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Lagrangians of hypergraphs: The Frankl-Füredi conjecture holds almost everywhere

Mykhaylo Tyomkyn

Abstract

Frankl and Füredi conjectured in 1989 that the maximum Lagrangian of all r-uniform hypergraphs of fixed size m is realised by the initial segment of the colexicographic order. In particular, in the principal case $m = \binom{t}{r}$ their conjecture states that the maximum is attained on the clique of order t.

We prove the latter statement for all $r \geq 4$ and large values of t (the case $r = 3$ was settled by Talbot in 2002). More generally, we show for any $r \geq 4$ that the Frankl-Füredi conjecture holds whenever $\binom{t-1}{r} \leq m \leq \binom{t}{r} - \gamma_r t^{r-2}$ for a constant $\gamma_r > 0$, thereby verifying it for 'most' $m \in \mathbb{N}$.

Furthermore, for $r = 3$ we make an improvement on the results of Talbot and of Tang, Peng, Zhang and Zhao.

1. *Introduction*

Multilinear polynomials are of central interest in most branches of modern mathematics, and extremal combinatorics is by no means an exception. In particular, a large number of hypergraph Turán problems reduce to calculating or estimating the Lagrangian of a hypergraph, which is a constrained maximum of the multilinear function naturally associated with the hypergraph.

To set the scene, we need a few definitions. We follow standard notation of extremal combinatorics (see e.g. [**1**]). In particular, for $n, r \in \mathbb{N}$, we write $[n]$ for the set $\{1, \dots, n\}$ and, given a set X, by $X^{(r)}$ we denote the set family $\{A \subseteq X : |A| = r\}$. Dealing with finite families of finite sets we will be freely switching between the set system and the hypergraph points of view: with no loss of generality, we can assume our hypergraphs to be defined on $\mathbb{N}$, yet we write $e(H)$ for the number of sets ('edges') in H.

For a finite r-uniform hypergraph $H \subseteq [n]^{(r)}$ and a vector of real numbers (referred as a *weighting*) $\vec{y} := (y_1, \dots, y_n)$ consider a multilinear polynomial function

$$L(H, \vec{y}) := \sum_{A \in H} \prod_{i \in A} y_i.$$

The *Lagrangian* of H is defined as its maximum on the standard simplex

$$\lambda(H) := \max\{L(H, \vec{y})\colon y_1, \dots, y_n \geq 0; \sum_{i=1}^{n} y_i = 1\};$$

note that, by compactness, the maximum does always exist (but need not be unique).

The above notion was introduced in 1965 by Motzkin and Strauss [**7**] for $r = 2$, that is for graphs, in order to give a new proof of Turán's theorem. Later it was extended to uniform

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hypergraphs, where the Lagrangian plays an important role in governing densities of blow-ups. In particular, using Lagrangians of r-graphs, Frankl and Rödl [**4**] disproved a conjecture of Erdős [**2**] by exhibiting infinitely many non-jumps for hypergraph Turán densities. In the following years the Lagrangian has found numerous applications in hypergraph Turán problems; for more details we refer to a survey by Keevash [**6**] and the references therein. Further results, which appeared after the publication of [**6**], include [**5**] and [**9**].

In this paper we address the problem of maximising the Lagrangian itself over all r-graphs with a fixed number of edges. Let $H^{m,r}$ be the subgraph of $\mathbb{N}^{(r)}$ consisting of the first m sets in the colexicographic order (recall that this is the ordering on $\mathbb{N}^{(r)}$ in which $A < B$ if $\max(A \triangle B) \in B$). In 1989 Frankl and Füredi [**3**] conjectured that the maximum Lagrangian of an r-graph on m edges is realised by $H^{m,r}$.

Conjecture 1.1 [**3**]. $\lambda(H^{m,r}) = \max\{\lambda(H) \colon H \subseteq \mathbb{N}^{(r)}, e(H) = m\}$.

In an important special case, which we refer to as the *principal case*, Conjecture 1.1 states that for $m = \binom{t}{r}$ the maximum Lagrangian is attained on $H^{m,r} = [t]^{(r)}$, where we have $\lambda(H^{m,r}) = \lambda([t]^{(r)}) = \frac{1}{t^r}\binom{t}{r}$. While initially the Frankl-Füredi conjecture was motivated by applications to hypergraph Turán problems, we think it also interesting in its own right, as it makes a natural and general statement about maxima of multilinear functions.

For $r = 2$ the validity of Conjecture 1.1 is easy to see and follows from the arguments of Motzkin and Strauss [**7**]. In fact, the Lagrangian of a graph H is attained by equi-distributing the weights between the vertices of the largest clique of H, resulting in $\lambda(H) = \frac{\omega(H)-1}{2\omega(H)}$. Since $H^{m,r}$ has the largest clique size over all graphs on m edges, Conjecture 1.1 holds.

On the other hand, the situation for hypergraphs is far more complex, since for $r \geq 3$, unlike in the graph case, no direct way of inferring $\lambda(H)$ from the structure of H is known. Hence one is confined to estimating the Lagrangians of different r-graphs against each other without calculating them directly.

For $r = 3$ Talbot [**8**] proved that Conjecture 1.1 holds whenever $\binom{t-1}{3} \leq m \leq \binom{t-1}{3} + \binom{t-2}{2} - (t-1) = \binom{t}{3} - (2t-3)$ for some $t \in \mathbb{N}$. Note that this range covers an asymptotic density 1 subset of $\mathbb{N}$, and also includes the principal case $m = \binom{t-1}{3}$. Recently Tang, Peng, Zhang and Zhao [**9**] extended the above range to $\binom{t-1}{3} \leq m \leq \binom{t-1}{3} + \binom{t-2}{2} - \frac{1}{2}(t-1)$. Furthermore, Conjecture 1.1 is known to hold when $\binom{t}{3} - m$ is a small constant, but for the remaining values of m it is still open.

In contrast to this, for $r \geq 4$ much less has been known so far, as Talbot's proof method for $r = 3$, perhaps surprisingly, does not immediately transfer. Talbot showed in the same paper [**8**] that for every $r \geq 4$ there is a constant $\gamma_r > 0$ such that if $\binom{t-1}{r} \leq m \leq \binom{t}{r} - \gamma_r t^{r-2}$ and H is supported on t vertices (that is, ignoring isolated vertices, H is a subgraph of $[t]^{(r)}$), then indeed $\lambda(H) \leq \lambda(H^{m,r})$. Still, for no value of m, apart from some trivial ones, Conjecture 1.1 has been known to hold. Our main goal in this article is to close this gap by confirming the Frankl-Füredi Conjecture for 'most' values of m for any given $r \geq 4$, including the principal case for large m.

