Self-referential options and linear stability



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MSc in Mathematical and Computational Finance

Trinity Term, 2012

Acknowledgements

I would like to thank Dr. Jeff Dewynne who accepted to supervise me for this thesis. He supported me throughout the term and always offered his help and wise inputs to keep my on the right track. I really enjoyed writing this dissertation under his guidance. Moreover, I would also like to thank Daniel Schwarz who took the time to present me his work on a Carbon credit model.

Contents

1 Introduction **1**

2 Separable solutions of Black-Scholes equation **6**

2.1 Separable solutions 6

2.2 General solution 7

2.3 Linear stability analysis 10

3 Forward-start calls / puts **12**

3.1 Linear forward-start call / put 12

3.2 Self-referential forward-start calls / puts 13

3.2.1 Self-referential forward-start call option 13

3.2.2 Self-referential forward-start put option 16

4 Linear stability of forward-start calls / puts **19**

4.1 Linear forward-start calls / puts 19

4.2 Self-referential calls / puts 20

4.2.1 Self-referential forward-start call option 22

4.2.2 Self-referential forward-start put option 25

5 Conclusion **30**

List of Figures

1 Existence and unicity of ξ_c 14
2 ξ_c as a function of α 15
3 Domain of existence and unicity of ξ_p 17
4 ξ_p as a function of α 18

5 Stability disc and actual circle covered by Z_c 23
6 Worst instability amplitude with respect to Fourier modes 24
7 Evolution of β_c with respect to α 26
8 Stability disc and actual circle covered by Z_p 28
9 Evolution of β_p with respect to α 29

1. Introduction

Carbon credit model

In order to regulate the growth of greenhouse gas emissions, governments have introduced assets which represent the right to eject carbon dioxide into the air. Thus, nowadays, companies willing to emit carbon dioxide have to cover their emission with allowances or pay a penalty. Moreover, the price of these allowances depend on the emissions. A simple model can be described as follows:
Let A_t be the price at time t of an allowance to emit one unit (one tonne) of carbon dioxide and E_t be the cumulative emissions up to time t. Imagine the trading period lasts until T and that at T, companies which are not able to produce enough allowances have to pay a penalty Π. This gives us the terminal value of the allowance price

$$A_T = \Pi \, \mathbb{1}_{[\Gamma,+\infty[}(E_T) \tag{1.1}$$

where Γ is the initial quantity of allowances issued on the markets (fixed at initiation). Indeed, if the final cumulative emissions are lower than the quantity of allowances, companies will not need to buy allowances and the price will be 0. However, if emissions are not covered by allowances, the price will be Π since this is the price they have to pay if they do not have enough allowances.

Let us assume that interest rates are zero and that A_t is a martingale under a risk-neutral measure $\mathbb{Q}$. In that case, the martingale representation theorem implies that A_t solves the backward stochastic differential equation

$$\mathrm{d}A_t = Z_t \, \mathrm{d}W_t$$

with terminal condition (1.1), where W_t a $\mathbb{Q}$-Brownian motion and Z_t is a predictable process adapted to the filtration generated by W_t.

Let us also assume that E_t follows the forward stochastic differential equation

$$\mathrm{d}E_t = \mu(E_t, A_t)\mathrm{d}t + \sigma \mathrm{d}W_t$$

with initial condition $E_0 = e$ with $e > 0$ represnting the emission level at the beginning of the trading period. Although σ is a constant here, μ is a function of both E_t and A_t since emissions are assumed to be dependant on the allowance prices (the higher the prices, the more careful companies will be to reduce their greenhouse gases emissions). We expect μ to be decreasing in both E and A since the purpose of these allowances is to reduce emissions.

In the end, we have two linked equations; one has to be solved forward and the other one backward, which introduces a feedback effect. If we assume we can write $A_t = u(t, E_t)$, Itô's formula gives us the equation u has to satisfy

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial E^2} + \mu(E, u)\frac{\partial u}{\partial E} = 0,$$

subject to a terminal condition

$$u(e, T) = \mathbb{1}_{[\Gamma, +\infty[}(e)$$

Assuming μ depends on u, this equation is non-linear and therefore it cannot be solved like a standard linear Black-Scholes equation. Moreover, the link between A_t and E_t lets us think that some market manipulation is possible since increasing your own carbon dioxide emissions will increase the allowance price and if the cost of the extra emission is lower than the increase in the allowance price, this will lead to some instability and over-pollution since companies would find financial interest in doing it indefinitely. This dissertation will present a highly simplified version of similar non-linear option problems, remaining in a Black-Scholes framework, and present results regarding their stability with respect to market manipulations.

Non-linear options

A standard forward-start call is a European option whose strike will be determined at a date T^* ($0 < T^* < T$) as $K = \alpha S_{T^*}$ in the general case, with $\alpha > 0$. From T^* onwards, it thus becomes a vanilla call with maturity T and strike $K = \alpha S_{T^*}$. Through continuity of the option price at T^*, we know that the option price before T^* is only the discounted expected option price of the vanilla option at T^*. Self-referential forward-start calls, on the other hand, are defined by the following non-linear equation

$$K = \alpha V(S, T^*; K, T), \tag{1.2}$$

where, again, α is a positive constant, K the option strike, T the option maturity and T^* the strike settlement date. That is, the strike is determined by the call's value at a given time, rather than by the underlying assets.
The non-linearity of these options comes from equation (1.2) since if we consider a portfolio Ω made of 2 similar forward-start calls C_{fs} following equation (1.2), then at T^*, the strike of the single call will be

$$K_c = \alpha C_{bs}(S, T^*; K_c, T)$$

while the strike of the portfolio will be

$$K_\Omega = 2\alpha C_{bs}(S, T^*; K_c, T) = 2K_c.$$

Thus, slightly after T^*, once the strike has been set, calls become vanilla calls and prices follow

$$C_{fs}(S, t, \alpha) = C_{bs}(S, t; K_c, T)$$

and

$$\Omega(S, t, \alpha) = 2C_{bs}(S, t; 2K_c, T)$$

Non-linearity comes from the fact that, since we consider cases where $K_c > 0$

$$2C_{bs}(S, t; 2K_c, T) \neq 2C_{bs}(S, t; K_c, T)$$

Although these options are not traded on the markets (and are not expected to appear at any time soon), their analysis is of interest since they might provide us with a better understanding of the non-linear feature presented in the carbon credit model introduced above. The study of non-linear European options is a new area in financial mathematics and this dissertation will try to contribute to its early stage by looking at the previously unexamined aspect of their linear stability and reaction to market manipulation.

We intend to present in this dissertation some results regarding the stability of these non-linear options. More precisely, we would like to understand how prices react to potential market manipulations. Intuitively, we have the feeling that the non-linear forward-start call option should be stable, because if we try to manipulate the option price (say, by pushing it up) before the strike settlement date T^*, the relation $K = \alpha C(S, T^*, K, T)$ suggests it will induct an increase of the strike which would then force the call value down after T^*. However, similar reasoning with the non-linear forward-start put leads to opposite conclusion since a decrease of the strike means a reduction of the option price and vice versa.

Throughout this dissertation, we will assume that the underlying asset's price, S_t, can be modelled as a geometric Brownian motion and satisfies the stochastic differential equation

$$\frac{dS_t}{S_t} = (\mu - y)dt + \sigma dW_t, \tag{1.3}$$

where μ is the constant instantaneous growth rate, y is the constant continuous dividend yield, σ represents the constant volatility and W_t is a standard Brownian motion. We also assume the risk-free rate r is a positive constant.

Finally, we assume the price process V_t of an option written on S_t can be expressed as a function of various variables (S, t) and parameters (r, σ, etc.) and satisfies the Black-Scholes equation (outside of certain dates corresponding to special events on which continuity of the option price nevertheless remains)

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - y)S\frac{\partial V}{\partial S} - rV = 0, \tag{1.4}$$

subject to a terminal condition

$$V(S, T, ...) = \text{Payoff}(S)$$

The linear stability is defined by the impact of a *small* perturbation on the option price, assuming the perturbed option price still follows the Black-Scholes equation (1.4) above. However, since the equation must be solved backwards, we need to adapt our methodology. Indeed, we will assume an arbitrary perturbation has happened at maturity, then solve the Black-Scholes equation to observe what this perturbation looked like at $T_1 < T$. Thus, this will give us the expression of the perturbation we can apply at T_1 in order to come up with the terminal perturbation at maturity. Finally, linear stability will be assessed in comparison to the linear version of the options, since our objective is to analyse the effect of the non-linear condition.

2. Separable solutions of Black-Scholes equation

In order to implement a stability analysis, it is helpful to have simple solutions of the governing partial differential equation which we may use as small perturbations. To this end, we first look for separable solutions of the Black-Scholes equation (1.4) and determine all such solutions which are bounded. We then show that these are general, in the context of a linear stability analysis, by demonstrating that it is possible to construct an arbitrary solution of the Black-Scholes equation using Fourier analysis.

