The Design of Inducement Prize Contests for Research and Innovation

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Submitted in partial fulfilment of the requirements for the degree of

Doctor of Philosophy in Economics

October 12, 2013

The Design of Inducement Prize Contests for Research and Innovation, Matthew Moore, Nuffield College. Submitted in Trinty Term 2013 for the degree of Doctor of Philosophy in Economics.

Abstract Inducement prize contests, where a monetary prize is offered for a specified technological achievement, are increasingly popular means of incentivising research and innovation. Such prizes are often modelled by the rent-seeking Tullock contest, or an all-pay auction. However, the direct application of such models can overlook some particular features of technological competition. This thesis addresses three such features.

The first model notes that preliminary prizes are often part of contest designs. A two-round Tullock model is used to investigate a potential motivation for offering such prizes when contestants have different productivities. A designer can identify and purchase the rights to the more productive technology where the award of the preliminary prize is conditional on the winner licensing his technology to other contestants. In this way, endowing a preliminary round results in the potential for increased second round productivity, but at the cost of a reduced second round prize. Such a structure is optimal when the productivities are sufficiently different.

In the second model, it is noted that expenditures in inducement prize contests are often too large to be explained by the cash prize alone. There usually exists a final consumer application of the research. This chapter examines how different types of prizes arise by considering the informational content of winning and the effect this has on quality differentiation in when there is an established quality leader. A purely informational

prize influences investment decisions and also the qualities offered in the market. The main result is that some prizes may aim to select the highest quality firm as often as possible, whereas other prizes may aim to reward the entrant only if a significant improvement in quality is made.

In the third model, prizes are not the only instruments available to contest designers. In particular, subsidy of spending may be possible. This chapter uses an intuitive interpretation of the Tullock contest to offer a matching-funds instrument to a budget-constrained principal. It is shown that symmetric prize/subsidy contest designs may be optimal even in the context of contestant asymmetry, in contrast to most existing contest design models. It is also shown that if only one subsidy is offered, it is always to the weaker contestant. The role of contest accuracy in these findings is also considered.

Word count: 50,000

Acknowledgements This thesis is founded on the conscientious and insightful supervision of Dr Margaret Meyer, who skilfully guided me from the development of the theme to the process of writing up. The generous funding of the ESRC is also gratefully acknowledged. Most importantly, I must thank my wife, Catherine, without whose support this work could never have existed.

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Chapter 1

Introduction to the thesis

1.1 Outline of the thesis

This thesis is an examination of some design problems relating to inducement prize contests (IPCs). These are prizes offered for some accomplishment, usually in engineering or scientific research. IPCs have specific prospective criteria for the award of the prize, and are thereby distinct from retrospective recognition prizes which typically reward the impact of an achievement, regardless of the form.

Three separate models are presented, each motivated by specific historical examples of IPC designs, with the aim of establishing some novel insights into both the unique properties of technical and scientific competitions, and the key tradeoffs or dilemmas that might face a potential IPC designer.

1.2 Prizes; ancient, modern and contemporary

Prizes have been important institutions since antiquity; but their form has evolved according to the needs and constraints of the time. An understanding of contemporary high-tech IPCs is enhanced by a brief review of their antecedents, and of the contemporary contest. Some of these prizes are extremely famous, almost household names. Others have been largely forgotten, surprisingly, considering their success in creating some ubiquitous and valuable technologies. Individual chapters will present the specifics of particular prizes; below is simply a sketch of their role in history to place them in context.

The most famous ancient prizes were of course awarded at the Olympics in ancient Greece, which rewarded military and athletic pursuits. The word "athletics" is in fact derived from the Ancient Greek "athlon", meaning prize (and the corresponding verb meaning to contest). Prizes eventually evolved into contests of intellect that dealt with technological problems, although to begin with these were usually still state-sponsored and focused on military ends.

The Longitude Prize was instituted when the British Parliament needed an accurate method for ships to determine their location at sea. The prize was created following a petition precipitated by the Scilly naval disaster of 1707. The location of the Prime Meridian through the Royal Observatory in Greenwich is testament to the success of the endeavour.

In 1783 the French Academy of Sciences offered a prize for the industrial production of alkali from sea salt. In 1791, the Leblanc process was patented. France continued to make use of such prizes after the revolution. During the first years of the Napoleonic Wars, the French government offered a prize for a means of preventing food for spoiling, in order to increase the campaigning season. The result was the invention of canning in 1809. Then in 1823 the French Society for the Encouragement of Industry instituted the Turbine Prize. The winning design was implemented across the textile industry. Nor did prize development abate across the Channel. The famous Stephenson's *Rocket* was not the first steam locomotive, but it was built for and won the Rainhill Trials, an open contest for a prize and a supply contract in 1829. The Royal Agricultural Society of England offered numerous annual prizes for innovation in agriculture throughout this period, and the French government offered the Napoleon III butter substitute prize, as a cheap food for the army, which led to the invention of margarine in 1869.

Although the use of prizes to solve military problems never disappeared, increasingly it was complemented by greater private and philanthropic involvement in prize offering. The celebrated Orteig prize for transatlantic flight was a waypoint on this journey, instituted as a means to improve international relations between France and the US. This is typical of the goals of modern philanthropic IPCs: they aim for a technological breakthrough that is believed will be the solution to a broader social problem.

Following the information revolution, the possibilities of IPCs have increased. Barriers to contestant entry are now extremely low, particularly in information technology, which

permits large numbers of amateurs and outsiders to challenge established producers. Further, the ability of large numbers of people to collaborate on technological problems and to compete without collocation has raised the relative efficacy of both IPCs and of prizes general. For example, McKinsey & Company [2009, pp 16-17] estimated the rate of growth in prize funds to be 18%. Moreover, in a data set of 219 continuing prizes with purses greater than $100,000, the value of that group increased fifteen times over in only 35 years. Of those, more than 60 were instituted since 2000, together totalling more than $250 million in prize money. In addition, the design of this set of prizes offered has undergone a significant shift. Prior to 1990 only 3% of prizes were inducement prize contests with specific ex ante criteria; (the balance being recognition prizes, awarded ex post by general criteria). By 2007, 78% of prizes were IPCs.

A part, although not all, of this growth has been created by governments and their agencies. For example, in the USA, three of the federal agencies with the largest procurement budgets are the National Aeronautics and Space Administration (NASA), the Department of Defense (DOD), and the Department of Energy (DOE). Each now offers at least one substantial IPC. NASA runs the Centennial Challenges; DOD has the DARPA Grand Challenge, and the DOE sponsors the H-Prize. Each offers prize moneys totalling over $10 million.

This expansion in number and technological potential has naturally led to an increase in the diversity of prize design. In many ways IPCs are well modelled using the rent-seeking models of political economy. But this can neglect important aspects of the competition

that arise from its nature as a technological exercise in research and development, rather than simply an all-pay auction. These aspects are the subject of the thesis. In each case, a feature of IPCs not shared with more general contest situations is identified. Each of the transferable nature of research methods, the commercial application of technological research, and the monetary basis for technological research effort is looked at. Then reasonable restrictions on more general models are put forward, and the results of the extra structure are examined.

1.3 Contests and Tullock Contests

A contest exists in economics whenever agents expend effort to get ahead of rivals in pursuit of some reward [Konrad, 2009, pp. 1]. In a simple contest, players compete for a single prize, which they may value differently, by exerting some level of effort, which they may find differently costly. These efforts are the inputs to the contest success function, which determines the probability of each contestant winning the prize.

More formally, simple contests are strategic-form games with:

1. *Players* Contestants $N = \{1, ..., n\}$, typical element i.

2. *Strategies* Each contestant $i \in N$ chooses an effort level x_i from a given set X_i of effort levels (usually non-negative), so $\mathbf{x} = (x_1, ..., x_n)$ is the strategy profile.

3. *Payoffs* $\pi_i(\mathbf{x}) = f_i(\mathbf{x})v_i(b) - C_i(x_i)$, where:

    - $f_i(\mathbf{x})$ is the contest success function, so $\sum_{i \in N} f_i(\mathbf{x}) = 1$ and $f_i(\mathbf{x}) \in [0, 1] \forall i \in$

$N, \mathbf{x} \in \mathbf{X}$ (where $\mathbf{X} = \times_{i \in N} X_i$).

- $v_i(b)$ is contestant i's valuation of the common prize b.
- $C_i(x_i)$ is the cost to contestant i of exerting effort x_i.

This thesis is motivated by considering situations in which a contest sponsor has a fixed amount of capital which is available for the contest design. Therefore, this survey excludes consideration of those models where the size of the prize is determined in part by the effort exerted (see, e.g. Cohen and Sela [2008] and Kaplan et al. [2002]). Further, this thesis only considers contests in which players pay their own bid in full, and only their own bid. That is, situations in which the payment made by any contestant is a function of other contestants' strategies are not considered.

Numerous and important elements of contests, such as questions of timing and participation, externalities and alliance formation, will be suppressed throughout. A recent book that includes all these elements and more is Konrad [2009].

1.3.1 The Tullock Contest

Two of the three substantive chapters of this thesis make extensive use of the Tullock contest. For conciseness, the major properties of this popular contest model are presented only once, here. A useful survey of the contest literature in general, and the Tullock contest in particular, can be found in Konrad [2009].

The Tullock Contest form is one of the most popular in the literature, particularly fo-

cused on rent-seeking applications, which is surveyed in Lockard and Tullock [2001]. It is tractable and familiar, which makes it a useful modelling tool when thinking about contests of all kinds. However, its foundation in rent-seeking analysis means that careful interpretation is necessary to translate it into technological contest applications, especially when considering the welfare implications. In rent-seeking contexts, such as political lobbying, contestant efforts may have no inherent social value: if all allocations of the resource are equally efficient, lobbying effort is pure waste. On the other hand, scientific research effort may be socially valuable even if it does not result in a prize-winning invention.

Further, as discussed above, additional assumptions or restrictions can be motivated in IPC contexts, which is what Chapters 2 and 4 exploit to add greater structure to the simple, and very general, model. Chapter 2 adds an additional round and the possibility of transferring contestant characteristics. Chapter 4 adds the possibility of directly increasing the bid of individual contestants. (Chapter 3 is not based on the Tullock contest, but looks at the market impact of the informational effects of prizes).