Theorem 1.2. *For every $r \geq 4$ there exists $\gamma_r > 0$ such that for all $\binom{t-1}{r} \leq m \leq \binom{t}{r} - \gamma_r t^{r-2}$ we have*

$$\lambda(H^{m,r}) = \max\{\lambda(H) \colon H \subseteq \mathbb{N}^{(r)}, e(H) = m\}.$$

COROLLARY 1.3. *For every $r \geq 4$ there exists $t_r \in \mathbb{N}$ such that for all $t \in \mathbb{N}$ with $t \geq t_r$ we have*

$$\max\left\{\lambda(H)\colon H \subseteq \mathbb{N}^{(r)}, e(H) = \binom{t}{r}\right\} = \lambda([t]^{(r)}) = \frac{1}{t^r}\binom{t}{r}.$$

By monotonicity, we obtain another immediate corollary, which can be viewed as a strong approximate version of Conjecture 1.1.

COROLLARY 1.4. *For every $r \geq 4$ there exists $t_r \in \mathbb{N}$ such that for all $t \geq t_r$ the following holds. Suppose that $\binom{t-1}{r} < m \leq \binom{t}{r}$ and that H is an r-graph with $e(H) = m$. Then*

$$\lambda(H) \leq \frac{1}{t^r}\binom{t}{r}.$$

When H is supported on $[t]$ we give a proof of a stronger statement, namely that in this case we can take $\gamma_r = (1 + o(1))/(r - 2)!$ in Theorem 1.2. More precisely, we claim the following.

THEOREM 1.5. *For every $r \geq 3$ there exists a constant $\delta_r > 0$ such that for all $\binom{t-1}{r} \leq m \leq \binom{t}{r} - \binom{t-2}{r-2} - \delta_r t^{r-9/4}$ we have*

$$\lambda(H^{m,r}) = \max\{\lambda(H)\colon H \subseteq [t]^{(r)}, e(H) = m\}.$$

For $r = 3$ it was implicitly shown by Talbot in [**8**] that for any $\binom{t-1}{3} < m \leq \binom{t}{3}$, that is for all $m \in \mathbb{N}$, the 3-graph maximising the Lagrangian amongst all m-edge 3-graphs can be assumed to be supported on $[t]$. Combined with Theorem 1.5, this yields, for large m, an improvement of the bounds in [**8**] and [**9**].

COROLLARY 1.6. *There exists a constant $\delta_3 > 0$ such that for all $\binom{t-1}{3} \leq m \leq \binom{t}{3} - (t - 2) - \delta_3 t^{3/4}$ we have*

$$\lambda(H^{m,r}) = \max\{\lambda(H)\colon H \subseteq \mathbb{N}^{(r)}, e(H) = m\}.$$

Overview of the proof

The proof of our main result, Theorem 1.2, uses a number of well-known properties of the Lagrangian, as well as induction on r and some facts about uniform set systems such as the Kruskal-Katona theorem. We begin by considering the r-graph $G \subseteq \mathbb{N}^{(r)}$ and the weighting $\vec{x}$ that (co-)achieve the largest Lagrangian amongst all r-graphs of size m. We assume that $\lambda(G)$ is strictly larger than $\lambda(H^{m,r})$, and aim to show that then m must lie outside the range specified in Theorem 1.2. This is carried out as follows.

First, we make a number of standard assumptions on G and $\vec{x}$. We assume that G covers pairs (meaning that any two vertices of G are contained in some edge) and has the minimum possible number of vertices (referred as the 'support'). Assuming by symmetry that the entries of $\vec{x}$ are listed in descending order, we can also claim that G is left-compressed. As a consequence, we obtain some bounds on the sizes of link hypergraphs of G, as stated in Proposition 2.1 (which is where we need the Kruskal-Katona theorem).

It turns out that the parameter we are most interested in is T, the support of G. By combining Proposition 2.8, which is a standard tool that relates $\lambda(G)$ to the Lagrangians of its link hypergraphs, with a number of further ideas such as induction of r and Proposition 2.1, we gradually establish better and better bounds on T as well as on related parameters such as x_1 and x_T (the largest and the smallest entries of $\vec{x}$). This part of the argument culminates in Lemma 5.1 and Lemma 6.2, where we show, respectively, that $T = t + C$ and $x_1 < 2x_{t-3\alpha}$ for some constants $0 \le C \le C_0(r)$ and $\alpha = \alpha(r)$.

In the final part of the proof the above bounds are applied to replace a number of 'bad' edges of G with some 'good' edges from $[t]^{(r)} \setminus G$ such that the resulting graph does G' not cover any pair in $\{t-1, \dots, T\}^{(2)}$, thus $\lambda(G') \le \lambda(H^{m,r}) < \lambda(G)$. The estimates on T and x_1 ensure that the good edges are reasonably heavy, so that, unless $\binom{t}{r} - m$ is small (in which case we might not find enough good edges), the total weight of the good edges is greater than that of the bad edges, resulting in $\lambda(G') > \lambda(G)$, a contradiction. Hence, $\binom{t}{r} - m$ has to be small, completing the proof.

2. *Notation and preliminaries*

Let $r \ge 2$ be an integer. Given an r-graph H and a set $S \subseteq \mathbb{N}$ with $|S| < r$, the $(r - |S|)$-uniform *link hypergraph* of S is defined as

$$H_S := \{A \in \mathbb{N}^{(r-|S|)} \colon A \cup S \in H\}.$$

To simplify notation we omit the parentheses and write, for instance, $H_{1,2}$ for $H_{\{1,2\}}$.

Let us now recall the following standard fact about left-compressed set systems. For the definition of left-compressions (also known as left-shifts) see e.g. [**1**].

PROPOSITION 2.1. *Let $n \in \mathbb{N}$ and let $H \subseteq [n]^{(r)}$ be left-compressed with $e(H) = \binom{x}{r}$ for some $n \ge x \ge r$ (x being not necessarily an integer). Then*

(i)

$$e(H_1) \ge \binom{x-1}{r-1} = \frac{r}{x} e(H)$$

(ii) *For all $1 \le j \le n$ we have*

$$e(H_j) \le \frac{r}{j} e(H).$$

Proof sketch. (i) is essentially equivalent to Kruskal-Katona theorem and is proved alongside the same lines, see e.g. [**1**]. (ii) follows by double-counting, using that H is left-compressed:

$$je(H_j) \le \sum_{i=1}^{j} e(H_i) \le \sum_{i=1}^{n} e(H_i) = re(H).$$

□

Next, we state a well-known inequality for elementary symmetric polynomials, which is a special case of Maclaurin's inequality.