2.1. Separable solutions

In this section, we would like to find solutions of the Black-Scholes equation (1.4) of the form

$$V(S,t) = F(S)G(t).$$

Substituting this expression in (1.4) gives

$$\frac{G'}{G} + \frac{1}{2}\sigma^2 S^2 \frac{F''}{F} + (r-y)S\frac{F'}{F} - r = 0$$

and, as usual, since $F = F(S)$ and $G = G(t)$, this implies that

$$\frac{1}{2}\sigma^2 S^2 \frac{F''}{F} + (r-y)S\frac{F'}{F} = m$$

and

$$\frac{G'}{G} = r - m$$

where m is independent of S and t.
Thus, as the ordinary differential equation for F is clearly an Euler equation, we look for solutions of the form $F(S) = S^n$, which gives

$$\frac{1}{2}\sigma^2 n(n-1) + (r-y)n = m$$

To be of any use for linear stability analysis, we require the solutions to be bounded for *all* values of S (and finite t) which implies that we must take $n = ik$ where $k \in \mathbb{R}$ and in which case

$$F(S) = S^{ik} = \cos(\log S) + i\sin(\log S)$$

and

$$m = -\frac{1}{2}\sigma^2 k^2 + ik(r - y - \frac{1}{2}\sigma^2)$$

so that

$$G(t) = e^{\lambda(T-t)}$$

with

$$\begin{aligned} \lambda &= m - r \\ &= -r - \frac{1}{2}\sigma^2 k^2 + ik(r - y - \frac{1}{2}\sigma^2). \end{aligned}$$

The fact that $F(S)$ and $G(t)$ are complex is not relevant as, firstly we can take the real or imaginary part of $F(S)G(t)$ and, secondly, we may use Fourier analysis to construct arbitrary real solutions from $F(S)G(t)$, as we demonstrate in the following section.

2.2. General solution

We would like to solve the following problem

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-y)S\frac{\partial V}{\partial S} - rV = 0,$$

subject to the terminal condition

$$V(S, T, ...) = \mathrm{P}(S),$$

where P is the payoff function.

Let us introduce the following functions

$$E(S,t;k) = S^{ik}e^{\lambda(T-t)},$$

where $k \in \mathbb{R}$ and

$$\begin{aligned} \lambda &= \frac{1}{2}\sigma^2 ik(ik-1) + ik(r-y) - r \qquad (2.1) \\ &= -\frac{1}{2}\sigma^2 k^2 - r + ik(r - y - \frac{1}{2}\sigma^2). \qquad (2.2) \end{aligned}$$

We have seen in the previous section that $E(S,t;k)$ is a solution of the Black-Scholes equation for all k.

Suppose now that we write

$$V(S,t) = \int_{-\infty}^{+\infty} \rho(k)E(S,t;k)\,\mathrm{d}k, \qquad (2.3)$$

with ρ being an arbitrary function we will determine later. Assuming the integral converges absolutely, $V(S,t)$ is also a solution of the Black-Scholes equation.

Moreover, the terminal condition indicates that

$$\begin{aligned} V(S,T) = P(S) &= \int_{-\infty}^{+\infty} \rho(k)E(S,T;k)\,\mathrm{d}k \\ &= \int_{-\infty}^{+\infty} \rho(k)S^{ik}\,\mathrm{d}k \\ &= \int_{-\infty}^{+\infty} \rho(k)e^{ik\log S}\,\mathrm{d}k, \end{aligned}$$

and by the change of variable $x = \log S$

$$P(e^x) = \int_{-\infty}^{+\infty} \rho(k)e^{ikx}\,\mathrm{d}k.$$

Recognizing this as a Fourier transform and using the Fourier inversion theorem gives us

$$\rho(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} P(e^x) e^{-ikx} \, \mathrm{d}x.$$

Substituting into (2.3), we finally obtain

$$\begin{aligned} V(S,t) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P(e^x) e^{-ikx} V(S,t;k) \, \mathrm{d}x \, \mathrm{d}k \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P(e^x) e^{-ikx} e^{ik \log S} e^{(-\frac{1}{2}\sigma^2 k^2 - r + ik(r - y - \frac{1}{2}\sigma^2))(T-t)} \, \mathrm{d}x \, \mathrm{d}k, \end{aligned}$$

which, by inverting the integration order, becomes

$$\begin{aligned} V(S,t) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} P(e^x) \int_{-\infty}^{+\infty} e^{-ikx} e^{ik \log S} e^{(-\frac{1}{2}\sigma^2 k^2 - r + ik(r - y - \frac{1}{2}\sigma^2))(T-t)} \, \mathrm{d}k \, \mathrm{d}x \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} P(e^x) Q(x) \, \mathrm{d}x. \end{aligned}$$

We do not need to know the payoff function P to work out $Q(x)$ as

$$\begin{aligned} Q(x) &= \int_{-\infty}^{+\infty} e^{-ikx} e^{ik \log S} e^{(-\frac{1}{2}\sigma^2 k^2 - r + ik(r - y - \frac{1}{2}\sigma^2))(T-t)} \, \mathrm{d}k \\ &= e^{-r(T-t)} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\sigma^2 (T-t)(k^2 - \frac{2ikd_1}{\sqrt{\sigma^2(T-t)}})} \, \mathrm{d}k \\ &= e^{-r(T-t)} e^{-\frac{d_1^2}{2}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\sigma^2 (T-t)(k - \frac{id_1}{\sqrt{\sigma^2(T-t)}})^2} \, \mathrm{d}k, \end{aligned}$$

where

$$d_1 = \frac{\log S - x + (r - y - \frac{1}{2}\sigma^2)(T-t)}{\sqrt{\sigma^2 (T-t)}}.$$

By introducing the change of variable

$$z = \sqrt{\sigma^2 (T-t)} k - i d_1,$$

it becomes

$$Q(x) = \frac{e^{-r(T-t)} e^{-d_1^2/2}}{\sqrt{\sigma^2 (T-t)}} \int_{-\infty - i d_1}^{+\infty - i d_1} e^{-\frac{z^2}{2}} \, \mathrm{d}z,$$

Since the integrand has no pole and vanishes rapidly as the real part of its argument becomes infinite, the Cauchy theorem for complex integration guarantees that

$$\begin{aligned} Q(x) &= \frac{e^{-r(T-t)}e^{-d_1^2/2}}{\sqrt{\sigma^2(T-t)}} \int_{-\infty}^{+\infty} e^{-\frac{z^2}{2}}\,\mathrm{d}z \\ &= \frac{\sqrt{2\pi}e^{-r(T-t)}e^{-d_1^2/2}}{\sqrt{\sigma^2(T-t)}}. \end{aligned}$$

Unwinding the changes of variable shows that the solution for a general payoff of an option satisfying the Black-Scholes equation, we can write

$$V(S,t) = \frac{e^{-r(T-t)}}{\sqrt{2\pi\sigma^2(T-t)}} \int_{-\infty}^{\infty} P(e^x) e^{-\frac{(\log S - x + (r - y - \frac{1}{2}\sigma^2)(T-t))^2}{2\sigma^2(T-t)}}\,\mathrm{d}x. \tag{2.4}$$

In order to get the price of a specific option, the only step left is to substitute the payoff function into equation (2.4) and calculate the last integral.

2.3. Linear stability analysis

We will conduct the linear stability analysis of option prices by introducing a perturbation and observe its effect on the price process. We consider that a perturbation will modify the final payoff according to the following

$$\tilde{V}(S,T,...) = V(S,T,...) + \epsilon\ e^{n\log S} = \text{Payoff}(S) + \epsilon\ e^{n\log S},\ \text{where } n \in i\mathbb{R} \tag{2.5}$$

and we will then be able to work out backwards the perturbation at any time. Pretending the final perturbation resulted from a perturbation which had occured at T_1 (with $T_1 < T$), we will finally know how a perturbation at time $0 \leqslant T_1 < T$ evolves. This form comes from the remark showed in the previous sections and is justified by Fourier analysis:
Indeed, if know the effect of a perturbation ϵS^n, with n in $i\mathbb{R}$, then we can deduce the impact of a perturbation $F(x) = \int_{-\infty}^{+\infty} f(k)e^{\imath kx}dk$, for any function f, and we recognise that F is the Fourier transform of f, but since f is arbitrary, then F is, too, and we therefore have the result for any perturbation.

Finally, the perturbation δ must solve the Black-Scholes equation (1.4) with terminal condition $\delta(S,T) = S^n$, which gives us, in the absence of any intermediary event (which might introduce boundary conditions)

$$\delta(t, S_t) = e^{\lambda(T-t)} S_t^n, \tag{2.6}$$

where

$$\begin{aligned} \lambda &= \frac{1}{2}\sigma^2 n(n-1) + (r-y)n - r & (2.7) \\ &= -(\frac{1}{2}\sigma^2 k^2 + r) + ik(r - y - \frac{1}{2}\sigma^2) & (2.8) \end{aligned}$$

where $n = ik$.

