

Packing random graphs and hypergraphs

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Abstract

We determine to within a constant factor the threshold for the property that two random k -uniform hypergraphs with edge probability p have an edge-disjoint packing into the same vertex set. More generally, we allow the hypergraphs to have different densities. In the graph case, we prove a stronger result, on packing a random graph with a fixed graph.

1 Introduction

Let G_1 and G_2 be two k -uniform hypergraphs of order n . We say that G_1 and G_2 can be *packed* if they can be placed onto the same vertex set so that their edge sets are disjoint.

In the graph case, quite a lot is known. Bollobás and Eldridge [2] and Catlin [5] independently conjectured that if $(\Delta(G_1) + 1)(\Delta(G_2) + 1) \leq n + 1$ then G_1 and G_2 can be packed. Sauer and Spencer [12] proved that graphs G_1 and G_2 of order n can be packed if $\Delta(G_1)\Delta(G_2) < n/2$. Let us note

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that the conjectured bound would be tight: suppose that $n = ab - 2$, and let $G_1 = (b - 1)K_a \cup K_{a-2}$ (the vertex-disjoint union of $b - 1$ complete graphs of order a and a complete graph of order $a - 2$) and $G_2 = (a - 1)K_b \cup K_{b-2}$. Then $(\Delta(G_1) + 1)(\Delta(G_2) + 1) = n + 2$, but G_1 and G_2 cannot be packed.

For fixed $k \geq 3$, the graph example given above is easy to generalize: suppose that $n = (a - 1)(b - 1)(k - 1) + a + b - 3$. Let G_1 be the vertex-disjoint union of $b - 1$ complete k -uniform graphs of order $(a - 1)(k - 1) + 1$ and $a - 2$ isolated vertices; let G_2 be the vertex-disjoint union of $a - 1$ complete k -uniform graphs of order $(b - 1)(k - 1) + 1$ and $b - 2$ isolated vertices. Then $\Delta(G_1)\Delta(G_2) = \Theta(a^{k-1}b^{k-1}) = \Theta(n^{k-1})$, but G_1 and G_2 cannot be packed. For another family of examples, choose $r < k$ and fix an r -set $A \subset [n]$. Let G_1 have all edges containing A , and G_2 be an (n, k, r) -design (these are now known to exist for suitable n : see Keevash [9]). G_1 and G_2 cannot be packed, and we have $\Delta(G_1) = \Theta(n^{k-r})$ and $\Delta(G_2) = \Theta(n^{r-1})$, and so again $\Delta(G_1)\Delta(G_2) = \Theta(n^{k-1})$. On the positive side, much less is known. Teirlinck [13] (see Alon [1] for further results and discussion) showed that, for $n \geq 7$, any two Steiner triple systems G_1, G_2 can be packed: note that these satisfy $\Delta(G_1)\Delta(G_2) = \Theta(n^2)$. There are also some nice results when one of G_1 and G_2 has very small maximal degree: see Rödl, Ruciński and Taraz [11] and Conlon [6].

In this paper, we consider what happens when G_1 and G_2 are *random* hypergraphs. For integers k, n and $p \in [0, 1]$, we write $\mathcal{G}(n, k, p)$ for the random k -uniform hypergraph on n vertices in which each possible edge is present independently with probability p ; when $k = 2$, we write $\mathcal{G}(n, p) = \mathcal{G}(n, 2, p)$. In the graph case, with $G_1 \in \mathcal{G}(n, p)$ and $G_2 \in \mathcal{G}(n, q)$, the extremal results mentioned above suggest that we should expect a condition of form $pqn \leq c$ (for suitable c) to be able to pack G_1 and G_2 . More generally, for k -uniform hypergraphs, we might hope for a condition of form $pqn^{k-1} \leq c$, as this would give $\Delta(G_1)\Delta(G_2) = O(n^{k-1})$ with high probability (i.e. with probability $1 - o(1)$ as $n \rightarrow \infty$) provided p, q are not extremely small (for instance $\min\{p, q\} \gg \log n / n^{k-1}$ is enough). In fact, we shall show here that it is possible to pack rather denser graphs: if G_1 and G_2 are both random then we can allow an additional factor $\log n$ in the product pqn^{k-1} , but not more. (We note that a similar phenomenon occurs when we try to minimize the overlap of two random hypergraphs: see Bollobás and Scott [3] and Ma, Naves and Sudakov [10].)

We will prove the following theorem.

Theorem 1. *Let $\delta \in (0, 1)$. For every $k \geq 2$, there exists $\varepsilon > 0$ such that the following holds. Let $p = p(n)$ and $q = q(n)$ be positive reals such that*

- $\max\{p, q\} \leq 1 - \delta$
- $pq \leq \varepsilon \log n / n^{k-1}$.