A brief discussion of the key features of the Tullock contest will be useful later, so that standard results can be applied. Throughout this thesis, contests with two contestants will be used. They may have different valuations of the prize:

$$v_1, v_2. \tag{1.1}$$

The contest success function is:

$$f_i(x_i, x_j) = \frac{x_i^r}{x_i^r + x_j^r}. \tag{1.2}$$

The cost function is linear:

$$C_i(x_i) = c_i x_i.$$

Therefore, the payoff function for contestant i is:

$$\pi_i(x_i, x_j) = v_i \frac{x_i^r}{x_i^r + x_j^r} - c_i x_i. \tag{1.3}$$

As well as parameterising the equilibrium, r has importance regarding its nature: it has a key role in determining whether the first order conditions in fact characterise the equilibrium, since $r \leq \frac{n}{n-1}$ is the condition that ensures the second order condition is satisfied and ensures a pure strategy equilibrium. This is shown in Nti [1999], where the equilibrium values are also given:

$$x_i^* = r \frac{v_j^r v_i^{1+r}}{(v_i^r + v_j^r)^2}.$$

$$f_i(x_i^*, x_j^*) = \frac{v_i^r}{v_i^r + v_j^r}.$$

$$\pi_i^* = \frac{v_i^{r+1}(v_i^r + v_j(1-r))}{(v_i^r + v_j^r)^2}.$$

Note that the contestant with the higher effective valuation wins with a higher probability. Note also that any increase in the scale of both effective valuations will not affect this equilibrium probability. The case where $r = 1$ is known as a *lottery contest*, since the probability of winning is exactly the proportion of total output produced, as when buying tickets in a lottery.

These results, with some modification or additional parameterisation, will be used repeatedly in the analysis that follows.

Chapter 2

Progress prizes in inducement prize contests

2.1 Introduction and motivation

This chapter argues that inducement prize contests may be designed with multiple contest stages in order to allow the productive approaches of successful contestants to be adopted by others. When contestants are heterogeneous in their productivities, there is an advantage to the designer in attempting to identify the most productive technology early and then allowing others to use it.

2.1.1 Outline of the chapter

The chapter begins by offering a definition of a progress prize and explores a number of pertinent examples in order to motivate the model and support the underlying intuition. Related work that can be reinterpreted as explaining progress prizes is examined, but the focus is on a few key papers, rather than an exhaustive literature survey.

A simple two-round contest model is then formulated, which analyses the award of progress prizes when they are conditional on the recipient licensing rivals to use his technology. For a second round effort maximizing designer, it is shown that the game is equivalent to an asymmetric one-round Tullock contest. The relevant incentive compatibility constraints are discussed and interpreted in some depth as key design considerations.

It is also shown that, when per-round research efforts are unrelated, the use of a progress prize is optimal if the contestants are sufficiently heterogeneous. Otherwise, the standard single grand prize model is optimal. Further, it is shown that increasing the extent to which per-round efforts are related increases the designer's payoff, provided the incentive compatibility constraint of the weaker contestant is not violated.

2.1.2 Examples

Features of progress prizes

Before turning to why contest designers might consider including progress prizes, a summary of what this thesis considers to be their defining characteristics is required. One element of such prizes is that they are awarded prior to the competition for some "grand" prize. That is, they necessarily exist within dynamic contests. Therefore, progress prizes are not runner-up prizes in the usual sense, although they may be awarded to a contestant who does not eventually win the grand prize. Further, progress prizes are awarded in the same overall contest as the grand prize. That is, there must be some element of strategic continuation from the progress prize contest to the grand prize contest. The two contests cannot be considered separately by either the designer or the contestants. There are several ways to interpret and capture this idea, but this thesis requires as a minimum that the set of competitors is identical. That is, progress prize contests are distinct from contests that result in the elimination of contestants for consideration for the grand prize. Failure to win a progress prize does not preclude a competitor from competing for the grand prize.

There are numerous examples of IPCs where the design exhibits these features. The ones below have been selected to help to motivate one particular story about why designers use progress prizes.

The Longitude Prize

In motivating an examination of IPCs it is not uncommon to make at least a passing reference to the Longitude Prize (for example, Konrad [2009, pp. 12], Moldovanu and Sela [2001, pp. 543] and McKinsey & Company [2009]). Arguably the most famous such contest, and directly responsible for the solution of one of the most important technological problems of the time, it was established by the British government in the Longitude Act of 1714 and offered three prizes for methods of determining longitude at sea. The use of this example is in some ways surprising, considering that none of these prizes were ever awarded, although this fact is usually suppressed in the contest literature. It is certainly true that the existence of these prizes incentivised a large proportion of the research that eventually led to a workable solution, and that the non-award of the prize was a matter of some controversy. However, in order for this paper to use the Longitude Prize as a motivating example, it requires more detail of the structure of the contest than is commonly presented. The detail is drawn from Sobel [1998].

The Longitude Act established the Board of Longitude to administer the scheme of prizes. The three main prizes were of £10,000, £15,000 and £20,000, offered for methods of determining longitude at sea within 60, 40 and 30 nautical miles respectively. These remained unclaimed when the Board was disbanded in 1828. Crucially however, the board also had the discretion to make awards of up to £5,000 for promising works in progress, significant contributions toward a solution, or to provide financial support to researchers.

The man who is generally considered to have finally solved the problem, John Harrison, was awarded a total of £23,065 over his lifetime, by the Board and Parliament. Parliament intervened when the Board rejected Harrison's designs; however, they made the award conditional on a full explanation of the workings of the solution, and that it be replicated by other craftsmen. Following his award, other individuals who noted his success worked on improvements to Harrison's designs and were also awarded amounts of up to £3,000.

An alternative solution to the longitude problem, the lunar distance method, was pioneered by Tobias Mayer. The board also awarded him £3,000 for his work, and subsequent improvements to this method by other researchers attracted awards of up to £1,317.

Much has been made of the success of the Longitude Prize in spurring the research that led to a solution. However, using this prize to motivate models of one-shot or winner-take-all contests suppresses important aspects of the design, and the way the research progressed over time. In particular, it is important that part of Harrison's award was conditional on the publication of his method, and that this information was a valuable input to subsequent research, also rewarded within the same contest.

The Netflix Prize

A more recent example of this type of contest design and contestant behaviour can be found in the Netflix Prize. Again, this has been used as an example of the success of

inducement prize contests. This prize was established by Netflix, Inc., an online DVD rental service, to incentivise research that would improve its algorithm for recommending films based on user ratings. The contest began in 2006, and, unlike the Longitude Prize, was explicit about the rules governing both the grand prize and any progress prizes. The grand prize of $1m was to be awarded to any entry that improved on the firm's existing algorithm by 10%. In every year that did not see the award of the grand prize, a progress prize of $50,000 was awarded to the best entry that year, provided it improved on the previous year's winner by at least 1%. In order to qualify for any prize, the winning algorithm had to be publicly explained and the source code non-exclusively licensed to Netflix.

The progression of the prize is again instructive. The first progress prize was awarded in 2007 to team "KorBell", composed of three researchers from AT&T Labs, who published their winning method as required. The second prize was won in 2008 by "BellKor in Big Chaos", a merger of this team and another front runner. Again, they published their work and received further improving support from another competitor. This final merger created the team "BellKor's Pragmatic Chaos", which won the grand prize in 2009. Here, the requirement of publication of the current best strategy led to other contestants identifying synergies with their own work and further improvements. The progress prizes highlighted and disseminated effective strategies, much as the Longitude Prize had three centuries before. In fact, the Netflix Prize went further, publishing up-to-date leader boards. When it became apparent that BellKor's Pragmatic Chaos were about to win the grand prize a merger of three front runners (each of which was

individually expanding rapidly in membership) was conducted in an attempt to overtake them. This team placed second by only the narrowest of margins. In both the Netflix and Longitude Prizes, more than one approach was highlighted and improved upon by multiple contestants.

Other progress prizes

Numerous other IPCs consist of multiple awards over time, and for the interested reader a few are mentioned here.

A particularly interesting progress prize design can be found in the MPrize Longevity Prize. This is a prize which aims to stimulate research into life extension therapies by awarding monetary prizes whenever the world record for oldest *mus musculus* mouse is broken. Each consecutive winning researcher receives only a proportion of the current prize fund, equal to (new record - old record)/(new record). Thus the degree of improvement is recognised, and the prize fund is never exhausted; and in fact grows on average due to continued donations.

Another set of notable prizes are the NASA centennial challenges. Some of these are simple grand prizes, but others are constructed in a number of "levels". For example, the lunar lander challenge requires non-governmental contestants to build a robot to vertically take off, fly 100m at an altitude of more than 50m, and land on a pad of 10m diameter. At Level 1, this pad is a flat surface, but at Level 2 it is a simulated lunar surface. In 2006 and 2007 no prizes were awarded, and only single team, Armadillo

Aerospace, managed any attempts. In 2008 two teams competed, including Armadillo Aerospace, who managed to win the Level 1 top prize of $350,000. An attempt at Level 2 was aborted. In 2010 Armadillo Aerospace and Masten Space Systems both qualified for Level 2 prizes; with the latter placing first and earning $1,000,000 due to better landing accuracy. The evident year-on-year improvement through the prize levels is evidence of a learning process.

Other IPCs that include levels or progress prize elements include: The NASA Glove Challenge; the Google Lunar X Prize; the N-Prize; Prize4Life; and the Schneider Trophy.

2.1.3 When and how should progress prizes be used?

The examples given above deliberately emphasised the role of progress prizes in changing the environment of the remainder of the contest to elicit better research. Progress prizes are frequently conditional on the publication of the winning solution. This publication potentially raises the productivity of the other competitors by giving them access to the winner's approaches and advances. In particular, the model of this chapter provides a theoretical treatment of this intuition, and provides insight into the following design questions:

1. When is publication a valid reason for establishing a progress prize in an IPC?
2. Does the designer always benefit from greater contestant productivity?
3. How should the criteria for the award of the progress prize relate to those for the

grand prize?

Other explanations for the use of progress prizes are possible and some alternatives are considered in the conclusion.

2.1.4 Related work

This section surveys papers in the contest literature that are relevant for the work presented in this thesis. The key theoretical antecedent, the Tullock contest, was considered in the introductory chapter. The papers surveyed here are those models that, directly or indirectly, deal with the concept of progress prizes.