Thus, this gives us the expression of the perturbations we can apply for any option, and since it is a solution of the Black-Scholes equation, we do already know its evolution outside of boundary conditions.

3. Forward-start calls / puts

3.1. Linear forward-start call / put

A linear forward-start call is defined by a strike settlement date T^*, a positive constant α such that $K = \alpha S_{T^*}$ and a maturity T (with $T^* < T$). Since for $t > T^*$, the strike is fixed, the option becomes a vanilla call with maturity T and strike αS_{T^*} and classic calculation using the Black-Scholes price of a call shows that

$$C_{fs}(S,t,\alpha) = \begin{cases} C_{bs}(S,t;\alpha S_{T^*},T) & \text{for} \quad T^* < t \leqslant T, \\ A_c(\alpha,\tau)Se^{-y(T^*-t)} & \text{for} \quad t \leqslant T^*. \end{cases}$$

with

$$A_c(\alpha,\tau) = e^{-y\tau}\mathcal{N}(d_+^\alpha) - \alpha e^{-r\tau}\mathcal{N}(d_-^\alpha)$$

where $\mathcal{N}$ is the normal distribution function

$$\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{u^2}{2}}\,\mathrm{d}u$$

and

$$d_\pm^\alpha = \frac{\log\alpha + (r - y \pm \frac{1}{2}\sigma^2)\tau}{\sqrt{\sigma^2\tau}}.$$

where $\tau = T - T^*$, C_{fs} denotes the forward-start call price function, C_{bs} the vanilla call Black-Scholes price function.

A similar reasoning will lead us to the price of a linear forward-start put

$$P_{fs}(S,t,\alpha) = \begin{cases} P_{bs}(S,t;\alpha S_{T^*},T) & \text{for} \quad T^* < t \leqslant T, \\ A_p(\alpha,\tau)Se^{-y(T^*-t)} & \text{for} \quad t \leqslant T^*. \end{cases}$$

where, using the same notations as above

$$A_p(\alpha, \tau) = \alpha e^{-r\tau}\mathcal{N}(-d_-^\alpha) - e^{-y\tau}\mathcal{N}(-d_+^\alpha).$$

The linearity of these options comes from the fact that the strike only depends on the underlying assets, which are independent from the number of options held.

3.2. Self-referential forward-start calls / puts

Non-linear calls/puts satisfy both the Black-Scholes equation (1.4) with the appropriate terminal condition (depending on the payoff) and the non-linear equation (1.2). Moreover, continuity at the strike settlement date T^* enables us, in theory, to obtain the option price at all time. This chapter will enable us to recall and summarize results shown in [1] regarding self referential forward-start options.

3.2.1. Self-referential forward-start call option

A self-referential forward-start call option (with price $C_{fs}(S, t, \alpha)$) is characterised by the following equations

$$\frac{\partial C_{fs}}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C_{fs}}{\partial S^2} + (r - y)S\frac{\partial C_{fs}}{\partial S} - rC_{fs} = 0 \quad \text{on} \quad [0, T[\backslash\{T^*\}, \tag{3.1}$$

$$K = \alpha C_{fs}(S, T^*, \alpha), \tag{3.2}$$

$$C_{fs}(S, T; \alpha) = \max(S - K, 0). \tag{3.3}$$

Once the strike is known (let us denote it K_c), a self-referential forward-start call becomes a vanilla call, solving equation (3.1) with terminal condition (3.3). Thus, for $t > T^*$, the option price is the Black-Scholes price of a vanilla call option

$$C_{fs}(S, t; \alpha) \;=\; C_{bs}(S, t; K_c, T).$$

Moreover, thanks to continuity at T^*, the non-linear condition (3.2) can be written

$$\begin{aligned} K_c &= \alpha C_{fs}(S, T^*, \alpha) \\ &= \alpha C_{bs}(S, T^*; K_c, T). \end{aligned}$$

Introducing $\tau = T - T^*$ and $\xi_c(\alpha, \tau) = \frac{S_{T^*}}{K_c}$, we observe that, since $K_c > 0$

$$C_{bs}(S, T^*; K_c, T) = K_c C_{bs}(\xi_c, T^*; 1, T)$$

and equation (3.2) finally becomes

$$C_{bs}(\xi_c, T^*; 1, T) = \frac{1}{\alpha} \tag{3.4}$$

which has a unique positive solution for all $\alpha > 0$ (see Figure 1) and this ξ_c does only depend on α and τ.

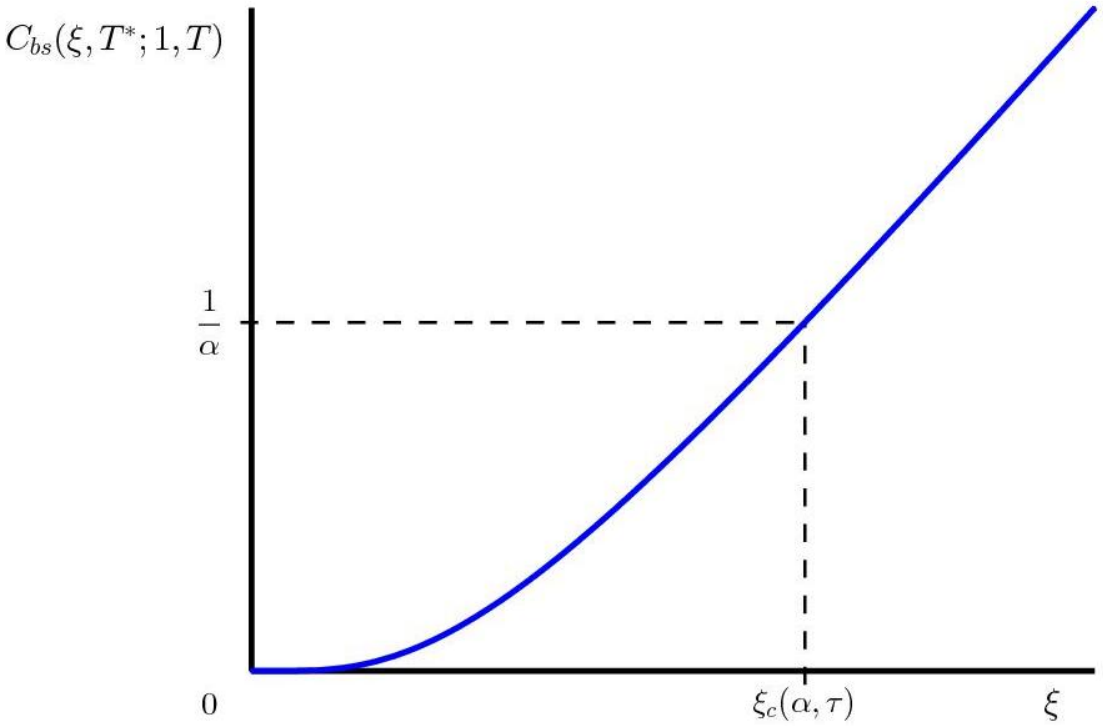


Figure 1: Existence and unicity of ξ_c

Rewriting (3.2) gives us

$$\begin{aligned} C_{fs}(S, T^*; \alpha) &= \frac{K_c}{\alpha} \\ &= \frac{S}{\alpha \xi_c(\alpha, \tau)}, \end{aligned}$$

and keeping in mind that C_{fs} solves the Black-Scholes equation (1.4), we can deduce the option price for all $t < T$

$$C_{fs}(S,t,\alpha) = \begin{cases} C_{bs}(S,t;K_c,T) & \text{for} \quad T^* < t \leqslant T, \\ \dfrac{Se^{-y(T^*-t)}}{\alpha\xi_c(\alpha,\tau)} & \text{for} \quad 0 \leqslant t \leqslant T^*. \end{cases}$$

where $\xi_c(\alpha,\tau)$ is the only positive solution of (3.4).