Let $G_1 \in \mathcal{G}(n, k, p)$ and $G_2 \in \mathcal{G}(n, k, q)$ be random k -uniform hypergraphs of order n . Then, with high probability, there is a packing of G_1 and G_2 .

Note that if $pq = \varepsilon \log n / n^{k-1}$ then with high probability G_1 and G_2 satisfy $\Delta(G_1)\Delta(G_2) = \Theta(n^{k-1} \log n)$.

The bound on pq in Theorem 1 is easily seen to be sharp up to the constant. Indeed, if $G_1 \in \mathcal{G}(n, k, p)$ and $G_2 \in \mathcal{G}(n, k, q)$ then the probability that G_1 and G_2 can be packed is at most the expected number of packings

$$n!(1 - pq)^{\binom{n}{k}} \leq \exp(n \log n - (1 + o(1))pq n^k / k!)$$

which is $o(1)$ if $pq \geq \alpha \log n / n^{k-1}$ for any constant $\alpha > k!$. In particular, if we take $p = q$, then combining this bound with Theorem 1 shows that the threshold density for two random k -uniform hypergraphs to be unpackable is $\Theta(\sqrt{\log n / n^{k-1}})$.

In the case of graphs, we will in fact prove a much stronger result: it turns out that we can take just *one* of the two graphs to be random. Indeed, we prove the following.

Theorem 2. *For all $\gamma, K > 0$ and $\delta \in (0, 1)$ there exists $\varepsilon > 0$ such that the following holds. Let $p = p(n)$ and $q = q(n)$ be positive reals such that*

- $p \leq 1 - \delta$
- $q \leq n^{-\gamma}$
- $pqn \leq \varepsilon \log n$.

Let G_1 be a graph of order n with maximal degree at most qn and let $G_2 \in \mathcal{G}(n, p)$. Then with failure probability $O(n^{-K})$ there is a packing of G_1 and G_2 .

The rest of the paper is organized as follows. In Section 2 we prove Theorem 2, and in Section 3 we prove the extension to hypergraphs. We conclude in Section 4 with some open problems.

2 Packing random graphs

The aim of this section is to prove Theorem 2. We begin by noting a couple of standard facts.

We will use the following Chernoff-type inequalities. Let X be a sum of Bernoulli random variables, and let $\mu = \mathbb{E}X$. Then for $t > 0$, we have

$$\mathbb{P}[X \leq \mathbb{E}X - t] \leq \exp(-t^2/2\mu) \quad (1)$$

and

$$\mathbb{P}[X \geq \mathbb{E}X + t] \leq \exp(-t^2/(2\mu + 2t/3)) \quad (2)$$

(see, e.g., [8, Theorems 2.1 and 2.8] or [4, Chapter 2]). Inequality (2) is often called Bernstein's inequality.

It will also be useful to note a simple (and standard) fact about the binomial distribution, see e.g., [8, Corollary 2.4].

Proposition 3. *For every $K > 0$ there is $\delta > 0$ such that if $x > 0$ and $X \sim \text{Bi}(n, p)$ is a binomial random variable with $np \leq \delta x$ then $\mathbb{P}[X \geq x] \leq e^{-Kx}$.*

Proof. This is standard; we include a proof for completeness. We have, assuming as we may that x is an integer,

$$\mathbb{P}[X \geq x] \leq \binom{n}{x} p^x \leq \left(\frac{enp}{x}\right)^x \leq (e\delta)^x$$

where we have used the standard bound $\binom{n}{k} \leq (en/k)^k$ in the second line. The result follows by choosing $\delta = e^{-K-1}$. \square

Our first lemma is the following, which shows that, if \mathcal{A} is a large, sparse set system then a random set (of suitable size) is quite likely to be disjoint from some member of \mathcal{A} .

Lemma 4. *For all $\delta, \gamma \in (0, 1)$ there is $\varepsilon > 0$ such that the following holds for all sufficiently large n . Let $d = n^{1-\gamma}$, let X be any set, and let \mathcal{A} be a set sequence in $\mathcal{P}(X)$ such that:*

- $|\mathcal{A}| \geq n$
- every element of X belongs to at most d sets from \mathcal{A}
- all sets in \mathcal{A} have size at most $\varepsilon \log n$.

Let $B \subset X$ be a random set where each element of X independently belongs to B with probability $1 - \delta$. Then B is disjoint from at least $n^{1-\gamma/4}$ sets of \mathcal{A} , with failure probability $O(\exp(-n^{\gamma/3}))$.