LEMMA 2.2. *For all $n \in \mathbb{N}$ and $y_1, \dots, y_n \ge 0$ with $\sum_{i \in [n]} y_i = Y$ one has*

$$\sum_{I \in [n]^{(r)}} \prod_{i \in I} y_i \le \binom{n}{r} \left(\frac{Y}{n}\right)^r < \frac{Y^r}{r!}. \tag{2.1}$$

In particular,

$$\lambda([n]^{(r)}) = \frac{1}{n^r}\binom{n}{r} < \frac{1}{r!}. \tag{2.2}$$

Viewing r as constant and n as tending to infinity, we shall make frequent use of the following asymptotics.

$$\begin{aligned} \frac{1}{n^r}\binom{n}{r} &= \frac{1}{r!}(1-\binom{r}{2}n^{-1}) + O(n^{-2}); \\ \frac{1}{(n-1)^r}\binom{n-1}{r} &= \frac{1}{r!}(1-\binom{r}{2}(n-1)^{-1}) + O(n^{-2}) = \frac{1}{r!}(1-\binom{r}{2}n^{-1}) + O(n^{-2}). \end{aligned} \tag{2.3}$$

A hypergraph H is said to *cover* a vertex pair $\{i,j\}$ if there exists an edge $A \in H$ with $\{i,j\} \subseteq A$. H is said to *cover pairs* if it covers every pair $\{i,j\} \subseteq \bigcup_{A\in H} A$. Let $H - i$ be the r-graph obtain from H by deleting vertex i and the corresponding edges.

PROPOSITION 2.3 [**3**]. *Let $n \in \mathbb{N}$ and $H \subseteq [n]^{(r)}$.*

(i) *Suppose that H does not cover the pair $\{i,j\}$. Then $\lambda(H) \leq \max\{\lambda(H-i), \lambda(H-j)\}$. In particular, $\lambda(H) \leq \lambda([n-1]^{(r)})$.*

(ii) *Suppose that $m, t \in \mathbb{N}$ satisfy $\binom{t-1}{r} \leq m \leq \binom{t}{r} - \binom{t-2}{r-2}$. Then*

$$\lambda(H^{m,r}) = \lambda([t-1]^{(r)}) = \frac{1}{(t-1)^r}\binom{t-1}{r}. \tag{2.4}$$

Proof sketch. (i) is obtained by considering all weights except y_i and y_j as fixed. This effectively turns $L(H, \vec{y})$ into a linear function in one variable, which is, ostensibly, maximised at an endpoint of its domain interval. (ii) follows from (i) since on the one hand $[t-1]^{(r)} \subseteq H^{m,r}$, and on the other hand for the above values of m the graph $H^{m,r}$ does not cover the pair $\{t-1, t\}$. □

A simple scaling gives the following fact about the Lagrangians of link hypergraphs.

LEMMA 2.4. *Suppose $H \subseteq [n]^{(r)}$ and $\vec{y} = (y_1, \ldots, y_n)$ is a weighting with $y_j \geq 0$ for all $j \in [n]$ and with $\sum_{j=1}^n y_j = 1$. Then for all $i \in [n]$ we have*

$$L(H_i, \vec{y}) \leq (1-y_i)^{r-1}\lambda(H_i). \tag{2.5}$$

Proof. If $y_i = 1$, then this is self-evident. Otherwise, define the new weighting $z_j = (1-y_i)^{-1}y_j$, for all $j \neq i$ and $z_i = 0$. We have $\sum_{j=1}^n z_j = 1$ and thus

$$\lambda(H_i) \geq L(H_i, \vec{z}) = (1-y_i)^{-(r-1)}L(H_i, \vec{y}).$$

□

Define $H_{i\setminus j} := \{A \in H_i \setminus H_j : j \notin A\}$. In other words, $H_{i\setminus j}$ contains precisely all $r-1$-sets A such that $A \cup \{i\} \in H$ but $A \cup \{j\} \notin H$. Note that having $H_{i\setminus j} = H_{j\setminus i} = \emptyset$ is equivalent to H having an automorphism via interchanging i and j. This implies the following straightforward fact.

PROPOSITION 2.5. *Suppose that $H_{i\setminus j} = H_{j\setminus i} = \emptyset$. Then $L(H, \vec{y}) \leq L(H, \vec{z})$, where $z_i = z_j = (y_i + y_j)/2$ and $z_\ell = y_\ell$ otherwise.*

Proof.

$$L(H,\vec{z}) - L(H,\vec{y}) = L(H_{i,j},\vec{y})(z_i z_j - y_i y_j) \geq 0.$$

□

From now on let $r \geq 4$ and suppose that $\binom{t-1}{r} \leq m \leq \binom{t}{r} - \binom{t-2}{r-2}$ for some $t \in \mathbb{N}$. Let G be a graph with $e(G) = m$ which satisfies $\lambda(G) = \max\{\lambda(H) \colon H \subseteq \mathbb{N}^{(r)}, e(H) = m\}$ and let $\vec{x}$ be a weighting attaining the Lagrangian of G, that is $x_i \geq 0$ for all i, $\sum x_i = 1$ and $L(G,\vec{x}) = \lambda(G)$ (note that in general G and $\vec{x}$ are not unique). Following the conventional notation (see e.g. [**8**]), we can assume by symmetry that the entries of $\vec{x}$ are listed in descending order, that is $x_i \geq x_j$ for all $i < j$. We shall furthermore assume that, subject to the above conditions, $\vec{x}$ has the minimum possible number of non-zero entries, and let T be this number. By the above, we have $x_1 \geq \cdots \geq x_T > 0$ and $x_1 + \cdots + x_T = 1$.

Suppose that G achieves a strictly larger Lagrangian than $H^{m,r}$. By (2.4) we have

$$\lambda(G) = L(G,\vec{x}) > \frac{1}{(t-1)^r}\binom{t-1}{r}, \tag{2.6}$$

which in turn implies $T \geq t$ (otherwise $\lambda(G) = \lambda(G \cap [t-1]^{(r)}) \leq \lambda([t-1]^{(r)})$, a contradiction). Our goal is to show that under these assumptions we must have $m > \binom{t}{r} - \gamma_r t^{r-2}$ for some constant $\gamma_r > 0$. Before we proceed, let us recall some well-known facts about the newly defined r-graph G and its Lagrangian.

Proposition 2.6. $\bigcup_{A\in G} A = [T]$.

Proof. If there exists some $i \in [T] \setminus \bigcup_{A\in G} A$, i.e. if there is a positive weight not used by any edge, then by re-distributing the weight x_i evenly between the vertices of some edge $A \in G$, we obtain a new weighting $\vec{z}$ with $\sum z_i = 1$ and $L(G,\vec{z}) > L(G,\vec{x}) = \lambda(G)$, a contradiction. Hence, $[T] \subseteq \bigcup_{A\in G} A$.

Conversely, suppose that there is some $A \in G$ with $A \not\subseteq [T]$, i.e. that an edge of G uses a zero weight. Then, since $T \geq t$ and $m < \binom{t}{r} \leq \binom{T}{r}$, there must be a set $B \in [T]^{(r)} \setminus G$. Now, the r-graph $G' = (G \setminus A) \cup B$ has m edges and satisfies $\lambda(G') \geq L(G',\vec{x}) > L(G,\vec{x}) = \lambda(G)$, contradicting the assumption on G. Hence, $\bigcup_{A\in G} A \subseteq [T]$. □

Proposition 2.7 [**3**]. *G can be assumed to be left-compressed and to cover pairs.*

Proof sketch. If G' is obtained from G by a series of left-shifts, it is clear that

$$\lambda(G) = L(G,\vec{x}) \leq L(G',\vec{x}) \leq \lambda(G').$$

Additionally, left-compressions cannot increase T. The fact that G covers pairs follows by Proposition 2.3(i) and the definition of T. □

The next statement holds in more generality, but we shall mainly need it for G and $\vec{x}$.

