

CHAPTER 1

Gradient flow reaction/diffusion models in phase transitions

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In this article we consider a nonlinear large reaction small diffusion problem which has two (or more) stable states, and we analyse it using two different methods.

- First method: an approach based on (asymptotic) expansions;
- and second method: an approach based on the notion of Γ -convergence.

The analysis of such a problem shows that the two methods are complementary. It is well-known that, for such a problem, time-dependent solutions are characterised by (moving) layers or vortices. Here, we are specially interested in the existence, the shape and the motion of such layers or vortices, with respect to the inhomogeneous coefficients appearing in the problem as well as the domain Ω . We generalise, in the limit of small diffusion, the usual motion by mean curvature laws found for homogeneous problems.

1. Introduction

The general form of the problem we are interested in is

$$(\mathcal{P}) \quad \begin{cases} \epsilon^2 \frac{\partial \mathbf{w}}{\partial t} = \epsilon^2 \operatorname{div}(k \nabla \mathbf{w}) + W_{\mathbf{w}}(x, \mathbf{w}) & \text{for all } x \in \Omega \text{ and } t > 0, \\ \frac{\partial \mathbf{w}}{\partial n} = 0 & \text{for all } x \in \partial\Omega \text{ and } t > 0, \\ \mathbf{w}(x, 0) = \psi(x) & \text{for all } x \in \Omega, \end{cases}$$

where the function W is a multi-well potential, $\mathbf{w} = \mathbf{w}(x, t)$ is a function of $x \in \Omega \subset \mathbb{R}^n$, $n \in \{1, 2\}$, and $t \in \mathbb{R}^+$.

Structure and Results

Let us first consider the simple scalar problem

$$(\mathcal{P}_{scal}) \quad \begin{cases} \epsilon^2 w_t = \epsilon^2 (kw_x)_x + f^2 w (g^2 - w^2) & \text{for all } x \in \Omega = (a, b) \subset \mathbb{R} \\ & \text{and } t > 0, \\ w_x(a, t) = w_x(b, t) = 0 & \text{for all } t > 0, \\ w(x, 0) = \psi(x) & \text{for all } x \in (a, b), \end{cases}$$

where the functions $f = f(x)$, $g = g(x)$, $k = k(x)$ and $\psi(x)$ are smooth. We take f, g, k strictly positive in $[a, b]$, and are interested in the case when ψ takes both signs in (a, b) . The potential $W(x, w)$ is equal to $\frac{f^2}{2}(g^2 - w^2)^2$ up to a term independent of w , and has two wells of equal depth located at $w = \pm g$.

Problem (\mathcal{P}_{scal}) has been considered by many mathematicians, see [2, 6, 7, 9] for instance. Traditionally their analyses are based on one of the two following approaches.

Approach based on the energy functional

The usual achievement of this method is the following result. (Note that for the sake of simplicity we provide in this introduction only a very restricted version. A more general version with weaker hypotheses is given at the beginning of Section 5.)

Assuming that

- the strictly positive functions $f, g, k \in C^\infty([a, b])$;
- g satisfies the homogeneous Neumann boundary conditions $g_x(a) = g_x(b) = 0$;
- ϵ is a small positive constant;
- w_0 , of the form $w_0 = \pm g\chi_{(a, c_1)} \pm g\chi_{(c_1, c_2)} \pm \dots \pm g\chi_{(c_d, b)}$ for a fixed $d \in \mathbb{N}$ and $a < c_1 < \dots < c_d < b$, is an isolated local minimiser of

$$E_0(\tilde{c}_1, \dots, \tilde{c}_d) = \frac{2}{3} \sum_{i=1}^d \sqrt{k(\tilde{c}_i)} f(\tilde{c}_i) g(\tilde{c}_i)^3$$

with respect to \tilde{c}_i , $i \in \{1, \dots, d\}$;

then there exist $\epsilon_0 > 0$ and a family $(w_\epsilon)_{0 < \epsilon < \epsilon_0}$ of steady solutions of Problem (\mathcal{P}_{scal}) such that for each $\epsilon \in (0, \epsilon_0)$

- $w_\epsilon \in C^2(a, b)$;
- $\|w_\epsilon - w_0\|_{L^1(a, b)} \rightarrow 0$ as $\epsilon \rightarrow 0$;
- w_ϵ is an asymptotically stable solution of Problem (\mathcal{P}_{scal}) , i.e. for any $\rho > 0$ small enough, there exists $\delta > 0$ such that for any $\psi \in C^2(a, b)$ with $\|\psi - w_\epsilon\|_{H^1(a, b)} < \delta$ we have

$$\|T(t)\psi - w_\epsilon\|_{H^1(a, b)} < \rho, \quad 0 < t < \infty.$$

Here $T(t)\psi$ denotes the solution of Problem (\mathcal{P}_{scal}) with $T(0)\psi = \psi$ as initial condition (satisfying the boundary conditions).

Moreover there exists a value $\tilde{\delta} > 0$ such that

$$\|\psi - w_\epsilon\|_{H^1(a, b)} < \tilde{\delta} \text{ implies that } \|T(t)\psi - w_\epsilon\|_{H^1(a, b)} \rightarrow 0 \text{ as } t \rightarrow \infty.$$

The steady solution w_ϵ of Problem (\mathcal{P}_{scal}) is an isolated local minimiser (in the Sobolev space $H^1(a, b)$) of the energy functional

$$E_\epsilon(w) = \int_a^b \frac{\epsilon}{2} k w_x^2 + \frac{1}{2\epsilon} f^2(w^2 - g^2)^2 dx,$$

which Γ -converges to E_0 as $\epsilon \rightarrow 0$.

The energy functional decreases (as time increases) on solutions $w(x, t) = T(t)\psi$ of the time-dependent problem. This is a gradient flow, and solutions decrease energy at the maximum rate by means of the relation

$$-\frac{\epsilon}{2} \|w_t\|_{L^2(\Omega)}^2 = DE_\epsilon(w)(w_t),$$

where DE_ϵ stands for the usual (variational) derivative of the energy functional E_ϵ . In this sense, the energy acts as a Lyapounov function for this dynamical system; the solution is quickly attracted to a slow manifold, and we wish to describe this slow manifold in the limit $\epsilon \rightarrow 0$.

Approach based on asymptotics

A serious drawback in the previous approach consists in the lack of information concerning the solution of Problem (\mathcal{P}_{scal}) that we denoted $T(t)\psi$, where ψ is the initial condition, and especially concerning the slow manifold mentioned above. Obviously, the solution $T(t)\psi$ depends on ψ . Nevertheless, for $\epsilon > 0$ small enough, it has been observed that the behaviour of the

time-dependent solution is roughly made of two distinct phases.

- The first phase is the evolution on a fast timescale of the solution towards a solution with "layer behaviour".
- The second phase is the motion of this (these) layer(s) towards a steady stable location (this is the slow manifold behaviour).