Proof. This can be proved in more than one way (an alternative proof pointed out by a referee runs an element exposure martingale on X and then applies the Hoeffding-Azuma inequality).

We may assume that $|\mathcal{A}| = n$. We choose a small $\varepsilon > 0$, and assume that n is large. We ignore below insignificant roundings to integers.

We begin by partitioning \mathcal{A} into sets of pairwise disjoint elements. Let G be the intersection graph of \mathcal{A} : so the vertices of G are the elements of \mathcal{A} , and G has edges AA' whenever $A \cap A'$ is nonempty. Since every vertex belongs to at most d sets from \mathcal{A} , and every set has size at most $\varepsilon \log n$, each set in \mathcal{A} meets at most $\varepsilon d \log n$ other sets. Thus G has maximal degree at most $\varepsilon d \log n$. It follows by a theorem of Hajnal and Szemerédi [7] that G has a colouring with at most $\varepsilon d \log n + 1$ colours in which the sizes of distinct colour classes differ by at most 1. Thus we may partition G into independent sets (and so \mathcal{A} into collections of pairwise disjoint sets) of size at least $n/(\varepsilon d \log n + 1) \geq n^{\gamma/2}$.

Let \mathcal{A}' be one of these collections of pairwise disjoint sets, and set $m = |\mathcal{A}'| \geq n^{\gamma/2}$. The random set B is disjoint from each member of \mathcal{A}' independently with probability at least $\delta^{\varepsilon \log n} = n^{-\varepsilon \log(1/\delta)} > n^{-0.01\gamma}$ provided we have chosen a sufficiently small ε ; it follows that the probability that B is disjoint from fewer than $m/n^{\gamma/4}$ sets in \mathcal{A}' is at most

$$\begin{aligned} \binom{m}{m/n^{\gamma/4}} (1 - n^{-0.01\gamma})^{m - m/n^{\gamma/4}} &\leq \left(\frac{em}{m/n^{\gamma/4}} \right)^{m/n^{\gamma/4}} \exp(-n^{-0.01\gamma} m/2) \\ &< e^{m \log n / n^{\gamma/4}} e^{-n^{-0.01\gamma} m/2} \\ &< e^{-n^{-0.01\gamma} m/4}, \end{aligned}$$

provided n is sufficiently large. There are $\varepsilon d \log n + 1 = o(n)$ colour classes, so with failure probability $o(ne^{-n^{-0.01\gamma} n^{\gamma/2}/4}) = O(e^{-n^{\gamma/3}})$, B is disjoint from at least a fraction $n^{-\gamma/4}$ of the sets in each colour class, and hence is disjoint from at least $n^{1-\gamma/4}$ sets in \mathcal{A} . \square

For positive integers m, n , and $p \in [0, 1]$ we write $\mathcal{S}(n, m, p)$ for a random sequence $(S_i)_{i=1}^m$ of m subsets of $[n]$, where the subsets are independent and each set independently contains each element of $[n]$ with probability p .

Equivalently, we could consider a random $m \times n$ matrix with entries 0 and 1, where each element independently takes value 1 with probability p . We shall refer to $S \in \mathcal{S}(n, m, p)$ as a *random set sequence*.

Given two random set sequences $\mathcal{A} \in \mathcal{S}(m, n, p)$ and $\mathcal{A}' \in \mathcal{S}(m, n, q)$, where $m \leq n$, it will be useful to pair up the sets from \mathcal{A} and \mathcal{A}' so that each pair is disjoint. For $A \in \mathcal{A}$ and $A' \in \mathcal{A}'$, the probability that A and A' are disjoint is $(1 - pq)^n \leq \exp(-npq)$, so if $pq > 2 \log n/n$ it is likely that we do not have any disjoint pairs at all. However, if $pq < c \log n/n$, for small enough c , we will show that such a pairing is possible. In fact we will prove a much stronger result: we can take just one of the set systems to be random, provided the other satisfies certain sparsity conditions.

Lemma 5. *For all $K > 0$ and $\eta, \gamma, \delta \in (0, 1)$ there is $\varepsilon > 0$ such that the following holds for all sufficiently large n . Suppose that $p = p(n), q = q(n) \in [0, 1]$ satisfy $0 \leq p < 1 - \delta$ and $pq < \varepsilon \log n/n$. Let $m \in [n^\eta, n]$ be an integer and set $d = m^{1-\gamma}$, and suppose that $\mathcal{A} = (A_i)_{i=1}^m$ is a sequence of subsets of $[n]$ such that*

- *every $i \in [n]$ belongs to at most d sets from \mathcal{A}*
- $\max_{A \in \mathcal{A}} |A| \leq qn$.