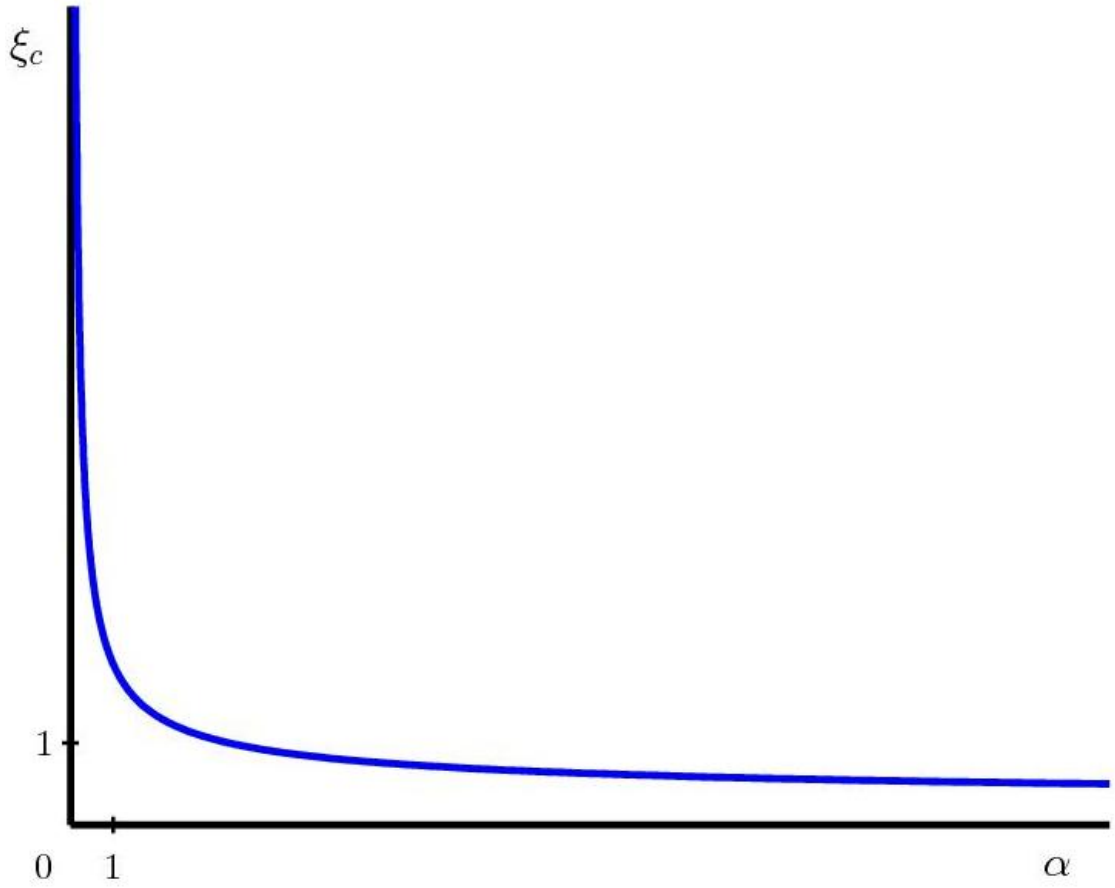


Figure 2: ξ_c as a function of α

Figure 2 illustrates the behaviour of $\xi_c(\alpha,\tau)$ with respect to α. As equation (3.4) and results in [1] prove, $\xi_c(\alpha,\tau) \to +\infty$ as $\alpha \to 0$ and $\xi_c(\alpha,\tau) \to 0$ as $\alpha \to +\infty$. In the end, obtaining the self-referential option price all comes down to solving the transcendental equation (3.4), which has a positive solution for all positive α. A similar process can be applied to the self-referential put.

3.2.2. Self-referential forward-start put option

A self-referential forward-start put option (with price $P_{fs}(S,t,\alpha)$ is characterised by the following equations

$$\frac{\partial P_{fs}}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P_{fs}}{\partial S^2} + (r-y)S\frac{\partial P_{fs}}{\partial S} - rP_{fs} = 0 \text{ on } [0,T[\backslash\{T^*\}, \tag{3.5}$$

$$K = \alpha P_{fs}(S,T^*,\alpha), \tag{3.6}$$

$$P_{fs}(S,T;\alpha) = \max(S-K,0). \tag{3.7}$$

Once the strike is known (let us denote it K_p), a self-referential forward-start put becomes a vanilla put, solving equation (3.5) with terminal condition (3.7). Thus, for $t > T^*$, the option price is the Black-Scholes price of a vanilla put option

$$P_{fs}(S,t;\alpha) \;=\; P_{bs}(S,t;K_p,T).$$

Moreover, thanks to continuity at T^*, the non-linear condition (3.6) can be written

$$\begin{aligned} K_p &= \alpha P_{fs}(S,T^*,\alpha) \\ &= \alpha P_{bs}(S,T^*;K_p,T). \end{aligned}$$

Introducing $\xi_p(\alpha,\tau) = \frac{S_{T^*}}{K_p}$, since we are only interested in $K_p > 0$ ($K_p = 0$ is always a solution, but we are interested in the non-trivial solution, when one exists), and keeping the same definition for τ as above, we observe that

$$P_{bs}(S,T^*;K_p,T) = K_p P_{bs}(\xi_p,T^*;1,T)$$

and equation (3.6) finally becomes

$$P_{bs}(\xi_p,T^*;1,T) = \frac{1}{\alpha} \tag{3.8}$$

which, since $P_{bs}(S,T^*;1,T) \leqslant e^{-r\tau}$, has a unique positive solution (see Figure 3) for

$$\alpha > \alpha_p = e^{r\tau} \tag{3.9}$$

and this ξ_p only depends on α and τ.

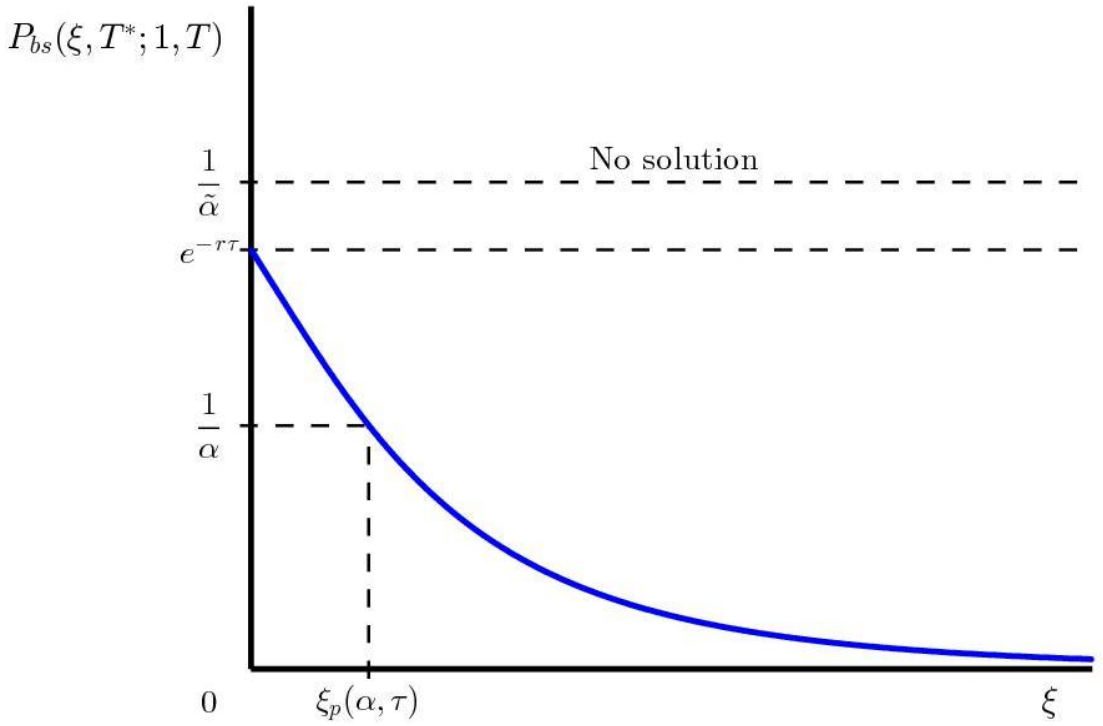


Figure 3: Domain of existence and unicity of ξ_p

Finally, rewriting (3.6) as

$$\begin{aligned} P_{fs}(S, T^*; \alpha) &= \frac{K_P}{\alpha} \\ &= \frac{S}{\alpha \xi_p(\alpha, \tau)}, \end{aligned}$$

and keeping in mind that P_{fs} solves the Black-Scholes equation (3.5), we can deduce the option price for all $t < T$

$$P_{fs}(S, t, \alpha) = \begin{cases} P_{bs}(S, t; K_p, T) & \text{for} \quad T^* < t \leqslant T, \\ \dfrac{S e^{-y(T^* - t)}}{\alpha \xi_p(\alpha, \tau)} & \text{for} \quad 0 \leqslant t \leqslant T^*. \end{cases}$$

where $\xi_p(\alpha, \tau)$ is the only positive solution of (3.8) for $\alpha > \alpha_p$.

Because of the definition of the put, the transcendental equation (3.6) will have a positive solution only for $\alpha > \alpha_p = e^{r(T - T^*)}$. Moreover, $\xi_p(\alpha, \tau) \to 0$ as $\alpha \to \alpha_p$

and $\xi_p(\alpha, \tau) \to +\infty$ as $\alpha \to +\infty$. Figure 4 shows the behaviour of ξ_p as a function of α.

