STUDIES IN IDENTIFICATION AND CONTROL

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Abstract.

The optimal steady-state control, and suboptimal adaptive control, of disturbed single-input-output systems are introduced, and the class of systems considered is defined.

It is noted that the stochastic tracking problem divides into a deterministic tracking problem and a stochastic regulator problem; the solutions to these two problems are shown to be independent but formally similar. The continuous regulator problem is approached via both frequency and time domain methods: the former method is extended to cover unstable systems; the latter method is extended to include systems with input delay. The two regulators are shown to be externally equivalent. The frequency domain method is briefly described for discrete systems, and shown to include the minimum variance regulator of Åström and Peterka as a special case. Some systems which allow measurement noise to be treated as a system disturbance for the purposes of optimal controller design are investigated.

A novel class of control laws is described in both continuous and discrete time; in the same way as the minimum variance regulator forms the basis of the self-tuning regulator of Åström & Wittenmark, these minimum variance controllers form the basis of

a self-tuning controller. These minimum variance controllers have a number of advantages over the minimum variance regulator, and are open to a number of interpretations including: a model following control law, and an extension of classical control laws to systems with delay. The optimality of a member of this class of control laws is investigated, and analogies drawn with the previously considered k-step-ahead control laws; some examples are given to illustrate the method.

An adaptive control law combining the above minimum variance controllers with a linear least-squares algorithm is proposed and shown to be self-tuning. These self-tuning controllers are only slightly more complex than the self-tuning regulator of Åström & Wittenmark, but have a number of advantages. Intuitive justification is given for the conjecture that some methods of Ljung, developed for the analysis of the self-tuning regulator, are applicable to the self-tuning controller. Simulated examples are given which compare and contrast the performance of the self-tuning controller with that of the self-tuning regulator. The first steps towards a quasi-continuous self-tuning controller are outlined.

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Notation

Polynomials are all of the same form, for example:

$$A(s) \triangleq a_0 + a_1 s + a_2 s^2 + \ldots + a_{n_a} s^{n_a}$$

Laplace transformed quantities are denoted by a bar, for example:

$$\bar{y}(s) \triangleq \text{Laplace transform of } \{y(t)\}$$

Arguments are often omitted where convenient: The loose, but convenient, notation is sometimes used whereby the symbols $A(s)y(t-T)$ are taken to mean the inverse Laplace transform of $e^{-sT} A(s)\bar{y}(s)$.

Symbols denoting vectors or matrices are underlined, those used in chapter 2 are defined on pages 55-58.

Block diagrams represent systems of equations - they must not be taken as wiring diagrams for analogue computers; for this reason, they may well contain differentiators or cancelling factors with roots outside the stability region.

$E\{\bullet\}$ denotes unconditional expectation: the ensemble average is over all possible realisations of the random processes affecting the argument.

$E\{\bullet \mid t\}$ denotes expectation conditioned on all measurable variables in the range $(-\infty, t]$: the ensemble average is over all possible realisations of the random processes affecting the argument which occur after time t.

Chapter 1.

Introduction.

1.1 Motivation.

Control theory is a rich and interesting field for the theoretician; it must not be forgotten, however, that its roots lie in the elucidation and solution of problems of practical engineering interest. Divorced from reality, control theory can become mere mathematical sophistry - separated from its context, by what criterion can a new development be judged? The theory presented in this thesis is, it is hoped, not only theoretically interesting but also practically applicable.

Historically, control theory grew up as a set of methods for analysing control systems selected using engineering intuition and judgement; this analytic design procedure has proved satisfactory for a wide range of problems. When the emphasis changes from designing a satisfactory control system to designing the 'best' control system - either for actual implementation or as a yardstick for measuring performance - such analytic methods become inappropriate; for this reason synthetic design procedures were introduced which, on the basis of known system and disturbance dynamics, automatically design controllers to minimise some criterion of performance.

As discussed in section 1.3, this thesis extends some of these methods to increase the range of systems to which they are applicable, and to eliminate some defects hitherto present. In particular, frequency-domain methods are extended to include unstable systems, and time-domain methods are extended to include continuous systems with input time-delay.

Many real-life systems have dynamics which cannot be easily deduced from physical laws, and may even be time varying. One approach to the control of such systems is to experimentally determine the dynamics of ('identify') the system, and on the basis of the results of such an experiment design a control law. Such a procedure is inappropriate if system dynamics change with time; moreover the design criteria of identification experiments are rarely matched to the performance of the resultant control law.

It has been suggested that a more useful approach is to use an adaptive control law which simultaneously identifies and controls the unknown system. Unfortunately the optimum solution to such a problem is computationally intractable, and so the optimum solution will, in most cases, represent an unobtainable performance bound. The field of adaptive control is thus distinguished by a plethora of suboptimum algorithms: unfortunately ',

it is also distinguished by the paucity of adaptive controllers in practical use.

A notable exeption to this rule is the self-tuning regulator (Åström & Wittenmark,1973); this strikes a happy mean between complexity and performance, and seems remarkably robust in practical situations. Applications have been reported in the fields of: paper making (Borrison & Wittenmark,1973), ship steering (Kallstrom,1975), ore crushing (Borrison & Syding,1974), and an enthalpy exchanger used in air conditioning (Jensen & Hansel,1974). In some of these cases tangible benefits have been obtained using the self-tuning regulator.

This thesis develops a more general class of self-tuning controllers (Clarke & Gawthrop,1975) which, whilst retaining the simplicity of the self-tuning regulator, are more powerful: set point following is included, without deterioration in regulator performance; control weighting is possible; and the closed loop system can be made to behave a prespecified system with respect to the set point, and the poles of the disturbance process arbitarily placed. This increased flexibility, without significant increase in complexity, makes the self-tuning controller a candidate for practical applications; in particular the algorithm is simple enough to be realised using a microprocessor (Clarke et al.,1975)

1.2 System models.

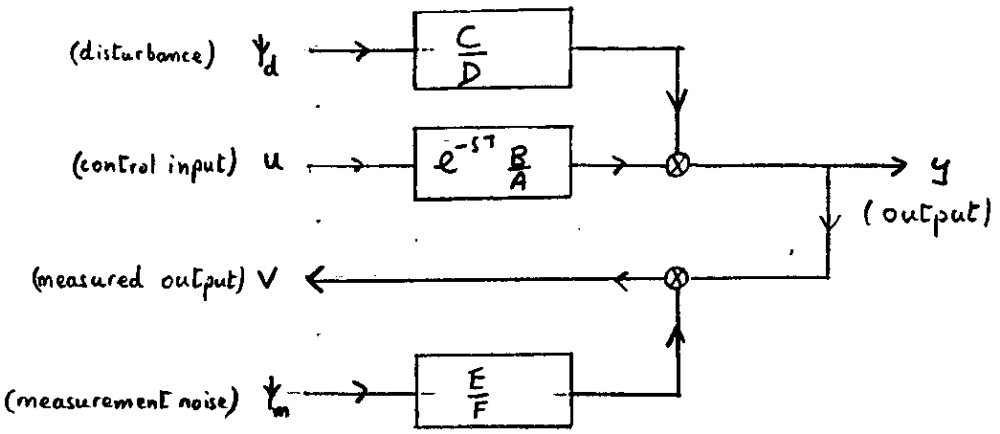

Fig 1.2.1 The system.

The continuous models considered in this thesis are as depicted in Fig 1.2.1, and will usually be expressed in transfer function notation as:

$$\bar{y}(s) = e^{-sT}\frac{B(s)}{A(s)}\bar{u}(s) + \frac{C(s)}{D(s)}\bar{\psi}_d(s) \qquad (1.2.1)$$

$$\bar{v}(s) = \bar{y}(s) + \frac{E(s)}{F(s)}\bar{\psi}_m(s) \qquad (1.2.2)$$

The control input, output measurement and output of the system are denoted by $\bar{u}$, $\bar{v}$ and $\bar{y}$ respectivly. The symbols ψ_d and ψ_m represent uncorrelated white noise processes, and A(s) - F(s) represent polynomials in the Laplace operator s of the form:

$$A(s) = a_0 + a_1 s + \ldots + a_{n_a} s^{n_a} \qquad (1.2.3)$$

Without loss of generality, a_o, c_o, d_o, e_o, and f_o are all taken to be unity, and the noise processes scaled accordingly. The system input delay is given by T.

The rigorous theory of stochastic processes, and the interpretation of white noise in terms of a Wiener process (Åström,1970; Melsa & Sage,1973), will not be required for the purposes of this thesis. The symbols $(C(s)/D(s))\dagger$ will normally merely imply a process with spectral density $C(s)C(-s)/D(s)D(-s)$; when the process is interpreted as a stochastic differential equation, it will always be possible (following Melsa & Sage,1973) to consider continuous white noise as a limiting case of discrete white noise - a conceptually much simpler process.

The discrete-time systems considered in this thesis are described by equations analagous to (1.2.1-3), except that the Laplace operator s is replaced by the z operator z^{-1}, and e^{-sT} is replaced by z^{-k} where k is the integer input delay. Discrete-time systems often arise from the sampled data control of continuous systems; the inter-sample behaviour of the underlying continuous system is often of practical importance. The consequences of this fact and the related problem of the choice of sample period, though of some importance, are beyond the scope of this thesis. The continuous to

discrete conversion of deterministic models (B/A) is discussed by Freeman (1965) and Kuo (1970); the different problem of the conversion of stochastic models (C/D & E/F) may be solved either via autocorrelation matching (Kuo,1970), or by integrating the stochastic differential equation (Åström,1970; Kwakernaak & Sivan;1972).

It is argued in section 2.2 that practical systems are such that the disturbance process denominator (D) contains the system denominator (A) as a factor - this observation leads to important results concerning the steady-state regulator problem. The system (1.2.1) can thus be rewritten as:

$$Dy = e^{-sT} B\tilde{D}' u + C\gamma_d \qquad (1.2.4)$$

where: $D = \tilde{D}'A$ (1.2.5)

It will be found that the case defined by C = 1 leads to particularly simple solutions to certain problems.

1.3 Steady-state optimal control of known systems.

The criterion of optimality for the design of continuous time control laws for known parameter systems is taken to be:

$$V = \lim_{t_0 \to \infty} \frac{1}{2t_0} \int_{-t_0}^{t_0} E\left\{(y(t+T) - w(t))^2 + \lambda u^2(t)\right\} dt \qquad (1.3.1)$$

The interpretation of this cost function as a minimisation of output error with a constraint on control effort is discussed by Kwakernaak & Sivan (1972). The symbol

w represents the set point - or desired value - of the system output (y); the argument of y is advanced by the system time-delay (T) to account for the fact that u(t) and w(t) cannot affect y until time t+T. The minimisation of V is constrained by the requirement that the minimising control law be realisable - in the sense that the control (u) must not have a functional dependence on future values of v or w. It is shown that the minimisation of V (2.3.1) decomposes into two independent problems: a deterministic servo problem, and a stochastic regulator problem.

The regulator problem has been previously approached via two main routes: the frequency-domain methods of Newton et al. (1957); and the time-domain methods of Kalman and Bellman (Athans,1971; Kwakernaak & Sivan,1972). Hitherto, neither method applied to all systems described by equations (1.1 & 2); the former method was inapplicable to unstable systems, the latter method was inapplicable to systems with input delay. (Some work has been reported (Alekal et al.,1971; Ross,1971) on certain deterministic systems with a time delay.) This thesis supplies the necessary extensions to each method to allow all systems described by (1.1.1 & 2) to be considered in a uniform manner.

As each method is applied to the same problem, it

would be expected that the resultant control laws would be the same; it is demonstrated that despite the different internal structure of the two control laws, the external input-output relationships are the same in each case. This demonstration hinges on relations between two different algebraic Riccati equations and spectral factorisation identities: one defining a state feedback gain, the other defining a Kalman filter gain. The former problem has been previously discussed (Brocket 1970; MacFarlane,1971); the latter problem is not the dual of the former - due to nonwhite measurement noise; however it is shown that a similar relationship also holds in this case.

The frequency-domain method is also used to obtain discrete-time results; it is demonstrated that these include the minimum variance regulator (Åström,1970; Peterka,1972) as a special case - when both measurement noise variance and control weight (λ) are zero.

This work highlights the importance of realistic disturbance modelling; unrealistic disturbance models are shown to lead to impracticable control laws when applying the frequency domain technique.

Having solved the regulator problem, the corresponding servo problem is found to have a formally similar solution. The two corresponding control laws are found to combine neatly to give a control law satisfying the overall

problem. It is noted that methods that do not distinguish between known set points and unknown disturbances (Newton et al.,1957; Youla et al.,1976) must, in general yield inferior control laws.

Finally, conditions are given which define systems for which the measurement noise may be treated as a disturbance - that is y in the cost function (1.3.1) is replaced by v - without deterioration of performance. These results are particularly useful when considering problems where, due to lack of a priori information, disturbance and measurement noise cannot be distinguished.

This work is described in chapter 2.

1.4 Adaptive control of unknown parameter systems.

The optimal control of unknown parameter systems - that is the coefficients of the polynomials A-F are unknown - is in most cases mathematically and computationally difficult. This has lead to a proliferation of suboptimal algorithms, some of which have been recently surveyed (Wittenmark,1975). The purpose of this section is to relate the self-tuning controllers of this thesis to some of these algorithms.

Consider the discrete-time system corresponding to 1.2.4, and with no measurement noise:

$$y(t) + a_1 y(t-1) + \ldots + a_n y(t-n) =$$

$$b_0 u(t-k) + b_1 u(t-k-1) + \ldots + b_n u(t-k-n)$$

$$+ \psi(t) + c_1 \psi(t-1) + \ldots + c_n \psi(t-n) \qquad (1.4.1)$$

The control of such systems can be usefully divided into two parts: estimation of unknown quantities, and generation of the control signal on the basis of such quantities. These two aspects are considered in turn.

As y can be exactly measured, and u is generated by the control law, the unknown parameters $a_1 - a_n$ and $b_0 - b_n$ are multiplied by variables known at tine t. The random numbers γ are not directly measurable, and must be deduced from measurements of y; as the parameters $c_1 - c_n$ are unknown a priori, the joint determination of γ and $c_1 - c_n$ leads to a non-linear estimation problem. Some strategies (e.g. Wieslander & Wittenmark,1971; Alster & Belanger,1974; Wittenmark,1975) avoid this non-linear problem by assuming that the c parameters are all zero;,the resultant linear problem is solved by the use of a Kalman filter (Åström & Wittenmark, 1971). Other strategies (e.g. Tse et al.,1973; Jacobs & Hughes,1975) explicitly include the estimation of $c_1 - c_n$ using some approximate non-linear filter, such as an extended Kalman filter (Sage & Melsa,1971).

The generation of the optimal control signal on the basis of current estimates is complicated by the fact that the current control affects future estimation performance - the control law is said to exercise the dual functions of estimation and control. Even if the estimation problem is linear, and the probability distributions specified by a finite number of sufficient

statistics, the minimisation of a multi-stage cost function is numerically formidable (e.g. Åström & Wittenmark, 1971). A common simplification is to remove the dual feature by using one-step cost functions, for which the effect of current control on future uncertainty is immaterial.

This simplification has two deleterious effects: identification may be poor, and even lead to 'sleeping' or 'turn off' (Åström & Wittenmark,1971; Hughes & Jacobs, 1974); the control may be poor in the sense that even if the system parameters were known, a one-step control law may give poor performance as measured by an infinite stage cost function. It is this latter effect that the extensions to the self-tuning regulator, given in this thesis, are designed to alleviate.

The self-tuning controller, presented in this thesis, is based on a rather different philosophy: it is considered that for many applications, where system parameters are constant or slowly varying, the long term steady-state performance is of greater importance than the short term start-up performance. Hence it is better to trade off transient - rather than long term - performance against complexity of the algorithm. This thesis develops a novel class of one step control laws which may be combined with the self-tuning regulator (Åström & Wittenmark,1973) in

such a way as to enhance its steady-state performance whilst retaining its essential simplicity.

The control laws corresponding to these one-step cost functions are shown to minimise the variance of the output (ϕ) of an auxiliary system; this is achieved by setting the least-squares prediction (ϕ^*) of ϕ to zero at each stage. It is shown that the auxiliary system may be written in the form:

$$\begin{aligned}\phi(t+k) &= \alpha_0 y(t) + \alpha_1 y(t-1) + \ldots + \alpha_n y(t-n) \\ &+ \beta_0 u(t) + \beta_1 u(t-1) + \ldots + \beta_n u(t-n) \\ &+ \gamma_0 w(t) + \gamma_1 w(t-1) + \ldots + \gamma_n w(t-n) \\ &- c_1 \phi^*(t+k-1) + c_2 \phi^*(t+k-2) + \ldots + c_n \phi^*(t+k-n) \\ &+ \epsilon^*(t+k) \qquad (1.4.2)\end{aligned}$$

where ϵ^* is a moving average process of order k-1 and given by:

$$\epsilon^*(t+k) = \psi(t+k) + e_1 \psi(t+k-1) + \ldots + e_n \psi(t+1) \qquad (1.4.3)$$

As in equation 1.4.2), due to lack of measurement noise, the output (y), input (u) and set point (w) are known quantities at time t; the corresponding terms are thus linear in unknown quantities. It is shown that the remaining non-linear terms - involving ψ and ϕ - can be ignored without affecting long term performance; the coefficients $\alpha_0 - \alpha_n$, $\beta_0 - \beta_n$, and are estimated using a linear estimation algorithm.

This useful behaviour is because the terms in ψ (1.4.3) are uncorrelated with y, u, and w as used in the the estimation, and because the control law is such as to set the current best estimate of ϕ^* to zero at each stage. Hence, although the transient behaviour of the simplified estimator is suboptimal, the long term performance is not affected.

The optimal control law based on parameter estimates would not only depend on the estimates themselves, but also on the estimate covariances or higher order statistics if the estimation were non-linear. If, however the system parameters are constant, the estimates will improve with time and the corresponding covariances decrease. Hence, in the long term, the control law would approximate to the certainty equivalent (Patchell & Jacobs,1971) control law based on the estimates alone. It is thus possible to trade offshort term performance against complexity by using the certainty equivalent control throughout; this also has the advantages of avoiding turn-off (Hughes & Jacobs,1974), and the need for a-priori knowledge of disturbance variance.

The development of the new one-step control laws is considered in chapter 3; the properties of the corresponding self-tuning controllers are discussed in chapter 4, and a number of simulated examples are given.

Chapter 2

Steady-state optimal control.

2.1 Introduction

In this chapter, the systems considered in section 1.2 are taken to have known parameters, and on the basis of these parameters the optimal steady-state control law is derived. Such theory is useful not only per se, but also for giving ultimate performance limits for steady-state control - thus providing a yardstick for the suboptimal control algorithms of later chapters.

As in section 1.3, the optimal control is taken to be that which minimises the cost function:

$$V = \underset{t_o \to \infty}{\text{Lt.}} \frac{1}{2t_o} \int_{-t_o}^{t_o} E\left\{ \left[y(t+T) - w(t) \right]^2 + \lambda u^2(t) \right\} dt \qquad (2.1.1)$$

for the system of Fig 2.1.1.

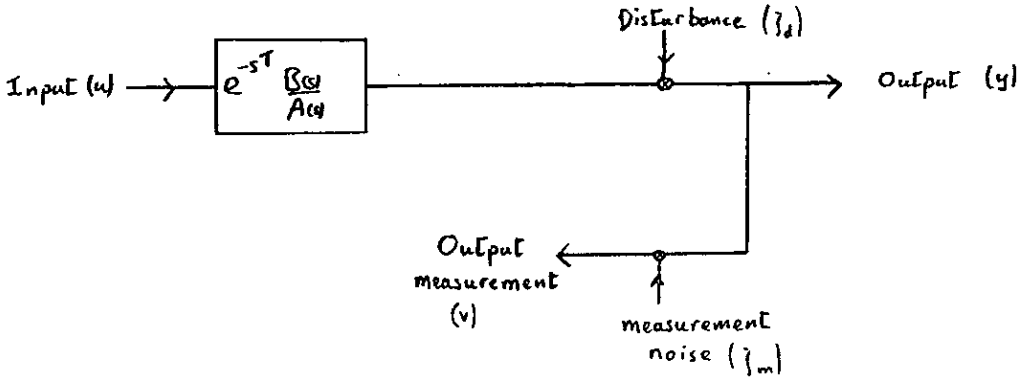

Fig 2.1.1 The system.

The signal w is the desired output or setpoint, and T the system time-delay.

As the system is linear, superposition may be used to rewrite the system of Fig 2.1.1 as the linear combination of the systems of Figs 2.1.2 & 3.†

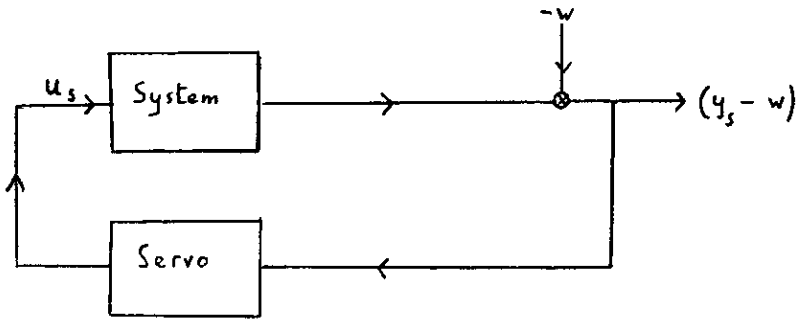

Fig 2.1.2 For the servo problem.

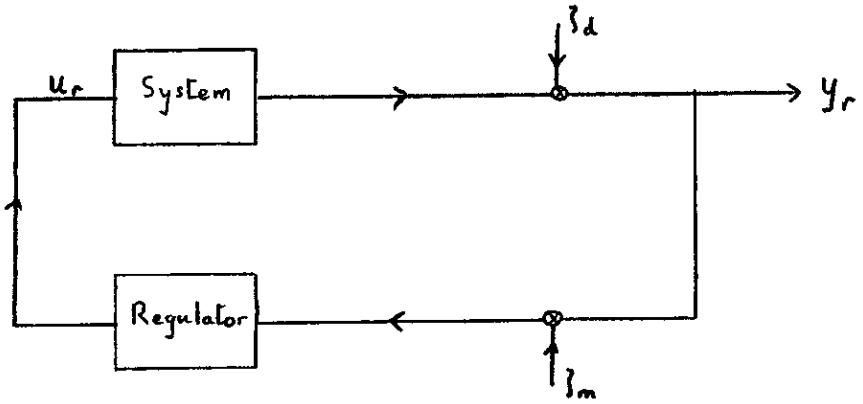

Fig 2.1.3 For the regulator problem.

Where:

$$y = y_s + y_r$$

$$u = u_s + u_r \qquad (2.1.2)$$

From Fig 2.1.2, it follows that u_s and y_s are functions

† This argument is expanded in Note 1, p236.

of w only; from Fig 2.1.3 it follows that u_r and y_r are functions of ζ_d and ζ_m only. The known set point w is taken to be independent of the zero mean stochastic processes ζ_d and ζ_m; hence, using (2.1.2) and this independence of y_s, u_s and y_r, u_r, the cost function (2.1.1) may be rewritten as:

$$V = V_s + V_r$$

where V_s and V_r are independent and given by:

$$V_s = \underset{t_o \to \infty}{\text{Lt.}} \frac{1}{2t_o} \int_{-t_o}^{t_o} E\left\{(y_s(t+T) - w(t))^2 + u_s^2(t)\right\} dt \qquad (2.1.3)$$

$$V_r = \underset{t_o \to \infty}{\text{Lt.}} \frac{1}{2t_o} \int_{-t_o}^{t_o} E\left\{y_r^2(t+T) + u_r^2(t)\right\} dt \qquad (2.1.4)$$

It follows that the control problem corresponding to (2.1.1) naturally divides into a servo problem (2.1.3) and a regulator problem (2.1.4). It will be shown that the solutions to these two problems are formally similar, and that the resultant control laws naturally combine into a control law for the complete problem. This control law cannot, in general, be written as feedback of (w - v), but rather it operates on -v and w in different ways; hence some previous solutions to this problem (eg Newton et al.,1957; Youla et al., 1976) which force the control law to operate on (w - v)

only, are not optimal.

There are two main methods of solution of the regulator problem: the transfer function - frequency domain methods of Newton et al. (1957), and the state space - time domain methods of Kalman (Athens,1971; Kwakernaak & Sivan,1972). However, for the class of systems described in section 1.2, these previous methods do not provide a complete solution: the Newton methods are inapplicable to unstable systems; the Kalman methods were not developed for continuous systems with a time-delay. (Some work on deterministic time-delay systems appears in Ross,1971, and Alekal et al., 1971)

The difficulty with the use of the frequency domain method of Newton et al. (1957) is that cancelling factors corresponding to the system poles appear in the closed loop system - hence if the system is unstable, these 'hidden modes' lead, in practice, to an unbounded output in closed loop. Lim & Bankoff (1970) attempted to overcome this problem by initially stabilising the system; by modifying the method to include this stabilising control signal, they derived a regulator giving rise to stable 'hidden modes'. It is considered that this method is theoretically artificial and computationally cumbersome. More recently the problem of cancelling modes has been overcome by Youla et al.

(1976); but by restricting the structure of the control law, their inclusion of set points is suboptimal. (See example 2.14.1). The method given here (Gawthrop, 1976) was derived independently of that of Youla et al., and is developed from a different point of view. Realising that the 'hidden modes' arising in the method of Newton are symptomatic of inappropriate system modelling, the method given here provides an intuitively appealing solution of the problem.

The regulator problem is also solved by state space methods. It is shown that the problem separates (in the strict sense) into a Kalman filter, a state predictor, and state prediction feedback corresponding to a certainty equivalent problem but with no time-delay. It is noted that the same 'hidden modes' can arise, but the usual formulation of the problem prevents this.

Using a relationship between spectral factorisation and the algebraic Riccati equation similar to that of MacFarlane (1971), the state space control law is written in transfer function form, and is found to be identical with that derived by the frequency domain method. This equivalence is demonstrated only for the continuous-time case; however it is conjectured that the discrete-time analogue of this result is

also true - this latter equivalence would contain previous results (Caines,1972; Hughes,1973; Watson, 1976),relating the Åström minimum variance regulator and the Kalman linear regulator, as a special case.

Finally these results are used to examine the question of when it is possible to treat measurement noise as a system disturbance, that is treat the measured system output, including measurement noise, as the actual system output. This question is of practical significance because sometimes, due to lack of a priori information, it is not possible to experimentally distinguish between the two noise processes.

2.2. THE SYSTEM AND DISTURBANCE MODEL

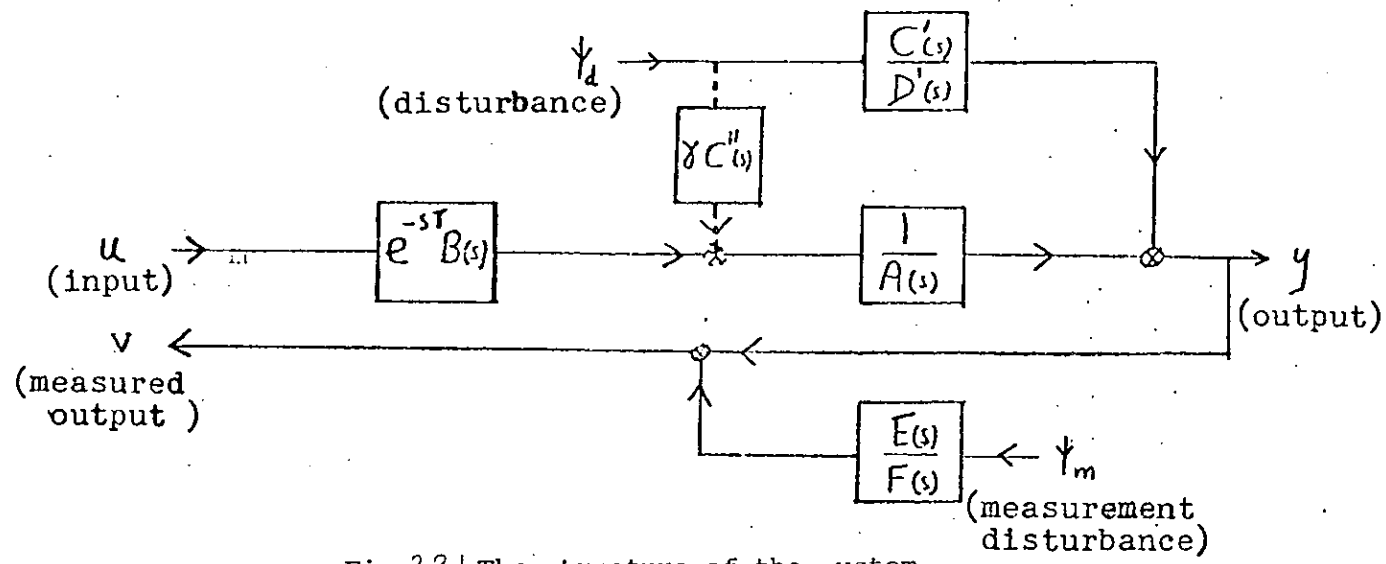

Fig.2.2.1 The structure of the system

The system (Fig.2.2.1) is assumed to be representable by a rational transfer function cascaded with a pure time delay. Disturbances are stationary and represented by the outputs of rational transfer functions driven by white noise; these disturbances, having been multiplied by the relevant post-disturbance system elements, are lumped together at the system output. The measured system output is corrupted with coloured noise represented in the same manner as the disturbances, and uncorrelated with them.

That is:

$$\bar{y}(s) = e^{-sT} \frac{B(s)}{A(s)} \bar{u}(s) + \frac{C'(s)}{D'(s)} \bar{\psi}_d(s) \qquad (2.2.1)$$

$$\bar{v}(s) = \bar{y}(s) + \frac{E(s)}{F(s)} \bar{\psi}_m(s) \qquad (2.2.2)$$

$\bar{u}$, $\bar{v}$, $\bar{y}$ are the Laplace transformed system input, measured output and output respectively. A(s), B(s) etc., are polynomials in the Laplace operator s, of order n_a, n_b etc., and with no cancelling factors. The orders are restricted by:

$$n_c < n_d, \; n_e < n_f \qquad (2.2.3)$$

The system time delay is T (≥ 0)

The white noise processes ψ_d and ψ_m are defined by:

$$E\{\psi_d(t)\ \psi_d(t+\tau)\} = \sigma_d\ \delta(\tau)$$

$$E\{\psi_m(t)\ \psi_m(t+\tau)\} = \sigma_m\ \delta(\tau) \qquad (2.2.4)$$

$$E\{\psi_m(t)\ \psi_d(t+\tau)\} = 0$$

where $E\{.\}$ denotes the expectation operator and $\delta(\tau)$ the Dirac delta function. Note that conditions (2.2.3) ensure that no white noise process appears at the system output.

As the measurement noise is stationary, all roots of $F(s)$ must have negative real parts. The stationary disturbance processes may, however, have been passed through the system, and thus when lumped together at the system output, $D'(s)$ may have roots corresponding to those of $A(s)$, which may have positive real parts. If $D'(s)$ does not have roots corresponding to all roots of $A(s)$ with positive real parts, the model has two serious defects. Firstly, if the system model has no input applied, the corresponding unstable modes will not appear at the system output – an unlikely situation in practice. Secondly, the resultant optimum control law designed by frequency - domain techniques will contain corresponding cancelling factors, and thus be infinitely sensitive to parameter variation.

To derive a more realistic model, let the modes of the system be excited by an infinitesimal amount of the disturbance process, as shown dotted in Fig. (2.2.1). The resultant output disturbance is then:

$$\overline{y}_d = \left(\frac{C'}{D'} + \frac{\gamma C''}{A}\right) \overline{\psi}_d$$

$$= \frac{C'\widetilde{A} + \gamma C''\widetilde{D}'}{D'\widetilde{A}}\ \overline{\psi}_d \qquad (2.2.5)$$

where $\widetilde{A}$ is a product of factors of A not common to D', $\widetilde{D}'$ is a product of factors of D' not common to A, and C'' is taken to contain no factors of $A(s)$, $A(-s)$ or $D(s)$. In the sequel γ is allowed to tend to zero, but for the present C' and D' are replaced by:

$$C \triangleq C'\widetilde{A} + \gamma C''\widetilde{D}' \qquad (2.2.6)$$

$$D \triangleq D'\widetilde{A} \qquad (2.2.7)$$

It is noted that if $\gamma \neq 0$, C and D have no common factors. It follows that the model obtained by letting the disturbance slightly excite the system modes has an output disturbance transfer function containing all unstable modes of the system. This new model has neither of the disadvantages associated with the previous model.

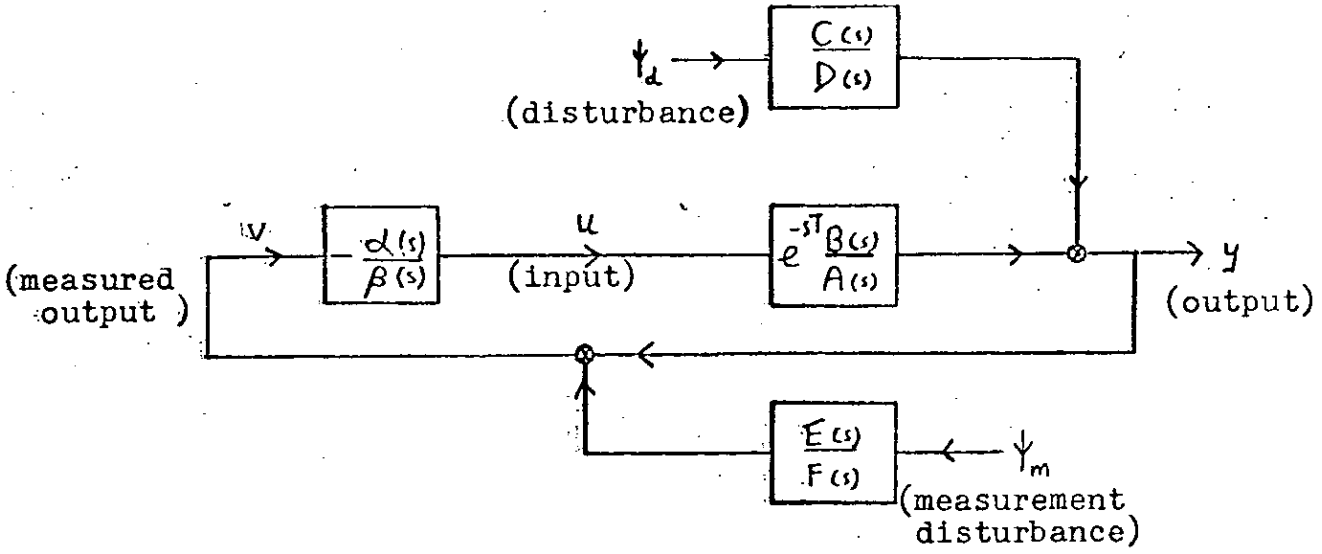

Fig.2.3.1 The modified system and the feedback controller

The regulator problem considered here, Fig.2.3.1, is to find a non-anticipatory control law $\frac{\alpha(s)}{\beta(s)}$ where $\alpha(s)$ and $\beta(s)$ are constant coefficient polynomials in the Laplace operator s. The criterion of optimality is the minimisation of the mean square output y, with a constraint on the mean square control input u, over an infinite time interval. Following Newton et al (1957), this problem reduces to minimising the value function:

$$V = \underset{t_o \to \infty}{\text{Lt.}} \frac{1}{2t_o} \int_{-t_o}^{t_o} E\left\{y^2(t+T) + \lambda u^2(t)\right\} dt \qquad (2.3.1)$$

with respect to all such control laws. The determination of the value of V corresponding to a given constraint on the control u is discussed by Newton et al and Lim and Bankoff (1970) so the discussion here is restricted to minimising (2.3.1) for any given value of λ .

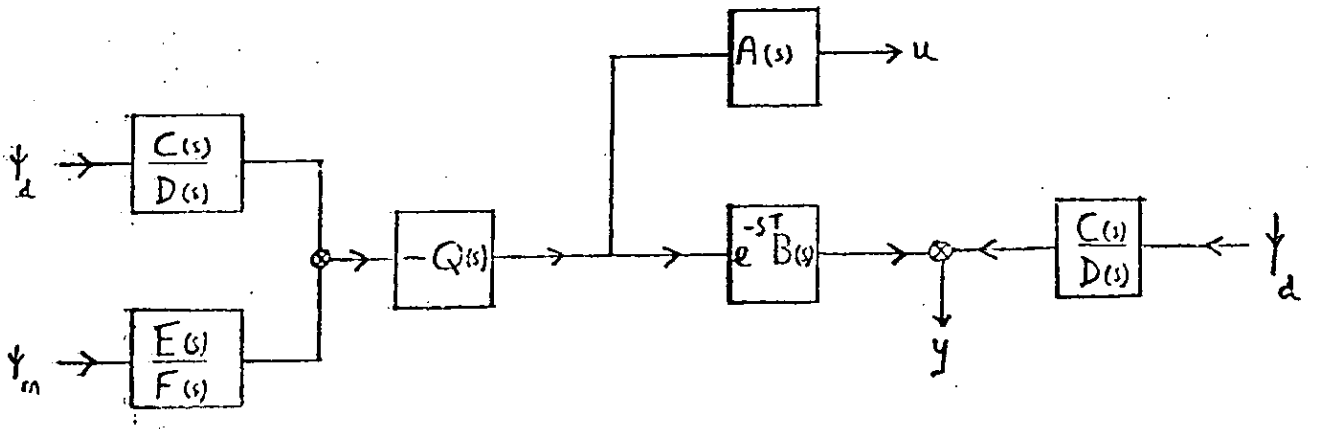

Fig.2.3.2 The equivalent cascade system

Following Newton et al (1957), the system is rewritten (Fig.2.3.2) as:

$$\bar{y} = \frac{C}{D}(1-e^{-sT_{BQ}})\,\bar{\Psi}_d - \frac{E}{F}\,e^{-sT_{BQ}}\,\bar{\Psi}_m \qquad (2.3.2)$$

$$\bar{u} = -\frac{C}{D}\,QA\,\bar{\Psi}_d - \frac{E}{F}\,QA\,\bar{\Psi}_m \qquad (2.3.3)$$

Q is the equivalent cascade compensation and given by:

$$Q = \frac{\alpha}{\beta A + e^{-sT}\,\alpha B} \qquad (2.3.4)$$

The method of solution used here is to define:

Q^o as the optimum cascade compensation,

Q' as an arbitrary cascade compensation,

then set $Q = Q^o + \epsilon Q'$. (2.3.5)

Having determined the constraints on Q^o and Q' imposed by the constraints on α/β, a variational argument in the frequency domain is used to find Q^o subject to these constraints. The corresponding feedback compensation $\frac{\alpha}{\beta}$ is then obtained using (2.3.4).

CONSTRAINTS ON THE CASCADE COMPENSATION, Q.

The solution of the regulator problem is dependent on the constraints imposed on Q^o and Q' by the problem formulation. Firstly, as the feedback compensation $\frac{\alpha}{\beta}$ is required to be physically realisable and the system itself is also realisable by definition, Q and hence Q^o and Q' must also be realisable. Secondly, the resultant closed loop system must be stable, that is both the system output y (2.3.2) and input u (2.3.3) must be bounded.

As the roots of D with positive real parts, which correspond to roots of A (section 2.2), are cancelled by the numerator factor A in (2.3.3), this equation imposes the condition that Q^o and Q' have no poles with positive real parts. Equation (2.3.2) together with (2.3.5) gives after slight rearrangement:

$$\bar{y} = \bar{y}^o + \epsilon \bar{y}' \qquad (2.3.6)$$

$$\text{where } \bar{y}^o = \frac{C}{D}(1 - e^{-sT}BQ^o)\,\bar{\psi}_d - \frac{E}{F}\,Q^o\,e^{-sT}\,B\,\bar{\psi}_m \qquad (2.3.7)$$

$$\text{and } \bar{y}' = -\frac{C}{D}\,e^{-sT}BQ'\,\bar{\psi}_d - \frac{E}{F}\,Q'\,e^{-sT}B\,\bar{\psi}_m \qquad (2.3.8)$$

If y is to be bounded for all ϵ, y^o and y' must be bounded, that is the expressions (2.3.7) and (2.3.8) must contain no poles with positive real parts. Now:

$$1 - e^{-sT}BQ = 1 - \frac{e^{-sT}B\alpha}{\beta A + e^{-st}\alpha B}$$

$$= \frac{\beta A}{\beta A + e^{-sT}\alpha B} \qquad (2.3.9)$$

The expression (2.3.9) has zeros corresponding to roots of A, and thus to the roots of D with positive real part. As F has no roots with positive real part (section 2.2), (2.3.7) and (2.3.9) require only that Q^o has no pole with a positive real part.

The first term of (2.3.8) imposes the additional condition that Q' have zeros corresponding to roots of D (or A) with positive real parts. To summarise, the constraints on Q^o and Q' are:

Constraint 1 - Q^o and Q' have no poles with positive real parts.

Constraint 2 - Q' has zeros corresponding to all roots of A with positive real parts.

2.4. DERIVATION OF THE OPTIMUM CASCADE COMPENSATION

From Parsevals theorem the value function V (2.3.1) becomes:

$$V = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} (\phi_{yy} + \lambda\phi_{uu})ds \qquad (2.4.1)$$

where ϕ_{yy} and ϕ_{uu} are the spectral densities of y and u respectively.

