Gradient of a straight line: An investigation into the effect that making different links between representations has on student understanding

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A Research & Development Project
Submitted for the
MSc Learning and Teaching 2015

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Abstract

The aim of this research is to explore whether using different approaches to finding the gradient of a straight line results in better conceptual understanding of gradient and tools to attempt questions about straight lines and linear relationships in different contexts.

The research was carried out in the context of a department that does not feel it teaches gradient of straight line well and is facing the prospect of greater emphasis on working with linear relationships in the new Mathematics GCSE.

An action research model was adopted with lessons taught by colleagues with the control groups observed and used as the basis for devising a series of intervention lessons. These lessons used different approaches to linear relationships including looking for increments in tables of values and describing the relationships between variables in sentences based on the definition of gradient as well as the traditional $\frac{rise}{tread}$ approach used with the control groups. Pre- and post- tests were used with both the control and intervention groups to investigate the effect of the intervention; these tests were analysed both qualitatively and quantitatively.

Analysis of the quantitative results did not show any significantly higher level of attainment for the intervention group either relative to the control groups or the baseline data for the whole cohort. However, qualitative analysis on an individual level suggests a greater breadth of understanding for the intervention group which puts them in a better position to attempt questions in a variety of contexts.

Contents

1. Introduction

The aim of this study is to explore how we, as a Mathematics department, teach straight-line graphs and to examine whether this could be done differently in order to enhance the students' conceptual understanding of how gradient relates to the rate of change between two variables. The main focus will be on the use of representations for linear functions, looking first at the representations (graphical, algebraic, tabular, and descriptive) that are currently used and the links made between them, and then going on to devise and use teaching materials that exploit different links between these representations.

The background to this study is that as a department we do not feel that we teach graph work well. Students struggle to calculate the gradient of a straight line reliably (especially if there is something unusual such as fractional or negative gradients) despite careful, procedural teaching and a lot of practice in Year 9. These problems are exposed in Year 10 where many students find working with parallel and perpendicular lines extremely difficult; they manage satisfactorily when working with standard equations but flounder when they have to select an alternative method or apply their skills in different scenarios. Evidence from the Sixth Form, where the students have to interpret gradient values when working with regression lines, suggests that even the most highly attaining mathematicians have little conceptual understanding of what gradient means especially when working in context.

In the new GCSE specification (DfE, 2013a) straight line gradients and their relationship to rate of change has a higher profile. The content is being expanded to re-introduce estimated gradient of curves using chords and tangents and interpreting gradients of lines presented in a variety of contexts. Other aspects of the graph content that have been transferred from the current GCSE specification are becoming of greater significance (evidenced by the new R strand covering ratio, proportion and rates of change). The fact that gradient of straight lines is now explicitly mentioned in both the Algebra strand (itself with an increased assessment proportion) and this new R strand points towards increasing importance of multiple representations for linear relationships and the need for students to have the ability to move between them with ease.

As Head of Department I will need to give guidance about how these new or higher priority topics should be approached. This is particularly true in my department because of the two colleagues who carry out the rest of the GCSE teaching, one is relatively early in their career and still in need of support and guidance, while the other is very experienced but does not teach A-level, so is understandably nervous about some of the more advanced aspects of the new syllabus.

The research will be carried out in a small girls' 3-18 independent school. Year 9 were chosen for this study because they are the year group that traditionally does the majority of the initial work on gradient calculation; in addition they will be the first cohort to be assessed at 16 via the new GCSE. In the school's most recent inspection report the overall contribution of teaching was rated as good with some excellent teaching being noted; following on from this inspection the school has an objective to raise the quality of all teaching to that of the best. Mathematics in the senior school is taught by a small team of specialists; although there is a great desire to support each other in meeting the challenges raised by the new curriculum there is no history of academic research within the department.

In Chapter 2, literature relating to straight-line graphs and gradient will be discussed; representational frameworks will be compared along with issues about covariance, the use of contexts, language, and the importance of graph scales. In Chapter 3, the literature guided by my initial thoughts will be used to devise a series of research questions. The methodology and research instruments used for my study are described in Chapter 4, as are the nature of the participants, ethical considerations, and the collaborative aspect of the research. Chapter 5 gives the results of both the qualitative and quantitative aspects of the research and in Chapter 6 these results will be discussed in conjunction with the findings of the literature review. Finally Chapter 7 will present the conclusions of this research and my thoughts about how these will guide me in the future, both as a teacher and Head of Department.

2. Literature Review

This review of literature focuses first on the role of representations in mathematics and more specifically in the teaching and learning of linear relationships that can be represented by straight-line graphs. It then examines the research that has been carried out to develop representational frameworks and models of levels of understanding as well as looking at how relationships between sets of values are examined and expressed. Transfer between representations and how this can be used to approach prediction tasks and rate of change analysis is discussed; and finally a number of additional but allied considerations (contextual problems, graphs on non-standard axes and the impact of language) are analysed.

Representations

Representations are "an inherent part of mathematics" (Dufour-Janvier et al., 1987, p110) which can be seen as multiple concretisations or parallel and equivalent forms of the same concept (Dufour-Janvier et al., 1987; Mason et al., 2005). Janvier (1987) expanded to describe a representation as "a combination of three components: symbols (written), real objects and mental images" (p68); it is assumed that printed images are included under the umbrella of real objects. Translation between different representations is the "psychological process involved in going from one mode of representation to another" (Janvier, 1987, p27). Fluent translation between representations is important because it enables students to compare the properties of different representations and choose the one most appropriate to a situation (Davis, 2007); it is crucial that students are also aware of the strengths and weaknesses, usefulness and limitations of different representations (Mason et al., 2005; Dufour-Janvier et al., 1987). Representational fluency is seen by many not only as an integral part of the study of mathematics (DfE, 2013b) but also as an essential tool in the problem solving process, although Gagatsis and Shiakalli (2004) concluded that while students who were better able to successfully translate between representations were also better at problem solving, this could only be counted as one of the factors that contributed to their problem solving success. Janvier (1987) explained that to complete a translation the *source representation* needs to be examined in terms of features of the *target representation*; for instance to write the equation of a straight line from its graph the student needs to consider the information that is required in the *target* algebraic representation (gradient, y-intercept) and decide how to extract these from the *source* graphical representation Concerns were, however, raised by Dufour-Janvier et al. (1987) that care is needed when introducing new representations because the student's focus on mastering a new representation may work to the detriment of understanding the mathematics that the representation is supposed to be helping with; they highlighted the particular problem that "certain representations lead to more difficulties rather than functioning as aids to learning" (p116).

Representational frameworks

For any mathematical construct there will be a number of relevant representations. Looking specifically at the representations and translations involved when working with straight-line graphs and functions Davis (2007) and Janvier (1987) have built complementary models (Figures 1 and 2) the first focusing on the representations themselves with the second more interested in the translation processes between them.

The figure originally presented here cannot be made freely available via ORA because of copyright. The figure was sourced at Davis, J. (2007) Real-world contexts, multiple representations, student-invented terminology, and y-intercept, Mathematical Thinking and Learning, 9(4), pp387-418

The figure originally presented here cannot be made freely available via ORA because of copyright. The figure was sourced at Janvier, C. (1987) Translation processes in mathematics education in Janvier, C. (ed) Problems of representation in the teaching and learning of mathematics, Lawrence Erlbaum: Hillsdale, NJ

The two frameworks are structurally different; Davis has six nodes each with a direct but undefined link to the others, whereas Janvier focuses on four generic representations and attempts to name the translation processes that link them.

Davis centres his model on real-world contexts because this is the emphasis given in the US standards-based curriculum where his research is located; in that curriculum student investigations are predominantly located in real-world contexts and they are then asked to translate in and out of other representations. In the current British national curriculum for KS3 mathematics (DfE, 2013b), the principal aims are fluency, mathematical reasoning and problem solving with an emphasis on developing "competence in solving increasingly sophisticated problems" (p2); however, there is no explicit mention of the context for these problems.

In Davis' representation based framework, the spoken (or natural) language used in the representations is separated from the contextual description of their situation. This was done because one focus for that study was students' use of natural language in straight-line graph work particularly when describing the starting point (y-intercept) of their graph. Because my research is looking in detail at gradient work, Davis' distinction of spoken language as a separate node has limited relevance although language issues are still important and are discussed later. Davis also identified two distinct families of graph, limited domain (LD) and full domain (FD). A limited domain graph is only valid for positive values of the independent variable with full domain graphs being valid for both positive and negative values of the independent variable. Once again this ties in with the contextual centre and spoken language for starting point elements of Davis' study because the relationship between y-intercept and beginning of a limited domain graph is rather more intuitive than for a full domain graph. In the setting of my research the distinction between the different domain graphs is relatively unimportant; the vast majority of our graph work is abstract and we try and work with full domain graphs from the outset to maximise the potential for practising negative number work. Contextual limited domain graphs are introduced as a subset of full domain rather than a separate entity.

In contrast Javier's framework places the emphasis on the translational processes rather than the representations themselves, with names and descriptions given to each of the translations between his four main representations. To do this he grouped all the descriptive elements (both context and language) together into "situation, verbal descriptions" as well as considering graphs as a single entity irrespective of their domain; the subtle differences between either the two descriptive elements or the two types of graph would not affect the translation processes to and from the other main representation so were of limited relevance.

Not only are each of the six possible links between the representations identified in Janvier's framework, but the descriptions given to complementary processes are not identical although they are identified as having some features in common. Janvier observed that there were practical difficulties with some of the direct links, with multiple indirect connections often made in preference; this is particularly true for the link from table to formula via graph and formula to graph via table. He noted that "a study of several mathematics programs showed that they exclusively develop the indirect version of many processes of the tables. Could we do otherwise? Would it be worthwhile?" (1987, p29).

Leinhardt et al. (1990) proposed that there were three types of function translation task: a) matching different representations of the same function, b) carrying out equivalent transformations of different representations of the same function and c) moving from one representation to another. The translations described by Davis and Janvier fall most easily into the first and last of those types with classroom activities frequently exploiting these translations. However, Mason et al. (2005) recommends building up a bank of known relationships and then transforming them as appropriate as a worthwhile method for developing understanding of functions, particularly those expressed as an equation. Although this is of relevance when looking at representations of functions as a whole it is of less relevance here where the use of representations to calculate and understand the gradient of linear functions is of greatest interest. Of most relevance are the different translation processes and type of translation tasks expressed by Janvier (1987) and Leinhardt et al. (1990) along with the question about direct and indirect links posed by Janvier; these will lead directly into the research questions.

Levels of understanding

As well as having access to models both of representations and of the translations between them, it is also important to consider how students' understanding of straight-line graphs and the relationships they demonstrate develops. One levelled framework was developed by Hitt (1998), drawing on a wealth of previous research. This framework (Figure 3) was devised to describe the understanding of the concept of function based on ability to transfer between different representations stating that "a central goal of mathematics teaching is thus taken to be that the students will be able to pass from one representation to another without falling into contradictions" (p125).

Level 1:	Imprecise ideas about a concept (incoherent mixture of different representations of the concept).
Level 2	Identification of different representations of a concept. Identification of systems of representations.
Level 3	Translation with preservation of meaning from one system to another.
Level 4:	Coherent articulation between two systems of representations.
Level 5:	Coherent articulation of different systems of representation in the solution of a problem.

Figure 3: Hitt's levels of understanding framework

(Hitt, 1998, p125)

Hitt's framework would have the simple matching of representations identified by Leinhardt et al. (1990) at Level 2 with moving from one representation to another (also identified by Leinhardt et al.) sitting comfortably in Level 3. Leinhardt's second type of translation (carrying out identical transformations on different representations of the same function) is more difficult to categorise in this framework but works best in Level 4 because of the idea of *coherent articulation*. From a teacher's perspective the objective would be to help students move up through the levels, ideally allowing them to identify the representations and develop methods for translation. The coherent articulation in Levels 4 and 5 carries the ideas of comparing properties of representations, being aware of limitations and making appropriate choices in the problem solving process as discussed earlier.

A second levels-of-understanding hierarchy was developed by Kerslake (1981) in an attempt to compare difficulties students found in graph work with all areas of the mathematics curriculum (Figure 4)

In contrast with Hitt's framework (Figure 3) this focuses instead on the constituent parts of the topic of straight-line graphs as they would be taught in school rather than on how representational understanding develops. It is much more like the type of list that would be produced by a teacher as part of a medium or long term planning exercise to identify the order that the parts of the topic need to be introduced, although it is interesting to note that *understanding* the relationship between graph and equation appears at the highest level although it is often introduced much earlier. This ties in with the assertion by Wavering (1989) that activities to develop one-to-one correspondence and pattern recognition should be used before formal graphing procedures and interpretation that requires quantitative understanding are taught. In addition some differences between teachers' perceived levels of difficulty of various aspects of straight-line graphs and the problems actually encountered by students were also identified by Hadjidemetriou and Williams (2002). They established that students demonstrated some covariational awareness at a relatively early stage which the authors contrasted with a perceived teacher belief that global aspects of graph interpretation (looking at patterns and rates shown by the whole graph) are intrinsically more difficult than local aspects (information given by individual points). With reference to teachers' practice a major conclusion of their study was that they should make opportunities for students to consider the behaviour of a graph in a global manner rather than just focusing on pointwise interpretation.

Covariance

Definitions and levels of understanding

Covariance is defined as the process of looking for patterns in two sets of values and finding a link between them to the point of being able to describe a rate of change (Confrey & Smith, 1995; Carlson et al., 2002; Ayalon et al., forthcoming).

Carlson et al. (2002), building on work by Saldanha and Thompson (1998), describe students' images of covariance as being developmental because they emerge in ordered levels; these levels were described in more detail by Thompson (1994) as "the development of images of rate starts with children's image of change in some quantity, progresses to a loosely coordinated image of two quantities, which progresses to an image of the covariation of two quantities so that their measures remain in constant ratio" (p232). A number of authors proposed understanding frameworks for covariance; Gomez et al. (1999) categorised students' ability to successfully answer rate of change questions using a range of representations whereas Carlson et al. (2002) proposed a covariation framework whereby the progression of understanding for linear function started with simply noticing a relationship, through observing the positive or negative nature of this relationship to quantifying the relationship.

In the context of this research into students' understanding of gradient and its relationship to rate of change, their ability to observe and assess covariance will be important. It is, however, also necessary to consider other methods of expressing linear relationships, particularly correspondence.

Correspondence vs covariance

A correspondence relationship is presented as the alternative to covariance; this is when a formal function is defined to describe the relationship from a value in one set (usually x) to a value in the other set (usually y) (Confrey & Smith, 1993). For instance, in the context of a value that changes with time in a linear fashion, a covariance approach would attempt to look at how the value changes over different time intervals whereas a correspondence relationship would give a rule or formula that allows the value to be calculated for any particular point in time. Ayalon et al. (forthcoming) present evidence from a variety of prior sources that many students find covariation harder to conceptualise than correspondence making particular note that "secondary students had considerable confusion concerning the numerator/denominator roles of the two quantities compromising a rate" (Ayalon et al, forthcoming, p5). As part of work on a graph comprehension study Mevarech and Kramarsky (1987) identified "poor understanding of the notion of covariance" (p255) as one of the possible sources of students' alternative conceptions; they recognised that understanding of covariance is often not as strong as expected because the emphasis is on change rather than individual values, and that there is a tendency to apply a preconceived degree of covariance between two events to a situation even if the evidence does not support this. The school mathematics curriculum gives correspondence a higher level of importance than covariance in terms of outcomes, for instance questions like *find the nth term rule of this sequence* or *state the equation for this line* with the resultant general rule being the preferred tool for answering questions that require interpolation or extrapolation. However, covariation is a tool that can be used while establishing correspondence relationships and is also an integral part of rate of change interpretation.

Approaches to covariance

A number of studies looked for different entry points to develop understanding of covariance. Confrey and Smith (1995) describe the use of tables as valuable especially when it leads to informal qualitative descriptions of related variations. Ayalon et al. (2014), while building on the work of Carlson et al. (2002), promote the use of spatial generalisation tasks as giving opportunities to explore covariation in relation to gradient. They initially worked with term-to-term interpretation of the output variable values in conjunction with the values taken by the input variable, having hypothesised that this would be easier to comprehend than position-to term interpretation. Contrary to expectations the majority of respondents looked for a correspondence (position-to – term) relationship as their first method of approach to the problem because it had been presented in a non-sequential manner to discourage reliance on recursive techniques.

A correspondence first approach is promoted by French (2002) as a desirable entry point to work on graphs; he recommends that students find the equation of a line by looking for the relationships between the x and y values in the coordinate points because it encourages them to "find meaning an algebraic symbols, expressions and equations and see purpose in what they are doing" (p37). However, The Royal Society (1995), in their review of algebra in the British national curriculum, were dismissive of over-use of generating expressions from patterns as this was not, in their view, an algebraic activity.