This section aims to show that some previous work on contests can be reinterpreted as finding a role for the use of progress prizes in the design of inducement prize contests. Again, note that this thesis is not considering elimination tournaments (see, for example Rosen [1986], Moldovanu and Sela [2006]). Consideration of tournaments with multiple stages involving the same set of competitors, and the potential use of those stages by the designer to influence contestant behaviour is fairly recent, with applications and motivations varying widely. However, in some cases, reinterpretation of the model as an inducement prize contest with a progress prize or prizes is straightforward. The following papers therefore present some potential reasons for the use of progress prizes, in addition to those considered in the new model of this chapter.

'Multi-battle contests' - Konrad and Kovenock [2009]

This paper is motivated in large part by a consideration of sporting contests, and focuses on games where overall victory (in the terminology of this thesis, the award of the grand prize) is determined by being the first player to reach a pre-specified number of victories in a sequence of all pay auctions. These component contests can also have prizes, termed intermediate prizes by the authors. The relevance for IPCs with progress prizes is clear. The authors focus on the dynamics of the contest: each possible subgame is defined by the number of victories held by each player at that point; the exact efforts exerted in the sequence of victories are sunk and irrelevant. Given this, the paper looks at which subgames in any given continutation game are reached with positive probability.

From the perspective of this thesis, of particular interest is their comparison of results regarding the presence or absence of the grand or progress prizes. The absence of a grand prize causes a multi-contest battle to degenerate into a sequence of unrelated contests. In the absence of any progress prizes, a sufficient lead in victories by one player can cause the other player to surrender competition for the grand prize, and exert zero effort for the remainder of the contests. For example, suppose that Player 1 requires only one more victory to win overall, whereas Player 2 requires two more. If Player 2 wins the current contest, he is then faced with a symmetric all-pay auction with complete information, where his expected payoff is zero. In contrast, Player 1 receives the grand prize from this victory. Player 2 will exert no effort, anticipating a zero-payoff continuation game.

In opposition to this, if progress prizes are non-zero throughout, then no contest occurs

where any player exerts zero effort, because the progress prizes prevent zero continuation values. This in turn means that at any contest, any subgame within the continuation game can be reached with positive probability. The authors term this property "pervasiveness" - and note that this prevents such contests from becoming boring to spectators, in the sense that the set of outcomes is as wide as possible. Moreover, the probability of a competitor who is down on component victories eventually winning the grand prize is increasing in the progress prizes. The authors conclude that "intermediate prizes are rather important for producing suspense" [Konrad and Kovenock, 2009, pp. 267]. McKinsey & Company [2009] note that the competitive, almost sporting, nature of IPCs helps to generate publicity, which might be of value to a designer.

'Tournaments with midterm reviews' - Gershkov and Perry [2009]

In this paper the authors note that in many tournaments, investments are made over time, and the designer can make a strategic decision about when and how to review progress. Most models permit only a single review - that at the end, which determines the winner. This more directly addresses the role of progress prizes than models in which it is privately optimal for the contestant to reveal some private information, as in Baker and Mezzetti [2005] or Gill [2008]. Progress prizes are used to induce behaviour that would not otherwise be optimal for the contestant, but is desired by the designer. This paper makes some assumptions which bring it close to the situation under consideration - in particular, the designer cares about total contestant effort, and the midterm review result is public knowledge. On the other hand, this model awards only a single fixed

prize - the effect of the review is to give information as to progression towards this prize, and to act as a partial basis for its award.

The main result of the paper is to show that the optimal prize-award rule incorporates a midterm review in some way. A midterm review where good performance increases the probability of winning has two countervailing effects: it tends to increase the agents' effort level at stage one, but to decrease it at stage two. Symmetric consideration of the midterm and final reviews in the award criteria results in an equilibrium strictly dominated for the designer by one in which only the final outcome is considered. However, using the midterm result minimally as a tie-breaking rule dominates final review only. The optimum weighting is somewhere between. The more effective first round effort is in influencing final round outcome, the smaller the optimal weight on the midterm review.

'Information feedback in a dynamic tournament' - Aoyagi [2010]

While it is common in the literature to formulate a tournament as a static game, tournaments in reality are often dynamic games in which agents make multiple effort decisions over time. In a multi-stage tournament, a key consideration is the informational dynamic, which may be within the control of the designer. This paper presents a model of a dynamic tournament in which the designer receives private information about intermediate performance, and has the option of reveal some or all of it to incentivise further effort. Unlike the model of this chapter, prize sizes are fixed exogenously. In the two

stage game, the optimal feedback policy varies totally with the concavity of the stage 2 marginal cost of effort. Given certain other assumptions including symmetry, concavity implies that full disclosure by the designer is optimal. Conversely, under convexity, no feedback is optimal. Certain of these assumptions can be relaxed if the agent's efforts are sufficiently complementary.

'Inventives versus Competitive Balance' - Möller [2012]

This is a recent paper that also posits the existence of a twice-played two-player Tullock contest, with a designer splitting a fixed endowment between the two rounds. Similarly to the model of this thesis, victory in the first round provides for an improvement in ability (in this case, a reduction in marginal cost, rather than an increase in productivity as modelled below). This however is the direct result of the prize won, which does not direct enter into the players' payoff function. Moreover, the designer cares about the weighted aggregate effort across both rounds. Both winner and loser prizes can be awarded in the first round, so the challenge for the designer is not only to maintain sufficient incentives to effort in the first round, but to prevent excessive asymmetry of ability in the second round. The main results deal with when it is optimal to offer multiple non-zero first round prizes. This model is an interesting complement to the one presented in this chaper. Although the majority of the modelling assumptions vary, the motivation underlying is very similar.

Summary

Since Lazear and Rosen [1981], tournaments have attracted considerable attention in economic theory. Numerous examples of situations which are well modelled by all-pay contests have been advanced, including rent-seeking, sporting competition, performance evaluation and labour markets. The majority of this work deals with static questions of prize allocation and comparisons with more general contracts. Where dynamics are considered, the typical form used is that of a sequential elimination contest. More recently, contests in which repeated competition between a given set of contestants occurs within a changing context have been explored. As argued in the introduction, this examination is timely considering the recent growth in the use of IPCs, where the dynamics are clearly important. This chapter is a further contribution to this strand of the literature. It focuses on the particular circumstances found in IPC contests, and examines different ways in which the structure of the contest might change round by round, depending upon the earlier actions of the players.

2.2 A model of technology switching using progress prizes

In the introductory examples, the award of a progress prize was often conditional on disclosure of the method or approach used by the winner. Designers believed the dissemination of this information to other contestants to be beneficial to the final outcome. That is, designers sought to raise the productivity of contestants by purchasing the rights to the most promising approach. Of course, selection of which approach to purchase and

distribute is then the key question. It is not unreasonable to suppose that, if the designer has opted to sponsor a contest as the best means of uncovering a solution, the circumstances are such that the same means - a contest - is the only way of identifying the most promising approach. The designer therefore splits his endowment between two contests, instituted simultaneously but run sequentially: the first identifies excellence, best practice or effective technologies and thereby sets the conditions for a more productive second contest, which rewards efforts focused more directly on the particular concerns of the designer.

This section sets out and analyses a simple two round model of such a situation, formalising the intuition that progress prizes are valuable when contestants are sufficiently heterogeneous. In particular, this section shows that, if the optimal progress prize is non-zero, the situation can be analysed using an asymmetric one-shot Tullock contest. Further, some remarks about the optimal nearness of the criteria for the award of the grand and progress prizes will be made.

2.2.1 Assumptions

Players and payoffs

Two contestants, i, j, are endowed with production technologies, labelled 1 and 2, for the conversion of research effort to output: e.g. $f_i(x_i) = p_1 x_i$. The scalar p_1 is the productivity of technology 1 which may be owned by either contestant. Without loss of generality, let $p_1 \leq p_2$. Technology rights belong to the contestant, and can be trans-

ferred. Designer D is endowed with a unit of prize funds and two research questions A and B. These questions are answered by the production of research output by contestants. The probability that any given contestant produces the best answer is his output as a proportion of the total output by both contestants. That is, the competition to produce the best answer is a lottery contest in output.

In addition to relevant output being produced by the same means, the questions are potentially related in another way. Some proportion $\alpha \in [0, 1]$ of the output produced researching an answer to A is useful in answering B and vice versa. That is, the two questions do not necessarily require orthogonal directions of research effort. Further discussion of some potential interpretations of this parameter is given later.

The contestants maximise expected payment minus their effort. The designer maximises the total output created answering question A. This is the research direction that he cares about. He is only interested in B in so far as it is useful to him in designing a mechanism by which to produce greater output for A.

Contestants have complete and perfect information. However, the designer can never observe a contestant's characteristics or actions. He can only identify the best answer to a question, and therefore can only distinguish between contestants by their wins. He knows that there are high and low productivities $\{p_1, p_2\}$ in the contestant population, and so has ex ante beliefs that $\Pr[(p_1, p_2) = (p_i, p_j)] = \Pr[(p_1, p_2) = (p_j, p_i)] = \frac{1}{2}$. From this point, we will refer to the contestant endowed with technology 1 as contestant 1. Crucially, however, the transfer of technology rights are verifiable, so payments can be

conditioned on this action.

Timing

This is a model of technology selection by means of a preliminary contest, and the progression of the various stages are assumed to be such as to allow this behaviour to occur, so that Question A is contested second. However, it should become clear that there would be no advantage anyway to the designer running simultaneous contests, or contests in which Question A is rewarded first. Such contests leave the conditions under which the contest A is conducted unchanged from a simple single contest for an answer to A.

The assumed structure of the contest is:

1. The designer commits to award prizes a, b for the best answers to research questions A and B, by means of two contests with rules as follows:

2. Contestants exert efforts $\{x_1, x_2\} \in \mathbb{R}^2_+$ in producing output answering question B, and prize b is awarded in a lottery contest in output - that is, a Tullock contest where $r = 1$ (see Chapter 1). (If $x_1 = x_2 = 0$, b is rolled forward and added to a).

3. The award of this prize is conditional on the contestant making his winning technology freely available for use by the other contestant. If the winner rejects, the prize is offered to the loser under the same conditions. If he also rejects, the prize is rolled over into the next round.

4. Contestants choose whether they will use their endowed technology or, if available, their rival's technology for the final contest.

5. Contestants exert efforts $\{y_1, y_2\} \in \mathbb{R}^2_+$ in producing output answering question A, and prize a is awarded in a lottery contest in relevant outputs $\{p_1^a z_1, p_2^a z_2\}$, where p_i^a is the productivity of the technology used by contestant i in this round, and where $z_i = y_i + \alpha x_i$ if $p_i^a = p_i$, or $z_i = y_i$ if $p_i^a = p_j$. That is, adopting a new technology is assumed to involve the loss of any previous progress. (If $z_1 = z_2 = 0$ no prize award is made).

