Dwork's method accessible to physicists and made an expansion of the Dwork operator $\mathcal{U}(\lambda)$ at small p-adic structure $\lambda = \infty$ (i.e. $\psi = 0$). At $\lambda = \infty$, the Dwork cohomology relations become simpler, making it easier to compute the action of the Dwork operator there. The λ dependence of $\mathcal{U}$ is determined by the periods ϖ_k, and hence by the Picard-Fuchs equation (1.4).

It would be of great interest to make a similar expansion at the large p-adic structure limit[2] $\lambda = 0$ (i.e. $\psi = \infty$), but this remains an open problem. The difficulty arises from the highly singular nature of the quintic at $\lambda = 0$, where it becomes a union of coordinate planes. The Dwork cohomology becomes singular at $\lambda = 0$ too, making it harder to meaningfully define the action of the Dwork operator there, as compared to $\lambda = \infty$.

The main theoretical result in this thesis is the development of a framework in which the action of $\mathcal{U}(0)$ can be defined at $\lambda = 0$ on a four dimensional subspace of the Dwork cohomology. It is then possible to make the desired expansion of $\mathcal{U}(\lambda)$ around

[1] Since the quintic has 204 complex structure parameters, it has 204 periods. Four of them, however, which it shares with the mirror quintic, satisfy the Picard-Fuchs equation (1.4).

[2] The Picard-Fuchs equation (1.4) has a point of maximal unipotent monodromy at $\lambda = 0$, where all four of its indices are 0. We call this the large p-adic structure point in analogy to the large complex structure limit of mirror symmetry.

this point. Based on numerical experiment, and the thesis of Kira Samol [7], we expect such an expansion to take the form

$$\mathcal{U}(\lambda) = E(\lambda^p)^{-1}\mathcal{U}(0)E(\lambda), \tag{1.5}$$

where

$$\mathcal{U}(0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ p\alpha & p & 0 & 0 \\ p^2\beta & p^2\alpha & p^2 & 0 \\ p^3\gamma & p^3\beta & p^3\alpha & p^3 \end{pmatrix}, \quad \text{and} \quad E_{jk}(\lambda) = \vartheta^k \varpi_j(\lambda)\Big|_{\log\lambda=0}. \tag{1.6}$$

Setting $\log\lambda = 0$ here is a formal operation and must be performed after taking λ derivatives. One can, in fact, leave the logarithmic terms in E, but it turns out that they cancel in (1.5), so we omit them from the outset. It will turn out that we can take α and β, but not γ, equal to zero. In fact, it was shown in [10] by motivic considerations that, in a certain natural basis, the Dwork operator (or rather, its inverse, the Frobenius map) at $\lambda = 0$ has at most one nonzero offdiagonal component, with powers of p on the diagonal. Shapiro [11] also showed this by computing $\mathcal{U}(\lambda)$ directly in Dwork cohomology for $\lambda \neq 0$, thus avoiding the singularity, before taking the limit $\lambda \to 0$ to obtain $\mathcal{U}(0)$. It was observed numerically, and then proved in [12], that this offdiagonal entry is a rational multiple of $\zeta_p(3)$.

We compute the action of the Dwork operator $\lambda = 0$ via a direct calculation at $\lambda = 0$ in a "logarithmic Dwork cohomology" space. This space, obtained by introducing formal logarithmic terms to the Dwork cohomology, remains well defined in the large p-adic structure limit. Computing $\mathcal{U}(0)$ and obtaining the expression for γ using our formalism is a key result of this thesis. We further observe that the correct multiple of $\zeta_p(3)$ is

$$\frac{\chi}{y}\zeta_p(3), \tag{1.7}$$

where χ is the Euler number and y is the value of the Yukawa coupling at the large structure point.

Though we perform this construction in the context of the (mirror) quintic, we conjecture that the expansion (1.5) of $\mathcal{U}(\lambda)$ at the large p-adic structure point also applies to other one parameter families of Calabi-Yau manifolds. All that is required to compute $\mathcal{U}(\lambda)$ for a given manifold is the value of $\mathcal{U}(0)$ and the periods, which determine the λ-dependence of $\mathcal{U}$.

We further conjecture that the offdiagonal component of $\mathcal{U}(0)$ is given by (1.7) not just for the quintic but for all one-parameter families of Calabi-Yau manifolds. We develop a numerical procedure with which we can deduce the correct value of the ratio $\rho = \chi/y$ from the series for $\mathcal{U}(\lambda)$. In all cases where χ and y are known for the manifold in question, the predicted value matches the true value.

The periods are the solutions of a Picard-Fuchs differential equation, which in the case of a Calabi-Yau manifold is said to have *Calabi-Yau type*. A comprehensive list [13] of Picard-Fuchs operators of Calabi-Yau type is now available online [14] and was provided to us in machine readable form by Duco van Straten. These operators provide all we need to obtain $\mathcal{U}(\lambda)$, and thus compute a factor of the zeta function.[3]

The relationship between the periods of a Calabi-Yau manifold and its zeta function is well studied. Both the explicit formulae of Candelas, de la Ossa and Rodriguez Villegas and the unit root method employed by Samol and van Straten link the periods to the zeta function. The former was used to count points of the quintic [5] and the latter to compute a quartic factor

$$R(T) = \det\left(1 - \frac{\mathcal{U}(\lambda)}{p}T\right) \tag{1.8}$$

of the zeta function for a number of one parameter families [8]. However, both these approaches scale badly as the prime increases and so are limited to small primes (less than 20) in practice. In this thesis, we compute up to $p = 97$ (inclusive), a feat which would not have been viable with the other methods. In fact it is possible to go to even higher primes, though this does necessitate increased series precision and correspondingly longer computation times.

One of the motivations for an efficient method of computing the zeta function is to investigate modularity conjectures. Where a variety becomes singular, the Weil conjectures still guarantee rationality but do not specify the precise form of the zeta function. However, in practice it typically takes some standard form and it is expected that an analogy to the Weil conjectures holds, with the mixed Hodge structure of the singular variety determining the degrees of the factors in the numerator and denominator of $\zeta_{\mathcal{V}}(T)$.

At a conifold point, for example, an eigenvalue of $\mathcal{U}(\lambda)$ goes to zero and the quartic factor $R(T)$ becomes cubic, factorising into a linear and quadratic piece,

$$R(T) = (1 - p\chi T)(1 - \alpha T + p^3T^2).$$

[3] For many operators in the list, the underlying manifold is not known (if it exists at all). Should the manifold not exist, it does not make sense to talk about counting points, but the Dwork operator $\mathcal{U}(\lambda)$ still has a suitable interpretation coming from the theory of motives.

There are fourteen one parameter families of Calabi-Yau manifolds whose Picard-Fuchs equations are of hypergeometric type [15, 16], including the quintic. Each such family has a conifold point where $R(T)$ takes the form above. Moreover it was found in [15] that the coefficient α for each p corresponds to the p^{th} coefficient in the q-expansion of a weight 4 classical modular form. Similar identifications have been made at the conifold points of various degree 2 operators of Calabi-Yau type by Samol and van Straten. However, the vast majority of cases remain open and it would be of great interest to identify modular forms in more cases. Of particular interest are non conifold singularities and irrational singular points, at which Hilbert modular forms are anticipated.[4]

We proceed as follows: in Chapter 2, we review the Dwork method for computing the zeta function, in order establish notation for Dwork cohomology, and to motivate our interest in computing the Dwork operator $\mathcal{U}(\lambda)$. In Chapter 3, we review the decomposition of the Dwork cohomology of the quintic into various representations under the automorphisms of the family (1.3). We explicitly compute a basis for each representation and derive the Picard-Fuchs equations they satisfy. We examine how these bases behave in the large structure limit $\lambda \to 0$, and discover how the Dwork cohomology breaks down there. Then, in Chapter 4, we reconstruct a four-dimensional subspace of the Dwork cohomology at $\lambda = 0$ by introducing formal logarithmic terms. Using this framework, we compute the action of the Dwork operator at the large structure point and thus obtain the desired expansion (1.5) of $\mathcal{U}(\lambda)$.

In Chapter 5, we describe our method for computing $R(T)$ from the periods, taking advantage of the fact that $\mathcal{U}(\lambda)$ becomes a rational function when reduced mod p^s for any $s \geq 1$. We describe the various types of singular points possessed by Calabi-Yau type operators of degree 2 and 3, giving examples where necessary. Our main results, given in Section 5.6, are the tables of singularities and corresponding modular forms, including a number of weight $[4, 4]$ Hilbert modular forms identified at irrational conifold points. Full tables of results for all operators considered, for $7 \leq p \leq 97$ and $1 \leq \lambda < p$, could not practically be included, but tables are given for a select few operators in Appendix A. The full data will be made available online at `https://cyarithmetic.gitlab.io`. Some modular forms identified in the course of this work cannot be easily looked up elsewhere and the relevant details of these

[4] In fact, Hilbert modular forms are only expected where the irrational singular points are real.

forms are given in Appendix B. In Appendix C we list the software used for our computations; though we do not include our code, it will also be made available online. Finally Appendix D provides a self contained review for physicists of relevant topics from number theory.

Chapter 2

The Dwork Method

In order to prove the rationality of the zeta function, Dwork showed that it could be computed as the superdeterminant of an operator on a differential complex of power series. What follows is a review of this approach following [17, 18]. A review accessible to physicists was given in [9]. Knowledge of fundamental concepts from number theory is assumed, but an accessible overview of relevant topics, including finite fields and p-adic numbers can be found in Appendix D.

Consider a family $\mathcal{V}_\lambda$ of varieties (not necessarily the quintic) defined by the vanishing locus of a single homogeneous, degree d polynomial $P_\lambda(x) \in \mathbb{F}_p[x]$ parametrized by $\lambda \in \mathbb{F}_p$. The zeta function is the generating function

$$\zeta_{\mathcal{V}_\lambda}(T) = \exp\left\{\sum_{r=1}^{\infty} N_r(\mathcal{V}_\lambda)\frac{T^r}{r}\right\} \tag{2.1}$$

of the numbers $N_r(\mathcal{V}_\lambda)$ counting the solutions of $P_\lambda(x) = 0$ in $\mathbb{F}_{p^r}\mathbb{P}^n$. The Dwork method does not compute ζ directly, but instead a closely related generating function

$$Z^*_{\mathcal{V}_\lambda}(T) = \exp\left\{\sum_{r=1}^{\infty} \frac{N^*_r(\mathcal{V}_\lambda)T^r}{r}\right\}, \qquad N^*_r(\mathcal{V}_\lambda) = \left|\left\{x \in (\mathbb{F}^*_{p^r})^n \colon P(x) = 0\right\}\right|. \tag{2.2}$$

As shown in [5, 7], one can recover $\zeta_{\mathcal{V}_\lambda}$ by summing $Z^*_{\mathcal{W}}$ over certain subvarieties $\mathcal{W}$ of $\mathcal{V}_\lambda$ (including $\mathcal{V}_\lambda$ itself).

The key innovation of Dwork's method, and the goal of this section, is to express $Z^*_{\mathcal{V}}(T)$ as the superdeterminant of an operator on a suitable vector space. The first step is to realise that we can count the solutions using finite characters (see Appendix D.4 for a definition).

The key property is equation (D.26), which we restate here for convenience,

$$\sum_{y\in\mathbb{F}_{p^r}} \chi(yz) = \begin{cases} p^r; & z = 0, \\ 0; & z \neq 0, \end{cases} \tag{2.3}$$

for $z \in \mathbb{F}_{p^r}$. Setting $z = P(x)$, defining $W(x) = x_0 P(x)$ and summing over $x_0 \in \mathbb{F}_{p^r}$ and $(x_1, \ldots, x_n) \in (\mathbb{F}_{p^r}^*)^n$, we obtain a point counting formula for each N_r^*,

$$p^r N_r^* = (p^r - 1)^n + \sum_{x\in(\mathbb{F}_{p^r}^*)^{n+1}} \chi(W(x)). \tag{2.4}$$

It remains to make a suitable choice of character so that we can make a link with an operator on a space of power series.

2.1 The Dwork Character

Choose $\pi \in \overline{\mathbb{Q}}_p$ such that $\pi^{p-1} = -p$ and consider the field extension $\mathbb{Q}_p(\pi)$. This is a ramified extension of $\mathbb{Q}_p$ and is the largest number field we shall require. Where we previously used a lower case x as a formal variable in polynomials over finite fields, we shall now use an upper case X as the formal variable in polynomials and power series over $\mathbb{Q}_p(\pi)$. The one exception to this convention is the parameter λ (and φ) which is used to mean both $\lambda \in \mathbb{F}_p$ and the generalised parameter in $\mathbb{Q}_p$ (the former being related to the latter by the Teichmüller character). It should always be clear from context which is meant.

We also define the *Teichmüller lift* of a polynomial

$$W(x) = \sum_{\mathbf{v}\in\Delta} w_{\mathbf{v}} x^{\mathbf{v}} \in \mathbb{F}_p[x]$$

to be the polynomial $W(X) \in K[X]$ given by

$$W(X) = \sum_{\mathbf{v}\in\Delta} W_{\mathbf{v}} X^{\mathbf{v}}, \quad W_{\mathbf{v}} = \mathrm{Teich}(w_{\mathbf{v}}). \tag{2.5}$$

The *Dwork exponential* $F(X)$ is given for $|X|_p < 1$ by

$$F(X) = \exp\left[\pi(X - X^p)\right] \in \mathbb{Q}_p(\pi)[\![X]\!]. \tag{2.6}$$

The factor of π ensures that the exponential converges for $|X|_p < 1$. To evaluate at Teichmüller points, where $|X|_p = 1$, we must define $F(X)$ via the series

$$F(X) = \sum_{n=0}^{\infty} c_n (\pi X)^n, \tag{2.7}$$

where the coefficients $c_n \in \mathbb{Q}_p$ are determined by expanding the exponential (2.6). It is easy to see by differentiating the series that the coefficients are given by

$$c_n = \begin{cases} 0 & n < 0, \\ 1 & n = 0, \\ \frac{1}{n}\left(c_{n-1} + c_{n-p}\right) & n > 0. \end{cases} \tag{2.8}$$

One can show [19, Theorem 4.1] that these coefficients are bounded,

$$\mathrm{ord}_p(c_n \pi^n) \geq \frac{p-1}{p^2} n > \mathrm{ord}_p \pi, \tag{2.9}$$

where the final inequality holds for $p > 3$. It follows from this bound that the series converges on the *closed* p-adic unit disc. Note, however, that it only converges to the exponential expression (2.6) in the *open* disc. In particular this means that for $X = \mathrm{Teich}(x)$, where $|X|_p = 1$, we do *not* have $F(X) = 1$, even though $X = X^p$. However, for such an X, since $|pX|_p < 1$, we find that $F(X)$ is a p-th root of unity,

$$F(X)^p = \exp\left[p\pi\left(X - X^p\right)\right] = 1. \tag{2.10}$$

Moreover, $F(1)$ is the unique p-th root such that

$$F(1) = 1 + \pi + \mathcal{O}\big(\pi^2\big). \tag{2.11}$$

This is not a trivial root of unity since the coefficient bound (2.9) together with the fact that $|\pi|_p < 1$ ensure that the higher order terms cannot cancel π. It is clear that $F(1)^n$ is distinct for each $n \in \mathbb{F}_p$ and thus $F(1)$ is a primitive p-th root of unity. It follows that the p-th roots of unity in Ω_p are in fact contained in $\mathbb{Q}_p(\pi)$.

Inspired by this, let us define the *Dwork Character* $\chi : \mathbb{F}_p \to \Omega_p^\times$ by

$$\chi(x) = F(1)^x = F(1)^{\mathrm{Teich}(x)}. \tag{2.12}$$

It follows from (2.10) that $\chi(p) = 1$, so this map is well defined on $\mathbb{F}_p$. Thus χ is a group homomorphism from the additive group $\mathbb{F}_p$ to the multiplicative group of p^{th} roots of unity in Ω_p.

Raising (2.11) to power $\mathrm{Teich}(x)$ we find

$$\chi(x) = 1 + \pi\ \mathrm{Teich}(x) + \mathcal{O}\big(\pi^2\big), \tag{2.13}$$

which is also equal to $F(\mathrm{Teich}(x))$ since the bound (2.9) for $|c_n|_p$ ensures that higher terms in the expansion do not interfere with the behaviour mod π^2. We thus have that [19, Theorem 4.4]

$$\chi(x) = F\big(\mathrm{Teich}(x)\big). \tag{2.14}$$

Note that for $x \in \mathbb{F}_{p^r}$ for $r > 1$, $\mathrm{Teich}(x)$ is not in $\mathbb{Q}_p$ but the extension field $\mathbb{Q}_p(\pi)$. For such x, we cannot evaluate $F(1)^{\mathrm{Teich}(x)}$ but can still define $\chi(x)$ using (2.13).

We extend the Dwork character to $\mathbb{F}_{p^r}$ by means of the trace map from $\mathbb{F}_{p^r}$ to $\mathbb{F}_p$ (see Appendix D.1.2),

$$\chi_r(x) = \chi\big(\mathrm{tr}(x)\big) = \chi\left(\sum_{l=0}^{r-1} x^{p^l}\right). \tag{2.15}$$

Dwork proved that χ_r is given by the product

$$\chi_r(x) = \prod_{l=0}^{r-1} \chi\left(x^{p^l}\right), \tag{2.16}$$

where χ is defined on elements of $\mathbb{F}_{p^r}$ as described above.

The point counting formula (2.4) with the Dwork character χ_r is

$$p^r N_r^* = (p^r - 1)^n + \sum_{x \in (\mathbb{F}_{p^r}^*)^{n+1}} \prod_{l=0}^{r-1} \chi\left(\big(W(x)\big)^{p^l}\right). \tag{2.17}$$

2.2 A Space of Power Series

Next, we must show that right hand side of equation (2.17) is a superdeterminant. Let us introduce a space of power series spanned by an infinite basis of monomials

$$X^{\mathbf{v}} = \prod_{i=0}^{5} X_i^{v_i}, \quad \mathbf{v} \in \mathbb{Z}_{\geq 0}. \tag{2.18}$$

The exponent of X_0 is excluded from the degree of monomials so that

$$\deg X^{\mathbf{v}} \equiv \deg \mathbf{v} = \sum_{i=1}^{5} v_i. \tag{2.19}$$

Consider the lattice Λ of degree d monomials $X^{\mathbf{v}}$ with $\deg \mathbf{v} = dv_0$,

$$\left\{ \pi^{v_0} X^{\mathbf{v}} = \pi^{v_0} \prod_{i=0}^{n} X_i^{v_i} \; : \; \deg \mathbf{v} = dv_0 \right\}.$$

Denote by $\Delta \subset \Lambda$ the set of monomials occuring in $W(X)$, so that

$$W(X) = \sum_{\mathbf{v} \in \Delta} w_{\mathbf{v}} X^{\mathbf{v}}.$$

The monomials in the power series expansion of $e^{\pi W(X)}$ make up the cone K over Δ in Λ. The desired space of power series is the $\Omega_p[\![\lambda]\!]$-span of K subject to a suitable convergence condition,

$$\mathcal{H} = \left\{ \sum_{\mathbf{v}\in K} a_{\mathbf{v}}(\pi X)^{\mathbf{v}} : a_{\mathbf{v}} \in \Omega_p[\![\lambda]\!],\ a_{\mathbf{v}} \to 0 \text{ as } \deg \mathbf{v} \to \infty \text{ for each } \lambda \right\}. \tag{2.20}$$

This property, called *overconvergence*, ensures that a power series in $\mathcal{H}$ converges on the *closed* p-adic unit disc. It also has important consequences for the cohomology, which will be discussed at the end of Section 2.4. The Dwork exponential is overconvergent [19, Theorem 4.1], though $e^{\pm\pi W}$ are not.

We follow [9] in borrowing notation and terminology from quantum mechanics, and shall call elements of $\mathcal{H}$ *states*. We use bra-ket notation to denote monomials (i.e. basis elements) and states,

$$|\mathbf{v}\rangle = X^{\mathbf{v}}, \quad |\Psi\rangle = \sum_{\mathbf{v}\in K} \Psi_{\mathbf{v}} X^{\mathbf{v}}, \tag{2.21}$$

and introduce a dual basis $\langle \mathbf{u}|$ such that

$$\langle \mathbf{u}|\mathbf{v}\rangle = \delta_{\mathbf{u}\mathbf{v}}. \tag{2.22}$$

Given the ring structure of $\mathcal{H}$, bear in mind that $\Psi(X) \in \mathcal{H}$ is both a state and an operator acting on states by multiplication.

Now, χ_r is a function on $\mathbb{F}_{p^r}$, additionally parametrized by $\lambda \in \mathbb{F}_p$, but to take derivatives (and hence compute cohomology) we will need to lift to a field on which we can do analysis, namely the p-adic numbers Ω_p. What we need is a power series $G_\lambda(X) \in \mathbb{Q}_p(\pi)[\![\lambda, X]\!]$ such that for $x \in \mathbb{F}_{p^r}$ and $\lambda \in \mathbb{F}_p$,

$$\chi_r(x;\lambda) = G_{\mathrm{Teich}(\lambda)}(\mathrm{Teich}(x)). \tag{2.23}$$

We also require that $G_\lambda(X)$ converges on the closed unit λ-disc and is overconvergent in x for each λ. An exponential expression for $G_\lambda(X)$ can be obtained from the expression (2.6) for $F(X)$, but care must be taken since this expression only applies for $|X|_p < 1$.

Since the coefficients $w_{\mathbf{v}}$ of $W(x)$ are in $\mathbb{F}_p$ we have that $w_{\mathbf{v}}^p = w_{\mathbf{v}}$ for any $l \geq 1$, and thus we find

$$\big(W(x)\big)^{p^l} = \sum_{\mathbf{v}\in\Delta} w_{\mathbf{v}} x^{p^l \mathbf{v}} = W\big(x^{p^l}\big).$$

We then have that

$$
\begin{aligned}
\chi(W(x^{p^l})) &= \chi\left(\sum_{\mathbf{v}\in\Delta} w_{\mathbf{v}} x^{p^l \mathbf{v}}\right) \\
&= \prod_{\mathbf{v}\in\Delta} \chi\left(w_{\mathbf{v}} x^{p^l \mathbf{v}}\right) \\
&= \lim_{\substack{X\to\mathrm{Teich}(x)\\ W\to\mathrm{Teich}(w)}} \prod_{\mathbf{v}\in\Delta} \exp\left[\pi\left(W_{\mathbf{v}} X^{p^l\mathbf{v}} - W_{\mathbf{v}}{}^p X^{p^{l+1}\mathbf{v}}\right)\right] \qquad (2.24)\\
&= \lim_{\substack{X\to\mathrm{Teich}(x)\\ \lambda\to\mathrm{Teich}(\bar{\lambda})}} \exp\left[\pi\left(W_\lambda(X^{p^l}) - W_{\lambda^p}(X^{p^{l+1}})\right)\right] \\
&= \lim_{\substack{X\to\mathrm{Teich}(x)\\ \lambda\to\mathrm{Teich}(\bar{\lambda})}} \exp\left[\pi\left(G_\lambda(X^{p^l})\right)\right],
\end{aligned}
$$

where $G(X)$ is given for $|X|_p < 1, |\lambda|_p < 1$ by

$$G_\lambda(X) = \exp\left[\pi\big(W_\lambda(X) - W_{\lambda^p}(X^p)\big)\right], \qquad (2.25)$$

and on the unit disc by analytic continuation, as for F.

Since the Teichmüller character embeds $\mathbb{F}_{p^r}^*$ as the (p^r-1)st roots of unity in Ω_p, we can rewrite the expression (2.17) for N_r^* in terms of $G_\lambda(X)$ as

$$p^r N_r^* = (p^r-1)^n + \sum_{(X^\alpha)^{p^r-1}=1} \prod_{l=0}^{r-1} G_{\mathrm{Teich}(\lambda)}(X^{p^l}). \qquad (2.26)$$

The product on the right hand side will turn out to be the determinant we seek.

2.3 The Dwork Operator and the Zeta Function

In order to define the Dwork map, we first must introduce the Frobenius map $\mathrm{Fr}_p : x \to x^p$, and its inverse, the *Atkin operator* $\mathcal{A}_p$,

$$\mathcal{A}_p X^{\mathbf{v}} = \begin{cases} X^{\mathbf{v}/p}; & p \mid \mathbf{v}, \\ 0; & \text{otherwise,} \end{cases} \qquad (2.27)$$

where $p \mid \mathbf{v}$ if and only if $p \mid v_i$ for all $0 \le i \le d$. It is not hard to prove that the Atkin operator satisfies the following properties for $\Psi(X) \in \mathcal{H}$,

$$
\begin{aligned}
(\mathcal{A}_p)^r &= \mathcal{A}_{p^r}, \\
\mathcal{A}_p \Psi(X^p) &= \Psi(X)\mathcal{A}_p, \qquad (2.28)\\
(\mathcal{A}_p \Psi(X))^r &= \mathcal{A}_{p^r} \Psi(X^p)\Psi(X^{p^2}) \cdots \Psi(X^{p^{r-1}}).
\end{aligned}
$$

In the basis K, the components of the operator $\mathcal{A}_{p^r}\Psi(X)$ are

$$\langle\mathbf{u}|\,\mathcal{A}_{p^r}\Psi(X)\,|\mathbf{v}\rangle = \Psi_{p^r\mathbf{u}-\mathbf{v}}, \quad \text{where} \quad \Psi_{\mathbf{w}} = 0 \quad \text{for} \quad \mathbf{w} \notin K, \tag{2.29}$$

and by summing over the Teichmüller points we find for $\Psi(X) \in \mathcal{H}$,

$$\begin{aligned} \sum_{(X^\alpha)^{p^r-1}=1} \Psi(X) &= \sum_{\mathbf{v}\in K} \Psi_{\mathbf{v}} \sum_{(X^\alpha)^{p^r-1}=1} X^{p^r\mathbf{v}} \\ &= (p^r-1)^{n+1} \sum_{\mathbf{v}\in K} \Psi_{(p^r-1)\mathbf{v}} \\ &= (p^r-1)^{n+1}\,\mathrm{Tr}\big(\mathcal{A}_{p^r}\Psi(X)\big). \end{aligned} \tag{2.30}$$

The expression in the second line comes from the terms in the sum where $p^r - 1 \mid \mathbf{v}$. All terms where this is not the case become sums over the (p^r-1)st roots of unity or some subgroup of them, and thus contribute nothing to the sum.

At last, we obtain a trace formula from the point counting formula (2.26),

$$\begin{aligned} p^r N_r^* &= (p^r-1)^n + (p^r-1)^{n+1}\,\mathrm{Tr}\left(\mathcal{A}_{p^r}\prod_{l=0}^{r-1} G_{\mathrm{Teich}(\bar\lambda)}(X^{p^l})\right) \\ &= \frac{(p^r-1)^n}{p^r} + \frac{(p^r-1)^{n+1}}{p^r}\,\mathrm{Tr}\,\mathcal{U}^r, \end{aligned} \tag{2.31}$$

where in the second line we applied the third property of (2.28) and defined the *Dwork operator*

$$\mathcal{U}(\lambda) = \mathcal{A}_p G_\lambda(X) = \mathrm{e}^{-\pi W_{\lambda^p}}\mathcal{A}_p \mathrm{e}^{\pi W_\lambda}. \tag{2.32}$$

Notice the power of p arising in the last expression. It is important to realise that λ here is a generalised parameter in K and thus does not have $\lambda^p = \lambda$, which will only be true in the limit at the Teichmüller points.

To obtain an expression for Z^* we need only substitute the expressions for N_r^* back into the definition, whence we find

$$Z^*(T) = \prod_{j=0}^{n}\big(1 - p^{j-1}T\big)^{(-1)^{n-j+1}\binom{n}{j}} \prod_{k=0}^{n+1} \det{}_{\mathcal{H}}\big(1 - p^{k-1}T\,\mathcal{U}\big)^{(-1)^{n-k}\binom{n+1}{k}}. \tag{2.33}$$

The second factor in this expression appears to be the determinant of an operator on an infinite dimensional vector space. However, it turns out that all but a finite number of the eigenvalues cancel leaving a finite superdeterminant over the differential Dwork complex.

2.4 Dwork Cohomology

The Dwork cohomology is a cohomology of power series and we define the exterior derivative on power series in the natural way following [20],

$$\mathrm{d}f = \mathrm{d}\left(\sum_{\mathbf{v}} f_{\mathbf{v}} X^{\mathbf{v}}\right) = \sum_{\mathbf{v}} f_{\mathbf{v}} dX^{\mathbf{v}}.$$

It will be useful to define logarithmic derivatives and one-forms,

$$\xi^{\alpha} = \mathrm{d}\log X^{\alpha} = \frac{\mathrm{d}X^{\alpha}}{X^{\alpha}}, \quad D_{\alpha} = X^{\alpha}\partial_{\alpha}, \tag{2.34}$$

where there is no contraction over indices. Since $\mathcal{A}_q(X^{\alpha})^q = X^{\alpha}$, it makes sense to extend $\mathcal{A}_p$ to act on forms by

$$\mathcal{A}_q \xi^a = \frac{\xi^a}{q}. \tag{2.35}$$

It follows that $[\mathrm{d}, \mathcal{A}_q] = 0$, since

$$\begin{aligned} \mathrm{d}\Big(\mathcal{A}_q\big(\chi_{\alpha_1\cdots\alpha_l}(X)\xi^{\alpha_1\cdots\alpha_l}\big)\Big) &= \mathrm{d}\big(\chi_{\alpha_1\cdots\alpha_l}(X^{1/q})\big) \wedge q^{-l}\xi^{\alpha_1\cdots\alpha_l} \\ &= q^{-(l+1)}\mathrm{d}\chi(X^{1/q}) \\ &= \mathcal{A}_q \mathrm{d}\chi(X). \end{aligned} \tag{2.36}$$

The *Dwork exterior derivative* is then defined to be compatible with $\mathcal{U}(\lambda)$,

$$\mathcal{D}(\lambda) = \mathrm{e}^{-\pi W_\lambda}\mathrm{d}\,\mathrm{e}^{\pi W_\lambda}, \tag{2.37}$$

which commutes with $\mathcal{U}(\lambda)$ in the sense that

$$[\mathcal{D}, \mathcal{U}] = \mathcal{D}(\lambda)\mathcal{U}(\lambda) - \mathcal{U}(\lambda)\mathcal{D}(\lambda^p) = 0. \tag{2.38}$$

In particular, this means that $\mathcal{U}$ is compatible with the cohomology of the complex $\mathcal{H}^\bullet$ under $\mathcal{D}$, which we call the *Dwork cohomology.* This cohomology complex can equivalently be viewed as the twisted de-Rham cohomology of $\mathcal{H}^\bullet$, that is the ordinary de-Rham cohomology of differential forms $\mathrm{e}^{-\pi W}\Psi$ for $\Psi \in \mathcal{H}^\bullet$. Note that if we did have $\exp(\pm\pi W) \in \mathcal{H}$ then Dwork cohomology would not differ from the usual de-Rham cohomology of $\mathcal{H}^\bullet$.

Now consider the superdeterminant over the power series complex $\mathcal{H}^\bullet$,

$$\mathrm{sdet}_{\mathcal{H}^\bullet}(1 - p^n\mathcal{U}T) = \prod_{l=0}^{n+1}\{\det_{\mathcal{H}^l}(1 - p^n\mathcal{U}T)\}^{(-1)^l}. \tag{2.39}$$

An eigenvalue $\Psi \in \mathcal{H}^0 \cong \mathcal{H}$ of $\mathcal{U}$ with eigenvalue μ contributes $(1 - p^n\mu T)$ to $\det_{\mathcal{H}^0}$. Since $\mathcal{U}$ maps any basis form $\xi^{\alpha_1\cdots\alpha_l}$ to a multiple of itself, it is not hard to see that all eigen-l-forms are of the form $\Psi\xi^{\alpha_1\cdots\alpha_l}$ for some eigenform $\Psi \in \mathcal{H}^0$. There are $\binom{n+1}{l}$ of these which contribute to $\det_{\mathcal{H}^l}$ a total of

$$(1 - p^{n-l}\mu T)^{(-1)^l\binom{n+1}{l}}.$$

Taking the product over all such terms, it can be seen that the superdeterminant is precisely the second factor in the expression (2.33) for $Z^*(T)$.

Since $\mathcal{U}$ commutes with $\mathcal{D}$, there corresponds to each eigen-l-form Ψ of $\mathcal{U}$ an eigen-$(l+1)$-form $\mathcal{D}\Psi$ with the same eigenvalue and opposite grading. The contributions to the superdeterminant of such eigenforms cancel, and thus the superdeterminant reduces to a determinant over $\mathcal{H}^\bullet/\mathcal{D}\mathcal{H}^\bullet$,

$$\operatorname{sdet}_{\mathcal{H}^\bullet}(1 - p^n\mathcal{U}T) = \operatorname{sdet}_{\mathcal{H}^\bullet/\mathcal{D}\mathcal{H}^\bullet}(1 - p^n\mathcal{U}T). \tag{2.40}$$

Dwork showed [3] that only the highest cohomology group $\mathcal{H}^{n+1}/\mathcal{D}\mathcal{H}^n$ is nontrivial. Thus we need only consider the action of $\mathcal{U}$ on closed modulo exact $(n+1)$-forms Ψ. If $\Psi \in \mathcal{H}$ has eigenvalue μ then $\Psi\xi = \Psi\xi^{0\cdots n} \in \mathcal{H}^{n+1}$ will have eigenvalue $p^{-(n+1)}\mu$, so in effect we are computing the determinant over $\mathcal{H}/\mathcal{D}\mathcal{H}$,

$$\det_{\mathcal{H}/\mathcal{D}\mathcal{H}}(1 - p^{-1}\mathcal{U}T), \tag{2.41}$$

giving the final point counting formula

$$Z^*(T) = \prod_{j=0}^{n}\left(1 - p^{j-1}T\right)^{(-1)^{n-j+1}\binom{n}{j}} \det_{\mathcal{H}/\mathcal{D}\mathcal{H}}\left(1 - p^{-1}\mathcal{U}(\lambda)T\right). \tag{2.42}$$

It will in fact prove useful to use a different convention for Dwork cohomology. Instead of considering the exterior derivative $\mathcal{D}$, let us define logarithmic differential operators

$$\mathcal{D}_\alpha(\lambda) = \mathrm{e}^{-\pi W}D_\alpha\mathrm{e}^{\pi W} = D_\alpha + \pi D_\alpha W. \tag{2.43}$$

Let K^S be the subspace of K containing only monomials with at least one power of of each X_i,

$$K^S = \{\mathbf{v} \in K\colon v_i > 0,\ 0 \le i \le n\}, \tag{2.44}$$

and define $\mathcal{H}^S$ analogously to $\mathcal{H}$,

$$\mathcal{H} = \left\{\sum_{\mathbf{v}\in K^S} a_{\mathbf{v}}(\pi X)^{\mathbf{v}}\colon a_{\mathbf{v}} \in \mathbb{Q}_p[\![\lambda]\!],\ a_{\mathbf{v}} \to 0 \text{ as } \deg\mathbf{v} \to \infty\right\}. \tag{2.45}$$

Then we can take the determinant over the space

$$\mathcal{H}^S \Big/ \bigcup \mathcal{D}_i(\lambda)\mathcal{H}, \tag{2.46}$$

which we shall often denote simply $\mathcal{H}^S/\mathcal{D}\mathcal{H}$ for brevity, or $\mathcal{H}^S/\mathcal{D}_\lambda\mathcal{H}$ if we wish to emphasise the λ-dependence. That is to say, we consider power series in $\mathcal{H}^S$, containing only monomials “divisible” by $\prod_{i=0}^5 X_i$, modulo twisted logarithmic total derivatives of power series in $\mathcal{H}$,

$$\begin{aligned} 0 &= \mathcal{D}_0\Psi = D_0\Psi + \pi W_\lambda \Psi, \\ 0 &= \mathcal{D}_i\Psi = D_i\Psi + (\pi D_i W_\lambda)\Psi. \end{aligned} \tag{2.47}$$

We now take a moment to comment on the significance of the overconvergence property to the Dwork cohomology. It is vital that $F \in \mathcal{H}$, and thus the convergence property must be sufficiently lenient to allow this. On the other hand, one can show that $e^{\pm\pi W} \notin \mathcal{H}$. This, too, is vital: the Dwork cohomology is the de-Rham cohomology of states $e^{-\pi W}\Psi$ for $\Psi \in \mathcal{H}$. Were $e^{\pi W}$ overconvergent, one could define $\tilde{\Psi} = e^{\pi W}\Psi$ and the Dwork cohomology would simply be the usual de Rham cohomology of $\mathcal{H}$. Overconvergence also ensures that the Poincaré Lemma holds, namely that in a contractible space, any closed form is exact. With a weaker convergence condition, it is possible to construct a convergent, closed form as the total derivative of a non convergent form (see [9, p12]).

It is important to remember that $\mathcal{D}_\lambda$, and hence the cohomology spaces, are λ-dependent. We will append a λ subscript to denote this,

$$|\Psi\rangle_\lambda \, ,$$

with the states at the large structure point being denoted $|\Psi\rangle_0$.

Indeed, in a punctured disc about the large structure point $\lambda = 0$, the Dwork cohomology groups $\mathcal{H}^S/\mathcal{D}_\lambda\mathcal{H}$ form a bundle over the parameter space. The *Gauss-Manin* connection provides a notion of parallel transport on this bundle, governing the behaviour of the states as λ varies. Its action on $\mathcal{H}$ is given by

$$\Theta\Psi = e^{-\pi W}\vartheta\big(e^{\pi W}\Psi\big). \tag{2.48}$$

The task of computing $Z^*(T)$ has thus been reduced to the task of computing the action of $\mathcal{U}$ on $\mathcal{H}^S/\mathcal{D}\mathcal{H}$. First, one writes down the cohomology relations (2.47) and finds a basis of monomials for the resulting space. Then the action of $\mathcal{U}$ can be evaluated on this basis and the determinant $\det_{\mathcal{H}^S/\mathcal{D}\mathcal{H}}(1 - p^{-1}\mathcal{U}T)$ computed to obtain the zeta function. In the next chapter, we will determine such a basis for the quintic.

Chapter 3

The Dwork Cohomology of the Quintic

We now turn our attention to the quintic threefold which is perhaps the most studied of all one-parameter families of Calabi-Yau manifolds. After making some basic definitions our main task is to find a basis for the Dwork cohomology of the quintic. Such a basis was described in [5], and our exposition is similar to that article, though we work directly in Dwork cohomology rather than with the periods.

The one-parameter family of quintics $\mathcal{M}_\varphi$ is defined on $\mathbb{F}_q$, for $q = p^r$, by the vanishing of the polynomial

$$P(x) = \varphi \sum_{i=1}^{5} x_i^5 - \prod_{i=1}^{5} x_i, \quad \varphi \in \mathbb{F}_q. \tag{3.1}$$

The parameter φ here is the reciprocal of that used in [5, 9], the present choice being more conducive to our goal of expanding the Dwork operator around the large structure point $\varphi = 0$. For the same reason, the present definition of $P(x)$ differs from that in the cited work by an overall factor φ. Both here and in the cited works, however, the same convention is used for the parameter λ, which is given in terms of the present φ as

$$\lambda = \varphi^5.$$

Given the potential for confusion around the meaning of φ, we generally use λ as much as possible, though φ is often a more convenient choice when working with the Dwork cohomology. However, we will always refer to $\lambda = 0$ as "large (p-adic) structure" and $\lambda = \infty$ as "small (p-adic) structure".

It will be useful to name the φ-dependent and independent parts of $W(x) = x_0 P(x)$,

$$W_0(x) = \sum_{i=1}^{5} x_i^5, \quad Q(x) = \prod_{i=0}^{5} x_i,$$

so that we can write

$$W = \varphi W_0 - Q. \tag{3.2}$$

The quintic has Hodge numbers $h^{3,0} = 1$ and $h^{2,1} = 101$, and thus its Dwork cohomology is $b^3 = 204$ dimensional. The corresponding Picard-Fuchs operator governing the λ-dependence of the states will thus be of order 204. Fortunately, this 204$^{\text{th}}$ order operator splits under a symmetry of the manifold.

Indeed, we are not considering a general quintic threefold but a specific one parameter family which possesses an automorphism group $\mathcal{A}$. That is, the action of $\mathcal{A}$ on $\mathbb{F}_p\mathbb{P}^4$ leaves the vanishing locus of $P(x)$ fixed. We will see that the monomials can be classified by their transformation properties under the action of this group, that is they split into representations of $\mathcal{A}$.

Let $\zeta \in \mathbb{C}_p$ be a fifth root of unity. Then the maps

$$x_i \to \zeta^{n_i} x_i, \quad 1 \le i \le 5, \tag{3.3}$$

for $n \in (\mathbb{Z}/5\mathbb{Z})^5$ such that $\sum_{i=1}^{5} n_i = 0 \mod 5$, defined up to an overall scale,

$$x_i \to \zeta^m x_i, \quad 1 \le i \le 5 \quad \text{for} \quad m \in \mathbb{Z}/5\mathbb{Z}, \tag{3.4}$$

form a group $\mathcal{Z} \cong (\mathbb{Z}/5\mathbb{Z})^3$. Together with the permutation group $\mathcal{P}$ of the homogeneous coordinates, this forms the automorphism group $\mathcal{A} = \mathcal{Z} \times \mathcal{P}$ of $\mathcal{M}_\varphi$.

Since Q is invariant under $\mathcal{A}$, it follows that the monomials $Q^k X^{\mathbf{v}}$ belong to the same representation for all integers $k \ge 1$. Thus each representation is a collection of monomials of the form

$$Q^k X^{\mathbf{v}}, \quad k \ge 1, \tag{3.5}$$

for some set of vectors $\mathbf{v} \in \mathbb{Z}_{\ge 0}^5$. In Section 3.1 we will enumerate the possible vectors $\mathbf{v}$, of which there turn out to be a finite number by virtue of the Dwork cohomology relations. We will derive identities between monomials to determine which ones in fact yield distinct representations. We will show that up to total derivatives, each representation of $\mathcal{A}$ is the span of $Q^k X^{\mathbf{v}}$ for a single vector $\mathbf{v}$.

Then in Section 3.2, we shall see that each representation is preserved by the action of the Gauss-Manin connection Θ. Consequently, the order 204 Picard-Fuchs operator splits into a product of lower order operators, with one corresponding to each representation. This observation reduces the difficulty of finding the Picard-Fuchs operators, as the representations turn out to have relatively low degree.

The Picard-Fuchs operator gives a linear relation between the monomials in a representation and thus determines the dimension of that representation. In Section 3.3, we summarise the results for each representation and give a natural basis for it. Then in Sections 3.5 and 3.6 we consider the limiting cases of small structure $\varphi \to \infty$ and large structure $\varphi \to 0$. In the latter case, the cohomology will break down and it will be the work of Chapter 4 to repair it.

In an introductory exposition of the quintic, it might seem customary to include a discussion of its periods. Certainly, series expansions for these periods will be vital later when we expand the Dwork operator. We will obtain these not by integrating over $\mathcal{M}_\varphi$, but by solving the Picard-Fuchs equations obtained from the Dwork cohomology. Consequently, we reserve this for the end of this chapter, Section 3.4, after we have obtained the Picard-Fuchs equations for the states, from which we can deduce the Picard-Fuchs equations satisfied by the periods, as given in [5].

3.1 Enumerating Monomials

A power series Ψ in the Dwork cohomology is comprised of monomials

$$X^{\mathbf{v}} = \pi^{v_0} X_0^{v_0} \prod_{i=1}^{5} X_i^{v_i}, \quad v_i \geq 1 \quad \text{for} \quad 1 \leq i \leq 5, \quad v_0 = \tfrac{1}{5} \sum_{i=1}^{5} v_i \in \mathbb{Z}, \tag{3.6}$$

defined modulo the cohomology relations

$$0 = \mathcal{D}_\alpha \Psi = D_\alpha \Psi + D_\alpha(\pi W)\Psi. \tag{3.7}$$

To keep algebraic manipulation tidy, the appropriate power of π is included in the definition of $X^{\mathbf{v}}$, unlike in Chapter 2.

With $P(x)$ defined for the quintic by equation (3.1), this gives the identities

$$(D_0 - \pi Q)\Psi = -\varphi \pi W_0 \Psi, \tag{3.8}$$

$$(D_i - \pi Q)\Psi = -5\varphi \pi X_0 X_i^5 \Psi. \tag{3.9}$$

The second relation can always be used, given a monomial $X^{\mathbf{v}}$ with some $v_i \geq 5$, to reduce the power of X_i by 5 at the cost of introducing an additional term with an increased power of Q. It follows that for any monomial $X^{\mathbf{v}}$, there is an identity

$$X^{\mathbf{v}} = \sum_j c_j (\pi Q)^{k_j} X^{\mathbf{w}_j}, \tag{3.10}$$

for some integers $k_j \geq 1$, coefficients $c_j \in \mathbb{Q}_p(\pi)[\![\varphi]\!]$ and monomials $X^{\mathbf{w}_j}$ with all $(\mathbf{w}_j)_i \leq 4$ and at least one zero. A basis for the Dwork cohomology must be some subset of the monomials occuring generically in the sum. Our task is to enumerate all such vectors $\mathbf{w}$, of which there are a finite number, and determine the cohomology relations between them. We will find five distinct types of representation of $\mathcal{A}$, each corresponding to a vector $\mathbf{w}$ and its permutations. Extending the bra-ket notation of the previous chapter, we will write these monomials as

$$(\pi Q)^k X^{\mathbf{w}} = |k; \mathbf{w}\rangle = |k; w_1, w_2, w_3, w_4, w_5\rangle\,.$$

We start by tabulating all vectors $\mathbf{w} \in (\mathbb{Z}/5\mathbb{Z})^5$ with at least one component zero, ordered by degree, up to permutations of the components. Each permutation corresponds to a distinct representation of $\mathcal{A}$ and the number of such permutations is given in the *permutations* column.

deg $\mathbf{w}$	$\mathbf{w}$	monomials	permutations
0	$(0,0,0,0,0)$	$(\pi Q)^k$	1
5	$(4,1,0,0,0)$	$(\pi Q)^k \pi X_0 X_1^4 X_2$	20
5	$(3,2,0,0,0)$	$(\pi Q)^k \pi X_0 X_1^3 X_2^2$	20
5	$(3,1,1,0,0)$	$(\pi Q)^k \pi X_0 X_1^3 X_2 X_3$	30
5	$(2,2,1,0,0)$	$(\pi Q)^k \pi X_0 X_1^2 X_2^2 X_3$	30
5	$(2,1,1,1,0)$	$(\pi Q)^k \pi X_0 X_1^2 X_2 X_3 X_4$	20
10	$(4,4,2,0,0)$	$(\pi Q)^k \pi^2 X_0^2 X_1^4 X_2^4 X_3^2$	30
10	$(4,4,1,1,0)$	$(\pi Q)^k \pi^2 X_0^2 X_1^4 X_2^4 X_3 X_4$	30
10	$(4,3,3,0,0)$	$(\pi Q)^k \pi^2 X_0^2 X_1^4 X_2^3 X_3^3$	30
10	$(4,2,2,2,0)$	$(\pi Q)^k \pi^2 X_0^2 X_1^4 X_2^2 X_3^2 X_4^2$	20
10	$(3,3,3,1,0)$	$(\pi Q)^k \pi^2 X_0^2 X_1^3 X_2^3 X_3^3 X_4$	20
10	$(3,3,2,2,0)$	$(\pi Q)^k \pi^2 X_0^2 X_1^3 X_2^3 X_3^2 X_4^2$	30
10	$(4,3,2,1,0)$	$(\pi Q)^k \pi^2 X_0^2 X_1^4 X_2^3 X_3^2 X_4$	120
15	$(4,4,4,3,0)$	$(\pi Q)^k \pi^3 X_0^3 X_1^4 X_2^4 X_3^4 X_4^3$	20

Table 3.1: All vectors $\mathbf{w} \in (\mathbb{Z}/5\mathbb{Z})^4 \times \{0\}$, up to permutation

Table 3.1 lists every possible $\mathbf{w}$ and so the listed monomials are the *only* monomials (up to total derivatives). There are equivalences, however, and it turns out that few of the monomials listed in the table are truly distinct. Once we remove this redundancy, we will be left with one vector $\mathbf{w}$ for each representation.

Note the distinction between the vectors $\mathbf{w}$, which represent monomial families and are only permitted to take the values listed in the table, and vectors $\mathbf{v} = \mathbf{w} + k\mathbf{1}$ for $k \geq 1$ which represent arbitrary monomials. It will be convenient to have a notation for the cohomology class of the monomials corresponding to some $\mathbf{w}$ in the table. Let us define

$$\langle \mathbf{w} \rangle = \left\{ \Psi \in \mathcal{H}^S/\mathcal{D}\mathcal{H} \,:\, \Psi = \sum_j c_j (\pi Q)^{k_j} X^{\mathbf{w}} + \mathcal{D}\chi \right\}. \tag{3.11}$$

In terms of a generic vector $\mathbf{v}$, the cohomology equivalence induced by $\mathcal{D}_1$ can be represented schematically as

$$\begin{array}{ccc} |k; v_1+5, v_2, v_3, v_4, v_5\rangle & \xrightarrow{D_1} & |k; v_1, v_2, v_3, v_4, v_5\rangle \\ \big\downarrow{\scriptstyle \pi Q} & & \\ |k+1; v_1, v_2, v_3, v_4, v_5+1\rangle & & \end{array}$$

Similar diagrams can be made for the other derivatives $\mathcal{D}_i$ for $2 \leq i \leq 5$. Such visualisations, which omit the precise factors involved, are extremely useful since they enable us to quickly create larger diagrams to reveal equivalences between monomials without having to compute exact expressions. The initial monomial can always be written as some $\mathbb{Q}_p(\pi)[\![\varphi]\!]$-linear combination of the terminating monomials, and one can trace back along the arrows to obtain the explicit coefficients if needed. The downward arrows always carry a coefficient of $\frac{1}{5\varphi}$, while the rightward arrows introduce a factor of $\frac{k+v_i}{5\varphi}$ (that is, the action of D_i on $|k; \mathbf{v}\rangle$).

We can also "borrow" from k to reduce a factor of X_i^4 in a monomial, since

$$|k+1; v_1, v_2, v_3, v_4, v_5\rangle = |k; v_1+1, v_2+1, v_3+1, v_4+1, v_5+1\rangle\,,$$

so that we can construct the following diagram.

$$\begin{array}{ccc} |k; 4, v_2, v_3, v_4, v_5\rangle & \xrightarrow{D_1} & |k-1; 0, v_2+1, v_3+1, v_4+1, v_5+1\rangle \\ \big\downarrow{\scriptstyle \pi Q} & & \\ |k; 0, v_2+1, v_3+1, v_4+1, v_5+1\rangle & & \end{array}$$

Of course, we do not allow monomials with no factors of Q so we have to be careful in the case that $k = 1$. Fortunately, the coefficient of the righthand term in this case is zero (since it equals $k - 1$).

We are now ready to write down, with the help of some diagrams, the relations between various monomials listed in the table. It will turn out that the vectors $\mathbf{w}$ of degree 0 and 5 are sufficient to label (almost) every representation.

3.1.1 Case 0

The first and simplest representation to consider is

$$\langle \mathbf{w}_0 \rangle = \langle (0,0,0,0,0) \rangle \, . \tag{3.12}$$

It contains all monomials which are powers of πQ alone, which are invariant under the action of $\mathcal{A}$. The monomials $\pi X_0 X_i^5$ also belong to $\langle \mathbf{w}_0 \rangle$, since they are equivalent to πQ in cohomology. There is nothing more to do until Section 3.2.1 where we will obtain a linear dependence between monomials $(\pi Q)^k$ and show that $\langle \mathbf{w}_0 \rangle$ is four-dimensional.

We note also that this space is shared by the mirror quintic [21], where it makes up the entire Dwork cohomology of dimension $b^3 = 4$ ($h^{3,0} = h^{2,1} = 1$). This, of course, means that the Picard-Fuchs operator and periods we later write down in this case are also common to the mirror quintic.

3.1.2 Case 1

This is the one exceptional case, where in fact two $\mathbf{w}$ of degree 5 are equivalent and both belong to this single representation. We will show that the monomials differ by a total derivative, but one can also check that they transform in the same way under $\mathcal{A}$.

Define the vectors

$$\begin{aligned} \mathbf{w}_1 &= (4,1,0,0,0), \\ \mathbf{w}_1' &= (0,2,1,1,1). \end{aligned} \tag{3.13}$$

The equivalence of monomials $(\pi Q)^k X^{\mathbf{w}}$ is revealed by a simple diagram.

$$\begin{array}{ccc}
\left|k;4,1,0,0,0\right\rangle & \xrightarrow{D_1} & \left|k-1;0,2,1,1,1\right\rangle \\
\big\downarrow{\scriptstyle \pi Q} & & \\
\left|k;0,2,1,1,1\right\rangle & &
\end{array}$$

As already mentioned, in the case $k = 1$, the rightmost term has coefficient zero, so no invalid state enters the identity, and the exact cohomology relation is

$$\left|1;\mathbf{w}_1\right\rangle = \tfrac{1}{5\varphi}\left|1;\mathbf{w}_1'\right\rangle .$$

The first and last degree 5 vectors $\mathbf{w}$ listed in the table are in fact equivalent. We will generally choose to use the first one, $\mathbf{w}_1 = (4, 1, 0, 0, 0)$, to represent this class. The diagram above can be reversed to rewrite a monomial $|k';\mathbf{w}_1'\rangle$ as a linear combination of monomials $|k;\mathbf{w}_1\rangle$ for $k \geq 1$ and thus show that $\langle \mathbf{w}_1' \rangle \subseteq \langle \mathbf{w}_1 \rangle$.

$$\begin{array}{ccc}
\left|k;\mathbf{w}_1'\right\rangle & \longrightarrow & \left|k;\mathbf{w}_1\right\rangle \\
\big\downarrow & & \\
\left|k-1;\mathbf{w}_1'\right\rangle & \longrightarrow & \left|k-1;\mathbf{w}-1\right\rangle \\
\big\downarrow & & \\
\vdots & & \\
\big\downarrow & & \\
\left|1;\mathbf{w}_1'\right\rangle & \longrightarrow & \left|1;\mathbf{w}_1\right\rangle
\end{array}$$

The span of the single degree 15 vector, $\langle(3, 0, 4, 4, 4)\rangle$ is also contained in $\langle \mathbf{w}_1 \rangle$. The following diagram shows how we can represent its elements as linear combinations of monomials $|k;\mathbf{w}_1\rangle$ for $k \geq 1$.

$$\begin{array}{ccccccc}
\left|k;3,0,4,4,4\right\rangle & \xrightarrow{D_3} & \left|k-1;4,1,0,5,5\right\rangle & \xrightarrow{D_4} & \left|k-1;4,1,0,0,5\right\rangle & \xrightarrow{D_5} & \left|k-1;\mathbf{w}_1\right\rangle \\
\big\downarrow{\scriptstyle \pi Q} & & \big\downarrow{\scriptstyle \pi Q} & & \big\downarrow{\scriptstyle \pi Q} & & \\
\left|k;4,1,0,5,5\right\rangle & \xrightarrow{D_4} & \left|k;4,1,0,0,5\right\rangle & \xrightarrow{D_5} & \left|k;\mathbf{w}_1\right\rangle & & \\
\big\downarrow{\scriptstyle \pi Q} & & \big\downarrow{\scriptstyle \pi Q} & & & & \\
\left|k+1;4,1,0,0,5\right\rangle & \xrightarrow{D_5} & \left|k+1;\mathbf{w}_1\right\rangle & & & & \\
\big\downarrow{\scriptstyle \pi Q} & & & & & & \\
\left|k+2;\mathbf{w}_1\right\rangle & & & & & &
\end{array}$$

We shall see that $\mathbf{w}_1'$ is the only degree 5 vector to be eliminated. Moreover, it will turn out that every monomial in the Dwork cohomology belongs to $\langle \mathbf{w} \rangle$ for some $\mathbf{w}$ of degree 0 or 5 (with one exception—see Section 3.1.6 below). The combinatorics already give some hints as to which monomials go together—multiplication by a power of πQ cannot change the number of permutations of $\mathbf{w}$. For example, $(4,4,2,0,0)$, with 30 permutations cannot be of the same type as $\mathbf{w}_1 = (4,1,0,0,0)$ which has only 20.

3.1.3 Case 2

Next, consider the representation

$$\langle \mathbf{w}_2 \rangle = \langle (3,2,0,0,0) \rangle \, .$$

We proceed, as before, by showing that monomials corresponding to various higher degree vectors $\mathbf{w}$ from the table (or rather permutations of them) belong to this space.

$$\begin{array}{ccccc}
|k+1;3,2,0,0,0\rangle & \xrightarrow{D_1} & |k-1;0,4,2,2,2\rangle & \xrightarrow{D_2} & |k-2;1,0,3,3,3\rangle \\
\big\downarrow{\scriptstyle \pi Q} & & \big\downarrow{\scriptstyle \pi Q} & & \\
|k;0,4,2,2,2\rangle & \xrightarrow{D_2} & |k-1;1,0,3,3,3\rangle & & \\
\big\downarrow{\scriptstyle \pi Q} & & & & \\
|k;1,0,3,3,3\rangle & & & &
\end{array}$$

These operations can be reversed to write the monomials

$$|k;1,0,3,3,3\rangle \, ,$$
$$|k;0,4,2,2,2\rangle$$

as linear combinations of $|k;\mathbf{w}_2\rangle$ for $k \geq 1$, whence we see that

$$\langle (1,0,3,3,3) \rangle \subseteq \langle \mathbf{w}_2 \rangle \, ,$$
$$\langle (0,4,2,2,2) \rangle \subseteq \langle \mathbf{w}_2 \rangle \, .$$

3.1.4 Case 3

Next, we consider the representation

$$\langle \mathbf{w}_3 \rangle = \langle (3,1,1,0,0) \rangle \, ,$$

and show that $\langle (2,0,0,4,4) \rangle \subseteq \langle \mathbf{w}_3 \rangle$.

$$
\begin{array}{ccccc}
|k;2,0,0,4,4\rangle & \xrightarrow{D_4} & |k-1;3,1,1,0,5\rangle & \xrightarrow{D_5} & |k-1;\mathbf{w}_3\rangle \\
\big\downarrow{\scriptstyle \pi Q} & & \big\downarrow{\scriptstyle \pi Q} & & \\
|k;3,1,1,0,5\rangle & \xrightarrow{D_5} & |k;\mathbf{w}_3\rangle & & \\
\big\downarrow{\scriptstyle \pi Q} & & & & \\
|k+1;\mathbf{w}_3\rangle & & & &
\end{array}
$$

We see similarly that $\langle(0,3,3,2,2)\rangle \subseteq \langle\mathbf{w}_3\rangle$, by reversing the following diagram.

$$
\begin{array}{ccc}
|k+1;3,1,1,0,0\rangle & \xrightarrow{D_1} & |k-1;0,3,3,2,2\rangle \\
\big\downarrow{\scriptstyle \pi Q} & & \\
|k;0,3,3,2,2\rangle & &
\end{array}
$$

3.1.5 Case 4

In this case, we show that the representation

$$\langle\mathbf{w}_4\rangle = \langle(2,2,1,0,0)\rangle$$

contains both $\langle(0,0,4,3,3)\rangle$ and $\langle(1,1,0,4,4)\rangle$.

$$
\begin{array}{ccccccc}
|k;0,0,4,3,3\rangle & \xrightarrow{D_3} & |k-1;1,1,0,4,4\rangle & \xrightarrow{D_4} & |k-2;2,2,1,0,5\rangle & \xrightarrow{D_5} & |k-2;\mathbf{w}_4\rangle \\
\big\downarrow{\scriptstyle \pi Q} & & \big\downarrow{\scriptstyle \pi Q} & & \big\downarrow{\scriptstyle \pi Q} & & \\
|k;1,1,0,4,4\rangle & \xrightarrow{D_4} & |k-1;2,2,1,0,5\rangle & \xrightarrow{D_5} & |k-1;\mathbf{w}_4\rangle & & \\
\big\downarrow{\scriptstyle \pi Q} & & \big\downarrow{\scriptstyle \pi Q} & & & & \\
|k;2,2,1,0,5\rangle & \xrightarrow{D_5} & |k;\mathbf{w}_4\rangle & & & & \\
\big\downarrow{\scriptstyle \pi Q} & & & & & & \\
|k+1;\mathbf{w}_4\rangle & & & & & &
\end{array}
$$

3.1.6 Case 5

Checking back to the table, we see that we have now accounted for every $\mathbf{w}$ but one, namely $\mathbf{w}_5 = (4,3,2,1,0)$. The monomial lives in a representation of its own, and in fact will turn out to be trivial in cohomology except at the conifold point (see Section 3.2.6). For now, we simply observe that what appear to be 120 distinct cases are actually only 24. The cohomology relation is as follows.

$$\begin{array}{ccc} |k;4,3,2,1,0\rangle & \xrightarrow{D_1} & |k-1;0,4,3,2,1\rangle \\ \big\downarrow{\scriptstyle \pi Q} & & \\ |k;0,4,3,2,1\rangle & & \end{array}$$

We see that only the relative ordering of the digits matters, and not the index of $\mathbf{w}$ where we place the 4. There are thus $120/5 = 24$ permutations.

3.1.7 Summary

Having eliminated all redundancy, we now restate the table from the start of the section retaining only the monomials required to label each unique representation.

case	deg $\mathbf{w}$	$\mathbf{w}$	monomials	permutations
0	0	$(0,0,0,0,0)$	$(\pi Q)^k$	1
1	5	$(4,1,0,0,0)$	$(\pi Q)^k \pi X_0 X_1^4 X_2$	20
2	5	$(3,2,0,0,0)$	$(\pi Q)^k \pi X_0 X_1^3 X_2^2$	20
3	5	$(3,1,1,0,0)$	$(\pi Q)^k \pi X_0 X_1^3 X_2 X_3$	30
4	5	$(2,2,1,0,0)$	$(\pi Q)^k \pi X_0 X_1^2 X_2^2 X_3$	30
5	10	$(4,3,2,1,0)$	$(\pi Q)^k \pi^2 X_0^2 X_1^4 X_2^3 X_3^2 X_4$	24

Table 3.2: The monomials $\mathbf{w}$ labelling the representations of $\mathcal{A}$

What remains is to establish the dimension of each representation. In the next section we will do this by showing that not all $|k;\mathbf{w}\rangle$ are linearly independent in each case. Closely related are the differential equations satisfied by certain states in each representation, which we shall also identify.

3.2 The Picard-Fuchs Equations

The dimension of each representation is determined by the order of the corresponding Picard-Fuchs equation, which is a factor of the 204^{th} order Picard-Fuchs equation of the quintic. To write down Picard-Fuchs equations on Dwork cohomology states, we require a λ derivative twisted by $e^{\pi W}$ which will be compatible with Dwork cohomology. This is precisely the Gauss-Manin connection Θ defined in equation (2.48). For the quintic, this becomes

$$\Theta\Psi = \left(\vartheta + \frac{1}{5}\varphi\pi W_0\right)\Psi. \tag{3.14}$$

Applying the cohomology relation (3.8), we can rewrite its action up to a total derivative as

$$\Theta\Psi = \left(\vartheta - \frac{1}{5}(D_0 - \pi Q)\right)\Psi. \tag{3.15}$$

Using equation (3.9), we can further replace the πQ with a D_i for any $1 \leq i \leq 5$,

$$\Theta\Psi = \left(\vartheta - \frac{1}{5}(D_0 - D_i) + \varphi\pi X_0 X_i^5\right)\Psi, \tag{3.16}$$

or in bra-ket notation,

$$\begin{aligned} 5\varphi \left|k; v_1+5, v_2, v_3, v_4, v_5\right\rangle &= \big(5\Theta - 5\vartheta + D_0 - D_1\big)\left|k; v_1, v_2, v_3, v_4, v_5\right\rangle \\ &= \big(5\Theta + v_0 - v_1\big)\left|k; v_1, v_2, v_3, v_4, v_5\right\rangle, \end{aligned} \tag{3.17}$$

where in the second line we used that $|k; v_1, v_2, v_3, v_4, v_5\rangle$ has no explicit λ-dependence.

3.2.1 Case 0

There is precisely one representation of the form

$$\langle \mathbf{w}_0 \rangle = \langle (0,0,0,0,0) \rangle, \tag{3.18}$$

which will turn out to be four-dimensional. We thus seek a fourth order differential equation satisfied by $|1; \mathbf{w}_0\rangle$, which will follow from a linear relation on $|k; \mathbf{w}_0\rangle$ for $1 \leq k \leq 5$.

Including the precise factors, the cohomology reduction on powers of X_1^5 can be written

$$|k; v_1+5, v_2, v_3, v_4, v_5\rangle = \frac{1}{5\varphi}\Big(\left|k+1; v_1, v_2, v_3, v_4, v_5\right\rangle - (v_1+k)\left|k; v_1, v_2, v_3, v_4, v_5\right\rangle\Big).$$

We start with $(\pi Q)^5$ and successively reduce each X_i^5 using the relation above and the analogous relation for each X_i.

$$\begin{aligned} |5; \mathbf{w}_0\rangle &= |0; 5, 5, 5, 5, 5\rangle \\ &= \tfrac{1}{5\varphi}\,|1; 0, 5, 5, 5, 5\rangle \\ &= \tfrac{1}{(5\varphi)^2}\big(\,|2; 0, 0, 5, 5, 5\rangle - |1; 0, 0, 5, 5, 5\rangle\,\big) \\ &= \tfrac{1}{(5\varphi)^3}\big(\,|3; 0, 0, 0, 5, 5\rangle - 3\,|2, 0, 0, 0, 5, 5\rangle + |1, 0, 0, 0, 5, 5\rangle\,\big) \\ &= \tfrac{1}{(5\varphi)^4}\big(\,|4; 0, 0, 0, 0, 5\rangle - 6\,|3; 0, 0, 0, 0, 5\rangle + 7\,|2; 0, 0, 0, 0, 5\rangle - |1; 0, 0, 0, 0, 5\rangle\,\big) \\ &= \tfrac{1}{(5\varphi)^5}\big(\,|5; \mathbf{w}_0\rangle - 10\,|4; \mathbf{w}_0\rangle + 25\,|3; \mathbf{w}_0\rangle - 15\,|2; \mathbf{w}_0\rangle + |1; \mathbf{w}_0\rangle\,\big). \end{aligned}$$

Rearranging gives the linear relation we sought,

$$(1-5^5\lambda)\,|5;\mathbf{w}_0\rangle - 10\,|4;\mathbf{w}_0\rangle + 25\,|3;\mathbf{w}_0\rangle - 15\,|2;\mathbf{w}_0\rangle + |1;\mathbf{w}_0\rangle = 0. \tag{3.19}$$

One can thus take $|k;\mathbf{w}_0\rangle$ for $1 \le k \le 4$ as a basis of $\langle \mathbf{w}_0 \rangle$. To obtain the Picard-Fuchs equation, we make use of the relationship between the action of Θ and of multiplication by πQ.

Borrowing further from the parlance of quantum mechanics, let us define the "base state",

$$|\Upsilon_0\rangle = |1;\mathbf{w}_0\rangle = |\pi Q\rangle\,. \tag{3.20}$$

Applying the expression (3.15) for the action of Θ on cohomology states, we find

$$\begin{aligned}
(5\Theta)\,|\Upsilon_0\rangle &= (\pi Q - 1)\,|\Upsilon_0\rangle\,,\\
(5\Theta)^2\,|\Upsilon_0\rangle &= \big((\pi Q)^2 - 3\pi Q + 1\big)\,|\Upsilon_0\rangle\,,\\
(5\Theta)^3\,|\Upsilon_0\rangle &= \big((\pi Q)^3 - 6(\pi Q)^2 + 7\pi Q - 1\big)\,|\Upsilon_0\rangle\,,\\
(5\Theta)^4\,|\Upsilon_0\rangle &= \big((\pi Q)^4 - 10(\pi Q)^3 + 25(\pi Q)^2 - 15\pi Q + 1\big)\,|\Upsilon_0\rangle\,.
\end{aligned} \tag{3.21}$$

Reversing this, we obtain explicit expressions for each $|k;\mathbf{w}_0\rangle$, $2 \le k \le 5$ as a linear combination of derivatives of $|\Upsilon_0\rangle$,

$$\begin{aligned}
|2;\mathbf{w}_0\rangle &= \big((5\Theta) + 1\big)\,|\Upsilon_0\rangle\,,\\
|3;\mathbf{w}_0\rangle &= \big((5\Theta)^2 + 3(5\Theta) + 2\big)\,|\Upsilon_0\rangle\,,\\
|4;\mathbf{w}_0\rangle &= \big((5\Theta)^3 + 6(5\Theta)^2 + 11(5\Theta) + 6\big)\,|\Upsilon_0\rangle\,,\\
|5;\mathbf{w}_0\rangle &= \big((5\Theta)^4 + 10(5\Theta)^3 + 35(5\Theta)^2 + 50(5\Theta) + 24\big)\,|\Upsilon_0\rangle\,.
\end{aligned} \tag{3.22}$$

At last, combining (3.19) and (3.22), we obtain the Picard-Fuchs equation satisfied by $|\Upsilon_0\rangle$.

$$\mathcal{P}_0\,|\Upsilon_0\rangle = \left(\Theta^4 - 5^5\lambda\prod_{i=1}^{4}\left(\Theta + \tfrac{i}{5}\right)\right)|\Upsilon_0\rangle = 0. \tag{3.23}$$

This four dimensional subspace is our primary focus. Indeed in Chapter 4, it is this subspace which we shall reconstruct using formal logarithmic states. Moreover this is the subspace on which $\mathcal{U}$ can be used to compute the quartic factor $R(T)$ of the zeta function. We thus denote this space

$$\mathcal{S}_\lambda = \langle \mathbf{w}_0 \rangle\,.$$

3.2.2 Case 1

Next, consider the representation

$$\langle \mathbf{w}_1 \rangle = \langle (4,1,0,0,0) \rangle \,,$$

and its 20 permutations, which we shall soon see to be two dimensional. We begin by seeking a linear relation between the first three spanning monomials. Reducing powers of X_i^5 as before, we find

$$\begin{aligned} |3;\mathbf{w}_1\rangle &= |2;5,2,1,1,1\rangle \\ &= \frac{1}{5\varphi}\big(\,|3;0,2,1,1,1\rangle - 2\,|2;0,2,1,1,1\rangle\,\big) \\ &= \frac{1}{5\varphi}\big(\,|3;\mathbf{w}_1'\rangle - 2\,|2;\mathbf{w}_1'\rangle\,\big), \end{aligned} \tag{3.24}$$

where we have used the alternative vector $\mathbf{w}_1' = (0,2,1,1,1)$. It is now necessary to express these $\mathbf{w}_1'$-monomials in terms of $\mathbf{w}_1$-monomials. We showed earlier that

$$|1;\mathbf{w}_1'\rangle = 5\varphi\,|1;\mathbf{w}_1\rangle \,. \tag{3.25}$$

We similarly find that

$$\begin{aligned} |2;\mathbf{w}_1\rangle &= |1;5,2,1,1,1\rangle \\ &= \frac{1}{5\varphi}\big(\,|2;0,2,1,1,1\rangle - |1;0,2,1,1,1\rangle\,\big) \\ &= \frac{1}{5\varphi}\big(\,|2;\mathbf{w}_1'\rangle - |1;\mathbf{w}_1'\rangle\,\big), \end{aligned} \tag{3.26}$$

which gives

$$|2;\mathbf{w}_1'\rangle = 5\varphi\big(\,|2;\mathbf{w}_1\rangle + |1;\mathbf{w}_1\rangle\,\big). \tag{3.27}$$

And finally,

$$\begin{aligned} |3;\mathbf{w}_1'\rangle &= |0;3,5,4,4,4\rangle \\ &= \frac{1}{5\varphi}\,|1;3,0,4,4,4\rangle \\ &= \frac{1}{5\varphi}\,|0;4,1,5,5,5\rangle \\ &= \frac{1}{(5\varphi)^2}\,|1;4,1,0,5,5\rangle \\ &= \frac{1}{(5\varphi)^3}\big(\,|2;4,1,0,0,5\rangle - |1;4,1,0,0,5\rangle\,\big) \\ &= \frac{1}{(5\varphi)^4}\big(\,|3;\mathbf{w}_1\rangle - 3\,|2;\mathbf{w}_1\rangle + |1;\mathbf{w}_1\rangle\,\big). \end{aligned} \tag{3.28}$$

Substituting (3.27) and (3.28) back into (3.24) we obtain the desired relation,

$$(1-5^5\lambda)\,|3;\mathbf{w}_1\rangle - (3+2\cdot 5^5\lambda)\,|2;\mathbf{w}_1\rangle + (1-2\cdot 5^5\lambda)\,|1;\mathbf{w}_1\rangle = 0. \tag{3.29}$$

From here, we can obtain the Picard-Fuchs equation as we did in the previous case. Define the "base state"

$$|\Upsilon_1\rangle = 5\varphi\,|1;\mathbf{w}_1\rangle = 5\varphi\pi^2 Q X_0 X_1^4 X_2, \tag{3.30}$$

where the factor of φ has been introduced as a matter of convention. Applying (3.17) for the action of Θ gives

$$\begin{aligned} 5\varphi\,|2;\mathbf{w}_1\rangle &= 5\varphi\,|1;5,2,1,1,1\rangle \\ &= 5\Theta\,|1;\mathbf{w}_1'\rangle + |1;\mathbf{w}_1'\rangle \\ &= 5\Theta\big(5\varphi\,|1;\mathbf{w}_1\rangle\big) + 5\varphi\,|1;\mathbf{w}\rangle\,, \end{aligned} \tag{3.31}$$

where we used that Θ is compatible with the cohomology. Similarly,

$$\begin{aligned} 5\varphi\,|3;\mathbf{w}_1\rangle &= 5\varphi\,|2;5,2,1,1,1\rangle \\ &= \big(5\Theta+1\big)\,|2;\mathbf{w}_1'\rangle \\ &= \big(5\Theta+1\big)\big(5\varphi\,|2;\mathbf{w}_1\rangle + 5\varphi\,|1;\mathbf{w}_1\rangle\big) \\ &= (5\Theta)^2\big(5\varphi\,|1;\mathbf{w}_1\rangle\big) + 3(5\Theta)\,|1;\mathbf{w}_1\rangle + 2\,|1;\mathbf{w}_1\rangle\,. \end{aligned} \tag{3.32}$$

Multiplying through by 5φ in (3.29) and substituting in equations (3.31) and (3.32), we obtain the Picard-Fuchs equation

$$\mathcal{P}_1\,|\Upsilon_1\rangle = \left(\Theta^2 - 5^5\lambda\left(\Theta+\frac{2}{5}\right)\left(\Theta+\frac{3}{5}\right)\right)|\Upsilon_1\rangle = 0. \tag{3.33}$$

3.2.3 Case 2

Consider the representation

$$\langle\mathbf{w}_2\rangle = \langle(3,2,0,0,0)\rangle\,,$$

of which there are 20 permutations, each being two dimensional. We proceed in a similar manner to Case 1.

$$\begin{aligned} |3;\mathbf{w}_2\rangle &= |1;5,4,2,2,2\rangle \\ &= \frac{1}{5\varphi}\big(\,|2;0,4,2,2,2\rangle - |1;0,4,2,2,2\rangle\,\big) \\ &= \frac{1}{5\varphi}\big(\,|1;1,5,3,3,3\rangle - |0;1,5,3,3,3\rangle\,\big) \\ &= \frac{1}{(5\varphi)^2}\big(\,|2;1,0,3,3,3\rangle - 2\,|1;1,0,3,3,3\rangle\,\big). \end{aligned} \tag{3.34}$$

For the first term, notice that

$$\begin{aligned}
|2;1,0,3,3,3\rangle &= |0;3,2,5,5,5\rangle \\
&= \frac{1}{5\varphi}\,|1;3,2,0,5,5\rangle \\
&= \frac{1}{(5\varphi)^2}\big(\,|2;3,2,0,0,5\rangle - |1;3,2,0,0,5\rangle\,\big) \\
&= \frac{1}{(5\varphi)^3}\big(\,|3;\mathbf{w}_2\rangle - 3\,|2;\mathbf{w}_2\rangle + |1;\mathbf{w}_2\rangle\,\big),
\end{aligned} \tag{3.35}$$

and for the second,

$$\begin{aligned}
|2;\mathbf{w}_2\rangle &= |0;5,4,2,2,2\rangle \\
&= \frac{1}{5\varphi}\,|1;0,4,2,2,2\rangle \\
&= \frac{1}{5\varphi}\,|0;1,5,3,3,3\rangle \\
&= \frac{1}{(5\varphi)^2}\,|1;1,0,3,3,3\rangle\,.
\end{aligned} \tag{3.36}$$

We thus obtain the linear relation,

$$(1-5^5\lambda)\,|3;\mathbf{w}_2\rangle - (3+2\cdot 5^5\lambda)\,|2;\mathbf{w}_2\rangle + |1;\mathbf{w}_2\rangle = 0. \tag{3.37}$$

Define the base state

$$|\Upsilon_2\rangle = 5\varphi\,|1;\mathbf{w}_2\rangle = \big|5\varphi\pi^2 Q X_0 X_1^3 X_2^2\big\rangle\,, \tag{3.38}$$

on which the action of Θ is given by

$$\begin{aligned}
5\Theta\,|\Upsilon_2\rangle &= \left(\varphi\frac{\mathrm{d}}{\mathrm{d}\varphi} + \pi Q - D_0\right) 5\varphi\,|1;3,2,0,0,0\rangle \\
&= 5\varphi\big(\,|2;\mathbf{w}_2\rangle - |1;\mathbf{w}_2\rangle\,\big),
\end{aligned} \tag{3.39}$$

$$\begin{aligned}
(5\Theta)^2\,|\Upsilon_2\rangle &= 5\Theta\left(\pi Q\,|\Upsilon_2\rangle - |\Upsilon_2\rangle\right) \\
&= 5\varphi\big(\,|1;\mathbf{w}_2\rangle - |2;\mathbf{w}_2\rangle\,\big) + \left(\varphi\frac{\mathrm{d}}{\mathrm{d}\varphi} + \pi Q - D_0\right)\big(5\varphi\,|2;3,2,0,0,0\rangle\,\big) \\
&= 5\varphi\big(\,|3;\mathbf{w}_2\rangle - 3\,|2;\mathbf{w}_2\rangle + |1;\mathbf{w}_2\rangle\,\big).
\end{aligned} \tag{3.40}$$

In reverse, we have

$$5\varphi\,|2;\mathbf{w}_2\rangle = \big(5\Theta+1\big)\,|\Upsilon_2\rangle\,, \tag{3.41}$$

$$5\varphi\,|3;\mathbf{w}_2\rangle = \big((5\Theta)^2 + 3\cdot 5\Theta + 2\big)\,|\Upsilon_2\rangle\,, \tag{3.42}$$

and thus the Picard-Fuchs equation is

$$\mathcal{P}_2\,|\Upsilon_2\rangle = \Big(\Theta^2 - 5^5\lambda\left(\Theta+\tfrac{1}{5}\right)\left(\Theta+\tfrac{4}{5}\right)\Big)\,|\Upsilon_2\rangle = 0. \tag{3.43}$$

3.2.4 Case 3

Consider the representation

$$\langle \mathbf{w}_3 \rangle = \langle (3,1,1,0,0) \rangle \,,$$

of which there are 30 permutations. Each is two more two-dimensional, which we demonstrate in a manner that is by now familiar.

$$|3;\mathbf{w}_3\rangle = |1;5,3,3,2,2\rangle = \frac{1}{5\varphi}\big(\,|2;0,3,3,2,2\rangle - |1;0,3,3,2,2\rangle\,\big).$$

For the last two terms, we have the following further equivalences in cohomology.

$$\begin{aligned}
|2;0,3,3,2,2\rangle &= |0;2,5,5,4,4\rangle \\
&= \frac{1}{5\varphi}\,|1;2,0,5,4,4\rangle \\
&= \frac{1}{(5\varphi)^2}\big(\,|2;2,0,0,4,4\rangle - |1;2,0,0,4,4\rangle\,\big) \\
&= \frac{1}{(5\varphi)^2}\big(\,|1;3,1,1,5,5\rangle - |0;3,1,1,5,5\rangle\,\big) \\
&= \frac{1}{(5\varphi)^3}\big(\,|2;3,1,1,0,5\rangle - 2\,|1;3,1,1,0,5\rangle\,\big) \\
&= \frac{1}{(5\varphi)^4}\big(\,|3;\mathbf{w}_3\rangle - 4\,|2;\mathbf{w}_3\rangle + 2\,|1;\mathbf{w}_3\rangle\,\big).
\end{aligned}$$

$$|2;\mathbf{w}_3\rangle = |0;5,3,3,2,2\rangle = \frac{1}{5\varphi}\,|1;0,3,3,2,2\rangle \,.$$

Combining these expressions, we obtain the linear relation,

$$(1-5^5\lambda)\,|3;\mathbf{w}_3\rangle - (4+5^5\lambda)\,|2;\mathbf{w}_3\rangle + (2-5^5\lambda)\,|1;\mathbf{w}_3\rangle = 0. \tag{3.44}$$

Next, define the base state

$$|\Upsilon_3\rangle = 5\vartheta\,|1;\mathbf{w}_3\rangle \,, \tag{3.45}$$

on which Θ acts as follows,

$$\begin{aligned}
5\Theta\,|\Upsilon_3\rangle &= 5\varphi\big(\,|2;\mathbf{w}_3\rangle - |1;\mathbf{w}_3\rangle\,\big), \\
(5\Theta)^2\,|\Upsilon_3\rangle &= 5\varphi\big(\,|3;\mathbf{w}_3\rangle - 3\,|2;\mathbf{w}_3\rangle + |1;\mathbf{w}_3\rangle\,\big),
\end{aligned}$$

or in reverse

$$\begin{aligned}
|2;\mathbf{w}_2\rangle &= \big(5\Theta+1\big)\,|\Upsilon_3\rangle \,, \\
|3;\mathbf{w}_2\rangle &= \big((5\Theta)^2 + 3\cdot 5\Theta + 2\big)\,|\Upsilon_3\rangle \,.
\end{aligned}$$

The Picard-Fuchs equation is thus

$$\mathcal{P}_3\,|\Upsilon_3\rangle = \Big(\Theta\big(\Theta-\tfrac{1}{5}\big) - 5^5\lambda\big(\Theta+\tfrac{1}{5}\big)\big(\Theta+\tfrac{3}{5}\big)\Big)\,|\Upsilon_3\rangle = 0. \tag{3.46}$$

3.2.5 Case 4

Consider the representation

$$\langle \mathbf{w}_4 \rangle = \langle (2,2,1,0,0) \rangle \,,$$

of which there are 30 permutations, each of which is two dimensional. We have

$$\begin{aligned}
|3;\mathbf{w}_4\rangle &= |0;5,5,4,3,3\rangle \\
&= \frac{1}{5\varphi}\,|1;0,5,4,3,3\rangle \\
&= \frac{1}{5\varphi}\,|0;1,6,5,4,4\rangle \\
&= \frac{1}{(5\varphi)^2}\,|1;1,6,0,4,4\rangle \\
&= \frac{1}{(5\varphi)^3}\big(\,|2;1,1,0,4,4\rangle - 2\,|1;1,1,0,4,4\rangle\,\big) \\
&= \frac{1}{(5\varphi)^3}\big(\,|1;2,2,1,5,5\rangle - 2\,|0;2,2,1,5,5\rangle\,\big) \\
&= \frac{1}{(5\varphi)^4}\big(\,|2;2,2,1,0,5\rangle - 3\,|1;2,2,1,0,5\rangle\,\big) \\
&= \frac{1}{(5\varphi)^5}\big(\,|3;\mathbf{w}_4\rangle - 5\,|2;\mathbf{w}_4\rangle + 3\,|1;\mathbf{w}_4\rangle\,\big),
\end{aligned}$$

which immediately yields the linear relation

$$(1-5^5\lambda)\,|3;\mathbf{w}_4\rangle - 5\,|2;\mathbf{w}_4\rangle + 3\,|1;\mathbf{w}_4\rangle = 0. \tag{3.47}$$

Define the base state

$$|\Upsilon_4\rangle = 5\varphi\,|1;\mathbf{w}_4\rangle \,, \tag{3.48}$$

on which the action of the Gauss-Manin connection is given by

$$\begin{aligned}
5\Theta\,|\Upsilon_4\rangle &= 5\varphi\big(\,|2;\mathbf{w}_4\rangle - |1;\mathbf{w}_4\rangle\,\big) \\
(5\Theta)^2\,|\Upsilon_4\rangle &= 5\varphi\big(\,|3;\mathbf{w}_4\rangle - 3\,|2;\mathbf{w}_4\rangle + |1;\mathbf{w}_4\rangle\,\big),
\end{aligned}$$

or in reverse,

$$\begin{aligned}
|2;\mathbf{w}_4\rangle &= \big(5\Theta+1\big)\,|\Upsilon_4\rangle \\
|3;\mathbf{w}_4\rangle &= \big((5\Theta)^2 + 3\cdot 5\Theta + 2\big)\,|\Upsilon_4\rangle \,.
\end{aligned}$$

The Picard-Fuchs equation is thus

$$\mathcal{P}_4\,|\Upsilon_4\rangle = \Big(\Theta\big(\Theta-\tfrac{2}{5}\big) - 5^5\lambda\big(\Theta+\tfrac{1}{5}\big)\big(\Theta+\tfrac{2}{5}\big)\Big)\,|\Upsilon_4\rangle = 0. \tag{3.49}$$

3.2.6 Case 5

The final case to consider is that of the 24 one-dimensional representations corresponding to the respective permutations (modulo cycles) of

$$\langle \mathbf{w}_5 \rangle = \langle (4,3,2,1,0) \rangle .$$

We have already seen that the components can be cycled using the cohomology relations,

$$|1; \mathbf{w}_5\rangle = |0; 5,4,3,2,1\rangle = \tfrac{1}{5\varphi} |1; 0,4,3,2,1\rangle .$$

Applying such a relation successively for each X_i we obtain the linear relation

$$(1 - 5^5\lambda) |1; \mathbf{w}_5\rangle = 0. \tag{3.50}$$

As mentioned before, these monomials are only nontrivial at the conifold point $\lambda = 5^{-5}$ where they do contribute. However, we shall be expanding around $\lambda = 0$ and $\lambda = \infty$ and will evaluate at $\lambda = 5^{-5}$ only by analytic continuation. Consequently, we have no need of these monomials and we will not consider them further.

3.3 A Basis of Dwork Cohomology

We have already seen that for each $\mathbf{w}$ in Table 3.2, $\langle \mathbf{w} \rangle$ is spanned by $(\pi Q)^k X^{\mathbf{w}}$ for a suitable range of $k \in \mathbb{Z}$, and moreover that *every* representation is characterised in this way. We also determined the dimension of each such space by obtaining a linear relation on its elements.

We have seen the relationship between the action of πQ as a multiplicative operator and the action of the Gauss-Manin connection Θ. It follows that an alternative basis consists of successive powers of Θ acting on the base state. In this section, we write down these bases and summarise the key facts regarding each class of representations.

In each case, we give the dimension d of representations, the defining vector $\mathbf{w}$ and the number of permutations of $\mathbf{w}$, which is also the number of representations in that case. We list the base state $|\Upsilon\rangle$ and the Picard-Fuchs operator $\mathcal{P}$ which annihilates it. The natural basis in each case is then

$$\left\{ (5\Theta)^k |\Upsilon\rangle : 0 \le k < d \right\}, \tag{3.51}$$

which is equivalent to the corresponding basis defined in Section 3.1,

$$\left\{ |k+1; \mathbf{w}\rangle = (\pi Q)^{k+1} X^{\mathbf{w}} \colon 0 \le k < d \right\}. \tag{3.52}$$

Case 0	1 permutation
d	4
$\mathbf{w}$	$(0,0,0,0,0)$
Υ	πQ
$\mathcal{P}$	$\Theta^4 - 5^5\lambda \prod_{i=1}^4 \left(\Theta + \frac{i}{5}\right)$

Case 1	20 permutations
d	2
$\mathbf{w}$	$(4,1,0,0,0)$
Υ	$5\varphi\pi^2 Q X_0 X_1^4 X_2$
$\mathcal{P}$	$\Theta^2 - 5^5\lambda \left(\Theta + \frac{2}{5}\right)\left(\Theta + \frac{3}{5}\right)$

Case 2	20 permutations
d	2
$\mathbf{w}$	$(3,2,0,0,0)$
Υ	$5\varphi\pi^2 Q X_0 X_1^3 X_2^2$
$\mathcal{P}$	$\Theta^2 - 5^5\lambda \left(\Theta + \frac{1}{5}\right)\left(\Theta + \frac{4}{5}\right)$

Case 3	30 permutations
d	2
$\mathbf{w}$	$(3,1,1,0,0)$
Υ	$5\varphi\pi^2 Q X_0 X_1^3 X_2 X_3$
$\mathcal{P}$	$\Theta\left(\Theta - \frac{1}{5}\right) - 5^5\lambda\left(\Theta + \frac{1}{5}\right)\left(\Theta + \frac{3}{5}\right)$

Case 4	30 permutations
d	2
$\mathbf{w}$	$(2,2,1,0,0)$
Υ	$5\varphi\pi^2 Q X_0 X_1^2 X_2^2 X_3$
$\mathcal{P}$	$\Theta\left(\Theta - \frac{2}{5}\Theta\right) - 5^5\lambda\left(\Theta + \frac{1}{5}\right)\left(\Theta + \frac{2}{5}\right)$

3.4 The Periods

We have now fully classified the monomials of the Dwork cohomology according to their transformation properties under $\mathcal{A}$, established linear dependence between monomials within each representation and identified the differential operator responsible for this dependence, which determines the dimension of each representation. Now we take a moment to discuss the story of the periods, which are intimately related to the monomials. We will only give the periods corresponding to the four dimensional space of Case 0. The periods and Picard-Fuchs equations for the other cases can be found in [5].

A Calabi-Yau manifold has an everywhere nonvanishing holomorphic 3-form Ω, the periods of which are a useful tool for studying complex structure variations of the manifold. The relation to the monomials we saw in the Dwork cohomology is revealed by integration. Indeed, for each $\mathbf{v} \in \mathbb{N}^5$, there is a 3-cycle $\gamma_{\mathbf{v}} \in H_3(\mathcal{M}_\varphi, \mathbb{C})$ such that

$$\varpi_{\mathbf{v}} = \int_{\gamma_{\mathbf{v}}} \Omega = \frac{1}{(2\pi \mathrm{i})^5} \int_C \frac{x^{\mathbf{v}}}{\left(\varphi^{-1} P(x)\right)^{v_0}}, \tag{3.53}$$

where C is a suitable 5-torus and here π is the usual $\pi \in \mathbb{C}$ rather than the p-adic $\pi \in \Omega_p$ chosen in Chapter 2.