While developing a competence model for graph interpretation, Clement (1989) looked at levels of understanding needed to establish both a static and dynamic view of the graph's content. For the static model the values of the two variables give rise to the location of one particular point on the graph; for the dynamic model the changes in the values of these variables give rise to covariation shown by the slope of the graph in a particular region. The point is made that basic static and dynamic interpretation of a graph can take place without knowing any particular values of the variables but that errors occur when students attempt to use a static interpretation of a graph when a dynamic interpretation would be more appropriate; for instance confusing the relative position of two points on a graph and the slope of two lines when trying to identify the faster object from a distance-time graph. This is an example of errors caused by "a misplaced link between a successfully isolated variable and an incorrect feature of the graph" (Clement, 1989, p7). This was expanded on by Leinhart et al. (1990) who stated that interpretation can be "global and general or local and specific" (p8). The student might be looking for a pattern, the opportunity for prediction or to establish a rate of change; alternatively the student might be looking to draw inferences or make comparisons relating to particular points on the graph. Prediction can be defined as "the action of conjecturing from a given part of a graph where other points (not explicitly given or plotted) of the graph should be located or how other parts of the graph should look" (p13). For functional relationships, predictions will require some sort of pattern spotting or covariational reasoning.

Transfer between representations

As discussed earlier, fluent transfer between representations is an important feature of both mathematical understanding as a whole (DfE, 2013b) and more specifically of work with linear relationships that may be expressed in a straight-line graph (Janvier, 1987). A number of studies noted the difficulties that students had when working with information presented in different representations, for instance Herbert and Pierce (2011) observed difficulties transferring information previously extracted from one representation to another. In addition to this Gagatsis and Shiakalli (2004) and Elia et al. (2007) both concluded that students struggle to recognise alternative representations of the same function, preferring to see them as distinct mathematical objects. There are opposing views on whether students can readily transfer between representations. On the one hand, Knuth (2000) observed that students did not effectively translate between representations, preferring to work in the algebraic form even when a graphical approach could be less computationally intense. On the other hand, Ayalon et al. (forthcoming) concluded that students were capable of adjusting their approach to rate of change problems depending on both the presentation of the information and the way in which the individual questions were expressed.

When working with algebraic and graphical representations of the same function, Leinhardt et al. (1990) noted the two representations do not convey the same information about the function and that neither set of information is complete; the "two symbolic systems are used to illuminate each other" (p3). This incompleteness of information in one particular representation supports the need for fluent translation skills as proposed by Dufont-Janvier et al. (1987) and Mason et al. (2005) amongst others. Thomas et al. (2010) established that the direction of translation between algebraic and graphical representations presented no difference in response level; this was identified by the authors to be in contrast with previous findings by Duval (2006) where it was suggested that graphical to algebraic should be more difficult that the opposite conversion.

Many students develop a preference for one representation over another although conflicting observations come from different pieces of research. Gagatsis and Shiakalli (2004), working in the Cypriot education system which itself has a strong emphasis on algebraic work, noted a lower success rate on function questions where the graphical representation was involved. However, Herbert and Pierce (2011), who had previously acknowledged representational transfer difficulties for their students, observed that many developed a preference for a numeric representation and would routinely convert graphical information into coordinate points before calculating rates from these numerical values rather than working with the gradient of a line.

Both the individual difficulties described above and the contrasting findings of different studies demonstrate potential problems in the use of representations for both students and teachers. It is important to establish which representations are most useful in different contexts and also to examine how students can be encouraged to make best use of different representations without causing additional confusion rather than clarity. There are also two particular areas where fluency of transfer between representations could be of particular use: prediction tasks and rate of change calculations.

Prediction tasks

As discussed earlier, a correspondence relationship or formula may be preferred for problems where some sort of prediction is required. However, Ayalon et al. (forthcoming) also demonstrated that students have the capability to use covariational reasoning for prediction tasks and often find this easier than establishing a general rule. When researching far-prediction tasks, Bieda and Nathan (2009) observed the transfer between representations looking specifically at how students approached a task when presented with a Cartesian graph containing the base information they needed to use. They categorised the responses at those who were *physically grounded* (only worked within the graphical representation initially presented), *spatially grounded* (could make near predictions by extending the axes of the graph) and *interpretatively grounded* (could transfer to an alternative representation to make predictions well outside the physical range of the original graph). With no explicit reference to gradient and only implied reference to rates of change they concluded that it is important for students to work at an early stage with far-prediction tasks and not be limited to the physical constraints of a particular Cartesian graph.

Greet (2009), in his review of Bieda and Nathan (2009) along with a number of other papers on representational fluency, observed that the authors did not note any incidences of misapplied proportional reasoning which seemed surprising given both the findings of much previous research on the problem and my own personal experience working with students on problems similar to this presented in either a graphical or a numerical manner. Indeed, Stacey (1989) found false proportional reasoning to be one of the main incorrect methods used by late primary and early secondary age students when attempting far prediction tasks for linear relationship problems presented in both diagrammatic and numeric formats. This finding was supported by the findings of Ayalon et al. (forthcoming) who, when looking at linear patterns presented through a variety of representations, observed that there was a tendency to assume false proportional reasoning until prompts in the question encouraged students to refine the view.

Rates of change

Students also struggled to extract rate of change values for linear relationships when presented with information in a variety of representations. Gomez et al. (1999) observed that, when students were presented with information on a multi-line distance time graph, their most common error was to use the coordinates of the end point of the line as the values to go into the rate calculation and ignore the y-intercept value in the process; this is another example of false-proportional reasoning as discussed in the context of prediction tasks.

Gomez et al. (1999) and Ayalon et al. (forthcoming) both looked at students' ability to make conclusions about rates of change where information about linear relationships are presented in a table, and came to different conclusions. While Gomez et al. (1999) established that students who could observe patterns within each column of data were generally unable to determine a covariance relationship between the two columns, Ayalon et al. (forthcoming) found that many students could overcome the difficulties presented by a change in the interval of input variable and correctly use covariational reasoning to predict the output variable. Both studies noted that errors occurred when students used incorrect additive relationships within one column and that, even when prompted by a checking question in Ayalon et al. (forthcoming), students were generally unable to go back and correct their previous error.

Additional considerations

The literature previously discussed has raised three additional allied areas of consideration: working in context, gradient calculations for graphs on non-homogenous axes, and the impact of language on understanding and solution of problems.

Working in context

Although the British national curriculum has less of an emphasis on contextual investigational work than the US standards based curriculum as described by Davis (2007), the ability to be able to work with linear relationships in a real-life situation is still important. In work done by Davis (2007) that was examining the effects of real-world contexts on the study of straight-line graphs one finding was that, in the main, the connection between slope and rate of change was made relatively easily, as was the transition between informal, student-generated notation and formal definitions of slope. In contrast to this, Herbert and Pierce (2011) established that, while students were happiest working with familiar contexts for rates of change such as speed, they did not always relate to these as a relationship between to variables preferring to view them as a single item. In addition, apparent understanding of a rate in terms of a familiar context did not automatically transfer to a new, unfamiliar context. A further alternative perspective is that of Kerslake (1981) who noted that when students were asked to plot both $W = 2h + 3$ having been given a context for W and h and guided through a series of questions that led them to calculate some initial points, and then $y = 2x + 3$ with no context given, there was little difference in the success rate between the contextual and abstract situations.

Graphs presented on non-homogenous axes

Homogenous axes have been defined to be those where the scale is the same on both axes and *non-homogenous* are those where the scales are different (Zaslavsky et al., 2002), terminology which will also be used in this research. An initial hypothesis leading to this research was that students can often struggle to apply their knowledge or method for gradient calculation when presented with a graph on non-homogenous axes. Scaling has been identified by Kaput (1987) as a critical aspect of graph work that is often overlooked. Indeed, Mason et al. (2005) stated that "for many learners there appears to be a rather uneasy relationship between graphs and their axes" (p240) and an appreciation of what visual features of a graph do and do not alter under a change of scale was identified by Leinhardt et al. (1990) as being an important component to fully understanding a graph. Zaslavsky et al. (2002) investigated whether students used an analytic approach to slope where it is "a property of the function and does not depend on the functions' representation" (p122) or a visual perspective where the slope is a "property of the line which may vary under change of scale" (p122).

The transfer from homogenous to non-homogenous axes means that students using a purely visual approach to slope will begin to get wrong answers when using a method that had previously delivered consistently correct answers; whereas those working analytically will still achieve success; those students using a combination of the two approaches will get contradictory results and may investigate further in an attempt to unpick the contradiction (Zaslavsky et al, 2002). Kerslake (1977 & 1981) observed in a pair of studies that a purely visual approach was a strong contributor to incorrect answers. When students were asked to find a pair of graphs representing the same information for lines presented on different axes many ignored the scales and selected the two visually equivalent lines (1977); when a different group of students were asked to determine whether two lines were parallel with one presented on axes and the other drawn alongside with the coordinates of its end-points given, many used the fact that they appeared parallel to make their decision rather than applying any sort of proportional reasoning, even though the numbers involved were part of what *should* have been a familiar pattern (1981).

Teaching methods may contribute to students' preference for a purely visual approach. Teuscher and Reys (2010), following a review of previous research on the subject, concluded that:

> Students are typically introduced to rate of change through the slope formula. Further, students tend to practise inputting numbers and calculating the slope with little or no focus on interpreting the meaning of the result within a given context and with little consideration for units of measure. Generally students are introduced first to slope and later to rate of change; perhaps students are not making the connection between these two concepts.
>
> (Teuscher & Reys, 2010, p519)

This view was supported by French (2002), who noted that "traditional approaches to teaching algebra are characterized by a lot of time being spent on learning skills before attempting to apply them to problems" (p6). His conclusion was that the desire for fluency is often manifested as routine practice without understanding or any apparent usefulness to the student meaning that a skill is forgotten and needs to be re-taught or revised.

The method that students use to calculate gradient, and the context within which it was initially taught and practised appears to have a significant impact on whether the students view gradient using a purely visual perspective or incorporate some analytical aspect and also whether their gradient calculations can be transferred to graphs on non-homogenous axes.

Language issues

Formal and informal language can have an impact on both how easily the student accesses a problem and also how they express their understanding of it. When Gomez et al. (1999) used a problem stated only in natural language they were surprised that the respondents used a variety of informal strategies to support their arguments rather than using one or more formal representation; the suggestion seems to be that if students see a problem expressed informally then they are less likely to use *proper* mathematical strategies to try and solve it although this may or may not impact on their ability to reach the correct answer. The use or misuse of formal language in graphing tasks was identified by Mevarech and Kramarsky (1997) as one of the origins of students' alternative conceptions. They recognised that formal language can "exert additional difficulties on students' understanding" (p254) particularly those words which have different meanings in mathematical and everyday language. This view is supported by Davis (2007) who stated that "semantic contamination" (p389) might mean that misunderstandings occur from students' use of natural language and the problems caused by use of the word slope to describe the steepness of a line as identified by Zaslavsky et al. (2002) were discussed earlier.

Stump (2001) also looked at language issues and concluded that the language used by students to describe slope also reflected the maturity of their conceptual understanding; the use of natural language implied intuitive understanding whereas more technical language implied that the student had been exposed to formal instruction on the topic although this did not guarantee mathematical maturity. The students who expressed ideas wholly in natural language struggled to conceptualise slope as a ratio or rate but those who correctly used formulae generally demonstrated a better conceptual understanding of slope either as a physical measure of steepness or as a rate of change between two unlike quantities. In addition to this, the main conclusion of Zaslavsky et al. (2002) was that “sloppy” terminology for slopes, coordinates and functions was a strong contributor to misconceptions and that putting more emphasis on rate of change as a descriptor would help clarify understanding compared with the word *slope* which makes particular links with visual representations. In Britain the word *gradient* is used in place of slope and carries less apparent potential for misunderstanding with it although there appears to be little research in this area.

The use of contexts clearly has the potential to enhance or confuse students’ understanding of rate of change and gradient, as does the language of the classroom and of the questions posed. Non-homogenous axes present a specific difficulty and are often themselves tied in with contextual problems. These are all areas worthy of investigation in relation to the students at my school.

3. Research questions

In the light of previous research described in the literature, considered alongside my experiences as a teacher of mathematics and my concerns as Head of Department, I have decided that this study should focus on two main areas: the use of representations that is currently made across the teaching in my department and the potential for better conceptual understanding of gradient if different links between representations were to be made.

Within these two strands of research a number of sub-questions have been identified.

How are representations used in the teaching of straight-line graphs in my department?

- What links between representations are currently used?
- How does our representational framework compare with those described in the literature?
- What other links between representations can possibly be used in the teaching of straight-line graphs?

Would making different links between representations develop better conceptual understanding of gradient?

- Are students who have explored different links better equipped to apply their knowledge in unfamiliar situations, particularly prediction tasks?
- Do students who have used methods other than the gradient formula perform better on tasks presented on non-homogenous axes?

4. Methodology

Overall study structure

This study was formed of four main blocks of work:

1. Lesson observations of colleagues teaching the research topic to the control groups
2. Series of lessons taught by the researcher to the intervention group
3. Analysis of test results to examine the effectiveness of the intervention lessons
4. Collaborative review of findings and incorporation into future planning

Each block was reflective in its nature, and the study as a whole formed an action research cycle. Cohen et al. (2011) argue that although action research is complex and multifaceted, "what unites different conceptions of action research is the desire for improvement to practice, based on a rigorous evidential trail of data and research" (p344). This piece of research was carried out in the context of a school which had improving teaching and learning at the heart of its development plan, and a mathematics department looking to improve teaching, and hence student understanding, of a topic that its teachers had previously identified as a source of concern. Although the teachers in the department appreciated the need to explore their practice on this particular topic, particularly in the light of the demands of the new GCSE course, there has been a reluctance to explore new teaching strategies without evidence of their success or benefits in case they were less effective than the methods they were currently employing.

One model of action research is cycles of:

- planning a change,
- activing and observing the consequences of the change,
- reflecting on these processes and consequences, and then
- re-planning,
- acting and observing,
- reflecting, and so on.

(Punch & Oancea, 2014, p173)

The action research model used within this piece of research is summarised in Figure 5 with a cycle of planning, acting and observing, reflecting and re-planning, as described by Punch and Oancea (2014) at its heart.

The figure originally presented here cannot be made freely available via ORA because of copyright. The figure was sourced at Punch, K. & Oancea, A. (2014) Introduction to research methods in education (2nd edition), Sage, London.

Embedded within this piece of action research was a piece of quasi-experimental work whereby all participants took part in the same pre-tests and post-tests, with two groups taught normally and forming the control groups while the third received the intervention lessons. This is “one of the most commonly used quasi-experimental designs in educational research” (Cohen et al., 2011, p323) because the participants were not allocated to the groups on a random basis so the groups were non-equivalent. This made it an appropriate model to use in a scenario where participants were to be taught in groups to which they have previously been allocated on the basis of prior mathematical attainment.

Pre- and post- test design and administration

The questions for the pre- and post- tests were selected from a variety of sources making use, as far as possible, of questions that had been used before by their designers; full copies of the pre- and post- tests are included in Appendix 1. This meant that they had been extensively trialled, although because the trials had not necessarily taken place in the exact context of this research this may, in turn, affect the usefulness of the results (Taber, 2007). Each question had a quantitative (either multiple choice or numeric) and qualitative (explanation of thinking) aspect. The questions were selected to explore a range of approaches to gradient and rate of change as well as applications of the equation of a straight line. Some questions were contextual while others were more abstract and focusing on numeric or algebraic representations. The questions for the pre-test were selected to cover content that had been covered in Years 7 and 8 or that which could be considered common-sense application of mathematics in different contexts. The post- questions included specific vocabulary that had been covered during the lessons on gradient and equation of a straight-line. In particular the pre-test questions did not include the word gradient although contextual rates of change were covered; gradient was, however, used more specifically in the post-test questions. There was some overlap of questions between the two tests; instructions for the post-test requested that the respondents attempt the repeated questions as if they had not seen them before rather than trying to remember their previous responses.

The pre-test was trialled on an anonymous basis with a Year 8 group who were asked to give feedback on the legibility and understandability of the questions as well as the amount of time available for the test. As a result of this feedback a number of questions were subtly re-written or re-formatted, and the instructions on the front of the test were re-written to make it clear that any form of explanation (word based, numeric or diagrammatic) would be acceptable.

The selection of questions for the tests was carried out in conjunction with the teachers of the other groups. As part of the discussion around writing the tests it was made clear that, although all three teachers (two control, one intervention) would have been party to the content of the post-test it was imperative that they should not explicitly prepare their groups for the questions in the test.