2.2.2 Interpretations

The interpretation of the model follows directly from the motivating examples, where the award of a progress prize was conditional on disclosure of the method or approach used to win. This is exactly the behaviour this model captures, as simply as possible. The designer cares about the amount of research output directed towards Question A. But he knows that the technology or know-how necessary to most efficiently produce this research output is not widely owned. This directly lowers the population productivity compared to if this technology were not proprietary. That is, less output is produced for any given set of efforts. But it also lowers effort itself by means of the well-known strategic effect in Tullock contests, whereby increased asymmetry lowers each individual's effort. The designer therefore splits his endowment and preludes the contest addressing Question A with a contest that (noisily) assesses contestant productivity, and makes the

prize contingent on the licensing of the winning technology to the other contestants. In this way, the designer overcomes the strong private incentive not to share technology, but at the cost of reduced prize funds for Question A.

2.3 Grand prize contest

The solution to this model is by backwards induction, and so the grand prize contest is considered first.

The grand prize contest is the final contest for prize a and the contest in which the designer values the output produced. This is a simple lottery contest in outputs, and period profit is given by:

$$\pi_i^a = a\frac{p_i^a z_i}{p_i^a z_i + p_j^a z_j} - y_i.$$

At this point, an important assumption is made. Assume that α is sufficiently small to ensure that the following condition always holds in equilibrium:

$$\alpha x_i \leq a\frac{p_i^a p_j^a}{\left(p_i^a + p_j^a\right)^2} \qquad i = 1, 2. \tag{2.1}$$

This ensures that the carried-forward efforts are within the intersection of the best response curves, so that the intersection gives the equilibrium. This means that the second round equilibrium efforts (though not profits) are independent of the first round efforts, which greatly increases tractability. The meaning of this assumption is discussed later.

The standard Tullock results then give that:

$$z_i^* = a\frac{p_i^a p_j^a}{\left(p_i^a + p_j^a\right)^2} \qquad i = 1, 2.$$

That is, $a\frac{p_i^a p_j^a}{\left(p_i^a+p_j^a\right)^2}$ is the target level of effort, and effort carried forward is topped up by y_i to achieve this.

The designer's equilibrium payoff is therefore the productivity-weighted sum of these two efforts:

$$\pi_D^* = a\frac{p_i^a p_j^a}{p_i^a + p_j^a},$$

which is increasing in both the grand prize a and in each competitor's productivity. This is the trade-off that the designer will face: whether to reduce a in the hope of increasing productivity.

The contestants face expected second round payoffs of:

$$\pi_i^{a*} = a\left(\frac{p_i^a}{p_i^a + p_j^a}\right)^2 + \alpha x_i,$$

which are increasing in a and own-productivity, but falling in rival productivity. This gives some initial insight into the first round decision of each contestant. A first round win has the potential to handicap the victorious contestant in the second round, regardless of their identity. The more productive contestant, on winning, potentially increases his rival's productivity. The less productive contestant, on winning, loses the opportunity to raise his own productivity.

2.4 Progress prize round

Given that this is the play of the final round, the designer's decision about the use of a progress prize can be considered. It is obvious that the adoption of p_1 is never optimal for 2 - it raises to cost of generating any given output and moreover causes him to lose his carried-forward effort. Therefore, the final round is played in one of two technological settings: either each contestant uses his endowed technology, or both contestants use the superior technology. Two necessary conditions for the latter setting to occur are now derived:

2.4.1 The Switching Condition

A necessary condition for a progress prize to be optimal is that it induces technology switching: if the more productive technology is available, the gain to the less productive contestant from adopting the better technology exceeds the cost of losing his progress so far.

$$\frac{a}{4} \geq a\left(\frac{p_1}{p_1+p_2}\right)^2 + \alpha x_1. \tag{2.2}$$

Without this, there is no possibility of reaching the superior technology situation in the second round. From this point on, unless otherwise stated, assume any discussed progress prize satisfies this condition. This is a further condition on the smallness of α. In fact, it is stricter than (2.1) if $p \leq \frac{1}{3}$. Otherwise, the converse is true. Therefore, the

analysis of this chapter holds for all α when $p \leq \frac{1}{3}$. But if not, there are α for which the analysis does not apply, but where the Switching Condition is not violated. This observation only matters at the very end of the chapter, when the optimal level of α is considered.

2.4.2 Winning and The Acceptance Condition

At this point, it is important to understand what combinations of wins, prize offers and prize acceptances are feasible, in order to clarify subsequent analysis. The identity of the winner determines the order in which the progress prize is offered to contestants. First, after a Contestant 2 win, 2's acceptance decision is independent of 1's subsequent acceptance decision, which does not affect the parameters of the second round. Moreover, 1 will always accept the prize in these circumstances, as he bares no cost from making his inferior technology available and has already lost the opportunity to gain access to the better. Therefore, Contestant 2 accepts if and only if:

$$b + \frac{a}{4} + \alpha x_2 \geq a\left(\frac{p_2}{p_1 + p_2}\right)^2 + \alpha x_2.$$

$$b + \frac{a}{4} - a\left(\frac{p_2}{p_1 + p_2}\right)^2 \geq 0. \tag{2.3}$$

Call this the Acceptance Condition.

Second, after a Contestant 1 win: Here, 1 does have to consider if 2 will accept the prize if he rejects it, as this alters his continuation payoff. The decision by 2 again depends

on if the Acceptance Condition (2.3) is satisfied. If (2.3) is not satisfied, there is no benefit to a strategic rejection of the progress prize by 1. If (2.3) is satisfied, however, contestant 1 will accept if and only if:

$$b + a\left(\frac{p_1}{p_1 + p_2}\right)^2 \geq \frac{a}{4}.$$

$$b + a\left(\frac{p_1}{p_1 + p_2}\right)^2 - \frac{a}{4} \geq 0. \tag{2.4}$$

It is straightforward to show that (2.3) implies (2.4). Therefore, in the only situation in which contestant 1 would consider strategic rejection, he never finds it worthwhile to do so. Thus, the Acceptance Condition is sufficient to characterise all acceptance behaviour. If satisfied, it implies that the winner always accepts the progress prize. If this condition fails, 2 will reject the progress prize if offered it, and 1 always accepts if offered. Therefore 1 always receives the progress prize if (2.3) is violated. Note that under no circumstance is the prize rolled forward (except in one very particular circumstance, discussed below).

2.5 Reformulation as a single-round contest

As shown above, the superior technology situation only occurs if the more productive contestant 2 wins the first round, *and* he accepts the prize, *and* contestant 1 choses to adopt the better technology. Otherwise the mixed technology situation prevails.

Since any posited $b > 0$ must reach the superior technology situation with positive probability if it is to improve on the $b = 0$ case, this reasoning gives three individually

necessary conditions on any optimal non-zero progress prize:

1. Contestant 2 must have a positive probability of being offered the progress prize.
2. Contestant 2 must accept the progress prize if offered - the Acceptance Condition.
3. Contestant 1 must adopt the new technology if able - the Switching Condition.

It will shortly be shown that conditions 1 and 2 are (almost) equivalent. Intuitively, Contestant 2 will not exert effort in pursuit of a prize he anticipates he will reject if offered.

If any of these conditions are not satisfied, the grand prize contest will necessarily be played under the mixed technology environment, just with a lower prize. If they are satisfied, however, the final round is played using the superior technology with positive probability. Clearly, the magnitude of this probability is crucial for the designer. The analysis now proceeds by considering the set of prizes that satisfy these conditions, and then comparing them it with the $b = 0$ situation.

Given that the Acceptance Condition (2.3) and the Switching Condition (2.2) hold, the more productive second round occurs if and only if 2 wins, and the whole contest can

be described as a one-round asymmetric Tullock contest with loser prizes:

$$\begin{aligned}
\pi_1 &= \frac{p_2 x_2}{p_1 x_1 + p_2 x_2}\frac{a}{4} + \frac{p_1 x_1}{p_1 x_1 + p_2 x_2}\left[b + a\left(\frac{p_1}{p_1+p_2}\right)^2 + \alpha x_1\right] - x_1. \\
&= \frac{a}{4} + \frac{p_1 x_1}{p_1 x_1 + p_2 x_2}\left[b + a\left(\frac{p_1}{p_1+p_2}\right)^2 - \frac{a}{4}\right] - \left(1 - \alpha\frac{p_1 x_1}{p_1 x_1 + p_2 x_2}\right)x_1. \\
\pi_2 &= \frac{p_1 x_1}{p_1 x_1 + p_2 x_2}\left[a\left(\frac{p_2}{p_1+p_2}\right)^2 + \alpha x_2\right] + \frac{p_2 x_2}{p_1 x_1 + p_2 x_2}\left[b + \frac{a}{4} + \alpha x_2\right] - x_1. \\
&= a\left(\frac{p_2}{p_1+p_2}\right)^2 + \frac{p_2 x_2}{p_1 x_1 + p_2 x_2}\left[b + \frac{a}{4} - a\left(\frac{p_2}{p_1+p_2}\right)^2\right] - (1-\alpha)x_1.
\end{aligned}$$

A few comments on this reinterpretation are warranted:

1. Each contestant is endowed with their continuation payoff from losing round 1. Their effective prize from exerting effort answering Question B is a chance of winning the progress prize, plus their alternative continuation payoff, less their endowment. Note therefore that each player perceives the prize to be worth less than b, since each realises he will be at a disadvantage in the second round compared to if he had lost the first round.

2. The Acceptance Condition is only satisfied if the effective prize for winning as perceived by Contestant 2 is positive. In this case, he will exert some non-zero effort and win with positive probability. This means that these two necessary conditions on the progress prize, above, are equivalent. The only exception to this statement is when $p_1 = 0$. In this case, prizes $\{a, b\}$ that satisfy the Acceptance Condition with equality will lead to no effort at all in the progress prize round: Contestant 1 is unable to produce output, and Contestant 2 is unwilling to. Therefore, in such a case, the designer rolls the prize forward, and the game is equivalent to a single

grand prize of $a + b$, which Contestant 2 wins without cost.