As ψ_m and ψ_d are uncorrelated (2.2.4), and y and u are bounded (section 2.3), (2.3.2) and (2.3.3) substituted into (2.4.1) gives:

$$V = \frac{\sigma_d}{2\pi j}\int_{-j\infty}^{j\infty} \frac{C(s)C(-s)}{D(s)D(-s)} \left[(1-e^{-sT}B(s)Q(s)).(1-e^{sT}B(-s)Q(-s)) + \lambda A(s)Q(s)A(-s)Q(-s)\right]ds$$

$$+ \frac{\sigma_m}{2\pi j}\int_{-j\infty}^{j\infty} \frac{E(s)E(-s)}{F(s)F(-s)} \left[B(s)Q(s)B(-s)Q(-s) + \lambda A(s)Q(s)A(-s)Q(-s)\right] ds. \qquad (2.4.2)$$

Setting $Q = Q^o + \epsilon Q'$ (2.3.5) and evaluating the first derivative of V with respect to ϵ at $\epsilon = 0$ gives;

$$\left.\frac{dV}{d\epsilon}\right|_{\epsilon=0} = \frac{2\sigma_d}{2\pi j}\int_{-j\infty}^{j\infty} \frac{C(s)C(-s)}{D(s)D(-s)} \left[-\left(1-e^{-sT}B(s)Q^o(s)\right)e^{sT}B(-s)Q'(-s) + \lambda A(s)Q^o(s)A(-s)Q'(-s)\right] ds$$

$$+ \frac{2\sigma_m}{2\pi j}\int_{-j\infty}^{j\infty} \frac{E(s)E(-s)}{F(s)F(-s)} \left[B(s)Q^o(s)B(-s)Q'(-s) + \lambda A(s)Q^o(s)A(-s)Q'(-s)\right] ds. \qquad (2.4.3)$$

Setting $\mu = \frac{\sigma_m}{\sigma_d}$ (2.4.4)

and rearranging, (2.4.3) becomes:

$$\left.\frac{dV}{d\epsilon}\right|_{\epsilon=0} = \frac{2\sigma_d}{2\pi j}\int_{-j\infty}^{j\infty} Q'(-s)\left[\left(\frac{C(s)C(-s)}{D(s)D(-s)} + \frac{\mu E(s)E(-s)}{F(s)F(-s)}\right)\left(B(s)B(-s) + \lambda A(s)A(-s)\right)Q^o(s) - \frac{C(s)C(-s)}{D(s)D(-s)}\, e^{sT}B(-s)\right] ds \qquad (2.4.5)$$

At this point the technique of spectral factorisation (Brockett, 1970) is introduced. Terms like A(s) A(-s) can be written as:

$$A(s)\,A(-s) = a_o \prod (s-a_i)\; a_o \prod (-s-a_i)$$

$$= a_o^2 \prod (-s^2 + a^2_i) \qquad (2.4.6)$$

where $a_i; 1 < i < n_a$ are the roots of A(s) and a_o is a constant. That is such terms, and sums of such terms, are polynomials in s^2, and thus their roots are in pairs of same modulus but opposite sign. Hence the following relationships uniquely define P(s) and R(s):

$$P(s)P(-s) = B(s)B(-s) + \lambda A(s)A(-s) \qquad (2.4.7)$$

$$R(s)R(-s) = C(s)C(-s)F(s)F(-s) + \mu E(s)E(-s)D(s)D(-s) \qquad (2.4.8)$$

where $P(s)$ and $R(s)$ have no roots with positive real parts.

Substituting (2.4.7) and (2.4.8) into (2.4.5):

$$\left.\frac{dV}{d\epsilon}\right|_{\epsilon=0} = \frac{2\sigma_d}{2\pi j}\int_{-j\infty}^{j\infty} Q'(-s)\left[\frac{R(s)R(-s)P(s)P(-s)}{D(s)D(-s)F(s)F(-s)}\ Q^o(s) - \frac{C(s)C(-s)e^{sT}B(-s)}{D(s)D(-s)}\right]ds \qquad (2.4.9)$$

In a similar manner it follows that:

$$\left.\frac{d^2V}{d\epsilon^2}\right|_{\epsilon=0} = \frac{2\sigma_d}{2\pi j}\int_{-j\infty}^{j\infty} Q'(s)Q'(-s)\frac{R(s)R(-s)P(s)P(-s)}{D(s)D(-s)F(s)F(-s)}\ ds \qquad (2.4.10)$$

The argument of the integral in (2.4.10) is of the form $G(s)\,G(-s)$ and thus, as σ_d is positive (2.2.3), $\left.\frac{d^2V}{d\epsilon^2}\right|_{\epsilon=0}$ is positive. If a stationary point of $V(\epsilon)$ is found at $\epsilon = 0$, it follows that it must be a minimum.

Rearranging (2.4.9), the right hand side becomes:

$$\frac{2\sigma_d}{2\pi j}\int_{-j\infty}^{j\infty} Q'(-s)\frac{R(-s)P(-s)}{D(-s)F(-s)}\left[\frac{R(s)P(s)}{D(s)F(s)}\ Q^o(s) - e^{sT}\frac{B(-s)C(s)C(-s)F(-s)}{P(-s)R(-s)D(s)}\right]ds \qquad (2.4.11)$$

At this stage it would seem possible to set $\left.\frac{dV}{d\epsilon}\right|_{\epsilon=0}$ to zero by choosing $Q^o(s)$ such that the argument of the integral in (2.4.11) was zero. However, this yields a form of Q that is unrealisable, in the sense that it anticipates the future. To avoid this effect, the term:

$$\bar{g}(s) \triangleq e^{sT}\frac{B(-s)C(s)C(-s)F(-s)}{P(-s)R(-s)D(s)} \qquad (2.4.12)$$

is divided into realisable and unrealisable components. That is:

$$\bar{g}(s) = \bar{g}_r(s) + \bar{g}_u(s)$$

where $g_r(t) = 0 \quad t < 0$

and $g_u(t) = 0 \quad t \geq 0$ (2.4.13)

To perform this decomposition (2.4.12) is rewritten as:

$$\bar{g}(s) = e^{sT}\frac{I(s)}{D(s)} + e^{sT}\frac{J(-s)}{P(-s)R(-s)} \qquad (2.4.14)$$

It is now shown that the first term in (2.4.14) represents a time function zero for $t \leq -T$, and the second term a time function zero $t \geq -T$. Setting

$t' = -t$, $s' = -s$ the inverse Laplace transform of (2.4.14) becomes:

$$g(t) = g_1(t + T) + g_2(t' - T) \tag{2.4.15}$$

$$\text{where } g_1(t + T) = \int_{r_1 - j\infty}^{r_1 + j\infty} e^{s(t + T)} \frac{I(s)}{D(s)} \, ds \tag{2.4.16}$$

$$\text{and } g_2(t' - T) = \int_{r_2 - j\infty}^{r_2 + j\infty} e^{s'(t' - T)} \frac{I(s')}{P(s')R(s')} \, ds' \tag{2.4.17}$$

The constants r_1 and r_2 lie to the right of all singularities. If $t' < T$, that is $t > -T$ the integral (2.4.17) vanishes around the right hand infinite semicircle, and as by definition no poles are enclosed by this contour and the path of integration, $g_2(t'-T)$ is zero if $t > -T$. The realisable part of (2.4.14) thus corresponds to that part of $g_1(t + T)$ occuring after $t = 0$. That is:

$$\bar{g}_r(s) = \int_0^\infty g_1(t + T) \, e^{-st} dt \tag{2.4.18}$$

The inverse Laplace transform (2.4.16) and the Laplace transform (2.4.18) are conveniently evaluated by rewriting $\frac{I(s)}{D(s)}$ as a partial fraction expansion. For example if D has no multiple roots, set:

$$\frac{I(s)}{D(s)} = \sum_1^n d \frac{r_i}{s - d_i} \tag{2.4.19}$$

where d_i are the roots of $D(s)$. Equation (2.4.16) then gives:

$$g_1(t+T) = \sum_1^n d \, r_i \, e^{d_i (t+T)} \tag{2.4.20}$$

Equation (2.4.18) becomes:

$$\bar{g}_r(s) = \sum_1^n d \frac{r_i e^{d_i T}}{s - d_i} \tag{2.4.21}$$

In general, defining:

$$\frac{I_r(s)}{D(s)} \triangleq \bar{g}_r(s) \tag{2.4.22}$$

and $$e^{sT}\frac{I_u(s)}{D(s)} \triangleq \frac{e^{sT}I(s) - I_r(s)}{D(s)} \qquad (2.4.23)$$

the decomposition becomes:

$$e^{sT}\frac{B(-s)\,C(s)\,C(-s)\,F(-s)}{P(-s)\,R(-s)\,D(s)} = \frac{e^{sT}I_u(s) + J_r(s)}{D(s)} + \frac{e^{sT}J(-s)}{P(-s)R(-s)} \qquad (2.4.24)$$

This decomposition is depicted in Fig. 2.4.1.

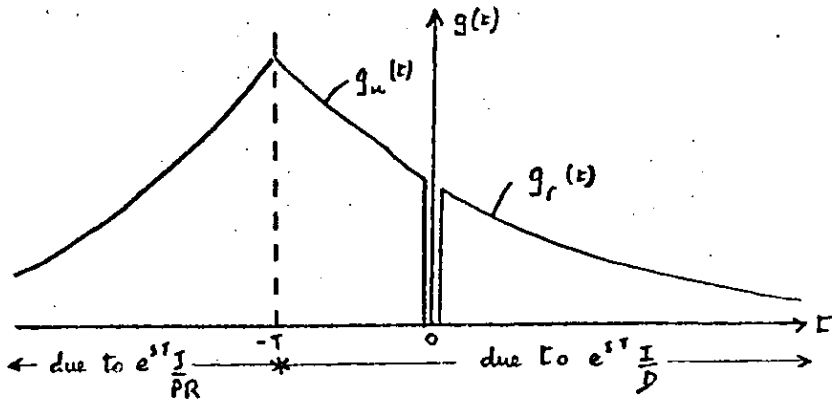

Fig.2.4.1 The realisability decomposition

A realisable Q^o can now be obtained by setting:

$$Q^o(s) = \frac{I_r(s)\,F(s)}{P(s)\,R(s)} \qquad (2.4.25)$$

It remains to show that this is the optimum realisable Q^o.

Substituting (2.4.25) in (2.4.11):

$$\left.\frac{dV}{d\epsilon}\right|_{\epsilon=0} = \frac{2\sigma_d}{2\pi j}\int_{-j\infty}^{j\infty} Q'(-s)\frac{R(-s)P(-s)}{D(-s)F(-s)}\left[\frac{e^{sT}I_u(s)}{D(s)} + \frac{e^{sT}J(-s)}{P(-s)R(-s)}\right] ds \qquad (2.4.26)$$

Condition 1 (section 2.3) implies that $Q'(-s)$ has no poles in the left hand plane, and condition 2 implies that $Q'(-s)$ has zeros corresponding to all left hand plane roots of $D(-s)$. By assumption $F(-s)$ has no left hand plane roots, and by construction neither do $P(-s)$ and $R(-s)$. By definition, $e^{sT}\frac{I_u}{D}$ is unrealisable, and thus its poles corresponding to realisable impulse responses must have zero residue, that is

$$I_u(d_i) = 0 \qquad (2.4.27)$$

d_i roots of $D(s)$.

Firstly consider the case T=0. From the conditions (2.2.3) and the definitions of P(2.4.7), R(2.4.8), Q(2.3.4), it follows that the numerator of the

of the argument of (2.4.26) is of degree two less than the numerator, and thus the integral vanishes around the left hand infinite semicircle. If T>0 this is clearly still true. As it has also been concluded that the left hand plane contains no poles with non-zero residue, (2.4.26) must vanish.

To summarise, it has been shown that with Q^o given by (2.4.25), $\frac{\partial V}{\partial \varepsilon}\Big|_{\varepsilon=0} = 0$, and $\frac{\partial^2 V}{\partial \varepsilon^2}\Big|_{\varepsilon=0} > 0$. It is thus concluded that Q^o is the optimum cascade compensation.

If the time-delay (T) is greater than zero, the integral in equation (2.4.26) vanishes whatever the (finite) degrees of the polynomials involved. Hence a formal solution exists even if $n_c \geqslant n_d$ or $n_e \geqslant n_f$ and white noise is present in the system. This point is not pursued further here as white noise does not appear in physical systems; it is taken up again in section 3.2 where attention is given to fictitious systems which may involve white noise processes and their derivatives.

2.5. THE OPTIMUM FEEDBACK CONTROL LAW

In section 2.4 the optimum cascade compensation was derived. The corresponding feedback compensation is now deduced. From (2.3.4) and (2.4.25), Q^o can be realised by setting:

$$\alpha = I_r F \tag{2.5.1}$$

$$\beta A = PR - e^{-sT} I_r FB \tag{2.5.2}$$

It is noted here that the cascade compensation could also be realised by using feedback compensation in the form:

$$\frac{\alpha'}{\beta'} = \frac{I_r FA}{PR - e^{-sT} I_r FB} \tag{2.5.3}$$

However (2.5.3) leads to a cancelling factor A in the denominator and numerator of Q^o. In practice if A contained roots with positive real parts, that is the system was unstable, this implies an infinitely sensitive control law. This is the source of the trouble in the basic method of Newton et al (1957).

It is now shown that because of the choice of disturbance model in section 2.2, the right hand side of (2.5.2) has a factor A, and so the control law can be realised in the form of (2.5.1) and (2.5.2), thus avoiding the problems associated with (2.5.3). From (2.4.24):

$$e^{sT} B(-s)\, C(s)\, C(-s)\, F(-s) = \left(e^{sT} I_u(s) + I_r(s)\right) P(-s)\, R(-s) + e^{sT} D(s)\, J(s) \tag{2.5.4}$$

In section 2.2, D(s) was forced to contain all roots of A(s) and from (2.4.23) $I_u(s)$ also contains all roots of A(s). Hence:

$$e^{a_i T} B(-a_i)\, C(a_i) C(-a_i) F(-a_i) = I_r(a_i) P(-a_i) R(-a_i) \tag{2.5.5}$$

but from the definitions of P and R (2.4.7), (2.4.8), noting that D(s) contains all roots of A(s) (Section 2.2):

$$P(a_i)P(-a_i) = B(a_i)B(-a_i) \tag{2.5.6}$$

$$R(a_i)R(-a_i) = C(a_i)C(-a_i)F(a_i)F(-a_i) \tag{2.5.7}$$

Substituting (2.5.6) and (2.5.7) into (2.5.5); and rearranging:

$$P(a_i)R(a_i) - e^{-a_i T} I_r(a_i)F(a_i)B(a_i) = 0 \qquad (2.5.8)$$

Comparing (2.5.8) and (2.5.2) it follows that the right hand side of (2.5.2) contains all roots of A(s). By considering multiple roots as the limiting case of nearby roots it also follows that the right hand side of (2.5.2) contains roots with the same multiplicity as those of A(s). It is concluded that β may be found from (2.5.2) by dividing the right hand side exactly by A(s).

Having now shown that the optimum cascade compensation Q^o can be realised directly, without cancelling factors, by the use of an equivalent feedback control law, it is now possible to derive expressions for the system input u and output y in closed loop.

Substituting (2.4.25) into (2.3.2), (2.3.3) and (2.3.4),

$$\bar{y} = \frac{C\beta A}{D\,PR}\bar{\psi}_d \quad -\frac{E}{F}e^{-sT}\frac{BFI_r}{PR}\bar{\psi}_m \qquad (2.5.9)$$

$$\bar{u} = -\frac{C}{D}\frac{\alpha A}{PR}\bar{\psi}_d \quad -\frac{E}{F}\frac{\alpha A}{PR}\bar{\psi}_m \qquad (2.5.10)$$

The leading terms of (2.5.9) and (2.5.10) contain cancelling factors corresponding to A(s). Hence if the system were actually simulated in the form of Fig. 2.3.1 and A(s) had roots with positive real parts, an unstable closed loop system would result. However, as described in section 2.2, the unstable modes of $\frac{C}{D}$ only arise because it is mathematically convenient to lump disturbances at the output, involving the multiplication of stable disturbance processes by possibly unstable post disturbance elements. Thus the system itself does not physically contain these cancelling factors, and so (2.5.9) and (2.5.10) may be safely written as:

$$\bar{y} = \frac{C\beta}{\tilde{D}\,PR}\bar{\psi}_d \quad -e^{-sT}\frac{BI_rE}{PR}\bar{\psi}_m \qquad (2.5.11)$$

$$\bar{u} = \frac{-C\alpha}{\tilde{D}PR}\bar{\psi}_d \quad -\frac{AI_rE}{PR}\bar{\psi}_m \qquad (2.5.12)$$

where $\tilde{D}$ is defined by:

$$D = \tilde{D}A \qquad (2.5.13)$$

2.6. COMPUTATION OF THE CONTROL LAW

In this section certain simplifications in the computation of the control law are introduced, and the problems associated with the realisation of e^{-sT} discussed. The equations specifying control are then, for convenience, collected together.

The realisability relations (2.4.18) and (2.4.16) involve the partial fraction expansion of the term 2.4.24:

$$\frac{B(-s)C(s)C(-s)F(-s)}{P(-s)R(-s)D(s)} \tag{2.6.1}$$

The only terms required , however, are those corresponding to D(s). Now from (2.4.8)

$$R(d_i)R(-d_i) = C(d_i)C(-d_i)F(d_i)F(-d_i)$$

where d_i is a root of D(s).

Hence for the purposes of the partial fraction expansion (2.6.1) may be replaced by:

$$\frac{B(-s)R(s)}{P(-s)F(s)D(s)} \tag{2.6.2}$$

If γ in equation (2.2.5) is set to zero, the spectral factorisation (2.4.8) simplifies to:

$$R(s) = R'(s)\,\tilde{A}^{+}(s)$$

$$\text{where } R'(s)R'(-s) = C'(s)C'(-s)F(s)F(-s) + \mu E(s)E(-s)D'(s)D'(-s) \tag{2.6.3}$$

$$\text{and } \tilde{A}^{+}(s)\,\tilde{A}^{+}(-s) = \tilde{A}(s)\tilde{A}(-s)$$

$R'(s)$, $\tilde{A}^{+}(s)$ do not contain roots with positive real parts.

Another simplification occurs if D=A. In this case the $n_d=n_a$ coefficients of $I_r(s)$ may be computed from the n_a equations associated with the fact that the right hand side of (2.6.1) must contain the n_a roots of A(s).

The function e^{-sT} cannot be exactly represented by a rational function. Hence in general if a control law in the form of a rational function of s is required, the optimum control law can only be approximated. However it is noted that in a particular case, the optimum

controller contains e^{-sT} explicitly and thus can be realised using a delay line. This occurs if R(s) has a factor A(s), as is the case if D(s) has no roots common to A(s), and A(s) has no roots with positive real parts (2.6.3). This implies from (2.5.8) that I_r also has a factor A(s) and so (2.5.2) becomes:

$$\beta = PR' - e^{-sT} I_r' BF \tag{2.6.4}$$

where $R = R'A$ and $I_r = I_r'A$

In general, however, e^{-sT} does not occur explicitly and it is suggested that it be approximated in the form:

$$\frac{G(s)}{H(s)} = \frac{1 + \sum g_i (sT)'}{1 + \sum h_i (sT)'} \doteq e^{-sT} \tag{2.6.5}$$

The 2n coefficients (2.6.5) may be, for instance, determined by making the approximation exact at n judiciously chosen complex values of s. The function (2.6.5) should be a close approximation for $s = j\omega$ over the bandwidth of u, and for $s = a_i$, a_i roots of A(s). The equations for the approximate feedback law then become:

$$\alpha = I_r FH \tag{2.6.6}$$

$$\beta A = PRH - I_r FBG \tag{2.6.7}$$

To summarise, the equations defining a rational approximation to the optimal control law are repeated.

a) Disturbance model conversion

The techniques of section 2.2 are applied to the original model (2.2.1) As γ tends to zero, (2.2.6) and (2.2.7) become:

$$C = C'\tilde{A} \tag{2.6.8}$$

$$D = D'\tilde{A} \tag{2.6.9}$$

b) Spectral factorisation

P(s) and R(s) are found from:

$$P(s)P(-s) = B(s)B(-s) + \lambda A(s)A(-s) \tag{2.6.10}$$

$$R(s)R(-s) = C(s)C(-s)F(s)F(-s) + \mu E(s)E(-s)D(s)D(-s) \qquad (2.6.11)$$

These factorisations may be accomplished by writing the right hand side as a polynomial in s^2 (2.4.6) and finding the roots of this polynomial. P (or R) is then found by forming a product of factors corresponding to the negative square roots of these roots, multiplying by a constant to satisfy (2.4.7) (or (2.4.8)). It may be convenient to replace (2.4.8) by (2.6.3) in some cases.

c) Realisability relation

The rational function (2.6.2) is expanded into partial fractions:

$$\frac{B(-s)R(s)}{P(-s)F(s)D(s)} = \Sigma \frac{r_i}{s-d_i} + \frac{J'(-s)}{F(s)P(-s)} \qquad (2.6.12)$$

It is not necessary to compute $J'(-s)$, and if D(s) has multiple roots, a more complicated expansion is necessary. I_r is then calculated from (2.4.18) and (2.4.22): in the case of distinct roots this becomes:

$$\frac{Ir}{D} = \Sigma \frac{r_i e^{d_i T}}{s-d_i} \qquad (2.6.13)$$

d) Approximate e^{-sT}

Find a rational function approximation to e^{-sT}:

$$\frac{G(s)}{H(s)} = \frac{1 + \Sigma g_i (sT)'}{1 + \Sigma h_i (sT)'} \doteq e^{-sT} \qquad (2.6.14)$$

This may be found as discussed above. A simple approximation is:

$$\frac{G(s)}{H(s)} = \frac{1 - 0.5\ sT}{1 + 0.5\ sT} \qquad (2.6.15)$$

In general the number n of coefficients necessary will depend on the ratio of T, the system time delay, to system time constant, and the degree of approximation required.

e) Calculate α and β

The approximate optimal control law α/β is computed from:

$$\alpha = I_r FH \qquad (2.6.16)$$

and

$$\beta A = PRH - GI_r FB \qquad (2.6.17)$$

Relation (2.6.7) implies algebraic long division of the right hand side by A(s). If the approximation to e^{-sT} is exact at roots of A(s), that is:

$$\frac{G(a_i)}{H(a_i)} = e^{-a_i T} \tag{2.6.18}$$

then the remainder will be zero.

These steps, a) - e), may be applied to all systems described by (2.2.1) without exception. However, if D=A, it may be more convenient to find I_r from the n_a equations:

$$P(a_i)R(a_i) - e^{-a_i T} I_r(a_i) F(a_i) B(a_i) = 0 . \tag{2.6.19}$$

If A(s) has multiple roots, the extra equations are given by setting derivatives of (2.6.19) to zero at the multiple roots.

Example 2.6.1

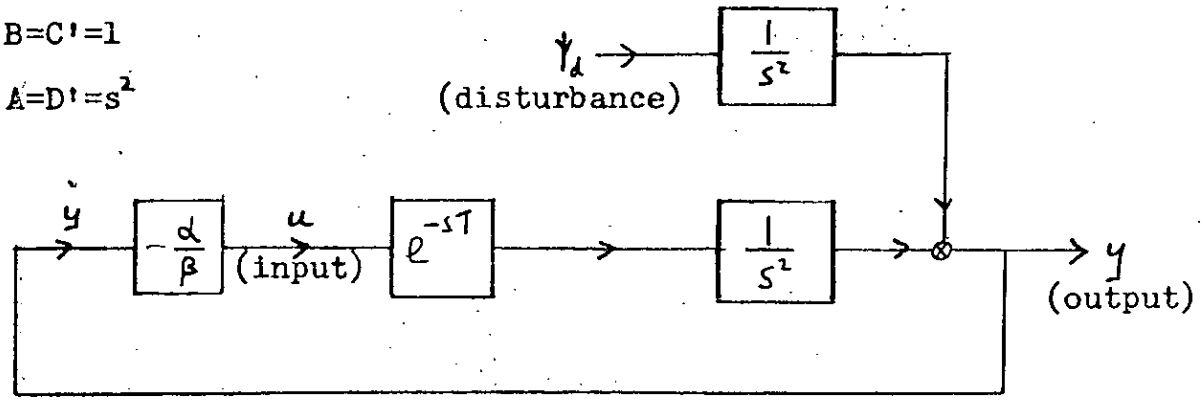

Fig.2.6.1 System for example 2.6.1

The system depicted in Fig.2.6.1 may be regarded as an inertia, rotating without friction, but disturbed by torques with a wide frequency spectrum. The control signal is delayed by a time T. Notice that this input disturbance has been transferred to the output to give a model in standard form.

a) Disturbance model conversion

As the system has a finite input disturbance, D contains all modes of A. Hence equations (2.6.8) and (2.6.9) leave the model unchanged.

b) Spectral factorisation

(2.6.10) becomes:

$$P(s)P(-s) = B(s)B(-s) + \lambda A(s)A(-s)$$
$$= 1 + \lambda s^4. \qquad (2.6.20)$$

The roots of (2.6.20) regarded as a polynomial in s^2 are:

$$s^2 = \pm j\lambda^{-\frac{1}{2}} ;$$

taking the square roots with negative real part:

$$P(s) = \left(\lambda^{\frac{1}{4}}s + \frac{(1+j)}{\sqrt{2}}\right)\left(\lambda^{\frac{1}{4}}s + \frac{(1-j)}{\sqrt{2}}\right)$$
$$= (\lambda^{\frac{1}{2}}s^2 + \sqrt{2}\lambda^{\frac{1}{4}}s + 1) \qquad (2.6.21)$$

μ is zero, and taking F=1,

$$R(s) = 1 \qquad (2.6.22)$$

c) Realisability relation

Equation (2.6.12) becomes:

$$\frac{B(-s)R(s)}{P(-s)F(s)D(s)} = \frac{1}{(\lambda^{\frac{1}{4}}s^2 - \sqrt{2}\lambda^{\frac{1}{4}}s+1)s^2}$$

$$= \frac{1 + \sqrt{2}\lambda^{\frac{1}{4}}s}{s^2} + \frac{J'(-s)}{P(-s)} \qquad (2.6.23)$$

(2.4.18) becomes:

$$g_r(t+T) = (t+T) + \sqrt{2}\lambda^{\frac{1}{4}}$$

$$\therefore \quad \frac{I_r}{s2} = \frac{1}{s2} + \frac{(T+\sqrt{2}\lambda^{\frac{1}{4}})}{s}$$

$$\therefore \quad I_r = 1 + (T + \sqrt{2}\lambda^{\frac{1}{4}})s \qquad (2.6.24)$$

d) Approximate e^{-sT}

For simplicity use (2.6.15).

$$e^{-sT} \doteq \frac{1 - 0.5\ sT}{1 + 0.5\ sT} = \frac{G(s)}{H(s)} \qquad (2.6.25)$$

Note that this is exact at s = 0, the root of A(s).

e) Calculate α and β

Equation (2.6.6) becomes:

$$\alpha = I_r FH$$

$$= (1 + (T + \sqrt{2}\lambda^{\frac{1}{4}})s)\ (1 + 0.5\ Ts) \qquad (2.6.26)$$

Equation (2.6.7) becomes:

$$\beta A = PRH - e^{-ST} I_r FBG$$

$$\therefore \quad \beta s^2 = (1+\sqrt{2}\lambda^{\frac{1}{4}}s + \lambda^{\frac{1}{2}}s^2)\ (1+0.5Ts)$$

$$- \{1 + (T + \sqrt{2}\lambda^{\frac{1}{4}})s\}\ (1-0.5sT)$$

$$= s^2\{(\lambda^{\frac{1}{2}} + \sqrt{2}\lambda^{\frac{1}{4}}T + 0.5\ T^2) + 0.5\ \lambda^{\frac{1}{4}}Ts\}$$

$$\therefore \quad \beta = (\lambda^{\frac{1}{2}} + \sqrt{2}\lambda^{\frac{1}{4}}T + 0.5T^2) + 0.5\lambda^{\frac{1}{4}}Ts \qquad (2.6.27)$$

As in this case D = A, and A has a double root at s = 0, I_r may also be found from (2.6.19) and its first derivative.

That is; when s = 0

$$PR - e^{-sT} I_r FB = 0 \qquad (2.6.28)$$

$$\frac{d}{ds}\left(PR - e^{-sT} I_r FB\right) = 0 \qquad (2.6.29)$$

setting $I_r = i_o + i_1 s$

(2.6.28) gives

$$1 - i_o = 0 .$$

(2.6.29) gives:

$$\sqrt{2}\,\lambda^{\frac{1}{2}} + T\, i_o - i_1 = 0 ,$$

hence $I_r = 1 + (T + \sqrt{2}\,\lambda^{\frac{1}{2}})s$. (2.6.30)

Note that (2.6.30) is the same as (2.6.24).

The control law given by (2.6.26) and (2.6.27) has a numerator of higher order than denominator. This means that is has a large high frequency gain, making it susceptible to high frequency measurement noise. However, if measurement noise has been explicitly included in the calculation, it would have been found that the orders of numerator and denominator in the optimal control law were equal.

2.7. DISCRETE-TIME SYSTEMS

The discrete-time system equivalent to (2.2.1) and (2.2.2) is given by:

$$\bar{y}(q) = q^{k}\frac{B(q)}{A(q)}\,\bar{u}(q) + \frac{C'(q)}{D'(q)}\,\bar{\psi}_d(q) \qquad (2.7.1)$$

$$\bar{v}(q) = \bar{y}(q) + \frac{E(q)}{F(q)}\,\bar{\psi}_m(q) \qquad (2.7.2)$$

u, v, and y are the discrete system input, sampled measured output, and discrete output respectively. A(q) etc are polynomials with non zero leading coefficients in the operator q, q given by:

$$q = z^{-1} = e^{-s\Delta} \qquad (2.7.3)$$

where s is the Laplace operator, z the z-transform operator and Δ the constant sample period. The sequences of uncorrelated random numbers ψ_d and ψ_m are defined by:

$$\begin{aligned} E\{\psi_d(i)\psi_d(i+j)\} &= \sigma_d\ \delta_k(j) \\ E\{\psi_m(i)\psi_m(i+j)\} &= \sigma_m\ \delta_k(j) \\ E\{\psi_m(i)\psi_d(i+j)\} &= 0 \end{aligned} \qquad (2.7.4)$$

where $E\{.\}$ denotes the expectation operator, and $\delta_k(j)$ the Kronecker delta function. Unlike the continuous case, discrete white noise does not have infinite variance, and may thus be allowed at the system output. Hence no restrictions on the orders of the system polynomials are required.

The optimum control is defined in a similar fashion to the continuous case. The derivation is also similar, and so only the salient points of the proof are outlined.

The loss function minimised (2.4.1) becomes (Åström, 1970):

$$V = \frac{1}{2\pi j}\oint_{\text{unit circle}} (\phi_{yy} + \phi_{uu})\,\frac{dq}{q} \qquad (2.7.5)$$

where ϕ_{yy} and ϕ_{uu} are discrete spectral densities. The spectral factorisation relationships (2.4.7) and (2.4.8) become:

$$P(q)P(q^{-1}) = B(q)B(q^{-1}) + \lambda A(q)A(q^{-1}) \qquad (2.7.6)$$

$$R(q)R(q^{-1}) = C(q)C(q^{-1})F(q)F(q^{-1}) + \mu E(q)E(q^{-1})D(q)D(q^{-1}) \quad (2.7.7)$$

where P and Q have all roots outside the unit disc. Note that $P(q)P(q^{-1})$ is a polynomial in $(q+q^{-1})$.

The realisability relation (2.4.14) becomes:

$$\bar{g}(q) = \frac{q^{-k}B(q^{-1})C(q)C(q^{-1})F(q^{-1})}{P(q-1)R(q^{-1})D(q)} = q^{-k}\frac{I(q)}{D(q)} + q^{-k}\frac{J(q^{-1})}{P(q^{-1})R(q^{-1})} \quad (2.7.8)$$

As no restriction was placed on the orders of the polynomials, (2.7.8) cannot always be expanded in terms of partial fractions only. In general the orders of I and J are given by:

$$n_I = n_c,\ n_J = n_b + n_c + n_f \quad (2.7.9)$$

If D(q) has distinct roots, in an analogous manner to (2.4.19) it follows that:

$$\frac{I(q)}{D(q)} = \sum_{i}^{nd} \frac{r_i}{q-d_i} + I'(q) \quad (2.7.10)$$

where I'(q) is of degree $n_c - n_d$ if $n_c \geq n_d$, and is zero otherwise.

The realisable part of g(q), (2.4.21) is then given by:

$$\bar{g}_r(q) = \frac{I_r(q)}{D(q)} = \sum_{i}^{nd} \frac{r_i d_i^{-k}}{q-d_i} + I_r'(q) \quad (2.7.11)$$

where $I'(q) = I_u'(q) + q^k I_r'(q)$ (2.7.12)

and I'_u is of degree k-1.

$I_u(q)$ is defined in a similar manner to (2.4.23) as:

$$\frac{I_u(q)}{D(q)} = \frac{I(q)}{D(q)} - q^k\frac{I_r(q)}{D(q)} \quad (2.7.13)$$

As before, (2.4.27), as $q^{-k}\frac{I_u}{D}$ is unrealisable, poles corresponding to realisable impulse responses must have zero residue, and so I_u must have a factor D. Also from (2.7.10), (2.7.11), (2.7.12) and (2.7.13), $I_u(q)$ is of degree $n_d + k-1$, and (2.7.13) becomes:

$$\frac{I(q)}{D(q)} = \tilde{I}_u(q) + q^k\frac{I_r(q)}{D(q)} \quad (2.7.14)$$

where $I_u = D\tilde{I}_u$ and $\tilde{I}_u$ is of degree k-1.

$\tilde{I}_u$ may be interpreted as the first k-1 terms of the weighting sequence expansion of $\frac{I}{D}$. Given I(q) from (2.7.8), $I_r(q)$ is thus found either via (2.7.9), (2.7.10), (2.7.11) and (2.7.12) or from (2.7.14).

The cascade compensation $Q^o(q)$ is given by an equation similar to (2.4.25):

$$Q^o(q) = \frac{I_r(q)F(q)}{P(q)R(q)} \tag{2.7.15}$$

This leads to an equation similar to (2.4.26):

$$\left.\frac{dV}{d\epsilon}\right|_{\epsilon=0} = \frac{2\sigma_d}{2\pi j}\oint_{\text{unit circle}} \frac{Q'(q^{-1})R(q^{-1})P(q^{-1})}{D(q^{-1})F(q^{-1})}\left[q^{-k}\tilde{I}_u(q) + q^{-k}\frac{I(q^{-1})}{P(q^{-1})F(q^{-1})}\right]\frac{dq}{q} \tag{2.7.16}$$

As in the continuous case it follows that the argument of the integral in (2.7.16) has no poles with non zero residue outside the unit disc. As Q'(q) is realisable, its denominator must have a non zero leading coefficient; by definition this is also true of D, F and P. Also $I_u(q)$ is of degree k-1, hence the integrand behaves as q^{-2} around the infinite circle, and thus the integral (2.7.16) vanishes. Once again it is concluded that (2.7.15) defines the optimum cascade compensation.

The equivalent feedback compensation is given by equations corresponding to (2.5.1) and (2.5.2):

$$\alpha = I_r F \tag{2.7.17}$$

$$\beta A = PR - q^k I_r FB \tag{2.7.18}$$

As in the continuous case, the right hand side of (2.7.18) has a factor A. Unlike the continuous case β is exactly given by a finite degree polynomial, hence the calculation of finite approximations (step (d), section 2.6) is avoided.

Example 2.7.1

To illustrate the solution of the discrete-time problem, the minimum variance controller of Åström and Peterka is obtained as a special case. The minimum variance control is that which for the model (2.7.1) with no measurement noise, unconditionally minimises the mean square output, that is as defined above with the conditions:

$$\lambda = \mu = 0 \tag{2.7.19}$$

a) Disturbance model conversion (2.6.8) and (2.6.9)

$$\begin{aligned} C &= C'\tilde{A} \\ D &= D'\tilde{A} \end{aligned} \tag{2.7.20}$$

b) Spectral factorisation (2.7.6) and (2.7.7)

$$\begin{aligned} P(q) &= B^{+}(q) \\ R(q) &= C^{+}(q)\tilde{A}^{+}(q) \end{aligned} \tag{2.7.21}$$

where $B^{+}(q)B^{+}(q^{-1}) = B(q)B(q^{-1})$, B^{+} has all roots outside the unit disc.

c) Realisability relation

From (2.7.8) and (2.7.21):

$$\frac{B(q^{-1})}{B^{+}(q^{-1})} \cdot \frac{C^{*}(q)}{D(q)} = \frac{I(q)}{D(q)} + \frac{I(q^{-1})}{B^{+}(q^{-1})} \tag{2.7.22}$$

where $n_I = n_c$, $n_J = n_b$, $C^{*} = C^{+}\tilde{A}^{+}$

from (2.7.21) this may be rewritten as:

$$\frac{B^{+}(q)}{B(q)} \cdot \frac{C^{*}(q)}{D(q)} = \frac{I(q)}{D(q)} + \frac{I'(q)}{B(q)} \tag{2.7.23}$$

where $\frac{I'(q)}{B(q)} = \frac{I(q^{-1})}{B^{+}(q^{-1})}$

From (2.7.14), (2.7.23) becomes:

$$\begin{aligned} B^{+}(q)C^{*}(q) &= D(q)\{I_u(q)B(q)+I'(q)\} + q^{k}I_r(q)B(q) \\ &= D(q)K(q) + q^{k}I_r(q)B(q) \end{aligned} \tag{2.7.24}$$

$K(q)$ is of degree $n_b + k-1$

e) Calculate α and β

From (2.7.17)

$$\alpha = I_r \tag{2.7.25}$$

From (2.7.18), (2.7.20) and (2.7.24)

$$\beta = \frac{B^+C^* - q^k I_r B}{A}$$

$$= \tilde{D}(q)K(q) \tag{2.7.26}$$

where $\tilde{D}A = D'\tilde{A} = D$

For the case $\tilde{D} = D'$, equations (2.7.24), (2.7.25) and (2.7.26) correspond to theorem 1 of Peterka (1972).

If $B^+ = B$, that is B has all roots outside the unit disc, (2.7.24) becomes:

$$C^* = I_u D + q^k I_r \tag{2.7.27}$$

and (2.7.26) becomes:

$$\beta = \tilde{D}BI_u \tag{2.7.28}$$

Equations (2.7.27), (2.7.25) and (2.7.28) correspond to the minimum variance strategy of Åström (1970).

2.8 On controllability.

This section considers the relationship between optimal controllers for controllable and uncontrollable systems, and in particular whether an uncontrollable system may be considered as a limiting case of a controllable system, for the purposes of controller design. The systems considered here (1.2.1 & 2) have two inputs affecting the system output y: the control input u, and the disturbance input y_d . There are thus two aspects of this problem: controllability with respect to the control input, and controllability with respect to the disturbance input.

As in section 2.2, a system fully controllable with respect to both the control input u, and the disturbance input y_d , is defined as:

$$\bar{y} = e^{-sT} \frac{B_c}{D} u + \frac{C_c}{D} y_d \qquad (2.8.1)$$

$$B_c = B\tilde{D}' + \gamma_1 B''\tilde{A} \qquad (2.8.2)$$

$$C_c = C'\tilde{A} + \gamma_2 C''\tilde{D}' \qquad (2.8.3)$$

where: $A\tilde{D}' = \tilde{A}D' = \tilde{}D$

and: $\tilde{A}$ contains all factors of A not common to D'

$\tilde{D}'$ " " " " D' " " " A

B" and C" are polynomials of the same order as B and C respectivly, containing no factors in common with B and C respectivly; the coefficients are otherwise arbitary.

† The use of the word 'controllable' is justified in Note 2, p238.

The problem can now be posed as: "Is the control law obtained in the limit as γ_1 and γ_2 tend to zero the same as that obtained at $\gamma_1 = \gamma_2 = 0$?" Note that when $\gamma_1 = \gamma_2 = 0$, equation (2.8.1) becomes the same as equation (1.2.1). The method given above covers all cases except that when $\gamma_2 = 0$, in which case - although the expression for the optimal cascade compensation is as given in equation (2.4.25) - it cannot be directly realised from equation (2.5.2); equation (2.5.3) must be used instead. The method then corresponds to that of Newton et al. (1957), and the closed loop system contains cancelling factors corresponding to the modes unaffected by the disturbance γ_d.

The method used to investigate this problem is to derive the optimal control law corresponding to (2.8.1) as γ_1 and γ_2 tend to zero, and compare this control law with that obtained for $\gamma_1 = \gamma_2 = 0$.

Using (2.8.2 & 3) the spectral factorisation (2.4.7) becomes:

$$\begin{aligned} P_c(s).P_c(-s) &= B_c(s).B_c(-s) + \lambda D(s).D(-s) \\ &= [B(s)\tilde{D}'(s) + \gamma_1 B''(s)A(s)] \\ &\quad * [B(-s)\tilde{D}'(-s) + \gamma_1 B''(-s)A(-s)] \\ &\quad + \lambda A'(s)\tilde{D}'(s)A(-s)\tilde{D}'(-s) \end{aligned} \qquad (2.8.4)$$

Define:

$$P(s).P(-s) = B(s).B(-s) + \lambda A(s).A(-s) \quad (2.8.5)$$

Letting γ_1 tend to zero, (2.8.4) becomes:

$$P_c(s).P_c(-s) = \tilde{D}(s).\tilde{D}(-s).P(s).P(-s) \quad (2.8.6)$$

Hence, iff D(s) has no roots in the right-hand plane, as γ_1 tends to zero:

$$P_c(s) = \tilde{D}(s).P(s) \quad (2.8.7)$$

Similarly iff A(s) has no roots in the right-hand plane, the spectral factorisation (2.4.8) becomes:

$$R_c(s) = \tilde{A}(s).R(s) \quad (2.8.8)$$

where:

$$R(s).R(-s) = C(s).C(-s).F(s).F(-s) + \lambda D'(s)D'(-s).E(s).E(-s) \quad (2.8.9)$$

The left-hand side of the realisability relation (2.6.2) becomes:

$$\frac{B_c(-s)}{P_c(-s)}.\frac{R_c(s)}{F(s)D(s)} \quad (2.8.10)$$

Letting γ_1 and γ_2 tend to zero, if D has no right-hand plane poles the expression (2.8.10) becomes:

$$\frac{\tilde{D}'(-s)B(-s)}{D'(-s)P(-s)}.\frac{\tilde{A}(s)R(s)}{F(s)D'(s)\tilde{A}(s)} \quad (2.8.11)$$

Define I_r in a similar manner to (2.6.2) as:

$$\frac{I_r'(s)}{D'(s)} = \text{realisable part of } \frac{B(-s)R(s)}{P(-s)F(s)\ D'(s)} \quad (2.8.12)$$

It follows that the realisable part of (2.8.11) is given by:

$$\frac{I_r\tilde{A}}{D'\tilde{A}} = \frac{I_r\tilde{A}}{D} \quad (2.8.13)$$

Hence the control law for the system given by (2.8.1) as γ_1 and γ_2 tend to zero, where D and A have no roots in the right-hand plane, is given by (2.5.1 & 2) as:

$$\alpha = I_r\tilde{A}F \quad (2.8.14)$$

$$\beta A = PR\tilde{A} - e^{-sT} I_r\tilde{A}FB \quad (2.8.15)$$

When $\gamma_1 = \gamma_2 = 0$, the optimal cascade compensation is given by (2.4.25) as:

$$Q^\circ = \frac{I_r F}{PR} \quad (2.8.16)$$

where I_r is given by (2.8.12). In section 2.5 it is shown (2.5.3) that this may be realised in feedback form as equations (2.8.14 & 15); this leads to cancelling factors $\tilde{A}$ in equation (2.8.16) - a practically feasible situation if $\tilde{A}$ has no right hand plane roots.

It has thus been shown that if $\tilde{A}$ and $\tilde{D}$ have no roots in the right-hand plane, the optimal control law parameters for the system given by (2.8.1) are continuous with respect to γ_1 and γ_2 at $\gamma_1 = \gamma_2 = 0$.

If, however, some unstable modes are not controllable with respect to the disturbance input - that is A has roots in the right-hand plane - equation (2.8.8) no longer holds as γ_i tends to zero; hence equations (2.8.14 & 15), which hold for $\gamma_i = 0$, do not hold as γ_i tends to zero. It follows that the control law parameters are discontinuous with respect to γ_i at $\gamma_i = 0$, in this particular case. Although the control law at $\gamma_i = 0$ may have a lower theoretical loss than the control law as γ_i tends to zero, the presence of the theoretically unexcited, unstable, closed loop modes makes it useless in practice. The method introduced in this chapter, for the case when γ_i tends to zero, is thus of some importance.