All three groups sat the pre-test at the same time but in their individual teaching groups and supervised by their normal teacher. The teachers read out the detailed information on the front of the test and made it clear that they could clarify a question but not explain how to answer it. Test conditions were enforced although the students were not made to sit at separate exam desks. The pre-test was sat in a 35 minute single lesson and students were given as much time as they needed. The post-tests were sat at different times dependent on the teaching schedule for the individual groups; these were sat at the start of a 70 minute double lesson because it was felt that the slightly longer test might require more time than available in a single lesson. Following the analysis of the post-test results it was determined that a number of the questions, while still considered meaningful in terms of structure, had not given sufficiently useful results because the values involved were too difficult. A second post-test was written in order to explore the same skills, but using easier numbers so that numerical difficulties did not preclude respondents demonstrating their understanding. This test was completed by all three groups at the same time; this was some time after their teaching on the subject had been completed but also after a period of school holiday when they should have been revising this topic, amongst others, for school internal exams.

Cohen et al. (2011) recommend that the pre-tests are carried out immediately before the intervention so that other external influences could be ruled out; this was not possible in this case because there was a delay between the pre-test and the intervention to allow for the lesson observations of the control groups to take place and be analysed. However, the content of the mathematics lessons received by the students in the intervention group between the pre-test and the start of the intervention lessons had no overlapping content with the work on straight-line graphs so could be considered to have minimal influence on the outcome of the intervention and thus the effect of the time delay could be ignored.

A number of arguments are discussed by Cohen at al. (2011) about the timing of post-tests; should they take place immediately after an intervention so that effects are not influenced by external factors in the intervening period or should there be a gap to allow for the maturation of ideas? It was decided that each group, two control and one intervention, should carry out the post-test immediately after their series of lessons to look at the immediate effect (uninfluenced by any subsequent revision for school exams); the second, initially unplanned, post-tests were carried out later as discussed above.

Control group lessons

The **lesson observations** of colleagues teaching the two control groups were carried out by recording lessons to watch at a subsequent date. The recordings were video taken from the front desk which made a visual record of the teacher and what she wrote or projected onto the board, as well as providing an audio record of lesson contributions by the students. This method for lesson observations was chosen largely for practical purposes because the researcher's own teaching timetable precluded direct observation of lessons, and the quantity of observations needed would not have been practical or realistic in real-time. Also, once the teachers and students had got used to the presence of the camera, the level of intrusion in the lesson was very low which has the advantage that the observer is not directly influencing what is going on in the classroom (Taber, 2007), although the presence of the camera may always influence behaviour by any participant in some way. This method also allowed for the researcher to focus directly on the parts of each lesson of greatest interest, returning to specific portions if necessary, and fast-forward over portions where there was less of interest happening.

The focus for watching the recordings was to examine the similarities and differences between the two individual teachers' approaches to structuring the topic of straight-line graphs and to look in some detail at how they each taught gradient. Instances of linking different representations were noted as were missed opportunities; questions, misconceptions and discussions between the teacher and individual students were also recorded where possible. Full transcription of the lessons was not attempted although accurate recording of short passages of dialogue between teacher and students was used when the actual words used added more useful detail than just notes. The recordings were watched with each teacher's lesson resources available to the researcher. Notes were made during the viewing of each lesson which were summarised and points of interest noted after the observations had been completed. This is an example of an unstructured observation to gather qualitative data, and look for clues and hunches that would need corroboration and expansion from other sources of information at a later point in time (Taber, 2007, Punch & Oancea, 2014).

Following the indirect observation of the control group lessons an informal group discussion took place between the researcher and the teachers who had been observed, using observation notes and lesson resources as a prompt. The purpose of this discussion was to expand on the lessons' intended and actual outcomes plus any reflections on the effectiveness of the resources used and the lessons as a whole. This discussion formed the feedback part of the three-phase observation cycle as identified by Hopkins (1993).

Intervention lessons

A series of six lessons (5x110 minutes, 1x35 minutes) were designed to introduce the intervention group to gradient of straight line and the equation of straight-line graphs; selected resources from the intervention lessons are included in Appendix 2. The first lesson revised the method for drawing straight lines by constructing a table of values, at the same time as starting to look for relationships between the coordinates of different pairs of points. The next three lessons centred on finding the gradient of a line and of examining the relationship between pairs of points, working with both the graph and coordinate values in parallel as well as verbally trying to describe the relationships observed. For the final two lessons the y-axis intercept and general equation of a straight line were introduced, along with students finally working on graphs of different real-life scenarios where they were required to find and interpret gradient and intercept values. The resources for the first four lessons were designed before the start of the intervention process although the detailed planning of the lessons took place on a rather more immediate basis to reflect the reality of what had actually happened in the previous lessons. The resources were mostly worked through by the researcher before the lessons were delivered which helped to ensure their appropriateness and avoid errors, although this was not completely successful. Pressures of time meant that the resources for the final two lessons were collated during the intervention process which meant that there was less time for trialling, checking and assessing before use.

The intervention lesson period included two pieces of homework. For the first of these the students' work was collected in and assessed before the next lesson; for the second the homework task assessment was carried out at the start of the next lesson. This pattern of assessment was the normal routine for the group.

Each lesson was recorded (video for teacher and whiteboard only, audio for students as per the procedure for the control groups) so that they could be referred back to and analysed in more detail after the lessons had taken place. After each lesson the researcher wrote a quick summary of what had happened along with notes to help the planning and delivery of subsequent lessons. The reflection that was carried out after each lesson followed this pattern of lesson evaluation:

- What did the students learn?
- Did the methods that I used for teaching them work well?
- How could I improve my teaching next time? (Menter et al, 2011, p4)

Supplemented by:

- Do I need to change my plans for the next lesson as a result?

Test analysis – quantitative and qualitative

Each question in the pre- and post- test had a *quantitative* aspect that was either multiple choice (A-D response) or numeric and also a *qualitative* aspect where the student was asked to explain the reasoning behind their quantitative response. For each multiple choice question a frequency analysis of the responses was carried out which identified the number of correct responses, as well as highlighting any frequently occurring incorrect responses. For the numeric response questions the number correct was recorded and any frequently occurring identical incorrect answers were noted. In addition the total score for each student was calculated to allow comparison with the post-test. For each set of quantitative responses the data was examined on an individual and group basis, looking for similarities and common differences both within a test and between the pre- and post- tests.

The first stage of the qualitative analysis of both tests was to read all the explanations looking for common themes and making notes with a view to subsequent coding of the data. A complication was that explanations in the form of calculations or diagrams had to be considered as part of the coding process. The second stage was to code the data for descriptive analysis (Savin-Baden & Major, 2013) using key-words identified during the first stage; this coded data was then subject to frequency analysis so that commonality for both individuals and within groups could be identified. Additionally any particularly insightful explanations (for both correct and incorrect responses) were individually noted so they could be referred to during the analysis and reporting of the data; these were, however, used with caution because "the things that students write only provide *indirect* evidence of what they think, know and understand" (Taber, 2007, p146).

Post-research action

At the end of the study the results were shared and discussed with the other teachers involved. The focus of these discussions was on the usefulness of both the overall principle that exploring different representations can enhance students' conceptual understanding, as well as the details of individual resources used as part of the study. Before the start of this research there had been a departmental discussion about whether it would be appropriate to move some element of the straight-line graph work into Year 8; this idea was discussed in more detail in the light of the outcomes of this study.

Participants

A small independent school for girls with two-form entry and approximately 40 students in each year was used for this study. Entry to the school is nominally academically selective with the general expectation that GCSE entry for mathematics is at Higher Tier and that all students should be achieving Grade C and above. The small size of the school involved meant that actually sampling students or staff for inclusion was not appropriate. All of Year 9 (36 students) were used in some way as part of either the control or intervention groups, and all the main specialist mathematics teachers in the school were also involved. A significant advantage of this approach was its ease for administrative purposes especially because the bulk of the school-based research would happen during normal teaching time. The small size of the sample was a disadvantage, especially with only 11 in the intervention teaching group, and this was considered during any quantitative data analysis with care taken to avoid making statistically inappropriate generalisations. Cohen at al. (2011) recognise that 30 is the smallest reasonable sample size if statistical analysis is going to take place – the quantity of data available in this study was not far from that lower limit and, at times, divided into three sub-groups, so extreme care was needed when working with anything beyond the most basic of summary statistics from the data. The small sample size was, however, practically a positive advantage for the process of coding, recording and analysing qualitative responses.

The intervention group was the middle of three groups set by prior mathematical attainment; the setting was initially based on the marks achieved in the entrance exam that all students sit mid-way through Year 6. Reviews had been carried out at regular intervals during Years 7 and 8 with some movement between sets taking place. Those decisions were based on both test marks and on teacher judgement with the student's personality as well as prior mathematical attainment taken into account when choosing the most appropriate set. It is anticipated that the vast majority of students have been allocated the most appropriate group for them by the start of Year 9 and after this point much fewer set changes take place.

The teachers used as part of the study were all the subject specialists who carry out the vast majority of the Key Stage 3 and 4 mathematics teaching in the school. They all started teaching these groups at the start of Year 9 having each taught different groups the previous year.

The use of control and intervention groups that were selected by ability (rather than on a random, mixed-ability basis) and from an ability-selected school population means that results of this study cannot be used on an inferential basis outside of this particular environment. It was important to remember that, despite the experimental nature of parts of this study, the sample of students used was not representative of the Year 9 student population as a whole. "In experimental research 'sample' refers to a subset that is representative of a larger population. In interpretative research it does not carry this connotation." (Thomas, 2009, p104)

Ethical considerations

This study was approved by the Central University Research Ethics Committee of the University of Oxford and permission received to work in the school by its Head-Teacher. Following the principles listed by BERA (2011) the following ethical considerations were taken into account when planning and undertaking the study:

- Testing before and after the teaching of a new topic can be considered normal practice so it was expected that all students would take part without explicit consent. The teaching intervention was an integral part of lessons so all students in the intervention group automatically took part. Resources developed as part of the intervention were made available to the teachers of the control groups so they could use them later in the GCSE course for their group if they felt it appropriate.

- The test results were only used as part of the study and not as part of any wider assessment of student, this was explained to the students before each test both verbally by the teacher administering the test and in the written instructions on the front of each test. Participant names were recorded on the test papers to allow for comparisons between the tests and cross-referencing with baseline data, but the analysis was conducted on an anonymous basis. Specific results were only shared amongst colleagues in the Mathematics department; general findings could be shared more widely in the future but only if relevant (i.e. with the Science department or the school's Director of Studies).
- There was a potential conflict of interests because the researcher was also line manager (in her role as Head of Department) for the colleagues whose lessons were recorded as part of the research. The teachers had the process explained to them before the start of the research and explicit consent sought for the recorded lessons. The consent form explained the conflict of interest issues and stated that the lesson observation data would only be used for the study and not for performance management purposes. The colleagues were involved in the qualitative analysis process for the tests carried out before and after the teaching of the topic.

Taber (2007) discusses the “ethical imperative” (p140) that should be a priority for all teacher-researchers to protect the interests of both the students and any junior colleagues involved in research. He also recognises the removal of one level of protection and safeguarding that occurs when an internal teacher-researcher undertakes work in school compared with an external academic-researcher and notes the importance of the teacher-researcher continuing to consider this need for protection.

Collaborative aspect

The collaborating teachers were involved at three main points in this piece of research:

1. Selection of questions to be included in the pre- and post- tests and subsequent refining of these questions to make them appropriate to the students involved.
2. Being observed teaching, sharing lesson resources for the observed lessons and taking part in a discussion about the rationale behind their lesson planning and the successfulness of the observed lessons.
3. Incorporating the findings of the research into ongoing collaborative work to refine the school’s current KS3 and KS4 schemes of work in light of the changes to GCSE.

5. Results

Observations of recorded lessons for control groups

The series of lessons for the two control groups (Control A and Control B) followed the same basic structure although the pace and extent of work ultimately covered was much greater for Control B, which was as to be expected given their prior mathematical attainment and the school's scheme of work for these groups. The first part of the block of lessons concentrated on recapping Year 7 and 8 work on straight-line graphs by plotting a number of lines and observing the relationship between the numbers in the equation and the behaviour of the lines. For Control A the word "backward" was introduced as the primary adjective to describe the behaviour of lines with a negative gradient. The graphs were plotted by using a table of values with y-coordinates calculated from the equations for pre-determined x-values. In the Control A lesson there was a lot of discussion about where these seemingly arbitrary x-values came from and whether they "had to be in a line". One student subsequently noticed that there was a pattern in y-values and commented on it – she was then told that this would help them all check their calculated values, but there was no further expansion on the relationship between the observed pattern, the behaviour of the graph and the values in the equation. In the Control B lesson the students were asked to give their own description of what gradient means which promoted a discussion with teacher prompts including "how could you find the gradient from a graph?" and "what controls the direction of the graph and why?" The student responses to these prompts turned into notes on gradient relating to steepness, positive/negative gradients and parallel lines but nothing about rates of change. There were opportunities in both classrooms to explore the meaning of gradient and the nature of steepness both in terms of the behaviour of the lines and the relationships between calculated values in tables that had been missed.

Following on from this initial block of work looking at the behaviour of straight lines and using gradient as an arbitrary value that controls steepness and direction both groups were introduced to $y = mx + c$ as the general equation for a straight line. The next activity for both groups was spent extracting gradient and y-intercept values from equations presented in both the standard format and when the order of terms had been changed; there was some work on extracting a fractional gradient when it was expressed as a division but no changing of the subject of the equation was involved. Forwards and backwards directions were still being used in Control A as the primary descriptor of direction. At this stage gradient was still an undefined concept that could be manipulated and talked about but with no real reference to what the numbers mean.

Gradient was formally defined in the third block of work. Control A watched a DVD that used a variety of contexts that gave rise to linear relationships that were displayed as straight-line graphs. Within this, gradient was explained from first principles, e.g. “for every 10 units in the x-direction the graph goes up by 100 units in the y-direction, or for every one unit in the x-direction the graph goes up 10 units in the y-direction”. Control B were given a similar description for a specific, non-contextual example drawn on homogenous axes “for every one I go across I go up four” which lead into an observation that gradient is a rate of change and that 4/1 and 8/2 triangles on the same graph give rise to the same gradient value.

Both groups were given a rule to use when calculating gradient:

Control A	$Gradient = \frac{change\ in\ y}{change\ in\ x} = \frac{rise}{tread}$
Control B	$Gradient = \frac{rise}{tread}$

Both groups dealt with negative gradients simply by observing and inserting a minus sign as appropriate into their answers although Control A wrote positive or negative on each graph before they started calculations.

Control A started off the gradient calculation practice on homogenous axes with dots on the lines indicating where the end-points of the gradient triangles should lie. Rise and tread was written on each triangle with their values before any calculations / division / fraction cancelling. Gradients, together with observed y-intercept values, were combined to give the equations of lines in the standard $y = mx + c$ format. Non-homogenous axes were introduced later, the importance of reading the rise and tread values off the axes rather than counting squares was emphasised. A couple of student questions that the observer considered to be interesting were glossed over – these were examining whether it is rise or tread that is negative for a negative gradient graph "just put the minus sign at the beginning, it doesn't really matter" and what units that gradient should take for a contextual question (miles per gallon for the case in question). More contextual graphs were included for Control A's final lesson including distance-time graphs and using gradient to find speed (where units were included).

Control B also practised finding gradient by working out the equations of lots of lines, starting first with examples presented on homogenous axes and moving on to non-homogenous axes without any prior warning or instruction; this gave rise to some differences of opinion and lively discussion between students. This group took their work on gradient much further that Control A. The first extension exercise explored drawing straight lines from their equations without using a table of values by starting at the y-intercept and then using the gradient to determine the direction of the line. Lines using integer gradients, both positive and negative, resulted in accurate responses from the majority of the group but fractional gradients required more unpicking.

> *Teacher: "How are we going to deal with gradient =* $\frac{1}{2}$ *?"*
>
> *Student: "we could go across 1 and up* $\frac{1}{2}$ *"*
>
> *Teacher: "yes, but we want to be really accurate and use whole numbers, how could we count in whole numbers?"*

This lead to $\frac{1}{2} = \frac{rise}{tread}, across\ 2\ and\ up\ 1$ being written on the board which was adapted to work with more complicated fractional gradients both positive and negative. The final part of this activity was an exploration of the relationship between lines such as $y = \frac{1}{2}x + 3$ and $y = 2x + 3$ with the intercept and gradient method of drawing the lines being used; the reversal of the horizontal and vertical portions of the gradient working, together with the sign change appeared to result in student observations about the behaviour of perpendicular lines.

The second extension exercise looked at finding the gradient of a line segment between two coordinate points. The recommended method was to draw a sketch diagram and then find the height and width of the gradient triangle by using the differences between the two coordinates. This appeared to work successfully within the lesson and is, indeed, the method I have always taught at GCSE rather than using $\frac{y_2 - y_1}{x_2 - x_1}$. However, this method does, in my experience, prove difficult for students who struggle to relate the relative position of two points without a coordinate grid, particularly where negative values are involved. The activities carried out by Control B went considerably further than I planned to attempt with the intervention group but there were aspects that could be modified and introduced, particularly the un-announced introduction of non-homogenous axes and the use of the intercept and gradient method for plotting lines.