To illustrate this, let us consider Problem (\mathcal{P}_{scal}) with $(a, b) = (-4, 4)$, $\epsilon = \frac{1}{\sqrt{10}}$, $g(x) = \frac{9}{10} - \frac{1}{10} \cos(\frac{\pi}{4}x)$, $\psi(x) = g(x) \tanh(\frac{x+1}{2})$, and $f \equiv k \equiv 1$. The solution is plotted in Figure 1. We clearly observe the formation of a layer in the time interval $(0, \bar{t})$ with $\bar{t} \sim \frac{1}{5}$, and then the motion of this layer towards its stable location (here at $x = 0$) in the time interval (\bar{t}, ∞) . We also observe that even if the first locations where layers occur depend

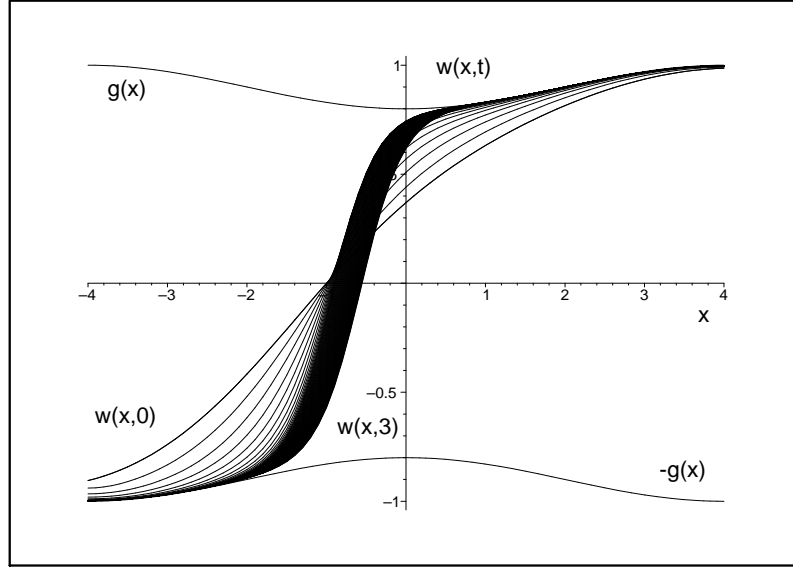


Fig. 1. Solution $w(x, t)$ of Problem (\mathcal{P}_{scal}) with $(a, b) = (-4, 4)$, $\epsilon = \frac{1}{\sqrt{10}}$, $g(x) = \frac{9}{10} - \frac{1}{10} \cos(\frac{\pi}{4}x)$, $\psi(x) = g(x) \tanh(\frac{x+1}{2})$, and $f \equiv k \equiv 1$. The curves in this figure are $w(x, 0), w(x, \frac{1}{25}), w(x, \frac{2}{25}), \dots, w(x, 3)$.

strongly on the initial condition, the initial condition does (almost) not influence the motion of the layers from the first location where they occur to their stable steady limit (if they have one). In Figure 1, the particular

initial condition $\psi(x) = g(x) \tanh(\frac{x+1}{2})$ creates a layer located at $x = -1$ which moves towards its stable steady limit located at $x = 0$.

As a consequence, for an isolated single layer, for $\epsilon > 0$ small enough we can model the second phase of the solution by an expansion of the form

$$w(x, t) = g(x) \tanh(\xi) + o(\epsilon^0),$$

where

$$\xi = \xi(x, t, \epsilon) = \frac{1}{\sqrt{2}\epsilon} \int_{S(t)}^x \frac{f(s)g(s)}{\sqrt{k(s)}} ds$$

is the fast spacescale, and where $S(t)$ (the zero of w , which corresponds to the location of the layer), satisfies

$$\frac{dS(t)}{dt} = -k(S(t)) \frac{d}{dx} \ln(\sqrt{k}fg^3)|_{x=S(t)} + o(\epsilon^0).$$

(Here $o(\epsilon^0)$ stands for terms vanishing in the limit $\epsilon \rightarrow 0$.)

Remark

When considering the method of (asymptotic) expansions for small values of ϵ , we commonly refer to problems stated without any initial conditions. This is to stress the fact that we consider the evolution of a layer once it has occurred (even if, from a purely mathematical point of view, such problems would be completely and correctly defined only with suitable initial conditions stated). The fact that a layer exists, and its location, imply that initial data suitable to form such a layer was involved.

Remark

From a purely abstract mathematical point of view, it is rather technical to give a precise definition of a layer. In the sequel, we consider that a layer is a fast transition between the neighbourhood of two stable states (also called phases). (For instance, in the example given in Figure 1, the layer is the fast transition between the neighbourhood of $-g(x)$ and that of $g(x)$.) As $\epsilon \rightarrow 0$, a solution (divided by g) with a layer converges in $L^1(\Omega)$ to a combination of indicator functions of sets with reasonable boundaries.

Let us now consider Problem (\mathcal{P}) , that is a generalised version of (\mathcal{P}_{scal}) . For the sake of simplicity, we assume that $\Omega \subset \mathbb{R}^2$. (The case $\Omega \subset \mathbb{R}$ can be deduced easily.)

Depending on the following characteristics of Problem (P), solutions w will be markedly different.

- The dimension of $\mathbf{w}(x, t)$ (either scalar or vector);
- the inhomogeneities of W (i.e. the dependence of W on x);
- the number of wells (possibly not finite);
- the depth of the wells.

In the next sections we cover, by developing both approaches (energy functional and expansions) as far as possible, some of the possible different combinations through one example of wells of the same depth. The case of wells of different depth does not create any difficulties, but is different. Indeed, if the difference of the depth of the wells does not vanish as $\epsilon \rightarrow 0$, then the motion takes place on another timescale and the use of the Γ -limit is no longer relevant.

We consider a vector problem (i.e. $\mathbf{w}(x, t)$ is a vector), of dimension 2: $\mathbf{w}(x, t) = (u(x, t), v(x, t))$

$$(\mathcal{P}_{vect}) \quad \begin{cases} \frac{\epsilon^2}{2} u_t = \frac{\epsilon^2}{2} \operatorname{div}(k \nabla u) + f^2 u (g^2 - u^2 - \alpha v^2), & \text{for all } x \in \Omega, t > 0, \\ \frac{\epsilon^2}{2} v_t = \frac{\epsilon^2}{2} \operatorname{div}(k \nabla v) + f^2 v (g^2 - v^2 - \alpha u^2), & \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 & \text{for all } x \in \partial\Omega, t > 0. \end{cases}$$

The factor $\frac{1}{2}$ has been added to simplify the notation later. Another reason is that the "shape" of the layer (i.e. \tanh in the expansion of a solution of Problem (\mathcal{P}_{scal}) given above), satisfies $\frac{1}{2} \tanh'' + \tanh(1 - \tanh^2) = 0$, which is the canonical equation for the interface or layer behaviour for this problem.

For $\alpha \in (0, 1)$ constant, the potential corresponding to Problem (\mathcal{P}_{vect}) has four wells of equal depth $\left(\pm \frac{g}{\sqrt{1+\alpha}}, \pm \frac{g}{\sqrt{1+\alpha}}\right)$. First we have the simple case where one component (either u or v) has a layer but the other does not. For interesting f, g, k we see that either the layer disappears in finite time, or that the layer is attracted to a stable steady limit.

The most interesting (and complicated) situation occurs when both components u and v have a layer and that both layers are forced by the x -dependent coefficients k, f and g to move towards the same location. The difficulty comes from the fact that the two layers (locally) repel each other. (Here and later "locally" means sufficiently close given a fixed $\epsilon > 0$.)