It turns out that the action of ϑ on the periods is equivalent to the action of Θ on the monomials. Just as Θ introduced additional powers of πQ into monomials, so too does ϑ for the periods. Indeed, applying ϑ to (3.53), we pull an extra power of Q out of the $P(x)$ in the denominator so that

$$\vartheta \varpi_{\mathbf{v}} = \varphi^{-1} v_0 \varpi_{\mathbf{v+1}},$$

relating the periods associated to $x^{\mathbf{v}}$ with those associated to $x^{\mathbf{v+1}} = Q x^{\mathbf{v}}$.

The periods split into representations under the action of $\mathcal{A}$, just as the monomials did. The action of ϑ preserves these for the same reason that Θ preserves the monomial representations, and so the Picard-Fuchs equation for the periods factorises. For each period representation, the Picard-Fuchs operator can be found by replacing Θ with ϑ in the Picard-Fuchs operator of the corresponding monomial representation.

Later, when expanding $\mathcal{U}(\lambda)$ around $\lambda = 0$ or $\lambda = \infty$, we will have need of series expansions for the period corresponding to each monomial. These can be obtained by solving the appropriate Picard-Fuchs equation. Moreover, if one starts with the series expansion and seeks a differential equation it satisifes, one often finds a higher order differential equation, of which the Picard-Fuchs equation will be a factor [5]. For the quintic, these higher order equations are fifth order and take a standard form depending only on the vector $\mathbf{w}$ labelling the representation.

The periods corresponding to the monomials of $\mathcal{S}_\lambda = \langle \mathbf{w}_0 \rangle$ are of particular interest and, like the corresponding monomials, are invariant under the action of $\mathcal{A}$. The Picard-Fuchs operator satisfied by these periods, analogous to equation (3.23), is well known and was stated previously in the introduction,

$$\mathcal{P} = \vartheta^4 - 5\lambda \prod_{i=1}^{4} (5\vartheta + i). \tag{3.54}$$

Its solutions can be found by the Frobenius method, where one substitutes the ansatz

$$\varpi(\lambda, \epsilon) = \sum_{n=0}^{\infty} A_n(\epsilon) \lambda^{n+\epsilon}, \tag{3.55}$$

and expands ϖ in powers of ϵ to obtain the periods,

$$\varpi(\lambda, \epsilon) = \sum_{k=0}^{\infty} \epsilon^k \varpi_k(\lambda). \tag{3.56}$$

We call $\varpi_0(\lambda) = \sum_{n=0}^{\infty} A_n(0)\lambda^n$ the *fundamental period*. Imposing $\mathcal{P}\varpi = 0$ at lowest order in λ yields the so called *indicial equation*, which in the case of the quintic is $\epsilon^4 = 0$. Its roots, the *indices*, are all zero, making $\lambda = 0$ a point of maximal unipotent monodromy. This has the effect that the periods contain successive powers of $\log\lambda$,

$$\begin{aligned}
\varpi_0(\lambda) &= f_0(\lambda), \\
\varpi_1(\lambda) &= f_1(\lambda) + \log\lambda\, f_0(\lambda), \\
\varpi_2(\lambda) &= f_2(\lambda) + \log\lambda\, f_1(\lambda) + \tfrac{1}{2}\log^2\lambda f_0(\lambda), \\
\varpi_3(\lambda) &= f_3(\lambda) + \log\lambda\, f_2(\lambda) + \tfrac{1}{2}\log^2\lambda f_1(\lambda) + \tfrac{1}{6}\log^3\lambda f_0(\lambda).
\end{aligned} \tag{3.57}$$

Here f_k is the holomorphic part of ϖ_k, obtained by formally setting $\log\lambda = 0$ in ϖ_k,

$$f_k(\lambda) = \varpi_k(\lambda)\Big|_{\log\lambda=0} = \frac{1}{k!}\sum_{n=0}^{\infty} A_n^{(k)}(0)\,\lambda^n, \quad 0 \le k \le 3. \tag{3.58}$$

The relationship between ϖ_k and f_k can be expressed concisely in matrix form as

$$\varpi = \lambda^N \mathbf{f}, \tag{3.59}$$

where we have defined

$$\varpi = (\varpi_0, \varpi_1, \varpi_2, \varpi_3)^\intercal, \quad \mathbf{f} = (f_0, f_1, f_2, f_3)^\intercal, \quad N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{3.60}$$

Imposing $\mathcal{P}\varpi = 0$ at each order in λ yields a recurrence relation on the coefficients $A_n(\epsilon)$. These can be taken to be

$$A_n(\epsilon) = \frac{\Gamma(1+5n+5\epsilon)}{\Gamma(1+5\epsilon)} \frac{\Gamma(1+\epsilon)^5}{\Gamma(1+n+\epsilon)^5}. \tag{3.61}$$

Here Γ is the usual Gamma function and not the p-adic one used frequently in this thesis. Thus $A_n(\epsilon)$ above is defined for complex ϵ. It can, however, be rewritten as a rational function in ϵ, thus allowing its extension to the p-adic integers $\mathbb{Z}_p$,

$$A_n(\epsilon) = \frac{(5n+5\epsilon)!}{(5\epsilon)!} \frac{\epsilon!^5}{(n+\epsilon)!^5} = \frac{(1+5\epsilon)_{5n}}{(1+\epsilon)_n^{\,5}}. \tag{3.62}$$

As mentioned before, another differential equation enters the story here, the GKZ system. This can be obtained by generic methods from the toric data of the quintic, and is given by

$$\mathcal{P}_{\mathrm{GKZ}} = \vartheta \mathcal{P} = \vartheta^5 - 5^5 \lambda \prod_{i=1}^{5} \left(\vartheta + \tfrac{i}{5}\right). \tag{3.63}$$

The periods clearly still satisfy this equation, but there is an additional solution, which we call the *semiperiod*. The semiperiod does not feature in our computation of the zeta function, but it did prove significant in [5].

3.5 The Small Structure Limit

Having completed our digression on the periods, we return to the study of Dwork cohomology by examining its behaviour in limiting cases. As $\varphi \to \infty$ in the small structure limit, the φ terms in the cohomology relations (3.8) and (3.9) dominate. For this section only, let us rescale the defining polynomial W by a factor of φ,

$$W = W_0 - \varphi^{-1} Q,$$

which clearly determines the same variety. Then the cohomology relations at $\varphi^{-1} = 0$ are

$$\begin{aligned} D_0 \Psi &= -\pi W_0 \Psi, \\ D_i \Psi &= -5\pi X_0 X_i^5 \Psi. \end{aligned} \tag{3.64}$$

This means that the cohomology diagrams of the previous sections can now be read sideways since the coefficients of all downward arrows are 0. Using the cohomology relations above, one can reduce any monomial to a monomial with no exponent greater than 4. Indeed, we have that

$$-\pi X_0 X_i^5 X^{\mathbf{v}} = \frac{v_i}{5} X^{\mathbf{v}}. \tag{3.65}$$

Clearly, these cohomology relations are much simpler, meaning that computing the action of the Dwork operator at $\varphi^{-1} = 0$ will be significantly easier than the general case. This is why it is so useful to expand the Dwork operator about $\varphi^{-1} = 0$ as was done in [9].

3.6 The Large Structure Limit

Returning to our usual convention for $W(x)$, let us consider the large structure limit $\varphi \to 0$. The cohomology relations (3.8) and (3.9) become

$$D_\alpha \left|\Psi\right\rangle_0 = \pi Q \left|\Psi\right\rangle_0 \quad \text{for} \quad 0 \leq \alpha \leq n. \tag{3.66}$$

Applying this to $\Psi = (\pi Q)^{\mu+n-1}$, one finds

$$\left|(\pi Q)^{\mu+n}\right\rangle_0 = (\mu + n - 1)\left|(\pi Q)^{\mu+n-1}\right\rangle_0 \quad \text{for} \quad n \geq 1,$$

and iterating this identity we obtain a relation for reducing powers of Q in a state,

$$\left|(\pi Q)^{\mu+n}\right\rangle_0 = (\mu)_n \left|(\pi Q)^{\mu}\right\rangle_0, \tag{3.67}$$

where $(\mu)_n$ denotes the rising factorial $\mu(\mu+1)\cdots(\mu+n-1)$. Setting $\mu = 1$, we see that integer powers of πQ become linearly dependent,

$$\left|(\pi Q)^{n+1}\right\rangle_0 = n!\left|\pi Q\right\rangle_0, \quad n \geq 1. \tag{3.68}$$

Worse still, since $1 \in \mathcal{H}$, we have in fact that

$$\mathcal{D}_0 \left|1\right\rangle_0 = D_0 \left|1\right\rangle_0 - \pi Q \left|1\right\rangle_0 = -\left|\pi Q\right\rangle_0.$$

It follows that in fact $\left|(\pi Q)^k\right\rangle_0 = 0$ in cohomology for all $k \geq 1$, and so $\mathcal{S}_0 = \mathcal{S}\big|_{\lambda=0}$ becomes zero dimensional.

In a similar manner, applying (3.66) to $\Psi = \left|X^{\mathbf{v}}\right\rangle_0$ (where $\mathbf{v} \neq v\mathbf{1}$) we find that in cohomology,

$$0 = (D_\alpha - D_\beta)\left|X^{\mathbf{v}}\right\rangle_0 = (v_\alpha - v_\beta)\left|X^{\mathbf{v}}\right\rangle_0,$$

whence

$$\left|X^{\mathbf{v}}\right\rangle_0 = 0, \tag{3.69}$$

since there are some i, j such that $v_i \neq v_j$. Thus all 204 basis monomials of the Dwork cohomology of the quintic vanish and the entire space $\mathcal{H}^S/\mathcal{D}(0)\mathcal{H}$ is zero dimensional. A major goal of the next chapter will be to construct a new space which does not suffer from this issue at $\varphi = 0$, though we will restrict our attention to the four dimensional space $\mathcal{S}_0$.

Chapter 4

Expanding the Dwork Operator

Having established a basis for the Dwork cohomology of the quintic, what remains is to compute the action of $\mathcal{U}(\lambda)$ on it. From this, the zeta function can be computed via the determinant formula (2.42). The denominator takes a standard form and the numerator is composed of a number of factors, each corresponding to one of the subspaces of the previous chapter. Indeed, it was shown in [6] that the zeta function is

$$\zeta(T) = \frac{R_1(T) R_A(p^\rho T^\rho)^{\frac{20}{\rho}} R_B(p^\rho T^\rho)^{\frac{30}{\rho}}}{(1-T)(1-pT)(1-p^2T)(1-p^3T)}, \tag{4.1}$$

where ρ is the smallest integer such that $5 \mid p^\rho - 1$ and R_1, R_A and R_B are quartics. In [9], the Dwork operator was computed using an expansion about the small structure point $\lambda \to \infty$. It was shown that R_1 corresponds to Case 0 of the previous chapter, while the powers of R_A and R_B arise from the remaining subspaces 1–4.

4.1 The Large Structure Expansion

The purpose of this chapter is to expand the Dwork operator around the large structure point $\lambda = 0$. We would like to find a matrix E such that

$$\mathcal{U}(\lambda) = E(\lambda^p)^{-1} \mathcal{U}(0) E(\lambda). \tag{4.2}$$

Recalling the definition of the Dwork operator,

$$\mathcal{U}(\lambda) = \mathrm{e}^{-\pi W_{\lambda^p}} \mathcal{A}_p \, \mathrm{e}^{\pi W_\lambda}, \tag{4.3}$$

we see that the natural choice is

$$E(\lambda) = \mathrm{e}^{\varphi \pi W_0}, \tag{4.4}$$

while the Dwork operator at the large structure point is given by

$$\mathcal{U}(0) = \mathrm{e}^{\pi Q} \mathcal{A}_p \mathrm{e}^{-\pi Q}. \tag{4.5}$$

A similar expansion can then be made for the Dwork exterior derivative,

$$\mathcal{D}(\lambda) = E(\lambda)^{-1}\, \mathcal{D}(0)\, E(\lambda), \quad \mathcal{D}(0) = \mathrm{e}^{\pi Q}\, \mathrm{d}\, \mathrm{e}^{-\pi Q}. \tag{4.6}$$

It is easy to see that $\exp(\varphi\pi W_0)$ maps $\mathcal{H}^S/\mathcal{D}(\lambda)\mathcal{H} \to \mathcal{H}^S/\mathcal{D}(0)\mathcal{H}$, since

$$\mathrm{e}^{\varphi\pi W_0}\big(\Psi + \mathcal{D}(\lambda)\alpha\big) = \mathrm{e}^{\varphi\pi W_0}\Psi + \mathcal{D}(0)\big(\mathrm{e}^{\varphi\pi W_0}\alpha\big). \tag{4.7}$$

In other words, the following diagram commutes.

$$\begin{array}{ccc}
\mathcal{H}^S/\mathcal{D}(\lambda)\mathcal{H} & \xrightarrow{\mathrm{e}^{\varphi\pi W_0}} & \mathcal{H}^S/\mathcal{D}(0)\mathcal{H} \\
\Big\downarrow {\scriptstyle \mathcal{U}(\lambda)} & & \Big\downarrow {\scriptstyle \mathcal{U}(0)} \\
\mathcal{H}^S/\mathcal{D}(\lambda^p)\mathcal{H} & \xleftarrow[\mathrm{e}^{-\varphi^p\pi W_0}]{} & \mathcal{H}^S/\mathcal{D}(0)\mathcal{H}
\end{array}$$

Note that $\mathcal{U} : \mathcal{H}^S/\mathcal{D}(\lambda)\mathcal{H} \to \mathcal{H}^S/\mathcal{D}(\lambda^p)\mathcal{H}$ is only an endomorphism when $\lambda = \lambda^p$, i.e. at the Teichmüller points. Note also that since $\mathrm{e}^{\pm\varphi\pi W_0}$ converges only for $|\varphi|_p < 1$, analytic continuation is required to extend $\mathcal{U}$ to the Teichmüller points.

We saw in Section 3.6 that $\mathcal{H}^S/\mathcal{D}(0)\mathcal{H}$ is zero dimensional. Thus the map $\mathrm{e}^{\varphi\pi W_0}$ is not invertible and the expansion (4.2) fails. In order to recover it, it will be necessary to "repair" the cohomology to obtain a modified space on which $\mathrm{e}^{\varphi\pi W_0}$ is invertible.

An expansion at the large structure point for Cases 1–4 has not yet been forthcoming. We shall henceforth restrict our attention to Case 0 and the corresponding 4×4 block of $\mathcal{U}$. We seek to compute the factor $R_1(T)$ of equation (4.1), henceforth simply $R(T)$,

$$R(T) = \det\left(1 - \frac{\mathcal{U}(\lambda)}{p}T\right). \tag{4.8}$$

Recall that we denoted the four-dimensional subspace of Dwork cohomology by $\mathcal{S}_\lambda$. This was given by the span

$$\mathcal{S}_\lambda = \mathrm{span}\left\{ |k\rangle_\lambda = (5\Theta)^k\, |\pi Q\rangle_\lambda \;:\; 0 \le k \le 3 \right\}.$$

Explicitly, this basis is

$$\begin{aligned}
|0\rangle_\lambda &= \quad |\pi Q\rangle_\lambda\,,\\
|1\rangle_\lambda &= -|\pi Q\rangle_\lambda + \quad \left|(\pi Q^2)\right\rangle_\lambda\,,\\
|2\rangle_\lambda &= \quad |\pi Q\rangle_\lambda - 3\left|(\pi Q^2)\right\rangle_\lambda + \quad \left|(\pi Q^3)\right\rangle_\lambda\,,\\
|3\rangle_\lambda &= -|\pi Q\rangle_\lambda + 7\left|(\pi Q^2)\right\rangle_\lambda - 6\left|(\pi Q^3)\right\rangle_\lambda + \left|(\pi Q^4)\right\rangle_\lambda\,.
\end{aligned} \tag{4.9}$$

In the small structure limit we need only compute the action of $\mathcal{U}(\lambda)$ on the cohomology classes of $(\pi Q)^{k+1}$ for $0 \leq k \leq 3$. In the case of large structure, however, we have already seen that every power of πQ is in the cohomology class of 0. We must repair the cohomology somehow in order to proceed. We thus seek a new four dimensional space $\tilde{\mathcal{S}}_\lambda$ which has similar structure to $\mathcal{S}_\lambda$ for $\lambda \neq 0$, but which does not become zero dimensional in the limit $\lambda \to 0$.

The breakdown itself gives us a hint as to how we might do this. We showed earlier that all *integer* powers of πQ are zero in cohomology at $\lambda = 0$. But what about non-integer powers? Let us introduce $\epsilon \in p\mathbb{Z}_p$ a p-adic (but not rational) integer which we assume to be p-adically small (i.e. highly divisible by p). Then for k a positive integer we have[1]

$$\left|(\pi Q)^k Q^\epsilon\right\rangle_0 = (\epsilon)_k \,|Q^\epsilon\rangle_0\,, \tag{4.10}$$

which cannot be expressed in terms of $\mathcal{D}\,|1\rangle_0$. The consideration of such expressions for "small" ϵ naturally leads to successive powers of $\log Q$. We will indeed see below that it is possible to construct a "logarithmic Dwork cohomology" which does not break down at $\lambda = 0$. Such a space should, of course, be four dimensional and, apart from its better behaviour in the large p-adic structure limit, have similar properties to the original space $\mathcal{S}_\lambda$.

4.2 Logarithmic Dwork Cohomology

There are two cohomology spaces, one λ-dependent and one λ-independent, related by the action of $E(\lambda)$. We shall frequently consider states $|\Psi\rangle_\lambda$ mapped into $\mathcal{S}_0$ by $E(\lambda)$ and it will be convenient to introduce notation for this. Indeed, for $|\Psi\rangle_\lambda \in \mathcal{S}_\lambda$ we denote the corresponding $E(\lambda)\,|\Psi\rangle_\lambda \in \mathcal{S}_0$ by

$$|\Psi;\lambda\rangle = \mathrm{e}^{\varphi\pi W_0}\,|\Psi\rangle_\lambda\,. \tag{4.11}$$

[1] Any $\epsilon \in \mathbb{Z}_p$ can be approximated by a sequence of rational integers. We define ϵ-powers of monomials via a limit using such a sequence.

Moreover, the Gauss-Manin connection commutes with E in a natural way,

$$|\Theta\Psi;\lambda\rangle = \mathrm{e}^{\varphi\pi W_0}\mathrm{e}^{-\varphi\pi W_0}\,\vartheta\,\mathrm{e}^{\varphi\pi W_0}\Psi = \vartheta\,|\Psi;\lambda\rangle\,. \tag{4.12}$$

Suppose then that we choose a state $|\Upsilon\rangle_\lambda$ such that $|\Upsilon;\lambda\rangle$ satisfies the ϑ-Picard-Fuchs equation (3.54). It is not hard to show, using equation (4.12), that $|\Upsilon\rangle_\lambda$ satisfies the "covariant" Picard-Fuchs equation (3.23),

$$\mathcal{P}_\Theta\,|\Upsilon\rangle_\lambda = \left(\Theta^4 - 5\prod_{i=1}^{4}(5\Theta+i)\right)|\Upsilon\rangle_\lambda = 0. \tag{4.13}$$

It follows that the $\mathbb{Q}_p(\pi)[\![\lambda]\!]$-module generated by

$$\left\{\Theta^k\,|\Upsilon\rangle_\lambda \;\middle|\; k\geq 0\right\} \tag{4.14}$$

is at most four dimensional. The problem of constructing a well defined basis for $\tilde{S}_0$ thus becomes the task of choosing $|\Upsilon\rangle_\lambda$ such that $|\Upsilon;\lambda\rangle$ satisfies the Picard-Fuchs equation and none of $\Theta^k\,|\Upsilon\rangle_\lambda$ $(0\leq k\leq 3)$ are equivalent in cohomology.

In [11], Shapiro took $|\Upsilon\rangle_\lambda = |\pi Q\rangle_\lambda$, which can be shown to satisfy the Picard-Fuchs equation. The resulting basis spans the original space $\mathcal{S}$,

$$\operatorname{span}\left\{\Theta^k\,|\pi Q\rangle_\lambda \;\middle|\; 0\leq k\leq 3\right\} = \operatorname{span}\left\{\left|(\pi Q)^{k+1}\right\rangle_\lambda \;\middle|\; 0\leq k\leq 3\right\},$$

which means, of course, that we cannot use this basis since it breaks down at $\lambda=0$. What we shall try to do is modify it just enough that the construction works. Recall, that the singularity at $\lambda=0$ is logarithmic and that successive powers of $\log\lambda$ arise in the solutions to the Picard-Fuchs equation.

Inspired by this, and our observations at the end of Section 4.1, we start by considering formal logarithmic states of the form

$$\left|\left(5\log\left(\frac{\varphi W_0}{Q}\right)\right)^i(\pi Q)^{j+1}\right\rangle_\lambda \quad \text{for} \quad i,j\geq 0. \tag{4.15}$$

That is, terms in the ϵ-expansion of

$$\left|\left(\frac{\varphi W_0}{Q}\right)^{5\epsilon}(\pi Q)^{j+1}\right\rangle_\lambda. \tag{4.16}$$

The ratio W_0/Q has been chosen to have degree zero so that the total degree of the state is still an integer and independent of ϵ. For brevity, we introduce the notation

$$L = 5\log\left(\frac{\varphi W_0}{Q}\right).$$

The states (4.15) should be considered formal objects generating a $\mathbb{Q}_p(\pi)[\![\lambda]\!]$-module, of which we will later choose a submodule of the form (4.14) for a certain natural choice of $|\Upsilon\rangle_\lambda$. Such a choice will become apparent, but first we must investigate the action of $\mathfrak{U}$ on (4.15), defined by its action on (4.16) subject to the cohomology relations (3.67) and (3.69).

Since the Dwork cohomology is a cohomology of power series, it is not clear what exactly is meant by the ratio W_0/Q in equation (4.16). Indeed, if we attempt to proceed with our calculations using it, we will encounter ambiguities. What we can do instead is write

$$L = 5\left(\log W_0 - \log Q + \log \varphi\right)$$

to express powers of L as linear combinations of powers of $\log W_0$ and $\log Q$. For example,

$$\left|L^3 \pi Q\right\rangle_\lambda = \frac{1}{2}\,\frac{\mathrm{d}^3}{\mathrm{d}\epsilon^3}\left(9\,|0,1;1\rangle_\lambda - 9\,|1,0;1\rangle_\lambda + |2,1;1\rangle_\lambda - |1,2;1\rangle_\lambda\right)\Bigg|_{\epsilon=0}. \tag{4.17}$$

Then the action of $\mathfrak{U}$ on these is obtained expanding in powers of ϵ its action on

$$|i,j;k\rangle_\lambda = \left|Q^{5i\epsilon}(\varphi W_0)^{5j\epsilon}(\pi Q)^k\right\rangle_\lambda, \tag{4.18}$$

and reducing with the cohomology relations (3.67) and (3.69). We will always require $k \geq 1$, i.e. at least one power of πQ.

One should not consider the objects (4.18) to be actual elements of some cohomology complex. Instead, these are merely computational tools used to define the action of $\mathfrak{U}$ on the states (4.15) via the cohomology relations (3.67) and (3.69).

With a certain prescience, we introduce slightly different states at $\lambda = 0$,

$$\left|\ell^i (\pi Q)^{j+1}\right\rangle_0 = \left|\left(5\log\left(\frac{W_0}{Q}\right)\right)^i (\pi Q)^{j+1}\right\rangle_\lambda \quad \text{for} \quad i,j \geq 0, \tag{4.19}$$

where ℓ is obtained by formally setting $\log\lambda = 0$ in L. In fact, one can choose to use L here again instead of ℓ, but since the logarithms cancel in $\mathfrak{U}(\lambda)$ we once again omit them from the outset. The action of $\mathfrak{U}$ on these states is given by its action on

$$|i,j;k\rangle_0 = \left|Q^{5i\epsilon} W_0^{5j\epsilon}(\pi Q)^k\right\rangle_0, \quad i,j \geq 0, k \geq 1, \tag{4.20}$$

up to the cohomology relations at $\lambda = 0$. These objects should be understood in the same sense as those in (4.18).

4.2.1 The Action of $E(\lambda)$

Using the notation introduced above, define

$$|i,j;k;\lambda\rangle = \mathrm{e}^{\varphi\pi W_0}\,|i,j;k\rangle_\lambda\,. \tag{4.21}$$

It is important to remember that $|\Psi;\lambda\rangle$ is defined up to a total $\mathcal{D}(0)$ derivative, even though we omit the 0 subscript for brevity. This is important since it means we can reduce these states using the simpler cohomology relations at $\lambda = 0$.

To obtain an expression for the action of E, we expand $\mathrm{e}^{\varphi\pi W_0}$,

$$|i,j;k;\lambda\rangle = \left|\sum_{n=0}^{\infty}\frac{1}{n!}(\varphi\pi W_0)^n Q^{5i\epsilon}(\varphi W_0)^{5j\epsilon}(\pi Q)^k\right\rangle_0\,.$$

Recall that $W_0 = X_0\sum_{i=1}^{5}X_i^5$ and expand W_0^μ for $\mu\in\mathbb{Z}_{\geq 0}$. By the cohomology relation (3.69), we know that the only terms in this expansion nontrivial in cohomology are those with equal powers of each X_i. If $5\nmid\mu$, there are no such terms; if $5\mid\mu$, the only such term is

$$X_0^\mu X_1^\mu\ldots X_n^\mu = Q^\mu,$$

whose multinomial coefficient $a_\mu = (5\mu)!/(\mu!)^5$ is also the μ^{th} expansion coefficient of the fundamental period ϖ_0. We can thus replace powers of W_0 with corresponding powers of Q at the cost of introducing a multinomial coefficient. So doing, we have

$$\begin{aligned} |i,j;k;\lambda\rangle &= \left|\sum_{n=0}^{\infty}\frac{\lambda^{n+j\epsilon}}{(5n)!}a_{n+j\epsilon}Q^{5(i+j)\epsilon}(\pi Q)^{5n+k}\right\rangle_0 \\ &= \left|\sum_{n=0}^{\infty}\frac{\lambda^{n+j\epsilon}}{(5n)!}a_{n+j\epsilon}\big(5(i+j)\epsilon+1\big)_{5n+k-1}Q^{5(i+j)\epsilon}\pi Q\right\rangle_0\,, \end{aligned} \tag{4.22}$$

where in the second line we reduced integer powers of πQ using the cohomology relation (3.67). We can also apply the above observation in reverse to replace powers of Q with powers of W_0, at the cost of another multinomial coefficient. Indeed,

$$|i,j;k;\lambda\rangle = \left|\sum_{n=0}^{\infty}\frac{\lambda^{n+j\epsilon}}{(5n)!}\frac{a_{n+j\epsilon}}{a_{j\epsilon}}\big(5(i+j)\epsilon+1\big)_{5n+k-1}Q^{5i\epsilon}{W_0}^{5j\epsilon}\pi Q\right\rangle_0\,. \tag{4.23}$$

Recognising the ratio $a_{n+j\epsilon}/a_{j\epsilon}$ to be the n^{th} expansion coefficient of the Frobenius period, this becomes

$$|i,j;k;\lambda\rangle = \sum_{n=0}^{\infty}\frac{(5(i+j)\epsilon+1)_{5n+k-1}}{(5n)!}A_n(j\epsilon)\lambda^{n+j\epsilon}\,|i,j;1\rangle_0\,. \tag{4.24}$$

We must be careful with convergence since the terms of the sum do not tend to zero p-adically as $n \to \infty$. We can, however, work with partial sums, and thankfully the problematic terms cancel in all relevant linear combinations (such as (4.17)), so we can still obtain well defined results in the limit. We find that

$$\left|L^4\pi Q;\lambda\right\rangle = \sum_{j=0}^{3} \varpi_j(\lambda) \left|\ell^{j+1}\pi Q\right\rangle_0. \tag{4.25}$$

Now, since $|\ell^{j+1}\pi Q\rangle_0$ is λ-independent and ϖ_j is a solution of the Picard-Fuchs equation for $0 \leq j \leq 3$, it follows that $|L^4\pi Q;\lambda\rangle_0$ is a solution too. In other words, a suitable choice of first basis element is

$$|\Upsilon\rangle_\lambda = \left|L^4\pi Q\right\rangle_\lambda. \tag{4.26}$$

We might ask whether there is any other suitable choice of $|\Upsilon\rangle_\lambda$. Since any state with no powers of L goes to 0 at $\lambda = 0$, it follows that four distinct powers of L are required to obtain a four dimensional space at $\lambda = 0$. We have that $\vartheta L = 5$, so successive application of Θ will produce terms with lower degrees in L and higher degrees in πQ. It thus follows that $|\Upsilon\rangle_\lambda$ must contain at least L^4.

Moreover, we can check that the corresponding expression for $|L^5\pi Q;\lambda\rangle$ has a term involving the semiperiod ϖ_4, which is not a solution of the Picard-Fuchs equation (although it is a solution of the fifth order equation $\vartheta\mathcal{P}f = 0$). It follows that $|L^5\pi Q;\lambda\rangle$ does not obey the Picard-Fuchs equation, and thus we must take exactly L^4. Recall also that application of Θ increases the power of πQ present by 1. We seek powers $(\pi Q)^{k+1}$ for $0 \leq k \leq 3$ since $(\pi Q)^5$ can be written as a linear combination of lower powers of πQ. (We *could* take a higher power of πQ in $|\Upsilon\rangle_\lambda$, but doing so would result in an equivalent basis.) Thus the natural choice for $|\Upsilon\rangle_\lambda$ is indeed (4.26).

4.2.2 The Basis

The goal has been to construct a four-dimensional space similar in structure to the original $\mathcal{S}_\lambda$ which does not break down at $\lambda = 0$. We denote this new space $\tilde{\mathcal{S}}_\lambda$, and define it to be the $\mathbb{Q}_p(\pi)[\![\lambda]\!]$ span of

$$|k\rangle_\lambda = \Theta^k\,|\Upsilon\rangle_\lambda = \Theta^k \left|L^4\pi Q\right\rangle_\lambda \quad \text{for} \quad 0 \leq k \leq 3. \tag{4.27}$$

Noting that $\vartheta L = \vartheta \log\lambda = 1$, successive application of Θ yields explicit expressions for these basis elements,

$$
\begin{aligned}
|0\rangle_\lambda &= \left|L^4 \pi Q\right\rangle_\lambda ,\\
|1\rangle_\lambda &= 4\left|L^3\pi Q\right\rangle_\lambda \ +\tfrac{1}{5}\left|L^4\pi Q\,(\pi Q-1)\right\rangle_\lambda ,\\
|2\rangle_\lambda &= 12\left|L^2\pi Q\right\rangle_\lambda + \tfrac{8}{5}\left|L^3\pi Q\,(\pi Q-1)\right\rangle_\lambda \ + \frac{1}{5^2}\left|L^4\pi Q\left((\pi Q)^2-3\pi Q+1\right)\right\rangle_\lambda ,\\
|3\rangle_\lambda &= 24\left|L\ \pi Q\right\rangle_\lambda + \tfrac{36}{5}\left|L^2\pi Q\,(\pi Q-1)\right\rangle_\lambda + \tfrac{12}{5^2}\left|L^3\pi Q\left((\pi Q)^2-3\pi Q+1\right)\right\rangle_\lambda\\
&\qquad\qquad\qquad + \tfrac{1}{5^3}\left|L^4\pi Q\left((\pi Q)^3-6(\pi Q)^2+7\pi Q-1\right)\right\rangle_\lambda .
\end{aligned}
\tag{4.28}
$$

At $\lambda = 0$ we define a different basis, which does not depend on λ,

$$
|k\rangle_0 = \vartheta^k\ \left|L^4\pi Q\right\rangle_0\Big|_{\log\lambda=0} = \frac{4!}{(4-k)!}\left|\ell^{4-k}\pi Q\right\rangle_0 \quad \text{for} \quad 0\le k\le 3, \tag{4.29}
$$

which we call the *fixed basis*. For each λ we can map the basis of $\tilde{\mathcal{S}}_\lambda$ into $\tilde{\mathcal{S}}_0$ using $E(\lambda)$. So doing, we obtain a λ-dependent basis $|k;\lambda\rangle$ for $\tilde{\mathcal{S}}_0$, which we call the *variable basis*. These bases are related by the action of $\mathrm{e}^{\varphi\pi W_0}$ and together fix the matrix of E. Indeed, with the bases so defined, writing $|0\rangle_\lambda$ in terms of the objects (4.18) and using (4.24), we find

$$
|0;\lambda\rangle = E(\lambda)\,|0\rangle_\lambda = \varpi_0(\lambda)\,|0\rangle_0 + \varpi_1(\lambda)\,|1\rangle_0 + \varpi_2(\lambda)\,|2\rangle_0 + \varpi_3(\lambda)\,|3\rangle_0\,, \tag{4.30}
$$

i.e. that the first column of E is given by

$$
E_{j0} = \varpi_j \quad \text{for} \quad 0\le j\le 3. \tag{4.31}
$$

Now, recalling how Θ commutes with E, it follows that

$$
|i;\lambda\rangle = \left|\Theta^i\Upsilon;\lambda\right\rangle = \vartheta^i\,|0;\lambda\rangle = |j\rangle_0\,\vartheta^i E_{j0}, \tag{4.32}
$$

since $|j\rangle_0$ does not depend on λ. Thus E takes the form of a Wronskian, as predicted in the introduction,

$$
E_{ji} = \vartheta^i E_{j0} = \vartheta^i\varpi_j. \tag{4.33}
$$

It is possible to verify this by computing every row of E_{ji} by a direct calculation using (4.28) to express each $|k\rangle_\lambda$ an ϵ-derivative of a linear combination of objects of the form (4.18).

As described in Section 3.4, the periods decompose into a logarithmic and holomorphic part. Indeed, if ϖ is a vector of the periods and $\mathbf{f} = \varpi\big|_{\log\lambda=0}$ is a vector of the holomorphic parts of the periods then

$$\varpi = \lambda^N \mathbf{f}, \tag{4.34}$$

where N is the matrix

$$N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{4.35}$$

so that $N^4 = 0$ and

$$\lambda^N = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \log\lambda & 1 & 0 & 0 \\ \frac{1}{2}\log^2\lambda & \log\lambda & 1 & 0 \\ \frac{1}{6}\log^3\lambda & \frac{1}{2}\log^2\lambda & \log\lambda & 1 \end{pmatrix}. \tag{4.36}$$

One can then check that E decomposes into a product of logarithmic and holomorphic parts,

$$E(\lambda) = \lambda^N E_{\text{hol}}(\lambda), \quad \text{where} \quad (E_{\text{hol}})_{ji} = \vartheta^i \varpi_j\bigg|_{\log\lambda=0}. \tag{4.37}$$

Direct calculation shows the components of E_{hol} to be

$$(E_{\text{hol}})_{ji} = \sum_{s=0}^{\min(j,i)} \frac{i!}{s!(i-s)!} \vartheta^{i-s} f_{j-s}(\lambda). \tag{4.38}$$

Notice that $|0\rangle_\lambda$, which contains a factor of $\log^4\lambda$, diverges as $\lambda \to 0$. It be made finite by dividing throughout by $\log^4\lambda$. Let us rescale the basis,

$$|k\rangle_\lambda = \frac{1}{\log^4\lambda} \Theta^k \,|0\rangle_\lambda \quad \text{for} \quad 0 \le k \le 3, \tag{4.39}$$

so that E becomes

$$E(\lambda) = E_0(\lambda) E_{\text{hol}}(\lambda), \tag{4.40}$$

$$E_0(\lambda) = \frac{\lambda^N}{\log^4\lambda} = \begin{pmatrix} \frac{1}{\log^4\lambda} & 0 & 0 & 0 \\ \frac{1}{\log^3\lambda} & \frac{1}{\log^4\lambda} & 0 & 0 \\ \frac{1}{2\log^2\lambda} & \frac{1}{\log^3\lambda} & \frac{1}{\log^4\lambda} & 0 \\ \frac{1}{6\log\lambda} & \frac{1}{2\log^2\lambda} & \frac{1}{\log^3\lambda} & \frac{1}{\log^4\lambda} \end{pmatrix}. \tag{4.41}$$

All we have done is make an overall rescaling to the basis in which we compute $\mathcal{U}(\lambda)$, which would not naïvely be expected to change the matrix. However, recall that $\mathcal{U}(\lambda)$ is defined for $|\lambda|_p < 1$ as a map between two distinct spaces,

$$\begin{aligned} &\mathcal{U}(\lambda) : \tilde{\mathcal{S}}_{\lambda^p} \to \tilde{\mathcal{S}}_\lambda \\ &\mathcal{U}(\lambda) = E(\lambda^p)^{-1}\mathcal{U}(0)E(\lambda), \end{aligned} \tag{4.42}$$

and is extended to an endomorphism of $\tilde{\mathcal{S}}_\lambda$ at the Teichmüller points $\lambda = \lambda^p$ by analytic continuation. A rescaling of the basis by a smooth function $f(\lambda)$ which is nonzero at the Teichmüller points will not change $\mathcal{U}(\lambda)$. But for a function such as $\log\lambda$ which is zero at the Teichmüller points, the rescaling can make a nontrivial contribution in the limit.

To see this, consider again the decomposition

$$\mathcal{U}(\lambda) = E_{\text{hol}}(\lambda^p)^{-1}E_0(\lambda^p)^{-1}\mathcal{U}(0)E_0(\lambda)E_{\text{hol}}(\lambda). \tag{4.43}$$

The rescaling contributes overall a factor of

$$\frac{\log^4\lambda^p}{\log^4\lambda} = p^4, \tag{4.44}$$

and thus the expansion becomes

$$\mathcal{U}(\lambda) = p^4E_{\text{hol}}(\lambda^p)^{-1}\lambda^{-pN}\mathcal{U}(0)\lambda^N E_{\text{hol}}(\lambda). \tag{4.45}$$

Once we obtain the matrix for $\mathcal{U}(0)$ in the next section, we will see that the logarithmic terms arising from λ^{-pN} and λ^N cancel.

4.3 The Action of $\mathcal{U}$

We are ready at last to compute the action of $\mathcal{U}(0)$ on $\tilde{\mathcal{S}}_0$, defined via its action on $|i,j;k\rangle_0$. First, let us establish some interesting properties of the expansion coefficients a_n, which will prove useful in the following.

For $n \in \mathbb{Z}$, we have already seen the multinomial coefficient

$$a_n = \frac{(5n)!}{(n!)^5}, \tag{4.46}$$

which is also the n^{th} expansion coefficient of the fundamental period ϖ_0. It cannot be extended smoothly to $\mathbb{Z}_p$, due to the bad p-adic behaviour of the factorial. However, it turns out that a ratio of such expressions is better behaved. For $\nu \in p\mathbb{Z}_p$, define

$$h(\nu) = \frac{a_\nu}{a_{\nu/p}}, \tag{4.47}$$

and use the factorial formula for Γ_p (equation (D.24) of Appendix D.3) to write this as

$$h(\nu) = \frac{(5\nu)!}{(5\nu/p)!}\left(\frac{\nu!}{(\nu/p)!}\right)^{-5} = \frac{\Gamma_p(5\nu+1)}{\Gamma_p(\nu+1)^5}. \tag{4.48}$$

Applying the recurrence relation for Γ_p (equation (D.12) of Appendix D.3), this becomes

$$h(\nu) = \frac{\Gamma_p(5\nu)}{\Gamma_p(\nu)^5}, \tag{4.49}$$

which we can use to extend $h(\nu)$ from $p\mathbb{Z}$ to $p\mathbb{Z}_p$.

We shall shortly derive relations involving $h(\nu)$ for $\nu \in p\mathbb{Z}_p$, whose derivations involve a_ν. The rationale for this is that the result is derived for $\nu \in p\mathbb{Z}$ and then the final expression (which contains $h(\nu)$ but no explicit occurences of a_ν) can be extended to $p\mathbb{Z}_p$.

It is easy to check that $h'(0) = 0$. The reflection formula for Γ_p (equation (D.14) of Appendix D.3) implies a reflection formula for h too,

$$h(\nu)h(-\nu) = 1 \quad \text{for} \quad \nu \in p\mathbb{Z}_p. \tag{4.50}$$

Taking successive derivatives at $\nu = 0$ and using the results from Appendix D.3 on $\Gamma_p^{(n)}(0)$, we find that h has an expansion of the form

$$h(\nu) = 1 + \frac{h^{(3)}(0)}{6}\nu^3 + \frac{h^{(5)}(0)}{120}\nu^5 + \frac{h^{(3)}(0)^2}{72}\nu^6 + \mathcal{O}(\nu^7). \tag{4.51}$$

The function $h(\nu)$ plays an important role in the computation of $\mathcal{U}(0)$ and it is in fact through $h(\nu)$ that the factor of $\zeta_p(3)$ emerges.

The Dwork operator $\mathcal{U}$ is given at $\lambda = 0$ by

$$\mathcal{U}(0) = \mathrm{e}^{-\pi Q}\mathcal{A}_p\mathrm{e}^{\pi Q}. \tag{4.52}$$

At the cost of a power of p, we can pull the first exponential through $\mathcal{A}_p$ to get

$$\mathcal{U}(0) = \mathcal{A}_p F(-Q) = \mathcal{A}_p \sum_{n=0}^{\infty} c_n(-\pi Q)^n. \tag{4.53}$$

and so we require an expression for the action of $\mathcal{A}_p$ on $|i,j;k\rangle_0$. A modicum of care is due since the Atkin operator $\mathcal{A}_p$ is not compatible with the cohomology, so we cannot simply "apply cohomology relations through the Atkin operator". That is, we must first apply $\mathcal{A}_p$ and then reduce the states using the cohomology relations (3.67, 3.69). It is true, however, that if a monomial $X^{\mathbf{v}}$ is a power of Q (or not), then the same is true of $\mathcal{A}_p X^{\mathbf{v}}$.

Indeed, expanding ${W_0}^{5j\epsilon}$ as before,

$$\begin{aligned}|i,j;k\rangle_0 &= \cdots + a_{j\epsilon}\,|i+j;k\rangle_0 + \cdots \\ &= \cdots + a_{j\epsilon}\pi^k \left|Q^{5(i+j)\epsilon+k}\right\rangle_0 + \cdots\,,\end{aligned}$$

where the terms in the ellipses are all monomials which are not powers of Q alone, all of which are zero in cohomology even after the application of $\mathcal{A}_p$. Thus

$$\mathcal{A}_p\,|i,j;k\rangle_0 = a_{j\epsilon}\pi^{k(1-1/p)}\left|\frac{i}{p}+\frac{j}{p};\frac{k}{p}\right\rangle_0\,,$$

and changing $j\epsilon/p$ powers of Q back into powers of W_0 at the cost of a factor of $a_{j\epsilon/p}$ in the denominator, we find

$$\mathcal{A}_p\,|i,j;k\rangle_0 = \begin{cases} h(j\epsilon)\pi^{k(1-1/p)}\left|\frac{i}{p},\frac{j}{p};\frac{k}{p}\right\rangle_0; & p\mid k,\\ 0; & p\nmid k.\end{cases} \tag{4.54}$$

Then for $1 \le k < p$, the action of $\mathcal{U}(0)$ can be found by expanding F as in equation (4.53),

$$\begin{aligned}\mathcal{U}(0)\,|i,j;k\rangle_0 &= \sum_{n=0}^{\infty} c_n(-1)^n\mathcal{A}_p\,|i,j;n+k\rangle_0 \\ &= h(j\epsilon)\sum_{n=0}^{\infty} c_{np+p-k}(-1)^{np+p-k}\pi^{(n+1)(p-1)}\left|\frac{i}{p},\frac{j}{p};n+1\right\rangle_0 \\ &= (-1)^k p h(j\epsilon)\sum_{n=0}^{\infty} c_{np+p-k}\; p^n\left(\frac{5\epsilon}{p}(i+j)+1\right)_n\left|\frac{i}{p},\frac{j}{p};1\right\rangle_0\,,\end{aligned}$$

where in the last line we used the cohomology relation (3.67) to reduce powers of πQ. Finally, we recognise in the above the expansion formula for Γ_p (equation (D.16) of Appendix D.3) and thus obtain the final expression for the action of $\mathcal{U}(0)$,

$$\mathcal{U}(0)\,|i,j;k\rangle_0 = (-1)^k p h(j\epsilon)\Gamma_p\left(5(i+j)\epsilon+k\right)\left|\frac{i}{p},\frac{j}{p};1\right\rangle_0\,. \tag{4.55}$$

Since $|i,j;k\rangle_0$ are related for each k by the cohomology reduction relation (3.67), and since $\mathcal{U}(0)$ is compatible with the cohomology, we find for $1 \le l \le k < p$ that

$$\begin{aligned}\mathcal{U}(0)\,|i,j;k\rangle_0 &= (5(i+j)\epsilon+l)_{k-l}\,\mathcal{U}(0)\,|i,j;l\rangle_0\\ &= (-1)^l ph(j\epsilon)\big(5(i+j)\epsilon+l\big)_{k-l}\Gamma_p\left(5(i+j)\epsilon+l\right)\left|\frac{i}{p},\frac{j}{p};1\right\rangle_0,\end{aligned}$$

from which we obtain the following consistency condition

$$\Gamma_p\left(5(i+j)\epsilon+k\right) = (-1)^{k-l}\big(5(i+j)\epsilon+l\big)_{k-l}\;\Gamma_p\left(5(i+j)\epsilon+l\right).$$

This holds by the recurrence relation for Γ_p (equation (D.12) of Appendix D.3), since for $\epsilon \in p\mathbb{Z}_p$ and $1 \le m < p$, $p \nmid 5\epsilon(i+j)+m$.

To find the matrix for $\mathcal{U}(0)$, we must express the basis $\left|\ell^{k+1}\pi Q\right\rangle_0$ in terms of ϵ-derivatives of linear combinations of $|i,j;k\rangle_0$ as we did before for $|L^4\pi Q\rangle_\lambda$. For example,

$$\left|\ell^4\pi Q\right\rangle_0 = \frac{\mathrm{d}^4}{\mathrm{d}\epsilon^4}\left\{9\,|0,1;1\rangle_0 + 9\,|1,0;1\rangle_0 - 9\,|1,1;1\rangle_0 - |2,1;1\rangle_0 - |1,2;1\rangle_0\right\}\Big|_{\epsilon=0}, \tag{4.56}$$

whence using equation (4.55) we find

$$\mathcal{U}(0)\left|\ell^4\pi Q\right\rangle_0 = p^{-3}\left|\ell^4\pi Q\right\rangle_0 + 24\gamma\,|\ell\pi Q\rangle_0, \tag{4.57}$$

where the constant γ is given by

$$\gamma = \frac{h'''(0)}{3!} = 20\left(\Gamma_p'''(0) - \Gamma_p'(0)^3\right). \tag{4.58}$$

The remaining basis elements are given by similar expressions,

$$\begin{aligned}|1\rangle_0 &= 2\,\frac{\mathrm{d}^3}{\mathrm{d}\epsilon^3}\left\{9\,|0,1;1\rangle_0 - 9\,|1,0;1\rangle_0 + |2,1;1\rangle_0 - |1,2;1\rangle_0\right\}\Big|_{\epsilon=0},\\ |2\rangle_0 &= 12\,\frac{\mathrm{d}^2}{\mathrm{d}\epsilon^2}\left\{2\,|0,1;1\rangle_0 + 2\,|1,0;1\rangle_0 - |1,1;1\rangle_0\right\}\Big|_{\epsilon=0},\\ |3\rangle_0 &= 24\,\frac{\mathrm{d}}{\mathrm{d}\epsilon}\left\{|0,1;1\rangle_0 - |1,0;1\rangle_0\right\}\Big|_{\epsilon=0},\end{aligned} \tag{4.59}$$

and a straightfoward calculation shows that

$$\mathcal{U}(0) = \begin{pmatrix} p^{-3} & 0 & 0 & 0\\ 0 & p^{-2} & 0 & 0\\ 0 & 0 & p^{-1} & 0\\ \gamma & 0 & 0 & 1\end{pmatrix}. \tag{4.60}$$

Having evaluated $\mathcal{U}(0)$ we can finally verify that the logarithmic terms in (4.45) cancel,

$$\lambda^{-pN}\mathcal{U}(0)\lambda^N = \mathcal{U}(0), \tag{4.61}$$

whence the final expression for the Dwork operator,

$$\frac{\mathcal{U}(\lambda)}{p} = p^3 E_{\mathrm{hol}}(\lambda^p)^{-1}\mathcal{U}(0)E_{\mathrm{hol}}(\lambda). \tag{4.62}$$

The single nonzero off-diagonal entry of $\mathcal{U}(0)$ is of significant interest. It can be checked that, up to a rescaling of the basis by powers of 5, our results agree with the findings of Shapiro [11, 12]. We add the additional observation that the factor of 20 in γ is in fact the ratio

$$-\frac{\chi}{2y},$$

where $\chi = -200$ is the Euler number and $y = 5$ is the value of the Yukawa coupling at $\lambda = 0$. Remarkably, we can rewrite γ in terms of the p-adic (Riemann) zeta function,

$$\gamma = \frac{\chi}{y}\zeta_p(3). \tag{4.63}$$

To do this, we make use of a recent identity relating the values of ζ_p and Γ_p,

$$\zeta_p(3) = -\frac{1}{2}\left(\Gamma_p'''(0) - \Gamma_p'(0)^3\right), \tag{4.64}$$

which was suspected on the basis of numerical calculation and before being proven by Shapiro [12] using an expansion of $\log\Gamma_p$ (see [22, p. 167]). Shapiro's formula features an additional factor of $\frac{p^3}{p^3-1}$ on the right hand side, but this is a matter of convention in the definition of $\zeta_p(3)$.

It is known that for any $k \geq 1$, $\mathcal{U}(\lambda)$ is a rational function mod p^s. This property is highly sensitive to the particular value of γ and it serves as a sanity check that with γ taken as above, numerical computation of $\mathcal{U}(\lambda)$ does indeed yield a rational function mod p^s.

There are a great many other Picard-Fuchs operators, similar in structure to (3.54), the so called *Calabi-Yau type* Picard-Fuchs operators, each of which has a point of maximal unipotent monodromy at $\lambda = 0$. A remarkable observation in practice is that for all such operators where χ and y are known, taking γ as in (4.63) we do indeed find that $\mathcal{U}(\lambda)$ is a rational function mod p^s. The task of evaluating $\mathcal{U}(\lambda)$ and using it to compute $R(T)$ for a given manifold is the focus of the next chapter.

Chapter 5

Zeta Functions and Modularity

In this chapter, we shall use the method of Dwork to compute a factor of the zeta function as a determinant, using the expansion for $\mathcal{U}(\lambda)$ developed in the previous chapter. First, we shall consider the quintic in order to establish notation and conventions, in particular focussing on the four-dimensional subspace S_λ of the Dwork cohomology introduced previously. Corresponding to this subspace is a quartic factor of the zeta function given by

$$R(T;\lambda) = \det\left(1 - \frac{\mathcal{U}(\lambda)}{p}\, T\right), \tag{5.1}$$

where the 4×4 matrix $\mathcal{U}(\lambda)$ is given by (4.62). Expanding the determinant gives

$$R(T) = 1 + aT + bpT^2 + p^3cT^3 + p^6dT^4, \tag{5.2}$$

where a, b, c, d are integer-valued functions of λ with

$$a(\lambda) = -\operatorname{tr}\mathcal{U}, \quad b(\lambda) = \frac{1}{2p}\left(\operatorname{tr}(\mathcal{U}^2) - (\operatorname{tr}\mathcal{U})^2\right). \tag{5.3}$$

Where the variety is nonsingular, the reflection formula of the Weil conjectures imposes $c = a$ and $d = 1$. At singular values of the parameter, one or more eigenvalues of $\mathcal{U}$ go to zero and the degree of $R(T)$ is reduced. At the conifold point $\lambda = 5^{-5}$ of the quintic, $R(T)$ becomes cubic and admits a factorisation into a linear and a quadratic factor,

$$R(T) = (1 - p\chi_p T)(1 - \alpha_p T + p^3T^2), \tag{5.4}$$

where $\chi_p = \pm 1$ is a character and α_p is an integer. The p subscript serves to disambiguate the character χ_p from the Euler number χ. For the quintic, α_p has been shown [6, 15] to be the p^{th} coefficient in the q-expansion of the weight 4 level 25 classical modular form (of trivial character) given by 25.4.1.b in the LMFDB labelling[1]

[1] `http://www.lmfdb.org/ModularForm/GL2/Q/holomorphic/25/4/1/b/`

(or 25/1 in Meyer's table).

All the information needed to compute $R(T)$ is captured by the periods and hence by the Picard-Fuchs equation (3.54), which is of so-called *Calabi-Yau* type. It has a large structure limit $\lambda = 0$, where all the indices vanish, making it a point of *maximal unipotent monodromy* (MUM). As already mentioned, there are many Picard-Fuchs operators of Calabi-Yau type besides that of the quintic. These have been constructed by various methods, and a list of several hundred exists [13, 14], though it is not certain that all of these correspond to actual Calabi-Yau manifolds. Nonetheless, such an operator contains almost[2] all the information necessary to compute $R(T)$ and thus identify any associated modular form for a given manifold.

A generic operator of Calabi-Yau type can be written in the standard form

$$\mathcal{P} = \sum_{k=0}^{d} \lambda^k P_k(\vartheta), \tag{5.5}$$

where each P_k is a polynomial in the logarithmic λ derivative ϑ and $P_0(\vartheta) = \vartheta^4$. We call d the *degree* of the operator and in the alternate form

$$\mathcal{P} = \sum_{i=0}^{4} R_i(\lambda)\vartheta^i, \tag{5.6}$$

it is the degree of R_4. A standard calculation yields a recurrence relation for the series expansion coefficients $A_n(\epsilon)$ of the Frobenius period $\varpi(\lambda;\epsilon)$ in terms of the P_k. From this, one can obtain a series for E and hence for $\mathcal{U}$. We will make use of the same expansion (4.62) for $\mathcal{U}(\lambda)$ as was made for the quintic. Evaluating this series at a given value of λ is not practical since the series converges slowly. However, as we shall soon see, it is possible to circumvent this process entirely by working mod p^4. Thus, given an operator, computing $R(T)$ is relatively straightforward and we can proceed as we did for the quintic. The behaviour at the singular points, however, does vary and we will encounter a number of forms for $R(T)$ with different modular behaviour.

The simplest cases are the fourteen (degree 1) hypergeometric operators [15, 16], foremost among them the quintic itself. Each of them has a conifold singularity to which a weight 4 modular form was associated in [15]. The task of Section 5.1 will be to develop a framework for computing $R(T)$ for each of these cases using the expansion (4.62).

[2]We also need the ratio χ/y. Though χ and y are known for the manifolds associated to some of the operators, this is certainly not always the case. Fortunately, we will be able to deduce the value of this ratio in the course of our computation.

Things are not so simple in the general case. We can, of course, always solve the Picard-Fuchs equation for the periods and thus obtain a power series for $\mathcal{U}(\lambda)$. This might seem to be all we need to compute $R(T)$. However, there are a number of subtleties. Not all singular points exhibit the same behaviour as the conifold points of the hypergeometric operators. In Section 5.2, we will describe a number of types of singular point and show how they can be classified according to their indices, using various degree 2 operators as examples. Some operators even have singular points which do not correspond to true singularities of the underlying manifold. We call these *apparent* singularities (or *pseudosingularities*) and we will have to account for such points. In Section 5.3, we present the generic method we have used to successfully compute $R(T)$ and identify modular forms for a large number of operators of degree less than four.

In certain cases, though, our procedure breaks down entirely and does not yield meaningful results. There are also cases where the computation of $R(T)$ is successful, but for various reasons it is not possible to identify a modular form. Some of these reasons relate to the availability of modular form data, and in Section 5.5 we discuss how such issues might be addressed in future work. Finally in Section 5.6 we present our main tables of results matching modular forms to singular points. Full tables of $R(T)$ for $7 \leq p \leq 97$ and $1 \leq \lambda < p$ would be too long to include for all operators, but they are provided for a select few in Appendix A. Details of the software used for the computation can be found in Appendix C.

5.1 The Hypergeometric Cases

We turn our attention first to the fourteen hypergeometric operators of degree 1. In all these cases, a weight 4 modular form was already identified in [15], against which we can check our results. We shall develop a framework for computing $R(T)$ in these cases which (except in the case of apparent singularities) generalises immediately to higher degree operators.

A degree one Picard-Fuchs operator can always be written in the standard form

$$\mathcal{P} = P_0(\vartheta) + \lambda P_1(\vartheta) = \sum_{k=0}^{4} R_k(\lambda)\vartheta^k, \tag{5.7}$$

where each R_k is a polynomial of degree one. This operator is singular at $\lambda = 0$, λ_0 and ∞, where λ_0 is the sole root of $R_4(\lambda)$. As always, $\lambda = 0$ is a MUM point;

moreover for the hypergeometric cases, $\lambda = \lambda_0$ is a conifold point: it has indices $\{0, 1, 1, 2\}$ and infinite monodromy. It is straightforward to obtain series expansions for the periods ϖ_k in terms of P_0 and P_1.

A priori, analytic continuation is required to evaluate $\mathcal{U}$ at the Teichmüller points via the series expansion. We can, however, exploit a useful property of $\mathcal{U}$ coming from p-adic analysis. Indeed, as can be verified computationally, to any fixed p-adic order, the components of $\mathcal{U}$ are rational functions. That is, for each $s \geq 1$, there exists a polynomial $g_s(\lambda)$ and a matrix of polynomials $\mathcal{U}_s$ such that

$$\mathcal{U}(\lambda) = \frac{\mathcal{U}_s(\lambda)}{g_s(\lambda)} \mod p^s. \tag{5.8}$$

For the hypergeometric cases, for $1 \leq s < p$, we can take g_s to be

$$g_s(\lambda) = \begin{cases} 1; & s \leq 4, \\ \Delta(\lambda)^{(s-4)p}; & s \geq 4, \end{cases} \tag{5.9}$$

where we have introduced the *discriminant*

$$\Delta = \lambda - \lambda_0. \tag{5.10}$$

This rule does in fact work for some $s > p$, but we observed that when $p \mid s$, a higher power of Δ is required. In practice, (5.9) has been used successfully with $s = 20$ at primes $7 \leq p < 20$ to compute the off-diagonal entry of $\mathcal{U}(0)$ for all operators considered.

It is a remarkable fact that we can write $\mathcal{U}(\lambda)$ in this way, and one that was vital in obtaining our results. Evaluating the series $\mathcal{U}(\lambda)$ is not practical as it converges very slowly. Instead, we can evaluate at the Teichmüller points by direct substitution into the rational function (5.8). Of course, this does not work when the denominator $g_s(\lambda)$ is zero, which for $s > 4$ occurs at the singular point λ_0. Fortunately, however, it will turn out that working to four p-adic digits (i.e. mod p^4) is sufficient to obtain an exact expression for $R(T)$.

Indeed, the Weil conjectures impose bounds on the roots α_j of $R(T)$ for a smooth variety, namely that $|\alpha_j| = p^{-3/2}$. It can then be shown [7] that for $p > 5$, a is constrained to a fixed range and for a fixed, b is constrained to a given range around some (a-dependent) value. Indeed,

$$-\frac{1}{2} \leq \frac{a}{p^3} \leq \frac{1}{2}, \quad -\frac{1}{2} \leq \frac{b - b_{\text{mid}}}{p^3} \leq \frac{1}{2}, \tag{5.11}$$

where we defined

$$b_{\text{mid}} = \frac{a^2}{8p} + |a|\lfloor\sqrt{p}\rfloor. \tag{5.12}$$

Given these bounds, it is clear that if a and b are known mod p^3 then they are known exactly. We shall see shortly, however, that to compute three p-adic digits of b, we require $\mathcal{U}(\lambda) \mod p^4$.

There are, in fact, stronger bounds on a and b than (5.11), but they are not necessary for our purposes. Indeed, it will be a good consistency check later to verify that, despite imposing only a weaker bound on a and b, the roots of the resulting polynomial $R(T)$ satisfy the Weil bound nonetheless.

The fact that four digits of p-adic precision is sufficient is worth labouring. This is not simply a matter of reduced computational intensity: it is *fundamental* to our procedure. Suppose that we needed 5 digits of precision. Then we would have to take for our denominator $g_5(\lambda) = \Delta(\lambda)$. But the points of greatest interest to us, at which we seek modular forms, are precisely the zeros of Δ, i.e. the singular points. As soon as we seek more than four digits of p-adic precision, we preclude any computation at these points using our procedure.

Recalling the formulae (5.3) for a and b and reducing mod p^3 we see that

$$a(\lambda) = -\operatorname{tr}\mathcal{U}_3, \quad b(\lambda) = \frac{1}{2p}\left(\operatorname{tr}(\mathcal{U}_4^2) - (\operatorname{tr}\mathcal{U}_4)^2\right). \tag{5.13}$$

Note that we require $\mathcal{U}_4$ to evaluate b mod p^3 on account of the factor of p appearing in the denominator of the expression for b. In practice, therefore, we evaluate a using

$$a(\lambda) = -\operatorname{tr}\mathcal{U}_4 \mod p^3, \tag{5.14}$$

since we must compute $\mathcal{U}_4$ anyway. For a given λ, we can evaluate a and b exactly by directly substituting into these expressions and taking the central lift.

Note that four p-adic digits of precision in $\mathcal{U}$ are not enough to evaluate c and d exactly since they occur in $R(T)$ multiplied by 3 and 6 powers of p respectively. It is thus important that their values are known in advance. We can, however, compute c mod p which provides a useful sanity check on our computation. If the variety is singular, we expect $c \equiv 0 \mod p$ and otherwise $c \equiv a \mod p$.

For each of the hypergeometric cases, at the conifold point $\lambda = \lambda_0$, the quartic $R(T)$ degenerates to a cubic which factorises in the form

$$R(T) = (1 - p\chi_p T)(1 - \alpha_p T + p^3 T^2). \tag{5.15}$$

Comparing with the general form (5.2) of $R(T)$, and noting that $\chi_p^2 = 1$, it is easy to see that

$$\chi_p = \frac{p^2 - p - b_p}{a_p}, \quad \alpha_p = -(a_p + p\chi_p). \tag{5.16}$$

It is a reassuring check to observe in practice that substituting computed values of a and b at $\lambda = \lambda_0$ yields $\chi_p = \pm 1$ and α_p integral. The first few values of χ_p and α_p at $\lambda = 1/3125$ for the quintic are given in Table 5.1.

p	χ_p	α_p
7	-1	6
11	1	-43
13	-1	-28
17	-1	91
19	1	-35
23	-1	162
29	1	160
31	1	42
37	-1	-314

Table 5.1: Values of χ_p and α_p for the quintic

As already noted, the coefficients α_p correspond to the level 4, weight 25 classical newform 25.4.1.b with q-expansion

$$q + q^2 + 7q^3 - 7q^4 + 7q^6 + 6q^7 - 15q^8 + 22q^9 + \mathcal{O}\big(q^{10}\big).$$

The modular forms for the hypergeometric operators, which as noted before were first identified in [15], are listed in Table 5.2.

operator	singularity	modular form
1	1/3125	25.4.1.b
2	1/800000	200.4/10
3	1/256	8.4.1.a
4	1/729	27.4.1.a
5	1/432	36.4.1.a
6	1/1024	16.4.1.a
7	1/65536	128.4/3
8	1/11664	108.4.1.a
9	1/2985984	864.4/4
10	1/4096	32.4.1.a
11	1/1728	9.4.1.a
12	1/27648	144.4/1
13	1/186624	216.4/4
14	1/6912	72.4.1.b

Table 5.2: Modular forms at the conifold points of the degree 1 operators

Notice that each singular point is a reciprocal integer and that the level divides this integer in each case (in both the LMFDB and Meyer labelling systems for classical modular forms, the first number in the label is the level—see Appendix D.5.4 for details). We will come back to this observation in Section 5.4.1.

In Section 5.2 we will examine the various types of singularity (other than conifold points) possessed by degree 2 and 3 operators. First, though, we make a short digression to introduce an efficiency improvement used in our code.

5.1.1 Fast E inversion

Before proceeding to higher degree operators, we introduce a useful trick for quickly computing the inverse of $E(\lambda)$. A priori, this matrix inversion is an expensive part of the computation. The components of E are power series and the inversion scales quadratically with the series precision. Nonetheless, we require a relatively high series precision to work with large primes. In particular, the series precision must be larger than the expected degree of the numerator $\mathcal{U}_4$. Fortunately, it turns out that E is almost orthogonal with respect to the natural symplectic structure of the underlying

Calabi-Yau manifold. It is thus possible to reduce the inversion to transposition and a little matrix multiplication. To see this, we start by making some definitions.

Associated to each Calabi-Yau threefold is a nowhere vanishing holomorphic 3-form $\Omega \in H^{(3,0)}$. Each successive derivative with respect to the complex structure parameter λ inroduces additional antiholomorphic terms, so for example $\Omega' \in H^{(3,0)} \oplus H^{(2,1)}$. It follows that

$$\int_{\mathcal{M}} \Omega\Omega' = \int_{\mathcal{M}} \Omega\Omega'' = 0, \quad \int_{\mathcal{M}} \Omega\Omega''' = \int_{\mathcal{M}} f, \tag{5.17}$$

where prime denotes λ-differentiation, and the scalar function f is the coefficient of the totally antiholomorphic part of Ω''',

$$\Omega'''\Big|_{(0,3)} = f\overline{\Omega}.$$

We can rewrite higher λ-derivatives of Ω using the Picard-Fuchs equation (5.6),

$$\vartheta^4\Omega = -\sum_{i=0}^{3} \frac{R_i(\lambda)}{R_4(\lambda)}\vartheta^i\Omega. \tag{5.18}$$

Now define the *Yukawa coupling*

$$y = \int_{\mathcal{M}} \Omega \frac{\mathrm{d}^3\Omega}{\mathrm{d}\lambda^3}. \tag{5.19}$$

It is easy to check that $\lambda^3 y$ satisfies a differential equation, which upon integration yields

$$y = \frac{1}{\lambda^3} \exp\left(-\frac{1}{2}\int \frac{-R_3}{\lambda R_4}\,\mathrm{d}\lambda\right). \tag{5.20}$$

Next, introduce a basis of periods $z^a(\lambda)$, $\mathcal{G}_a(\lambda)$ such that

$$\Omega = z^a\alpha_a - \mathcal{G}_a\beta^a, \tag{5.21}$$

where α_a, β^a is a corresponding basis of $H^3(\mathcal{M},\mathbb{Z})$ dual to a symplectic basis of $H_3(\mathcal{M},\mathbb{Z})$,

$$\int_{\mathcal{M}} \alpha_a\beta^b = \delta_a{}^b. \tag{5.22}$$

The integral basis Π of periods is related to the Frobenius basis ϖ by the matrix ρ,

$$\Pi = \rho\varpi, \quad \rho = \begin{pmatrix} -\frac{1}{3}Y_{000} & -\frac{1}{2}Y_{001} & 0 & Y_{111} \\ -\frac{1}{2}Y_{001} & -Y_{011} & -Y_{111} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{5.23}$$

where Y_{abc} are constants and in particular $Y_{111} = y(0)$, the classical value of the Yukawa coupling. The Wronskian matrix $\mathcal{E}$ of the integral periods is related to E, the Wronskian matrix of the Frobenius periods, by

$$\mathcal{E} = \rho E.$$

Now to reveal the symplectic structure, introduce the standard matrices

$$\Sigma = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \tag{5.24}$$

which are related by conjugation by ρ,

$$\rho^{\intercal} \Sigma \rho = -y(0)\sigma. \tag{5.25}$$

It then follows that

$$-E^{\intercal}\sigma E = \tfrac{1}{y(0)}\mathcal{E}^{\intercal}\Sigma\mathcal{E}. \tag{5.26}$$

Denoting the matrix on the right hand side by W, we obtain the following formula for the inverse of E,

$$E^{-1} = -W^{-1}E^{\intercal}\sigma. \tag{5.27}$$

The matrix W can be expressed in terms of the holomorphic 3-form by means of equation (5.21) and has components

$$W_{ij} = \frac{-1}{y(0)} \int_{\mathcal{M}} \vartheta^i \Omega \vartheta^j \Omega.$$

Then, using the Picard-Fuchs equation and some identities obtained by applying ϑ to equation (5.17), we find that

$$W^{-1} = \frac{y(0)}{\lambda^3 y} \begin{pmatrix} 0 & w & r & 1 \\ -w & 0 & -1 & 0 \\ -r & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \tag{5.28}$$

where we have introduced functions w, r of λ given by

$$r = \frac{R_3}{2R_4}, \quad w = \frac{R_2}{R_4} - r^2 - \vartheta r. \tag{5.29}$$

Note that the Yukawa coupling y should be normalised such that $y(0) = H^3$, which is known in some cases (including the hypergeometric cases), but not in general. Fortunately, it can be seen above that its value does not come into the expression for W. In our code, therefore, we normalise such that $y(0) = 1$.

5.2 Types of Singular Point

As we saw in the hypergeometric cases, the degree of $R(T)$ falls at singular points, since one or more eigenvalues of $\mathcal{U}$ goes to zero. Typically, a quadratic factor arises in $R(T)$ regardless of its overall degree (though there are exceptions), and the specific form of this factor determines the weight of the associated modular form.

A common type of singularity is the conifold point, which we already encountered in the hypergeometric cases. Not all singular points are conifolds, however, and each type of singularity yields a different quadratic in $R(T)$ and thus different modular behaviour. In practice, singular points can be almost entirely classified according to their indices.

In this section, we introduce the various types of singularity encountered at degrees 2 and 3, using suitable degree 2 operators as examples. At degree 4 and higher, additional types of singularity emerge, but we leave such cases to future work. Though the results were not ready at the time of writing, we had preliminary success in identifying modular forms for degree 4 operators of known type.

The table below summarises the behaviour of each type of singularity encountered. There are, however, a number of exceptional cases which are discussed further below. It should be noted that we are only talking about *rational* singular points. Irrational singularities are discussed separately in Section 5.2.5.

case	indices	$\deg R(T)$	weight	notes
0	$\{0,1,1,2\}$	3	4	conifold
1	$\{0,r,1-r,1\}$	2	4	$0<r<1$
2	$\{0,r,1-r,1\}$	2	2	$r<0$
3	$\{0,0,1,1\}$	2	3	$r=0$
4	$\{0,r,1+r,1\}$	–	–	$r\neq 0,1$

The conifold singularity is the most commonly encountered in the list, having indices $\{0,1,1,2\}$. We have already seen what happens in this case since all the hypergeometric operators have conifold singularities. Indeed, at these singular points $R(T)$ becomes a cubic of the form

$$R(T) = (1-p\chi_p T)(1-\alpha_p T + p^3T^2). \tag{5.30}$$

The coefficient α_p corresponds to the p^{th} coefficient in the q-expansion of a weight 4 classical modular form. We do not give an example in this case since any of the hypergeometric (i.e. degree 1) cases covered in Section 5.1 will suffice.

Cases 1–3 comprise all singular points with indices $\{0, r, 1-r, 1\}$ for rational r. Such singularities are typically multiplicity 2 roots of R_4 and yield a quadratic $R(T)$. The details depend on the sign of r. Our computations failed at all examples of case 4.

5.2.1 Case 1

Here, $R(T)$ takes the same form as the quadratic factor that arises in case 0,

$$R(T) = 1 - \alpha_p T + p^3 T^2. \tag{5.31}$$

As in case 0, the coefficients α_p correspond to a weight 4 classical modular form. Take, for example, operator 2.17 which has singularities at 0, 1/256 and ∞. The indices at 1/256 are $\left\{0, \frac{1}{2}, \frac{1}{2}, 1\right\}$, i.e. $r = \frac{1}{2}$. The first few coefficients α_p are given in Table 5.3 and correspond to the classical modular form 16.4.1.a.

p	α_p
7	−24
11	44
13	22
17	50
19	−44
23	56
29	198
31	160
37	−162

Table 5.3: Values of α_p for operator 2.17 at $\lambda = 1/256$

5.2.2 Case 2

This is the first case where we encounter problems and where we cannot distinguish behaviour based on the indices alone. In those cases where the computation succeeds, henceforth referred to as case 2a, $R(T)$ takes the form

$$R(T) = 1 - p\beta_p T + p^3 T^2 = 1 - \beta_p(pT) + p(pT)^2, \tag{5.32}$$

and the coefficients β_p correspond to a weight 2 classical modular form. For example, operator 2.41 has singularities at $0, 1/2916, \infty$. The indices at 1/2916 are $\left\{-\frac{1}{2}, 0, 1, \frac{3}{2}\right\}$, i.e. $r = -\frac{1}{2}$. The coefficients β_p, the first few of which are given in Table 5.4, correspond to the classical modular form 54.2.1.b.

p	β_p
7	-1
11	3
13	-4
17	0
19	2
23	6
29	-6
31	5
37	2

Table 5.4: Values of β_p for operator 2.41 at $\lambda = 1/2916$

However, at certain singular points with the same indices, things start to go wrong. In such cases, henceforth called case 2b, we find that $\mathcal{U}(\lambda)$ only reduces to a rational function mod p^4 for a subset of the primes $7 \leq p \leq 97$. In each such case considered there is an integer D such that the "good" primes are precisely those for which the polynomial $x^2 - D$ factorises over $\mathbb{F}_p$. We expect that some higher degree operators exhibit similar behaviour with a polynomial of degree greater than 2.

In each of these problem cases (at degree 2), factors of $\sqrt{D}$ occur in the monodromy matrix in the Frobenius basis at the singular point. This is not the case for the type 2a examples, such as 2.41 (i.e. those where the calculation succeeds). Thus to completely classify singularities of type 2, we must look at the monodromy as well as the indices. Table 5.5 lists each degree 2 and 3 operator of type 2b along with the value of D. These values, which we discovered experimentally, were separately predicted by Duco van Straten.

operator	$\sqrt{D}$
2.39	$\sqrt{5}$
2.42	$\sqrt{3}$
2.43	$\sqrt{2}$
2.44	$\sqrt{2}$
2.45	$\sqrt{3}$
2.48	$\sqrt{6}$
2.49	$\sqrt{2}$
3.30	$\sqrt{2}$
3.35	$\sqrt{2}$

Table 5.5: Irrational values occurring in the monodromy of certain operators

For example, operator 2.43 has singularities at $0, 1/4096, \infty$ and the indices at $1/4096$ are $\{-\frac{1}{2}, 0, 1, \frac{3}{2}\}$, i.e. $r = -\frac{1}{2}$. The matrix $\mathcal{U}(\lambda)$ reduces to a polynomial mod p^4 for $p \equiv \pm 1 \mod 8$, i.e. we have $D = 2$. We thus obtain a table of coefficients for a reduced set of primes. We might hope that, at $\lambda = 1/4096$, $R(T)$ still takes the form (5.32) for the "good primes" and that a weight 2 modular form can still be found. Unfortunately, though, the problems do not stop here.

The first few coefficients of $R(T)$ for 2.43 are given in Table 5.6. Note that $p \mid a$ and $p^2 \mid b$, which seems to be the case for all type 2b singular points (of course, this is also true for type 2a singular points).

p	a/p	b/p^2
7	3	1
17	−14	1
23	104	70
31	−108	63
41	365	452
47	600	1035
71	−419	356
73	−884	1461
79	−2077	7032

Table 5.6: Values of a and b for operator 2.43 at $\lambda = 1/4096$

These do not match any of our expectations for a quadratic factor, nor do they yield a factorisation of a cubic into linear and quadratic terms. However, notice that in each case $b/p^2 \equiv 1 \mod p$. Thus if we reduce mod p^3 we do have $R(T)$ of the form (5.32),

$$R(T) = 1 - p\beta_p T + p^3 T^2.$$

(Recall that not b but $b - b_{\text{mid}}$ is required to be in the range $[-p^3/2, p^3/2]$.) However, the coefficients $\beta_p = -a_p/p$ did not yield a matching weight 2 modular form. Moreover, at nonsingular values of λ, the quartic polynomial $R(T)$ does not satisfy the Weil bound as it should (its roots over $\mathbb{C}$ do not all have the same absolute value).

It is possible that these problems are caused by p-adic precision issues. If so there may be no solution, since we cannot do better than mod p^4 at the singular points. We could try to perform the computation over $\mathbb{Q}(\sqrt{2})$ and $\mathbb{F}_p(\sqrt{2})$, which would at least allow us to obtain $R(T)$ for all primes, but our code does not support this at present.

5.2.3 Case 3

At degree 2, there are only two operators with a singular point whose indices are $\{0, 0, 1, 1\}$, namely 2.63 and 2.66. The latter is a special case and at its singularity $\lambda = 1/432$, $R(T)$ splits into two linear factors

$$R(T) = (1 - \chi_p T)(1 - p^2 \chi_p T), \tag{5.33}$$

where the first few values of the character χ_p are given in Table 5.7. There is of course no modular form in this case.

p	χ_p
7	1
11	1
13	−1
17	−1
19	1
23	1
29	−1
31	1
37	−1

Table 5.7: Values of χ_p for operator 2.66 at $\lambda = 1/432$

Operator 2.63 is a more typical example whose singularity $\lambda = 1/64$ behaves the same as the type 3 singularities of degree 3 operators. Here, $R(T)$ takes the form

$$R(T) = 1 - \alpha_p T + p^2 \chi_p T^2, \tag{5.34}$$

where χ_p is a character. In this case we expect a weight 3 classical modular form which for 2.63 is 8.3.3.a. Recall that the second "3" in this label indicates the presence of a nontrivial character[3] in the transformation relation of the modular form. In this case, this is precisely the character χ_p appearing in $R(T)$. The first few values of α_p and χ_p are given in Table 5.8.

p	χ_p	α_p
7	-1	0
11	1	14
13	-1	0
17	1	2
19	1	-34
23	-1	0
29	-1	0
31	-1	0
37	-1	0

Table 5.8: Values of χ_p and α_p for operator 2.63 at $\lambda = 1/64$

5.2.4 Case 4

Problems were encountered at all singular points with indices $\{0, r, 1+r, 1\}$ for some r. At degree 2 the relevant operators are 2.67, 2.68 and 2.70, and many more exist at degree 3. As in case 2b, $\mathcal{U}(\lambda)$ only reduces to a rational functional mod p^4 for a subset of the primes $7 \leq p \leq 97$. As in case 2b, these primes corresponded to the existence of a quadratic residue D, but for $D < 0$ in this case.

As in case 2b, even at the "good" primes, $R(T)$ did not satisfy the Weil bound at nonsingular λ. The values of the coefficient c of $R(T)$ computed mod p also indicated a problem: we did not have $c \equiv a \mod p$ at nonsingular λ (though we did find that $c \equiv 0 \mod p$ at singular points). As in case 2b, these failures may suggest precision issues, though it is not clear to the author why more precision should be required in this case. Regardless, no recognized form of $R(T)$ could be identified at singular points of this type, so no modular forms could be identified either.

Another possibility is that such operators do not in fact correspond to actual Calabi-Yau manifolds. In the absence of an underlying manifold, of course, the Weil conjectures do not apply and we have no reason to expect a particular form for $R(T)$ (or

[3]As noted in Appendix D.5.3, there are no classical modular forms of $\Gamma_0(N)$ of odd weight and trivial character. This, perhaps, explains why a character arises in $R(T)$ in this case.

indeed that $\zeta(T)$ be a rational function at all). However, the fact that $\mathcal{U}(\lambda)$ reduces mod p^4 for certain p suggests that at least some underlying structure exists.

5.2.5 Irrational Singular Points

So far, the focus has been on rational singular points, but a more interesting question is the behaviour at irrational singularities. Such singularities necessitate working over an algebraic extension K of $\mathbb{Q}$. Though our code only works over $\mathbb{Q}$ at present, we have already seen a workaround. Namely, we can restrict our attention to the primes at which the singularity in question exists in $\mathbb{F}_p$, and thus obtain a table of coefficients of $R(T)$ for some subset of the primes $7 \leq p \leq 97$.

The form of $R(T)$ is determined by the indices in the same way as for rational singular points. So, for example, at irrational conifold singularities (with indices $\{0, 1, 1, 2\}$) we have

$$R(T) = (1 - p\chi_p T)(1 - \alpha_p T + p^3 T^2), \tag{5.35}$$

for some character χ_p and integer α_p. Next, we must ask what the conjecture for modular forms is in this case. We seek a weight 4 form as before, but can no longer expect a classical modular form over $\mathbb{Q}$. Instead, we must seek a Hilbert (or Bianchi) modular form of parallel weight 4 over the extension field K (see Appendix D.6 for a definition).

To identify a classical modular form, a list of q-expansion coefficients α_p for some primes p is needed. For a rational singular point λ_0 and a given prime p, one simply reduces λ_0 mod p to determine the correct value of the parameter $\lambda \in \mathbb{F}_p$ at which to count points. Then, one equates the appropriate coefficient of $R(T)$ with the coefficient α_p of the q-expansion.

The process of identifying Hilbert modular forms at a real irrational singular point λ_0 is similar. The "q-expansion" coefficients of a Hilbert modular form are labelled by prime ideals $\mathfrak{p}$ in the ring of integers $\mathbb{Z}_K$ of K. Such ideals have ideal norm p where p is a split prime in $\mathbb{Z}_K$. That is, $x^2 - p$ factors in $\mathbb{F}_p$ and the ideal (p) generated by p in $\mathbb{Z}_K$ factors into a product of prime ideals $\mathfrak{p}_1\mathfrak{p}_2$. Given an irrational singular point in K, we reduce mod $\mathfrak{p}$ to obtain $\lambda \in \mathbb{F}_p$. Then we compute $R(T)$ for $p = \mathrm{Norm}(\mathfrak{p})$ and $\lambda = \lambda_0 \mod \mathfrak{p}$ to obtain the coefficient $\alpha_\mathfrak{p}$.

If K is a degree 2 extension of $\mathbb{Q}$, there will be a pair of prime ideals $\mathfrak{p}_1, \mathfrak{p}_2$ of any given norm p, for p a split prime. Suppose an operator has a conjugate pair of singularities

$\lambda_1, \lambda_2 \in K$ which reduce to $\overline{\lambda}_1, \overline{\lambda}_2$ under the residue map of $\mathfrak{p}_1$. Then, under the residue map of $\mathfrak{p}_2$, the order is reversed, i.e.

$$\lambda_1 \mapsto \overline{\lambda}_2, \quad \lambda_2 \mapsto \overline{\lambda}_1.$$

This has the consequence that any coefficients $\alpha_{\mathfrak{p}_1}, \alpha_{\mathfrak{p}_2}$ of $R(T)$ at λ_1 will be reversed at λ_2. In the following examples, therefore, we shall only list the coefficients for one singularity since those of its conjugate can be deduced by reversing each pair of coefficients.

First, take operator 2.55 which has singularities at $0, \frac{3}{16} \pm \frac{1}{8}\sqrt{2}, \infty$. The pair of irrational singular points have indices $\{0, 1, 1, 2\}$ (making them conifold singularities) and minimal polynomial

$$256\lambda^2 - 96\lambda + 1.$$

This polynomial factors over $\mathbb{F}_p$ whenever $p \equiv \pm 1 \mod 8$, and so these are the primes for which the computation of $R(T)$ can proceed. The manifold is defined over $K = \mathbb{Q}(\sqrt{2})$. We obtain the desired expression (5.35) for $R(T)$ and the first few coefficients $\chi_\mathfrak{p}, \alpha_\mathfrak{p}$ are given in Table 5.9 for $\frac{3}{16} + \frac{1}{8}\sqrt{2}$.

	$\frac{3}{16} + \frac{1}{8}\sqrt{2}$	
$\mathfrak{p}$	$\chi_\mathfrak{p}$	$\alpha_\mathfrak{p}$
$[7, 7, w + 3]$	1	-8
$[7, 7, w + 4]$	-1	-8
$[17, 17, w + 6]$	-1	-14
$[17, 17, w + 11]$	-1	-14
$[23, 23, w + 5]$	1	-152
$[23, 23, w + 18]$	-1	-152
$[31, 31, w + 8]$	-1	224
$[31, 31, w + 23]$	1	224

Table 5.9: Values of $\chi_\mathfrak{p}$ and $\alpha_\mathfrak{p}$ at a singular point of operator 2.55

Notice that in this case, the value of $\alpha_\mathfrak{p}$ at each pair of ideals with the same norm p are equal. When this occurs (or occurs up to sign), then it is sometimes possible to identify a *classical* modular form associated with the coefficients. In this case, however, no such match could be found in our tables of classical forms. We did find a match, however, for the Hilbert modular form 2–1.4/1 at both singular points.

Note that all Hilbert modular forms referenced in this section can be looked up in Appendix B to find their level and coefficients.

Next, consider operator 2.5, with a pair of irrational conifold singularities $-\frac{11}{32} \pm \frac{5}{32}\sqrt{5}$ defined over $K = \mathbb{Q}(\sqrt{5})$. In this case, the coefficients at a given p are equal up to sign, and the first few are listed in Table 5.10. We identified for the respective singular points the Hilbert modular forms 5–256.4/1 and 5–256.4/2 of level $[256, 16, 16]$.

	$-\frac{11}{32} + \frac{5}{32}\sqrt{5}$	
$\mathfrak{p}$	$\chi_\mathfrak{p}$	$\alpha_\mathfrak{p}$
$[11, 11, w + 3]$	-1	-20
$[11, 11, w + 7]$	-1	20
$[19, 19, w + 4]$	1	84
$[19, 19, w + 14]$	1	-84
$[29, 29, w + 5]$	-1	6
$[29, 29, w + 23]$	-1	6
$[31, 31, w + 12]$	-1	-224
$[31, 31, w + 18]$	-1	224

Table 5.10: Values of $\chi_\mathfrak{p}$ and $\alpha_\mathfrak{p}$ at a singular point of operator 2.5

Finally, consider operator 2.6 with a pair of conifold singularities at $-\frac{11}{54} \pm \frac{5}{54}\sqrt{5}$. This time the pairs of coefficients are entirely distinct. As always, the pairs are reversed between the two singular points. For the respective singular points we identified the modular forms 5–729.4/1 and 5–729.4/2 of level $[729, 27, 27]$.

	$-\frac{11}{54} + \frac{5}{54}\sqrt{5}$	
$\mathfrak{p}$	$\chi_\mathfrak{p}$	$\alpha_\mathfrak{p}$
$[11, 11, w + 3]$	-1	-18
$[11, 11, w + 7]$	-1	27
$[19, 19, w + 4]$	-1	-19
$[19, 19, w + 14]$	-1	80
$[29, 29, w + 5]$	1	126
$[29, 29, w + 23]$	1	-198
$[31, 31, w + 12]$	1	-262
$[31, 31, w + 18]$	1	-37

Table 5.11: Values of $\chi_\mathfrak{p}$ and $\alpha_\mathfrak{p}$ at a singular point of operator 2.6

We proceed the same way for all other real irrational conifold singularities. The results can be found in Appendix A, and the coefficients of the modular forms in Appendix B. Note that the level of every Hilbert modular form mentioned in this section is generated by a single (rational) integer, e.g. $[256, 16, 16] = (16)$. This is not always the case—for example at the conifold singularities of operator 3.24, we find Hilbert modular forms over $\mathbb{Q}(\sqrt{2})$ of levels $[392, 196, 20 + 2\sqrt{2}]$ and $[392, 196, 176 + 2\sqrt{2}]$.

Though we did not encounter other types of real irrational singular points among the operators considered for this thesis, they do exist at higher degrees. In such cases, too, we still expect Hilbert modular forms, but of differing weights.

5.3 The General Procedure

We can now establish the general procedure for obtaining $R(T)$ given a Calabi-Yau type Picard-Fuchs operator from the list. Before we do so, however, there are a few subtleties which must be addressed. Recall that certain higher degree operators have singularities which are not true singularities of the underlying manifold, called *apparent singularities*. Where they occur, we cannot simply take $g_4(\lambda) = 1$.

Additionally, to compute $\mathcal{U}(0)$ the ratio χ/y, henceforth denoted ρ, is required. In some cases, $\chi = c_3$ and $y = H^3$ are individually known from the topology of the underlying manifold, but in many cases they are not. Fortunately, the correct value of ρ (but not the individual topological numbers) can be deduced in the course of the computation.

For a general Picard-Fuchs operator, denote the roots of R_4 by λ_i, $0 \leq i \leq r$ (we could have $r < d$ since we do not count multiplicity). Then the singular points are $0, \lambda_1, \ldots, \lambda_r, \infty$. The discriminant is given by

$$\Delta(\lambda) = \prod_{i=1}^{r} (\lambda - \lambda_i)^{\mu_i}, \tag{5.36}$$

where μ_i is the multiplicity of the root λ_i. With the discriminant so defined, we can in many cases take $g_s(\lambda)$ as before, and thus in particular $g_4(\lambda) = 1$. However, in some cases the denominator g_s becomes nontrivial for $s \geq 3$. This will be discussed in Section 5.3.1.

5.3.1 Apparent Singular Points

We cannot always take the denominator g_4 to be identically 1 as we did for the hypergeometric cases. As mentioned already, for certain operators, there are singularities of the operator which do not correspond to true singularities of the manifold. The operators where our previous formula for g_s is insufficient are precisely those with apparent singularities. Note that, in fact, no operator of degree less than 4 has an apparent singular point. The formula in this section was developed during experimentation with higher degree operators. Though we do not consider such operators in this thesis, they will be the target of future work. We still describe the formula here so that the method, as written in this thesis, is fully applicable to generic operators of degree greater than 3.

The astute reader might wonder how apparent singular points are defined in cases where this manifold is not known (or does not exist). This is somewhat murky but we have developed a numerical procedure for identifying the apparent singularities which seems to work in practice.

Recall the matrix W introduced as part of the fast E inversion procedure of Section 5.1.1. It has been observed in practice that the nonzero roots of the denominator of W^{-1} are precisely the apparent singularities. This observation was first made experimentally for the one parameter family with Hodge numbers $(h^{11}, h^{21}) = (1, 4)$ (see [23] for the definition of this manifold), but seems to work in other cases as well.

Now, getting back to the question of the denominator, let us define the "apparent discriminant",

$$\tilde{\Delta}(\lambda) = \prod_i \left(\lambda - \tilde{\lambda}_i\right)^{\tilde{\mu}_i}, \tag{5.37}$$

where $\tilde{\lambda}_i$ are the apparent singularities and $\tilde{\mu}_i$ their multiplicities. We now define the discriminant to contain only true singularities so that

$$\Delta(\lambda)\tilde{\Delta}(\lambda) = R_4(\lambda).$$

With the discriminant and apparent discriminant so defined, we can take the denominator to be the product of the original g_s and the polynomial

$$\tilde{g}_s(\lambda) = \begin{cases} 1; & s \leq 2, \\ \tilde{\Delta}(\lambda)^{p(s-2)}; & s \geq 2. \end{cases} \tag{5.38}$$

Based on numerical experiment, we conjecture that, mod p^s for $1 \leq s < p$, $\mathcal{U}(\lambda)$ is still a rational function with the denominator as given above. In fact, this denominator also seems to work for some larger s, but in particular when $p \mid s$, it seems higher powers of Δ and $\tilde{\Delta}$ are required. We do not have a hard rule for this and, since we only require $s = 4$, we do not discuss it further.