Reflections on lessons for the intervention group

The main objective for the first intervention lesson was to explore relationships in a series of linear graphs by using and linking a variety of representations. This was placed within the constraint of needing to revise plotting straight-line graphs from their equation using a table of values. At the start of the activity there was confusion between the different relationships and the students did not make observations in the natural, informal manner that I have experienced when teaching them prior to this study; nor were their observations unprompted in the way that I saw in the observed lessons for the control groups. The students, used to a rather more didactic style, appeared uneasy about this type of exploratory activity. They and I were also getting used to having the lesson recorded which may have influenced both their contributions and my handling of periods of quiet or confusion. The students were also more interested in writing the correct answer than observing relationships and noting what they saw, evidenced by them checking with each other, as well as with me, whether they had written the correct answer. Drawing staircases with different sized steps lead to a number of observations about the behaviour of the sloped line. As time went on the observations from some members of the group showed that they were noticing the relationships between the differences in x-values and y-values. The homework activity set at the end of this lesson was to plot a further three lines and to complete at least one of the observational sections. By far the most popular observational section was answering the questions like “when x increases by 3 what happens to y?” which lead to the gradient expressed as the behaviour of y for a unit increase in x. A number of students also attempted to look at the coordinates for other points that lay on the line and checking that their previous observations still held. Some students were also able to describe accurately the slope of the line in words or give correct descriptions of relationships that would give a negative gradient.

The second lesson revisited the lines plotted in the first lesson and asked students to spend more time exploring the relationships without any reference to the formal equation of each line. The objective was to look at three different representations in tandem:

- Visual representation by drawing a series of different sized staircases (gradient triangles) and calculate $\frac{steps\ up/down}{steps\ right}$ for each one
- The numeric representation by recording the coordinates of the points being used and exploring the relationships between them.
- The covariance relationship by using sentences to describe the relationships and “cancelling” them down to find the behaviour of y as x increases by one.

Negative gradients were approached by drawing staircases above the line thus forcing the x-change to always remain positive and give a decrease in y for each step. In practice many students naturally reverted to drawing the steps under each line irrespective of its direction or, having drawn the step above the line still failed to notice that the y-change would be negative.

The students made use of the covariance sentences although many left them un-cancelled in the style of "when x increases by 2, y decreases by 4"; there was little evidence that they realised they could, and should, cancel them down to a unit x-increase and thus the gradient. Non-integer gradients caused particular problems with this approach, for instance the transfer from "when x increases by 2, y increases by 3" to "when x increases by 1, y increases by $\frac{3}{2}$".

Working with tables of coordinate values caused problems because many of the students made numeric mistakes when calculating the difference between two numbers when one was positive and the other negative. The transfer from a table which had x-coordinates above y-coordinates to a rule using $\frac{y-change}{x-change}$ was also starting to cause problems.

By the end of the lesson each student had the three equivalent methods for calculating gradient recorded in their exercise book and was ready to attempt the homework exercise, which was to calculate the gradient of three new lines using each method and checking their values against each other looking for inconsistencies. The first two questions were on homogenous scales and were included for reinforcement of the lesson's work but the third was a non-homogenous scale and was included without prior warning as I had seen happen in Control B's lesson.

At the start of the third lesson considerable confusion over the homework was brought to my attention. Although a small number of students had made errors in the first two questions the gradient calculation for graphs on homogenous axes was largely accurate. It was, however, the third question that caused the problems, as it had been designed to do, because staircases / gradient triangles that used square counting gave a different answer to using coordinate values. After a discussion a consensus was reached that the coordinate value approach yielded the correct answer and that the calculation for gradient triangle needed to be $\frac{y-change}{x-change}$ looking at coordinate values rather than $\frac{rise}{treat}$ purely using a square counting approach.

Once the problem of non-homogenous axes had been overcome the students then consolidated their skills working with pre-drawn graphs on a variety of axes. Negative gradients were still causing intermittent problems although the word 'decreased' was being used more frequently. There was still some evidence of transposition of the y-change and x-change values in the gradient calculation. Use of a covariance sentence in conjunction with one of the other methods gave clarification of correctness or indication of a problem but the students still had to remember to put the x increment first in their sentence.

This work on gradients of pre-drawn lines was extended to develop a method for finding the gradient of a line segment between two points. The students were given the option to plot formally first before latterly transferring into sketch; those who were the most confident working with coordinate values in tables did not use the diagram and were happy working just with the numbers. There were still identification issues for positive and negative gradients; confusion between 2 and $\frac{1}{2}$ was also becoming apparent.

Eventually I taught them all a formal method for finding the gradient of a line segment without using any sort of diagram; I insisted on recording the point with the smaller x-coordinate first so that the x-change was always positive thus matching the approach for the staircases used on graphs. They had the choice of using a division formula or a covariance sentence. By the end of this section of work all students had developed a preferred method that they could work with although there was still not complete reliability in differentiating between gradients of m, 1/m, -m and -1/m.

An attempt was made to introduce drawing lines of a given gradient onto squared grids without the use of coordinate values. Many students struggled to draw lines with positive integer gradients and demonstrated less than secure knowledge of left and right directions as well as a lack of awareness of the increased accuracy that would be achieved by plotting a series of points rather than just two or three. After some consolidation and repetitive practice that allowed all students to demonstrate increased ability with the entry-level lines, negative gradients were introduced; once again many students' ability to reliably draw accurate lines with positive or negative gradients was low. Finally contextual graphs were introduced and the gradient used to calculate rates of change between two specified variables. Because the students had previously had a lot of exposure to graphs on non-homogenous axes they adapted well to graphs in context. The use of the covariance sentences and constant reinforcement of the basic definition of gradient meant that they should have been well placed to start interpreting gradient values in the context of the question; this was variably true in practice.

Analysis of pre- and post- test results

Quantitative summary

The allocation of control and intervention groups was based on the existing mathematics groups which were organised on the basis of prior attainment at the beginning of the academic year. The intervention group was the middle of three sets with the top and bottom sets forming the control groups. Table 1 shows the mean average score for the pre- and combined post- test for each group along with a baseline measure of mathematical ability (MidYIS 7 Maths score).

	Control A (Maths Set 3) n=7	Control B (Maths Set 1) n=15	Intervention (Maths Set 2) n=10
MidYIS 7 Maths	100	117	108
Pre-test (%age)	31	61	46
Post-tests (%age)	34	44	33

Table 1: baseline measure of mathematical ability and mean average percentage for pre- and post- tests

Comparison of the pre-test results with the baseline data indicates average performance from the three groups that is roughly in line with the standardised baseline scores. The combined post-test scores suggest that the intervention group have not made the expected progress; in fact they appear to have fallen behind Control A. However, while the pre-test standard deviation values for Control A and Intervention were roughly equal (2.05 and 2.06) the values for the post- tests were rather different (2.5 and 3.3) indicating greater variation between the performance of individual respondents in the intervention group.

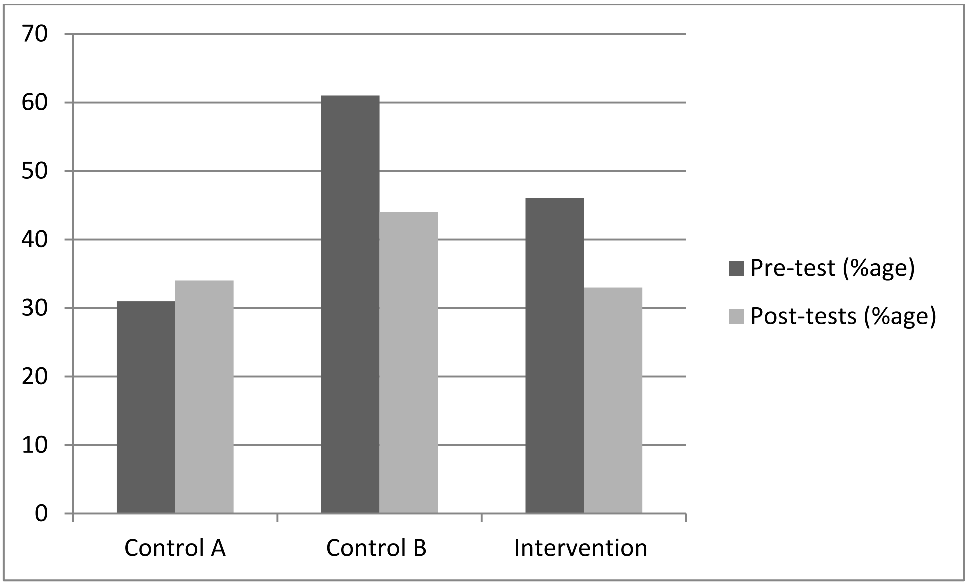

Figure 6: pre- and post- test marks for the control and intervention groups

The small sample sizes involved mean that these values cannot be taken to give global predictions but they do give broad-brush indications of the success or otherwise of the intervention on a local level. They are still, however, heavily influenced by each respondent. Question-by-question analysis looking for patterns of response on an individual basis are needed to assess the successes and limitations of the intervention; these summary statistics do indicate that, while a number of students in the intervention group made progress, a number failed to answer even the most straightforward questions successfully on the post-tests despite showing potential on the pre-test.

Qualitative analysis of selected test questions

The pre- and post- tests were designed to examine a wide range of aspects of gradient and equations of straight lines as covered in the Year 9 programme of study. Detailed analysis was carried out for all questions but only those that covered aspects of gradient that are relevant to the research questions are included here. They have been organised into three main areas:

- Working with contextual values and rates
- Working with coordinate points and patterns
- Working with information presented on graphs with non-homogenous axes

Within each area there might be questions that appeared on both the pre- and first post- test or on just one of the pre-, first post- and second post- test. The post- test was done in two sections and the results combined to give an overall score for the comparative quantitative analysis; this aggregation was carried out because the second post- test was much shorter than the first so it was not appropriate to compare the two sets of results. The actual tests included a number of other questions that were outside the scope of this research; the complete tests are included in Appendix A.

Working with contextual values and rates

Lift-floors question (question 1 in both pre- and first post-test)

Question	Control A (7)	Control B (15)	Intervention (10)
Pre	1	12	5
Post	3	11	8

Table 2: correct answers on part a of the lift-floor question from both tests

Of those answering correctly, the most common explanation (12 for pre- and 16 for post-) explicitly stated that this was because the lift travelled at a rate of 2 floors per second. The proportion citing the rate of travel as the reason for their answer increased from both the pre- to the post- tests; the success rate for each group also correlated with its prior mathematical attainment. It is, however, not clear whether this is an indication of the increased sophistication of mathematical ideas or of the language used. It is also possible that a greater proportion of respondents talked about rates in the post- test because they remembered the question style from the pre- test. Most of the other correct responses mentioned halving in the context of both the number of seconds and the number of floors with a small number giving reasons that did not mention rates or halving even though they had got the correct answer.

Of the incorrect answers the majority of respondents did not use or refer to the change in time interval and had continued to apply the 'subtracting four floors' rule. There was, however, evidence that some respondents had noticed that the time interval had changed but did not apply this change correctly to the number of floors; a number attempted to replicate the time pattern by subtracting 1 from the floor interval rather than halving.

The majority of respondents gave the correct answer to part b (26 for the pre- test and 28 for the post- test) with the breakdown of correct answers per set given in the table below:

	Control A (7)	Control B (15)	Intervention (10)	Total (32)
Pre	3	13	10	26
Post	6	13	9	28

Table 3: correct answers on part b of the lift-floor question from both tests

Of these almost all gave an explanation for their answer similar to:

- "Because it does 4 floors per 2 seconds so it would be 2 floors per 1 second."
- "The lift goes down four floors every two seconds and 2 is half of four and 1 is half of two."

Some respondents looked specifically at one pair of values to make their calculation with the implication being that the pattern would follow for any combinations of floors (that the rate was constant):

- "14 to 10 is a difference of 4, but at a difference of 2 seconds. So as seconds increase by 1, floors must decrease by 2."

- “From 2 to 4 seconds the lift goes from 10 to 6. 10-6=4. 4÷2=2. Therefore it is two floors per second.”

Detailed examination of the pre-test scripts suggests that many respondents had been prompted to change their answer to part a, possibly as a result of working on part b or through the reflective process. Only five respondents got both parts of the question correct with no evident corrections to their response to part a. A further 14 indicated a correct response to both parts of the question but had clearly corrected their answer to part a at some stage. Seven respondents gave correct answers to part b having given incorrect answers to part a; of these most stated that their two statements said the same thing, although one did say that she thought her answer to part a was probably incorrect but did not go back and change it. There was a minority of respondents who got both parts of the question wrong, and all of these consistently stated that the subtracting four pattern needed to be continued:

- “Because every second you are moving down four floors.”
- “Because you go down from 14-10-6-2.”

Working with coordinate points and patterns

<u>Two points on a line, find the third (question 4 on both pre- and first post- test)</u>

	Control A (7)	Control B (15)	Intervention (10)
Pre	2	8	3
Post	0	5	2

Table 4: correct answers for the given two points on a line, find the third question

Although the same question was included in both tests, the pre- test also included the instruction “You may want to use the space below to draw a sketch diagram to help you visualise the points” with a half-page of blank space under the question. All but one of the correct answers included a reasonably accurate diagram.

For the incorrect answers, five used a sketch diagram but the scaling of the axes led to incorrect answers. Of the rest of the respondents who gave an incorrect answer most attempted to use a diagram with a number of diagrams featuring multiple lines; there was also a small number of respondents who attempted to answer the question by pattern spotting. None attempted to put the numbers into a table.

In the post- test the question was stated in the same way but did not have the additional suggestion about using a sketch diagram and there was less space given with the question. Of those who achieved a correct answer, three used a sketch diagram (of those only one had successfully answered the question on the pre- test) and two gave rather imprecise explanations. The two remaining correct answers had not used diagrams and appear to have used some sort of rate of change / pattern spotting.

For the incorrect answers only two now had conceptually correct but badly executed sketch diagrams and none had diagrams that could be used to answer the question. The majority of the incorrect answers attempted to use pattern spotting or gave no reason at all. Two attempted to calculate and apply a gradient but with no success. Once again no students put the coordinates for the three points into a table, although one did use a table of values as part of their attempt to find the gradient.

This question was re-worked for the second post- test to exploit the same mathematical features but with easier numbers (question 3 on second post- test)

Control A (7)	Control B (15)	Intervention (10)
1	3	2

Table 5: correct answers from the re-worked 'given two points on a line find the third' question

Once again there were few correct responses to this question and many struggled to access the problem. From Control A the one correct answer was obtained by drawing an accurate diagram. Another two responses of five and seven were obtained by correctly working with an inaccurate diagram. The remaining responses used half-drawn or inaccurate diagrams. For the three correct responses from Control B the first was from the student who used a complete series of coordinate points by filling in the gap between the first two given points and extrapolating to complete the coordinates for the third point. The second correct response used a diagram but also correctly calculated the gradient between the two given points, crossed this out and did not appear to make use of the information, while the third used a diagram that showed the two given points and then obtained the correct answer with no further working. Of the incorrect responses the strategy of spotting a pattern between the first two given y-coordinates and applying it without reference to the x-increment was the most popular. There were also unsuccessful attempts to use tables of values and coordinate grids.

Of the two successful responses from the intervention group the first was from a student who achieved considerable success on all aspects of the second post- test – this time the gradient was calculated from the two complete coordinate points using a descriptive sentence and then applied to extrapolate via an incremental table of values. The second correct answer was from a student who had been consistently inverting x-change and y-change in all previous questions but, because the actual gradient was not required here, managed to correctly use an incremental table of values. Of the incorrect responses one had attempted an additive rather than multiplicative relationship to scale the y-increment, another had correctly calculated but misapplied the y-increment and the third had started a table of values and correctly noted x- and y-increments but failed to complete the question. The remaining students had failed to access the question and demonstrated this by showing either very sketchy working, or no working at all or by not giving any answer.

<u>Completing coordinates of a point on a line given the gradient and another point on the line</u>

A question of this style appeared on both the pre- and first post- tests but with values too difficult to allow access for all students. It was re-worked with easier numbers for the second post- test (question 2).

Control A (7)	Control B (15)	Intervention (10)
0	1	3

Table 6: correct answers for 'given coordinates of a point and the gradient of the line find a second point' question

Despite the easier numbers this question was still answered poorly with few correct answers. Control A were unable to apply what they had been taught about gradient of a line to this scenario. Some attempted sketch diagrams but got little further, one attempted to apply an incorrect proportional relationship. The correct answer from Control B listed coordinate points that fitted the correct gradient relationship and continued this as far as required to answer the question. Four of the incorrect answers incorrectly used the rule $y = x + 2$ to obtain a y-coordinate of 7, while the rest had attempts at diagrams but made little progress.