Proposition 2.8 [**4**],[**8**]. *Let G, T and $\vec{x}$ be as defined above. Then*

(i) *For all $1 \leq i \leq T$ one has*

$$L(G_i,\vec{x}) = r\lambda(G). \tag{2.7}$$

(ii) *For all* $1 \leq i < j \leq T$ *one has*

$$(x_i - x_j)L(G_{i,j}, \vec{x}) = L(G_{i \setminus j}, \vec{x}). \tag{2.8}$$

Proof sketch. Notice that $L(G_i, \vec{x})$ is the partial derivative of $L(G, \vec{x})$ with respect to x_i. By considering Lagrange multipliers, one obtains $L(G_i, \vec{x}) = L(G_j, \vec{x})$ for all $i \neq j$. Hence, Proposition 2.6 yields

$$L(G, \vec{x}) = \frac{1}{r}\sum_{j=1}^{T} L(G_j, \vec{x})x_i = \frac{1}{r}L(G_i, \vec{x})\sum_{j=1}^{T} x_j = \frac{1}{r}L(G_i, \vec{x}),$$

proving (1). To see that (2) also holds, note again that $L(G_i, \vec{x}) = L(G_j, \vec{x})$ and use the fact that G is left-compressed. □

In order to prove Theorem 1.2 we apply induction on r, assuming Corollary 1.4 for $r-1$ as the induction hypothesis. Since Theorem 1.2 is concerned with large values of m, we will assume t to be greater than any given number (which may also depend on r), whenever we need it. We do not attempt to optimise γ_r.

As our induction base we take Corollary 1.4 for $r = 3$, which is known to hold by Talbot's theorem [**8**]. Note though, that our proof does not crucially rely on [**8**], as, alternatively, we could start at $r = 3$, taking the trivial $r = 2$ case as the induction base; this would also give a new proof of a slightly weaker form of Talbot's theorem for $r = 3$.

The rest of the paper is organised as follows. In Section 3 we establish first upper bounds on T and x_1. With this information we show in Section 4 that if T is greater than t by some additive term, then x_T is less than $1/t$ by a similar multiplicative term. This implies that, for a certain index $q < t$, x_q is significantly smaller than $1/t$. This fact will, in turn, be applied in Section 5, where we prove that $T - t \leq C_0$ for a constant $C_0(r)$, which allows in Section 6, using a refinement of an argument from [**9**], to bound $\binom{t}{r} - m$, concluding the proof of Theorem 1.2. In Section 7 we apply a version of the argument in Section 6 in order to prove Theorem 1.5. In the final section we discuss possible ways of extending our results.

3. *Some coarse bounds*

In this section we prove some, rather crude, first bounds on T and x_1. They will be required for establishing tighter bounds later on.

Lemma 3.1. *$T < 10t$.*

Proof. We have

$$\frac{1}{(t-1)^r}\binom{t-1}{r} \overset{(2.6)}{<} L(G, \vec{x}) \overset{(2.7)}{=} \frac{1}{r}L(G_T, \vec{x}) \overset{(2.5)}{\leq} \frac{1}{r}(1 - x_T)^{r-1}\lambda(G_T) \leq \frac{\lambda(G_T)}{r}.$$

So,

$$\lambda(G_T) \geq \frac{r}{(t-1)^r}\binom{t-1}{r}. \tag{3.1}$$

Let $s \in \mathbb{N}$ be such that

$$\binom{s-1}{r-1} < e(G_T) \leq \binom{s}{r-1}. \tag{3.2}$$

Since, by Proposition 2.7, G is left-compressed, Proposition 2.1(ii) implies

$$e(G_T) \leq \frac{r}{T}m < \frac{r}{T}\binom{t}{r} \leq \frac{r}{t}\binom{t}{r} = \binom{t-1}{r-1}, \tag{3.3}$$

which means $s < t$. We claim that s cannot be too small either, or, in other words, that $e(G_T)$ is 'reasonably large'.

Claim 3.2.

$$s \geq (1+o(1))\frac{r-2}{r}t.$$

Proof. By the induction hypothesis and monotonicity, we have

$$\frac{r}{(t-1)^r}\binom{t-1}{r} \overset{(3.1)}{\leq} \lambda(G_T) \leq \frac{1}{{s_1}^{r-1}}\binom{s_1}{r-1},$$

where $s_1 = \max\{s, t_{r-1}\}$ and t_{r-1} is as in Corollary 1.4. Since we can assume that $t > 10t_{r-1}$, so that $t_{r-1} < t/10$, it follows that

$$\frac{r}{(t-1)^r}\binom{t-1}{r} \leq \frac{1}{s_2^{r-1}}\binom{s_2}{r-1}, \tag{3.4}$$

where $s_2 = \max\{s, t/10\}$. Suppose now that $s < t/10$. Then $s_2 = t/10$ and we get

$$\begin{aligned} \frac{1}{(r-1)!}\cdot(1-\binom{r}{2}\frac{1}{t}) + O(t^{-2}) &\overset{(2.3)}{=} \frac{r}{(t-1)^r}\binom{t-1}{r} \overset{(3.4)}{\leq} \frac{1}{(t/10)^{r-1}}\binom{t/10}{r-1} \\ &\overset{(2.3)}{=} \frac{1}{(r-1)!}\cdot(1-\binom{r-1}{2}\frac{10}{t}) + O(t^{-2}), \end{aligned} \tag{3.5}$$

which results in a contradiction, as $10\binom{r-1}{2} > \binom{r}{2}$. Thus we must have $s \geq t/10$, so that $s_2 = s$ and

$$\frac{r}{(t-1)^r}\binom{t-1}{r} \leq \frac{1}{s^{r-1}}\binom{s}{r-1}.$$

Applying (2.3) and the just established fact that $t/10 \leq s \leq t$, i.e. $s = \Theta(t)$, similarly to (3.5) we obtain

$$1 - \binom{r-1}{2}\frac{1}{s} \geq 1 - \binom{r}{2}\frac{1}{t} + O\left(t^{-2}\right).$$

Equivalently,

$$\binom{r}{2}s \geq \binom{r-1}{2}t + O(1) = (1+o(1))\binom{r-1}{2}t.$$

Thus,

$$s \geq (1+o(1))\frac{\binom{r-1}{2}}{\binom{r}{2}}t = (1+o(1))\frac{r-2}{r}t.$$

□

Notice now that

$$\frac{r}{T}\binom{t}{r} \overset{(3.3)}{\geq} e(G_T) \overset{(3.2)}{\geq} \binom{s-1}{r-1} = \frac{r}{s}\binom{s}{r}.$$

Thus, using Claim 3.2, we obtain

$$T \le \frac{s\binom{t}{r}}{\binom{s}{r}} = (1+o(1))s \cdot \left(\frac{t}{s}\right)^r = (1+o(1))t \cdot \left(\frac{t}{s}\right)^{r-1} \le (1+o(1))t\left(\frac{r}{r-2}\right)^{r-1}$$
$$= (1+o(1))t \cdot \left(1 - \frac{2}{r}\right)^{-r+1} < 10t,$$

where the last inequality is due to the fact that $e^2 < 8$, with the latter replaced by 10 to account for small values of r. This completes the proof of Lemma 3.1. □

Next, we give an upper bound on x_1.