3. The effort carry-forward αx_i can be reinterpreted as reducing the marginal cost of effort this round, with second round prizes as if there was no carry forward. Contestant 1 has a higher marginal cost, since he is not certain of this carry-forward. Since Contestant 2 anticipates always reducing his second round effort by αx_2, the marginal cost of effort in the equivalent one-shot game is $(1 - \alpha)$. Contestant 1, on the other hand, anticipates retaining this effort only if he wins - therefore his marginal cost of effort is falling in the probability that he wins the progress prize (that is, in x_1).

2.6 The special case of unrelated contests: $\alpha = 0$

Consider the special case of $\alpha = 0$. That is, the situation where research output for Question B is orthogonal to that of Question A, and the two questions are related only by the technologies used to produce output. In this case, the Switiching Condition is trivially satisfied.

Applying standard Tullock contest results on the single round reinterpretation give the equilibrium probability of a first round win by 2:

$$\Pr(2 \text{ wins } b) = \frac{B_2 p_2}{B_1 p_1 + B_2 p_2}.$$

Where B_i is the effective prize faced by i:

$$B_1 = b + a\left(\frac{p_1}{p_1+p_2}\right)^2 - \frac{a}{4},$$
$$B_2 = b + \frac{a}{4} - a\left(\frac{p_2}{p_1+p_2}\right)^2.$$

The designer's payoff is given by:

$$\pi_D = a\Pr(2 \text{ wins } b)\frac{p_2}{2} + a\Pr(1 \text{ wins } b)\frac{p_1 p_2}{p_1+p_2}. \tag{2.5}$$

Two simplifications to the model can now occur. First, it is clear that the optimal contest design has $a+b=1$. Scalar increases in a and b leave the Acceptance Condition unaffected, as well as $\Pr(2 \text{ wins } b)$, and therefore increase the designer's payoff. Second while the designer has preferences over the levels of $\{p_1, p_2\}$, with scalar increases in both increasing the designer's payoff in the same proportion, it is without loss of generality regarding strategy to consider only the ratio $\frac{p_1}{p_2} \in [0,1]$. Call this ratio p

Using $a+b=1$ to eliminate b from $\Pr(2 \text{ wins } b)$, it can be seen that it is falling in a, that is, increasing in the progress prize b.

$$\frac{\partial \Pr(2 \text{ wins } b)}{\partial a} = -\frac{8p(p^2-1)^2}{\left(\{p[p(p+13)+11]+7\}a - 4(p+1)^3\right)^2}.$$

As noted above, 1 always faces a larger effective progress prize, because his second round handicap (difference in continuation values) from a first round win is smaller. However, as a falls and b increases, the size of the handicap becomes relatively less important, and the ratio of effective prizes tends to one. This provides some initial clarity to the designer's tradeoff. Reductions in a reduce effort in the second round, but they

raise the probability that 2 wins the first round and therefore that the second is more productive.

Note also that both effective prizes are increasing in p. In each case, p determines the payoff in the mixed technology state. For Contestant 1, winning the progress prize entails the loss of the opportunity to improve his technology in the second round. This is less important if his existing technology is not too bad. For Contestant 2, winning and accepting the progress prize entails the loss of his technological advantage in the second round. This is less of a cost if his rival's technology is already close to his own.

Proposition 2.1. *For $\alpha = 0$, in the set of prizes that satisfy the Acceptance Condition, there exists a unique optimum for the designer, given by $a^\dagger(p) = \frac{g(p)}{h(p)}$ where*

$$\begin{aligned} g(p) =& 4(2p^4 + 23p^3 + 19p^2 + 13p + 7)(p+1)^3 \\ & -4\sqrt{2}\sqrt{(1-p)^3 p(p+1)^5 (p(p(p(2p+23)+19)+13)+7)}, \end{aligned}$$

and

$$h(p) = (p^3 + 13p^2 + 11p + 7)(2p^4 + 23p^3 + 19p^2 + 13p + 7).$$

and the corresponding $b^\dagger(p) = 1 - a^\dagger(p)$. Therefore $a^\dagger(0) = \frac{4}{9}$ and $a^\dagger(1) = 1$.

Proof. Equation (2.5), together with the fact that $a^* + b^* = 1$ has:

$$\pi_D = a\frac{B_2}{B_1 p + B_2}\frac{1}{2} + a\left(1 - \frac{B_2}{B_1 p + B_2}\right)\frac{p}{p+1}.$$

The first order condition is:

$$16(2p^2+p+1)(p+1)^5+\{p[p(p+13)+11]+7\}$$
$$\frac{(p\{p[p(2p+23)+19]+13\}+7)a^2-8(p\{p[p(2p+23)+19]+13\}+7)(p+1)^3a}{2(p+1)\left(\{p[p(p+13)+11]+7\}a-4(p+1)^3\right)^2}=0.$$

The denominator is zero precisely when $B_1p+B_2=0$ - that is, when the payoff function is not valid anyway. The numerator is quadratic in a:

$$(p^3+13p^2+11p+7)(2p^4+23p^3+19p^2+13p+7)a^2$$
$$-8(2p^4+23p^3+19p^2+13p+7)(p+1)^3a$$
$$+16(2p^2+p+1)(p+1)^5=0.$$

Therefore, there are two solutions to this first order condition. Examination of the quadratic determinant shows that these solutions are both real for all $p\geq 0$

Consideration of the payoff function and probability function can identify a maximum and a minimum. It is important to stress that some of the analysis takes place outside the domain of a in which these functions have meaning as payoffs or probabilities, but since there are two turning points, we have to determine which is in the domain and whether it is a maximum.

The profit function is simply a probability-weighted average of two possible payoffs: $\frac{a}{2}$ in the high tech state and $\frac{ap}{1+p}$ in the mixed tech state. When the probability function is between 0 and 1, the payoff function lies between these two lines. Otherwise, it lies outside.

Consider the interval $a\in[0,\infty)$, and recall that both B_1 and B_2 are falling in a everywhere in this interval, and that $B_1>B_2$. At $a=0$, $B_1=B_2=1$ and $\Pr(2 \text{ wins } b)=$

$\frac{1}{1+p}$. This is the standard Tullock result in the progress prize round. As a increases, we know that Pr(2 wins b) falls to zero (the Acceptance Condition). At this point, the payoff function no longer lies between the two lines, but continues to fall, with an asymptote at $B_1 p + B_2 = 0$, the same as the probability function, at some point becoming negative. From this, we know there is a maximum in this interval.

To the right of the asymptote, both the numerator and the denominator are negative - "probabilities" are initially extremely large. But the derivative of probability is negative everywhere (except at the asymptote where it is obviously undefined), and it continues to fall, with a limit as $x \to +\infty$ of $\frac{3p^2+6p+7}{p^3+13p^2+11p+7}$, which is in $(0, 1)$. Therefore, the payoff function falls between the lines at some point, but then remains between these increasing lines, so it must itself be increasing, implying a minimum to the right of the asymptote.

These then are the two stationary points. Since we know that for real solutions, the discriminant is positive, and since the denominator of the solutions is positive, the solution with the negative square root term is the smaller, and therefore represents the maximum. □

Evidently the analytic form of this solution has little intuitive content, and the remaining analysis does not make much use of the specific form. A stylised plot of the solution is given by Figure 2.1

Proposition 2.2. *For $\alpha = 0$, there exists a unique value of p, above which no progress prize is the optimal design, and below which the optimal design involves a non-zero progress prize. That is, for $\alpha = 0$ $\exists \hat{p} \in (0, 1)$ such that $p < \hat{p} \implies a^* = a^\dagger$ and*

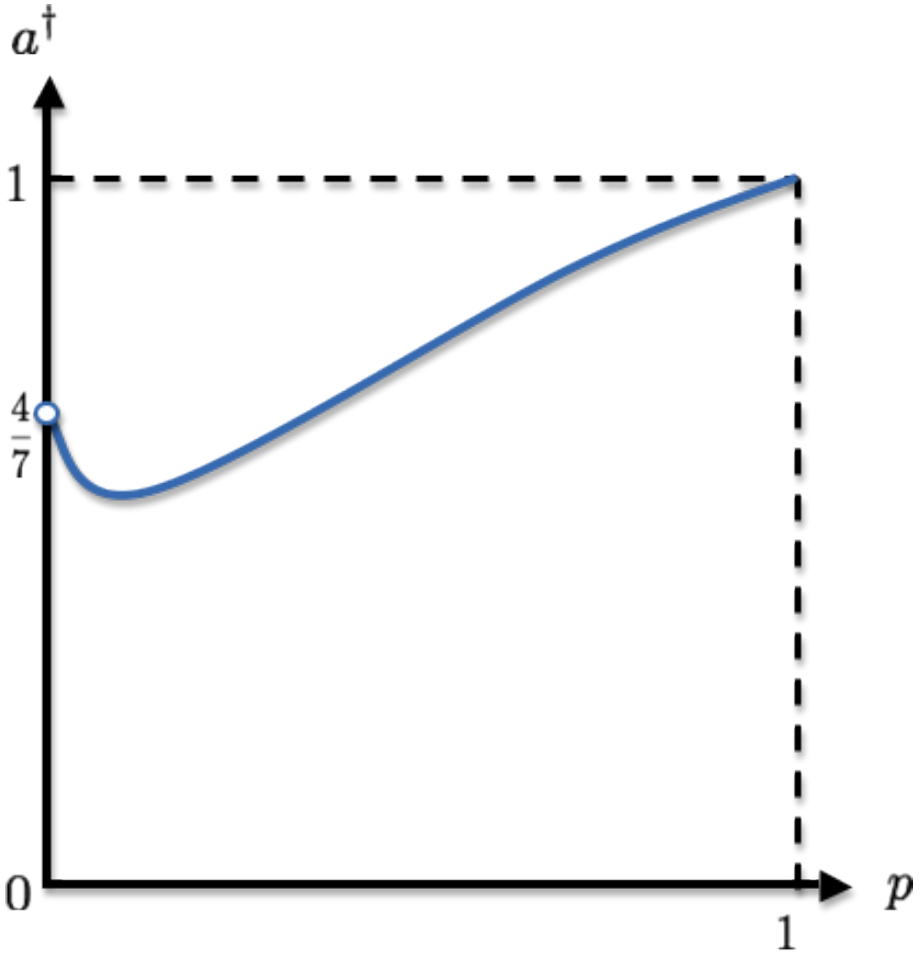


Figure 2.1: The grand optimal prize that satisfies the Acceptance Condition, $a^\dagger$

$p \geq \hat{p} \implies a^* = 1$.