The three correct answers from the intervention group used a variety of related strategies: one gave a table of values for all x values from 1 to 5 and y-values stepping by 2 each time and also gave a further table with just the two points of interest; the second used a sketch diagram with four incremental steps from (1,5) to (5,11) and the third produced a table of values which they then used with a value calculated to give the y-change and thus the correct gradient. Of the incorrect responses only three had any real attempt at the question: the first had used a table of a values but had misapplied the gradient by inverting the x- and y-values; the second had applied the gradient to increase the y-value by two without making any reference to the x-increment and the third had increased y by four to mirror the increase in x.

Working with information presented on graphs with non-homogenous axes

Delivery cost question (question 7 on pre- and first post- test)

This was a contextual question. In part a, respondents were asked to find the incremental delivery cost for each extra book in the parcel. This was generally answered well although there were clearly some conceptual problems as well as calculation errors. The distribution of correct answers was roughly the same for both the pre- and the post-test.

	Control A (7)	Control B (15)	Intervention (10)
Pre	2	13	7
Post	2	14	7

Table 7: correct answers for the delivery cost question

Of the correct answers there was a roughly equal split between those who worked out the increment by finding the difference between two costs (most commonly £3.40-£2.80) and those who counted squares and then worked out what each square was worth. There were a number of incorrect answers that appeared to come from conceptually correct logic but with mathematical errors either in the subtraction of values or in working out how much each small square was worth. The respondents in

Control A struggled the most with this question with three indicating that they did not understand what was being asked either by not attempting the question or by attempting to give the delivery cost for one book rather than the incremental value. This may be an indication of literacy rather than numeracy problems for these individual students; the baseline data indicates that these three have some of the lowest vocabulary scores in the cohort.

There was little progress from the pre- to the post- test for this question. In Control B almost all students had been correct in the first place and one additional student answered correctly second time. The final student, who had not attempted the question first time gave an answer that indicated conceptual lack of understanding. In Control A and Intervention the vast majority replicated their answer from the pre- test although a small number gave a different answer (both incorrect to correct and the other way round). The distribution of methods was roughly the same although there were a very small number of students attempting to use gradient with mixed success.

<u>Finding a positive integer gradient on non-homogenous axes (question 1c on second post- test)</u>

Control A (7)	Control B (15)	Intervention (10)
1	2	3

Table 8: correct answers for the 'find a positive gradient on non-homogenous axes' question

This question was included in the second post- test in an attempt to isolate causes of difficultly. It was answered poorly by all three groups, particularly so by both control groups. The three correct answers across the two control groups all stated $\frac{rise}{tread} = \frac{10}{2}$.

For these groups the majority incorrect answer had purely counted squares for the $\frac{rise}{tread}$ calculation without any reference to the scales on the axes.

While the intervention group still answered the question poorly the rate of success was comparatively high. The three correct answers were all achieved by students who had also answered the previous two questions correctly and each used the same method as before (two table of values, one triangle drawn onto diagram but with "measurements" being scale appropriate rather than square counting). Another student (who had also been successful with previous questions) used a table of values but had included one incorrect coordinate point so correctly calculated an incorrect answer. A further two students had produced correct tables of values but had inverted y-change and x-change (consistent with their performance on previous questions) with the remainder using square counting or producing an answer than indicated complete miscomprehension of the question.

<u>Interpretation of given gradient for a contextual graph on non-homogenous axes (question 4 on second post-test)</u>

Control A (7)	Control B (15)	Intervention (10)	
4	5	0	Fully correct solution – gradient value interpreted in context
0	1	2	Partially correct solution – gradient value interpreted without context
0	7	1	Factually correct description of gradient with no reference to the given value or of the context

Table 9: correct and partially correct answers for the gradient interpretation question

The group that gave the highest proportion of fully correct answers to this question was Control A who had been introduced to the concept of gradient via the DVD which used a lot of contextual examples with the gradient defined from first principles. Within Control B there were some students who were able to correctly interpret the gradient but the vast majority ignored (or misinterpreted) the wording of the question and gave a factually correct description that gradient represented the steepness of the line but ignored both the given value and the context. The success rate within the intervention group was very low with no completely correct answers, and two partially correct answers which omitted the context “This means that when x is increased by 1, y is increased by 0.2”. Within all three groups there were a number of responses that had misinterpreted the scale, for instance “For every unit of gas the cost is £2”. There were also a small number who interpreted the question as meaning that they needed to calculate the gradient for the line (or verify that it was the given value) by working with the graph whereas the graph was not in fact needed to answer the question but was provided to give additional contrast.

6. Discussion

The objective for this research was to investigate the links made between representations of straight-line gradient and to see whether making different links between representations would lead to a better understanding of gradient in different situations. Each of these two research objectives will be considered in turn looking in detail at the individual research questions that were identified at the outset.

How are representations used in the teaching of straight-line graphs in my department?

- **What links between representations are currently used?**
- **How does our framework compare with those described in the literature?**

The curriculum audit revealed that students are introduced to drawing straight-line graphs from their equation by constructing a table of values during Year 7, and in Year 8 this is extended by including more complicated gradients (negative and non-integer); additionally observational work about the effect of different gradient and intercept values on both the graph and the equation is used in both years and this is extended to matching equations to their graphs by using the *clues* that gradient and intercept give without any formal calculation of gradient. Up to this point, different representations are used as a tool for the production of a graph; Janvier's (1987) framework (Figure 2) includes translating from equation to table as *computing* and from the table to a graph as *plotting* but because these are routine skills, I struggle to recognise them as a translation between representations. However, the work matching graphs and equations by using features and comparisons would qualify under the first of Leinhardt's (1990) definitions of translation tasks. The equation of a straight line (in the form

$y = mx + c$) is not formally introduced until Year 9 along with the term gradient and a method for its calculation. Gradient, as a measure of slope, is traditionally calculated working entirely from the graphical representation using the slope formula with little apparent understanding, as previously observed by Teuscher & Reys (2010). Work on non-homogenous axes and in context is bolted on, as is any work when the graphical form is not given (e.g. finding the gradient of a line segment between two coordinate points) although this is generally not introduced until Year 10 alongside formal work on parallel and perpendicular lines. Interpretation of rate of change values gathered from any of the representations does not feature in any detail; students are introduced to the formal definition of gradient in passing and occasional reference to a rate of change is made in contextual work. Where there are links made between representations they are generally one-directional and can be described by Level 3 of the framework devised by Hitt (1998) with an emphasis on the *traditional*, indirect routes described by Janvier (1987).

The lesson observations for the control groups demonstrated that little is made of the link between numerical values (either given in a table or as a set of coordinate points that may or may not be extracted from a graph) and gradient calculations. The test results for the lift-floor question revealed that the students have an ability to extract rate of change information and use it to make predictions. This confirmed the findings of Ayalon et al. (forthcoming) who had originally devised and used the question, and had established that students could work with rates of change despite not having been formally introduced to them; Herbert & Pierce (2011) had also found the numeric representation to be the strongest for their students and conversion from graphical to numeric being a frequently used starting point for solving problems.

These representations and links are summarized in Figure 7.

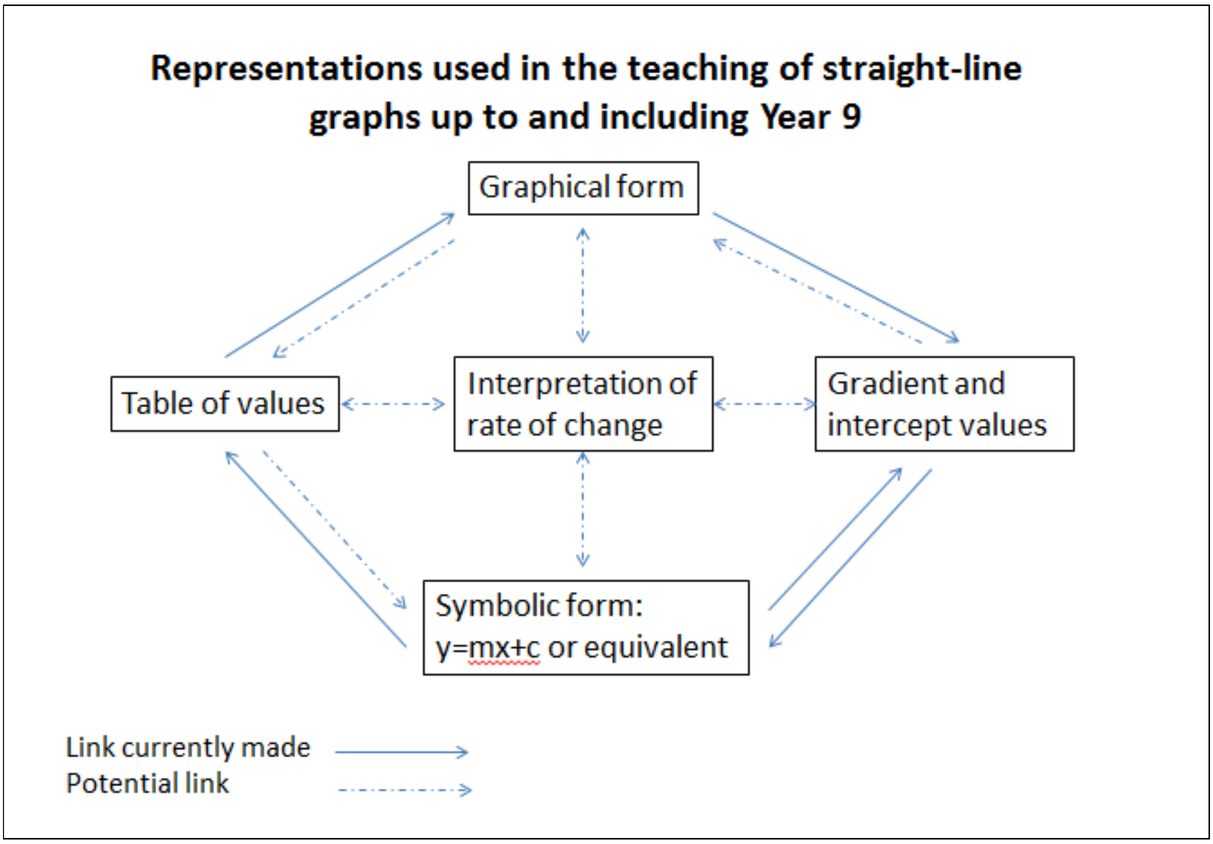

Figure 7: diagram of straight-line graph representations

This shows that the strongest links between representations are in the cycles:

- equation ⇨ table ⇨ graph, and
- graph ⇨ gradient and intercept values ⇨ equation,

with additional links being made in both directions between the equation and its component parts. The lesson observations and Janvier's (1987) framework suggest that further links are possible; firstly those in the reverse direction to the cycle described above and secondly direct links between opposite corners of the diagram: equation ⬄ graph and table ⬄ gradient and intercept values.

- **What other links between representations can possibly be used in the teaching of straight-line graphs?**

The diagram of the links between representations currently used at my school (figure 7) along with Janvier's framework (figure 2) identify a number of alternative links that could be used when teaching straight-line graphs. In the first teaching activity for the intervention group (Appendix 2a) the students were encouraged to look for links between the values they calculated in the table and the behaviour of the line that they subsequently drew. This was an attempt to tap into their natural ability to work with covariance and develop their analytical skills through the different stages of understanding linear relationships as described by Carlson et al. (2002). The table of values was identified by Confrey and Smith (1993) as a useful starting point for the analysis of covariance relations. It was also hoped that if the students could describe the relationship in words 'as x increases by 2, y increases by 6' and cancel this down to a change in y for a unit change in x they would gain greater conceptual understanding of gradient. The students universally found the activity difficult quite possibly because it was trying to attempt too many things at once and, as the teacher, I found guiding the discussion difficult because I knew what I wanted the students to say.

The second activity (Appendix 2b) approached the same linear relationships again but this time starting with the graphical representation with the objective of relating both the graphical picture (the visual perspective as described by Zaslavsky et al. (2002)) and the table of values and covariance relationship (Zaslavsky's analytic perspective). Students were encouraged to work with both perspectives in parallel so that they got used to the alternative methods and were able to appreciate the relationships between them, and also in preparation for the contradictory results they would meet when they were suddenly introduced to graphs on non-homogenous axes (Appendix 2c). One significant problem with the calculation of gradient from differences between values calculated in a table was that the table had the x-values on top and y-values underneath whereas the gradient calculation rule needed them the other way up. It had been hoped that also working with a covariance description sentence would help the students understand which way the values needed to be related but the inversion problem persisted.

Additional intended consequences of working with multiple representations (graphical, numeric and descriptive) and moving between them was that some of the difficult aspects of gradients (negative and / or non-integer values) would be overcome more easily and that the transition to working purely with coordinate points (for calculating the gradient of a line segment) would be made more naturally. The worded description of the relationship shown by a line with a negative gradient helped to an extent primarily because the word 'decrease' was used so prominently but, when working independently, identification of a negative gradient was still inconsistent. Non-integer gradients presented even more problems because their appeared to be a tendency to flip the values in the gradient calculation. There was, however, a relatively easy transition to calculating the gradient of line segments (Appendices 2d and 3e) purely from the coordinate values as this was a seemingly natural extension of the work the students had previously encountered rather than needing a whole new method.

Would making different links between representations develop better conceptual understanding of gradient?

The students' conceptual understanding of gradient was examined in the context of two specific areas of difficulty that had been identified at the start of the research; prediction tasks with information presented non-graphically and work on non-homogenous axes. Given that the majority of the students had demonstrated that they could naturally work with rates of change calculated from values in a contextual situation the over-riding objective was to assess whether this covariational ability could be transferred to a more abstract situation.

- **Are students who have explored different links better equipped to apply their knowledge in unfamiliar situations, particularly prediction tasks?**

The students were presented with two questions that required them to make predictions based on a selection of coordinate and other information presented as values rather than graphically. The quantitative assessment demonstrated a low success rate for both questions for students from all groups (control and intervention) although there were interesting differences depending on how the question was presented and also some evidence of correct thinking using different representations from some members of the intervention group.

The first question required students to make a prediction about the position of a third point on a straight-line having been given coordinates for two other points that lie on the line. The results of the lift-floor question discussed earlier suggest that the majority of students involved in the study were capable of using covariational reasoning to predict a value from a linear pattern even if the interval for the other variable changed.

However, when presented with largely equivalent information in the form of coordinates the vast majority of students could not answer the question correctly. When explicitly told that they could use a diagram those who did were more successful at answering the question, although they generally needed a fairly accurate sketch diagram to do so, suggesting that they are *physically* or *spatially grounded* (Bieda & Nathan, 2009). For those not using a diagram many demonstrated misapplied proportional reasoning as discussed by Stacey (1989). When the students were presented with the same task, or even a moderated version of it using easier numbers, without any explicit reference to use of a diagram there were very few correct or partially correct solutions from the control groups, indeed the majority appeared unable to start. In contrast, some students in the intervention group were able to answer the question, or demonstrate skills that would lead to a correct answer barring numerical errors, by putting the values into a table and using covariational reasoning. These students would have been described as interpretatively grounded by Bieda & Nathan (2009) and demonstrated a preference for a numeric representation as discussed by Herbert & Pierce (2011).

The same general pattern of performance was continued into the second question where the students were asked to complete the coordinates of a point on a line when given another point on the line and its gradient. Even given the re-working of the question for the second post-test to make the numbers more accessible, the success rate across all three groups was poor. The successful responses all converted the information into a numeric representation to a greater or lesser extent; by way of a contrast from the previous questions the majority of incorrect solutions had attempted to derive a $y = f(x)$ correspondence relationship, as advocated by French (2002) but were working from insufficient information for this to be a valid strategy in this case.

My surprise at the students' inability to access this problem even when the numbers had been made easy mirrors that of Kerslake (1981) "It was thought that most children would know the [.....] relationship" (p130) although the students' unsuccessful attempts to draw straight lines with a given gradient, no matter how easy the values appeared, should have given some prior warning. However, if the question had been presented in a form like the lift-floors question discussed earlier and in Ayalon et al. (forthcoming) there is still a strong feeling that it would have been answered much better.

Overall two conclusions can be drawn from the investigations around this research question. Firstly, students need instruction and practice at interpreting questions presented in coordinate form and in considering how the points and values will relate to each other without needing to plot points in any sort of accurate manner. Secondly, working within a numeric representation and making use of inherent covariational understanding offers a useful solution strategy for students of all levels of prior mathematical attainment that can be applied directly to prediction tasks and has potential for application to other unusual problem types.