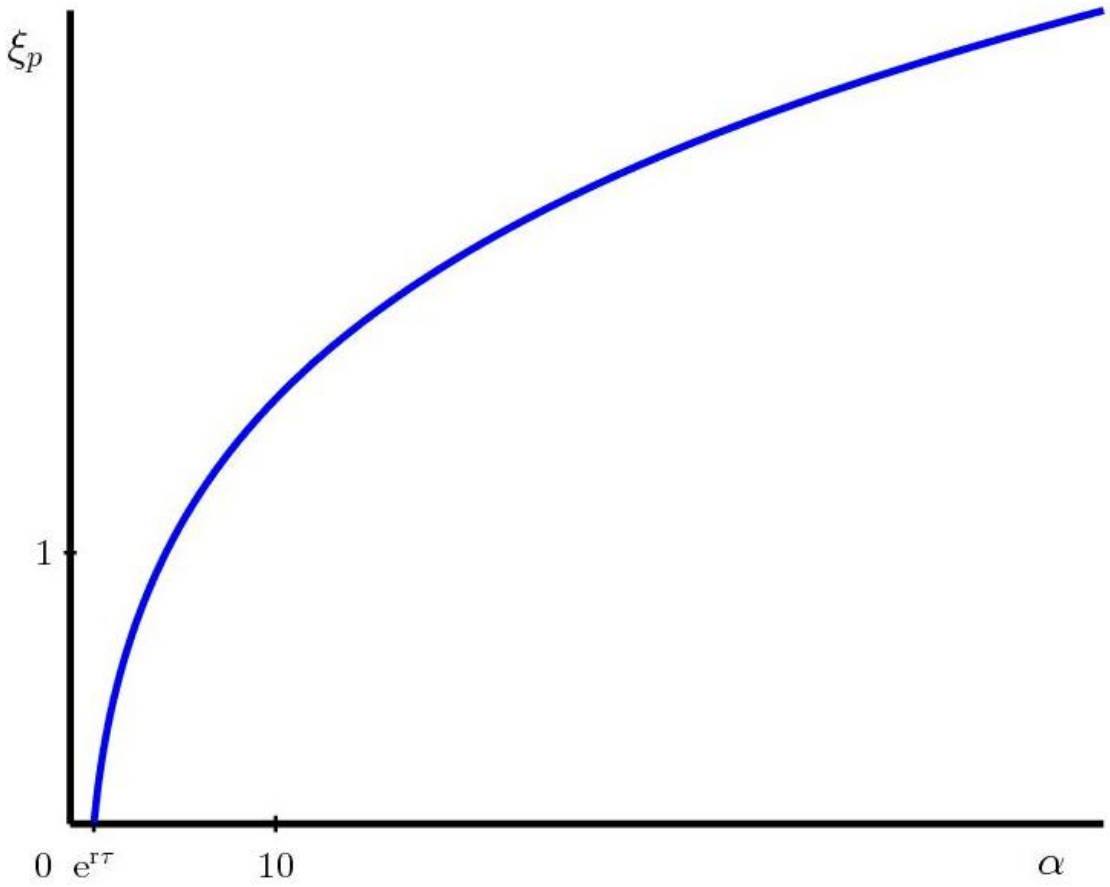


Figure 4: ξ_p as a function of α

Further results have been exhibited in [1] regarding self-referential digital options, but no closed formula can be obtained since the transcendental equation cannot be reduced as easily as previously by introducing $\xi = \frac{S}{K}$. However, the option prices can be obtained numerically. We will in the next section analyse the forward stability conditions in the general case and show numerical results for the self-referential call and put presented above.

4. Linear stability of forward-start calls / puts

4.1. Linear forward-start calls / puts

Let us check the linear stability of the solution. Let us assume a perturbation has been introduced at $T_1 < T^*$ and results, according to equation (2.5), in a perturbation of the terminal option price of the form

$$\tilde{C}_{fs}(S,T,\alpha) = C_{fs}(S,T,\alpha) + \epsilon\ S^n \tag{4.1}$$

where $\tilde{C}$ is the perturbed option price, C the original one and $n \in i\mathbb{R}$.
$\tilde{C}$ satisfies (1.4) with terminal condition (4.1) and it is therefore straightforward to show that

$$\tilde{C}_{fs}(S,t,\alpha) = C_{fs}(S,t,\alpha) + \epsilon\delta(t,S), \quad \text{for } T^* < t \leqslant T,$$

with δ and λ as in (2.6) and (2.8).
Moreover, $\tilde{C}$ also satisfies (1.4) on $[0,T^*]$ and is continuous at T^*, which allows us to write

$$\tilde{C}_{fs}(S,t,\alpha) = C_{fs}(S,t,\alpha) + \epsilon\delta(t,S), \quad \text{for } T_1 \leqslant t \leqslant T,$$

It is noticeable that if we could introduce a perturbation (which seems unlikely in the linear forward-start call case because the price is a closed formula of given parameters), it would grow according to $\epsilon\delta$. However, the forward-start feature does not alter the effect of the perturbation and this perturbation does not depend on the payoff as long as we stay within a linear framework. From now on, we will

consider a perturbation $\epsilon\delta(T_1, S)$ has been applied at $T_1 < T^*$ and will study its evolution. We will consider that an option is linearly stable if the final perturbation is in absolute value lower than $\epsilon\delta(T, S)$, the final perturbation in the linear case.

4.2. Self-referential calls / puts

Let us consider a self-referential option (general case for the moment) to which we apply the same perturbation as in the linear case at $T_1 < T^*$, which is possible because $\epsilon\delta(t, S)$ solves the Black-Scholes equation. In the linear case, we know it will evolve acording to $\epsilon\delta(t, S)$ stated above. Moreover, before the strike settlement occurs, the perturbation is similar in the linear and non-linear cases. But how does the non-linear forward-start condition affect it after T^*?

Because of the condition

$$K = \alpha V(S, T^*; K, T),$$

we must assume the perturbation will also modify the strike.

Let us write the perturbed option price

$$\tilde{V} = \tilde{V}(S, t; \tilde{K}, T),$$

where $\tilde{K}$ is the perturbed strike which can be written $\tilde{K} = K + \epsilon\hat{K}$.
We then conduct a first order Taylor expansion, in which we neglect terms of order $O(\epsilon^2)$ and higher since ϵ is supposed to be very small in our linear stability analysis. The forward-start condition thus becomes

$$\begin{aligned} K + \epsilon\hat{K} &= \alpha\tilde{V}(S, T^*; K + \epsilon\hat{K}, T) \\ &= \alpha[V(S, T^*; K + \epsilon\hat{K}, T) + \epsilon\delta(T^*, S)] \\ &\approx \alpha[V(S, T^*; K, T) + \epsilon\hat{K}\frac{\partial V}{\partial K}(S, T^*; K, T) + \epsilon\delta(T^*, S)], \end{aligned}$$

while we still have

$$K = \alpha V(S, T^*, K, T),$$

We therefore obtain

$$\hat{K} = \frac{\alpha\delta(T^*)}{1 - \alpha\frac{\partial V_{bs}}{\partial K}(S, T^*; K, T)}$$

and thus the perturbed option price for $t > T^*$

$$\begin{aligned}\tilde{V}(S,t,\alpha) &= V_{bs}(S,t;K+\epsilon\tilde{K},T) + \epsilon\delta(t,S) \\ &= V_{bs}(S,t;K,T) + \epsilon\delta(t,S) + \epsilon\hat{K}\frac{\partial V_{bs}}{\partial K}(S,t;K,T).\end{aligned}$$

We have seen previously that the first two terms would appear without the non-linear forward-start condition in any case. Thus, the question is whether the term added by the non-linearity attenuates the perturbation or reinforces it, which is determined by the inequality between

$$|\delta(t,S)| \text{ and } |\delta(t,S) + \tilde{K}\frac{\partial V_{bs}}{\partial K}(S,t;K,T)|,$$

i.e. the sign of

$$1 - |1 + \frac{\hat{K}}{\delta(t,S)}\frac{\partial V_{bs}}{\partial K}(S,t;K,T)|,$$

which can be written

$$1 - |1 + \frac{\alpha}{1 - \alpha\frac{\partial V_{bs}}{\partial K}(S,T^*;K,T)}\frac{\delta(T^*,S)}{\delta(t,S)}\frac{\partial V_{bs}}{\partial K}(S,t;K,T)|.$$

It suffices to focus on the final perturbation in our linear stability analysis, i.e. on the sign of

$$1 - |1 + \frac{\alpha}{1 - \alpha\frac{\partial V_{bs}}{\partial K}(S,T^*;K,T)}\frac{\delta(T^*,S)}{\delta(T,S)}\frac{\partial V_{bs}}{\partial K}(S,T;K,T)|.$$

We have shown that δ may be assumed to be of the form

$$\begin{aligned}\delta(t,S) &= e^{\lambda(T-t)}S_t^{\imath k} \quad \text{for } k \in \mathbb{R} \\ &= e^{-(\frac{1}{2}\sigma^2k^2+r)(T-t)}e^{\imath k(\log S + (r-y-\frac{1}{2}\sigma^2)(T-t))}\end{aligned}$$

In order to complete our analysis, we need to consider specific payoff, which will give us the form of $\frac{\partial V_{bs}}{\partial K}(S,t;K,T)$ and allow us to determine whether or not our initial intuitions are confirmed.