In practice, when working mod p^4, we take

$$g_4(\lambda) = \tilde{\Delta}(\lambda)^{2p},$$

which of course reduces to $g_4(\lambda) = 1$ when there are no apparent singularities. Importantly, though, this means that we cannot evaluate $\mathcal{U}(\lambda)$ at apparent singular points.

5.3.2 Deducing ρ

Pick a p-adic precision s and work mod p^s. Leaving ρ unspecified, we compute $\mathcal{U}(\lambda)$ and multiply by the expected denominator $g_s(\lambda)$. The resulting power series will, of course, not yet be a low degree polynomial. However, its coefficients will depend on the value of ρ. If we compute to $\mathcal{O}(\lambda^{2500})$ and take the coefficient of the highest term, it can be set to zero and solved for ρ. Assuming $\mathcal{U}(\lambda)$ reduces to a rational function as expected, doing so will set all other high degree terms in λ to zero as well. We can thus obtain an expression for ρ to as many digits of p-adic precision as we desire, but to compute $\mathcal{U}_4$ it is sufficient to know ρ mod p.

This has the disadvantage that ρ must be separately computed for each p. This computational overhead could be avoided if we obtained an exact expression. Since $\rho = \chi/y$, it is generally a rational number of low height. It is thus possible to obtain an exact value with relatively few digits of p-adic precision.

In our code, we pick a prime $7 \leq p \leq 19$, obtain 10 p-adic digits for ρ (working to 20 p-adic digits for the intermediate steps) and then use the PARI/GP `bestappr` function to find the exact value. It might be that this operation fails, for example if the chosen p divides the denominator of ρ. Should this happen, we try again with the next prime and generally at least one prime in the range $7 \leq p \leq 19$ is suitable. It is best to use the smallest primes possible, since the degree of $\mathcal{U}_4$ increases as the prime increases, necessitating a higher series precision. In cases where χ and y are individually known (including the hypergeometric cases and many degree 2 cases), they provide a useful sanity check on the result.

5.3.3 The Computation

We are now ready to outline the general procedure which we have used to compute $R(T)$ for (almost) all operators of degree less than 4, for $7 \le p \le 97$ and $1 \le \lambda < p$. Sample output from the computation for four operators is listed in Appendix A. Details of the software used can be found in Appendix C.

Before the main computation, there are a few preliminary steps:

- take as input polynomials $P_k(\vartheta)$ determining the Picard-Fuchs operator

$$\mathcal{P} = \sum_k \lambda^k P_k(\vartheta);$$

- compute polynomials R_i such that $\mathcal{P} = \sum_i R_i(\lambda)\vartheta^i$;
- compute W^{-1} as in Section 5.1.1;
- find the roots of the denominator of W^{-1}, which are the apparent singular points;
- find the roots of $R_4(\lambda)$, and separate them into true and apparent singular points;
- compute the ratio $\rho = \chi/y$ as in Section 5.3.2.

Having done this, the computation of $R(T)$ for each $7 \le p \le 97$ and $0 \le \lambda < p$ proceeds as follows:

- compute the expansion coefficients of the periods ϖ_k from P_k;
- compute $E(\lambda)$ from $\varpi_k(\lambda)$;
- compute $E(\lambda)^{-1}$ as in Section 5.1.1;
- compute the numerator

$$\mathcal{U}_4 = \mathcal{U}(\lambda)g_4(\lambda) \mod p^4;$$

- evaluate $\mathcal{U}_4$ at each $1 \le \lambda < p$ by direct substitution;
- obtain a and b from $\mathcal{U}_4$ using equation (5.3).

The tables will not always be complete since we cannot compute at primes of bad reduction, or at values of the parameter which coincide with apparent singularities mod p.

At each singular point of the operator (excluding 0 and ∞), there are a few more steps:

- determine defining field of the expected modular form (this will typically be the smallest field containing both $\mathbb{Q}$ and the singular point);
- reduce the singular point λ_0 mod p if it is rational or mod $\mathfrak{p}$ otherwise to find the appropriate value of $\lambda \in \mathbb{F}_p$ (where $\mathfrak{p}$ is a prime ideal of $\mathbb{Z}_K$);
- isolate the coefficients a, b for the prime p (or Norm($\mathfrak{p}$)) and parameter λ;
- identify the correct form of $R(T)$ based on the values of a, b;
- determine the expected weight of modular form from $R(T)$;
- identify the modular form whose q-expansion matches the coefficients.

This final step is not always straightfoward and, in Section 5.4, we discuss how exactly the modular form should be identified. Note that we do not use the indices to predict the form of $R(T)$ but instead deduce it solely from the values of a and b. As such, it is reassuring that the results we obtain can be so neatly categorized according to the indices.

5.3.4 Causes of Failure

We already saw in Section 5.2 that singularities of types 2b and 4 tend to cause our computational procedure to fail. For operators of degree higher than 2, such singularities typically arise alongside singularities of other types. Unfortunately, the "bad" singularities tend to cause problems for the whole operator. In many cases where such a singularity is present, the computation also breaks down at the operator's other singular points. Numerous such cases exist at degree 3. Each of the following operators has a conifold point and a case 4 singularity and computation of $R(T)$ failed at both points: 3.12, 3.14, 3.22, 3.23, 3.27, 3.28 and 3.29. On the other hand, operators 3.6 and 3.13 also have a conifold point and a case 4 singularity, but in these two cases a weight 4 classical modular form *was* identified at the conifold point.

Finally for 3.21, the standard conifold factorisation for $R(T)$ was found, but a modular form could not be matched to the coefficients. Since this conifold point is at $\lambda = 1/4624$, this may simply be because the level is too high. It thus seems that the precision (or other) issues associated with case 4 singular points sometimes, but not always, jeopordise the identification of modular forms at other singular points.

Where there is a conifold and a type 2b singular point (recall the "b" suffix means the computation fails at the type 2 point), $R(T)$ does sometimes take the standard cubic form expected at the conifold point. However, it is not always possible to find a unique matching modular form, since α_p is only available for a reduced set of primes.

To illustrate this point, let us consider the operators 3.30 and 3.35, for which computation can only proceed where $x^2 - 2$ has roots in $\mathbb{F}_p$, i.e. when the Legendre symbol $(2|p) = 1$. At each manifold's conifold point, $R(T)$ becomes cubic and factorises for such p. However, the resulting coefficients α_p cannot uniquely determine a modular form. Let us take 3.30 as an example. Two candidate modular forms are identified, 112.4.1.d and 448.4/8. The second is the twist of the first by a character whose value at a prime p is precisely the Legendre symbol $(2|p)$. This poses a problem for us: the primes at which these forms' coefficients differ are precisely those at which we cannot compute $R(T)$. The same happens with 3.35 with the two forms 48.4.1.a and 192.4/12. The obvious solution here is to generalise our PARI/GP code to work over field extensions to obtain the coefficients α_p at the primes where $(2|p) = -1$.

As the degree increases, the chance that a given operator has at least one problematic singular point increases. Thus the overall failure rate of our method also increases as the degree of operators increases.

One other cause of failure is worth mentioning. In some cases, for whatever reason, we may simply be unable to find a suitable value of ρ for which $\mathcal{U}_4$ becomes a low degree polynomial. This occurs at operator 3.31, which is omitted from our tables of results, since in the absence of ρ we could not compute any tables for this operator. This is the first operator which requires working over a degree 3 extension of $\mathbb{Q}$. The only primes less than 100 for which all three roots exist mod p are 47 and 53, and so these are the only ones where we expect $\mathcal{U}(\lambda)$ to reduce to a rational function mod p^4. Our code for deducing ρ thus fails, since it only tries primes up to 19. Using larger primes is possible but would require a greater series precision since increasing the prime increases the degree of $\mathcal{U}_4$.

This operator has one real and two complex singular points which are the roots of

$$2816\lambda^3 - 896\lambda^2 + 80\lambda - 1$$

and so we expect modular forms defined in the field extension of $\mathbb{Q}$ obtained by adjoining an appropriate root of this polynomial. Such forms cannot yet be computed using MAGMA, and so we did not attempt to perform the computation on this operator by computing ρ at a larger prime.

5.4 Identifying Modular Forms

The practice of identifying modular forms given a q-expansion series has been mentioned frequently and, given a table of known q-expansion coefficients, finding a match is straightforward. The issue is obtaining the table in the first place. In this section, we specify our sources for q-expansion data.

In the case of weight 4 classical newforms, there is a comprehensive table by Meyer [24], which lists the q-expansion coefficients of almost all such forms up to level 2000. For classical newforms more generally, there is the LMFDB [25] which provides a great deal of information (including q-expansion coefficients) of newforms of various weights, levels and character. However, the maximum levels are not as high as those in Meyer's table.

Where we expect forms of higher levels (or have failed to find a match at lower level), we must compute additional newforms ourselves. All modular forms computed this way are listed in Appendix B. Facilities for computing the q-expansions of classical modular forms are readily available in SageMath [26] and MAGMA [27]. However, as the level increases, the dimension of spaces of forms, and the computational intensity along with it, becomes large. The task would be significantly easier were we able to predict the possible levels of the expected form in advance. We are not aware of any hard rules for this, but we do have some heuristics which we describe in Section 5.4.1.

In the case of Hilbert modular forms, there is also a great deal of data on the LMFDB, but only for those forms of parallel weight 2. All the real irrational singular points of operators considered in this thesis are of conifold type, and so we anticipate weight 4 forms, which do not feature in the LMFDB database at present. Moreover, the database queries become quite large, so that performing the computations locally becomes relatively more attractive than querying the remote database. Routines for computing coefficients of Hilbert modular forms of arbitrary parallel weight (but only over field extensions of even degree) are available in MAGMA. As in the case of classical forms, informed guesses for the expected level help save unecessary computation. In fact, most of the Hilbert modular forms listed in Appendix B were found with the aid of informed guesses for the level.

For some operators, Hilbert modular forms over field extensions of odd degree are expected. The Hecke eigenvalues of such forms cannot currently be computed in MAGMA except in a few special cases. Fortunately, the minimum operator degree at which they occur is 5. It is worth noting that the singularities of the degree 3 operator 3.31 *do* exist in suitable cubic extensions of $\mathbb{Q}$, but that forms over such fields are not Hilbert modular forms and cannot yet be computed in MAGMA.

The prospects for identifying Bianchi modular forms are even more limited. As with Hilbert modular forms, only those of parallel weight 2 are listed on the LMFDB and, in this case, MAGMA can only compute weight 2 forms as well. Consequently, no Bianchi modular forms have been identified at conifold singularities in the course of our computations. However, there is some hope of identifying them at singularities of type 2a (i.e. indices $\{0, r, 1-r, r\}$ for $r < 0$) where weight 2 forms are anticipated. Such singularities only arise in operators of degree 4 or higher.

5.4.1 Predicting the Level

There are no strict rules known governing the level of the identified forms. Nevertheless some heuristics were pointed out to us by Duco van Straten which proved extremely helpful. The idea is that when two singular points of a given operator coincide mod p, something special is going on! It is thus suggested that any prime p at which this occurs may feature in the prime factorisation of the level N of the associated modular form. In a few cases, the primes 2 or 3 (or both) may occur regardless of the above rule. We conjecture that these are the *only* primes permitted in the level. Experimentally, this rule holds in all cases where we have successfully identified classical modular forms and, moreover, has proved instrumental in minimizing computation when searching for them.

Suppose $\lambda_0 = \frac{a}{b}$ is a rational singular point in lowest terms. Now if $p \mid a$ then $\lambda_0 \equiv 0 \mod p$ and if $p \mid b$ then $\lambda_0 \equiv \infty \mod p$. Since both 0 and ∞ are singular points of all of the operators, it follows that any p dividing a or b may feature in the level.

Of course, if there is another singular point $\frac{c}{d}$ then primes such that $\frac{a}{b} \equiv \frac{c}{d} \mod p$ can also be expected. A special case of this rule was noted in [8] by Samol and van Straten, who observed that the rational singularities are often reciprocal integers, and that the level of the form usually divides this integer. The unexpected occurence of 2 or 3 can break this pattern, as happens with operator 3.1. At $\lambda = 1, 1/9$ we find a level 6 form and at $\lambda = 1/25$ a level 30 form.

In the case of Hilbert and Bianchi modular forms over a field K, where the levels are not rational integers but ideals in the ring of integers $\mathbb{Z}_K$, it is not entirely clear how to generalise the rule. We could try to apply the same rule with *prime integer* replaced by *prime ideal*, but we took a simpler approach. We applied the rule directly to those ideals which are generated by a single prime integer. Consider operator 2.5, whose singular points are

$$\frac{11}{32} \pm \frac{5}{32}\sqrt{5} \equiv \infty \mod 16.$$

Inspired by this, a guess of the level $[256, 16, 16]$ generated by 16 yielded the matching Hilbert modular forms 5–256.4/1 and 5–256.4/2.

Of course, not all Hilbert modular forms have a level generated by a rational integer. More generally, we can use the rule to constrain the ideal norm of the expected level rather than the level itself. Consider operator 2.56 whose singular points are

$$-\frac{1}{12} \pm \frac{1}{18}\sqrt{3} \equiv \infty \mod 6.$$

By considering only forms whose level norm is $2^m 3^n$, we reduced the size of the search space and identified the forms 3–18.4/1 and 3–18.4/2 of level $[18, 6, 3 + 3\sqrt{3}]$.

5.5 Current Limitations and Future Work

Classical modular forms of weight 4 have already been identified at degree 1 by Rodriguez Villegas [15] and at a number of conifold points of degree 2 operators by Samol and van Straten. In the degree 2 cases, the computation of the zeta function was performed using the unit root method and thus constrained to low primes. Using the method in this thesis, we were able to verify these identifications up to $p = 97$.

We were also able to identify modular forms in many more cases than had previously been considered. This included classical modular forms of weights 2, 3 and 4 at various types of singular point and at operators of degrees 2 and 3. Most significantly, we were able for the first time to identify parallel weight 4 Hilbert modular forms corresponding to the coefficients of $R(T)$ at irrational conifold singularities.

The simplest generalisation is to continue to higher degree Picard-Fuchs operators in the list until it is completely exhausted. Computations have already proceeded to degree 4 with some success and in time results are expected in at least some cases. The method of Section 5.3 can already compute $R(T)$ for higher degree operators. The difficulty is in recognizing the form of $R(T)$ at singular points, identifying modular forms, and dealing with any other issues that arise with more complex operators. At least those singular points of the same type as those already encountered at lower degrees should be identifiable. Further considerations might be required for new types of singularity. As noted previously, as the degree, and thus the number of singular points, increases, there is an increasing risk that a single "bad" singularity will jeopordise the computation for the whole operator.

Most other promising avenues for future work arise from present computational limitations. Some of these may be overcome simply with more time for thought, but many depend on external constraints. These include both machine time (which limits the number of modular forms we can compute and search) and human time (which limits the software implementations available for computing spaces of forms).

In some cases, $R(T)$ is known at the singular point, but its coefficients could not be matched to a modular form of the expected weight. Typically in such cases, there is reason to expect the level to be large (e.g. a large integer in the denominator of the singular point). Many of the classical modular forms listed in Appendix B started this way and the fact that these forms were subsequently identified is encouraging. Nonetheless cases such as operator 2.8 remain. It has a pair of irrational conifold singularities at which weight 4 Hilbert modular forms are expected. As yet, these forms remain unidentified, but could perhaps be found with an intelligent guess as to the possible levels and a few days of computer time. As the potential levels get higher, such issues may tend towards intractability. However, as computing power increases over time (or we devise a stricter rule for guessing the level), these problems are likely to solve themselves.

Another exciting prospect is the identification of Hilbert modular forms of weight $[4, 4, 4]$. As noted previously, these cannot in general be computed in MAGMA at the time of writing. However, our code readily computes the coefficients of $R(T)$ in cases where we expect them (i.e. real irrational conifold points whose minimal polynomial is cubic). Such an example is given by 5.7 whose conifold singularities are the roots of

$$\lambda^3 - 289\lambda^2 - 57\lambda + 1.$$

(The additional two roots of the quintic polynomial R_4 are given by an apparent singularity of multiplicity 2.) Once Hilbert modular forms of weight $[4, 4, 4]$ can be computed, identifying the correct forms in cases such as 5.7 becomes a realistic possibility.

The story at imaginary conifold points is similar. Here, we expect Bianchi modular forms of parallel weight 4. The coefficients of $R(T)$ in such cases have already been found in this thesis in various cases (e.g. 2.20–2.23, 2.54 and 2.57). Once implementations for computing coefficients of parallel weight 4 Bianchi modular forms exist, all that remains is to compute them and search for matches. Of course, as always, this search will be limited by available computational resources. The same applies to other types of imaginary singular point where we expect Bianchi modular forms of parallel weight 3 and 4.

There is more immediate hope at imaginary singular points of type 2, where parallel weight 2 Bianchi modular forms (already supported in MAGMA) are anticipated. Indeed, preliminary results suggest success here for operator 4.6 at its type 2 singular points $\lambda = \frac{11}{2000} \pm \frac{i}{1000}$. These results were not ready for inclusion at the time of writing, but may feature in a future paper. In Section 5.4, we already mentioned operator 3.31 at whose conifold points we hope to find parallel weight 4 forms once the appropriate implementations are added to MAGMA.

More study of certain types of singular point is also warranted. At degree 4 and higher, new types of singular point arise which may exhibit different behaviour, or at which we cannot compute at all. We already saw how the computation of $R(T)$ breaks down for singular points of type 2b and 4. At present, the reasons for this are not understood, but this merits further study. These are the only outstanding singularities of degree 2 operators which remain unclassified despite the necessary modular forms tooling already existing.

5.6 Results

The main computational result of this thesis is the identification of modular forms at the singular points of various Calabi-Yau type operators, tables of which are given below. See Appendix D.5.4 and Appendix D.6.1 for explanations of the labels of modular forms, and Appendix B for tables of coefficients for modular forms computed specifically for this thesis. All other forms can be looked up in Meyer's thesis [24] or the LMFDB [25].

Where $R(T)$ was computed successfully, but does not match any recognized expression for $R(T)$ expected at a singular point (perhaps due to p-adic precision issues), the modular form and $R(T)$ columns are marked "–". If a good expression for $R(T)$ was found but no modular form could be matched to the coefficients, the modular form column is marked T.k where k is the expected weight, and T is the expected type of the form—C for classical modular forms, H for Hilbert modular forms and B for Bianchi modular forms.

Of course, the identification of the modular forms first required the computation $R(T)$ at the singular points. It would be impractical to include full tables of $R(T)$ for all $7 \leq p \leq 97$ and $1 \leq \lambda < p$, but such tables are given for select examples in Appendix A. In cases where $R(T)$ could not be computed (e.g. at an apparent singular point or a prime of bad reduction), the appropriate rows are omitted from those tables.

degree 1

index	singularity	case	indices	modular form	$R(T)$
1	$\frac{1}{3125}$	0	$\{0,1,1,2\}$	25.4.1.b	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
2	$\frac{1}{800000}$	0	$\{0,1,1,2\}$	200.4/10	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
3	$\frac{1}{256}$	0	$\{0,1,1,2\}$	8.4.1.a	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
4	$\frac{1}{729}$	0	$\{0,1,1,2\}$	27.4.1.a	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
5	$\frac{1}{432}$	0	$\{0,1,1,2\}$	36.4.1.a	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
6	$\frac{1}{1024}$	0	$\{0,1,1,2\}$	16.4.1.a	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
7	$\frac{1}{65536}$	0	$\{0,1,1,2\}$	128.4/3	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
8	$\frac{1}{11664}$	0	$\{0,1,1,2\}$	108.4.1.a	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
9	$\frac{1}{2985984}$	0	$\{0,1,1,2\}$	864.4/4	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
10	$\frac{1}{4096}$	0	$\{0,1,1,2\}$	32.4.1.a	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
11	$\frac{1}{1728}$	0	$\{0,1,1,2\}$	9.4.1.a	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
12	$\frac{1}{27648}$	0	$\{0,1,1,2\}$	144.4/1	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
13	$\frac{1}{186624}$	0	$\{0,1,1,2\}$	216.4/4	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
14	$\frac{1}{6912}$	0	$\{0,1,1,2\}$	72.4.1.b	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$

degree 2

index	singularity	case	indices	modular form	$R(T)$
1	$-\frac{1}{16}$	0	$\{0,1,1,2\}$	8.4.1.a	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
1	$\frac{1}{128}$	0	$\{0,1,1,2\}$	64.4.1.b	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
2	$-\frac{1}{27}$	0	$\{0,1,1,2\}$	27.4.1.a	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
2	$\frac{1}{216}$	0	$\{0,1,1,2\}$	54.4.1.a	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
3	$\frac{1}{512}$	0	$\{0,1,1,2\}$	256.4/3	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
3	$-\frac{1}{64}$	0	$\{0,1,1,2\}$	32.4.1.a	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
4	$-\frac{1}{432}$	0	$\{0,1,1,2\}$	216.4/4	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
4	$\frac{1}{3456}$	0	$\{0,1,1,2\}$	1728.4/16	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
5	$\frac{5}{32}\sqrt{5}-\frac{11}{32}$	0	$\{0,1,1,2\}$	5–256.4/1	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
5	$-\frac{5}{32}\sqrt{5}-\frac{11}{32}$	0	$\{0,1,1,2\}$	5–256.4/2	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
6	$\frac{5}{54}\sqrt{5}-\frac{11}{54}$	0	$\{0,1,1,2\}$	5–729.4/1	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
6	$-\frac{5}{54}\sqrt{5}-\frac{11}{54}$	0	$\{0,1,1,2\}$	5–729.4/2	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
7	$\frac{5}{128}\sqrt{5}-\frac{11}{128}$	0	$\{0,1,1,2\}$	5–4096.4/1	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
7	$-\frac{5}{128}\sqrt{5}-\frac{11}{128}$	0	$\{0,1,1,2\}$	5–4096.4/2	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
8	$-\frac{5}{864}\sqrt{5}-\frac{11}{864}$	0	$\{0,1,1,2\}$	H.4	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
8	$\frac{5}{864}\sqrt{5}-\frac{11}{864}$	0	$\{0,1,1,2\}$	H.4	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
9	$\frac{1}{144}$	0	$\{0,1,1,2\}$	48.4.1.b	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
9	$\frac{1}{16}$	0	$\{0,1,1,2\}$	16.4.1.a	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$

degree 2 (cont.)

index	singularity	case	indices	modular form	$R(T)$
10	$\frac{1}{243}$	0	$\{0,1,1,2\}$	243.4/1	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
10	$\frac{1}{27}$	0	$\{0,1,1,2\}$	27.4.1.b	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
11	$\frac{1}{576}$	0	$\{0,1,1,2\}$	576.4/3	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
11	$\frac{1}{64}$	0	$\{0,1,1,2\}$	64.4.1.a	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
12	$\frac{1}{3888}$	0	$\{0,1,1,2\}$	3888.4.1	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
12	$\frac{1}{432}$	0	$\{0,1,1,2\}$	432.4/9	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
13	$\frac{1}{128}$	0	$\{0,1,1,2\}$	64.4.1.c	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
13	$\frac{1}{64}$	0	$\{0,1,1,2\}$	32.4.1.c	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
14	$\frac{1}{216}$	0	$\{0,1,1,2\}$	9.4.1.a	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
14	$\frac{1}{108}$	0	$\{0,1,1,2\}$	108.4.1.d	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
15	$\frac{1}{256}$	0	$\{0,1,1,2\}$	128.4/4	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
15	$\frac{1}{512}$	0	$\{0,1,1,2\}$	256.4/1	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
16	$\frac{1}{3456}$	0	$\{0,1,1,2\}$	576.4/8	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
16	$\frac{1}{1728}$	0	$\{0,1,1,2\}$	864.4/3	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
17	$\frac{1}{256}$	1	$\{0,1,\frac{1}{2},\frac{1}{2}\}$	16.4.1.a	$1-\alpha_p T+p^3T^2$
18	$\frac{1}{432}$	1	$\{0,1,\frac{1}{2},\frac{1}{2}\}$	12.4.1.a	$1-\alpha_p T+p^3T^2$
19	$\frac{1}{6912}$	1	$\{0,1,\frac{1}{2},\frac{1}{2}\}$	144.4/4	$1-\alpha_p T+p^3T^2$

degree 2 (cont.)

index	singularity	case	indices	modular form	$R(T)$
20	$-\frac{1}{96}i\sqrt{\frac{1}{3}}+\frac{1}{96}$	0	$\{0,1,1,2\}$	B.4	$(1-p\chi_pT)(1-\alpha_pT+p^3T^2)$
20	$\frac{1}{96}i\sqrt{\frac{1}{3}}+\frac{1}{96}$	0	$\{0,1,1,2\}$	B.4	$(1-p\chi_pT)(1-\alpha_pT+p^3T^2)$
21	$\frac{1}{162}i\sqrt{\frac{1}{3}}+\frac{1}{162}$	0	$\{0,1,1,2\}$	B.4	$(1-p\chi_pT)(1-\alpha_pT+p^3T^2)$
21	$-\frac{1}{162}i\sqrt{\frac{1}{3}}+\frac{1}{162}$	0	$\{0,1,1,2\}$	B.4	$(1-p\chi_pT)(1-\alpha_pT+p^3T^2)$
22	$-\frac{1}{384}i\sqrt{\frac{1}{3}}+\frac{1}{384}$	0	$\{0,1,1,2\}$	B.4	$(1-p\chi_pT)(1-\alpha_pT+p^3T^2)$
22	$\frac{1}{384}i\sqrt{\frac{1}{3}}+\frac{1}{384}$	0	$\{0,1,1,2\}$	B.4	$(1-p\chi_pT)(1-\alpha_pT+p^3T^2)$
23	$-\frac{1}{2592}i\sqrt{\frac{1}{3}}+\frac{1}{2592}$	0	$\{0,1,1,2\}$	B.4	$(1-p\chi_pT)(1-\alpha_pT+p^3T^2)$
23	$\frac{1}{2592}i\sqrt{\frac{1}{3}}+\frac{1}{2592}$	0	$\{0,1,1,2\}$	B.4	$(1-p\chi_pT)(1-\alpha_pT+p^3T^2)$
24	$\frac{1}{144}$	0	$\{0,1,1,2\}$	24.4.1.a	$(1-p\chi_pT)(1-\alpha_pT+p^3T^2)$
24	$\frac{1}{128}$	0	$\{0,1,1,2\}$	64.4.1.d	$(1-p\chi_pT)(1-\alpha_pT+p^3T^2)$
25	$\frac{1}{216}$	0	$\{0,1,1,2\}$	54.4.1.d	$(1-p\chi_pT)(1-\alpha_pT+p^3T^2)$
25	$\frac{1}{243}$	0	$\{0,1,1,2\}$	243.4/2	$(1-p\chi_pT)(1-\alpha_pT+p^3T^2)$
26	$\frac{1}{576}$	0	$\{0,1,1,2\}$	288.4/10	$(1-p\chi_pT)(1-\alpha_pT+p^3T^2)$
26	$\frac{1}{512}$	0	$\{0,1,1,2\}$	256.4/4	$(1-p\chi_pT)(1-\alpha_pT+p^3T^2)$
27	$\frac{1}{3888}$	0	$\{0,1,1,2\}$	1944.4/5	$(1-p\chi_pT)(1-\alpha_pT+p^3T^2)$
27	$\frac{1}{3456}$	0	$\{0,1,1,2\}$	1728.4/15	$(1-p\chi_pT)(1-\alpha_pT+p^3T^2)$

degree 2 (cont.)

index	singularity	case	indices	modular form	$R(T)$
28	$\frac{1}{432}$	1	$\{0, 1, \frac{1}{3}, \frac{2}{3}\}$	144.4/7	$1 - \alpha_p T + p^3 T^2$
29	$\frac{1}{729}$	1	$\{0, 1, \frac{1}{3}, \frac{2}{3}\}$	27.4.1.b	$1 - \alpha_p T + p^3 T^2$
30	$\frac{1}{11664}$	1	$\{0, 1, \frac{1}{3}, \frac{2}{3}\}$	432.4/3	$1 - \alpha_p T + p^3 T^2$
31	$\frac{1}{1024}$	1	$\{0, 1, \frac{1}{4}, \frac{3}{4}\}$	8.4.1.a	$1 - \alpha_p T + p^3 T^2$
32	$\frac{1}{1728}$	1	$\{0, 1, \frac{1}{4}, \frac{3}{4}\}$	9.4.1.a	$1 - \alpha_p T + p^3 T^2$
33	$\frac{1}{4096}$	1	$\{0, 1, \frac{1}{4}, \frac{3}{4}\}$	64.4.1.a	$1 - \alpha_p T + p^3 T^2$
34	$\frac{1}{27648}$	1	$\{0, 1, \frac{1}{4}, \frac{3}{4}\}$	72.4.1.d	$1 - \alpha_p T + p^3 T^2$
35	$\frac{1}{6912}$	1	$\{0, 1, \frac{1}{6}, \frac{5}{6}\}$	144.4/4	$1 - \alpha_p T + p^3 T^2$
36	$\frac{1}{11664}$	1	$\{0, 1, \frac{1}{6}, \frac{5}{6}\}$	108.4.1.d	$1 - \alpha_p T + p^3 T^2$
37	$\frac{1}{27648}$	1	$\{0, 1, \frac{1}{6}, \frac{5}{6}\}$	576.4/22	$1 - \alpha_p T + p^3 T^2$
38	$\frac{1}{186624}$	1	$\{0, 1, \frac{1}{6}, \frac{5}{6}\}$	432.4/9	$1 - \alpha_p T + p^3 T^2$
39	$\frac{1}{12500}$	2b	$\{-\frac{1}{2}, 0, 1, \frac{3}{2}\}$	–	–
40	$\frac{1}{3200000}$	2a	$\{-\frac{1}{2}, 0, 1, \frac{3}{2}\}$	800.2.1	$1 - p\beta_b T + p^3 T^2$
41	$\frac{1}{2916}$	2a	$\{-\frac{1}{2}, 0, 1, \frac{3}{2}\}$	54.2.1.b	$1 - p\beta_b T + p^3 T^2$
42	$\frac{1}{1728}$	2b	$\{-\frac{1}{2}, 0, 1, \frac{3}{2}\}$	–	–

degree 2 (cont.)

index	singularity	case	indices	modular form	$R(T)$
43	$\frac{1}{4096}$	2b	$\{-\frac{1}{2}, 0, 1, \frac{3}{2}\}$	–	–
44	$\frac{1}{262144}$	2b	$\{-\frac{1}{2}, 0, 1, \frac{3}{2}\}$	–	–
45	$\frac{1}{46656}$	2b	$\{-\frac{1}{2}, 0, 1, \frac{3}{2}\}$	–	–
46	$\frac{1}{11943936}$	2a	$\{-\frac{1}{2}, 0, 1, \frac{3}{2}\}$	3456.2.1	$1 - p\beta_b T + p^3T^2$
47	$\frac{1}{16384}$	2a	$\{-\frac{1}{2}, 0, 1, \frac{3}{2}\}$	128.2.1.b	$1 - p\beta_b T + p^3T^2$
48	$\frac{1}{6912}$	2b	$\{-\frac{1}{2}, 0, 1, \frac{3}{2}\}$	–	–
49	$\frac{1}{110592}$	2b	$\{-\frac{1}{2}, 0, 1, \frac{3}{2}\}$	–	–
50	$\frac{1}{746496}$	2a	$\{-\frac{1}{2}, 0, 1, \frac{3}{2}\}$	864.2.1	$1 - p\beta_b T + p^3T^2$
51	$\frac{1}{27648}$	2a	$\{-\frac{1}{2}, 0, 1, \frac{3}{2}\}$	288.2.1.c	$1 - p\beta_b T + p^3T^2$
52	$\frac{1}{64}$	0	$\{0, 1, 1, 2\}$	6.4.1.a	$(1 - p\chi_p T)(1 - \alpha_p T + p^3T^2)$
52	$\frac{1}{16}$	0	$\{0, 1, 1, 2\}$	12.4.1.a	$(1 - p\chi_p T)(1 - \alpha_p T + p^3T^2)$
53	$3\sqrt{2} + \frac{17}{4}$	0	$\{0, 1, 1, 2\}$	2–18.4/1	$(1 - p\chi_p T)(1 - \alpha_p T + p^3T^2)$
53	$-3\sqrt{2} + \frac{17}{4}$	0	$\{0, 1, 1, 2\}$	2–18.4/2	$(1 - p\chi_p T)(1 - \alpha_p T + p^3T^2)$
54	$-\frac{1}{81}i\sqrt{2} + \frac{7}{324}$	0	$\{0, 1, 1, 2\}$	B.4	$(1 - p\chi_p T)(1 - \alpha_p T + p^3T^2)$
54	$\frac{1}{81}i\sqrt{2} + \frac{7}{324}$	0	$\{0, 1, 1, 2\}$	B.4	$(1 - p\chi_p T)(1 - \alpha_p T + p^3T^2)$
55	$\frac{1}{4}\sqrt{\frac{1}{2}} + \frac{3}{16}$	0	$\{0, 1, 1, 2\}$	2–1.4/1	$(1 - p\chi_p T)(1 - \alpha_p T + p^3T^2)$

degree 2 (cont.)

index	singularity	case	indices	modular form	$R(T)$
55	$-\frac{1}{4}\sqrt{\frac{1}{2}}+\frac{3}{16}$	0	$\{0,1,1,2\}$	2–1.4/1	$(1-p\chi_pT)(1-\alpha_pT+p^3T^2)$
56	$\frac{1}{6}\sqrt{\frac{1}{3}}-\frac{1}{12}$	0	$\{0,1,1,2\}$	3–18.4/1	$(1-p\chi_pT)(1-\alpha_pT+p^3T^2)$
56	$-\frac{1}{6}\sqrt{\frac{1}{3}}-\frac{1}{12}$	0	$\{0,1,1,2\}$	3–18.4/2	$(1-p\chi_pT)(1-\alpha_pT+p^3T^2)$
57	$-\frac{1}{250}i+\frac{11}{500}$	0	$\{0,1,1,2\}$	B.4	$(1-p\chi_pT)(1-\alpha_pT+p^3T^2)$
57	$\frac{1}{250}i+\frac{11}{500}$	0	$\{0,1,1,2\}$	B.4	$(1-p\chi_pT)(1-\alpha_pT+p^3T^2)$
58	$\frac{1}{108}$	1	$\{0,1,\frac{1}{6},\frac{5}{6}\}$	18.4.1.a	$1-\alpha_pT+p^3T^2$
59	$\frac{1}{1728}$	2a	$\{-\frac{1}{6},0,1,\frac{7}{6}\}$	72.2.1.a	$1-p\beta_bT+p^3T^2$
60	$-\frac{1}{16}$	0	$\{0,1,1,2\}$	40.4.1.a	$(1-p\chi_pT)(1-\alpha_pT+p^3T^2)$
60	$\frac{1}{64}$	0	$\{0,1,1,2\}$	80.4.1.b	$(1-p\chi_pT)(1-\alpha_pT+p^3T^2)$
61	$-\frac{1}{4}$	0	$\{0,1,1,2\}$	28.4.1.a	$(1-p\chi_pT)(1-\alpha_pT+p^3T^2)$
61	$\frac{1}{108}$	0	$\{0,1,1,2\}$	252.4/3	$(1-p\chi_pT)(1-\alpha_pT+p^3T^2)$
62	$\frac{1}{64}$	0	$\{0,1,1,2\}$	14.4.1.b	$(1-p\chi_pT)(1-\alpha_pT+p^3T^2)$
62	1	0	$\{0,1,1,2\}$	7.4.1.a	$(1-p\chi_pT)(1-\alpha_pT+p^3T^2)$
63	$\frac{1}{64}$	3	$\{0,0,1,1\}$	8.3.3.a	$1-\alpha_pT+p^2T^2$
64	$\frac{1}{27}$	0	$\{0,1,1,2\}$	33.4.1.a	$(1-p\chi_pT)(1-\alpha_pT+p^3T^2)$
64	$\frac{1}{16}$	0	$\{0,1,1,2\}$	22.4.1.c	$(1-p\chi_pT)(1-\alpha_pT+p^3T^2)$
65	$-\frac{1}{64}$	0	$\{0,1,1,2\}$	16.4.1.a	$(1-p\chi_pT)(1-\alpha_pT+p^3T^2)$

degree 2 (cont.)

index	singularity	case	indices	modular form	$R(T)$
65	$-\frac{1}{48}$	0	$\{0, 1, 1, 2\}$	72.4.1.d	$(1 - p\chi_p T)(1 - \alpha_p T + p^3 T^2)$
66	$\frac{1}{432}$	3	$\{0, 0, 1, 1\}$	None	$(1 - \chi_p T)(1 - p^2 \chi_p T)$
67	$\frac{1}{108}$	4	$\{0, 1, \frac{1}{6}, \frac{7}{6}\}$	–	–
68	$\frac{1}{256}$	4	$\{0, 1, \frac{1}{4}, \frac{5}{4}\}$	–	–
69	$\frac{1}{27}$	0	$\{0, 1, 1, 2\}$	15.4.1.a	$(1 - p\chi_p T)(1 - \alpha_p T + p^3 T^2)$
69	$\frac{1}{32}$	0	$\{0, 1, 1, 2\}$	5.4.1.a	$(1 - p\chi_p T)(1 - \alpha_p T + p^3 T^2)$
70	$\frac{1}{256}$	4	$\{-\frac{3}{4}, 0, 1, \frac{1}{4}\}$	–	–

degree 3

index	singularity	case	indices	modular form	$R(T)$
1	1	0	$\{0,1,1,2\}$	6.4.1.a	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
1	$\frac{1}{25}$	0	$\{0,1,1,2\}$	30.4.1.a	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
1	$\frac{1}{9}$	0	$\{0,1,1,2\}$	6.4.1.a	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
2	$-\frac{1}{1296}$	0	$\{0,1,1,2\}$	108.4.1.a	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
2	$-\frac{1}{1728}$	1	$\{0,1,\frac{1}{2},\frac{1}{2}\}$	144.4/1	$1-\alpha_p T+p^3T^2$
3	$-\frac{1}{192}$	0	$\{0,1,1,2\}$	9.4.1.a	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
3	$-\frac{1}{256}$	1	$\{0,1,\frac{1}{2},\frac{1}{2}\}$	32.4.1.a	$1-\alpha_p T+p^3T^2$
4	$-\frac{1}{81}$	0	$\{0,1,1,2\}$	27.4.1.a	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
4	$-\frac{1}{108}$	1	$\{0,1,\frac{1}{2},\frac{1}{2}\}$	9.4.1.a	$1-\alpha_p T+p^3T^2$
5	$\frac{1}{196}$	0	$\{0,1,1,2\}$	126.4/2	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
5	$\frac{1}{4}$	1	$\{0,1,\frac{1}{4},\frac{3}{4}\}$	6.4.1.a	$1-\alpha_p T+p^3T^2$
6	$\frac{1}{784}$	0	$\{0,1,1,2\}$	196.4/4	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
6	$-\frac{1}{512}$	4	$\{0,1,\frac{1}{2},\frac{3}{2}\}$	–	–
7	$\frac{1}{324}$	0	$\{0,1,1,2\}$	54.4.1.a	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
7	$-\frac{1}{108}$	3	$\{0,0,1,1\}$	36.3.19.a	$1-\alpha_p T+p^2T^2$
8	$\frac{1}{49}$	0	$\{0,1,1,2\}$	21.4.1.a	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
8	$-\frac{1}{32}$	1	$\{0,1,\frac{1}{2},\frac{1}{2}\}$	6.4.1.a	$1-\alpha_p T+p^3T^2$
9	$\frac{1}{121}$	0	$\{0,1,1,2\}$	55.4.1.a	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$

degree 3 (cont.)

index	singularity	case	indices	modular form	$R(T)$
9	$-\frac{1}{4}$	1	$\{0,1,\frac{1}{2},\frac{1}{2}\}$	5.4.1.a	$1-\alpha_pT+p^3T^2$
10	$\frac{1}{100}$	0	$\{0,1,1,2\}$	10.4.1.a	$(1-p\chi_pT)(1-\alpha_pT+p^3T^2)$
10	$\frac{1}{36}$	1	$\{0,1,\frac{1}{2},\frac{1}{2}\}$	18.4.1.a	$1-\alpha_pT+p^3T^2$
11	$\frac{1}{289}$	0	$\{0,1,1,2\}$	17.4.1.a	$(1-p\chi_pT)(1-\alpha_pT+p^3T^2)$
11	$\frac{1}{288}$	1	$\{0,1,\frac{1}{2},\frac{1}{2}\}$	18.4.1.a	$1-\alpha_pT+p^3T^2$
12	$\frac{1}{1296}$	0	$\{0,1,1,2\}$	–	–
12	$\frac{1}{1728}$	4	$\{0,1,\frac{1}{2},\frac{3}{2}\}$	–	–
13	$\frac{1}{1600}$	0	$\{0,1,1,2\}$	1600.4/16	$(1-p\chi_pT)(1-\alpha_pT+p^3T^2)$
13	$\frac{1}{576}$	4	$\{0,1,\frac{1}{2},\frac{3}{2}\}$	–	–
14	$\frac{1}{1936}$	0	$\{0,1,1,2\}$	–	–
14	$-\frac{1}{64}$	4	$\{0,1,\frac{1}{2},\frac{3}{2}\}$	–	–
15	$-\frac{1}{200}$	0	$\{0,1,1,2\}$	80.4.1.b	$(1-p\chi_pT)(1-\alpha_pT+p^3T^2)$
15	$\frac{1}{16}$	1	$\{0,1,\frac{1}{6},\frac{5}{6}\}$	8.4.1.a	$1-\alpha_pT+p^3T^2$
16	$\frac{1}{196}$	0	$\{0,1,1,2\}$	14.4.1.b	$(1-p\chi_pT)(1-\alpha_pT+p^3T^2)$
16	$\frac{1}{324}$	3	$\{0,0,1,1\}$	72.3.19	$1-\alpha_pT+p^2T^2$
17	$\frac{1}{400}$	0	$\{0,1,1,2\}$	60.4.1.a	$(1-p\chi_pT)(1-\alpha_pT+p^3T^2)$
17	$\frac{1}{256}$	3	$\{0,0,1,1\}$	12.3.5.a	$1-\alpha_pT+p^2T^2$
18	$\frac{1}{1156}$	0	$\{0,1,1,2\}$	102.4.1.c	$(1-p\chi_pT)(1-\alpha_pT+p^3T^2)$

degree 3 (cont.)

index	singularity	case	indices	modular form	$R(T)$
18	$\frac{1}{4}$	3	$\{0,0,1,1\}$	24.3.5.b	$1-\alpha_p T+p^2T^2$
19	$\frac{1}{484}$	0	$\{0,1,1,2\}$	22.4.1.a	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
19	$\frac{1}{500}$	3	$\{0,0,1,1\}$	100.3.51	$1-\alpha_p T+p^2T^2$
20	$\frac{1}{169}$	0	$\{0,1,1,2\}$	13.4.1.a	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
20	$\frac{1}{196}$	1	$\{0,1,\frac{1}{6},\frac{5}{6}\}$	49.4.1.c	$1-\alpha_p T+p^3T^2$
21	$\frac{1}{4624}$	0	$\{0,1,1,2\}$	C.4	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
21	$\frac{1}{4608}$	4	$\{0,1,\frac{1}{2},\frac{3}{2}\}$	–	–
22	$\frac{1}{676}$	0	$\{0,1,1,2\}$	–	–
22	$-\frac{1}{108}$	4	$\{0,1,\frac{1}{3},\frac{4}{3}\}$	–	–
23	$-\frac{1}{800}$	0	$\{0,1,1,2\}$	–	–
23	$-\frac{1}{864}$	4	$\{0,1,\frac{1}{3},\frac{4}{3}\}$	–	–
24	$\frac{1}{7}\sqrt{2}+\frac{5}{28}$	0	$\{0,1,1,2\}$	2–392.4/1	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
24	$-\frac{1}{7}\sqrt{2}+\frac{5}{28}$	0	$\{0,1,1,2\}$	2–392.4/2	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
24	$\frac{1}{16}$	0	$\{0,1,1,2\}$	8.4.1.a	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
25	$\frac{1}{10}i-\frac{1}{20}$	0	$\{0,1,1,2\}$	B.4	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
25	$-\frac{1}{48}$	0	$\{0,1,1,2\}$	36.4.1.a	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
25	$-\frac{1}{10}i-\frac{1}{20}$	0	$\{0,1,1,2\}$	B.4	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$
26	$-\frac{1}{64}$	0	$\{0,1,1,2\}$	6.4.1.a	$(1-p\chi_p T)(1-\alpha_p T+p^3T^2)$

degree 3 (cont.)

index	singularity	case	indices	modular form	$R(T)$
26	$-\frac{1}{100}$	2a	$\{-\frac{1}{2}, 0, 1, \frac{3}{2}\}$	150.2.1.a	$1 - p\beta_b T + p^3T^2$
27	$-\frac{1}{144}$	0	$\{0, 1, 1, 2\}$	–	–
27	$\frac{1}{256}$	4	$\{0, 1, \frac{1}{4}, \frac{5}{4}\}$	–	–
28	$\frac{1}{81}$	0	$\{0, 1, 1, 2\}$	–	–
28	$\frac{1}{32}$	4	$\{0, 1, \frac{3}{4}, \frac{7}{4}\}$	–	–
29	$\frac{1}{81}$	0	$\{0, 1, 1, 2\}$	–	–
29	$\frac{1}{32}$	4	$\{0, 1, \frac{1}{4}, \frac{5}{4}\}$	–	–
30	$\frac{1}{16}$	0	$\{0, 1, 1, 2\}$	C.4	$(1 - p\chi_p T)(1 - \alpha_p T + p^3T^2)$
30	$-\frac{1}{768}$	2b	$\{-\frac{1}{2}, 0, 1, \frac{3}{2}\}$	–	–
32	$\frac{1}{512}$	0	$\{0, 1, 1, 2\}$	6.4.1.a	$(1 - p\chi_p T)(1 - \alpha_p T + p^3T^2)$
32	$-\frac{1}{64}$	2a	$\{-\frac{1}{2}, 0, 1, \frac{3}{2}\}$	24.2.1.a	$1 - p\beta_b T + p^3T^2$
33	$-\frac{1}{16}$	0	$\{0, 1, 1, 2\}$	144.4/3	$(1 - p\chi_p T)(1 - \alpha_p T + p^3T^2)$
33	$-\frac{1}{64}$	2a	$\{-\frac{1}{2}, 0, 1, \frac{3}{2}\}$	192.2.1.d	$1 - p\beta_b T + p^3T^2$
34	$\frac{1}{144}$	0	$\{0, 1, 1, 2\}$	12.4.1.a	$(1 - p\chi_p T)(1 - \alpha_p T + p^3T^2)$
34	$\frac{1}{128}$	1	$\{0, 1, \frac{1}{2}, \frac{1}{2}\}$	16.4.1.a	$1 - \alpha_p T + p^3T^2$
35	$\frac{1}{256}$	0	$\{0, 1, 1, 2\}$	C.4	$(1 - p\chi_p T)(1 - \alpha_p T + p^3T^2)$
35	$-\frac{1}{2048}$	2b	$\{-\frac{1}{2}, 0, 1, \frac{3}{2}\}$	–	–

Appendix A

Full Tables of $R(T)$ for Select Operators

Here we list full tables of $R(T)$ and the coefficients a, b for $7 \leq p \leq 97$ and $1 \leq \lambda < p$ for select operators, namely 2.1, 2.17, 2.41 and 2.63. One example is given for each type of singular point described in Section 5.2 except for cases 2b and 4, where the computation fails. These are excluded since the results in such cases may not be meaningful due to precision issues. Including such tables for all operators in [14] of degree less than 4 would be impractical, but the full data will be made available online at `https://cyarithmetic.gitlab.io`.

A.1 Case 0: Operator 2.1

$p = 7$

λ	a	b	$R(T)$
1	2	-46	$p^6T^4 + 2p^3T^3 - 46pT^2 + 2T + 1$
2	-8	2	$p^6T^4 - 8p^3T^3 + 2pT^2 - 8T + 1$
3	-17	25	$(pT + 1)(p^3T^2 - 24T + 1)$
4	31	73	$(pT + 1)(p^3T^2 + 24T + 1)$
5	10	50	$p^6T^4 + 10p^3T^3 + 50pT^2 + 10T + 1$
6	32	98	$p^6T^4 + 32p^3T^3 + 98pT^2 + 32T + 1$

$p = 11$

λ	a	b	$R(T)$
1	56	290	$p^6T^4 + 56p^3T^3 + 290pT^2 + 56T + 1$
2	33	77	$-(pT - 1)(p^3T^2 + 44T + 1)$
3	-16	2	$p^6T^4 - 16p^3T^3 + 2pT^2 - 16T + 1$
4	6	26	$p^6T^4 + 6p^3T^3 + 26pT^2 + 6T + 1$
5	16	98	$p^6T^4 + 16p^3T^3 + 98pT^2 + 16T + 1$
6	12	114	$p^6T^4 + 12p^3T^3 + 114pT^2 + 12T + 1$
7	26	106	$p^6T^4 + 26p^3T^3 + 106pT^2 + 26T + 1$
8	33	77	$-(pT - 1)(p^3T^2 + 44T + 1)$
9	-8	2	$p^6T^4 - 8p^3T^3 + 2pT^2 - 8T + 1$
10	-36	210	$p^6T^4 - 36p^3T^3 + 210pT^2 - 36T + 1$

$p = 13$

λ	a	b	$R(T)$
1	-8	270	$p^6T^4 - 8p^3T^3 + 270pT^2 - 8T + 1$
2	20	-106	$p^6T^4 + 20p^3T^3 - 106pT^2 + 20T + 1$
3	-4	86	$p^6T^4 - 4p^3T^3 + 86pT^2 - 4T + 1$
4	-35	191	$-(pT-1)(p^3T^2 - 22T + 1)$
5	22	-30	$p^6T^4 + 22p^3T^3 - 30pT^2 + 22T + 1$
6	9	147	$-(pT-1)(p^3T^2 + 22T + 1)$
7	-34	50	$p^6T^4 - 34p^3T^3 + 50pT^2 - 34T + 1$
8	-16	302	$p^6T^4 - 16p^3T^3 + 302pT^2 - 16T + 1$
9	58	146	$p^6T^4 + 58p^3T^3 + 146pT^2 + 58T + 1$
10	18	34	$p^6T^4 + 18p^3T^3 + 34pT^2 + 18T + 1$
11	84	406	$p^6T^4 + 84p^3T^3 + 406pT^2 + 84T + 1$
12	56	206	$p^6T^4 + 56p^3T^3 + 206pT^2 + 56T + 1$

$p = 17$

λ	a	b	$R(T)$
1	-33	239	$(pT+1)(p^3T^2 - 50T + 1)$
2	-33	239	$(pT+1)(p^3T^2 - 50T + 1)$
3	-24	542	$p^6T^4 - 24p^3T^3 + 542pT^2 - 24T + 1$
4	44	166	$p^6T^4 + 44p^3T^3 + 166pT^2 + 44T + 1$
5	210	1218	$p^6T^4 + 210p^3T^3 + 1218pT^2 + 210T + 1$
6	24	-178	$p^6T^4 + 24p^3T^3 - 178pT^2 + 24T + 1$
7	-100	278	$p^6T^4 - 100p^3T^3 + 278pT^2 - 100T + 1$
8	22	50	$p^6T^4 + 22p^3T^3 + 50pT^2 + 22T + 1$
9	-4	70	$p^6T^4 - 4p^3T^3 + 70pT^2 - 4T + 1$
10	52	470	$p^6T^4 + 52p^3T^3 + 470pT^2 + 52T + 1$
11	-84	342	$p^6T^4 - 84p^3T^3 + 342pT^2 - 84T + 1$
12	48	-34	$p^6T^4 + 48p^3T^3 - 34pT^2 + 48T + 1$
13	22	-334	$p^6T^4 + 22p^3T^3 - 334pT^2 + 22T + 1$
14	18	258	$p^6T^4 + 18p^3T^3 + 258pT^2 + 18T + 1$
15	184	974	$p^6T^4 + 184p^3T^3 + 974pT^2 + 184T + 1$
16	-56	302	$p^6T^4 - 56p^3T^3 + 302pT^2 - 56T + 1$

$p = 19$

λ	a	b	$R(T)$
1	-16	354	$p^6T^4 - 16p^3T^3 + 354pT^2 - 16T + 1$
2	-8	-126	$p^6T^4 - 8p^3T^3 - 126pT^2 - 8T + 1$
3	132	658	$p^6T^4 + 132p^3T^3 + 658pT^2 + 132T + 1$
4	-70	410	$p^6T^4 - 70p^3T^3 + 410pT^2 - 70T + 1$
5	112	834	$p^6T^4 + 112p^3T^3 + 834pT^2 + 112T + 1$
6	10	410	$p^6T^4 + 10p^3T^3 + 410pT^2 + 10T + 1$
7	100	434	$p^6T^4 + 100p^3T^3 + 434pT^2 + 100T + 1$
8	36	-110	$p^6T^4 + 36p^3T^3 - 110pT^2 + 36T + 1$
9	-30	154	$p^6T^4 - 30p^3T^3 + 154pT^2 - 30T + 1$
10	122	762	$p^6T^4 + 122p^3T^3 + 762pT^2 + 122T + 1$
11	76	722	$p^6T^4 + 76p^3T^3 + 722pT^2 + 76T + 1$
12	-104	258	$p^6T^4 - 104p^3T^3 + 258pT^2 - 104T + 1$
13	-25	317	$(pT + 1)(p^3T^2 - 44T + 1)$
14	66	538	$p^6T^4 + 66p^3T^3 + 538pT^2 + 66T + 1$
15	-25	317	$(pT + 1)(p^3T^2 - 44T + 1)$
16	20	-238	$p^6T^4 + 20p^3T^3 - 238pT^2 + 20T + 1$
17	-4	-334	$p^6T^4 - 4p^3T^3 - 334pT^2 - 4T + 1$
18	-30	-38	$p^6T^4 - 30p^3T^3 - 38pT^2 - 30T + 1$

$p = 23$

λ	a	b	$R(T)$
1	170	818	$p^6T^4 + 170p^3T^3 + 818pT^2 + 170T + 1$
2	14	-430	$p^6T^4 + 14p^3T^3 - 430pT^2 + 14T + 1$
3	0	-94	$p^6T^4 - 94pT^2 + 1$
4	-12	66	$p^6T^4 - 12p^3T^3 + 66pT^2 - 12T + 1$
5	-34	434	$p^6T^4 - 34p^3T^3 + 434pT^2 - 34T + 1$
6	66	978	$p^6T^4 + 66p^3T^3 + 978pT^2 + 66T + 1$
7	24	482	$p^6T^4 + 24p^3T^3 + 482pT^2 + 24T + 1$

$p = 23$ (cont.)

λ	a	b	$R(T)$
8	4	290	$p^6T^4 + 4p^3T^3 + 290pT^2 + 4T + 1$
9	-26	-238	$p^6T^4 - 26p^3T^3 - 238pT^2 - 26T + 1$
10	33	473	$-(pT - 1)(p^3T^2 + 56T + 1)$
11	22	-142	$p^6T^4 + 22p^3T^3 - 142pT^2 + 22T + 1$
12	0	930	$p^6T^4 + 930pT^2 + 1$
13	120	482	$p^6T^4 + 120p^3T^3 + 482pT^2 + 120T + 1$
14	56	674	$p^6T^4 + 56p^3T^3 + 674pT^2 + 56T + 1$
15	48	482	$p^6T^4 + 48p^3T^3 + 482pT^2 + 48T + 1$
16	-79	585	$-(pT - 1)(p^3T^2 - 56T + 1)$
17	172	1090	$p^6T^4 + 172p^3T^3 + 1090pT^2 + 172T + 1$
18	8	674	$p^6T^4 + 8p^3T^3 + 674pT^2 + 8T + 1$
19	74	754	$p^6T^4 + 74p^3T^3 + 754pT^2 + 74T + 1$
20	-244	1346	$p^6T^4 - 244p^3T^3 + 1346pT^2 - 244T + 1$
21	32	418	$p^6T^4 + 32p^3T^3 + 418pT^2 + 32T + 1$
22	82	530	$p^6T^4 + 82p^3T^3 + 530pT^2 + 82T + 1$

$p = 29$

λ	a	b	$R(T)$
1	126	1466	$p^6T^4 + 126p^3T^3 + 1466pT^2 + 126T + 1$
2	-136	542	$p^6T^4 - 136p^3T^3 + 542pT^2 - 136T + 1$
3	-164	886	$p^6T^4 - 164p^3T^3 + 886pT^2 - 164T + 1$
4	260	1382	$p^6T^4 + 260p^3T^3 + 1382pT^2 + 260T + 1$
5	-108	-186	$p^6T^4 - 108p^3T^3 - 186pT^2 - 108T + 1$
6	-284	2278	$p^6T^4 - 284p^3T^3 + 2278pT^2 - 284T + 1$
7	0	1502	$p^6T^4 + 1502pT^2 + 1$
8	114	1034	$p^6T^4 + 114p^3T^3 + 1034pT^2 + 114T + 1$
9	-169	643	$(pT + 1)(p^3T^2 - 198T + 1)$
10	122	602	$p^6T^4 + 122p^3T^3 + 602pT^2 + 122T + 1$
11	288	1470	$p^6T^4 + 288p^3T^3 + 1470pT^2 + 288T + 1$
12	52	-490	$p^6T^4 + 52p^3T^3 - 490pT^2 + 52T + 1$

$p = 29$ (cont.)

λ	a	b	$R(T)$
13	-40	782	$p^6T^4 - 40p^3T^3 + 782pT^2 - 40T + 1$
14	-52	758	$p^6T^4 - 52p^3T^3 + 758pT^2 - 52T + 1$
15	-28	-634	$p^6T^4 - 28p^3T^3 - 634pT^2 - 28T + 1$
16	32	-34	$p^6T^4 + 32p^3T^3 - 34pT^2 + 32T + 1$
17	227	1039	$(pT + 1)(p^3T^2 + 198T + 1)$
18	82	650	$p^6T^4 + 82p^3T^3 + 650pT^2 + 82T + 1$
19	112	1054	$p^6T^4 + 112p^3T^3 + 1054pT^2 + 112T + 1$
20	184	1742	$p^6T^4 + 184p^3T^3 + 1742pT^2 + 184T + 1$
21	-76	758	$p^6T^4 - 76p^3T^3 + 758pT^2 - 76T + 1$
22	168	366	$p^6T^4 + 168p^3T^3 + 366pT^2 + 168T + 1$
23	-34	122	$p^6T^4 - 34p^3T^3 + 122pT^2 - 34T + 1$
24	118	1130	$p^6T^4 + 118p^3T^3 + 1130pT^2 + 118T + 1$
25	256	1630	$p^6T^4 + 256p^3T^3 + 1630pT^2 + 256T + 1$
26	-180	1286	$p^6T^4 - 180p^3T^3 + 1286pT^2 - 180T + 1$
27	60	-378	$p^6T^4 + 60p^3T^3 - 378pT^2 + 60T + 1$
28	-88	494	$p^6T^4 - 88p^3T^3 + 494pT^2 - 88T + 1$

$p = 31$

λ	a	b	$R(T)$
1	96	34	$p^6T^4 + 96p^3T^3 + 34pT^2 + 96T + 1$
2	-380	2498	$p^6T^4 - 380p^3T^3 + 2498pT^2 - 380T + 1$
3	60	770	$p^6T^4 + 60p^3T^3 + 770pT^2 + 60T + 1$
4	-8	-958	$p^6T^4 - 8p^3T^3 - 958pT^2 - 8T + 1$
5	300	1922	$p^6T^4 + 300p^3T^3 + 1922pT^2 + 300T + 1$
6	-44	1602	$p^6T^4 - 44p^3T^3 + 1602pT^2 - 44T + 1$
7	344	2498	$p^6T^4 + 344p^3T^3 + 2498pT^2 + 344T + 1$
8	-129	801	$(pT + 1)(p^3T^2 - 160T + 1)$
9	18	-1118	$p^6T^4 + 18p^3T^3 - 1118pT^2 + 18T + 1$
10	-98	738	$p^6T^4 - 98p^3T^3 + 738pT^2 - 98T + 1$
11	200	1154	$p^6T^4 + 200p^3T^3 + 1154pT^2 + 200T + 1$

$p = 31$ (cont.)

λ	a	b	$R(T)$
12	122	546	$p^6T^4 + 122p^3T^3 + 546pT^2 + 122T + 1$
13	-180	1282	$p^6T^4 - 180p^3T^3 + 1282pT^2 - 180T + 1$
14	24	-62	$p^6T^4 + 24p^3T^3 - 62pT^2 + 24T + 1$
15	-40	834	$p^6T^4 - 40p^3T^3 + 834pT^2 - 40T + 1$
16	-116	-382	$p^6T^4 - 116p^3T^3 - 382pT^2 - 116T + 1$
17	-90	418	$p^6T^4 - 90p^3T^3 + 418pT^2 - 90T + 1$
18	-156	1474	$p^6T^4 - 156p^3T^3 + 1474pT^2 - 156T + 1$
19	136	962	$p^6T^4 + 136p^3T^3 + 962pT^2 + 136T + 1$
20	-164	770	$p^6T^4 - 164p^3T^3 + 770pT^2 - 164T + 1$
21	200	1218	$p^6T^4 + 200p^3T^3 + 1218pT^2 + 200T + 1$
22	32	1730	$p^6T^4 + 32p^3T^3 + 1730pT^2 + 32T + 1$
23	-140	1346	$p^6T^4 - 140p^3T^3 + 1346pT^2 - 140T + 1$
24	20	-766	$p^6T^4 + 20p^3T^3 - 766pT^2 + 20T + 1$
25	340	2658	$p^6T^4 + 340p^3T^3 + 2658pT^2 + 340T + 1$
26	-22	-958	$p^6T^4 - 22p^3T^3 - 958pT^2 - 22T + 1$
27	272	1346	$p^6T^4 + 272p^3T^3 + 1346pT^2 + 272T + 1$
28	274	2114	$p^6T^4 + 274p^3T^3 + 2114pT^2 + 274T + 1$
29	191	1121	$(pT + 1)\,(p^3T^2 + 160T + 1)$
30	-100	1346	$p^6T^4 - 100p^3T^3 + 1346pT^2 - 100T + 1$

$p = 37$

λ	a	b	$R(T)$
1	-128	1470	$p^6T^4 - 128p^3T^3 + 1470pT^2 - 128T + 1$
2	288	2126	$p^6T^4 + 288p^3T^3 + 2126pT^2 + 288T + 1$
3	172	758	$p^6T^4 + 172p^3T^3 + 758pT^2 + 172T + 1$
4	64	1022	$p^6T^4 + 64p^3T^3 + 1022pT^2 + 64T + 1$
5	-64	1998	$p^6T^4 - 64p^3T^3 + 1998pT^2 - 64T + 1$
6	108	1190	$p^6T^4 + 108p^3T^3 + 1190pT^2 + 108T + 1$
7	184	366	$p^6T^4 + 184p^3T^3 + 366pT^2 + 184T + 1$
8	-252	694	$p^6T^4 - 252p^3T^3 + 694pT^2 - 252T + 1$

$p = 37$ (cont.)

λ	a	b	$R(T)$
9	0	−2626	$p^6T^4 - 2626pT^2 + 1$
10	−60	−26	$p^6T^4 - 60p^3T^3 - 26pT^2 - 60T + 1$
11	152	2030	$p^6T^4 + 152p^3T^3 + 2030pT^2 + 152T + 1$
12	234	610	$p^6T^4 + 234p^3T^3 + 610pT^2 + 234T + 1$
13	−156	1846	$p^6T^4 - 156p^3T^3 + 1846pT^2 - 156T + 1$
14	42	2410	$p^6T^4 + 42p^3T^3 + 2410pT^2 + 42T + 1$
15	−140	2294	$p^6T^4 - 140p^3T^3 + 2294pT^2 - 140T + 1$
16	−200	750	$p^6T^4 - 200p^3T^3 + 750pT^2 - 200T + 1$
17	−36	2086	$p^6T^4 - 36p^3T^3 + 2086pT^2 - 36T + 1$
18	32	1070	$p^6T^4 + 32p^3T^3 + 1070pT^2 + 32T + 1$
19	136	1790	$p^6T^4 + 136p^3T^3 + 1790pT^2 + 136T + 1$
20	630	5170	$p^6T^4 + 630p^3T^3 + 5170pT^2 + 630T + 1$
21	64	446	$p^6T^4 + 64p^3T^3 + 446pT^2 + 64T + 1$
22	280	1070	$p^6T^4 + 280p^3T^3 + 1070pT^2 + 280T + 1$
23	46	−990	$p^6T^4 + 46p^3T^3 - 990pT^2 + 46T + 1$
24	−199	1531	$-(pT-1)(p^3T^2 - 162T + 1)$
25	−154	−486	$p^6T^4 - 154p^3T^3 - 486pT^2 - 154T + 1$
26	−244	1334	$p^6T^4 - 244p^3T^3 + 1334pT^2 - 244T + 1$
27	−180	1910	$p^6T^4 - 180p^3T^3 + 1910pT^2 - 180T + 1$
28	302	3242	$p^6T^4 + 302p^3T^3 + 3242pT^2 + 302T + 1$
29	260	1670	$p^6T^4 + 260p^3T^3 + 1670pT^2 + 260T + 1$
30	125	1207	$-(pT-1)(p^3T^2 + 162T + 1)$
31	−158	1946	$p^6T^4 - 158p^3T^3 + 1946pT^2 - 158T + 1$
32	−148	230	$p^6T^4 - 148p^3T^3 + 230pT^2 - 148T + 1$
33	−84	1462	$p^6T^4 - 84p^3T^3 + 1462pT^2 - 84T + 1$
34	100	1382	$p^6T^4 + 100p^3T^3 + 1382pT^2 + 100T + 1$
35	16	894	$p^6T^4 + 16p^3T^3 + 894pT^2 + 16T + 1$
36	338	2706	$p^6T^4 + 338p^3T^3 + 2706pT^2 + 338T + 1$

$p = 41$

λ	a	b	$R(T)$
1	-300	1238	$p^6T^4 - 300p^3T^3 + 1238pT^2 - 300T + 1$
2	-276	1382	$p^6T^4 - 276p^3T^3 + 1382pT^2 - 276T + 1$
3	22	-838	$p^6T^4 + 22p^3T^3 - 838pT^2 + 22T + 1$
4	462	3282	$p^6T^4 + 462p^3T^3 + 3282pT^2 + 462T + 1$
5	392	3470	$p^6T^4 + 392p^3T^3 + 3470pT^2 + 392T + 1$
6	196	2246	$p^6T^4 + 196p^3T^3 + 2246pT^2 + 196T + 1$
7	38	2234	$p^6T^4 + 38p^3T^3 + 2234pT^2 + 38T + 1$
8	-24	2190	$p^6T^4 - 24p^3T^3 + 2190pT^2 - 24T + 1$
9	190	1010	$p^6T^4 + 190p^3T^3 + 1010pT^2 + 190T + 1$
10	-144	1598	$p^6T^4 - 144p^3T^3 + 1598pT^2 - 144T + 1$
11	-388	2294	$p^6T^4 - 388p^3T^3 + 2294pT^2 - 388T + 1$
12	-240	1022	$p^6T^4 - 240p^3T^3 + 1022pT^2 - 240T + 1$
13	20	-2122	$p^6T^4 + 20p^3T^3 - 2122pT^2 + 20T + 1$
14	-194	3530	$p^6T^4 - 194p^3T^3 + 3530pT^2 - 194T + 1$
15	128	1582	$p^6T^4 + 128p^3T^3 + 1582pT^2 + 128T + 1$
16	190	-526	$p^6T^4 + 190p^3T^3 - 526pT^2 + 190T + 1$
17	28	-778	$p^6T^4 + 28p^3T^3 - 778pT^2 + 28T + 1$
18	-24	-114	$p^6T^4 - 24p^3T^3 - 114pT^2 - 24T + 1$
19	-166	2210	$p^6T^4 - 166p^3T^3 + 2210pT^2 - 166T + 1$
20	78	1362	$p^6T^4 + 78p^3T^3 + 1362pT^2 + 78T + 1$
21	36	374	$p^6T^4 + 36p^3T^3 + 374pT^2 + 36T + 1$
22	14	-1078	$p^6T^4 + 14p^3T^3 - 1078pT^2 + 14T + 1$
23	239	1879	$(pT + 1)(p^3T^2 + 198T + 1)$
24	380	2630	$p^6T^4 + 380p^3T^3 + 2630pT^2 + 380T + 1$
25	-64	-226	$p^6T^4 - 64p^3T^3 - 226pT^2 - 64T + 1$
26	40	238	$p^6T^4 + 40p^3T^3 + 238pT^2 + 40T + 1$
27	-326	2402	$p^6T^4 - 326p^3T^3 + 2402pT^2 - 326T + 1$
28	-92	1718	$p^6T^4 - 92p^3T^3 + 1718pT^2 - 92T + 1$
29	36	-474	$p^6T^4 + 36p^3T^3 - 474pT^2 + 36T + 1$
30	652	5174	$p^6T^4 + 652p^3T^3 + 5174pT^2 + 652T + 1$
31	204	3110	$p^6T^4 + 204p^3T^3 + 3110pT^2 + 204T + 1$
32	32	1886	$p^6T^4 + 32p^3T^3 + 1886pT^2 + 32T + 1$
33	239	1879	$(pT + 1)(p^3T^2 + 198T + 1)$

$p = 41$ (cont.)

λ	a	b	$R(T)$
34	104	238	$p^6T^4 + 104p^3T^3 + 238pT^2 + 104T + 1$
35	332	2486	$p^6T^4 + 332p^3T^3 + 2486pT^2 + 332T + 1$
36	-594	5394	$p^6T^4 - 594p^3T^3 + 5394pT^2 - 594T + 1$
37	56	2414	$p^6T^4 + 56p^3T^3 + 2414pT^2 + 56T + 1$
38	280	3502	$p^6T^4 + 280p^3T^3 + 3502pT^2 + 280T + 1$
39	336	2174	$p^6T^4 + 336p^3T^3 + 2174pT^2 + 336T + 1$
40	-210	3090	$p^6T^4 - 210p^3T^3 + 3090pT^2 - 210T + 1$

$p = 43$

λ	a	b	$R(T)$
1	56	3170	$p^6T^4 + 56p^3T^3 + 3170pT^2 + 56T + 1$
2	-228	178	$p^6T^4 - 228p^3T^3 + 178pT^2 - 228T + 1$
3	344	3810	$p^6T^4 + 344p^3T^3 + 3810pT^2 + 344T + 1$
4	302	3098	$p^6T^4 + 302p^3T^3 + 3098pT^2 + 302T + 1$
5	324	3922	$p^6T^4 + 324p^3T^3 + 3922pT^2 + 324T + 1$
6	356	1554	$p^6T^4 + 356p^3T^3 + 1554pT^2 + 356T + 1$
7	-108	2290	$p^6T^4 - 108p^3T^3 + 2290pT^2 - 108T + 1$
8	-9	1797	$(pT + 1)(p^3T^2 - 52T + 1)$
9	-176	3458	$p^6T^4 - 176p^3T^3 + 3458pT^2 - 176T + 1$
10	456	3586	$p^6T^4 + 456p^3T^3 + 3586pT^2 + 456T + 1$
11	-330	3802	$p^6T^4 - 330p^3T^3 + 3802pT^2 - 330T + 1$
12	-160	1922	$p^6T^4 - 160p^3T^3 + 1922pT^2 - 160T + 1$
13	76	1554	$p^6T^4 + 76p^3T^3 + 1554pT^2 + 76T + 1$
14	16	-1438	$p^6T^4 + 16p^3T^3 - 1438pT^2 + 16T + 1$
15	-4	1266	$p^6T^4 - 4p^3T^3 + 1266pT^2 - 4T + 1$
16	128	386	$p^6T^4 + 128p^3T^3 + 386pT^2 + 128T + 1$
17	-432	4258	$p^6T^4 - 432p^3T^3 + 4258pT^2 - 432T + 1$
18	438	3482	$p^6T^4 + 438p^3T^3 + 3482pT^2 + 438T + 1$
19	2	-1590	$p^6T^4 + 2p^3T^3 - 1590pT^2 + 2T + 1$
20	24	-542	$p^6T^4 + 24p^3T^3 - 542pT^2 + 24T + 1$

$p = 43$ (cont.)

λ	a	b	$R(T)$
21	280	2882	$p^6T^4 + 280p^3T^3 + 2882pT^2 + 280T + 1$
22	180	1042	$p^6T^4 + 180p^3T^3 + 1042pT^2 + 180T + 1$
23	-832	7394	$p^6T^4 - 832p^3T^3 + 7394pT^2 - 832T + 1$
24	224	2114	$p^6T^4 + 224p^3T^3 + 2114pT^2 + 224T + 1$
25	-48	514	$p^6T^4 - 48p^3T^3 + 514pT^2 - 48T + 1$
26	-88	-1662	$p^6T^4 - 88p^3T^3 - 1662pT^2 - 88T + 1$
27	324	3058	$p^6T^4 + 324p^3T^3 + 3058pT^2 + 324T + 1$
28	320	3650	$p^6T^4 + 320p^3T^3 + 3650pT^2 + 320T + 1$
29	-574	4170	$p^6T^4 - 574p^3T^3 + 4170pT^2 - 574T + 1$
30	152	1826	$p^6T^4 + 152p^3T^3 + 1826pT^2 + 152T + 1$
31	340	3954	$p^6T^4 + 340p^3T^3 + 3954pT^2 + 340T + 1$
32	382	2490	$p^6T^4 + 382p^3T^3 + 2490pT^2 + 382T + 1$
33	-234	26	$p^6T^4 - 234p^3T^3 + 26pT^2 - 234T + 1$
34	24	-2654	$p^6T^4 + 24p^3T^3 - 2654pT^2 + 24T + 1$
35	-128	2018	$p^6T^4 - 128p^3T^3 + 2018pT^2 - 128T + 1$
36	-86	-822	$p^6T^4 - 86p^3T^3 - 822pT^2 - 86T + 1$
37	-136	866	$p^6T^4 - 136p^3T^3 + 866pT^2 - 136T + 1$
38	266	330	$p^6T^4 + 266p^3T^3 + 330pT^2 + 266T + 1$
39	-36	-1358	$p^6T^4 - 36p^3T^3 - 1358pT^2 - 36T + 1$
40	456	4642	$p^6T^4 + 456p^3T^3 + 4642pT^2 + 456T + 1$
41	-2	1178	$p^6T^4 - 2p^3T^3 + 1178pT^2 - 2T + 1$
42	-9	1797	$(pT + 1)(p^3T^2 - 52T + 1)$

$p = 47$

λ	a	b	$R(T)$
1	32	2882	$p^6T^4 + 32p^3T^3 + 2882pT^2 + 32T + 1$
2	156	2114	$p^6T^4 + 156p^3T^3 + 2114pT^2 + 156T + 1$
3	-4	1346	$p^6T^4 - 4p^3T^3 + 1346pT^2 - 4T + 1$
4	-16	130	$p^6T^4 - 16p^3T^3 + 130pT^2 - 16T + 1$
5	-394	1282	$p^6T^4 - 394p^3T^3 + 1282pT^2 - 394T + 1$
6	-432	4482	$p^6T^4 - 432p^3T^3 + 4482pT^2 - 432T + 1$
7	626	5122	$p^6T^4 + 626p^3T^3 + 5122pT^2 + 626T + 1$
8	462	3522	$p^6T^4 + 462p^3T^3 + 3522pT^2 + 462T + 1$
9	-216	3842	$p^6T^4 - 216p^3T^3 + 3842pT^2 - 216T + 1$
10	96	66	$p^6T^4 + 96p^3T^3 + 66pT^2 + 96T + 1$
11	560	4034	$p^6T^4 + 560p^3T^3 + 4034pT^2 + 560T + 1$
12	-20	-1310	$p^6T^4 - 20p^3T^3 - 1310pT^2 - 20T + 1$
13	-46	-1246	$p^6T^4 - 46p^3T^3 - 1246pT^2 - 46T + 1$
14	328	3394	$p^6T^4 + 328p^3T^3 + 3394pT^2 + 328T + 1$
15	284	2242	$p^6T^4 + 284p^3T^3 + 2242pT^2 + 284T + 1$
16	252	962	$p^6T^4 + 252p^3T^3 + 962pT^2 + 252T + 1$
17	-692	5474	$p^6T^4 - 692p^3T^3 + 5474pT^2 - 692T + 1$
18	481	1681	$-(pT - 1)\,(p^3T^2 + 528T + 1)$
19	-156	-126	$p^6T^4 - 156p^3T^3 - 126pT^2 - 156T + 1$
20	36	-766	$p^6T^4 + 36p^3T^3 - 766pT^2 + 36T + 1$
21	-212	386	$p^6T^4 - 212p^3T^3 + 386pT^2 - 212T + 1$
22	50	482	$p^6T^4 + 50p^3T^3 + 482pT^2 + 50T + 1$
23	72	2754	$p^6T^4 + 72p^3T^3 + 2754pT^2 + 72T + 1$
24	252	1538	$p^6T^4 + 252p^3T^3 + 1538pT^2 + 252T + 1$
25	16	4034	$p^6T^4 + 16p^3T^3 + 4034pT^2 + 16T + 1$
26	-60	-2494	$p^6T^4 - 60p^3T^3 - 2494pT^2 - 60T + 1$
27	-346	2690	$p^6T^4 - 346p^3T^3 + 2690pT^2 - 346T + 1$
28	-360	4418	$p^6T^4 - 360p^3T^3 + 4418pT^2 - 360T + 1$
29	-436	2882	$p^6T^4 - 436p^3T^3 + 2882pT^2 - 436T + 1$
30	-92	3266	$p^6T^4 - 92p^3T^3 + 3266pT^2 - 92T + 1$
31	188	3650	$p^6T^4 + 188p^3T^3 + 3650pT^2 + 188T + 1$
32	346	2978	$p^6T^4 + 346p^3T^3 + 2978pT^2 + 346T + 1$
33	396	3842	$p^6T^4 + 396p^3T^3 + 3842pT^2 + 396T + 1$

$p = 47$ (cont.)

λ	a	b	$R(T)$
34	12	−1854	$p^6T^4 + 12p^3T^3 - 1854pT^2 + 12T + 1$
35	432	3266	$p^6T^4 + 432p^3T^3 + 3266pT^2 + 432T + 1$
36	58	3554	$p^6T^4 + 58p^3T^3 + 3554pT^2 + 58T + 1$
37	728	6914	$p^6T^4 + 728p^3T^3 + 6914pT^2 + 728T + 1$
38	320	3970	$p^6T^4 + 320p^3T^3 + 3970pT^2 + 320T + 1$
39	254	962	$p^6T^4 + 254p^3T^3 + 962pT^2 + 254T + 1$
40	176	578	$p^6T^4 + 176p^3T^3 + 578pT^2 + 176T + 1$
41	24	−190	$p^6T^4 + 24p^3T^3 - 190pT^2 + 24T + 1$
42	−346	1154	$p^6T^4 - 346p^3T^3 + 1154pT^2 - 346T + 1$
43	−320	4034	$p^6T^4 - 320p^3T^3 + 4034pT^2 - 320T + 1$
44	−575	2737	$-(pT-1)(p^3T^2 - 528T + 1)$
45	−130	386	$p^6T^4 - 130p^3T^3 + 386pT^2 - 130T + 1$
46	426	2178	$p^6T^4 + 426p^3T^3 + 2178pT^2 + 426T + 1$

$p = 53$

λ	a	b	$R(T)$
1	36	1766	$p^6T^4 + 36p^3T^3 + 1766pT^2 + 36T + 1$
2	188	1318	$p^6T^4 + 188p^3T^3 + 1318pT^2 + 188T + 1$
3	−726	6122	$p^6T^4 - 726p^3T^3 + 6122pT^2 - 726T + 1$
4	−158	1106	$p^6T^4 - 158p^3T^3 + 1106pT^2 - 158T + 1$
5	−126	5082	$p^6T^4 - 126p^3T^3 + 5082pT^2 - 126T + 1$
6	−28	−3290	$p^6T^4 - 28p^3T^3 - 3290pT^2 - 28T + 1$
7	288	894	$p^6T^4 + 288p^3T^3 + 894pT^2 + 288T + 1$
8	624	6302	$p^6T^4 + 624p^3T^3 + 6302pT^2 + 624T + 1$
9	−60	1766	$p^6T^4 - 60p^3T^3 + 1766pT^2 - 60T + 1$
10	172	−778	$p^6T^4 + 172p^3T^3 - 778pT^2 + 172T + 1$
11	−236	2822	$p^6T^4 - 236p^3T^3 + 2822pT^2 - 236T + 1$
12	520	6254	$p^6T^4 + 520p^3T^3 + 6254pT^2 + 520T + 1$
13	−462	3090	$p^6T^4 - 462p^3T^3 + 3090pT^2 - 462T + 1$
14	408	1118	$p^6T^4 + 408p^3T^3 + 1118pT^2 + 408T + 1$

$p = 53$ (cont.)

λ	a	b	$R(T)$
15	-12	3782	$p^6T^4 - 12p^3T^3 + 3782pT^2 - 12T + 1$
16	-694	6338	$p^6T^4 - 694p^3T^3 + 6338pT^2 - 694T + 1$
17	-202	3482	$p^6T^4 - 202p^3T^3 + 3482pT^2 - 202T + 1$
18	816	6446	$p^6T^4 + 816p^3T^3 + 6446pT^2 + 816T + 1$
19	-120	1950	$p^6T^4 - 120p^3T^3 + 1950pT^2 - 120T + 1$
20	288	318	$p^6T^4 + 288p^3T^3 + 318pT^2 + 288T + 1$
21	-52	-2618	$p^6T^4 - 52p^3T^3 - 2618pT^2 - 52T + 1$
22	536	2734	$p^6T^4 + 536p^3T^3 + 2734pT^2 + 536T + 1$
23	644	4598	$p^6T^4 + 644p^3T^3 + 4598pT^2 + 644T + 1$
24	526	3562	$p^6T^4 + 526p^3T^3 + 3562pT^2 + 526T + 1$
25	234	-862	$p^6T^4 + 234p^3T^3 - 862pT^2 + 234T + 1$
26	-258	866	$p^6T^4 - 258p^3T^3 + 866pT^2 - 258T + 1$
27	152	2174	$p^6T^4 + 152p^3T^3 + 2174pT^2 + 152T + 1$
28	292	4070	$p^6T^4 + 292p^3T^3 + 4070pT^2 + 292T + 1$
29	-248	3854	$p^6T^4 - 248p^3T^3 + 3854pT^2 - 248T + 1$
30	38	5426	$p^6T^4 + 38p^3T^3 + 5426pT^2 + 38T + 1$
31	-384	4286	$p^6T^4 - 384p^3T^3 + 4286pT^2 - 384T + 1$
32	-182	1514	$p^6T^4 - 182p^3T^3 + 1514pT^2 - 182T + 1$
33	-8	-3586	$p^6T^4 - 8p^3T^3 - 3586pT^2 - 8T + 1$
34	-706	6914	$p^6T^4 - 706p^3T^3 + 6914pT^2 - 706T + 1$
35	16	-994	$p^6T^4 + 16p^3T^3 - 994pT^2 + 16T + 1$
36	336	5406	$p^6T^4 + 336p^3T^3 + 5406pT^2 + 336T + 1$
37	308	1990	$p^6T^4 + 308p^3T^3 + 1990pT^2 + 308T + 1$
38	-188	4070	$p^6T^4 - 188p^3T^3 + 4070pT^2 - 188T + 1$
39	-740	7094	$p^6T^4 - 740p^3T^3 + 7094pT^2 - 740T + 1$
40	54	-1510	$p^6T^4 + 54p^3T^3 - 1510pT^2 + 54T + 1$
41	-189	2567	$(pT + 1)(p^3T^2 - 242T + 1)$
42	108	3638	$p^6T^4 + 108p^3T^3 + 3638pT^2 + 108T + 1$
43	295	3051	$(pT + 1)(p^3T^2 + 242T + 1)$
44	588	3894	$p^6T^4 + 588p^3T^3 + 3894pT^2 + 588T + 1$
45	544	4094	$p^6T^4 + 544p^3T^3 + 4094pT^2 + 544T + 1$
46	-28	-1114	$p^6T^4 - 28p^3T^3 - 1114pT^2 - 28T + 1$
47	68	4646	$p^6T^4 + 68p^3T^3 + 4646pT^2 + 68T + 1$

$p = 53$ (cont.)

λ	a	b	$R(T)$
48	-364	470	$p^6T^4 - 364p^3T^3 + 470pT^2 - 364T + 1$
49	376	3374	$p^6T^4 + 376p^3T^3 + 3374pT^2 + 376T + 1$
50	310	4786	$p^6T^4 + 310p^3T^3 + 4786pT^2 + 310T + 1$
51	176	3134	$p^6T^4 + 176p^3T^3 + 3134pT^2 + 176T + 1$
52	40	3086	$p^6T^4 + 40p^3T^3 + 3086pT^2 + 40T + 1$

$p = 59$

λ	a	b	$R(T)$
1	-116	338	$p^6T^4 - 116p^3T^3 + 338pT^2 - 116T + 1$
2	4	242	$p^6T^4 + 4p^3T^3 + 242pT^2 + 4T + 1$
3	0	-94	$p^6T^4 - 94pT^2 + 1$
4	-948	8274	$p^6T^4 - 948p^3T^3 + 8274pT^2 - 948T + 1$
5	888	10050	$p^6T^4 + 888p^3T^3 + 10050pT^2 + 888T + 1$
6	609	2813	$-(pT-1)(p^3T^2 + 668T + 1)$
7	-172	338	$p^6T^4 - 172p^3T^3 + 338pT^2 - 172T + 1$
8	-380	6866	$p^6T^4 - 380p^3T^3 + 6866pT^2 - 380T + 1$
9	-248	3650	$p^6T^4 - 248p^3T^3 + 3650pT^2 - 248T + 1$
10	24	2786	$p^6T^4 + 24p^3T^3 + 2786pT^2 + 24T + 1$
11	609	2813	$-(pT-1)(p^3T^2 + 668T + 1)$
12	668	6674	$p^6T^4 + 668p^3T^3 + 6674pT^2 + 668T + 1$
13	-376	1058	$p^6T^4 - 376p^3T^3 + 1058pT^2 - 376T + 1$
14	94	922	$p^6T^4 + 94p^3T^3 + 922pT^2 + 94T + 1$
15	568	3746	$p^6T^4 + 568p^3T^3 + 3746pT^2 + 568T + 1$
16	-256	-574	$p^6T^4 - 256p^3T^3 - 574pT^2 - 256T + 1$
17	-78	3146	$p^6T^4 - 78p^3T^3 + 3146pT^2 - 78T + 1$
18	340	4178	$p^6T^4 + 340p^3T^3 + 4178pT^2 + 340T + 1$
19	-1230	12810	$p^6T^4 - 1230p^3T^3 + 12810pT^2 - 1230T + 1$
20	-204	6098	$p^6T^4 - 204p^3T^3 + 6098pT^2 - 204T + 1$
21	492	3218	$p^6T^4 + 492p^3T^3 + 3218pT^2 + 492T + 1$
22	744	5666	$p^6T^4 + 744p^3T^3 + 5666pT^2 + 744T + 1$

$p = 59$ (cont.)

λ	a	b	$R(T)$
23	-92	-46	$p^6T^4 - 92p^3T^3 - 46pT^2 - 92T + 1$
24	-836	6706	$p^6T^4 - 836p^3T^3 + 6706pT^2 - 836T + 1$
25	48	3330	$p^6T^4 + 48p^3T^3 + 3330pT^2 + 48T + 1$
26	616	3458	$p^6T^4 + 616p^3T^3 + 3458pT^2 + 616T + 1$
27	536	6434	$p^6T^4 + 536p^3T^3 + 6434pT^2 + 536T + 1$
28	200	6530	$p^6T^4 + 200p^3T^3 + 6530pT^2 + 200T + 1$
29	238	5882	$p^6T^4 + 238p^3T^3 + 5882pT^2 + 238T + 1$
30	286	-1766	$p^6T^4 + 286p^3T^3 - 1766pT^2 + 286T + 1$
31	-40	3938	$p^6T^4 - 40p^3T^3 + 3938pT^2 - 40T + 1$
32	-444	6098	$p^6T^4 - 444p^3T^3 + 6098pT^2 - 444T + 1$
33	-428	1298	$p^6T^4 - 428p^3T^3 + 1298pT^2 - 428T + 1$
34	56	1186	$p^6T^4 + 56p^3T^3 + 1186pT^2 + 56T + 1$
35	-908	8210	$p^6T^4 - 908p^3T^3 + 8210pT^2 - 908T + 1$
36	184	-3614	$p^6T^4 + 184p^3T^3 - 3614pT^2 + 184T + 1$
37	258	-1878	$p^6T^4 + 258p^3T^3 - 1878pT^2 + 258T + 1$
38	366	1562	$p^6T^4 + 366p^3T^3 + 1562pT^2 + 366T + 1$
39	78	2138	$p^6T^4 + 78p^3T^3 + 2138pT^2 + 78T + 1$
40	-82	-3302	$p^6T^4 - 82p^3T^3 - 3302pT^2 - 82T + 1$
41	232	578	$p^6T^4 + 232p^3T^3 + 578pT^2 + 232T + 1$
42	504	6242	$p^6T^4 + 504p^3T^3 + 6242pT^2 + 504T + 1$
43	196	2930	$p^6T^4 + 196p^3T^3 + 2930pT^2 + 196T + 1$
44	56	866	$p^6T^4 + 56p^3T^3 + 866pT^2 + 56T + 1$
45	158	-838	$p^6T^4 + 158p^3T^3 - 838pT^2 + 158T + 1$
46	-828	8274	$p^6T^4 - 828p^3T^3 + 8274pT^2 - 828T + 1$
47	-548	2098	$p^6T^4 - 548p^3T^3 + 2098pT^2 - 548T + 1$
48	546	7178	$p^6T^4 + 546p^3T^3 + 7178pT^2 + 546T + 1$
49	-366	3722	$p^6T^4 - 366p^3T^3 + 3722pT^2 - 366T + 1$
50	708	7250	$p^6T^4 + 708p^3T^3 + 7250pT^2 + 708T + 1$
51	272	2914	$p^6T^4 + 272p^3T^3 + 2914pT^2 + 272T + 1$
52	376	3938	$p^6T^4 + 376p^3T^3 + 3938pT^2 + 376T + 1$
53	1026	9482	$p^6T^4 + 1026p^3T^3 + 9482pT^2 + 1026T + 1$
54	-136	1378	$p^6T^4 - 136p^3T^3 + 1378pT^2 - 136T + 1$
55	776	7586	$p^6T^4 + 776p^3T^3 + 7586pT^2 + 776T + 1$

$p = 59$ (cont.)