Let $\mathcal{B} = (B_i)_{i=1}^m \in \mathcal{S}(n, m, p)$ be a random set sequence, and let H be the bipartite graph with vertex classes \mathcal{A} and \mathcal{B} , where we join A_i to B_j if $A_i \cap B_j \neq \emptyset$. Then, with failure probability $O(n^{-K})$, H has minimal degree at least $m^{1-\gamma/4}$; furthermore, H has a perfect matching.

Proof. Let $\varepsilon, \varepsilon' > 0$ be fixed, small quantities (with $\varepsilon \ll \varepsilon'$) that we shall choose later. We generate \mathcal{B} in two steps: we first choose a random set sequence $\mathcal{B}' = (B'_i)_{i=1}^m \in \mathcal{S}(n, m, (1 + \delta)p)$, and then obtain \mathcal{B} from \mathcal{B}' by deleting each element from each set B'_i independently with probability $\delta' = \delta/(1 + \delta)$.

Note first that for any i, j , the distribution of the intersection $|A_i \cap B'_j|$ is stochastically dominated by a binomial $\text{Bi}(nq, p(1 + \delta))$. So for fixed $\varepsilon' > 0$, it follows from Proposition 3 that we have $|A_i \cap B'_j| < \varepsilon' \log m$ for all i and j , with failure probability $O(n^{-K})$, provided ε is small enough in terms of ε' . We may therefore assume from now on that this event occurs, and condition on the choice of \mathcal{B}' (so \mathcal{B}' is fixed and \mathcal{B} is still random).

Now consider the bipartite graph H . We need to prove that H has a perfect matching. We shall apply Hall's condition to \mathcal{B} , so it is enough to show that for every subset $S \subset \mathcal{B}$ we have $|\Gamma_H(S)| \geq |S|$.

Consider $B'_i \in \mathcal{B}'$, and let $\mathcal{A}'_i = (A_j \cap B'_i)_{j=1}^m$ be the restriction of \mathcal{A} to B'_i . Then every vertex belongs to at most d sets from \mathcal{A}'_i and $\max_j |A_j \cap B'_i| < \varepsilon' \log m$, so provided ε' is sufficiently small we can apply Lemma 4 to deduce that with failure probability $O(e^{-m^{\gamma/3}})$ the set B_i is disjoint from at least $m^{1-\gamma/4}$ sets from \mathcal{A}'_i . This occurs independently for each i (recall that we are conditioning on \mathcal{B}'), so with failure probability $O(me^{-m^{\gamma/3}}) = O(n^{-K})$ every vertex in \mathcal{B} has degree at least $m^{1-\gamma/4}$ in H , and so Hall's condition holds for every $S \subset \mathcal{B}$ with $|S| < m^{1-\gamma/4}$.

Now consider an element $A_i \in \mathcal{A}$. Each B'_j meets A_i in at most $\varepsilon' \log m$ vertices, and so each B_j independently is disjoint from A_i with probability at least $(\delta')^{\varepsilon' \log m} > m^{-\gamma/6}$, provided ε' is sufficiently small. The number of B_j disjoint from A_i is thus a binomial with expectation at least $m^{1-\gamma/6}$ and so, by (1), is at least $m^{1-\gamma/6}/2 > m^{1-\gamma/4}$, with failure probability $O(e^{-m^{1-\gamma/6}/8})$. So with failure probability $O(me^{-m^{1-\gamma/6}/8}) = O(n^{-K})$ every vertex in \mathcal{A} has degree at least $m^{1-\gamma/4}$ in H , and so Hall's condition holds for every $S \subset \mathcal{B}$ with $|S| > m - m^{1-\gamma/4}$.

We have now shown that H has minimal degree at least $m^{1-\gamma/4}$. All that remains is to verify Hall's condition for sets $S \subset \mathcal{B}$ of size between $m^{1-\gamma/4}$ and $m - m^{1-\gamma/4}$. Let $t \in [m^{1-\gamma/4}, m - m^{1-\gamma/4}]$: we shall bound the probability that there is any subset of \mathcal{B} of size t with t or fewer neighbours in \mathcal{A} . Suppose that $S \subset \mathcal{B}$ has size t and $T \subset \mathcal{A}$ has size $m - t$. For any fixed $B'_i \in S$, the set sequence $\mathcal{A}' = (A \cap B'_i)_{A \in T}$ has $\max_{A' \in \mathcal{A}'} |A'| \leq \varepsilon' \log m$ and every vertex belongs to at most d sets from \mathcal{A}' , where $d = m^{1-\gamma} \leq |\mathcal{A}'|^{1-\gamma/4}$. So by Lemma 4, the probability that B_i intersects every set in T is at most $\exp(-(m - t)^{\gamma/12})$. Thus the probability that (in the graph H) S has no neighbours in T is at most $\exp(-t \cdot (m - t)^{\gamma/12})$. Since there are at most $n^{2t} = \exp(2t \log n)$ choices for the pair (S, T) , we deduce that the probability that there is any set S of size t with at most t neighbours is bounded by $\exp(2t \log n) \exp(-t \cdot (m - t)^{\gamma/12}) = O(n^{-(K+1)})$, uniformly in t . Summing over t , we see that Hall's condition holds with failure probability $O(n^{-K})$.