A system containing unstable modes unaffected by the control input (u), that is D has right-hand plane roots, can never be stabilised. Hence, for all cases of practical interest (where D has no right-hand plane roots) uncontrollable systems should always, for practical controller design, be considered as the limiting case of a controllable system.

Example 2.8.1

To illustrate these ideas, consider the discrete system given by:

$$y = \frac{qu + \gamma\psi}{1 - aq} + \frac{1}{1 - dq}\psi$$

$$= \frac{q}{1 - aq}u + \frac{(1+\gamma) - (d+\gamma a)q}{(1 - aq)(1 - dq)} \qquad (2.8.19)$$

where: $a>1$, $|d|<1$, and $q = z^{-1}$

Note that when $\gamma = 0$, ψ does not excite all system modes; and so when $\gamma = 0$, the system is uncontrollable with respect to the disturbance. For simplicity, the optimum control law is considered for the case when both the control weight and the measurement noise are zero. This minimum variance regulator (Åström, 1970) has been outlined in example 2.7.1 ; for the case when γ tends to zero, it was shown that the control law is given by:

$$\alpha = I_r \qquad (2.7.25)$$

$$\beta = \tilde{D}B\tilde{I}_u \qquad (2.7.26)$$

where:

$$C^* = \tilde{I}_u D + q^k I_r \qquad (2.7.27)$$

and I_r is of degree $k - 1$.

In this case, as $a > 1$, C^* is given by:

$$C^*(q).C^*(q^{-1}) = (1 - aq)(1 - aq^{-1}) \qquad (2.8.20)$$

where $C^*(q)$ has no roots within the unit circle. Hence:

$$C^*(q) = A^*(q) = a - q \qquad (2.8.21)$$

Equation (2.7.27) then becomes:

$$(a - q) = I_u(1 - aq)(1 - dq) + qI_r$$

giving: $I_u = a$, $I_r = a(a+d-adq) - 1$

Substituting the preceeding equations into (2.8.19), and letting γ tend to zero, the closed loop system is given by:

$$y = \frac{\beta A}{BA^*D} \cdot \gamma$$

$$= a\frac{A}{A^*} \qquad (2.8.22)$$

The spectral density of y is then given by:

$$\Phi_{yy} = a\frac{A(q)}{A^*(q)} . a\frac{A(q^{-1})}{A^*(q^{-1})} = a^2$$

Hence the loss function is given by:

$$V \triangleq E\{y^2\} = a^2$$

Repeating example (2.7.1) when $\gamma = 0$ leads to:

$$\alpha = I_r A \qquad (2.8.23)$$

$$\beta = \tilde{D}B\tilde{I}_u \qquad (2.8.24)$$

where:

$$C' = 1 = \tilde{I}_u D + q^k I_r \qquad (2.8.25)$$

This gives: $I_u = 1$, $I_r = d$

Substituting these equations into (2.8.19) and setting

$\gamma = 0$, gives the closed loop system:

$$y = \frac{\beta A}{DAB} \gamma$$

$$= \frac{A}{A} \gamma \qquad (2.8.26)$$

Theoretically A cancels in (2.8.26), giving a loss of unity; however in practice, the unstable mode corresponding to A will be excited - leading to an infinite loss. Hence this control law, though giving a lower theoretical loss than the control law based on the limiting controllable system, is useless in practice.

2.9 State-space formulation.

The regulator minimising the cost function (2.1.4) has, in previous sections, been derived by a frequency domain method: in this section, the same cost function is minimised over all linear realisable control laws, expressed in state-space form, by time domain methods. The solution of the same problem by two different methods is not mere mathematical sophistry: the two methods each give a different insight into the problem and its solution. The resultant forms of the regulators are of very different internal structure, but in section 2.13 it is shown that - as expected - the

external characteristics of the two regulators are identical.

The problem of minimising the cost function (2.1.4) is unusual, from the state-space point of viey, in two respects: the measurement noise is non-white, and the system has an input time-delay. The former problem has been treated previously (Bryson & Johansen,1965; Sage & Melsa,1971; Kwakernaak & Sivan, 1972); the contribution of section 2.10 is to provide a relationship between spectral factorisation and the algebraic Ricatti equation, similar to that for the standard Kalman filter problem. (Brockett,1970; MacFarlane,1971). The latter problem is shown, in section 2.11, to involve the insertion of a state predictor between the usual Kalman filter and state feedback stages (Fig 2.9.1).

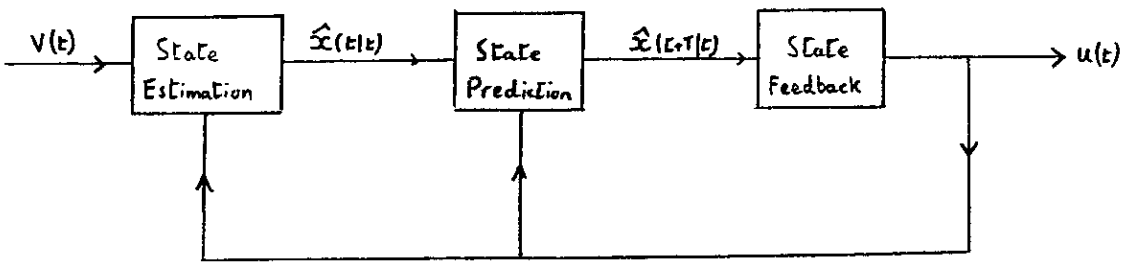

Fig 2.9.1 The structure of the state-space regulator.

It is convenient to consider the regulator as three distinct elements (Fig 2.9.1): an estimator of current states (a Kalman filter), a predictor of future states, and feedback from predicted future states. It is shown (section 2.11) that these three elements are seperable in the strict sense (Patchell & Jacobs,1971) - only estimates of the state itself are passed forward from one element to the next, no higher order information is required. These three elements are treated in sections 2.10-13.

Note that the internal feedback loop from controller output (u) to the state predictor can allow the state feedback gain to be infinite,whilst retaining a finite closed loop gain; it will be seen (section 3.7) that such a situation corresponds to a continuous analogue of the minimum variance regulator (Åström,1970).

The transfer function formulation of the regulator problem is now restated in state-space notation.

The system equations (2.2.1 & 2) are repeated here as:

$$\bar{y} = e^{-sT} \frac{B}{A}.\bar{u} + \frac{C'}{D'}\bar{\gamma} \qquad (2.9.1)$$

$$\bar{v} = \bar{y} + \frac{E}{F}\bar{\gamma} \qquad (2.9.2)$$

It is recalled that A, B, C, D, E and F are polynomials

in the Laplace operator s. The following development assumes that the system has no direct link, that is $n_b < n_a$.

The state-space equivalent of (2.9.1) is given by:

$$y(t) = \underline{M}_d \underline{x}(t) \tag{2.9.3}$$

$$\dot{\underline{x}}_d(t) = \underline{A}_d \underline{x}_d(t) + \underline{B}\, u(t-T) + \underline{C}_d \psi(t) \tag{2.9.4}$$

where: $\underline{M}_d$ is a n_d row vector.

$\underline{x}_d$, $\underline{B}$, and $\underline{C}_d$ are n_d column vectors.

$\underline{A}_d$ is a $n_d \times n_d$ matrix.

n_d is the degree of D (= $\tilde{D}'A = D'\tilde{A}$)

The components of the matrices are such that:

$$\underline{M}_d(s\underline{1} - \underline{A}_d)^{-1}\underline{B} = \frac{B(s)}{A(s)} \tag{2.9.5}$$

$$\underline{M}_d(s\underline{1} - A)^{-1}\underline{C}_d = \frac{C'(s)}{D'(s)} \tag{2.9.6}$$

The following results (Kwakernaak & Sivan,1972) are noted:

$$\underline{M}_d\underline{B} = b_{n_b}, \quad \underline{M}\,\underline{C}_d = c_{n_c} \tag{2.9.7}$$

where b_{n_b} and c_{n_c} are the highest order coefficients of B(s) and C'(s) respectively.

In particular, if the highest order coefficient of C'(s) is non-zero , it may be taken as unity and the

noise variance scaled accordingly; it then follows that:

$$\underline{M}\,\underline{C} = 1 \qquad (2.9.8)$$

Unless particular care is taken during matrix manipulations, even though (2.9.1) might be uncontrollable with respect to either input, numerical inaccuracies could well make (2.9.3 & 4) controllable with respect to both inputs. If this were so, the results obtained would apply to the limit as this controllable system tends to the uncontrollable system - rather than to the uncontrollable system itself. However, as pointed out in section 2.8, this yields the required control law; and is presumably a reason why the problem of cancelling unstable modes, found in frequency domain studies, does not appear in the state-space literature. Throughout the following sections, then, the system will be taken to be the limiting controllable system (2.8.1) rather than the possibly uncontrollable system (2.9.1).

Following the same procedure, the state-space equivalent of (2.9.2) is given by:

$$v(t) = y(t) + \underline{M}_m \underline{x}_m(t) \qquad (2.9.9)$$

$$\dot{\underline{x}}_m(t) = \underline{A}_m \underline{x}(t) + \underline{C}_m f(t) \qquad (2.9.10)$$

2.10 The Kalman filter.

The problem considered in this section is to find the least-squares estimate of those states which affect the system output. The problem is different to that originally considered by Kalman, in that the measurement noise is coloured and of finite variance - as opposed to the standard case of white measurement noise of infinite variance. The approach taken here is to augment the state vector affecting the system output with those states corresponding to the measurement noise , and estimate the $n_d + n_t$ components of the augmented vector. That is, find the least-squares estimate of $\underline{x}$ where:

$$\underline{v}(t) = \underline{M}\,\underline{x}(t) \qquad (2.10.1)$$

$$\dot{\underline{x}}(t) = \underline{A}\,\underline{x}(t) + \underline{B}\,u(t-T) + \underline{C}\,\underline{\dagger}(t) \qquad (2.10.2)$$

where: $\underline{M} = [\underline{M}_d \; \underline{M}_m]$, $\underline{A} = \begin{bmatrix} \underline{A}_d & \underline{0} \\ \underline{0} & \underline{A}_m \end{bmatrix}$

$$\underline{B} = \begin{bmatrix} \underline{B} \\ \underline{0} \end{bmatrix}, \quad \underline{C} = \begin{bmatrix} \underline{C}_d & \underline{0} \\ \underline{0} & \underline{C}_m \end{bmatrix}, \quad \underline{\dagger} = \begin{bmatrix} \dagger_d \\ \dagger_m \end{bmatrix}$$

See section 2.9 for further definitions.

Initially, the particular case will be considered where the orders of the disturbance and measurement processes are restricted by:

$$n_c = n_d - 1 \ , \quad n_e = n_f - 1$$

This augmented problem is singular in that there is no measurement noise; a standard technique (Bryson & Johansen,1965; Sage & Melsa,1971; Kwakernaak & Sivan, 1972) is to differentiate (2.10.1) and substitute (2.10.2) to give:

$$\dot{\underline{v}} = \underline{M}\,\underline{A}\,\underline{x} + \underline{M}\,\underline{C}\,\underline{\gamma} + \underline{M}\,\underline{B}\,u \qquad (2.10.3)$$

This measurement equation now contains a white noise term, which using (2.9.7) is given by:

$$\underline{M}\,\underline{C}\,\underline{\gamma} = \underline{\gamma}_d + \underline{\gamma}_m \qquad (2.10.4)$$

The solution to this non-singular problem, with correlated 'plant' and 'measurement' noise, is standard (Sage & Melsa,1971); it is of the form given in Fig 2.10.1.

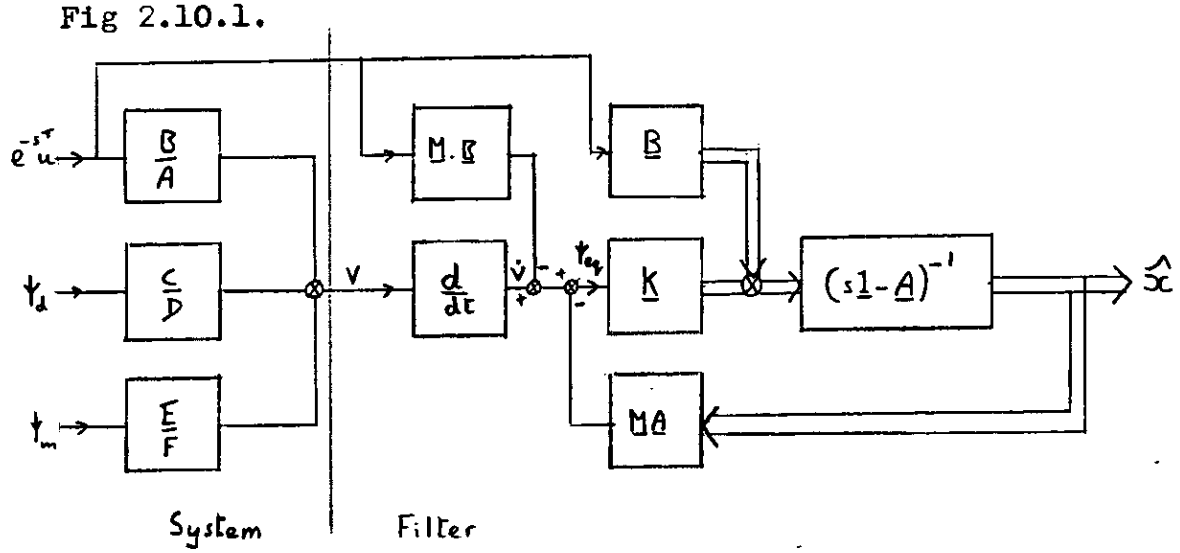

Fig 2.10.1 The structure of the Kalman filter.

Riccati

The filter gain $\underline{K}$ is given by the algebraic Ricatti equation:

$$\underline{K} = (\underline{S}\,\underline{A}^T\underline{M}^T + \underline{C}\,\underline{S}_{dm})\,S_{mm}^{-1} \qquad (2.10.5)$$

$$\underline{A}\,\underline{S} + \underline{S}\,\underline{A}^T + \underline{C}\,\underline{S}_{dd}\,\underline{C}^T - \underline{K}\,\underline{S}_{mm}\underline{K}^T = \underline{Q} \qquad (2.10.6)$$

where:
$$S_{mm} = \frac{1}{E\{\gamma_d^2\}} E\left\{(\gamma_d + \gamma_m)^2\right\} = (1+\mu)$$

$$\underline{S}_{dm} = \frac{1}{E\{\gamma_d^2\}} E\left\{\begin{bmatrix}\gamma_d\\ \gamma_m\end{bmatrix}(\gamma_d+\gamma_m)\right\} = \begin{bmatrix}1\\ \mu\end{bmatrix} \qquad (2.10.7)$$

$$\underline{S}_{dd} = \frac{1}{E\{\gamma_d^2\}} E\left\{\begin{bmatrix}\gamma_d\\ \gamma_m\end{bmatrix}[\gamma_d\ \gamma_m]\right\} = \begin{bmatrix}1 & 0\\ 0 & \mu\end{bmatrix}$$

μ is the ratio of the variance of γ_m to that of γ_d.

It has been shown (MacFarlane ,1971) that the non-singular filter problem, with uncorrelated plant and measurement noise, may be solved via a spectral factorisation involving the return difference of the Kalman filter. It is now shown that the singular filter problem associated with equations (2.10.1 & 2) - having been transformed into the non-singular problem associated with equations (2.10.3 & 2), with correlated plant and measurement noise - may be solved in a similar manner.

Adding $s\underline{S}\,\underline{1} - s\underline{S}\,\underline{1}$ to (2.10.6), premultiplying by: $\underline{M}\,\underline{A}\,(-s\underline{1} - \underline{A})^{-1}$ and postmultiplying by:

$\left[\underline{M}\,\underline{A}\,(-s\underline{1} - \underline{A})^{-1}\right]^T$ this equation becomes:

$$\underline{M}\,\underline{A}\,\underline{S}(-s\underline{1} - \underline{A}^T)^{-1}\underline{A}^T\underline{M}^T + \underline{M}\,\underline{A}(s\underline{1} - \underline{A})^{-1}\underline{S}\,\underline{A}^T\underline{M}^T$$

$$- \underline{M}\,\underline{A}(s\underline{1} - \underline{A})^{-1}(\underline{C}\,\underline{S}_{dd}\,\underline{C}^T + \underline{K}\,\underline{S}_{mm}\underline{K}^T)(-s\underline{1} - \underline{A}^T)^{-1}\underline{A}^T\underline{M}^T$$

$$= 0 \qquad (2.10.8)$$

From equations (2.10.5 & 7) the following equations hold:

$$\underline{M}\,\underline{A}\,\underline{S} = \underline{S}_{mm}^T\underline{K}^T - \underline{S}_{dm}^T\underline{C}^T$$

$$= (1+\mu)\underline{K}^T - \underline{C}_d^T - \mu\underline{C}_m^T$$

$$\underline{C}\,\underline{S}_{dd}\,\underline{C}^T = \underline{C}_d\underline{C}_d^T + \mu\,\underline{C}_m\underline{C}_m^T$$

$$\underline{K}\,S_{mm}\underline{K}^T = (1+\mu)\underline{K}\,\underline{K}^T$$

Substituting these equations into (2.10.8), and adding $(1+\mu)$ and subtracting 1 and μ from the resultant equation:

$$(a)\quad (1+\mu)\left[1 + \underline{M}\,\underline{A}(s\underline{1} - \underline{A})^{-1}\underline{K}\right]\left[1 + \underline{K}^T(-s\underline{1} - \underline{A}^T)^{-1}\underline{A}^T\underline{M}^T\right] =$$

$$(b)\quad \left[1 + \underline{M}\,\underline{A}(s\underline{1} - \underline{A})^{-1}\underline{C}_d\right]\left[1 + \underline{C}_d^T(-s\underline{1} - \underline{A}^T)^{-1}\underline{A}^T\underline{M}^T\right] +$$

$$(c)\quad \mu\left[1 + \underline{M}\,\underline{A}(s\underline{1} - \underline{A})^{-1}\underline{C}_m\right]\left[1 + \underline{C}_m^T(-s\underline{1} - \underline{A}^T)^{-1}\underline{A}^T\underline{M}^T\right]$$

(2.10.9)

The contents of the left-hand parentheses of terms (a), (b), and (c) of equation (2.10.9) are all of the same form, and each has two interpretations.

Firstly, from Fig 2.10.1, they are the return differences of the Kalman filter with gains $\underline{K}$, $\underline{C}_d$ and $\underline{C}_n$ respectively. Secondly, from equations (2.10.2 & 3) they are transfer functions representing derivatives of stochastic processes, which for terms (b) and (c) are described by white noise passing into transfer functions C/D and E/F respectively.

Define the polynomial R(s) by:

$$\frac{I}{(1+\mu)^{\frac{1}{2}}}.\frac{sR(s)}{D(s)F(s)} = 1 + \underline{M}\,\underline{A}(s\underline{1} - \underline{A})^{-1}\underline{K} \qquad (2.10.11)$$

Note that the right hand term is the return difference of the Kalman filter. Equation (2.10.9) becomes:

$$\frac{R(s)R(-s)}{D(s)F(s)D(-s)F(-s)} = \frac{C(s)C(-s)}{D(s)D(-s)} + \mu\frac{E(s)E(-s)}{F(s)F(-s)} \qquad (2.10.12)$$

Note that (2.10.12) corresponds to the spectal factorisation relationship (2.4.8). As the closed loop transfer function of the filter must be stable, R(s) must have no roots in the right-hand plane; hence the spectral factorisation implied by (2.10.11) is unique, and the $n_d + n_f$ roots of R(s) define the $n_d + n_f$ components of the Kalman gain vector $\underline{K}$.

For example if the eigenvalues of $\underline{A}$ are distinct, the system equations (2.10.1 & 2) may be written in

normal form, and from (2.10.11) $\underline{K}$ is given by:

$$1 + \underline{M}\,\underline{A}(s\underline{1} - \underline{A})^{-1}\,\underline{K} = 1 + \sum \frac{a_i k_i}{(s-a_i)}$$

$$= \frac{1}{(1+\mu)^{\frac{1}{2}}} \cdot \frac{sR(s)}{D(s)F(s)} \qquad (2.10.13)$$

where k_i is the ith component of $\underline{K}$ and a_i the ith root of $D(s)F(s)$ - the ith eigenvalue of $\underline{A}$. Equation (2.10.13) is satisfied by choosing:

$$k_i = \frac{1}{(1+\mu)^{\frac{1}{2}}} \cdot r_i \qquad (2.10.14)$$

where r_i is the residue of $R(s)/F(s)D(s)$ at $s = a_i$.

The process given by:

$$\frac{1}{(1+\mu)^{\frac{1}{2}}} \cdot \frac{R(s)}{D(s)F(s)} \psi_{eq} \qquad \text{Where: } E\{\psi_{eq}^2\} = (1+\mu)E\{\psi_d^2\} \qquad (2.10.15)$$

has the same spectral density as the process given by:

$$\frac{C(s)}{D(s)}\psi_d + \frac{E(s)}{F(s)}\psi_m \qquad (2.10.16)$$

In this sense, the processes (2.10.15 & 16) may be considered to be equivalent - and yield the same Kalman filter. Referring to Fig 2.10.1, it is seen that the Kalman filter - in the steady state - generates ψ_{eq} by comparing $(\dot{\underline{v}} - \underline{M}\,\underline{B}\,u)$ with its predicted value.

The white noise process, ψ_{eq}, then drives a model of the system corresponding to (2.10.15) to yield an asymptotically exact estimate of the equivalent state corresponding to this system. It is concluded that the exact estimate of this equivalent state is the best estimate of the actual state corresponding to (2.10.16) and (2.10.1 & 2).

Formally, then, the Kalman filter may be written as:

$$\underline{\hat{x}}(t|t) = (s\underline{1} - \underline{A})^{-1}\left[\underline{K}\,\psi_{eq} + \underline{B}u(t-T)\right] \qquad (2.10.17)$$

$$\psi_{eq} = (1+\mu)^{\frac{1}{2}}\frac{D(s)F(s)}{R(s)}\left[v - \frac{B(s)}{A(s)}u(t-T)\right] \qquad (2.10.18)$$

It is emphasised that equations (2.10.17 & 18) do not correspond physically to the filter, properly realised in the form of Fig 2.10.1 ; but rather they provide an external description of the filter, involving theoretically exact mathematical cancellation.

Attention now turns to another particular case defined by:

$$n_c = n_d - 1$$

$$n_a < n_t - 1$$

In this case , equation (2.10.4) becomes:

$$\underline{M}\,\underline{C}\,\underline{Y} = \psi_d \qquad (2.10.19)$$

Equations (2.10.7) then become:

$$S_{mm} = 1$$

$$\underline{S}_{dm} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad (2.10.20)$$

$$\underline{S}_{dd} = \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix}$$

Equation (2.10.19) is replaced by:

$$\left[1 + \underline{M}\,\underline{A}(s\underline{1} - \underline{A})^{-1}\underline{K}\right]\left[1 + \underline{K}^{T}(-s\underline{1} - \underline{A}^{T})^{-1}\underline{A}^{T}\underline{M}^{T}\right] =$$

$$\left[1 + \underline{M}\,\underline{A}(s\underline{1} - \underline{A})^{-1}\underline{C}_{d}\right]\left[1 + \underline{C}_{d}^{T}(-s\underline{1} - \underline{A}^{T})^{-1}\underline{A}^{T}\underline{M}^{T}\right] +$$

$$\underline{M}\,\underline{A}(s\underline{1} - \underline{A})^{-1}\underline{C}_{m}\underline{C}_{m}^{T}(-s\underline{1} - \underline{A}^{T})^{-1}\underline{A}^{T}\underline{M}^{T} \qquad (2.10.21)$$

The argument then proceeds exactly as before, except that the noise process ψ_{eq} now has a covariance $E\{\psi_{d}^{1}\}$ rather than $(1+\mu)E\{\psi_{d}^{1}\}$

The derivation for the case defined by:

$$n_c < n_d - 1$$

$$n_d = n_f - 1$$

is analogous, ψ_{eq} having a covariance $\mu E\{\psi_{d}^{1}\}$. It remains to consider the general case.

If none of the previous three cases holds, the measured system output (v) is differentiated N times - effectively increasing the degrees of C(s) and E(s) by N - until one of the three sets of conditions

holds. The derivation then proceeds as before, but using the new polynomials $s^N C(s)$ and $s^N E(s)$. The return difference of the Kalman filter becomes:

$$\frac{s}{\sqrt{\nu}}^{N+1} \frac{R(s)}{D(s)F(s)} \quad \text{where } \sqrt{\nu} = (1+\mu)^{\frac{1}{2}}, \ 1, \text{ or } \mu^{\frac{1}{2}}. \quad (2.10.22)$$

The N+1 differentiations are thus cancelled by N+1 integrations; hence equations (2.10.14) and (2.10.17 & 18) are unchanged, but for the factor ν.

The derivation of the optimum filter for the augmented state is now complete; it is a standard result (Sage & Melsa, 1971) that the first n_A elements of the estimated augmented vector form the best estimate of the required state vector $\underline{x}_A$. From equation (2.10.17) the required state estimate may be written as:

$$\dot{\hat{\underline{x}}}_A(t|t) = \underline{A}_A \hat{\underline{x}}_A(t|t) + \underline{B}\, u(t-T) + \underline{K}_A \gamma_{eq}(t) \quad (2.10.23)$$

where $\underline{K}_A$ is a n_A column vector formed from the first n_A elements of $\underline{K}$, and γ is generated using (2.10.18).

2.11 The state predictor.

In the previous section, the optimum estimate ($\hat{\underline{x}}_d(t|t)$) of the current system state ($\underline{x}_d(t)$) was derived; this section is devoted to finding the optimum prediction ($\hat{\underline{x}}_d(t+T|t)$) of the future state ($\underline{x}_d(t+T)$) time T ahead. In most standard treatments of this subject, the effect of the control input on the system state is often omitted as being deterministic and thus easily accounted for. However, as will be seen in this section, this deterministic control signal can lead to the effect of cancelling poles and zeros noted in the realisability relation of the frequency domain treatment (2.4.27).

It can be shown (Sage & Melsa,1971) that the optimum (least-squares) state prediction is obtained from the state equation (2,10.2), with the noise term omitted, and using the current state estimate ($\hat{\underline{x}}_d(t|t)$) as initial condition. That is, integrating equation (2.10.2):

$$\hat{\underline{x}}_d(t+T|t) = e^{\underline{A}_d T}\,\hat{\underline{x}}_d(t|t) + \int_t^{t+T} e^{\underline{A}_d.(t+T-\alpha)}\,\underline{B}\,u(\alpha-T)\,d\alpha \qquad (2.11.1)$$

The integral term in (2.11.1) renders it unsuitable for on-line use; so to obtain a recursive solution, (2.11.1) is differentiated with respect to t to give:

$$\hat{\underline{x}}_d(t+T|t) = e^{\underline{A}_d T}\hat{\underline{x}}_d(t|t) + \underline{B}u(t) - e^{\underline{A}_d T}\underline{B}u(t-T) + \underline{A}_d \int_t^{t+T} e^{\underline{A}_d(t+T-\alpha)}\underline{B}u(\alpha-T)\,d\alpha \quad (2.11.2)$$

Multiplying both sides of (2.11.1) by $\underline{A}_d$, and subtracting the result from (2.11.2):

$$\dot{\hat{\underline{x}}}_d(t+T|t) - \underline{A}_d\hat{\underline{x}}_d(t+T|t) = e^{\underline{A}_d T}\dot{\hat{\underline{x}}}_d(t|t) - \underline{A}_d e^{\underline{A}_d T}\hat{\underline{x}}_d(t|t) + \underline{B}u(t) - e^{-\underline{A}_d T}\underline{B}u(t-T) \quad (2.11.3)$$

Once again the steady-state solution is of interest, and Laplace transforming (2.11.3) :

$$\hat{\underline{x}}_d(t+T|t) = e^{\underline{A}_d T}\hat{\underline{x}}_d(t|t) + (s\underline{1} - \underline{A}_d)^{-1}(\underline{1} - e^{-(s\underline{1}-\underline{A}_d)T})\underline{B}u(t) \quad (2.11.4)$$

To examine the second term of the right-hand side of equation (2.11.4) more closely, consider the case where $\underline{A}_d$ has distinct eigenvalues and is written in normal form; the ith component then becomes:

$$\frac{1 - e^{-(s-a_i)T}}{s - a_i}\cdot b_i\, u(t) \quad (2.11.5)$$

where a_i is the ith eigenvalue of $\underline{A}_d$ and b_i the ith component of $\underline{B}$. Note that $s = a_i$ is a root of both numerator and denominator of (2.11.5), and thus if

not explicitly cancelled will remain as a hidden mode. Hence if the predictor is to be actually realised as such , this cancellation must be explicitly performed. The fact that the predictor cannot be represented by a rational transfer function, and contains no poles, is analogous to the properties of the unrealisable transfer function I_*/D encountered in section 2.4.

It is a standard result (Sage & Melsa,1971; Kwakernaak & Sivan,1972) that both the state estimate ($\underline{\hat{x}}_d(t|t)$) generated by the Kalman filter, and the control input ($u(t)$), are uncorrelated with the estimation error; this is the property underlying the separation principle (Kwakernaak & Sivan,1972) for the optimal control of systems with no time delay. It is now shown that a similar property holds for the state predictor developed in this section; it will be seen that this implies separation between state feedback and prediction in this more general case.

Integrating the state equation (2.9.4) from t to t+T :

$$\underline{x}_d(t+T) = e^{\underline{A}_d T}\underline{x}_d(t) + \int_t^{t+T} e^{\underline{A}_d(t+T-\alpha)}\underline{B}u(\alpha-T)\,d\alpha + \int_t^{t+T} e^{\underline{A}_d(t+T-\alpha)}\underline{C}_d(\alpha)\,d\alpha \qquad (2.11.6)$$

Defining the prediction error $\underline{\epsilon}(t+T|t)$ as:

$$\underline{\epsilon}(t+T|t) = \underline{\hat{x}}_d(t+T|t) - \underline{x}_d(t+T) \qquad (2.11.7)$$

and subtracting (2.11.6) from (2.11.1) it follows that:

$$\underline{\epsilon}(t+T|t) = e^{\underline{A}_d T}(\underline{x}_d(t|t) - \underline{x}_d(t)) - \int_t^{t+T} e^{\underline{A}_d.(t+T-\alpha)} \underline{C}_d(\alpha)\, d\alpha \qquad (2.11.8)$$

It is a standard result (Sage & Melsa,1971) concerning the Kalman filter that the first term of (2.11.8) is uncorrelated with $\underline{\hat{x}}_d(t|t)$ and independent of u(t); the second term contains white noise processes occurring after time t and is hence also uncorrelated with $\underline{\hat{x}}_d(t|t)$ and u(t). It follows from multiplying (2.11.1 & 7) and taking expectations that:

$$E\left\{\underline{\epsilon}(t+T|t).\underline{\hat{x}}_d^T(t+T|t)\right\} = 0 \qquad (2.11.9)$$

Moreover, it follows from (2.11.8) and the above conditions that:

u(t) and $\underline{\epsilon}(t+T|t)$ are independent (2.11.10)

The cost function to be minimised by the control law is given by equation(2.1.4); the argument of this integral is:

$$E\left\{y_r^2(t+T) + \lambda u^2(t)\right\} \qquad (2.11.11)$$

Substituting (2.9.3) into (2.11.11), the argument becomes:

$$E\{\underline{x}_d^T(t+T)\ \underline{M}_d^T\underline{M}_d\underline{x}_d(t+T) + \lambda u^2(t)\} \qquad (2.11.12)$$

Using equations (2.11.7), (2.11.12) becomes:

$$\begin{aligned}(i)\quad & E\{\hat{\underline{x}}_d^T(t+T|t)\ \underline{M}_d^T\underline{M}_d\hat{\underline{x}}_d(t+T|t) + \lambda u^2(t)\\ (ii)\quad & + 2\,\underline{\epsilon}^T(t+T|t)\ \underline{M}_d^T\underline{M}_d\hat{\underline{x}}_d(t+T|t)\\ (iii)\quad & + \underline{\epsilon}^T(t+T|t)\ \underline{M}_d^T\underline{M}_d\underline{\epsilon}(t+T|t)\}\end{aligned} \qquad (2.11.13)$$

From equation (2.11.9) the second term of (2.11.13) is zero, and from equation (2.11.10) the third term is independent of u(t). It follows that the minimisation of V (2.1.4) can be replaced by the minimisation of:

$$V' = \underset{t_o\to\infty}{\text{Lt.}}\ \frac{1}{2t_o}\int_{-t_o}^{t_o} E\{\hat{\underline{x}}_d(t+T|t)\ \underline{M}_d^T\underline{M}_d\hat{\underline{x}}_d(t+T|t) + \lambda u^2(t)\}\,dt \qquad (2.11.14)$$

The functions of state prediction and state feedback are thus seperable.

Using equations (2.11.3) and (2.10.23), the prediction can be expressed as:

$$\dot{\hat{\underline{x}}}_d(t+T|t) = \underline{A}_d\hat{\underline{x}}_d(t+T|t) + \underline{B}u(t) + e^{\underline{A}_dT}\underline{K}_d\psi_{eq} \qquad (2.11.15)$$

where $\underline{K}_d$ is the Kalman gain vector and ψ_{eq} a white noise

process. The minimisation of the cost function (2.11.14), with $\underline{x}$ given by equation (2.11.15), is a standard problem (Kwakernaak & Sivan,1972); the best linear solution is that associated with the certainty-equivalent (Patchell & Jacobs,1971) deterministic problem obtained by removing the expectation operator from (2.11.14), and setting $\underline{\psi}$ to zero in equation (2.11.15). Note that this certainty-equivalent system has the same dynamics as the system (2.9.4) and, as initial conditions do not affect the steady-state solution, the time-delay does not formally enter the problem. That is the state feedback element of the control law is the same as that which would be obtained for the system (2.9.3 & 4) with no disturbance and no time-delay; hence:

$$u(t) = \underline{K}_c \hat{\underline{x}}_d(t+T|t) \qquad (2.11.16)$$

where $\underline{K}_c$ is obtained on the basis of the deterministic system:

$$\dot{\underline{x}}'(t) = \underline{A}_d \underline{x}'(t) + \underline{B}u(t) \qquad (2.11.17)$$

2.12 State feedback.

In the previous section, it was shown that the state feedback element of the control law (Fig 2.9.1) is identical to that obtained from the minimisation of the cost function (2.11.14), repeated here as:

$$V' = \underset{t_o \to \infty}{\text{Lt.}} \frac{1}{2t_o - t_i} \int_{t_i}^{t_o} \underline{x}^T(t) \underline{M}_1^T \underline{M}_1 \underline{x}(t) + \lambda u^2(t)\, dt \quad (2.12.1)$$

where the system is given by (2.11.17) as:

$$\dot{\underline{x}}(t) = \underline{A}_d \underline{x}(t) + \underline{B}\, u(t) \quad (2.12.2)$$

It is a standard result (Kwakernaak & Sivan, 1972) that the state feedback vector $\underline{K}_c$, associated with (2.12.1 & 2), is given by the solution of the algebraic Riccati equation:

$$\underline{K}_c = \frac{1}{\lambda} \underline{B}^T \underline{S} \quad (2.12.3)$$

$$\underline{S}\, \underline{A}_d + \underline{A}_d^T \underline{S} + \underline{M}_1^T \underline{M}_1 - \underline{K}_c \lambda \underline{K}_c^T = \underline{0} \quad (2.12.4)$$

To avoid difficulties with the solution of (2.12.3 & 4), the results of section 2.8 are invoked to replace the - possibly uncontrollable - system (2.12.2) with a nearby controllable system; that is $\underline{B}$ in equations (2.12.2-4) is replaced by $\underline{B}_c$, corresponding to the numerator polynomial $B_c(s)$ defined in (2.8.2)

Comparing equations (2.12.3 & 4) with equations (2.10.5 & 6), it follows that this control problem is not the dual of the filter problem discussed in section 2.10. However it has been shown that (MacFarlane,1971), by using similar manipulations to those given in section 2.10, the solution of the Riccati equation (2.12.3 & 4) may be expressed in terms of a spectral factorisation. That is, defining the return difference associated with the closed loop control as:

$$\frac{1}{\lambda^2}\frac{P_c(s)}{D(s)} = 1 + \underline{K}_c(s\underline{1} - \underline{A}_d)^{-1}\underline{B}_c \qquad (2.12.5)$$

$P_c(s)$ is given by the spectral factorisation relationship:

$$P_c(s)P_c(-s) = B_c(s)B_c(-s) + \lambda D(s)D(-s) \qquad (2.12.6)$$

where $B_c(s)$ is given by (2.8.2) as:

$$B_c(s) = B(s)\tilde{D}'(s) + \gamma_1 B''(s)\tilde{A}(s)$$

If the roots of D are distinct, the system may be written in normal coordinates, and (2.12.5) becomes:

$$\frac{1}{\lambda^2}\frac{P_c(s)}{D(s)} = 1 + \frac{k_i b_i}{s-d_i} \qquad (2.12.7)$$

where d_i is the ith root of D(s), and k_i and b_i are

the ith components of $\underline{K}_c$ and $\underline{B}_c$ expressed in normal coordinates; in particular b_i is given by:

$$b_i = \text{residue of } \frac{B_c(s)}{D(s)} \text{ at } s = d_i \qquad (2.12.8)$$

Also, as B_c/D represents a controllable system, $B_c(d_i)$ is non-zero and so from (2.12.7 & 8) k_i is given by:

$$k_i = \frac{1}{\lambda^2}\frac{P_c(d_i)}{B_c(d_i)} \qquad (2.12.9)$$

Again using the fact that $B_c(d_i)$ is non-zero, it also follows from (2.12.6) that:

$$\frac{P_c(d_i)}{B_c(d_i)} = \frac{B_c(-d_i)}{P_c(-d_i)} \qquad (2.12.10)$$

$$\therefore \quad k_i = \frac{1}{\lambda^2}\frac{B_c(-d_i)}{P_c(-d_i)} \qquad (2.12.11)$$

As in section 2.4 (2.4.7), P(s) is defined as:

$$P(s)P(-s) = B(s)B(-s) + \lambda A(s)A(-s) \qquad (2.12.12)$$

where P(s) has no right-hand plane roots. Then letting γ_i tend to zero in equation (2.8.2), the ith element of the gain vector for the limiting controllable system (2.12.11) becomes:

$$k_i = \frac{1}{\lambda^2}\frac{D'(-d_i)B(-d_i)}{D'(-d_i)P(-d_i)}$$

$$= \frac{1}{\lambda^2}\frac{B(-d_i)}{P(-d_i)} \qquad (2.12.13)$$

It is concluded that although this method does not define the state feedback vector $\underline{K}$. for systems uncontrollable with respect to the control input u, it does give $\underline{K}$. for the limiting controllable system. With the results of section 2.8 in mind, it is conjectured that (2.12.13) is also valid for the uncontrollable system (2.9.1).

2.13 Comparison of state space and transfer function solutions.

Having derived the optimum (in the sense of section 2.1) regulator via both state space and transfer function methods, it is possible to show, by direct comparison, that both methods do indeed yield a regulator with the same external behaviour. A similar comparison has previously been made (Caines,1972; Hughes,1973; Watson,1976) for discrete-time systems with no measurement noise, and no weighting on control effort. The comparison given in this section is restricted to the continous-time case, but the control weighting may be non-zero and coloured measurement noise may be present; it is conjectured that discrete-time results are analogous.

It is recalled that the transfer function derivation involved carrying cancelling factors through to the final calculation. To permit direct comparison, the

state space formulae are written in a form involving such cancellations; it is emphasised that this is for mathematical simplicity, and does not imply that the components of the control law are physically realised in this form. Collecting together results from sections (2.10-12) - in the 'uncancelled' form where appropriate - the three components of the control law (Fig 2.9.1) become:

State feedback (2.11.16) :

$$u = -\underline{K}_c \hat{\underline{x}}_d(t+T|t) \tag{2.13.1}$$

State prediction (2.11.4) :

$$\hat{\underline{x}}_d(t+T|t) = e^{\underline{A}\,T}\hat{\underline{x}}_d(t|t) + (s\underline{1} - \underline{A}_d)^{-1}(\underline{1} - e^{-(s\underline{1}-\underline{A}_d)T})\underline{B}u(t) \tag{2.13.2}$$

State estimation (2.10.23) :

$$\hat{\underline{x}}_d(t|t) = (s\underline{1} - \underline{A}_d)^{-1}(\underline{K}_d \underline{y}_{eq} + \underline{B}u(t-T)) \tag{2.13.3}$$

$$= \nu \frac{D(s)F(s)}{R(s)}\left[v - e^{-sT}\frac{B(s)}{A(s)}u\right] \tag{2.13.4}$$

where $\nu = (1+\mu)^{\frac{1}{2}}$, 1, or $\mu^{\frac{1}{2}}$ depending on the relative degrees of C, D, E and F (see section 2.10).

Noting that $(s\underline{1} - \underline{A}_d)^{-1}$ and $e^{-(s\underline{1}-\underline{A}_d)T}$ commute, the first three equations combine to give:

$$u = -\underline{K}_c(s\underline{1} - \underline{A}_d)^{-1}(e^{\underline{A}_d T}\underline{K}_d\downarrow_{eq} + \underline{B}u) \qquad (2.13.5)$$

It is assumed that $\underline{A}_d$ has distinct eigenvalues, and thus, working in normal coordinates:

$$\underline{K}_c(s\underline{1} - \underline{A})^{-1} e^{\underline{A}_d T} \underline{K}_d = \frac{k_c^i e^{-d_i T} k_d^i}{s-d_i} \qquad (2.13.6)$$

where d_i is the ith root of $D(s)$ - ith eigenvalue of $\underline{A}$ - and k_c^i and k_d^i are the ith components of $\underline{K}_c$ and $\underline{K}_d$ respectively.