λ	a	b	$R(T)$
56	-414	42	$p^6T^4 - 414p^3T^3 + 42pT^2 - 414T + 1$
57	-280	-1054	$p^6T^4 - 280p^3T^3 - 1054pT^2 - 280T + 1$
58	136	1346	$p^6T^4 + 136p^3T^3 + 1346pT^2 + 136T + 1$

$p = 61$

λ	a	b	$R(T)$
1	-108	5062	$p^6T^4 - 108p^3T^3 + 5062pT^2 - 108T + 1$
2	-128	-2562	$p^6T^4 - 128p^3T^3 - 2562pT^2 - 128T + 1$
3	-560	4926	$p^6T^4 - 560p^3T^3 + 4926pT^2 - 560T + 1$
4	816	5950	$p^6T^4 + 816p^3T^3 + 5950pT^2 + 816T + 1$
5	-508	3878	$p^6T^4 - 508p^3T^3 + 3878pT^2 - 508T + 1$
6	-198	5498	$p^6T^4 - 198p^3T^3 + 5498pT^2 - 198T + 1$
7	420	2566	$p^6T^4 + 420p^3T^3 + 2566pT^2 + 420T + 1$
8	-372	6598	$p^6T^4 - 372p^3T^3 + 6598pT^2 - 372T + 1$
9	102	458	$p^6T^4 + 102p^3T^3 + 458pT^2 + 102T + 1$
10	154	5114	$p^6T^4 + 154p^3T^3 + 5114pT^2 + 154T + 1$
11	28	-90	$p^6T^4 + 28p^3T^3 - 90pT^2 + 28T + 1$
12	158	186	$p^6T^4 + 158p^3T^3 + 186pT^2 + 158T + 1$
13	-536	2222	$p^6T^4 - 536p^3T^3 + 2222pT^2 - 536T + 1$
14	68	-1498	$p^6T^4 + 68p^3T^3 - 1498pT^2 + 68T + 1$
15	-560	4926	$p^6T^4 - 560p^3T^3 + 4926pT^2 - 560T + 1$
16	816	5566	$p^6T^4 + 816p^3T^3 + 5566pT^2 + 816T + 1$
17	-16	-514	$p^6T^4 - 16p^3T^3 - 514pT^2 - 16T + 1$
18	1140	12070	$p^6T^4 + 1140p^3T^3 + 12070pT^2 + 1140T + 1$
19	-611	4271	$-(pT - 1)(p^3T^2 - 550T + 1)$
20	92	2006	$p^6T^4 + 92p^3T^3 + 2006pT^2 + 92T + 1$
21	152	1166	$p^6T^4 + 152p^3T^3 + 1166pT^2 + 152T + 1$
22	-256	2526	$p^6T^4 - 256p^3T^3 + 2526pT^2 - 256T + 1$
23	-376	-322	$p^6T^4 - 376p^3T^3 - 322pT^2 - 376T + 1$
24	446	3282	$p^6T^4 + 446p^3T^3 + 3282pT^2 + 446T + 1$

$p = 61$ (cont.)

λ	a	b	$R(T)$
25	372	1414	$p^6T^4 + 372p^3T^3 + 1414pT^2 + 372T + 1$
26	−288	6526	$p^6T^4 - 288p^3T^3 + 6526pT^2 - 288T + 1$
27	228	7270	$p^6T^4 + 228p^3T^3 + 7270pT^2 + 228T + 1$
28	180	454	$p^6T^4 + 180p^3T^3 + 454pT^2 + 180T + 1$
29	−64	4446	$p^6T^4 - 64p^3T^3 + 4446pT^2 - 64T + 1$
30	−256	2366	$p^6T^4 - 256p^3T^3 + 2366pT^2 - 256T + 1$
31	−190	2954	$p^6T^4 - 190p^3T^3 + 2954pT^2 - 190T + 1$
32	810	8410	$p^6T^4 + 810p^3T^3 + 8410pT^2 + 810T + 1$
33	500	1190	$p^6T^4 + 500p^3T^3 + 1190pT^2 + 500T + 1$
34	464	638	$p^6T^4 + 464p^3T^3 + 638pT^2 + 464T + 1$
35	642	6154	$p^6T^4 + 642p^3T^3 + 6154pT^2 + 642T + 1$
36	454	2346	$p^6T^4 + 454p^3T^3 + 2346pT^2 + 454T + 1$
37	692	7670	$p^6T^4 + 692p^3T^3 + 7670pT^2 + 692T + 1$
38	−394	2370	$p^6T^4 - 394p^3T^3 + 2370pT^2 - 394T + 1$
39	−276	5494	$p^6T^4 - 276p^3T^3 + 5494pT^2 - 276T + 1$
40	−20	−186	$p^6T^4 - 20p^3T^3 - 186pT^2 - 20T + 1$
41	−288	6238	$p^6T^4 - 288p^3T^3 + 6238pT^2 - 288T + 1$
42	−26	−3958	$p^6T^4 - 26p^3T^3 - 3958pT^2 - 26T + 1$
43	−576	4414	$p^6T^4 - 576p^3T^3 + 4414pT^2 - 576T + 1$
44	−44	3078	$p^6T^4 - 44p^3T^3 + 3078pT^2 - 44T + 1$
45	−16	−130	$p^6T^4 - 16p^3T^3 - 130pT^2 - 16T + 1$
46	512	7646	$p^6T^4 + 512p^3T^3 + 7646pT^2 + 512T + 1$
47	94	−5638	$p^6T^4 + 94p^3T^3 - 5638pT^2 + 94T + 1$
48	1386	15154	$p^6T^4 + 1386p^3T^3 + 15154pT^2 + 1386T + 1$
49	−476	1638	$p^6T^4 - 476p^3T^3 + 1638pT^2 - 476T + 1$
50	−676	7910	$p^6T^4 - 676p^3T^3 + 7910pT^2 - 676T + 1$
51	489	3171	$-(pT - 1)\,(p^3T^2 + 550T + 1)$
52	−1218	12634	$p^6T^4 - 1218p^3T^3 + 12634pT^2 - 1218T + 1$
53	−416	4206	$p^6T^4 - 416p^3T^3 + 4206pT^2 - 416T + 1$
54	−208	5598	$p^6T^4 - 208p^3T^3 + 5598pT^2 - 208T + 1$
55	530	7722	$p^6T^4 + 530p^3T^3 + 7722pT^2 + 530T + 1$
56	450	1090	$p^6T^4 + 450p^3T^3 + 1090pT^2 + 450T + 1$
57	388	422	$p^6T^4 + 388p^3T^3 + 422pT^2 + 388T + 1$

$p = 61$ (cont.)

λ	a	b	$R(T)$
58	1220	11814	$p^6T^4 + 1220p^3T^3 + 11814pT^2 + 1220T + 1$
59	-100	-3034	$p^6T^4 - 100p^3T^3 - 3034pT^2 - 100T + 1$
60	-320	3102	$p^6T^4 - 320p^3T^3 + 3102pT^2 - 320T + 1$

$p = 67$

λ	a	b	$R(T)$
1	544	4290	$p^6T^4 + 544p^3T^3 + 4290pT^2 + 544T + 1$
2	118	1386	$p^6T^4 + 118p^3T^3 + 1386pT^2 + 118T + 1$
3	288	-1310	$p^6T^4 + 288p^3T^3 - 1310pT^2 + 288T + 1$
4	-368	7650	$p^6T^4 - 368p^3T^3 + 7650pT^2 - 368T + 1$
5	298	3290	$p^6T^4 + 298p^3T^3 + 3290pT^2 + 298T + 1$
6	-876	7186	$p^6T^4 - 876p^3T^3 + 7186pT^2 - 876T + 1$
7	408	7330	$p^6T^4 + 408p^3T^3 + 7330pT^2 + 408T + 1$
8	428	4466	$p^6T^4 + 428p^3T^3 + 4466pT^2 + 428T + 1$
9	108	5650	$p^6T^4 + 108p^3T^3 + 5650pT^2 + 108T + 1$
10	-174	3322	$p^6T^4 - 174p^3T^3 + 3322pT^2 - 174T + 1$
11	-121	4301	$(pT + 1)\,(p^3T^2 - 188T + 1)$
12	-160	6434	$p^6T^4 - 160p^3T^3 + 6434pT^2 - 160T + 1$
13	-4	7826	$p^6T^4 - 4p^3T^3 + 7826pT^2 - 4T + 1$
14	416	9122	$p^6T^4 + 416p^3T^3 + 9122pT^2 + 416T + 1$
15	-48	4514	$p^6T^4 - 48p^3T^3 + 4514pT^2 - 48T + 1$
16	1144	12002	$p^6T^4 + 1144p^3T^3 + 12002pT^2 + 1144T + 1$
17	-56	290	$p^6T^4 - 56p^3T^3 + 290pT^2 - 56T + 1$
18	-14	4922	$p^6T^4 - 14p^3T^3 + 4922pT^2 - 14T + 1$
19	276	-2222	$p^6T^4 + 276p^3T^3 - 2222pT^2 + 276T + 1$
20	824	6146	$p^6T^4 + 824p^3T^3 + 6146pT^2 + 824T + 1$
21	388	3122	$p^6T^4 + 388p^3T^3 + 3122pT^2 + 388T + 1$
22	-694	6458	$p^6T^4 - 694p^3T^3 + 6458pT^2 - 694T + 1$
23	58	-646	$p^6T^4 + 58p^3T^3 - 646pT^2 + 58T + 1$
24	-210	4714	$p^6T^4 - 210p^3T^3 + 4714pT^2 - 210T + 1$

$p = 67$ (cont.)

λ	a	b	$R(T)$
25	272	4802	$p^6T^4 + 272p^3T^3 + 4802pT^2 + 272T + 1$
26	-304	7554	$p^6T^4 - 304p^3T^3 + 7554pT^2 - 304T + 1$
27	-288	4834	$p^6T^4 - 288p^3T^3 + 4834pT^2 - 288T + 1$
28	808	8834	$p^6T^4 + 808p^3T^3 + 8834pT^2 + 808T + 1$
29	534	2122	$p^6T^4 + 534p^3T^3 + 2122pT^2 + 534T + 1$
30	-408	3650	$p^6T^4 - 408p^3T^3 + 3650pT^2 - 408T + 1$
31	-16	6146	$p^6T^4 - 16p^3T^3 + 6146pT^2 - 16T + 1$
32	-232	2	$p^6T^4 - 232p^3T^3 + 2pT^2 - 232T + 1$
33	936	11362	$p^6T^4 + 936p^3T^3 + 11362pT^2 + 936T + 1$
34	-120	1474	$p^6T^4 - 120p^3T^3 + 1474pT^2 - 120T + 1$
35	584	6914	$p^6T^4 + 584p^3T^3 + 6914pT^2 + 584T + 1$
36	76	7154	$p^6T^4 + 76p^3T^3 + 7154pT^2 + 76T + 1$
37	-568	7362	$p^6T^4 - 568p^3T^3 + 7362pT^2 - 568T + 1$
38	-162	5450	$p^6T^4 - 162p^3T^3 + 5450pT^2 - 162T + 1$
39	760	7874	$p^6T^4 + 760p^3T^3 + 7874pT^2 + 760T + 1$
40	-496	3938	$p^6T^4 - 496p^3T^3 + 3938pT^2 - 496T + 1$
41	14	1994	$p^6T^4 + 14p^3T^3 + 1994pT^2 + 14T + 1$
42	136	7874	$p^6T^4 + 136p^3T^3 + 7874pT^2 + 136T + 1$
43	-14	-2950	$p^6T^4 - 14p^3T^3 - 2950pT^2 - 14T + 1$
44	740	4818	$p^6T^4 + 740p^3T^3 + 4818pT^2 + 740T + 1$
45	-604	8082	$p^6T^4 - 604p^3T^3 + 8082pT^2 - 604T + 1$
46	-121	4301	$(pT + 1)(p^3T^2 - 188T + 1)$
47	-104	-3358	$p^6T^4 - 104p^3T^3 - 3358pT^2 - 104T + 1$
48	576	8546	$p^6T^4 + 576p^3T^3 + 8546pT^2 + 576T + 1$
49	-690	3754	$p^6T^4 - 690p^3T^3 + 3754pT^2 - 690T + 1$
50	156	-206	$p^6T^4 + 156p^3T^3 - 206pT^2 + 156T + 1$
51	96	-3038	$p^6T^4 + 96p^3T^3 - 3038pT^2 + 96T + 1$
52	118	3114	$p^6T^4 + 118p^3T^3 + 3114pT^2 + 118T + 1$
53	-84	6226	$p^6T^4 - 84p^3T^3 + 6226pT^2 - 84T + 1$
54	-346	330	$p^6T^4 - 346p^3T^3 + 330pT^2 - 346T + 1$
55	212	5810	$p^6T^4 + 212p^3T^3 + 5810pT^2 + 212T + 1$
56	764	4274	$p^6T^4 + 764p^3T^3 + 4274pT^2 + 764T + 1$
57	480	7906	$p^6T^4 + 480p^3T^3 + 7906pT^2 + 480T + 1$

$p = 67$ (cont.)

λ	a	b	$R(T)$
58	768	6242	$p^6T^4 + 768p^3T^3 + 6242pT^2 + 768T + 1$
59	-122	-1974	$p^6T^4 - 122p^3T^3 - 1974pT^2 - 122T + 1$
60	216	-3902	$p^6T^4 + 216p^3T^3 - 3902pT^2 + 216T + 1$
61	10	1370	$p^6T^4 + 10p^3T^3 + 1370pT^2 + 10T + 1$
62	876	9490	$p^6T^4 + 876p^3T^3 + 9490pT^2 + 876T + 1$
63	-1426	16202	$p^6T^4 - 1426p^3T^3 + 16202pT^2 - 1426T + 1$
64	-270	5050	$p^6T^4 - 270p^3T^3 + 5050pT^2 - 270T + 1$
65	-624	5954	$p^6T^4 - 624p^3T^3 + 5954pT^2 - 624T + 1$
66	-216	-3262	$p^6T^4 - 216p^3T^3 - 3262pT^2 - 216T + 1$

$p = 71$

λ	a	b	$R(T)$
1	400	4802	$p^6T^4 + 400p^3T^3 + 4802pT^2 + 400T + 1$
2	144	7490	$p^6T^4 + 144p^3T^3 + 7490pT^2 + 144T + 1$
3	-48	3746	$p^6T^4 - 48p^3T^3 + 3746pT^2 - 48T + 1$
4	360	5762	$p^6T^4 + 360p^3T^3 + 5762pT^2 + 360T + 1$
5	657	4313	$-(pT - 1)\,(p^3T^2 + 728T + 1)$
6	122	6514	$p^6T^4 + 122p^3T^3 + 6514pT^2 + 122T + 1$
7	144	-286	$p^6T^4 + 144p^3T^3 - 286pT^2 + 144T + 1$
8	548	7682	$p^6T^4 + 548p^3T^3 + 7682pT^2 + 548T + 1$
9	-140	3778	$p^6T^4 - 140p^3T^3 + 3778pT^2 - 140T + 1$
10	-60	-2302	$p^6T^4 - 60p^3T^3 - 2302pT^2 - 60T + 1$
11	-160	1442	$p^6T^4 - 160p^3T^3 + 1442pT^2 - 160T + 1$
12	-360	-2302	$p^6T^4 - 360p^3T^3 - 2302pT^2 - 360T + 1$
13	-136	5282	$p^6T^4 - 136p^3T^3 + 5282pT^2 - 136T + 1$
14	210	3666	$p^6T^4 + 210p^3T^3 + 3666pT^2 + 210T + 1$
15	-794	9074	$p^6T^4 - 794p^3T^3 + 9074pT^2 - 794T + 1$
16	-8	-2398	$p^6T^4 - 8p^3T^3 - 2398pT^2 - 8T + 1$
17	318	6834	$p^6T^4 + 318p^3T^3 + 6834pT^2 + 318T + 1$
18	-776	5474	$p^6T^4 - 776p^3T^3 + 5474pT^2 - 776T + 1$

$p = 71$ (cont.)

λ	a	b	$R(T)$
19	280	3554	$p^6T^4 + 280p^3T^3 + 3554pT^2 + 280T + 1$
20	152	-1886	$p^6T^4 + 152p^3T^3 - 1886pT^2 + 152T + 1$
21	768	7778	$p^6T^4 + 768p^3T^3 + 7778pT^2 + 768T + 1$
22	-256	8674	$p^6T^4 - 256p^3T^3 + 8674pT^2 - 256T + 1$
23	736	10658	$p^6T^4 + 736p^3T^3 + 10658pT^2 + 736T + 1$
24	576	10082	$p^6T^4 + 576p^3T^3 + 10082pT^2 + 576T + 1$
25	1418	15890	$p^6T^4 + 1418p^3T^3 + 15890pT^2 + 1418T + 1$
26	1032	11106	$p^6T^4 + 1032p^3T^3 + 11106pT^2 + 1032T + 1$
27	-232	5954	$p^6T^4 - 232p^3T^3 + 5954pT^2 - 232T + 1$
28	768	6050	$p^6T^4 + 768p^3T^3 + 6050pT^2 + 768T + 1$
29	-518	9458	$p^6T^4 - 518p^3T^3 + 9458pT^2 - 518T + 1$
30	-746	7762	$p^6T^4 - 746p^3T^3 + 7762pT^2 - 746T + 1$
31	-799	5769	$-(pT - 1)(p^3T^2 - 728T + 1)$
32	304	-478	$p^6T^4 + 304p^3T^3 - 478pT^2 + 304T + 1$
33	-300	6594	$p^6T^4 - 300p^3T^3 + 6594pT^2 - 300T + 1$
34	-482	5906	$p^6T^4 - 482p^3T^3 + 5906pT^2 - 482T + 1$
35	374	1394	$p^6T^4 + 374p^3T^3 + 1394pT^2 + 374T + 1$
36	56	-6686	$p^6T^4 + 56p^3T^3 - 6686pT^2 + 56T + 1$
37	-80	4514	$p^6T^4 - 80p^3T^3 + 4514pT^2 - 80T + 1$
38	572	6434	$p^6T^4 + 572p^3T^3 + 6434pT^2 + 572T + 1$
39	48	290	$p^6T^4 + 48p^3T^3 + 290pT^2 + 48T + 1$
40	-232	2786	$p^6T^4 - 232p^3T^3 + 2786pT^2 - 232T + 1$
41	542	6770	$p^6T^4 + 542p^3T^3 + 6770pT^2 + 542T + 1$
42	264	2594	$p^6T^4 + 264p^3T^3 + 2594pT^2 + 264T + 1$
43	-64	-3934	$p^6T^4 - 64p^3T^3 - 3934pT^2 - 64T + 1$
44	-592	4898	$p^6T^4 - 592p^3T^3 + 4898pT^2 - 592T + 1$
45	-226	-4846	$p^6T^4 - 226p^3T^3 - 4846pT^2 - 226T + 1$
46	-476	9986	$p^6T^4 - 476p^3T^3 + 9986pT^2 - 476T + 1$
47	-832	7970	$p^6T^4 - 832p^3T^3 + 7970pT^2 - 832T + 1$
48	-174	7058	$p^6T^4 - 174p^3T^3 + 7058pT^2 - 174T + 1$
49	912	12834	$p^6T^4 + 912p^3T^3 + 12834pT^2 + 912T + 1$
50	-684	10818	$p^6T^4 - 684p^3T^3 + 10818pT^2 - 684T + 1$
51	-924	7490	$p^6T^4 - 924p^3T^3 + 7490pT^2 - 924T + 1$

$p = 71$ (cont.)

λ	a	b	$R(T)$
52	808	9122	$p^6T^4 + 808p^3T^3 + 9122pT^2 + 808T + 1$
53	-176	-2398	$p^6T^4 - 176p^3T^3 - 2398pT^2 - 176T + 1$
54	-682	6866	$p^6T^4 - 682p^3T^3 + 6866pT^2 - 682T + 1$
55	810	11378	$p^6T^4 + 810p^3T^3 + 11378pT^2 + 810T + 1$
56	-64	1058	$p^6T^4 - 64p^3T^3 + 1058pT^2 - 64T + 1$
57	812	8450	$p^6T^4 + 812p^3T^3 + 8450pT^2 + 812T + 1$
58	-144	-5214	$p^6T^4 - 144p^3T^3 - 5214pT^2 - 144T + 1$
59	-16	-94	$p^6T^4 - 16p^3T^3 - 94pT^2 - 16T + 1$
60	1328	13730	$p^6T^4 + 1328p^3T^3 + 13730pT^2 + 1328T + 1$
61	-24	6882	$p^6T^4 - 24p^3T^3 + 6882pT^2 - 24T + 1$
62	-908	12674	$p^6T^4 - 908p^3T^3 + 12674pT^2 - 908T + 1$
63	1042	12338	$p^6T^4 + 1042p^3T^3 + 12338pT^2 + 1042T + 1$
64	-152	6530	$p^6T^4 - 152p^3T^3 + 6530pT^2 - 152T + 1$
65	-76	-382	$p^6T^4 - 76p^3T^3 - 382pT^2 - 76T + 1$
66	296	-1054	$p^6T^4 + 296p^3T^3 - 1054pT^2 + 296T + 1$
67	116	-1022	$p^6T^4 + 116p^3T^3 - 1022pT^2 + 116T + 1$
68	272	1250	$p^6T^4 + 272p^3T^3 + 1250pT^2 + 272T + 1$
69	-14	7634	$p^6T^4 - 14p^3T^3 + 7634pT^2 - 14T + 1$
70	208	3298	$p^6T^4 + 208p^3T^3 + 3298pT^2 + 208T + 1$

$p = 73$

λ	a	b	$R(T)$
1	-1144	14030	$p^6T^4 - 1144p^3T^3 + 14030pT^2 - 1144T + 1$
2	1116	12422	$p^6T^4 + 1116p^3T^3 + 12422pT^2 + 1116T + 1$
3	-376	-946	$p^6T^4 - 376p^3T^3 - 946pT^2 - 376T + 1$
4	-227	5483	$-(pT - 1)\,(p^3T^2 - 154T + 1)$
5	-532	3110	$p^6T^4 - 532p^3T^3 + 3110pT^2 - 532T + 1$
6	-376	2894	$p^6T^4 - 376p^3T^3 + 2894pT^2 - 376T + 1$
7	812	5142	$p^6T^4 + 812p^3T^3 + 5142pT^2 + 812T + 1$
8	12	3494	$p^6T^4 + 12p^3T^3 + 3494pT^2 + 12T + 1$

$p = 73$ (cont.)

λ	a	b	$R(T)$
9	-180	-5722	$p^6T^4 - 180p^3T^3 - 5722pT^2 - 180T + 1$
10	-72	8302	$p^6T^4 - 72p^3T^3 + 8302pT^2 - 72T + 1$
11	164	2054	$p^6T^4 + 164p^3T^3 + 2054pT^2 + 164T + 1$
12	-556	9302	$p^6T^4 - 556p^3T^3 + 9302pT^2 - 556T + 1$
13	-68	2454	$p^6T^4 - 68p^3T^3 + 2454pT^2 - 68T + 1$
14	-270	8146	$p^6T^4 - 270p^3T^3 + 8146pT^2 - 270T + 1$
15	-388	8694	$p^6T^4 - 388p^3T^3 + 8694pT^2 - 388T + 1$
16	-208	6206	$p^6T^4 - 208p^3T^3 + 6206pT^2 - 208T + 1$
17	126	7210	$p^6T^4 + 126p^3T^3 + 7210pT^2 + 126T + 1$
18	-568	3662	$p^6T^4 - 568p^3T^3 + 3662pT^2 - 568T + 1$
19	1338	13594	$p^6T^4 + 1338p^3T^3 + 13594pT^2 + 1338T + 1$
20	604	2006	$p^6T^4 + 604p^3T^3 + 2006pT^2 + 604T + 1$
21	-616	9294	$p^6T^4 - 616p^3T^3 + 9294pT^2 - 616T + 1$
22	-764	2934	$p^6T^4 - 764p^3T^3 + 2934pT^2 - 764T + 1$
23	436	2454	$p^6T^4 + 436p^3T^3 + 2454pT^2 + 436T + 1$
24	762	5146	$p^6T^4 + 762p^3T^3 + 5146pT^2 + 762T + 1$
25	-1360	14270	$p^6T^4 - 1360p^3T^3 + 14270pT^2 - 1360T + 1$
26	244	2438	$p^6T^4 + 244p^3T^3 + 2438pT^2 + 244T + 1$
27	1328	14654	$p^6T^4 + 1328p^3T^3 + 14654pT^2 + 1328T + 1$
28	-552	10030	$p^6T^4 - 552p^3T^3 + 10030pT^2 - 552T + 1$
29	672	5422	$p^6T^4 + 672p^3T^3 + 5422pT^2 + 672T + 1$
30	660	2630	$p^6T^4 + 660p^3T^3 + 2630pT^2 + 660T + 1$
31	-360	9454	$p^6T^4 - 360p^3T^3 + 9454pT^2 - 360T + 1$
32	-4	-2746	$p^6T^4 - 4p^3T^3 - 2746pT^2 - 4T + 1$
33	290	-1390	$p^6T^4 + 290p^3T^3 - 1390pT^2 + 290T + 1$
34	-724	11382	$p^6T^4 - 724p^3T^3 + 11382pT^2 - 724T + 1$
35	-164	9606	$p^6T^4 - 164p^3T^3 + 9606pT^2 - 164T + 1$
36	292	2742	$p^6T^4 + 292p^3T^3 + 2742pT^2 + 292T + 1$
37	-380	7158	$p^6T^4 - 380p^3T^3 + 7158pT^2 - 380T + 1$
38	548	6134	$p^6T^4 + 548p^3T^3 + 6134pT^2 + 548T + 1$
39	-328	5822	$p^6T^4 - 328p^3T^3 + 5822pT^2 - 328T + 1$
40	1052	11654	$p^6T^4 + 1052p^3T^3 + 11654pT^2 + 1052T + 1$
41	-227	5483	$-(pT - 1)\,(p^3T^2 - 154T + 1)$

$p = 73$ (cont.)

λ	a	b	$R(T)$
42	1384	15582	$p^6T^4 + 1384p^3T^3 + 15582pT^2 + 1384T + 1$
43	66	-5102	$p^6T^4 + 66p^3T^3 - 5102pT^2 + 66T + 1$
44	-546	1066	$p^6T^4 - 546p^3T^3 + 1066pT^2 - 546T + 1$
45	1204	11190	$p^6T^4 + 1204p^3T^3 + 11190pT^2 + 1204T + 1$
46	350	4914	$p^6T^4 + 350p^3T^3 + 4914pT^2 + 350T + 1$
47	-110	7442	$p^6T^4 - 110p^3T^3 + 7442pT^2 - 110T + 1$
48	-34	-4686	$p^6T^4 - 34p^3T^3 - 4686pT^2 - 34T + 1$
49	96	-2786	$p^6T^4 + 96p^3T^3 - 2786pT^2 + 96T + 1$
50	-204	-1258	$p^6T^4 - 204p^3T^3 - 1258pT^2 - 204T + 1$
51	-464	3710	$p^6T^4 - 464p^3T^3 + 3710pT^2 - 464T + 1$
52	-340	3350	$p^6T^4 - 340p^3T^3 + 3350pT^2 - 340T + 1$
53	208	5454	$p^6T^4 + 208p^3T^3 + 5454pT^2 + 208T + 1$
54	432	5182	$p^6T^4 + 432p^3T^3 + 5182pT^2 + 432T + 1$
55	-22	8570	$p^6T^4 - 22p^3T^3 + 8570pT^2 - 22T + 1$
56	504	-146	$p^6T^4 + 504p^3T^3 - 146pT^2 + 504T + 1$
57	-228	3206	$p^6T^4 - 228p^3T^3 + 3206pT^2 - 228T + 1$
58	-552	4078	$p^6T^4 - 552p^3T^3 + 4078pT^2 - 552T + 1$
59	-228	-3130	$p^6T^4 - 228p^3T^3 - 3130pT^2 - 228T + 1$
60	510	7402	$p^6T^4 + 510p^3T^3 + 7402pT^2 + 510T + 1$
61	412	6150	$p^6T^4 + 412p^3T^3 + 6150pT^2 + 412T + 1$
62	100	6374	$p^6T^4 + 100p^3T^3 + 6374pT^2 + 100T + 1$
63	176	-4962	$p^6T^4 + 176p^3T^3 - 4962pT^2 + 176T + 1$
64	632	8558	$p^6T^4 + 632p^3T^3 + 8558pT^2 + 632T + 1$
65	-16	8894	$p^6T^4 - 16p^3T^3 + 8894pT^2 - 16T + 1$
66	606	-86	$p^6T^4 + 606p^3T^3 - 86pT^2 + 606T + 1$
67	-226	306	$p^6T^4 - 226p^3T^3 + 306pT^2 - 226T + 1$
68	268	-3130	$p^6T^4 + 268p^3T^3 - 3130pT^2 + 268T + 1$
69	908	8102	$p^6T^4 + 908p^3T^3 + 8102pT^2 + 908T + 1$
70	926	7218	$p^6T^4 + 926p^3T^3 + 7218pT^2 + 926T + 1$
71	296	5774	$p^6T^4 + 296p^3T^3 + 5774pT^2 + 296T + 1$
72	-790	10106	$p^6T^4 - 790p^3T^3 + 10106pT^2 - 790T + 1$

$p = 79$

λ	a	b	$R(T)$
1	-568	4034	$p^6T^4 - 568p^3T^3 + 4034pT^2 - 568T + 1$
2	422	-702	$p^6T^4 + 422p^3T^3 - 702pT^2 + 422T + 1$
3	1194	15586	$p^6T^4 + 1194p^3T^3 + 15586pT^2 + 1194T + 1$
4	600	898	$p^6T^4 + 600p^3T^3 + 898pT^2 + 600T + 1$
5	-256	-4798	$p^6T^4 - 256p^3T^3 - 4798pT^2 - 256T + 1$
6	-976	10626	$p^6T^4 - 976p^3T^3 + 10626pT^2 - 976T + 1$
7	356	7106	$p^6T^4 + 356p^3T^3 + 7106pT^2 + 356T + 1$
8	-616	4290	$p^6T^4 - 616p^3T^3 + 4290pT^2 - 616T + 1$
9	-1314	15650	$p^6T^4 - 1314p^3T^3 + 15650pT^2 - 1314T + 1$
10	1144	12290	$p^6T^4 + 1144p^3T^3 + 12290pT^2 + 1144T + 1$
11	336	6082	$p^6T^4 + 336p^3T^3 + 6082pT^2 + 336T + 1$
12	-840	5698	$p^6T^4 - 840p^3T^3 + 5698pT^2 - 840T + 1$
13	428	-1726	$p^6T^4 + 428p^3T^3 - 1726pT^2 + 428T + 1$
14	-1348	14210	$p^6T^4 - 1348p^3T^3 + 14210pT^2 - 1348T + 1$
15	-202	-7806	$p^6T^4 - 202p^3T^3 - 7806pT^2 - 202T + 1$
16	-692	11522	$p^6T^4 - 692p^3T^3 + 11522pT^2 - 692T + 1$
17	-250	9282	$p^6T^4 - 250p^3T^3 + 9282pT^2 - 250T + 1$
18	-160	5570	$p^6T^4 - 160p^3T^3 + 5570pT^2 - 160T + 1$
19	-52	6114	$p^6T^4 - 52p^3T^3 + 6114pT^2 - 52T + 1$
20	368	10946	$p^6T^4 + 368p^3T^3 + 10946pT^2 + 368T + 1$
21	120	962	$p^6T^4 + 120p^3T^3 + 962pT^2 + 120T + 1$
22	146	-8862	$p^6T^4 + 146p^3T^3 - 8862pT^2 + 146T + 1$
23	134	482	$p^6T^4 + 134p^3T^3 + 482pT^2 + 134T + 1$
24	48	6466	$p^6T^4 + 48p^3T^3 + 6466pT^2 + 48T + 1$
25	-1020	15362	$p^6T^4 - 1020p^3T^3 + 15362pT^2 - 1020T + 1$
26	744	4738	$p^6T^4 + 744p^3T^3 + 4738pT^2 + 744T + 1$
27	-208	5186	$p^6T^4 - 208p^3T^3 + 5186pT^2 - 208T + 1$
28	-120	5314	$p^6T^4 - 120p^3T^3 + 5314pT^2 - 120T + 1$
29	944	9026	$p^6T^4 + 944p^3T^3 + 9026pT^2 + 944T + 1$
30	-464	9410	$p^6T^4 - 464p^3T^3 + 9410pT^2 - 464T + 1$
31	-144	4354	$p^6T^4 - 144p^3T^3 + 4354pT^2 - 144T + 1$
32	844	8642	$p^6T^4 + 844p^3T^3 + 8642pT^2 + 844T + 1$
33	1182	9730	$p^6T^4 + 1182p^3T^3 + 9730pT^2 + 1182T + 1$

$p = 79$ (cont.)

λ	a	b	$R(T)$
34	592	6338	$p^6T^4 + 592p^3T^3 + 6338pT^2 + 592T + 1$
35	606	6370	$p^6T^4 + 606p^3T^3 + 6370pT^2 + 606T + 1$
36	942	14914	$p^6T^4 + 942p^3T^3 + 14914pT^2 + 942T + 1$
37	-456	3842	$p^6T^4 - 456p^3T^3 + 3842pT^2 - 456T + 1$
38	326	5538	$p^6T^4 + 326p^3T^3 + 5538pT^2 + 326T + 1$
39	-152	-702	$p^6T^4 - 152p^3T^3 - 702pT^2 - 152T + 1$
40	728	7682	$p^6T^4 + 728p^3T^3 + 7682pT^2 + 728T + 1$
41	-844	3266	$p^6T^4 - 844p^3T^3 + 3266pT^2 - 844T + 1$
42	-272	-6558	$p^6T^4 - 272p^3T^3 - 6558pT^2 - 272T + 1$
43	1692	19522	$p^6T^4 + 1692p^3T^3 + 19522pT^2 + 1692T + 1$
44	-1468	18434	$p^6T^4 - 1468p^3T^3 + 18434pT^2 - 1468T + 1$
45	-424	450	$p^6T^4 - 424p^3T^3 + 450pT^2 - 424T + 1$
46	-320	1410	$p^6T^4 - 320p^3T^3 + 1410pT^2 - 320T + 1$
47	-442	11394	$p^6T^4 - 442p^3T^3 + 11394pT^2 - 442T + 1$
48	16	-3390	$p^6T^4 + 16p^3T^3 - 3390pT^2 + 16T + 1$
49	916	11714	$p^6T^4 + 916p^3T^3 + 11714pT^2 + 916T + 1$
50	-577	5585	$(pT + 1)\,(p^3T^2 - 656T + 1)$
51	102	-926	$p^6T^4 + 102p^3T^3 - 926pT^2 + 102T + 1$
52	178	642	$p^6T^4 + 178p^3T^3 + 642pT^2 + 178T + 1$
53	1432	18050	$p^6T^4 + 1432p^3T^3 + 18050pT^2 + 1432T + 1$
54	-112	7682	$p^6T^4 - 112p^3T^3 + 7682pT^2 - 112T + 1$
55	702	12802	$p^6T^4 + 702p^3T^3 + 12802pT^2 + 702T + 1$
56	872	3906	$p^6T^4 + 872p^3T^3 + 3906pT^2 + 872T + 1$
57	340	5826	$p^6T^4 + 340p^3T^3 + 5826pT^2 + 340T + 1$
58	840	8194	$p^6T^4 + 840p^3T^3 + 8194pT^2 + 840T + 1$
59	-1158	11842	$p^6T^4 - 1158p^3T^3 + 11842pT^2 - 1158T + 1$
60	-234	2978	$p^6T^4 - 234p^3T^3 + 2978pT^2 - 234T + 1$
61	-980	6914	$p^6T^4 - 980p^3T^3 + 6914pT^2 - 980T + 1$
62	-1040	7746	$p^6T^4 - 1040p^3T^3 + 7746pT^2 - 1040T + 1$
63	436	5378	$p^6T^4 + 436p^3T^3 + 5378pT^2 + 436T + 1$
64	1060	8994	$p^6T^4 + 1060p^3T^3 + 8994pT^2 + 1060T + 1$
65	796	6050	$p^6T^4 + 796p^3T^3 + 6050pT^2 + 796T + 1$
66	-160	-1918	$p^6T^4 - 160p^3T^3 - 1918pT^2 - 160T + 1$

$p = 79$ (cont.)

λ	a	b	$R(T)$
67	-320	386	$p^6T^4 - 320p^3T^3 + 386pT^2 - 320T + 1$
68	-10	5570	$p^6T^4 - 10p^3T^3 + 5570pT^2 - 10T + 1$
69	-1524	15362	$p^6T^4 - 1524p^3T^3 + 15362pT^2 - 1524T + 1$
70	84	-2366	$p^6T^4 + 84p^3T^3 - 2366pT^2 + 84T + 1$
71	766	8226	$p^6T^4 + 766p^3T^3 + 8226pT^2 + 766T + 1$
72	510	10114	$p^6T^4 + 510p^3T^3 + 10114pT^2 + 510T + 1$
73	686	3074	$p^6T^4 + 686p^3T^3 + 3074pT^2 + 686T + 1$
74	735	6897	$(pT + 1)(p^3T^2 + 656T + 1)$
75	814	2210	$p^6T^4 + 814p^3T^3 + 2210pT^2 + 814T + 1$
76	132	2626	$p^6T^4 + 132p^3T^3 + 2626pT^2 + 132T + 1$
77	-432	1666	$p^6T^4 - 432p^3T^3 + 1666pT^2 - 432T + 1$
78	1084	14978	$p^6T^4 + 1084p^3T^3 + 14978pT^2 + 1084T + 1$

$p = 83$

λ	a	b	$R(T)$
1	340	5618	$p^6T^4 + 340p^3T^3 + 5618pT^2 + 340T + 1$
2	476	3506	$p^6T^4 + 476p^3T^3 + 3506pT^2 + 476T + 1$
3	-460	-1198	$p^6T^4 - 460p^3T^3 - 1198pT^2 - 460T + 1$
4	-128	-6142	$p^6T^4 - 128p^3T^3 - 6142pT^2 - 128T + 1$
5	-1176	14210	$p^6T^4 - 1176p^3T^3 + 14210pT^2 - 1176T + 1$
6	1032	13506	$p^6T^4 + 1032p^3T^3 + 13506pT^2 + 1032T + 1$
7	1408	17762	$p^6T^4 + 1408p^3T^3 + 17762pT^2 + 1408T + 1$
8	312	-478	$p^6T^4 + 312p^3T^3 - 478pT^2 + 312T + 1$
9	-1928	21506	$p^6T^4 - 1928p^3T^3 + 21506pT^2 - 1928T + 1$
10	-352	3938	$p^6T^4 - 352p^3T^3 + 3938pT^2 - 352T + 1$
11	-664	2114	$p^6T^4 - 664p^3T^3 + 2114pT^2 - 664T + 1$
12	-544	11714	$p^6T^4 - 544p^3T^3 + 11714pT^2 - 544T + 1$
13	388	12146	$p^6T^4 + 388p^3T^3 + 12146pT^2 + 388T + 1$
14	306	602	$p^6T^4 + 306p^3T^3 + 602pT^2 + 306T + 1$
15	38	10826	$p^6T^4 + 38p^3T^3 + 10826pT^2 + 38T + 1$

$p = 83$ (cont.)

λ	a	b	$R(T)$
16	188	1202	$p^6T^4 + 188p^3T^3 + 1202pT^2 + 188T + 1$
17	498	13274	$p^6T^4 + 498p^3T^3 + 13274pT^2 + 498T + 1$
18	784	9250	$p^6T^4 + 784p^3T^3 + 9250pT^2 + 784T + 1$
19	288	1826	$p^6T^4 + 288p^3T^3 + 1826pT^2 + 288T + 1$
20	236	5714	$p^6T^4 + 236p^3T^3 + 5714pT^2 + 236T + 1$
21	−1306	18250	$p^6T^4 - 1306p^3T^3 + 18250pT^2 - 1306T + 1$
22	−376	8834	$p^6T^4 - 376p^3T^3 + 8834pT^2 - 376T + 1$
23	1408	14498	$p^6T^4 + 1408p^3T^3 + 14498pT^2 + 1408T + 1$
24	−319	7125	$-(pT-1)\,(p^3T^2 - 236T + 1)$
25	−312	−2494	$p^6T^4 - 312p^3T^3 - 2494pT^2 - 312T + 1$
26	34	506	$p^6T^4 + 34p^3T^3 + 506pT^2 + 34T + 1$
27	−232	9122	$p^6T^4 - 232p^3T^3 + 9122pT^2 - 232T + 1$
28	−536	12962	$p^6T^4 - 536p^3T^3 + 12962pT^2 - 536T + 1$
29	26	−2566	$p^6T^4 + 26p^3T^3 - 2566pT^2 + 26T + 1$
30	154	−70	$p^6T^4 + 154p^3T^3 - 70pT^2 + 154T + 1$
31	−8	1538	$p^6T^4 - 8p^3T^3 + 1538pT^2 - 8T + 1$
32	488	8450	$p^6T^4 + 488p^3T^3 + 8450pT^2 + 488T + 1$
33	−688	12962	$p^6T^4 - 688p^3T^3 + 12962pT^2 - 688T + 1$
34	−920	12226	$p^6T^4 - 920p^3T^3 + 12226pT^2 - 920T + 1$
35	52	4178	$p^6T^4 + 52p^3T^3 + 4178pT^2 + 52T + 1$
36	332	−2158	$p^6T^4 + 332p^3T^3 - 2158pT^2 + 332T + 1$
37	560	5090	$p^6T^4 + 560p^3T^3 + 5090pT^2 + 560T + 1$
38	−1052	16082	$p^6T^4 - 1052p^3T^3 + 16082pT^2 - 1052T + 1$
39	1690	19514	$p^6T^4 + 1690p^3T^3 + 19514pT^2 + 1690T + 1$
40	380	6770	$p^6T^4 + 380p^3T^3 + 6770pT^2 + 380T + 1$
41	1804	21842	$p^6T^4 + 1804p^3T^3 + 21842pT^2 + 1804T + 1$
42	806	8074	$p^6T^4 + 806p^3T^3 + 8074pT^2 + 806T + 1$
43	−402	3050	$p^6T^4 - 402p^3T^3 + 3050pT^2 - 402T + 1$
44	964	7730	$p^6T^4 + 964p^3T^3 + 7730pT^2 + 964T + 1$
45	848	9058	$p^6T^4 + 848p^3T^3 + 9058pT^2 + 848T + 1$
46	328	−2590	$p^6T^4 + 328p^3T^3 - 2590pT^2 + 328T + 1$
47	−688	3682	$p^6T^4 - 688p^3T^3 + 3682pT^2 - 688T + 1$
48	−322	−118	$p^6T^4 - 322p^3T^3 - 118pT^2 - 322T + 1$

$p = 83$ (cont.)

λ	a	b	$R(T)$
49	-560	-1342	$p^6T^4 - 560p^3T^3 - 1342pT^2 - 560T + 1$
50	904	6850	$p^6T^4 + 904p^3T^3 + 6850pT^2 + 904T + 1$
51	-136	-6014	$p^6T^4 - 136p^3T^3 - 6014pT^2 - 136T + 1$
52	-744	5570	$p^6T^4 - 744p^3T^3 + 5570pT^2 - 744T + 1$
53	888	10754	$p^6T^4 + 888p^3T^3 + 10754pT^2 + 888T + 1$
54	-360	4418	$p^6T^4 - 360p^3T^3 + 4418pT^2 - 360T + 1$
55	-488	5570	$p^6T^4 - 488p^3T^3 + 5570pT^2 - 488T + 1$
56	-404	466	$p^6T^4 - 404p^3T^3 + 466pT^2 - 404T + 1$
57	-319	7125	$-(pT - 1)\,(p^3T^2 - 236T + 1)$
58	326	3722	$p^6T^4 + 326p^3T^3 + 3722pT^2 + 326T + 1$
59	-334	506	$p^6T^4 - 334p^3T^3 + 506pT^2 - 334T + 1$
60	-260	5938	$p^6T^4 - 260p^3T^3 + 5938pT^2 - 260T + 1$
61	-776	11362	$p^6T^4 - 776p^3T^3 + 11362pT^2 - 776T + 1$
62	72	13698	$p^6T^4 + 72p^3T^3 + 13698pT^2 + 72T + 1$
63	426	7674	$p^6T^4 + 426p^3T^3 + 7674pT^2 + 426T + 1$
64	2008	24290	$p^6T^4 + 2008p^3T^3 + 24290pT^2 + 2008T + 1$
65	248	-478	$p^6T^4 + 248p^3T^3 - 478pT^2 + 248T + 1$
66	-306	-3862	$p^6T^4 - 306p^3T^3 - 3862pT^2 - 306T + 1$
67	1498	14138	$p^6T^4 + 1498p^3T^3 + 14138pT^2 + 1498T + 1$
68	216	-1630	$p^6T^4 + 216p^3T^3 - 1630pT^2 + 216T + 1$
69	-64	3650	$p^6T^4 - 64p^3T^3 + 3650pT^2 - 64T + 1$
70	478	-4342	$p^6T^4 + 478p^3T^3 - 4342pT^2 + 478T + 1$
71	-264	10178	$p^6T^4 - 264p^3T^3 + 10178pT^2 - 264T + 1$
72	480	6434	$p^6T^4 + 480p^3T^3 + 6434pT^2 + 480T + 1$
73	-1712	17570	$p^6T^4 - 1712p^3T^3 + 17570pT^2 - 1712T + 1$
74	-174	-5158	$p^6T^4 - 174p^3T^3 - 5158pT^2 - 174T + 1$
75	328	7106	$p^6T^4 + 328p^3T^3 + 7106pT^2 + 328T + 1$
76	-1376	12994	$p^6T^4 - 1376p^3T^3 + 12994pT^2 - 1376T + 1$
77	310	8650	$p^6T^4 + 310p^3T^3 + 8650pT^2 + 310T + 1$
78	848	2498	$p^6T^4 + 848p^3T^3 + 2498pT^2 + 848T + 1$
79	400	10850	$p^6T^4 + 400p^3T^3 + 10850pT^2 + 400T + 1$
80	1382	14986	$p^6T^4 + 1382p^3T^3 + 14986pT^2 + 1382T + 1$
81	810	8826	$p^6T^4 + 810p^3T^3 + 8826pT^2 + 810T + 1$

$p = 83$ (cont.)

λ	a	b	$R(T)$
82	-210	-5590	$p^6T^4 - 210p^3T^3 - 5590pT^2 - 210T + 1$

$p = 89$

λ	a	b	$R(T)$
1	1592	21038	$p^6T^4 + 1592p^3T^3 + 21038pT^2 + 1592T + 1$
2	758	5474	$p^6T^4 + 758p^3T^3 + 5474pT^2 + 758T + 1$
3	214	2170	$p^6T^4 + 214p^3T^3 + 2170pT^2 + 214T + 1$
4	1526	17762	$p^6T^4 + 1526p^3T^3 + 17762pT^2 + 1526T + 1$
5	992	9374	$p^6T^4 + 992p^3T^3 + 9374pT^2 + 992T + 1$
6	40	-3682	$p^6T^4 + 40p^3T^3 - 3682pT^2 + 40T + 1$
7	-100	4982	$p^6T^4 - 100p^3T^3 + 4982pT^2 - 100T + 1$
8	28	12550	$p^6T^4 + 28p^3T^3 + 12550pT^2 + 28T + 1$
9	1050	16506	$p^6T^4 + 1050p^3T^3 + 16506pT^2 + 1050T + 1$
10	-284	-8266	$p^6T^4 - 284p^3T^3 - 8266pT^2 - 284T + 1$
11	-1028	14662	$p^6T^4 - 1028p^3T^3 + 14662pT^2 - 1028T + 1$
12	-20	1334	$p^6T^4 - 20p^3T^3 + 1334pT^2 - 20T + 1$
13	-316	8390	$p^6T^4 - 316p^3T^3 + 8390pT^2 - 316T + 1$
14	220	-1690	$p^6T^4 + 220p^3T^3 - 1690pT^2 + 220T + 1$
15	-416	11486	$p^6T^4 - 416p^3T^3 + 11486pT^2 - 416T + 1$
16	-625	7207	$(pT + 1)\,(p^3T^2 - 714T + 1)$
17	460	8230	$p^6T^4 + 460p^3T^3 + 8230pT^2 + 460T + 1$
18	816	8318	$p^6T^4 + 816p^3T^3 + 8318pT^2 + 816T + 1$
19	-316	4070	$p^6T^4 - 316p^3T^3 + 4070pT^2 - 316T + 1$
20	-1340	17654	$p^6T^4 - 1340p^3T^3 + 17654pT^2 - 1340T + 1$
21	560	7550	$p^6T^4 + 560p^3T^3 + 7550pT^2 + 560T + 1$
22	278	5666	$p^6T^4 + 278p^3T^3 + 5666pT^2 + 278T + 1$
23	24	-5650	$p^6T^4 + 24p^3T^3 - 5650pT^2 + 24T + 1$
24	-680	14254	$p^6T^4 - 680p^3T^3 + 14254pT^2 - 680T + 1$
25	812	7526	$p^6T^4 + 812p^3T^3 + 7526pT^2 + 812T + 1$
26	-588	3462	$p^6T^4 - 588p^3T^3 + 3462pT^2 - 588T + 1$

$p = 89$ (cont.)

λ	a	b	$R(T)$
27	1848	21998	$p^6T^4 + 1848p^3T^3 + 21998pT^2 + 1848T + 1$
28	-690	4106	$p^6T^4 - 690p^3T^3 + 4106pT^2 - 690T + 1$
29	-1134	10674	$p^6T^4 - 1134p^3T^3 + 10674pT^2 - 1134T + 1$
30	-314	5018	$p^6T^4 - 314p^3T^3 + 5018pT^2 - 314T + 1$
31	-348	10118	$p^6T^4 - 348p^3T^3 + 10118pT^2 - 348T + 1$
32	908	7334	$p^6T^4 + 908p^3T^3 + 7334pT^2 + 908T + 1$
33	-80	12622	$p^6T^4 - 80p^3T^3 + 12622pT^2 - 80T + 1$
34	-108	8534	$p^6T^4 - 108p^3T^3 + 8534pT^2 - 108T + 1$
35	476	1462	$p^6T^4 + 476p^3T^3 + 1462pT^2 + 476T + 1$
36	1512	18510	$p^6T^4 + 1512p^3T^3 + 18510pT^2 + 1512T + 1$
37	668	9206	$p^6T^4 + 668p^3T^3 + 9206pT^2 + 668T + 1$
38	-1172	18230	$p^6T^4 - 1172p^3T^3 + 18230pT^2 - 1172T + 1$
39	-76	-8554	$p^6T^4 - 76p^3T^3 - 8554pT^2 - 76T + 1$
40	676	-1738	$p^6T^4 + 676p^3T^3 - 1738pT^2 + 676T + 1$
41	-212	902	$p^6T^4 - 212p^3T^3 + 902pT^2 - 212T + 1$
42	-1080	14862	$p^6T^4 - 1080p^3T^3 + 14862pT^2 - 1080T + 1$
43	-452	-778	$p^6T^4 - 452p^3T^3 - 778pT^2 - 452T + 1$
44	-202	8930	$p^6T^4 - 202p^3T^3 + 8930pT^2 - 202T + 1$
45	336	15806	$p^6T^4 + 336p^3T^3 + 15806pT^2 + 336T + 1$
46	1026	11250	$p^6T^4 + 1026p^3T^3 + 11250pT^2 + 1026T + 1$
47	-154	1346	$p^6T^4 - 154p^3T^3 + 1346pT^2 - 154T + 1$
48	484	-746	$p^6T^4 + 484p^3T^3 - 746pT^2 + 484T + 1$
49	176	7550	$p^6T^4 + 176p^3T^3 + 7550pT^2 + 176T + 1$
50	-625	7207	$(pT + 1)\,(p^3T^2 - 714T + 1)$
51	-258	4394	$p^6T^4 - 258p^3T^3 + 4394pT^2 - 258T + 1$
52	272	1646	$p^6T^4 + 272p^3T^3 + 1646pT^2 + 272T + 1$
53	1862	22658	$p^6T^4 + 1862p^3T^3 + 22658pT^2 + 1862T + 1$
54	-114	4682	$p^6T^4 - 114p^3T^3 + 4682pT^2 - 114T + 1$
55	24	15086	$p^6T^4 + 24p^3T^3 + 15086pT^2 + 24T + 1$
56	72	-4498	$p^6T^4 + 72p^3T^3 - 4498pT^2 + 72T + 1$
57	326	578	$p^6T^4 + 326p^3T^3 + 578pT^2 + 326T + 1$
58	1488	13934	$p^6T^4 + 1488p^3T^3 + 13934pT^2 + 1488T + 1$
59	-10	5242	$p^6T^4 - 10p^3T^3 + 5242pT^2 - 10T + 1$

$p = 89$ (cont.)

λ	a	b	$R(T)$
60	164	−5770	$p^6T^4 + 164p^3T^3 - 5770pT^2 + 164T + 1$
61	52	−10826	$p^6T^4 + 52p^3T^3 - 10826pT^2 + 52T + 1$
62	1496	17038	$p^6T^4 + 1496p^3T^3 + 17038pT^2 + 1496T + 1$
63	968	7918	$p^6T^4 + 968p^3T^3 + 7918pT^2 + 968T + 1$
64	−1306	11522	$p^6T^4 - 1306p^3T^3 + 11522pT^2 - 1306T + 1$
65	660	−954	$p^6T^4 + 660p^3T^3 - 954pT^2 + 660T + 1$
66	156	−5434	$p^6T^4 + 156p^3T^3 - 5434pT^2 + 156T + 1$
67	−888	15374	$p^6T^4 - 888p^3T^3 + 15374pT^2 - 888T + 1$
68	412	13510	$p^6T^4 + 412p^3T^3 + 13510pT^2 + 412T + 1$
69	−784	5246	$p^6T^4 - 784p^3T^3 + 5246pT^2 - 784T + 1$
70	754	14834	$p^6T^4 + 754p^3T^3 + 14834pT^2 + 754T + 1$
71	704	4958	$p^6T^4 + 704p^3T^3 + 4958pT^2 + 704T + 1$
72	−448	11294	$p^6T^4 - 448p^3T^3 + 11294pT^2 - 448T + 1$
73	−316	−778	$p^6T^4 - 316p^3T^3 - 778pT^2 - 316T + 1$
74	2336	30494	$p^6T^4 + 2336p^3T^3 + 30494pT^2 + 2336T + 1$
75	674	5234	$p^6T^4 + 674p^3T^3 + 5234pT^2 + 674T + 1$
76	−580	15190	$p^6T^4 - 580p^3T^3 + 15190pT^2 - 580T + 1$
77	−1064	17774	$p^6T^4 - 1064p^3T^3 + 17774pT^2 - 1064T + 1$
78	−670	4970	$p^6T^4 - 670p^3T^3 + 4970pT^2 - 670T + 1$
79	−198	7482	$p^6T^4 - 198p^3T^3 + 7482pT^2 - 198T + 1$
80	−1052	13622	$p^6T^4 - 1052p^3T^3 + 13622pT^2 - 1052T + 1$
81	98	14186	$p^6T^4 + 98p^3T^3 + 14186pT^2 + 98T + 1$
82	−738	6122	$p^6T^4 - 738p^3T^3 + 6122pT^2 - 738T + 1$
83	−352	11870	$p^6T^4 - 352p^3T^3 + 11870pT^2 - 352T + 1$
84	−336	10622	$p^6T^4 - 336p^3T^3 + 10622pT^2 - 336T + 1$
85	−312	5006	$p^6T^4 - 312p^3T^3 + 5006pT^2 - 312T + 1$
86	−250	10586	$p^6T^4 - 250p^3T^3 + 10586pT^2 - 250T + 1$
87	432	−2050	$p^6T^4 + 432p^3T^3 - 2050pT^2 + 432T + 1$
88	−472	8654	$p^6T^4 - 472p^3T^3 + 8654pT^2 - 472T + 1$

$p = 97$

λ	a	b	$R(T)$
1	-618	-3566	$p^6T^4 - 618p^3T^3 - 3566pT^2 - 618T + 1$
2	-320	14334	$p^6T^4 - 320p^3T^3 + 14334pT^2 - 320T + 1$
3	2084	26486	$p^6T^4 + 2084p^3T^3 + 26486pT^2 + 2084T + 1$
4	88	462	$p^6T^4 + 88p^3T^3 + 462pT^2 + 88T + 1$
5	-1590	22354	$p^6T^4 - 1590p^3T^3 + 22354pT^2 - 1590T + 1$
6	381	8931	$-(pT-1)(p^3T^2 + 478T + 1)$
7	-1444	19014	$p^6T^4 - 1444p^3T^3 + 19014pT^2 - 1444T + 1$
8	320	-6402	$p^6T^4 + 320p^3T^3 - 6402pT^2 + 320T + 1$
9	-232	5966	$p^6T^4 - 232p^3T^3 + 5966pT^2 - 232T + 1$
10	64	-3474	$p^6T^4 + 64p^3T^3 - 3474pT^2 + 64T + 1$
11	-56	-2706	$p^6T^4 - 56p^3T^3 - 2706pT^2 - 56T + 1$
12	1312	18494	$p^6T^4 + 1312p^3T^3 + 18494pT^2 + 1312T + 1$
13	576	11582	$p^6T^4 + 576p^3T^3 + 11582pT^2 + 576T + 1$
14	810	7858	$p^6T^4 + 810p^3T^3 + 7858pT^2 + 810T + 1$
15	-374	6738	$p^6T^4 - 374p^3T^3 + 6738pT^2 - 374T + 1$
16	-506	13106	$p^6T^4 - 506p^3T^3 + 13106pT^2 - 506T + 1$
17	-534	-974	$p^6T^4 - 534p^3T^3 - 974pT^2 - 534T + 1$
18	1212	12550	$p^6T^4 + 1212p^3T^3 + 12550pT^2 + 1212T + 1$
19	-1476	12022	$p^6T^4 - 1476p^3T^3 + 12022pT^2 - 1476T + 1$
20	-148	-1930	$p^6T^4 - 148p^3T^3 - 1930pT^2 - 148T + 1$
21	592	8798	$p^6T^4 + 592p^3T^3 + 8798pT^2 + 592T + 1$
22	64	5886	$p^6T^4 + 64p^3T^3 + 5886pT^2 + 64T + 1$
23	-1332	7894	$p^6T^4 - 1332p^3T^3 + 7894pT^2 - 1332T + 1$
24	-420	8518	$p^6T^4 - 420p^3T^3 + 8518pT^2 - 420T + 1$
25	984	14030	$p^6T^4 + 984p^3T^3 + 14030pT^2 + 984T + 1$
26	-24	8270	$p^6T^4 - 24p^3T^3 + 8270pT^2 - 24T + 1$
27	-122	13106	$p^6T^4 - 122p^3T^3 + 13106pT^2 - 122T + 1$
28	-700	9750	$p^6T^4 - 700p^3T^3 + 9750pT^2 - 700T + 1$
29	-488	5358	$p^6T^4 - 488p^3T^3 + 5358pT^2 - 488T + 1$
30	702	10906	$p^6T^4 + 702p^3T^3 + 10906pT^2 + 702T + 1$
31	-212	-7258	$p^6T^4 - 212p^3T^3 - 7258pT^2 - 212T + 1$
32	126	-3518	$p^6T^4 + 126p^3T^3 - 3518pT^2 + 126T + 1$
33	1000	6830	$p^6T^4 + 1000p^3T^3 + 6830pT^2 + 1000T + 1$

$p = 97$ (cont.)

λ	a	b	$R(T)$
34	868	7494	$p^6T^4 + 868p^3T^3 + 7494pT^2 + 868T + 1$
35	-402	7522	$p^6T^4 - 402p^3T^3 + 7522pT^2 - 402T + 1$
36	-320	-9730	$p^6T^4 - 320p^3T^3 - 9730pT^2 - 320T + 1$
37	-568	1886	$p^6T^4 - 568p^3T^3 + 1886pT^2 - 568T + 1$
38	286	6906	$p^6T^4 + 286p^3T^3 + 6906pT^2 + 286T + 1$
39	432	-6130	$p^6T^4 + 432p^3T^3 - 6130pT^2 + 432T + 1$
40	-448	14142	$p^6T^4 - 448p^3T^3 + 14142pT^2 - 448T + 1$
41	628	566	$p^6T^4 + 628p^3T^3 + 566pT^2 + 628T + 1$
42	686	3450	$p^6T^4 + 686p^3T^3 + 3450pT^2 + 686T + 1$
43	14	6818	$p^6T^4 + 14p^3T^3 + 6818pT^2 + 14T + 1$
44	-1578	22162	$p^6T^4 - 1578p^3T^3 + 22162pT^2 - 1578T + 1$
45	1352	17454	$p^6T^4 + 1352p^3T^3 + 17454pT^2 + 1352T + 1$
46	884	10982	$p^6T^4 + 884p^3T^3 + 10982pT^2 + 884T + 1$
47	980	16982	$p^6T^4 + 980p^3T^3 + 16982pT^2 + 980T + 1$
48	-664	1838	$p^6T^4 - 664p^3T^3 + 1838pT^2 - 664T + 1$
49	240	-3746	$p^6T^4 + 240p^3T^3 - 3746pT^2 + 240T + 1$
50	432	7198	$p^6T^4 + 432p^3T^3 + 7198pT^2 + 432T + 1$
51	-512	15326	$p^6T^4 - 512p^3T^3 + 15326pT^2 - 512T + 1$
52	2616	35566	$p^6T^4 + 2616p^3T^3 + 35566pT^2 + 2616T + 1$
53	-1792	16830	$p^6T^4 - 1792p^3T^3 + 16830pT^2 - 1792T + 1$
54	-1600	25086	$p^6T^4 - 1600p^3T^3 + 25086pT^2 - 1600T + 1$
55	778	15026	$p^6T^4 + 778p^3T^3 + 15026pT^2 + 778T + 1$
56	1040	9422	$p^6T^4 + 1040p^3T^3 + 9422pT^2 + 1040T + 1$
57	1372	11094	$p^6T^4 + 1372p^3T^3 + 11094pT^2 + 1372T + 1$
58	-592	2334	$p^6T^4 - 592p^3T^3 + 2334pT^2 - 592T + 1$
59	650	15218	$p^6T^4 + 650p^3T^3 + 15218pT^2 + 650T + 1$
60	-1460	15974	$p^6T^4 - 1460p^3T^3 + 15974pT^2 - 1460T + 1$
61	-828	15478	$p^6T^4 - 828p^3T^3 + 15478pT^2 - 828T + 1$
62	-258	-446	$p^6T^4 - 258p^3T^3 - 446pT^2 - 258T + 1$
63	184	14126	$p^6T^4 + 184p^3T^3 + 14126pT^2 + 184T + 1$
64	1132	19238	$p^6T^4 + 1132p^3T^3 + 19238pT^2 + 1132T + 1$
65	1134	11362	$p^6T^4 + 1134p^3T^3 + 11362pT^2 + 1134T + 1$
66	-152	18350	$p^6T^4 - 152p^3T^3 + 18350pT^2 - 152T + 1$

$p = 97$ (cont.)

λ	a	b	$R(T)$
67	330	10450	$p^6T^4 + 330p^3T^3 + 10450pT^2 + 330T + 1$
68	-672	12910	$p^6T^4 - 672p^3T^3 + 12910pT^2 - 672T + 1$
69	-896	-66	$p^6T^4 - 896p^3T^3 - 66pT^2 - 896T + 1$
70	-324	17542	$p^6T^4 - 324p^3T^3 + 17542pT^2 - 324T + 1$
71	136	14238	$p^6T^4 + 136p^3T^3 + 14238pT^2 + 136T + 1$
72	381	8931	$-(pT - 1)(p^3T^2 + 478T + 1)$
73	816	8734	$p^6T^4 + 816p^3T^3 + 8734pT^2 + 816T + 1$
74	-84	-3434	$p^6T^4 - 84p^3T^3 - 3434pT^2 - 84T + 1$
75	668	966	$p^6T^4 + 668p^3T^3 + 966pT^2 + 668T + 1$
76	-568	18014	$p^6T^4 - 568p^3T^3 + 18014pT^2 - 568T + 1$
77	-296	1166	$p^6T^4 - 296p^3T^3 + 1166pT^2 - 296T + 1$
78	984	16190	$p^6T^4 + 984p^3T^3 + 16190pT^2 + 984T + 1$
79	192	1342	$p^6T^4 + 192p^3T^3 + 1342pT^2 + 192T + 1$
80	892	12086	$p^6T^4 + 892p^3T^3 + 12086pT^2 + 892T + 1$
81	348	14662	$p^6T^4 + 348p^3T^3 + 14662pT^2 + 348T + 1$
82	1208	19182	$p^6T^4 + 1208p^3T^3 + 19182pT^2 + 1208T + 1$
83	1052	8246	$p^6T^4 + 1052p^3T^3 + 8246pT^2 + 1052T + 1$
84	348	8950	$p^6T^4 + 348p^3T^3 + 8950pT^2 + 348T + 1$
85	660	13462	$p^6T^4 + 660p^3T^3 + 13462pT^2 + 660T + 1$
86	382	-3774	$p^6T^4 + 382p^3T^3 - 3774pT^2 + 382T + 1$
87	-22	6162	$p^6T^4 - 22p^3T^3 + 6162pT^2 - 22T + 1$
88	-2304	31870	$p^6T^4 - 2304p^3T^3 + 31870pT^2 - 2304T + 1$
89	1166	19490	$p^6T^4 + 1166p^3T^3 + 19490pT^2 + 1166T + 1$
90	-472	5486	$p^6T^4 - 472p^3T^3 + 5486pT^2 - 472T + 1$
91	1608	20078	$p^6T^4 + 1608p^3T^3 + 20078pT^2 + 1608T + 1$
92	-18	6682	$p^6T^4 - 18p^3T^3 + 6682pT^2 - 18T + 1$
93	190	8514	$p^6T^4 + 190p^3T^3 + 8514pT^2 + 190T + 1$
94	-1104	11806	$p^6T^4 - 1104p^3T^3 + 11806pT^2 - 1104T + 1$
95	1052	7110	$p^6T^4 + 1052p^3T^3 + 7110pT^2 + 1052T + 1$
96	-496	2142	$p^6T^4 - 496p^3T^3 + 2142pT^2 - 496T + 1$

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$p = 7$

λ	a	b	$R(T)$
1	-24	34	$p^6T^4 - 24p^3T^3 + 34pT^2 - 24T + 1$
2	24	49	$p^3T^2 + 24T + 1$
3	-4	82	$p^6T^4 - 4p^3T^3 + 82pT^2 - 4T + 1$
4	-24	98	$p^6T^4 - 24p^3T^3 + 98pT^2 - 24T + 1$
5	-8	-30	$p^6T^4 - 8p^3T^3 - 30pT^2 - 8T + 1$
6	-12	18	$p^6T^4 - 12p^3T^3 + 18pT^2 - 12T + 1$

$p = 11$

λ	a	b	$R(T)$
1	-32	194	$p^6T^4 - 32p^3T^3 + 194pT^2 - 32T + 1$
2	20	-30	$p^6T^4 + 20p^3T^3 - 30pT^2 + 20T + 1$
3	56	162	$p^6T^4 + 56p^3T^3 + 162pT^2 + 56T + 1$
4	-44	121	$p^3T^2 - 44T + 1$
5	8	34	$p^6T^4 + 8p^3T^3 + 34pT^2 + 8T + 1$
6	-16	130	$p^6T^4 - 16p^3T^3 + 130pT^2 - 16T + 1$
7	20	98	$p^6T^4 + 20p^3T^3 + 98pT^2 + 20T + 1$
8	-44	98	$p^6T^4 - 44p^3T^3 + 98pT^2 - 44T + 1$
9	-48	130	$p^6T^4 - 48p^3T^3 + 130pT^2 - 48T + 1$
10	-40	162	$p^6T^4 - 40p^3T^3 + 162pT^2 - 40T + 1$

$p = 13$

λ	a	b	$R(T)$
1	-16	190	$p^6T^4 - 16p^3T^3 + 190pT^2 - 16T + 1$
2	4	-154	$p^6T^4 + 4p^3T^3 - 154pT^2 + 4T + 1$
3	-22	169	$p^3T^2 - 22T + 1$
4	-28	-26	$p^6T^4 - 28p^3T^3 - 26pT^2 - 28T + 1$
5	92	358	$p^6T^4 + 92p^3T^3 + 358pT^2 + 92T + 1$
6	28	182	$p^6T^4 + 28p^3T^3 + 182pT^2 + 28T + 1$
7	4	230	$p^6T^4 + 4p^3T^3 + 230pT^2 + 4T + 1$
8	-12	54	$p^6T^4 - 12p^3T^3 + 54pT^2 - 12T + 1$
9	48	318	$p^6T^4 + 48p^3T^3 + 318pT^2 + 48T + 1$
10	8	174	$p^6T^4 + 8p^3T^3 + 174pT^2 + 8T + 1$
11	-36	102	$p^6T^4 - 36p^3T^3 + 102pT^2 - 36T + 1$
12	100	486	$p^6T^4 + 100p^3T^3 + 486pT^2 + 100T + 1$

$p = 17$

λ	a	b	$R(T)$
1	-50	289	$p^3T^2 - 50T + 1$
2	12	-26	$p^6T^4 + 12p^3T^3 - 26pT^2 + 12T + 1$
3	-76	118	$p^6T^4 - 76p^3T^3 + 118pT^2 - 76T + 1$
4	-112	414	$p^6T^4 - 112p^3T^3 + 414pT^2 - 112T + 1$
5	44	486	$p^6T^4 + 44p^3T^3 + 486pT^2 + 44T + 1$
6	-4	118	$p^6T^4 - 4p^3T^3 + 118pT^2 - 4T + 1$
7	-12	-26	$p^6T^4 - 12p^3T^3 - 26pT^2 - 12T + 1$
8	24	-178	$p^6T^4 + 24p^3T^3 - 178pT^2 + 24T + 1$
9	76	102	$p^6T^4 + 76p^3T^3 + 102pT^2 + 76T + 1$
10	52	246	$p^6T^4 + 52p^3T^3 + 246pT^2 + 52T + 1$
11	116	742	$p^6T^4 + 116p^3T^3 + 742pT^2 + 116T + 1$
12	-4	374	$p^6T^4 - 4p^3T^3 + 374pT^2 - 4T + 1$
13	80	542	$p^6T^4 + 80p^3T^3 + 542pT^2 + 80T + 1$
14	44	102	$p^6T^4 + 44p^3T^3 + 102pT^2 + 44T + 1$
15	88	462	$p^6T^4 + 88p^3T^3 + 462pT^2 + 88T + 1$
16	12	230	$p^6T^4 + 12p^3T^3 + 230pT^2 + 12T + 1$

$p = 19$

λ	a	b	$R(T)$
1	40	386	$p^6T^4 + 40p^3T^3 + 386pT^2 + 40T + 1$
2	108	322	$p^6T^4 + 108p^3T^3 + 322pT^2 + 108T + 1$
3	-108	546	$p^6T^4 - 108p^3T^3 + 546pT^2 - 108T + 1$
4	-48	-158	$p^6T^4 - 48p^3T^3 - 158pT^2 - 48T + 1$
5	-168	770	$p^6T^4 - 168p^3T^3 + 770pT^2 - 168T + 1$
6	-24	-126	$p^6T^4 - 24p^3T^3 - 126pT^2 - 24T + 1$
7	16	354	$p^6T^4 + 16p^3T^3 + 354pT^2 + 16T + 1$
8	8	642	$p^6T^4 + 8p^3T^3 + 642pT^2 + 8T + 1$
9	24	2	$p^6T^4 + 24p^3T^3 + 2pT^2 + 24T + 1$
10	48	354	$p^6T^4 + 48p^3T^3 + 354pT^2 + 48T + 1$
11	0	-94	$p^6T^4 - 94pT^2 + 1$
12	-56	-126	$p^6T^4 - 56p^3T^3 - 126pT^2 - 56T + 1$
13	-144	866	$p^6T^4 - 144p^3T^3 + 866pT^2 - 144T + 1$
14	4	482	$p^6T^4 + 4p^3T^3 + 482pT^2 + 4T + 1$
15	108	322	$p^6T^4 + 108p^3T^3 + 322pT^2 + 108T + 1$
16	-64	418	$p^6T^4 - 64p^3T^3 + 418pT^2 - 64T + 1$
17	44	361	$p^3T^2 + 44T + 1$
18	-148	834	$p^6T^4 - 148p^3T^3 + 834pT^2 - 148T + 1$

$p = 23$

λ	a	b	$R(T)$
1	-8	802	$p^6T^4 - 8p^3T^3 + 802pT^2 - 8T + 1$
2	-16	-286	$p^6T^4 - 16p^3T^3 - 286pT^2 - 16T + 1$
3	16	482	$p^6T^4 + 16p^3T^3 + 482pT^2 + 16T + 1$
4	-72	482	$p^6T^4 - 72p^3T^3 + 482pT^2 - 72T + 1$
5	176	994	$p^6T^4 + 176p^3T^3 + 994pT^2 + 176T + 1$
6	0	354	$p^6T^4 + 354pT^2 + 1$
7	168	1186	$p^6T^4 + 168p^3T^3 + 1186pT^2 + 168T + 1$

$p = 23$ (cont.)

λ	a	b	$R(T)$
8	-56	529	$p^3T^2 - 56T + 1$
9	-32	-158	$p^6T^4 - 32p^3T^3 - 158pT^2 - 32T + 1$
10	44	466	$p^6T^4 + 44p^3T^3 + 466pT^2 + 44T + 1$
11	-28	-622	$p^6T^4 - 28p^3T^3 - 622pT^2 - 28T + 1$
12	120	290	$p^6T^4 + 120p^3T^3 + 290pT^2 + 120T + 1$
13	-264	1762	$p^6T^4 - 264p^3T^3 + 1762pT^2 - 264T + 1$
14	-20	466	$p^6T^4 - 20p^3T^3 + 466pT^2 - 20T + 1$
15	-24	-94	$p^6T^4 - 24p^3T^3 - 94pT^2 - 24T + 1$
16	56	34	$p^6T^4 + 56p^3T^3 + 34pT^2 + 56T + 1$
17	-92	786	$p^6T^4 - 92p^3T^3 + 786pT^2 - 92T + 1$
18	-8	-286	$p^6T^4 - 8p^3T^3 - 286pT^2 - 8T + 1$
19	-88	674	$p^6T^4 - 88p^3T^3 + 674pT^2 - 88T + 1$
20	-20	594	$p^6T^4 - 20p^3T^3 + 594pT^2 - 20T + 1$
21	-284	1810	$p^6T^4 - 284p^3T^3 + 1810pT^2 - 284T + 1$
22	-96	866	$p^6T^4 - 96p^3T^3 + 866pT^2 - 96T + 1$

$p = 29$

λ	a	b	$R(T)$
1	52	-378	$p^6T^4 + 52p^3T^3 - 378pT^2 + 52T + 1$
2	36	854	$p^6T^4 + 36p^3T^3 + 854pT^2 + 36T + 1$
3	244	2054	$p^6T^4 + 244p^3T^3 + 2054pT^2 + 244T + 1$
4	216	782	$p^6T^4 + 216p^3T^3 + 782pT^2 + 216T + 1$
5	152	1422	$p^6T^4 + 152p^3T^3 + 1422pT^2 + 152T + 1$
6	104	302	$p^6T^4 + 104p^3T^3 + 302pT^2 + 104T + 1$
7	-204	1158	$p^6T^4 - 204p^3T^3 + 1158pT^2 - 204T + 1$
8	-92	342	$p^6T^4 - 92p^3T^3 + 342pT^2 - 92T + 1$
9	-232	1678	$p^6T^4 - 232p^3T^3 + 1678pT^2 - 232T + 1$
10	12	-170	$p^6T^4 + 12p^3T^3 - 170pT^2 + 12T + 1$
11	12	518	$p^6T^4 + 12p^3T^3 + 518pT^2 + 12T + 1$
12	12	518	$p^6T^4 + 12p^3T^3 + 518pT^2 + 12T + 1$

$p = 29$ (cont.)

λ	a	b	$R(T)$
13	-40	270	$p^6T^4 - 40p^3T^3 + 270pT^2 - 40T + 1$
14	12	-554	$p^6T^4 + 12p^3T^3 - 554pT^2 + 12T + 1$
15	164	1622	$p^6T^4 + 164p^3T^3 + 1622pT^2 + 164T + 1$
16	180	1414	$p^6T^4 + 180p^3T^3 + 1414pT^2 + 180T + 1$
17	-36	-586	$p^6T^4 - 36p^3T^3 - 586pT^2 - 36T + 1$
18	140	214	$p^6T^4 + 140p^3T^3 + 214pT^2 + 140T + 1$
19	12	1366	$p^6T^4 + 12p^3T^3 + 1366pT^2 + 12T + 1$
20	-128	798	$p^6T^4 - 128p^3T^3 + 798pT^2 - 128T + 1$
21	-220	1110	$p^6T^4 - 220p^3T^3 + 1110pT^2 - 220T + 1$
22	180	902	$p^6T^4 + 180p^3T^3 + 902pT^2 + 180T + 1$
23	-198	841	$p^3T^2 - 198T + 1$
24	-12	1542	$p^6T^4 - 12p^3T^3 + 1542pT^2 - 12T + 1$
25	384	2846	$p^6T^4 + 384p^3T^3 + 2846pT^2 + 384T + 1$
26	116	1206	$p^6T^4 + 116p^3T^3 + 1206pT^2 + 116T + 1$
27	-12	1030	$p^6T^4 - 12p^3T^3 + 1030pT^2 - 12T + 1$
28	-12	-506	$p^6T^4 - 12p^3T^3 - 506pT^2 - 12T + 1$

$p = 31$

λ	a	b	$R(T)$
1	-8	1026	$p^6T^4 - 8p^3T^3 + 1026pT^2 - 8T + 1$
2	32	898	$p^6T^4 + 32p^3T^3 + 898pT^2 + 32T + 1$
3	-472	3458	$p^6T^4 - 472p^3T^3 + 3458pT^2 - 472T + 1$
4	-160	961	$p^3T^2 - 160T + 1$
5	-120	258	$p^6T^4 - 120p^3T^3 + 258pT^2 - 120T + 1$
6	284	1810	$p^6T^4 + 284p^3T^3 + 1810pT^2 + 284T + 1$
7	-216	1730	$p^6T^4 - 216p^3T^3 + 1730pT^2 - 216T + 1$
8	-96	-126	$p^6T^4 - 96p^3T^3 - 126pT^2 - 96T + 1$
9	328	2562	$p^6T^4 + 328p^3T^3 + 2562pT^2 + 328T + 1$
10	96	1666	$p^6T^4 + 96p^3T^3 + 1666pT^2 + 96T + 1$
11	100	1042	$p^6T^4 + 100p^3T^3 + 1042pT^2 + 100T + 1$

$p = 31$ (cont.)

λ	a	b	$R(T)$
12	-92	-878	$p^6T^4 - 92p^3T^3 - 878pT^2 - 92T + 1$
13	-156	402	$p^6T^4 - 156p^3T^3 + 402pT^2 - 156T + 1$
14	-32	-126	$p^6T^4 - 32p^3T^3 - 126pT^2 - 32T + 1$
15	-152	642	$p^6T^4 - 152p^3T^3 + 642pT^2 - 152T + 1$
16	-24	450	$p^6T^4 - 24p^3T^3 + 450pT^2 - 24T + 1$
17	288	1922	$p^6T^4 + 288p^3T^3 + 1922pT^2 + 288T + 1$
18	-160	898	$p^6T^4 - 160p^3T^3 + 898pT^2 - 160T + 1$
19	-168	962	$p^6T^4 - 168p^3T^3 + 962pT^2 - 168T + 1$
20	-72	770	$p^6T^4 - 72p^3T^3 + 770pT^2 - 72T + 1$
21	32	-126	$p^6T^4 + 32p^3T^3 - 126pT^2 + 32T + 1$
22	156	658	$p^6T^4 + 156p^3T^3 + 658pT^2 + 156T + 1$
23	-164	1426	$p^6T^4 - 164p^3T^3 + 1426pT^2 - 164T + 1$
24	-164	786	$p^6T^4 - 164p^3T^3 + 786pT^2 - 164T + 1$
25	96	1666	$p^6T^4 + 96p^3T^3 + 1666pT^2 + 96T + 1$
26	-104	898	$p^6T^4 - 104p^3T^3 + 898pT^2 - 104T + 1$
27	96	898	$p^6T^4 + 96p^3T^3 + 898pT^2 + 96T + 1$
28	24	194	$p^6T^4 + 24p^3T^3 + 194pT^2 + 24T + 1$
29	-220	1042	$p^6T^4 - 220p^3T^3 + 1042pT^2 - 220T + 1$
30	88	130	$p^6T^4 + 88p^3T^3 + 130pT^2 + 88T + 1$

$p = 37$

λ	a	b	$R(T)$
1	-124	2150	$p^6T^4 - 124p^3T^3 + 2150pT^2 - 124T + 1$
2	148	1606	$p^6T^4 + 148p^3T^3 + 1606pT^2 + 148T + 1$
3	-60	742	$p^6T^4 - 60p^3T^3 + 742pT^2 - 60T + 1$
4	168	974	$p^6T^4 + 168p^3T^3 + 974pT^2 + 168T + 1$
5	404	3526	$p^6T^4 + 404p^3T^3 + 3526pT^2 + 404T + 1$
6	100	598	$p^6T^4 + 100p^3T^3 + 598pT^2 + 100T + 1$
7	272	2590	$p^6T^4 + 272p^3T^3 + 2590pT^2 + 272T + 1$
8	-276	1622	$p^6T^4 - 276p^3T^3 + 1622pT^2 - 276T + 1$

$p = 37$ (cont.)

λ	a	b	$R(T)$
9	196	1254	$p^6T^4 + 196p^3T^3 + 1254pT^2 + 196T + 1$
10	-304	2974	$p^6T^4 - 304p^3T^3 + 2974pT^2 - 304T + 1$
11	4	358	$p^6T^4 + 4p^3T^3 + 358pT^2 + 4T + 1$
12	162	1369	$p^3T^2 + 162T + 1$
13	-148	598	$p^6T^4 - 148p^3T^3 + 598pT^2 - 148T + 1$
14	12	-1850	$p^6T^4 + 12p^3T^3 - 1850pT^2 + 12T + 1$
15	-60	742	$p^6T^4 - 60p^3T^3 + 742pT^2 - 60T + 1$
16	388	2662	$p^6T^4 + 388p^3T^3 + 2662pT^2 + 388T + 1$
17	188	486	$p^6T^4 + 188p^3T^3 + 486pT^2 + 188T + 1$
18	228	2646	$p^6T^4 + 228p^3T^3 + 2646pT^2 + 228T + 1$
19	-276	982	$p^6T^4 - 276p^3T^3 + 982pT^2 - 276T + 1$
20	-28	1366	$p^6T^4 - 28p^3T^3 + 1366pT^2 - 28T + 1$
21	-204	1350	$p^6T^4 - 204p^3T^3 + 1350pT^2 - 204T + 1$
22	108	214	$p^6T^4 + 108p^3T^3 + 214pT^2 + 108T + 1$
23	196	230	$p^6T^4 + 196p^3T^3 + 230pT^2 + 196T + 1$
24	364	2774	$p^6T^4 + 364p^3T^3 + 2774pT^2 + 364T + 1$
25	-280	2382	$p^6T^4 - 280p^3T^3 + 2382pT^2 - 280T + 1$
26	384	2174	$p^6T^4 + 384p^3T^3 + 2174pT^2 + 384T + 1$
27	-24	-2226	$p^6T^4 - 24p^3T^3 - 2226pT^2 - 24T + 1$
28	180	838	$p^6T^4 + 180p^3T^3 + 838pT^2 + 180T + 1$
29	-28	-170	$p^6T^4 - 28p^3T^3 - 170pT^2 - 28T + 1$
30	-152	-434	$p^6T^4 - 152p^3T^3 - 434pT^2 - 152T + 1$
31	-284	2646	$p^6T^4 - 284p^3T^3 + 2646pT^2 - 284T + 1$
32	188	486	$p^6T^4 + 188p^3T^3 + 486pT^2 + 188T + 1$
33	-252	2918	$p^6T^4 - 252p^3T^3 + 2918pT^2 - 252T + 1$
34	128	2686	$p^6T^4 + 128p^3T^3 + 2686pT^2 + 128T + 1$
35	-116	1990	$p^6T^4 - 116p^3T^3 + 1990pT^2 - 116T + 1$
36	168	462	$p^6T^4 + 168p^3T^3 + 462pT^2 + 168T + 1$

$p = 41$

λ	a	b	$R(T)$
1	-420	2694	$p^6T^4 - 420p^3T^3 + 2694pT^2 - 420T + 1$
2	296	2382	$p^6T^4 + 296p^3T^3 + 2382pT^2 + 296T + 1$
3	-492	3558	$p^6T^4 - 492p^3T^3 + 3558pT^2 - 492T + 1$
4	364	1766	$p^6T^4 + 364p^3T^3 + 1766pT^2 + 364T + 1$
5	412	2566	$p^6T^4 + 412p^3T^3 + 2566pT^2 + 412T + 1$
6	-68	-1914	$p^6T^4 - 68p^3T^3 - 1914pT^2 - 68T + 1$
7	-220	2182	$p^6T^4 - 220p^3T^3 + 2182pT^2 - 220T + 1$
8	40	1358	$p^6T^4 + 40p^3T^3 + 1358pT^2 + 40T + 1$
9	-24	1742	$p^6T^4 - 24p^3T^3 + 1742pT^2 - 24T + 1$
10	-224	2398	$p^6T^4 - 224p^3T^3 + 2398pT^2 - 224T + 1$
11	220	1974	$p^6T^4 + 220p^3T^3 + 1974pT^2 + 220T + 1$
12	-196	518	$p^6T^4 - 196p^3T^3 + 518pT^2 - 196T + 1$
13	220	1206	$p^6T^4 + 220p^3T^3 + 1206pT^2 + 220T + 1$
14	444	3590	$p^6T^4 + 444p^3T^3 + 3590pT^2 + 444T + 1$
15	36	1670	$p^6T^4 + 36p^3T^3 + 1670pT^2 + 36T + 1$
16	-36	-122	$p^6T^4 - 36p^3T^3 - 122pT^2 - 36T + 1$
17	372	3254	$p^6T^4 + 372p^3T^3 + 3254pT^2 + 372T + 1$
18	-96	1118	$p^6T^4 - 96p^3T^3 + 1118pT^2 - 96T + 1$
19	172	2790	$p^6T^4 + 172p^3T^3 + 2790pT^2 + 172T + 1$
20	104	462	$p^6T^4 + 104p^3T^3 + 462pT^2 + 104T + 1$
21	-292	1670	$p^6T^4 - 292p^3T^3 + 1670pT^2 - 292T + 1$
22	164	2438	$p^6T^4 + 164p^3T^3 + 2438pT^2 + 164T + 1$
23	-80	702	$p^6T^4 - 80p^3T^3 + 702pT^2 - 80T + 1$
24	-292	694	$p^6T^4 - 292p^3T^3 + 694pT^2 - 292T + 1$
25	176	-834	$p^6T^4 + 176p^3T^3 - 834pT^2 + 176T + 1$
26	220	-74	$p^6T^4 + 220p^3T^3 - 74pT^2 + 220T + 1$
27	-84	1510	$p^6T^4 - 84p^3T^3 + 1510pT^2 - 84T + 1$
28	-12	1462	$p^6T^4 - 12p^3T^3 + 1462pT^2 - 12T + 1$
29	-140	-714	$p^6T^4 - 140p^3T^3 - 714pT^2 - 140T + 1$
30	-92	902	$p^6T^4 - 92p^3T^3 + 902pT^2 - 92T + 1$
31	-16	1598	$p^6T^4 - 16p^3T^3 + 1598pT^2 - 16T + 1$
32	156	2054	$p^6T^4 + 156p^3T^3 + 2054pT^2 + 156T + 1$
33	108	2790	$p^6T^4 + 108p^3T^3 + 2790pT^2 + 108T + 1$

$p = 41$ (cont.)

λ	a	b	$R(T)$
34	276	998	$p^6T^4 + 276p^3T^3 + 998pT^2 + 276T + 1$
35	-268	2870	$p^6T^4 - 268p^3T^3 + 2870pT^2 - 268T + 1$
36	-420	2950	$p^6T^4 - 420p^3T^3 + 2950pT^2 - 420T + 1$
37	198	1681	$p^3T^2 + 198T + 1$
38	572	4742	$p^6T^4 + 572p^3T^3 + 4742pT^2 + 572T + 1$
39	108	2278	$p^6T^4 + 108p^3T^3 + 2278pT^2 + 108T + 1$
40	496	3646	$p^6T^4 + 496p^3T^3 + 3646pT^2 + 496T + 1$

$p = 43$

λ	a	b	$R(T)$
1	-80	1666	$p^6T^4 - 80p^3T^3 + 1666pT^2 - 80T + 1$
2	76	-574	$p^6T^4 + 76p^3T^3 - 574pT^2 + 76T + 1$
3	92	-1790	$p^6T^4 + 92p^3T^3 - 1790pT^2 + 92T + 1$
4	128	194	$p^6T^4 + 128p^3T^3 + 194pT^2 + 128T + 1$
5	-8	-350	$p^6T^4 - 8p^3T^3 - 350pT^2 - 8T + 1$
6	-464	3842	$p^6T^4 - 464p^3T^3 + 3842pT^2 - 464T + 1$
7	-392	2722	$p^6T^4 - 392p^3T^3 + 2722pT^2 - 392T + 1$
8	272	1666	$p^6T^4 + 272p^3T^3 + 1666pT^2 + 272T + 1$
9	-328	2786	$p^6T^4 - 328p^3T^3 + 2786pT^2 - 328T + 1$
10	-320	2242	$p^6T^4 - 320p^3T^3 + 2242pT^2 - 320T + 1$
11	-232	1442	$p^6T^4 - 232p^3T^3 + 1442pT^2 - 232T + 1$
12	-152	1378	$p^6T^4 - 152p^3T^3 + 1378pT^2 - 152T + 1$
13	304	3714	$p^6T^4 + 304p^3T^3 + 3714pT^2 + 304T + 1$
14	-152	1058	$p^6T^4 - 152p^3T^3 + 1058pT^2 - 152T + 1$
15	-88	-1950	$p^6T^4 - 88p^3T^3 - 1950pT^2 - 88T + 1$
16	104	-2014	$p^6T^4 + 104p^3T^3 - 2014pT^2 + 104T + 1$
17	216	2466	$p^6T^4 + 216p^3T^3 + 2466pT^2 + 216T + 1$
18	-180	1858	$p^6T^4 - 180p^3T^3 + 1858pT^2 - 180T + 1$
19	-100	2946	$p^6T^4 - 100p^3T^3 + 2946pT^2 - 100T + 1$
20	144	3714	$p^6T^4 + 144p^3T^3 + 3714pT^2 + 144T + 1$

$p = 43$ (cont.)