- **Do students who have used methods other than the gradient formula perform better on tasks presented on non-homogenous axes?**

The students were given three tests questions involving graphs presented on non-homogenous axes. As for previous questions the quantitative analysis revealed very mixed results across the groups from which no obvious conclusions could be drawn. When asked to calculate the gradient of a line presented on non-homogenous axes with no context attached the success rate was low for all groups, although students in the intervention group who were more used to converting coordinates to numbers in a table than using $gradient = \frac{rise}{tread}$ direct from the graph were more successful than the control groups. This, using the definitions of Zaslavsky et al. (2002) gives some evidence that the students in the intervention group who had translated to the numeric representation were taking a more analytic approach to gradient calculations whereas those in the control group were still dependent on the visual perspective. The findings match those of Kerslake (1977) who established that the visual perspective was strongest when students were asked to match graphs representing the same information. Students from both the intervention and control groups had demonstrated in lessons much greater success rates with routine problems like this one, but by the time they were tested were using part-remembered procedures. This could be seen as evidence that without a problem solving context, routine skills are less memorable or become less well embedded for the student as discussed by French (2002).

The other two problems using graphs on non-homogenous axes were, however, presented in context and had a semi-realistic problem attached to them. For the first of these the students were asked to extract a per-unit delivery cost from a graph; this had a relatively high success rate for both the Intervention and Control B groups although Control A (the lower attaining set) struggled. It is worth noting that the same question was used on both the pre- and post- test and that the success rate was approximately the same both times. This suggests that success at this question had rather more to do with prior mathematical experience and attainment, particularly problem solving skills and literacy levels, than on the graph interpretation experiences for any of the groups. These outcomes compare favourably with those of Gomez (1999) who found that the majority of her respondents struggled to calculate a rate of change from a contextual graph with many of the incorrect responses using the coordinates of the end point as the only input to their rate calculation; that error was not seen here although a number selected either the starting point (y-intercept) or the cost for one unit instead of using two points to calculate a rate. Both of these errors were discussed in the literature and identified as common misconceptions (Clement, 1989; Hadjidemetriou & Williams, 2002).

As a further contrast the third problem (used only on the second post-test) presented another contextual graph but this time the students were given a gradient value and asked to interpret it in the context of the graph. The results from this question were rather unexpected with Control A having the highest success rate followed by Control B and Intervention thus contradicting the interpretation of previous results in light of prior mathematical attainment and literacy levels. One possible explanation is that Control A were initially introduced to gradient calculations via a DVD that included a number of contextual examples with discussions about interpreting gradient values. Another point of interest from this question was the proportion of Control B who misinterpreted the question and gave factually correct interpretations of gradient without any reference to the value they were given or to the context of the question.

Looking at the results of these three questions alongside one another suggests that while the students' inherent ability to process rate of change information is good, as seen earlier in the lift-floor questions and in Ayalon et al. (forthcoming) the introduction of the word gradient causes difficulty. It would be interesting to directly follow the successfully answered question about unit cost of book delivery with the same graph stripped of its context and ask the students to calculate the gradient of the line. There is evidence in the literature (Elia et al, 2007; Gagatsis & Shiakalli, 2004) that students struggle to recognise alternative representations of the same function when presented in consecutive questions although, by way of a contrast, Kerslake (1981) noted that when students were asked to plot the same graph both in a contextual setting and as an abstract graph the success rate was very similar.

The problems caused by introduction of formal or technical language correspond with those found by Davis (2007) and Mevarech & Kramarsky (1997) although in the context of preparing these students for GCSE examination the introduction of formal language cannot be delayed any longer. Stump (2001) found some correlation between use of formal language and maturity of understanding; there does not appear to be the same correlation in these results although the inability of Control B to interpret a gradient value in the context of the question may have more to do with their desire to over- or under-complicate the question (or purely not reading it properly) than their understanding level.

Once again, while there was no dramatically improved performance for students in the intervention group, the methods they had used offer potential as part of a range of problem solving strategies. It seems to be important that the teaching of gradient does not occur entirely in the visual perspective because the analytic perspective and numeric representations offer greater potential for success and self-correction when working on non-homogenous axes. It is also worth noting that while many students can easily interpret a contextual rate of change presented graphically, this skill is not universal and that the introduction of technical language can prove an impediment for many.

7. Conclusions

While the quantitative results of this research did not reveal any startling outcomes, the qualitative analysis revealed great potential for strategies to be incorporated into our teaching of straight-line graphs in the future. Of particular interest were the numerically based methods for calculating the gradient of a line, moving away from reliance on $gradient = \frac{rise}{tread}$ as the only method; one colleague has already started using $gradient = \frac{change\ in\ y}{change\ in\ x}$ as her preferred terminology. Additionally, recognising that our students can work well on problems like the lift-floor question offers potential new strategies for approaching tasks that require similar logic, even if they are presented using different representations.

As a department we had already been discussing how we could re-structure our teaching of this topic, in conjunction with revising all Key Stage 3 schemes of work in light of the demands of the new mathematics GCSE. As a result of this study we have decided to spend a block of time in Year 8 working with linear rates of change presented in a variety of manners including formally teaching about gradient; this will be followed in Year 9 by introducing the equation of a straight line and extending into work on parallel and perpendicular lines. This will enable us to make use of and expand our students' natural ability to perform covariational reasoning and build on work done with linear sequences in Year 7. The intention is that when $y = mx + c$ and other forms of equation for a straight line are introduced in Year 9 gradient will be a concept that all students are comfortable with.

The methodology employed for this study generally proved to be sound although I should have listened more to my colleagues when they expressed concern that the questions I had selected for the pre-test and especially for the post-test were too difficult. The qualitative aspect of the data collected from the tests was the most useful which worked against my instincts as a mathematician to use quantitative analysis as much as possible. My original plan had been to include some clinical interviews of students working on problems, particularly making predictions or using non-homogenous axes, so I could ask some probing questions that would attempt to understand the strategies they were using and mistakes they were making. The need to carry out a second set of post-tests for all students meant that these interviews had to be abandoned which was a shame because I felt they would had added colour to the rest of my findings.

The collaborative aspect of this research has not been easy. While my school has an emphasis on improving teaching and learning there is little history of lesson based study; this is particularly true in the mathematics department. Everyone is busy and we are feeling increasing pressure from the demands of the curriculum to push on with new content leaving less and less time for interesting diversions, either for the students or the teachers. My colleagues were naturally nervous about having their lessons recorded and observed and had a tendency to appear defensive whenever I talked about the teaching I had seen. My personal feeling started with “well, I never mind having my lessons watched” but I actually found recording my own lessons very nervewracking due to the twin pressures of generating results for this research as well as experimenting with teaching methods for a topic I knew the students would find difficult. I feel that as a relatively new head of department these experiences will be positive for my future work to help my colleagues develop their teaching and for us to embed a culture where we can all learn from each other in a supportive environment. This, in turn, has implications for the development of teaching and learning in the school as a whole.

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Appendix 1a: Pre- test questions

Question 1

You are staying in a hotel on its 14th floor. You are going to use the lift to go down to the parking level. The hotel has a ground level numbered zero, and there are several parking levels underneath the zero floor

The table below shows what floor you reach after a number of seconds.

Number of seconds	Floor number
0	14
2	10
4	6
6	2
7	?

1a) Where will the lift be after seven seconds?

A. 1	B. -2	C. 0	D. -1

Explain your answer:………

……

……

1b) At what rate does the lift descend?

A. 0.5 floors per second	B. Four floors per second	C. Two floors per second	D. One floor per second

Explain your answer:………

……

……

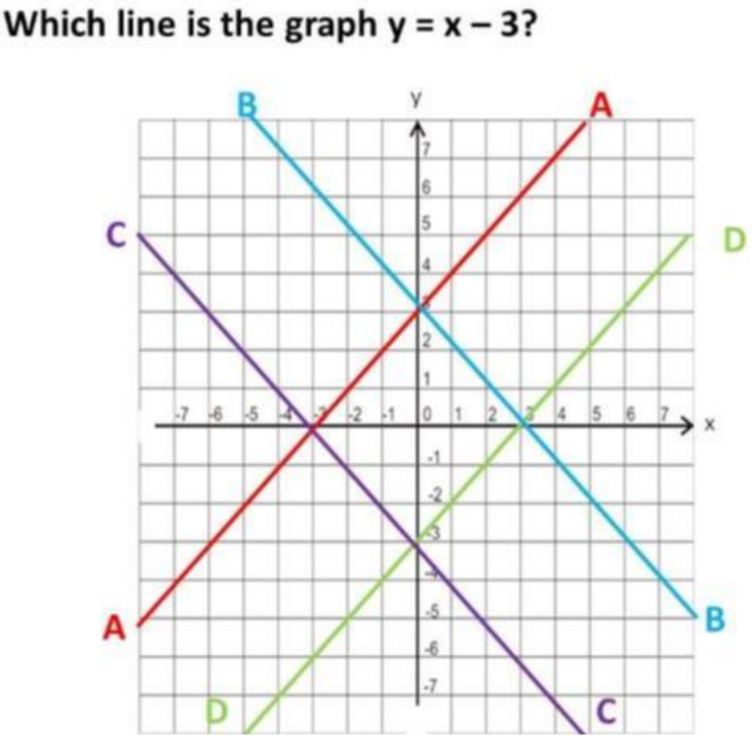
1c) Does your answer to question 1b fit with your answer to question 1a?

……

……

[Source: Ayalon et al. (forthcoming)]

Question 2.



A.	B.	C.	D.

Explain your answer:……

……

……

[Source: www.diagnosticquestions.com/Questions/Go#/1666]

Question 3

These three points all lie on a straight line:

(1, 5) (3, 11) (5, 17)

Which equation describes the line?

A. $y = 2x + 3$	B. $y = 3x + 2$	C. $y = 5x$	D. $y = 3x - 2$

Explain your answer:……

……

……

[Source: www.diagnosticquestions.com/Questions/Go#/733]

Question 4

The straight line joining points A (0, 8) and B (-4, 0) passes through the point C (p, -4).

What is the value of p?

A. -8	B. -6	C. -2	D. 6

Explain your answer:………

……

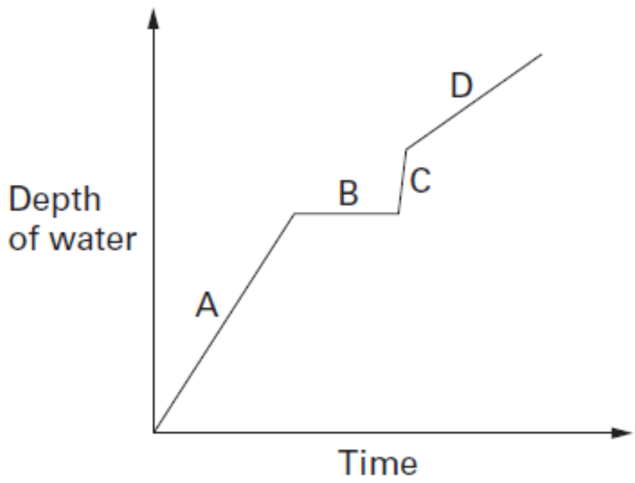
……

[Source: www.diagnosticquestions.com/Questions/Go#/3381]

Question 5

Alice starts to have a bath

The simplified graph shows the depth of water in the bath



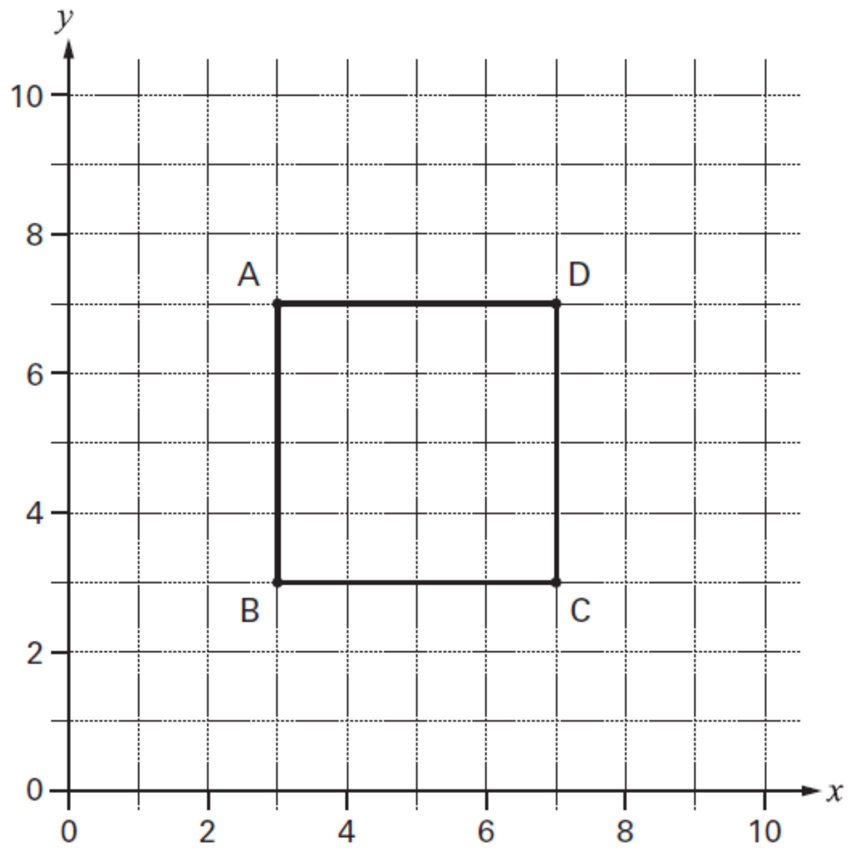
Each line on the graph is labelled with a letter. Write down the letter which best matches the descriptions below. Explain how you made your decision each time.

	Graph section	Explanation
The cold tap is off. The hot tap is off.		
Alice gets in the bath.		
The cold tap is off. The hot tap is full on.		
The cold tap is full on. The hot tap is full on.		

[Source: Year 7 optional 2004 Mathematics level 4-6 paper 1, question 18]

Question 6

The graph shows square ABCD



The equation of the straight line through C and D is x=7

6a) What is the equation of the straight line through B and C? ………………………………..

Explain your answer:………

………

………

6b) What is the equation of the straight line through B and D? ………………………………..

Explain your answer:………

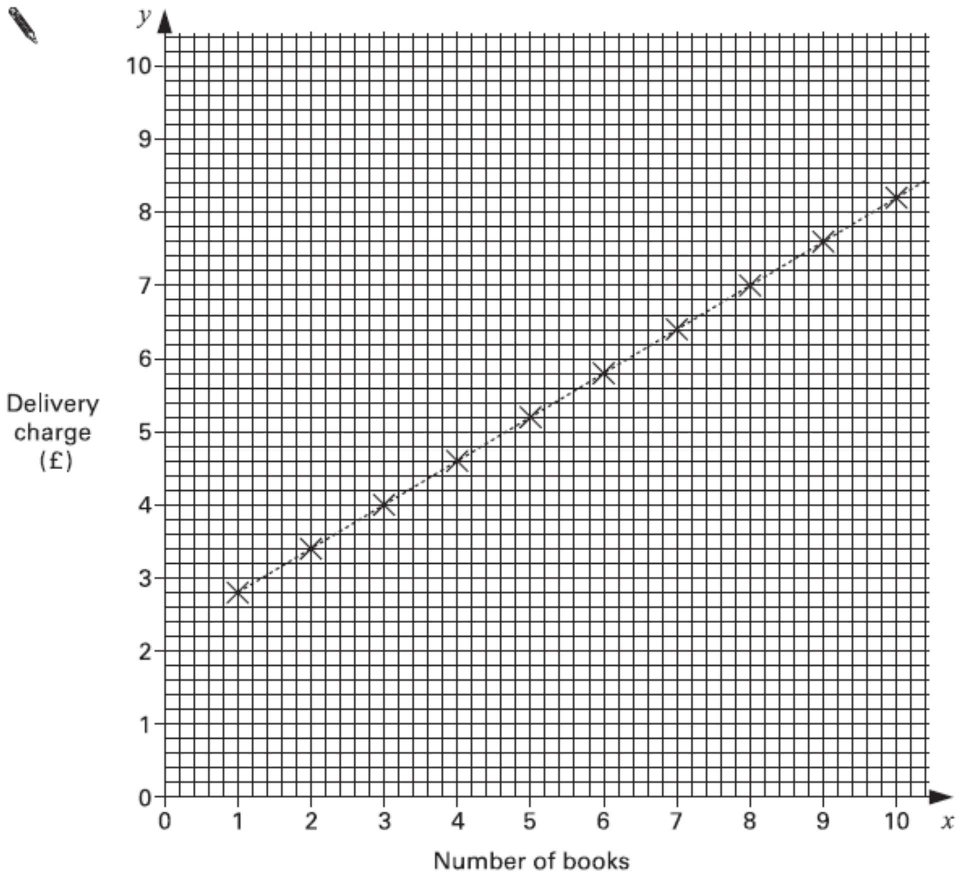
………

………

[Source: SATS KS3 Year 8 2005 Level 4-6 Paper 1, question 23]

Question 7

A company sells books using the internet. The graph below shows their delivery charges.