From equation (2.12.13) k_c^i is given by:

$$k_c^i = \frac{l_1}{\lambda^2}\frac{B(-d_i)}{P(-d_i)} \qquad (2.13.7)$$

and from equation (2.10.13) k_d^i is given by:

$$k_d^i = \frac{1}{\nu}.\text{residue of}\left[\frac{R(s)}{D(s)F(s)}\right] \text{at } s = d_i \qquad (2.13.8)$$

In section 2.4 - equations (2.4.21 & 22) - the realisable transfer function was defined as:

$$\frac{I_r(s)}{D(s)} = \frac{e^{-d_i T}.r_i}{s-d_i} \qquad (2.13.9)$$

where: r_i = residue of $\left[\frac{B(-s)}{P(-s)}.\frac{R(s)}{F(s)D(s)}\right]$ at $s = d_i$

It follows from equations (2.13.6-9) that:

$$\underline{K}_c(s\underline{1} - \underline{A}_d)^{-1} e^{\underline{A}_d T} \underline{K}_d = \frac{1}{\sqrt{\lambda_2}} \frac{I_r(s)}{D(s)} \qquad (2.13.10)$$

From the results of section 2.12, equation (2.12.3) may be written as:

$$\underline{K}_c(s\underline{1} - \underline{A}_d)^{-1} \underline{B} = \frac{1}{\lambda^2} \frac{P(s)}{A(s)} \underline{1} \qquad (2.13.11)$$

Combining the equation (2.13.5) with equations (2.13. 10 & 11) the control law is given by:

$$u = -\frac{1}{\sqrt{\lambda^2}} \cdot \frac{I_r(s)}{D(s)} \Big|_{eq} - \left(\frac{1}{\lambda^2} \frac{P(s)}{A(s)} - 1\right)u$$

$$\therefore P(s)u = -\frac{I_r(s)A(s)}{\sqrt{\ } D(s)} \qquad (2.13.12)$$

Using equation (2.13.4), (2.13.12) becomes:

$$P(s)u = -\frac{I_r(s)A(s)F(s)}{R(s)} \left[v - e^{-sT} \frac{B(s)}{A(s)} u \right] \qquad (2.13.13)$$

Collecting terms, the control law (2.13.13) becomes:

$$u = -\frac{\alpha}{\beta} v \qquad (2.13.14)$$

where:

$$\alpha = I_r F \qquad (2.13.15)$$

$$\beta = \frac{1}{A}(PR - e^{-sT} B\, I_r F) \qquad (2.13.16)$$

Equations (2.13.15 & 16) are identical to equations (2.5.1 & 2) defining the control law derived via transfer function - frequency domain methods. It is concluded that - as expected - the state space and transfer function solutions are externally equivalent.

2.14 Set point following.

Having solved the regulator problem, attention turns to the servo problem posed in section 2.1. That is, for the system of Fig 2.1.2 given by:

$$(y(t+T) - w(t)) = \frac{B(s)}{A(s)}u(t) - w(t) \qquad (2.14.1)$$

the problem is to minimise V_s of equation (2.1.4) repeated here as:

$$V_s = \underset{t_o \to o}{\text{Lt.}} \frac{1}{2t_o} \int_{-t_o}^{t_o} E\left\{(y_s(t+T) - w(t))^2 + \lambda u_s^2(t)\right\} dt \qquad (2.14.2)$$

Note that because of the way that the cost function is chosen, the system of equation (2.14.1) effectively has no time-delay; w(t) acts as a disturbance process. The problem is approached via the frequency-domain formulation of sections 2.2-7; the state space method would follow that outlined in sections 2.9-12, but is omitted here.

As in the regulator problem, it is necessary to assign some structural properties to the 'disturbance' process w. This may be done in two ways: firstly, following Newton et al.(1957), the setpoint may be described as a stochastic process with rational spectral density - such a description is appropriate to a setpoint continuously varying in a random manner; secondly, following Chang (1961), the setpoint may be described as a deterministic process with rational Laplace transform, for instance a step, ramp, or sinusoidal function. That is, w is described by:

$$\bar{w}(s) = \frac{C^w(s)}{D^w(s)} \bar{\gamma}(s) \qquad (2.14.3)$$

where $\bar{\gamma}(s)$ is either a white noise process or an impulse function. For example, if $C^w = 1$ and $D^w = s$, the setpoint is either a drifting stochastic process with uncorrelated increments (Wiener process), or else a step function.

Formally, the solution of this servo problem is very similar to that of the regulator problem. The system is put into equivalent cascade form - as in Fig 3.1 ; as Parsevals theorem is formally the same for either a stochastic process driven by white noise, or a deterministic system driven by an impulse, the proof proceeds as for the regulator problem.

There is, however, an important difference between the realisation of the resultant cascade system of the regulator, and that of the servo problem. The disturbance process occuring in the regulator problem is not directly known, and can only be deduced from measurements of the system output; hence realisation in the feedback configuration is obligatory. The 'disturbance' process occuring in the servo problem is - however described - a known variable; hence the control law may be realised in either feedback or cascade form. It will be seen below that the servo control law can be neatly realised in cascade form as a feedforward term added to the feedback regulator solution. This cascade realisation does not have the problem of cancelling factors associated with feedback realisation, and thus no disturbance conversion - to obtain a limiting controllable system - is necessary. This result is pleasing, for whereas there are theoretical reasons for letting the disturbance process excite all system modes, there is no reason why the setpoint should be in anyway matched to the system.

The solution of the servo problem is thus given by:

$$u = -Q^{\circ} A\, w \tag{2.14.4}$$

$$Q^{\circ} = \frac{I_r^w}{PR} \tag{2.14.5}$$

where I_r^w is obtained from the realisability relation:

$$\frac{B(-s)}{P(-s)} \cdot \frac{R^w(s)}{D^w(s)} = \frac{I_r^w(s)}{D^w(s)} + \frac{J(-s)}{P(-s)} \tag{2.14.6}$$

and $R^w(s)$ is obtainedd from the spectral factorisation:

$$R^w(s)R^w(-s) = C^w(s)C^w(-s) \tag{2.14.7}$$

Equations (2.14.4-7) correspond to equations (2.3.2), (2.4.25), (2.4.14) and (2.4.8) respectively. P(s) is as defined in the regulator problem and is given by the spectral factorisation relation (2.4.7).

The solution to the regulator problem - equations (2.5.1 & 2) - can be combined with that of the servo problem as:

$$\beta u = -\alpha v + R.\frac{I_r^w}{R^w} w \tag{2.14.8}$$

where:

$$\alpha = I_r F \tag{2.5.1}$$

$$\beta A = PR - e^{-sT} I_r FB \tag{2.5.2}$$

and: $$R(s)R(-s) = C(s)C(-s)F(s)F(-s) + \mu E(s)E(-s)D(s)D(-s) \tag{2.4.8}$$

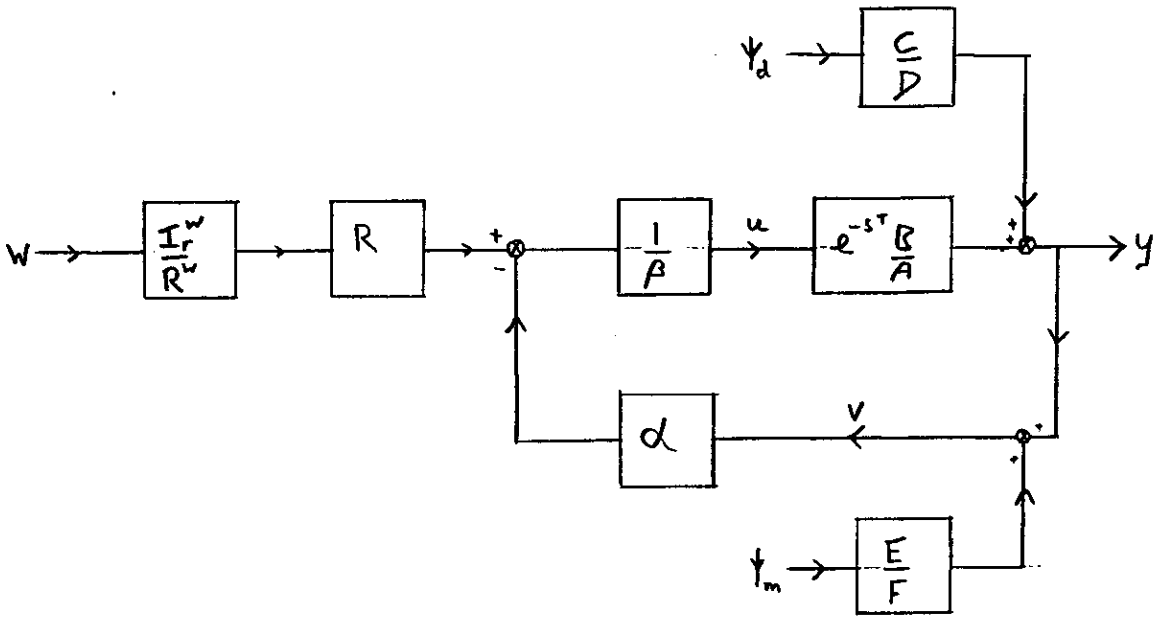

Fig 2.14.1 The complete control configuration.

It is readily verified that the control law (2.14.8) satisfies the regulator and servo problems simultaneously; this complete solution is represented in Fig 2.14.1. It is noted that this solution does not, in general, only involve feedback of the output error (v-w), but contains a feedforward term; thus solutions which are forced to be of pure feedback form (e.g. Newton et al.,1957; Youla et al.,1976) are suboptimal.

The discrete-time servo problem has an analogous solution, corresponding to the discrete regulator problem discussed in section 2.7; these solutions then combine in a manner formally the same as equation (2.14.8).

Example 2.14.1.

To illustrate the application of the preceeding theory, consider a problem described previously (Youla et al.,1976). The system is given by:

$$y = e^{-0.1s}\frac{(s-1)}{s(s-2)}u + \frac{1}{(s+10)}\gamma_d$$

$$v = y + \gamma_m$$

where γ_d and γ_m are unit white noise processes. The theory developed in this chapter does not consider the physically impossible case of white measurement noise. However the white measurement noise in this example presumably represents an approximation to wide bandwidth noise; if the measurement noise is considered as such, the preceeding theory also applies to this case.

In this example the cost function (2.1.1) is taken to have a control weight $\lambda=4$; the setpoint is taken to be a step function. The regulator problem is considered first ; from section 2.1 this is independent of the servo problem. This is in distinction to the method of Youla et al. in which the problems interact due to an unnecessary constraint on the form of the control law. For simplicity, the derivation follows the steps outlined in section 2.6.

a) Disturbance conversion.

$$C = C'\tilde{A} = s(s-2)$$

$$D = D'\tilde{A} = (s+10).s(s-2)$$

b) Spectral factorisation

$$P(s)P(-s) = B(s)B(-s) + \lambda A(s)A(-s)$$

$$= (s-1)(-s-1) + 4s(s-2).-s(-s-2)$$

$$= 1 - 17s^2 + 4s^4$$

$$= 4(s^2 + 4.190)(s^2 + 0.05966)$$

$$P(s) = 2(s + \sqrt{4.190})(s + \sqrt{0.05966})$$

$$= 1 + 4.583s + 2s^2$$

In a similar manner it follows that:

$$R(s) = s(20.10 + 12.05s + s^2)$$

c) Realisability relation

Is R(-s) / right?

$$\frac{B(-s)}{P(-s)} \cdot \frac{R(-s)}{F(s)D(s)} = \frac{(-1-s)}{(1-4.583s+2s^2)} \cdot \frac{s(20.1+12.05s+s^2)}{1.s(s-2)(s+10)}$$

$$= \frac{0}{s} + \frac{72.95}{(s-2)} - \frac{1.265\ 10^{-3}}{(s+10)} + \frac{J(-s)}{P(-s)}$$

From equation (2.6.13):

$$\frac{I_r}{D} = \frac{0}{s} + e^{0.2} \cdot \frac{72.95}{(s-2)} - e^{-1} \frac{1.265\ 10^{-3}}{(s+10)}$$

$$I_r = s(891.1 + 89.12s)$$

d) Approximate e^{-sT}

The method of Youla et al. (1976) does not explicitly include a system time-delay; hence they derived the exact controller for the system approximated using a second order Pade approximation. For comparison the same approximation is used here, but this method gives an approximate controller for the exact system. The approximation is given by:

$$e^{-0.1s} \doteq \frac{1 - 0.05s + 0.0008333s^2}{1 + 0.05s + 0.0008333s^2} \triangleq \frac{G(s)}{H(s)}$$

e) Calculate the control law.

$$\alpha = I_r FH$$

$$= s(891.1 + 89.12s).1.(1 + 0.05s + 0.0008333s^2)$$

$$= s(891.1 + 133.7s + 5.200s^2 + 0.07426s^3)$$

$$\beta A = PRH - I_r BFG$$

$$= s(911.2 - 741.4s + 53.36s^2 + 37.38s^3$$

$$+ 3.440s^4 + 0.1239s^5 + 0.001667s^6)$$

The right-hand side of this equation clearly has a

factor s; it can also be verified that it has a factor (s-2). These are the two factors of A; cancelling them gives:

$$\beta = -455.6 + 142.9s + 44.77s^2 + 3.695s^3 + 0.1272s^4 + 0.001667s^5$$

The solution to the servo problem is similar. The step setpoint is modelled by:

$$\frac{C^{\sim}(s)}{D^{\sim}(s)} = \frac{1}{s}$$

It has been noted that step a) does not apply to the servo problem.

b) Spectral factorisation

From (2.14.7):

$$R^{\sim}(s)R^{\sim}(-s) = C^{\sim}(s)C^{\sim}(-s)$$

$$\therefore R^{\sim}(s) = 1$$

c) Realisability relation:

This is given by (2.14.6) as:

$$\frac{B(-s)}{P(-s)}\cdot\frac{R^{\sim}(s)}{D^{\sim}(s)} = \frac{-s-1}{1-4.583s+2s^2}\cdot\frac{1}{s}$$

$$= -\frac{1}{s} + \frac{J(-s)}{P(-s)}$$

$$\therefore I_r^{\sim} = -1$$

Note that B(0) and P(0) have the same magnitude in this case because the system contains an integrator, and is thus matched to the setpoint.

Finally, from equation (2.14.8), the control law is given by:

$$u = \frac{-d}{\bar{\beta}} v + \frac{I}{R} \cdot \frac{R}{\bar{\beta}} w$$

Using a standard root finding routine, it follows that the entire control law is given by:

$$u = \frac{-k_1 s(s+\alpha_1)(s+\alpha_2)(s+\bar{\alpha}_2)v + k_2 s(s+\gamma_1)(s+\bar{\gamma}_2)w}{(s+\beta_1)(s+\beta_2)(s+\beta_3)(s+\beta_4)(s+\bar{\beta}_4)}$$

where $\bar{}$ denotes complex conjugate. The roots are tabulated in Table 3.14.1. This control law should be compared with that of Youla et al. (1976) given by:

$$u = -k_1 \frac{(s+\alpha_0)(s+\alpha_1)(s+\alpha_2)(s+\bar{\alpha}_2)}{(s+\beta_1)(s+\beta_2)(s+\beta_3)(s+\beta_4)(s+\bar{\beta}_4)} \cdot (v-w)$$

The roots of the polynomials in this control law are also given in Table 3.14.1.

The differences between these two control laws seem to be due to two main factors: the optimal control law treats set points and disturbances separately, that of Youla et al. does not; the optimal control law explicitly treats the system time-delay and includes it in the cost function, that of Youla et al. does

not actually treat time-delays, but rather systems which contain a rational approximation to a time-delay.

	Optimal	Youla et al.
k_1	44.58	67.22
α_0	0	0.01487
α_1	-10.00	-10.00
α_2	-30.00 + j17.32	-30.00 + j17.32
k_2	599.9	-
γ_0	0	-
γ_1	-2.000	-
γ_2	-10.05	-
β_1	1.887	2.413
β_2	-9.957	-9.981
β_3	-21.11	-33.65
β_4	-23.58 + j11.54	-18.06 + j14.99

Table 2.14.1 Control law constants.

Finally, it is noted that the control law derived in this example is independent of the noise and set point amplitudes; that of Youla et al. - derived for a unit step - depends on the relative amplitude of the disturbance process and the set point.

2.15 On measurement noise.

In some situations, due to the absence of a-priori information about the statistics of the disturbance and measurement noise, it may be necessary to minimise a cost function containing the measured output - rather than the actual output - of the system. That is, although the system is actually of the form of equations (1.2.1 & 2):

$$\bar{y} = e^{-sT} \frac{B}{A} \bar{u} + \frac{C}{D} \bar{\nu}_d \qquad (2.15.1)$$

$$\bar{v} = \bar{y} + \frac{E}{F} \bar{\nu}_m \qquad (2.15.2)$$

it may be necessary to minimise a cost function containing v rather than y, $E/F\,\bar{\nu}_m$ being regarded as a disturbance. This section examines the implications of such a procedure, when associated with the optimal control laws developed in this chapter.

Consider the cost function (2.1.1):

$$V = \underset{t_o \to \infty}{\text{Lt.}} \frac{1}{2t_o} \int_{-t_o}^{t_o} E\left\{(y(t+T) - w(t))^2 + \lambda u^2(t)\right\} dt \qquad (2.15.3)$$

and also the cost function:

$$V' = \underset{t_o \to \infty}{\text{Lt.}} \frac{1}{2t_o} \int_{-t_o}^{t_o} E\left\{(v(t+T) - w(t))^2 + \lambda u^2(t)\right\} dt \qquad (2.15.4)$$

For the purposes of minimising V' (2.15.4), the measurement noise is treated as a disturbance and combined with the disturbance noise to give a process described by:

$$v = e^{-sT}\frac{B}{A}u + \frac{R}{DF}\gamma_{eq} \qquad (2.15.5)$$

The symbol γ_{eq} represents a white noise process , and the polynomial R is given by the spectral factorisation:

$$R(s)R(-s) = C(s)C(-s)F(s)F(-s) + \mu E(s)E(-s)D(s)D(-s) \qquad (2.15.6)$$

Note that the autocorrelation of v given by (2.15.5) is identical to that given by (2.15.1 & 2) when u = 0.

From the results of section 2.14, the control laws corresponding to (2.15.1-3) and (2.15.4-6) may both be written down as:

$$\alpha y + \beta u + \Gamma w = 0 \qquad (2.15.7)$$

In the former case, minimising a function of y, the control law polynomials are given by:

$$\alpha = I_r F \qquad (2.15.8)$$

$$\beta A = PR - e^{-sT} I_r FB \qquad (2.15.9)$$

$$\Gamma = R.\frac{I_r}{R^*} \qquad (2.15.10)$$

where R is given by (2.15.6), R^{ν} by (2.14.7), and I_r by: (2.6.12) as:

$$\frac{B(-s)}{P(-s)}\cdot\frac{R(s)}{D(s)F(s)} = \frac{I_u + e^{-sT} I_r}{D(s)} + \frac{J(-s)}{P(-s)F(s)} \qquad (2.15.11)$$

In the latter case, minimising a function of v, the control law is given by:

$$\alpha' y + \beta' u + \Gamma' w = 0 \qquad (2.15.12)$$

where:

$$\alpha' = I_r' \qquad (2.15.13)$$

$$\beta' = PR - e^{-sT} I_r' B \qquad (2.15.14)$$

$$\Gamma' = \Gamma = R.\frac{I_r^{\nu}}{R^{\nu}} \qquad (2.15.15)$$

Once again R is given by (2.15.6), R^{ν} by (2.14.7), and I_r^{ν} by (2.14.6), but I_r' is given by (2.6.12) as:

$$\frac{B(-s)}{P(-s)}\cdot\frac{R(s)}{D(s)F(s)} = \frac{I_u' + e^{-sT} I_r'}{D(s)F(s)} + \frac{J'(-s)}{P(-s)} \qquad (2.15.16)$$

Comparing equations (2.15.7-11) with equations (2.15.12-16), it follows firstly that the closed loop characteristic polynomial is identical in each case and given by PR. Secondly, the differences appear in the expressions for α and β, and are associated with the expression for I_r (2.15.11) and with that for I_r' (2.15.16). In particular, if:

$$I_r' = I_r F \tag{2.15.17}$$

the expressions are identical. Assuming that DF has distinct roots and using equation (2.6.13), I_r and I_r' may be explicitly written as:

$$\frac{I_r}{D} = \sum_{i}^{n_d} e^{-d_i T} \left(\frac{r_i}{(s-d_i)}\right) \tag{2.15.18}$$

$$\frac{I_r'}{DF} = e^{-d_i T} \left(\frac{r_i}{(s-d_i)}\right) + e^{-f_j T} \left(\frac{\rho_j}{(s-f_j)}\right) \tag{2.15.19}$$

where r_i and ρ_j are the residues of the expressions on the left-hand side of (2.15.11 & 16) at the n_d roots of D, and the n_f roots of F, respectively. If the second term on the right-hand side of equation (2.15.19) is small, equation (2.15.17) is satisfied; this occurs if:

$$e^{-f_j T} \ll e^{-d_i T} \qquad j = 1,n_f \; ; \; i = 1,n_d \tag{2.15.20}$$

From the expression for R (2.15.6):

$$R(f_j)R(-f_j) = \mu E(f_j)E(-f_j)D(f_j)D(-f_j) \tag{2.15.21}$$

Equation (2.15.21) implies that if μ is small, $R(f_j)$ is small, and hence ρ_j is small. That is, if the ratio of the variance of the measurement noise to that of the disturbance is small, equation (2.15.17) is approximately true and hence the two control laws are approximately equal.

These two conditions - high frequency or small variance measurement noise - are perhaps fairly obvious: it is now shown that if the poles of the measurement noise transfer function are close to those of the disturbance, then an interesting third possibility arises.

Consider the case where D containes all factors of F; then as, by definition, F has no right-hand plane roots, R (2.15.6) also has a factor F and the equivalent noise process becomes:

$$\frac{R}{DF} = \frac{R'F}{D\,F}$$

$$= \frac{R'}{D} \qquad (2.15.22)$$

where R' is defined by: $R = R'F$.

The expression for I_r' (2.15.16) now becomes:

$$\frac{B(-s)}{P(-s)}.\frac{R'(s)}{D(s)} = \frac{I_u' + e^{-sT} I_r'}{D(s)} + \frac{J'(-s)}{P(-s)} \qquad (2.15.23)$$

and that for β' (2.15.14) becomes:

$$\beta'A = PR' - e^{-sT} I_r'B \qquad (2.15.24)$$

The expression for I_r(2.15.11) becomes:

$$\frac{B(-s)}{P(-s)}.\frac{R'(s)}{D(s)} = \frac{I_u + e^{-sT} I_r}{D(s)} + \frac{J(-s)}{P(-s)} \qquad (2.15.25)$$

and that for (2.15.9) becomes:

$$\beta A = PR'F - e^{-sT} I_r FB \qquad (2.15.26)$$

Comparing equations (2.15.23 & 25) it follows that:

$$I_r' = I_r \qquad (2.15.27)$$

From equations (2.15.8, 26 & 10) it follows that:

$$\begin{aligned} \alpha &= \alpha' F \\ \beta &= \beta' F \\ \Gamma &= \Gamma' F \end{aligned} \qquad (2.15.28)$$

Hence the two control laws are identical, except for a common factor which - by definition - contains no right-hand plane roots.

To summarise, control laws minimising the cost functions (2.15.3 & 4) - identical but for the system output being replaced by its measured value - for the system of equations (2.15.1 & 2) become identical if either:

a) The measurement noise variance is much smaller than that of the disturbance process .

or b) The time constants associated with the measurement noise are much smaller than those of the disturbance process , and are much smaller than the system time-delay.

or c) . The denominator polynimial (F) of the measurement process is a factor of the denominator polynomial (D) of the disturbance process. (Recall that the polynomial D contains the system denominator polynomial (A)as a factor.)

In all cases, the closed loop characteristic polynomial is the same for both control laws.

2.16 Summary.

The problem of tracking a set point in the presence of disturbances and measurement noise of known rational spectral densities, and where the system dynamics are known, has been considered with respect to the usual steady-state quadratic criterion. It was shown that the stochastic tracking problem can be considered as the sum of two independent problems: a deterministic servo problem, and a stochastic regulator problem.

The latter problem was solved via both transfer function - frequency domain techniques, and state-space - time domain techniques. It was shown that by considering a system, uncontrollable with respect to the disturbance input, as the limiting case of a controllable system, the frequency domain technique could be extended to cover all systems of the type considered in a unified manner. A physical justification of this method was given; and it was shown that unstable 'hidden modes' associated with earlier methods do not arise. A complete derivation was given for the continuous-time problem; the analogous discrete-time results were outlined.

State space methods were extended - by the use of a state predictor - to include systems with an input time-delay. It was noted that because of the way that the system was written in state space form, the above

transformation of the disturbance model occurs implicitly. By the use of certain relationships between spectral factorisation and the algebraic Riccati equation - extended to include the case of coloured measurement noise - the state space control law was found to be externally equivalent to that derived by the transfer function approach. The derivation was restricted to the continuous-time problem; it was conjectured that discrete -time results are similar.

The solution of the deterministic servo problem has been shown to follow directly from the solution of the stochastic regulator problem; and the two solutions were combined to give the final control law. It was noted that some previous solutions of this problem, by forcing the control law to operate on the error signal only, are suboptimal.

Finally, some conditions were given which define a class of systems for which measurement noise may be treated as a system disturbance, whilst still producing an optimum control law.

Chapter 3

Minimum Variance Control.†

3.1 Introduction.

The minimum variance regulator, developed by Åström for the control of discrete-time systems (Åström, 1970), has proved useful in practice; in particular it has been shown to form the basis of a self-tuning regulator (Åström & Wittenmark, 1973). However it suffers from a number of drawbacks: control effort can only be altered by manipulating the sample rate; systems with zeros outside the unit circle cannot be regulated; the inclusion of a set-point can lead to a deterioration in regulator performance; and the excessive control effort can lead to undesirable inter-sample behaviour. The purpose of this chapter is to develop a class of control laws which, whilst retaining the properties necessary for self-tuning, go some way towards alleviating the disadvantages of the minimum variance regulator.

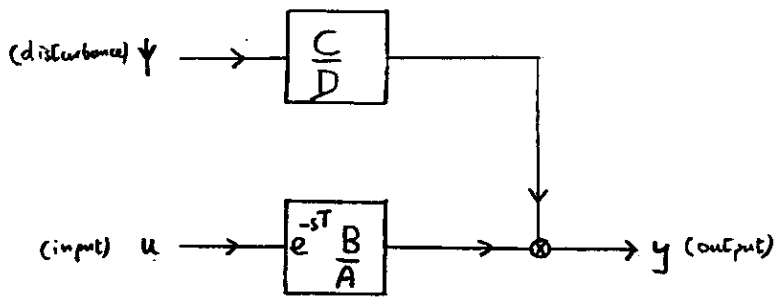

Fig. 3.1.1 The system

† The stability properties of this control method are discussed on p111 - equation (3.3.26) onwards.

The development is mainly concerned with the continuous systems (Fig. 3.1.1) described by:

$$y = e^{-sT} \frac{B(s)}{A(s)} u + \frac{C(s)}{D(s)} \xi \qquad (3.1.1)$$

The system output, control input, and disturbance input are y,u and ξ respectively; A,B,C and D are polynomials in the Laplace operator s. As in section 2.2, the system is taken to be excited by the disturbance process, and thus D always has a factor A; this avoids the problem of cancelled unstable poles discussed in section 2.8. The consideration of continuous - as opposed to discrete - problems is not only of interest in itself, but also elucidates some aspects made obscure in the discrete-time treatment. The discrete-time results are, however, similar and are summarised in section 3.10.

The cost function considered in this section is:

$$V = E\{\phi^2(t+T) \mid t\} \qquad (3.1.2)^{\dagger}$$

$$\phi(t+T) = \Lambda^y y(t+T) + \Lambda^u u(t) + \Lambda^w w(t) \qquad (3.1.3)$$

The symbols Λ^y, Λ^u, and Λ^w denote rational transfer functions in s; T and w denote the system time-delay and set point respectively. The symbol $E\{\cdot \mid t\}$ denotes expectation conditional on data available up to time t; this conditioning is equivalent to a realisability constraint (Åström, 1970).

The cost function V (3.1.2) is closely related to the cost function minimised by the minimum variance

† $E\{\cdot \mid t\}$ denotes expectation conditioned on all measurable variables in the range $(-\infty, t]$: the ensemble average is over all possible realisations of the random processes affecting the output which occur after time t.

regulator given by:

$$V' = E\{y^2(t+T)|t\} \qquad (3.1.4)$$

For this reason, the control laws considered here are termed 'minimum variance controllers'. The minimum variance regulator is based on the optimum prediction of the future system output $y(t+T)$; in the same way the minimum variance controller is based on the optimum prediction of the future value of the auxiliary function $\phi(t+T)$. Hence the main part of the development concerns a continuous analogue of the previously derived discrete predictor (Åström,1970).

The utility of the cost function V (3.1.2) is unclear at first sight. However it is shown that this cost function generates a number of useful control laws. These include: a model following control law; an extension of classical control laws to systems with input delay; and a control law which under certain circumstances approximates to the optimal control law of chapter 2. Some attention is given as to which systems can be well controlled by the approximate optimal controller, and it is shown that this problem is closely connected with the relative performance of optimal infinitesimal and infinite horizon control laws. This latter problem is the continuous-time analogue of the relative performance of optimal k-step-ahead and infinite step ahead control laws which has been considered previously (Hughes & Jacobs,1973; Hughes,1975; Ashton, 1975).

Summary of sections 3.6-3.9.

There are three classes of control laws considered in sedtions 3.6-3.9: those corresponding to the cost functions:-

a. Infinite horizon.

$$V = \operatorname*{Lt}_{t_o \to \infty} \int_{-t_o}^{t_o} E\left\{(y(t+T)-w(t))^2 + \lambda u^2(t)\right\} dt$$

The minimisation of this cost function with respect to realisable control laws is considered in chapter 2.

b. Infinitesimal horizon.

$$V_\delta = \operatorname*{Lt}_{\delta \to 0} \left[E\left\{y(t+T+\delta) - w(t+)\right)^2\right\} + \int_t^{t+\delta} E\left\{(y(\alpha+T) - w(t))^2 + \lambda_\delta u^2(\alpha)\right\} d\alpha \right]$$

The first temm of this cost function represents the terminal cost. The minimisation of this cost function with respect to realisable control laws is considered in section 3.6.

c. Proportional minimum variance.

$$V_o = E\left\{(y(t+T) - w(t) + \lambda_o u(t))^2 \mid t\right\}$$

The minimisation of this cost function is considered in section 3.5.

The expectation appearing in cost functions a and b is unconditional: the ensemble average is over all possible realisations of the random variables involved. The expectation in c is conditioned on all measured variables in the range $[t,-\infty)$: the ensemble average is over all random variables occuring after time t, conditioned on variables occuring up to and including time t.

The sections may be summarised as follows:-

3.6. The cost function V_δ (b.) is introduced and the resultant control law compared with that corresponding to V_o (c.). It is shown that if the system transfer function:

$$\frac{b_o + b_1 s + \ldots + b_n s^n}{a_o + a_1 s + \ldots + a_n s^n}$$

is such that $b_n = 0$, the two control laws are equal if

$$\lambda_o = \frac{\lambda_\delta}{b_{n-1}} \qquad (3.6.20)$$

3.7. The control law corresponding to cost function c is discussed in a state-space context. Standard results concerning the cost function b are used to verify equation (3.6.20).

3.8. The performance of control laws corresponding to a and c are compared; root-locus conditions are

derived which characterise systems for which the performances of the two control laws are similar. It is argued that if (3.6.20) holds, and so the control laws corresponding to b and c are identical, the results of section 3.8 also relate the control laws corresponding to a and b. This latter comparison is the continuous-time analogue of previous work in discrete time.

3.9. The results of 3.8 are illustrated using a first order example.

3.2 Optimal least-squares prediction.

The theory of minimum variance control revolves around the theory of least-squares prediction. Prediction has been previously treated (Åström,1970) in connexion with the discrete-time minimum variance regulator; in this section a theory is developed in a similar way for continuous systems, and the discrete-time results briefly pointed out in section 3.10.

Consider the continuous system:

$$y(s) = e^{-sT} \frac{B(s)}{A(s)} u(s) + \frac{C(s)}{D(s)} \psi(s) \qquad (3.2.1)$$

In the sequel, fictitious systems are considered which may have white noise processes appearing at the output, hence no restriction is placed on the relative orders of C and D. As in section 2.2, D is forced to contain all roots of A.

Define a new disturbance process given by:

$$\psi' = \frac{C}{R} \psi \qquad (3.2.2)$$

where R has no roots in the right hand plane and is given by the spectral factorisation relationship:

$$R(s).R(-s) = C(s).C(-s) \qquad (3.2.3)$$

Using a standard relationship (Åström,1970), the spectral density of ψ' (Φ') is given in terms of the spectral density of ψ (Φ) by:

$$\Phi' = \frac{C(s).C(-s)}{R(s).R(-s)} \Phi \qquad (3.2.4)$$

Using (3.2.3) it follows that the spectral densities

are equal. Hence as γ is white so is γ'; using this fact (3.2.1) may be rewritten as:

$$y = e^{-sT} \frac{B(s)}{A(s)} u + \frac{R(s)}{D(s)} \gamma' \qquad (3.2.5)$$

As in section 2.4 the disturbance transfer function is divided into realisable and unrealisable components. However in this case the degree of $R(n_r)$ may be greater than that of $D(n_d)$, and so first of all it is rewritten as:

$$\frac{R(s)}{D(s)} = R'(s) + \frac{R''(s)}{D(s)} \qquad (3.2.6)$$

where the order of $R'(n_{r'})$ is given by: $n_{r'} = n_r - n_d$ if $n_r \geq n_d$, and $R'=0$ if $n_r < n_d$. Following section 2.4, $\frac{R''(s)}{D(s)}$ is divided into realisable and unrealisable components as:

$$e^{sT} \frac{R''(s)}{D(s)} = e^{-sT} \frac{I'_u(s)}{D(s)} + \frac{I_r(s)}{D(s)} \qquad (3.2.7)$$

Once again I'_u has a factor D and so:

$$e^{sT} \frac{R(s)}{D(s)} = e^{sT} (R'(s) + \tilde{I}'_u(s)) + \frac{I_r(s)}{D(s)} \qquad (3.2.8)$$

where: $I'_u = D\,\tilde{I}'_u$. The decomposition 3.2.8 is depicted in Fig. 3.2.1.

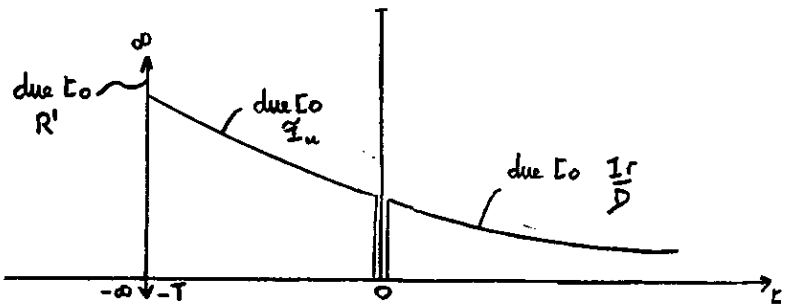

Fig. 3.2.1 The realisability decomposition.

By including R'(s) in the unrealisable response, it is tacitly assumed that the time-delay (T) is non-zero, and hence that the corresponding impulse and its derivatives occur before time t=0. This leads to a 'well-behaved' realisable response, and is related to the comments at the end of section 2.4.

Substituting (3.2.8) into (3.2.5):

$$y(t+T) = \underset{\bar{A}}{B}\, u(t) + I_{\underline{r}}\, \psi'(t) + \tilde{I}'_u(t+T) \qquad (3.2.9)$$

where: $\tilde{I}_u \triangleq R' + \tilde{I}'_u$

As $I_u \psi'(t+T)$ is a function of future (>t) values of ψ' only, and as ψ' is white, it is uncorrelated with both $\psi'(t)$ and $u(t)$, that is:

$$E\{I_u \psi'(t+T).\psi'(t)|t\} = 0;\ E\{I_u \psi'(t+T).u(t)|t\} = 0 \qquad (3.2.10)$$

Let $y^*(t+T|t)$ be the least squares estimate of $y(t+T)$ based on data up to time t, and let $\epsilon(t+T)$ be the associated error. Subtracting y^* from both sides of (3.2.9):

$$\epsilon(t+T) = \underset{\bar{A}}{Bu}(t) + I_{\underline{r}}\, \psi'(t) - y^*(t+T|t) + I_u \psi'(t+T) \qquad (3.2.11)$$

Squaring, taking expectations and using (3.2.10):

$$E\{\epsilon^2\} = \left[\underset{\bar{A}}{Bu}(t) + I_{\underline{r}}\, \psi'(t) - y^*(t+T|t)\right]^2 + E\{(I_u \psi'(t+T))^2|t\} \qquad (3.2.12)$$

As the terms on the right-hand side of (3.2.12) are positive and mutually independent, the minimising value of y^* is that which sets the first term to zero. That is:

$$y^*(t+T|t) = \frac{B}{A}\, u(t) + \frac{I}{D}\, \psi(t) \qquad (3.2.13)$$

Subtracting (3.2.13) from (3.2.9), the minimal error (ϵ^*) is given by:

$$\epsilon^* \triangleq y(t+T) - y^*(t+T|t) = \tilde{I}_u\, \psi'(t+T) \qquad (3.2.14)$$

From (3.2.10) it follows that:

$$E\{y^*(t+T|t).\epsilon^*\} = 0 \qquad (3.2.15)$$

It is noted that if $\tilde{I}_u \psi'$ contains white noise processes, and is thus of infinite variance, the minimisation of (3.2.12) is rather dubious. However this problem may be circumvented by minimising ϵ^2 over a finite bandwidth - leading to a finite variance. Equation (3.2.12) can thus be considered as a limiting case as the bandwidth goes to infinity.

Substituting for ψ' (3.2.5) in (3.2.13):

$$y^*(t+T|t) = \frac{I_r}{R}\, y(t) + \frac{B}{AR}\,[R - e^{-sT} I_r]\, u(t) \qquad (3.2.16)$$

but from (3.2.8 & 9):

$$R - e^{-sT} I_r = \tilde{I}_u D \qquad (3.2.17)$$

also, as by construction D always has a factor A, $\tilde{D}$ is defined by:

$$D = \tilde{D}.A \qquad (3.2.18)$$

Substituting (3.2.17 & 18) into (3.2.16):

$$y^*(t+T|t) = \frac{I}{R}\, y(t) + \frac{BDI}{R}\, u(t) \qquad (3.2.19)$$

The cancellation of A in the denominator of (3.2.16) is only possible because the system is controllable with respect to γ, that is D has a factor A. This is similar to the results of section 2.8.

It is also noted that if the above were repeated with the original system (3.2.1), the noise process numerator (C) would have replaced R in (3.2.19). The conversion to (3.2.5) is thus necessary to ensure stability for all C. This result is similar to that derived in section 2.10, where it was found that the Riccati equation for the Kalman filter had many solutions, only one of which produced a stable filter - corresponding to the minimum-phase disturbance representation.

3.3 The minimum variance controller as a regulator

In this section it is shown that a minimum variance controller acting on a system may be considered as a minimum variance regulator acting on an associated system. It has been shown (Åström & Wittenmark,1973) that the minimum variance regulator can form the basis of a self-tuning regulator; in chapter 4 it will be seen that the result of this section leads to the more general minimum variance controller also forming the basis of a self-tuning algorithm.

The cost function minimised by the minimum variance controller is given by (3.1.2 & 3) and repeated here as:

$$V = E\{(\Lambda^{y} y(t+T) + \Lambda^{u} u(t) + \Lambda^{w} w(t))^2 | t\} \qquad (3.3.1)$$

Defining an auxiliary function ϕ_y by:

$$\phi_y(t+T) = \Lambda^{y} y(t+T) \qquad (3.3.2)$$

the system (3.2.1) may be rewritten as:

$$\phi_y = e^{-sT} \Lambda^{y} \frac{B}{A} u + \Lambda^{y} \frac{C}{D} \psi \qquad (3.3.3)$$

Equation (3.3.3) is of the same form as (3.2.1) and as no restrictions have been placed on the relative orders of the numerator and denominator of Λ^{y}, ϕ_y may contain white noise processes. Define the numerator (Λ^{y}_{n}) and denominator (Λ^{y}_{d}) polynomials of Λ^{y} as:

$$\Lambda^{y} = \Lambda^{y}_{n} / \Lambda^{y}_{d} \qquad (3.3.4)$$

Using the results of section 3.2 the least-squares prediction (ϕ^{*}_{y}) of ϕ_y is given by:

$$\phi^{*}_{y}(t+T|t) = I \frac{\phi(t)}{\frac{r}{R}} + \frac{B \Lambda^{y} \tilde{D} I \, u(t)}{\frac{n \quad u}{R}}$$

$$= I \frac{\Lambda^{y} y(t)}{\frac{r}{R} \frac{\tilde{n}}{\Lambda^{y}_{d}}} + \frac{B \Lambda^{y} \tilde{D} I \, u(t)}{\frac{n \quad u}{R}} \qquad (3.3.5)$$

The polynomial R is obtained from the spectral factorisation (3.2.3) and given by:

$$R(s).R(-s) = C(s).\Lambda_n^y(s).C(-s).\Lambda_n^y(-s) \qquad (3.3.6)$$

and I_r and $\tilde{I}_u$ from the realisability relations (3.2.6 & 7):

$$\frac{R(s)}{D(s)\Lambda_d^y(s)} = R'(s) + \frac{R''(s)}{D(s)\Lambda_d^y(s)} \qquad (3.3.7)$$

where the degree of R' is the difference between the numerator and denominator degrees of the left hand side of (3.3.7). The remaining relations are:

$$e^{sT}\frac{R''}{D\Lambda_d^y} \doteq e^{sT}\frac{I'}{\frac{u}{D\Lambda_d^y}} + \frac{I}{\frac{r}{D\Lambda_d^y}} \qquad (3.3.8)$$

$$\tilde{I}_u = \tilde{I}'_{\frac{u}{D\Lambda_d^y}} + R' \qquad (3.3.9)$$

note that the polynomial division in (3.3.9) is exact.

It is recalled that in section 3.2, it was found that the estimate (ϕ^*) was uncorrelated with the estimation error (ϵ^*). Using this result the cost function V (3.1.1) becomes:

$$V = \left(\phi_y^*(t+T|t) + \Lambda^u u(t) + \Lambda^w w(t)\right)^2 + E\{(\epsilon^*(t+T))^2|t\} \qquad (3.3.10)$$

As the two terms are positive and mutually independent, the minimising value of u(t) is obtained from:

$$\phi_y^*(t+T|t) + \Lambda^u u(t) + \Lambda^w w(t) = 0 \qquad (3.3.11)$$

Note that $\Lambda^{u} u(t)$ and $\Lambda^{w} w(t)$ are known quantities at time t, and so defining:

$$\phi(t+T) = \phi_{y}(t+T) + \Lambda^{u} u(t) + \Lambda^{w} w(t) \qquad (3.3.12)$$

it follows directly that the least-squares estimate (ϕ^{*}) of $\phi(t+T)$ is given by:

$$\phi^{*}(t+T|t) = \phi_{y}^{*}(t+T|t) + \Lambda^{u} u(t) + \Lambda^{w} w(t) \qquad (3.3.13)$$

The associated error is given by:

$$\begin{aligned} \phi(t+T) - \phi^{*}(t+T|t) &= \phi_{y}(t+T) - \phi_{y}^{*}(t+T|t) \\ &= \epsilon^{*}(t+T) \end{aligned} \qquad (3.3.14)$$

Effectively the preceeding equations imply that the auxiliary system (3.3.3) may be decomposed into a deterministic term (ϕ^{*}) and a completely unpredictable term (ϵ^{*}), that is:

$$\phi(t+T) = \phi^{*}(t+T|t) + \epsilon^{*}(t+T) \qquad (3.3.15)$$

The control law (3.3.11), minimising the expected variance of (3.3.1), involves setting the deterministic component to zero, that is:

$$\phi^{*}(t+T|t) = 0 \qquad (3.3.16)$$

The cost function V (3.1.1) and its associated control law (3.3.16) correspond to those of the minimum variance regulator applied to the auxiliary system (3.3.3) whose output is ϕ. For the purposes of this thesis, the important result of this section is that when the control law (3.3.16) is applied, the steady-state output of the auxiliary system is given by:

$$\phi(t+T) = \epsilon^*(t+T) \tag{3.3.17}$$

where $\epsilon^*(t+T)$ is uncorrelated with y(t), u(t) and w(t). The error process (ϵ^*) is given by white noise passing into the 'unrealisable' polynomial I_u, that is:

$$\epsilon^*(t+T) = I_u \psi(t+T) \tag{3.3.18}$$

The impulse response corresponding to I_u is depicted in Fig. 3.3.1.