λ	a	b	$R(T)$
21	52	1849	$p^3T^2 + 52T + 1$
22	-332	4194	$p^6T^4 - 332p^3T^3 + 4194pT^2 - 332T + 1$
23	664	6050	$p^6T^4 + 664p^3T^3 + 6050pT^2 + 664T + 1$
24	24	-94	$p^6T^4 + 24p^3T^3 - 94pT^2 + 24T + 1$
25	-24	-478	$p^6T^4 - 24p^3T^3 - 478pT^2 - 24T + 1$
26	160	1218	$p^6T^4 + 160p^3T^3 + 1218pT^2 + 160T + 1$
27	28	258	$p^6T^4 + 28p^3T^3 + 258pT^2 + 28T + 1$
28	-496	3714	$p^6T^4 - 496p^3T^3 + 3714pT^2 - 496T + 1$
29	184	-94	$p^6T^4 + 184p^3T^3 - 94pT^2 + 184T + 1$
30	12	-318	$p^6T^4 + 12p^3T^3 - 318pT^2 + 12T + 1$
31	-16	1154	$p^6T^4 - 16p^3T^3 + 1154pT^2 - 16T + 1$
32	436	3426	$p^6T^4 + 436p^3T^3 + 3426pT^2 + 436T + 1$
33	-48	130	$p^6T^4 - 48p^3T^3 + 130pT^2 - 48T + 1$
34	-164	1154	$p^6T^4 - 164p^3T^3 + 1154pT^2 - 164T + 1$
35	-472	3874	$p^6T^4 - 472p^3T^3 + 3874pT^2 - 472T + 1$
36	176	3714	$p^6T^4 + 176p^3T^3 + 3714pT^2 + 176T + 1$
37	-628	5954	$p^6T^4 - 628p^3T^3 + 5954pT^2 - 628T + 1$
38	-288	2242	$p^6T^4 - 288p^3T^3 + 2242pT^2 - 288T + 1$
39	-392	4258	$p^6T^4 - 392p^3T^3 + 4258pT^2 - 392T + 1$
40	64	1730	$p^6T^4 + 64p^3T^3 + 1730pT^2 + 64T + 1$
41	-192	706	$p^6T^4 - 192p^3T^3 + 706pT^2 - 192T + 1$
42	564	4194	$p^6T^4 + 564p^3T^3 + 4194pT^2 + 564T + 1$

$p = 47$

λ	a	b	$R(T)$
1	-272	2114	$p^6T^4 - 272p^3T^3 + 2114pT^2 - 272T + 1$
2	-264	3842	$p^6T^4 - 264p^3T^3 + 3842pT^2 - 264T + 1$
3	56	-2046	$p^6T^4 + 56p^3T^3 - 2046pT^2 + 56T + 1$
4	-696	6466	$p^6T^4 - 696p^3T^3 + 6466pT^2 - 696T + 1$
5	-104	-1342	$p^6T^4 - 104p^3T^3 - 1342pT^2 - 104T + 1$
6	296	2818	$p^6T^4 + 296p^3T^3 + 2818pT^2 + 296T + 1$
7	-280	2562	$p^6T^4 - 280p^3T^3 + 2562pT^2 - 280T + 1$
8	112	2114	$p^6T^4 + 112p^3T^3 + 2114pT^2 + 112T + 1$
9	528	2209	$p^3T^2 + 528T + 1$
10	540	5010	$p^6T^4 + 540p^3T^3 + 5010pT^2 + 540T + 1$
11	-108	1682	$p^6T^4 - 108p^3T^3 + 1682pT^2 - 108T + 1$
12	-592	3906	$p^6T^4 - 592p^3T^3 + 3906pT^2 - 592T + 1$
13	156	2834	$p^6T^4 + 156p^3T^3 + 2834pT^2 + 156T + 1$
14	48	-190	$p^6T^4 + 48p^3T^3 - 190pT^2 + 48T + 1$
15	-164	3730	$p^6T^4 - 164p^3T^3 + 3730pT^2 - 164T + 1$
16	624	5442	$p^6T^4 + 624p^3T^3 + 5442pT^2 + 624T + 1$
17	-120	2114	$p^6T^4 - 120p^3T^3 + 2114pT^2 - 120T + 1$
18	-144	-702	$p^6T^4 - 144p^3T^3 - 702pT^2 - 144T + 1$
19	264	1474	$p^6T^4 + 264p^3T^3 + 1474pT^2 + 264T + 1$
20	16	-702	$p^6T^4 + 16p^3T^3 - 702pT^2 + 16T + 1$
21	-360	1090	$p^6T^4 - 360p^3T^3 + 1090pT^2 - 360T + 1$
22	-560	4930	$p^6T^4 - 560p^3T^3 + 4930pT^2 - 560T + 1$
23	-44	-2798	$p^6T^4 - 44p^3T^3 - 2798pT^2 - 44T + 1$
24	-88	2818	$p^6T^4 - 88p^3T^3 + 2818pT^2 - 88T + 1$
25	88	-190	$p^6T^4 + 88p^3T^3 - 190pT^2 + 88T + 1$
26	268	786	$p^6T^4 + 268p^3T^3 + 786pT^2 + 268T + 1$
27	-184	4418	$p^6T^4 - 184p^3T^3 + 4418pT^2 - 184T + 1$
28	368	2114	$p^6T^4 + 368p^3T^3 + 2114pT^2 + 368T + 1$
29	24	-2366	$p^6T^4 + 24p^3T^3 - 2366pT^2 + 24T + 1$
30	-300	3730	$p^6T^4 - 300p^3T^3 + 3730pT^2 - 300T + 1$
31	72	2754	$p^6T^4 + 72p^3T^3 + 2754pT^2 + 72T + 1$
32	-328	2050	$p^6T^4 - 328p^3T^3 + 2050pT^2 - 328T + 1$
33	-52	1938	$p^6T^4 - 52p^3T^3 + 1938pT^2 - 52T + 1$

$p = 47$ (cont.)

λ	a	b	$R(T)$
34	-400	1346	$p^6T^4 - 400p^3T^3 + 1346pT^2 - 400T + 1$
35	136	3522	$p^6T^4 + 136p^3T^3 + 3522pT^2 + 136T + 1$
36	304	3906	$p^6T^4 + 304p^3T^3 + 3906pT^2 + 304T + 1$
37	152	2114	$p^6T^4 + 152p^3T^3 + 2114pT^2 + 152T + 1$
38	-304	2882	$p^6T^4 - 304p^3T^3 + 2882pT^2 - 304T + 1$
39	-424	1986	$p^6T^4 - 424p^3T^3 + 1986pT^2 - 424T + 1$
40	80	-190	$p^6T^4 + 80p^3T^3 - 190pT^2 + 80T + 1$
41	-508	4114	$p^6T^4 - 508p^3T^3 + 4114pT^2 - 508T + 1$
42	48	-190	$p^6T^4 + 48p^3T^3 - 190pT^2 + 48T + 1$
43	132	1682	$p^6T^4 + 132p^3T^3 + 1682pT^2 + 132T + 1$
44	-240	4418	$p^6T^4 - 240p^3T^3 + 4418pT^2 - 240T + 1$
45	-564	3858	$p^6T^4 - 564p^3T^3 + 3858pT^2 - 564T + 1$
46	580	5394	$p^6T^4 + 580p^3T^3 + 5394pT^2 + 580T + 1$

$p = 53$

λ	a	b	$R(T)$
1	388	1126	$p^6T^4 + 388p^3T^3 + 1126pT^2 + 388T + 1$
2	-236	5878	$p^6T^4 - 236p^3T^3 + 5878pT^2 - 236T + 1$
3	772	6246	$p^6T^4 + 772p^3T^3 + 6246pT^2 + 772T + 1$
4	212	2694	$p^6T^4 + 212p^3T^3 + 2694pT^2 + 212T + 1$
5	-460	2182	$p^6T^4 - 460p^3T^3 + 2182pT^2 - 460T + 1$
6	-152	4622	$p^6T^4 - 152p^3T^3 + 4622pT^2 - 152T + 1$
7	340	2950	$p^6T^4 + 340p^3T^3 + 2950pT^2 + 340T + 1$
8	620	5638	$p^6T^4 + 620p^3T^3 + 5638pT^2 + 620T + 1$
9	724	6534	$p^6T^4 + 724p^3T^3 + 6534pT^2 + 724T + 1$
10	388	2150	$p^6T^4 + 388p^3T^3 + 2150pT^2 + 388T + 1$
11	-188	998	$p^6T^4 - 188p^3T^3 + 998pT^2 - 188T + 1$
12	60	230	$p^6T^4 + 60p^3T^3 + 230pT^2 + 60T + 1$
13	-176	2014	$p^6T^4 - 176p^3T^3 + 2014pT^2 - 176T + 1$
14	404	758	$p^6T^4 + 404p^3T^3 + 758pT^2 + 404T + 1$

$p = 53$ (cont.)

λ	a	b	$R(T)$
15	-112	2654	$p^6T^4 - 112p^3T^3 + 2654pT^2 - 112T + 1$
16	400	4702	$p^6T^4 + 400p^3T^3 + 4702pT^2 + 400T + 1$
17	-136	-82	$p^6T^4 - 136p^3T^3 - 82pT^2 - 136T + 1$
18	220	4214	$p^6T^4 + 220p^3T^3 + 4214pT^2 + 220T + 1$
19	-332	3718	$p^6T^4 - 332p^3T^3 + 3718pT^2 - 332T + 1$
20	-660	7174	$p^6T^4 - 660p^3T^3 + 7174pT^2 - 660T + 1$
21	364	5126	$p^6T^4 + 364p^3T^3 + 5126pT^2 + 364T + 1$
22	-676	4854	$p^6T^4 - 676p^3T^3 + 4854pT^2 - 676T + 1$
23	-76	6	$p^6T^4 - 76p^3T^3 + 6pT^2 - 76T + 1$
24	144	3166	$p^6T^4 + 144p^3T^3 + 3166pT^2 + 144T + 1$
25	-456	4142	$p^6T^4 - 456p^3T^3 + 4142pT^2 - 456T + 1$
26	236	-1274	$p^6T^4 + 236p^3T^3 - 1274pT^2 + 236T + 1$
27	236	774	$p^6T^4 + 236p^3T^3 + 774pT^2 + 236T + 1$
28	336	-34	$p^6T^4 + 336p^3T^3 - 34pT^2 + 336T + 1$
29	-520	5294	$p^6T^4 - 520p^3T^3 + 5294pT^2 - 520T + 1$
30	348	-138	$p^6T^4 + 348p^3T^3 - 138pT^2 + 348T + 1$
31	-460	6150	$p^6T^4 - 460p^3T^3 + 6150pT^2 - 460T + 1$
32	180	3590	$p^6T^4 + 180p^3T^3 + 3590pT^2 + 180T + 1$
33	660	6134	$p^6T^4 + 660p^3T^3 + 6134pT^2 + 660T + 1$
34	436	4742	$p^6T^4 + 436p^3T^3 + 4742pT^2 + 436T + 1$
35	60	230	$p^6T^4 + 60p^3T^3 + 230pT^2 + 60T + 1$
36	-444	2534	$p^6T^4 - 444p^3T^3 + 2534pT^2 - 444T + 1$
37	-764	7014	$p^6T^4 - 764p^3T^3 + 7014pT^2 - 764T + 1$
38	84	4486	$p^6T^4 + 84p^3T^3 + 4486pT^2 + 84T + 1$
39	-532	5382	$p^6T^4 - 532p^3T^3 + 5382pT^2 - 532T + 1$
40	-44	3974	$p^6T^4 - 44p^3T^3 + 3974pT^2 - 44T + 1$
41	12	150	$p^6T^4 + 12p^3T^3 + 150pT^2 + 12T + 1$
42	336	2014	$p^6T^4 + 336p^3T^3 + 2014pT^2 + 336T + 1$
43	84	134	$p^6T^4 + 84p^3T^3 + 134pT^2 + 84T + 1$
44	480	3710	$p^6T^4 + 480p^3T^3 + 3710pT^2 + 480T + 1$
45	20	-3082	$p^6T^4 + 20p^3T^3 - 3082pT^2 + 20T + 1$
46	260	-1178	$p^6T^4 + 260p^3T^3 - 1178pT^2 + 260T + 1$
47	242	2809	$p^3T^2 + 242T + 1$

$p = 53$ (cont.)

λ	a	b	$R(T)$
48	348	2806	$p^6T^4 + 348p^3T^3 + 2806pT^2 + 348T + 1$
49	-32	-3458	$p^6T^4 - 32p^3T^3 - 3458pT^2 - 32T + 1$
50	-188	22	$p^6T^4 - 188p^3T^3 + 22pT^2 - 188T + 1$
51	4	2150	$p^6T^4 + 4p^3T^3 + 2150pT^2 + 4T + 1$
52	56	2094	$p^6T^4 + 56p^3T^3 + 2094pT^2 + 56T + 1$

$p = 59$

λ	a	b	$R(T)$
1	384	7298	$p^6T^4 + 384p^3T^3 + 7298pT^2 + 384T + 1$
2	-768	7298	$p^6T^4 - 768p^3T^3 + 7298pT^2 - 768T + 1$
3	-668	3481	$p^3T^2 - 668T + 1$
4	64	-894	$p^6T^4 + 64p^3T^3 - 894pT^2 + 64T + 1$
5	-192	4610	$p^6T^4 - 192p^3T^3 + 4610pT^2 - 192T + 1$
6	-220	-2014	$p^6T^4 - 220p^3T^3 - 2014pT^2 - 220T + 1$
7	248	930	$p^6T^4 + 248p^3T^3 + 930pT^2 + 248T + 1$
8	-604	4258	$p^6T^4 - 604p^3T^3 + 4258pT^2 - 604T + 1$
9	488	5154	$p^6T^4 + 488p^3T^3 + 5154pT^2 + 488T + 1$
10	-220	2722	$p^6T^4 - 220p^3T^3 + 2722pT^2 - 220T + 1$
11	656	8642	$p^6T^4 + 656p^3T^3 + 8642pT^2 + 656T + 1$
12	-776	7074	$p^6T^4 - 776p^3T^3 + 7074pT^2 - 776T + 1$
13	408	4066	$p^6T^4 + 408p^3T^3 + 4066pT^2 + 408T + 1$
14	64	-2942	$p^6T^4 + 64p^3T^3 - 2942pT^2 + 64T + 1$
15	376	418	$p^6T^4 + 376p^3T^3 + 418pT^2 + 376T + 1$
16	128	4610	$p^6T^4 + 128p^3T^3 + 4610pT^2 + 128T + 1$
17	424	2082	$p^6T^4 + 424p^3T^3 + 2082pT^2 + 424T + 1$
18	60	3202	$p^6T^4 + 60p^3T^3 + 3202pT^2 + 60T + 1$
19	-1016	10338	$p^6T^4 - 1016p^3T^3 + 10338pT^2 - 1016T + 1$
20	-960	10370	$p^6T^4 - 960p^3T^3 + 10370pT^2 - 960T + 1$
21	504	2530	$p^6T^4 + 504p^3T^3 + 2530pT^2 + 504T + 1$
22	176	5570	$p^6T^4 + 176p^3T^3 + 5570pT^2 + 176T + 1$

$p = 59$ (cont.)

λ	a	b	$R(T)$
23	124	-4222	$p^6T^4 + 124p^3T^3 - 4222pT^2 + 124T + 1$
24	36	-94	$p^6T^4 + 36p^3T^3 - 94pT^2 + 36T + 1$
25	496	1986	$p^6T^4 + 496p^3T^3 + 1986pT^2 + 496T + 1$
26	-344	4642	$p^6T^4 - 344p^3T^3 + 4642pT^2 - 344T + 1$
27	-344	6690	$p^6T^4 - 344p^3T^3 + 6690pT^2 - 344T + 1$
28	184	5282	$p^6T^4 + 184p^3T^3 + 5282pT^2 + 184T + 1$
29	392	-414	$p^6T^4 + 392p^3T^3 - 414pT^2 + 392T + 1$
30	-116	-1598	$p^6T^4 - 116p^3T^3 - 1598pT^2 - 116T + 1$
31	-284	5794	$p^6T^4 - 284p^3T^3 + 5794pT^2 - 284T + 1$
32	508	3842	$p^6T^4 + 508p^3T^3 + 3842pT^2 + 508T + 1$
33	88	1250	$p^6T^4 + 88p^3T^3 + 1250pT^2 + 88T + 1$
34	-280	5922	$p^6T^4 - 280p^3T^3 + 5922pT^2 - 280T + 1$
35	-688	5314	$p^6T^4 - 688p^3T^3 + 5314pT^2 - 688T + 1$
36	-688	5314	$p^6T^4 - 688p^3T^3 + 5314pT^2 - 688T + 1$
37	-48	962	$p^6T^4 - 48p^3T^3 + 962pT^2 - 48T + 1$
38	36	-94	$p^6T^4 + 36p^3T^3 - 94pT^2 + 36T + 1$
39	332	5698	$p^6T^4 + 332p^3T^3 + 5698pT^2 + 332T + 1$
40	-112	-574	$p^6T^4 - 112p^3T^3 - 574pT^2 - 112T + 1$
41	-128	1538	$p^6T^4 - 128p^3T^3 + 1538pT^2 - 128T + 1$
42	576	4226	$p^6T^4 + 576p^3T^3 + 4226pT^2 + 576T + 1$
43	-680	6882	$p^6T^4 - 680p^3T^3 + 6882pT^2 - 680T + 1$
44	-244	834	$p^6T^4 - 244p^3T^3 + 834pT^2 - 244T + 1$
45	720	7874	$p^6T^4 + 720p^3T^3 + 7874pT^2 + 720T + 1$
46	-8	5602	$p^6T^4 - 8p^3T^3 + 5602pT^2 - 8T + 1$
47	40	2594	$p^6T^4 + 40p^3T^3 + 2594pT^2 + 40T + 1$
48	448	1154	$p^6T^4 + 448p^3T^3 + 1154pT^2 + 448T + 1$
49	-56	-5278	$p^6T^4 - 56p^3T^3 - 5278pT^2 - 56T + 1$
50	40	2082	$p^6T^4 + 40p^3T^3 + 2082pT^2 + 40T + 1$
51	-720	4546	$p^6T^4 - 720p^3T^3 + 4546pT^2 - 720T + 1$
52	-348	4770	$p^6T^4 - 348p^3T^3 + 4770pT^2 - 348T + 1$
53	-624	6850	$p^6T^4 - 624p^3T^3 + 6850pT^2 - 624T + 1$
54	548	4258	$p^6T^4 + 548p^3T^3 + 4258pT^2 + 548T + 1$
55	-216	-2014	$p^6T^4 - 216p^3T^3 - 2014pT^2 - 216T + 1$

$p = 59$ (cont.)

λ	a	b	$R(T)$
56	-284	4130	$p^6T^4 - 284p^3T^3 + 4130pT^2 - 284T + 1$
57	440	4578	$p^6T^4 + 440p^3T^3 + 4578pT^2 + 440T + 1$
58	-832	9346	$p^6T^4 - 832p^3T^3 + 9346pT^2 - 832T + 1$

$p = 61$

λ	a	b	$R(T)$
1	-608	8158	$p^6T^4 - 608p^3T^3 + 8158pT^2 - 608T + 1$
2	-148	6982	$p^6T^4 - 148p^3T^3 + 6982pT^2 - 148T + 1$
3	56	6222	$p^6T^4 + 56p^3T^3 + 6222pT^2 + 56T + 1$
4	420	1126	$p^6T^4 + 420p^3T^3 + 1126pT^2 + 420T + 1$
5	-440	2414	$p^6T^4 - 440p^3T^3 + 2414pT^2 - 440T + 1$
6	620	4422	$p^6T^4 + 620p^3T^3 + 4422pT^2 + 620T + 1$
7	-404	-362	$p^6T^4 - 404p^3T^3 - 362pT^2 - 404T + 1$
8	-196	-3082	$p^6T^4 - 196p^3T^3 - 3082pT^2 - 196T + 1$
9	1108	11462	$p^6T^4 + 1108p^3T^3 + 11462pT^2 + 1108T + 1$
10	236	4246	$p^6T^4 + 236p^3T^3 + 4246pT^2 + 236T + 1$
11	-68	2550	$p^6T^4 - 68p^3T^3 + 2550pT^2 - 68T + 1$
12	-176	5182	$p^6T^4 - 176p^3T^3 + 5182pT^2 - 176T + 1$
13	-96	-34	$p^6T^4 - 96p^3T^3 - 34pT^2 - 96T + 1$
14	404	4422	$p^6T^4 + 404p^3T^3 + 4422pT^2 + 404T + 1$
15	212	3526	$p^6T^4 + 212p^3T^3 + 3526pT^2 + 212T + 1$
16	32	2270	$p^6T^4 + 32p^3T^3 + 2270pT^2 + 32T + 1$
17	-148	2838	$p^6T^4 - 148p^3T^3 + 2838pT^2 - 148T + 1$
18	-100	-538	$p^6T^4 - 100p^3T^3 - 538pT^2 - 100T + 1$
19	1224	12398	$p^6T^4 + 1224p^3T^3 + 12398pT^2 + 1224T + 1$
20	-44	5062	$p^6T^4 - 44p^3T^3 + 5062pT^2 - 44T + 1$
21	-252	918	$p^6T^4 - 252p^3T^3 + 918pT^2 - 252T + 1$
22	340	7878	$p^6T^4 + 340p^3T^3 + 7878pT^2 + 340T + 1$
23	132	-618	$p^6T^4 + 132p^3T^3 - 618pT^2 + 132T + 1$
24	-172	-778	$p^6T^4 - 172p^3T^3 - 778pT^2 - 172T + 1$

$p = 61$ (cont.)

λ	a	b	$R(T)$
25	32	-4898	$p^6T^4 + 32p^3T^3 - 4898pT^2 + 32T + 1$
26	852	7670	$p^6T^4 + 852p^3T^3 + 7670pT^2 + 852T + 1$
27	-696	8558	$p^6T^4 - 696p^3T^3 + 8558pT^2 - 696T + 1$
28	-44	2550	$p^6T^4 - 44p^3T^3 + 2550pT^2 - 44T + 1$
29	-68	4982	$p^6T^4 - 68p^3T^3 + 4982pT^2 - 68T + 1$
30	-172	5622	$p^6T^4 - 172p^3T^3 + 5622pT^2 - 172T + 1$
31	724	6214	$p^6T^4 + 724p^3T^3 + 6214pT^2 + 724T + 1$
32	-172	1782	$p^6T^4 - 172p^3T^3 + 1782pT^2 - 172T + 1$
33	388	150	$p^6T^4 + 388p^3T^3 + 150pT^2 + 388T + 1$
34	-44	966	$p^6T^4 - 44p^3T^3 + 966pT^2 - 44T + 1$
35	1028	10902	$p^6T^4 + 1028p^3T^3 + 10902pT^2 + 1028T + 1$
36	-172	-1338	$p^6T^4 - 172p^3T^3 - 1338pT^2 - 172T + 1$
37	316	2678	$p^6T^4 + 316p^3T^3 + 2678pT^2 + 316T + 1$
38	796	8422	$p^6T^4 + 796p^3T^3 + 8422pT^2 + 796T + 1$
39	-72	1358	$p^6T^4 - 72p^3T^3 + 1358pT^2 - 72T + 1$
40	-940	7750	$p^6T^4 - 940p^3T^3 + 7750pT^2 - 940T + 1$
41	84	3270	$p^6T^4 + 84p^3T^3 + 3270pT^2 + 84T + 1$
42	148	3910	$p^6T^4 + 148p^3T^3 + 3910pT^2 + 148T + 1$
43	340	-186	$p^6T^4 + 340p^3T^3 - 186pT^2 + 340T + 1$
44	516	3174	$p^6T^4 + 516p^3T^3 + 3174pT^2 + 516T + 1$
45	-120	-1298	$p^6T^4 - 120p^3T^3 - 1298pT^2 - 120T + 1$
46	120	6606	$p^6T^4 + 120p^3T^3 + 6606pT^2 + 120T + 1$
47	212	-2618	$p^6T^4 + 212p^3T^3 - 2618pT^2 + 212T + 1$
48	648	7918	$p^6T^4 + 648p^3T^3 + 7918pT^2 + 648T + 1$
49	376	6606	$p^6T^4 + 376p^3T^3 + 6606pT^2 + 376T + 1$
50	-1172	12102	$p^6T^4 - 1172p^3T^3 + 12102pT^2 - 1172T + 1$
51	-964	9590	$p^6T^4 - 964p^3T^3 + 9590pT^2 - 964T + 1$
52	-220	4710	$p^6T^4 - 220p^3T^3 + 4710pT^2 - 220T + 1$
53	620	3094	$p^6T^4 + 620p^3T^3 + 3094pT^2 + 620T + 1$
54	108	6470	$p^6T^4 + 108p^3T^3 + 6470pT^2 + 108T + 1$
55	212	326	$p^6T^4 + 212p^3T^3 + 326pT^2 + 212T + 1$
56	-550	3721	$p^3T^2 - 550T + 1$
57	-236	4166	$p^6T^4 - 236p^3T^3 + 4166pT^2 - 236T + 1$

$p = 61$ (cont.)

λ	a	b	$R(T)$
58	144	2750	$p^6T^4 + 144p^3T^3 + 2750pT^2 + 144T + 1$
59	4	2022	$p^6T^4 + 4p^3T^3 + 2022pT^2 + 4T + 1$
60	-236	70	$p^6T^4 - 236p^3T^3 + 70pT^2 - 236T + 1$

$p = 67$

λ	a	b	$R(T)$
1	0	5602	$p^6T^4 + 5602pT^2 + 1$
2	152	-1918	$p^6T^4 + 152p^3T^3 - 1918pT^2 + 152T + 1$
3	88	3970	$p^6T^4 + 88p^3T^3 + 3970pT^2 + 88T + 1$
4	-168	2178	$p^6T^4 - 168p^3T^3 + 2178pT^2 - 168T + 1$
5	164	-1950	$p^6T^4 + 164p^3T^3 - 1950pT^2 + 164T + 1$
6	328	-1470	$p^6T^4 + 328p^3T^3 - 1470pT^2 + 328T + 1$
7	20	-6110	$p^6T^4 + 20p^3T^3 - 6110pT^2 + 20T + 1$
8	812	7874	$p^6T^4 + 812p^3T^3 + 7874pT^2 + 812T + 1$
9	8	-254	$p^6T^4 + 8p^3T^3 - 254pT^2 + 8T + 1$
10	584	7746	$p^6T^4 + 584p^3T^3 + 7746pT^2 + 584T + 1$
11	60	6530	$p^6T^4 + 60p^3T^3 + 6530pT^2 + 60T + 1$
12	-1004	10914	$p^6T^4 - 1004p^3T^3 + 10914pT^2 - 1004T + 1$
13	-1408	14818	$p^6T^4 - 1408p^3T^3 + 14818pT^2 - 1408T + 1$
14	880	8738	$p^6T^4 + 880p^3T^3 + 8738pT^2 + 880T + 1$
15	768	8034	$p^6T^4 + 768p^3T^3 + 8034pT^2 + 768T + 1$
16	24	-382	$p^6T^4 + 24p^3T^3 - 382pT^2 + 24T + 1$
17	152	5250	$p^6T^4 + 152p^3T^3 + 5250pT^2 + 152T + 1$
18	104	322	$p^6T^4 + 104p^3T^3 + 322pT^2 + 104T + 1$
19	-816	9506	$p^6T^4 - 816p^3T^3 + 9506pT^2 - 816T + 1$
20	-16	-734	$p^6T^4 - 16p^3T^3 - 734pT^2 - 16T + 1$
21	1208	12994	$p^6T^4 + 1208p^3T^3 + 12994pT^2 + 1208T + 1$
22	-208	6690	$p^6T^4 - 208p^3T^3 + 6690pT^2 - 208T + 1$
23	-848	9250	$p^6T^4 - 848p^3T^3 + 9250pT^2 - 848T + 1$
24	-56	-1982	$p^6T^4 - 56p^3T^3 - 1982pT^2 - 56T + 1$

$p = 67$ (cont.)

λ	a	b	$R(T)$
25	216	4226	$p^6T^4 + 216p^3T^3 + 4226pT^2 + 216T + 1$
26	128	482	$p^6T^4 + 128p^3T^3 + 482pT^2 + 128T + 1$
27	-540	4578	$p^6T^4 - 540p^3T^3 + 4578pT^2 - 540T + 1$
28	124	-1662	$p^6T^4 + 124p^3T^3 - 1662pT^2 + 124T + 1$
29	-640	7522	$p^6T^4 - 640p^3T^3 + 7522pT^2 - 640T + 1$
30	-424	386	$p^6T^4 - 424p^3T^3 + 386pT^2 - 424T + 1$
31	332	3394	$p^6T^4 + 332p^3T^3 + 3394pT^2 + 332T + 1$
32	-220	866	$p^6T^4 - 220p^3T^3 + 866pT^2 - 220T + 1$
33	-896	10594	$p^6T^4 - 896p^3T^3 + 10594pT^2 - 896T + 1$
34	-492	3362	$p^6T^4 - 492p^3T^3 + 3362pT^2 - 492T + 1$
35	368	3618	$p^6T^4 + 368p^3T^3 + 3618pT^2 + 368T + 1$
36	-632	6402	$p^6T^4 - 632p^3T^3 + 6402pT^2 - 632T + 1$
37	-440	2050	$p^6T^4 - 440p^3T^3 + 2050pT^2 - 440T + 1$
38	128	-2590	$p^6T^4 + 128p^3T^3 - 2590pT^2 + 128T + 1$
39	188	4489	$p^3T^2 + 188T + 1$
40	568	7362	$p^6T^4 + 568p^3T^3 + 7362pT^2 + 568T + 1$
41	-784	6434	$p^6T^4 - 784p^3T^3 + 6434pT^2 - 784T + 1$
42	488	3906	$p^6T^4 + 488p^3T^3 + 3906pT^2 + 488T + 1$
43	-860	6626	$p^6T^4 - 860p^3T^3 + 6626pT^2 - 860T + 1$
44	-384	7650	$p^6T^4 - 384p^3T^3 + 7650pT^2 - 384T + 1$
45	1124	12258	$p^6T^4 + 1124p^3T^3 + 12258pT^2 + 1124T + 1$
46	280	5250	$p^6T^4 + 280p^3T^3 + 5250pT^2 + 280T + 1$
47	-424	5250	$p^6T^4 - 424p^3T^3 + 5250pT^2 - 424T + 1$
48	420	3810	$p^6T^4 + 420p^3T^3 + 3810pT^2 + 420T + 1$
49	-448	482	$p^6T^4 - 448p^3T^3 + 482pT^2 - 448T + 1$
50	-40	2434	$p^6T^4 - 40p^3T^3 + 2434pT^2 - 40T + 1$
51	-656	6434	$p^6T^4 - 656p^3T^3 + 6434pT^2 - 656T + 1$
52	-576	4578	$p^6T^4 - 576p^3T^3 + 4578pT^2 - 576T + 1$
53	380	4610	$p^6T^4 + 380p^3T^3 + 4610pT^2 + 380T + 1$
54	-704	10722	$p^6T^4 - 704p^3T^3 + 10722pT^2 - 704T + 1$
55	-432	2338	$p^6T^4 - 432p^3T^3 + 2338pT^2 - 432T + 1$
56	264	3842	$p^6T^4 + 264p^3T^3 + 3842pT^2 + 264T + 1$
57	576	4578	$p^6T^4 + 576p^3T^3 + 4578pT^2 + 576T + 1$

$p = 67$ (cont.)

λ	a	b	$R(T)$
58	468	6050	$p^6T^4 + 468p^3T^3 + 6050pT^2 + 468T + 1$
59	-144	546	$p^6T^4 - 144p^3T^3 + 546pT^2 - 144T + 1$
60	-1208	11266	$p^6T^4 - 1208p^3T^3 + 11266pT^2 - 1208T + 1$
61	-364	4258	$p^6T^4 - 364p^3T^3 + 4258pT^2 - 364T + 1$
62	384	3554	$p^6T^4 + 384p^3T^3 + 3554pT^2 + 384T + 1$
63	-428	4770	$p^6T^4 - 428p^3T^3 + 4770pT^2 - 428T + 1$
64	-456	9410	$p^6T^4 - 456p^3T^3 + 9410pT^2 - 456T + 1$
65	208	-3806	$p^6T^4 + 208p^3T^3 - 3806pT^2 + 208T + 1$
66	232	3906	$p^6T^4 + 232p^3T^3 + 3906pT^2 + 232T + 1$

$p = 71$

λ	a	b	$R(T)$
1	-1176	10786	$p^6T^4 - 1176p^3T^3 + 10786pT^2 - 1176T + 1$
2	-1176	12322	$p^6T^4 - 1176p^3T^3 + 12322pT^2 - 1176T + 1$
3	-832	11170	$p^6T^4 - 832p^3T^3 + 11170pT^2 - 832T + 1$
4	360	-414	$p^6T^4 + 360p^3T^3 - 414pT^2 + 360T + 1$
5	296	7010	$p^6T^4 + 296p^3T^3 + 7010pT^2 + 296T + 1$
6	-280	-2526	$p^6T^4 - 280p^3T^3 - 2526pT^2 - 280T + 1$
7	124	3154	$p^6T^4 + 124p^3T^3 + 3154pT^2 + 124T + 1$
8	176	-2014	$p^6T^4 + 176p^3T^3 - 2014pT^2 + 176T + 1$
9	240	-3806	$p^6T^4 + 240p^3T^3 - 3806pT^2 + 240T + 1$
10	128	7074	$p^6T^4 + 128p^3T^3 + 7074pT^2 + 128T + 1$
11	-1424	16930	$p^6T^4 - 1424p^3T^3 + 16930pT^2 - 1424T + 1$
12	-664	2658	$p^6T^4 - 664p^3T^3 + 2658pT^2 - 664T + 1$
13	-1280	12194	$p^6T^4 - 1280p^3T^3 + 12194pT^2 - 1280T + 1$
14	-520	9698	$p^6T^4 - 520p^3T^3 + 9698pT^2 - 520T + 1$
15	-496	5666	$p^6T^4 - 496p^3T^3 + 5666pT^2 - 496T + 1$
16	360	6754	$p^6T^4 + 360p^3T^3 + 6754pT^2 + 360T + 1$
17	-1364	15698	$p^6T^4 - 1364p^3T^3 + 15698pT^2 - 1364T + 1$
18	368	9506	$p^6T^4 + 368p^3T^3 + 9506pT^2 + 368T + 1$

$p = 71$ (cont.)

λ	a	b	$R(T)$
19	424	3874	$p^6T^4 + 424p^3T^3 + 3874pT^2 + 424T + 1$
20	480	1442	$p^6T^4 + 480p^3T^3 + 1442pT^2 + 480T + 1$
21	836	4114	$p^6T^4 + 836p^3T^3 + 4114pT^2 + 836T + 1$
22	−136	−30	$p^6T^4 - 136p^3T^3 - 30pT^2 - 136T + 1$
23	−464	3106	$p^6T^4 - 464p^3T^3 + 3106pT^2 - 464T + 1$
24	−792	7266	$p^6T^4 - 792p^3T^3 + 7266pT^2 - 792T + 1$
25	−256	9122	$p^6T^4 - 256p^3T^3 + 9122pT^2 - 256T + 1$
26	888	8162	$p^6T^4 + 888p^3T^3 + 8162pT^2 + 888T + 1$
27	80	6178	$p^6T^4 + 80p^3T^3 + 6178pT^2 + 80T + 1$
28	384	−2142	$p^6T^4 + 384p^3T^3 - 2142pT^2 + 384T + 1$
29	1376	14754	$p^6T^4 + 1376p^3T^3 + 14754pT^2 + 1376T + 1$
30	32	162	$p^6T^4 + 32p^3T^3 + 162pT^2 + 32T + 1$
31	−908	9746	$p^6T^4 - 908p^3T^3 + 9746pT^2 - 908T + 1$
32	−408	5666	$p^6T^4 - 408p^3T^3 + 5666pT^2 - 408T + 1$
33	824	12002	$p^6T^4 + 824p^3T^3 + 12002pT^2 + 824T + 1$
34	120	7138	$p^6T^4 + 120p^3T^3 + 7138pT^2 + 120T + 1$
35	−64	−4190	$p^6T^4 - 64p^3T^3 - 4190pT^2 - 64T + 1$
36	−152	−4062	$p^6T^4 - 152p^3T^3 - 4062pT^2 - 152T + 1$
37	−280	8738	$p^6T^4 - 280p^3T^3 + 8738pT^2 - 280T + 1$
38	728	5041	$p^3T^2 + 728T + 1$
39	772	6802	$p^6T^4 + 772p^3T^3 + 6802pT^2 + 772T + 1$
40	−48	−7134	$p^6T^4 - 48p^3T^3 - 7134pT^2 - 48T + 1$
41	−84	82	$p^6T^4 - 84p^3T^3 + 82pT^2 - 84T + 1$
42	−324	4306	$p^6T^4 - 324p^3T^3 + 4306pT^2 - 324T + 1$
43	−904	11746	$p^6T^4 - 904p^3T^3 + 11746pT^2 - 904T + 1$
44	436	5522	$p^6T^4 + 436p^3T^3 + 5522pT^2 + 436T + 1$
45	368	1314	$p^6T^4 + 368p^3T^3 + 1314pT^2 + 368T + 1$
46	452	9874	$p^6T^4 + 452p^3T^3 + 9874pT^2 + 452T + 1$
47	−1084	12178	$p^6T^4 - 1084p^3T^3 + 12178pT^2 - 1084T + 1$
48	232	−2974	$p^6T^4 + 232p^3T^3 - 2974pT^2 + 232T + 1$
49	−328	7074	$p^6T^4 - 328p^3T^3 + 7074pT^2 - 328T + 1$
50	−768	4002	$p^6T^4 - 768p^3T^3 + 4002pT^2 - 768T + 1$
51	−656	1570	$p^6T^4 - 656p^3T^3 + 1570pT^2 - 656T + 1$

$p = 71$ (cont.)

λ	a	b	$R(T)$
52	840	9314	$p^6T^4 + 840p^3T^3 + 9314pT^2 + 840T + 1$
53	428	9170	$p^6T^4 + 428p^3T^3 + 9170pT^2 + 428T + 1$
54	232	5218	$p^6T^4 + 232p^3T^3 + 5218pT^2 + 232T + 1$
55	-8	-2590	$p^6T^4 - 8p^3T^3 - 2590pT^2 - 8T + 1$
56	388	2194	$p^6T^4 + 388p^3T^3 + 2194pT^2 + 388T + 1$
57	472	9442	$p^6T^4 + 472p^3T^3 + 9442pT^2 + 472T + 1$
58	464	8226	$p^6T^4 + 464p^3T^3 + 8226pT^2 + 464T + 1$
59	128	-94	$p^6T^4 + 128p^3T^3 - 94pT^2 + 128T + 1$
60	-616	3234	$p^6T^4 - 616p^3T^3 + 3234pT^2 - 616T + 1$
61	-136	-1566	$p^6T^4 - 136p^3T^3 - 1566pT^2 - 136T + 1$
62	-148	4818	$p^6T^4 - 148p^3T^3 + 4818pT^2 - 148T + 1$
63	236	5586	$p^6T^4 + 236p^3T^3 + 5586pT^2 + 236T + 1$
64	-160	-606	$p^6T^4 - 160p^3T^3 - 606pT^2 - 160T + 1$
65	-260	1490	$p^6T^4 - 260p^3T^3 + 1490pT^2 - 260T + 1$
66	40	-1694	$p^6T^4 + 40p^3T^3 - 1694pT^2 + 40T + 1$
67	500	146	$p^6T^4 + 500p^3T^3 + 146pT^2 + 500T + 1$
68	-572	5906	$p^6T^4 - 572p^3T^3 + 5906pT^2 - 572T + 1$
69	-272	9762	$p^6T^4 - 272p^3T^3 + 9762pT^2 - 272T + 1$
70	-212	7634	$p^6T^4 - 212p^3T^3 + 7634pT^2 - 212T + 1$

$p = 73$

λ	a	b	$R(T)$
1	-708	8134	$p^6T^4 - 708p^3T^3 + 8134pT^2 - 708T + 1$
2	-154	5329	$p^3T^2 - 154T + 1$
3	-260	2886	$p^6T^4 - 260p^3T^3 + 2886pT^2 - 260T + 1$
4	188	8390	$p^6T^4 + 188p^3T^3 + 8390pT^2 + 188T + 1$
5	852	9462	$p^6T^4 + 852p^3T^3 + 9462pT^2 + 852T + 1$
6	124	-954	$p^6T^4 + 124p^3T^3 - 954pT^2 + 124T + 1$
7	-12	4262	$p^6T^4 - 12p^3T^3 + 4262pT^2 - 12T + 1$
8	12	-858	$p^6T^4 + 12p^3T^3 - 858pT^2 + 12T + 1$

$p = 73$ (cont.)

λ	a	b	$R(T)$
9	-48	5374	$p^6T^4 - 48p^3T^3 + 5374pT^2 - 48T + 1$
10	-356	4678	$p^6T^4 - 356p^3T^3 + 4678pT^2 - 356T + 1$
11	-1396	14886	$p^6T^4 - 1396p^3T^3 + 14886pT^2 - 1396T + 1$
12	728	3310	$p^6T^4 + 728p^3T^3 + 3310pT^2 + 728T + 1$
13	516	6470	$p^6T^4 + 516p^3T^3 + 6470pT^2 + 516T + 1$
14	340	5750	$p^6T^4 + 340p^3T^3 + 5750pT^2 + 340T + 1$
15	-252	2886	$p^6T^4 - 252p^3T^3 + 2886pT^2 - 252T + 1$
16	-192	6558	$p^6T^4 - 192p^3T^3 + 6558pT^2 - 192T + 1$
17	844	11542	$p^6T^4 + 844p^3T^3 + 11542pT^2 + 844T + 1$
18	-372	7078	$p^6T^4 - 372p^3T^3 + 7078pT^2 - 372T + 1$
19	-872	10350	$p^6T^4 - 872p^3T^3 + 10350pT^2 - 872T + 1$
20	268	1446	$p^6T^4 + 268p^3T^3 + 1446pT^2 + 268T + 1$
21	1252	14614	$p^6T^4 + 1252p^3T^3 + 14614pT^2 + 1252T + 1$
22	700	1398	$p^6T^4 + 700p^3T^3 + 1398pT^2 + 700T + 1$
23	712	6542	$p^6T^4 + 712p^3T^3 + 6542pT^2 + 712T + 1$
24	392	8974	$p^6T^4 + 392p^3T^3 + 8974pT^2 + 392T + 1$
25	188	6854	$p^6T^4 + 188p^3T^3 + 6854pT^2 + 188T + 1$
26	356	2454	$p^6T^4 + 356p^3T^3 + 2454pT^2 + 356T + 1$
27	-500	8358	$p^6T^4 - 500p^3T^3 + 8358pT^2 - 500T + 1$
28	-180	2070	$p^6T^4 - 180p^3T^3 + 2070pT^2 - 180T + 1$
29	1108	11766	$p^6T^4 + 1108p^3T^3 + 11766pT^2 + 1108T + 1$
30	-580	5494	$p^6T^4 - 580p^3T^3 + 5494pT^2 - 580T + 1$
31	-396	-1626	$p^6T^4 - 396p^3T^3 - 1626pT^2 - 396T + 1$
32	-256	-5090	$p^6T^4 - 256p^3T^3 - 5090pT^2 - 256T + 1$
33	700	7798	$p^6T^4 + 700p^3T^3 + 7798pT^2 + 700T + 1$
34	28	5958	$p^6T^4 + 28p^3T^3 + 5958pT^2 + 28T + 1$
35	1352	16014	$p^6T^4 + 1352p^3T^3 + 16014pT^2 + 1352T + 1$
36	80	-4098	$p^6T^4 + 80p^3T^3 - 4098pT^2 + 80T + 1$
37	-628	1446	$p^6T^4 - 628p^3T^3 + 1446pT^2 - 628T + 1$
38	396	1446	$p^6T^4 + 396p^3T^3 + 1446pT^2 + 396T + 1$
39	668	3270	$p^6T^4 + 668p^3T^3 + 3270pT^2 + 668T + 1$
40	84	3062	$p^6T^4 + 84p^3T^3 + 3062pT^2 + 84T + 1$
41	-112	382	$p^6T^4 - 112p^3T^3 + 382pT^2 - 112T + 1$

$p = 73$ (cont.)

λ	a	b	$R(T)$
42	188	5238	$p^6T^4 + 188p^3T^3 + 5238pT^2 + 188T + 1$
43	-628	1318	$p^6T^4 - 628p^3T^3 + 1318pT^2 - 628T + 1$
44	-396	6566	$p^6T^4 - 396p^3T^3 + 6566pT^2 - 396T + 1$
45	444	7542	$p^6T^4 + 444p^3T^3 + 7542pT^2 + 444T + 1$
46	72	7310	$p^6T^4 + 72p^3T^3 + 7310pT^2 + 72T + 1$
47	844	9494	$p^6T^4 + 844p^3T^3 + 9494pT^2 + 844T + 1$
48	268	678	$p^6T^4 + 268p^3T^3 + 678pT^2 + 268T + 1$
49	-616	8302	$p^6T^4 - 616p^3T^3 + 8302pT^2 - 616T + 1$
50	-68	2246	$p^6T^4 - 68p^3T^3 + 2246pT^2 - 68T + 1$
51	-428	2934	$p^6T^4 - 428p^3T^3 + 2934pT^2 - 428T + 1$
52	84	-1802	$p^6T^4 + 84p^3T^3 - 1802pT^2 + 84T + 1$
53	-836	7542	$p^6T^4 - 836p^3T^3 + 7542pT^2 - 836T + 1$
54	728	4846	$p^6T^4 + 728p^3T^3 + 4846pT^2 + 728T + 1$
55	-448	926	$p^6T^4 - 448p^3T^3 + 926pT^2 - 448T + 1$
56	-540	10134	$p^6T^4 - 540p^3T^3 + 10134pT^2 - 540T + 1$
57	652	5030	$p^6T^4 + 652p^3T^3 + 5030pT^2 + 652T + 1$
58	332	22	$p^6T^4 + 332p^3T^3 + 22pT^2 + 332T + 1$
59	524	2982	$p^6T^4 + 524p^3T^3 + 2982pT^2 + 524T + 1$
60	-124	6726	$p^6T^4 - 124p^3T^3 + 6726pT^2 - 124T + 1$
61	8	-5106	$p^6T^4 + 8p^3T^3 - 5106pT^2 + 8T + 1$
62	-1404	14406	$p^6T^4 - 1404p^3T^3 + 14406pT^2 - 1404T + 1$
63	356	-746	$p^6T^4 + 356p^3T^3 - 746pT^2 + 356T + 1$
64	-128	1822	$p^6T^4 - 128p^3T^3 + 1822pT^2 - 128T + 1$
65	1040	9342	$p^6T^4 + 1040p^3T^3 + 9342pT^2 + 1040T + 1$
66	-996	8646	$p^6T^4 - 996p^3T^3 + 8646pT^2 - 996T + 1$
67	1292	16038	$p^6T^4 + 1292p^3T^3 + 16038pT^2 + 1292T + 1$
68	756	7846	$p^6T^4 + 756p^3T^3 + 7846pT^2 + 756T + 1$
69	892	5958	$p^6T^4 + 892p^3T^3 + 5958pT^2 + 892T + 1$
70	-824	12686	$p^6T^4 - 824p^3T^3 + 12686pT^2 - 824T + 1$
71	-68	-2362	$p^6T^4 - 68p^3T^3 - 2362pT^2 - 68T + 1$
72	-260	6214	$p^6T^4 - 260p^3T^3 + 6214pT^2 - 260T + 1$

$p = 79$

λ	a	b	$R(T)$
1	232	962	$p^6T^4 + 232p^3T^3 + 962pT^2 + 232T + 1$
2	472	5250	$p^6T^4 + 472p^3T^3 + 5250pT^2 + 472T + 1$
3	−280	7746	$p^6T^4 - 280p^3T^3 + 7746pT^2 - 280T + 1$
4	728	3714	$p^6T^4 + 728p^3T^3 + 3714pT^2 + 728T + 1$
5	−1008	15298	$p^6T^4 - 1008p^3T^3 + 15298pT^2 - 1008T + 1$
6	−224	11202	$p^6T^4 - 224p^3T^3 + 11202pT^2 - 224T + 1$
7	−1544	19266	$p^6T^4 - 1544p^3T^3 + 19266pT^2 - 1544T + 1$
8	−432	10946	$p^6T^4 - 432p^3T^3 + 10946pT^2 - 432T + 1$
9	−432	−4414	$p^6T^4 - 432p^3T^3 - 4414pT^2 - 432T + 1$
10	−1096	11970	$p^6T^4 - 1096p^3T^3 + 11970pT^2 - 1096T + 1$
11	−352	10178	$p^6T^4 - 352p^3T^3 + 10178pT^2 - 352T + 1$
12	576	450	$p^6T^4 + 576p^3T^3 + 450pT^2 + 576T + 1$
13	576	9666	$p^6T^4 + 576p^3T^3 + 9666pT^2 + 576T + 1$
14	−672	6082	$p^6T^4 - 672p^3T^3 + 6082pT^2 - 672T + 1$
15	−1816	22594	$p^6T^4 - 1816p^3T^3 + 22594pT^2 - 1816T + 1$
16	720	9922	$p^6T^4 + 720p^3T^3 + 9922pT^2 + 720T + 1$
17	−528	−2878	$p^6T^4 - 528p^3T^3 - 2878pT^2 - 528T + 1$
18	−664	12738	$p^6T^4 - 664p^3T^3 + 12738pT^2 - 664T + 1$
19	992	6082	$p^6T^4 + 992p^3T^3 + 6082pT^2 + 992T + 1$
20	1168	11202	$p^6T^4 + 1168p^3T^3 + 11202pT^2 + 1168T + 1$
21	−312	4994	$p^6T^4 - 312p^3T^3 + 4994pT^2 - 312T + 1$
22	56	5314	$p^6T^4 + 56p^3T^3 + 5314pT^2 + 56T + 1$
23	328	2434	$p^6T^4 + 328p^3T^3 + 2434pT^2 + 328T + 1$
24	248	1346	$p^6T^4 + 248p^3T^3 + 1346pT^2 + 248T + 1$
25	−656	6241	$p^3T^2 - 656T + 1$
26	440	−3390	$p^6T^4 + 440p^3T^3 - 3390pT^2 + 440T + 1$
27	−716	3986	$p^6T^4 - 716p^3T^3 + 3986pT^2 - 716T + 1$
28	−344	−1214	$p^6T^4 - 344p^3T^3 - 1214pT^2 - 344T + 1$
29	−1364	13330	$p^6T^4 - 1364p^3T^3 + 13330pT^2 - 1364T + 1$
30	−92	−8430	$p^6T^4 - 92p^3T^3 - 8430pT^2 - 92T + 1$
31	264	10370	$p^6T^4 + 264p^3T^3 + 10370pT^2 + 264T + 1$
32	968	10114	$p^6T^4 + 968p^3T^3 + 10114pT^2 + 968T + 1$
33	100	3090	$p^6T^4 + 100p^3T^3 + 3090pT^2 + 100T + 1$

$p = 79$ (cont.)

λ	a	b	$R(T)$
34	-4	-1518	$p^6T^4 - 4p^3T^3 - 1518pT^2 - 4T + 1$
35	-524	2450	$p^6T^4 - 524p^3T^3 + 2450pT^2 - 524T + 1$
36	-664	5570	$p^6T^4 - 664p^3T^3 + 5570pT^2 - 664T + 1$
37	440	-1470	$p^6T^4 + 440p^3T^3 - 1470pT^2 + 440T + 1$
38	-1032	14274	$p^6T^4 - 1032p^3T^3 + 14274pT^2 - 1032T + 1$
39	-1024	10690	$p^6T^4 - 1024p^3T^3 + 10690pT^2 - 1024T + 1$
40	1632	17090	$p^6T^4 + 1632p^3T^3 + 17090pT^2 + 1632T + 1$
41	-588	11026	$p^6T^4 - 588p^3T^3 + 11026pT^2 - 588T + 1$
42	-864	12226	$p^6T^4 - 864p^3T^3 + 12226pT^2 - 864T + 1$
43	-76	-3566	$p^6T^4 - 76p^3T^3 - 3566pT^2 - 76T + 1$
44	976	9922	$p^6T^4 + 976p^3T^3 + 9922pT^2 + 976T + 1$
45	-728	8898	$p^6T^4 - 728p^3T^3 + 8898pT^2 - 728T + 1$
46	-368	-1854	$p^6T^4 - 368p^3T^3 - 1854pT^2 - 368T + 1$
47	-276	-110	$p^6T^4 - 276p^3T^3 - 110pT^2 - 276T + 1$
48	364	-5358	$p^6T^4 + 364p^3T^3 - 5358pT^2 + 364T + 1$
49	-488	6018	$p^6T^4 - 488p^3T^3 + 6018pT^2 - 488T + 1$
50	-240	9410	$p^6T^4 - 240p^3T^3 + 9410pT^2 - 240T + 1$
51	-624	8386	$p^6T^4 - 624p^3T^3 + 8386pT^2 - 624T + 1$
52	-112	-8254	$p^6T^4 - 112p^3T^3 - 8254pT^2 - 112T + 1$
53	176	-7486	$p^6T^4 + 176p^3T^3 - 7486pT^2 + 176T + 1$
54	816	3778	$p^6T^4 + 816p^3T^3 + 3778pT^2 + 816T + 1$
55	-1912	23682	$p^6T^4 - 1912p^3T^3 + 23682pT^2 - 1912T + 1$
56	-332	7058	$p^6T^4 - 332p^3T^3 + 7058pT^2 - 332T + 1$
57	628	3602	$p^6T^4 + 628p^3T^3 + 3602pT^2 + 628T + 1$
58	304	-318	$p^6T^4 + 304p^3T^3 - 318pT^2 + 304T + 1$
59	56	11842	$p^6T^4 + 56p^3T^3 + 11842pT^2 + 56T + 1$
60	-400	10434	$p^6T^4 - 400p^3T^3 + 10434pT^2 - 400T + 1$
61	1068	14226	$p^6T^4 + 1068p^3T^3 + 14226pT^2 + 1068T + 1$
62	-128	10690	$p^6T^4 - 128p^3T^3 + 10690pT^2 - 128T + 1$
63	-84	7314	$p^6T^4 - 84p^3T^3 + 7314pT^2 - 84T + 1$
64	-576	5314	$p^6T^4 - 576p^3T^3 + 5314pT^2 - 576T + 1$
65	-128	2498	$p^6T^4 - 128p^3T^3 + 2498pT^2 - 128T + 1$
66	296	8002	$p^6T^4 + 296p^3T^3 + 8002pT^2 + 296T + 1$

$p = 79$ (cont.)

λ	a	b	$R(T)$
67	-216	-5438	$p^6T^4 - 216p^3T^3 - 5438pT^2 - 216T + 1$
68	888	6978	$p^6T^4 + 888p^3T^3 + 6978pT^2 + 888T + 1$
69	172	1298	$p^6T^4 + 172p^3T^3 + 1298pT^2 + 172T + 1$
70	228	658	$p^6T^4 + 228p^3T^3 + 658pT^2 + 228T + 1$
71	188	9618	$p^6T^4 + 188p^3T^3 + 9618pT^2 + 188T + 1$
72	-552	4738	$p^6T^4 - 552p^3T^3 + 4738pT^2 - 552T + 1$
73	408	9090	$p^6T^4 + 408p^3T^3 + 9090pT^2 + 408T + 1$
74	1148	12434	$p^6T^4 + 1148p^3T^3 + 12434pT^2 + 1148T + 1$
75	-652	9362	$p^6T^4 - 652p^3T^3 + 9362pT^2 - 652T + 1$
76	504	8130	$p^6T^4 + 504p^3T^3 + 8130pT^2 + 504T + 1$
77	748	10002	$p^6T^4 + 748p^3T^3 + 10002pT^2 + 748T + 1$
78	-24	3138	$p^6T^4 - 24p^3T^3 + 3138pT^2 - 24T + 1$

$p = 83$

λ	a	b	$R(T)$
1	-424	9282	$p^6T^4 - 424p^3T^3 + 9282pT^2 - 424T + 1$
2	-80	-3742	$p^6T^4 - 80p^3T^3 - 3742pT^2 - 80T + 1$
3	-1280	11170	$p^6T^4 - 1280p^3T^3 + 11170pT^2 - 1280T + 1$
4	-824	4546	$p^6T^4 - 824p^3T^3 + 4546pT^2 - 824T + 1$
5	-744	5186	$p^6T^4 - 744p^3T^3 + 5186pT^2 - 744T + 1$
6	-148	5954	$p^6T^4 - 148p^3T^3 + 5954pT^2 - 148T + 1$
7	-184	-3134	$p^6T^4 - 184p^3T^3 - 3134pT^2 - 184T + 1$
8	792	9794	$p^6T^4 + 792p^3T^3 + 9794pT^2 + 792T + 1$
9	360	4482	$p^6T^4 + 360p^3T^3 + 4482pT^2 + 360T + 1$
10	1552	19810	$p^6T^4 + 1552p^3T^3 + 19810pT^2 + 1552T + 1$
11	-1264	11746	$p^6T^4 - 1264p^3T^3 + 11746pT^2 - 1264T + 1$
12	236	6889	$p^3T^2 + 236T + 1$
13	-748	14754	$p^6T^4 - 748p^3T^3 + 14754pT^2 - 748T + 1$
14	-1000	11330	$p^6T^4 - 1000p^3T^3 + 11330pT^2 - 1000T + 1$
15	-868	10626	$p^6T^4 - 868p^3T^3 + 10626pT^2 - 868T + 1$

$p = 83$ (cont.)

λ	a	b	$R(T)$
16	400	4450	$p^6T^4 + 400p^3T^3 + 4450pT^2 + 400T + 1$
17	104	−4222	$p^6T^4 + 104p^3T^3 - 4222pT^2 + 104T + 1$
18	416	11170	$p^6T^4 + 416p^3T^3 + 11170pT^2 + 416T + 1$
19	−36	2818	$p^6T^4 - 36p^3T^3 + 2818pT^2 - 36T + 1$
20	−568	2690	$p^6T^4 - 568p^3T^3 + 2690pT^2 - 568T + 1$
21	−488	5378	$p^6T^4 - 488p^3T^3 + 5378pT^2 - 488T + 1$
22	148	10530	$p^6T^4 + 148p^3T^3 + 10530pT^2 + 148T + 1$
23	−104	4866	$p^6T^4 - 104p^3T^3 + 4866pT^2 - 104T + 1$
24	−1364	14786	$p^6T^4 - 1364p^3T^3 + 14786pT^2 - 1364T + 1$
25	−920	8578	$p^6T^4 - 920p^3T^3 + 8578pT^2 - 920T + 1$
26	−816	14306	$p^6T^4 - 816p^3T^3 + 14306pT^2 - 816T + 1$
27	−680	13890	$p^6T^4 - 680p^3T^3 + 13890pT^2 - 680T + 1$
28	400	2402	$p^6T^4 + 400p^3T^3 + 2402pT^2 + 400T + 1$
29	912	12642	$p^6T^4 + 912p^3T^3 + 12642pT^2 + 912T + 1$
30	−216	−6270	$p^6T^4 - 216p^3T^3 - 6270pT^2 - 216T + 1$
31	−920	9090	$p^6T^4 - 920p^3T^3 + 9090pT^2 - 920T + 1$
32	−424	2370	$p^6T^4 - 424p^3T^3 + 2370pT^2 - 424T + 1$
33	−32	9122	$p^6T^4 - 32p^3T^3 + 9122pT^2 - 32T + 1$
34	−96	4002	$p^6T^4 - 96p^3T^3 + 4002pT^2 - 96T + 1$
35	−164	8322	$p^6T^4 - 164p^3T^3 + 8322pT^2 - 164T + 1$
36	−752	5602	$p^6T^4 - 752p^3T^3 + 5602pT^2 - 752T + 1$
37	136	8386	$p^6T^4 + 136p^3T^3 + 8386pT^2 + 136T + 1$
38	1104	13154	$p^6T^4 + 1104p^3T^3 + 13154pT^2 + 1104T + 1$
39	684	−574	$p^6T^4 + 684p^3T^3 - 574pT^2 + 684T + 1$
40	24	5378	$p^6T^4 + 24p^3T^3 + 5378pT^2 + 24T + 1$
41	416	5026	$p^6T^4 + 416p^3T^3 + 5026pT^2 + 416T + 1$
42	−1428	13634	$p^6T^4 - 1428p^3T^3 + 13634pT^2 - 1428T + 1$
43	516	12898	$p^6T^4 + 516p^3T^3 + 12898pT^2 + 516T + 1$
44	1176	15426	$p^6T^4 + 1176p^3T^3 + 15426pT^2 + 1176T + 1$
45	172	−1598	$p^6T^4 + 172p^3T^3 - 1598pT^2 + 172T + 1$
46	224	−3166	$p^6T^4 + 224p^3T^3 - 3166pT^2 + 224T + 1$
47	136	5506	$p^6T^4 + 136p^3T^3 + 5506pT^2 + 136T + 1$
48	−488	−766	$p^6T^4 - 488p^3T^3 - 766pT^2 - 488T + 1$

$p = 83$ (cont.)

λ	a	b	$R(T)$
49	-480	6050	$p^6T^4 - 480p^3T^3 + 6050pT^2 - 480T + 1$
50	364	-5694	$p^6T^4 + 364p^3T^3 - 5694pT^2 + 364T + 1$
51	336	482	$p^6T^4 + 336p^3T^3 + 482pT^2 + 336T + 1$
52	340	6818	$p^6T^4 + 340p^3T^3 + 6818pT^2 + 340T + 1$
53	-1428	13634	$p^6T^4 - 1428p^3T^3 + 13634pT^2 - 1428T + 1$
54	608	-94	$p^6T^4 + 608p^3T^3 - 94pT^2 + 608T + 1$
55	904	5506	$p^6T^4 + 904p^3T^3 + 5506pT^2 + 904T + 1$
56	364	3522	$p^6T^4 + 364p^3T^3 + 3522pT^2 + 364T + 1$
57	240	7010	$p^6T^4 + 240p^3T^3 + 7010pT^2 + 240T + 1$
58	-552	12098	$p^6T^4 - 552p^3T^3 + 12098pT^2 - 552T + 1$
59	88	-3006	$p^6T^4 + 88p^3T^3 - 3006pT^2 + 88T + 1$
60	444	8450	$p^6T^4 + 444p^3T^3 + 8450pT^2 + 444T + 1$
61	-1056	9122	$p^6T^4 - 1056p^3T^3 + 9122pT^2 - 1056T + 1$
62	-992	10146	$p^6T^4 - 992p^3T^3 + 10146pT^2 - 992T + 1$
63	-1016	11970	$p^6T^4 - 1016p^3T^3 + 11970pT^2 - 1016T + 1$
64	-1408	12706	$p^6T^4 - 1408p^3T^3 + 12706pT^2 - 1408T + 1$
65	576	8610	$p^6T^4 + 576p^3T^3 + 8610pT^2 + 576T + 1$
66	-208	-6814	$p^6T^4 - 208p^3T^3 - 6814pT^2 - 208T + 1$
67	944	7522	$p^6T^4 + 944p^3T^3 + 7522pT^2 + 944T + 1$
68	-880	13282	$p^6T^4 - 880p^3T^3 + 13282pT^2 - 880T + 1$
69	1560	19010	$p^6T^4 + 1560p^3T^3 + 19010pT^2 + 1560T + 1$
70	-256	-5214	$p^6T^4 - 256p^3T^3 - 5214pT^2 - 256T + 1$
71	108	-3646	$p^6T^4 + 108p^3T^3 - 3646pT^2 + 108T + 1$
72	-376	8066	$p^6T^4 - 376p^3T^3 + 8066pT^2 - 376T + 1$
73	-892	15586	$p^6T^4 - 892p^3T^3 + 15586pT^2 - 892T + 1$
74	392	13698	$p^6T^4 + 392p^3T^3 + 13698pT^2 + 392T + 1$
75	1624	16386	$p^6T^4 + 1624p^3T^3 + 16386pT^2 + 1624T + 1$
76	124	-3198	$p^6T^4 + 124p^3T^3 - 3198pT^2 + 124T + 1$
77	-928	10146	$p^6T^4 - 928p^3T^3 + 10146pT^2 - 928T + 1$
78	640	1442	$p^6T^4 + 640p^3T^3 + 1442pT^2 + 640T + 1$
79	1840	19810	$p^6T^4 + 1840p^3T^3 + 19810pT^2 + 1840T + 1$
80	-516	2562	$p^6T^4 - 516p^3T^3 + 2562pT^2 - 516T + 1$
81	328	5058	$p^6T^4 + 328p^3T^3 + 5058pT^2 + 328T + 1$

$p = 83$ (cont.)

λ	a	b	$R(T)$
82	-572	8546	$p^6T^4 - 572p^3T^3 + 8546pT^2 - 572T + 1$

$p = 89$

λ	a	b	$R(T)$
1	-864	10142	$p^6T^4 - 864p^3T^3 + 10142pT^2 - 864T + 1$
2	96	4638	$p^6T^4 + 96p^3T^3 + 4638pT^2 + 96T + 1$
3	772	6358	$p^6T^4 + 772p^3T^3 + 6358pT^2 + 772T + 1$
4	-36	7558	$p^6T^4 - 36p^3T^3 + 7558pT^2 - 36T + 1$
5	-1000	11694	$p^6T^4 - 1000p^3T^3 + 11694pT^2 - 1000T + 1$
6	588	-858	$p^6T^4 + 588p^3T^3 - 858pT^2 + 588T + 1$
7	716	4902	$p^6T^4 + 716p^3T^3 + 4902pT^2 + 716T + 1$
8	-714	7921	$p^3T^2 - 714T + 1$
9	1164	11942	$p^6T^4 + 1164p^3T^3 + 11942pT^2 + 1164T + 1$
10	140	9894	$p^6T^4 + 140p^3T^3 + 9894pT^2 + 140T + 1$
11	464	-2626	$p^6T^4 + 464p^3T^3 - 2626pT^2 + 464T + 1$
12	444	2438	$p^6T^4 + 444p^3T^3 + 2438pT^2 + 444T + 1$
13	-764	4182	$p^6T^4 - 764p^3T^3 + 4182pT^2 - 764T + 1$
14	372	12454	$p^6T^4 + 372p^3T^3 + 12454pT^2 + 372T + 1$
15	-100	-6986	$p^6T^4 - 100p^3T^3 - 6986pT^2 - 100T + 1$
16	348	6790	$p^6T^4 + 348p^3T^3 + 6790pT^2 + 348T + 1$
17	-756	14246	$p^6T^4 - 756p^3T^3 + 14246pT^2 - 756T + 1$
18	92	-10618	$p^6T^4 + 92p^3T^3 - 10618pT^2 + 92T + 1$
19	844	3878	$p^6T^4 + 844p^3T^3 + 3878pT^2 + 844T + 1$
20	344	8238	$p^6T^4 + 344p^3T^3 + 8238pT^2 + 344T + 1$
21	-232	7598	$p^6T^4 - 232p^3T^3 + 7598pT^2 - 232T + 1$
22	1680	17470	$p^6T^4 + 1680p^3T^3 + 17470pT^2 + 1680T + 1$
23	-68	5622	$p^6T^4 - 68p^3T^3 + 5622pT^2 - 68T + 1$
24	-1092	13062	$p^6T^4 - 1092p^3T^3 + 13062pT^2 - 1092T + 1$
25	1036	6566	$p^6T^4 + 1036p^3T^3 + 6566pT^2 + 1036T + 1$
26	404	3318	$p^6T^4 + 404p^3T^3 + 3318pT^2 + 404T + 1$

$p = 89$ (cont.)

λ	a	b	$R(T)$
27	-780	17318	$p^6T^4 - 780p^3T^3 + 17318pT^2 - 780T + 1$
28	-780	3766	$p^6T^4 - 780p^3T^3 + 3766pT^2 - 780T + 1$
29	-268	15286	$p^6T^4 - 268p^3T^3 + 15286pT^2 - 268T + 1$
30	332	-3290	$p^6T^4 + 332p^3T^3 - 3290pT^2 + 332T + 1$
31	-68	-6666	$p^6T^4 - 68p^3T^3 - 6666pT^2 - 68T + 1$
32	-1312	14622	$p^6T^4 - 1312p^3T^3 + 14622pT^2 - 1312T + 1$
33	-820	6438	$p^6T^4 - 820p^3T^3 + 6438pT^2 - 820T + 1$
34	-88	-1266	$p^6T^4 - 88p^3T^3 - 1266pT^2 - 88T + 1$
35	-1564	15878	$p^6T^4 - 1564p^3T^3 + 15878pT^2 - 1564T + 1$
36	1960	24846	$p^6T^4 + 1960p^3T^3 + 24846pT^2 + 1960T + 1$
37	-1164	12454	$p^6T^4 - 1164p^3T^3 + 12454pT^2 - 1164T + 1$
38	-796	10246	$p^6T^4 - 796p^3T^3 + 10246pT^2 - 796T + 1$
39	32	-2914	$p^6T^4 + 32p^3T^3 - 2914pT^2 + 32T + 1$
40	652	15014	$p^6T^4 + 652p^3T^3 + 15014pT^2 + 652T + 1$
41	-1676	22694	$p^6T^4 - 1676p^3T^3 + 22694pT^2 - 1676T + 1$
42	396	678	$p^6T^4 + 396p^3T^3 + 678pT^2 + 396T + 1$
43	300	1110	$p^6T^4 + 300p^3T^3 + 1110pT^2 + 300T + 1$
44	284	-1274	$p^6T^4 + 284p^3T^3 - 1274pT^2 + 284T + 1$
45	800	5790	$p^6T^4 + 800p^3T^3 + 5790pT^2 + 800T + 1$
46	1068	13654	$p^6T^4 + 1068p^3T^3 + 13654pT^2 + 1068T + 1$
47	-100	2310	$p^6T^4 - 100p^3T^3 + 2310pT^2 - 100T + 1$
48	1044	15094	$p^6T^4 + 1044p^3T^3 + 15094pT^2 + 1044T + 1$
49	-600	15118	$p^6T^4 - 600p^3T^3 + 15118pT^2 - 600T + 1$
50	-100	5638	$p^6T^4 - 100p^3T^3 + 5638pT^2 - 100T + 1$
51	-468	8790	$p^6T^4 - 468p^3T^3 + 8790pT^2 - 468T + 1$
52	-436	9766	$p^6T^4 - 436p^3T^3 + 9766pT^2 - 436T + 1$
53	28	13318	$p^6T^4 + 28p^3T^3 + 13318pT^2 + 28T + 1$
54	812	13142	$p^6T^4 + 812p^3T^3 + 13142pT^2 + 812T + 1$
55	-2676	35494	$p^6T^4 - 2676p^3T^3 + 35494pT^2 - 2676T + 1$
56	372	-1882	$p^6T^4 + 372p^3T^3 - 1882pT^2 + 372T + 1$
57	-116	-346	$p^6T^4 - 116p^3T^3 - 346pT^2 - 116T + 1$
58	332	7206	$p^6T^4 + 332p^3T^3 + 7206pT^2 + 332T + 1$
59	1124	10502	$p^6T^4 + 1124p^3T^3 + 10502pT^2 + 1124T + 1$

$p = 89$ (cont.)

λ	a	b	$R(T)$
60	372	7334	$p^6T^4 + 372p^3T^3 + 7334pT^2 + 372T + 1$
61	−108	9974	$p^6T^4 - 108p^3T^3 + 9974pT^2 - 108T + 1$
62	356	14086	$p^6T^4 + 356p^3T^3 + 14086pT^2 + 356T + 1$
63	1300	12022	$p^6T^4 + 1300p^3T^3 + 12022pT^2 + 1300T + 1$
64	−96	−5730	$p^6T^4 - 96p^3T^3 - 5730pT^2 - 96T + 1$
65	588	422	$p^6T^4 + 588p^3T^3 + 422pT^2 + 588T + 1$
66	−68	11766	$p^6T^4 - 68p^3T^3 + 11766pT^2 - 68T + 1$
67	928	12702	$p^6T^4 + 928p^3T^3 + 12702pT^2 + 928T + 1$
68	1288	9038	$p^6T^4 + 1288p^3T^3 + 9038pT^2 + 1288T + 1$
69	−120	6222	$p^6T^4 - 120p^3T^3 + 6222pT^2 - 120T + 1$
70	500	3494	$p^6T^4 + 500p^3T^3 + 3494pT^2 + 500T + 1$
71	168	8974	$p^6T^4 + 168p^3T^3 + 8974pT^2 + 168T + 1$
72	12	−90	$p^6T^4 + 12p^3T^3 - 90pT^2 + 12T + 1$
73	912	11838	$p^6T^4 + 912p^3T^3 + 11838pT^2 + 912T + 1$
74	1468	15862	$p^6T^4 + 1468p^3T^3 + 15862pT^2 + 1468T + 1$
75	1852	23302	$p^6T^4 + 1852p^3T^3 + 23302pT^2 + 1852T + 1$
76	−636	9942	$p^6T^4 - 636p^3T^3 + 9942pT^2 - 636T + 1$
77	−124	5206	$p^6T^4 - 124p^3T^3 + 5206pT^2 - 124T + 1$
78	−1632	17822	$p^6T^4 - 1632p^3T^3 + 17822pT^2 - 1632T + 1$
79	−676	6278	$p^6T^4 - 676p^3T^3 + 6278pT^2 - 676T + 1$
80	12	−1626	$p^6T^4 + 12p^3T^3 - 1626pT^2 + 12T + 1$
81	92	134	$p^6T^4 + 92p^3T^3 + 134pT^2 + 92T + 1$
82	−100	10934	$p^6T^4 - 100p^3T^3 + 10934pT^2 - 100T + 1$
83	372	4262	$p^6T^4 + 372p^3T^3 + 4262pT^2 + 372T + 1$
84	1368	13870	$p^6T^4 + 1368p^3T^3 + 13870pT^2 + 1368T + 1$
85	−48	−2626	$p^6T^4 - 48p^3T^3 - 2626pT^2 - 48T + 1$
86	−452	7046	$p^6T^4 - 452p^3T^3 + 7046pT^2 - 452T + 1$
87	−628	12966	$p^6T^4 - 628p^3T^3 + 12966pT^2 - 628T + 1$
88	1420	15014	$p^6T^4 + 1420p^3T^3 + 15014pT^2 + 1420T + 1$

$p = 97$

λ	a	b	$R(T)$
1	652	10726	$p^6T^4 + 652p^3T^3 + 10726pT^2 + 652T + 1$
2	-1256	9358	$p^6T^4 - 1256p^3T^3 + 9358pT^2 - 1256T + 1$
3	220	-7098	$p^6T^4 + 220p^3T^3 - 7098pT^2 + 220T + 1$
4	988	2118	$p^6T^4 + 988p^3T^3 + 2118pT^2 + 988T + 1$
5	404	16054	$p^6T^4 + 404p^3T^3 + 16054pT^2 + 404T + 1$
6	-672	9150	$p^6T^4 - 672p^3T^3 + 9150pT^2 - 672T + 1$
7	412	-6858	$p^6T^4 + 412p^3T^3 - 6858pT^2 + 412T + 1$
8	424	14702	$p^6T^4 + 424p^3T^3 + 14702pT^2 + 424T + 1$
9	732	12870	$p^6T^4 + 732p^3T^3 + 12870pT^2 + 732T + 1$
10	548	17478	$p^6T^4 + 548p^3T^3 + 17478pT^2 + 548T + 1$
11	-1396	21478	$p^6T^4 - 1396p^3T^3 + 21478pT^2 - 1396T + 1$
12	-180	-5274	$p^6T^4 - 180p^3T^3 - 5274pT^2 - 180T + 1$
13	-108	7974	$p^6T^4 - 108p^3T^3 + 7974pT^2 - 108T + 1$
14	-1892	20806	$p^6T^4 - 1892p^3T^3 + 20806pT^2 - 1892T + 1$
15	668	7990	$p^6T^4 + 668p^3T^3 + 7990pT^2 + 668T + 1$
16	-288	4286	$p^6T^4 - 288p^3T^3 + 4286pT^2 - 288T + 1$
17	-2268	27206	$p^6T^4 - 2268p^3T^3 + 27206pT^2 - 2268T + 1$
18	-468	11942	$p^6T^4 - 468p^3T^3 + 11942pT^2 - 468T + 1$
19	548	14406	$p^6T^4 + 548p^3T^3 + 14406pT^2 + 548T + 1$
20	1564	18758	$p^6T^4 + 1564p^3T^3 + 18758pT^2 + 1564T + 1$
21	-356	-5578	$p^6T^4 - 356p^3T^3 - 5578pT^2 - 356T + 1$
22	-544	-7490	$p^6T^4 - 544p^3T^3 - 7490pT^2 - 544T + 1$
23	-236	9254	$p^6T^4 - 236p^3T^3 + 9254pT^2 - 236T + 1$
24	76	4198	$p^6T^4 + 76p^3T^3 + 4198pT^2 + 76T + 1$
25	-372	5094	$p^6T^4 - 372p^3T^3 + 5094pT^2 - 372T + 1$
26	1316	21062	$p^6T^4 + 1316p^3T^3 + 21062pT^2 + 1316T + 1$
27	220	11334	$p^6T^4 + 220p^3T^3 + 11334pT^2 + 220T + 1$
28	804	18646	$p^6T^4 + 804p^3T^3 + 18646pT^2 + 804T + 1$
29	-100	1078	$p^6T^4 - 100p^3T^3 + 1078pT^2 - 100T + 1$
30	2204	29766	$p^6T^4 + 2204p^3T^3 + 29766pT^2 + 2204T + 1$
31	1240	9742	$p^6T^4 + 1240p^3T^3 + 9742pT^2 + 1240T + 1$
32	-36	13894	$p^6T^4 - 36p^3T^3 + 13894pT^2 - 36T + 1$
33	732	2630	$p^6T^4 + 732p^3T^3 + 2630pT^2 + 732T + 1$

$p = 97$ (cont.)

λ	a	b	$R(T)$
34	1556	24118	$p^6T^4 + 1556p^3T^3 + 24118pT^2 + 1556T + 1$
35	652	-1562	$p^6T^4 + 652p^3T^3 - 1562pT^2 + 652T + 1$
36	478	9409	$p^3T^2 + 478T + 1$
37	-100	-3386	$p^6T^4 - 100p^3T^3 - 3386pT^2 - 100T + 1$
38	-492	14390	$p^6T^4 - 492p^3T^3 + 14390pT^2 - 492T + 1$
39	932	18774	$p^6T^4 + 932p^3T^3 + 18774pT^2 + 932T + 1$
40	1036	20390	$p^6T^4 + 1036p^3T^3 + 20390pT^2 + 1036T + 1$
41	1204	8550	$p^6T^4 + 1204p^3T^3 + 8550pT^2 + 1204T + 1$
42	412	-2490	$p^6T^4 + 412p^3T^3 - 2490pT^2 + 412T + 1$
43	48	3870	$p^6T^4 + 48p^3T^3 + 3870pT^2 + 48T + 1$
44	92	-2234	$p^6T^4 + 92p^3T^3 - 2234pT^2 + 92T + 1$
45	-1492	17766	$p^6T^4 - 1492p^3T^3 + 17766pT^2 - 1492T + 1$
46	-1100	17254	$p^6T^4 - 1100p^3T^3 + 17254pT^2 - 1100T + 1$
47	-164	16710	$p^6T^4 - 164p^3T^3 + 16710pT^2 - 164T + 1$
48	220	-6586	$p^6T^4 + 220p^3T^3 - 6586pT^2 + 220T + 1$
49	88	-9458	$p^6T^4 + 88p^3T^3 - 9458pT^2 + 88T + 1$
50	-144	5790	$p^6T^4 - 144p^3T^3 + 5790pT^2 - 144T + 1$
51	1060	19526	$p^6T^4 + 1060p^3T^3 + 19526pT^2 + 1060T + 1$
52	-228	-3514	$p^6T^4 - 228p^3T^3 - 3514pT^2 - 228T + 1$
53	1128	10478	$p^6T^4 + 1128p^3T^3 + 10478pT^2 + 1128T + 1$
54	-948	14438	$p^6T^4 - 948p^3T^3 + 14438pT^2 - 948T + 1$
55	-1364	13910	$p^6T^4 - 1364p^3T^3 + 13910pT^2 - 1364T + 1$
56	-1884	23878	$p^6T^4 - 1884p^3T^3 + 23878pT^2 - 1884T + 1$
57	1180	11574	$p^6T^4 + 1180p^3T^3 + 11574pT^2 + 1180T + 1$
58	-100	17222	$p^6T^4 - 100p^3T^3 + 17222pT^2 - 100T + 1$
59	-852	12630	$p^6T^4 - 852p^3T^3 + 12630pT^2 - 852T + 1$
60	-212	998	$p^6T^4 - 212p^3T^3 + 998pT^2 - 212T + 1$
61	-1428	12838	$p^6T^4 - 1428p^3T^3 + 12838pT^2 - 1428T + 1$
62	-1120	7998	$p^6T^4 - 1120p^3T^3 + 7998pT^2 - 1120T + 1$
63	916	6454	$p^6T^4 + 916p^3T^3 + 6454pT^2 + 916T + 1$
64	1040	10078	$p^6T^4 + 1040p^3T^3 + 10078pT^2 + 1040T + 1$
65	2088	25710	$p^6T^4 + 2088p^3T^3 + 25710pT^2 + 2088T + 1$
66	472	9230	$p^6T^4 + 472p^3T^3 + 9230pT^2 + 472T + 1$

$p = 97$ (cont.)

λ	a	b	$R(T)$
67	148	11446	$p^6T^4 + 148p^3T^3 + 11446pT^2 + 148T + 1$
68	164	-10282	$p^6T^4 + 164p^3T^3 - 10282pT^2 + 164T + 1$
69	-756	12070	$p^6T^4 - 756p^3T^3 + 12070pT^2 - 756T + 1$
70	-104	-11378	$p^6T^4 - 104p^3T^3 - 11378pT^2 - 104T + 1$
71	-1124	20166	$p^6T^4 - 1124p^3T^3 + 20166pT^2 - 1124T + 1$
72	364	4902	$p^6T^4 + 364p^3T^3 + 4902pT^2 + 364T + 1$
73	-272	-11362	$p^6T^4 - 272p^3T^3 - 11362pT^2 - 272T + 1$
74	2220	29526	$p^6T^4 + 2220p^3T^3 + 29526pT^2 + 2220T + 1$
75	1248	13502	$p^6T^4 + 1248p^3T^3 + 13502pT^2 + 1248T + 1$
76	668	15414	$p^6T^4 + 668p^3T^3 + 15414pT^2 + 668T + 1$
77	164	-4794	$p^6T^4 + 164p^3T^3 - 4794pT^2 + 164T + 1$
78	-1876	24150	$p^6T^4 - 1876p^3T^3 + 24150pT^2 - 1876T + 1$
79	844	9062	$p^6T^4 + 844p^3T^3 + 9062pT^2 + 844T + 1$
80	292	15702	$p^6T^4 + 292p^3T^3 + 15702pT^2 + 292T + 1$
81	1888	24510	$p^6T^4 + 1888p^3T^3 + 24510pT^2 + 1888T + 1$
82	292	-4538	$p^6T^4 + 292p^3T^3 - 4538pT^2 + 292T + 1$
83	556	8422	$p^6T^4 + 556p^3T^3 + 8422pT^2 + 556T + 1$
84	1300	9142	$p^6T^4 + 1300p^3T^3 + 9142pT^2 + 1300T + 1$
85	-1060	20550	$p^6T^4 - 1060p^3T^3 + 20550pT^2 - 1060T + 1$
86	-420	11590	$p^6T^4 - 420p^3T^3 + 11590pT^2 - 420T + 1$
87	-972	12902	$p^6T^4 - 972p^3T^3 + 12902pT^2 - 972T + 1$
88	432	30	$p^6T^4 + 432p^3T^3 + 30pT^2 + 432T + 1$
89	604	-3258	$p^6T^4 + 604p^3T^3 - 3258pT^2 + 604T + 1$
90	-1236	20070	$p^6T^4 - 1236p^3T^3 + 20070pT^2 - 1236T + 1$
91	96	-4674	$p^6T^4 + 96p^3T^3 - 4674pT^2 + 96T + 1$
92	820	5222	$p^6T^4 + 820p^3T^3 + 5222pT^2 + 820T + 1$
93	-944	2270	$p^6T^4 - 944p^3T^3 + 2270pT^2 - 944T + 1$
94	-408	13038	$p^6T^4 - 408p^3T^3 + 13038pT^2 - 408T + 1$
95	536	17038	$p^6T^4 + 536p^3T^3 + 17038pT^2 + 536T + 1$
96	-608	-1218	$p^6T^4 - 608p^3T^3 - 1218pT^2 - 608T + 1$

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$p = 7$

λ	a	b	$R(T)$
1	20	93	$p^6T^4 + 20p^3T^3 + 93pT^2 + 20T + 1$
2	7	49	$p^3T^2 + pT + 1$
3	-12	16	$p^6T^4 - 12p^3T^3 + 16pT^2 - 12T + 1$
4	24	70	$p^6T^4 + 24p^3T^3 + 70pT^2 + 24T + 1$
5	5	42	$p^6T^4 + 5p^3T^3 + 42pT^2 + 5T + 1$
6	-36	100	$p^6T^4 - 36p^3T^3 + 100pT^2 - 36T + 1$

$p = 11$

λ	a	b	$R(T)$
1	-33	121	$p^3T^2 - 3pT + 1$
2	-24	134	$p^6T^4 - 24p^3T^3 + 134pT^2 - 24T + 1$
3	-24	134	$p^6T^4 - 24p^3T^3 + 134pT^2 - 24T + 1$
4	-15	50	$p^6T^4 - 15p^3T^3 + 50pT^2 - 15T + 1$
5	66	266	$p^6T^4 + 66p^3T^3 + 266pT^2 + 66T + 1$
6	12	50	$p^6T^4 + 12p^3T^3 + 50pT^2 + 12T + 1$
7	3	-82	$p^6T^4 + 3p^3T^3 - 82pT^2 + 3T + 1$
8	3	80	$p^6T^4 + 3p^3T^3 + 80pT^2 + 3T + 1$
9	12	53	$p^6T^4 + 12p^3T^3 + 53pT^2 + 12T + 1$
10	12	-4	$p^6T^4 + 12p^3T^3 - 4pT^2 + 12T + 1$

$p = 13$

λ	a	b	$R(T)$
1	38	-6	$p^6T^4 + 38p^3T^3 - 6pT^2 + 38T + 1$
2	42	226	$p^6T^4 + 42p^3T^3 + 226pT^2 + 42T + 1$
3	-48	253	$p^6T^4 - 48p^3T^3 + 253pT^2 - 48T + 1$
4	24	-17	$p^6T^4 + 24p^3T^3 - 17pT^2 + 24T + 1$
5	-63	238	$p^6T^4 - 63p^3T^3 + 238pT^2 - 63T + 1$
6	-4	276	$p^6T^4 - 4p^3T^3 + 276pT^2 - 4T + 1$
7	-28	-36	$p^6T^4 - 28p^3T^3 - 36pT^2 - 28T + 1$
8	-9	-140	$p^6T^4 - 9p^3T^3 - 140pT^2 - 9T + 1$
9	50	87	$p^6T^4 + 50p^3T^3 + 87pT^2 + 50T + 1$
10	52	169	$p^3T^2 + 4pT + 1$
11	-4	168	$p^6T^4 - 4p^3T^3 + 168pT^2 - 4T + 1$
12	-36	211	$p^6T^4 - 36p^3T^3 + 211pT^2 - 36T + 1$

$p = 17$

λ	a	b	$R(T)$
1	9	92	$p^6T^4 + 9p^3T^3 + 92pT^2 + 9T + 1$
2	0	289	$p^3T^2 + 1$
3	-36	470	$p^6T^4 - 36p^3T^3 + 470pT^2 - 36T + 1$
4	-108	578	$p^6T^4 - 108p^3T^3 + 578pT^2 - 108T + 1$
5	99	368	$p^6T^4 + 99p^3T^3 + 368pT^2 + 99T + 1$
6	-72	416	$p^6T^4 - 72p^3T^3 + 416pT^2 - 72T + 1$
7	-63	44	$p^6T^4 - 63p^3T^3 + 44pT^2 - 63T + 1$
8	-36	11	$p^6T^4 - 36p^3T^3 + 11pT^2 - 36T + 1$
9	72	503	$p^6T^4 + 72p^3T^3 + 503pT^2 + 72T + 1$
10	-36	416	$p^6T^4 - 36p^3T^3 + 416pT^2 - 36T + 1$
11	0	-340	$p^6T^4 - 340pT^2 + 1$
12	72	206	$p^6T^4 + 72p^3T^3 + 206pT^2 + 72T + 1$
13	72	557	$p^6T^4 + 72p^3T^3 + 557pT^2 + 72T + 1$
14	45	200	$p^6T^4 + 45p^3T^3 + 200pT^2 + 45T + 1$
15	144	713	$p^6T^4 + 144p^3T^3 + 713pT^2 + 144T + 1$
16	-144	746	$p^6T^4 - 144p^3T^3 + 746pT^2 - 144T + 1$

$p = 19$

λ	a	b	$R(T)$
1	92	309	$p^6T^4 + 92p^3T^3 + 309pT^2 + 92T + 1$
2	-156	904	$p^6T^4 - 156p^3T^3 + 904pT^2 - 156T + 1$
3	24	40	$p^6T^4 + 24p^3T^3 + 40pT^2 + 24T + 1$
4	-75	94	$p^6T^4 - 75p^3T^3 + 94pT^2 - 75T + 1$
5	42	418	$p^6T^4 + 42p^3T^3 + 418pT^2 + 42T + 1$
6	-112	654	$p^6T^4 - 112p^3T^3 + 654pT^2 - 112T + 1$
7	0	655	$p^6T^4 + 655pT^2 + 1$
8	173	876	$p^6T^4 + 173p^3T^3 + 876pT^2 + 173T + 1$
9	24	310	$p^6T^4 + 24p^3T^3 + 310pT^2 + 24T + 1$
10	-1	-270	$p^6T^4 - p^3T^3 - 270pT^2 - T + 1$
11	104	519	$p^6T^4 + 104p^3T^3 + 519pT^2 + 104T + 1$
12	11	228	$p^6T^4 + 11p^3T^3 + 228pT^2 + 11T + 1$
13	-4	114	$p^6T^4 - 4p^3T^3 + 114pT^2 - 4T + 1$
14	-48	-68	$p^6T^4 - 48p^3T^3 - 68pT^2 - 48T + 1$
15	-12	40	$p^6T^4 - 12p^3T^3 + 40pT^2 - 12T + 1$
16	-4	-21	$p^6T^4 - 4p^3T^3 - 21pT^2 - 4T + 1$
17	-38	361	$p^3T^2 - 2pT + 1$
18	0	574	$p^6T^4 + 574pT^2 + 1$

$p = 23$

λ	a	b	$R(T)$
1	-12	-49	$p^6T^4 - 12p^3T^3 - 49pT^2 - 12T + 1$
2	-129	506	$p^6T^4 - 129p^3T^3 + 506pT^2 - 129T + 1$
3	240	1409	$p^6T^4 + 240p^3T^3 + 1409pT^2 + 240T + 1$
4	87	938	$p^6T^4 + 87p^3T^3 + 938pT^2 + 87T + 1$
5	-3	-346	$p^6T^4 - 3p^3T^3 - 346pT^2 - 3T + 1$
6	-192	842	$p^6T^4 - 192p^3T^3 + 842pT^2 - 192T + 1$
7	105	140	$p^6T^4 + 105p^3T^3 + 140pT^2 + 105T + 1$

$p = 23$ (cont.)