LEMMA 3.3. $x_1 \le r/t$.

Proof. Observe that

$$L(G,\vec{x}) \overset{(2.7)}{=} \frac{1}{r}L(G_1,\vec{x}) \overset{(2.5)}{\le} \frac{1}{r}(1-x_1)^{r-1}\lambda(G_1) \overset{(2.2)}{<} \frac{(1-x_1)^{r-1}}{r!}. \tag{3.6}$$

Hence, by (2.6), we must have

$$\frac{1}{(t-1)^r}\binom{t-1}{r} < \frac{(1-x_1)^{r-1}}{r!},$$

which, due to (2.3), means

$$(1-x_1)^{r-1} > 1 - \binom{r}{2}\frac{1}{t} + O(t^{-2}).$$

Since $(1-x_1)^{r-1}$ is a decreasing function of x_1 and $(1-r/t)^{r-1} = 1 - 2\binom{r}{2}/t + O(t^{-2}) < 1 - \binom{r}{2}/t + O(t^{-2})$, we must have $x_1 \le r/t$. □

4. *Bounding the tails*

With the above information we aim to establish some upper bounds on x_i for large values of i. Put $\Delta := T - t$, so that by Lemma 3.1 we have $\Delta \le 9t$. For technical reasons we will assume here that $\Delta \ge 1$; the case $\Delta = 0$ will be dealt with in Section 6.

LEMMA 4.1. *If $\Delta \ge 1$ then*

$$x_T \le \frac{10}{\Delta^{1/(r-1)}t}. \tag{4.1}$$

Proof. Recall from Proposition 2.8 that

$$(x_1 - x_T)L(G_{1,T},\vec{x}) = L(G_{1\setminus T},\vec{x}),$$

and note that, by Lemma 2.2, we have $L(G_{1,T},\vec{x}) \le 1/(r-2)!$. Combining these facts with Lemma 3.3, we obtain

$$\frac{r}{t(r-2)!} \ge \frac{x_1}{(r-2)!} \ge (x_1 - x_T)L(G_{1,T},\vec{x}) = L(G_{1\setminus T},\vec{x}) \ge e(G_{1\setminus T})x_T^{r-1}.$$

Thus,

$$x_T \le \left(\frac{r}{t(r-2)! \cdot e(G_{1/T})}\right)^{1/(r-1)}. \tag{4.2}$$

Now, observe that, since G is left-compressed, we have

$$\begin{aligned} e(G_{1\setminus T}) &= e(G_1) - e(G_{1,T}) - e(G_1 \cap G_T) = e(G_1) - e(G_{1,T}) - (e(G_T) - e(G_{1,T})) \\ &= e(G_1) - e(G_T). \end{aligned} \tag{4.3}$$

Let x be the real number satisfying $m = \binom{x}{r}$, so that $t-1 \leq x < t$. By Proposition 2.1(i) we have

$$e(G_1) \geq \frac{rm}{x} > \frac{rm}{t}.$$

On the other hand, by Proposition 2.1(ii), we have $e(G_T) \leq rm/T$. Together with Lemma 3.1 this implies

$$\begin{aligned} e(G_{1\setminus T}) &\overset{(4.3)}{=} e(G_1) - e(G_T) \geq (\frac{1}{t} - \frac{1}{T})rm = \frac{\Delta rm}{tT} \geq \frac{\Delta rm}{10t^2} > \frac{\Delta r}{10t^2} \cdot \frac{3}{4}\binom{t}{r} \\ &> \frac{\Delta t^{r-2}}{20(r-1)!}. \end{aligned} \tag{4.4}$$

Hence, combining (4.2) and (4.4), we obtain

$$x_T \leq \left(\frac{r}{t(r-2)!} \cdot \frac{20(r-1)!}{\Delta t^{r-2}}\right)^{1/(r-1)} = \left(\frac{20r(r-1)}{\Delta t^{r-1}}\right)^{1/(r-1)} < \frac{10}{\Delta^{1/(r-1)}t}.$$

□

Next, we want to establish a similar upper bound for x_q where q is ‘somewhat smaller’ than t; this will prove crucial in due course. More precisely, let $q \in \mathbb{N}$ be such that

$$\binom{q-1}{r-1} \leq \frac{t}{T-1}\binom{t-1}{r-1} < \binom{q}{r-1}. \tag{4.5}$$

The following technical lemma bounds q from above and below.

LEMMA 4.2. *We have*
(i)

$$t \cdot \left(\frac{t}{T}\right)^{\frac{1}{r-1}} - r < q < t \cdot \left(\frac{t}{T}\right)^{\frac{1}{r-1}} + r.$$

In particular,

$$q = (1 + o(1))t \left(\frac{t}{T}\right)^{\frac{1}{r-1}} = \Theta(t). \tag{4.6}$$

(ii) *If* $\Delta > 4r$ *then* $t - q < \Delta$.

Proof. For the upper bound in (i) observe that

$$\frac{(q-r)^{r-1}}{(r-1)!} < \binom{q-1}{r-1} \overset{(4.5)}{\leq} \frac{t}{T-1}\binom{t-1}{r-1} < \frac{t}{T}\frac{t^{r-1}}{(r-1)!},$$

where the last inequality uses $(t-1)/(T-1) \leq t/T$. The lower bound in (i) follows similarly:

$$\frac{q^{r-1}}{(r-1)!} > \binom{q}{r-1} \overset{(4.5)}{>} \frac{t}{T-1}\binom{t-1}{r-1} > \frac{t}{T}\frac{(t-r)^{r-1}}{(r-1)!}.$$

With these bounds, $q = \Theta(t)$ follows from $T = \Theta(t)$, which we know to hold by Lemma 3.1.

To show (ii), notice that this is self-evident when $\Delta \geq t$, and for $\Delta < t$ it suffices to verify the inequality $\binom{t-\Delta}{r-1} < \frac{t}{T-1}\binom{t-1}{r-1}$. Substituting $T = t + \Delta$ yields

$$\frac{t}{T-1}\binom{t-1}{r-1} - \binom{t-\Delta}{r-1} > \frac{t(t-r)^{r-1} - (t+\Delta)(t-\Delta)^{r-1}}{(T-1)(r-1)!}.$$

Since $(t+\Delta)(t-\Delta)^{r-1} = (t^2-\Delta^2)(t-\Delta)^{r-2}$ is a decreasing function of Δ in the domain $\Delta \in [0,t]$, for $\Delta > 4r$ we get

$$\begin{aligned} t(t-r)^{r-1} - (t+\Delta)(t-\Delta)^{r-1} &> t(t-r)^{r-1} - (t+4r)(t-4r)^{r-1} \\ &= (4r(r-2) - r(r-1))t^{r-2} + O(t^{r-3}) > 0. \end{aligned}$$

□

Now let us estimate x_q.