We conclude by noting that we can choose first ε' and then ε sufficiently small for the estimates above to hold. \square

We are now ready to prove Theorem 2.

Proof of Theorem 2. Let $\eta = \gamma/2$, $t = \lceil (K + 2)/\eta \rceil$, and let G_1 have vertex set V and G_2 have vertex set W . We begin by finding a partition of V into sets V_1, V_2, \dots of size $\Theta(n^\eta)$ such that:

- V_i is an independent set in G_1 for every i ,
- Every vertex in V has fewer than t neighbours in each set V_j .

Indeed, we first colour V randomly with $n^{1-\eta}$ colours, giving each vertex a colour selected uniformly at random and independently. It follows from (1) and (2) that, with failure probability $O(n^{-K})$, every colour class has size $(1+o(1))n^\eta$. Consider a vertex $v \in V$, say with degree d . Then by assumption $d \leq qn \leq n^{1-\gamma}$. So the probability that v has a set of t neighbours, all with the same colour, is at most

$$\binom{d}{t} (1/n^{1-\eta})^{t-1} \leq d^t n^{\eta t - t + 1} \leq n^{1+t\eta-t\gamma} = n^{1-t\eta} = O(n^{-(K+1)}).$$

It follows that, with failure probability $O(n^{-K})$, no vertex has t neighbours of the same colour. Each colour class now induces a subgraph with maximum degree less than t , so we can apply the Hajnal-Szemerédi Theorem [7] to each class, splitting it into $O(t)$ independent sets of (almost) the same size. The vertex classes are now independent, have size $\Theta(n^\eta)$, and no vertex has t neighbours in any other class.

Reordering if necessary, we may assume that $|V_1| \geq |V_2| \geq \dots$. Now let $W = W_1 \cup W_2 \cup \dots$ be an arbitrary partition of W (chosen before revealing G_2) such that $|W_i| = |V_i|$ for every i . We construct a bijection between V and W that defines a packing (i.e., does not map any edge of G_1 to an edge of G_2) by constructing suitable bijections between V_i and W_i for $i = 1, 2, \dots$.

For $i = 1$, we choose an arbitrary bijection between V_1 and W_1 . (Recall that V_1 is independent.) For $i > 1$, we set $S_i = \bigcup_{j < i} V_j$ and $T_i = \bigcup_{j < i} W_j$, and suppose that we have found a bijection $\varphi_i : S_i \rightarrow T_i$. The neighbourhoods of vertices in V_i and W_i define set sequences $\mathcal{A} = (\Gamma(v) \cap S_i)_{v \in V_i}$ in S_i and $\mathcal{B} = (\Gamma(v) \cap T_i)_{v \in W_i}$ in T_i , and the bijection φ_i allows us to identify S_i and T_i . We now check that these two set sequences satisfy the conditions of Lemma 5, which we will then apply to obtain a bijection between V_i and W_i . Let

$$\begin{aligned} \tilde{n} &= |S_i| = |T_i| = \Theta(in^\eta), \\ \tilde{m} &= |V_i| = |W_i| = \Theta(n^\eta), \end{aligned}$$

and note that $|\mathcal{A}| = |\mathcal{B}| = \tilde{m}$ and $\tilde{m} \in [\tilde{n}^{\eta/2}, \tilde{n}]$. By construction of the partition $(V_j)_{j \geq 1}$, no vertex belongs to t sets from \mathcal{A} , as each vertex in S_i has fewer than t neighbours in V_i . Let $\tilde{q} = \max_{A \in \mathcal{A}} |A|/\tilde{n}$. Each set in \mathcal{A}

has size at most qn and so $\tilde{q} \leq qn/\tilde{n} = O(qn^{1-\eta}/i)$. The set sequence \mathcal{B} is random with $\mathcal{B} \in S(\tilde{n}, \tilde{m}, p)$, and depends only on the edges between W_i and T_i . Furthermore,

$$pq\tilde{n} \leq p \cdot (qn/\tilde{n}) \cdot \tilde{n} = pqn \leq \varepsilon \log n = O(\varepsilon \log \tilde{n}).$$

We can therefore apply Lemma 5, to deduce that if ε is sufficiently small then with failure probability $O(n^{-(K+1)})$ there is a bijection between the two set sequences for which the corresponding pairs are disjoint; this corresponds to a bijection between V_i and W_i so that there are no common edges in the bipartite graphs between (V_i, S_i) and (W_i, T_i) where S_i and T_i are identified by φ_i . Extending φ_i with this bijection, we obtain a bijection $\varphi_{i+1} : S_{i+1} \rightarrow T_{i+1}$.