λ	a	b	$R(T)$
8	-48	518	$p^6T^4 - 48p^3T^3 + 518pT^2 - 48T + 1$
9	-138	529	$p^3T^2 - 6pT + 1$
10	-48	-142	$p^6T^4 - 48p^3T^3 - 142pT^2 - 48T + 1$
11	204	1220	$p^6T^4 + 204p^3T^3 + 1220pT^2 + 204T + 1$
12	114	938	$p^6T^4 + 114p^3T^3 + 938pT^2 + 114T + 1$
13	168	1046	$p^6T^4 + 168p^3T^3 + 1046pT^2 + 168T + 1$
14	-21	1004	$p^6T^4 - 21p^3T^3 + 1004pT^2 - 21T + 1$
15	-12	-76	$p^6T^4 - 12p^3T^3 - 76pT^2 - 12T + 1$
16	-120	329	$p^6T^4 - 120p^3T^3 + 329pT^2 - 120T + 1$
17	-156	1154	$p^6T^4 - 156p^3T^3 + 1154pT^2 - 156T + 1$
18	42	113	$p^6T^4 + 42p^3T^3 + 113pT^2 + 42T + 1$
19	78	1058	$p^6T^4 + 78p^3T^3 + 1058pT^2 + 78T + 1$
20	-147	788	$p^6T^4 - 147p^3T^3 + 788pT^2 - 147T + 1$
21	-48	452	$p^6T^4 - 48p^3T^3 + 452pT^2 - 48T + 1$
22	60	830	$p^6T^4 + 60p^3T^3 + 830pT^2 + 60T + 1$

$p = 29$

λ	a	b	$R(T)$
1	-24	653	$p^6T^4 - 24p^3T^3 + 653pT^2 - 24T + 1$
2	273	2084	$p^6T^4 + 273p^3T^3 + 2084pT^2 + 273T + 1$
3	-105	1112	$p^6T^4 - 105p^3T^3 + 1112pT^2 - 105T + 1$
4	-60	1169	$p^6T^4 - 60p^3T^3 + 1169pT^2 - 60T + 1$
5	-132	707	$p^6T^4 - 132p^3T^3 + 707pT^2 - 132T + 1$
6	192	1463	$p^6T^4 + 192p^3T^3 + 1463pT^2 + 192T + 1$
7	-186	1466	$p^6T^4 - 186p^3T^3 + 1466pT^2 - 186T + 1$
8	-159	1004	$p^6T^4 - 159p^3T^3 + 1004pT^2 - 159T + 1$
9	-168	1223	$p^6T^4 - 168p^3T^3 + 1223pT^2 - 168T + 1$
10	237	1682	$p^6T^4 + 237p^3T^3 + 1682pT^2 + 237T + 1$
11	-204	602	$p^6T^4 - 204p^3T^3 + 602pT^2 - 204T + 1$
12	-6	1034	$p^6T^4 - 6p^3T^3 + 1034pT^2 - 6T + 1$

$p = 29$ (cont.)

λ	a	b	$R(T)$
13	-78	-262	$p^6T^4 - 78p^3T^3 - 262pT^2 - 78T + 1$
14	-168	764	$p^6T^4 - 168p^3T^3 + 764pT^2 - 168T + 1$
15	-204	1736	$p^6T^4 - 204p^3T^3 + 1736pT^2 - 204T + 1$
16	75	-640	$p^6T^4 + 75p^3T^3 - 640pT^2 + 75T + 1$
17	165	356	$p^6T^4 + 165p^3T^3 + 356pT^2 + 165T + 1$
18	30	-238	$p^6T^4 + 30p^3T^3 - 238pT^2 + 30T + 1$
19	264	1790	$p^6T^4 + 264p^3T^3 + 1790pT^2 + 264T + 1$
20	174	841	$p^3T^2 + 6pT + 1$
21	12	-694	$p^6T^4 + 12p^3T^3 - 694pT^2 + 12T + 1$
22	-24	950	$p^6T^4 - 24p^3T^3 + 950pT^2 - 24T + 1$
23	84	-319	$p^6T^4 + 84p^3T^3 - 319pT^2 + 84T + 1$
24	48	-370	$p^6T^4 + 48p^3T^3 - 370pT^2 + 48T + 1$
25	66	1331	$p^6T^4 + 66p^3T^3 + 1331pT^2 + 66T + 1$
26	-312	2384	$p^6T^4 - 312p^3T^3 + 2384pT^2 - 312T + 1$
27	192	872	$p^6T^4 + 192p^3T^3 + 872pT^2 + 192T + 1$
28	48	35	$p^6T^4 + 48p^3T^3 + 35pT^2 + 48T + 1$

$p = 31$

λ	a	b	$R(T)$
1	-117	1138	$p^6T^4 - 117p^3T^3 + 1138pT^2 - 117T + 1$
2	-16	-87	$p^6T^4 - 16p^3T^3 - 87pT^2 - 16T + 1$
3	60	-710	$p^6T^4 + 60p^3T^3 - 710pT^2 + 60T + 1$
4	-144	274	$p^6T^4 - 144p^3T^3 + 274pT^2 - 144T + 1$
5	330	1882	$p^6T^4 + 330p^3T^3 + 1882pT^2 + 330T + 1$
6	53	936	$p^6T^4 + 53p^3T^3 + 936pT^2 + 53T + 1$
7	104	-210	$p^6T^4 + 104p^3T^3 - 210pT^2 + 104T + 1$
8	200	1182	$p^6T^4 + 200p^3T^3 + 1182pT^2 + 200T + 1$
9	-136	2043	$p^6T^4 - 136p^3T^3 + 2043pT^2 - 136T + 1$
10	150	1693	$p^6T^4 + 150p^3T^3 + 1693pT^2 + 150T + 1$
11	23	-858	$p^6T^4 + 23p^3T^3 - 858pT^2 + 23T + 1$

$p = 31$ (cont.)

λ	a	b	$R(T)$
12	158	330	$p^6T^4 + 158p^3T^3 + 330pT^2 + 158T + 1$
13	24	262	$p^6T^4 + 24p^3T^3 + 262pT^2 + 24T + 1$
14	104	1221	$p^6T^4 + 104p^3T^3 + 1221pT^2 + 104T + 1$
15	38	1506	$p^6T^4 + 38p^3T^3 + 1506pT^2 + 38T + 1$
16	-155	961	$p^3T^2 - 5pT + 1$
17	-192	1612	$p^6T^4 - 192p^3T^3 + 1612pT^2 - 192T + 1$
18	-84	-62	$p^6T^4 - 84p^3T^3 - 62pT^2 - 84T + 1$
19	-220	1869	$p^6T^4 - 220p^3T^3 + 1869pT^2 - 220T + 1$
20	114	-62	$p^6T^4 + 114p^3T^3 - 62pT^2 + 114T + 1$
21	-120	-548	$p^6T^4 - 120p^3T^3 - 548pT^2 - 120T + 1$
22	131	222	$p^6T^4 + 131p^3T^3 + 222pT^2 + 131T + 1$
23	-36	-320	$p^6T^4 - 36p^3T^3 - 320pT^2 - 36T + 1$
24	276	1936	$p^6T^4 + 276p^3T^3 + 1936pT^2 + 276T + 1$
25	-75	910	$p^6T^4 - 75p^3T^3 + 910pT^2 - 75T + 1$
26	-300	1990	$p^6T^4 - 300p^3T^3 + 1990pT^2 - 300T + 1$
27	23	1032	$p^6T^4 + 23p^3T^3 + 1032pT^2 + 23T + 1$
28	-136	1530	$p^6T^4 - 136p^3T^3 + 1530pT^2 - 136T + 1$
29	119	858	$p^6T^4 + 119p^3T^3 + 858pT^2 + 119T + 1$
30	-144	1246	$p^6T^4 - 144p^3T^3 + 1246pT^2 - 144T + 1$

$p = 37$

λ	a	b	$R(T)$
1	-324	2689	$p^6T^4 - 324p^3T^3 + 2689pT^2 - 324T + 1$
2	312	2830	$p^6T^4 + 312p^3T^3 + 2830pT^2 + 312T + 1$
3	-48	859	$p^6T^4 - 48p^3T^3 + 859pT^2 - 48T + 1$
4	-84	-221	$p^6T^4 - 84p^3T^3 - 221pT^2 - 84T + 1$
5	-136	1890	$p^6T^4 - 136p^3T^3 + 1890pT^2 - 136T + 1$
6	-81	664	$p^6T^4 - 81p^3T^3 + 664pT^2 - 81T + 1$
7	240	1534	$p^6T^4 + 240p^3T^3 + 1534pT^2 + 240T + 1$
8	27	2500	$p^6T^4 + 27p^3T^3 + 2500pT^2 + 27T + 1$

$p = 37$ (cont.)

λ	a	b	$R(T)$
9	455	3192	$p^6T^4 + 455p^3T^3 + 3192pT^2 + 455T + 1$
10	158	1626	$p^6T^4 + 158p^3T^3 + 1626pT^2 + 158T + 1$
11	-232	606	$p^6T^4 - 232p^3T^3 + 606pT^2 - 232T + 1$
12	-490	3570	$p^6T^4 - 490p^3T^3 + 3570pT^2 - 490T + 1$
13	105	1966	$p^6T^4 + 105p^3T^3 + 1966pT^2 + 105T + 1$
14	416	3198	$p^6T^4 + 416p^3T^3 + 3198pT^2 + 416T + 1$
15	-39	-572	$p^6T^4 - 39p^3T^3 - 572pT^2 - 39T + 1$
16	-624	5098	$p^6T^4 - 624p^3T^3 + 5098pT^2 - 624T + 1$
17	212	2598	$p^6T^4 + 212p^3T^3 + 2598pT^2 + 212T + 1$
18	-28	-540	$p^6T^4 - 28p^3T^3 - 540pT^2 - 28T + 1$
19	-147	562	$p^6T^4 - 147p^3T^3 + 562pT^2 - 147T + 1$
20	-328	2976	$p^6T^4 - 328p^3T^3 + 2976pT^2 - 328T + 1$
21	-74	1369	$p^3T^2 - 2pT + 1$
22	80	-540	$p^6T^4 + 80p^3T^3 - 540pT^2 + 80T + 1$
23	54	1258	$p^6T^4 + 54p^3T^3 + 1258pT^2 + 54T + 1$
24	320	1572	$p^6T^4 + 320p^3T^3 + 1572pT^2 + 320T + 1$
25	293	1356	$p^6T^4 + 293p^3T^3 + 1356pT^2 + 293T + 1$
26	-324	2365	$p^6T^4 - 324p^3T^3 + 2365pT^2 - 324T + 1$
27	-216	1771	$p^6T^4 - 216p^3T^3 + 1771pT^2 - 216T + 1$
28	80	1269	$p^6T^4 + 80p^3T^3 + 1269pT^2 + 80T + 1$
29	-108	934	$p^6T^4 - 108p^3T^3 + 934pT^2 - 108T + 1$
30	-4	2274	$p^6T^4 - 4p^3T^3 + 2274pT^2 - 4T + 1$
31	92	2388	$p^6T^4 + 92p^3T^3 + 2388pT^2 + 92T + 1$
32	-112	870	$p^6T^4 - 112p^3T^3 + 870pT^2 - 112T + 1$
33	158	1167	$p^6T^4 + 158p^3T^3 + 1167pT^2 + 158T + 1$
34	24	2641	$p^6T^4 + 24p^3T^3 + 2641pT^2 + 24T + 1$
35	411	3586	$p^6T^4 + 411p^3T^3 + 3586pT^2 + 411T + 1$
36	0	1150	$p^6T^4 + 1150pT^2 + 1$

$p = 41$

λ	a	b	$R(T)$
1	294	770	$p^6T^4 + 294p^3T^3 + 770pT^2 + 294T + 1$
2	-147	1472	$p^6T^4 - 147p^3T^3 + 1472pT^2 - 147T + 1$
3	78	650	$p^6T^4 + 78p^3T^3 + 650pT^2 + 78T + 1$
4	-84	-187	$p^6T^4 - 84p^3T^3 - 187pT^2 - 84T + 1$
5	-192	2675	$p^6T^4 - 192p^3T^3 + 2675pT^2 - 192T + 1$
6	591	4484	$p^6T^4 + 591p^3T^3 + 4484pT^2 + 591T + 1$
7	-129	446	$p^6T^4 - 129p^3T^3 + 446pT^2 - 129T + 1$
8	312	2066	$p^6T^4 + 312p^3T^3 + 2066pT^2 + 312T + 1$
9	240	1595	$p^6T^4 + 240p^3T^3 + 1595pT^2 + 240T + 1$
10	-570	4847	$p^6T^4 - 570p^3T^3 + 4847pT^2 - 570T + 1$
11	-30	2162	$p^6T^4 - 30p^3T^3 + 2162pT^2 - 30T + 1$
12	-264	2390	$p^6T^4 - 264p^3T^3 + 2390pT^2 - 264T + 1$
13	24	-1012	$p^6T^4 + 24p^3T^3 - 1012pT^2 + 24T + 1$
14	-192	3134	$p^6T^4 - 192p^3T^3 + 3134pT^2 - 192T + 1$
15	-372	3362	$p^6T^4 - 372p^3T^3 + 3362pT^2 - 372T + 1$
16	60	905	$p^6T^4 + 60p^3T^3 + 905pT^2 + 60T + 1$
17	-300	3254	$p^6T^4 - 300p^3T^3 + 3254pT^2 - 300T + 1$
18	240	1514	$p^6T^4 + 240p^3T^3 + 1514pT^2 + 240T + 1$
19	51	-1996	$p^6T^4 + 51p^3T^3 - 1996pT^2 + 51T + 1$
20	564	4943	$p^6T^4 + 564p^3T^3 + 4943pT^2 + 564T + 1$
21	24	542	$p^6T^4 + 24p^3T^3 + 542pT^2 + 24T + 1$
22	-21	2282	$p^6T^4 - 21p^3T^3 + 2282pT^2 - 21T + 1$
23	78	-1417	$p^6T^4 + 78p^3T^3 - 1417pT^2 + 78T + 1$
24	-264	1202	$p^6T^4 - 264p^3T^3 + 1202pT^2 - 264T + 1$
25	213	-256	$p^6T^4 + 213p^3T^3 - 256pT^2 + 213T + 1$
26	429	2594	$p^6T^4 + 429p^3T^3 + 2594pT^2 + 429T + 1$
27	-111	2162	$p^6T^4 - 111p^3T^3 + 2162pT^2 - 111T + 1$
28	204	1148	$p^6T^4 + 204p^3T^3 + 1148pT^2 + 204T + 1$
29	-156	2714	$p^6T^4 - 156p^3T^3 + 2714pT^2 - 156T + 1$
30	195	1148	$p^6T^4 + 195p^3T^3 + 1148pT^2 + 195T + 1$
31	-246	2930	$p^6T^4 - 246p^3T^3 + 2930pT^2 - 246T + 1$
32	87	608	$p^6T^4 + 87p^3T^3 + 608pT^2 + 87T + 1$
33	-246	1681	$p^3T^2 - 6pT + 1$

$p = 41$ (cont.)

λ	a	b	$R(T)$
34	204	1472	$p^6T^4 + 204p^3T^3 + 1472pT^2 + 204T + 1$
35	24	662	$p^6T^4 + 24p^3T^3 + 662pT^2 + 24T + 1$
36	-408	2486	$p^6T^4 - 408p^3T^3 + 2486pT^2 - 408T + 1$
37	-102	2066	$p^6T^4 - 102p^3T^3 + 2066pT^2 - 102T + 1$
38	60	2714	$p^6T^4 + 60p^3T^3 + 2714pT^2 + 60T + 1$
39	-48	-94	$p^6T^4 - 48p^3T^3 - 94pT^2 - 48T + 1$
40	-48	-1471	$p^6T^4 - 48p^3T^3 - 1471pT^2 - 48T + 1$

$p = 43$

λ	a	b	$R(T)$
1	140	1527	$p^6T^4 + 140p^3T^3 + 1527pT^2 + 140T + 1$
2	101	-204	$p^6T^4 + 101p^3T^3 - 204pT^2 + 101T + 1$
3	-535	4038	$p^6T^4 - 535p^3T^3 + 4038pT^2 - 535T + 1$
4	504	4870	$p^6T^4 + 504p^3T^3 + 4870pT^2 + 504T + 1$
5	62	3258	$p^6T^4 + 62p^3T^3 + 3258pT^2 + 62T + 1$
6	32	609	$p^6T^4 + 32p^3T^3 + 609pT^2 + 32T + 1$
7	59	-282	$p^6T^4 + 59p^3T^3 - 282pT^2 + 59T + 1$
8	-144	982	$p^6T^4 - 144p^3T^3 + 982pT^2 - 144T + 1$
9	15	-290	$p^6T^4 + 15p^3T^3 - 290pT^2 + 15T + 1$
10	32	2175	$p^6T^4 + 32p^3T^3 + 2175pT^2 + 32T + 1$
11	-36	2251	$p^6T^4 - 36p^3T^3 + 2251pT^2 - 36T + 1$
12	-103	2094	$p^6T^4 - 103p^3T^3 + 2094pT^2 - 103T + 1$
13	24	3058	$p^6T^4 + 24p^3T^3 + 3058pT^2 + 24T + 1$
14	-12	331	$p^6T^4 - 12p^3T^3 + 331pT^2 - 12T + 1$
15	-100	2421	$p^6T^4 - 100p^3T^3 + 2421pT^2 - 100T + 1$
16	430	1849	$p^3T^2 + 10pT + 1$
17	60	-830	$p^6T^4 + 60p^3T^3 - 830pT^2 + 60T + 1$
18	-181	1044	$p^6T^4 - 181p^3T^3 + 1044pT^2 - 181T + 1$
19	62	-2142	$p^6T^4 + 62p^3T^3 - 2142pT^2 + 62T + 1$
20	-228	1222	$p^6T^4 - 228p^3T^3 + 1222pT^2 - 228T + 1$

$p = 43$ (cont.)

λ	a	b	$R(T)$
21	20	−1257	$p^6T^4 + 20p^3T^3 - 1257pT^2 + 20T + 1$
22	180	172	$p^6T^4 + 180p^3T^3 + 172pT^2 + 180T + 1$
23	−292	3795	$p^6T^4 - 292p^3T^3 + 3795pT^2 - 292T + 1$
24	332	1287	$p^6T^4 + 332p^3T^3 + 1287pT^2 + 332T + 1$
25	60	3490	$p^6T^4 + 60p^3T^3 + 3490pT^2 + 60T + 1$
26	−300	1978	$p^6T^4 - 300p^3T^3 + 1978pT^2 - 300T + 1$
27	−360	2980	$p^6T^4 - 360p^3T^3 + 2980pT^2 - 360T + 1$
28	302	2634	$p^6T^4 + 302p^3T^3 + 2634pT^2 + 302T + 1$
29	−48	−344	$p^6T^4 - 48p^3T^3 - 344pT^2 - 48T + 1$
30	−156	3328	$p^6T^4 - 156p^3T^3 + 3328pT^2 - 156T + 1$
31	492	3922	$p^6T^4 + 492p^3T^3 + 3922pT^2 + 492T + 1$
32	−252	442	$p^6T^4 - 252p^3T^3 + 442pT^2 - 252T + 1$
33	204	−506	$p^6T^4 + 204p^3T^3 - 506pT^2 + 204T + 1$
34	113	−714	$p^6T^4 + 113p^3T^3 - 714pT^2 + 113T + 1$
35	668	5709	$p^6T^4 + 668p^3T^3 + 5709pT^2 + 668T + 1$
36	−400	1770	$p^6T^4 - 400p^3T^3 + 1770pT^2 - 400T + 1$
37	−264	1816	$p^6T^4 - 264p^3T^3 + 1816pT^2 - 264T + 1$
38	140	2445	$p^6T^4 + 140p^3T^3 + 2445pT^2 + 140T + 1$
39	−360	2818	$p^6T^4 - 360p^3T^3 + 2818pT^2 - 360T + 1$
40	−498	3733	$p^6T^4 - 498p^3T^3 + 3733pT^2 - 498T + 1$
41	18	−638	$p^6T^4 + 18p^3T^3 - 638pT^2 + 18T + 1$
42	263	2388	$p^6T^4 + 263p^3T^3 + 2388pT^2 + 263T + 1$

$p = 47$

λ	a	b	$R(T)$
1	606	6146	$p^6T^4 + 606p^3T^3 + 6146pT^2 + 606T + 1$
2	-114	554	$p^6T^4 - 114p^3T^3 + 554pT^2 - 114T + 1$
3	-96	3473	$p^6T^4 - 96p^3T^3 + 3473pT^2 - 96T + 1$
4	156	3335	$p^6T^4 + 156p^3T^3 + 3335pT^2 + 156T + 1$
5	-168	3416	$p^6T^4 - 168p^3T^3 + 3416pT^2 - 168T + 1$
6	-24	-2062	$p^6T^4 - 24p^3T^3 - 2062pT^2 - 24T + 1$
7	-636	5309	$p^6T^4 - 636p^3T^3 + 5309pT^2 - 636T + 1$
8	-240	2690	$p^6T^4 - 240p^3T^3 + 2690pT^2 - 240T + 1$
9	-15	2474	$p^6T^4 - 15p^3T^3 + 2474pT^2 - 15T + 1$
10	183	1934	$p^6T^4 + 183p^3T^3 + 1934pT^2 + 183T + 1$
11	264	3416	$p^6T^4 + 264p^3T^3 + 3416pT^2 + 264T + 1$
12	-24	-1441	$p^6T^4 - 24p^3T^3 - 1441pT^2 - 24T + 1$
13	-384	2930	$p^6T^4 - 384p^3T^3 + 2930pT^2 - 384T + 1$
14	-492	3362	$p^6T^4 - 492p^3T^3 + 3362pT^2 - 492T + 1$
15	-231	1286	$p^6T^4 - 231p^3T^3 + 1286pT^2 - 231T + 1$
16	-240	2285	$p^6T^4 - 240p^3T^3 + 2285pT^2 - 240T + 1$
17	-600	4553	$p^6T^4 - 600p^3T^3 + 4553pT^2 - 600T + 1$
18	-132	3095	$p^6T^4 - 132p^3T^3 + 3095pT^2 - 132T + 1$
19	372	2822	$p^6T^4 + 372p^3T^3 + 2822pT^2 + 372T + 1$
20	-33	692	$p^6T^4 - 33p^3T^3 + 692pT^2 - 33T + 1$
21	12	-1819	$p^6T^4 + 12p^3T^3 - 1819pT^2 + 12T + 1$
22	21	476	$p^6T^4 + 21p^3T^3 + 476pT^2 + 21T + 1$
23	75	-604	$p^6T^4 + 75p^3T^3 - 604pT^2 + 75T + 1$
24	282	2209	$p^3T^2 + 6pT + 1$
25	102	2525	$p^6T^4 + 102p^3T^3 + 2525pT^2 + 102T + 1$
26	-276	3794	$p^6T^4 - 276p^3T^3 + 3794pT^2 - 276T + 1$
27	426	4010	$p^6T^4 + 426p^3T^3 + 4010pT^2 + 426T + 1$
28	-6	-742	$p^6T^4 - 6p^3T^3 - 742pT^2 - 6T + 1$
29	-24	98	$p^6T^4 - 24p^3T^3 + 98pT^2 - 24T + 1$
30	-60	-1660	$p^6T^4 - 60p^3T^3 - 1660pT^2 - 60T + 1$
31	-276	1094	$p^6T^4 - 276p^3T^3 + 1094pT^2 - 276T + 1$
32	102	-1174	$p^6T^4 + 102p^3T^3 - 1174pT^2 + 102T + 1$
33	-204	692	$p^6T^4 - 204p^3T^3 + 692pT^2 - 204T + 1$

$p = 47$ (cont.)

λ	a	b	$R(T)$
34	-24	179	$p^6T^4 - 24p^3T^3 + 179pT^2 - 24T + 1$
35	210	2258	$p^6T^4 + 210p^3T^3 + 2258pT^2 + 210T + 1$
36	318	1121	$p^6T^4 + 318p^3T^3 + 1121pT^2 + 318T + 1$
37	732	6119	$p^6T^4 + 732p^3T^3 + 6119pT^2 + 732T + 1$
38	336	4310	$p^6T^4 + 336p^3T^3 + 4310pT^2 + 336T + 1$
39	-285	2096	$p^6T^4 - 285p^3T^3 + 2096pT^2 - 285T + 1$
40	48	-2254	$p^6T^4 + 48p^3T^3 - 2254pT^2 + 48T + 1$
41	228	-118	$p^6T^4 + 228p^3T^3 - 118pT^2 + 228T + 1$
42	-69	-10	$p^6T^4 - 69p^3T^3 - 10pT^2 - 69T + 1$
43	-213	1934	$p^6T^4 - 213p^3T^3 + 1934pT^2 - 213T + 1$
44	-330	962	$p^6T^4 - 330p^3T^3 + 962pT^2 - 330T + 1$
45	120	962	$p^6T^4 + 120p^3T^3 + 962pT^2 + 120T + 1$
46	651	4958	$p^6T^4 + 651p^3T^3 + 4958pT^2 + 651T + 1$

$p = 53$

λ	a	b	$R(T)$
1	477	2809	$p^3T^2 + 9pT + 1$
2	-36	3836	$p^6T^4 - 36p^3T^3 + 3836pT^2 - 36T + 1$
3	189	710	$p^6T^4 + 189p^3T^3 + 710pT^2 + 189T + 1$
4	432	3431	$p^6T^4 + 432p^3T^3 + 3431pT^2 + 432T + 1$
5	0	4430	$p^6T^4 + 4430pT^2 + 1$
6	0	-937	$p^6T^4 - 937pT^2 + 1$
7	-198	4106	$p^6T^4 - 198p^3T^3 + 4106pT^2 - 198T + 1$
8	-252	974	$p^6T^4 - 252p^3T^3 + 974pT^2 - 252T + 1$
9	-864	8054	$p^6T^4 - 864p^3T^3 + 8054pT^2 - 864T + 1$
10	162	-673	$p^6T^4 + 162p^3T^3 - 673pT^2 + 162T + 1$
11	-513	4052	$p^6T^4 - 513p^3T^3 + 4052pT^2 - 513T + 1$
12	-576	4592	$p^6T^4 - 576p^3T^3 + 4592pT^2 - 576T + 1$
13	540	4025	$p^6T^4 + 540p^3T^3 + 4025pT^2 + 540T + 1$
14	-648	4916	$p^6T^4 - 648p^3T^3 + 4916pT^2 - 648T + 1$

$p = 53$ (cont.)

λ	a	b	$R(T)$
15	0	3329	$p^6T^4 + 3329pT^2 + 1$
16	360	5294	$p^6T^4 + 360p^3T^3 + 5294pT^2 + 360T + 1$
17	270	650	$p^6T^4 + 270p^3T^3 + 650pT^2 + 270T + 1$
18	792	8210	$p^6T^4 + 792p^3T^3 + 8210pT^2 + 792T + 1$
19	216	4430	$p^6T^4 + 216p^3T^3 + 4430pT^2 + 216T + 1$
20	-81	332	$p^6T^4 - 81p^3T^3 + 332pT^2 - 81T + 1$
21	270	170	$p^6T^4 + 270p^3T^3 + 170pT^2 + 270T + 1$
22	189	-208	$p^6T^4 + 189p^3T^3 - 208pT^2 + 189T + 1$
23	-135	3194	$p^6T^4 - 135p^3T^3 + 3194pT^2 - 135T + 1$
24	540	4841	$p^6T^4 + 540p^3T^3 + 4841pT^2 + 540T + 1$
25	-144	-403	$p^6T^4 - 144p^3T^3 - 403pT^2 - 144T + 1$
26	-360	3620	$p^6T^4 - 360p^3T^3 + 3620pT^2 - 360T + 1$
27	27	-802	$p^6T^4 + 27p^3T^3 - 802pT^2 + 27T + 1$
28	756	5921	$p^6T^4 + 756p^3T^3 + 5921pT^2 + 756T + 1$
29	-396	2486	$p^6T^4 - 396p^3T^3 + 2486pT^2 - 396T + 1$
30	252	920	$p^6T^4 + 252p^3T^3 + 920pT^2 + 252T + 1$
31	54	4706	$p^6T^4 + 54p^3T^3 + 4706pT^2 + 54T + 1$
32	540	5942	$p^6T^4 + 540p^3T^3 + 5942pT^2 + 540T + 1$
33	189	5462	$p^6T^4 + 189p^3T^3 + 5462pT^2 + 189T + 1$
34	-567	4922	$p^6T^4 - 567p^3T^3 + 4922pT^2 - 567T + 1$
35	729	7022	$p^6T^4 + 729p^3T^3 + 7022pT^2 + 729T + 1$
36	-756	7157	$p^6T^4 - 756p^3T^3 + 7157pT^2 - 756T + 1$
37	864	8021	$p^6T^4 + 864p^3T^3 + 8021pT^2 + 864T + 1$
38	9	-1132	$p^6T^4 + 9p^3T^3 - 1132pT^2 + 9T + 1$
39	-612	5294	$p^6T^4 - 612p^3T^3 + 5294pT^2 - 612T + 1$
40	-216	-430	$p^6T^4 - 216p^3T^3 - 430pT^2 - 216T + 1$
41	-180	1946	$p^6T^4 - 180p^3T^3 + 1946pT^2 - 180T + 1$
42	0	2978	$p^6T^4 + 2978pT^2 + 1$
43	-432	3998	$p^6T^4 - 432p^3T^3 + 3998pT^2 - 432T + 1$
44	198	3890	$p^6T^4 + 198p^3T^3 + 3890pT^2 + 198T + 1$
45	144	-538	$p^6T^4 + 144p^3T^3 - 538pT^2 + 144T + 1$
46	-216	1601	$p^6T^4 - 216p^3T^3 + 1601pT^2 - 216T + 1$
47	-738	7778	$p^6T^4 - 738p^3T^3 + 7778pT^2 - 738T + 1$

$p = 53$ (cont.)

λ	a	b	$R(T)$
48	36	1514	$p^6T^4 + 36p^3T^3 + 1514pT^2 + 36T + 1$
49	-432	2789	$p^6T^4 - 432p^3T^3 + 2789pT^2 - 432T + 1$
50	36	2972	$p^6T^4 + 36p^3T^3 + 2972pT^2 + 36T + 1$
51	-189	5402	$p^6T^4 - 189p^3T^3 + 5402pT^2 - 189T + 1$
52	324	5759	$p^6T^4 + 324p^3T^3 + 5759pT^2 + 324T + 1$

$p = 59$

λ	a	b	$R(T)$
1	-660	6827	$p^6T^4 - 660p^3T^3 + 6827pT^2 - 660T + 1$
2	-12	-1690	$p^6T^4 - 12p^3T^3 - 1690pT^2 - 12T + 1$
3	-444	1982	$p^6T^4 - 444p^3T^3 + 1982pT^2 - 444T + 1$
4	-48	6206	$p^6T^4 - 48p^3T^3 + 6206pT^2 - 48T + 1$
5	-912	7745	$p^6T^4 - 912p^3T^3 + 7745pT^2 - 912T + 1$
6	-219	-3298	$p^6T^4 - 219p^3T^3 - 3298pT^2 - 219T + 1$
7	312	4559	$p^6T^4 + 312p^3T^3 + 4559pT^2 + 312T + 1$
8	420	-232	$p^6T^4 + 420p^3T^3 - 232pT^2 + 420T + 1$
9	-84	5261	$p^6T^4 - 84p^3T^3 + 5261pT^2 - 84T + 1$
10	105	6800	$p^6T^4 + 105p^3T^3 + 6800pT^2 + 105T + 1$
11	321	4154	$p^6T^4 + 321p^3T^3 + 4154pT^2 + 321T + 1$
12	-174	2009	$p^6T^4 - 174p^3T^3 + 2009pT^2 - 174T + 1$
13	456	4640	$p^6T^4 + 456p^3T^3 + 4640pT^2 + 456T + 1$
14	-741	8204	$p^6T^4 - 741p^3T^3 + 8204pT^2 - 741T + 1$
15	312	1010	$p^6T^4 + 312p^3T^3 + 1010pT^2 + 312T + 1$
16	510	4370	$p^6T^4 + 510p^3T^3 + 4370pT^2 + 510T + 1$
17	60	2075	$p^6T^4 + 60p^3T^3 + 2075pT^2 + 60T + 1$
18	-336	-934	$p^6T^4 - 336p^3T^3 - 934pT^2 - 336T + 1$
19	-192	2345	$p^6T^4 - 192p^3T^3 + 2345pT^2 - 192T + 1$
20	582	6842	$p^6T^4 + 582p^3T^3 + 6842pT^2 + 582T + 1$
21	-156	3479	$p^6T^4 - 156p^3T^3 + 3479pT^2 - 156T + 1$
22	-480	2669	$p^6T^4 - 480p^3T^3 + 2669pT^2 - 480T + 1$

$p = 59$ (cont.)

λ	a	b	$R(T)$
23	-3	2264	$p^6T^4 - 3p^3T^3 + 2264pT^2 - 3T + 1$
24	1176	11324	$p^6T^4 + 1176p^3T^3 + 11324pT^2 + 1176T + 1$
25	-300	-1543	$p^6T^4 - 300p^3T^3 - 1543pT^2 - 300T + 1$
26	708	3481	$p^3T^2 + 12pT + 1$
27	15	-178	$p^6T^4 + 15p^3T^3 - 178pT^2 + 15T + 1$
28	132	-2461	$p^6T^4 + 132p^3T^3 - 2461pT^2 + 132T + 1$
29	510	7178	$p^6T^4 + 510p^3T^3 + 7178pT^2 + 510T + 1$
30	-228	2576	$p^6T^4 - 228p^3T^3 + 2576pT^2 - 228T + 1$
31	-444	2198	$p^6T^4 - 444p^3T^3 + 2198pT^2 - 444T + 1$
32	465	3128	$p^6T^4 + 465p^3T^3 + 3128pT^2 + 465T + 1$
33	-732	5396	$p^6T^4 - 732p^3T^3 + 5396pT^2 - 732T + 1$
34	312	3506	$p^6T^4 + 312p^3T^3 + 3506pT^2 + 312T + 1$
35	-462	3317	$p^6T^4 - 462p^3T^3 + 3317pT^2 - 462T + 1$
36	-192	-382	$p^6T^4 - 192p^3T^3 - 382pT^2 - 192T + 1$
37	-399	7178	$p^6T^4 - 399p^3T^3 + 7178pT^2 - 399T + 1$
38	132	6530	$p^6T^4 + 132p^3T^3 + 6530pT^2 + 132T + 1$
39	42	-598	$p^6T^4 + 42p^3T^3 - 598pT^2 + 42T + 1$
40	96	-3838	$p^6T^4 + 96p^3T^3 - 3838pT^2 + 96T + 1$
41	564	3506	$p^6T^4 + 564p^3T^3 + 3506pT^2 + 564T + 1$
42	-12	-5308	$p^6T^4 - 12p^3T^3 - 5308pT^2 - 12T + 1$
43	-516	5018	$p^6T^4 - 516p^3T^3 + 5018pT^2 - 516T + 1$
44	123	-2218	$p^6T^4 + 123p^3T^3 - 2218pT^2 + 123T + 1$
45	204	4331	$p^6T^4 + 204p^3T^3 + 4331pT^2 + 204T + 1$
46	-417	2198	$p^6T^4 - 417p^3T^3 + 2198pT^2 - 417T + 1$
47	-120	3062	$p^6T^4 - 120p^3T^3 + 3062pT^2 - 120T + 1$
48	-120	-1690	$p^6T^4 - 120p^3T^3 - 1690pT^2 - 120T + 1$
49	132	2669	$p^6T^4 + 132p^3T^3 + 2669pT^2 + 132T + 1$
50	-273	6152	$p^6T^4 - 273p^3T^3 + 6152pT^2 - 273T + 1$
51	-174	-2203	$p^6T^4 - 174p^3T^3 - 2203pT^2 - 174T + 1$
52	564	6044	$p^6T^4 + 564p^3T^3 + 6044pT^2 + 564T + 1$
53	420	3170	$p^6T^4 + 420p^3T^3 + 3170pT^2 + 420T + 1$
54	-444	1172	$p^6T^4 - 444p^3T^3 + 1172pT^2 - 444T + 1$
55	96	1280	$p^6T^4 + 96p^3T^3 + 1280pT^2 + 96T + 1$

$p = 59$ (cont.)

λ	a	b	$R(T)$
56	375	1778	$p^6T^4 + 375p^3T^3 + 1778pT^2 + 375T + 1$
57	384	293	$p^6T^4 + 384p^3T^3 + 293pT^2 + 384T + 1$
58	-174	2642	$p^6T^4 - 174p^3T^3 + 2642pT^2 - 174T + 1$

$p = 61$

λ	a	b	$R(T)$
1	950	8331	$p^6T^4 + 950p^3T^3 + 8331pT^2 + 950T + 1$
2	464	6252	$p^6T^4 + 464p^3T^3 + 6252pT^2 + 464T + 1$
3	-144	4366	$p^6T^4 - 144p^3T^3 + 4366pT^2 - 144T + 1$
4	-211	3444	$p^6T^4 - 211p^3T^3 + 3444pT^2 - 211T + 1$
5	-488	3721	$p^3T^2 - 8pT + 1$
6	-400	3552	$p^6T^4 - 400p^3T^3 + 3552pT^2 - 400T + 1$
7	150	-3710	$p^6T^4 + 150p^3T^3 - 3710pT^2 + 150T + 1$
8	1112	12138	$p^6T^4 + 1112p^3T^3 + 12138pT^2 + 1112T + 1$
9	884	7842	$p^6T^4 + 884p^3T^3 + 7842pT^2 + 884T + 1$
10	-532	8010	$p^6T^4 - 532p^3T^3 + 8010pT^2 - 532T + 1$
11	585	6958	$p^6T^4 + 585p^3T^3 + 6958pT^2 + 585T + 1$
12	132	-3683	$p^6T^4 + 132p^3T^3 - 3683pT^2 + 132T + 1$
13	-408	421	$p^6T^4 - 408p^3T^3 + 421pT^2 - 408T + 1$
14	-640	7470	$p^6T^4 - 640p^3T^3 + 7470pT^2 - 640T + 1$
15	-984	9682	$p^6T^4 - 984p^3T^3 + 9682pT^2 - 984T + 1$
16	852	8305	$p^6T^4 + 852p^3T^3 + 8305pT^2 + 852T + 1$
17	248	2688	$p^6T^4 + 248p^3T^3 + 2688pT^2 + 248T + 1$
18	-292	-1470	$p^6T^4 - 292p^3T^3 - 1470pT^2 - 292T + 1$
19	116	3177	$p^6T^4 + 116p^3T^3 + 3177pT^2 + 116T + 1$
20	128	5790	$p^6T^4 + 128p^3T^3 + 5790pT^2 + 128T + 1$
21	-400	2040	$p^6T^4 - 400p^3T^3 + 2040pT^2 - 400T + 1$
22	24	4093	$p^6T^4 + 24p^3T^3 + 4093pT^2 + 24T + 1$
23	369	6040	$p^6T^4 + 369p^3T^3 + 6040pT^2 + 369T + 1$
24	140	960	$p^6T^4 + 140p^3T^3 + 960pT^2 + 140T + 1$

$p = 61$ (cont.)

λ	a	b	$R(T)$
25	140	3714	$p^6T^4 + 140p^3T^3 + 3714pT^2 + 140T + 1$
26	15	2122	$p^6T^4 + 15p^3T^3 + 2122pT^2 + 15T + 1$
27	−493	6708	$p^6T^4 - 493p^3T^3 + 6708pT^2 - 493T + 1$
28	−9	−1844	$p^6T^4 - 9p^3T^3 - 1844pT^2 - 9T + 1$
29	411	6334	$p^6T^4 + 411p^3T^3 + 6334pT^2 + 411T + 1$
30	248	204	$p^6T^4 + 248p^3T^3 + 204pT^2 + 248T + 1$
31	−48	2878	$p^6T^4 - 48p^3T^3 + 2878pT^2 - 48T + 1$
32	872	8712	$p^6T^4 + 872p^3T^3 + 8712pT^2 + 872T + 1$
33	−981	9982	$p^6T^4 - 981p^3T^3 + 9982pT^2 - 981T + 1$
34	−792	8578	$p^6T^4 - 792p^3T^3 + 8578pT^2 - 792T + 1$
35	−300	70	$p^6T^4 - 300p^3T^3 + 70pT^2 - 300T + 1$
36	24	7198	$p^6T^4 + 24p^3T^3 + 7198pT^2 + 24T + 1$
37	45	6364	$p^6T^4 + 45p^3T^3 + 6364pT^2 + 45T + 1$
38	423	7282	$p^6T^4 + 423p^3T^3 + 7282pT^2 + 423T + 1$
39	734	7602	$p^6T^4 + 734p^3T^3 + 7602pT^2 + 734T + 1$
40	−588	3958	$p^6T^4 - 588p^3T^3 + 3958pT^2 - 588T + 1$
41	398	2010	$p^6T^4 + 398p^3T^3 + 2010pT^2 + 398T + 1$
42	24	2986	$p^6T^4 + 24p^3T^3 + 2986pT^2 + 24T + 1$
43	−525	1096	$p^6T^4 - 525p^3T^3 + 1096pT^2 - 525T + 1$
44	32	−3144	$p^6T^4 + 32p^3T^3 - 3144pT^2 + 32T + 1$
45	−264	2851	$p^6T^4 - 264p^3T^3 + 2851pT^2 - 264T + 1$
46	−46	−1494	$p^6T^4 - 46p^3T^3 - 1494pT^2 - 46T + 1$
47	302	2823	$p^6T^4 + 302p^3T^3 + 2823pT^2 + 302T + 1$
48	240	−794	$p^6T^4 + 240p^3T^3 - 794pT^2 + 240T + 1$
49	−804	6847	$p^6T^4 - 804p^3T^3 + 6847pT^2 - 804T + 1$
50	−88	2982	$p^6T^4 - 88p^3T^3 + 2982pT^2 - 88T + 1$
51	−292	−174	$p^6T^4 - 292p^3T^3 - 174pT^2 - 292T + 1$
52	−169	2172	$p^6T^4 - 169p^3T^3 + 2172pT^2 - 169T + 1$
53	−88	5412	$p^6T^4 - 88p^3T^3 + 5412pT^2 - 88T + 1$
54	−532	6552	$p^6T^4 - 532p^3T^3 + 6552pT^2 - 532T + 1$
55	546	2986	$p^6T^4 + 546p^3T^3 + 2986pT^2 + 546T + 1$
56	636	7873	$p^6T^4 + 636p^3T^3 + 7873pT^2 + 636T + 1$
57	−589	2472	$p^6T^4 - 589p^3T^3 + 2472pT^2 - 589T + 1$

$p = 61$ (cont.)

λ	a	b	$R(T)$
58	180	73	$p^6T^4 + 180p^3T^3 + 73pT^2 + 180T + 1$
59	429	2338	$p^6T^4 + 429p^3T^3 + 2338pT^2 + 429T + 1$
60	-684	6607	$p^6T^4 - 684p^3T^3 + 6607pT^2 - 684T + 1$

$p = 67$

λ	a	b	$R(T)$
1	-880	10254	$p^6T^4 - 880p^3T^3 + 10254pT^2 - 880T + 1$
2	1238	14154	$p^6T^4 + 1238p^3T^3 + 14154pT^2 + 1238T + 1$
3	396	6124	$p^6T^4 + 396p^3T^3 + 6124pT^2 + 396T + 1$
4	752	6486	$p^6T^4 + 752p^3T^3 + 6486pT^2 + 752T + 1$
5	288	-1382	$p^6T^4 + 288p^3T^3 - 1382pT^2 + 288T + 1$
6	-436	573	$p^6T^4 - 436p^3T^3 + 573pT^2 - 436T + 1$
7	158	546	$p^6T^4 + 158p^3T^3 + 546pT^2 + 158T + 1$
8	-733	6594	$p^6T^4 - 733p^3T^3 + 6594pT^2 - 733T + 1$
9	-328	5298	$p^6T^4 - 328p^3T^3 + 5298pT^2 - 328T + 1$
10	-48	8407	$p^6T^4 - 48p^3T^3 + 8407pT^2 - 48T + 1$
11	60	1468	$p^6T^4 + 60p^3T^3 + 1468pT^2 + 60T + 1$
12	-1200	11782	$p^6T^4 - 1200p^3T^3 + 11782pT^2 - 1200T + 1$
13	185	5028	$p^6T^4 + 185p^3T^3 + 5028pT^2 + 185T + 1$
14	200	-897	$p^6T^4 + 200p^3T^3 - 897pT^2 + 200T + 1$
15	-306	-2246	$p^6T^4 - 306p^3T^3 - 2246pT^2 - 306T + 1$
16	-183	-2690	$p^6T^4 - 183p^3T^3 - 2690pT^2 - 183T + 1$
17	-328	-210	$p^6T^4 - 328p^3T^3 - 210pT^2 - 328T + 1$
18	-156	6166	$p^6T^4 - 156p^3T^3 + 6166pT^2 - 156T + 1$
19	296	2745	$p^6T^4 + 296p^3T^3 + 2745pT^2 + 296T + 1$
20	240	3520	$p^6T^4 + 240p^3T^3 + 3520pT^2 + 240T + 1$
21	-48	1954	$p^6T^4 - 48p^3T^3 + 1954pT^2 - 48T + 1$
22	200	2262	$p^6T^4 + 200p^3T^3 + 2262pT^2 + 200T + 1$
23	-938	4489	$p^3T^2 - 14pT + 1$
24	-792	9094	$p^6T^4 - 792p^3T^3 + 9094pT^2 - 792T + 1$

$p = 67$ (cont.)

λ	a	b	$R(T)$
25	-340	183	$p^6T^4 - 340p^3T^3 + 183pT^2 - 340T + 1$
26	80	6498	$p^6T^4 + 80p^3T^3 + 6498pT^2 + 80T + 1$
27	-1224	11632	$p^6T^4 - 1224p^3T^3 + 11632pT^2 - 1224T + 1$
28	-220	3570	$p^6T^4 - 220p^3T^3 + 3570pT^2 - 220T + 1$
29	96	1495	$p^6T^4 + 96p^3T^3 + 1495pT^2 + 96T + 1$
30	293	4326	$p^6T^4 + 293p^3T^3 + 4326pT^2 + 293T + 1$
31	941	8376	$p^6T^4 + 941p^3T^3 + 8376pT^2 + 941T + 1$
32	960	10486	$p^6T^4 + 960p^3T^3 + 10486pT^2 + 960T + 1$
33	636	7138	$p^6T^4 + 636p^3T^3 + 7138pT^2 + 636T + 1$
34	-217	2016	$p^6T^4 - 217p^3T^3 + 2016pT^2 - 217T + 1$
35	-436	3381	$p^6T^4 - 436p^3T^3 + 3381pT^2 - 436T + 1$
36	-4	2571	$p^6T^4 - 4p^3T^3 + 2571pT^2 - 4T + 1$
37	42	901	$p^6T^4 + 42p^3T^3 + 901pT^2 + 42T + 1$
38	-444	8002	$p^6T^4 - 444p^3T^3 + 8002pT^2 - 444T + 1$
39	834	8002	$p^6T^4 + 834p^3T^3 + 8002pT^2 + 834T + 1$
40	153	-2138	$p^6T^4 + 153p^3T^3 - 2138pT^2 + 153T + 1$
41	-517	8160	$p^6T^4 - 517p^3T^3 + 8160pT^2 - 517T + 1$
42	828	10390	$p^6T^4 + 828p^3T^3 + 10390pT^2 + 828T + 1$
43	887	8700	$p^6T^4 + 887p^3T^3 + 8700pT^2 + 887T + 1$
44	456	7084	$p^6T^4 + 456p^3T^3 + 7084pT^2 + 456T + 1$
45	-36	-4136	$p^6T^4 - 36p^3T^3 - 4136pT^2 - 36T + 1$
46	-228	2116	$p^6T^4 - 228p^3T^3 + 2116pT^2 - 228T + 1$
47	420	-638	$p^6T^4 + 420p^3T^3 - 638pT^2 + 420T + 1$
48	384	-44	$p^6T^4 + 384p^3T^3 - 44pT^2 + 384T + 1$
49	-93	-962	$p^6T^4 - 93p^3T^3 - 962pT^2 - 93T + 1$
50	-1	612	$p^6T^4 - p^3T^3 + 612pT^2 - T + 1$
51	-139	6648	$p^6T^4 - 139p^3T^3 + 6648pT^2 - 139T + 1$
52	-684	9958	$p^6T^4 - 684p^3T^3 + 9958pT^2 - 684T + 1$
53	740	7986	$p^6T^4 + 740p^3T^3 + 7986pT^2 + 740T + 1$
54	104	7242	$p^6T^4 + 104p^3T^3 + 7242pT^2 + 104T + 1$
55	-136	4419	$p^6T^4 - 136p^3T^3 + 4419pT^2 - 136T + 1$
56	168	1846	$p^6T^4 + 168p^3T^3 + 1846pT^2 + 168T + 1$
57	-109	6282	$p^6T^4 - 109p^3T^3 + 6282pT^2 - 109T + 1$

$p = 67$ (cont.)

λ	a	b	$R(T)$
58	-792	7474	$p^6T^4 - 792p^3T^3 + 7474pT^2 - 792T + 1$
59	-1413	15142	$p^6T^4 - 1413p^3T^3 + 15142pT^2 - 1413T + 1$
60	-381	3034	$p^6T^4 - 381p^3T^3 + 3034pT^2 - 381T + 1$
61	347	438	$p^6T^4 + 347p^3T^3 + 438pT^2 + 347T + 1$
62	234	778	$p^6T^4 + 234p^3T^3 + 778pT^2 + 234T + 1$
63	384	8164	$p^6T^4 + 384p^3T^3 + 8164pT^2 + 384T + 1$
64	200	2991	$p^6T^4 + 200p^3T^3 + 2991pT^2 + 200T + 1$
65	-244	5769	$p^6T^4 - 244p^3T^3 + 5769pT^2 - 244T + 1$
66	902	8634	$p^6T^4 + 902p^3T^3 + 8634pT^2 + 902T + 1$

$p = 71$

λ	a	b	$R(T)$
1	720	10763	$p^6T^4 + 720p^3T^3 + 10763pT^2 + 720T + 1$
2	126	-2494	$p^6T^4 + 126p^3T^3 - 2494pT^2 + 126T + 1$
3	-144	-4903	$p^6T^4 - 144p^3T^3 - 4903pT^2 - 144T + 1$
4	720	5606	$p^6T^4 + 720p^3T^3 + 5606pT^2 + 720T + 1$
5	-306	5330	$p^6T^4 - 306p^3T^3 + 5330pT^2 - 306T + 1$
6	-198	773	$p^6T^4 - 198p^3T^3 + 773pT^2 - 198T + 1$
7	765	10352	$p^6T^4 + 765p^3T^3 + 10352pT^2 + 765T + 1$
8	-936	9650	$p^6T^4 - 936p^3T^3 + 9650pT^2 - 936T + 1$
9	756	7409	$p^6T^4 + 756p^3T^3 + 7409pT^2 + 756T + 1$
10	-333	5930	$p^6T^4 - 333p^3T^3 + 5930pT^2 - 333T + 1$
11	612	8732	$p^6T^4 + 612p^3T^3 + 8732pT^2 + 612T + 1$
12	-936	12350	$p^6T^4 - 936p^3T^3 + 12350pT^2 - 936T + 1$
13	72	-3898	$p^6T^4 + 72p^3T^3 - 3898pT^2 + 72T + 1$
14	-333	-1744	$p^6T^4 - 333p^3T^3 - 1744pT^2 - 333T + 1$
15	-36	-2878	$p^6T^4 - 36p^3T^3 - 2878pT^2 - 36T + 1$
16	720	7334	$p^6T^4 + 720p^3T^3 + 7334pT^2 + 720T + 1$
17	549	6842	$p^6T^4 + 549p^3T^3 + 6842pT^2 + 549T + 1$
18	-90	7685	$p^6T^4 - 90p^3T^3 + 7685pT^2 - 90T + 1$

$p = 71$ (cont.)

λ	a	b	$R(T)$
19	360	6842	$p^6T^4 + 360p^3T^3 + 6842pT^2 + 360T + 1$
20	504	6113	$p^6T^4 + 504p^3T^3 + 6113pT^2 + 504T + 1$
21	54	4250	$p^6T^4 + 54p^3T^3 + 4250pT^2 + 54T + 1$
22	180	8144	$p^6T^4 + 180p^3T^3 + 8144pT^2 + 180T + 1$
23	-576	2852	$p^6T^4 - 576p^3T^3 + 2852pT^2 - 576T + 1$
24	-387	1442	$p^6T^4 - 387p^3T^3 + 1442pT^2 - 387T + 1$
25	108	2387	$p^6T^4 + 108p^3T^3 + 2387pT^2 + 108T + 1$
26	828	7442	$p^6T^4 + 828p^3T^3 + 7442pT^2 + 828T + 1$
27	-432	551	$p^6T^4 - 432p^3T^3 + 551pT^2 - 432T + 1$
28	-108	-5470	$p^6T^4 - 108p^3T^3 - 5470pT^2 - 108T + 1$
29	1152	13922	$p^6T^4 + 1152p^3T^3 + 13922pT^2 + 1152T + 1$
30	144	6275	$p^6T^4 + 144p^3T^3 + 6275pT^2 + 144T + 1$
31	-1008	11654	$p^6T^4 - 1008p^3T^3 + 11654pT^2 - 1008T + 1$
32	-72	-4714	$p^6T^4 - 72p^3T^3 - 4714pT^2 - 72T + 1$
33	-63	-1420	$p^6T^4 - 63p^3T^3 - 1420pT^2 - 63T + 1$
34	720	7598	$p^6T^4 + 720p^3T^3 + 7598pT^2 + 720T + 1$
35	612	2792	$p^6T^4 + 612p^3T^3 + 2792pT^2 + 612T + 1$
36	234	2657	$p^6T^4 + 234p^3T^3 + 2657pT^2 + 234T + 1$
37	-972	11621	$p^6T^4 - 972p^3T^3 + 11621pT^2 - 972T + 1$
38	-1080	11378	$p^6T^4 - 1080p^3T^3 + 11378pT^2 - 1080T + 1$
39	459	2522	$p^6T^4 + 459p^3T^3 + 2522pT^2 + 459T + 1$
40	288	-3790	$p^6T^4 + 288p^3T^3 - 3790pT^2 + 288T + 1$
41	1035	8138	$p^6T^4 + 1035p^3T^3 + 8138pT^2 + 1035T + 1$
42	-351	6140	$p^6T^4 - 351p^3T^3 + 6140pT^2 - 351T + 1$
43	-1197	14402	$p^6T^4 - 1197p^3T^3 + 14402pT^2 - 1197T + 1$
44	-576	368	$p^6T^4 - 576p^3T^3 + 368pT^2 - 576T + 1$
45	360	7355	$p^6T^4 + 360p^3T^3 + 7355pT^2 + 360T + 1$
46	720	7064	$p^6T^4 + 720p^3T^3 + 7064pT^2 + 720T + 1$
47	-225	7220	$p^6T^4 - 225p^3T^3 + 7220pT^2 - 225T + 1$
48	-432	5789	$p^6T^4 - 432p^3T^3 + 5789pT^2 - 432T + 1$
49	144	3359	$p^6T^4 + 144p^3T^3 + 3359pT^2 + 144T + 1$
50	99	-2278	$p^6T^4 + 99p^3T^3 - 2278pT^2 + 99T + 1$
51	315	-2716	$p^6T^4 + 315p^3T^3 - 2716pT^2 + 315T + 1$

$p = 71$ (cont.)

λ	a	b	$R(T)$
52	729	6626	$p^6T^4 + 729p^3T^3 + 6626pT^2 + 729T + 1$
53	-1152	10514	$p^6T^4 - 1152p^3T^3 + 10514pT^2 - 1152T + 1$
54	468	2873	$p^6T^4 + 468p^3T^3 + 2873pT^2 + 468T + 1$
55	-396	6626	$p^6T^4 - 396p^3T^3 + 6626pT^2 - 396T + 1$
56	153	7976	$p^6T^4 + 153p^3T^3 + 7976pT^2 + 153T + 1$
57	0	5041	$p^3T^2 + 1$
58	396	3122	$p^6T^4 + 396p^3T^3 + 3122pT^2 + 396T + 1$
59	-900	6038	$p^6T^4 - 900p^3T^3 + 6038pT^2 - 900T + 1$
60	396	8414	$p^6T^4 + 396p^3T^3 + 8414pT^2 + 396T + 1$
61	-792	7166	$p^6T^4 - 792p^3T^3 + 7166pT^2 - 792T + 1$
62	180	2738	$p^6T^4 + 180p^3T^3 + 2738pT^2 + 180T + 1$
63	180	-2548	$p^6T^4 + 180p^3T^3 - 2548pT^2 + 180T + 1$
64	-108	-5875	$p^6T^4 - 108p^3T^3 - 5875pT^2 - 108T + 1$
65	612	5660	$p^6T^4 + 612p^3T^3 + 5660pT^2 + 612T + 1$
66	-468	2852	$p^6T^4 - 468p^3T^3 + 2852pT^2 - 468T + 1$
67	-36	308	$p^6T^4 - 36p^3T^3 + 308pT^2 - 36T + 1$
68	-423	-826	$p^6T^4 - 423p^3T^3 - 826pT^2 - 423T + 1$
69	-360	7712	$p^6T^4 - 360p^3T^3 + 7712pT^2 - 360T + 1$
70	-972	10082	$p^6T^4 - 972p^3T^3 + 10082pT^2 - 972T + 1$

$p = 73$

λ	a	b	$R(T)$
1	-648	11050	$p^6T^4 - 648p^3T^3 + 11050pT^2 - 648T + 1$
2	-58	8943	$p^6T^4 - 58p^3T^3 + 8943pT^2 - 58T + 1$
3	324	5299	$p^6T^4 + 324p^3T^3 + 5299pT^2 + 324T + 1$
4	420	121	$p^6T^4 + 420p^3T^3 + 121pT^2 + 420T + 1$
5	573	5602	$p^6T^4 + 573p^3T^3 + 5602pT^2 + 573T + 1$
6	-166	-4638	$p^6T^4 - 166p^3T^3 - 4638pT^2 - 166T + 1$
7	567	4246	$p^6T^4 + 567p^3T^3 + 4246pT^2 + 567T + 1$
8	-648	5461	$p^6T^4 - 648p^3T^3 + 5461pT^2 - 648T + 1$

$p = 73$ (cont.)

λ	a	b	$R(T)$
9	-232	10110	$p^6T^4 - 232p^3T^3 + 10110pT^2 - 232T + 1$
10	740	8652	$p^6T^4 + 740p^3T^3 + 8652pT^2 + 740T + 1$
11	492	9814	$p^6T^4 + 492p^3T^3 + 9814pT^2 + 492T + 1$
12	-868	6594	$p^6T^4 - 868p^3T^3 + 6594pT^2 - 868T + 1$
13	465	4306	$p^6T^4 + 465p^3T^3 + 4306pT^2 + 465T + 1$
14	212	-1074	$p^6T^4 + 212p^3T^3 - 1074pT^2 + 212T + 1$
15	188	972	$p^6T^4 + 188p^3T^3 + 972pT^2 + 188T + 1$
16	-679	4488	$p^6T^4 - 679p^3T^3 + 4488pT^2 - 679T + 1$
17	320	2760	$p^6T^4 + 320p^3T^3 + 2760pT^2 + 320T + 1$
18	511	5329	$p^3T^2 + 7pT + 1$
19	-298	9018	$p^6T^4 - 298p^3T^3 + 9018pT^2 - 298T + 1$
20	-1432	17550	$p^6T^4 - 1432p^3T^3 + 17550pT^2 - 1432T + 1$
21	-340	4170	$p^6T^4 - 340p^3T^3 + 4170pT^2 - 340T + 1$
22	956	11784	$p^6T^4 + 956p^3T^3 + 11784pT^2 + 956T + 1$
23	24	-554	$p^6T^4 + 24p^3T^3 - 554pT^2 + 24T + 1$
24	563	816	$p^6T^4 + 563p^3T^3 + 816pT^2 + 563T + 1$
25	-480	8086	$p^6T^4 - 480p^3T^3 + 8086pT^2 - 480T + 1$
26	-760	6432	$p^6T^4 - 760p^3T^3 + 6432pT^2 - 760T + 1$
27	632	7086	$p^6T^4 + 632p^3T^3 + 7086pT^2 + 632T + 1$
28	-219	-1472	$p^6T^4 - 219p^3T^3 - 1472pT^2 - 219T + 1$
29	-399	6844	$p^6T^4 - 399p^3T^3 + 6844pT^2 - 399T + 1$
30	740	8166	$p^6T^4 + 740p^3T^3 + 8166pT^2 + 740T + 1$
31	627	1444	$p^6T^4 + 627p^3T^3 + 1444pT^2 + 627T + 1$
32	-768	7870	$p^6T^4 - 768p^3T^3 + 7870pT^2 - 768T + 1$
33	105	3550	$p^6T^4 + 105p^3T^3 + 3550pT^2 + 105T + 1$
34	80	8640	$p^6T^4 + 80p^3T^3 + 8640pT^2 + 80T + 1$
35	24	-2849	$p^6T^4 + 24p^3T^3 - 2849pT^2 + 24T + 1$
36	134	1674	$p^6T^4 + 134p^3T^3 + 1674pT^2 + 134T + 1$
37	-382	1842	$p^6T^4 - 382p^3T^3 + 1842pT^2 - 382T + 1$
38	-912	9679	$p^6T^4 - 912p^3T^3 + 9679pT^2 - 912T + 1$
39	-471	6628	$p^6T^4 - 471p^3T^3 + 6628pT^2 - 471T + 1$
40	-534	202	$p^6T^4 - 534p^3T^3 + 202pT^2 - 534T + 1$
41	1022	12939	$p^6T^4 + 1022p^3T^3 + 12939pT^2 + 1022T + 1$

$p = 73$ (cont.)

λ	a	b	$R(T)$
42	428	−1992	$p^6T^4 + 428p^3T^3 - 1992pT^2 + 428T + 1$
43	632	1740	$p^6T^4 + 632p^3T^3 + 1740pT^2 + 632T + 1$
44	276	−2714	$p^6T^4 + 276p^3T^3 - 2714pT^2 + 276T + 1$
45	−436	−534	$p^6T^4 - 436p^3T^3 - 534pT^2 - 436T + 1$
46	−1296	15667	$p^6T^4 - 1296p^3T^3 + 15667pT^2 - 1296T + 1$
47	1104	13054	$p^6T^4 + 1104p^3T^3 + 13054pT^2 + 1104T + 1$
48	−336	3955	$p^6T^4 - 336p^3T^3 + 3955pT^2 - 336T + 1$
49	432	8431	$p^6T^4 + 432p^3T^3 + 8431pT^2 + 432T + 1$
50	80	270	$p^6T^4 + 80p^3T^3 + 270pT^2 + 80T + 1$
51	−243	1492	$p^6T^4 - 243p^3T^3 + 1492pT^2 - 243T + 1$
52	459	9538	$p^6T^4 + 459p^3T^3 + 9538pT^2 + 459T + 1$
53	104	60	$p^6T^4 + 104p^3T^3 + 60pT^2 + 104T + 1$
54	536	3489	$p^6T^4 + 536p^3T^3 + 3489pT^2 + 536T + 1$
55	−652	7782	$p^6T^4 - 652p^3T^3 + 7782pT^2 - 652T + 1$
56	320	4218	$p^6T^4 + 320p^3T^3 + 4218pT^2 + 320T + 1$
57	−120	−1499	$p^6T^4 - 120p^3T^3 - 1499pT^2 - 120T + 1$
58	−741	3442	$p^6T^4 - 741p^3T^3 + 3442pT^2 - 741T + 1$
59	620	3402	$p^6T^4 + 620p^3T^3 + 3402pT^2 + 620T + 1$
60	78	−4118	$p^6T^4 + 78p^3T^3 - 4118pT^2 + 78T + 1$
61	−264	−581	$p^6T^4 - 264p^3T^3 - 581pT^2 - 264T + 1$
62	−696	3118	$p^6T^4 - 696p^3T^3 + 3118pT^2 - 696T + 1$
63	729	10888	$p^6T^4 + 729p^3T^3 + 10888pT^2 + 729T + 1$
64	956	11298	$p^6T^4 + 956p^3T^3 + 11298pT^2 + 956T + 1$
65	−216	−4934	$p^6T^4 - 216p^3T^3 - 4934pT^2 - 216T + 1$
66	−16	6438	$p^6T^4 - 16p^3T^3 + 6438pT^2 - 16T + 1$
67	−4	−8499	$p^6T^4 - 4p^3T^3 - 8499pT^2 - 4T + 1$
68	−220	−2856	$p^6T^4 - 220p^3T^3 - 2856pT^2 - 220T + 1$
69	456	8005	$p^6T^4 + 456p^3T^3 + 8005pT^2 + 456T + 1$
70	−108	1195	$p^6T^4 - 108p^3T^3 + 1195pT^2 - 108T + 1$
71	−1020	7033	$p^6T^4 - 1020p^3T^3 + 7033pT^2 - 1020T + 1$
72	−193	1356	$p^6T^4 - 193p^3T^3 + 1356pT^2 - 193T + 1$

$p = 79$

λ	a	b	$R(T)$
1	-1060	12027	$p^6T^4 - 1060p^3T^3 + 12027pT^2 - 1060T + 1$
2	942	5938	$p^6T^4 + 942p^3T^3 + 5938pT^2 + 942T + 1$
3	491	798	$p^6T^4 + 491p^3T^3 + 798pT^2 + 491T + 1$
4	366	5857	$p^6T^4 + 366p^3T^3 + 5857pT^2 + 366T + 1$
5	-640	7065	$p^6T^4 - 640p^3T^3 + 7065pT^2 - 640T + 1$
6	96	1726	$p^6T^4 + 96p^3T^3 + 1726pT^2 + 96T + 1$
7	167	8304	$p^6T^4 + 167p^3T^3 + 8304pT^2 + 167T + 1$
8	-522	8290	$p^6T^4 - 522p^3T^3 + 8290pT^2 - 522T + 1$
9	-208	-3762	$p^6T^4 - 208p^3T^3 - 3762pT^2 - 208T + 1$
10	450	5050	$p^6T^4 + 450p^3T^3 + 5050pT^2 + 450T + 1$
11	6	-2135	$p^6T^4 + 6p^3T^3 - 2135pT^2 + 6T + 1$
12	-360	8776	$p^6T^4 - 360p^3T^3 + 8776pT^2 - 360T + 1$
13	-292	2607	$p^6T^4 - 292p^3T^3 + 2607pT^2 - 292T + 1$
14	506	11730	$p^6T^4 + 506p^3T^3 + 11730pT^2 + 506T + 1$
15	-792	5806	$p^6T^4 - 792p^3T^3 + 5806pT^2 - 792T + 1$
16	-1425	14902	$p^6T^4 - 1425p^3T^3 + 14902pT^2 - 1425T + 1$
17	180	9208	$p^6T^4 + 180p^3T^3 + 9208pT^2 + 180T + 1$
18	-1170	14122	$p^6T^4 - 1170p^3T^3 + 14122pT^2 - 1170T + 1$
19	680	42	$p^6T^4 + 680p^3T^3 + 42pT^2 + 680T + 1$
20	248	2202	$p^6T^4 + 248p^3T^3 + 2202pT^2 + 248T + 1$
21	-630	8722	$p^6T^4 - 630p^3T^3 + 8722pT^2 - 630T + 1$
22	344	9138	$p^6T^4 + 344p^3T^3 + 9138pT^2 + 344T + 1$
23	-940	13893	$p^6T^4 - 940p^3T^3 + 13893pT^2 - 940T + 1$
24	440	4770	$p^6T^4 + 440p^3T^3 + 4770pT^2 + 440T + 1$
25	140	6549	$p^6T^4 + 140p^3T^3 + 6549pT^2 + 140T + 1$
26	-640	3582	$p^6T^4 - 640p^3T^3 + 3582pT^2 - 640T + 1$
27	-265	-6330	$p^6T^4 - 265p^3T^3 - 6330pT^2 - 265T + 1$
28	600	4912	$p^6T^4 + 600p^3T^3 + 4912pT^2 + 600T + 1$
29	-427	1446	$p^6T^4 - 427p^3T^3 + 1446pT^2 - 427T + 1$
30	276	4750	$p^6T^4 + 276p^3T^3 + 4750pT^2 + 276T + 1$
31	456	3670	$p^6T^4 + 456p^3T^3 + 3670pT^2 + 456T + 1$
32	1328	15243	$p^6T^4 + 1328p^3T^3 + 15243pT^2 + 1328T + 1$
33	-36	-1268	$p^6T^4 - 36p^3T^3 - 1268pT^2 - 36T + 1$

$p = 79$ (cont.)

λ	a	b	$R(T)$
34	683	4068	$p^6T^4 + 683p^3T^3 + 4068pT^2 + 683T + 1$
35	329	-498	$p^6T^4 + 329p^3T^3 - 498pT^2 + 329T + 1$
36	-616	-282	$p^6T^4 - 616p^3T^3 - 282pT^2 - 616T + 1$
37	302	-174	$p^6T^4 + 302p^3T^3 - 174pT^2 + 302T + 1$
38	-1413	16714	$p^6T^4 - 1413p^3T^3 + 16714pT^2 - 1413T + 1$
39	-408	5830	$p^6T^4 - 408p^3T^3 + 5830pT^2 - 408T + 1$
40	788	3633	$p^6T^4 + 788p^3T^3 + 3633pT^2 + 788T + 1$
41	707	7764	$p^6T^4 + 707p^3T^3 + 7764pT^2 + 707T + 1$
42	24	646	$p^6T^4 + 24p^3T^3 + 646pT^2 + 24T + 1$
43	-130	6954	$p^6T^4 - 130p^3T^3 + 6954pT^2 - 130T + 1$
44	896	10491	$p^6T^4 + 896p^3T^3 + 10491pT^2 + 896T + 1$
45	-632	6241	$p^3T^2 - 8pT + 1$
46	-36	-9449	$p^6T^4 - 36p^3T^3 - 9449pT^2 - 36T + 1$
47	456	9340	$p^6T^4 + 456p^3T^3 + 9340pT^2 + 456T + 1$
48	305	9360	$p^6T^4 + 305p^3T^3 + 9360pT^2 + 305T + 1$
49	42	-7886	$p^6T^4 + 42p^3T^3 - 7886pT^2 + 42T + 1$
50	-147	6154	$p^6T^4 - 147p^3T^3 + 6154pT^2 - 147T + 1$
51	-964	8037	$p^6T^4 - 964p^3T^3 + 8037pT^2 - 964T + 1$
52	-952	8814	$p^6T^4 - 952p^3T^3 + 8814pT^2 - 952T + 1$
53	708	10312	$p^6T^4 + 708p^3T^3 + 10312pT^2 + 708T + 1$
54	-300	5236	$p^6T^4 - 300p^3T^3 + 5236pT^2 - 300T + 1$
55	636	4615	$p^6T^4 + 636p^3T^3 + 4615pT^2 + 636T + 1$
56	960	9610	$p^6T^4 + 960p^3T^3 + 9610pT^2 + 960T + 1$
57	479	1416	$p^6T^4 + 479p^3T^3 + 1416pT^2 + 479T + 1$
58	-468	5968	$p^6T^4 - 468p^3T^3 + 5968pT^2 - 468T + 1$
59	-181	8496	$p^6T^4 - 181p^3T^3 + 8496pT^2 - 181T + 1$
60	-1507	18942	$p^6T^4 - 1507p^3T^3 + 18942pT^2 - 1507T + 1$
61	-900	5752	$p^6T^4 - 900p^3T^3 + 5752pT^2 - 900T + 1$
62	-184	-9165	$p^6T^4 - 184p^3T^3 - 9165pT^2 - 184T + 1$
63	312	3994	$p^6T^4 + 312p^3T^3 + 3994pT^2 + 312T + 1$
64	504	7426	$p^6T^4 + 504p^3T^3 + 7426pT^2 + 504T + 1$
65	504	11314	$p^6T^4 + 504p^3T^3 + 11314pT^2 + 504T + 1$
66	-480	10690	$p^6T^4 - 480p^3T^3 + 10690pT^2 - 480T + 1$

$p = 79$ (cont.)

λ	a	b	$R(T)$
67	45	5698	$p^6T^4 + 45p^3T^3 + 5698pT^2 + 45T + 1$
68	−156	7558	$p^6T^4 - 156p^3T^3 + 7558pT^2 - 156T + 1$
69	317	−1176	$p^6T^4 + 317p^3T^3 - 1176pT^2 + 317T + 1$
70	−1380	14902	$p^6T^4 - 1380p^3T^3 + 14902pT^2 - 1380T + 1$
71	−36	−6776	$p^6T^4 - 36p^3T^3 - 6776pT^2 - 36T + 1$
72	548	7551	$p^6T^4 + 548p^3T^3 + 7551pT^2 + 548T + 1$
73	140	393	$p^6T^4 + 140p^3T^3 + 393pT^2 + 140T + 1$
74	410	7602	$p^6T^4 + 410p^3T^3 + 7602pT^2 + 410T + 1$
75	491	4956	$p^6T^4 + 491p^3T^3 + 4956pT^2 + 491T + 1$
76	−417	−1190	$p^6T^4 - 417p^3T^3 - 1190pT^2 - 417T + 1$
77	528	4858	$p^6T^4 + 528p^3T^3 + 4858pT^2 + 528T + 1$
78	1964	23394	$p^6T^4 + 1964p^3T^3 + 23394pT^2 + 1964T + 1$

$p = 83$

λ	a	b	$R(T)$
1	−240	8402	$p^6T^4 - 240p^3T^3 + 8402pT^2 - 240T + 1$
2	1344	18314	$p^6T^4 + 1344p^3T^3 + 18314pT^2 + 1344T + 1$
3	1416	16019	$p^6T^4 + 1416p^3T^3 + 16019pT^2 + 1416T + 1$
4	408	1898	$p^6T^4 + 408p^3T^3 + 1898pT^2 + 408T + 1$
5	642	170	$p^6T^4 + 642p^3T^3 + 170pT^2 + 642T + 1$
6	−348	3974	$p^6T^4 - 348p^3T^3 + 3974pT^2 - 348T + 1$
7	−1392	14966	$p^6T^4 - 1392p^3T^3 + 14966pT^2 - 1392T + 1$
8	264	1034	$p^6T^4 + 264p^3T^3 + 1034pT^2 + 264T + 1$
9	−1392	12806	$p^6T^4 - 1392p^3T^3 + 12806pT^2 - 1392T + 1$
10	1083	11534	$p^6T^4 + 1083p^3T^3 + 11534pT^2 + 1083T + 1$
11	1182	10106	$p^6T^4 + 1182p^3T^3 + 10106pT^2 + 1182T + 1$
12	516	6674	$p^6T^4 + 516p^3T^3 + 6674pT^2 + 516T + 1$
13	−348	4082	$p^6T^4 - 348p^3T^3 + 4082pT^2 - 348T + 1$
14	372	10862	$p^6T^4 + 372p^3T^3 + 10862pT^2 + 372T + 1$
15	−24	−13090	$p^6T^4 - 24p^3T^3 - 13090pT^2 - 24T + 1$

$p = 83$ (cont.)

λ	a	b	$R(T)$
16	372	7811	$p^6T^4 + 372p^3T^3 + 7811pT^2 + 372T + 1$
17	-168	3842	$p^6T^4 - 168p^3T^3 + 3842pT^2 - 168T + 1$
18	516	788	$p^6T^4 + 516p^3T^3 + 788pT^2 + 516T + 1$
19	381	4490	$p^6T^4 + 381p^3T^3 + 4490pT^2 + 381T + 1$
20	930	11834	$p^6T^4 + 930p^3T^3 + 11834pT^2 + 930T + 1$
21	-618	6890	$p^6T^4 - 618p^3T^3 + 6890pT^2 - 618T + 1$
22	192	5324	$p^6T^4 + 192p^3T^3 + 5324pT^2 + 192T + 1$
23	-186	9077	$p^6T^4 - 186p^3T^3 + 9077pT^2 - 186T + 1$
24	-1185	12590	$p^6T^4 - 1185p^3T^3 + 12590pT^2 - 1185T + 1$
25	-591	4190	$p^6T^4 - 591p^3T^3 + 4190pT^2 - 591T + 1$
26	-1320	16934	$p^6T^4 - 1320p^3T^3 + 16934pT^2 - 1320T + 1$
27	336	12131	$p^6T^4 + 336p^3T^3 + 12131pT^2 + 336T + 1$
28	1128	9485	$p^6T^4 + 1128p^3T^3 + 9485pT^2 + 1128T + 1$
29	1920	22118	$p^6T^4 + 1920p^3T^3 + 22118pT^2 + 1920T + 1$
30	48	-4771	$p^6T^4 + 48p^3T^3 - 4771pT^2 + 48T + 1$
31	408	7082	$p^6T^4 + 408p^3T^3 + 7082pT^2 + 408T + 1$
32	1119	15992	$p^6T^4 + 1119p^3T^3 + 15992pT^2 + 1119T + 1$
33	-807	4082	$p^6T^4 - 807p^3T^3 + 4082pT^2 - 807T + 1$
34	-996	13964	$p^6T^4 - 996p^3T^3 + 13964pT^2 - 996T + 1$
35	-600	2114	$p^6T^4 - 600p^3T^3 + 2114pT^2 - 600T + 1$
36	-24	-8041	$p^6T^4 - 24p^3T^3 - 8041pT^2 - 24T + 1$
37	120	35	$p^6T^4 + 120p^3T^3 + 35pT^2 + 120T + 1$
38	300	575	$p^6T^4 + 300p^3T^3 + 575pT^2 + 300T + 1$
39	-69	-3178	$p^6T^4 - 69p^3T^3 - 3178pT^2 - 69T + 1$
40	-456	9998	$p^6T^4 - 456p^3T^3 + 9998pT^2 - 456T + 1$
41	-384	7946	$p^6T^4 - 384p^3T^3 + 7946pT^2 - 384T + 1$
42	-186	-1558	$p^6T^4 - 186p^3T^3 - 1558pT^2 - 186T + 1$
43	-240	-2206	$p^6T^4 - 240p^3T^3 - 2206pT^2 - 240T + 1$
44	345	9350	$p^6T^4 + 345p^3T^3 + 9350pT^2 + 345T + 1$
45	642	10538	$p^6T^4 + 642p^3T^3 + 10538pT^2 + 642T + 1$
46	-24	4352	$p^6T^4 - 24p^3T^3 + 4352pT^2 - 24T + 1$
47	-537	10808	$p^6T^4 - 537p^3T^3 + 10808pT^2 - 537T + 1$
48	1650	20498	$p^6T^4 + 1650p^3T^3 + 20498pT^2 + 1650T + 1$

$p = 83$ (cont.)

λ	a	b	$R(T)$
49	1506	16586	$p^6T^4 + 1506p^3T^3 + 16586pT^2 + 1506T + 1$
50	-1032	6002	$p^6T^4 - 1032p^3T^3 + 6002pT^2 - 1032T + 1$
51	696	7865	$p^6T^4 + 696p^3T^3 + 7865pT^2 + 696T + 1$
52	-573	3356	$p^6T^4 - 573p^3T^3 + 3356pT^2 - 573T + 1$
53	-60	-11224	$p^6T^4 - 60p^3T^3 - 11224pT^2 - 60T + 1$
54	-240	-1750	$p^6T^4 - 240p^3T^3 - 1750pT^2 - 240T + 1$
55	-132	9968	$p^6T^4 - 132p^3T^3 + 9968pT^2 - 132T + 1$
56	-987	10484	$p^6T^4 - 987p^3T^3 + 10484pT^2 - 987T + 1$
57	417	1142	$p^6T^4 + 417p^3T^3 + 1142pT^2 + 417T + 1$
58	903	6218	$p^6T^4 + 903p^3T^3 + 6218pT^2 + 903T + 1$
59	-204	6785	$p^6T^4 - 204p^3T^3 + 6785pT^2 - 204T + 1$
60	1704	22334	$p^6T^4 + 1704p^3T^3 + 22334pT^2 + 1704T + 1$
61	-96	7001	$p^6T^4 - 96p^3T^3 + 7001pT^2 - 96T + 1$
62	-780	12074	$p^6T^4 - 780p^3T^3 + 12074pT^2 - 780T + 1$
63	354	2138	$p^6T^4 + 354p^3T^3 + 2138pT^2 + 354T + 1$
64	-456	12995	$p^6T^4 - 456p^3T^3 + 12995pT^2 - 456T + 1$
65	-1428	18017	$p^6T^4 - 1428p^3T^3 + 18017pT^2 - 1428T + 1$
66	-1356	13454	$p^6T^4 - 1356p^3T^3 + 13454pT^2 - 1356T + 1$
67	-375	4112	$p^6T^4 - 375p^3T^3 + 4112pT^2 - 375T + 1$
68	-249	6889	$p^3T^2 - 3pT + 1$
69	-276	3518	$p^6T^4 - 276p^3T^3 + 3518pT^2 - 276T + 1$
70	-492	9269	$p^6T^4 - 492p^3T^3 + 9269pT^2 - 492T + 1$
71	624	13046	$p^6T^4 + 624p^3T^3 + 13046pT^2 + 624T + 1$
72	-132	2732	$p^6T^4 - 132p^3T^3 + 2732pT^2 - 132T + 1$
73	993	5192	$p^6T^4 + 993p^3T^3 + 5192pT^2 + 993T + 1$
74	-393	13292	$p^6T^4 - 393p^3T^3 + 13292pT^2 - 393T + 1$
75	-942	7133	$p^6T^4 - 942p^3T^3 + 7133pT^2 - 942T + 1$
76	-609	8486	$p^6T^4 - 609p^3T^3 + 8486pT^2 - 609T + 1$
77	-672	8402	$p^6T^4 - 672p^3T^3 + 8402pT^2 - 672T + 1$
78	-141	3842	$p^6T^4 - 141p^3T^3 + 3842pT^2 - 141T + 1$
79	-132	7106	$p^6T^4 - 132p^3T^3 + 7106pT^2 - 132T + 1$
80	588	2762	$p^6T^4 + 588p^3T^3 + 2762pT^2 + 588T + 1$
81	-1212	17747	$p^6T^4 - 1212p^3T^3 + 17747pT^2 - 1212T + 1$

$p = 83$ (cont.)

λ	a	b	$R(T)$
82	-231	1898	$p^6T^4 - 231p^3T^3 + 1898pT^2 - 231T + 1$

$p = 89$

λ	a	b	$R(T)$
1	-711	8552	$p^6T^4 - 711p^3T^3 + 8552pT^2 - 711T + 1$
2	-828	12797	$p^6T^4 - 828p^3T^3 + 12797pT^2 - 828T + 1$
3	-324	6986	$p^6T^4 - 324p^3T^3 + 6986pT^2 - 324T + 1$
4	-423	-4084	$p^6T^4 - 423p^3T^3 - 4084pT^2 - 423T + 1$
5	585	-2896	$p^6T^4 + 585p^3T^3 - 2896pT^2 + 585T + 1$
6	1089	12602	$p^6T^4 + 1089p^3T^3 + 12602pT^2 + 1089T + 1$
7	-45	-9430	$p^6T^4 - 45p^3T^3 - 9430pT^2 - 45T + 1$
8	-1026	9362	$p^6T^4 - 1026p^3T^3 + 9362pT^2 - 1026T + 1$
9	900	6263	$p^6T^4 + 900p^3T^3 + 6263pT^2 + 900T + 1$
10	144	1991	$p^6T^4 + 144p^3T^3 + 1991pT^2 + 144T + 1$
11	-720	13418	$p^6T^4 - 720p^3T^3 + 13418pT^2 - 720T + 1$
12	873	16328	$p^6T^4 + 873p^3T^3 + 16328pT^2 + 873T + 1$
13	711	16874	$p^6T^4 + 711p^3T^3 + 16874pT^2 + 711T + 1$
14	-18	12122	$p^6T^4 - 18p^3T^3 + 12122pT^2 - 18T + 1$
15	-1800	23402	$p^6T^4 - 1800p^3T^3 + 23402pT^2 - 1800T + 1$
16	-72	-7183	$p^6T^4 - 72p^3T^3 - 7183pT^2 - 72T + 1$
17	783	9848	$p^6T^4 + 783p^3T^3 + 9848pT^2 + 783T + 1$
18	1008	13823	$p^6T^4 + 1008p^3T^3 + 13823pT^2 + 1008T + 1$
19	-612	-3922	$p^6T^4 - 612p^3T^3 - 3922pT^2 - 612T + 1$
20	-1584	16658	$p^6T^4 - 1584p^3T^3 + 16658pT^2 - 1584T + 1$
21	1224	13574	$p^6T^4 + 1224p^3T^3 + 13574pT^2 + 1224T + 1$
22	684	-142	$p^6T^4 + 684p^3T^3 - 142pT^2 + 684T + 1$
23	-972	11792	$p^6T^4 - 972p^3T^3 + 11792pT^2 - 972T + 1$
24	-207	8660	$p^6T^4 - 207p^3T^3 + 8660pT^2 - 207T + 1$
25	-342	9443	$p^6T^4 - 342p^3T^3 + 9443pT^2 - 342T + 1$
26	333	2990	$p^6T^4 + 333p^3T^3 + 2990pT^2 + 333T + 1$

$p = 89$ (cont.)

λ	a	b	$R(T)$
27	-72	776	$p^6T^4 - 72p^3T^3 + 776pT^2 - 72T + 1$
28	756	5474	$p^6T^4 + 756p^3T^3 + 5474pT^2 + 756T + 1$
29	-990	15146	$p^6T^4 - 990p^3T^3 + 15146pT^2 - 990T + 1$
30	1332	14600	$p^6T^4 + 1332p^3T^3 + 14600pT^2 + 1332T + 1$
31	-450	-622	$p^6T^4 - 450p^3T^3 - 622pT^2 - 450T + 1$
32	360	-34	$p^6T^4 + 360p^3T^3 - 34pT^2 + 360T + 1$
33	-1296	15194	$p^6T^4 - 1296p^3T^3 + 15194pT^2 - 1296T + 1$
34	1116	11447	$p^6T^4 + 1116p^3T^3 + 11447pT^2 + 1116T + 1$
35	-45	-5434	$p^6T^4 - 45p^3T^3 - 5434pT^2 - 45T + 1$
36	936	11873	$p^6T^4 + 936p^3T^3 + 11873pT^2 + 936T + 1$
37	765	-1000	$p^6T^4 + 765p^3T^3 - 1000pT^2 + 765T + 1$
38	108	3530	$p^6T^4 + 108p^3T^3 + 3530pT^2 + 108T + 1$
39	576	1370	$p^6T^4 + 576p^3T^3 + 1370pT^2 + 576T + 1$
40	-72	-4327	$p^6T^4 - 72p^3T^3 - 4327pT^2 - 72T + 1$
41	-1044	6614	$p^6T^4 - 1044p^3T^3 + 6614pT^2 - 1044T + 1$
42	360	-1141	$p^6T^4 + 360p^3T^3 - 1141pT^2 + 360T + 1$
43	1926	23570	$p^6T^4 + 1926p^3T^3 + 23570pT^2 + 1926T + 1$
44	144	-1702	$p^6T^4 + 144p^3T^3 - 1702pT^2 + 144T + 1$
45	144	11690	$p^6T^4 + 144p^3T^3 + 11690pT^2 + 144T + 1$
46	252	6182	$p^6T^4 + 252p^3T^3 + 6182pT^2 + 252T + 1$
47	540	12278	$p^6T^4 + 540p^3T^3 + 12278pT^2 + 540T + 1$
48	-153	1808	$p^6T^4 - 153p^3T^3 + 1808pT^2 - 153T + 1$
49	-288	5966	$p^6T^4 - 288p^3T^3 + 5966pT^2 - 288T + 1$
50	-450	9794	$p^6T^4 - 450p^3T^3 + 9794pT^2 - 450T + 1$
51	-720	2342	$p^6T^4 - 720p^3T^3 + 2342pT^2 - 720T + 1$
52	-612	9632	$p^6T^4 - 612p^3T^3 + 9632pT^2 - 612T + 1$
53	-828	17003	$p^6T^4 - 828p^3T^3 + 17003pT^2 - 828T + 1$
54	-288	5420	$p^6T^4 - 288p^3T^3 + 5420pT^2 - 288T + 1$
55	-792	11873	$p^6T^4 - 792p^3T^3 + 11873pT^2 - 792T + 1$
56	252	-6676	$p^6T^4 + 252p^3T^3 - 6676pT^2 + 252T + 1$
57	792	14741	$p^6T^4 + 792p^3T^3 + 14741pT^2 + 792T + 1$
58	1305	11906	$p^6T^4 + 1305p^3T^3 + 11906pT^2 + 1305T + 1$
59	-315	-8884	$p^6T^4 - 315p^3T^3 - 8884pT^2 - 315T + 1$

$p = 89$ (cont.)