Proof. A non-zero progress prize is optimal whenever there exists an $a \in (0,1)$ such that:

$$a \Pr(2 \text{ wins } b)\frac{1}{2} + a \Pr(1 \text{ wins b})\frac{p_1}{p_1+1} \geq \frac{p_1}{p_1+1}.$$

That is, some choice of a generates a $\Pr(2 \text{ wins } b)$ that satisfies:

$$\Pr(2 \text{ wins } b) - 2\frac{1-a}{a}\frac{p}{1-p} \geq 0.$$

Of course, those a that imply that $pB_1 + B_2 < 0$, and thus are outside the region defined by the Acceptance Condition where the payoff function is valid, but imply large positive "probabilities", must be excluded. However, this condition has the following form:

$$\frac{(p\{p[p(2p+23)+19]+13\}+7)a^2 - 2(p\{p[p(4p+11)+23]+17\}+9)a + 8(p+1)^3}{2(p+1)\{p[p(p+13)+11]+7\}a - 8(p+1)^4} \geq 0.$$

The denominator is less than zero precisely when $pB_1 + B_2 > 0$. Thus, simply taking the denominator as negative ensures the only solutions to the problem are those relevant for solving the game:

$$-((p\{p[p(2p+23)+19]+13\}+7)a^2 - 2(p\{p[p(4p+11)+23]+17\}+9)a + 8(p+1)^3) \geq 0.$$

(If the denominator is greater than zero, the inequality is obviously reversed). This inequality is quadratic in a, with a negative term on a^2. Therefore, such an a exists if and only if the analogous equality has solutions. That is, if the discriminant is non-negative. (If the inequality were reversed, there would always be solutions, due to the large positive probabilities beyond the asymptote).

Rewrite the condition as:

$$f(p)a^2 + g(p)a + h(p) \geq 0.$$

Then the condition on the discriminant $\Delta(p)$ is:

$$\Delta(p) = g(p)^2 - 4f(p)h(p) \geq 0.$$

The discriminant factorises significantly, and by taking these factors out the final analysis is shortened substantially:

$$\frac{(1-p)^3}{4(1+p)^4}\delta(p) = \Delta(p) \quad p \in [0,1].$$

Note that $1-p$ is a factor of the discriminant because at $p = 1$ there is a unique solution to the quadratic equality. This is because at this point $a^\dagger = 1$, so the optimal progress prize level coincides with the payoff from no progress prize.

Ignoring positive factors, the conditions are the same:

$$\Delta(p) \geq 0 \iff \delta(p) \geq 0.$$

And then putting in the functional forms to obtain the final analytic condition for a positive progress prize to be optimal.

$$\delta(p) = 9p^5 + 25p^4 + 35p^3 + 7p^2 - 8p - 4 \geq 0.$$

Note that $\delta''(p) > 0, \quad \delta'(0) < 0, \quad \delta'(1) > 0$. Together these imply that $\delta'(p)$ has a single root in $p \in (0,1)$. Further, $\delta(0) < 0, \delta(1) > 0$, which implies that $\delta(p)$ has a single

root $\hat{p}$ in $p \in (0,1)$, and therefore:

$$\begin{cases} \delta(p) > 0 & \text{if } p \in [0,\hat{p}) \\ \delta(p) = 0 & \text{if } p = \hat{p} \\ \delta(p) < 0 & \text{if } p \in (\hat{p},1) \end{cases} \Longrightarrow \begin{cases} \Delta(p) > 0 & \text{if } p \in [0,\hat{p}) \\ \Delta(p) = 0 & \text{if } p = \hat{p} \\ \Delta(p) < 0 & \text{if } p \in (\hat{p},1) \end{cases} .$$

Thus the discriminant is non-negative if $p < \hat{p}$. In this case, there are a that improve on the $a = 1$ outcome, and from Proposition 1, we know the best such is $\alpha^{\dagger}$. Otherwise, there are no such a, and $a = 1$ is the optimum. (Note again that $\alpha^{\dagger}$ is again the optimum at $p = 1$, but here the optimal progress prize is not distinct from offering no progress prize.) □

The intuition for this result can be developed via two benchmarks.

Benchmark 1: $p = 1$ In this case, the two contestants are equally productive. The two possible technology states are the same, regardless of who wins any progress prize round. Therefore, the designer's payoff is maximised by setting the grand prize as large as possible: $a^*(1) = 1$.

Benchmark 2: $p = 0$ In this case, the less productive contestant is completely unproductive. In the absence of a progress prize, there is no output (since Contestant 2 can win the grand prize with marginal effort), and the designer's payoff is zero. If b and a (strictly) satisfy the Acceptance Condition, then 2 wins with certainty. In this case, there is no advantage to the designer increasing b beyond that level necessary to induce acceptance. Therefore, the optimal progress prize that that which satisfies the

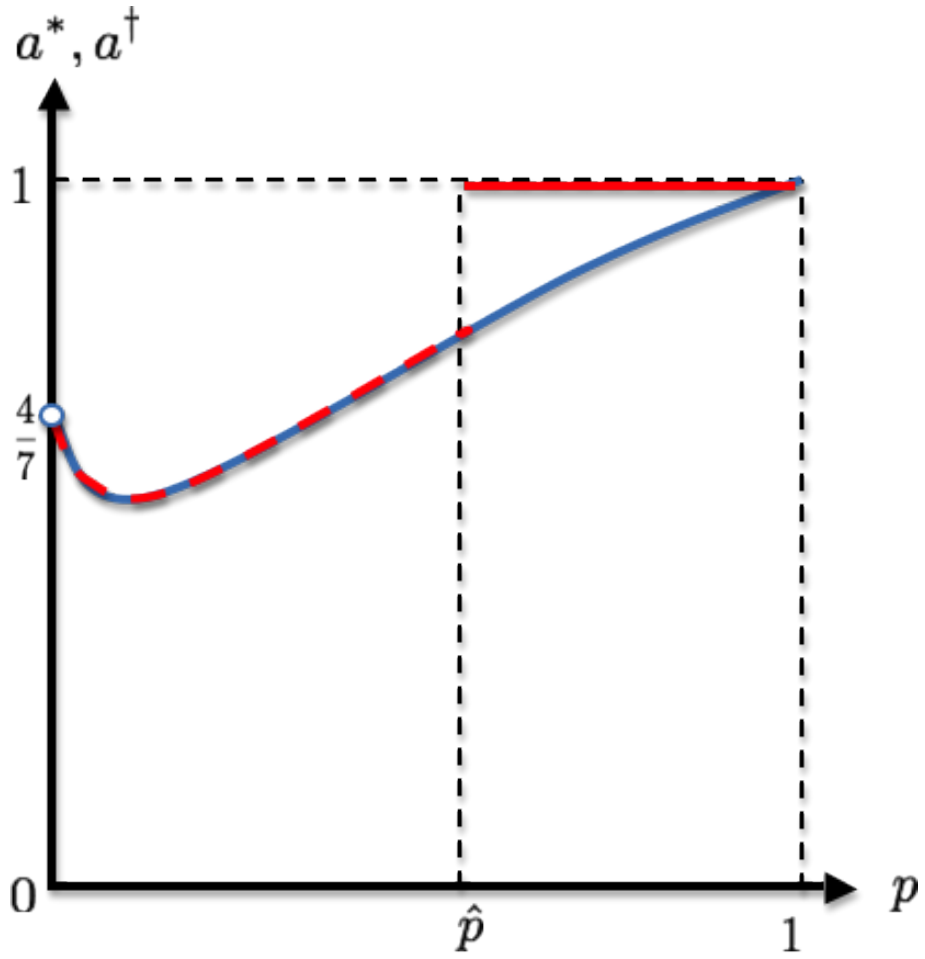


Figure 2.2: The optimal grand prize a^*, in red; and the optimal grand prize satisfying the acceptance condition $a^\dagger$, in blue

Acceptance Condition with equality, so $b^*(0) = \frac{3}{7} + \epsilon, \pi_D^* = \frac{2}{7}$, where ϵ simply indicates that the prize has to be slightly better than that needed to generate acceptance if the prize were simply offered - in order to induce a marginal effort and avoid rolling over the prize.

Essentially, the designer buys the productive approach, with no additional identification cost (2 can prove he has the productive technology by a marginal output), and then distributes it before the final round. This benchmark also emphasises the more general point that the designer must be committed to offering the prize to the runner up before adding it to the main prize pot, otherwise the incentive for the productive contestant to accept may be compromised, as in this case.

Therefore, in the one extreme, there is no incentive to introduce a progress prize, and at the other, the lack of a progress prize means a zero payoff. As p increases, the importance and efficacy of a progress prize are reduced, in the sense that the absence of a progress prize returns a higher payoff, and any given level of b induces a lower equilibrium Pr(2 wins b).

Figure 2.3 shows the key tradeoff: The payoff in both the high and low technology states (blue lines) increases as the progress prize increases. But the probability of reaching the high technology state is increased as the progress prize is increased beyond the Acceptance Condition (Acc). The red line shows the combined effects.

The following proposition shows that there is a region in which D's payoff is falling in p. So while contestant productivity is increasing, the increased difficulty of selecting the

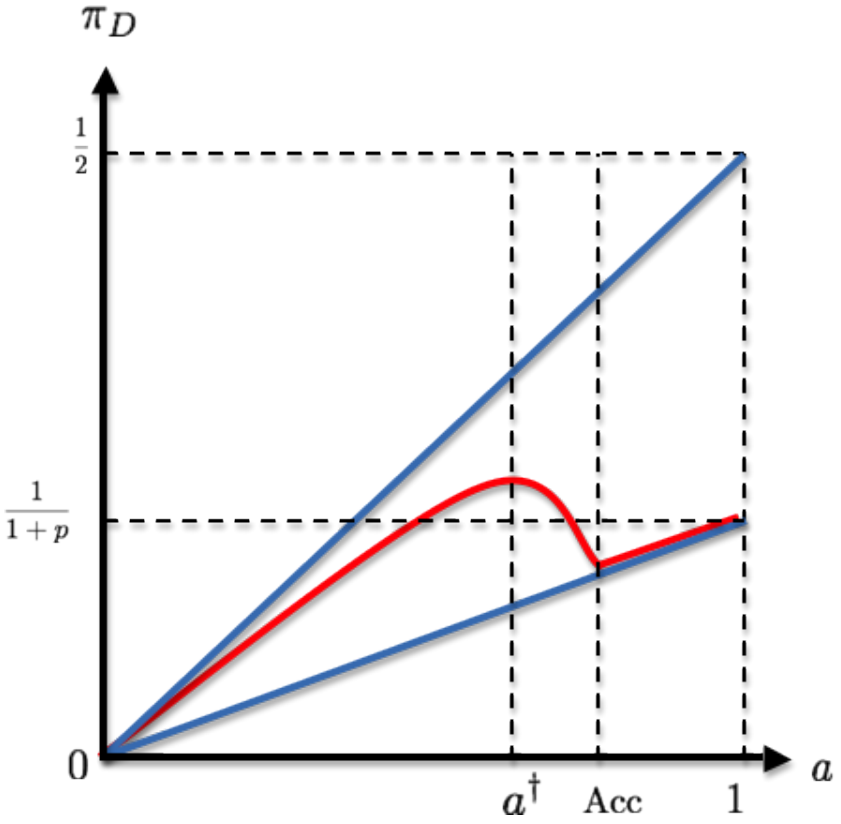


Figure 2.3: Designer's payoff as a function of a for $p < \hat{p}$, in red; and the designer's payoff by state, in blue

better technology and reaching the high technology case outweighs the increased payoff in the low technology case:

Proposition 2.3. *The designer's maximised payoff is not monotone in p: specifically, it is falling in p for small p, and rising in p for large p.*

Proof. Let $\pi_D^*(p)$ be the maximised value of $\pi_D(p)$, for any given p, and $a^*(p)$ be the prize level that maximises it. By the envelope theorem, $\frac{\partial \pi_D^*(p)}{\partial p} = \frac{\partial \pi_D(a^*(p),p)}{\partial p}$. At $p = 0$, the payoff function and the derivative are not defined. But we can sign the local derivative by considering the limit as $p \to 0^+$. (The one-sided limit is appropriate because the parameter space is [0,1]).

$$\lim_{p\to 0^+} \frac{\partial \pi_D^*(p)}{\partial p} = \lim_{p\to 0^+} \frac{\partial \pi_D(a^*(p),p)}{\partial p}.$$

$$\begin{aligned}
\pi_D(a^*(p),p) &= a^*(p)\Pr(2 \text{ wins } b)\frac{p_2}{2} + a^*(p)\Pr(1 \text{ wins b})\frac{p_1 p_2}{p_1+p_2},\\
\frac{\partial \pi_D(a^*(p),p)}{\partial p} &= \frac{a^*(p)}{2}\frac{\partial \Pr(2 \text{ wins } b)}{\partial p}\frac{1-p}{1+p} + \frac{1}{1+p}\Pr(1 \text{ wins } b),\\
\lim_{p\to 0^+} \frac{\partial \pi_D(a^*(p),p)}{\partial p} &= \frac{1}{2}\lim_{p\to 0^+} a^*(p).\lim_{p\to 0^+}\frac{\partial \Pr(2 \text{ wins } b)}{\partial p} + \lim_{p\to 0^+}\Pr(1 \text{ wins } b).
\end{aligned}$$

First, the term $\lim_{p\to 0^+} a^*(p)$. We know, from benchmark 2, that this is $\frac{4}{7}$. Alternatively, this is the limit of $a^\dagger(p)$ (although $a^\dagger(0)$ is undefined), and for low p, $a^\dagger = a^*$ by Proposition 2.

It is important to note that this limit is approached from below: $\lim_{p\to 0^+} a^*(p) = \frac{4}{7}^-$. To see this, take the limit of the first derivative of $a^\dagger$, which is $-\infty$:

For conciseness, express $a^\dagger$ as a function of polynomial functions:

$$a^\dagger = \frac{4A(p) - 4\sqrt{2}\sqrt{B(p)}}{C(p)},$$
$$\frac{\partial a^\dagger}{\partial p} = \frac{4C(p)A'(p) - 2\sqrt{2}\frac{B'(p)}{\sqrt{B(p)}} - 4A(p)C'(p) + 4\sqrt{2}\sqrt{B(p)}C'(p)}{C(p)^2}.$$

Then, taking the limits of these terms:

$$\lim_{p\to 0^+} \frac{\partial a^\dagger}{\partial p} = \frac{4 \times 49 \times 0 - \frac{2\sqrt{2}\times 7\times 49}{\to 0^+} - 4 \times 7 \times 168 + 4 \times 168 \times \sqrt{2} \times 0}{\to 0^+} = \frac{-\infty}{\to 0^+} = -\infty.$$

This also shows incidentally that the optimal progress prize is not always falling in p.

Differentiation of the expression for optimised designer payoff gives:

$$\frac{\partial \pi_D(a^*(p), p)}{\partial p} = -\frac{(((p^2+2p)(3a^*(p)-4) + 7a^*(p) - 4)(p^2(a^*(p)-4) + 2p(5a^*(p)-4) + 5a^*(p) - 4))}{(p^3(a^*(p)-4) + p^2(13a^*(p)-12) + p(11a^*(p)-12) + 7a^*(p) - 4)^2}.$$

The limit is:

$$\lim_{p\to 0^+} \frac{\partial \Pr(2 \text{ wins } b)}{\partial p} = \frac{4 - 5\lim_{p\to 0^+} a^*(p)}{7\lim_{p\to 0^+} a^*(p) - 4} = \frac{\to \frac{8}{7}}{\to 0^-} = -\infty.$$

Since we know that $\Pr(1 \text{ wins } b) \in [0, 1]$, showing that $\frac{\partial \Pr(2 \text{ wins } b)}{\partial p}$ is large and negative is sufficient to show that the overall expression is negative. Therefore an improvement in the weaker contestant's technology can lower the designer's payoff.

It is obvious that the designer can benefit from increased p: this is true for all $p > \hat{p}$ for example. □

The purpose of this proposition is simply to show that the designer's payoff is not monotone in technology. That is, the cost imposed by increased difficulty of selection can outweigh the benefit of an improved low-tech outcome. However, from Proposition

2, we know that this effect does not cause multiple regions in which progress prizes are optimal.

2.7 General case

When $\alpha > 0$, the more productive contestant has an additional advantage (continuing to assume that α is sufficiently low that the conditions (2.1) and (2.2) are satisfied). His marginal cost is always lower, and this results in an increased equilibrium probability of his winning the progress prize:

Proposition 2.4. *If conditions* (2.1) *and* (2.2) *remain satisfied, an increase in* α *raises the designer's payoff for any given* a, b.

Proof. In a Tullock contest, the ratio of efforts equals the ratio of prize values multiplied by the reciprocal of the ratio of marginal costs. Therefore, provided the Acceptance Condition and the Switching Condition remain satisfied, the equivalent single-round reformulation of the contest describes contestant behaviour and gives that:

$$\frac{x_1^{b*}}{x_2^{b*}} = \frac{B_1}{B_2}\frac{1-\alpha}{1-\alpha(1-\Pr(2 \text{ wins } b)^{*2})},$$

$$\alpha = \left[1 - \frac{B_2 \Pr(2 \text{ wins } b)^{*2}}{B_2 - B_1 \frac{x_2^{b*}}{x_1^{b*}}}\right]^{-1}. \tag{2.6}$$

We characterised the designer's payoff function as being a probability-weighted average of two different technology states. Altering α does not alter the payoff from either state for a given p and a. However, the probabilities with which the two states occur does

change.

Since $\Pr(2 \text{ wins } b)^*$ and $\frac{x_2^{b*}}{x_1^{b*}}$ move in the same direction by the definition of the CSF, inspection of (2.6) shows that an increase in α causes an increase in $\Pr(2)^*$, holding a, b, p (and therefore B_1, B_2) constant. Therefore the higher-payoff subgame occurs with greater probability. Therefore, the designer's payoff increases. □

Furthermore, the region in which a progress prize is optimal is not reduced by a higher α:

Proposition 2.5. *If conditions* (2.1) *and* (2.2) *remain satisfied, non-zero progress prizes remain optimal in the region defined by Proposition 2 when* $\alpha > 0$.

Proof. Let $\tilde{p} < \hat{p}$, so when $\alpha = 0$, $b^* > 0$, by Proposition 2.

Define the optimal grand prize for a general α as $a^*(\tilde{p}, \alpha) = \arg\max_a \pi_D(a; \tilde{p}, \alpha)$. This definition implies that it is better to have this prize than no progress prize when $\alpha = 0$:

$$\implies \pi_D(a^*(\tilde{p}, 0), \tilde{p}, 0) \geq \pi_D(1, \tilde{p}, 0).$$

If there is no progress prize, the level of α is irrelevant to the designer's payoff:

$$\pi_D(1, \tilde{p}, 0) = \pi_D(1, \tilde{p}, \alpha).$$

And by Proposition 4, the designer's payoff is increased for a given prize level by an increase in α, if (2.1) and (2.2) remain satisfied:

$$\pi_D(a^*(\tilde{p}, 0), \tilde{p}, \alpha) \geq \pi_D(a^*(\tilde{p}, 0), \tilde{p}, 0).$$

Combining these two inequalities and substituting using the equality, we have the result that:

$$\pi_D(a^*(\tilde{p}, 0), \tilde{p}, \alpha) \geq \pi_D(0, \tilde{p}, \alpha).$$

So some non-negative progress prize improves on the zero progress prize. □

Given these two results, it is possible to make some comment on the optimal "nearness" of the Question A to Question B. It is a simple matter to generalise the game so that some costless control over α is part of the designer's strategy space, occurring before the choice of prizes. Such control is likely to be due to some discretion in the precise Question B used to assess the progress prize.

The designer benefits from increasing α, that is, from making the research direction of B nearer to that of A. This increases the first-round advantage of the more productive contestant. However, if α becomes too large, the Switching Condition will be violated, and the investment in the first round will be too valuable for 1 to switch technology. The designer must therefore ensure that the research direction used for assessment of technology is not too similar to the direction he is interested in, and optimally maximises α, subject to the Switching Condition still being satisfied. This is the one case in this chapter when condition (2.1) becomes important. The analysis here is not valid if this condition is not satisfied, although it is true that designer payoff is increasing in α in the region where the condition holds.

2.8 Conclusion and discussion

2.8.1 Implications for prize design

This chapter has shown how a preliminary contest can be used as a means of identifying a superior technological approach and distributing it for another contest, and thereby increase the output generated in it. Under the argument of this model, progress prizes need not produce any output of direct value to the designer: this is not their function. They are a tool for selecting from among competitors those with the best ways of working. But more than this, they are compensation for being so selected. Since the contest for the grand prize is uncertain, but the offer of the progress prize is not, this structure can induce productive contestants to share their productivities. It is this combination of selection and conditional reward that makes progress prizes such a simple means for prize designers to improve the productivity of a population of researchers.