6a) For every extra book you buy, how much more must you pay for delivery?

…………………………..

Explain your answer:……

……

……

6b) What is the fixed charge for deliveries that applies irrespective of the number of books delivered?

…………………………..

Explain your answer:……

……

……

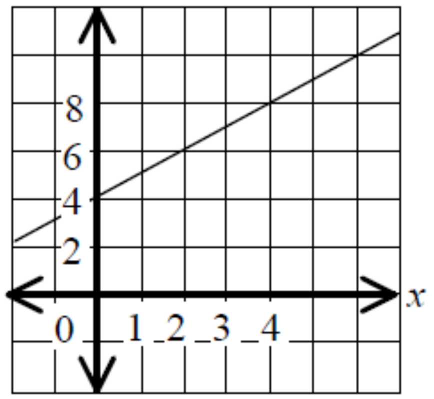
[Source: adapted from SATS KS3 Year 9 2005 Level 5-7 Paper 1, question 5]

Question 8

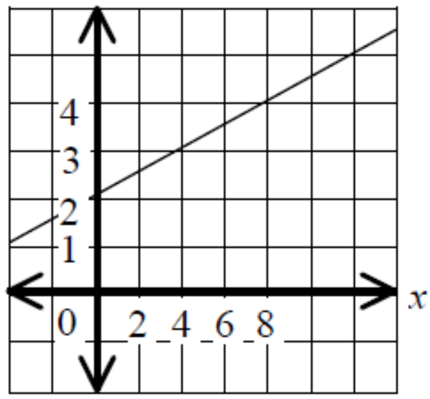
Two of the graphs below represent the same function or relationship.

Please circle these two graphs and then explain how you determined your answer.

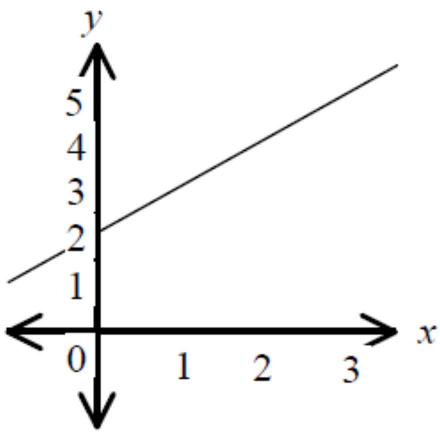
a.



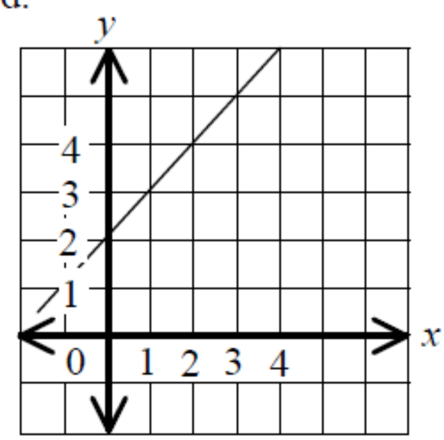
b.



c.



d.



Explain your answer:……

………

………

[Source: Ronda (2004)]

Appendix 1b: First post- test

Question 1

You are staying in a hotel on its 14^{th} floor. You are going to use the lift to go down to the parking level. The hotel has a ground level numbered zero, and there are several parking levels underneath the zero floor

The table below shows what floor you reach after a number of seconds.

Number of seconds	Floor number
0	14
2	10
4	6
6	2
7	?

1a) Where will the lift be after seven seconds?

A. 1	B. -2	C. 0	D. -1

Explain your answer:……

……

……

1b) At what rate does the lift descend?

A. 0.5 floors per second	B. Four floors per second	C. Two floors per second	D. One floor per second

Explain your answer:……

……

……

1c) Does your answer to question 1b fit with your answer to question 1a?

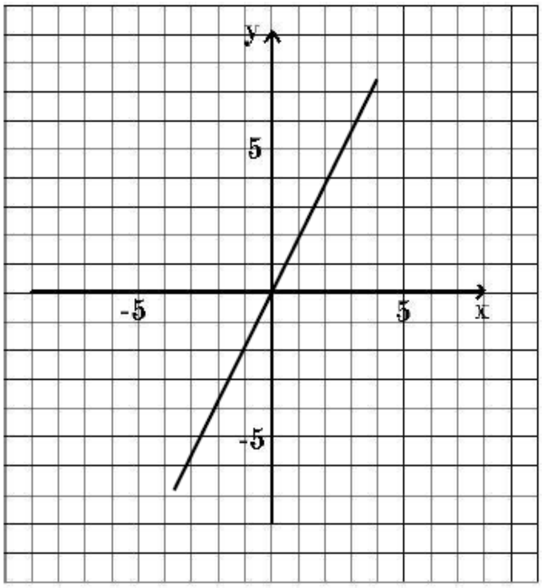
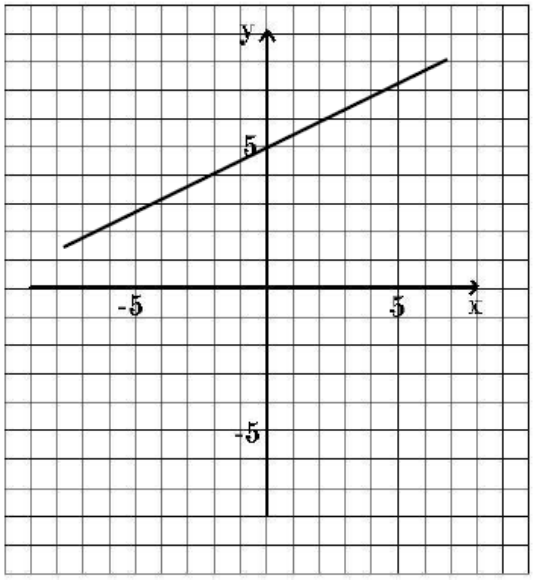
……

……

[Source: Ayalon et al. (forthcoming)]

Question 2

Below are four straight lines. Two are in the form of equations, and two are in the form of graphs.

1. $y = 2x + 9$	2. $y = 5x + 2$
3. 	4. 

2a) Indicate all the straight lines given above that are parallel to $y = 2x + 5$. There can be more than one answer.

2b) How do you know that your answer to question 2a is correct?

Explain in as much detail and mathematical language as you can.

………

………

………

………

………

………

………

[Source: Ayalon et al. (forthcoming)]

Question 3

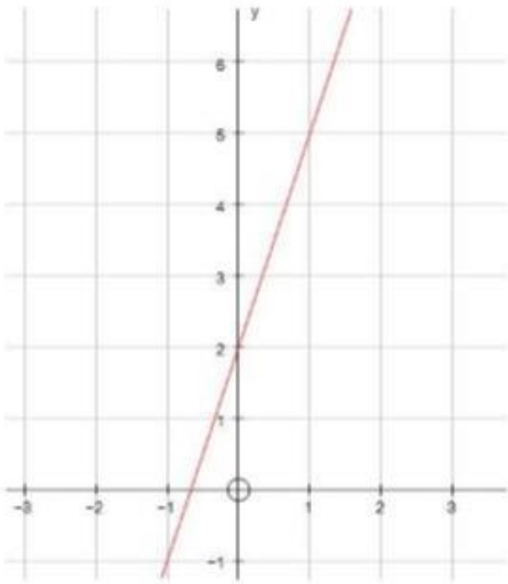
What is the equation of the line?

A) $y = 2x + 3$

B) $y = 3x + 2$

C) $y = x + 2$

D) $y = 3x$



A.	**B.**	**C.**	**D.**

Explain your answer:……

………

………

[Source: www.diagnosticquestions.com/Questions/Go#/2303]

Question 4

The straight line joining points A (0, 8) and B (-4, 0) passes through the point C (p, -4).

What is the value of p?

A. -8	B. -6	C. -2	D. 6

Explain your answer:……

………

………

[Source: www.diagnosticquestions.com/Questions/Go#/3381]

Question 5

The line joining the points (-2, -3) and (6, k) has gradient $\frac{2}{3}$

What is the value of k?

A. $\frac{14}{3}$	B. $\frac{7}{3}$	C. $-\frac{1}{3}$	D. -9

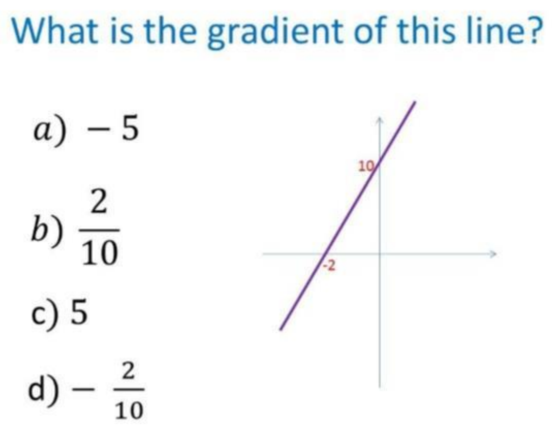
Explain your answer:……

………

………

[Source: www.diagnosticquestions.com/Questions/Go#/3168]

Question 6



A.	B.	C.	D.

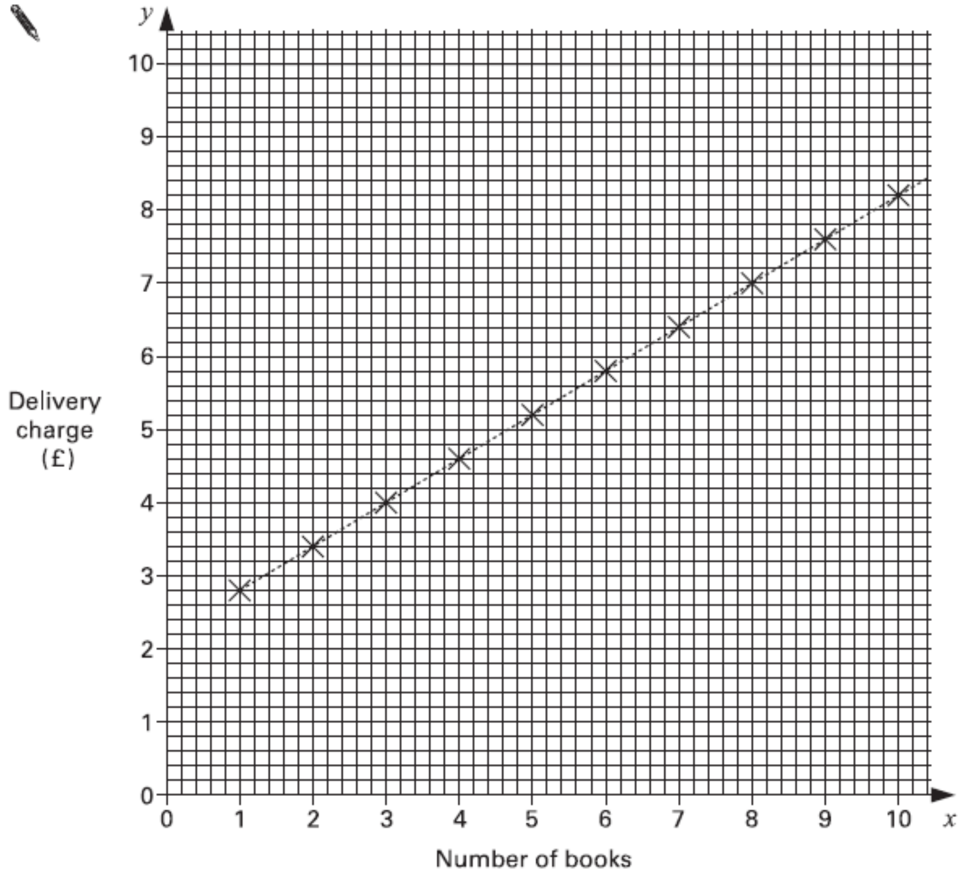
Explain your answer:……

………

[Source: www.diagnosticquestions.com/Questions/Go#/629]

Question 7

A company sells books using the internet. The graph below shows their delivery charges.



6a) For every extra book you buy, how much more must you pay for delivery?

…………………………….

Explain your answer:………

………

………

6b) What is the fixed charge for deliveries that applies irrespective of the number of books delivered?

…………………………….

Explain your answer:………

………

………

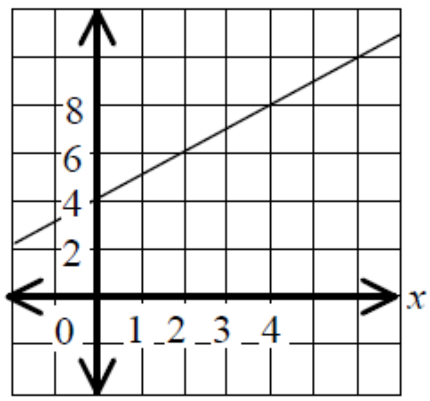
[Source: adapted from SATS KS3 Year 9 2005 Level 5-7 Paper 1, question 5]

Question 8

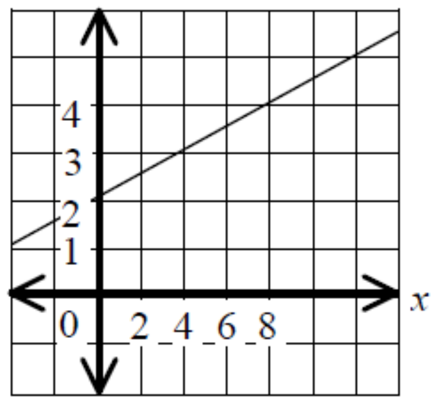
Two of the graphs below represent the same function or relationship.

Please circle these two graphs and then explain how you determined your answer.

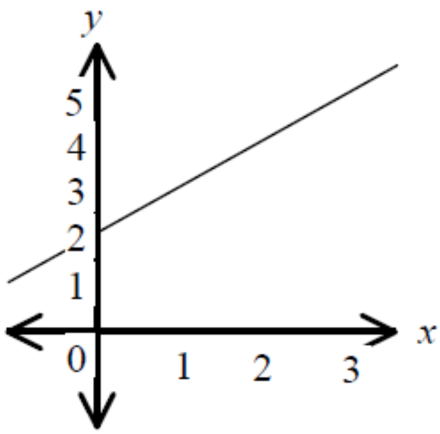
a.



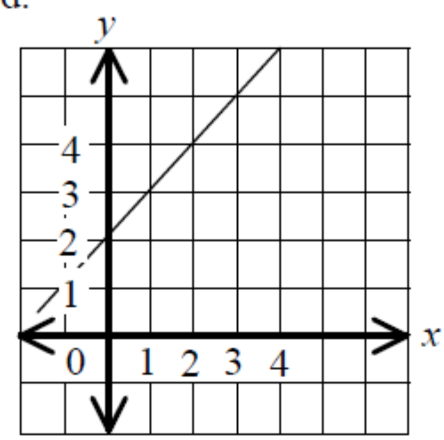
b.



c.



d.



Explain your answer:……

………

………

[Source: Ronda (2004)]

Question 9

A straight line has equation $10y = 3x + 15$

Which of the following is true?

A. The gradient is 3 and the y-intercept is 15
B. The gradient is 0.3 and the y-intercept is 1.5
C. The gradient is 15 and the y-intercept is 3
D. The gradient is 1.5 and the y-intercept is 0.3

Explain your answer:………

………

………

[Source: www.diagnosticquestions.com/Questions/Go#/285]

Question 10

The relationship between x and y in Table 1 is $y = 2x + 1$

In Table 2, the values of x and y from Table 1 have been swapped.

Please write the equation which shows the relationship between x and y in Table 2.

Table 1

x	y
0	1
1	3
2	5
3	7
4	9

Table 2

x	y
1	0
3	1
5	2
7	3
9	4

$y = 2x + 1$ ……………………………..