Indeed, it can be shown that the configuration which consists of two layers located exactly at the same place, is unstable. Therefore, for the limiting steady solution, we have a trade-off between the fact that the coefficients k, f and g force the layers to move towards a common place and the fact that they locally repel each other. See the following example for an illustration.

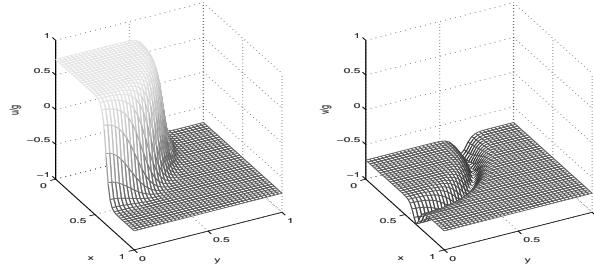
Example

In the case $\Omega = (0, 1) \times (0, 1)$, $\alpha = 0.9$, $\epsilon = \frac{1}{200}$, $f \equiv k \equiv 1$, and

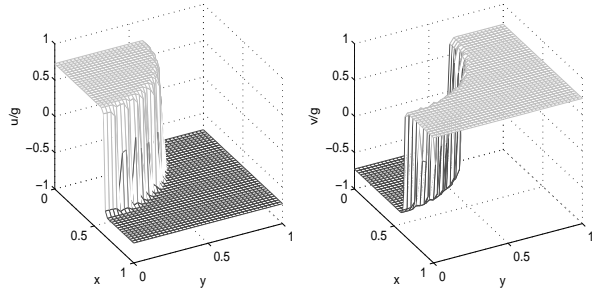
$$g(x, y) = \begin{cases} 1 - 3 \cosh(\pi r)^2 \exp(-\frac{1}{1-r}) \exp(-\frac{1}{r}) & \text{if } r := \sqrt{x^2 + y^2} \leq 1; \\ 1 & \text{otherwise,} \end{cases}$$

we find four steady solutions for Problem (\mathcal{P}_{vect}) .

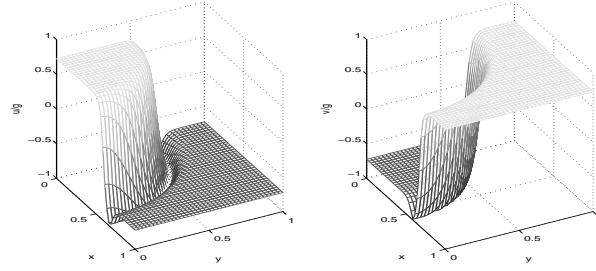
First and second (stable) solutions, the latter by interchanging u and v :
(note that v has a "ridge" where u has a layer)



third (unstable) solution:
(here $u = -v$)



fourth (stable) solution:
(here u and v both have layers and nearby ridges)



In the approach based on expansions for small values of $\epsilon > 0$, this difficulty (of the competition between larger scale attraction and local repulsion of pairs of layers) appears in the following fact that what is a stable local minimiser under the constraint $u = \pm v$ turns out to be a saddle point without any constraint, with additional local minimisers created. More technically, there is no solution to the canonical equations that arise for the double interface

$$\frac{1}{2}U_0\xi\xi + U_0(1 - U_0^2 - \alpha V_0^2) = 0 \quad \text{and} \quad \frac{1}{2}V_0\xi\xi + V_0(1 - V_0^2 - \alpha U_0^2) = 0,$$

for $-\infty < \xi < \infty$, such that $V_0(\xi) = U_0(-\xi)$ for all $\xi \in \mathbb{R}$, that

$$\lim_{\xi \rightarrow \infty} U_0(\xi) = - \lim_{\xi \rightarrow -\infty} U_0(\xi) = \frac{1}{\sqrt{1 + \alpha}},$$

and such that the unique zero of U_0 is different from 0. Thus to model a situation as the one illustrated in Figure 2 (right panel), we can only use for the "shape" of a layer, instead of \tanh , the solution $(U_0, V_0) = (\mathcal{U}, \mathcal{V})$ illustrated in Figure 2 (left panel) of the canonical equations. But the boundary conditions are

$$\lim_{\xi \rightarrow \infty} \mathcal{U}(\xi) = - \lim_{\xi \rightarrow -\infty} \mathcal{U}(\xi) = \lim_{\xi \rightarrow \infty} \mathcal{V}(\xi) = \lim_{\xi \rightarrow -\infty} \mathcal{V}(\xi) = \frac{1}{\sqrt{1 + \alpha}},$$

and when matching two expansions (one in the neighbourhood of $S^u(t, \epsilon)$ and one in the neighbourhood of $S^v(t, \epsilon)$), we realise that the distance between the two limiting steady curves $S_\infty^u(\epsilon) = \lim_{t \rightarrow \infty} S^u(t, \epsilon)$ and $S_\infty^v(\epsilon) = \lim_{t \rightarrow \infty} S^v(t, \epsilon)$ must be of larger order in ϵ than the order of

the width of a layer (i.e. $\mathcal{O}(\epsilon)$). This indicates that very accurate expansions for the time-independent and time-dependent solutions will be needed. Especially we note that the accurate expansion for the time-independent problem can be used to get a more explicit expression for the Γ -limit of the associated energy functional than what we can obtain directly from a result on the relaxation of functionals given by Fonseca and Müller³.

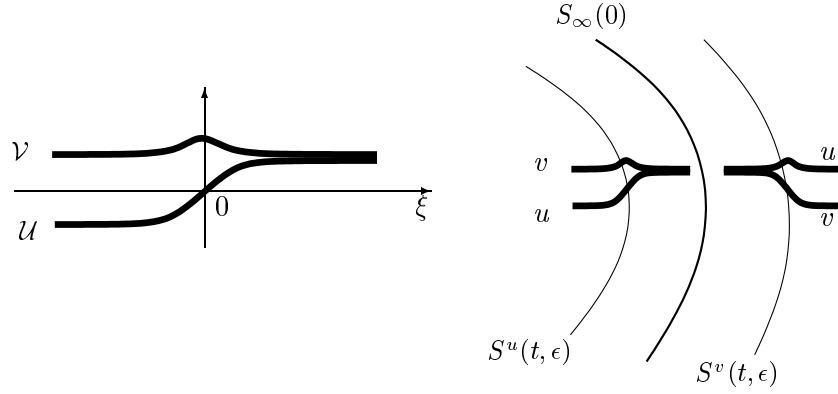


Fig. 2. Scheme of the two matched expansions in the case of two colliding layers. $S^u(t, \epsilon)$ (resp. $S^v(t, \epsilon)$) represents the nodal curve of $u(x, t)$: $\{x \in \Omega : u(x, t) = 0\}$ (resp. $v(x, t)$: $\{x \in \Omega : v(x, t) = 0\}$). $S_\infty(0)$ represents the common location of both nodal curves in the limit $t \rightarrow \infty$ and $\epsilon \rightarrow 0$.

- In Section 2 we consider the canonical equations (i.e. the steady one space dimensional homogeneous version of Problem (\mathcal{P}_{vect})). The existence and the properties of new (families of) solutions are discussed. The lack of a better solution than $(\mathcal{U}, \mathcal{V})$ to model two repelling layers, as well as the existence and properties of $(\mathcal{U}, \mathcal{V})$, are also commented upon.