λ	a	b	$R(T)$
60	-1071	8450	$p^6T^4 - 1071p^3T^3 + 8450pT^2 - 1071T + 1$
61	468	2882	$p^6T^4 + 468p^3T^3 + 2882pT^2 + 468T + 1$
62	-324	12764	$p^6T^4 - 324p^3T^3 + 12764pT^2 - 324T + 1$
63	-468	-1006	$p^6T^4 - 468p^3T^3 - 1006pT^2 - 468T + 1$
64	1224	19838	$p^6T^4 + 1224p^3T^3 + 19838pT^2 + 1224T + 1$
65	414	15794	$p^6T^4 + 414p^3T^3 + 15794pT^2 + 414T + 1$
66	279	-1324	$p^6T^4 + 279p^3T^3 - 1324pT^2 + 279T + 1$
67	-270	2234	$p^6T^4 - 270p^3T^3 + 2234pT^2 - 270T + 1$
68	1836	22322	$p^6T^4 + 1836p^3T^3 + 22322pT^2 + 1836T + 1$
69	-1044	6311	$p^6T^4 - 1044p^3T^3 + 6311pT^2 - 1044T + 1$
70	1368	13952	$p^6T^4 + 1368p^3T^3 + 13952pT^2 + 1368T + 1$
71	18	4826	$p^6T^4 + 18p^3T^3 + 4826pT^2 + 18T + 1$
72	-1602	7921	$p^3T^2 - 18pT + 1$
73	585	-2356	$p^6T^4 + 585p^3T^3 - 2356pT^2 + 585T + 1$
74	72	13574	$p^6T^4 + 72p^3T^3 + 13574pT^2 + 72T + 1$
75	-207	11258	$p^6T^4 - 207p^3T^3 + 11258pT^2 - 207T + 1$
76	-1188	13142	$p^6T^4 - 1188p^3T^3 + 13142pT^2 - 1188T + 1$
77	900	8714	$p^6T^4 + 900p^3T^3 + 8714pT^2 + 900T + 1$
78	-612	6641	$p^6T^4 - 612p^3T^3 + 6641pT^2 - 612T + 1$
79	144	3374	$p^6T^4 + 144p^3T^3 + 3374pT^2 + 144T + 1$
80	-1170	13682	$p^6T^4 - 1170p^3T^3 + 13682pT^2 - 1170T + 1$
81	-1143	7364	$p^6T^4 - 1143p^3T^3 + 7364pT^2 - 1143T + 1$
82	-288	8552	$p^6T^4 - 288p^3T^3 + 8552pT^2 - 288T + 1$
83	-252	-9592	$p^6T^4 - 252p^3T^3 - 9592pT^2 - 252T + 1$
84	792	13067	$p^6T^4 + 792p^3T^3 + 13067pT^2 + 792T + 1$
85	360	2618	$p^6T^4 + 360p^3T^3 + 2618pT^2 + 360T + 1$
86	603	3962	$p^6T^4 + 603p^3T^3 + 3962pT^2 + 603T + 1$
87	-1368	18218	$p^6T^4 - 1368p^3T^3 + 18218pT^2 - 1368T + 1$
88	135	13736	$p^6T^4 + 135p^3T^3 + 13736pT^2 + 135T + 1$

$p = 97$

λ	a	b	$R(T)$
1	-1980	20989	$p^6T^4 - 1980p^3T^3 + 20989pT^2 - 1980T + 1$
2	-805	9060	$p^6T^4 - 805p^3T^3 + 9060pT^2 - 805T + 1$
3	708	14911	$p^6T^4 + 708p^3T^3 + 14911pT^2 + 708T + 1$
4	780	5893	$p^6T^4 + 780p^3T^3 + 5893pT^2 + 780T + 1$
5	915	8242	$p^6T^4 + 915p^3T^3 + 8242pT^2 + 915T + 1$
6	-46	9954	$p^6T^4 - 46p^3T^3 + 9954pT^2 - 46T + 1$
7	321	12076	$p^6T^4 + 321p^3T^3 + 12076pT^2 + 321T + 1$
8	-628	3117	$p^6T^4 - 628p^3T^3 + 3117pT^2 - 628T + 1$
9	923	12840	$p^6T^4 + 923p^3T^3 + 12840pT^2 + 923T + 1$
10	969	15370	$p^6T^4 + 969p^3T^3 + 15370pT^2 + 969T + 1$
11	-427	18996	$p^6T^4 - 427p^3T^3 + 18996pT^2 - 427T + 1$
12	-1008	5842	$p^6T^4 - 1008p^3T^3 + 5842pT^2 - 1008T + 1$
13	-1101	11644	$p^6T^4 - 1101p^3T^3 + 11644pT^2 - 1101T + 1$
14	753	12994	$p^6T^4 + 753p^3T^3 + 12994pT^2 + 753T + 1$
15	-1048	13434	$p^6T^4 - 1048p^3T^3 + 13434pT^2 - 1048T + 1$
16	-1210	5874	$p^6T^4 - 1210p^3T^3 + 5874pT^2 - 1210T + 1$
17	680	-4170	$p^6T^4 + 680p^3T^3 - 4170pT^2 + 680T + 1$
18	-900	4519	$p^6T^4 - 900p^3T^3 + 4519pT^2 - 900T + 1$
19	884	17670	$p^6T^4 + 884p^3T^3 + 17670pT^2 + 884T + 1$
20	236	498	$p^6T^4 + 236p^3T^3 + 498pT^2 + 236T + 1$
21	-192	4030	$p^6T^4 - 192p^3T^3 + 4030pT^2 - 192T + 1$
22	-1440	12430	$p^6T^4 - 1440p^3T^3 + 12430pT^2 - 1440T + 1$
23	1239	22120	$p^6T^4 + 1239p^3T^3 + 22120pT^2 + 1239T + 1$
24	-1596	21877	$p^6T^4 - 1596p^3T^3 + 21877pT^2 - 1596T + 1$
25	-1884	27439	$p^6T^4 - 1884p^3T^3 + 27439pT^2 - 1884T + 1$
26	-417	9376	$p^6T^4 - 417p^3T^3 + 9376pT^2 - 417T + 1$
27	-88	-11247	$p^6T^4 - 88p^3T^3 - 11247pT^2 - 88T + 1$
28	-1048	10788	$p^6T^4 - 1048p^3T^3 + 10788pT^2 - 1048T + 1$
29	447	4354	$p^6T^4 + 447p^3T^3 + 4354pT^2 + 447T + 1$
30	1233	10810	$p^6T^4 + 1233p^3T^3 + 10810pT^2 + 1233T + 1$
31	-130	8682	$p^6T^4 - 130p^3T^3 + 8682pT^2 - 130T + 1$
32	8	6606	$p^6T^4 + 8p^3T^3 + 6606pT^2 + 8T + 1$
33	1883	27012	$p^6T^4 + 1883p^3T^3 + 27012pT^2 + 1883T + 1$

$p = 97$ (cont.)

λ	a	b	$R(T)$
34	1316	10650	$p^6T^4 + 1316p^3T^3 + 10650pT^2 + 1316T + 1$
35	24	-10118	$p^6T^4 + 24p^3T^3 - 10118pT^2 + 24T + 1$
36	96	14101	$p^6T^4 + 96p^3T^3 + 14101pT^2 + 96T + 1$
37	440	13572	$p^6T^4 + 440p^3T^3 + 13572pT^2 + 440T + 1$
38	2192	29472	$p^6T^4 + 2192p^3T^3 + 29472pT^2 + 2192T + 1$
39	33	-4718	$p^6T^4 + 33p^3T^3 - 4718pT^2 + 33T + 1$
40	-1587	19204	$p^6T^4 - 1587p^3T^3 + 19204pT^2 - 1587T + 1$
41	789	1438	$p^6T^4 + 789p^3T^3 + 1438pT^2 + 789T + 1$
42	909	442	$p^6T^4 + 909p^3T^3 + 442pT^2 + 909T + 1$
43	-624	2005	$p^6T^4 - 624p^3T^3 + 2005pT^2 - 624T + 1$
44	240	14155	$p^6T^4 + 240p^3T^3 + 14155pT^2 + 240T + 1$
45	236	16050	$p^6T^4 + 236p^3T^3 + 16050pT^2 + 236T + 1$
46	2516	33306	$p^6T^4 + 2516p^3T^3 + 33306pT^2 + 2516T + 1$
47	-252	9217	$p^6T^4 - 252p^3T^3 + 9217pT^2 - 252T + 1$
48	-130	12786	$p^6T^4 - 130p^3T^3 + 12786pT^2 - 130T + 1$
49	302	6306	$p^6T^4 + 302p^3T^3 + 6306pT^2 + 302T + 1$
50	830	16239	$p^6T^4 + 830p^3T^3 + 16239pT^2 + 830T + 1$
51	801	7678	$p^6T^4 + 801p^3T^3 + 7678pT^2 + 801T + 1$
52	1260	15238	$p^6T^4 + 1260p^3T^3 + 15238pT^2 + 1260T + 1$
53	672	17071	$p^6T^4 + 672p^3T^3 + 17071pT^2 + 672T + 1$
54	-910	11898	$p^6T^4 - 910p^3T^3 + 11898pT^2 - 910T + 1$
55	-2304	27550	$p^6T^4 - 2304p^3T^3 + 27550pT^2 - 2304T + 1$
56	1304	20430	$p^6T^4 + 1304p^3T^3 + 20430pT^2 + 1304T + 1$
57	356	-1524	$p^6T^4 + 356p^3T^3 - 1524pT^2 + 356T + 1$
58	-282	-5582	$p^6T^4 - 282p^3T^3 - 5582pT^2 - 282T + 1$
59	680	2580	$p^6T^4 + 680p^3T^3 + 2580pT^2 + 680T + 1$
60	-750	2842	$p^6T^4 - 750p^3T^3 + 2842pT^2 - 750T + 1$
61	1104	17314	$p^6T^4 + 1104p^3T^3 + 17314pT^2 + 1104T + 1$
62	1104	3193	$p^6T^4 + 1104p^3T^3 + 3193pT^2 + 1104T + 1$
63	-508	-3738	$p^6T^4 - 508p^3T^3 - 3738pT^2 - 508T + 1$
64	-900	3709	$p^6T^4 - 900p^3T^3 + 3709pT^2 - 900T + 1$
65	-316	4365	$p^6T^4 - 316p^3T^3 + 4365pT^2 - 316T + 1$
66	600	15586	$p^6T^4 + 600p^3T^3 + 15586pT^2 + 600T + 1$

$p = 97$ (cont.)

λ	a	b	$R(T)$
67	−1035	12430	$p^6T^4 - 1035p^3T^3 + 12430pT^2 - 1035T + 1$
68	276	12886	$p^6T^4 + 276p^3T^3 + 12886pT^2 + 276T + 1$
69	−628	6276	$p^6T^4 - 628p^3T^3 + 6276pT^2 - 628T + 1$
70	−1372	19941	$p^6T^4 - 1372p^3T^3 + 19941pT^2 - 1372T + 1$
71	735	−560	$p^6T^4 + 735p^3T^3 - 560pT^2 + 735T + 1$
72	1382	9762	$p^6T^4 + 1382p^3T^3 + 9762pT^2 + 1382T + 1$
73	950	11058	$p^6T^4 + 950p^3T^3 + 11058pT^2 + 950T + 1$
74	−336	574	$p^6T^4 - 336p^3T^3 + 574pT^2 - 336T + 1$
75	−832	16350	$p^6T^4 - 832p^3T^3 + 16350pT^2 - 832T + 1$
76	2076	28438	$p^6T^4 + 2076p^3T^3 + 28438pT^2 + 2076T + 1$
77	560	−4524	$p^6T^4 + 560p^3T^3 - 4524pT^2 + 560T + 1$
78	140	4200	$p^6T^4 + 140p^3T^3 + 4200pT^2 + 140T + 1$
79	317	17832	$p^6T^4 + 317p^3T^3 + 17832pT^2 + 317T + 1$
80	−1696	24828	$p^6T^4 - 1696p^3T^3 + 24828pT^2 - 1696T + 1$
81	97	9409	$p^3T^2 + pT + 1$
82	764	2718	$p^6T^4 + 764p^3T^3 + 2718pT^2 + 764T + 1$
83	248	−13080	$p^6T^4 + 248p^3T^3 - 13080pT^2 + 248T + 1$
84	−856	12816	$p^6T^4 - 856p^3T^3 + 12816pT^2 - 856T + 1$
85	20	−366	$p^6T^4 + 20p^3T^3 - 366pT^2 + 20T + 1$
86	−444	6919	$p^6T^4 - 444p^3T^3 + 6919pT^2 - 444T + 1$
87	−940	15108	$p^6T^4 - 940p^3T^3 + 15108pT^2 - 940T + 1$
88	−1020	8701	$p^6T^4 - 1020p^3T^3 + 8701pT^2 - 1020T + 1$
89	−1276	21666	$p^6T^4 - 1276p^3T^3 + 21666pT^2 - 1276T + 1$
90	32	12408	$p^6T^4 + 32p^3T^3 + 12408pT^2 + 32T + 1$
91	−2064	28762	$p^6T^4 - 2064p^3T^3 + 28762pT^2 - 2064T + 1$
92	−1156	13812	$p^6T^4 - 1156p^3T^3 + 13812pT^2 - 1156T + 1$
93	1196	13734	$p^6T^4 + 1196p^3T^3 + 13734pT^2 + 1196T + 1$
94	816	6946	$p^6T^4 + 816p^3T^3 + 6946pT^2 + 816T + 1$
95	−336	1870	$p^6T^4 - 336p^3T^3 + 1870pT^2 - 336T + 1$
96	−792	13969	$p^6T^4 - 792p^3T^3 + 13969pT^2 - 792T + 1$

A.4 Case 3: Operator 2.63

$p = 7$

λ	a	b	$R(T)$
1	0	-7	$-(pT-1)(pT+1)$
2	12	34	$p^6T^4 + 12p^3T^3 + 34pT^2 + 12T + 1$
3	2	66	$p^6T^4 + 2p^3T^3 + 66pT^2 + 2T + 1$
4	-8	2	$p^6T^4 - 8p^3T^3 + 2pT^2 - 8T + 1$
5	-18	34	$p^6T^4 - 18p^3T^3 + 34pT^2 - 18T + 1$
6	20	66	$p^6T^4 + 20p^3T^3 + 66pT^2 + 20T + 1$

$p = 11$

λ	a	b	$R(T)$
1	32	194	$p^6T^4 + 32p^3T^3 + 194pT^2 + 32T + 1$
2	60	242	$p^6T^4 + 60p^3T^3 + 242pT^2 + 60T + 1$
3	-36	242	$p^6T^4 - 36p^3T^3 + 242pT^2 - 36T + 1$
4	-16	98	$p^6T^4 - 16p^3T^3 + 98pT^2 - 16T + 1$
5	-14	11	$p^2T^2 - 14T + 1$
6	-2	74	$p^6T^4 - 2p^3T^3 + 74pT^2 - 2T + 1$
7	-8	-94	$p^6T^4 - 8p^3T^3 - 94pT^2 - 8T + 1$
8	-42	170	$p^6T^4 - 42p^3T^3 + 170pT^2 - 42T + 1$
9	40	194	$p^6T^4 + 40p^3T^3 + 194pT^2 + 40T + 1$
10	-2	-118	$p^6T^4 - 2p^3T^3 - 118pT^2 - 2T + 1$

$p = 13$

λ	a	b	$R(T)$
1	-12	-26	$p^6T^4 - 12p^3T^3 - 26pT^2 - 12T + 1$
2	28	278	$p^6T^4 + 28p^3T^3 + 278pT^2 + 28T + 1$
3	-16	318	$p^6T^4 - 16p^3T^3 + 318pT^2 - 16T + 1$
4	-112	510	$p^6T^4 - 112p^3T^3 + 510pT^2 - 112T + 1$
5	14	-102	$p^6T^4 + 14p^3T^3 - 102pT^2 + 14T + 1$
6	58	146	$p^6T^4 + 58p^3T^3 + 146pT^2 + 58T + 1$
7	6	202	$p^6T^4 + 6p^3T^3 + 202pT^2 + 6T + 1$
8	36	166	$p^6T^4 + 36p^3T^3 + 166pT^2 + 36T + 1$
9	36	-26	$p^6T^4 + 36p^3T^3 - 26pT^2 + 36T + 1$
10	-8	206	$p^6T^4 - 8p^3T^3 + 206pT^2 - 8T + 1$
11	-16	30	$p^6T^4 - 16p^3T^3 + 30pT^2 - 16T + 1$
12	0	-13	$-(pT - 1)(pT + 1)$

$p = 17$

λ	a	b	$R(T)$
1	84	470	$p^6T^4 + 84p^3T^3 + 470pT^2 + 84T + 1$
2	-108	470	$p^6T^4 - 108p^3T^3 + 470pT^2 - 108T + 1$
3	-12	470	$p^6T^4 - 12p^3T^3 + 470pT^2 - 12T + 1$
4	-2	17	$p^2T^2 - 2T + 1$
5	76	422	$p^6T^4 + 76p^3T^3 + 422pT^2 + 76T + 1$
6	-112	446	$p^6T^4 - 112p^3T^3 + 446pT^2 - 112T + 1$
7	-22	-22	$p^6T^4 - 22p^3T^3 - 22pT^2 - 22T + 1$
8	80	542	$p^6T^4 + 80p^3T^3 + 542pT^2 + 80T + 1$
9	24	110	$p^6T^4 + 24p^3T^3 + 110pT^2 + 24T + 1$
10	-18	2	$p^6T^4 - 18p^3T^3 + 2pT^2 - 18T + 1$
11	-18	578	$p^6T^4 - 18p^3T^3 + 578pT^2 - 18T + 1$
12	-24	398	$p^6T^4 - 24p^3T^3 + 398pT^2 - 24T + 1$
13	-16	-34	$p^6T^4 - 16p^3T^3 - 34pT^2 - 16T + 1$
14	-22	170	$p^6T^4 - 22p^3T^3 + 170pT^2 - 22T + 1$
15	32	350	$p^6T^4 + 32p^3T^3 + 350pT^2 + 32T + 1$
16	76	230	$p^6T^4 + 76p^3T^3 + 230pT^2 + 76T + 1$

$p = 19$

λ	a	b	$R(T)$
1	-72	34	$p^6T^4 - 72p^3T^3 + 34pT^2 - 72T + 1$
2	-52	-78	$p^6T^4 - 52p^3T^3 - 78pT^2 - 52T + 1$
3	18	298	$p^6T^4 + 18p^3T^3 + 298pT^2 + 18T + 1$
4	-64	162	$p^6T^4 - 64p^3T^3 + 162pT^2 - 64T + 1$
5	-120	514	$p^6T^4 - 120p^3T^3 + 514pT^2 - 120T + 1$
6	-72	418	$p^6T^4 - 72p^3T^3 + 418pT^2 - 72T + 1$
7	-16	66	$p^6T^4 - 16p^3T^3 + 66pT^2 - 16T + 1$
8	-174	1066	$p^6T^4 - 174p^3T^3 + 1066pT^2 - 174T + 1$
9	44	306	$p^6T^4 + 44p^3T^3 + 306pT^2 + 44T + 1$
10	168	898	$p^6T^4 + 168p^3T^3 + 898pT^2 + 168T + 1$
11	34	19	$p^2T^2 + 34T + 1$
12	46	122	$p^6T^4 + 46p^3T^3 + 122pT^2 + 46T + 1$
13	32	546	$p^6T^4 + 32p^3T^3 + 546pT^2 + 32T + 1$
14	4	-46	$p^6T^4 + 4p^3T^3 - 46pT^2 + 4T + 1$
15	18	-470	$p^6T^4 + 18p^3T^3 - 470pT^2 + 18T + 1$
16	100	338	$p^6T^4 + 100p^3T^3 + 338pT^2 + 100T + 1$
17	176	834	$p^6T^4 + 176p^3T^3 + 834pT^2 + 176T + 1$
18	-50	314	$p^6T^4 - 50p^3T^3 + 314pT^2 - 50T + 1$

$p = 23$

λ	a	b	$R(T)$
1	-72	-94	$p^6T^4 - 72p^3T^3 - 94pT^2 - 72T + 1$
2	-4	-94	$p^6T^4 - 4p^3T^3 - 94pT^2 - 4T + 1$
3	16	386	$p^6T^4 + 16p^3T^3 + 386pT^2 + 16T + 1$
4	188	1058	$p^6T^4 + 188p^3T^3 + 1058pT^2 + 188T + 1$
5	100	578	$p^6T^4 + 100p^3T^3 + 578pT^2 + 100T + 1$
6	-64	2	$p^6T^4 - 64p^3T^3 + 2pT^2 - 64T + 1$
7	-92	578	$p^6T^4 - 92p^3T^3 + 578pT^2 - 92T + 1$

$p = 23$ (cont.)

λ	a	b	$R(T)$
8	24	−94	$p^6T^4 + 24p^3T^3 - 94pT^2 + 24T + 1$
9	0	−23	$-(pT-1)(pT+1)$
10	−186	1346	$p^6T^4 - 186p^3T^3 + 1346pT^2 - 186T + 1$
11	−130	770	$p^6T^4 - 130p^3T^3 + 770pT^2 - 130T + 1$
12	−72	1058	$p^6T^4 - 72p^3T^3 + 1058pT^2 - 72T + 1$
13	64	98	$p^6T^4 + 64p^3T^3 + 98pT^2 + 64T + 1$
14	140	386	$p^6T^4 + 140p^3T^3 + 386pT^2 + 140T + 1$
15	138	1058	$p^6T^4 + 138p^3T^3 + 1058pT^2 + 138T + 1$
16	−24	194	$p^6T^4 - 24p^3T^3 + 194pT^2 - 24T + 1$
17	6	770	$p^6T^4 + 6p^3T^3 + 770pT^2 + 6T + 1$
18	−44	482	$p^6T^4 - 44p^3T^3 + 482pT^2 - 44T + 1$
19	−16	866	$p^6T^4 - 16p^3T^3 + 866pT^2 - 16T + 1$
20	24	−94	$p^6T^4 + 24p^3T^3 - 94pT^2 + 24T + 1$
21	−110	866	$p^6T^4 - 110p^3T^3 + 866pT^2 - 110T + 1$
22	138	482	$p^6T^4 + 138p^3T^3 + 482pT^2 + 138T + 1$

$p = 29$

λ	a	b	$R(T)$
1	−76	902	$p^6T^4 - 76p^3T^3 + 902pT^2 - 76T + 1$
2	50	−286	$p^6T^4 + 50p^3T^3 - 286pT^2 + 50T + 1$
3	−158	770	$p^6T^4 - 158p^3T^3 + 770pT^2 - 158T + 1$
4	−32	−418	$p^6T^4 - 32p^3T^3 - 418pT^2 - 32T + 1$
5	0	−29	$-(pT-1)(pT+1)$
6	308	1670	$p^6T^4 + 308p^3T^3 + 1670pT^2 + 308T + 1$
7	−20	470	$p^6T^4 - 20p^3T^3 + 470pT^2 - 20T + 1$
8	−150	−46	$p^6T^4 - 150p^3T^3 - 46pT^2 - 150T + 1$
9	12	566	$p^6T^4 + 12p^3T^3 + 566pT^2 + 12T + 1$
10	−216	1358	$p^6T^4 - 216p^3T^3 + 1358pT^2 - 216T + 1$
11	−320	1982	$p^6T^4 - 320p^3T^3 + 1982pT^2 - 320T + 1$
12	300	2294	$p^6T^4 + 300p^3T^3 + 2294pT^2 + 300T + 1$

$p = 29$ (cont.)

λ	a	b	$R(T)$
13	-120	1070	$p^6T^4 - 120p^3T^3 + 1070pT^2 - 120T + 1$
14	72	206	$p^6T^4 + 72p^3T^3 + 206pT^2 + 72T + 1$
15	190	1178	$p^6T^4 + 190p^3T^3 + 1178pT^2 + 190T + 1$
16	260	1670	$p^6T^4 + 260p^3T^3 + 1670pT^2 + 260T + 1$
17	-54	530	$p^6T^4 - 54p^3T^3 + 530pT^2 - 54T + 1$
18	64	62	$p^6T^4 + 64p^3T^3 + 62pT^2 + 64T + 1$
19	-10	266	$p^6T^4 - 10p^3T^3 + 266pT^2 - 10T + 1$
20	-36	566	$p^6T^4 - 36p^3T^3 + 566pT^2 - 36T + 1$
21	-84	-10	$p^6T^4 - 84p^3T^3 - 10pT^2 - 84T + 1$
22	-120	1070	$p^6T^4 - 120p^3T^3 + 1070pT^2 - 120T + 1$
23	-316	2438	$p^6T^4 - 316p^3T^3 + 2438pT^2 - 316T + 1$
24	60	-10	$p^6T^4 + 60p^3T^3 - 10pT^2 + 60T + 1$
25	168	494	$p^6T^4 + 168p^3T^3 + 494pT^2 + 168T + 1$
26	234	1682	$p^6T^4 + 234p^3T^3 + 1682pT^2 + 234T + 1$
27	56	1070	$p^6T^4 + 56p^3T^3 + 1070pT^2 + 56T + 1$
28	-32	350	$p^6T^4 - 32p^3T^3 + 350pT^2 - 32T + 1$

$p = 31$

λ	a	b	$R(T)$
1	-32	2	$p^6T^4 - 32p^3T^3 + 2pT^2 - 32T + 1$
2	-312	2338	$p^6T^4 - 312p^3T^3 + 2338pT^2 - 312T + 1$
3	-42	1138	$p^6T^4 - 42p^3T^3 + 1138pT^2 - 42T + 1$
4	188	1890	$p^6T^4 + 188p^3T^3 + 1890pT^2 + 188T + 1$
5	-252	2146	$p^6T^4 - 252p^3T^3 + 2146pT^2 - 252T + 1$
6	-72	514	$p^6T^4 - 72p^3T^3 + 514pT^2 - 72T + 1$
7	120	1282	$p^6T^4 + 120p^3T^3 + 1282pT^2 + 120T + 1$
8	-32	-1150	$p^6T^4 - 32p^3T^3 - 1150pT^2 - 32T + 1$
9	-184	258	$p^6T^4 - 184p^3T^3 + 258pT^2 - 184T + 1$
10	200	1794	$p^6T^4 + 200p^3T^3 + 1794pT^2 + 200T + 1$
11	140	1986	$p^6T^4 + 140p^3T^3 + 1986pT^2 + 140T + 1$

$p = 31$ (cont.)

λ	a	b	$R(T)$
12	180	1474	$p^6T^4 + 180p^3T^3 + 1474pT^2 + 180T + 1$
13	-22	-1038	$p^6T^4 - 22p^3T^3 - 1038pT^2 - 22T + 1$
14	-136	1314	$p^6T^4 - 136p^3T^3 + 1314pT^2 - 136T + 1$
15	-32	-1150	$p^6T^4 - 32p^3T^3 - 1150pT^2 - 32T + 1$
16	0	-31	$-(pT - 1)(pT + 1)$
17	114	562	$p^6T^4 + 114p^3T^3 + 562pT^2 + 114T + 1$
18	228	1378	$p^6T^4 + 228p^3T^3 + 1378pT^2 + 228T + 1$
19	16	-94	$p^6T^4 + 16p^3T^3 - 94pT^2 + 16T + 1$
20	92	354	$p^6T^4 + 92p^3T^3 + 354pT^2 + 92T + 1$
21	-158	1010	$p^6T^4 - 158p^3T^3 + 1010pT^2 - 158T + 1$
22	120	1282	$p^6T^4 + 120p^3T^3 + 1282pT^2 + 120T + 1$
23	-214	2226	$p^6T^4 - 214p^3T^3 + 2226pT^2 - 214T + 1$
24	-42	946	$p^6T^4 - 42p^3T^3 + 946pT^2 - 42T + 1$
25	-184	1410	$p^6T^4 - 184p^3T^3 + 1410pT^2 - 184T + 1$
26	94	1586	$p^6T^4 + 94p^3T^3 + 1586pT^2 + 94T + 1$
27	-12	-62	$p^6T^4 - 12p^3T^3 - 62pT^2 - 12T + 1$
28	304	2210	$p^6T^4 + 304p^3T^3 + 2210pT^2 + 304T + 1$
29	-52	1218	$p^6T^4 - 52p^3T^3 + 1218pT^2 - 52T + 1$
30	14	114	$p^6T^4 + 14p^3T^3 + 114pT^2 + 14T + 1$

$p = 37$

λ	a	b	$R(T)$
1	100	-346	$p^6T^4 + 100p^3T^3 - 346pT^2 + 100T + 1$
2	-416	2846	$p^6T^4 - 416p^3T^3 + 2846pT^2 - 416T + 1$
3	152	2286	$p^6T^4 + 152p^3T^3 + 2286pT^2 + 152T + 1$
4	160	830	$p^6T^4 + 160p^3T^3 + 830pT^2 + 160T + 1$
5	-214	930	$p^6T^4 - 214p^3T^3 + 930pT^2 - 214T + 1$
6	186	706	$p^6T^4 + 186p^3T^3 + 706pT^2 + 186T + 1$
7	196	1766	$p^6T^4 + 196p^3T^3 + 1766pT^2 + 196T + 1$
8	300	1846	$p^6T^4 + 300p^3T^3 + 1846pT^2 + 300T + 1$

$p = 37$ (cont.)

λ	a	b	$R(T)$
9	52	1190	$p^6T^4 + 52p^3T^3 + 1190pT^2 + 52T + 1$
10	-224	1598	$p^6T^4 - 224p^3T^3 + 1598pT^2 - 224T + 1$
11	0	-37	$-(pT - 1)(pT + 1)$
12	-40	-786	$p^6T^4 - 40p^3T^3 - 786pT^2 - 40T + 1$
13	170	-414	$p^6T^4 + 170p^3T^3 - 414pT^2 + 170T + 1$
14	-48	382	$p^6T^4 - 48p^3T^3 + 382pT^2 - 48T + 1$
15	-232	2766	$p^6T^4 - 232p^3T^3 + 2766pT^2 - 232T + 1$
16	-36	2614	$p^6T^4 - 36p^3T^3 + 2614pT^2 - 36T + 1$
17	-58	1626	$p^6T^4 - 58p^3T^3 + 1626pT^2 - 58T + 1$
18	-180	1078	$p^6T^4 - 180p^3T^3 + 1078pT^2 - 180T + 1$
19	-162	1162	$p^6T^4 - 162p^3T^3 + 1162pT^2 - 162T + 1$
20	100	230	$p^6T^4 + 100p^3T^3 + 230pT^2 + 100T + 1$
21	336	1822	$p^6T^4 + 336p^3T^3 + 1822pT^2 + 336T + 1$
22	-48	-2	$p^6T^4 - 48p^3T^3 - 2pT^2 - 48T + 1$
23	204	310	$p^6T^4 + 204p^3T^3 + 310pT^2 + 204T + 1$
24	-14	242	$p^6T^4 - 14p^3T^3 + 242pT^2 - 14T + 1$
25	56	174	$p^6T^4 + 56p^3T^3 + 174pT^2 + 56T + 1$
26	68	1734	$p^6T^4 + 68p^3T^3 + 1734pT^2 + 68T + 1$
27	596	4998	$p^6T^4 + 596p^3T^3 + 4998pT^2 + 596T + 1$
28	64	-130	$p^6T^4 + 64p^3T^3 - 130pT^2 + 64T + 1$
29	-154	1818	$p^6T^4 - 154p^3T^3 + 1818pT^2 - 154T + 1$
30	-328	3246	$p^6T^4 - 328p^3T^3 + 3246pT^2 - 328T + 1$
31	126	2698	$p^6T^4 + 126p^3T^3 + 2698pT^2 + 126T + 1$
32	-232	2766	$p^6T^4 - 232p^3T^3 + 2766pT^2 - 232T + 1$
33	-36	310	$p^6T^4 - 36p^3T^3 + 310pT^2 - 36T + 1$
34	-124	966	$p^6T^4 - 124p^3T^3 + 966pT^2 - 124T + 1$
35	-390	2818	$p^6T^4 - 390p^3T^3 + 2818pT^2 - 390T + 1$
36	108	502	$p^6T^4 + 108p^3T^3 + 502pT^2 + 108T + 1$

$p = 41$

λ	a	b	$R(T)$
1	-428	3158	$p^6T^4 - 428p^3T^3 + 3158pT^2 - 428T + 1$
2	48	2750	$p^6T^4 + 48p^3T^3 + 2750pT^2 + 48T + 1$
3	-6	3002	$p^6T^4 - 6p^3T^3 + 3002pT^2 - 6T + 1$
4	-32	-1570	$p^6T^4 - 32p^3T^3 - 1570pT^2 - 32T + 1$
5	340	2390	$p^6T^4 + 340p^3T^3 + 2390pT^2 + 340T + 1$
6	-90	3362	$p^6T^4 - 90p^3T^3 + 3362pT^2 - 90T + 1$
7	540	4550	$p^6T^4 + 540p^3T^3 + 4550pT^2 + 540T + 1$
8	104	2030	$p^6T^4 + 104p^3T^3 + 2030pT^2 + 104T + 1$
9	60	-2362	$p^6T^4 + 60p^3T^3 - 2362pT^2 + 60T + 1$
10	-228	518	$p^6T^4 - 228p^3T^3 + 518pT^2 - 228T + 1$
11	10	-550	$p^6T^4 + 10p^3T^3 - 550pT^2 + 10T + 1$
12	-482	3890	$p^6T^4 - 482p^3T^3 + 3890pT^2 - 482T + 1$
13	-290	3506	$p^6T^4 - 290p^3T^3 + 3506pT^2 - 290T + 1$
14	10	602	$p^6T^4 + 10p^3T^3 + 602pT^2 + 10T + 1$
15	-228	2246	$p^6T^4 - 228p^3T^3 + 2246pT^2 - 228T + 1$
16	60	2246	$p^6T^4 + 60p^3T^3 + 2246pT^2 + 60T + 1$
17	-136	2510	$p^6T^4 - 136p^3T^3 + 2510pT^2 - 136T + 1$
18	-420	2822	$p^6T^4 - 420p^3T^3 + 2822pT^2 - 420T + 1$
19	102	-670	$p^6T^4 + 102p^3T^3 - 670pT^2 + 102T + 1$
20	148	470	$p^6T^4 + 148p^3T^3 + 470pT^2 + 148T + 1$
21	-128	-610	$p^6T^4 - 128p^3T^3 - 610pT^2 - 128T + 1$
22	378	3002	$p^6T^4 + 378p^3T^3 + 3002pT^2 + 378T + 1$
23	-432	4478	$p^6T^4 - 432p^3T^3 + 4478pT^2 - 432T + 1$
24	52	-298	$p^6T^4 + 52p^3T^3 - 298pT^2 + 52T + 1$
25	46	41	$p^2T^2 + 46T + 1$
26	444	3398	$p^6T^4 + 444p^3T^3 + 3398pT^2 + 444T + 1$
27	144	3038	$p^6T^4 + 144p^3T^3 + 3038pT^2 + 144T + 1$
28	102	1634	$p^6T^4 + 102p^3T^3 + 1634pT^2 + 102T + 1$
29	52	-1066	$p^6T^4 + 52p^3T^3 - 1066pT^2 + 52T + 1$
30	352	2750	$p^6T^4 + 352p^3T^3 + 2750pT^2 + 352T + 1$
31	44	230	$p^6T^4 + 44p^3T^3 + 230pT^2 + 44T + 1$
32	-280	1262	$p^6T^4 - 280p^3T^3 + 1262pT^2 - 280T + 1$
33	-48	2174	$p^6T^4 - 48p^3T^3 + 2174pT^2 - 48T + 1$

$p = 41$ (cont.)

λ	a	b	$R(T)$
34	248	974	$p^6T^4 + 248p^3T^3 + 974pT^2 + 248T + 1$
35	-148	2150	$p^6T^4 - 148p^3T^3 + 2150pT^2 - 148T + 1$
36	-80	-610	$p^6T^4 - 80p^3T^3 - 610pT^2 - 80T + 1$
37	148	1622	$p^6T^4 + 148p^3T^3 + 1622pT^2 + 148T + 1$
38	486	3362	$p^6T^4 + 486p^3T^3 + 3362pT^2 + 486T + 1$
39	-96	-706	$p^6T^4 - 96p^3T^3 - 706pT^2 - 96T + 1$
40	-324	3398	$p^6T^4 - 324p^3T^3 + 3398pT^2 - 324T + 1$

$p = 43$

λ	a	b	$R(T)$
1	248	2850	$p^6T^4 + 248p^3T^3 + 2850pT^2 + 248T + 1$
2	-74	170	$p^6T^4 - 74p^3T^3 + 170pT^2 - 74T + 1$
3	-356	2930	$p^6T^4 - 356p^3T^3 + 2930pT^2 - 356T + 1$
4	180	1234	$p^6T^4 + 180p^3T^3 + 1234pT^2 + 180T + 1$
5	288	1474	$p^6T^4 + 288p^3T^3 + 1474pT^2 + 288T + 1$
6	-136	1314	$p^6T^4 - 136p^3T^3 + 1314pT^2 - 136T + 1$
7	-62	-934	$p^6T^4 - 62p^3T^3 - 934pT^2 - 62T + 1$
8	530	4698	$p^6T^4 + 530p^3T^3 + 4698pT^2 + 530T + 1$
9	-204	1618	$p^6T^4 - 204p^3T^3 + 1618pT^2 - 204T + 1$
10	136	3170	$p^6T^4 + 136p^3T^3 + 3170pT^2 + 136T + 1$
11	-568	3906	$p^6T^4 - 568p^3T^3 + 3906pT^2 - 568T + 1$
12	90	1978	$p^6T^4 + 90p^3T^3 + 1978pT^2 + 90T + 1$
13	64	1442	$p^6T^4 + 64p^3T^3 + 1442pT^2 + 64T + 1$
14	-432	3298	$p^6T^4 - 432p^3T^3 + 3298pT^2 - 432T + 1$
15	68	402	$p^6T^4 + 68p^3T^3 + 402pT^2 + 68T + 1$
16	-48	-1694	$p^6T^4 - 48p^3T^3 - 1694pT^2 - 48T + 1$
17	288	1474	$p^6T^4 + 288p^3T^3 + 1474pT^2 + 288T + 1$
18	164	1938	$p^6T^4 + 164p^3T^3 + 1938pT^2 + 164T + 1$
19	-198	2362	$p^6T^4 - 198p^3T^3 + 2362pT^2 - 198T + 1$
20	-96	322	$p^6T^4 - 96p^3T^3 + 322pT^2 - 96T + 1$

$p = 43$ (cont.)

λ	a	b	$R(T)$
21	-240	-158	$p^6T^4 - 240p^3T^3 - 158pT^2 - 240T + 1$
22	-102	-902	$p^6T^4 - 102p^3T^3 - 902pT^2 - 102T + 1$
23	-28	3474	$p^6T^4 - 28p^3T^3 + 3474pT^2 - 28T + 1$
24	-336	2146	$p^6T^4 - 336p^3T^3 + 2146pT^2 - 336T + 1$
25	0	-1598	$p^6T^4 - 1598pT^2 + 1$
26	-796	7314	$p^6T^4 - 796p^3T^3 + 7314pT^2 - 796T + 1$
27	96	2626	$p^6T^4 + 96p^3T^3 + 2626pT^2 + 96T + 1$
28	-176	3074	$p^6T^4 - 176p^3T^3 + 3074pT^2 - 176T + 1$
29	762	6394	$p^6T^4 + 762p^3T^3 + 6394pT^2 + 762T + 1$
30	344	3234	$p^6T^4 + 344p^3T^3 + 3234pT^2 + 344T + 1$
31	472	3074	$p^6T^4 + 472p^3T^3 + 3074pT^2 + 472T + 1$
32	84	-1454	$p^6T^4 + 84p^3T^3 - 1454pT^2 + 84T + 1$
33	-142	-1446	$p^6T^4 - 142p^3T^3 - 1446pT^2 - 142T + 1$
34	-158	-166	$p^6T^4 - 158p^3T^3 - 166pT^2 - 158T + 1$
35	200	1218	$p^6T^4 + 200p^3T^3 + 1218pT^2 + 200T + 1$
36	240	994	$p^6T^4 + 240p^3T^3 + 994pT^2 + 240T + 1$
37	276	2386	$p^6T^4 + 276p^3T^3 + 2386pT^2 + 276T + 1$
38	316	3314	$p^6T^4 + 316p^3T^3 + 3314pT^2 + 316T + 1$
39	-650	5930	$p^6T^4 - 650p^3T^3 + 5930pT^2 - 650T + 1$
40	-184	1986	$p^6T^4 - 184p^3T^3 + 1986pT^2 - 184T + 1$
41	-14	43	$p^2T^2 - 14T + 1$
42	198	2026	$p^6T^4 + 198p^3T^3 + 2026pT^2 + 198T + 1$

$p = 47$

λ	a	b	$R(T)$
1	-432	4418	$p^6T^4 - 432p^3T^3 + 4418pT^2 - 432T + 1$
2	-232	3170	$p^6T^4 - 232p^3T^3 + 3170pT^2 - 232T + 1$
3	-48	962	$p^6T^4 - 48p^3T^3 + 962pT^2 - 48T + 1$
4	336	4418	$p^6T^4 + 336p^3T^3 + 4418pT^2 + 336T + 1$
5	518	5138	$p^6T^4 + 518p^3T^3 + 5138pT^2 + 518T + 1$
6	500	3554	$p^6T^4 + 500p^3T^3 + 3554pT^2 + 500T + 1$
7	348	3554	$p^6T^4 + 348p^3T^3 + 3554pT^2 + 348T + 1$
8	-480	4706	$p^6T^4 - 480p^3T^3 + 4706pT^2 - 480T + 1$
9	-280	578	$p^6T^4 - 280p^3T^3 + 578pT^2 - 280T + 1$
10	-402	1682	$p^6T^4 - 402p^3T^3 + 1682pT^2 - 402T + 1$
11	-184	2498	$p^6T^4 - 184p^3T^3 + 2498pT^2 - 184T + 1$
12	-248	866	$p^6T^4 - 248p^3T^3 + 866pT^2 - 248T + 1$
13	-498	4562	$p^6T^4 - 498p^3T^3 + 4562pT^2 - 498T + 1$
14	132	3554	$p^6T^4 + 132p^3T^3 + 3554pT^2 + 132T + 1$
15	-396	4418	$p^6T^4 - 396p^3T^3 + 4418pT^2 - 396T + 1$
16	404	4706	$p^6T^4 + 404p^3T^3 + 4706pT^2 + 404T + 1$
17	-200	2498	$p^6T^4 - 200p^3T^3 + 2498pT^2 - 200T + 1$
18	-48	2114	$p^6T^4 - 48p^3T^3 + 2114pT^2 - 48T + 1$
19	584	4034	$p^6T^4 + 584p^3T^3 + 4034pT^2 + 584T + 1$
20	-356	4034	$p^6T^4 - 356p^3T^3 + 4034pT^2 - 356T + 1$
21	-132	98	$p^6T^4 - 132p^3T^3 + 98pT^2 - 132T + 1$
22	684	5570	$p^6T^4 + 684p^3T^3 + 5570pT^2 + 684T + 1$
23	18	-1198	$p^6T^4 + 18p^3T^3 - 1198pT^2 + 18T + 1$
24	152	2402	$p^6T^4 + 152p^3T^3 + 2402pT^2 + 152T + 1$
25	-200	962	$p^6T^4 - 200p^3T^3 + 962pT^2 - 200T + 1$
26	114	-622	$p^6T^4 + 114p^3T^3 - 622pT^2 + 114T + 1$
27	-200	3650	$p^6T^4 - 200p^3T^3 + 3650pT^2 - 200T + 1$
28	384	4706	$p^6T^4 + 384p^3T^3 + 4706pT^2 + 384T + 1$
29	-12	-190	$p^6T^4 - 12p^3T^3 - 190pT^2 - 12T + 1$
30	28	-574	$p^6T^4 + 28p^3T^3 - 574pT^2 + 28T + 1$
31	154	1874	$p^6T^4 + 154p^3T^3 + 1874pT^2 + 154T + 1$
32	-48	-2494	$p^6T^4 - 48p^3T^3 - 2494pT^2 - 48T + 1$
33	48	3266	$p^6T^4 + 48p^3T^3 + 3266pT^2 + 48T + 1$

$p = 47$ (cont.)

λ	a	b	$R(T)$
34	136	-670	$p^6T^4 + 136p^3T^3 - 670pT^2 + 136T + 1$
35	240	962	$p^6T^4 + 240p^3T^3 + 962pT^2 + 240T + 1$
36	0	-47	$-(pT - 1)(pT + 1)$
37	76	98	$p^6T^4 + 76p^3T^3 + 98pT^2 + 76T + 1$
38	-730	5714	$p^6T^4 - 730p^3T^3 + 5714pT^2 - 730T + 1$
39	-210	2258	$p^6T^4 - 210p^3T^3 + 2258pT^2 - 210T + 1$
40	382	2066	$p^6T^4 + 382p^3T^3 + 2066pT^2 + 382T + 1$
41	88	-1342	$p^6T^4 + 88p^3T^3 - 1342pT^2 + 88T + 1$
42	104	-958	$p^6T^4 + 104p^3T^3 - 958pT^2 + 104T + 1$
43	-94	146	$p^6T^4 - 94p^3T^3 + 146pT^2 - 94T + 1$
44	114	3410	$p^6T^4 + 114p^3T^3 + 3410pT^2 + 114T + 1$
45	58	3602	$p^6T^4 + 58p^3T^3 + 3602pT^2 + 58T + 1$
46	-124	-958	$p^6T^4 - 124p^3T^3 - 958pT^2 - 124T + 1$

$p = 53$

λ	a	b	$R(T)$
1	84	2054	$p^6T^4 + 84p^3T^3 + 2054pT^2 + 84T + 1$
2	-266	1562	$p^6T^4 - 266p^3T^3 + 1562pT^2 - 266T + 1$
3	346	3746	$p^6T^4 + 346p^3T^3 + 3746pT^2 + 346T + 1$
4	180	-1978	$p^6T^4 + 180p^3T^3 - 1978pT^2 + 180T + 1$
5	-432	3134	$p^6T^4 - 432p^3T^3 + 3134pT^2 - 432T + 1$
6	136	3470	$p^6T^4 + 136p^3T^3 + 3470pT^2 + 136T + 1$
7	48	-2914	$p^6T^4 + 48p^3T^3 - 2914pT^2 + 48T + 1$
8	-606	3890	$p^6T^4 - 606p^3T^3 + 3890pT^2 - 606T + 1$
9	328	1934	$p^6T^4 + 328p^3T^3 + 1934pT^2 + 328T + 1$
10	-148	1238	$p^6T^4 - 148p^3T^3 + 1238pT^2 - 148T + 1$
11	-144	-2914	$p^6T^4 - 144p^3T^3 - 2914pT^2 - 144T + 1$
12	-396	3782	$p^6T^4 - 396p^3T^3 + 3782pT^2 - 396T + 1$
13	-284	2726	$p^6T^4 - 284p^3T^3 + 2726pT^2 - 284T + 1$
14	-904	8462	$p^6T^4 - 904p^3T^3 + 8462pT^2 - 904T + 1$

$p = 53$ (cont.)

λ	a	b	$R(T)$
15	220	4790	$p^6T^4 + 220p^3T^3 + 4790pT^2 + 220T + 1$
16	-64	3518	$p^6T^4 - 64p^3T^3 + 3518pT^2 - 64T + 1$
17	336	2270	$p^6T^4 + 336p^3T^3 + 2270pT^2 + 336T + 1$
18	276	3782	$p^6T^4 + 276p^3T^3 + 3782pT^2 + 276T + 1$
19	-920	8942	$p^6T^4 - 920p^3T^3 + 8942pT^2 - 920T + 1$
20	476	3734	$p^6T^4 + 476p^3T^3 + 3734pT^2 + 476T + 1$
21	-414	5618	$p^6T^4 - 414p^3T^3 + 5618pT^2 - 414T + 1$
22	-492	3782	$p^6T^4 - 492p^3T^3 + 3782pT^2 - 492T + 1$
23	476	2966	$p^6T^4 + 476p^3T^3 + 2966pT^2 + 476T + 1$
24	612	4358	$p^6T^4 + 612p^3T^3 + 4358pT^2 + 612T + 1$
25	336	3422	$p^6T^4 + 336p^3T^3 + 3422pT^2 + 336T + 1$
26	554	6914	$p^6T^4 + 554p^3T^3 + 6914pT^2 + 554T + 1$
27	686	7178	$p^6T^4 + 686p^3T^3 + 7178pT^2 + 686T + 1$
28	528	2270	$p^6T^4 + 528p^3T^3 + 2270pT^2 + 528T + 1$
29	0	-53	$-(pT - 1)(pT + 1)$
30	92	-106	$p^6T^4 + 92p^3T^3 - 106pT^2 + 92T + 1$
31	-282	4538	$p^6T^4 - 282p^3T^3 + 4538pT^2 - 282T + 1$
32	-74	-2470	$p^6T^4 - 74p^3T^3 - 2470pT^2 - 74T + 1$
33	-736	6302	$p^6T^4 - 736p^3T^3 + 6302pT^2 - 736T + 1$
34	6	-2950	$p^6T^4 + 6p^3T^3 - 2950pT^2 + 6T + 1$
35	32	-994	$p^6T^4 + 32p^3T^3 - 994pT^2 + 32T + 1$
36	-492	3206	$p^6T^4 - 492p^3T^3 + 3206pT^2 - 492T + 1$
37	-816	6878	$p^6T^4 - 816p^3T^3 + 6878pT^2 - 816T + 1$
38	-484	5462	$p^6T^4 - 484p^3T^3 + 5462pT^2 - 484T + 1$
39	-108	-826	$p^6T^4 - 108p^3T^3 - 826pT^2 - 108T + 1$
40	512	3902	$p^6T^4 + 512p^3T^3 + 3902pT^2 + 512T + 1$
41	258	3890	$p^6T^4 + 258p^3T^3 + 3890pT^2 + 258T + 1$
42	-188	2534	$p^6T^4 - 188p^3T^3 + 2534pT^2 - 188T + 1$
43	816	8606	$p^6T^4 + 816p^3T^3 + 8606pT^2 + 816T + 1$
44	-56	3086	$p^6T^4 - 56p^3T^3 + 3086pT^2 - 56T + 1$
45	-12	-3130	$p^6T^4 - 12p^3T^3 - 3130pT^2 - 12T + 1$
46	-48	-1186	$p^6T^4 - 48p^3T^3 - 1186pT^2 - 48T + 1$
47	-236	2534	$p^6T^4 - 236p^3T^3 + 2534pT^2 - 236T + 1$

$p = 53$ (cont.)

λ	a	b	$R(T)$
48	414	4970	$p^6T^4 + 414p^3T^3 + 4970pT^2 + 414T + 1$
49	−44	1766	$p^6T^4 - 44p^3T^3 + 1766pT^2 - 44T + 1$
50	302	−310	$p^6T^4 + 302p^3T^3 - 310pT^2 + 302T + 1$
51	510	4970	$p^6T^4 + 510p^3T^3 + 4970pT^2 + 510T + 1$
52	136	−1906	$p^6T^4 + 136p^3T^3 - 1906pT^2 + 136T + 1$

$p = 59$

λ	a	b	$R(T)$
1	648	3074	$p^6T^4 + 648p^3T^3 + 3074pT^2 + 648T + 1$
2	746	6938	$p^6T^4 + 746p^3T^3 + 6938pT^2 + 746T + 1$
3	−776	7106	$p^6T^4 - 776p^3T^3 + 7106pT^2 - 776T + 1$
4	548	3602	$p^6T^4 + 548p^3T^3 + 3602pT^2 + 548T + 1$
5	−408	3074	$p^6T^4 - 408p^3T^3 + 3074pT^2 - 408T + 1$
6	6	4298	$p^6T^4 + 6p^3T^3 + 4298pT^2 + 6T + 1$
7	−552	3938	$p^6T^4 - 552p^3T^3 + 3938pT^2 - 552T + 1$
8	356	7058	$p^6T^4 + 356p^3T^3 + 7058pT^2 + 356T + 1$
9	40	2594	$p^6T^4 + 40p^3T^3 + 2594pT^2 + 40T + 1$
10	266	986	$p^6T^4 + 266p^3T^3 + 986pT^2 + 266T + 1$
11	−456	3938	$p^6T^4 - 456p^3T^3 + 3938pT^2 - 456T + 1$
12	82	59	$p^2T^2 + 82T + 1$
13	−462	6458	$p^6T^4 - 462p^3T^3 + 6458pT^2 - 462T + 1$
14	−16	5570	$p^6T^4 - 16p^3T^3 + 5570pT^2 - 16T + 1$
15	−84	1202	$p^6T^4 - 84p^3T^3 + 1202pT^2 - 84T + 1$
16	684	6962	$p^6T^4 + 684p^3T^3 + 6962pT^2 + 684T + 1$
17	224	5474	$p^6T^4 + 224p^3T^3 + 5474pT^2 + 224T + 1$
18	560	5954	$p^6T^4 + 560p^3T^3 + 5954pT^2 + 560T + 1$
19	−408	4226	$p^6T^4 - 408p^3T^3 + 4226pT^2 - 408T + 1$
20	−48	−94	$p^6T^4 - 48p^3T^3 - 94pT^2 - 48T + 1$
21	−84	50	$p^6T^4 - 84p^3T^3 + 50pT^2 - 84T + 1$
22	376	7106	$p^6T^4 + 376p^3T^3 + 7106pT^2 + 376T + 1$

$p = 59$ (cont.)

λ	a	b	$R(T)$
23	-728	8738	$p^6T^4 - 728p^3T^3 + 8738pT^2 - 728T + 1$
24	350	3434	$p^6T^4 + 350p^3T^3 + 3434pT^2 + 350T + 1$
25	328	-478	$p^6T^4 + 328p^3T^3 - 478pT^2 + 328T + 1$
26	-120	3074	$p^6T^4 - 120p^3T^3 + 3074pT^2 - 120T + 1$
27	-316	3218	$p^6T^4 - 316p^3T^3 + 3218pT^2 - 316T + 1$
28	72	1922	$p^6T^4 + 72p^3T^3 + 1922pT^2 + 72T + 1$
29	-316	5138	$p^6T^4 - 316p^3T^3 + 5138pT^2 - 316T + 1$
30	300	2354	$p^6T^4 + 300p^3T^3 + 2354pT^2 + 300T + 1$
31	-186	2570	$p^6T^4 - 186p^3T^3 + 2570pT^2 - 186T + 1$
32	6	4874	$p^6T^4 + 6p^3T^3 + 4874pT^2 + 6T + 1$
33	418	4922	$p^6T^4 + 418p^3T^3 + 4922pT^2 + 418T + 1$
34	-682	5066	$p^6T^4 - 682p^3T^3 + 5066pT^2 - 682T + 1$
35	128	1250	$p^6T^4 + 128p^3T^3 + 1250pT^2 + 128T + 1$
36	560	5954	$p^6T^4 + 560p^3T^3 + 5954pT^2 + 560T + 1$
37	-106	-310	$p^6T^4 - 106p^3T^3 - 310pT^2 - 106T + 1$
38	-830	6074	$p^6T^4 - 830p^3T^3 + 6074pT^2 - 830T + 1$
39	-570	3722	$p^6T^4 - 570p^3T^3 + 3722pT^2 - 570T + 1$
40	198	1418	$p^6T^4 + 198p^3T^3 + 1418pT^2 + 198T + 1$
41	-180	3506	$p^6T^4 - 180p^3T^3 + 3506pT^2 - 180T + 1$
42	-220	-1390	$p^6T^4 - 220p^3T^3 - 1390pT^2 - 220T + 1$
43	120	1634	$p^6T^4 + 120p^3T^3 + 1634pT^2 + 120T + 1$
44	356	1682	$p^6T^4 + 356p^3T^3 + 1682pT^2 + 356T + 1$
45	312	3938	$p^6T^4 + 312p^3T^3 + 3938pT^2 + 312T + 1$
46	88	3266	$p^6T^4 + 88p^3T^3 + 3266pT^2 + 88T + 1$
47	-418	2474	$p^6T^4 - 418p^3T^3 + 2474pT^2 - 418T + 1$
48	-8	-1342	$p^6T^4 - 8p^3T^3 - 1342pT^2 - 8T + 1$
49	40	4898	$p^6T^4 + 40p^3T^3 + 4898pT^2 + 40T + 1$
50	-84	-2254	$p^6T^4 - 84p^3T^3 - 2254pT^2 - 84T + 1$
51	-192	5378	$p^6T^4 - 192p^3T^3 + 5378pT^2 - 192T + 1$
52	312	5090	$p^6T^4 + 312p^3T^3 + 5090pT^2 + 312T + 1$
53	128	-3358	$p^6T^4 + 128p^3T^3 - 3358pT^2 + 128T + 1$
54	176	1730	$p^6T^4 + 176p^3T^3 + 1730pT^2 + 176T + 1$
55	300	50	$p^6T^4 + 300p^3T^3 + 50pT^2 + 300T + 1$

$p = 59$ (cont.)

λ	a	b	$R(T)$
56	492	4658	$p^6T^4 + 492p^3T^3 + 4658pT^2 + 492T + 1$
57	-736	7394	$p^6T^4 - 736p^3T^3 + 7394pT^2 - 736T + 1$
58	-174	5306	$p^6T^4 - 174p^3T^3 + 5306pT^2 - 174T + 1$

$p = 61$

λ	a	b	$R(T)$
1	-540	1990	$p^6T^4 - 540p^3T^3 + 1990pT^2 - 540T + 1$
2	-308	4022	$p^6T^4 - 308p^3T^3 + 4022pT^2 - 308T + 1$
3	232	494	$p^6T^4 + 232p^3T^3 + 494pT^2 + 232T + 1$
4	-552	7246	$p^6T^4 - 552p^3T^3 + 7246pT^2 - 552T + 1$
5	-64	2718	$p^6T^4 - 64p^3T^3 + 2718pT^2 - 64T + 1$
6	114	5602	$p^6T^4 + 114p^3T^3 + 5602pT^2 + 114T + 1$
7	-168	-3602	$p^6T^4 - 168p^3T^3 - 3602pT^2 - 168T + 1$
8	526	4730	$p^6T^4 + 526p^3T^3 + 4730pT^2 + 526T + 1$
9	-148	3030	$p^6T^4 - 148p^3T^3 + 3030pT^2 - 148T + 1$
10	10	4754	$p^6T^4 + 10p^3T^3 + 4754pT^2 + 10T + 1$
11	54	4618	$p^6T^4 + 54p^3T^3 + 4618pT^2 + 54T + 1$
12	-452	1142	$p^6T^4 - 452p^3T^3 + 1142pT^2 - 452T + 1$
13	804	8134	$p^6T^4 + 804p^3T^3 + 8134pT^2 + 804T + 1$
14	-1276	13350	$p^6T^4 - 1276p^3T^3 + 13350pT^2 - 1276T + 1$
15	-460	5094	$p^6T^4 - 460p^3T^3 + 5094pT^2 - 460T + 1$
16	-1024	9246	$p^6T^4 - 1024p^3T^3 + 9246pT^2 - 1024T + 1$
17	-226	-486	$p^6T^4 - 226p^3T^3 - 486pT^2 - 226T + 1$
18	386	6210	$p^6T^4 + 386p^3T^3 + 6210pT^2 + 386T + 1$
19	-64	7326	$p^6T^4 - 64p^3T^3 + 7326pT^2 - 64T + 1$
20	-444	7558	$p^6T^4 - 444p^3T^3 + 7558pT^2 - 444T + 1$
21	704	7806	$p^6T^4 + 704p^3T^3 + 7806pT^2 + 704T + 1$
22	-300	454	$p^6T^4 - 300p^3T^3 + 454pT^2 - 300T + 1$
23	210	1570	$p^6T^4 + 210p^3T^3 + 1570pT^2 + 210T + 1$
24	534	3082	$p^6T^4 + 534p^3T^3 + 3082pT^2 + 534T + 1$

$p = 61$ (cont.)

λ	a	b	$R(T)$
25	-396	646	$p^6T^4 - 396p^3T^3 + 646pT^2 - 396T + 1$
26	-48	6430	$p^6T^4 - 48p^3T^3 + 6430pT^2 - 48T + 1$
27	-48	1726	$p^6T^4 - 48p^3T^3 + 1726pT^2 - 48T + 1$
28	-160	5502	$p^6T^4 - 160p^3T^3 + 5502pT^2 - 160T + 1$
29	232	1934	$p^6T^4 + 232p^3T^3 + 1934pT^2 + 232T + 1$
30	446	6234	$p^6T^4 + 446p^3T^3 + 6234pT^2 + 446T + 1$
31	586	4178	$p^6T^4 + 586p^3T^3 + 4178pT^2 + 586T + 1$
32	312	-530	$p^6T^4 + 312p^3T^3 - 530pT^2 + 312T + 1$
33	166	-598	$p^6T^4 + 166p^3T^3 - 598pT^2 + 166T + 1$
34	-52	534	$p^6T^4 - 52p^3T^3 + 534pT^2 - 52T + 1$
35	98	3906	$p^6T^4 + 98p^3T^3 + 3906pT^2 + 98T + 1$
36	-300	454	$p^6T^4 - 300p^3T^3 + 454pT^2 - 300T + 1$
37	-292	-1578	$p^6T^4 - 292p^3T^3 - 1578pT^2 - 292T + 1$
38	-308	4022	$p^6T^4 - 308p^3T^3 + 4022pT^2 - 308T + 1$
39	-344	5102	$p^6T^4 - 344p^3T^3 + 5102pT^2 - 344T + 1$
40	-544	2814	$p^6T^4 - 544p^3T^3 + 2814pT^2 - 544T + 1$
41	0	-61	$-(pT - 1)(pT + 1)$
42	276	3142	$p^6T^4 + 276p^3T^3 + 3142pT^2 + 276T + 1$
43	482	4866	$p^6T^4 + 482p^3T^3 + 4866pT^2 + 482T + 1$
44	-72	-2066	$p^6T^4 - 72p^3T^3 - 2066pT^2 - 72T + 1$
45	-4	5910	$p^6T^4 - 4p^3T^3 + 5910pT^2 - 4T + 1$
46	172	6518	$p^6T^4 + 172p^3T^3 + 6518pT^2 + 172T + 1$
47	516	6022	$p^6T^4 + 516p^3T^3 + 6022pT^2 + 516T + 1$
48	136	-1618	$p^6T^4 + 136p^3T^3 - 1618pT^2 + 136T + 1$
49	-528	4606	$p^6T^4 - 528p^3T^3 + 4606pT^2 - 528T + 1$
50	-292	-1578	$p^6T^4 - 292p^3T^3 - 1578pT^2 - 292T + 1$
51	682	4562	$p^6T^4 + 682p^3T^3 + 4562pT^2 + 682T + 1$
52	-360	2638	$p^6T^4 - 360p^3T^3 + 2638pT^2 - 360T + 1$
53	-412	2982	$p^6T^4 - 412p^3T^3 + 2982pT^2 - 412T + 1$
54	748	9398	$p^6T^4 + 748p^3T^3 + 9398pT^2 + 748T + 1$
55	320	2814	$p^6T^4 + 320p^3T^3 + 2814pT^2 + 320T + 1$
56	136	4142	$p^6T^4 + 136p^3T^3 + 4142pT^2 + 136T + 1$
57	1272	13582	$p^6T^4 + 1272p^3T^3 + 13582pT^2 + 1272T + 1$

$p = 61$ (cont.)

λ	a	b	$R(T)$
58	620	3798	$p^6T^4 + 620p^3T^3 + 3798pT^2 + 620T + 1$
59	-270	802	$p^6T^4 - 270p^3T^3 + 802pT^2 - 270T + 1$
60	-256	-1506	$p^6T^4 - 256p^3T^3 - 1506pT^2 - 256T + 1$

$p = 67$

λ	a	b	$R(T)$
1	1176	10882	$p^6T^4 + 1176p^3T^3 + 10882pT^2 + 1176T + 1$
2	-306	4186	$p^6T^4 - 306p^3T^3 + 4186pT^2 - 306T + 1$
3	-84	946	$p^6T^4 - 84p^3T^3 + 946pT^2 - 84T + 1$
4	-276	4018	$p^6T^4 - 276p^3T^3 + 4018pT^2 - 276T + 1$
5	80	3618	$p^6T^4 + 80p^3T^3 + 3618pT^2 + 80T + 1$
6	400	2	$p^6T^4 + 400p^3T^3 + 2pT^2 + 400T + 1$
7	740	3666	$p^6T^4 + 740p^3T^3 + 3666pT^2 + 740T + 1$
8	-824	10946	$p^6T^4 - 824p^3T^3 + 10946pT^2 - 824T + 1$
9	92	6258	$p^6T^4 + 92p^3T^3 + 6258pT^2 + 92T + 1$
10	-640	8514	$p^6T^4 - 640p^3T^3 + 8514pT^2 - 640T + 1$
11	-238	4362	$p^6T^4 - 238p^3T^3 + 4362pT^2 - 238T + 1$
12	848	10146	$p^6T^4 + 848p^3T^3 + 10146pT^2 + 848T + 1$
13	188	882	$p^6T^4 + 188p^3T^3 + 882pT^2 + 188T + 1$
14	-8	1250	$p^6T^4 - 8p^3T^3 + 1250pT^2 - 8T + 1$
15	-604	8658	$p^6T^4 - 604p^3T^3 + 8658pT^2 - 604T + 1$
16	352	98	$p^6T^4 + 352p^3T^3 + 98pT^2 + 352T + 1$
17	416	1218	$p^6T^4 + 416p^3T^3 + 1218pT^2 + 416T + 1$
18	-276	8626	$p^6T^4 - 276p^3T^3 + 8626pT^2 - 276T + 1$
19	972	11698	$p^6T^4 + 972p^3T^3 + 11698pT^2 + 972T + 1$
20	942	10906	$p^6T^4 + 942p^3T^3 + 10906pT^2 + 942T + 1$
21	72	4642	$p^6T^4 + 72p^3T^3 + 4642pT^2 + 72T + 1$
22	-62	67	$p^2T^2 - 62T + 1$
23	320	-318	$p^6T^4 + 320p^3T^3 - 318pT^2 + 320T + 1$
24	-648	2434	$p^6T^4 - 648p^3T^3 + 2434pT^2 - 648T + 1$

$p = 67$ (cont.)

λ	a	b	$R(T)$
25	-1080	10786	$p^6T^4 - 1080p^3T^3 + 10786pT^2 - 1080T + 1$
26	-848	8834	$p^6T^4 - 848p^3T^3 + 8834pT^2 - 848T + 1$
27	-716	8210	$p^6T^4 - 716p^3T^3 + 8210pT^2 - 716T + 1$
28	-578	7322	$p^6T^4 - 578p^3T^3 + 7322pT^2 - 578T + 1$
29	232	194	$p^6T^4 + 232p^3T^3 + 194pT^2 + 232T + 1$
30	-196	-2958	$p^6T^4 - 196p^3T^3 - 2958pT^2 - 196T + 1$
31	-234	250	$p^6T^4 - 234p^3T^3 + 250pT^2 - 234T + 1$
32	-606	3274	$p^6T^4 - 606p^3T^3 + 3274pT^2 - 606T + 1$
33	-416	1634	$p^6T^4 - 416p^3T^3 + 1634pT^2 - 416T + 1$
34	-32	-286	$p^6T^4 - 32p^3T^3 - 286pT^2 - 32T + 1$
35	512	8514	$p^6T^4 + 512p^3T^3 + 8514pT^2 + 512T + 1$
36	-448	-1086	$p^6T^4 - 448p^3T^3 - 1086pT^2 - 448T + 1$
37	1148	11634	$p^6T^4 + 1148p^3T^3 + 11634pT^2 + 1148T + 1$
38	-442	3258	$p^6T^4 - 442p^3T^3 + 3258pT^2 - 442T + 1$
39	72	3874	$p^6T^4 + 72p^3T^3 + 3874pT^2 + 72T + 1$
40	-648	7042	$p^6T^4 - 648p^3T^3 + 7042pT^2 - 648T + 1$
41	-58	-2694	$p^6T^4 - 58p^3T^3 - 2694pT^2 - 58T + 1$
42	-84	-3662	$p^6T^4 - 84p^3T^3 - 3662pT^2 - 84T + 1$
43	998	10746	$p^6T^4 + 998p^3T^3 + 10746pT^2 + 998T + 1$
44	986	9258	$p^6T^4 + 986p^3T^3 + 9258pT^2 + 986T + 1$
45	410	618	$p^6T^4 + 410p^3T^3 + 618pT^2 + 410T + 1$
46	902	6906	$p^6T^4 + 902p^3T^3 + 6906pT^2 + 902T + 1$
47	108	1330	$p^6T^4 + 108p^3T^3 + 1330pT^2 + 108T + 1$
48	230	-390	$p^6T^4 + 230p^3T^3 - 390pT^2 + 230T + 1$
49	744	8482	$p^6T^4 + 744p^3T^3 + 8482pT^2 + 744T + 1$
50	-360	4738	$p^6T^4 - 360p^3T^3 + 4738pT^2 - 360T + 1$
51	-538	7482	$p^6T^4 - 538p^3T^3 + 7482pT^2 - 538T + 1$
52	-4	2418	$p^6T^4 - 4p^3T^3 + 2418pT^2 - 4T + 1$
53	-496	9378	$p^6T^4 - 496p^3T^3 + 9378pT^2 - 496T + 1$
54	-304	-1374	$p^6T^4 - 304p^3T^3 - 1374pT^2 - 304T + 1$
55	-80	-1918	$p^6T^4 - 80p^3T^3 - 1918pT^2 - 80T + 1$
56	16	7682	$p^6T^4 + 16p^3T^3 + 7682pT^2 + 16T + 1$
57	-414	5194	$p^6T^4 - 414p^3T^3 + 5194pT^2 - 414T + 1$

$p = 67$ (cont.)

λ	a	b	$R(T)$
58	-138	3322	$p^6T^4 - 138p^3T^3 + 3322pT^2 - 138T + 1$
59	-676	3186	$p^6T^4 - 676p^3T^3 + 3186pT^2 - 676T + 1$
60	88	8546	$p^6T^4 + 88p^3T^3 + 8546pT^2 + 88T + 1$
61	-290	-2470	$p^6T^4 - 290p^3T^3 - 2470pT^2 - 290T + 1$
62	820	9746	$p^6T^4 + 820p^3T^3 + 9746pT^2 + 820T + 1$
63	736	7394	$p^6T^4 + 736p^3T^3 + 7394pT^2 + 736T + 1$
64	-360	7042	$p^6T^4 - 360p^3T^3 + 7042pT^2 - 360T + 1$
65	-408	3874	$p^6T^4 - 408p^3T^3 + 3874pT^2 - 408T + 1$
66	-112	-606	$p^6T^4 - 112p^3T^3 - 606pT^2 - 112T + 1$

$p = 71$

λ	a	b	$R(T)$
1	264	-286	$p^6T^4 + 264p^3T^3 - 286pT^2 + 264T + 1$
2	736	3650	$p^6T^4 + 736p^3T^3 + 3650pT^2 + 736T + 1$
3	180	-4894	$p^6T^4 + 180p^3T^3 - 4894pT^2 + 180T + 1$
4	736	6722	$p^6T^4 + 736p^3T^3 + 6722pT^2 + 736T + 1$
5	696	4610	$p^6T^4 + 696p^3T^3 + 4610pT^2 + 696T + 1$
6	-880	7874	$p^6T^4 - 880p^3T^3 + 7874pT^2 - 880T + 1$
7	1068	10370	$p^6T^4 + 1068p^3T^3 + 10370pT^2 + 1068T + 1$
8	-112	194	$p^6T^4 - 112p^3T^3 + 194pT^2 - 112T + 1$
9	1128	13538	$p^6T^4 + 1128p^3T^3 + 13538pT^2 + 1128T + 1$
10	0	-71	$-(pT - 1)(pT + 1)$
11	90	4034	$p^6T^4 + 90p^3T^3 + 4034pT^2 + 90T + 1$
12	-120	-2590	$p^6T^4 - 120p^3T^3 - 2590pT^2 - 120T + 1$
13	-46	1730	$p^6T^4 - 46p^3T^3 + 1730pT^2 - 46T + 1$
14	-102	5186	$p^6T^4 - 102p^3T^3 + 5186pT^2 - 102T + 1$
15	660	8930	$p^6T^4 + 660p^3T^3 + 8930pT^2 + 660T + 1$
16	-668	2402	$p^6T^4 - 668p^3T^3 + 2402pT^2 - 668T + 1$
17	-1156	10754	$p^6T^4 - 1156p^3T^3 + 10754pT^2 - 1156T + 1$
18	-24	2018	$p^6T^4 - 24p^3T^3 + 2018pT^2 - 24T + 1$

$p = 71$ (cont.)

λ	a	b	$R(T)$
19	56	−286	$p^6T^4 + 56p^3T^3 - 286pT^2 + 56T + 1$
20	60	−6622	$p^6T^4 + 60p^3T^3 - 6622pT^2 + 60T + 1$
21	112	8738	$p^6T^4 + 112p^3T^3 + 8738pT^2 + 112T + 1$
22	206	−6238	$p^6T^4 + 206p^3T^3 - 6238pT^2 + 206T + 1$
23	1184	11042	$p^6T^4 + 1184p^3T^3 + 11042pT^2 + 1184T + 1$
24	−880	12482	$p^6T^4 - 880p^3T^3 + 12482pT^2 - 880T + 1$
25	−972	8930	$p^6T^4 - 972p^3T^3 + 8930pT^2 - 972T + 1$
26	−26	2018	$p^6T^4 - 26p^3T^3 + 2018pT^2 - 26T + 1$
27	−360	−2302	$p^6T^4 - 360p^3T^3 - 2302pT^2 - 360T + 1$
28	−274	2018	$p^6T^4 - 274p^3T^3 + 2018pT^2 - 274T + 1$
29	−928	7202	$p^6T^4 - 928p^3T^3 + 7202pT^2 - 928T + 1$
30	−380	6242	$p^6T^4 - 380p^3T^3 + 6242pT^2 - 380T + 1$
31	684	8066	$p^6T^4 + 684p^3T^3 + 8066pT^2 + 684T + 1$
32	224	674	$p^6T^4 + 224p^3T^3 + 674pT^2 + 224T + 1$
33	−276	3458	$p^6T^4 - 276p^3T^3 + 3458pT^2 - 276T + 1$
34	−198	2306	$p^6T^4 - 198p^3T^3 + 2306pT^2 - 198T + 1$
35	−1082	11810	$p^6T^4 - 1082p^3T^3 + 11810pT^2 - 1082T + 1$
36	1032	13538	$p^6T^4 + 1032p^3T^3 + 13538pT^2 + 1032T + 1$
37	360	10082	$p^6T^4 + 360p^3T^3 + 10082pT^2 + 360T + 1$
38	−424	1634	$p^6T^4 - 424p^3T^3 + 1634pT^2 - 424T + 1$
39	−618	7202	$p^6T^4 - 618p^3T^3 + 7202pT^2 - 618T + 1$
40	224	−3934	$p^6T^4 + 224p^3T^3 - 3934pT^2 + 224T + 1$
41	112	674	$p^6T^4 + 112p^3T^3 + 674pT^2 + 112T + 1$
42	666	6914	$p^6T^4 + 666p^3T^3 + 6914pT^2 + 666T + 1$
43	280	3938	$p^6T^4 + 280p^3T^3 + 3938pT^2 + 280T + 1$
44	−312	8930	$p^6T^4 - 312p^3T^3 + 8930pT^2 - 312T + 1$
45	−24	6626	$p^6T^4 - 24p^3T^3 + 6626pT^2 - 24T + 1$
46	−982	10370	$p^6T^4 - 982p^3T^3 + 10370pT^2 - 982T + 1$
47	−522	4898	$p^6T^4 - 522p^3T^3 + 4898pT^2 - 522T + 1$
48	−12	866	$p^6T^4 - 12p^3T^3 + 866pT^2 - 12T + 1$
49	400	−478	$p^6T^4 + 400p^3T^3 - 478pT^2 + 400T + 1$
50	−588	4322	$p^6T^4 - 588p^3T^3 + 4322pT^2 - 588T + 1$
51	−580	3074	$p^6T^4 - 580p^3T^3 + 3074pT^2 - 580T + 1$

$p = 71$ (cont.)

λ	a	b	$R(T)$
52	304	−862	$p^6T^4 + 304p^3T^3 - 862pT^2 + 304T + 1$
53	−446	1154	$p^6T^4 - 446p^3T^3 + 1154pT^2 - 446T + 1$
54	−1320	13826	$p^6T^4 - 1320p^3T^3 + 13826pT^2 - 1320T + 1$
55	−502	6338	$p^6T^4 - 502p^3T^3 + 6338pT^2 - 502T + 1$
56	−84	3458	$p^6T^4 - 84p^3T^3 + 3458pT^2 - 84T + 1$
57	−416	2114	$p^6T^4 - 416p^3T^3 + 2114pT^2 - 416T + 1$
58	1072	13346	$p^6T^4 + 1072p^3T^3 + 13346pT^2 + 1072T + 1$
59	−734	7874	$p^6T^4 - 734p^3T^3 + 7874pT^2 - 734T + 1$
60	−64	1826	$p^6T^4 - 64p^3T^3 + 1826pT^2 - 64T + 1$
61	996	8642	$p^6T^4 + 996p^3T^3 + 8642pT^2 + 996T + 1$
62	−8	7778	$p^6T^4 - 8p^3T^3 + 7778pT^2 - 8T + 1$
63	−504	8930	$p^6T^4 - 504p^3T^3 + 8930pT^2 - 504T + 1$
64	100	9314	$p^6T^4 + 100p^3T^3 + 9314pT^2 + 100T + 1$
65	1532	17666	$p^6T^4 + 1532p^3T^3 + 17666pT^2 + 1532T + 1$
66	492	5762	$p^6T^4 + 492p^3T^3 + 5762pT^2 + 492T + 1$
67	726	6626	$p^6T^4 + 726p^3T^3 + 6626pT^2 + 726T + 1$
68	286	4898	$p^6T^4 + 286p^3T^3 + 4898pT^2 + 286T + 1$
69	72	866	$p^6T^4 + 72p^3T^3 + 866pT^2 + 72T + 1$
70	−42	866	$p^6T^4 - 42p^3T^3 + 866pT^2 - 42T + 1$

$p = 73$

λ	a	b	$R(T)$
1	412	7814	$p^6T^4 + 412p^3T^3 + 7814pT^2 + 412T + 1$
2	1180	12038	$p^6T^4 + 1180p^3T^3 + 12038pT^2 + 1180T + 1$
3	−208	−4194	$p^6T^4 - 208p^3T^3 - 4194pT^2 - 208T + 1$
4	−72	3310	$p^6T^4 - 72p^3T^3 + 3310pT^2 - 72T + 1$
5	−160	3198	$p^6T^4 - 160p^3T^3 + 3198pT^2 - 160T + 1$
6	−356	5894	$p^6T^4 - 356p^3T^3 + 5894pT^2 - 356T + 1$
7	−164	5510	$p^6T^4 - 164p^3T^3 + 5510pT^2 - 164T + 1$
8	142	73	$p^2T^2 + 142T + 1$

$p = 73$ (cont.)

λ	a	b	$R(T)$
9	604	8582	$p^6T^4 + 604p^3T^3 + 8582pT^2 + 604T + 1$
10	-1044	13606	$p^6T^4 - 1044p^3T^3 + 13606pT^2 - 1044T + 1$
11	-1032	10894	$p^6T^4 - 1032p^3T^3 + 10894pT^2 - 1032T + 1$
12	1360	14846	$p^6T^4 + 1360p^3T^3 + 14846pT^2 + 1360T + 1$
13	74	4314	$p^6T^4 + 74p^3T^3 + 4314pT^2 + 74T + 1$
14	-360	3598	$p^6T^4 - 360p^3T^3 + 3598pT^2 - 360T + 1$
15	-318	682	$p^6T^4 - 318p^3T^3 + 682pT^2 - 318T + 1$
16	1064	11406	$p^6T^4 + 1064p^3T^3 + 11406pT^2 + 1064T + 1$
17	-932	7046	$p^6T^4 - 932p^3T^3 + 7046pT^2 - 932T + 1$
18	-444	-1802	$p^6T^4 - 444p^3T^3 - 1802pT^2 - 444T + 1$
19	-328	78	$p^6T^4 - 328p^3T^3 + 78pT^2 - 328T + 1$
20	-184	9006	$p^6T^4 - 184p^3T^3 + 9006pT^2 - 184T + 1$
21	-418	114	$p^6T^4 - 418p^3T^3 + 114pT^2 - 418T + 1$
22	-126	4714	$p^6T^4 - 126p^3T^3 + 4714pT^2 - 126T + 1$
23	-128	254	$p^6T^4 - 128p^3T^3 + 254pT^2 - 128T + 1$
24	32	2526	$p^6T^4 + 32p^3T^3 + 2526pT^2 + 32T + 1$
25	-496	6750	$p^6T^4 - 496p^3T^3 + 6750pT^2 - 496T + 1$
26	708	9334	$p^6T^4 + 708p^3T^3 + 9334pT^2 + 708T + 1$
27	-460	3222	$p^6T^4 - 460p^3T^3 + 3222pT^2 - 460T + 1$
28	-42	4354	$p^6T^4 - 42p^3T^3 + 4354pT^2 - 42T + 1$
29	174	8914	$p^6T^4 + 174p^3T^3 + 8914pT^2 + 174T + 1$
30	-26	-1246	$p^6T^4 - 26p^3T^3 - 1246pT^2 - 26T + 1$
31	-544	3966	$p^6T^4 - 544p^3T^3 + 3966pT^2 - 544T + 1$
32	-372	934	$p^6T^4 - 372p^3T^3 + 934pT^2 - 372T + 1$
33	-648	6286	$p^6T^4 - 648p^3T^3 + 6286pT^2 - 648T + 1$
34	450	5674	$p^6T^4 + 450p^3T^3 + 5674pT^2 + 450T + 1$
35	704	10398	$p^6T^4 + 704p^3T^3 + 10398pT^2 + 704T + 1$
36	124	-1978	$p^6T^4 + 124p^3T^3 - 1978pT^2 + 124T + 1$
37	404	-3114	$p^6T^4 + 404p^3T^3 - 3114pT^2 + 404T + 1$
38	72	-1106	$p^6T^4 + 72p^3T^3 - 1106pT^2 + 72T + 1$
39	58	2042	$p^6T^4 + 58p^3T^3 + 2042pT^2 + 58T + 1$
40	212	-5610	$p^6T^4 + 212p^3T^3 - 5610pT^2 + 212T + 1$
41	-652	11670	$p^6T^4 - 652p^3T^3 + 11670pT^2 - 652T + 1$

$p = 73$ (cont.)

λ	a	b	$R(T)$
42	1018	12218	$p^6T^4 + 1018p^3T^3 + 12218pT^2 + 1018T + 1$
43	24	526	$p^6T^4 + 24p^3T^3 + 526pT^2 + 24T + 1$
44	−268	6294	$p^6T^4 - 268p^3T^3 + 6294pT^2 - 268T + 1$
45	700	6662	$p^6T^4 + 700p^3T^3 + 6662pT^2 + 700T + 1$
46	−640	6174	$p^6T^4 - 640p^3T^3 + 6174pT^2 - 640T + 1$
47	834	4906	$p^6T^4 + 834p^3T^3 + 4906pT^2 + 834T + 1$
48	−408	8686	$p^6T^4 - 408p^3T^3 + 8686pT^2 - 408T + 1$
49	972	8998	$p^6T^4 + 972p^3T^3 + 8998pT^2 + 972T + 1$
50	176	6558	$p^6T^4 + 176p^3T^3 + 6558pT^2 + 176T + 1$
51	1318	13346	$p^6T^4 + 1318p^3T^3 + 13346pT^2 + 1318T + 1$
52	1902	22546	$p^6T^4 + 1902p^3T^3 + 22546pT^2 + 1902T + 1$
53	−618	10114	$p^6T^4 - 618p^3T^3 + 10114pT^2 - 618T + 1$
54	−268	−4074	$p^6T^4 - 268p^3T^3 - 4074pT^2 - 268T + 1$
55	36	−1994	$p^6T^4 + 36p^3T^3 - 1994pT^2 + 36T + 1$
56	128	8958	$p^6T^4 + 128p^3T^3 + 8958pT^2 + 128T + 1$
57	−256	10014	$p^6T^4 - 256p^3T^3 + 10014pT^2 - 256T + 1$
58	−848	7454	$p^6T^4 - 848p^3T^3 + 7454pT^2 - 848T + 1$
59	258	3562	$p^6T^4 + 258p^3T^3 + 3562pT^2 + 258T + 1$
60	158	−1614	$p^6T^4 + 158p^3T^3 - 1614pT^2 + 158T + 1$
61	−328	2382	$p^6T^4 - 328p^3T^3 + 2382pT^2 - 328T + 1$
62	−184	9006	$p^6T^4 - 184p^3T^3 + 9006pT^2 - 184T + 1$
63	36	9718	$p^6T^4 + 36p^3T^3 + 9718pT^2 + 36T + 1$
64	−1124	10886	$p^6T^4 - 1124p^3T^3 + 10886pT^2 - 1124T + 1$
65	−280	4878	$p^6T^4 - 280p^3T^3 + 4878pT^2 - 280T + 1$
66	−126	2986	$p^6T^4 - 126p^3T^3 + 2986pT^2 - 126T + 1$
67	72	−4178	$p^6T^4 + 72p^3T^3 - 4178pT^2 + 72T + 1$
68	−694	9882	$p^6T^4 - 694p^3T^3 + 9882pT^2 - 694T + 1$
69	112	1214	$p^6T^4 + 112p^3T^3 + 1214pT^2 + 112T + 1$
70	−24	−7442	$p^6T^4 - 24p^3T^3 - 7442pT^2 - 24T + 1$
71	−84	1510	$p^6T^4 - 84p^3T^3 + 1510pT^2 - 84T + 1$
72	220	9350	$p^6T^4 + 220p^3T^3 + 9350pT^2 + 220T + 1$

$p = 79$

λ	a	b	$R(T)$
1	224	7362	$p^6T^4 + 224p^3T^3 + 7362pT^2 + 224T + 1$
2	-1560	16834	$p^6T^4 - 1560p^3T^3 + 16834pT^2 - 1560T + 1$
3	288	1858	$p^6T^4 + 288p^3T^3 + 1858pT^2 + 288T + 1$
4	-588	4834	$p^6T^4 - 588p^3T^3 + 4834pT^2 - 588T + 1$
5	200	3810	$p^6T^4 + 200p^3T^3 + 3810pT^2 + 200T + 1$
6	602	12114	$p^6T^4 + 602p^3T^3 + 12114pT^2 + 602T + 1$
7	-298	-3438	$p^6T^4 - 298p^3T^3 - 3438pT^2 - 298T + 1$
8	24	6370	$p^6T^4 + 24p^3T^3 + 6370pT^2 + 24T + 1$
9	1388	13026	$p^6T^4 + 1388p^3T^3 + 13026pT^2 + 1388T + 1$
10	-256	-7230	$p^6T^4 - 256p^3T^3 - 7230pT^2 - 256T + 1$
11	96	-62	$p^6T^4 + 96p^3T^3 - 62pT^2 + 96T + 1$
12	-156	9538	$p^6T^4 - 156p^3T^3 + 9538pT^2 - 156T + 1$
13	-952	8418	$p^6T^4 - 952p^3T^3 + 8418pT^2 - 952T + 1$
14	714	5458	$p^6T^4 + 714p^3T^3 + 5458pT^2 + 714T + 1$
15	390	4882	$p^6T^4 + 390p^3T^3 + 4882pT^2 + 390T + 1$
16	-360	-542	$p^6T^4 - 360p^3T^3 - 542pT^2 - 360T + 1$
17	-4	-4926	$p^6T^4 - 4p^3T^3 - 4926pT^2 - 4T + 1$
18	224	-7614	$p^6T^4 + 224p^3T^3 - 7614pT^2 + 224T + 1$
19	-136	-6462	$p^6T^4 - 136p^3T^3 - 6462pT^2 - 136T + 1$
20	0	-6206	$p^6T^4 - 6206pT^2 + 1$
21	0	-79	$-(pT - 1)(pT + 1)$
22	28	4322	$p^6T^4 + 28p^3T^3 + 4322pT^2 + 28T + 1$
23	440	11970	$p^6T^4 + 440p^3T^3 + 11970pT^2 + 440T + 1$
24	-752	9410	$p^6T^4 - 752p^3T^3 + 9410pT^2 - 752T + 1$
25	44	4194	$p^6T^4 + 44p^3T^3 + 4194pT^2 + 44T + 1$
26	-1080	6850	$p^6T^4 - 1080p^3T^3 + 6850pT^2 - 1080T + 1$
27	764	9666	$p^6T^4 + 764p^3T^3 + 9666pT^2 + 764T + 1$
28	-196	10434	$p^6T^4 - 196p^3T^3 + 10434pT^2 - 196T + 1$
29	-30	4690	$p^6T^4 - 30p^3T^3 + 4690pT^2 - 30T + 1$
30	-682	11730	$p^6T^4 - 682p^3T^3 + 11730pT^2 - 682T + 1$
31	140	1122	$p^6T^4 + 140p^3T^3 + 1122pT^2 + 140T + 1$
32	-1232	12098	$p^6T^4 - 1232p^3T^3 + 12098pT^2 - 1232T + 1$
33	1036	13250	$p^6T^4 + 1036p^3T^3 + 13250pT^2 + 1036T + 1$

$p = 79$ (cont.)

λ	a	b	$R(T)$
34	1022	14802	$p^6T^4 + 1022p^3T^3 + 14802pT^2 + 1022T + 1$
35	-1348	12738	$p^6T^4 - 1348p^3T^3 + 12738pT^2 - 1348T + 1$
36	152	-8766	$p^6T^4 + 152p^3T^3 - 8766pT^2 + 152T + 1$
37	-1394	13970	$p^6T^4 - 1394p^3T^3 + 13970pT^2 - 1394T + 1$
38	640	4322	$p^6T^4 + 640p^3T^3 + 4322pT^2 + 640T + 1$
39	254	-366	$p^6T^4 + 254p^3T^3 - 366pT^2 + 254T + 1$
40	-68	-1822	$p^6T^4 - 68p^3T^3 - 1822pT^2 - 68T + 1$
41	1218	14674	$p^6T^4 + 1218p^3T^3 + 14674pT^2 + 1218T + 1$
42	-616	1986	$p^6T^4 - 616p^3T^3 + 1986pT^2 - 616T + 1$
43	122	-3054	$p^6T^4 + 122p^3T^3 - 3054pT^2 + 122T + 1$
44	632	1218	$p^6T^4 + 632p^3T^3 + 1218pT^2 + 632T + 1$
45	736	8162	$p^6T^4 + 736p^3T^3 + 8162pT^2 + 736T + 1$
46	920	13890	$p^6T^4 + 920p^3T^3 + 13890pT^2 + 920T + 1$
47	-1298	16658	$p^6T^4 - 1298p^3T^3 + 16658pT^2 - 1298T + 1$
48	320	6594	$p^6T^4 + 320p^3T^3 + 6594pT^2 + 320T + 1$
49	504	2530	$p^6T^4 + 504p^3T^3 + 2530pT^2 + 504T + 1$
50	-1232	15554	$p^6T^4 - 1232p^3T^3 + 15554pT^2 - 1232T + 1$
51	104	9186	$p^6T^4 + 104p^3T^3 + 9186pT^2 + 104T + 1$
52	288	10306	$p^6T^4 + 288p^3T^3 + 10306pT^2 + 288T + 1$
53	178	-1774	$p^6T^4 + 178p^3T^3 - 1774pT^2 + 178T + 1$
54	-24	-1982	$p^6T^4 - 24p^3T^3 - 1982pT^2 - 24T + 1$
55	-1280	15074	$p^6T^4 - 1280p^3T^3 + 15074pT^2 - 1280T + 1$
56	360	-830	$p^6T^4 + 360p^3T^3 - 830pT^2 + 360T + 1$
57	-90	-2222	$p^6T^4 - 90p^3T^3 - 2222pT^2 - 90T + 1$
58	-216	5698	$p^6T^4 - 216p^3T^3 + 5698pT^2 - 216T + 1$
59	926	12114	$p^6T^4 + 926p^3T^3 + 12114pT^2 + 926T + 1$
60	148	3650	$p^6T^4 + 148p^3T^3 + 3650pT^2 + 148T + 1$
61	-30	7954	$p^6T^4 - 30p^3T^3 + 7954pT^2 - 30T + 1$
62	352	6626	$p^6T^4 + 352p^3T^3 + 6626pT^2 + 352T + 1$
63	658	4754	$p^6T^4 + 658p^3T^3 + 4754pT^2 + 658T + 1$
64	-312	706	$p^6T^4 - 312p^3T^3 + 706pT^2 - 312T + 1$
65	-12	610	$p^6T^4 - 12p^3T^3 + 610pT^2 - 12T + 1$
66	-414	1234	$p^6T^4 - 414p^3T^3 + 1234pT^2 - 414T + 1$

$p = 79$ (cont.)