It follows that, with failure probability $O(n^{-K})$, we succeed at every step and construct the desired bijection. \square

Finally in this section, we note that Theorem 2 can be used to pack several random graphs.

Corollary 6. *Let $\gamma, K > 0$, let $\delta \in (0, 1)$, and let t be a positive integer. Then there exists $\varepsilon > 0$ such that the following holds. Let $p_0(n), \dots, p_t(n)$ satisfy*

- $\max_i p_i \leq 1 - \delta$
- $p_0 \leq n^{-\gamma}$
- $\max_{i < j} p_i p_j n \leq \varepsilon \log n$.

Let G_0 be a graph of order n with maximal degree at most $p_0 n$ and, for $i = 2, \dots, t$, let $G_i \in \mathcal{G}(n, p_i)$. Then with failure probability $O(n^{-K})$ there is a packing of G_0, \dots, G_t .

Proof. We may assume that $p_1 \leq \dots \leq p_t$. Thus, by the second and third conditions above, we have $\sum_{i=0}^{t-1} p_i = O(n^{-\min\{\gamma, 1/3\}})$. We first pack G_0 and G_1 , then add in the remaining graphs one at a time, applying Theorem 2 at each stage. Thus at the i th stage we have packed G_0, \dots, G_i to obtain a graph H_i : it follows easily from Proposition 3 that with high probability the maximum degree condition of Theorem 2 is satisfied by H_i (with a slightly smaller γ). Provided ε is sufficiently small, we get that with failure probability $O(n^{-K})$ we can pack H_i with G_{i+1} . \square

3 Packing hypergraphs

In this section, we will prove Theorem 1.

Proof of Theorem 1. Note that the case $k = 2$ follows immediately from Theorem 2, so we can assume $k \geq 3$. Let $\eta = 1/5$, $t = 15k$, and let $\varepsilon, \varepsilon' > 0$ be small constants and K, K' large constants; we will choose $\varepsilon, \varepsilon'$ and K, K' later. (In fact, we will first choose ε' ; K' will be determined by ε' ; we then choose K and finally ε .) We may assume that $q \leq p$, and so in particular $q = O(\sqrt{\log n/n^{k-1}}) < n^{-1/2}$ (for large n). We may also assume that $q \geq \varepsilon \log n/n^{k-1}$, or increase to this value.

Our argument will follow a similar strategy to Theorem 2, but there are some additional complications. It will be helpful to reveal the edges of G_1 and G_2 in several steps. This time we let V be the vertex set of G_2 and W the vertex set of G_1 .

We first generate a partition of V into sets V_1, V_2, \dots by colouring V randomly with $n^{1-\eta}$ colours, giving each vertex a colour selected uniformly at random and independently. It follows from (1) and (2) that, with failure probability $o(1)$, every colour class V_i has size $(1+o(1))n^\eta$, so we may assume that this holds. Reordering if necessary, we may assume that $|V_1| \geq |V_2| \geq \dots$. Let $W = W_1 \cup W_2 \cup \dots$ be a random partition of W such that $|W_i| = |V_i|$ for every i . For $i \geq 1$, we set $S_i = \bigcup_{j < i} V_j$ and $T_i = \bigcup_{j < i} W_j$ (note that $S_1 = T_1 = \emptyset$; also $S_L = V$ and $T_L = W$, where $L = n^{1-\eta} + 1$).

As before, we will construct a bijection between V and W by constructing bijections between V_i and W_i for $i = 1, 2, \dots$. However, we need to be a little more careful than in the graph case, as there are more ways for edges to intersect the classes V_i and W_i . For $j = 1, \dots, k$, and any i , we say that an edge is *of type j for V_i or W_i* if it has j vertices in V_i or W_i , and the remaining $k-j$ vertices in S_i or T_i .