- In Section 3 we provide an extension of Norbury and Yeh's expansion, see [7,8,10], together with another expansion based on the particular coordinate $z(x, t)$ defined below in (1). These two different expansions illustrate the trade-off we face between the lack of smoothness (of the first two leading terms), the lack of information as $t \rightarrow \infty$, and the limitation of the domain of validity in Ω of such (asymptotic) expansions as $\epsilon \rightarrow 0$. More precisely, for Problem (\mathcal{P}_{scal}) for instance, the extended version of Norbury

and Yeh's expansion for the time-independent solutions is such that

- either the expansion is valid only in a neighbourhood of the nodal curve $S(t)$ of bandwidth $\mathcal{O}(\epsilon |\ln \epsilon|)$;
- or the expansion is valid in the entire domain Ω (without matching), but has a lack of smoothness. The second term in the expansion $w(x) = g(x) \tanh(\xi) + \epsilon U_1(\xi, x) + o(\epsilon)$, where $\xi = \xi(x, \epsilon)$ is the fast length-scale, is such that

$$\frac{\epsilon^2}{2} \left[\operatorname{div}(k \nabla \epsilon U_1(\xi(x, \epsilon), x)) \right]_{S_{\infty}^-(\epsilon)}^{S_{\infty}^+(\epsilon)} = \mathcal{O}(\epsilon^2).$$

The only smooth expansion such that the leading order term is $\mathcal{O}(\epsilon)$ -accurate uniformly in the entire domain Ω is the following second expansion, corresponding only to time-dependent solutions:

$$w(x, t) = g(x) \tanh \left(\frac{f(x)g(x)}{\sqrt{k(x)}|\nabla z|} \xi \right) + \mathcal{O}(\epsilon),$$

where $\xi = \xi(x, t, \epsilon) = \frac{z(x, t)}{\epsilon}$, and where z satisfies

$$\frac{z_t}{|\nabla z|} = \frac{\sqrt{k}}{fg^3} \operatorname{div} \left(\sqrt{k} f g^3 \frac{\nabla z}{|\nabla z|} \right). \quad (1)$$

(As $\operatorname{div} \left(\frac{\nabla z}{|\nabla z|} \right)$ is the mean curvature of the level curve of z and as $\frac{z_t}{|\nabla z|}$ is the normal velocity of such a level curve with respect to the normal $-\frac{\nabla z}{|\nabla z|}$, this corresponds to an extended motion by mean curvature law where the motion is influenced by the inhomogeneous terms $\sqrt{k}fg^3$ and \sqrt{k} .)

Comparisons and numerical computations show the necessity of considering both expansions.

- In Section 4 we develop the approach based on the energy functional. We first give the main theoretical results we need, and then we give the Γ -limit of the energy functional associated with the steady version of Problem (\mathcal{P}_{vect}) . This gives us a necessary condition for a curve to be the limit, as $\epsilon \rightarrow 0$, of a steady nodal curve.

The structure of this Γ -convergence theory enables us to avoid the analysis of the differences between the two possibilities in the first expansion, but then does not allow us to analyse properly all the details of the solutions as $\epsilon \rightarrow 0$. The approach based on the (asymptotic) expansion of the solutions

gives a complete understanding of the problem for solutions quickly forming one or several layers, their motion being perfectly well characterised, whereas the theory based on the Γ -convergence idea justifies formally the results obtained with the expansions. The main results attained with the Γ -convergence theory for steady problems are formal proofs of existence of solutions under some basic assumptions, as well as the convergence (in $L^1(\Omega)$) of these solutions to a limit in $BV(\Omega)$ (the Banach space of bounded variation functions on Ω) as $\epsilon \rightarrow 0$.

- Finally, in Section 5 we extend the results of Section 4 to time-dependent problems, to get laws for the motion of layers. It is unusual to use the notion of Γ -limit for time-dependent problems. Nevertheless this gives interesting results.

Remark

In the case $\alpha = 1$, the potential W admits a circle $\{u^2 + v^2 = g^2\}$ as the set of local minimisers. Layers do not occur anymore, but vortices occur, say at a_1, \dots, a_d . (A vortex is a point singularity and we say that a solution (u_ϵ, v_ϵ) admits a vortex at $a_i \in \Omega$ if, in the limit $\epsilon \rightarrow 0$, the solution (u_0, v_0) is twice continuously differentiable in a neighbourhood of a_i excluding a_i , that $u_0^2 + v_0^2 = g^2$ holds for all x in this neighbourhood (except at $x = a_i$), and that the winding number of (u_0, v_0) around a_i is different from zero.) Similarly to what has been done for the layers, by giving an expansion for small values of $\epsilon > 0$ of a solution with vortices, and using this expansion with the energy functional approach, the motion of distinct vortices can be computed with respect to the coefficients f, g and k . We get, for $a_i = a_i(t)$,

$$\frac{da_i}{dt} = -k\nabla(kg^2)|_{x=a_i}, \quad i = 1, \dots, d.$$

Note that we also have a trade-off here, as two vortices a_i, a_j , $i \neq j$, can be forced to move towards a common location by the coefficients k, f, g , but repel each other locally. (The repelling effect for vortices when f, g, k are constant has been characterised by Bethuel, Brezis and Hélein¹.)

2. Canonical Equations

Let us consider the canonical problem:

$$(\mathcal{P}_\alpha) \quad \begin{cases} \frac{1}{2}u_{xx} + u[1 - u^2 - \alpha v^2] = 0, \\ \frac{1}{2}v_{xx} + v[1 - v^2 - \alpha u^2] = 0, \\ \lim_{x \rightarrow \pm\infty} u(x), v(x) = 0. \end{cases} \text{ for all } x \in (-\infty, \infty),$$

This is a perturbed Ginzburg-Landau system (or two coupled Allen-Cahn equations), in which all the coefficients are constant and which is defined over the entire real line $\Omega = \mathbb{R}$.

Any solution (u, v) of Problem (\mathcal{P}_α) is in fact $C^\infty(-\infty, \infty) \times C^\infty(-\infty, \infty)$. Moreover, if (u, v) is a solution of Problem (\mathcal{P}_α) , then

$$(u, v), \quad (u, -v), \quad (-u, v), \quad (-u, -v)$$

are solutions, as well as

$$(u(-x), v(-x)) \quad \text{and} \quad (u(x+c), v(x+c)),$$

where $c \in \mathbb{R}$ is a constant. Thus we introduce the notion of family of solutions.

Definition

By a family of solutions we mean a set of solutions $\{(u_j, v_j), j \in J\}$ such that any solution (u_{j_0}, v_{j_0}) can be mapped into any other solution (u_{j_1}, v_{j_1}) of the same family by using one or more of the mappings: reflection with respect to the x -axis or the y -axis, exchange of the names u and v , and shift along the x -axis.

In the sequel, we shall identify a family with one of its representatives.

The analysis of this problem provides necessary results for the understanding of the more general case and shows different numerical solutions such as those illustrated in Figures 3 and 4.

Remark

When we refer to the first, second and third families of solutions, we point out that we don't have any list of all existing solutions. The question of a classification of all solutions with respect to a particular characteristic (such as the total number of zeros of u and v for instance), remains an open issue.