LEMMA 4.3. *If* $\Delta \geq 1$ *then*

$$x_q \leq \frac{10r\Delta^{-1/(r-1)^2}}{t}.$$

Proof. By Proposition 2.1(ii) we have

$$e(G_{T-1}) \leq \frac{r}{T-1}m \leq \frac{r}{T-1}\binom{t}{r} = \frac{t}{T-1}\binom{t-1}{r-1} \overset{(4.5)}{<} \binom{q}{r-1}. \tag{4.7}$$

Since, by Proposition 2.7, G covers pairs, there exists some $A \in G$ with $\{T-1, T\} \subseteq A$; note that $L(A, \vec{x}) \leq x_1^{r-2}x_{T-1}x_T$. On the other hand, since, also by Proposition 2.7, G is left-compressed, for the set $B = \{q-r+2, \ldots, q\} \cup \{T-1\}$ we must have $B \notin G$, for otherwise we would have $[q]^{(r-1)} \subseteq G_{T-1}$, contradicting (4.7) (this is the reasoning behind the definition of q).

Note that if we had $L(B,\vec{x}) > L(A,\vec{x})$, then for the r-graph $G' = (G \setminus \{A\}) \cup \{B\}$, with $e(G') = m$, we would have $\lambda(G') \geq L(G',\vec{x}) > L(G,\vec{x}) = \lambda(G)$, a contradiction. Therefore,

$$x_1^{r-2} x_{T-1} x_T \geq L(A,\vec{x}) \geq L(B,\vec{x}) \geq x_q^{r-1} x_{T-1}.$$

Together with Lemma 3.3 and Lemma 4.1, this gives

$$x_q \leq (x_1^{r-2} x_T)^{1/(r-1)} \leq \left(\frac{r^{r-2}}{t^{r-2}} \cdot \frac{10}{\Delta^{1/(r-1)} t} \right)^{1/(r-1)} \leq \frac{10r}{\Delta^{1/(r-1)^2} t}.$$

□

Combining Lemma 4.2 and Lemma 4.3 we obtain the following upper bound on the sum of all 'small' weights.

Corollary 4.4. *If $\Delta > 4r$ then*

$$\sum_{i=q+1}^{T} x_i \leq 2\Delta x_q \leq \frac{20r\Delta^{1-\frac{1}{(r-1)^2}}}{t}.$$

5. *A better estimate for T*

Our next task will be to prove that Δ is in fact bounded above by some (large) constant.

Lemma 5.1. *There exists a constant $C_0(r)$ such that $\Delta \leq C_0(r)$.*

Proof. If $\Delta \leq 4r$, then there is nothing to prove. Thus, suppose that $\Delta > 4r$, so that Lemma 4.2 and Corollary 4.4 apply. It turns out that in this situation we can generously add all the missing edges to G and show that $L([T]^{(r)},\vec{x}) \leq \lambda([t-1]^{(r)})$, unless $\Delta \leq C_0$.

More precisely, let $S := \sum_{i=q+1}^{T} x_i$, so that by Corollary 4.4 we have $S \leq 20r\Delta^{1-\frac{1}{(r-1)^2}}/t$. Let $\vec{z}$ be the weighting defined by $z_1 = \cdots = z_q = (1-S)/q$ and $z_{q+1} = \cdots = z_T = S/(T-q)$. It is an immediate consequence of Proposition 2.5 that

$$L([T]^{(r)},\vec{z}) = \max\{L([T]^{(r)},\vec{y}) \colon y_1,\ldots,y_T \geq 0, \sum_{i=1}^{q} y_i = 1-S, \sum_{i=q+1}^{T} y_i = S\},$$

so that, in particular, $L([T]^{(r)},\vec{x}) \leq L([T]^{(r)},\vec{z})$. Therefore, we have

$$\lambda(G) = L(G,\vec{x}) \leq L([T]^{(r)},\vec{x}) \leq L([T]^{(r)},\vec{z}). \tag{5.1}$$

Estimating the latter, we obtain

$$\begin{aligned} L([T]^{(r)},\vec{z}) &= \binom{q}{r} \cdot \frac{1}{q^r}(1-S)^r + \sum_{p=1}^{r} \binom{T-q}{p}\binom{q}{r-p} \frac{S^p}{(T-q)^p} \frac{(1-S)^{r-p}}{q^{r-p}} \\ &\overset{(2.3),(4.6)}{\leq} \frac{1}{r!}(1-S)^r\left(1-\binom{r}{2}q^{-1}\right) + O(t^{-2}) + \sum_{p=1}^{r} \frac{1}{p!(r-p)!} S^p (1-S)^{r-p} \\ &\leq \frac{1}{r!}\left(-(1-S)^r \binom{r}{2} q^{-1} + \sum_{p=0}^{r} \binom{r}{p} S^p (1-S)^{r-p}\right) + O(t^{-2}) \\ &= \frac{1}{r!}\left(1-(1-S)^r\binom{r}{2}q^{-1}\right) + O(t^{-2}). \end{aligned} \tag{5.2}$$

On the other hand, as before, we have

$$\lambda(G) \overset{(2.6)}{>} \frac{1}{(t-1)^r}\binom{t-1}{r} \overset{(2.3)}{=} \frac{1}{r!}(1-\binom{r}{2}t^{-1}) + O(t^{-2}). \tag{5.3}$$

Hence, (5.1), (5.2) and (5.3) together imply

$$-(1-S)^r/q \geq -1/t + O(t^{-2}),$$

which, due to $q = \Theta(t)$ by (4.6), yields

$$q \geq t(1-S)^r + O(1) > t(1-rS) + O(1).$$

Invoking Lemma 4.2(i) and Corollary 4.4 we obtain

$$r + t(1+\Delta/t)^{-1/(r-1)} \geq t(1 - 20r^2\Delta^{1-1/(r-1)^2}t^{-1}) + O(1),$$

thus

$$(1+\Delta/t)^{-1/(r-1)} \geq 1 - 20r^2\Delta^{1-1/(r-1)^2}t^{-1} + O(t^{-1}). \tag{5.4}$$

Now, on the one hand, if $\Delta = \Omega(t)$, then the left hand side of (5.4) is $1-\Omega(1)$ while the right hand side is $1-o(1)$, a contradiction. On the other hand, for $\Delta = o(t)$ we can write

$$(1+\Delta/t)^{-1/(r-1)} < 1 - \frac{\Delta}{2t(r-1)} < 1 - \frac{\Delta}{2rt}.$$

With this, (5.4) implies

$$1 - \frac{\Delta}{2rt} \geq 1 - \frac{20r^2\Delta^{1-1/(r-1)^2}}{t} + O(t^{-1}),$$

or, equivalently,

$$\Delta \leq 40r^3\Delta^{1-1/(r-1)^2} + O(1).$$

The last inequality can only hold if Δ is bounded. Hence, $\Delta \leq C_0(r)$. □

6. *The small support case*

Lemma 5.1 entails $T = t + C$, where C is at most a constant: $0 \leq C \leq C_0(r)$. In this section we apply this fact to complete the proof of Theorem 1.2.