We now reveal all type 1 edges in G_2 . For a $(k-1)$ -set $A \subset S_i$, the probability that V_i contains t vertices v such that $A \cup \{v\}$ is an edge of G_2 is at most

$$\binom{2n^\eta}{t} q^t = O(n^{\eta t - t/2}) = o(n^{-k}).$$

It follows that, with high probability, for every integer i and every $(k-1)$ -set $A \subset S_i$, V_i contains fewer than t vertices that can be added to A to obtain an edge of G_2 . In other words, each $(k-1)$ -set in S_i is contained in fewer than t type 1 edges for V_i .

For each vertex $v \in V_i$, we define the *type 1 neighbourhood of v* to be the $(k-1)$ -uniform hypergraph on S_i with edge set

$$\{A \subset S_i : A \cup \{v\} \text{ is a type 1 edge for } V_i\};$$

similarly, for vertices in W_i , the type 1 neighbourhood is a $(k-1)$ -uniform hypergraph on T_i .

At the first step of the partitioning process, we take a random bijection between V_1 and W_1 . The expected number of common edges is at most $pqn^{k\eta} = o(1)$, and so with high probability there are no common edges.

Now consider a later stage of the partitioning process: suppose we have constructed a bijection $\varphi_i : S_i \rightarrow T_i$ and wish to extend this to a bijection $\varphi_i : S_{i+1} \rightarrow T_{i+1}$. In constructing our bijection, we will only consider edges of type 1 and 2; we will consider edges of type 3 at the end of the argument.

We first consider type 1 edges in V_i and W_i . For each $v \in V_i$, we consider the type 1 neighbourhood of v as a subset of $S_i^{(k-1)}$ (rather than as a k -uniform hypergraph on S_{i+1}). The collection of type 1 neighbourhoods of vertices in V_i then defines a set sequence \mathcal{A} of subsets of $S_i^{(k-1)}$; similarly, the collection of type 1 neighbourhoods of vertices in W_i defines a set sequence \mathcal{B} of subsets of $T_i^{(k-1)}$; and the bijection φ_i allows us to identify $S_i^{(k-1)}$ and $T_i^{(k-1)}$. As in the proof of Theorem 2, we wish to apply Lemma 5, so we need to check that its conditions are satisfied.

Let

$$\begin{aligned}\tilde{n} &= |S_i^{(k-1)}| = |T_i^{(k-1)}| = \Theta(i^{k-1}n^{\eta(k-1)}), \\ \tilde{m} &= |V_i| = |W_i| = (1 + o(1))n^\eta,\end{aligned}$$

and note that $|\mathcal{A}| = |\mathcal{B}| = \tilde{m}$ and $\tilde{m} \in [\tilde{n}^{\eta/k}, \tilde{n}]$.

By construction of the partition $(V_j)_{j \geq 1}$, no element of $S_i^{(k-1)}$ is contained in t sets from \mathcal{A} , as each $(k-1)$ -set $A \subset S_i$ is contained in fewer than t type 1 edges for V_i . The size of each set in \mathcal{A} has distribution $\text{Bi}(\tilde{n}, q)$. Choose a small $\varepsilon' > 0$, let $K' = 2/(\eta\varepsilon')$, and then choose a large K . Let $\tilde{q} = \max\{Kq, \varepsilon'(\log \tilde{n})/\tilde{n}\}$. It follows from Proposition 3 that, provided K is large enough (depending on K'), every set in \mathcal{A} has size at most $\tilde{n}\tilde{q}$, with failure probability at most

$$\tilde{m}e^{-K'\tilde{n}\tilde{q}} \leq ne^{-K'\varepsilon'\log \tilde{n}} \leq n^{1-K'\varepsilon'\eta} = o(1/n).$$

Furthermore, since $\tilde{n} \leq n^{k-1}$, by choosing ε small enough we get

$$pKq \leq K\varepsilon \frac{\log n^{k-1}}{n^{k-1}} \leq \varepsilon' \frac{\log \tilde{n}}{\tilde{n}}$$

and hence $p\tilde{q} \leq \varepsilon'(\log \tilde{n})/\tilde{n}$. We can therefore apply Lemma 5, to deduce that if ε' is sufficiently small then with failure probability $O(n^{-2})$ we get the following:

- a bijection $\varphi^* : V_i \rightarrow W_i$ such that the corresponding pairs in the two set sequences are disjoint. This corresponds to a bijection between V_i and W_i so that there are no collisions between type 1 edges for V_i and W_i . Also:
- for all distinct $u, v \in V_i$ and $x, y \in W_i$, a bijection

$$\varphi^{**} : V_i \setminus \{u, v\} \rightarrow W_i \setminus \{x, y\}$$

such that there are no collisions of type 1 edges for V_i and W_i , except possibly for edges containing u, v, x or y .