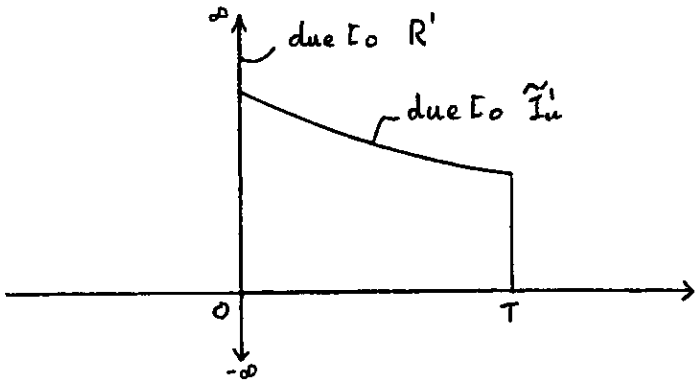

Fig. 3.3.1 The error transfer function impulse response

Using the explicit formulae (3.3.5 & 13) for the predictor of ϕ, the control law (3.3.16) becomes:

$$\phi^*(t+T|t) = \frac{I_r \Lambda_n^y}{R \Lambda_d^y} y(t) + \frac{B \tilde{D} \tilde{I}_u \Lambda_u^y}{R} u(t) + \Lambda^u u(t) + \Lambda^w w(t) = 0 \tag{3.3.19}$$

That is:

$$\alpha y' + \beta u' + \Gamma w' = 0 \tag{3.3.20}$$

where:

$$\alpha = I_r \Lambda_n^u \tag{3.3.21}$$

$$\beta = (B \Lambda_n^y \tilde{D} \tilde{I}_u \Lambda_d^u + \Lambda_n^u R) \tag{3.3.22}$$

$$\Gamma = R \tag{3.3.23}$$

and: $y' = \frac{1}{\Lambda_d^y} y$, $u' = \frac{1}{\Lambda_d^u} u$, $w' = \Lambda^w w$ (3.3.24)

Note that if Λ_n^u has no right hand plane roots, it is a factor of R(s) and thus may be cancelled from the control law.

The closed loop characteristic polynomial is given by:

$$P_c(s) = \beta \Lambda_d^u A + e^{-sT} I \alpha \Lambda_d^u B$$

$$= \Lambda_d^u(\Lambda_n^u \Lambda_d^u B\tilde{D}\tilde{I}_u + \Lambda_n^u R) + e^{-sT} \Lambda_u^u \Lambda_d^u I_r B \qquad (3.3.25)$$

using the realisability relations (3.3.7 & 8):

$$P_c(s) = R(\Lambda_n^u \Lambda_d^u B + \Lambda_d^u \Lambda_n^u A) \qquad (3.3.26)$$

As R is obtained from the spectral factorisation relationship (3.3.6), it has no right hand plane roots. The stability of the closed loop is thus determined by the term in parentheses. This term corresponds to the numerator polynomial of the auxiliary system (3.3.12) with ϕ as output; the condition for stability is thus that this polynomial has no right hand plane roots. This result is analogous to a similar result concerning the discrete-time minimum variance regulator (Åström, 1970).

As in chapter 2, the control law (3.3.20) cannot be exactly realised in terms of rational transfer functions. In this case, the offending term is $\tilde{I}_u'$ which is given by equations (3.3.8 & 9) as:

$$\tilde{I}_u' = \frac{R'' - e^{-sT} I_r'}{D \Lambda_d^u} \qquad (3.3.27)$$

As in equation (2.4.21), if $D\Lambda_d^y$ has distinct zeros (p_i), (3.3.4) becomes:

$$\tilde{I}_u' = \sum_i^n r_i \left(\frac{1 - e^{-(s - p_i)T}}{(s - p_i)}\right) \qquad (3.3.28)$$

where r_i is the residue of $\frac{R^u}{D\Lambda_d^y}$ at p_i.

Using the series expansion of e , (3.3.28) becomes:

$$\tilde{I}_u' = T\sum_i^n r_i\left(1 - (s - p_i)\frac{T}{2!} + (s - p_i)^2\frac{T^2}{3!} - \ldots\right) \qquad (3.3.29)$$

The expression (3.3.29) can be approximated either by truncation or by some rational approximation. For example using the first order Pade approximation for $e^{-(s - p_i)T}$:

$$\tilde{I}_u' = \sum_i^n \frac{r_i}{(s - p_i)}\left(1 - \frac{(1 - \frac{1}{2}(s - p_i)T)}{(1 + \frac{1}{2}(s - p_i)T)}\right)$$

$$= T\sum \frac{r_i}{(1 + \frac{1}{2}(s - p_i)T)} \qquad (3.3.30)$$

3.4 The minimum variance controller as a model follower

In this section, the subclass of minimum variance controllers with no control weighting ($\Lambda^u = 0$) is considered. It is shown that apart from the unavoidable system time-delay, the closed loop system behaves as an arbitrary model, specified by the cost function, with respect to the set point. The autocorrelation function of the error corresponds to a transfer function with the same poles as the model, but the zeros are system

dependent. This result provides the basis for a self-tuning algorithm which makes the closed loop system largely independent of possibly varying open loop parameters.

Consider the cost function (3.3.1) with $\Lambda^u = 0$, that is:

$$V = E\left\{\left(\Lambda^y y(t+T) + \Lambda^w w(t)\right)^2 \mid t\right\} \qquad (3.4.1)$$

In section 3.3 it is shown that this cost function is minimised by setting the least squares prediction of an auxiliary function to zero (3.3.15). That is:

$$\phi^*(t+T \mid t) = 0 \qquad (3.4.2)$$

where in this case:

$$\phi(t+T) = \Lambda^y y(t+T) + \Lambda^w w(t) \qquad (3.4.3)$$

It is also shown that the resultant closed loop system is specified by equation (3.3.17) as:

$$\phi(t+T) = \epsilon^*(t+T) \qquad (3.4.4)$$

Combining (3.4.3 & 4) and rearranging:)

$$y(t+T) = -\frac{\Lambda^w}{\Lambda^y} w(t) + \frac{1}{\Lambda^y} \epsilon^*(t+T) \qquad (3.4.5)$$

That is, in closed loop, the system behaves as an arbitrary model $-\frac{\Lambda^w}{\Lambda^y}$ with respect to the set point, and as an arbitrary model $\frac{1}{\Lambda^y}$ with respect to the prediction error .

The question immediately arises as to whether the control law corresponding to (3.4.1) is practicable.

For instance it might require excessive control signals or excessive component gains. Using equations (3.3.20-24) the control law (3.4.2) becomes:

$$u(t) = -\frac{I_r \Lambda^y_n y(t) + \Lambda^y_d \Lambda^w R\, w(t)}{\Lambda^y_n \Lambda^y_d \, BD\tilde{I}_u} \qquad (3.4.6)$$

From (3.4.5) it follows that the numerator of Λ^y must have no roots in the right hand plane if the model followed is to be stable. In this case the spectral factorisation (3.3.6) becomes:

$$R(s) = \Lambda^y_n(s).C^*(s) \qquad (3.4.7)$$

where:

$$C^*(s).C^*(-s) = C(s).C(-s)$$

Substituting (3.4.7) into (3.4.6) yields the slightly simpler expression:

$$u(t) = -\frac{I_r y(t) + \Lambda^y_d C^* \Lambda^w w(t)}{\Lambda^y_d BD\, \tilde{I}_u} \qquad (3.4.8)$$

The control law (3.4.8) has a factor $\tilde{I}_u$ in the denominator; hence it is necessary that $\tilde{I}_u$ is not a zero polynomial. From the expression for the closed loop system (3.4.5) it follows that if $\tilde{I}_u$ is non-zero and ψ is non-zero then the model following error is non-zero. It is concluded that (as would be expected) it is not possible to eliminate the effects of disturbances. It is recalled from equation (3.3.9) that $\tilde{I}_u$ is the sum of two terms : the first (R^1) depending on the relative degrees of the disturbance transfer function ($\frac{C}{D}$) and the weighting transfer function (Λ^y) ; the second tending to zero as the system time-delay (T) tends to zero (3.3.29). It follows that either the time delay (T) be non-zero or the auxiliary system output (ϕ) must contain white

noise processes.

The above conditions ensure that the coefficients of the control law are finite. However the control law may still produce unacceptably large control signals, for instance when trying to make a system follow a step input. Hence the model to be followed, and the set point variations, must be realistic with regard to the system to be controlled and the control effort available, In this context it is noted that the set point weighting (Λ^w) appears directly in the control law as a filter operating on w. It can thus be used to filter out high frequency components of the set point, whilst enabling the system to follow slowly varying set points in a suitable manner.

From the results of section 3.3, the closed loop characteristic polynomial is given by (3.3.26) as:

$$P_c = RB\Lambda_n^u \qquad (3.4.9)$$

As R (by definition, 3.4.7) and Λ_n^u (by choice) have no roots in the right hand plane, the stability condition is that B has no right hand plane roots.

Example 3.4.1

Consider the system (Fig. 3.4.1) given by:

$$y = \frac{1}{s+a} u + \frac{1}{s+d} v$$

For the purposes of this example, both a and d are

taken to be positive (though the former condition is not essential). It is desired to make the system follow the model:

$$M(s) = \frac{1}{1 + T_c\zeta s + T_c^2 s^2} \quad \text{where } T_c>0, \zeta>0$$

Hence choose the cost function specified by:

$$\Lambda^y = 1 + T_c\zeta s + T_c^2 s^2 \; ; \; \Lambda^u = 0 \; ; \; \Lambda^w = -1 \; .$$

For clarity the derivation follows the steps given in section 2.6.

a) Noise conversion

$$C = s + a \; ; \; D = (s + a)(s + d)$$

b) Spectral factorisation

From equation (3.4.7):

$$C^* = s + a \; ; \; R = (s + a)(1 + T_c\zeta s + T_c^2 s^2)$$

c) Realisability relation

The system is assumed to have a vanishingly small (but non-zero) time-delay. From equation (3.3.7):

$$R'(s) = T_c(\zeta - T_c d) + T_c s^2 = \tilde{I}_u$$

$$R''(s) = [1 - T_c d(\zeta - T_c d)](s + a) = I_r$$

d) Approximate $\tilde{I}_u$

As the time-delay is infinitesimal, $\tilde{I}_u$ is taken to be zero.

e) Calculate control law

From equation (3.4.8):

$$u = -\frac{I_r y + C^* \Lambda^w w}{BD\tilde{I}_u}$$

Hence the control law is as given by Fig. 3.4.1

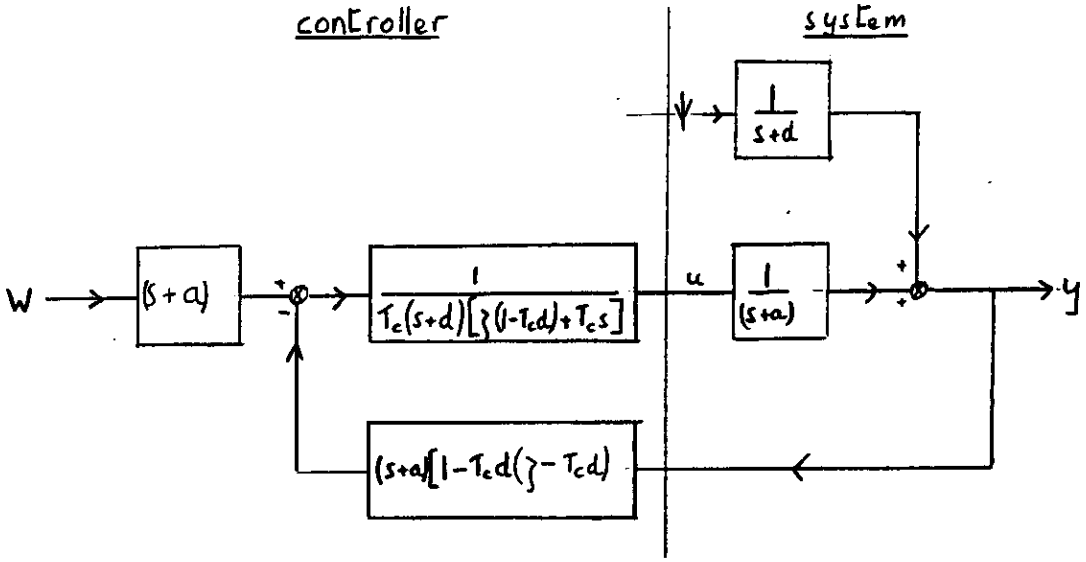

Fig. 3.4.1 The closed loop system

It can be verified from Fig 3.4.1 that:

$$\bar{y} = \frac{1}{1 + T_c\zeta s + T_c^2 s^2} w + \frac{T_c[(\zeta - T_c d) + T_c s]}{1 + T_c\zeta s + T_c^2 s^2} \psi$$

This result agrees with equation(3.4.5).

At first sight it would seem that the same result would have been obtained with the weightings:

$$\Lambda^y = 1 \; ; \; \Lambda^u = 0 \; ; \; \Lambda^w = \frac{-1}{1 + T_c\zeta s + T_c^2 s^2}$$

Repeating the above calculations with these weightings gives:

$$I_r = s + a \; ; \; \tilde{I}_u = 0$$

The latter equation leads to an infinite loop gain; making the system infinitely fast. Λ^w then filters the set point to give the required response. Such a strategy - eliminating the disturbance - is impracticable due to the infinite gain required.

3.5 The minimum variance controller as a classical control law.

In this section is considered the subclass of minimum variance control laws with unity weighting on the output error, and arbitrary weighting on the control signal. That is the cost function V is given by:

$$V = E\{(y(t+T) - w(t) + \Lambda^u u(t))^2 | t\} \qquad (3.5.1)$$

It will be seen that the resultant control law is a more sophisticated version of a classical control law involving a 'Smith predictor' (Smith,1959).

From equation (3.3.11), the control law corresponding to (3.5.1) is given by:

$$u(t) = -\frac{1}{\Lambda^u}(y^*(t+T|t) - w(t)) \qquad (3.5.2)$$

Firstly consider the case of zero time-delay, then as the least-squares estimate of y(t),at time t, is y itself, (3.5.2) becomes:

$$u(t) = -\frac{1}{\Lambda^u}(y(t) - w(t)) \qquad (3.5.3)$$

This is a classical control law with transfer function $1/\Lambda^u$, for example the PI control law:

$$\frac{1}{\Lambda^u} = K(1 + \frac{1}{sT_i}) \qquad (3.5.4)$$

where K is the 'proportional gain', and T_i the integral time constant. Comparing (3.5.2 & 3) it follows that the former is a generalisation of the latter appropriate to time-delay systems. From equation (3.3.26), the closed loop characteristic polynomial is given by:

$$P_c(s) = C^*(s)[\Lambda_d^u B + \Lambda_n^u A] \qquad (3.5.5)$$

where Λ_n^u and Λ_d^u are the numerator and denominator polynomials of the control weighting transfer function Λ^u. As the closed loop polynomial is independent of the system time-delay (T), the effect of the predictor is to allow a classical design ignoring the system time-delay.

From the practical point of view, although sensible choice of the transfer function $1/\Lambda^u$ requires at most approximate knowledge of the system, the predictor design requires accurate knowledge. This latter fact is a drawback to the design of an invariant control law, but when incorporated into a self-tuning strategy, the predictor parameters are automatically detemined and this problem no longer arises.

The idea of designing a control law to operate on a predicted output - rather than the output itself - seems to be due to Smith (Smith,1959). His predictor generates $y^s(t+T|t)$ where:

$$y^s(t+T|t) = y(t) + \frac{B}{A}(1 - e^{-sT})\, u(t) \qquad (3.5.6)$$

This should be compared with the predictor of section 3.2 given by:

$$y^*(t+T|t) = \frac{I_r}{R} y(t) + \frac{B\tilde{D}\tilde{I}_u}{R} u(t) \qquad (3.2.19)$$

If the system is stable, and the noise transfer function minimum phase, (3.2.19) becomes:

$$y^*(t+T|t) = \frac{I_r}{C} y(t) + \frac{B}{C}.\frac{\tilde{AD}}{A}.\tilde{I}_u u(t) \qquad (3.5.7)$$

Using the realisability relation (3.2.9):

$$\tilde{AD}\tilde{I}_u = I_u = C - e^{-sT} I_r \qquad (3.5.8)$$

Combining (3.5.7 & 8):

$$y^*(t+T|t) = \frac{I_r}{C} y(t) + \frac{B}{A}\left(1 - e^{-sT}\frac{I_r}{C}\right) u(t) \qquad (3.5.9)$$

If all the system time constants are long compared with the time-delay (T), from the realisability relation (3.2.9) it follows that: $I_r \doteqdot C$. Inthis case the Smith predictor (3.5.6) approximates to the optimal least-squares predictor (3.5.9). It is emphasised, however, that the introduction of cancelling A polynomials in (3.5.7) is only possible if A has no right hand plane roots.

It is concluded that for stable systems, disturbed by minimum phase noise processes, both slow with respect to the time-delay, the Smith predictor approximates to the optimal least-sqares predictor.

Example 3.5.1

Consider the system:

$$y = \frac{e^{-sT}}{s+a} u + \frac{1}{s+a} \gamma$$

From section 3.2, the optimal predictor of y is given by:

$$y^*(t+T|t) = e^{-aT} y(t) + \tilde{I}_u u(t)$$

where: $\tilde{I}_u = \frac{1 - e^{-(s+a)T}}{s+a}$, the division being taken to be exact. From (3.3.30) if $T \ll a$, $\tilde{I}_u$ may be approximated by:

$$\tilde{I}_u \doteq \frac{T}{1 + \frac{1}{2}(s+a)T}$$

The generalisation of a proportional control law is given by (3.5.2) as:

$$u = -\frac{1}{\lambda}(y^*(t+T|t) - w(t))$$

$$\therefore u(t) = -\frac{e^{-aT} y(t) - w(t)}{\lambda + \tilde{I}_u}$$

From the cost function (3.5.1), infinite proportional gain ($\lambda = 0$) corresponds to the continuous minimum variance regulator. Note that if $T>0$, $\tilde{I}_u$ is non-zero and hence the control law does not involve an infinite gain.

3.6 An infinitesimal horizon control law

In this section is considered the subclass of minimum variance controllers discussed in section 3.5, but with scalar weighting on control. That is the control law is such as to minimmise the cost function V_o where:

$$V_o = E\{(y(t+T) - w(t) + \lambda_o u(t))^2 | t\} \qquad (3.6.1)$$

In section 3.5, this control law was interpreted as proportional feedback of the predicted output error;

hence it will be called the proportional minimum variance control law.

The performance of this control law will, in section 3.8, be compared with that of the optimal control law of chapter 2. Previous work on discrete-time control systems has compared the performance of k-step-ahead and infinite-step-ahead control laws (Hughes & Jacobs,1973). In this section it is demonstrated that, under certain circumstances, the proportional minimum variance control law is equivalent to the infinitesimal horizon control law - the continuous analogue of the discrete-time k-step-ahead control law. Hence, under certain circumstances, the comparison of the proportional minimum variance control law and the optimal control law of chapter 2 becomes the continuous analogue of the problem considered by Hughes & Jacobs.

It is now shown that under certain circumstances, and by suitable choice of λ_o, the control law minimising V_o (3.6.1) is identical to that minimising V_δ, where V_δ is given by:

$$V_\delta = \lim_{\delta \to 0} E\left\{ \left(y(t+\delta+T) - w(t+\delta) \right)^2 + \int_t^{t+\delta} \left[\left(y(\alpha+T) - w(\alpha) \right)^2 + \lambda_\delta u^2(\alpha) \right] d\alpha \right\} \quad (3.6.2)^\dagger$$

The cost function V corresponds to that of the optimal control problem where the control horizon is infinitesimal; this will be called the infinitesimal horizon control

† The first term represents the terminal cost.

problem. It is noted that if δ in equation (3.6.2) becomes infinite, the term outside the integral would become negligible; the cost function would then correspond to that of the optimal infinite-horizon control law (2.1.1).

As in section 3.3, the expectation operator in (3.6.2) can be removed by setting:

$$y(t+\delta+T) = y^*(t+\delta+T|t+\delta) + \epsilon^*(t+\delta) \qquad (3.6.3)$$

The term y^* is the least-squares prediction of y as given by (3.2.19), and ϵ^* is the associated error. Assuming that y ,u and w are continuous between t and $t+\delta$, (3.6.2) becomes:

$$\begin{aligned} V_\delta = \underset{\delta\to o}{\text{Lt.}}\Big[& \left(y^*(t+\delta+T|t+\delta) - w(t+\delta)\right)^2 \\ & + \delta\left(y^*(t+T|t) - w(t)\right)^2 + \delta\lambda_\delta u^2(t) \\ & + o(\delta^2) + (1+\delta)\, E(\epsilon^{*2})\Big] \end{aligned} \qquad (3.6.4)$$

Differentiating (3.6.4) with respect to u(t) and setting the result to zero gives:

$$\begin{aligned} \underset{\delta\to o}{\text{Lt.}}\Big[& 2\left(y^*(t+\delta+T|t+\delta) - w(t+\delta)\right).\frac{\partial}{\partial u(t)} y^*(t+\delta+T|t+\delta) \\ & + 2.\delta\left(y^*(t+T|t) - w(t)\right).\frac{\partial}{\partial u(t)} y^*(t+T|t) \\ & + 2.\delta\lambda_\delta u(t) + o(\delta^2)\Big] = 0 \end{aligned} \qquad (3.6.5)$$

y^* may be expanded in the form of a Taylor series as:

$$y^*(t+\delta+T\,|\,t+\delta) = y^*(t+T\,|\,t) + \delta\,\dot{y}^*(t+T\,|\,t) + o(\delta^2) \qquad (3.6.6)$$

There are thus three cases to consider:

a) $\frac{\partial y^*}{\partial u} \neq 0$

b) $\frac{\partial y^*}{\partial u} = 0$, $\frac{\partial \dot{y}^*}{\partial u} = 0$

c) $\frac{\partial y^*}{\partial u} = \frac{\partial \dot{y}^*}{\partial u} = 0$

The step response of systems corresponding to cases a) ,b) and c) are depicted in Fig. 3.6.1.

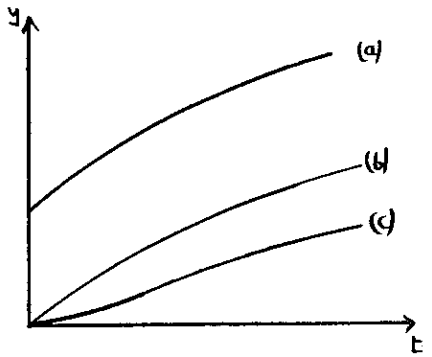

Fig. 3.6.1 Initial response to a step input.

In case a), setting $\delta \neq 0$ gives the minimising control law:

$$y^*(t+T\,|\,t) - w(t) = 0 \qquad (3.6.7)$$

In case b) setting $\delta \neq 0$ gives the minimising control law:

$$\left(y^*(t+T\,|\,t) - w(t)\right)\frac{\partial \dot{y}^*}{\partial u} + \lambda_\delta u(t) = 0 \qquad (3.6.8)$$

Incase c) setting $\delta \neq 0$ gives the minimising control law:

$$\lambda_\delta u(t) = 0 \qquad (3.6.9)$$

From equation (3.2.16), the prediction (y^*) may be expressed in its 'uncancelled' form as:

$$y^*(t+T|t) = \frac{I_r}{R} y(t) + \frac{B}{A} u(t) - \frac{BI_r}{AR} u(t-T) \qquad (3.6.10)$$

where: $$\frac{B(s)}{A(s)} = \frac{b_0 + b_1 s + \ldots + b_n s^n}{a_0 + a_1 s + \ldots + a_n s^n}$$

$$= b_n + \frac{1}{s}(b_{n-1} - b_n a_{n-1}) + \frac{1}{s} f(s) \qquad (3.6.11)$$

Now:

$$\frac{\partial}{\partial u(t)} \int_{t_0}^{t} u(\alpha)\, d\alpha = \frac{\partial}{\partial u(t)} \underset{\delta \to 0}{\text{Lt.}} \left[\delta u(t) + \int_{t_0}^{t-\delta} u(\alpha) d\alpha \right]$$

$$= \underset{\delta \to 0}{\text{Lt.}} [\delta] = 0 \qquad (3.6.12)$$

and similarly for higher integrals of u(t). From equations (3.6.10-12) it follows that:

$$\frac{\partial y^*}{\partial u(t)}(t+T|t) = b_n \qquad (3.6.13)$$

also if $b_n = 0$ it follows that:

$$\frac{\partial \dot{y}^*}{\partial u(t)}(t+T|t) = b_{n-1} \qquad (3.6.14)$$

Substituting (3.6.13 & 14) into (3.6.7-9), it follows that the control law is given by:

$$y(t+T|t) - w(t) = 0 \qquad \text{if } b_n \neq 0 \qquad (3.6.15)$$

$$y(t+T|t) - w(t) + \frac{\lambda_\delta}{b_{n-1}} u(t) = 0 \qquad \text{if } b_n = 0 \qquad (3.6.16)$$

$$\lambda_\delta u(t) = 0 \qquad \text{if } b_n = b_{n-1} = 0 \qquad (3.6.17)$$

Compare equations (3.6.15-17) with the proportional minimum variance control law given by (3.5.2) as:

$$y^*(t+T|t) - w(t) + \lambda_0 u(t) = 0 \qquad (3.6.18)$$

It is seen that the control laws are identical when:

$$b_n \neq 0 \ , \ \lambda_0 = 0 \qquad (3.6.19)$$

$$b_n = 0 \ , \ \lambda_0 = \frac{\lambda_1}{b_{n-1}} \qquad (3.6.20)$$

It is concluded that the particular minimum variance control law corresponding to V_0 (3.6.1) is closely related to the infinitesimal horizon control law corresponding to V_1 (3.6.2).

3.7 State space interpretation of a minimum variance controller.

In this section a state space interpretation of the proportional minimum variance controller (as defined in section 3.6) is given.

As in section 2.9, the system under consideration (3.1.1) may be rewritten in state space termininology as:

$$\dot{\underline{x}}(t) = \underline{A}.\underline{x}(t) + \underline{B}.u(t-T) + \underline{C}\psi(t) \qquad (3.7.1)$$

$$y(t) = \underline{M}.\underline{x}(t) \qquad (3.7.2)$$

where $\underline{A}$ is an $n \times n$ matrix, $\underline{B}$ and $\underline{C}$ column n-vectors, and $\underline{M}$ a row n-vector. In a similar manner, the set point (w) is taken to be representable by:

$$\dot{\underline{x}}_w(t) = \underline{A}_w \underline{x}_w(t) + \underline{C}_w \psi_w(t) \qquad (3.7.3)$$

$$w(t) = \underline{M}_w \underline{x}_w(t) \qquad (3.7.4)$$

From section 3.5, the proportional minimum variance control law is given by:

$$u(t) = \frac{1}{\lambda_o}\left(y^*(t+T|t) - w(t)\right) \qquad (3.7.5)$$

It is recalled from section 3.2 that $y^*(t+T|t)$ is the least-squares prediction of $y(t+T)$ based on data up to time t; the important characteristic being that the prediction error is uncorrelated with the prediction. Similarly, in chapter 2 the least-sqares prediction, $\hat{\underline{x}}(t+T|t)$, of the state $\underline{x}(t+T)$ was found; again the prediction error was uncorrelated with the prediction. That is:

$$\underline{x}(t+T) - \hat{\underline{x}}(t+T|t) = \hat{\underline{\epsilon}}(t+T) \qquad (3.7.6)$$

where: $$E\left[\hat{\underline{x}} \cdot \hat{\underline{\epsilon}}^T\right] = 0 \qquad (3.7.7)$$

premultiplying by $\underline{M}$, and using (3.7.2):

$$y - \underline{M}\,\hat{\underline{x}} = \underline{M}\,\hat{\underline{\epsilon}} \qquad (3.7.8)$$

substituting into (3.7.7):

$$E\left[\underline{M}\,\underline{x}.\hat{\underline{\epsilon}}^T\underline{M}^T\right] = 0 \qquad (3.7.8)$$

It is concluded that as $\underline{M}\,\hat{\underline{x}}$ is uncorrelated with $\underline{M}\,\underline{\epsilon}$, it is also the least sqares prediction of y. That is:

$$y^*(t+T|t) = \underline{M}\,\underline{x}(t+T|t) \qquad (3.7.10)$$

This is a well known result (Sage & Melsa,1971).

As w is a known quantity, it follows from section 2.10 that $\underline{x}$ can be exactly determined in the steady state. It is thus possible using (3.7.4 & 10) to rewrite the control law (3.7.5) as:

$$u(t) = \frac{1}{\lambda_0} \underline{M}\, \hat{\underline{x}}(t+T|t) + \underline{M}_w \underline{x}_w \qquad (3.7.11)$$

That is the state feedback vector is the measurement vector ($\underline{M}$) of the combined state equations (3.7.1-4), divided by λ_0. If the system has distinct roots and and is written in normal form, the elements of $\underline{M}$ and $\underline{M}_w$ are all unity, and thus the state feedback vector ($\underline{K}_c$) is given by:

$$\underline{K}_c = \left(\frac{1}{\lambda_0}, \frac{1}{\lambda_0}, \ldots, \frac{1}{\lambda_0}\right) \qquad (3.7.12)$$

In the literature (Kwakernaak & Sivan, 1972) it is shown that the state space solution of the infinitesimal horizon regulator problem (section 3.6), for the system (3.7.1 & 2) but with no time-delay, is given by:

$$u(t) = \frac{1}{\lambda_0} \underline{B}^T \underline{M}^T \underline{M}\, \hat{\underline{x}}(t|t) \qquad (3.7.13)$$

This state feedback vector corresponds to the initial condition of the associated Riccati equation. Comparing (3.7.10 & 13) and noting (section 2.9) that:

$$\underline{B}^T \underline{M}^T = b_{n-1} \qquad (2.9.7)$$

it follows that the infinitesimal horizon control law and the proportional minimum variance control law are identical if: $\lambda_0 = \frac{\lambda_0}{b_{n-1}}$. This agrees with section 3.6.

3.8 On the optimality of a minimum variance controller.

In this section the performance of the proportional minimum variance controller (section 3.6) is considered. It is recalled that the proportional minimum variance controller minimises the cost function:

$$V_o = E\left[\left(y(t+T) - w(t) + \lambda_o u(t)\right)^2 / t\right] \qquad (3.8.1)$$

The corresponding control law is given by (3.6.18) as:

$$u(t) = \frac{1}{\lambda_o}(y(t+T|t) - w(t)) \qquad (3.8.2)$$

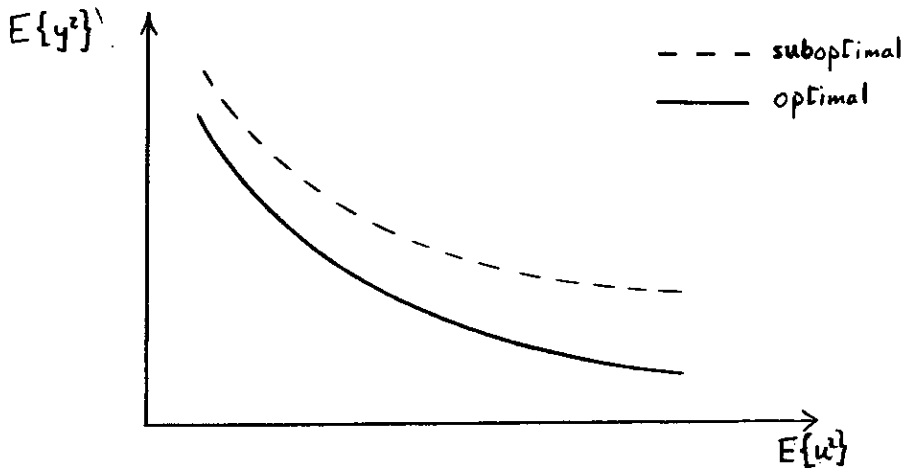

Fig 3.8.1 Controller performance curves.

The performance criterion used here is depicted in Fig 3.8.1;each curve represents a locus of system output error variance against the corresponding input control variance - obtained with a particular system, controlled by a given class of control laws, as a control law design parameter is varied. The full curve represents an optimum series of control laws, in the

sense that for every value of control input variance, the output error variance is the least possible for all steady-state realisable control laws operating on the particular system.

This optimality criterion may be stated mathematically using the Lagrange multiplier technique (Kwakernaak & Sivan, 1972) as the minimisation of the cost function:

$$V = \lim_{t_o \to \infty} \frac{1}{2t_o} \int_{-t_o}^{t_o} E\left\{\left(y(t+T) - w(t)\right)^2 + \lambda_o u^2(t)\right\} dt \quad (3.8.3)$$

This is the cost function (2.1.1) considered in chapter 2.

The dashed line in Fig 3.8.1 is obtained by replacing the optimal control law by the proportional minimum variance control law (3.8.2), using a range of values of λ_o. The criterion of performance of this control law, operating on a particular system, is taken to be some measure of the closeness of the two curves displayed in Fig.3.8.1.

In section 3.6 it was noted that in some circumstances, the proportional minimum variance controller corresponded to the optimal control law with infinitesimal horizon (3.6.2): in chapter 2 it was found that the control law minimising V (3.8.3) corresponded to the optimal control law with infinite horizon. Thus in some cases, the comparison of the proportional minimum variance controller and the optimal controller can be translated

into the comparison of steady-state infinitesimal and infinite horizon controllers. The discrete-time analogue of the latter comparison (with zero set point) has been considered previously (Ashton,1974; Hughes & Jacobs,1973; Hughes,1974). It was shown that for first order systems, the two control laws had identical performance; but apart from this result, work was aimed at numerical determination of performance curves related to those of Fig.3.8.1. In some cases it was found that the performance curves were widely seperated, and for other systems the curves lay close together. In his summary of this work, Hughes states: "It is yet to be established what characterises such systems." - this section supplies this characterisation for the continuous-time problem, the discrete-time results are discussed in section 3.10.

Firstly, the formula for the proportional minimum variance control law is rewritten in a similar manner to that of the optimal control law. Substituting: $\Lambda^y = 1$, $\Lambda^u = \lambda_o$ and $\Lambda^w = -1$ into equations (3.3.20-24) the proportional minimum variance control law is given by:

$$\alpha^o y + \beta^o u + \Gamma^o w = 0 \qquad (3.8.4)$$

Where:

$$\alpha^o = I_r^o \qquad (3.8.5)$$

$$\beta^{\circ} = B\tilde{D}\tilde{I}_{u}^{\circ} + \lambda_{o}C^{*} \tag{3.8.6}$$

$$\Gamma^{\circ} = C^{*} \tag{3.8.7}$$

I_r° and $\tilde{I}_u$ are given by (3.3.7-9) as:

$$\frac{C^{*}}{D} = \tilde{I}_{u}^{\circ} + e^{-sT}\frac{I_r^{\circ}}{D} \tag{3.8.8}$$

where C^{*} has no roots in the right-hand plane and is given by:

$$C^{*}(s).C^{*}(-s) = C(s).C(-s) \tag{3.8.9}$$

Substituting (3.8.8) into (3.8.6), β° may be written as:

$$\beta^{\circ}A = C^{*}(B + \lambda_{o}A) - e^{-sT}I_r^{\circ}B \tag{3.8.10}$$

Define P° as:

$$P^{\circ} = (B + \lambda_{o}A) \tag{3.8.11}$$

then (3.8.10) becomes:

$$\beta^{\circ}A = C^{*}P^{\circ} - e^{-sT}I_r^{\circ}B \tag{3.8.12}$$

This should be compared with the expression for the optimal control law (2.5.1 & 2) repeated here as:

$$\alpha^{\infty}y + \beta^{\infty}u + \Gamma^{\infty}w = 0 \tag{3.8.13}$$

$$\alpha^{\infty} = I_r^{\infty} \tag{3.8.14}$$

$$\beta^{\infty}A = C^{*}P^{\infty} - e^{-sT}I_r^{\infty}B \tag{3.8.15}$$

$$\Gamma^{\infty} = C^{*}.\frac{I_r^{w}}{C^{w}} \tag{3.8.16}$$

P^{∞} has no roots within the right-hand plane and is given by (2.4.7) as:

$$P^{\infty}(s).P^{\infty}(-s) = B(s).B(-s) + \lambda_{\infty}A(s).A(-s) \quad (3.8.18)$$

I^{∞}_r is given by (2.4.24) as:

$$\frac{B(-s)}{P(-s)}.\frac{C^{*}(s)}{D(s)} = \frac{I(s)}{D(s)} + \frac{J(-s)}{P(-s)} \quad (3.8.19)$$

$$\frac{I(s)}{D(s)} = \tilde{I}_{\infty}(s) + e^{-sT}\frac{I_r(s)}{D(s)} \quad (3.8.20)$$

and I is given by:

$$\frac{B(-s)}{P(-s)}.\frac{C^{\nu}(s)}{D^{\nu}(s)} = \frac{I^{\nu}_r(s)}{D^{\nu}(s)} + \frac{J^{\nu}(-s)}{P(-s)} \quad (3.8.21)$$

Note that the coefficients of the control laws (3.8.4&13) may be multiplied by the same factor and still yield the same control signal; to overcome this non-uniqueness it is convenient to normalise the polynomials P° and P^{∞} such that the highest coefficient of each is unity.

If the control laws (3.8.4 & 13) are to have approximately the same coefficients, the following conditions must hold (3.8.5 & 14):

$$\frac{I^{\infty}_r}{p^{\infty}_n} = \frac{I^{\circ}_r}{p^{\circ}_n} \quad (3.8.22)$$

which from equations (3.8.8 & 19) implies that:

$$\frac{1}{p^{\infty}_n}.\frac{B(-d_i)}{P^{\infty}(-d_i)} = \frac{1}{p^{\circ}_n} \qquad i = 1, n \quad (3.8.23)$$

where d_i is the i^{th} root of D(s).

If the system (3.1.1) has no direct link, and as the highest coefficient of A is normalised to unity, it follows from (3.8.11 & 18) that: $p_w^\infty = \lambda_\infty^{\frac{1}{2}}$ and $p_w^\circ = \lambda_\circ$. For such systems it follows (section 2.12) that the left-hand side of (3.8.23) represents the i^{th} element - in normal coordinates - of the optimal state feedback vector: from section 3.7 it follows that the right-hand side of (3.8.23) represents the i^{th} element of the suboptimal state feedback vector. Condition (3.8.23) can thus be interpreted as the frequency-domain analogue of the time-domain condition that the state feedback vectors of the two control laws be close.

In a similar manner it follows from (3.8.7 & 16) that:

$$\frac{1}{p_w^\circ}\frac{B(-d_i^w)}{P^\infty(-d_i^w)} = \frac{1}{p_w^\circ} \qquad i = 1, n_d \qquad (3.8.24)$$

From (3.8.12 & 15) a further condition is that:

$$\frac{P^\circ(s)}{p_n^\circ} = \frac{P^\infty(s)}{p_n^\infty} \qquad (3.8.25)$$

By inspection of the defining equations, conditions (3.8.23-25) are both necessary and sufficient for the parameters of the two control laws to be close. Conditions (3.8.22 & 24) are not, however, independent. From equation (3.8.18), when d_i is a root of A(s), the left-hand side of (3.8.23)becomes:

$$\frac{1}{p_n^{\infty}}\frac{B(-a_i)}{P^{\infty}(-a_i)} = \frac{1}{p_n^{\infty}}\frac{P^{\infty}(a_i)}{B(a_i)} \qquad (3.8.26)$$

If (3.8.25) holds, and using the definition of P° (3.8.11), (3.8.26) becomes:

$$\frac{1}{p_n^{\infty}}\frac{P^{\infty}(a_i)}{B(a_i)} = \frac{1}{p_n^{\circ}}\frac{P^{\circ}(a_i)}{B(a_i)}$$

$$= \frac{1}{p_n^{\circ}}\frac{B(a_i) + \lambda . A(a_i)}{B(a_i)}$$

$$= \frac{1}{p_n^{\circ}} \qquad (3.8.27)$$

It is concluded that if (3.8.25) holds, condition (3.8.23) must also hold for $d_i = a_i$. Hence if the system is fully controllable, that is d_i and $d_i^{\vee}$ are roots of A(s) for all i (1-n), condition (3.8.25) is sufficient for the control laws to have close coefficients; that is the characteristic polynomial associated with the closed loop system fully defines the state feedback gains. A similar result relates equations (3.8.24 & 25).

In general, condition (3.8.25) is investigated using root-locus techniques, and then conditions (3.8.23 & 24) are checked at the uncontrollable system poles - from the previous paragraph it follows that if (3.8.25) holds, and the uncontrollable poles are 'close' to the controllable poles, conditions (3.8.23 & 24) will also hold.

Having determined conditions on the system polynomial, which ensure that the suboptimal control law parameters are close to those of the optimal control law, the fact that the optimal control law is insensitive to parameter variation (Kwakernaak & Sivan, 1972) is invoked to deduce that the performance curves (Fig. 3.8.1) will be close at this point. Note that the preceeding conditions are independent of the system time-delay.

These conditions are qualitative, and provide a useful 'rule of thumb' for characterising systems for which the suggested suboptimal control law is near optimal; more detailed design would use the previously suggested numerical methods (Hughes & Jacobs,1973; Ashton,1974). The application of the above techniques is best illustrated by a specific example.