First, we claim that Lemma 5.1 yields a stronger upper bound on x_1.

LEMMA 6.1. *$x_1 < 1/(t-\alpha)$ for some constant $\alpha = \alpha(r) \in \mathbb{N}$.*

Proof. We have

$$L(G,\vec{x}) \overset{(2.7),(2.5)}{\leq} \frac{1}{r}(1-x_1)^{r-1}\lambda(G_1) \leq \frac{1}{r}(1-x_1)^{r-1}\lambda([T-1]^{(r-1)})$$
$$\overset{(2.2)}{=} \frac{1}{r}\left(\frac{1-x_1}{T-1}\right)^{r-1}\binom{T-1}{r-1}.$$

Hence, by (2.6) we must have

$$\frac{1}{(t-1)^r}\binom{t-1}{r} < \frac{1}{r}\left(\frac{1-x_1}{T-1}\right)^{r-1}\binom{T-1}{r-1}.$$

Applying (2.3), Lemma 3.3 and Lemma 5.1, we obtain

$$\begin{aligned}1-(r-1)x_1+O(t^{-2}) &> \left(1-\binom{r}{2}\frac{1}{t}+O(t^{-2})\right)\left(1-\binom{r-1}{2}\frac{1}{t+C}+O(t^{-2})\right)^{-1}\\ &= \left(1-\binom{r}{2}\frac{1}{t}+O(t^{-2})\right)\left(1-\binom{r-1}{2}\frac{1}{t}+O(t^{-2})\right)^{-1}\\ &= \left(1-\binom{r}{2}\frac{1}{t}+O(t^{-2})\right)\left(1+\binom{r-1}{2}\frac{1}{t}+O(t^{-2})\right)\\ &= 1-\binom{r}{2}\frac{1}{t}+\binom{r-1}{2}\frac{1}{t}+O(t^{-2})\\ &= 1-(r-1)\frac{1}{t}+O(t^{-2}).\end{aligned}$$

Thus,

$$x_1<\frac{1}{t}+O(t^{-2})<\frac{1}{t-\alpha}.$$

for some constant $\alpha=\alpha(r)$. □

Without loss of generality, we can assume that $\alpha \geq C$, for otherwise rename α to be $\max\{\alpha, C\}$.

LEMMA 6.2. *In the above setting, $x_1 < 2x_{t-3\alpha}$.*

Proof. Suppose otherwise. Then

$$2\alpha x_1 \geq 4\alpha x_{t-3\alpha} \geq (3\alpha+C)x_{t-3\alpha} \geq \sum_{i=t-3\alpha+1}^{T} x_i,$$

and therefore

$$(t-\alpha)x_1=(t-3\alpha)x_1+2\alpha x_1 \geq \sum_{i=1}^{t-3\alpha} x_i + \sum_{i=t-3\alpha+1}^{T} x_i = \sum_{i=1}^{T} x_i = 1,$$

a contradiction. □

As an immediate consequence of Lemma 6.2 we obtain

$$x_1^{r-2}x_{t-1}x_t < 2^{r-2}x_{t-3\alpha}^{r-2}x_{t-1}x_t. \tag{6.1}$$

LEMMA 6.3. *Let*

$\mathcal{A} := \{A \in G\colon |A \cap \{t-1,\dots,T\}| \geq 2\}$,

$\mathcal{B} := \{B \in [t]^{(r)} \setminus G\colon |B \cap \{t-1,t\}| \leq 1$ *and* $|B \cap \{t-3\alpha+1,\dots,t\}| \leq 2\}$.

Then for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$ we have

$$L(B,\vec{x}) > L(A,\vec{x})/2^{r-2}.$$

Proof.

$$L(A,\vec{x}) \leq x_1^{r-2}x_{t-1}x_t \overset{(6.1)}{<} 2^{r-2}x_{t-3\alpha}^{r-2}x_{t-1}x_t \leq 2^{r-2}L(B,\vec{x}).$$

□

Now we are ready to conclude our main theorem.

Proof of Theorem 1.2. Let G, T and $\vec{x}$ be as defined in Section 2, in particular, (2.6) holds. By Lemma 5.1 this implies $T = t + C$ where $C < C_0(r)$. In this case, with the notation of Lemma 6.3, we can assume that $\mathcal{A} \neq \emptyset$, for otherwise G does not cover pairs, contradicting Proposition 2.7. Suppose now that $|\mathcal{B}| \geq 2^{r-2}|\mathcal{A}|$, and let $G' := (G \setminus \mathcal{A}) \cup \mathcal{B}$ (it does not matter that $e(G') > m$). Then Lemma 6.3 entails

$$\lambda(G') \geq L(G', \vec{x}) > L(G, \vec{x}) = \lambda(G).$$

However, G' does not cover any pair in $\{t-1, \dots, T\}^{(2)}$, therefore, by applying Proposition 2.3(i) $C+1$ times, we obtain $\lambda([t-1]^{(r)}) \geq \lambda(G') > \lambda(G)$, a contradiction. Thus,

$$|\mathcal{B}| < 2^{r-2}|\mathcal{A}| \leq 2^{r-2}\binom{C+2}{2} \cdot \binom{T-2}{r-2} \leq 2^{2r}C^2\binom{t-2}{r-2}.$$

Hence, we must have

$$\begin{aligned}\binom{t}{r} - m &= |[t]^{(r)}| - e(G) \leq |[t]^{(r)} \setminus G| \\ &\leq |\mathcal{B}| + |\{C \in [t]^{(r)} : \{t-1, t\} \subseteq C\}| + |\{D \in [t]^{(r)} : |\{t - 3\alpha + 1, \dots, t\} \cap D| \geq 3\}| \\ &\leq (2^{2r}C^2 + 1)\binom{t-2}{r-2} + O(t^{r-3}) \leq \gamma_r t^{r-2},\end{aligned}$$

as claimed. Thus the proof of Theorem 1.2 is completed. □

7. *A refinement for $T \leq t$*

Here we prove Theorem 1.5. In this section let G be a graph on $[t]$ with $e(G) = m$ and $\binom{t-1}{r} \leq m \leq \binom{t}{r} - \binom{t-2}{r-2}$, maximising the Lagrangian amongst all r-graphs on $[t]$ with m edges. Suppose that $\lambda(G) > \lambda(H^{m,r}) = \frac{1}{(t-1)^r}\binom{t-1}{r}$. Our aim is to show that then $m \geq \binom{t}{r} - \binom{t-2}{r-2} - \delta_r t^{r-9/4}$ for a constant $\delta_r > 0$.