λ	a	b	$R(T)$
67	-764	2018	$p^6T^4 - 764p^3T^3 + 2018pT^2 - 764T + 1$
68	208	12098	$p^6T^4 + 208p^3T^3 + 12098pT^2 + 208T + 1$
69	612	10306	$p^6T^4 + 612p^3T^3 + 10306pT^2 + 612T + 1$
70	-1782	19474	$p^6T^4 - 1782p^3T^3 + 19474pT^2 - 1782T + 1$
71	-130	9042	$p^6T^4 - 130p^3T^3 + 9042pT^2 - 130T + 1$
72	1908	23650	$p^6T^4 + 1908p^3T^3 + 23650pT^2 + 1908T + 1$
73	-108	-2462	$p^6T^4 - 108p^3T^3 - 2462pT^2 - 108T + 1$
74	-904	14658	$p^6T^4 - 904p^3T^3 + 14658pT^2 - 904T + 1$
75	-732	1858	$p^6T^4 - 732p^3T^3 + 1858pT^2 - 732T + 1$
76	1552	19778	$p^6T^4 + 1552p^3T^3 + 19778pT^2 + 1552T + 1$
77	-408	7618	$p^6T^4 - 408p^3T^3 + 7618pT^2 - 408T + 1$
78	1108	14402	$p^6T^4 + 1108p^3T^3 + 14402pT^2 + 1108T + 1$

$p = 83$

λ	a	b	$R(T)$
1	-792	7010	$p^6T^4 - 792p^3T^3 + 7010pT^2 - 792T + 1$
2	-1118	7082	$p^6T^4 - 1118p^3T^3 + 7082pT^2 - 1118T + 1$
3	-560	9026	$p^6T^4 - 560p^3T^3 + 9026pT^2 - 560T + 1$
4	680	8450	$p^6T^4 + 680p^3T^3 + 8450pT^2 + 680T + 1$
5	-210	-838	$p^6T^4 - 210p^3T^3 - 838pT^2 - 210T + 1$
6	-922	10778	$p^6T^4 - 922p^3T^3 + 10778pT^2 - 922T + 1$
7	-80	12866	$p^6T^4 - 80p^3T^3 + 12866pT^2 - 80T + 1$
8	324	-46	$p^6T^4 + 324p^3T^3 - 46pT^2 + 324T + 1$
9	1320	10466	$p^6T^4 + 1320p^3T^3 + 10466pT^2 + 1320T + 1$
10	-1184	15266	$p^6T^4 - 1184p^3T^3 + 15266pT^2 - 1184T + 1$
11	-520	6050	$p^6T^4 - 520p^3T^3 + 6050pT^2 - 520T + 1$
12	192	10754	$p^6T^4 + 192p^3T^3 + 10754pT^2 + 192T + 1$
13	1146	11402	$p^6T^4 + 1146p^3T^3 + 11402pT^2 + 1146T + 1$
14	-582	2186	$p^6T^4 - 582p^3T^3 + 2186pT^2 - 582T + 1$
15	132	11474	$p^6T^4 + 132p^3T^3 + 11474pT^2 + 132T + 1$

$p = 83$ (cont.)

λ	a	b	$R(T)$
16	456	11618	$p^6T^4 + 456p^3T^3 + 11618pT^2 + 456T + 1$
17	-624	9314	$p^6T^4 - 624p^3T^3 + 9314pT^2 - 624T + 1$
18	956	4082	$p^6T^4 + 956p^3T^3 + 4082pT^2 + 956T + 1$
19	-636	2258	$p^6T^4 - 636p^3T^3 + 2258pT^2 - 636T + 1$
20	-856	7682	$p^6T^4 - 856p^3T^3 + 7682pT^2 - 856T + 1$
21	96	-5374	$p^6T^4 + 96p^3T^3 - 5374pT^2 + 96T + 1$
22	1064	8834	$p^6T^4 + 1064p^3T^3 + 8834pT^2 + 1064T + 1$
23	-232	8354	$p^6T^4 - 232p^3T^3 + 8354pT^2 - 232T + 1$
24	-1086	10538	$p^6T^4 - 1086p^3T^3 + 10538pT^2 - 1086T + 1$
25	680	2	$p^6T^4 + 680p^3T^3 + 2pT^2 + 680T + 1$
26	824	5282	$p^6T^4 + 824p^3T^3 + 5282pT^2 + 824T + 1$
27	-1296	16226	$p^6T^4 - 1296p^3T^3 + 16226pT^2 - 1296T + 1$
28	-424	11042	$p^6T^4 - 424p^3T^3 + 11042pT^2 - 424T + 1$
29	804	6866	$p^6T^4 + 804p^3T^3 + 6866pT^2 + 804T + 1$
30	572	-3214	$p^6T^4 + 572p^3T^3 - 3214pT^2 + 572T + 1$
31	-576	6146	$p^6T^4 - 576p^3T^3 + 6146pT^2 - 576T + 1$
32	792	2114	$p^6T^4 + 792p^3T^3 + 2114pT^2 + 792T + 1$
33	36	-4654	$p^6T^4 + 36p^3T^3 - 4654pT^2 + 36T + 1$
34	-6	8522	$p^6T^4 - 6p^3T^3 + 8522pT^2 - 6T + 1$
35	-582	1610	$p^6T^4 - 582p^3T^3 + 1610pT^2 - 582T + 1$
36	912	13922	$p^6T^4 + 912p^3T^3 + 13922pT^2 + 912T + 1$
37	-184	-4606	$p^6T^4 - 184p^3T^3 - 4606pT^2 - 184T + 1$
38	208	11330	$p^6T^4 + 208p^3T^3 + 11330pT^2 + 208T + 1$
39	-360	-5950	$p^6T^4 - 360p^3T^3 - 5950pT^2 - 360T + 1$
40	420	3410	$p^6T^4 + 420p^3T^3 + 3410pT^2 + 420T + 1$
41	-896	13730	$p^6T^4 - 896p^3T^3 + 13730pT^2 - 896T + 1$
42	326	3482	$p^6T^4 + 326p^3T^3 + 3482pT^2 + 326T + 1$
43	-306	8954	$p^6T^4 - 306p^3T^3 + 8954pT^2 - 306T + 1$
44	-232	-4702	$p^6T^4 - 232p^3T^3 - 4702pT^2 - 232T + 1$
45	160	12962	$p^6T^4 + 160p^3T^3 + 12962pT^2 + 160T + 1$
46	1186	12458	$p^6T^4 + 1186p^3T^3 + 12458pT^2 + 1186T + 1$
47	600	10178	$p^6T^4 + 600p^3T^3 + 10178pT^2 + 600T + 1$
48	-158	83	$p^2T^2 - 158T + 1$

$p = 83$ (cont.)

λ	a	b	$R(T)$
49	244	1682	$p^6T^4 + 244p^3T^3 + 1682pT^2 + 244T + 1$
50	76	4274	$p^6T^4 + 76p^3T^3 + 4274pT^2 + 76T + 1$
51	−272	11330	$p^6T^4 - 272p^3T^3 + 11330pT^2 - 272T + 1$
52	406	−2662	$p^6T^4 + 406p^3T^3 - 2662pT^2 + 406T + 1$
53	404	2450	$p^6T^4 + 404p^3T^3 + 2450pT^2 + 404T + 1$
54	296	10370	$p^6T^4 + 296p^3T^3 + 10370pT^2 + 296T + 1$
55	−526	−2518	$p^6T^4 - 526p^3T^3 - 2518pT^2 - 526T + 1$
56	216	−4798	$p^6T^4 + 216p^3T^3 - 4798pT^2 + 216T + 1$
57	−1446	19466	$p^6T^4 - 1446p^3T^3 + 19466pT^2 - 1446T + 1$
58	−6	−5302	$p^6T^4 - 6p^3T^3 - 5302pT^2 - 6T + 1$
59	244	8978	$p^6T^4 + 244p^3T^3 + 8978pT^2 + 244T + 1$
60	−950	13706	$p^6T^4 - 950p^3T^3 + 13706pT^2 - 950T + 1$
61	−1272	15074	$p^6T^4 - 1272p^3T^3 + 15074pT^2 - 1272T + 1$
62	268	434	$p^6T^4 + 268p^3T^3 + 434pT^2 + 268T + 1$
63	−192	13058	$p^6T^4 - 192p^3T^3 + 13058pT^2 - 192T + 1$
64	1188	13778	$p^6T^4 + 1188p^3T^3 + 13778pT^2 + 1188T + 1$
65	864	13058	$p^6T^4 + 864p^3T^3 + 13058pT^2 + 864T + 1$
66	816	7010	$p^6T^4 + 816p^3T^3 + 7010pT^2 + 816T + 1$
67	1284	18386	$p^6T^4 + 1284p^3T^3 + 18386pT^2 + 1284T + 1$
68	144	7010	$p^6T^4 + 144p^3T^3 + 7010pT^2 + 144T + 1$
69	−20	7730	$p^6T^4 - 20p^3T^3 + 7730pT^2 - 20T + 1$
70	792	2114	$p^6T^4 + 792p^3T^3 + 2114pT^2 + 792T + 1$
71	586	13898	$p^6T^4 + 586p^3T^3 + 13898pT^2 + 586T + 1$
72	436	4370	$p^6T^4 + 436p^3T^3 + 4370pT^2 + 436T + 1$
73	−582	2186	$p^6T^4 - 582p^3T^3 + 2186pT^2 - 582T + 1$
74	282	2762	$p^6T^4 + 282p^3T^3 + 2762pT^2 + 282T + 1$
75	−120	10466	$p^6T^4 - 120p^3T^3 + 10466pT^2 - 120T + 1$
76	−142	4394	$p^6T^4 - 142p^3T^3 + 4394pT^2 - 142T + 1$
77	−712	6050	$p^6T^4 - 712p^3T^3 + 6050pT^2 - 712T + 1$
78	−748	10514	$p^6T^4 - 748p^3T^3 + 10514pT^2 - 748T + 1$
79	−842	11546	$p^6T^4 - 842p^3T^3 + 11546pT^2 - 842T + 1$
80	736	2594	$p^6T^4 + 736p^3T^3 + 2594pT^2 + 736T + 1$
81	460	−718	$p^6T^4 + 460p^3T^3 - 718pT^2 + 460T + 1$

$p = 83$ (cont.)

λ	a	b	$R(T)$
82	-1292	12818	$p^6T^4 - 1292p^3T^3 + 12818pT^2 - 1292T + 1$

$p = 89$

λ	a	b	$R(T)$
1	-364	7766	$p^6T^4 - 364p^3T^3 + 7766pT^2 - 364T + 1$
2	4	-3850	$p^6T^4 + 4p^3T^3 - 3850pT^2 + 4T + 1$
3	1314	10442	$p^6T^4 + 1314p^3T^3 + 10442pT^2 + 1314T + 1$
4	-1708	19286	$p^6T^4 - 1708p^3T^3 + 19286pT^2 - 1708T + 1$
5	640	5822	$p^6T^4 + 640p^3T^3 + 5822pT^2 + 640T + 1$
6	714	10874	$p^6T^4 + 714p^3T^3 + 10874pT^2 + 714T + 1$
7	-956	15926	$p^6T^4 - 956p^3T^3 + 15926pT^2 - 956T + 1$
8	188	-58	$p^6T^4 + 188p^3T^3 - 58pT^2 + 188T + 1$
9	1008	8030	$p^6T^4 + 1008p^3T^3 + 8030pT^2 + 1008T + 1$
10	208	-4162	$p^6T^4 + 208p^3T^3 - 4162pT^2 + 208T + 1$
11	-580	-3130	$p^6T^4 - 580p^3T^3 - 3130pT^2 - 580T + 1$
12	-192	-3202	$p^6T^4 - 192p^3T^3 - 3202pT^2 - 192T + 1$
13	630	15842	$p^6T^4 + 630p^3T^3 + 15842pT^2 + 630T + 1$
14	288	2558	$p^6T^4 + 288p^3T^3 + 2558pT^2 + 288T + 1$
15	1056	11774	$p^6T^4 + 1056p^3T^3 + 11774pT^2 + 1056T + 1$
16	-72	-7378	$p^6T^4 - 72p^3T^3 - 7378pT^2 - 72T + 1$
17	-1488	16094	$p^6T^4 - 1488p^3T^3 + 16094pT^2 - 1488T + 1$
18	-720	-34	$p^6T^4 - 720p^3T^3 - 34pT^2 - 720T + 1$
19	-522	866	$p^6T^4 - 522p^3T^3 + 866pT^2 - 522T + 1$
20	-240	6302	$p^6T^4 - 240p^3T^3 + 6302pT^2 - 240T + 1$
21	2300	27974	$p^6T^4 + 2300p^3T^3 + 27974pT^2 + 2300T + 1$
22	-272	9854	$p^6T^4 - 272p^3T^3 + 9854pT^2 - 272T + 1$
23	864	2558	$p^6T^4 + 864p^3T^3 + 2558pT^2 + 864T + 1$
24	-196	4166	$p^6T^4 - 196p^3T^3 + 4166pT^2 - 196T + 1$
25	-1052	14390	$p^6T^4 - 1052p^3T^3 + 14390pT^2 - 1052T + 1$
26	1482	19514	$p^6T^4 + 1482p^3T^3 + 19514pT^2 + 1482T + 1$

$p = 89$ (cont.)

λ	a	b	$R(T)$
27	304	7646	$p^6T^4 + 304p^3T^3 + 7646pT^2 + 304T + 1$
28	116	7574	$p^6T^4 + 116p^3T^3 + 7574pT^2 + 116T + 1$
29	−414	1226	$p^6T^4 - 414p^3T^3 + 1226pT^2 - 414T + 1$
30	292	9398	$p^6T^4 + 292p^3T^3 + 9398pT^2 + 292T + 1$
31	−652	9878	$p^6T^4 - 652p^3T^3 + 9878pT^2 - 652T + 1$
32	−146	89	$p^2T^2 - 146T + 1$
33	−956	9014	$p^6T^4 - 956p^3T^3 + 9014pT^2 - 956T + 1$
34	1152	9182	$p^6T^4 + 1152p^3T^3 + 9182pT^2 + 1152T + 1$
35	722	10730	$p^6T^4 + 722p^3T^3 + 10730pT^2 + 722T + 1$
36	−564	−4570	$p^6T^4 - 564p^3T^3 - 4570pT^2 - 564T + 1$
37	−2112	26750	$p^6T^4 - 2112p^3T^3 + 26750pT^2 - 2112T + 1$
38	−614	1370	$p^6T^4 - 614p^3T^3 + 1370pT^2 - 614T + 1$
39	−180	6950	$p^6T^4 - 180p^3T^3 + 6950pT^2 - 180T + 1$
40	−372	10406	$p^6T^4 - 372p^3T^3 + 10406pT^2 - 372T + 1$
41	888	12494	$p^6T^4 + 888p^3T^3 + 12494pT^2 + 888T + 1$
42	−288	542	$p^6T^4 - 288p^3T^3 + 542pT^2 - 288T + 1$
43	84	−1834	$p^6T^4 + 84p^3T^3 - 1834pT^2 + 84T + 1$
44	−292	8198	$p^6T^4 - 292p^3T^3 + 8198pT^2 - 292T + 1$
45	404	86	$p^6T^4 + 404p^3T^3 + 86pT^2 + 404T + 1$
46	438	2594	$p^6T^4 + 438p^3T^3 + 2594pT^2 + 438T + 1$
47	−720	16094	$p^6T^4 - 720p^3T^3 + 16094pT^2 - 720T + 1$
48	1244	11462	$p^6T^4 + 1244p^3T^3 + 11462pT^2 + 1244T + 1$
49	692	1814	$p^6T^4 + 692p^3T^3 + 1814pT^2 + 692T + 1$
50	1548	21926	$p^6T^4 + 1548p^3T^3 + 21926pT^2 + 1548T + 1$
51	62	6290	$p^6T^4 + 62p^3T^3 + 6290pT^2 + 62T + 1$
52	−1064	16142	$p^6T^4 - 1064p^3T^3 + 16142pT^2 - 1064T + 1$
53	880	15230	$p^6T^4 + 880p^3T^3 + 15230pT^2 + 880T + 1$
54	−860	11702	$p^6T^4 - 860p^3T^3 + 11702pT^2 - 860T + 1$
55	−720	6878	$p^6T^4 - 720p^3T^3 + 6878pT^2 - 720T + 1$
56	822	3746	$p^6T^4 + 822p^3T^3 + 3746pT^2 + 822T + 1$
57	104	−9394	$p^6T^4 + 104p^3T^3 - 9394pT^2 + 104T + 1$
58	46	4850	$p^6T^4 + 46p^3T^3 + 4850pT^2 + 46T + 1$
59	614	16322	$p^6T^4 + 614p^3T^3 + 16322pT^2 + 614T + 1$

$p = 89$ (cont.)

λ	a	b	$R(T)$
60	930	16202	$p^6T^4 + 930p^3T^3 + 16202pT^2 + 930T + 1$
61	-30	1226	$p^6T^4 - 30p^3T^3 + 1226pT^2 - 30T + 1$
62	-154	7874	$p^6T^4 - 154p^3T^3 + 7874pT^2 - 154T + 1$
63	-722	5810	$p^6T^4 - 722p^3T^3 + 5810pT^2 - 722T + 1$
64	-472	974	$p^6T^4 - 472p^3T^3 + 974pT^2 - 472T + 1$
65	438	3746	$p^6T^4 + 438p^3T^3 + 3746pT^2 + 438T + 1$
66	1440	16382	$p^6T^4 + 1440p^3T^3 + 16382pT^2 + 1440T + 1$
67	-2300	28406	$p^6T^4 - 2300p^3T^3 + 28406pT^2 - 2300T + 1$
68	-80	7166	$p^6T^4 - 80p^3T^3 + 7166pT^2 - 80T + 1$
69	-440	13166	$p^6T^4 - 440p^3T^3 + 13166pT^2 - 440T + 1$
70	1406	13010	$p^6T^4 + 1406p^3T^3 + 13010pT^2 + 1406T + 1$
71	396	-3418	$p^6T^4 + 396p^3T^3 - 3418pT^2 + 396T + 1$
72	-1420	17558	$p^6T^4 - 1420p^3T^3 + 17558pT^2 - 1420T + 1$
73	-100	1286	$p^6T^4 - 100p^3T^3 + 1286pT^2 - 100T + 1$
74	-864	4862	$p^6T^4 - 864p^3T^3 + 4862pT^2 - 864T + 1$
75	246	-2014	$p^6T^4 + 246p^3T^3 - 2014pT^2 + 246T + 1$
76	-268	-3178	$p^6T^4 - 268p^3T^3 - 3178pT^2 - 268T + 1$
77	860	6854	$p^6T^4 + 860p^3T^3 + 6854pT^2 + 860T + 1$
78	564	-2410	$p^6T^4 + 564p^3T^3 - 2410pT^2 + 564T + 1$
79	864	12062	$p^6T^4 + 864p^3T^3 + 12062pT^2 + 864T + 1$
80	-24	6446	$p^6T^4 - 24p^3T^3 + 6446pT^2 - 24T + 1$
81	-1140	10406	$p^6T^4 - 1140p^3T^3 + 10406pT^2 - 1140T + 1$
82	396	4646	$p^6T^4 + 396p^3T^3 + 4646pT^2 + 396T + 1$
83	154	-3814	$p^6T^4 + 154p^3T^3 - 3814pT^2 + 154T + 1$
84	-624	8606	$p^6T^4 - 624p^3T^3 + 8606pT^2 - 624T + 1$
85	204	38	$p^6T^4 + 204p^3T^3 + 38pT^2 + 204T + 1$
86	-2350	29162	$p^6T^4 - 2350p^3T^3 + 29162pT^2 - 2350T + 1$
87	-1440	16670	$p^6T^4 - 1440p^3T^3 + 16670pT^2 - 1440T + 1$
88	1828	18998	$p^6T^4 + 1828p^3T^3 + 18998pT^2 + 1828T + 1$

$p = 97$

λ	a	b	$R(T)$
1	−980	12710	$p^6T^4 - 980p^3T^3 + 12710pT^2 - 980T + 1$
2	−816	6334	$p^6T^4 - 816p^3T^3 + 6334pT^2 - 816T + 1$
3	388	−394	$p^6T^4 + 388p^3T^3 - 394pT^2 + 388T + 1$
4	1832	25134	$p^6T^4 + 1832p^3T^3 + 25134pT^2 + 1832T + 1$
5	194	15450	$p^6T^4 + 194p^3T^3 + 15450pT^2 + 194T + 1$
6	572	17286	$p^6T^4 + 572p^3T^3 + 17286pT^2 + 572T + 1$
7	−1354	18258	$p^6T^4 - 1354p^3T^3 + 18258pT^2 - 1354T + 1$
8	−424	2862	$p^6T^4 - 424p^3T^3 + 2862pT^2 - 424T + 1$
9	876	10918	$p^6T^4 + 876p^3T^3 + 10918pT^2 + 876T + 1$
10	774	16306	$p^6T^4 + 774p^3T^3 + 16306pT^2 + 774T + 1$
11	108	13606	$p^6T^4 + 108p^3T^3 + 13606pT^2 + 108T + 1$
12	−852	13990	$p^6T^4 - 852p^3T^3 + 13990pT^2 - 852T + 1$
13	186	12394	$p^6T^4 + 186p^3T^3 + 12394pT^2 + 186T + 1$
14	−676	−4410	$p^6T^4 - 676p^3T^3 - 4410pT^2 - 676T + 1$
15	−1736	18542	$p^6T^4 - 1736p^3T^3 + 18542pT^2 - 1736T + 1$
16	−308	−2458	$p^6T^4 - 308p^3T^3 - 2458pT^2 - 308T + 1$
17	−10	−5742	$p^6T^4 - 10p^3T^3 - 5742pT^2 - 10T + 1$
18	2160	28414	$p^6T^4 + 2160p^3T^3 + 28414pT^2 + 2160T + 1$
19	586	−6838	$p^6T^4 + 586p^3T^3 - 6838pT^2 + 586T + 1$
20	−176	−15106	$p^6T^4 - 176p^3T^3 - 15106pT^2 - 176T + 1$
21	−398	12986	$p^6T^4 - 398p^3T^3 + 12986pT^2 - 398T + 1$
22	−480	574	$p^6T^4 - 480p^3T^3 + 574pT^2 - 480T + 1$
23	476	198	$p^6T^4 + 476p^3T^3 + 198pT^2 + 476T + 1$
24	−1060	5574	$p^6T^4 - 1060p^3T^3 + 5574pT^2 - 1060T + 1$
25	0	−6530	$p^6T^4 - 6530pT^2 + 1$
26	1330	15290	$p^6T^4 + 1330p^3T^3 + 15290pT^2 + 1330T + 1$
27	1504	22430	$p^6T^4 + 1504p^3T^3 + 22430pT^2 + 1504T + 1$
28	−1526	16394	$p^6T^4 - 1526p^3T^3 + 16394pT^2 - 1526T + 1$
29	1248	17182	$p^6T^4 + 1248p^3T^3 + 17182pT^2 + 1248T + 1$
30	758	8850	$p^6T^4 + 758p^3T^3 + 8850pT^2 + 758T + 1$
31	268	5606	$p^6T^4 + 268p^3T^3 + 5606pT^2 + 268T + 1$
32	−536	5774	$p^6T^4 - 536p^3T^3 + 5774pT^2 - 536T + 1$
33	668	12486	$p^6T^4 + 668p^3T^3 + 12486pT^2 + 668T + 1$

$p = 97$ (cont.)

λ	a	b	$R(T)$
34	-774	9514	$p^6T^4 - 774p^3T^3 + 9514pT^2 - 774T + 1$
35	-1428	22822	$p^6T^4 - 1428p^3T^3 + 22822pT^2 - 1428T + 1$
36	-1236	11686	$p^6T^4 - 1236p^3T^3 + 11686pT^2 - 1236T + 1$
37	6	-13262	$p^6T^4 + 6p^3T^3 - 13262pT^2 + 6T + 1$
38	300	9766	$p^6T^4 + 300p^3T^3 + 9766pT^2 + 300T + 1$
39	-1240	10830	$p^6T^4 - 1240p^3T^3 + 10830pT^2 - 1240T + 1$
40	374	14802	$p^6T^4 + 374p^3T^3 + 14802pT^2 + 374T + 1$
41	572	19014	$p^6T^4 + 572p^3T^3 + 19014pT^2 + 572T + 1$
42	770	-1830	$p^6T^4 + 770p^3T^3 - 1830pT^2 + 770T + 1$
43	-472	11310	$p^6T^4 - 472p^3T^3 + 11310pT^2 - 472T + 1$
44	-1148	21110	$p^6T^4 - 1148p^3T^3 + 21110pT^2 - 1148T + 1$
45	-1536	23710	$p^6T^4 - 1536p^3T^3 + 23710pT^2 - 1536T + 1$
46	364	7142	$p^6T^4 + 364p^3T^3 + 7142pT^2 + 364T + 1$
47	94	97	$p^2T^2 + 94T + 1$
48	768	1150	$p^6T^4 + 768p^3T^3 + 1150pT^2 + 768T + 1$
49	-496	-2466	$p^6T^4 - 496p^3T^3 - 2466pT^2 - 496T + 1$
50	288	-962	$p^6T^4 + 288p^3T^3 - 962pT^2 + 288T + 1$
51	-1740	23830	$p^6T^4 - 1740p^3T^3 + 23830pT^2 - 1740T + 1$
52	1326	13666	$p^6T^4 + 1326p^3T^3 + 13666pT^2 + 1326T + 1$
53	1052	11718	$p^6T^4 + 1052p^3T^3 + 11718pT^2 + 1052T + 1$
54	-684	-170	$p^6T^4 - 684p^3T^3 - 170pT^2 - 684T + 1$
55	-472	6990	$p^6T^4 - 472p^3T^3 + 6990pT^2 - 472T + 1$
56	-852	2854	$p^6T^4 - 852p^3T^3 + 2854pT^2 - 852T + 1$
57	-978	15778	$p^6T^4 - 978p^3T^3 + 15778pT^2 - 978T + 1$
58	856	3566	$p^6T^4 + 856p^3T^3 + 3566pT^2 + 856T + 1$
59	864	13342	$p^6T^4 + 864p^3T^3 + 13342pT^2 + 864T + 1$
60	10	6410	$p^6T^4 + 10p^3T^3 + 6410pT^2 + 10T + 1$
61	-588	18454	$p^6T^4 - 588p^3T^3 + 18454pT^2 - 588T + 1$
62	-1148	16502	$p^6T^4 - 1148p^3T^3 + 16502pT^2 - 1148T + 1$
63	-4	5190	$p^6T^4 - 4p^3T^3 + 5190pT^2 - 4T + 1$
64	684	15526	$p^6T^4 + 684p^3T^3 + 15526pT^2 + 684T + 1$
65	-1248	19006	$p^6T^4 - 1248p^3T^3 + 19006pT^2 - 1248T + 1$
66	-136	12654	$p^6T^4 - 136p^3T^3 + 12654pT^2 - 136T + 1$

$p = 97$ (cont.)

λ	a	b	$R(T)$
67	382	18626	$p^6T^4 + 382p^3T^3 + 18626pT^2 + 382T + 1$
68	374	978	$p^6T^4 + 374p^3T^3 + 978pT^2 + 374T + 1$
69	−668	−2314	$p^6T^4 - 668p^3T^3 - 2314pT^2 - 668T + 1$
70	−832	11742	$p^6T^4 - 832p^3T^3 + 11742pT^2 - 832T + 1$
71	92	10182	$p^6T^4 + 92p^3T^3 + 10182pT^2 + 92T + 1$
72	1332	22294	$p^6T^4 + 1332p^3T^3 + 22294pT^2 + 1332T + 1$
73	1628	18630	$p^6T^4 + 1628p^3T^3 + 18630pT^2 + 1628T + 1$
74	176	5694	$p^6T^4 + 176p^3T^3 + 5694pT^2 + 176T + 1$
75	196	−4618	$p^6T^4 + 196p^3T^3 - 4618pT^2 + 196T + 1$
76	−778	6738	$p^6T^4 - 778p^3T^3 + 6738pT^2 - 778T + 1$
77	−386	1154	$p^6T^4 - 386p^3T^3 + 1154pT^2 - 386T + 1$
78	776	2766	$p^6T^4 + 776p^3T^3 + 2766pT^2 + 776T + 1$
79	−96	−194	$p^6T^4 - 96p^3T^3 - 194pT^2 - 96T + 1$
80	−84	−8666	$p^6T^4 - 84p^3T^3 - 8666pT^2 - 84T + 1$
81	−4	13830	$p^6T^4 - 4p^3T^3 + 13830pT^2 - 4T + 1$
82	−476	−3850	$p^6T^4 - 476p^3T^3 - 3850pT^2 - 476T + 1$
83	1044	15766	$p^6T^4 + 1044p^3T^3 + 15766pT^2 + 1044T + 1$
84	778	12362	$p^6T^4 + 778p^3T^3 + 12362pT^2 + 778T + 1$
85	240	−2306	$p^6T^4 + 240p^3T^3 - 2306pT^2 + 240T + 1$
86	328	9806	$p^6T^4 + 328p^3T^3 + 9806pT^2 + 328T + 1$
87	174	14050	$p^6T^4 + 174p^3T^3 + 14050pT^2 + 174T + 1$
88	2028	27046	$p^6T^4 + 2028p^3T^3 + 27046pT^2 + 2028T + 1$
89	−1192	20526	$p^6T^4 - 1192p^3T^3 + 20526pT^2 - 1192T + 1$
90	1332	17686	$p^6T^4 + 1332p^3T^3 + 17686pT^2 + 1332T + 1$
91	−1636	17094	$p^6T^4 - 1636p^3T^3 + 17094pT^2 - 1636T + 1$
92	750	3490	$p^6T^4 + 750p^3T^3 + 3490pT^2 + 750T + 1$
93	−684	7510	$p^6T^4 - 684p^3T^3 + 7510pT^2 - 684T + 1$
94	−192	9982	$p^6T^4 - 192p^3T^3 + 9982pT^2 - 192T + 1$
95	−304	4062	$p^6T^4 - 304p^3T^3 + 4062pT^2 - 304T + 1$
96	1056	20542	$p^6T^4 + 1056p^3T^3 + 20542pT^2 + 1056T + 1$

Appendix B

Tables of Modular Forms

Here we list the expansions of all modular forms referenced in this thesis and not found on the LMFDB [25] or in Meyer's table [24]. Each table lists modular forms defined over a particular field (recall that classical modular forms are defined over $\mathbb{Q}$ while Hilbert modular forms are defined over totally real extensions of $\mathbb{Q}$).

For classical forms, the labels take the form $N.k.j$ where N is the level, k is the weight and j labels the character (see Appendix D.5.4 for details). For Hilbert modular forms, the label is D–$N.k/i$ where N is the level norm, k is the parallel weight, $\mathbb{Q}(\sqrt{D})$ is the field over which the form is defined, and i is an arbitrary numeric label. Since the label of a Hilbert modular form does not uniquely determine its level, the level of each form is also given in the table.

$\mathbb{Q}$

p	800.2.1	864.2.1	3456.2.1	3888.4.1	72.3.19	100.3.51
2	0	0	0	0	2	−2
3	1	0	0	0	0	0
5	0	1	3	11	0	0
7	−2	−3	5	−30	0	0
11	5	−3	−1	4	−14	0
13	0	0	6	16	0	24
17	5	−4	−6	70	−2	16
19	5	−6	0	−5	−34	0
23	6	6	8	−53	0	0
29	4	2	−2	−15	0	−42
31	10	−9	7	−74	0	0
37	−10	−2	−8	−156	0	−24
41	5	10	−4	366	46	−18
43	4	−6	−2	−292	14	0
47	−8	6	−6	201	0	0
53	−10	−13	9	557	0	−56
59	0	−12	−4	364	82	0
61	−10	8	−8	−518	0	22
67	3	−6	2	7	62	0
71	0	12	10	473	0	0
73	−5	9	−7	−359	−142	−96
79	10	0	0	−1024	0	0
83	−1	−3	−9	−1078	−158	0
89	−9	−14	−14	1044	−146	78
97	10	−9	3	−193	−94	−144

$\mathbb{Q}(\sqrt{3})$

	3–18.4/1	3–18.4/2
$\mathfrak{p}$	$[18, 6, 3w + 3]$	$[18, 6, 3w + 3]$
$[11, 11, w + 5]$	-60	12
$[11, 11, w + 6]$	12	-60
$[13, 13, w + 4]$	-70	74
$[13, 13, w + 9]$	74	-70
$[23, 23, w + 7]$	-120	-120
$[23, 23, w + 16]$	-120	-120
$[37, 37, w + 15]$	-322	-34
$[37, 37, w + 22]$	-34	-322
$[47, 47, w + 12]$	480	-240
$[47, 47, w + 35]$	-240	480
$[59, 59, w + 11]$	204	-588
$[59, 59, w + 48]$	-588	204
$[61, 61, w + 8]$	-250	326
$[61, 61, w + 53]$	326	-250
$[71, 71, w + 28]$	-72	216
$[71, 71, w + 43]$	216	-72
$[73, 73, w + 21]$	-682	-970
$[73, 73, w + 52]$	-970	-682
$[83, 83, w + 13]$	-204	-996
$[83, 83, w + 70]$	-996	-204
$[97, 97, w + 10]$	1406	-1186
$[97, 97, w + 87]$	-1186	1406

$\mathbb{Q}(\sqrt{2})$

	2–18.4/1	2–18.4/2	2–1.4/1	2–392.4/1	2–392.4/2
$\mathfrak{p}$	$[18, 6, 3w]$	$[18, 6, 3w]$	$[1, 1, 1]$	$[392, 196, 20 + 2w]$	$[392, 196, 176 + 2w]$
$[7, 7, w + 3]$	-16	32	-8	0	24
$[7, 7, w + 4]$	32	-16	-8	24	0
$[17, 17, w + 6]$	-78	114	-14	-46	-66
$[17, 17, w + 11]$	114	-78	-14	-66	-46
$[23, 23, w + 5]$	-72	24	-152	-8	-88
$[23, 23, w + 18]$	24	-72	-152	-88	-8
$[31, 31, w + 8]$	296	-136	224	-48	320
$[31, 31, w + 23]$	-136	296	224	320	-48
$[41, 41, w + 17]$	-390	-198	-70	246	-422
$[41, 41, w + 24]$	-198	-390	-70	-422	246
$[47, 47, w + 7]$	-288	-384	336	192	400
$[47, 47, w + 40]$	-384	-288	336	400	192
$[71, 71, w + 12]$	840	168	-72	-904	-104
$[71, 71, w + 59]$	168	840	-72	-104	-904
$[73, 73, w + 32]$	-358	-742	-294	-678	-698
$[73, 73, w + 41]$	-742	-358	-294	-698	-678
$[79, 79, w + 9]$	632	200	-464	-624	880
$[79, 79, w + 70]$	200	632	-464	880	-624
$[89, 89, w + 25]$	-1014	-246	266	-1322	-822
$[89, 89, w + 64]$	-246	-1014	266	-822	-1322
$[97, 97, w + 14]$	-1438	1634	994	-1138	98
$[97, 97, w + 83]$	1634	-1438	994	98	-1138

$\mathbb{Q}(\sqrt{5})$

	5–256.4/1	5–256.4/2	5–729.4/1	5–729.4/2	5–4096.4/1	5–4096.4/2
$\mathfrak{p}$	$[256, 16, 16]$	$[256, 16, 16]$	$[729, 27, 27]$	$[729, 27, 27]$	$[4096, 64, 64]$	$[4096, 64, 64]$
$[11, 11, w+3]$	-20	20	-18	27	-68	12
$[11, 11, w+7]$	20	-20	27	-18	12	-68
$[19, 19, w+4]$	84	-84	-19	80	-4	-52
$[19, 19, w+14]$	-84	84	80	-19	-52	-4
$[29, 29, w+5]$	6	6	126	-198	38	-230
$[29, 29, w+23]$	6	6	-198	126	-230	38
$[31, 31, w+12]$	-224	224	-262	-37	-72	232
$[31, 31, w+18]$	224	-224	-37	-262	232	-72
$[41, 41, w+6]$	-266	-266	-306	-243	-348	100
$[41, 41, w+34]$	-266	-266	-243	-306	100	-348
$[59, 59, w+25]$	28	-28	342	189	-348	84
$[59, 59, w+33]$	-28	28	189	342	84	-348
$[61, 61, w+17]$	-182	-182	-802	386	248	680
$[61, 61, w+43]$	-182	-182	386	-802	680	248
$[71, 71, w+8]$	-408	408	-684	-315	976	240
$[71, 71, w+62]$	408	-408	-315	-684	240	976
$[79, 79, w+29]$	48	-48	1106	341	-136	-984
$[79, 79, w+49]$	-48	48	341	1106	-984	-136
$[89, 89, w+9]$	-1526	-1526	504	954	262	902
$[89, 89, w+79]$	-1526	-1526	954	504	902	262

Appendix C

Implementation and Software Details

Calculations were performed using a variety of software. SageMath [26] was used for data management, processing and presentation. Computation of p-adic series for $\mathcal{U}(\lambda)$ was performed in PARI/GP [28] via the SageMath interface to the GP calculator. Recursive computations in PARI/GP were memoized using a third party script [29]. The coefficients of modular forms were computed using MAGMA [27], via the SageMath MAGMA interface. Details of the code will be published to `https://cyarithmetic.gitlab.io`.

Appendix D

Number Theory Primer

The study of the arithmetic properties of manifolds and their zeta functions requires a working knowledge of the theory of finite fields and p-adic numbers. The proof of the Weil conjectures, and specifically Dwork's method for computing the zeta function as a determinant, rely upon machinery from arithmetic geometry and algebraic number theory. This thesis, however, is written for a mixed audience and is intended to be understandable without advanced knowledge in these areas. To keep the thesis self-contained, we review in this chapter all the background material needed to understand it.

The material is mostly of an elementary nature and can be found in standard textbooks. For field theory we recommend [30, 31]. For the theory of p-adic numbers there is an excellent introductory text by Koblitz [17], which goes so far as to cover Dwork's proof of the rationality of the zeta function. For a gentler introduction which covers the basics in more detail, we recommend [32]. An introduction to some of these topics aimed at physicists is given in [5].

To keep this review brief we focus mainly on results used directly in the body of the text. However, we also try to give enough background and context that the unfamiliar reader might gain a little intuition for the subject. We provide explanations and derivations for results where they are brief but omit longer proofs. We attempt to give relevant terminology where appropriate so that the interested reader might look up further details elsewhere.

D.1 Basic Field Theory

Here we define finite fields and field extensions and describe some of their properties. This is the most elementary material of all in this chapter, and can be safely skipped by those familiar with it.

A field K is a set with an "addition" operation $+$ and a multiplication operation $\times$ which obey the standard properties with which we are familar from the real or complex numbers. In particular, $(K, +)$ and $(K \setminus \{0\}, \times)$ must be abelian groups and multiplication must distribute over addition,

$$a \times (b + c) = a \times b + a \times c \quad \text{for} \quad a, b, c \in K.$$

We call the set $K^\times = K \setminus \{0\}$ of nonzero elements of K, along with the multiplication operator $\times$, the *group of multiplicative units* of K.

Field Characteristic

The *characteristic* $\mathrm{char}(K)$ of a field K is the number of times the multiplicative identity $1 \in K$ must be added to itself to obtain the additive identity $0 \in K$. Of course, one may never get back to zero, in which case we define $\mathrm{char}(K) = 0$. The familiar fields $\mathbb{R}$ and $\mathbb{C}$ (and $\mathbb{Q}$) have characteristic zero. So too do the p-adic number fields we construct in Appendix D.2. In Appendix D.1.2, though, we introduce finite fields, which necessarily have nonzero characteristic.

D.1.1 Field Extensions

Given a field $(L, +, \times)$, a subset $K \subset L$ is called *subfield* if it is a field with respect to the same operations and is closed under them. We then call L a *field extension* of K and write L/K. Then L is a K-vector field, whose dimension we call the *degree* of the extension and denote $[L : K]$. This degree can be finite (e.g. $[\mathbb{C} : \mathbb{R}] = 2$), but is not always (e.g. $[\mathbb{R} : \mathbb{Q}]$).

Given a field extension L/K and a subset $S \subset L$, we can construct the smallest subfield of L containing both the field K and the set S. We call this the field *generated over K by S*, and denote it by $K(S)$. It is of course a field extension of K. We call this extension *simple* if the set S contains only a single element s, which is then called a *primitive element*.

Algebraic Extensions

A natural motivation for field extensions is the pursuit of polynomial roots. We denote by $K[x]$ the ring of polynomials in the formal variable x with coefficients in K. Let L/K be a field extension as before. If an element $\alpha \in L$ is the root of a polynomial in $K[x]$ then we call α *algebraic over* K. The lowest degree monic polynomial in $K[x]$ having α as a root is called the *minimal polynomial* of s over K. If $S \subset L$ is a set of algebraic elements over K then the extension $K(S)$ is called *algebraic*.

A field K is called *algebraically closed* if it contains the roots of every polynomial in $K[x]$. Equivalently this means that all polynomials in $K[x]$ factorise completely into linear factors in $K[x]$. If K is not algebraically closed, it is possible to construct an algebraically closed field $\overline{K}$ by adjoining to K the roots of successive polynomials in $K[x]$ irreducible over K until there are no more irreducible polynomials. The resulting field $\overline{K}$ is a field extension of K called the *algebraic closure* of K. It is unique up to isomorphism.

Examples

To gain some intuition for field extensions, let us consider some familiar fields. As a field extension of $\mathbb{R}$, $\mathbb{C}$ is both simple and algebraic since it can be written

$$\mathbb{C} = \mathbb{R}(\mathrm{i}),$$

where i is a root of the irreducible (over $\mathbb{R}$) monic polynomial $x^2 + 1$. The extension thus has degree $[\mathbb{C} : \mathbb{R}] = 2$ as can also be seen from the fact that $\mathbb{C}$ is two dimensional as an $\mathbb{R}$-vector space.

On the other hand, $\mathbb{R}$ is not an algebraic extension of $\mathbb{Q}$ since it cannot be obtained from $\mathbb{Q}$ by adjoining polynomial roots. We can of course construct $\overline{\mathbb{Q}} \subset \mathbb{C}$ which contains the roots of all rational polynomials (so in particular $\mathrm{i} \in \overline{\mathbb{Q}}$), but this field is smaller than $\mathbb{C}$. What $\overline{\mathbb{Q}}$ is missing is the limit points of certain Cauchy sequences (more on this later).

Simple Algebraic Extensions

So far this may seem a little abstract, but fortunately we will mostly be concerned with simple algebraic extensions, which can be understood in a very practical manner.

Let L/K be a field extension and $\alpha \in L$ be algebraic over K with minimal polynomial $P(x) \in K[x]$ of degree s. The simple algebraic extension $K(\alpha)$ has degree

$$[K(\alpha) : K] = s,$$

and a basis for $K(\alpha)$ as a K-vector space is given by $\{1, \alpha, \ldots, \alpha^{s-1}\}$. In other words, the elements of $K(\alpha)$ are polynomials in α of degree strictly less than s with coefficients in K.

To see this, note that if the proposed basis vectors were linearly dependent then there would exist $a_i \in K$ not all zero such that

$$a_0 + a_1\alpha + \cdots + a_{s-1}\alpha^{s-1} = 0.$$

But this is a polynomial in $K[x]$ of degree strictly less than s having α as a root—a contradiction since by construction the minimal polynomial $P(x)$ of α was of minimal degree.

The extension $K(\alpha)$ can be expressed another way as a quotient of polynomial rings. The ideal $\langle P(x)\rangle$ of $K[x]$ generated by $P(x)$,

$$\langle P(x)\rangle = \{Q(x)P(x) \, : \, Q(x) \in K[x]\}, \tag{D.1}$$

contains precisely the polynomials in $K[x]$ having $P(x)$ as a factor. Then we can define $K(x)$ as a ring quotient,

$$K(x) = K[x]/\langle P(x)\rangle.$$

This means that elements of $K(x)$ are polynomials in $K[x]$ defined modulo $P(x) = 0$.

D.1.2 Finite Fields

The order $|K|$ of a field K is the number of elements it contains. If $|K|$ is finite then we call K a *finite field*. Such a field must have nonzero characteristic or else it would contain as a subset the positive integers

$$\{1, 1+1, 1+1+1, \ldots\}.$$

Suppose that $\text{char}(K) = mn$ for some integers $m, n > 1$. Since $m, n < mn$ it follows that they are in K. But this means we have two nonzero elements of K whose product is zero (by definition of the character), in violation of the field axioms. We cannot have $\text{char}(K) = 1$ since then we would have $1 = 0$, which is generally forbidden as a field axiom. We conclude that a finite field must have prime characteristic $\text{char}(K) = p$.

Prime Fields

The simplest field of characteristic p is the integers with the usual operations of addition and multiplication defined mod p,

$$\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} = \{1, \ldots, p-1\}.$$

Additive inversion is simple, since we have $-n = p - n$ for all nonzero n. For multiplicative inversion, we define $1/n$ to be the unique integer $1 \leq m < p$ such that $mn \equiv 1 \bmod p$ (which always exists by Euclid's algorithm since $(n, p) = 1$).

As a multiplicative group, $\mathbb{F}_p^\times$ has order $p - 1$ and is cyclic. This means that there exists an element $\alpha \in \mathbb{F}_p^\times$ such that every $x \in \mathbb{F}_p^\times$ can be written as α^u for some $u \in \mathbb{Z}/(p-1)\mathbb{Z}$. The following table shows how this works in $\mathbb{F}_7$ where every element can be written as 3^u for some $u \in \mathbb{Z}/6\mathbb{Z}$.

n	1	2	3	4	5	6
n^{-1}	1	4	5	2	3	6
u	0	2	1	4	5	3

Finite Extension Fields

Now, any finite field K of characteristic p must contain $\mathbb{F}_p$ as a subfield, called the *prime field* of K. In other words, every finite field is a field extension of its prime field $\mathbb{F}_p$, having some finite degree $s = [K : \mathbb{F}_p]$. Recall that this means K is an s-dimensional $\mathbb{F}_p$-vector space, from which it follows that $|K| = p^s$.

Moreover, this extension must be algebraic, that is, every element is the root of a polynomial in $\mathbb{F}_p[x]$. To see this, note that no $s + 1$ elements of K can be linearly independent, so for $x \in K$, there exist $a_0, \ldots, a_{s-1}, a_s \in \mathbb{F}_p$ not all zero such that

$$a_0 + a_1 x + \cdots + a_{s-1} x^{s-1} + a_s x^s = 0,$$

giving a polynomial in $\mathbb{F}_p[x]$ of which x is a root.

It can further be shown that this extension is simple. So every finite field has order $q = p^s$ for some prime p and positive integer s and can be written

$$\mathbb{F}_q = \mathbb{F}_p(\alpha),$$

where α is the root of a monic irreducible degree s polynomial $P(x) \in \mathbb{F}_p[x]$.

Like $\mathbb{F}_p^\times$, the multiplicative group $\mathbb{F}_q^\times$ is cyclic. Up to isomorphism, $\mathbb{F}_q$ is unique and so it makes sense to talk about *the* finite field of order q. Its elements can be viewed as polynomials in variable α with coefficients in $\mathbb{F}_p$, of degree strictly less than s,

$$a = a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_{s-1}\alpha^{s-1} \in \mathbb{F}_q, \quad a_0, \ldots, a_{s-1} \in \mathbb{F}_p.$$

As an example we give the table for for $\mathbb{F}_9 = \mathbb{F}_{3^2}$, analogous to the earlier table for $\mathbb{F}_7$. We need to pick an irreducible polynomial in $\mathbb{F}_3[x]$ whose root we shall use to extend $\mathbb{F}_3$. Take $x^2 + 1$, which has no roots in $\mathbb{F}_3$, as can be checked by testing each element $0, 1, 2$. Let α be a root of this polynomial so that $\alpha^2 = -1$. Then an element of $\mathbb{F}_3$ takes the form $a + b\alpha$, $a, b \in \mathbb{F}_3$ with addition and multiplication defined in the natural way.[1] It is easy to check that every element of $\mathbb{F}_{3^2}$ can be written as β^u for $\beta = 1 + \alpha$ and $u \in \mathbb{Z}/8\mathbb{Z}$.

n	1	2	α	$1+\alpha$	$2+\alpha$	2α	$1+2\alpha$	$2+2\alpha$
n^{-1}	1	2	2α	$2+\alpha$	$1+\alpha$	α	$2+2\alpha$	$1+2\alpha$
u	0	4	6	1	7	2	3	5

Properties of Finite Fields

Since $\mathbb{F}_q$ has characteristic p (i.e. $p = 0$ in $\mathbb{F}_q$), it follows that when raising expressions to the power p, the "Freshman's dream" holds,

$$(x + y)^p = x^p + y^p \quad \text{for} \quad x, y \in \mathbb{F}_q.$$

One can also show that

$$x^{q-1} = 1, \quad x \in \mathbb{F}_q^\times,$$

which can be rewritten to include 0,

$$x^q = x, \quad x \in \mathbb{F}_q.$$

In the case $q = p$ this states that $x^p = x \mod p$, a result known as *Fermat's Little Theorem*, which is often used as a quick check of primality (or rather non-primality, since the condition is necessary, but not sufficient).

[1] Equivalently, one can consider elements of $\mathbb{F}_{3^2}$ as polynomials in $\mathbb{F}_p[x]$ modulo $x^2 + 1 = 0$.

The Frobenius Map

The *Frobenius map* is given for $x \in \mathbb{F}_q$ by

$$\mathrm{Fr}_p : x \to x^p. \tag{D.2}$$

It is easy to see that this preserves not only multiplication but also addition in $\mathbb{F}_q$ (because of the "Freshman's dream"). Moreover, it follows from Fermat's Little Theorem that Fr_p leaves the subfield $\mathbb{F}_p \subset \mathbb{F}_q$ fixed. Such a map, which preserves the field structure of $\mathbb{F}_q$ and leaves the subfield $\mathbb{F}_p$ fixed is called an $\mathbb{F}_p$-*homomorphism*.

The Trace Map

The *trace map* from $\mathbb{F}_q$ to its prime field $\mathbb{F}_p$ is given by

$$\mathrm{tr}(a) = \sum_{r=0}^{s-1} a^{p^r}, \quad x \in \mathbb{F}_q. \tag{D.3}$$

By virtue of the previous paragraph, to check that indeed $\mathrm{tr}(a) \in \mathbb{F}_p$ we need only verify that $\mathrm{tr}(a)^p = \mathrm{tr}(a)$. We have (thanks to the "Freshman's dream") that

$$\mathrm{tr}(a)^p = a^p + a^{p^2} + \cdots + a^{p^{s-1}} + a^{p^s} = \mathrm{tr}(a),$$

since for any element $a \in \mathbb{F}_q$ we have $a^{p^s} = a^q = a$.

We also have that $\mathrm{tr}(0) = 0$ and $\mathrm{tr}(x + y) = \mathrm{tr}(x) + \mathrm{tr}(y)$. In other words, tr is a group homomorphism from $\mathbb{F}_q$ to $\mathbb{F}_p$.

Quadratic Residues

A common question is whether, for a given p, one can find roots of a polynomial in $\mathbb{F}_p$. We will restrict our attention to polynomials $x^2 - a = 0$ for $a \in \mathbb{F}_p$. If this polynomial has roots over $\mathbb{F}_p$ then a is called a *quadratic residue* mod p.

There is a useful function for counting quadratic residues, called the *Legendre symbol*. It is denoted $\left(\frac{a}{p}\right)$ or $(a|p)$ and for $a \in \mathbb{F}_p$ its value is 1 if a is a quadratic residue mod p, 0 if a is 0, and -1 otherwise. This may not seem immediately useful but it has a number of properties that make it easy to compute. For example, it is multiplicative in its top argument a. For some a it is known explicitly, such as $a = 5$ where we have for $p > 2$ that

$$\left(\frac{5}{p}\right) = \begin{cases} 1 & p \equiv \pm 1 \mod 5, \\ -1 & p \equiv \pm 3 \mod 5. \end{cases} \tag{D.4}$$

We can use this to determine when $\sqrt{5}$ exists in a given finite field $\mathbb{F}_p$.

D.2 p-adic Numbers

Before we introduce the p-adic numbers, we briefly review the construction of the complex numbers. Starting with the natural numbers $\mathbb{N}$ we include zero and the negative integers to obtain the ring $\mathbb{Z}$, so that we may solve equations of the form $x - a = b$ with $a, b \in \mathbb{N}$. We notice that we still cannot solve equations of the form $ax = b$ for $a, b \in \mathbb{Z}$ (and a nonzero). Seeking to solve such equations we arrive at the field $\mathbb{Q}$ of rational numbers.

We could keep seeking solutions to polynomial equations and indeed if we did we would arrive at the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$, but neither $\mathbb{Q}$ nor its algebraic closure is complete. That is, there are sequences in $\mathbb{Q}$ which converge to a limit not in $\mathbb{Q}$.[2] Including all such limits (by considering equivalence classes of Cauchy sequences), we arrive at the real numbers $\mathbb{R}$, a field extension of $\mathbb{Q}$ of non-finite degree.

At this point we return to the task of solving polynomial equations and observe that there are polynomials, e.g. $x^2 + 1$, which do not have roots in $\mathbb{R}$. Adjoining a root of this polynomial, which we call i, to $\mathbb{R}$ we obtain the complex numbers

$$\mathbb{C} = \mathbb{R}(\mathrm{i}) = \overline{\mathbb{R}}.$$

Miraculously, this simple algebraic extension of $\mathbb{R}$ is algebraically closed—all polynomials in $\mathbb{C}[x]$ factorise completely into linear factors (the Fundamental Theorem of Algebra). The p-adic case will not prove so simple.

This construction is familiar and seems natural, but one might ask whether it is unique or if we unwittingly made some choices along the way. And indeed we did: the choice of norm on $\mathbb{Q}$ used to define limits, convergence and Cauchy sequences. Recall that a norm on a field K is a function $\|\cdot\| : K \to \mathbb{R}$ such that

[2] If we want to make this statement without appealing to a larger field in which such a sequence converges to a limit, we can say that there are Cauchy sequences in $\mathbb{Q}$ which do not converge.

- $\|x\| \geq 0 \quad \text{for} \quad x \in K$;
- $\|x\| = 0 \quad \text{if and only if} \quad x = 0$;
- $\|xy\| = \|x\|\|y\| \quad \text{for} \quad x, y \in K$;
- (triangle inequality) $\quad \|x + y\| \leq \|x\| + \|y\| \quad \text{for} \quad x, y \in K$.

In constructing $\mathbb{R}$ we took the usual Euclidean absolute value $|\cdot|$, but it turns out this choice was not unique: for a given prime we can define a *p-adic norm* which gives rise to a different notion of convergence.

D.2.1 Constructing the p-adic Numbers

We start by defining the *p-adic ordinal* $\text{ord}_p n$ of a nonzero integer $n \in \mathbb{Z}$ to be the largest $l \in \mathbb{Z}$ such that $p^l \mid n$, so $n = p^l m$ for some $m \in \mathbb{Z}$ coprime with p. This is then extended to $\mathbb{Q}^\times$ by defining

$$\text{ord}_p\left(\frac{a}{b}\right) = \text{ord}_p a - \text{ord}_p b.$$

We define the p-adic norm on $\mathbb{Q}$ to be

$$|x|_p = \begin{cases} p^{-\text{ord}_p x} & x \neq 0, \\ 0 & x = 0. \end{cases}$$

Under this norm, increasing powers of the prime p are considered to be successively smaller. This naturally leads to "p-adic expansions" of numbers with digits in the range 0 to $p-1$ in ascending powers of p, just as real numbers are expanded decimally in successive powers of $1/10$.

It is easy to check that $|\cdot|_p$ satisfies the first three properties of a norm listed above. As for the triangle inequality, it turns out it satisfies an even stronger version. Indeed for $x, y \in \mathbb{Q}$ we have

$$\text{ord}_p(x + y) \geq \min\left(\text{ord}_p x, \text{ord}_p y\right),$$

from which it follows that

$$|x + y|_p \leq \max\left(|x|_p, |y|_p\right) \leq |x|_p + |y|_p. \tag{D.5}$$

This stronger property is called the *non-Archimedean triangle inequality*, and the p-adic norm is called *non-Archimedean*. On account of this property, $|\cdot|_p$ has very different properties to the usual Euclidean norm, which we call *Archimedean*.

Non-Archimedean Norms

A remarkable consequence of this stronger triangle inequality is the *isosceles triangle principle*. For $x, y \in \mathbb{Q}$ with $|x|_p < |y|_p$, we have that $|y|_p = |x+y|_p$.

Another counter-intuitive property of the p-adic norm (and non-Archimedean norms in general) is that every point of a disc (open or closed) is its center. Indeed choose a point $x \in \mathbb{Q}$ and a radius $r > 0$ and let $y \in \overline{D}(x,r)$ be a point in the closed p-adic disc, $|x-y|_p \leq r$. Then for any $z \in \overline{D}(y,r)$, we have

$$|z-x|_p \leq \max\left(|z-y|_p, |y-x|_p\right) \leq r.$$

It is easy to check that the reverse is also true and hence that $\overline{D}(x,r) = \overline{D}(y,r)$. That is, we can take any point of a p-adic disc to be its center. Since this includes points on the boundary, it follows that there exist discs centered at the boundary which are entirely contained within $\overline{D}(x,r)$. That is, in some sense, the boundary of the disc has a "thickness". If the reader wishes to learn more about the intriguing and surprising properties of non-Archimedean norms, we recommend the references given at the start of the chapter.

$\mathbb{Q}_p$ and p-adic Expansions

Just as $\mathbb{Q}$ is completed to $\mathbb{R}$ via the usual Euclidean norm, the p-adic norm for each prime p can be used to obtain the p-adic completion $\mathbb{Q}_p$. Technically this field is comprised of equivalence classes of Cauchy sequences in $\mathbb{Q}$, but fortunately there is a more practical approach analogous to the decimal expansions of real numbers. Indeed a p-adic number $X \in \mathbb{Q}_p$ can be written uniquely as a p-adic expansion,

$$X = \sum_{l=r}^{\infty} X_l p^l, \quad X_l \in \mathbb{Z}/p\mathbb{Z}, \quad X_r \neq 0, \tag{D.6}$$

for some $r \in \mathbb{Z}$. We define $\text{ord}_p X = r$. If the sum is in fact finite, then $X \in \mathbb{Q}$. Moreover any p-adic number can be approximated to a given p-adic precision by a rational number. If $r \geq 0$, then call X a *p-adic integer* and write $X \in \mathbb{Z}_p$. We thus have

$$\mathbb{Z}_p = \left\{X \in \mathbb{Q}_p \colon |X|_p \leq 1\right\}.$$

To avoid ambiguity, we refer to elements of $\mathbb{Z}$ as *rational integers*. The group of p-adic units $\mathbb{Z}_p^\times$ is the group of elements of $\mathbb{Z}_p$ with a multiplicative inverse also in $\mathbb{Z}_p$. These are precisely the elements with p-adic norm 1, i.e. with $X_0 \neq 0$. The complement of $\mathbb{Z}_p^\times$ in $\mathbb{Z}_p$ is $p\mathbb{Z}_p$. The group quotient $\mathbb{Z}_p/p\mathbb{Z}_p$ gives the finite field $\mathbb{F}_p$.

Constructing $\mathbb{C}_p$

Just like $\mathbb{R}$, $\mathbb{Q}_p$ is not algebraically closed. There exist polynomials with coefficients in $\mathbb{Q}_p$ which do not factorise completely over $\mathbb{Q}_p$. Unlike $\mathbb{C}$, the algebraic closure of $\mathbb{Q}_p$ is not simple. In fact we must adjoin an infinite number of polynomial roots to $\mathbb{Q}_p$ to obtain the algebraically closed field $\overline{\mathbb{Q}}_p$.

In a further divergence from the familiar story of the complex numbers, in closing $\mathbb{Q}_p$ to $\overline{\mathbb{Q}}_p$ we lose completeness. (The p-adic norm extends in a unique way from $\mathbb{Q}_p$ to $\overline{\mathbb{Q}}_p$ and so can be used to define limits and Cauchy sequences on $\overline{\mathbb{Q}}_p$.) We add limit points of sequences to obtain the complete field $\mathbb{C}_p$ (sometimes called Ω_p). Fortunately $\mathbb{C}_p$ is algebraically closed, so at this point we are done. It is a huge field with rich properties and it would be out of scope to cover it further in this review. For our purposes, $\mathbb{Q}_p$ and a few finite degree algebraic extensions of it will prove sufficient. We refer the interested reader to the book by Koblitz as a starting point for further study.

At this point one might wonder how many other exotic fields can be constructed by choosing different norms with which to complete $\mathbb{Q}$. However, *Ostrowski's theorem* quickly puts an end to this. It states that any nontrivial norm on $\mathbb{Q}$ is equivalent to either the Euclidean absolute value or the p-adic norm for some prime p. In other words, starting with $\mathbb{Q}$, the only complete and algebraically closed fields one can obtain are $\mathbb{C}$ or $\mathbb{C}_p$ for some prime p. Thus the journey from $\mathbb{N}$ to a complete and algebraically closed field is as follows.

$$\begin{array}{ccccccccccc}
\mathbb{N} & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Q} & \longrightarrow & \mathbb{R} & \longrightarrow & \mathbb{C} & & \\
 & & & & & \searrow & & & & & \\
 & & & & & & \mathbb{Q}_p & \longrightarrow & \overline{\mathbb{Q}}_p & \longrightarrow & \mathbb{C}_p
\end{array}$$

D.2.2 p-adic Series

It is easy to check whether a p-adic series converges by virtue of the non-Archimedean triangle inequality. Indeed,

$$\left| \sum_{n=0}^{N} a_n \right|_p \leq \max_n |a_n|_p,$$

and so if $|a_n|_p \to 0$ as $n \to \infty$ then the series, too, converges. It follows that there is no notion of conditional convergence. A series converges if and only if

$$\sum_{n=0}^{\infty} |a_n|_p$$

does. This is a stark constrast to the Euclidean case where, for example, the series $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n}$ converges conditionally but not absolutely.

p-adic Power Series

We denote the ring of power series in formal variable X with coefficients in $\mathbb{Q}_p$ by $\mathbb{Q}_p[\![X]\!]$ and write such a power series

$$\Psi = \sum_{n=0}^{\infty} a_n X^n, \quad a_n \in \mathbb{Q}_p.$$

This series converges for all $|X|_p < r$ where the *radius of convergence* is given by

$$r = \frac{1}{\limsup_{n\to\infty} |a_n|_p^{1/n}}. \tag{D.7}$$

At the boundary $|x|_p = r$, a series converges either everywhere or nowhere, for the same reason that there is no such thing as p-adic conditional convergence.

The p-adic Logarithm and Exponential Functions

One can define the p-adic logarithm and exponential functions by the same series as their complex equivalents,

$$\log_p(1+X) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} X^n}{n}, \quad \exp_p(X) = \sum_{n=0}^{\infty} \frac{X^n}{n!}. \tag{D.8}$$

The series for $\log_p(1+X)$ converges for X in the unit p-adic disc $D(0,1)$, but can be extended uniquely to a function on the whole of $\mathbb{C}_p^\times$ such that $\log_p p = 0$ and $\log_p(XY) = \log_p X + \log_p Y$. This extension is called the *Iwasawa logarithm*.

The exponential function is less well behaved, since the factor of $1/n!$ becomes large p-adically as $n \to \infty$. This series converges only on the open disc $D(0, p^{-1/(p-1)})$, though it is possible to fix the situation by making a modified definition of $\exp_p$, the *Artin-Hasse* exponential.

So defined, $\log_p$ and $\exp_p$ are mutually inverse isomorphisms between the multiplicative group $D(1, p^{-1/(p-1)})$ and the additive group $D(0, p^{-1/(p-1)})$.

D.3 The p-adic Gamma Function

The classical Gamma function is given for $s \in \mathbb{C}$ by

$$\Gamma(s) = \int_0^\infty x^{s-1} \mathrm{e}^{-x} \, \mathrm{d}x \, . \tag{D.9}$$

It interpolates the factorial $n \mapsto n!$ on the positive integers, though it is not the only function to do so. There is a similar interpolating function on the p-adic integers $\mathbb{Z}_p$, which is in fact unique.

The factorial is not continuous in a p-adic sense. To see this, consider $x = 1$ and $y = 1 + p^N$ for some large integer N. Then $|x - y|_p = p^{-N}$ can be made as small as we like by choosing N sufficiently large. However, $p^N \mid (1 + p^N)!$ so $\left|(1 + p^N)!\right|_p \leq p^{-N} < 1$ and by the isosceles triangle principle

$$\left|1! - (1 + p^N)!\right|_p = 1.$$

So no matter how small we make $|x - y|_p$, we cannot make $|x! - y!|_p < \epsilon$ for a given $\epsilon > 0$. As a result, the factorial is not conducive to interpolation, but with some small changes this can be rectified.

Indeed, by omitting certain terms from the factorial product we can obtain a p-adically well-behaved interpolating function. For $n \in \mathbb{N}$, define

$$\Gamma_p(n + 1) = (-1)^{n+1} \prod_{1 \leq i \leq n, \ p \nmid i} i. \tag{D.10}$$

The product contains no factor divisible by p, so $p \nmid \Gamma_p(n+1)$. That is, $\Gamma_p(n+1) \in \mathbb{Z}_p^\times$. Now, $\mathbb{N}$ is dense in $\mathbb{Z}_p$, so Γ_p can be extended uniquely to a continuous function $\mathbb{Z}_p \to \mathbb{Z}_p^\times$ by directly taking the limit,

$$\Gamma_p(z) = \lim_{\substack{n \to z \\ n \in \mathbb{N}}} \Gamma_p(n), \quad z \in \mathbb{Z}_p. \tag{D.11}$$

We can thus prove results about Γ_p in $\mathbb{N}$ and automatically carry them to $\mathbb{Z}_p$ by the continuity of Γ_p.

Recurrence Relation

The first such result is the recurrence relation satisfied by the p-adic Gamma function,

$$\Gamma_p(X + 1) = -\Gamma_p(X) \begin{cases} X; & X \in \mathbb{Z}_p^\times, \\ 1; & X \in p\mathbb{Z}_p. \end{cases} \tag{D.12}$$

Reflection Formula

It follows from the recurrence relation that

$$\frac{\Gamma_p(X+1)\Gamma_p(-X)}{\Gamma_p(X)\Gamma_p(1-X)} = \begin{cases} -1; & X \in \mathbb{Z}_p^\times, \\ 1; & X \in p\mathbb{Z}_p. \end{cases}$$

Since $\Gamma_p(1)\Gamma_p(0) = -1$, induction on the natural numbers yields the *reflection formula*

$$\Gamma_p(X)\Gamma_p(1-X) = (-1)^{X_0}, \tag{D.13}$$

where X_0 is the unique element of $\{1, 2, \ldots, p\}$ equal to X mod p. Again, by continuity this holds for all $X \in \mathbb{Z}_p$. Using the recurrence relation again, we obtain

$$\Gamma_p(pX)\Gamma_p(-pX) = 1. \tag{D.14}$$

Taking successive X derivatives at $X = 0$ yields relations between the $\Gamma_p^{(n)}(0)$ for various n, the first of which is

$$\Gamma_p''(0) = \Gamma_p'(0)^2. \tag{D.15}$$

Expansion Formula

When it comes to computing Γ_p and its derivatives the following expansion formula [22] in terms of the expansion coefficients c_n of the Dwork exponential, for $0 \leq a < p$ and $z \in \mathbb{Z}_p$, is extremely useful,

$$\Gamma_p(pz - a) = \sum_{n=0}^{\infty} c_{np+a}\, p^n (z)_n. \tag{D.16}$$

This follows from a more general expansion due to Robert [33].

Relation to the Factorial

We finish this section by restoring the link between the p-adic Gamma and the factorial function. Including the missing factors in the definition of Γ_p we have

$$n! = (-1)^{n+1}\Gamma_p(n+1)\, p^{\lfloor n/p \rfloor} \left\lfloor \frac{n}{p} \right\rfloor ! \tag{D.17}$$

where $\lfloor x \rfloor$ denotes the greatest integer $m \leq x$. Expanding $n \in \mathbb{Z}$ p-adically,

$$n = \sum_{j=0}^{l} n_j p^j, \quad n_j \in \mathbb{Z}, \quad 0 \leq n_j < p, \tag{D.18}$$

it is easy to see that

$$\left\lfloor \frac{\lfloor n/p^k \rfloor}{p} \right\rfloor = \left\lfloor \frac{n}{p^{k+1}} \right\rfloor. \tag{D.19}$$

Then, applying equation (D.17) recursively, we obtain the factorial formula

$$n! = (-1)^{l+1+\sum_{j=0}^{l} \left\lfloor \frac{n}{p^j} \right\rfloor} p^{\sum_{j=1}^{l} \left\lfloor \frac{n}{p^j} \right\rfloor} \prod_{j=0}^{l} \Gamma_p \left(\left\lfloor \frac{n}{p^j} \right\rfloor + 1 \right). \tag{D.20}$$

If we introduce the sum of digits function

$$S_n = \sum_{j=0}^{l} n_j, \tag{D.21}$$

and observe that

$$\frac{n - S_n}{p-1} = n_1 + n_2(p+1) + \cdots + n_l(p^{l-1} + \cdots + 1) = \sum_{j=1}^{l} \left\lfloor \frac{n}{p^j} \right\rfloor, \tag{D.22}$$

then we can rewrite (D.20) as

$$n! = (-1)^{l+1+n} (-p)^{\frac{n-S_n}{p-1}} \prod_{j=0}^{l} \Gamma_p \left(\left\lfloor \frac{n}{p^j} \right\rfloor + 1 \right). \tag{D.23}$$

By taking a ratio of such expressions we find that for $p > 2$ (i.e. p odd)

$$\frac{(pn)!}{n!} = -(-p)^n \Gamma_p(pn+1). \tag{D.24}$$

It is worth taking a moment to appreciate the significance of this result. We have already described how the factorial function is not even continuous p-adically. And yet the right hand side, given in terms of Γ_p, is smooth. The ratio on the left hand side is better behaved than its consituent parts. This fact will prove crucial in our Dwork cohomology calculations in Chapter 4.

D.4 Finite Characters

Given a field K and group G, a K-valued *character* on G is a group homomorphism from G to the multiplicative group $K^\times$. Commonly one takes $K = \mathbb{C}$, but we shall take K to be some p-adic number field. There are two characters of particular interest to us: the Teichmüller character (Appendix D.4.1) and the Dwork character (Section 2.1).

If χ is a K-valued character of a *finite* group G then it has a property which will prove particularly useful to us. For $h \in G$,

$$\chi(h) \sum_{g \in G} \chi(g) = \sum_{g \in G} \chi(hg) = \sum_{g \in G} \chi(g),$$

where the last equality follows because it is simply a re-ordering of the sum. If we take $h \neq e_G$ so that $\chi(h) \neq 1 \in K$ we see that

$$\sum_{g \in G} \chi(g) = 0. \tag{D.25}$$

Now suppose in particular we have a K-valued character of the additive group $\mathbb{F}_q$. Any nonzero $y \in \mathbb{F}_q^\times$ gives an automorphism $x \mapsto yx$ of $\mathbb{F}_q$ so that

$$\sum_{x \in \mathbb{F}_q} \chi(yx) = 0.$$

But if $y = 0$ then $\chi(yx) = 1$ for all $x \in \mathbb{F}_q$. We thus have for $y \in \mathbb{F}_q$ that

$$\sum_{x \in \mathbb{F}_q} \chi(yx) = \begin{cases} q & \text{if} \quad y = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{D.26}$$

In Chapter 2 we showed how we can use this property to count the points of a variety over a finite field using the Dwork character.

D.4.1 The Teichmüller Character

Once we have a point counting formula in terms of the Dwork character we still need to make a connection between this and suitable smooth functions on a field such as $\mathbb{C}_p$ where we can do analysis. In other words, we need a means of embedding the finite field $\mathbb{F}_q$ inside $\mathbb{C}_p$ in a natural way.

We will start with the prime fields $\mathbb{F}_p$, which are easier to work with. One might be tempted to simply use the natural identification of elements of $\mathbb{F}_p$ with rational integers in $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{Q}_p$. This does not, however, preserve the group structure: in $\mathbb{F}_p$ arithmetic is performed mod p, which is certainly not the case in $\mathbb{Q}_p$. Moreover, in $\mathbb{Q}_p$, Fermat's Little Theorem only holds to zeroth order in p,

$$x^{p-1} = 1 + \mathcal{O}(p), \quad x \in \mathbb{Q}_p^\times. \tag{D.27}$$

We can build on this approximate relation to obtain an embedding which preserves the key properties of the $\mathbb{F}_p^\times$ as a multiplicative group.

Indeed, taking successive powers of p we achieve a better and better approximation to the true result,

$$x^{p^n(p-1)} = 1 + \mathcal{O}\big(p^{n+1}\big). \tag{D.28}$$

Each x^{p^n} is a representative of x mod p and is successively closer to obeying Fermat's Little Theorem exactly.

Inspired by this we define the *Teichmüller character* on $\mathbb{F}_p^\times$. Let $\bar{x} \in \mathbb{F}_p$ and $x \in \mathbb{Z} \subset \mathbb{Z}_p$ be the smallest positive representative of $\bar{x}$. Then define

$$\text{Teich}(\bar{x}) = \lim_{n\to\infty} x^{p^n} \ , \quad \bar{x} \in \mathbb{F}_p. \tag{D.29}$$

The distinction between $\bar{x}$ and x may seem rather technical, given they are equivalent as integers, but it is an important point. Since $\bar{x}^{p^n} = \bar{x}$ exactly, it follows that $\lim_{n\to\infty} \bar{x}^{p^n} = \bar{x}$ in $\mathbb{F}_p$, so it is important that the limit is taken in $\mathbb{Z}_p$, where $x^p = x$ only to $\mathcal{O}(p)$. Having made this distinction we shall henceforth omit the bar and denote by x both the element of $\mathbb{F}_p$ and the corresponding element of $\mathbb{Z} \subset \mathbb{Z}_p$. It follows from the limiting process that Teich(x) is a $(p-1)$st root of unity,

$$\text{Teich}(x)^{p-1} = 1. \tag{D.30}$$

Namely, it obeys Fermat's Little Theorem exactly in $\mathbb{Z}_p$. It must also have $|\text{Teich}(x)|_p = 1$. It follows immediately from the definition that

$$\text{Teich}(xy) = \text{Teich}(x)\text{Teich}(y), \tag{D.31}$$

and so Teich is a group homomorphism from $\mathbb{F}_p^\times$ to the multiplicative group of $(p-1)st$ roots of unity in $\mathbb{Z}_p^\times$. In other words, Teich is a $\mathbb{Q}_p$-valued character on $\mathbb{F}_p^\times$.

We can extend Teich to a function $\mathbb{F}_p \to \mathbb{Z}_p$ by defining Teich$(0) = 0$. Defined in this way, Teich respects addition on $\mathbb{F}_p$ to zeroth order in p,

$$\text{Teich}(x+y) = \lim_{n\to\infty} (x+y)^{p^n} = \text{Teich}(x) + \text{Teich}(y) + \mathcal{O}(p). \tag{D.32}$$

We will also need a definition of Teich on the extension fields $\mathbb{F}_q$. Starting from the fact that

$$x^{q-1} = 1 + \mathcal{O}(p)$$

for $x \in \mathbb{F}_q^\times$, we can perform a similar limiting process and define

$$\text{Teich}(x) = \lim_{n\to\infty} x^{q^n}. \tag{D.33}$$

As before this is a group homomorphism from the multiplicative group $\mathbb{F}_q^\times$ to the multiplicative group of $(q-1)st$ roots of unity. However, for $q = p^s$ with $s > 1$, these roots of unity do not belong to $\mathbb{Z}_p$, but some algebraic extension K of $\mathbb{Q}_p$. (Such an extension, obtained by adjoining an m^{th} root of unity to $\mathbb{Q}_p$, where $p \nmid m$, is called an *unramified* extension.) Of course, these roots still have unit p-adic norm, so even though they are not p-adic integers, they are contained in the so-called *integral closure* of $\mathbb{Z}_p$ in K,

$$A = \left\{ x \in K \colon |x|_p \leq 1 \right\}.$$

Thus Teich gives a K-valued character of $\mathbb{F}_q$, and in particular allows us to embed $\mathbb{F}_q$ in $\mathbb{C}_p$ while preserving the most important properties of $\mathbb{F}_q$.

D.5 Classical Modular Forms

Finally, we introduce modular forms whose relations with Calabi-Yau manifolds are the main subject of this thesis. We keep this section brief and only provide the definitions and results necessary to support our study of the modularity properties of Calabi-Yau manifolds. For a full introduction to the theory of modular forms we refer the interested reader to [34]. There are also some good definitions and summaries to be found on the LMFDB [25].

D.5.1 The Modular Group

The *Modular group*

$$\Gamma_1 = \mathrm{SL}(2, \mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\} \tag{D.34}$$

gives the (orientation preserving) transformations between bases of integer lattices (note that any $\gamma \in \mathrm{GL}(2, \mathbb{Z})$ must have $\det \gamma = \pm 1$ in order that γ^{-1} also be integral). Alternatively $\gamma \in \Gamma_1$ acts on the upper half plane $\mathbb{H}$ by Möbius transformations

$$z \mapsto \gamma z = \frac{az + b}{cz + d}, \tag{D.35}$$

which preserves $\mathbb{H}$ since

$$\mathrm{Im}\, \gamma z = \frac{\mathrm{Im}\, z}{|cz + d|^2}. \tag{D.36}$$

The modular group is generated by the elements

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{D.37}$$

which act on $\mathbb{H}$ as follows

$$S : z \mapsto -1/z, \qquad T : z \mapsto z + 1. \tag{D.38}$$

We will also have use of certain finite index[3] subgroups of Γ_1. The *principal congruence subgroup* is

$$\Gamma(N) = \{\gamma \in \Gamma_1 : \gamma \equiv \mathbb{1} \mod N\}. \tag{D.39}$$

We also define

$$\Gamma_1(N) = \left\{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1 : \gamma \equiv \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mod N\right\}, \tag{D.40}$$

$$\Gamma_0(N) = \left\{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1 : \gamma \equiv \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mod N\right\}. \tag{D.41}$$

D.5.2 Modular Forms

Classical modular forms are holomorphic functions on $\mathbb{H}$ which transform in a specific way under the action of Γ_1 or a subgroup $\Gamma_0(N)$ or $\Gamma_1(N)$ for some $N \in \mathbb{N}$. Indeed, let G be such a group and let $k \in \mathbb{Z}$. Then a modular form f on G of weight k is a holomorphic function on $\mathbb{H}$ such that for all $z \in \mathbb{H}$ and $\gamma \in G$,

$$f(\gamma z) = (cz + d)^k f(z). \tag{D.42}$$

One or more points must be added at $\mathrm{i}\infty$ to make $\mathbb{H}$ complete. Such points are called *cusps* and f must be holomorphic at such points. If f is zero at the cusps then it is called a *cusp form*.

Observe that every possible group G contains the element T and thus it follows that modular forms are periodic along the real axis,

$$f(z) = f(z + 1) \quad \text{for all} \quad z \in \mathbb{H}.$$

It follows that such forms can be Fourier expanded

$$f(z) = \sum_{n=0}^{\infty} a_n q^n, \quad q = \mathrm{e}^{2\mathrm{i}\pi z}. \tag{D.43}$$

[3]The index of a subgroup H of G is the number of times H "fits into" G. That is, it is the order of the quotient group, G/H (or more generally the left coset space G/H if H is not a normal subgroup).

The terms for $n < 0$ are all zero by the holomorphicity requirement. The cusps correspond to $q = 0$ and so f is a cusp form if and only if $a_0 = 0$.

It is this Fourier expansion, henceforth called the *q-expansion*, which is of primary interest to us. The coefficients in these expansions arise in many surprising places in mathematics and in particular they occur as coefficients in the zeta functions of certain arithmetic varieties.

D.5.3 Spaces of Modular Forms

One can show that there are no modular forms of $\Gamma_0(N)$ of negative or odd weight. There are no nontrivial modular forms of weight zero either, but such forms do exist if the holomorphicity requirement is relaxed. Meromorphic weight zero modular forms are called *modular functions*.

The space of modular forms of weight k on G is denoted $M_k(G)$. We have already established that this has dimension zero for $k \leq 0$ and k odd (for $G = \Gamma_0(N)$). For k even, it is finite dimensional, which makes the task of proving modular form identities at a given weight and level a purely computational task. Likewise, when we try to identify modular forms corresponding to coefficients in the zeta functions of Calabi-Yau manifolds, there is only a finite search space for a given weight and level.

The *level* of a modular form on G is the least integer N such that the principal congruence subgroup $\Gamma(N)$ is contained in G. In particular modular forms on $\Gamma_0(N)$ or $\Gamma_1(N)$ have level N. Let N, M, d be integers with $N = Md$. If $f(\tau)$ is a level M modular form then $f(d\tau)$ is a level N form. Since this form arises from a form of lower level, it is called an *oldform*. Modular forms which do not arise in this way are called *newforms*, and it is newforms which we have identified at the singular points of Calabi-Yau operators.

At a given level N, the elements of $M_k(\Gamma_1(N))$ split into subspaces according to their transformation properties under the larger group $\Gamma_0(N)$. Indeed, we have that

$$M_k(N) = M_k(\Gamma_1(N)) = \bigoplus_{\chi} M_k(N, \chi),$$

where the sum is taken over all Dirichlet characters $\chi : \mathbb{Z} \to \mathbb{C}$ mod N. The space $M_k(N, \chi)$ consists of modular forms on $\Gamma_1(N)$ which for $\gamma \in \Gamma_0(N)$ transform as

$$f(\gamma z) = \chi(d)(cz + d)^k f(z).$$

Then f is called a modular form with character (or nebentypus) on $\Gamma_0(N)$. If χ is the trivial character then $M_k(N,\chi) = M_k(\Gamma_0(N))$. Note that $M_k(N,\chi)$ for χ nontrivial and k odd need not be zero dimensional.

D.5.4 Labelling Modular Forms

Before we start talking about particular modular forms, we need a convenient way to distinguish them. Though their q-expansions are unique and can be used for this purpose, a labelling system will be more practical. There are three distinct labelling conventions used for classical modular forms in this thesis, because there are three sources of data for such modular forms: the LMFDB [25], the thesis of Christian Meyer [24] and our own computations (Appendix B). We will follow the LMFDB labelling system wherever possible, since it is the most complete and convenient source of knowledge on modular forms and related topics.

A space of modular forms can be uniquely identified by its level N, its weight k and the character χ. The character is assigned a numeric label j using a numbering system introduced by Brian Conrey [35]. There is a (nonstandard, i.e. not packaged with SageMath) package [36] for working with this labelling system, and j is such that the character is given by `DirichletGroup_conrey(N)[j]` in SageMath. Note that $j = 1$ always refers to the trivial character.

If a given space $M_k(N,\chi)$ has dimension greater than 1, we must also distinguish between independent forms in this space. For this purpose we use a letter a, b, c, etc. which here we denote generically by x. These labels are chosen to match the LMFDB, which labels forms according to the lexicographical ordering of the traces of their Hecke eigenvalues.

In summary the complete LMFDB label for a given modular form of weight k and level N is

$$N.k.j.\mathrm{x},$$

which uniquely determines it and can be used to look it up on the LMFDB.

Meyer's thesis lists a great many newforms of weight 4 and trivial character. In his tables they are labelled simply N/l where N is the level and l is a numeric label for the particular form of that level. We preserve the slash to avoid ambiguity with LMFDB labels, but also include the weight. Thus forms from Meyer's tables will be labelled in this thesis as $N.4/l$.

We must also label newforms computed for the purpose of this thesis, which cannot easily be found elsewhere, but can be looked up in Appendix B. Such forms will be denoted $N.k.j$ where N is the level, k the weight and j the label for the character following the LMFDB rules. Such labels emulate the LMFDB labels, but cannot be confused with them due to the absence of an alphabetic orbit label. We do not require this label since no two forms listed in Appendix B share the same weight, level and character.

D.6 Modular Forms Over Field Extensions

In the previous section we considered classical modular forms on $\mathbb{Q}$ which transformed in a specific way under the action of $\mathrm{SL}(2,\mathbb{Z})$. But certain Calabi-Yau manifolds must be considered over some algebraic extension K of $\mathbb{Q}$ with ring of integers $\mathbb{Z}_K$. In such cases, we expect modular forms which transform under the action of $\mathrm{GL}(2,\mathbb{Z}_K)$, or finite index subgroups of it. If K is contained within $\mathbb{R}$ then it is called *totally real* and we call the corresponding forms *Hilbert modular forms*; if the extension is imaginary we call them *Bianchi modular forms*. When talking about such forms it is important to specify the particular field extension over which they are defined.

It is outside the scope of this thesis to discuss such modular forms in any detail. We refer the interested reader to the LMFDB for an overview and the book [34] for a more in-depth study of Hilbert modular forms. Below we give a brief definition and intuitive explanation of Hilbert modular forms. No Bianchi forms have been identified in this thesis so we do not discuss them further.

D.6.1 Hilbert Modular Forms

Let K be a totally real degree n algebraic extension of $\mathbb{Q}$ with defining polynomial P, and let α be a root of P. Then a basis of K as a $\mathbb{Q}$-vector space is given by $\{1,\alpha,\ldots,\alpha^{n-1}\}$. The ring of integers $\mathbb{Z}_K$ of K is a $\mathbb{Z}$-module spanned by the same basis. Likewise the group $\mathrm{GL}(2,\mathbb{Z}_K)$ can be viewed as an $\mathrm{GL}(2,\mathbb{Z})$-module spanned by this basis. In other words, $\mathrm{GL}(2,\mathbb{Z}_K)$ is intuitively given by n copies of $\mathrm{GL}(2,\mathbb{Z})$ with some additional structure. We can write an element $\gamma\in\mathrm{GL}(2,\mathbb{Z}_K)$ as

$$\gamma=\sum_{i=1}^{n}\gamma_i\alpha^{i-1},\quad \gamma_i\in\mathrm{GL}(2,\mathbb{Z}).$$

The conguence subgroups of $\mathrm{GL}(2, \mathbb{Z}_K)$ are labelled by ideals of $\mathbb{Z}_K$. For an ideal $\mathfrak{N}$, define

$$\Gamma_0(\mathfrak{N}) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z}_F) \colon c \in \mathfrak{N},\ \det\gamma \gg 0 \right\}. \tag{D.44}$$

By $\det\gamma \gg 0$ we mean that $\det\gamma$ is *totally positive*. This means that no matter how one embeds K in $\mathbb{R}$ the image of $\det\gamma$ is always positive. Notice the similarity to the congruence subgroups of the classical case, where we required $c \equiv 0 \mod N$, that is $c \in N\mathbb{Z}$ where $N\mathbb{Z} = \{Nm : m \in \mathbb{Z}\}$ is an ideal of $\mathbb{Z}$.

A Hilbert modular form on K of parallel weight $k \in \mathbb{Z}$ and level $\mathfrak{N}$ is a function $f : \mathbb{H}^n \to \mathbb{C}$ such that for all $z \in \mathbb{H}^n$ and $\gamma \in \Gamma_0(\mathfrak{N})$,

$$f(\gamma z) = f\left(\frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}, \ldots, \frac{a_n z_n + b_n}{c_n z_n + d_n}\right) = \prod_{i=1}^{n} \left(\frac{(c_i z_i + d_i)^k}{\det\gamma_i^{k-1}}\right) f(z). \tag{D.45}$$

We see that, in a suitable basis, each of the n copies of $\mathrm{SL}(2, \mathbb{Z})$ in $\mathrm{SL}(2, \mathbb{Z}_K)$ acts on one of the n variables of the function f.

Hilbert modular forms have coefficients analogous to the q-expansions of classical modular forms, which also arise in the zeta functions of certain varieties. These coefficients are labelled not by prime integers but prime ideals in $\mathbb{Z}_K$. We label such ideals in line with the LMFDB, writing $[N, m, \alpha]$, where N is the norm of the ideal (i.e. the order of the quotient $\mathbb{Z}_K/I$), m is the smallest rational integer in the ideal and α is an element such that m and α together generate the ideal. When $K = \mathbb{Q}(\sqrt{D})$ is a quadratic extension, we denote the generator of K by $w = \sqrt{D}$.

Labelling

At the time of writing, the LMFDB lists only Hilbert modular forms of parallel weight 2. They are labelled $N.i$-x where N is the ideal norm of the level and i-x is a somewhat arbitrary label for each particular form amongst all forms of the same level norm. In particular, i is a numeric label for the ideal and x is an alphabetic label for the newforms at a particular level. The full label on the LMFDB also includes a prefix which identifies the field over which the forms are defined.

The Hilbert forms identified in this thesis are all defined over quadratic extensions of $\mathbb{Q}$. They have parallel weight 4 and so cannot currently be found on the LMFDB. We list the levels and coefficients of these forms in Appendix B and use our own labelling system,

$$D{-}N.k/l,$$

where $\mathbb{Q}(\sqrt{D})$ is the field extension, N is the level norm, k is the parallel weight and l is an arbitrary numeric label. The coefficients for each form are sufficient to identify it uniquely and so these forms can be identified with LMFDB labels once the forms are added to the database.

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