Example 3.8.1

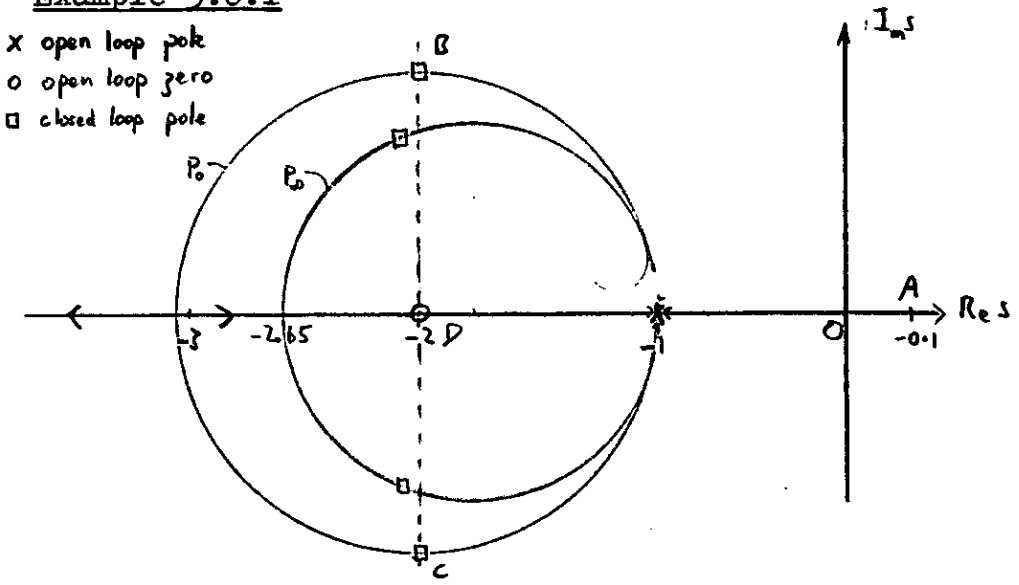

Fig. 3.8.2. The comparative root-locus diagram.

Consider the system given by:

$$\dot{y} = e^{-sT}\frac{(2+s)}{(1+s)^2}u + \frac{1}{s+d}\psi$$

The polynomial corresponding to the optimal control law is given by (3.8.18) as:

$$P^{\infty}(s).P^{\infty}(-s) = (2+s)(2-s) + \lambda_{\rho}(1+s)^2(1-s)^2$$

$$= 4 + \lambda_{\rho} - (1+2\lambda_{\rho})s^2 + \lambda_{\rho}s^4$$

The roots of this polynomial are given by:

$$p^2 = \frac{1+2\lambda_{\rho}}{2\lambda_{\rho}} \pm \frac{(1+12\lambda_{\rho})^{\frac{1}{2}}}{2\lambda_{\rho}}$$

The polynomial corresponding to the suboptimal control law is given by (3.8.11) as:

$$P^{o}(s) = (2+s) + \lambda_{o}(1+s)^2$$

$$= (2+\lambda_{o}) + (1+2\lambda_{o})s + \lambda_{o}s^2$$

The roots of P^{o} and $P^{\infty}(s).P^{\infty}(-s)$, as λ_{o} and λ_{ρ} are varied, can be plotted using conventional root-locus techniques (Jacobs,1974); this is done in Fig 3.8.2. Only the loci corresponding to $P^{\infty}(s)$ are shown: those corresponding to $P^{\infty}(-s)$ are the mirror images in the imaginary axis.

For a particular application, closed loop poles at $-2 \pm j$ are considered acceptable, these correspond to:

$$\lambda_o = 0.5 \ , \quad \underset{\lambda_o}{P^o} = s^2 + 4s + 5$$

The problem is to determine whether there exist points on the root-locus of the optimum control law close to these poles. From Fig 3.8.2, it is seen that the loci are, in fact, fairly close. By trial and error it is found that a close optimum polynomial is given by:

$$\lambda_\infty = 0.15 \ , \quad \underset{\lambda_\infty}{P^\infty} = s^2 + 4.38s + 5.26$$

with roots at: $p = -2.19 \pm 0.68j$

These roots are marked in Fig 3.8.2.

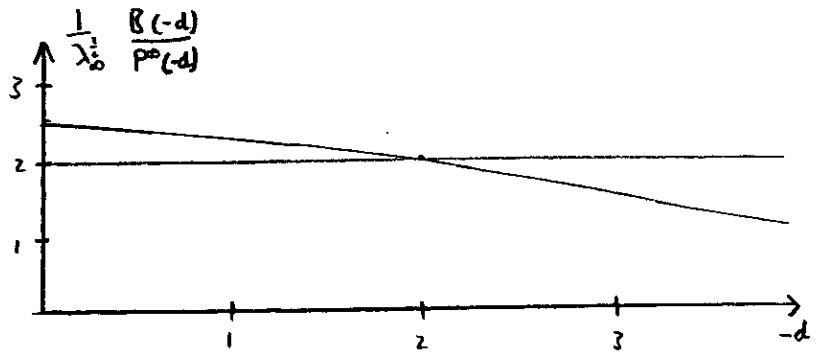

Fig 3.8.3 Optimum gain against position of pole.

It has been found that the controllable part of the system is such that the suboptimal characteristic

polynomial P^o is close to the optimal characteristic polynomial P^p (condition 3.8.25). It remains to examine condition (3.8.23). Fig 3.8.3 is a plot of $\frac{B(-d)}{P(-d)}$ against the position of the uncontrollable pole -d ; the requirement of (3.8.23) is that this should be close to $\frac{1}{\lambda_o} = 2$ — at d = -2 this condition is satisfied.

It is noted that this quantity can be calculated from either the expressions for P and B or directly from the root-locus diagram. For example if d = -0.1 :

$$\frac{B(0.1)}{\lambda_o^t P(0.1)} = \frac{1}{\lambda_p} \cdot \frac{\overline{AD}}{\overline{AB}.\overline{AC}}$$

where $\overline{AB}$ denotes the length of the line AB in Fig 3.8.2

Straightforward, but tedious, calculation yeilds the input and output variances corresponding to the closed loop systems when d = -2, these are given in table 3.8.1. for the case when T=0.

	Optimal	Suboptimal
$E\{y^2\}$	0.1273	0.1500
$E\{u^2\}$	0.5090	0.6000

Table 3.8.1 Optimal & Subotimal variances.

The entries of this table verify that the two control laws acting on the particular system, and for the particular values of λ, do indeed have similar

performance. Curves plotted for a range of values of λ_∞ and λ_0 - as in Fig 3.8.1 - are, however, necessary for a detailed comparison.

This example considers the regulator problem; identical techniques would be applied to the servo problem of making a noise-free system follow a set point described by a transfer function:

$$\frac{1}{s + d}$$

Although the particular case T = 0 was chosen, for simplicity, when actual variances were calculated; it is emphasised that the broad conclusions drawn concerning the system in this example are independent of the system time-delay. This point is further examined in the next section.

3.9 Performance of regulators acting on first-order systems.

In previous sections the relative performance of the optimal (in the sense of chapter 2) control law and the proportional minimum variance control law has been considered. In this section the theory is applied to the regulation of a first-order system with time-delay; this problem is simple enough to yeild explicit expressions for the control signal and system output variances, and yet it provides useful insight into the relationship of the two control laws.

Consider the system:

$$y = e^{-sT} \frac{s+b}{s+a} u + \frac{1}{s+a} \gamma \qquad (3.9.1)$$

This class of systems can be usefully divided into four catagories:

1. $a > 0$, $b > 0$
2. $a < 0$, $b > 0$
3. $a > 0$, $b < 0$
4. $a < 0$, $b < 0$

Each catagory is subdivided into:

(a) $|b| > |a|$

(b) $|b| < |a|$

The response of the system to a unit step is shown in Fig 3.9.1.

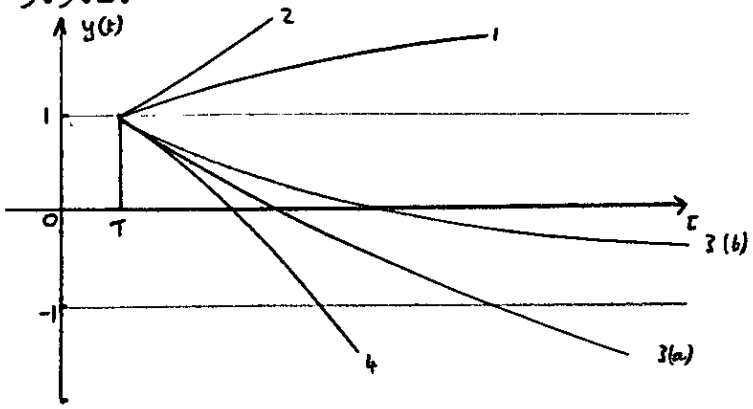

Fig 3.9.1 The system step response.

The system (3.9.1) is, effectivly, controllable with respect to the input u; from the results of the previous section, this implies that the two control laws are defined by the roots of the closed loop polynomials P^{o} and P^{∞} (3.8.11 & 18). The comparitive root-locus diagrams for the eight catagories are

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x open loop pole — reflected open loop pole

o open loop zero — reflected open loop zero

→ locus of root of P^{∞} as λ_∞ increases from 0 to ∞ .

-→- locus of root of P^{0} as $|\lambda_0|$ increases from 0 to ∞ ,
sign of λ_0 as indicated.

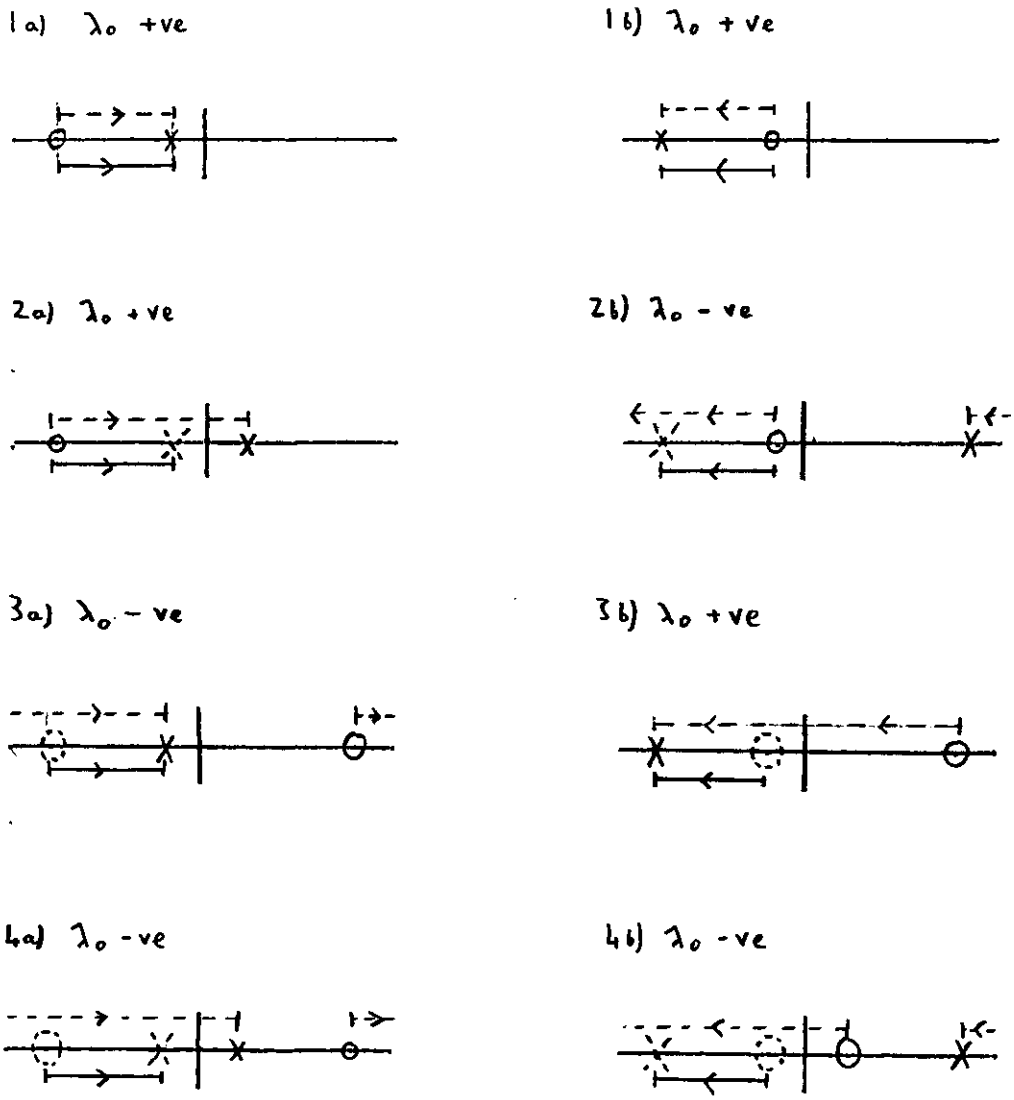

Fig 3.9.2 Comparative root locus diagrams.

shown in Fig 3.9.2: the weighting (λ) in the optimal cost function is varied from 0 to ∞; the weighting (λ_o) in the suboptimal cost is chosen so that its modulus varies from 0 to ∞, and the sign is indicated separately on each diagram. In every case it is seen that there exists a value of λ_o (possibly negative) yeilding a root of P° corresponding to a root of P^∞, for all $\lambda_\infty \geq 0$. In general, the sign of λ° is the same as the sign of the larger of 1 and b/a for stable systems: the reverse is true for unstable systems. Note that for stable systems, the step response is characterised by: $y(0) = 1$ and $y(\infty) = b/a$; hence the sign of is the same as the sign of the steady-state gain, or the initial step response, whichever is larger in magnitude.

Simple calculation gives the following table of values of λ_o corresponding to : $\lambda_\infty = 0$ (minimum variance of y), and to $\lambda_\infty = \infty$ (minimum variance of u), for the four catagories.

	$\lambda_\infty = 0$	$\lambda_\infty = \infty$
1	0	∞
2	0	$\frac{1}{2}(-\frac{b}{a} - 1)$
3	$-\frac{2b}{b+a}$	∞
4	$-\frac{2b}{b+a}$	$\frac{1}{2}(-\frac{b}{a} - 1)$

Table 3.9.1 Range of λ_o corresponding to optimal control.

To examine this behaviour further, the values of the system output and control input are derived for the system regulated by the suboptimal controller. The closed loop polynomial is given by (3.8.11) as:

$$P^{\circ} = B + \lambda_o A$$
$$= (b + \lambda_o a) + (1 + \lambda_o)s \qquad (3.9.2)$$

As the system (3.9.1) is controllable, the coefficients of the control law may, as in section 2.6, be computed directly from the identity:

$$P^{\circ}(s) = \beta(s).A(s) + e^{-sT}\alpha(s).B(s) \qquad (3.9.3)$$

From section 3.2 α is of zero degree; substituting $s = -a$ into (3.9.3) and using the fact that $P^{\circ}(-a) = B(-a)$:

$$\alpha = e^{-a\,T} \qquad (3.9.4)$$

Elementary manipulation gives the closed loop equation:

$$u = -\frac{\alpha}{P}\,y$$
$$= -\frac{e^{-aT}}{(b+\lambda_o a) + (1+\lambda_o)s}\,y \qquad (3.9.5)$$

Using standard techniques, the variance of the control input is thus given by:

$$V_u \triangleq E\{u^2\} = \frac{e^{-2at}}{2} \cdot \frac{1}{(b+\lambda_o a)(1+\lambda_o)} \qquad (3.9.6)$$

if the expression (3.9.6) is positive; $V_u = \infty$ otherwise.

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—— $E\{\dot{y}^2\}$ } to different scales.

---- $E\{u^2\}$ }

⟺ useful range of λ_o

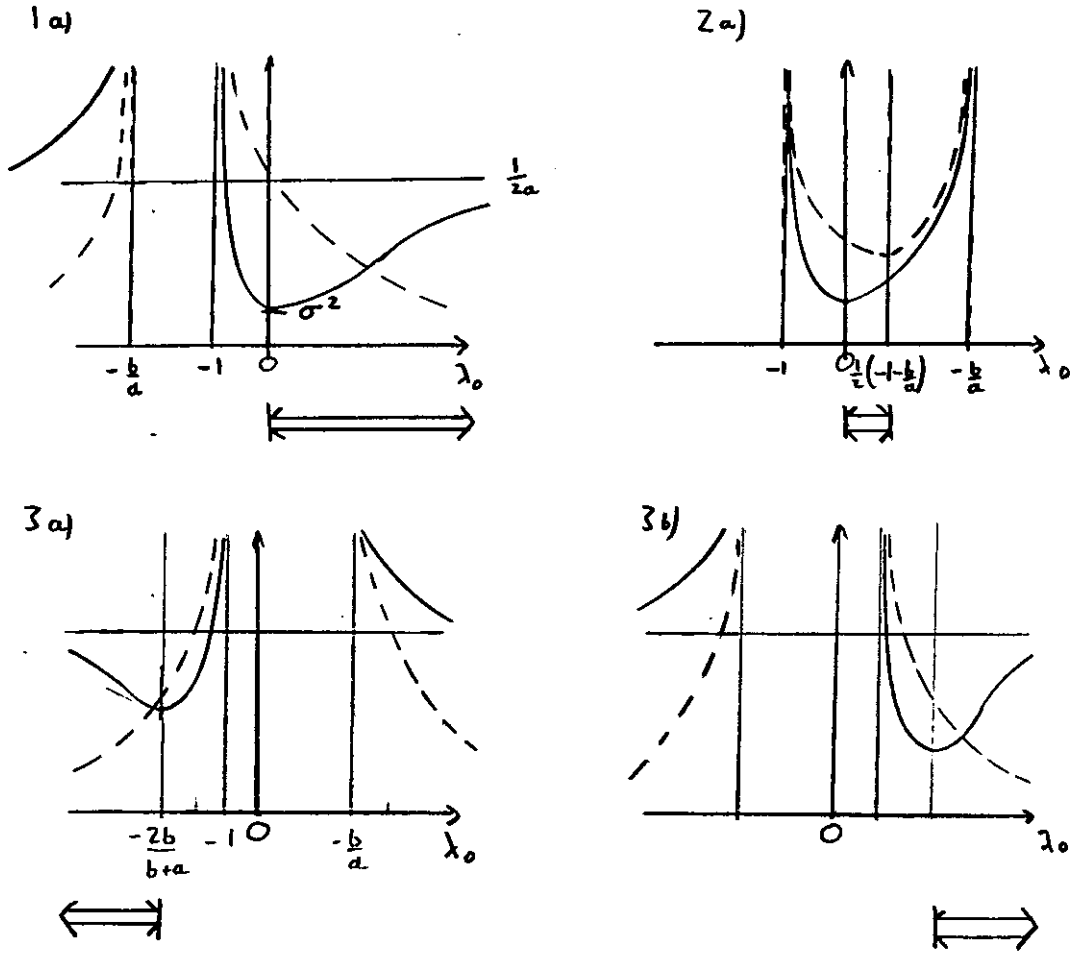


Fig 3.9.3 Input and output variances against λ_o.

From equation (3.3.17), the system output (y) is given by:

$$\phi(t+T) = y(t+T) + \lambda_0 u(t) = \epsilon^*(t+T) \qquad (3.9.7)$$

where $\epsilon^*(t+T)$ is uncorrelated with u(t). Subtracting $\lambda_0 u$ from both sides of (3.9.7), squaring, and taking expected values:

$$E\{y^2\} = V_y = \sigma^2 + \lambda^2 V_u \qquad (3.9.8)$$

where σ^2 is the variance of ϵ^* and is given by:

$$\sigma^2 \triangleq E\{\epsilon^{*2}\} = \frac{1}{2a}(1 - e^{-2aT}) \qquad (3.9.9)$$

Curves of V_y and V_u are sketched in Fig 3.9.3 for a selection of the catagories into which the class of sysems considered has been divided.

In conclusion, the following results of this section are noted:

The performance curves, that is plots of V_y against V_u with λ_0 as parameter, are such that for each finite value of V_y, there are two corresponding values of V_u - and vice versa.

The lower part of the performance curve - between minimum V_y and minimum V_u - corresponds to that of the optimal controller as λ_0 goes from zero to infinity.

Whereas the optimal control law decreases the output variance for all $\lambda_0 \geqslant 0$, the proportional minimum variance control law can produce worse results than those obtained using zero control, and can also induce instability; the choice of λ_0 is thus critical.

3.10 Discrete-time systems.

The theory presented so far in this chapter concerns continuous-time systems: this section contains brief discussions of the analogous discrete-time results, which are mainly concerned with highlighting the points at which the theory differs. Further details appear in Appendix 1, which contains a reproduction of a paper by Clarke & Gawthrop (1975).

Optimal least-squares prediction.

The system (3.2.1) becomes:

$$y(z^{-1}) = z^{-k}\frac{B(z^{-1})}{A(z^{-1})}.u(z^{-1}) + \frac{C(z^{-1})}{D(z^{-1})}.\xi(z^{-1}) \qquad (3.10.1)$$

where k is the integer time delay, and z the z-transform operator.

The spectral factorization (3.2.3) becomes:

$$R(z^{-1}).R(z) = C(z^{-1}).C(z) \qquad (3.10.2)$$

The realisability relations (3.2.6-8) are - as in section 2.7 - replaced by:

$$\frac{R(z^{-1})}{D(z^{-1})} = \tilde{I}_u(z^{-1}) + z^{-k}\frac{I_r(z^{-1})}{D(z^{-1})} \qquad (3.10.3)$$

where $\tilde{I}_u$ is of order k - 1. Unlike the continuous case,

$\tilde{I}_u$ is a finite degree polynomial (in z^{-1}) and represents a moving average stochastic process of finite variance. Moreover as k is always at least unity, $\tilde{I}_u$ is always non-zero.

The remaining equations are formally the same as in the continuous case, but with the real number T being replaced by the integer time-delay k.

The minimum variance controller as a regulator.

The cost function used is formally that of (3.3.1), but the weighting terms Λ^y, Λ^u and Λ^v are now transfer functions in z^{-1}. The method of solution of the discrete-time problem is formally similar to that of the continuous-time problem, but an important difference is that $\tilde{I}_u$ is now of finite degree and so does not have to be approximated. The prediction error $\epsilon^*(t+k|t) = \tilde{I}_u \psi(t+k)$ becomes a moving average process of order k - 1, uncorrelated with y(t), u(t) and w(t); the pulse response of a typical defining transfer function ($\tilde{I}_u$) is depicted in Fig 3.10.1.

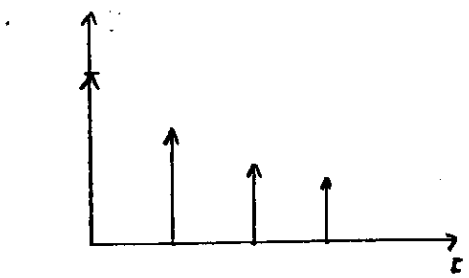


Fig 3.10.1 The pulse response of I (k = 4)

A different derivation restricted to Λ^y, Λ^u and Λ^w being polynomials - as opposed to transfer functions - is given by Clarke & Gawthrop (1975), and is reproduced in Appendix 1.

The minimum variance controller as a model follower.

As in the continuous case, the discrete model following control law makes the closed loop system follow an arbitary model with respect to the set point, and the poles of the error transfer function are prescribed. It was noted previously that, in the discrete case, $\tilde{I}_u$ is non-zero; hence, from the discussion following equation (3.4.8), there is no restriction on the model to be followed. A major disadvantage of the discrete-time formulation is that inter-sample behaviour is not considered, and may be poor.

Note that the minimum variance regulator (Åström, 1970) is a special case where the model to be followed is z^{-k}, and the set point is zero.

The minimum variance controller as a classical control law.

The discrete -time system (3.10.1) has a delay of at least one sample interval; the generalisation of classical control laws to include prediction is thus always useful. In particular, it is noted (see Example 3.5.1) that the minimum variance regulator (Astrom,1970) is the generalisation of a proportional feedback regulator with infinite gain.

Example 3.10.1

Consider the system:

$$y = \frac{z^{-1}}{1 - az^{-1}}.u + \text{disturbance}$$

$$= z^{-1}.\frac{z}{z - a}.u + \text{disturbance}$$

A standard proportional control law :

$$u(t) = -\frac{1}{\lambda}.y(t)$$

gives the closed loop characteristic polynomial:

$$P(z) = 1 + \lambda(z - a)$$

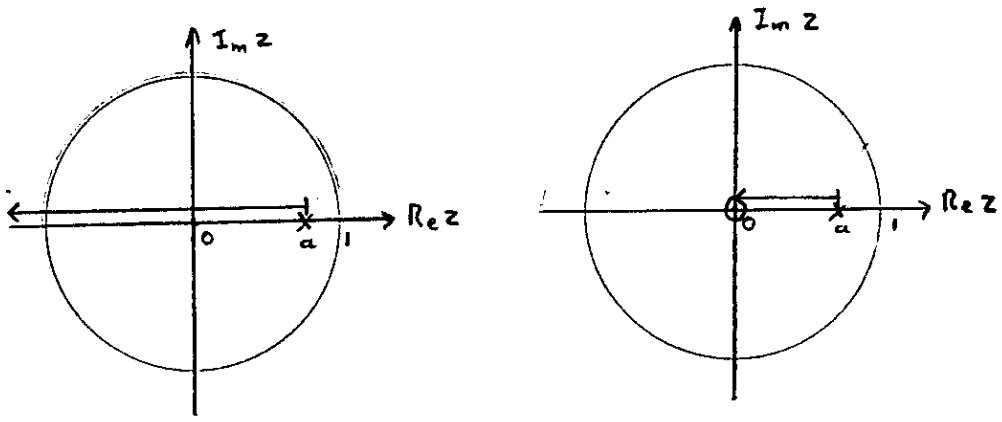
The corresponding root-locus diagram is given in Fig 3.10.2(a). The proportional feedback of prediction:

$$u(t) = -\frac{1}{\lambda}.y^*(t+k|t)$$

gives the closed loop characteristic polynomial as if k = 0, and given by:

$$P(z) = z + \lambda(z - a)$$

The corresponding root-locus diagram is given in Fig 3.10.2 (b) .



(a) proportional feedback

(b) proportional feedback of prediction

Fig 3.10.2 Root-locus diagrams.

From Fig 3.10.2 (a), the conventional control law leads to an oscillatory, and then unstable, response as the gain $(1/\lambda)$ increases; from Fig 3.10.2 (b), the predictor control law merely leads to a faster system as the gain increases; $\lambda = 0$ gives the minimum variance regulator, or dead-beat controller.

Clarke & Hastings-James (1971) considered a related class of control laws which minimised a one-stepcost function; in the next subsection it is shown that such control laws can also be interpreted as proportional feedback of predicted output. (However their method is not applicable to unstable systems, for the reasons discussed in section 2.8). In a similar manner (Appendix 1), extensions to the one-step control law given by Ashton (1973) can be considered

as minimum variance control laws - which can, in turn, be interpreted as generalisations of classical control laws.

Example 3.10.2

A differential control law (Ashton,1973) has been proposed, which minimises the cost function:

$$V = E\left\{(y(t+k) - w(t))^2 + \lambda(1 - z^{-1})^2 u(t)\right\}$$

from Appendix 1 , this is equivalent to minimising:

$$V' = E\left\{(y(t+k) - w(t) + \lambda'(1 - z^{-1})u(t))^2\right\}$$

where $\lambda' = \lambda \times$constant. The suggested control law can thus be written in the form:

$$u(t) = \frac{1}{\lambda'(1-z^{-1})}(y^*(t+k|t) - w(t))$$

Such a control law, containing no proportional term, would be expected to have poor performance - this was experimentally found to be so.

A one-step control law.

A discrete analogue of the infinitesimal horizon control law (section 3.6) is the one-step (or k-step-ahead) control law. That is the cost function (3.6.2) is replaced by:

$$V_1 = E\left[(y(t+k) - w(t))^2 + \lambda_1 u^2(t)\right] \qquad (3.10.4)$$

Similar cost functions have been considered previously. (Clarke & Hastings-James,1971; Ashton, 1973; Hughes & Jacobs,1973). As in section 3.6, using the fact that the prediction error (ϵ^*) is uncorrelated with y(t), u(t), and w(t), the cost function (3.10.4) becomes:

$$V_1 = (y^*(t+k|t) - w(t))^2 + \lambda_1 u^2(t) + \sigma^2 \qquad (3.10.5)$$

where the variance of the prediction error is σ^2. Unlike the continuous case, $\partial y^*(t+k|t)/\partial u(t)$ is never zero - for k is chosen in such a way that the leading coefficient (b_0) of $B(z^{-1})$ is non-zero. V_1 (3.10.4) is thus minimised by choosing u(t) such that:

$$(y^*(t+k|t) - w(t)).b_0 + \lambda_1 u(t) = 0 \qquad (3.10.6)$$

The proportional minimum variance control law is:

$$u(t) = -\frac{1}{\lambda}(y^*(t+k|t) - w(t)) \qquad (3.10.7)$$

Equations (3.10.7 & 8) are identical if:

$$\lambda = \frac{\lambda_1}{b_0} \qquad (3.10.8)$$

Hence, unlike the continuous-time case, the one-step and proportional minimum variance controllers are always equivalent - results about the optimality of one control law imply the same results about the optimality of the other.

On the optimality of a minimum variance controller.

The discrete-time comparison of the proportional minimum variance and optimal control laws is based on the sampled behaviour only; this leads to directly analogous results to those found for the continuous case. The comparative root-locus diagrams are drawn in the z - as opposed to s - plane; and the spectral factorisation is into roots inside and outside the unit circle. This comparison does not, however, provide information about inter-sample behaviour.

Example 3.10.3

Consider the system:

$$y = z^{-k}\frac{(1 - 0.5z^{-1})}{(1 - 0.9z^{-1})^2}.u + \frac{1}{(1 - dz^{-1})}\gamma$$

$$= z^{-k}\frac{z(z - 0.5)}{(z - 0.9)}.u + \frac{z}{(z - d)}\gamma$$

The comparative root-loci are defined by:

$$P^{o}(z) = z(z - 0.5) + \lambda_{o}(z - 0.9)^2$$

and $$P^{\infty}(z).P^{\infty}(z^{-1}) = \lambda z(z - 0.5)z^{-1}(z^{-1} - 0.5) + \lambda_{\infty}(z - 0.9)^2(z^{-1} - 0.9)^2$$

These loci are depicted in Fig 3.10.3.

open loop poles x

" " zeros o

reflected open loop poles ×

" " " zeros ○

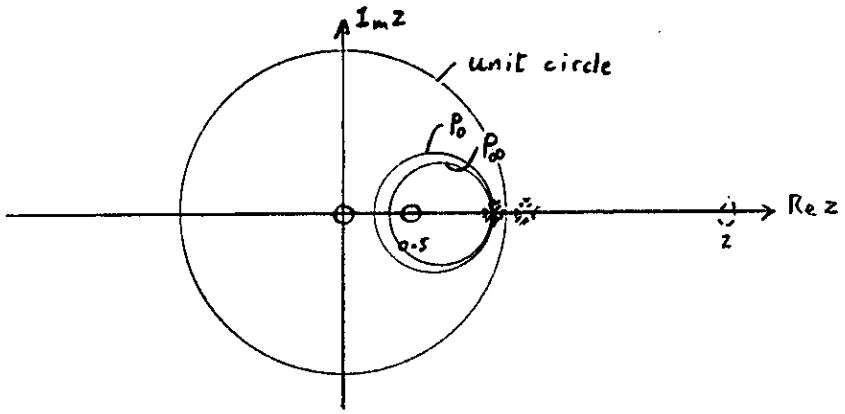


Fig 3.10.3 Comparative root-locus diagram.

More detailed analysis is similar to that of Example 3.8.1; once again, the comparison is independent of the system time-delay, k.

3.11 Summary.

A cost function, closely allied to that of the minimum variance regulator of Åström, has been considered in both continuous and discrete time, and shown to be open to a number of useful interpretations:

1) A model following control law which enables the closed-loop system response, to a filtered set point, to be prespecified subject to certain constraints. It is shown that it is not possible to eliminate disturbances; but it is shown that the poles of the closed-loop disturbance process are those of the prespecified closed-loop system.

2) A classical control law which operates on the least-squares prediction of the system output, a time ahead corresponding to the system time-delay. This allows classical design methods for systems with no dead-time to be directly applied to systems with dead-time.

3) A special case of 2), proportional feedback of predicted output error, is interpreted as an approximation to the optimal control law of chapter 2; conditions are given for a system to be well controlled by this control law. Analogies are drawn with a previously studied problem involving comparison of discrete-time k-step-ahead and infinite horizon control laws.

The important feature of these control laws is that they form the basis of a self-tuning algorithm.

Chapter 4

A self-tuning controller.

4.1 Introduction.

In the previous chapter was considered the control of systems described by transfer functions with known coefficients: in this chapter is considered the control of systems with the same structure, but the transfer function coefficients are unknown.

The discrete-time systems considered in this chapter are of the form:

$$y = z^{-k}\frac{B}{A}u + \frac{C}{D}\nu \qquad (4.4.1)$$

where y, u, and ν are the discrete system output, control input, and noise input respectively. A, B, C, and D are polynomials, in the z-transform operator z^{-1}, of known degree but unknown coefficients; the integer time-delay k ($\geqslant 1$) is assumed known.

A 'self-tuning regulator' has been proposed (Åström & Wittenmark,1973) which combines a linear least-squares estimation algorithm with a linear feedback control law based on the parameter estimates; this is done in such a way that, asymptotically, the control law becomes the minimum variance regulator (Åström, 1970) designed using prior knowledge of the values

of the transfer function coefficients. This property, whereby the control law parameters asymptotically correspond to those of some prespecified control law based on prior knowledge of system parameters, has been termed 'self-tuning' (Borisson,1975).

In this chapter, it is shown that the more general class of minimum variance controllers (discussed in chapter 3) can be combined with a least-squares estimation scheme, in such a way that the overall algorithm is - in this sense - self-tuning. The exposition is largely based on previous work (Ljung & Wittenmark, 1974); the central result (Ljung,1974) is quoted without proof, but some intuitive justification is given

As pointed out in chapter 3, the minimum variance controller has a number of advantages over the basic minimum variance regulator, in particular with regard to control weighting and set point following. The self-tuning controller is only slightly more complex than the basic self-tuning regulator. The combination of these two facts make the self-tuning controller a valuable extension of the self-tuning regulator.

4.2 The self-tuning controller.

In chapter 3 (equations 3.3.19-24), it was shown that the system (4.4.1) could be written in terms of the auxiliary output (ϕ) as:

$$R\phi^*(t+k|t) = \alpha y'(t) + \beta u'(t) + \Gamma w'(t) \qquad (4.2.1)$$

$$\phi(t+k) = \phi^*(t+k|t) + \epsilon^*(t+k) \qquad (4.2.2)$$

where ϕ is the auxiliary system output, given by:

$$\phi(t+k) = \frac{\Lambda_n^y}{\Lambda_d^y} y(t+k) + \frac{\Lambda_n^u}{\Lambda_d^u} u(t) - w'(t) \qquad (4.2.3)$$

and: $y' = 1/\Lambda_d^y . y$, $u' = 1/\Lambda_d^u . u$, $w' = \Lambda w$. R is given by (3.3.6); but note that if Λ_n^y has no roots outside the stability region, this factor may be cancelled from $R, \alpha, \beta,$ and Γ.

The control law defined by setting the right-hand side of equation (4.2.1) to zero, was shown to minimise the cost function:

$$V = E\left\{\phi^2(t+k) \middle| t\right\} \qquad (4.2.4)$$

Define the column vectors $\underline{X}$ and $\underline{\Theta}$ as:

$$\underline{X}(t) = \begin{bmatrix} y'(t) \\ \vdots \\ y'(t-n_\alpha) \\ u'(t) \\ \vdots \\ u'(t-n_\beta) \\ w'(t) \\ \vdots \\ w'(t-n_\gamma) \end{bmatrix} \quad \underline{\Theta} = \begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_{n_\alpha} \\ \beta_0 \\ \vdots \\ \beta_{n_\beta} \\ \gamma_0 \\ \vdots \\ \gamma_{n_\gamma} \end{bmatrix} \qquad (4.2.5)$$

where $\alpha_1 \ldots \alpha_{n_\alpha}$ are the n_α coefficients of α etc.

The equations (4.2.1 & 2) can now be rewritten as:

$$\phi(t+k) = \sum_{i=-\phi}^{t} \rho_{t-i} \underline{X}^T(i).\underline{\Theta} + \epsilon^*(t+k) \qquad (4.2.6)$$

where: $$\frac{1}{R} = \sum_{i=0}^{\phi} \rho_i z^{-i} \qquad (4.2.7)$$

Without loss of generality, ρ_0 and R can be chosen such that $\rho_0 = 1$, ϵ^* and ψ being scaled accordingly.

Consider first the case when $\rho_0 = R = 1$, that is:

$$\rho_i = 0 \qquad i \geqslant 1 \qquad (4.2.8)$$

Then equation (4.2.4) becomes:

$$\phi(t+k) = \underline{X}^T(t).\underline{\Theta} + \epsilon^*(t+k) \qquad (4.2.9)$$

The parameter vector $\underline{\Theta}$ is unknown; whereas the data vector $\underline{X}$, being formed from measured values of y, u, and w, is known at time t. If the time-delay k is unity, $\epsilon^*(t+k)$ is a discrete white noise process (section 3.10); in this case it has been shown (Astrom & Wittenmark, 1971) that $\underline{\Theta}$ is optimally estimated - in a least-squares sense - by a recursive least-squares algorithm. The least-squares algorithm for the system (4.2.9) can be written in a number of forms (Astrom & Eykhoff, 1971); a selection of these are now presented.

The basic non-recursive form of the least-squares algorithm, based on N measurements, is given by:

$$\hat{\underline{\Theta}}(N) = \underline{P}(N)\left[\sum_{t=1}^{N} \underline{X}(t-k)\phi(t)\right] \qquad (4.2.10)$$

where:

$$P(N) \triangleq \left[\sum_{t=1}^{N} \underline{X}(t-k).\underline{X}^{T}(t-k)\right]^{-1} \qquad (4.2.11)$$

These equations may be put into recursive form as:

$$\hat{\underline{\Theta}}(t) = \hat{\underline{\Theta}}(t-1) + \underline{K}(t)\left[\phi(t) - \underline{X}^{T}(t-k).\hat{\underline{\Theta}}(t-1)\right] \qquad (4.2.12)$$

$$\underline{K}(t) = \underline{P}(t)\underline{X}(t-k) \qquad (4.2.13)$$

$$\underline{P}^{-1}(t) = \underline{P}^{-1}(t-1) + \underline{X}(t-k).\underline{X}^{T}(t-k) \qquad (4.2.14)$$

Using the matrix inversion lemma (Sage & Melsa, 1971), equation (4.2.14) may be converted to give $\underline{P}$ directly as:

$$\underline{P}(t) = \underline{P}(t-1) - \frac{\underline{P}(t-1)\underline{X}(t-k)\underline{X}^{T}(t-k)\underline{P}(t-1)}{1 + \underline{X}^{T}(t-k)P(t-1)\underline{X}(t-k)} \qquad (4.2.15)$$

Substituting (4.2.15) into (4.2.13), $\underline{K}$ may be written in a more usual form as:

$$\underline{K}(t) = \frac{\underline{P}(t-1).\underline{X}(t-k)}{1 + \underline{X}^{T}(t-k)\underline{P}(t-1)X(t-k)} \qquad (4.2.16)$$

Equations (4.2.15 & 16) may, in practice, be updated

using efficient square-root techniques (Clarke et al., 1975); for theoretical purposes, however, equations (4.2.10-14) are more convenient.

The above recursive equations are closely related (Åström & Eykoff,1971) to those of the Kalman filter, $\underline{\Theta}$ being regarded as the (constant) state and equation (4.2.9) as the measurement equation. It can be shown that $\underline{P}(t)$ is proportional to the a posteriori covariance matrix of the estimation error at time t; because the state ($\underline{\Theta}$) is constant, $\underline{P}(t-1)$ is proportional to the a priori covariance matrix at time t. Equations (4.2.13 & 16) thus represent the alternative forms of the Kalman gain based on the a posteriori and a priori covariances respectively.

The proposed self-tuning strategy can now be stated as follows:

a) Using the measured values of y,u,and w form the auxiliary output (4.2.3).

b) Using the measured values of y,u, and w form the data vector $\underline{X}(t-k)$ and use a set of equations equivalent to (4.2.12-14) to update the parameter estimate $\underline{\theta}(t)$.

c) Use the current estimate $\hat{\underline{\Theta}}$ and current measurements to choose the current control input u(t) such that:

$$\underline{X}^{T}(t).\hat{\underline{\theta}}(t) = 0 \qquad (4.2.17)$$

Steps a)-c) are repeated at each sample instant. If, in step a), $\phi(t) = y(t)$, the above algorithm becomes that of the self-tuning regulator (Åström & Wittenmark, 1973); the additional complexity of the self-tuning controller is thus small.

In the above strategy, the least-squares estimator is - in general - not operating under conditions for which it was designed: because of step c) feedback is present; the system is, in general, given by equation (4.2.6) rather than equation (4.2.9); and if $k > 1$, the noise term (ϵ^*) is non-white. The problem now becomes one of analysing the behaviour of the estimator under the actual operating conditions, and making ad hoc modifications to improve performance.

4.3 Steady state properties.

In the previous section, the self-tuning scheme was stated; it was noted that the estimation algorithm was not operating under its design conditions. For the proposed scheme to be self-tuning in the sense of section 4.1, it is necessary that the estimate $\hat{\underline{\theta}}$ both converges to some constant value, and also that this value corresponds to the desired control law. In this section convergence is assumed and the latter condition examined.

In the original paper on the self-tuning regulator (Åström & Wittenmark,1973), this latter property was verified for the particular case $\phi(t) = y(t)$; this proof can be modified to fit the more general case, but a more direct method is given here.

Consider the self-tuning scheme of the previous section, but with step c) replaced by the constant parameter control law:

$$\underline{X}^T(t).\bar{\underline{\Theta}} = 0 \qquad (4.3.1)$$

where $\bar{\underline{\Theta}}$ is of the same form as $\underline{\Theta}$ (4.2.5); this will be the steady state situation if $\hat{\underline{\Theta}}$ converges to $\bar{\underline{\Theta}}$. Firstly it is noted that, as from (4.3.1) the components of $\underline{X}$ are linearly related, the matrix $\underline{P}^{-1}$ (4.2.11) is singular and thus $\hat{\underline{\Theta}}(N)$ is not uniquely determined. In practice, this would mean that the components of $\hat{\underline{\Theta}}$ would drift, but would eventually become in a constant ratio to eachother. This would not affect the control signal generated by equation (4.2.17), but could lead to numerical difficulties. Following the approach of Åström & Wittenmark (1973), one parameter is fixed at some prespecified value, and the remaining parameters estimated. The following definitions are made:

$\underline{\Theta}_i$ is $\underline{\Theta}$ but with the ith element (θ_i) deleted.
$\hat{\underline{\Theta}}_i$ " $\hat{\underline{\Theta}}$ " " ($\hat{\theta}_i$) "
$\underline{X}_i$ " $\underline{X}$ " " (x_i) "

The estimator equations (4.2.10 & 11) become:

$$\hat{\underline{\Theta}}_i(N) = \underline{P}_i(N)\left[\sum_{t=1}^{N}\underline{X}_i(t-k)\left(\phi(t) + \hat{\theta}_i x_i(t-k)\right)\right] \quad (4.3.2)$$

$$\underline{P}_i^{-1}(N) = \sum_{t=1}^{N}\underline{X}_i(t-k)\underline{X}_i^T(t-k) \quad (4.3.3)$$

where $\hat{\theta}_i$ is the ith component of $\hat{\underline{\Theta}}_i$ and chosen a priori.