Shape of a layer

Problem (\mathcal{P}_α) admits a family of solutions for which one of its representa-

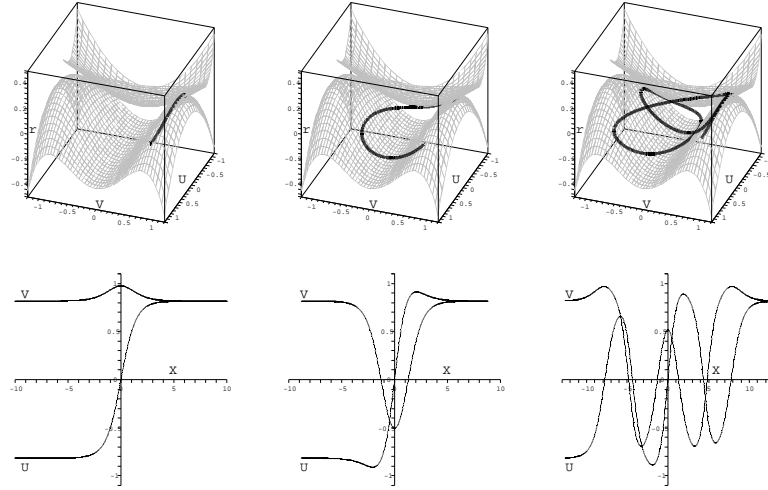


Fig. 3. First, second and third families of solutions, for $\alpha = \frac{1}{2}$, in the (u, v, r) -space (upper panels) and in the usual plane (lower panels). The surfaces represent forbidden zones, and the dark curves orbits of solutions that begin and end as $x \rightarrow \pm\infty$. (The third variable r is defined up to a constant by $r_x = \frac{1}{2}(u^2vv_x - v^2uu_x)$.)

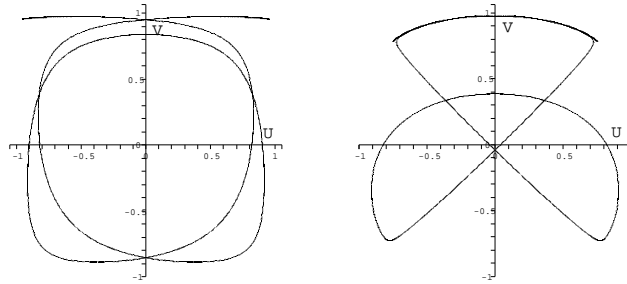


Fig. 4. Third (family of) solution(s) for $\alpha = 0.1$ (left) and $\alpha = 0.688$ (right), plotted in the (u, v) -plane.

tives, say $(\mathcal{U}, \mathcal{V})$, satisfies

$$\mathcal{U}(x) \leq 0 \text{ for } x \leq 0 \text{ and } \mathcal{U}(x) \geq 0 \text{ for } x \geq 0,$$

$$\mathcal{V}(x) \geq 0 \text{ for all } x \in \mathbb{R},$$

$$\lim_{x \rightarrow \pm\infty} \mathcal{U}(x) = \pm \frac{1}{\sqrt{1+\alpha}}, \quad \lim_{x \rightarrow \pm\infty} \mathcal{V}(x) = \frac{1}{\sqrt{1+\alpha}}.$$

$(\mathcal{U}, \mathcal{V})$ also satisfies the equality

$$\frac{1}{4}(\mathcal{U}_x^2 + \mathcal{V}_x^2) = \frac{1}{4}(\mathcal{U}^4 + \mathcal{V}^4) - \frac{1}{2}(\mathcal{U}^2 + \mathcal{V}^2) + \frac{\alpha}{2}\mathcal{U}^2\mathcal{V}^2 + \frac{1}{2(1+\alpha)}. \quad (2)$$

See Figure 5 for an illustration of \mathcal{U} and \mathcal{V} when $\alpha = 0.1, 0.5$ and 0.9 .

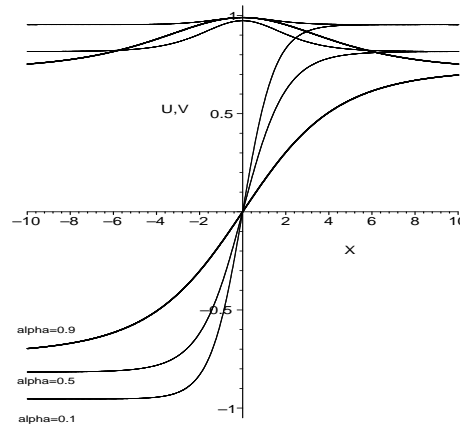


Fig. 5. First family of solutions for $\alpha = 0.1, 0.5$ and 0.9 .

Remark

There is no solution to Problem (\mathcal{P}_α) such that

$$\frac{1}{\sqrt{1+\alpha}} = -\lim_{x \rightarrow -\infty} u(x) = \lim_{x \rightarrow \infty} u(x),$$

$v(x) = u(-x)$, and such that the unique zero of u is different from 0.

(The observation from Norbury and Yeh's result⁸, tells us that if such a (stable) solution did exist, then it should be such that $v_x(S^u) = 0$, but this would mean that u (resp. v) is odd (resp. even) about $x = S^u$, which is in contradiction with the fact that v has only one zero distinct from S^u .)

Then we get for the time-dependent problem the following result.

Travelling waves

The solutions of the time-dependent problem

$$\begin{aligned} \frac{1}{2}u_t &= \frac{1}{2}u_{xx} + u(1 - u^2 - \alpha v^2), \\ \frac{1}{2}v_t &= \frac{1}{2}v_{xx} + v(1 - v^2 - \alpha u^2), \end{aligned}$$

for all $x \in \mathbb{R}$, $\alpha \in (0, 1)$, with boundary conditions

$$\lim_{x \rightarrow \pm\infty} u(x)^2, v(x)^2 = \frac{1}{1 + \alpha},$$

and starting with the initial conditions

$$\begin{pmatrix} u(x, 0) \\ v(x, 0) \end{pmatrix} = \begin{pmatrix} \mathcal{U}(x - S^u(0)) \\ \mathcal{V}(x - S^u(0)) \end{pmatrix} + \begin{pmatrix} \mathcal{V}(S^u(0) - x) \\ \mathcal{U}(S^u(0) - x) \end{pmatrix} - \begin{pmatrix} \frac{1}{\sqrt{1+\alpha}} \\ \frac{1}{\sqrt{1+\alpha}} \end{pmatrix},$$

where $0 < -S^u(0) = S^v(0)$ is large, behave like travelling waves for large t , and

$$\lim_{t \rightarrow \infty} -S^u(t) = \lim_{t \rightarrow \infty} S^v(t) = \infty.$$

This is what we call hereafter the "repelling effect".