Let $\vec{x}$ be a weighting attaining $\lambda(G)$; we can assume that $\vec{x}$ has exactly t non-zero entries (otherwise $\lambda(G) \leq \lambda([t-1]^{(r)})$, a contradiction) and that the entries of $\vec{x}$ are listed in descending order. It is a straightforward check that Propositions 2.7 and 2.8 (with $T = t$) remain valid for the just defined G and $\vec{x}$. Hence, analogously to Lemma 6.1 (with no need to establish Lemma 5.1) we obtain the following upper bound on x_1.

LEMMA 7.1. $x_1 < 1/(t-r+1)$.

Proof. We have

$$\begin{aligned}\frac{1}{(t-1)^r}\binom{t-1}{r} < L(G, \vec{x}) &\overset{(2.7),(2.5)}{\leq} \frac{1}{r}(1-x_1)^{r-1}\lambda(G_1) \leq \frac{1}{r}(1-x_1)^{r-1}\lambda([t-1]^{(r-1)}) \\ &\overset{(2.2)}{=} \frac{1}{r}\left(\frac{1-x_1}{t-1}\right)^{r-1}\binom{t-1}{r-1}.\end{aligned}$$

This translates to

$$(1-x_1)^{r-1} > \frac{t-r}{t-1}.$$

Since $(1-x_1)^{r-1}$ is a decreasing function of x_1, it suffices to verify that $(1 - 1/(t-r+1))^{r-1} < (t-r)/(t-1)$, or, equivalently, that $(t-r)^{r-2}(t-1) < (t-r+1)^{r-1}$. The latter inequality holds by AM-GM. □

Now, with hindsight, select an integer $k \sim t^{1/4}$.

LEMMA 7.2.

$$x_1 < \frac{k+1}{k} x_{t-(k+1)r}. \tag{7.1}$$

Proof. Suppose otherwise. Then

$$rkx_1 \geq r(k+1)x_{t-(k+1)r} \geq \sum_{i=t-(k+1)r+1}^{t} x_i, \tag{7.2}$$

and therefore

$$(t-r+1)x_1 > (t-r(k+1))x_1 + rkx_1 \overset{(7.2)}{\geq} \sum_{i=1}^{t-r(k+1)} x_i + \sum_{i=t-r(k+1)+1}^{t} x_i = \sum_{i=1}^{t} x_i = 1,$$

a contradiction. □

LEMMA 7.3. *Let*

$\mathcal{A} := \{A \in G\colon \{t-1, t\} \subseteq A\},$
$\mathcal{B} := \{B \in [t]^{(r)} \setminus G\colon |B \cap \{t-1,t\}| \leq 1$ *and* $|B \cap \{t-(k+1)r+1, \dots, t\}| \leq 2\}.$

Then for all $A \in \mathcal{A}$ *and* $B \in \mathcal{B}$ *we have*

$$L(B, \vec{x}) > L(A, \vec{x}) \cdot \left(\frac{k}{k+1}\right)^{r-2}.$$

Proof.

$$L(A, \vec{x}) \leq x_1^{r-2} x_{t-1} x_t \overset{(7.1)}{<} \left(\frac{k+1}{k}\right)^{r-2} x_{t-(k+1)r}^{r-2} x_{t-1} x_t \leq \left(\frac{k+1}{k}\right)^{r-2} L(B, \vec{x}).$$

□

Proof of Theorem 1.5. If $\mathcal{A} = \emptyset$ then G does not cover pairs, contradicting Proposition 2.7; so, we can assume that $\mathcal{A} \neq \emptyset$. Suppose that $|\mathcal{B}| \geq (\frac{k+1}{k})^{r-2}|\mathcal{A}|$, and let $G' := (G \setminus \mathcal{A}) \cup \mathcal{B}$ (it does not matter that $e(G') > m$). Then Lemma 7.3 entails

$$\lambda(G') \geq L(G', \vec{x}) > L(G, \vec{x}) = \lambda(G).$$

However, G' does not cover the pair $\{t-1, t\}$, therefore, by Proposition 2.3, $\lambda([t-1]^{(r)}) \geq \lambda(G') > \lambda(G)$, a contradiction. So

$$|\mathcal{B}| < \left(\frac{k+1}{k}\right)^{r-2} |\mathcal{A}| \leq \left(1 + \frac{r}{k}\right) |\mathcal{A}|.$$

Thus,

$$\begin{aligned}\binom{t}{r} - m &= |[t]^{(r)} \setminus G| \\ &\leq |\mathcal{B}| + |\{C \in [t]^{(r)} : \{t-1, t\} \subseteq C\} \setminus \mathcal{A}| \\ &\quad + |\{D \in [t]^{(r)} : |\{t-(k+1)r+1, \dots, t\} \cap D| \geq 3\}| \\ &\leq \left(1 + \frac{r}{k}\right)|\mathcal{A}| + \binom{t-2}{r-2} - |\mathcal{A}| + O(k^3 t^{r-3}) \\ &\leq \left(1 + \frac{r}{k}\right)\binom{t-2}{r-2} + O(k^3 t^{r-3}) \leq \binom{t-2}{r-2} + \delta_r t^{r-9/4}.\end{aligned}$$

The last inequality is the explanation for the choice of k. □

8. *Concluding remarks*

Closing the remaining gap of the Frankl-Füredi Conjecture is a challenging open problem even for $r = 3$. As remarked by Talbot in [**8**] (with emphasis on $r = 3$) the values of m split in two different regimes: $R_1 = \{m : \exists t \in \mathbb{N} : \binom{t-1}{r} \leq m \leq \binom{t}{r} - \binom{t-2}{r-2}\}$ and $R_2 = \mathbb{N} \setminus R_1$. This split is explained by the fact that in R_1 we have $\lambda(H^{m,r}) = \lambda([t-1]^{(r)})$, while in R_2 the value $\lambda(H^{m,r})$ 'jumps' with every increasing m. In the present paper we dealt solely with R_1, having confirmed Conjecture 1.1 for all but the $O(t^{r-2})$ largest values of m in each of its intervals (where the interval itself is of length $O(t^{r-1})$). We are wondering, if a version of our argument can be used to deduce $T \leq t$, in which case Theorem 1.5 will further reduce the above range as it already does for $r = 3$. With this said, it seems that both completely solving R_1 and tackling R_2 will require some new ideas.

Talbot's theorem for $r = 3$ applies also to small values of t. It would be interesting to find an argument that would confirm Conjecture 1.1 in the principal case for *all* $m = \binom{t}{r}$.

With regard to blow-up densities, we have determined, for all m specified by Theorem 1.2, the size of the asymptotically largest blow-up amongst all r-graphs of size m. We hope that this will find applications in hypergraph Turán problems, as this was the initial motivation behind considering the Lagrangian.

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Mykhaylo Tyomkyn
School of Mathematics
Tel Aviv University
Tel Aviv 69978
Israel

tyomkynm@post.tau.ac.il