Substituting (4.2.2) into (4.3.2) and premultiplying by $\underline{P}_i^{-1}(N)$:

$$\underline{P}_i^{-1}(N)\hat{\underline{\Theta}}_i(N) = \sum_{t=1}^{N}\underline{X}_i(t-k)\left[\phi^*(t) + \epsilon^*(t) + \hat{\Theta}_i(t)x_i(t-k)\right] \quad (4.3.4)$$

In section 3.3 it was shown that $\epsilon^*(t)$ was uncorrelated with y, u, and w occuring at and before time t-k, and so uncorrelated with $\underline{X}_i(t-k)$; hence, taking expected values of both sides of (4.3.4):

$$E\left\{\underline{P}_i^{-1}(N)\hat{\underline{\Theta}}_i(N)\right\} = E\left\{\sum_{t=1}^{N}\underline{X}_i(t-k)\left[\phi^*(t) + \hat{\theta}_i x_i(t-k)\right]\right\} \quad (4.3.5)$$

Using the control law (4.3.1) and the expression for $\underline{P}_i^{-1}$ (4.3.3), equation (4.3.5) becomes:

$$E\left\{\underline{P}_i^{-1}(N)\hat{\underline{\Theta}}_i(N)\right\} = E\left\{\sum_{t=1}^{N}\underline{X}_i(t-k)\phi^*(t)\right\} + \frac{\hat{\Theta}_i}{\bar{\Theta}_i}E\left\{\underline{P}_i^{-1}(N)\,\bar{\underline{\Theta}}_i\right\} \quad (4.3.6)$$

If equation (4.3.6) is to represent the asymptotic self-tuning scheme, the constant control law must correspond to the estimated parameter vector; that is:

$$x_i(t) = \frac{1}{\hat{\theta}_i}\underline{X}_i^T(t)\hat{\underline{\theta}}_i(t) = \frac{1}{\bar{\theta}_i}\underline{X}_i^T(t)\bar{\underline{\theta}}_i \qquad (4.3.7)$$

Substituting (4.3.7) into (4.3.6), the asymptotic estimated control law must also be such that:

$$E\left\{\sum_{t=1}^{N}\underline{X}_i(t-k)\phi^*(t)\right\} = 0 \qquad (4.3.8)$$

From (4.2.1) it follows that $\underline{X}_i(t-k)$ and $\phi^*(t)$ are, in general, correlated; hence if the components of $\underline{X}$ are not always zero, the estimated control law must be such that:

$$\phi^*(t) = 0$$

$$\therefore \hat{\underline{\theta}}_i(t) = \frac{\hat{\theta}_i}{\theta_i}\underline{\theta}_i \qquad (4.3.9)$$

Hence, if the self-tuning controller converges to give a constant control law, this constant control law must correspond to the minimum variance controller defined by the weighting transfer functions Λ^y, Λ^u, and Λ^w - as used in the generation of ϕ.

Note that as, in general, the self-tuning scheme takes some time to converge, $\phi^*(t)$ cannot initially be zero: the self tuning scheme can only make the right-hand side of (4.3.8) zero by generating further non-zero terms to compensate. In practice, this is found to lead to slow convergence of the estimate $\underline{\theta}_1$. This bias of the estimate after some finite time t=N can be reduced - at the expense of increased variance of the estimate - by using 'exponential forgetting' (Eykhoff,1975). This leads to equation (4.3.8)

being replaced by:

$$E\left[\sum_{t=1}^{N}(1-\delta)^{N+1-t}\,\underline{X}_t(t-k)\phi^*(t)\right] = 0\ ,\ 0<\delta<1 \quad (4.3.10)$$

where $(1-\delta)$ is the forgetting factor, δ usually being of the order of 0.01. With this modified estimator, initial non zero values of ϕ^* have coefficients which tend to zero as N grows.

To summarise, by assuming that the estimate $\hat{\underline{\Theta}}$ converges to some constant value, it has been shown that if one element of $\hat{\underline{\theta}}$ is fixed a priori, and the least-squares estimator is modified to weight out past data, this constant value gives the minimum variance control law specified by Λ^u, Λ^w, and Λ^v - used to generate the auxiliary output ϕ . That is, the proposed scheme is self-tuning.

It has been noted that it is useful to weight out past data using, for instance, equation (4.3.10); this prevents the estimator gain $\underline{K}$ (4.2.13) from becoming zero, and so prevents the estimate $\hat{\underline{\theta}}$ converging to a constant value. It has been suggested (Borrison,1975), that by letting the forgetting factor in (4.3.10) tend to unity ($\delta \to 0$) according to some predetermined scheme, the estimator gain eventually becomes zero, and the estimate $\hat{\underline{\theta}}$ can settle down to some constant value. It is now shown that the estimate wandering about its correct value does not, in practice, lead

to poor control; convergence to a constant value is thus not necessary, and the forgetting factor does not have to be changed on-line. The following argument has been given previously (Clarke & Gawthrop,1975).

The cost function minimised by the modified least-squares estimator is :

$$V_E(\hat{\underline{\theta}}\ (N)) = \frac{1}{\sum_{t=1}^{N}\gamma(t)}\sum_{t=1}^{N}\gamma(t)(\phi(t) - \underline{X}^T(t-k)\hat{\underline{\theta}}\ (N))^2 \qquad (4.3.11)$$

Where $\gamma(t)$ is some weighting factor, for example $(1-\delta)^{(N-t)}$ as in equation (4.3.10). The cost function minimised by the control law can be written as:

$$V_c(\bar{\underline{\theta}}) = E\ (\phi(t) - \underline{X}^T(t-k)\ \bar{\underline{\theta}}\ )^2 \qquad (4.3.12)$$

Although $\underline{X}(t)$ is not a stationary sequence, and thus ergodicity cannot be invoked to equate V_E and V_c, if the estimates $\hat{\underline{\theta}}(t)$ do not diverge, and the control law gives a stable closed loop system, the two cost functions become approximately equivalent. Hence, if $V_E(\hat{\underline{\theta}})$ is such that it does not change rapidly in some direction in parameter space - and so $\hat{\underline{\theta}}$ tends to wander in such a direction - the control law according to $V_c(\bar{\underline{\theta}})$ will be insensitive to such variations. That is wandering estimates do not imply poor control.

To summarise, the effect of exponential forgetting is to reduce - at any given time - the bias of the estimate due to initial non-minimum variance feedback; but this is at the expense of increased variance of the estimate. This increased variance has been shown to be relatively unimportant.

4.4 Convergence.

In the previous section it was shown that the proposed adaptive control algorithm was self-tuning - on the assumtion that the estimated parameters converged to some constant value. In this section attention turns to the question of under what conditions the self-tuning controller converges. A large body of results for the self-tuning regulator ($\phi(t) = y(t)$) exists in the literature (Ljung,1974 a,b; Ljung & Wittenmark,1974 a,b); because of the similarity between the self-tuning controller and the self-tuning regulator, a large part of this theory would seem to apply directly to the more general problem. The purpose of this section is to give an intuitive overview of this previous work in relation to the more general self-tuning controller.

To give some idea of the problems involved in the analysis of the self-tuning controller, consider the particular case were the system has an integer time-delay (k) of unity. The recursive form of the estimator (4.3.2 & 3) is given by:

$$\hat{\underline{\theta}}_i(t) = \hat{\underline{\theta}}_i(t-1) + \underline{K}_i(t)\left[\phi(t) - \underline{x}_i^T(t-1)\hat{\underline{\theta}}_i(t-1) - x_i(t-1)\hat{\theta}_i\right] \quad (4.4.1)$$

where $\underline{K}_i$ is given by equations similar to (4.2.13 & 14). The feedback control law (4.2.17) is such as to set the sum of the last two terms of (4.4.1) to zero.

Using also the expression for $\phi(t)$ (4.2.6), equation (4.4.1) becomes:

$$\hat{\underline{\theta}}_i(t) = \hat{\underline{\theta}}_i(t-1) + \underline{K}_i(t)\left[\sum_{j=0}^{r} \rho_{r-j}\underline{X}_i^T(j-1)\underline{\theta} + \epsilon^*(t)\right] \tag{4.4.2}$$

Define: $$\tilde{\underline{\theta}}_i(t) = \hat{\underline{\theta}}_i(t) - \frac{\hat{\theta}_i}{\theta_i}\underline{\theta}_i \tag{4.4.3}$$

and define $\tilde{\underline{\theta}}$ in a similar manner. Then using the control law (4.2.17), equation (4.4.2) becomes:

$$\tilde{\underline{\theta}}_i(t) = \tilde{\underline{\theta}}_i(t-1) - \frac{\theta_i}{\hat{\theta}_i}\underline{K}_i(t)\left[\sum_{j=0}^{r} \rho_{r-j}\underline{X}_i(j-1)\tilde{\underline{\theta}}(j-1)\right] + \underline{K}_i(t)\epsilon^*(t) \tag{4.4.4}$$

This equation (4.4.4) is, in general, stochastic, non-linear, and time varying; moreover $\tilde{\underline{\theta}}_i$ does not define the state of the system described by (4.4.4); the analysis of this equation is thus formidable. It is, however, possible to draw some rough conclusions about the behaviour of its solution. Firstly, if $\rho(j) = 0$, $j \geqslant 1$ (that is R is unity) and $\hat{\underline{\theta}}_i = \underline{\theta}_i$ (that is $\underline{\theta}_i$ is exactly known a priori), equation (4.4.4) becomes the standard least-squares update equation for $\tilde{\underline{\theta}}_i$ with $\underline{K}_i(t)$ the optimum gain vector.

If the former condition holds but $\hat{\underline{\theta}}_i = \mu\underline{\theta}_i$, the optimum gain vector $\underline{K}$ is multiplied by a factor $\frac{1}{\mu}$. If $\mu < 1$, the actual gain is greater than optimum, leading to 'overshoot' or even a diverging sequence of estimates: if $\mu > 1$, the actual gain is less than optimal leading to slow convergence.

If the latter condition holds, the behaviour of

the solution of equation (4.4.4) depends on the magnitudes of ρ_j, $j \geq 1$; hence if the roots of R(s) -which corresponds to the numerator of the system disturbance if $1/A$ is stable - are near the unit circle, the behaviour of the solutions of equation (4.4.4) will, due to the far from zero values of ρ_j, $j \geq 1$, be far from optimal in some sense.

The contribution of Ljung (1974 a,b) was to analyse similar equations, and hence show that the convergence of the self-tuning regulator was assured if:

a) The output and input sequences y(t) and u(t) are not always unbounded; that is the closed loop system is not always unstable. And

b) An associated ordinary (non-stochastic) non-linear differential equation converges.

It would appear that the above conditions also hold for the more general self-tuning controller. Ljung's proof is long and involves advanced techniques, and so no rigorous proof of this conjecture is given here; however, an intuitive justification is given here, which closely follows one given previously (Ljung & Wittenmark,1974 a,b) for the above conditions applied to the self-tuning regulator.

Using the estimator equations (4.4.1) and (4.2.13), the estimate $\hat{\underline{\theta}}_i$ is updated according to:

$$\underline{\hat{\theta}}_i(t) = \underline{\hat{\theta}}_i(t-1) + \underline{P}_i(t)\underline{X}_i(t-k)\left[\phi(t) - \underline{X}_i^T(t-k)\underline{\hat{\theta}}_i(t-1) - x_1(t-1)\hat{\theta}_i\right]$$

$$\underline{P}_i^{-1}(t) = \underline{P}_i^{-1}(t-1) + \underline{X}_i(t-k)\underline{X}_i^T(t-k) \qquad (4.4.5)$$

The latter equation may be suitably modified to weight out past data, but this does not essentially change the following argument. As the quantity $\underline{P}_i$ gets small it requires more time for $\underline{\hat{\theta}}_i$ to change by a given small amount; if also the closed loop system is stable, the sequence $\underline{X}_i(t)$ becomes approximately stationary. Using some 'law of large numbers' it would seem possible to replace quantities by their expected values - as if they were actually generated over a long period of time with $\underline{\hat{\theta}}_i$ constant. The following quantities are now defined:

$$\underline{f}(\underline{\hat{\theta}}_i) = E\left\{\underline{X}_i(t-k)\phi(t)\right\} \qquad (4.4.6)$$

$$\underline{G}(\underline{\hat{\theta}}_i) = E\left\{\underline{X}_i(t-k)\underline{X}_i^T(t-k)\right\} \qquad (4.4.7)$$

where the expectations are taken for the steady state system when the feedback control law is constant and given by:

$$\underline{X}_i(t)\,\underline{\hat{\theta}}_i = 0 \qquad (4.4.8)$$

Substituting equations (4.4.6-8) into equations (4.4.5), according to the preceeding non-rigorous argument:

$$\hat{\underline{\theta}}_i(t+T) - \hat{\underline{\theta}}_i(t) \doteq T\,\bar{\underline{P}}(t)\,\underline{f}(\hat{\underline{\theta}}_i(t)) \qquad (4.4.9)$$

$$\bar{\underline{P}}_i^{-1}(t+T) - \bar{\underline{P}}_i^{-1}(t) \doteq T\,\underline{G}(\hat{\underline{\theta}}_i(t)) \qquad (4.4.10)$$

Equations (4.4.9 & 10) are, formally, difference approximations to the differential equations:

$$\frac{d}{dt}\hat{\underline{\theta}}_i(t) = \underline{P}(t)\,\underline{f}(\hat{\underline{\theta}}_i(t)) \qquad (4.4.11)$$

$$\frac{d}{dt}\bar{\underline{P}}_i^{-1}(t) = \underline{G}(\hat{\underline{\theta}}_i(t)) \qquad (4.4.12)$$

In the work of Ljung and Wittenmark (1974 a,b) these equations are expressed in a different form as:

$$\frac{d}{d\tau}\hat{\underline{\theta}}_i(\tau) = \underline{S}(\tau)\,\underline{f}(\hat{\underline{\theta}}_i(\tau)) \qquad (4.4.13)$$

$$\frac{d}{d\tau}\underline{S}^{-1}(\tau) = \underline{G}(\hat{\underline{\theta}}_i(\tau)) - \underline{S}^{-1}(\tau) \qquad (4.4.14)$$

where: $\frac{d\tau}{dt} = \frac{1}{t}$, $\underline{S}(t) = tP(t)$

The above intuitive argument suggests that if the closed loop system is stable, equations (4.4.11 & 12) describe the actual trajectories arbitrarily closely as $t \to \infty$, and thus that their convergence also implies convergence of the algorithm described by equations (4.4.5).

Ljung (1974 a,b) has shown rigorously that conditions a) and b) , where the associated ordinary differential equation is given by (4.4.13 & 14), do imply convergence of the self-tuning regulator; the preceeding argument suggests that this result can be extended to the more general self-tuning controller. Attention now turns to the practical verification of these conditions.

The verification of condition a) involves showing that if the estimated parameters are such as to make the closed loop system unstable, the estimates are always driven towards values giving a stable closed-loop system. This has been considered previously (Ljung & Wittenmark,1974 a,b); similar methods are used here, but are extended to include consideration of set point parameters, which are not present in the self-tuning regulator.

If the closed loop system is unstable, the system output (y) and input (u) will become much larger than the disturbance and set point processes, which may then be neglected in comparison. There are two cases to consider: firstly a coefficient of y or u is not fixed at aconstant value; secondly a coefficient of y or u is fixed.

In the former case, as the measurements of y and u are effectively noise free, the corresponding parameter estimates will rapidly converge towards

those based on the system parameters, but ignoring the disturbance and set point dynamics. Such a control law, though suboptimal, gives a closed loop system with characteristic polynomial P_c , obtained from equation (3.3.26) by setting the disturbance polynomials C' and D' to unity. If this polynomial - corresponding to the numerator of the auxiliary system (3.3.12) - has no right-hand plane roots, the closed loop system is stable. That is, if the numerator of the auxiliary system (3.3.12) has no right-hand plane roots, estimates outside the closed loop stability region are forced inside the stability region.

If, on the other hand, a coefficient of y or u is fixed at a constant value, the resultant estimator of the remaining coefficients of y and u is not a least squares estimator, and the estimates do not necessarily converge to the above stabilising control law. In this latter case it can be shown (Ljung & Wittenmark,1974 a) that the estimates will converge to those of the above suboptimal control law if:

$$\frac{\hat{\theta}_i'}{\theta_i'} > 0.5 \quad \text{and} \quad k = l \tag{4.4.15}$$

where $\hat{\theta}_i'$ is the value to which the coefficient θ_i' of the above suboptimal control law is actually fixed.

To summarise, condition a) is satisfied if the closed loop polynomial obtained from equation (3.3.26) with R set to unity:

$$P_c' = \Lambda_m^y \Lambda_d^u B + \Lambda_d^y \Lambda_m^u A \qquad (4.4.16)$$

has no roots in the right-hand plane; and either:

1) No coefficient of y or u is fixed.

or 2) A coefficient of y or u is fixed, the system time-delay (k) is unity, and the fixed parameter is not less than half of its actual value in the suboptimal control law which ignores set point and disturbance dynamics.

More general cases have not, as yet, been investigated.

The verification of the convergence of the ordinary differential equation (4.4.11 & 12) - condition b) - seems to be difficult in general; methods suggested (Ljung & Wittenmark,1974) include: numerical solution for a range of initial conditions; construction of a suitable Liapunov function; and linearisation about the required steady-state solution. This last method has been used (Ljung & Wittenmark,1974 a) to construct a system for which the self-tuning regulator, although satisfying condition a), does not have a convergent ordinary differential equation; it is noted that a feature of this system is complex roots of the noise numerator near the unit circle, it has been noted above that the performance of the self-tuning controller would be expected to be poor for such a system.

4.5 Steady-state offset.

It is often an important practical requirement that a control algorithm be such that the mean steady-state output of the system under control has zero offset with respect to the set point - even if the disturbance process and setpoint have a non-zero mean value. The purpose of this section is twofold: the theory of least-squares prediction (chapter 3) is extended to include disturbances with a known constant component, the self-tuning controller being modified accordingly; and conditions are derived which ensure that the minimum variance controller - that is the steady-state self-tuning controller - has zero offset.

A known constant disturbance is included in the model (3.10.1) to give:

$$y = z^{-k} \frac{B}{A} u + \frac{C}{D} \gamma + d \qquad (4.5.1)$$

where d is a known constant. Define:

$$y_o = y - d \qquad (4.5.2)$$

$$\phi_o = \phi - \Lambda^u(1)\, d \qquad (4.5.3)$$

where $\Lambda^u(1)\, d$ is the steady-state value of $\Lambda^u(z^{-1})\, d$. The equations for y_o and ϕ_o correspond to those

discussed in chapter 3 with no constant term; hence using equation (3.3.19), the least- squares predictor for ϕ_o may be written down as:

$$\phi_o^*(t+k|t) = \frac{1}{R}(\alpha y_o' + \beta u' + \Gamma w') \qquad (4.5.4)$$

where α, β, Γ, y', u' ,w' are given by equations (3.3.20-24). Substituting equations (4.5.2 & 3) into (4.5.4), the predictor for ϕ itself is given by:

$$\phi^*(t+k|t) = \frac{1}{R}(\alpha y' + \beta u' + \Gamma w' + d') \qquad (4.5.5)$$

where d' is a constant given by:

$$d' = \left[R(1) - \frac{\alpha(1)}{\lambda_1(1)}\right]d \qquad (4.5.6)$$

The prediction error associated with (4.5.5) is identical to that associated with (3.3.19) and is thus independent of the constant d; hence the predictor model (4.2.1) is replaced by (4.5.5), whilst equation (4.2.2) is unchanged. Augmenting the data vector $\underline{X}$ (4.2.5) with a unit element, and the parameter vector $\underline{\Theta}$ (4.2.5) with a constant element d', the modified system system (4.5.1) may be written in predictor form (4.5.5) as equation (4.2.6). The self-tuning controller is thus simply modified by including a unit element in $\underline{X}$, and estimating the constant (but now unknown) term d'.

It is now assumed that the modified self-tuning controller has converged to the corresponding minimum variance control law, and the resultant offset investigated. The minimum variance controller has the property (3.3.17) that in the steady state, the auxiliary output (ϕ) is given by:

$$\phi(t+k) = \Lambda^{y} y(t+k) + \Lambda^{u} u(t) + \Lambda^{w} w(t)$$

$$= \epsilon^{*}(t+k) \qquad (4.5.7)$$

Denoting the mean values of y, u and w by the subscript m and noting that ϵ^{*}(3.3.18) has zero mean, in the steady-state:

$$y_m = \frac{-1}{\Lambda^{y}(1)} (\Lambda^{u}(1)\, u_m + \Lambda^{w}(1)\, w_m) \qquad (4.5.8)$$

As the system coefficients are unknown a priori, u_m is unknown a priori; under these circumstances, the offset $(y_m - w_m)$ will always be zero iff:

$$\text{a)} \quad \Lambda^{u}(1) = 0 \qquad (4.5.9)$$

$$\text{and} \quad \text{b)} \quad \frac{\Lambda^{w}(1)}{\Lambda^{y}(1)} = -1 \qquad (4.5.10)$$

The former condition can be ommited if it is known a priori that u_m is zero - for example if the system is known to contain an integrator. Some particular cases where conditions a) and b) are satisfied are

now given.

1) Model following control law.

Such control laws were discussed in section 3.4. By definition, Λ^{u} was taken to be zero, and so condition a) is automatically satisfied. Condition b) then merely states that the chosen model $(\Lambda^{y}/\Lambda^{v})$ has unity gain.

2) Extended classical control law.

Such control laws were discussed in section 3.5. By definition, $\Lambda^{v} = 1$ and $\Lambda^{y} = -1$, and so condition b) is automatically satisfied. Condition a) then states that the feedback compensation $(1/\Lambda^{u})$ must contain a discrete-time integrator - this result is expected from classical control theory.

4.6 Simulation of the self-tuning controller.

Unlike the invariant control laws presented in chapters 2 & 3, the behaviour of the non-linear self-tuning controller cannot be accurately predicted in advance. This section presents a selection of simulated examples of the self-tuning controller operating under various conditions; these examples are designed to illustrate the preceeding theory - rather than provide a detailed statistical verification.

The simulation package was designed for use on a D.E.C. PDP 11/45 digital computer and written in FORTRAN, making fulluse of the rapid output afforded by an associated visual display unit. The main design criteria adopted were flexibility and ease of setting up simulation runs, and provision of comprehensive graphical and numerical of selected generated data.

An experiment or run is defined by one set of parameters representing the system; and up to five sets of parameters, each defining a controller, set point and noise sequence. Up to five controllers - which may be different algorithms, or the same algorithms but with different parameters - may thus be simulated independently, in parallel, on identical systems with either the same or different nise sequences. This allows easy comparison of algorithms with clearly defined

defined differences in experimental conditions. The package consists of three interconnected programs, data being passed via disk files. These programs are now briefly described.

The program 'SIMSET' sets up the six sets of defining parameters described above. When an experiment has been set up, it is stored on a disk file; its name and date of entry are entered in a file directory, enabling easy retrieval at a future date. Once an experiment has been entered manually, or read from disk, any one of these six parameter sets can be displayed on the visual display screen; the system display has a fixed format, but the controller display depends on the type of controller.

The parameters being displayed can be changed in three different ways. Firstly, associated with each type of controller is a standard set of constants - which may be system dependent - which may be loaded on command. Secondly, any controller parameter set may be replaced by that of any other of the controllers in the particular experiment. Thirdly, by entering the first two letters of a parameter name, followed by its required value, any parameter under display can be individually changed. The combination of these three entry methods leads to a very flexible, yet simple, set up procedure.

The program 'SIMRUN' reads the required experiment from disk, the five controller parameter sets being loaded into the named COMMON areas, /COM1/-/COM5/. The program then calls, in turn, up to five subroutines - identical but for the named COMMON. Each subroutine, according to an indexing parameter, calls one of a number of possible controller subroutines; the corresponding parameter set being interpreted by an EQUIVALENCE statement. This allows up to five independent simulations to be run in parallel.

The discrete-time white noise processes are generated using a D.E.C. supplied pseudo-random number generator which produces a sequence of uncorrelated numbers with a uniform distribution. The mean of twelve of such numbers is suitably normalised to produce - via the central limit theorem (Melsa & Sage,1973) - a new sequence with an approximate gaussian distribution. This was not because the theory relies on such a distribution, but rather because it was felt that a gaussian distribution was more realistic than a uniform distribution.

The display program, 'SIMDSP', uses the display terminal to plot graphs of any required data generated, and stored on disk file, by the previous program. Similarly, data may be tabulated; or any data set correlated with any other to provide statistical diagnostics.

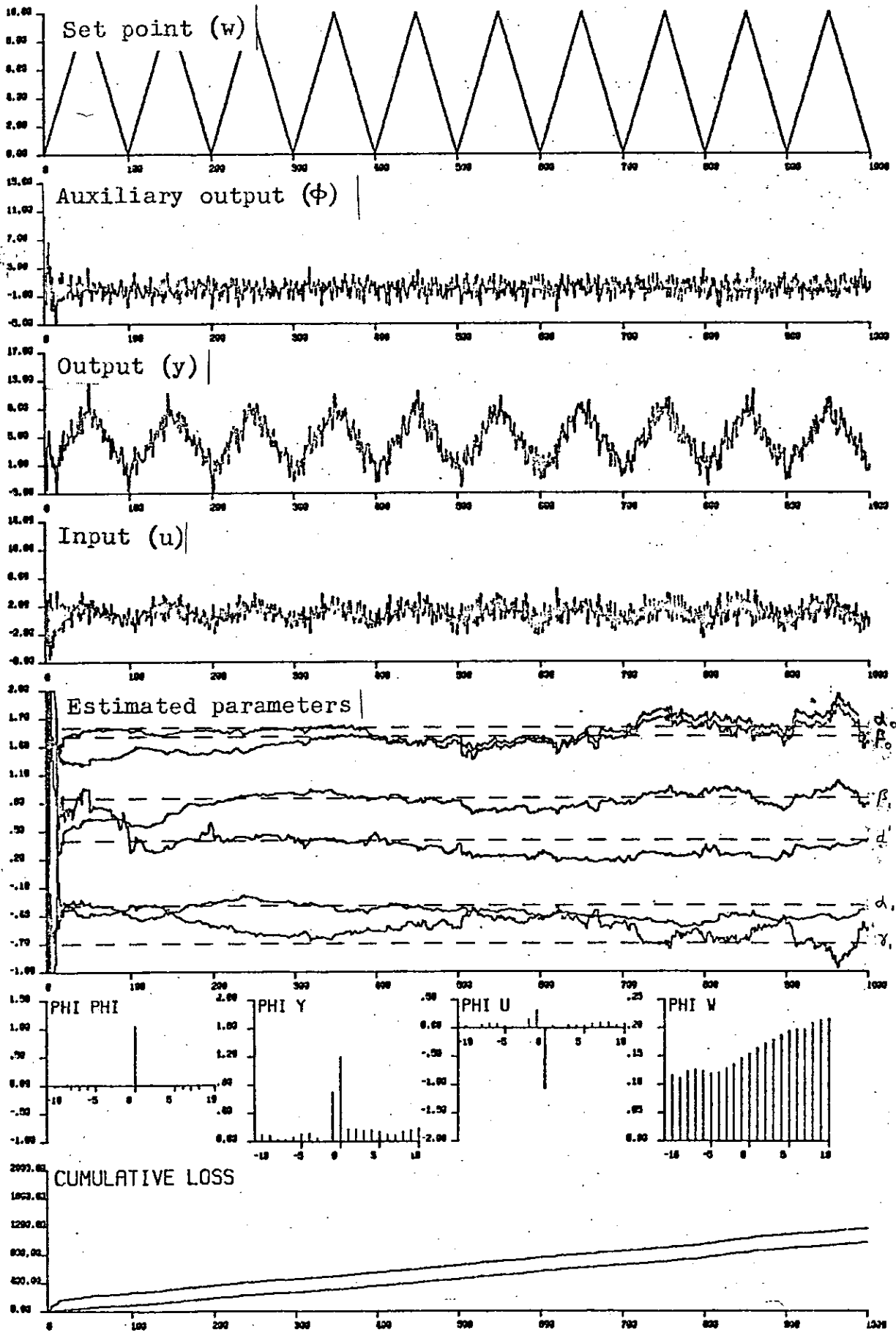


Fig.4.6.1 A general example.

Example 4.6.1 A general example.

This example (Fig 4.6.1) is intended to give a general view of a self-tuning controller in operation, and to point out typical features.

System.

$$y = \frac{z^{-1}(1 + 0.5z^{-1})u + (1 + 0.7z^{-1})\xi}{(1 - 0.9z^{-1} + 0.3z^{-2})}$$

White noise statistics.

Mean = 1.0, Standard deviation = 1.0

Cost function.

$$\Lambda^{y} = 1, \quad \Lambda^{u} = 0.5, \quad \Lambda^{\nu} = -1$$

Minimum variance control law. (3.3.20-24)

$$\alpha = 1.6 - 0.3z^{-1}$$

$$\beta = 1.5 + 0.85z^{-1}$$

$$\Lambda^{\nu}\Gamma = -1.0 - 0.7z^{-1}$$

$$d' = 0.4 \qquad (4.5.6)$$

Self-tuning controller parameters.

γ_0 fixed at -1.0, forgetting factor (4.3.10) 0.995

$$\hat{\underline{\theta}} = \text{col.}[\hat{\alpha}_0, \hat{\alpha}_1, \hat{\beta}_0, \hat{\beta}_1, -\hat{\gamma}_1, \hat{d}'], \quad \underline{P}(0) = 100.\underline{1}$$

The first four graphs are all self-explanatory, and refer to the self-tuning controller. The fifth

graph gives the parameters estimated by the self-tuning controller, the asymptotes correspond to the parameters of the minimum variance control law - to which the estimated parameters should converge. The four graphs entitled $\phi\phi$, ϕy, ϕu, and ϕw are the cross correlations of each two variables taken over the last five hundred points shown. The final graph,'Cumulative loss', gives the quantity:

$$\sum_{t=1}^{N} \phi^2(t)$$

plotted against N. The upper curve is based on the self-tuning controller data; for comparison the lower curve is based on the corresponding minimum variance controller operating on the same system with identical noise sequence. The relative increase of these two quantities is a measure of the degree of convergence of the self-tuning controller.

From section 3.3, the minimum variance control law - asymptotic self-tuning controller - is characterised by the auxiliary output (ϕ) being a moving average process of order k-1; in this case (k=1) a white noise process. This is verified by the sample autocorrelation over the last five hundred points, given in graph $\phi\phi$. Similarly graphs ϕy, ϕu, ϕw verify that ϕ is approximately uncorrelated with $y(t-i)$, $u(t-i)$, and $w(t-i)$ for $i > k$.

The estimated parameters wander about their ideal values (as shown by dashed lines); however they wander in unison and, as the cross correlations indicate that the control law has the required properties, it is conjectured that this is due to the control law being insensitive to such deviations. This behaviour is discussed at the end of section 4.3.

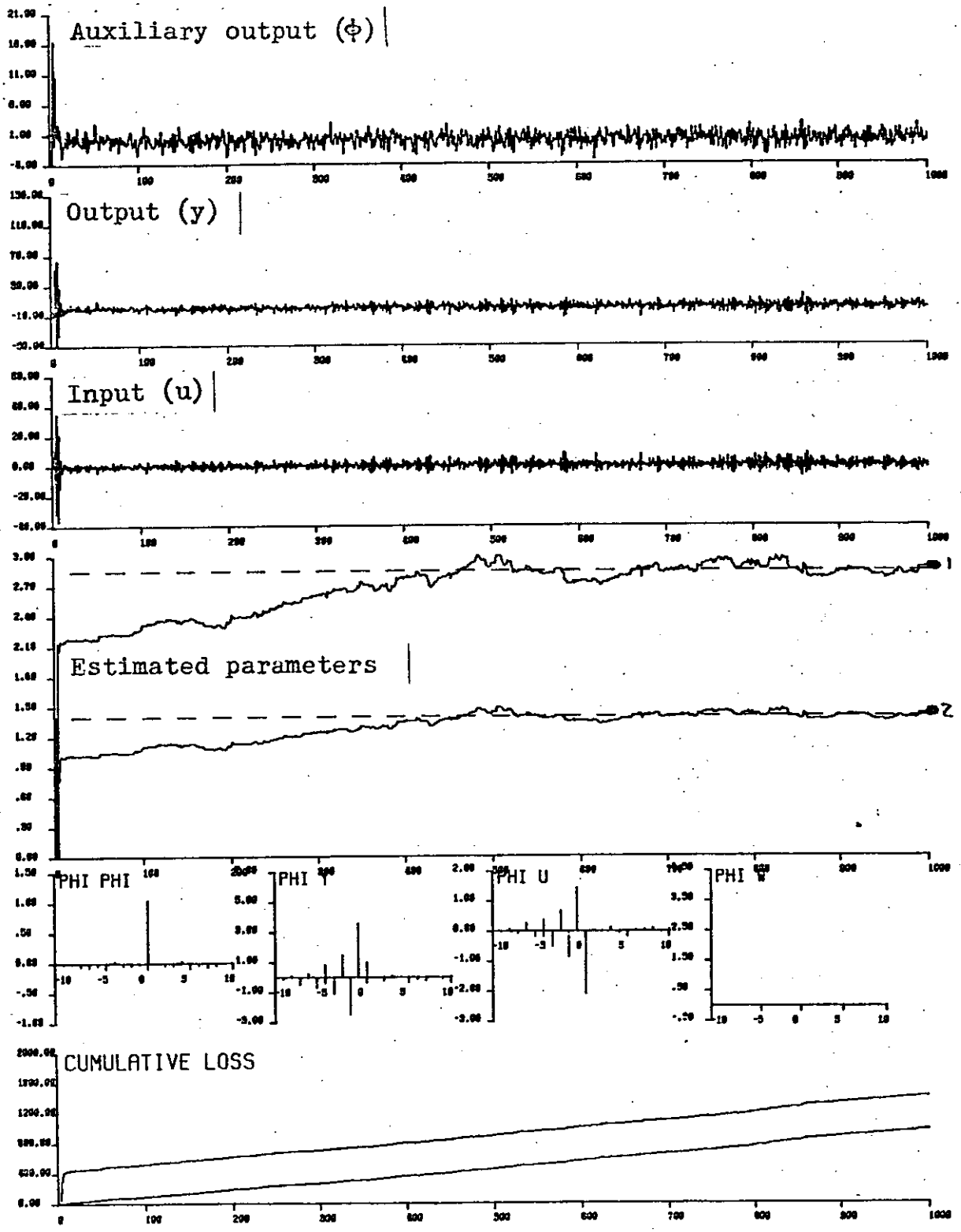


Fig 4.6.2 A non-minimum phase system.

Example 4.6.2 A non-minimum phase system.

This example demonstrates the use of control weighting to succesfully control a system with a zero outside the unit circle. Such systems cannot be controlled using the basic self-tuning regulator.

System.

$$y = \frac{z^{-1}(-1 + 2z^{-1})u + (1 + 0.5z^{-1})Y}{(1 - 0.9z^{-1})}$$

White noise statistics.

Mean = 0, Standard deviation = 1.0

Cost function.

$\Lambda^y = 1$, $\Lambda^u = 1.7$; the set point is zero.

Minimum variance control law.

$$\alpha = 1.4$$

$$\beta = 0.7 + 2.85z^{-1}$$

Self-tuning controller parameters.

β_0 fixed at 0.7, forgetting factor = 0.99

$\hat{\underline{\theta}} = \text{col.}[\hat{\alpha}_0, \hat{\beta}_1]$, $\underline{P}(0) = 100\,\underline{1}$

The graphs are of the same format as those of example 4.6.1.

The self-tuning controller operates succesfully in this example because Λ^u is such that the auxiliary

system is given by:

$$\phi = y + z^{-1} 1.7\, u$$

$$= \frac{z^{-1}(-1 + 2z^{-1} + 1.7(1-0.9z^{-1}))\, u + (1+0.5z^{-1})\gamma}{(1 - 0.9z^{-1})}$$

$$= \frac{z^{-1}(0.7 + 0.47z^{-1})u + (1 + 0.5z^{-1})\gamma}{(1 - 0.9z^{-1})}$$

This system has no zeros outside the unit circle, hence as discussed in section 3.3, the minimum variance controller operates succesfully on this system.

An interesting feature of this example is that the initial large excursions of the system output and input cause - as discussed in section 4.4 - the parameter estimates to move close to those pertaining to the case where the noise numerator polynomial is replaced by unity, that is: $\alpha_0' = 0.9$, $\beta_1' = 2.0$. After some time, however, the estimated parameters tend to their indicated ideal values.

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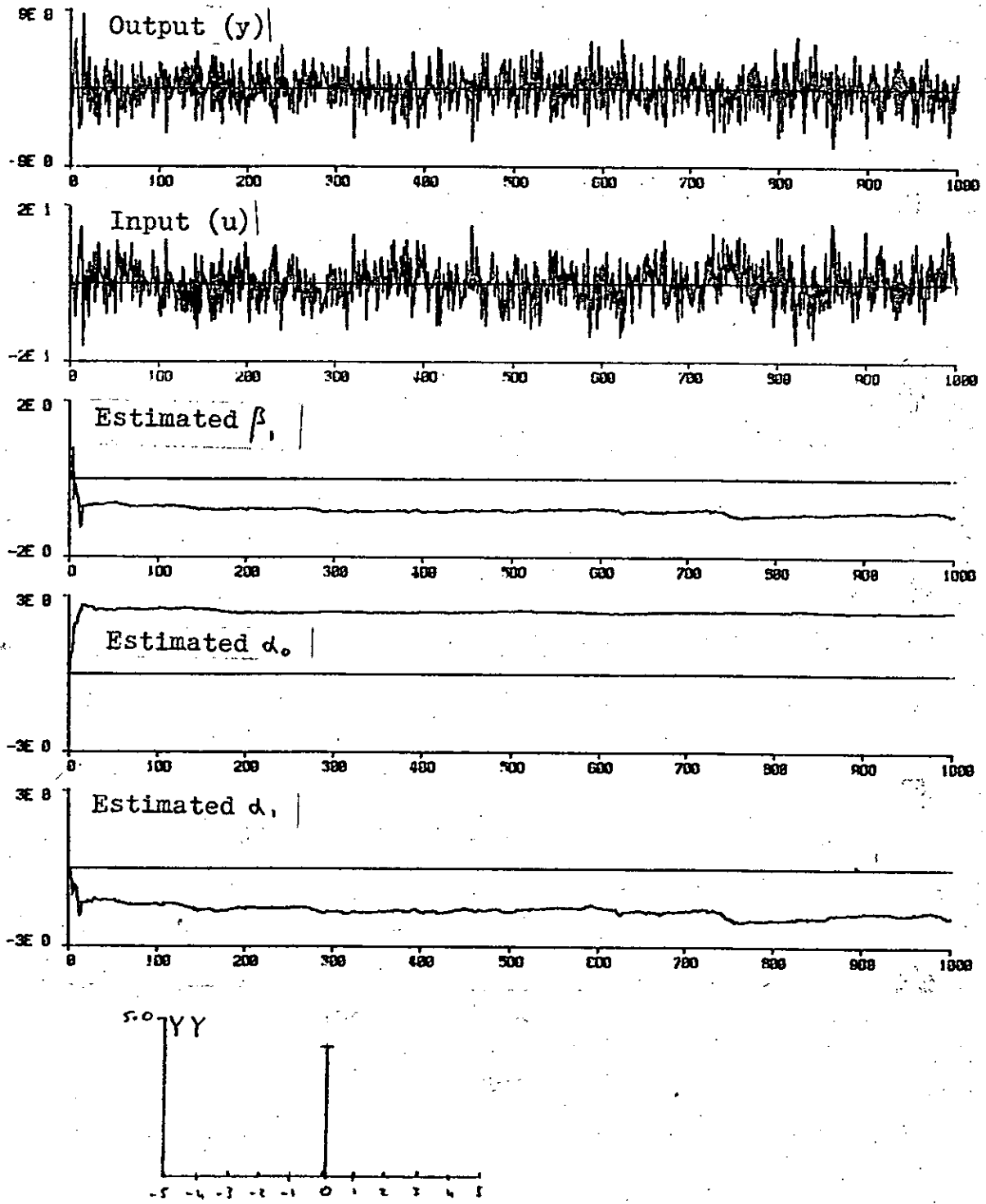

Fig 4.6.3 An unstable system.

Example 4.6.3 An unstable system.

This example demonstrates the behaviour of the self-tuning regulator operating on an unstable system; the control of this system is also considered in example 2.8.1.

System.

$$y = \frac{1}{1 - 2z^{-1}} u + \frac{1}{1 - 0.9z^{-1}} \psi$$

White noise statistics

Mean = 0.0, Standard deviation = 1.0

Cost function.

$\Lambda^y = 1$, $\Lambda^u = 0$; the set point is zero.

Minimum variance control law.

$$\alpha = 2.4 - 1.8z^{-1}$$

$$\beta = 1.0 - 0.9z^{-1}$$

Self-tuning controller parameters.

β_o fixed at 1.0, forgetting factor = 0.995, $\underline{P}(0) = \underline{1}$

In this example, the unstable system leads to a converted disturbance process with a zero outside the unit circle given by (2.6.8) as:

$$C = C'A = 1 - 2z^{-1}$$

As the control weighting (Λ^u) is zero, the minimum

variance controller becomes the minimum variance regulator discussed in example 2.8.1. There it was shown that although it is theoretically possible to achieve a white noise ouput (y) of unit variance; such a control law involves cancellation of a pole outside the unit circle, and is thus impracticable. The optimum insensitive control law (given overleaf) was shown to be given by replacing C by C^* given by:

$$C^* = 2 - z^{-1}$$

This latter control law gives a white noise output with variance four.

In this example it is demonstrated that the self-tuning regulator does, as expected, converge to this insensitive control law; the sample autocorrelation does indeed approximate to that of white noise with a variance of four.