Explain your answer:………

………

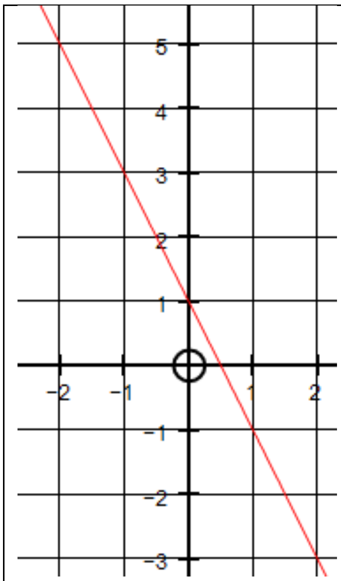
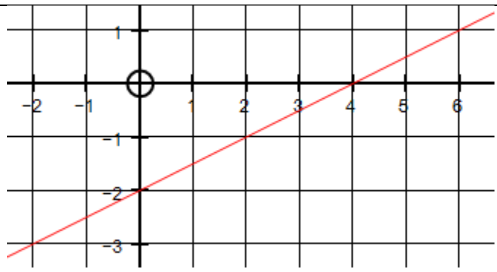
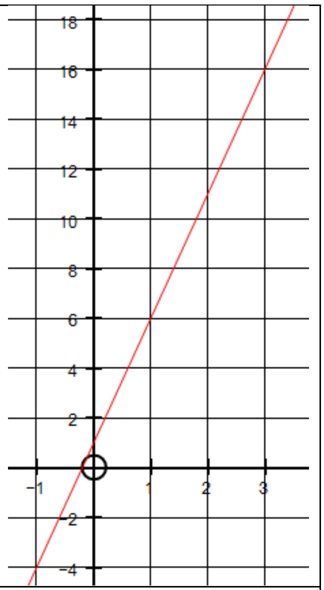
………

[Source: Ronda (2004)]

Appendix 1c: Second post- test

Some more questions about straight lines

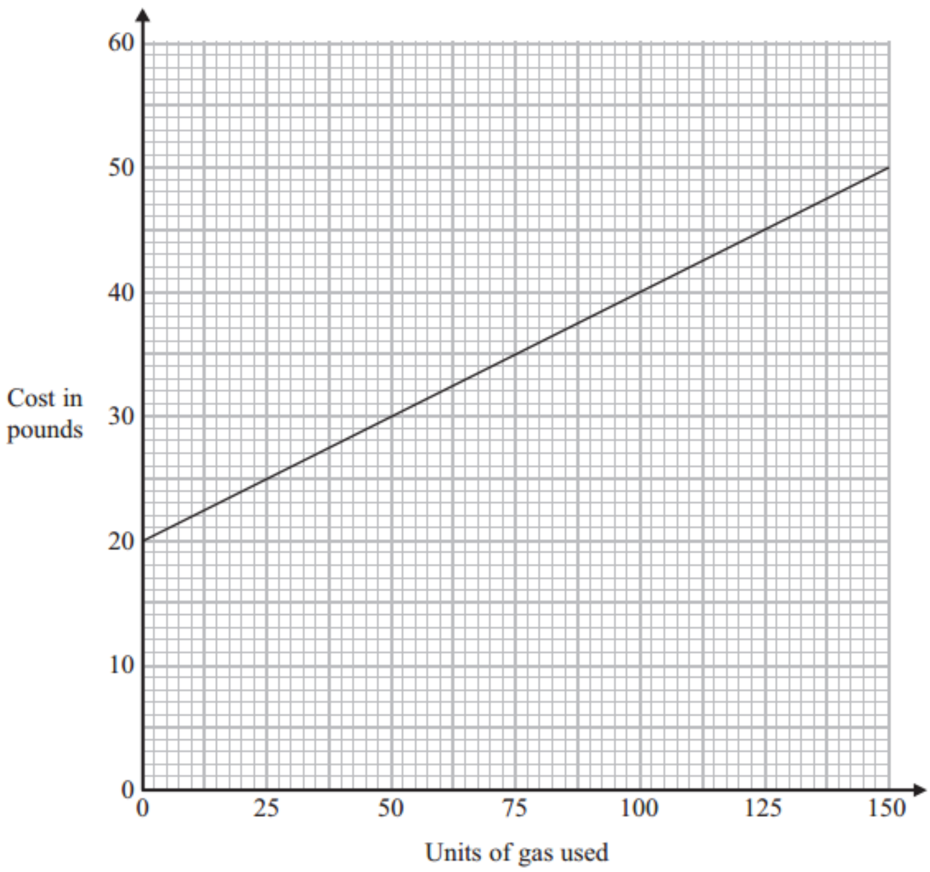
1. Find the gradient of these lines – use the boxes below to show how you worked each one out

		
Gradient =	Gradient =	Gradient =

2. A line with a gradient of 2 passes through the point (1,3) and (5, m). Find the value of m.

3. A line passes through (1,1), (3,7) and (n, 16). Find the value of n.

4. This graph shows the cost of gas



The gradient of this line is 0.2. Explain what this number means in the context of the graph.

………

………

[Source: all questions derived by researcher, final question image from a past GCSE paper]

Appendix 2a: First teaching resource

1a) Complete the table of values for $y = x + 3$

x	-6	-3	-2	0	1	4	6
y							

b) Answer the following questions:

- when x increases by 3 what happens to y?...

....

- when x increases by 2 what happens to y?...

....

- when x increases by 1 what happens to y?...

....

c) Plot the graph on the grid opposite -*remember to continue to the line to the edge of the grid and label the line with its equation*

d) Describe the slope of the line in words – *think about in across and up or down terms*)

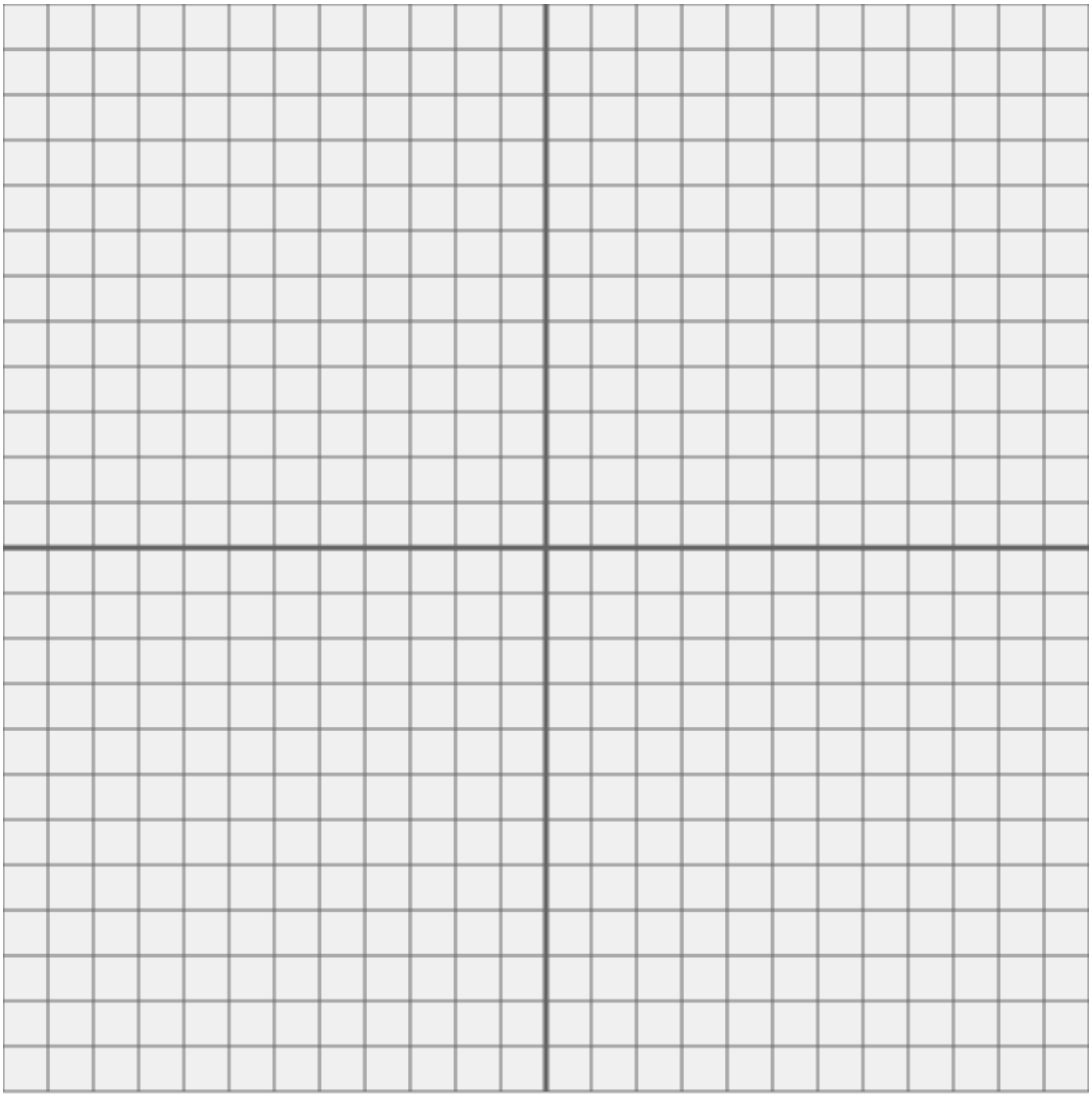
....

....

e) For each of the x-coordinate rules given write down two new points on the line and state what happens to the y-coordinates

x-coordinate rule	coordinates	y-coordinate rule
increase by 3		
increase by 2		
increase by 1		

$$y = x + 3$$



f) Can you see any connection between your answers to b, d and e?

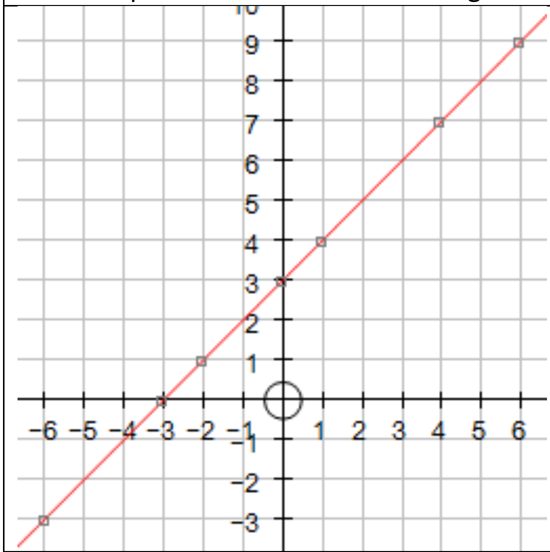
………

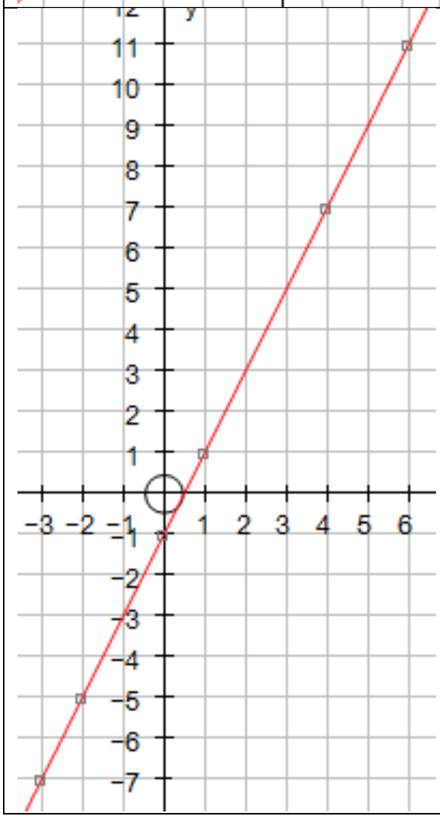
………

………

Appendix 2b: Second teaching resource

- For each graph draw a right angled-triangle that joins up two of the marked points – for each triangle go right from the first point until you are level with the second and then up or down to complete the triangle.
- Work out the steps right and the steps up or down for your triangle and calculate $\frac{steps\ up/down}{steps\ right}$.
- Repeat for a different triangle – do you notice anything about your two answers.
- Also, use a table to record the pairs of points you used each time – can you see any pattern in the change in x-coordinate and the change in y-coordinate?
- Complete the sentence describing the slope.





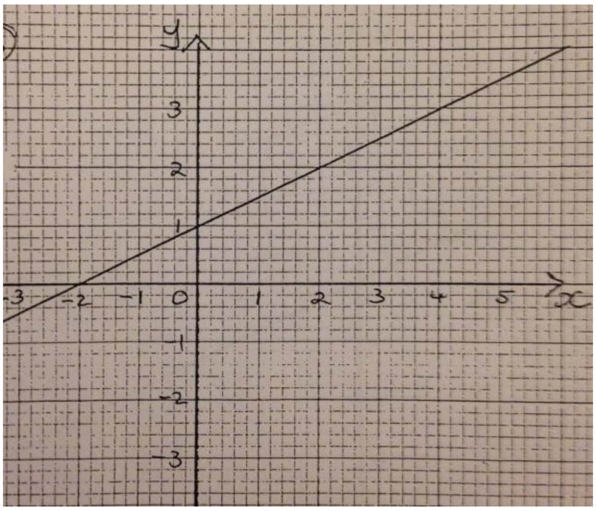
Appendix 2c: Third teaching resource

For each line:

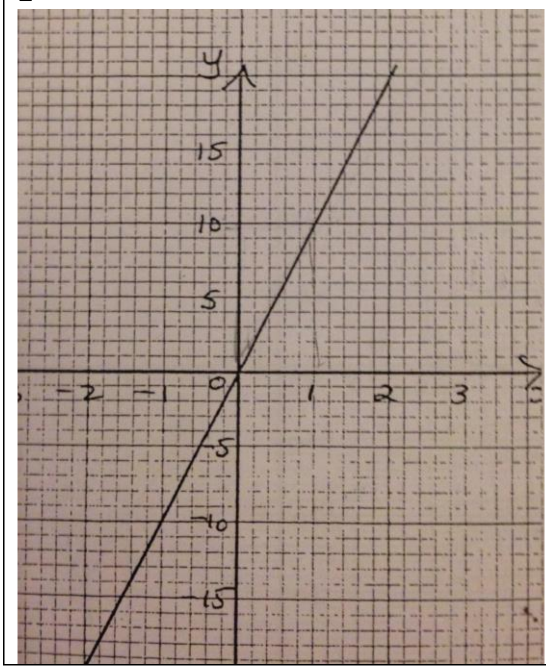
- pick two points to make a gradient triangle, draw in the triangle and calculate the gradient in your (showing all your working out).
- put the coordinates of both points into a coordinate table and compare x-change and y-change.
- write a sentence describing the gradient of each line.

THINK do your three ways of describing the gradient of each line all say the same thing? If not, why not? Is one of them wrong? Can you use the other information to go back and correct your work?

1



2

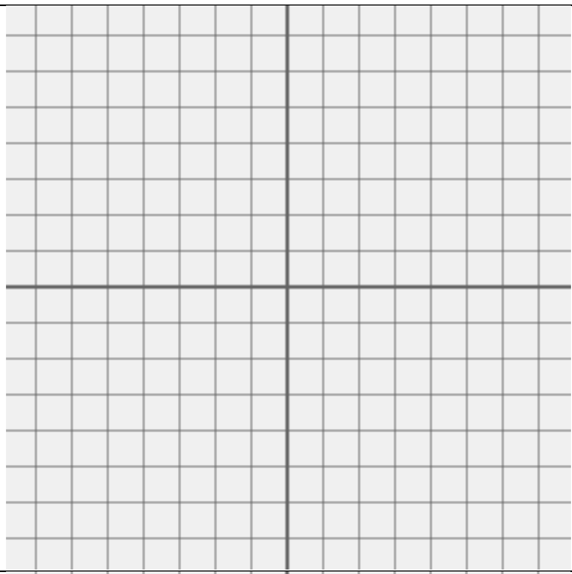
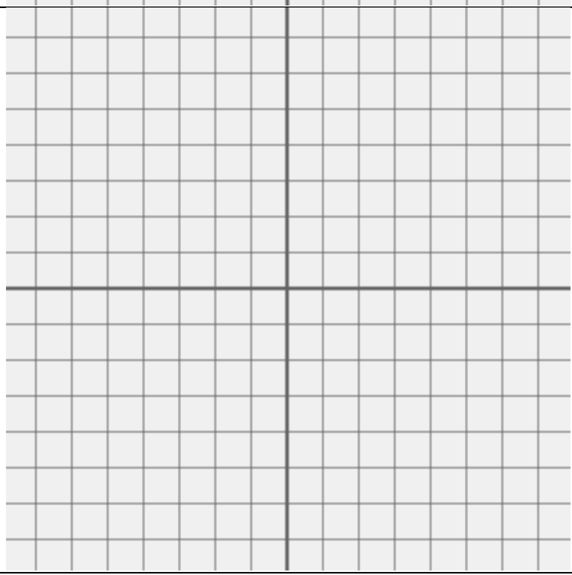


Appendix 2d: Fourth teaching resource

For each pair of points

- plot on the coordinate axes
- draw in the line that passes through the points and calculate its gradient by drawing on a triangle
- put the coordinates in a table and calculate the changes in x and y-values – use these to calculate the gradient
- write a sentence describing the gradient
- check that all three answers say the same thing

At the end – think which of the methods of finding the gradient you find most useful

1 (1, -2) and (-3, 2)	
2 (4, 5) and (3, -5)	

Appendix 2e: Fifth teaching resource

Activity 2D

For each pair of points

- sketch the relative position of the two points, draw in a gradient triangle and calculate the gradient
- use a coordinate table to calculate the gradient
- describe the gradient in words
- check that all three answers say the same thing

At the end – think which of the methods of finding the gradient you find most useful

1 (3, -2) and (5, 10)	
2 (2, -2) and (0, 2)	
3 (1, 7) and (2, 4)	
4 (3, 1) and (2, 0)	