3. Expansions

Notation

Let $\gamma_{x,p}$ stand for a curve with endpoints x and p . We use the metric $\frac{fg}{\sqrt{k}}$ in the sense that we define the signed distance between two points x and p , computed along the (oriented) curve $\gamma_{x,p}$, by

$$d_{\gamma_{x,p}}(x, p) = \int_{\gamma_{x,p}} \frac{fg}{\sqrt{k}} ds,$$

where s stands for the arc length. We define the minimal positive distance $d(x, p)$ between two points x and p as the minimum of the positive distance $|d_{\gamma_{x,p}}(x, p)|$ over all possible curves $\gamma_{x,p}$ starting at x and ending at p . For any point x and smooth curve S , we denote by $p(x, S)$ the projection assumed unique of x on S , i.e. the point of S whose positive distance to x is the least. (The results don't change if the uniqueness does not hold and the precise formulation then only adds technical details.) We denote the signed distance between x and S by $d(x, S)$ (such that $|d(x, S)| = d(x, p(x, S))$). We also define the two curves \hat{c} and \check{c} as follows. For S and a particular point $p \in S$, $\hat{c}(p, S)$ stands for the set of points x whose projection on S is p , i.e.

$$\hat{c}(p, S) := \{x \in \Omega : p(x, S) = p\},$$

and \check{c} stands for the parallel curve to S passing through x , i.e.

$$\check{c}(x, S) := \{y \in \Omega : d(y, S) = d(x, S)\}.$$

In order to simplify our integrations later, we introduce, for $p \in S$ and $x \in \hat{c}(p, S)$, the notation $\hat{c}_S(p, x)$ denoting the part of the curve $\hat{c}(p, S)$ located between the two points p and x . Finally $S(t, \epsilon)$ (with $S_\infty(\epsilon) := \lim_{t \rightarrow \infty} S(t, \epsilon)$), stands for the nodal curve of a solution (i.e. the set of zeros of w for Problem (\mathcal{P}_{scal})), $n = n(x, t, \epsilon)$ stands for the unit normal to the curve $\check{c}(x, S(t, \epsilon))$, $\nu = \nu(x, t, \epsilon)$ for the normal velocity of $\check{c}(x, S(t, \epsilon))$, and $\kappa = \kappa(x, t, \epsilon)$ for the mean curvature of the curve $\check{c}(x, S(t, \epsilon))$. (For the orientation, n is the outward unit normal if $\check{c}(x, S(t, \epsilon))$ is closed, or either of the two possible orientations otherwise. The normal velocity ν refers to n , as well as the mean curvature $\kappa = \text{div}(n)$.) See Figure 6 for an illustration.

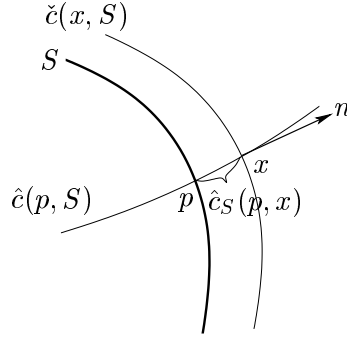


Fig. 6. Illustration of the notation. If $S = S(t, \epsilon)$, then $n(x, t, \epsilon)$ corresponds to n in this figure.

Then the following two expansions hold.

First expansion

A solution $(u(x, t), v(x, t))$ of Problem (\mathcal{P}_{vect}) , with a single layer for u , admits an expansion of the form

$$\begin{aligned} u(x, t) &= \pm g(x) \mathcal{U}(\xi) + \underbrace{\epsilon U_1(\xi, x, t) + o(\epsilon)}_{R_u(\xi, x, t, \epsilon)}, \\ v(x, t) &= \pm g(x) \mathcal{V}(\xi) + \underbrace{\epsilon V_1(\xi, x, t) + o(\epsilon)}_{R_v(\xi, x, t, \epsilon)}, \end{aligned}$$

where

$$\xi = \xi(x, t, \epsilon) = \frac{1}{\epsilon} \int_{\hat{c}_{S^u(t, \epsilon)}(p(x, S^u(t, \epsilon)), x)} \frac{fg}{\sqrt{k}} ds,$$

and

$$\nu_u = -k \left(\frac{\partial}{\partial n} \ln(\sqrt{k} f g^3) + \kappa_u \right) \Big|_{x \in S^u(t, \epsilon)} + o(1). \quad (3)$$

Here

- either the functions $U_1(\cdot, x, t), V_1(\cdot, x, t)$ are unbounded, but under the assumption that $d(x, S^u(t, \epsilon)) = \mathcal{O}(\epsilon |\ln \epsilon|)$ we have $R_{u,v} = \mathcal{O}(\epsilon)$ with the accuracy in (3) being $\mathcal{O}(\epsilon |\ln \epsilon|)$ instead of $o(1)$;
- or the functions $U_1(\cdot, x, t), V_1(\cdot, x, t)$ are bounded, $R_{u,v}(\xi, x, t, \epsilon) = \mathcal{O}(\epsilon)$, the accuracy in (3) is $\mathcal{O}(\epsilon)$ instead of $o(1)$ but $U_1(\xi(x, t, \epsilon), x, t), V_1(\xi(x, t, \epsilon), x, t) \in C^2(\Omega \setminus S^u(t, \epsilon)) \cap C^1(\Omega)$ for all $t > 0$, and the jump of the second derivative is of order ϵ^2 :

$$\begin{aligned} \frac{\epsilon^2}{2} \left[\operatorname{div}(k \nabla \epsilon U_1(\xi(x, t, \epsilon), x, t)) \right]_{S^{u-}(t, \epsilon)}^{S^{u+}(t, \epsilon)} &= \mathcal{O}(\epsilon^2), \\ \frac{\epsilon^2}{2} \left[\operatorname{div}(k \nabla \epsilon V_1(\xi(x, t, \epsilon), x, t)) \right]_{S^{u-}(t, \epsilon)}^{S^{u+}(t, \epsilon)} &= \mathcal{O}(\epsilon^2). \end{aligned}$$

This first expansion provides in general a good approximation of the solutions, especially for the steady solutions. Nevertheless, it is natural to look for a smooth and accurate expansion. The next expansion provides an answer to this issue.

Second expansion

Solutions of Problem (\mathcal{P}_{vect}) , with a single layer for u , admit an expansion of the form

$$\begin{aligned} u(x, t, \epsilon) &= g(x) U_0(\xi, x, t) + \epsilon U_1(\xi, x, t) + \mathcal{O}(\epsilon^2), \\ v(x, t, \epsilon) &= g(x) V_0(\xi, x, t) + \epsilon V_1(\xi, x, t) + \mathcal{O}(\epsilon^2), \end{aligned}$$

where

$$\begin{aligned} U_0(\xi, x, t) &= \pm \mathcal{U} \left(\frac{f(x)g(x)}{\sqrt{k(x)}|\nabla z|} \xi \right), \\ V_0(\xi, x, t) &= \pm \mathcal{V} \left(\frac{f(x)g(x)}{\sqrt{k(x)}|\nabla z|} \xi \right), \\ \xi &= \xi(x, t, \epsilon) = \frac{z(x, t)}{\epsilon}, \end{aligned}$$

and where $z(x, t)$ is defined (implicitly) by

$$\frac{z_t}{|\nabla z|} = \frac{\sqrt{k}}{fg^3} \operatorname{div} \left(\sqrt{k} f g^3 \frac{\nabla z}{|\nabla z|} \right), \quad (4)$$

together with some initial conditions $z(x, 0) = z_{init}(x)$ for all $x \in \Omega$ and some boundary condition $z(x, t) = z_{bc}(x, t)$ for all x belonging to a specified subset of $\partial\Omega$ and all $t > 0$. The functions $U_1(\xi, x, t), V_1(\xi, x, t)$ are uniformly bounded and smooth with respect to $\xi \in \mathbb{R}$, but this expansion is valid only under the assumption that the solution z of (4) is $\mathcal{O}(1)$.