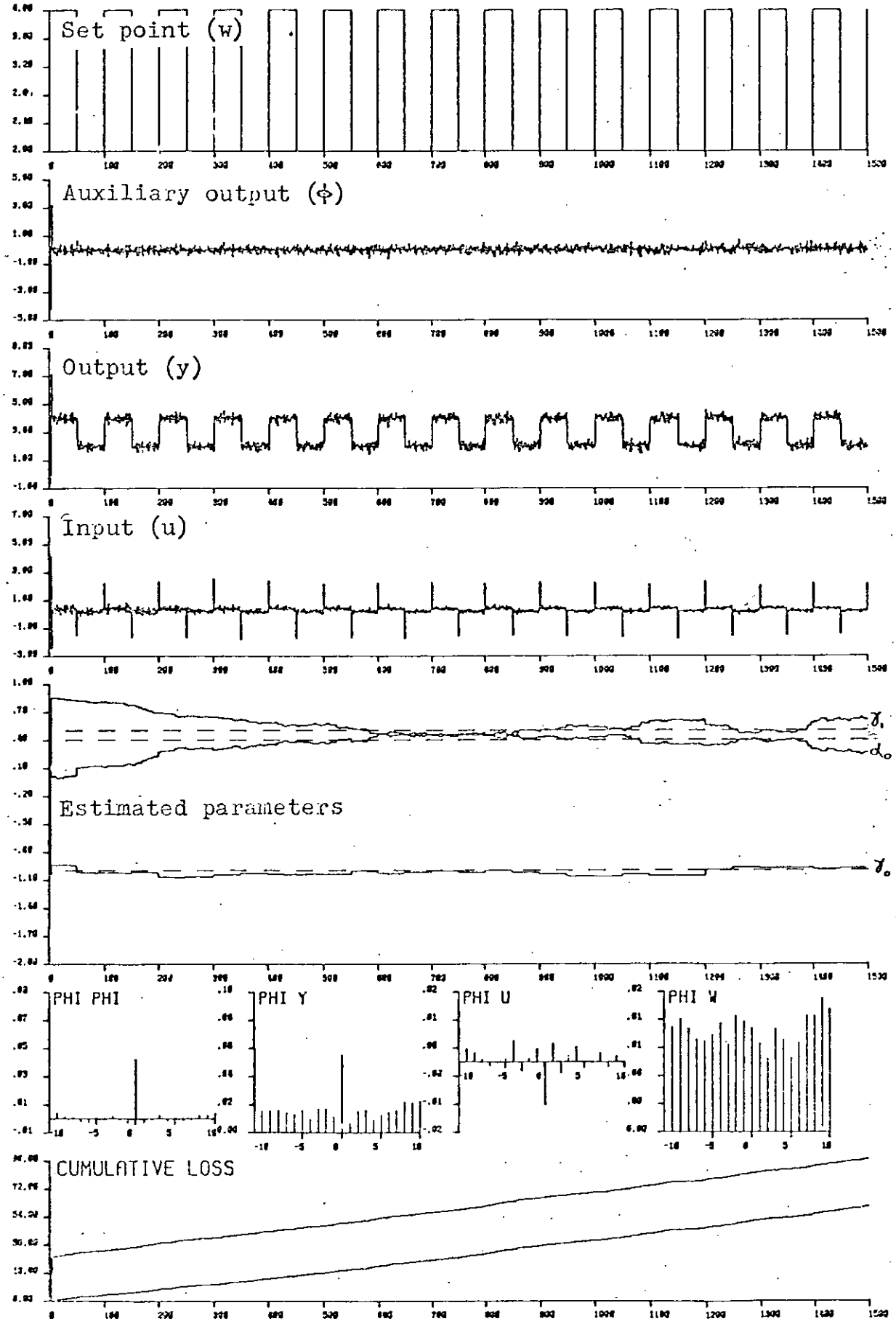

Fig 4.6.4 a) The self-tuning controller.

Example 4.6.4 A stochastic tracking problem.

This example demonstrates the superiority of the self-tuning controller over the self-tuning regulator, when applied to a stochastic tracking problem. As the weighting on control (λ^u) is zero, the minimum variance controller corresponds to the optimal control laws discussed in chapter 2; whereas the self-tuning regulator of this example corresponds to the suboptimal control laws, discussed in section 2.1, which treat the set point as a disturbance.

System.

$$y = \frac{u + (1 - 0.5z^{-1})\psi}{(1 - 0.9z^{-1})}$$

White noise statistics.

Mean = 0.0, Standard deviation = 0.2

Cost function

$$\lambda^y = 1, \quad \lambda^u = 0, \quad \lambda^\nu = -1$$

Minimum variance control law.

$$\alpha = 0.4$$

$$\beta = 1.0$$

$$\lambda^y \Gamma = -1.0 + 0.5z^{-1}$$

Self-tuning parameters.

β_o fixed at 1.0, forgetting factor = 0.995, $P(0) = 100.\underline{1}$

The graphs are of the same format as those of example 4.6.1

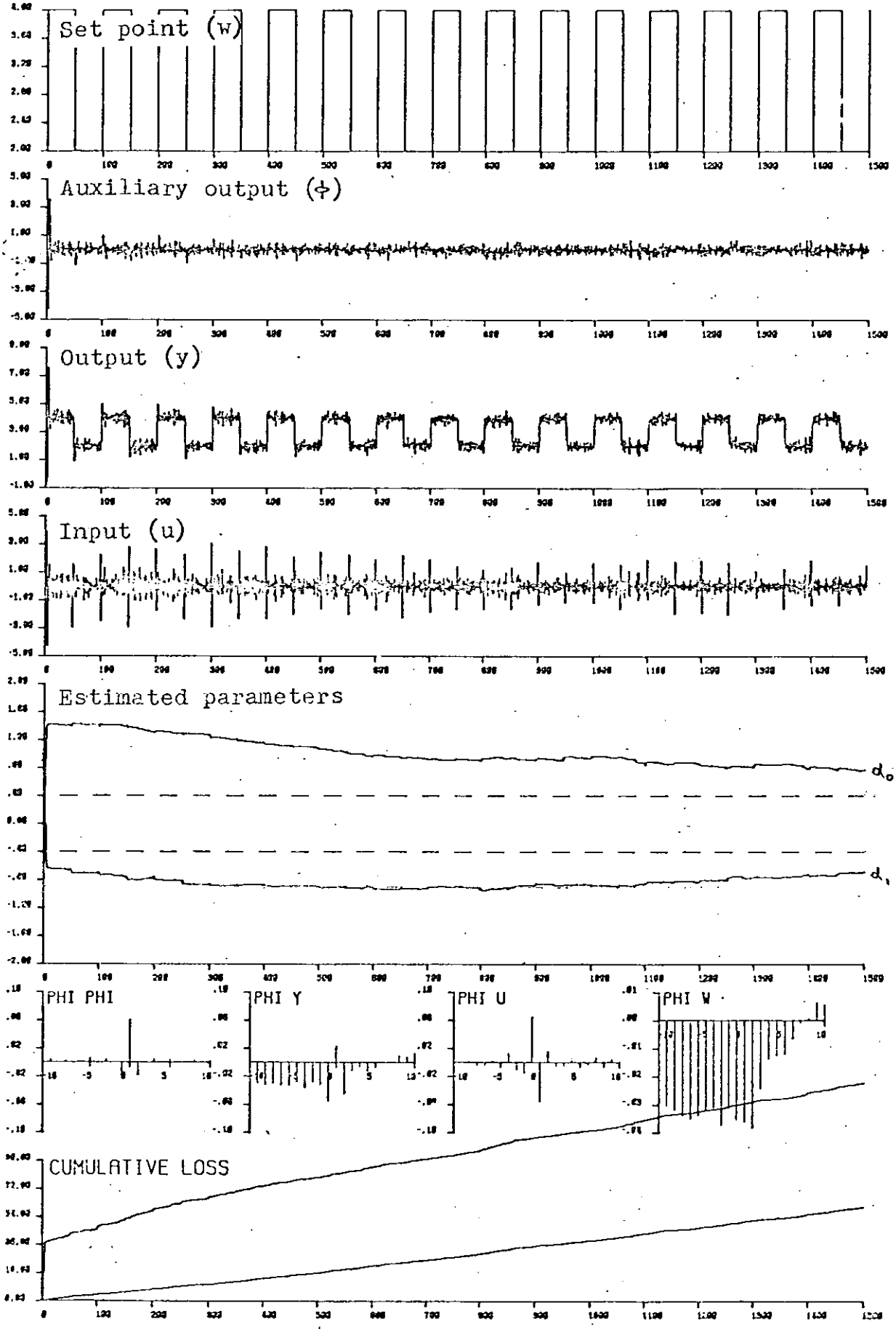

Fig 4.6.4 b) The modified self-tuning regulator.

Fig 4.6.4 b) shows the behaviour of a modified self-tuning regulator, operating on the same system with identical set point and disturbance. This modification, which ensures zero offset, is given by Wittenmark (1973) and is depicted in Fig 4.6.4 c).

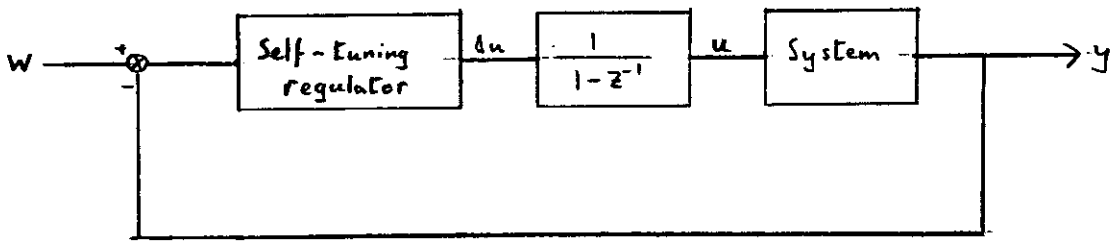

Fig 4.6.4 c) The modified self-tuning regulator.

The integrator ensures no steady-state offset, but - as discussed in chapter 2, -treating the set point and disturbance in the same manner reduces the amount of information available to the controller, and performance is adversely affected. The parameters of this suboptimal regulator depend on the magnitude and pattern of the set point; the dashed lines in Fig 4.6.4 b) are for zero set point. The self-tuning parameters are as for the self-tuning controller.

From the auto correlation ($\phi\phi$) taken over the last 500 points - as for the self-tuning controller - it can be seen that the average loss is 0.6, compared

with the minimum possible of 0.4 attained by the self-tuning controller (Fig 4.6.4 a)). This is also reflected in the fact that the cumulative loss of this modified self-tuning regulator increases more rapidly than that of the self-tuning controller.

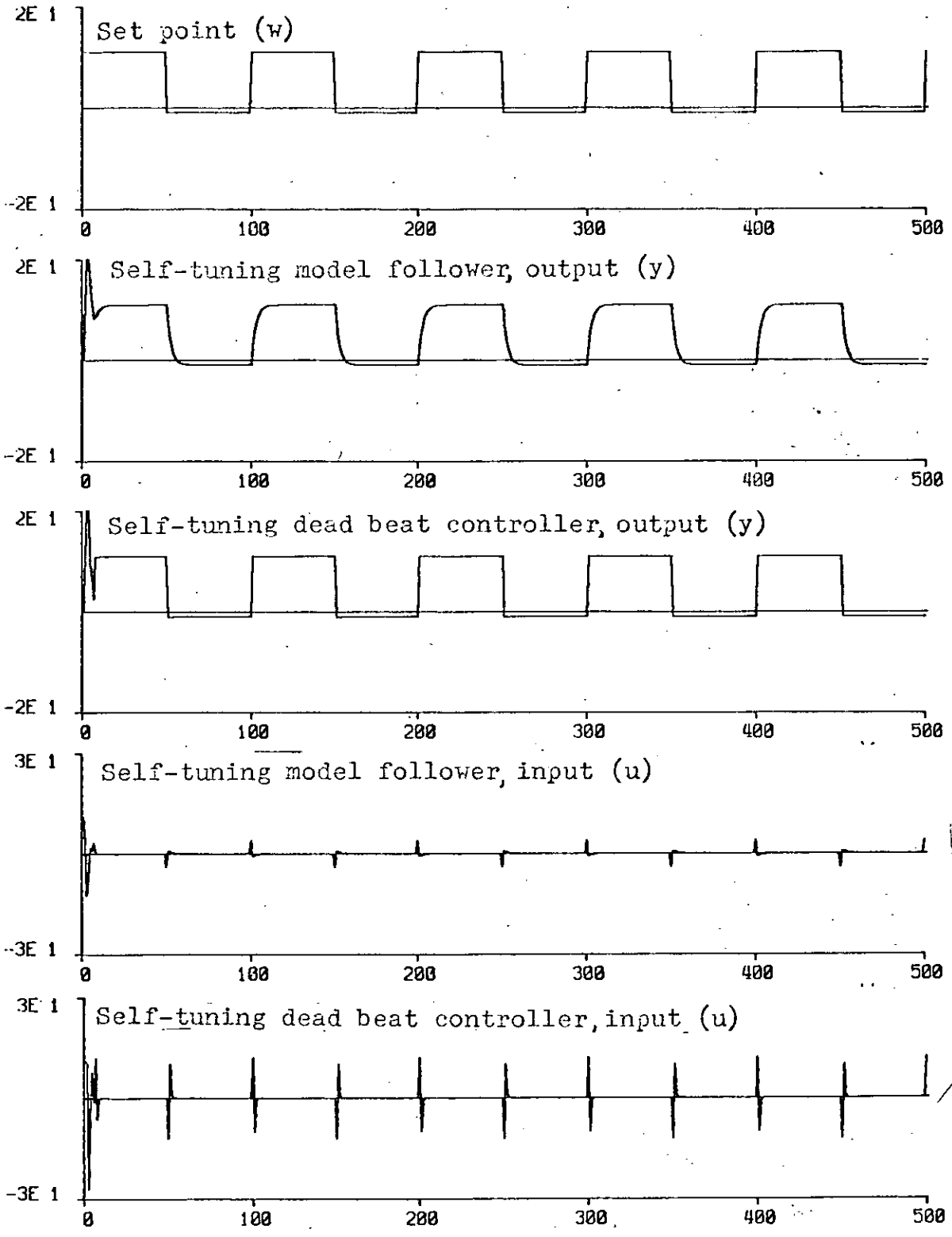

Fig 4.6.5 a) Self-tuning model following.

Example 4.6.5 Model following.

This example demonstrates the utility of the model following control laws (section 3.4) when included in a self-tuning scheme. Fig 4.6.5 a) applies to a servo problem with no disturbance; Fig 4.6.5 b) applies to a stochastic regulator problem.

System.

$$y = \frac{z^{-1}(1 - 0.9z^{-1})u + (1 - 0.5z^{-1})v}{(1 - 0.95z^{-1})^2}$$

White noise statistics.

Mean = 0.0; a) Standard deviation = 0.0

b) " " = 1.0

Cost function.

$$\Lambda^y = 3 - 2z^{-1}, \quad \Lambda^u = 0, \quad \Lambda^w = -1$$

Minimum variance control law.

$$\alpha = 2.2 - 1.7075z^{-1}$$

$$\beta = 3(1 - 0.9z^{-1})$$

$$\Lambda^w\Gamma = -1 + 0.5\,z^{-1}$$

Self-tuning controller parameters.

a) γ_0 fixed at 1, forgetting factor = 0.995, $\underline{P}(0) = \underline{1}$

b) β_0 " " 3, " " " " "

For comparison, a self tuning controller identical but for $\Lambda^u = 1$ was also simulated; for case a) this corresponds

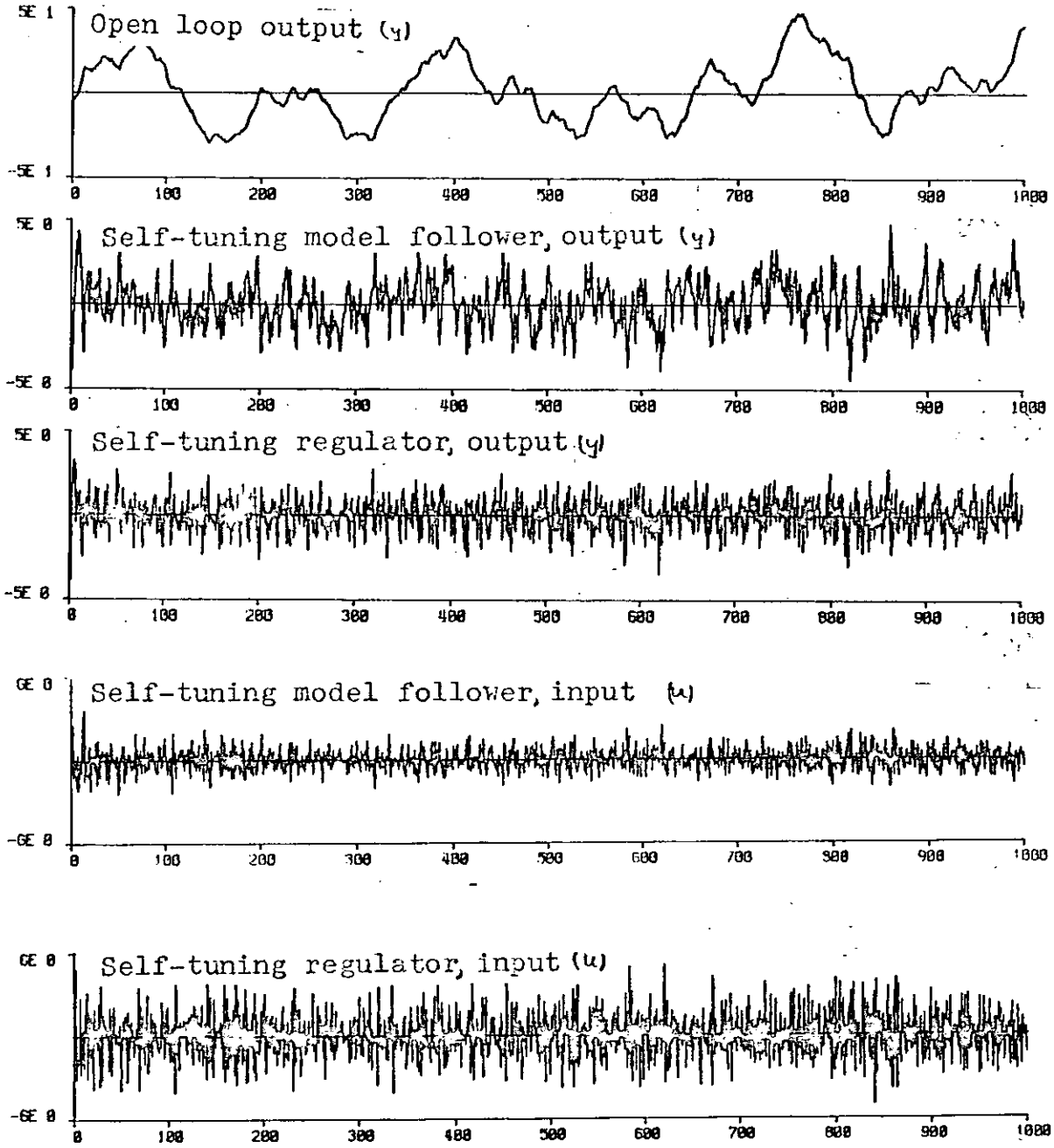

Fig 4.6.5 b) Self-tuning model following.

to a self-tuning dead beat controller, and for case b) it corresponds to a self-tuning regulator. It can be seen from Figs 4.6.5 a) & b) that the model following control law trades off a decrease in output performance against a relatively large decrease in control effort.

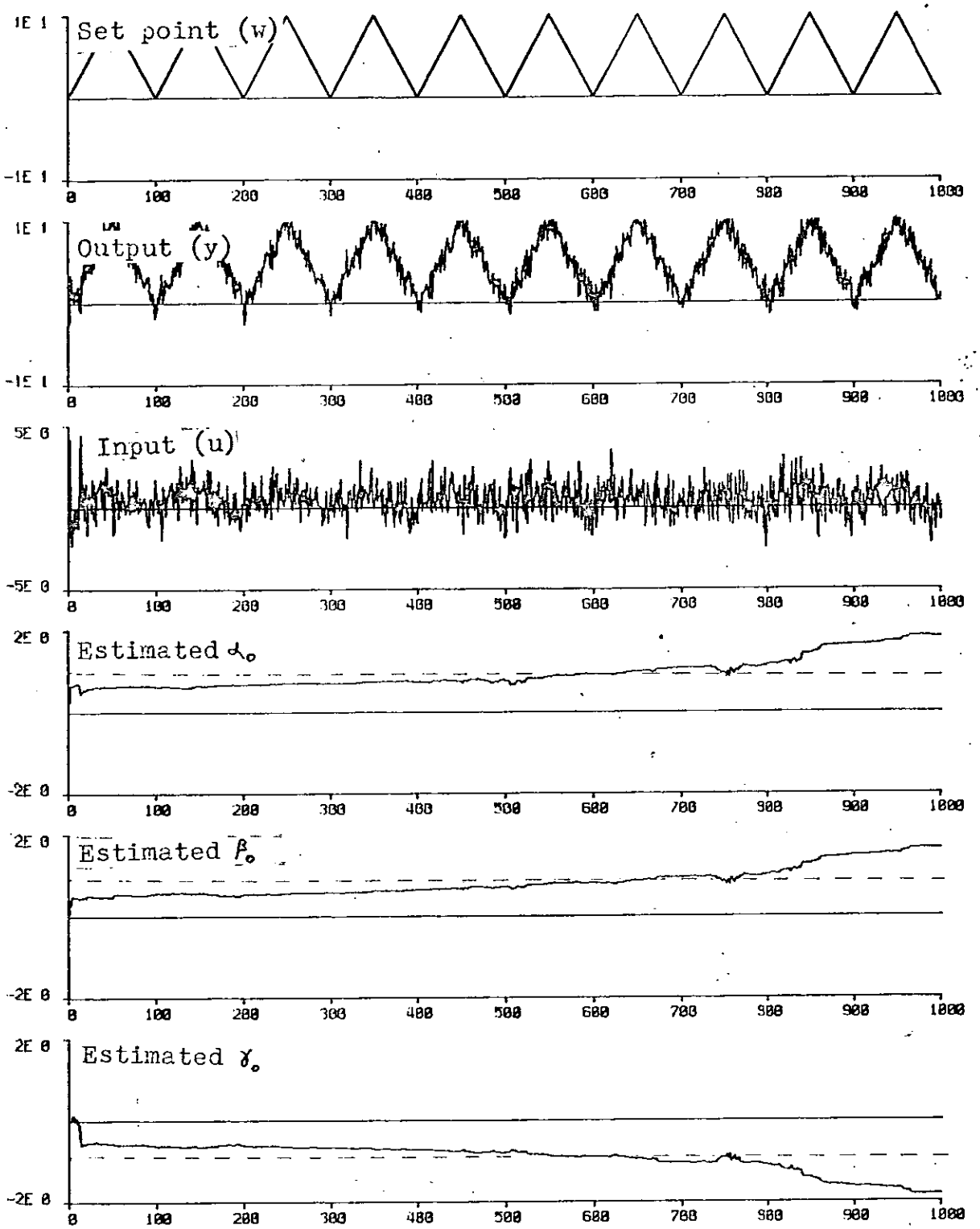

Fig 4.6.6. No parameter fixed.

Example 4.6.6 The effect of not fixing a parameter.

This example demonstrates the effect of not fixing an estimated parameter at a prespecified constant value; this was discussed in section 4.3.

System.

$$y = \frac{z^{-1} u + \psi}{1 - 0.9z^{-1}}$$

White noise statistics.

Mean = 0.0, Standard deviation = 1.0

Cost function.

$$\Lambda^y = 1, \quad \Lambda^u = 0, \quad \Lambda^v = -1$$

Minimum variance control law.

$$\alpha = 0.9$$

$$\beta = 1.0$$

$$\Lambda^v \Gamma = -1.0$$

Self-tuning controller parameters.

No parameter fixed. Forgetting factor = 0.995.

$$\underline{\hat{\theta}} = \text{col}[\hat{\alpha}_o, \hat{\beta}_o, -\hat{\gamma}_o] \ , \ \underline{P}(0) = \underline{1}$$

The self-tuning controller depicted in Fig 4.6.6 estimates all three controller parameters - no parameter is fixed at a constant value. It is seen that the parameters drift in unison; there is no tendency for

them to settle down to constant values. As discussed in section 4.3, this does not effect the control signal; but if the run were continued, the parameters could take on very large or small values - this could lead to numerical problems. Also, if the previously constant parameters changed, the control law would become time varying and hence identifiable; if the parameters had drifted to very large values, controller performance could be poor as the parameters were retuned.

4.7 A quasi-continuous self-tuning controller.

In chapter 2 the optimal control of continuous systems was considered, and the optimum control signal was found to be a continuous function of the system output and input - a well known result. It follows that a sampled data control system, involving discrete sampling of output and piecewise constant control signal, is suboptimal. This point has been considered previously (Åström,1963) in connexion with the discrete control of linear systems with no time-delay and designed with respect to a continuous-time cost function; it was shown that as the sample period decreases, the performance of the control law improves.

The discrete-time control laws considered in this thesis are based on discrete-time cost functions; this can lead to poor inter-sample behaviour, for example 'hidden oscillations' (Kwakernaak & Sivan,1972). It seems, once again, that decreasing the sample period, and changing the control law parameters accordingly, leads to improved continuous performance.

For a number of situations, then, it seems that a small sample period is better than a long sample period. The basic self tuning controller described in earlier sections has a lower limit on the sample period imposed by two main factors. Firstly, as the sample

period decreases, measured values at one sample instant differ by a decreasingly small amount from those at the previous instant,,and hence the discrete system and controller will have poles and zeroes close to the unit circle. This leads to an increasingly sensitive control law, and possible convergence and numerical problems.

Secondly, if the continuous system has a finite time delay T - due to transport or computational lag - the integer time-delay of the corresponding discrete-time system is given by:

$$k = \text{Integer part of}\left[\frac{T}{\Delta}\right] + 1 \qquad (4.7.1)$$

where Δ is the sample period. Hence as Δ decreases, k grows and the number of parameters in the corresponding discrete-time control law grows - in practice $k > 5$ gives excessive computation time for the self-tuning controller.

The approach taken in this section is to consider a continuous analogue of the discrete-time self-tuning controller, and then discretise the algorithm for digital computation. This leads to a self-tuning controller with the following properties:

a) The number and value of the controller coefficients are largely independent of the sample period.

b) The control law may be realised in analogue form, or digital approximation with small sample period; the more complex estimation phase being realised digitally at a sample period independent of that of the control law.

The theory of this class of quasi-continuous self-tuning controllers is not as advanced as that of the discrete version, and is a subject of current research. In this section some basic results are given, and topics requiring further elucidation are pointed out.

Firstly consider systems with zero time-delay; from equations (3.3.20-24), the predictor is in terms of finite degree polynomials, hence analogously to equations (4.2.1 & 2) the auxiliary system may be written as:

$$R\,\phi^*(t+T\ t) = \alpha y(t) + \beta u(t) + \Gamma w(t) \qquad (4.7.2)$$

$$\phi(t+T) = \phi^*(t+T|t) + \epsilon^*(t+T) \qquad (4.7.3)$$

Where α, β, and Γ are now polynomials in the Laplace operator s. The formally equivalent procedure to that of section 4.2 would be to form a vector ($\underline{X}$) of y,,u, w and their derivatives, and use this as the measurement vector in an estimation algorithm corresponding to equations (4.2.12-14). However y, u, and w are not, in general, differentiable the requisite number of times - in the sense that the derivatives might contain

white noise processes. One approach which avoids this problem is to rewrite the polynomials α, β, and Γ in terms of (s+a), as opposed to s, where a is some constant; this gives a new set of polynomials $\alpha'(s+a)$, $\beta'(s+a)$ and $\Gamma'(s+a)$. Defining N as the highest degree of α', β', and Γ', equation (4.7.2) may be rewritten as:

$$\frac{R\,\phi^*(t+T|t)}{(s+a)^N} = \sum_{i=0}^{N} \frac{\alpha_i'}{(s+a)^i} y + \cdots + \frac{\beta_i'}{(s+a)^i} u + \frac{\gamma_i'}{(s+a)^i} w \tag{4.7.4}$$

where α_i', β_i', and γ_i' are the ith coefficients of α', β' and γ' respectively. The vectors $\underline{X}$ and $\underline{\Theta}$ can now be defined - as in (4.2.5) - by:

$$\underline{X}(t) \triangleq \begin{bmatrix} y(t) \\ \vdots \\ \frac{y(t)}{(s+a)^N} \\ u(t) \\ \vdots \\ \frac{u(t)}{(s+a)^N} \\ w(t) \\ \vdots \\ \frac{w(t)}{(s+a)^N} \end{bmatrix} \qquad \underline{\Theta} = \begin{bmatrix} \alpha_N' \\ \vdots \\ \alpha_0' \\ \beta_N' \\ \vdots \\ \beta_0' \\ \gamma_N' \\ \vdots \\ \gamma_0' \end{bmatrix} \tag{4.7.5}$$

The generation of $\underline{X}$ is depicted in Fig 4.7.1.

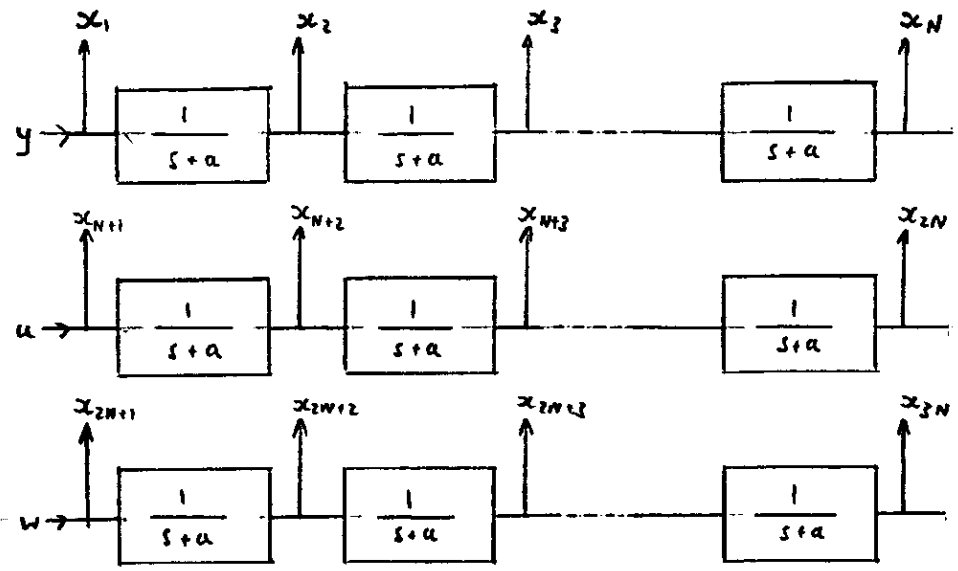

Fig 4.7.1. Generation of the X vector.

Equation (4.7.4) then becomes:

$$\phi^*(t+T|t) = \frac{(s+a)^N}{R} \underline{X}(t).\underline{\Theta} \qquad (4.7.6)$$

It is not clear that this is the 'best' form of the data vector; for instance, filters of different forms could be used to make the components of $\underline{X}$ as different as possible, in some sense. This is a matter for further research.

The components of $\underline{X}$ are now well behaved, and a continuous analogue of the discrete self-tuning controller is given by:

a) Using the measured values of y, u and w, form the auxiliary output $\phi(t)$

b) Pass the measured values of y, u, and w through a bank of low-pass filters (Fig 4.7.1) to generate the vector $\underline{X}$; then update the parameter estimate $\hat{\underline{\Theta}}(t)$ using a continuous Kalman filter.

c) Use the current estimate $\hat{\underline{\Theta}}$ and the data vector $\underline{X}$ to choose U(t) such that:

$$\underline{X}^T(t).\hat{\underline{\Theta}}(t) = 0 \qquad (4.7.7)$$

For practical purposes this continuous algorithm would be discretised to give a quasi-continuous algorithm.

In general, step a) involves differentiation of the system output (y) and input (u); the continuous Kalman filter integrates the variable ϕ once. Hence, if the estimate $\hat{\underline{\Theta}}$ is not to have a white noise component, ϕ must not have any component involving differentiated white noise. This restricts the degrees of the polynomials defining ϕ, depending on the degrees of the system polynomials.

A similar restriction arises from the method by which the data vector $\underline{X}$ is generated. From equation (4.7.5) it follows that if N is greater than the degree of R, ϕ^* will contain derivatives of $\underline{X}$ $\underline{\Theta}$. In the previous chapter it was stated that ϕ^* does not contain white noise processes, hence if any element of $\underline{X}$ cannot be differentiated the requisite number of times, the corresponding element of $\underline{\Theta}$ must be zero. The existence of such zero elements is not known a priori, hence white noise processes can appear in the estimated value of given by $\underline{X}^T\hat{\underline{\Theta}}$; this could lead to difficulties in the estimation of $\underline{\Theta}$. It would seem, however, that the fact that the algorithm is, in practice, a discrete-time approximation would ameliorate these effects. This is a matter for further research.

If the system time delay is non-zero, the minimum variance control law cannot be represented in terms of finite degree polynomials; it can, however, be approximated by rational transfer functions. In this case the self tuning controller would involve a parameter vector of finite dimensions corresponding to this approximate control law. It is a matter for further research as to what degree of approximation is required to obtain a satisfactory self-tuning controller; however the number of parameters used in the quasi-continuous approximation would be independent of the sample period used.

4.8 Summary.

An adaptive control scheme has been proposed, which is applicable to the control of systems of known structure and with unknown - but constant - coefficients. This control law uses a recursive least-squares algorithm to directly estimate the parameters of a minimum variance control law, of the type discussed in chapter 3. It was shown that although the adaptive controller is suboptimal, it has the property of being self-tuning - in the sense that if the estimates converge, they converge to the control law parameters that would be obtained if the system parameters were known a priori.

It was conjectured that as this self-tuning controller is similar to the self-tuning regulator (Åström & Wittenmark,1973), the methods of Ljung (1974 a,b) - developed for the analysis of the latter algorithm - are also applicable to the self-tuning controller. Some intuitive justification for this conjecture was given.

A selection of simulated examples were given which illustrate the behaviour of the of the self-tuning controller, and demonstrate some of its advantages as comared to the self-tuning regulator.

Finally, the first steps of some research into a quasi-continuous self-tuning controller were given;

it is hoped that such algorithms will be beneficial at high sample rates.

Chapter 5

Conclusion.

5.1 Introduction.

This chapter summarises the results of previous chapters, and suggests possible directions of future research.

The systems considered in this thesis are single input - single output: the dynamics are described by a rational transfer function cascaded with a pure time delay; the system is disturbed by a stochastic process with rational spectral density. In chapter 2, the output measurement is taken to be corrupted by noise described in a similar fashion to the disturbance; in chapters 3 & 4 the output is taken to be directly measurable - the effect of the latter assumtion when the former condition holds is treated in section 2.15.

A possibility for further research is the extension of the results of this thesis to the multivariable case.

5.2 Steady-state optimal control.

The steady-state control of the systems considered in this thesis, where the objective is to follow some set point in the presence of disturbances has been

considered previously via two main routes: the frequency domain - transfer function methods, and the time domain - state space methods. The fact that the solution to such problems involves both feedback and feedforward components, the former being independent of the set point process, has been noted in state space literature (Kwakernaak & Sivan, 1972), but seems to have escaped attention in the frequency domain literature. In section 2.1, the stochastic tracking problem is shown to comprise two independent problems: a deterministic tracking problem, and a stochastic regulator problem. It is pointed out that the two problems have formally similar solutions.

The frequency domain methods of Newton et al. (1957) did not apply to unstable systems; this defect is remedied in chapter 2 by the realisation that the unstable 'hidden modes' of previous treatments were due to inappropriate disturbance modelling. A discrete time derivation is also outlined and shown to give the minimum variance regulator (Åström,1970; Peterka,1972) as a special case. The problem of unstable 'hidden modes' has been recently treated by Youla et al. (1976), but their method is different and perhaps rather more forced than that given here; moreover their method suboptimally includes set points and does not treat time delay systems.

A possibility for further research would be to extend the results of this thesis to the multivariable case. This has been done by Youla et al. for their method, but as their method does not include time-delays the problem of different delays on each input is avoided.

Solutions to the stochastic regulator problem via state space methods did not hitherto include systems with input delay. In chapter 2, the optimal state space control law is shown to involve: a Kalman filter generating an estimate of the current state, a predictor of the state the time delay ahead, and feedback of the predicted state. These three elements are shown to be separable in that only state estimates are passed between them - no higher orded statistics are required. The state feedback is shown to correspond to that for a certainty equivalent system but with no time-delay. Using relations between the algebraic Riccati equation and spectral factorisation, it is demonstrated that the state space control law is externally equivalent to the transfer function control law derived by frequency domain methods.

Further research could take two main directions: non-steady-state problems, and multivariable problems. The derivation of the filter in section 2.10 is perhaps rather long winded; the innovations approach to this problem (Kailath & Geasey,1973) would possibly be more direct.

5.3 Minimum variance control.

In chapter 3 are developed a novel class of control laws for both continuous and discrete systems; these can all be interpreted as regulators minimising the output of some auxiliary system, and are thus similar to the minimum variance regulator (Åström,1970). In the same way as the minimum variance regulator forms the basis of a self-tuning regulator, these control form the basis of a self-tuning controller. These minimum variance controllers are shown to be open to a number of interpretations; these include: a model following control law, an extension of classical control laws to time-delay systems, and an infinitesimal horizon control law. This latter control law has been previously considered (Clarke & Hastings-James,1971) and compared with the optimal infinite horizon control law (Ashton, 1973; Hughes and Jacobs,1973); conditions, involving root-locus diagrams in the s or z plane, are given in section 3.8 which characterise systems for which the infinitesimal horizon control law approximates the infinite horizon control law.

It would seem that the model following control law could also be considered as an approximate infinite

horizon control law - this is a matter for further research. The discrete control law of Clarke & Hastings-James has been extended to multivariable systems (Ashton, 1974), further research is needed to apply this to the more general class of discrete control laws considered in this thesis, and also to the more complicated [illegible] continuous multivariable case.

5.4 A self-tuning controller.

In chapter 4, the minimum variance control laws of chapter 3 were shown to form the basis of a self-tuning algorithm. This self-tuning controller is only slightly more complex than the self-tuning regulator (Åström & Wittenmark, 1973), and as from chapter 3 it has a number of advantages, it would seem to be a useful extension to the self-tuning regulator. The question of convergence of the algorithm is complicated, but intuitive justification is given to support a conjecture that the methods developed by Ljung (1974 a,b) are applicable to the self-tuning controller. Some examples are given to illustrate the potential of the self-tuning controller.

It is envisaged that the self-tuning controller will follow the self-tuning regulator in becoming a useful tool for the control of industrial processes.

Once again, a possible direction of further research is into the self-tuning control of multivariable systems; the self-tuning regulator has been extended (Borrison, 1975) to the multivariable case where all inputs have the same delay. The self-tuning controller can presumably extended in the same manner, but the extension to systems with different time delays remains unsolved.

An important gap in the field of suboptimal adaptive control is the lack of a unified treatment of the various algorithms proposed; further research could be directed towards the elucidation of the connexions and differences between the many adaptive control laws appearing in the literature.

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Additional notes.

Note 1 expands on the superposition argument of section 2.1. p 236

Note 2 examines the use of the word 'controllable' as used in section 2.8. p 238

Note 1 (section 2.1).

The control input to the system (Fig.2.1.1.) is, by assumption, generated by linear transfer functions operating on the measured output (v) and set point (w). As discussed by Horowitz†, although there are many ways to write such control laws, they are all essentially equivalent and may be written as:

$$u = G_1 w + G_2 v \qquad (N1.1)$$

where G_1 and G_2 are transfer functions (FigN1.1).

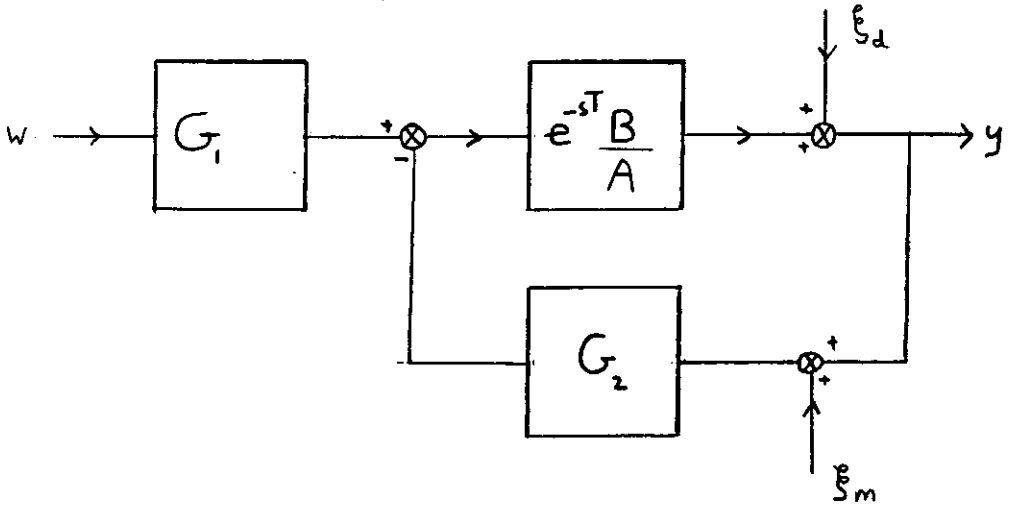

Fig. N1.1. A possible control configuration.

There are two 'degrees of freedom'† available to realise desired system characteristics: the loop gain may be fixed by a suitable choice of G_2, the transmission

† Horowitz, I.M. (1963): "Synthesis of feedback systems" Academic Press. Section 6.1, pp. 246-249.

from w to y may then be independently manipulated using G_1.

The control law (N1.1) and the system (2.2.1) are both linear, and so superposition may be invoked to write the variables y and u as the sum of the effects of the variables ξ_d, ξ_m and w. One way of doing this is given by equations (2.1.2). This decomposition may be diagrammed in many forms; that given in Figs. 2.1.2 & 3 is convenient for the purposes of this chapter, and demonstrates that the regulator and servo problems are similar in principle.

The fact that the blocks labelled 'servo' and 'regulator' can be independently realised is a consequence of the previously mentioned 'two degrees of freedom' inherent in the control law (N1.1). This fact is explicitly used in section 2.14, where the control laws of Figs. 2.1.2 & 3 are recombined in a form similar to that of Fig.N1.1.

Note 2 (section 2.8).

Section 2.8 stands between sections couched in transfer function language and sections couched in state-space language; it is inevitable that the vocabulary is a mixture of the two. This note gives justification for the replacement of the phrase: "all modes excited by", by the phrase:"controllable with respect to".

Consider the following state-space description of a system:

$$\underline{\dot{x}} = \begin{bmatrix} -a_1 & 0 & 0 \\ 0 & -a_2 & 0 \\ 0 & 0 & -a_3 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \psi \qquad \text{(N2.1)}$$

$$y = [1\ 1\ 1]\,\underline{x}, \quad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

where a_1, a_2 and a_3 are constants.

This system may also be written in transfer function form as:

$$y = \frac{(a_1 + a_2) + 2s}{(a_1 + s)(a_2 + s)}\, u + \frac{(a_2 + a_3) + 2s}{(a_2 + s)(a_3 + s)}\, \psi \qquad \text{(N2.2)}$$

The first two states (x_1 & x_2) of the system of equation (N2.1) are clearly controllable with respect to u, and the last two states (x_2 & x_3) are controllable with respect to ψ. This is reflected in equation (N2.2) by the fact that the corresponding poles appear in

respective transfer functions - the usual convention has been adopted that identical poles and zeros are cancelled.

It is seen from this example that, for the class of systems considered in this thesis, there is a direct connexion between poles with non-zero residues in the transfer function description and controllable states in the state-space description. The use of the word "controllable" to distinguish transfer function poles with non-zero residue thus corresponds to state-space usage, and is unambiguous.