4.2.1. Self-referential forward-start call option

Applying the above result and observing that

$$\frac{\partial C_{bs}}{\partial K}(S,t;K_c,T) = -C^d_{bs}(S,t;K_c,T) \leqslant 0$$

and

$$C^d_{bs}(S,T^*;K_c,T) = C^d_{bs}(\xi_c,T^*;1,T)$$

where $\xi_c(\alpha,\tau)$ is defined in Section 3.2.1 and C^d_{bs} represents the Black-Scholes price of a digital call, we can write the effect of the perturbation on the strike as

$$\hat{K} = \frac{\alpha\delta(T^*,S)}{1+\alpha C^d_{bs}(\xi_c,T^*;1,T)}$$

This confirms our initial guess that the strike will be pushed up if the initial perturbation is positive.

Moreover, the price perturbation can be written

$$\tilde{C}_{fs}(S,t;\alpha) - C_{fs}(S,t;\alpha) = \begin{cases} \epsilon\delta(t,S) & \text{for} \quad T_1 < t \leqslant T^*, \\ \epsilon\delta(t,S) - \dfrac{\epsilon\alpha\delta(T^*,S)C^d_{bs}(S,t;K_c,T)}{1+\alpha C^d_{bs}(\xi_c,T^*;1,T)} & \text{for} \quad T^* < t \leqslant T. \end{cases}$$

Thus, the final perturbation can be simplified to

$$\begin{cases} \epsilon\delta(T,S) & \text{if} \quad S_T < K_c = \frac{S_{T^*}}{\xi_c} \\ \epsilon(\delta(T,S) - \dfrac{\alpha\delta(T^*,S)}{1+\alpha C^d_{bs}(\xi_c,T^*;1,T}) & \text{if} \quad S_T \geqslant K_c \ . \end{cases}$$

Therefore, if $S_T < K_c$, the non-linear condition does not alter the perturbation. However, its effect comes into play if $S_T \geqslant K_c$. In this case, replacing δ by its expression, linear stability comes down to the sign of

$$1 - |1 - \beta_c e^{-X} e^{\imath\theta}|. \tag{4.2}$$

where

$$\begin{aligned} \beta_c &= \frac{\alpha}{1+\alpha C^d_{bs}(\xi_c,T^*;1,T)} \\ X &= (\frac{1}{2}\sigma^2k^2 + r)(T-T^*) \\ \theta &= k(\log(\frac{S_{T^*}}{S_T}) + (r - y - \frac{1}{2}\sigma^2)(T-T^*)). \end{aligned}$$

Introducing $Z_c = \beta_c e^{-X} e^{\imath\theta}$, equation (4.2) above tells us that stability will be guaranteed if Z_c stays within the complex disc centered around $(1, 0)$ and of radius 1.

Furthermore, once α and k are fixed, β_c and X will be fixed too. However, despite the fact that in this situation, $S_T \geqslant K_c$, θ will still be able to take values in an infinite interval. Thus, Z_c will be found on the complex circle centered on the origin and with radius $\beta_c e^{-X}$.

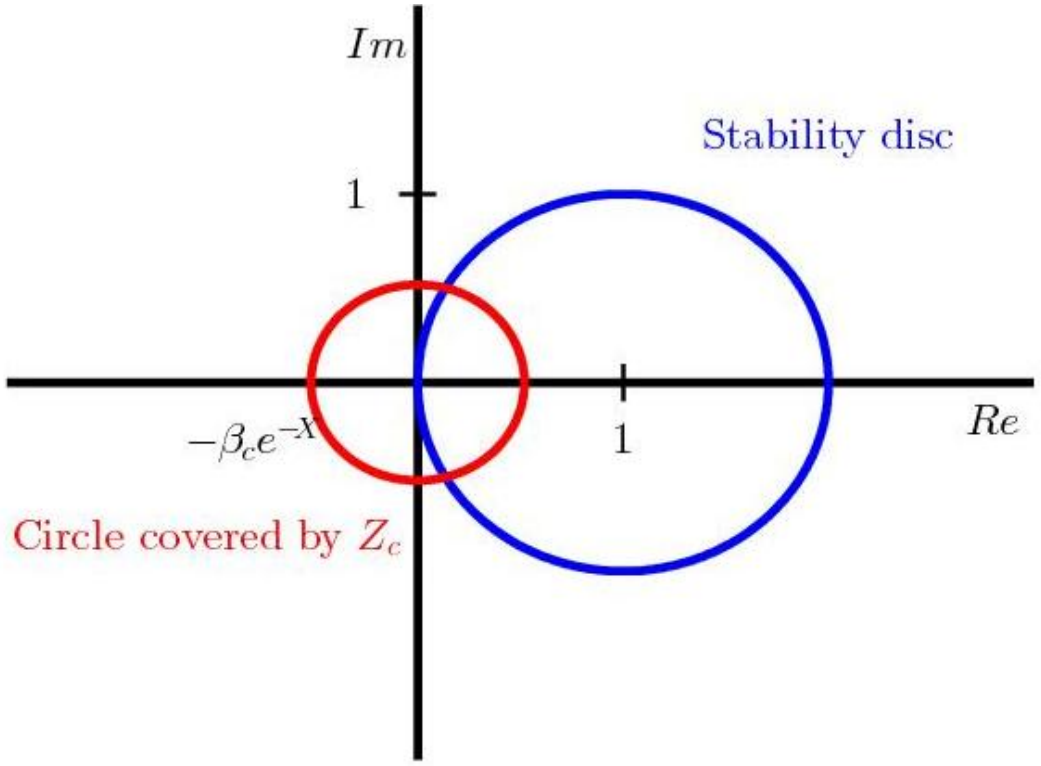


Figure 5: Stability disc and actual circle covered by Z_c

As Figure 5 illustrates, the circle covered by Z_c will never be entirely in the stability area. However, it is important to observe that it remains bounded by a worst case scenario defined by $Z_c = -\beta_c e^{-X}$. Moreover, given the expressions of β_c and X above, we observe the amplitude of the worst case scenario decreases as k increases (for α fixed). Since k represents the frequency in the Fourier mode,

we can conclude the high frequency modes (in the Fourier decomposition of the perturbation) are less unstable than the low frequency ones.

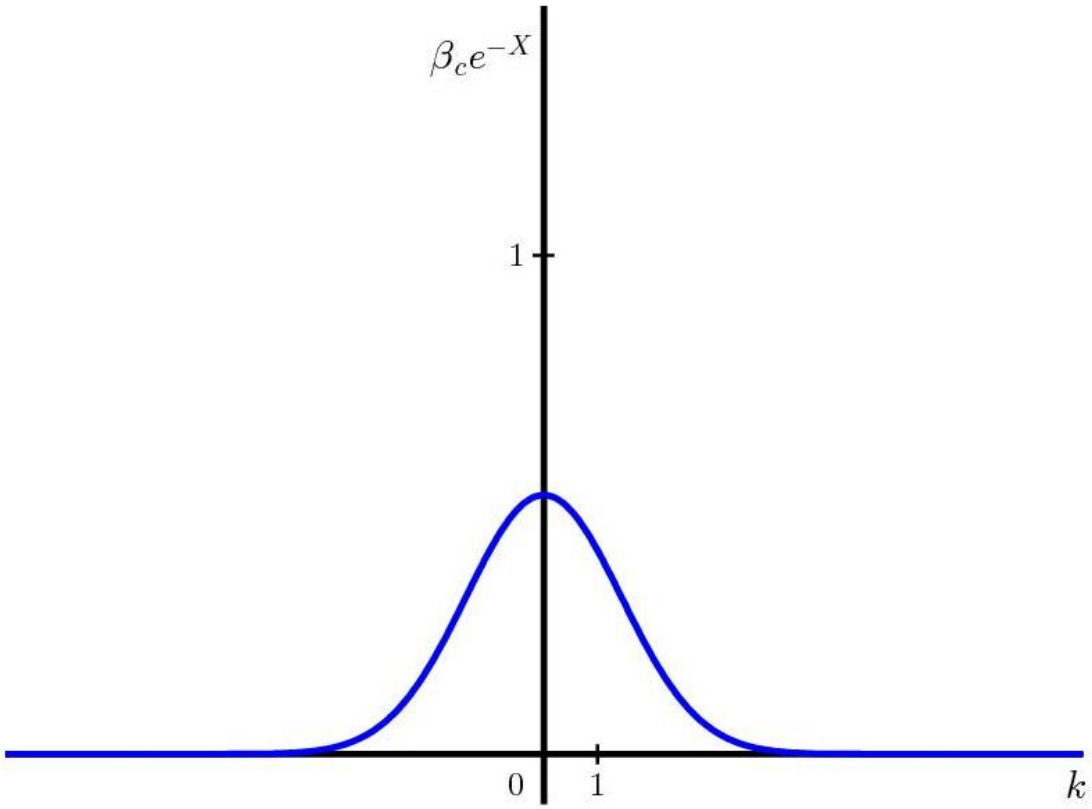


Figure 6: Worst instability amplitude with respect to Fourier modes

Finally, the value of β_c only depends on α and τ and a simple function analysis shows us β_c is an increasing function of α and has the same limits as α. Thus, β_c will remain finite for reasonable values of α.

$$\begin{aligned}
\beta_c &= \frac{\alpha}{1+\alpha C^d_{bs}(\xi_c, T^*; 1, T)} \\
&= \frac{1}{\frac{1}{\alpha} + C^d_{bs}(\xi_c, T^*; 1, T)} \\
&= \frac{1}{C_{bs}(\xi_c, T^*; 1, T) + C^d_{bs}(\xi_c, T^*; 1, T)} \\
&= \frac{1}{\xi_c e^{-y(T-T^*)}\mathcal{N}(d_+)}
\end{aligned}$$

where

$$d_+ = \frac{\log \xi_c + (r - y + \frac{1}{2}\sigma^2)(T-T^*)}{\sqrt{\sigma^2(T-T^*)}}.$$

To conclude, unlike our initial guess, the non-linear boundary condition does not provide stability, despite the fact that it indeed introduces a feedback effect on the strike. Given the condition obtained in the first section and the results for the call, we can expect the put to be unstable too.