This expansion is uniformly convergent on the space \times time domain $\Omega \times (0, \infty)$. It consists in deriving a new system of coordinates, which enables us to compute the motion of layers and the evolution of the solutions, but does not allow us to compute details of the time-independent solutions (i.e. to get anything better than a correctly located step function in the limit $t \rightarrow \infty$).

Remark

Recalling that the term

$$\frac{z_t}{|\nabla z|}$$

corresponds to the normal velocity of the curves $\{x \in \Omega : z(x, t) = c\}$ for any fixed $t > 0$ and any constant $c \in \mathbb{R}$, with respect to the unit normal $-\frac{\nabla z}{|\nabla z|}$, we observe that

$$\nu(x, t) = -\frac{z_t}{|\nabla z|}$$

is the normal velocity of such a level set with respect to the unit vector $\frac{\nabla z}{|\nabla z|}$. Recalling also that (for each fixed $t > 0$), the term

$$\operatorname{div} \left(\frac{\nabla z}{|\nabla z|} \right)$$

is the mean curvature $\kappa(x, t)$ of the curves $\{x \in \Omega : z(x, t) = c\}$, we observe that (4) gives (3) when evaluated on $S^u(t, \epsilon)$. Moreover, in one dimension $\Omega \subset \mathbb{R}$, (4) can be solved by computing its characteristics as a first order hyperbolic PDE for $z(x, t)$.

4. Γ -Convergence - Steady Problem

Γ -convergence

We say that the functionals E_ϵ defined on $L^1(\Omega) \times L^1(\Omega)$ Γ -converge to the functional E_0 as ϵ tends to 0 if the next two conditions hold.

- **The lower bound:** for any fixed $(u, v) \in L^1(\Omega) \times L^1(\Omega)$ and for any sequences $(p_j, q_j)_{j \geq 1} \in L^1(\Omega) \times L^1(\Omega)$ and $(\epsilon_j)_{j \geq 1} \subset \mathbb{R}_+$ such that, as $j \rightarrow \infty$: $\epsilon_j \rightarrow 0$ and

$$(p_j, q_j) \rightarrow (u, v) \in L^1(\Omega) \times L^1(\Omega),$$

we have the inequality

$$\liminf_{j \rightarrow \infty} E_{\epsilon_j}(p_j, q_j) \geq E_0(u, v);$$

- **Attaining the lower bound:** for any fixed $(u, v) \in L^1(\Omega) \times L^1(\Omega)$ and for any sequence $(\epsilon_j)_{j \geq 1} \in \mathbb{R}_+$ such that $\epsilon_j \rightarrow 0$ as $j \rightarrow \infty$, there exists at least one sequence $(P_j, Q_j)_{j \geq 1} \in L^1(\Omega) \times L^1(\Omega)$ satisfying

$$(P_j, Q_j) \rightarrow (u, v) \in L^1(\Omega) \times L^1(\Omega) \text{ as } j \rightarrow \infty,$$

and

$$\lim_{j \rightarrow \infty} E_{\epsilon_j}(P_j, Q_j) = E_0(u, v).$$

The link between the minimisers of a family of functionals E_ϵ and its Γ -limit has been established by Kohn and Sternberg⁶ in the case of functionals defined on $L^1(\Omega)$. The extension of their result to the case of our functionals defined on $L^1(\Omega) \times L^1(\Omega)$ below is direct.

Properties of the Γ -limit

Suppose that the functionals E_ϵ Γ -converge to the functional E_0 and that the two following hypotheses are satisfied:

- every sequence $(p_j, q_j)_{j \geq 1}$ and positive $\epsilon_j \rightarrow 0$ as $j \rightarrow \infty$ for which $E_{\epsilon_j}(p_j, q_j) \leq C < \infty$ for all j has a $L^1(\Omega) \times L^1(\Omega)$ -convergent subsequence;
- there exists an isolated local (or global) minimiser $(u_0, v_0) \in L^1(\Omega) \times L^1(\Omega)$ of E_0 .

Then there exists $\epsilon_0 > 0$ and a family $(u_\epsilon, v_\epsilon)_{0 < \epsilon \leq \epsilon_0}$ such that:

- (u_ϵ, v_ϵ) is a local minimiser of E_ϵ ;
- $\|(u_\epsilon, v_\epsilon) - (u_0, v_0)\|_{L^1(\Omega) \times L^1(\Omega)} \rightarrow 0$ as $\epsilon \rightarrow 0$.

Here is the energy functional we are looking for. (Note that for this energy functional, we can prove that the first hypothesis in the properties of the Γ -limit given above is satisfied.)

Energy functional associated with Problem (\mathcal{P}_{vect})

If Ω is an open bounded domain whose boundary is C^{2+m} , $k, f^2, g^2 \in C^m(\bar{\Omega})$, where $m \in \mathbb{N}$ with $m \geq 2$, then an isolated local minimiser $(u_\epsilon, v_\epsilon) \in H^1(\Omega) \times H^1(\Omega)$ of the functional E_ϵ , defined on $L^1(\Omega) \times L^1(\Omega)$ for $\epsilon > 0$ by

$$E_\epsilon(u, v) = \begin{cases} \int_\Omega \frac{\epsilon}{4} k (|\nabla u|^2 + |\nabla v|^2) + \frac{1}{\epsilon} W(x, u, v) dx & \text{if } u, v \in H^1(\Omega), \\ \infty & \text{otherwise,} \end{cases}$$

is a steady solution of Problem (\mathcal{P}_{vect}) . Moreover the minimiser $(u_\epsilon, v_\epsilon) \in C^m(\bar{\Omega}) \times C^m(\bar{\Omega})$.

In order to simplify our notation, we define

$$\bar{g}(x) := \frac{g(x)}{\sqrt{1+\alpha}}, \quad \mathcal{A}_u := \{x : u(x) = \bar{g}(x)\}, \quad \mathcal{A}_v := \{x : v(x) = \bar{g}(x)\}.$$

With such a notation, we can give explicitly the Γ -limit E_0 of the functionals E_ϵ .

Γ -limit E_0

The Γ -limit E_0 of the functionals E_ϵ is given by

$$E_0(u, v) = \begin{cases} K(\alpha) \left(\int_\Omega \sqrt{k} f g^3 |D\chi_{\mathcal{A}_u}| + \int_\Omega \sqrt{k} f g^3 |D\chi_{\mathcal{A}_v}| \right) & \text{for } u, v \in BV(\Omega) \text{ s.t.} \\ & (u, v) \in \{(\pm \bar{g}, \pm \bar{g})\} \\ & \mathcal{L}^2\text{-a.e.,} \\ \infty & \text{otherwise,} \end{cases}$$

where by definition

$$\begin{aligned} K(\alpha) &= \Psi\left(-\frac{1}{\sqrt{1+\alpha}}, \frac{1}{\sqrt{1+\alpha}}\right) = \Psi\left(\frac{1}{\sqrt{1+\alpha}}, -\frac{1}{\sqrt{1+\alpha}}\right) \\ &= 2 \int_{-\infty}^{\infty} \mathcal{W}(\mathcal{U}(\xi), \mathcal{V}(\xi)) d\xi, \\ \Psi(u, v) &= \inf \left\{ \int_{-1}^1 \sqrt{\mathcal{W}(l(s))} |l'(s)| ds : l \text{ is piecewise } C^1, \right. \\ &\quad \left. l(-1) = \left(-\frac{1}{\sqrt{1+\alpha}}, -\frac{1}{\sqrt{1+\alpha}}\right), l(1) = (u, v) \right\}, \end{aligned}$$

and

$$\mathcal{W}(u, v) = \frac{1}{4}(u^4 + v^4) - \frac{1}{2}(u^2 + v^2) + \frac{\alpha}{2}u^2v^2 + \frac{1}{2(1+\alpha)}.$$