The mapping φ^* deals with collisions between type 1 edges. However, we must also consider type 2 edges for V_i and W_i . We do not reveal type 2 edges at this stage, but only the number of collisions between type 2 edges created by the mapping φ^* . There are at most $n^{k-2+2\eta}$ type 2 edges for V_i and W_i , and so the probability that φ^* maps any type 2 edge for V_i in G_2 to a type 2 edge in G_1 is at most $pqn^{k-2+2\eta} \leq \log n/n^{1-2\eta}$; the probability that there are at least two collisions is $O(\log^2 n/n^{2-2\eta}) = o(1/n)$ (which is small enough to ignore). If there are no collisions, then we use φ^* to extend φ_i .

This leaves the case when there is one collision between type 2 edges. We reveal the edge where this occurs: say $A \cup \{u, v\}$ maps to $A \cup \{x, y\}$ under φ^* . We thus condition on the existence of these two edges in G_2 and G_1 , and on this being the only collision. We shall show the existence of another mapping φ^{**} from V_i to W_i that avoids collisions for both type 1 and type 2 edges with probability at least $1 - O(\log n/\sqrt{n})$. Then the probability that we get collisions for both φ^* and φ^{**} is $O((\log n/n^{1-2\eta}) \cdot \log n/\sqrt{n})$, which is $o(1/n)$.

Let $D = \lceil 6(\log \tilde{n})/\delta \rceil$. We choose distinct vertices $x_1, \dots, x_D, y_1, \dots, y_D$ in W_i such that the type 1 neighbourhood of u is edge-disjoint from the type

1 neighbourhoods of x_1, \dots, x_D , and the type 1 neighbourhood of v is edge-disjoint from the type 1 neighbourhoods of y_1, \dots, y_D (the existence of these vertices follows from the minimal degree condition on H in Lemma 5).

We reveal the edges $A \cup \{x_\ell, y_\ell\}$ for each $\ell \leq D$: since $p \leq 1 - \delta$, it follows that with probability $1 - o(1/n)$ there is some ℓ such that $A \cup \{x_\ell, y_\ell\}$ is not present in G_1 . We then use the appropriate mapping φ^{**} from $V_i \setminus \{u, v\}$ to $W_i \setminus \{x_\ell, y_\ell\}$ that we found above, and extend it by setting $\varphi^{**}(u) = x_\ell$ and $\varphi^{**}(v) = y_\ell$ so that we have a mapping from V_i to W_i . The mapping φ^{**} does not cause any collision of type 1 edges. Finally, we reexamine the type 2 edges for collisions. We have ensured that $A \cup \{u, v\}$ does not collide with anything; the probability of a collision involving any edge of form $A \cup \{x_j, y_j\}$ is at most $qD = O(\log n / \sqrt{n})$; and the probability of any other collision is at most $\log n / n^{1-2\eta} = O(1/\sqrt{n})$, as before. (More formally: we have conditioned on the edges $A \cup \{x_j, y_j\}$, on the event that a particular pair of type 2 edges collide, and the event that no other collisions occur. If we resample all type 2 edges that are not in the colliding pair or of form $A \cup \{x_j, y_j\}$, the number of collisions under φ^{**} stochastically dominates the number before resampling, giving the same bound.) So the probability that φ^{**} yields a collision is $O(\log n / \sqrt{n})$, as required.

It follows that, with probability $1 - o(1/n)$, we are able to find a good bijection between V_i and W_i , and extend φ_i to φ_{i+1} . Continuing in this way, we find a bijection from V to W in which there are no collisions between type 1 or 2 edges for any V_i, W_i .

Finally, we reveal all edges of type 3 or more. There are at most $n^{k-2+2\eta}$ possible edges of type 3 or more, and so the probability that any of these is an edge in both hypergraphs is at most $pqn^{k-2+2\eta} = o(1)$. The algorithm therefore succeeds with probability $1 - o(1)$. \square

4 Conclusion

We conclude by mentioning a few open questions.

- The bound in Theorem 1 is sharp to within a constant factor. It is natural to expect that there is some $c = c(k) > 0$ such that almost surely a pair of random k -uniform hypergraphs $G_1, G_2 \in \mathcal{G}(n, k, p)$ are packable if $p < (c - \varepsilon)\sqrt{\log n / n^{k-1}}$ and are unpackable if $p > (c + \varepsilon)\sqrt{\log n / n^{k-1}}$. Is this correct? If so, what is the value of c ?

- What happens with the results above if we take $G_1 = G_2$? We would expect this to make no difference.
- All our examples of unpackable k -uniform hypergraphs G_1, G_2 have $\Delta(G_1)\Delta(G_2) = \Omega(n^{k-1})$. What is the correct bound here?

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