4.2.2. Self-referential forward-start put option

A similar reasoning can be made for the non-linear forward-start put option, with the exception that

$$\begin{aligned}
\frac{\partial P_{bs}}{\partial K}(S, t; K_p, T) &= P^d_{bs}(S, T^*; K_p, T) \\
&= P^d_{bs}(\xi_p, t; 1, T) \geqslant 0
\end{aligned}$$

where $\xi_p(\alpha, \tau)$ is defined in Section 3.2.2 and P^d_{bs} represents the Black-Scholes price of a digital call. This allows us to write the strike perturbation for the put

$$\tilde{K} = \frac{\alpha\delta(T^*, S)}{1 - \alpha P^d_{bs}(\xi_p, T^*; 1, T)}$$

Observing that

$$\alpha P^d_{bs}(\xi_p, T^*; 1, T) = \frac{P^d_{bs}(\xi_p, T^*; 1, T)}{P_{bs}(\xi_p, T^*; 1, T)} \geqslant 1$$

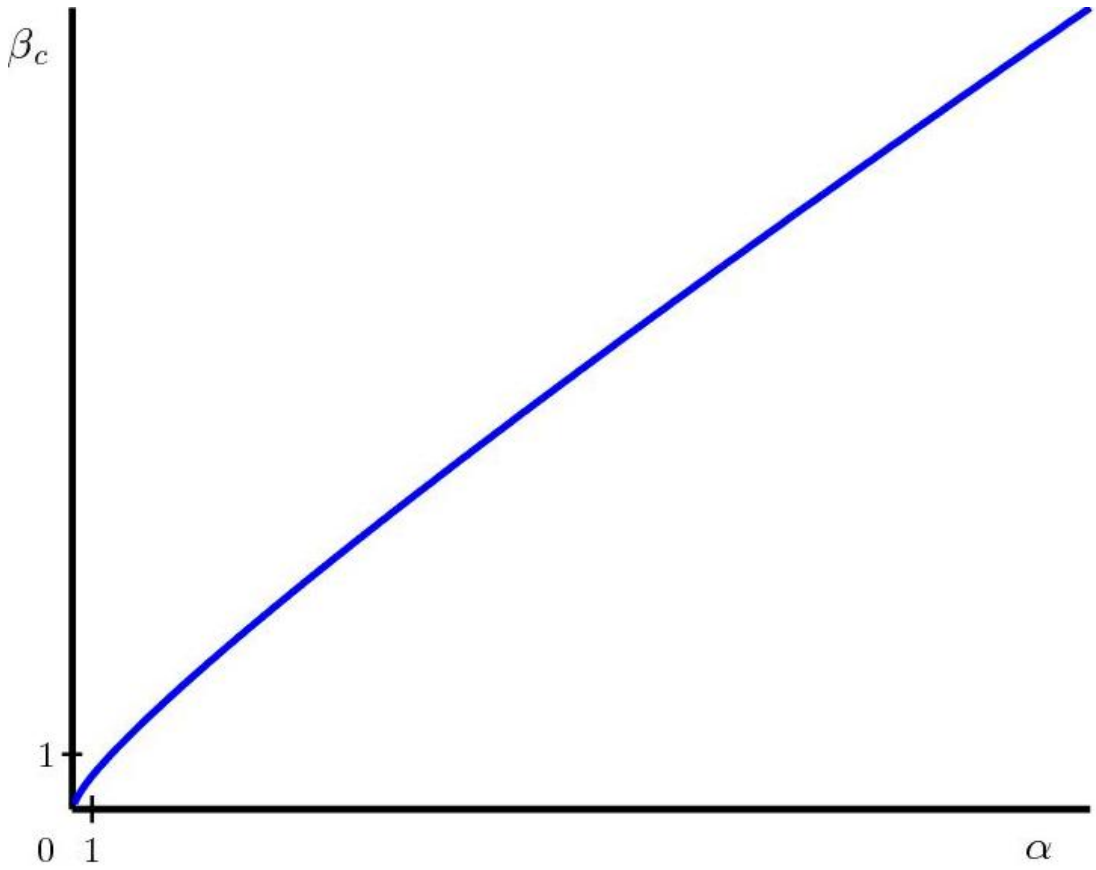


Figure 7: Evolution of β_c with respect to α

that tells us $\tilde{K} \leqslant 0$ and the strike will thus be pushed down by a positive perturbation, which goes against our initial guess. So, let us pursue the analysis and write the price perturbation

$$\tilde{P}_{fs}(S,t,\alpha) - P_{fs}(S,t,\alpha) = \begin{cases} \epsilon\delta(t,S) & \text{for} \quad T_1 < t \leqslant T^*, \\ \epsilon\delta(t,S) - \dfrac{\epsilon\delta(T^*,S)\alpha P^d_{bs}(S,t;K_p,T)}{\alpha P^d_{bs}(\xi_p,T^*;1,T) - 1} & \text{for} \quad T^* < t \leqslant T. \end{cases}$$

Thus, the final perturbation can be simplified to

$$\begin{cases} \epsilon\delta(T,S) & \text{if} \quad S_T > K_p = \frac{S_{T^*}}{\xi_p} \\ \epsilon(\delta(T,S) - \dfrac{\alpha\delta(T^*,S)}{\alpha P^d_{bs}(\xi_p,T^*;1,T} - 1) & \text{if} \quad S_T \leqslant K_p \ . \end{cases}$$

Therefore, if $S_T > K_p$, the non-linear condition does not alter the perturbation. However, its effect comes into play if $S_T \leqslant K_p$. In this case, replacing δ by its

expression, linear stability comes down to the sign of

$$1 - |1 - \beta_p e^{-X} e^{\imath\theta}|. \tag{4.3}$$

where

$$\begin{aligned}
\beta_p &= \frac{\alpha}{\alpha C^d_{bs}(\xi_c, T^*; 1, T) - 1} \\
X &= (\frac{1}{2}\sigma^2 k^2 + r)(T - T^*) \\
\theta &= k(\log(\frac{S_{T^*}}{S_T}) + (r - y - \frac{1}{2}\sigma^2)(T - T^*)).
\end{aligned}$$

Introducing $Z_p = \beta_p e^{-X} e^{\imath\theta}$, equation (4.3) above tells us that stability will be guaranteed if Z_p stays within the complex disc centered around $(1, 0)$ and of radius 1.

Furthermore, following a similar reasoning as for the call, once α and k are fixed, β_p and X will be fixed too. However, despite the fact that in this situation, $S_T \leqslant K_p$, θ will still be able to take values in an infinite interval. Thus, Z_p will be found on the complex circle centered on the origin and with radius $\beta_p e^{-X}$.