5. Γ -Convergence - Time-Dependent Problem

Let us assume for Problem (\mathcal{P}_{vect}) that

- Ω is a smooth (C^4) bounded connected domain of \mathbb{R}^2 ;
- $f^2, g^2 \in C^{0,\beta}(\bar{\Omega})$, $\sqrt{k} \in C^{1,\beta}(\bar{\Omega})$, $\beta \in (0, 1)$, $g \in BV(\Omega)$;
- g satisfies the homogeneous Neumann boundary conditions;
- ϵ and α are constant, with ϵ small and $\alpha \in (0, 1)$;
- (u_0, v_0) is a local isolated minimiser of E_0 (defined above);

then there exist $\epsilon_0 > 0$ and a family $(u_\epsilon, v_\epsilon)_{0 < \epsilon < \epsilon_0}$ of steady solutions of Problem (\mathcal{P}_{vect}) such that, for each $\epsilon \in (0, \epsilon_0)$

- $(u_\epsilon, v_\epsilon) \in C^{2,\beta}(\bar{\Omega}) \times C^{2,\beta}(\bar{\Omega})$;
- $\|u_\epsilon - u_0\|_{L^1(\Omega)} + \|v_\epsilon - v_0\|_{L^1(\Omega)} \rightarrow 0$ as $\epsilon \rightarrow 0$;
- for each $\lambda > 0$ small enough, we have that $\mathcal{L}^2(\Omega_\epsilon^\lambda) \rightarrow 0$ as $\epsilon \rightarrow 0$, where

$$\begin{aligned} \Omega_\epsilon^\lambda &= \{x \in \Omega : -\frac{g(x)}{\sqrt{1+\alpha}} + \lambda < u_\epsilon(x) < \frac{g(x)}{\sqrt{1+\alpha}} - \lambda\} \\ &\cup \{x \in \Omega : -\frac{g(x)}{\sqrt{1+\alpha}} + \lambda < v_\epsilon(x) < \frac{g(x)}{\sqrt{1+\alpha}} - \lambda\}; \end{aligned}$$

- (u_ϵ, v_ϵ) is an asymptotically stable solution of Problem (\mathcal{P}_{vect}) .

Looking for the motion of the limits of the nodal curves $S^u(t, 0) =$

$\lim_{\epsilon \rightarrow 0} S^u(t, \epsilon), S^v(t, 0) = \lim_{\epsilon \rightarrow 0} S^v(t, \epsilon)$ of the solutions $(u, v) = (T_1(t)(\psi_1, \psi_2), T_2(t)(\psi_1, \psi_2))$ of Problem (\mathcal{P}_{vect}) , we find the following result.

Motion of nodal curves

The nodal curves $S^u(t, 0), S^v(t, 0)$ move according to

$$\begin{aligned}\nu_u &= -k \frac{\partial}{\partial n_u} \ln(\sqrt{k} f g^3) - k \kappa_u + o(\epsilon^0), \\ \nu_v &= -k \frac{\partial}{\partial n_v} \ln(\sqrt{k} f g^3) - k \kappa_v + o(\epsilon^0),\end{aligned}$$

where by definition ν_u (resp. ν_v) is the normal velocity of the curve $S^u(t, 0)$ (resp. $S^v(t, 0)$) and n_u (resp. n_v) is the unit normal to the curve $S^u(t, 0)$ (resp. $S^v(t, 0)$) pointing in the same direction as ν_u (resp. ν_v). κ_u (resp. κ_v) is the signed curvature of the nodal curve $S^u(t, 0)$ (resp. $S^v(t, 0)$) at the point x .

The result above proves that the coupled problem ($\alpha \neq 0$) behaves like the uncoupled one ($\alpha = 0$) as $\epsilon \rightarrow 0$, from the point of view of the nodal curves. Moreover, we can find a necessary condition for a curve to be the stable limit as $t \rightarrow \infty$ of a nodal curve.

Steady stable nodal curves

A necessary condition for a nodal curve $S_\infty^{u,v}(0)$ to be the steady stable limit as $t \rightarrow \infty$ of the (limit) nodal curves $S^{u,v}(t, 0)$ is

$$\kappa_{u,v} = -\frac{1}{\sqrt{k} f g^3} \frac{\partial}{\partial n} (\sqrt{k} f g^3) = -\frac{\partial}{\partial n} \ln(\sqrt{k} f g^3),$$

which is one of Norbury and Yeh's results⁷ for the uncoupled problem, obtained by asymptotic expansion methods. It expresses the possible steady limits of nodal curves as (extremal) weighted geodesics of Ω using the metric $\sqrt{k} f g^3$ (strictly positive in Ω). To be stable we require this geodesic to be minimal in the sense that any variation of the extremal geodesic leads to an increase of the energy functional.

6. Conclusion

As we have seen, there are essentially two ways to analyse the type of multi-well large reaction small diffusion problems we have considered.

One way is based on the notion of Γ -convergence, which appears to be very efficient for nonlocal analyses of the solutions, providing proofs of results on the existence and the convergence of steady solutions for $\epsilon > 0$. It helps in analysing the time-dependent problems in some particular cases, as for example the motion of distinct vortices located at $\{a_i(t)\}_{i=1}^d \in \Omega$ and given by

$$\frac{da_i(t)}{dt} = -k \nabla \ln(kg^2) \Big|_{x=a_i(t)} \quad \text{for } i = 1, \dots, d,$$

but fails when the inner layer behaviour (characterised by the fast length-scale ξ) is not known from any of the possible optimal sequences in the definition of the Γ -limit.

The other way, which consists in transforming the large reaction small diffusion problem into an optimal coordinate $z(x, t)$ problem of the form

$$\frac{z_t}{|\nabla z|} = \frac{\sqrt{k}}{fg^3} \operatorname{div} \left(\sqrt{k} f g^3 \frac{\nabla z}{|\nabla z|} \right),$$

is a powerful method for computing and/or producing examples of interfaces and models with desired properties for the interfaces between the states. The above formula generalises the motion by mean curvature results that have been established for homogeneous vectorial problems by Rubinstein, Sternberg and Keller ⁹.

At this stage, one “challenging” problem remains open for the theory presented here. It is the number of families of solutions of Problem (P_α) , with respect to $\alpha \in (-1, \infty)$. If it were possible to find the number of such families, together with a characterisation of some subspaces in which they appear as stable, then these solutions would be applicable to the transmission of data through optical fibres, and lead to a massive improvement of the speed of transmission. (See [4,5] for instance.) Such a result is difficult, but should attract future attention.

This work has been partially submitted in three separate articles (which include proofs of results stated here):

- New solutions for perturbed steady Ginzburg-Landau equations in \mathbb{R} ,
Journal of Differential Equations;

- New steady solutions (and their stability) for a large reaction small diffusion problem, *Nonlinearity, Institute of Physics*;
- New coordinates for time-dependent nonlinear reaction diffusion problems, *Nonlinearity, Institute of Physics*.

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