Similarly to the call, Figure 8 illustrates, the circle covered by Z_p will never be entirely in the stability area. However, it is important to observe that it remains bounded by a worst case scenario defined by $Z_p = -\beta_p e^{-X}$. The frequency analysis will be exactly the same as the one conducted for the call previously. However, given the expression of β_p, it can be of interest to study its behaviour with respect to α

$$\begin{aligned}
\beta_p &= \frac{\alpha}{\alpha P^d_{bs}(\xi_p, T^*; 1, T) - 1} \\
&= \frac{1}{P^d_{bs}(\xi_p, T^*; 1, T) - \frac{1}{\alpha}} \\
&= \frac{1}{P^d_{bs}(\xi_p, T^*; 1, T) - P_{bs}(\xi_p, T^*; 1, T)} \\
&= \frac{1}{\xi_p e^{-y(T-T^*)} \mathcal{N}(-d_+)}
\end{aligned}$$

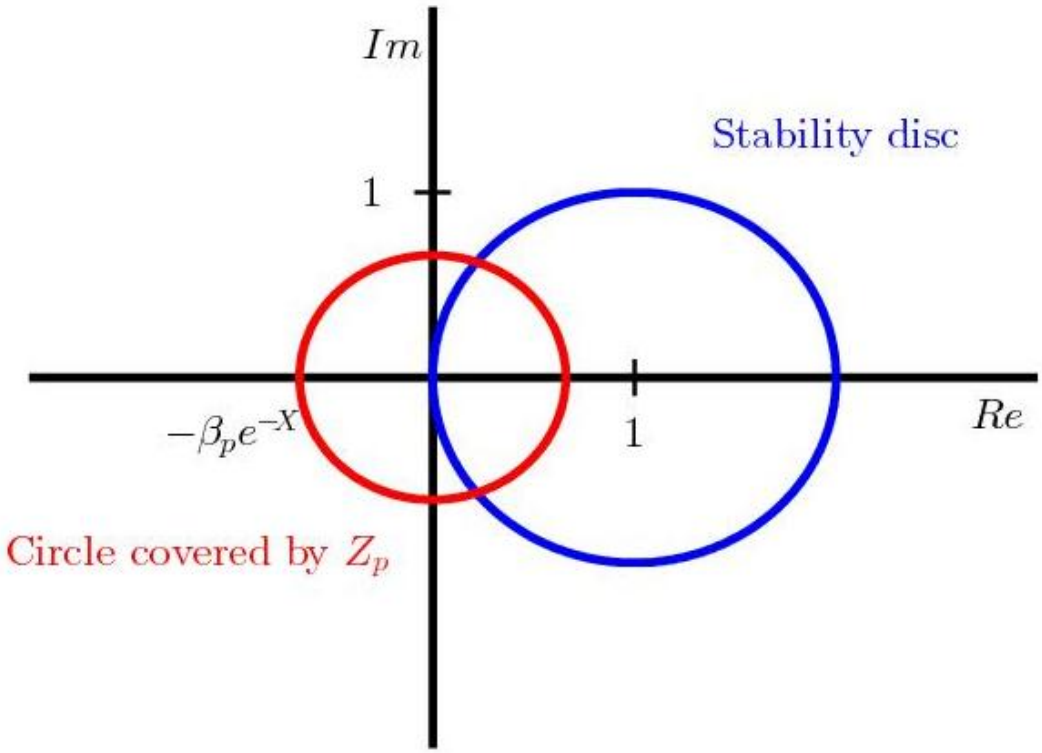


Figure 8: Stability disc and actual circle covered by Z_p

where

$$d_+ = \frac{\log \xi_p + (r - y + \frac{1}{2}\sigma^2)(T - T^*)}{\sqrt{\sigma^2(T - T^*)}}.$$

Thus, as α gets closer to $\alpha_p = e^{r(T-T^*)}$, instability will get stronger and stronger, as Figure 9 shows.

As expected, the self-referential put is unstable, but the feedback effect on the strike is not what we anticipated. To that extent, the call and the put option behave similarly. The main difference takes place in the amplitude of the instability with respect to α since the put becomes highly unstable if $\alpha \to \alpha_p = e^{r(T-T^*)}$ while the call remains bounded for acceptable values of α.

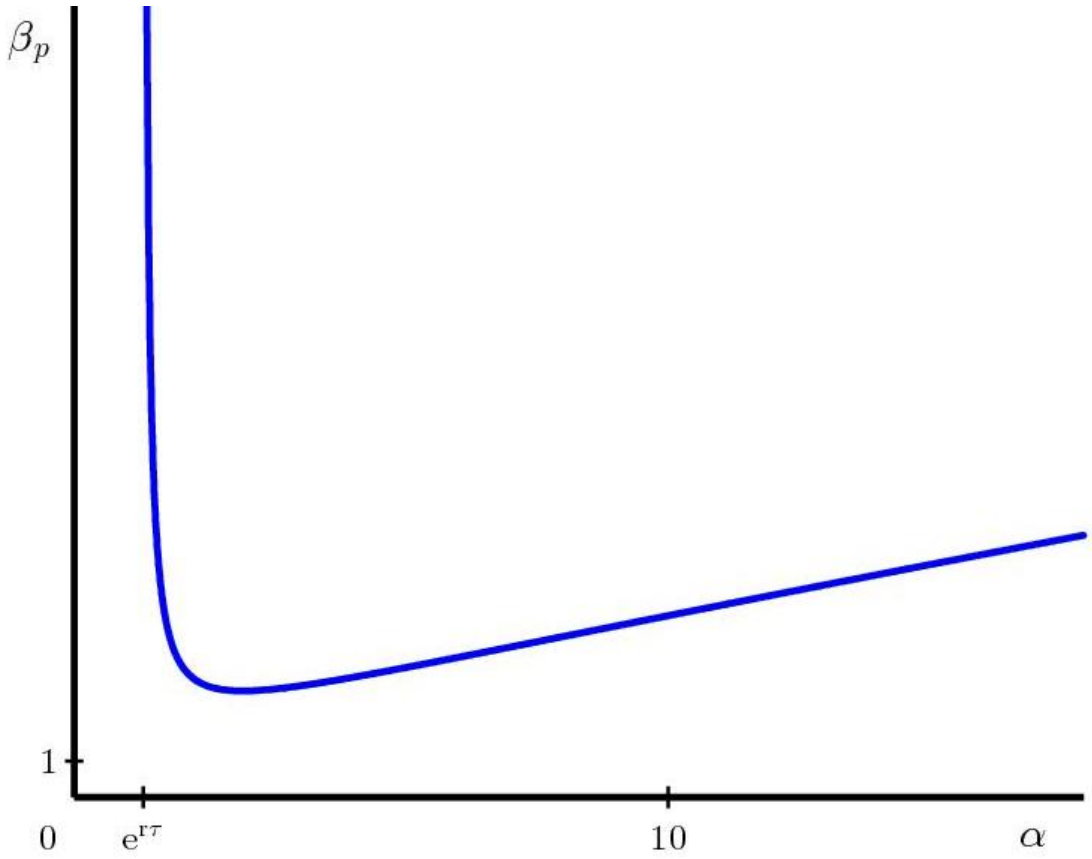


Figure 9: Evolution of β_p with respect to α

5. Conclusion

In the first part of this dissertation, we have found separable solutions of the Black-Scholes equation and showed they constitute a basis for general solutions of the equation. Beyond the fact that it shows us an alternative way to compute option prices, we mainly used these new results as a justification to our linear stability analysis. Indeed, considering a perturbation of the payoff of the form ϵS^n with $n \in i\mathbb{R}$ comes from Fourier analysis and that a combination of these modes (as n changes) can create any bounded perturbation.

Following this, we have conducted our linear stability analysis considering a self-referential option would be stable if the final perturbation was the same as the one the same initial perturbation would have created for its linear counterparty. However, in the non-linear case, the final perturbation depends on the final stock price in a way that *some* trajectories of S would lead to a greater final perturbation than the linear version of the option. Indeed, the feedback effect introduced by the non-linear condition tends to push the strike to reduce the option price, but this feedback effect can become too strong. Our initial reasoning assumed the strike would evolve permanently and, in the end, go back to the stable value after some cross evolution of the option price and the strike. However, the strike is settled once and will not evolve afterwards, once T^* is reached. The perturbation will evolve according to the Black-Scholes dynamics only, without any possible further evolution of the strike.

Moreover, it must be mentioned again that even though the perturbations (strike or option price) presented in this dissertation are complex, this is only because we considered Fourier modes one by one, in order to simplify the calculation,

but we need to keep in mind that if we wanted the formal result, we would need to pick the form of the perturbation, break it down into its Fourier modes and invert the process after having applied our results to each mode. In the end, our method does provide general results, without any loss of generality, despite the suprising complex numbers appearing in financial assets prices.

Although we have presented the results for the call and put only, the condition presented in Section 4.1 would lead to similar conclusions for the digital call and put. Finally, the self-referential forward-start call and put presented on that paper have been chosen because they represent the simplest non-linear options within a Black-Scholes model, but it can be interesting to see [1] for more options including the non-linear Asian options which solve a non-linear Black-Scholes equation, similar to the one presented in the introduction for the Carbon credit model.

References

[1] *J.N. Dewynne and S.D. Howison*, Nonlinear Options, OCIAM, Mathematical Institute, Oxford University, Forthcoming.

[2] *D. Schwarz and S.D. Howison*, Risk-Neutral Pricing of Financial Instruments in Emission Markets: A Structural Approach, To appear in SIAM Journal on Financial Mathematics.

[3] *R. Carmona, F. Delarue, G.-E. Espinosa, N. Touzi*, Singular Forward-Backward Stochastic Differential Equations and Emissions Derivatives, hal-00555591, version 2, 2011.

[4] *S. Chandrasekhar*, Hydrodynamic and Hydromagnetic Stability, Dover, 1981.

[5] *S. Bochner and K. Chandrasekharan*, Fourier Transforms, Princetion University Press, 1949.