In such a situation, the designer must have access to a reasonable signal of contestant ability. Moreover, that ability must be transferable, and the transfer must be verifiable. These requirements make sense in the context of particular types of technological IPC, but not others. Research programmes where output is determined in part by the technology or working practices of the researcher are attractive settings for progress prizes. For example, the efficiency of a technology for testing prototypes can be shared. So can systems of collaboration or information technology. However, the higher productivity of a successful mathematician is likely in large part the result of non-transferable human

capital.

Many contests serve as focal points for researcher expertise and information exchange. Competitors can hardly avoid learning from others, as well as contributing to their rivals' endeavours. Progress prizes that require publication formalise the structure of this contest feature. However, care should be taken that in doing so, more effective incentives for early collaboration are not compromised. By forcing researchers into direct competition at an early stage, the informal processes of cooperation might be weakened.

2.8.2 Discussion of modelling choices

This chapter took as given that a contest was the only available way to learn about contestant ability. Moreover, the use of a Tullock lottery was taken as given. This was done to highlight the dynamic considerations of a single design decision (the use of the progress prize). It also raises clarity and tractability substantially, allowing the direct use of the familiar and simple Tullock lottery results in each subgame.

The model also constrains the designer to only offer a single prize in each round. In usual Tullock settings this is innocuous, since effort is maximized by setting the loser prize to zero. However, it requires more justification here as a reduced form of an optimal designer decision. Readers may have noted that a designer who was not so constrained could offer each contestant the same, fixed amount for the license to their technology, dispensing with the first contest (or equivalently, offering equal winner and

loser prizes). This ensures the better technology becomes available for the second round, and can be preferred to the single progress prize design. Although the designer can only differentiate by a win, he knows the other contestant holds the other technology. To motivate the award of a single prize, we can imagine that 1 and 2 are simply the only productive contestants in a large population of unproductive rent-seekers, who moreover cannot use a technology even if offered it. Then, the designer can identify the winner as a productive contestant, but cannot tell which of a large number of others produced effort. This removes the possibility of paying all the losers for their technologies.

This leads directly to an examination of why only two productive contestants are considered. Partly this is driven by tractability, ensuring a pure strategy equilibrium. Additionally, with only two contestants, the entire productivity distribution, from a design perspective, can be captured in a single variable. With a larger number of contestants, the potential gains are larger (since the winning technology can be applied to all those contestants who would benefit from it) but selection might become more difficult.

Two assumptions that perhaps sit a little uneasily together are those of limited designer information and assigned technologies. It is difficult to conceive of a case where the fact that low and high technologies exist is known, but the technology assignment is not: in real situations, knowledge of the level of a technology is usually embedded in the knowledge of what and where the technology is. However, this assumption gets directly to the core of the problem facing the designer: there are productive and unproductive contestants.

Regarding the assumed timing, it should be obvious that making the simultaneity and/or order of the contests endogenous adds nothing substantial to the designers strategy set. A simultaneous contest, or sequential contest in which Question A is answered first, leaves the technological environment on the Question A contest unchanged. Therefore, the optimal division of prize funds in such a case is $a = 1$. But this is identical to the structure above in outcome. Therefore, this chapter takes the timing as exogenously determined for ease of strategy exposition.

It should be noted that, aside from the assumptions about technology and effort carry-forward, which simply alter the parameters of the Tullock contest, the assumption about what happens when no effort is exerted is different from the standard Tullock case. Usually such a case involves a random (1/2,1/2) assignment of the prize. This makes sense in, for instance, rent-seeking applications with binary decision or sporting contests where someone has to win. Here, we permit (in fact, require) the designer to reserve the right not to award prize funds in the event, essentially, that no-one competes for them. This assumption aids analysis but is not unreasonable in an IPC contest. Many have certain minimum qualifying quality thresholds for competitor entry. Since the focus of this model is not the strategic effects of such thresholds, we can interpret this assumption as being that the designer can only differentiate between zero and positive effort, and the nature of effort is otherwise unobservable, except through the operation of the CSF.

2.8.3 Alternative reasons for progress prizes

Unrelated to the technology switching considered above is the use of progress prizes as a basis for contestant merger. Alliance formation is a key part of the contest literature, as well as the motivating examples in the introduction above. It seems that the award of progress prizes might help contestants to identify useful alliances in which to contest the final round. Another important idea, highlighted by McKinsey & Company [2009], is that early stage prizes can provide vital cash to capital-constrained but otherwise productive competitors. In such cases, as in this chapter, the designer wants to identify promising approaches early, but cannot directly observe them.

Early prizes might also allow contestants to signal to each other about their relative or absolute advantages. A contestant reasonably confident of winning the grand prize may redouble their efforts if an underdog rival unexpectedly wins the progress prize. For example, in the Netflix Prize, the revealed strength of the leader served to intensify the efforts of the other competitors. Alternatively, a contestant developing a particular technology might switch and cultivate an alternative approach if another contestant is shown to have a relative advantage in that area, and thereby avoid wasteful duplication of research. Conversely, such prizes may carry costs for designers if the approach itself is shown to be particularly fruitful, which may result in "groupthink" and the loss of diversity in research approaches.

More generally, this chapter has raised the question of why contest designers choose to implement progress prizes. As noted in the introduction, a number of possible motiva-

tions are possible, which standard models of contests fail to capture, as the aim of such a prize is to alter the circumstances of competition, rather than prelude an identical replay. The term "progress prize" in the context of this model is something of a misnomer, as the designer is actually rewarding a signal of technological advantage; that is, the means of producing output, not the output produced itself, which may be irrelevant or largely irrelevant to the designer's purposes. An alternative model might be structured as a race, rather than a contest, and actual progress identified and distributed.

Chapter 3

Prizes as quality verification mechanisms

3.1 Introduction and motivation

Inducement prize contests involving research and development do not occur in isolation. Often, the prize is not offered for the sake of the research alone, but because the research has some specific application or applications. Numerous technological IPCs mention future markets for the technology in their goals. This chapter aims to capture some limited elements of the interaction between research prizes and the markets for the resulting applications. In particular, it examines how the informational content of prizes can increase the incentive to research, and alter the offered qualities, and explores the determinants of the optimal difficulty of winning such a prize.

3.1.1 Outline of the chapter

This chapter analyses the welfare-optimal difficulty of prizes; that is, how good an entry must be in order to win. When there exists a quality leader, providing a binary signal about follower quality can improve welfare; both because it can give an incentive to follower research, and because it can alter the resulting qualities offered.

The chapter begins by offering a definition of such a "verification prize" and presents examples of IPCs where the competitor's research subsequently became a commercial venture. Indeed, the stated aims of the prizes looked at market development. These examples are used to motivate a model of quality information transfer in a vertically differentiated market with an established quality leader. Related work on strategic disclosure is discussed, and the broader context of a principal setting the conditions for communication are also considered.

A simple model is formulated, based on the vertical quality differentiation game of Shaked and Sutton [1982]. This is used as the basis for the final product market, while research and quality decisions are framed in the context of a binary signal sent from the entrant-follower to the incumbent-leader, where the signal type is determined by the prize designer.

The main results show that in such environment a number of different prize types emerge. Some focus on selecting high quality, while others attempt to stimulate intense price competition. Slightly counter-intuitively, others aim to discourage research altogether. In other cases, the absence of a prize is socially optimal - when research costs are particu-

larly low, or particularly high.

3.1.2 Features of verification prizes

There are numerous reasons why contestants might compete for a prize, independent of the size of the purse offered. Aside from the intrinsic desire to win, contestants may face other extrinsic monetary incentives related to success in an IPC. It is obvious that publicly winning a difficult prize is an informative signal, in what may be a very uncertain environment of cutting-edge research. But what is the content of that information? Who learns from it? And how does it create incentives for research? These are some of the questions that motivate the reading of the following examples and the construction of this chapter's model.

It is clear that for the informational content of a prize to create incentives to win it, there must be a post-prize reward environment. It is natural to consider this to be the market for the innovation in question. This also helps explain the observation of seemingly excessive spending by prize contestants, who are often noted to have spent more in pursuit of the prize that the prize itself, as in the RASE prizes below.

Moreover, the observation that the prize has been won must resolve some uncertainty for other market participants. Looking at the nature of IPCs, it is clear that they often rank an innovation characteristic along a single dimension, or against some single benchmark. Given that the ability to create some such unambiguous ranking is a precondition to offering a prize, this same measurement naturally suggests itself as a space for vertical

differentiation in the final market application of the innovation.

At this point, any model of verification prizes could reasonably suggest that the prize informs any or all of consumers, investors, or potential acquirers. However, this chapter considers a verification prize to be a signal to *rival producers* about the quality of the contestant's product. It requires that the research has some specific, predictable or expected commercial application. It assumes that the researcher and the producer are the same agent. (Or that they act so; for example, the researcher might sell forward the rights to commercialise the research, but this is not modelled explicitly). The following examples provide illustrations of prizes where these mechanisms might reasonably be supposed to have been at work.

3.1.3 Examples

Prizes offered by the Royal Agricultural Society of England (1839-1939) - Brunt et al. [2012]

This example holds the unique distinction and advantage in this thesis of having been the subject of an empirical analysis, in Brunt et al. [2012]. From 1839 the RASE awarded prizes for improvements to agricultural implements at its annual show, the first such being located in Oxford. These prizes could be monetary, or simple public recognition in the form of medals. The criteria for winning these prizes varied from year to year, but were usually announced a full year in advance. The authors of the paper creatively tie the award of nearly 2,000 of these prizes to the records at the British Patent Office

and to the products that followed, finding that the offer of a prize, whether monetary or not, had a significant impact on the quantity and quality of the patents that followed. In fact, the prestigious medals had greater effects on the number of entries than the monetary prizes. The authors present a hypothesis as to why this was: "Interestingly, entrants were also attracted to purely non-pecuniary prize competitions by the offer of medals. Part of the payoff to entrants came in the form of free advertising that entry (and particularly winning) conferred on the invention." [Brunt et al., 2012, p. 669]. That is, winning the prize produced an informational payoff for the contestant.

In fact, the post-prize market for the innovation must have formed part of the entrant's calculation, since even "[t]he monetary prizes, although substantial, certainly did not fully cover the average costs of development." [Brunt et al., 2012, p. 668]. There was an undoubted quality signalling advantage in the final consumer market resulting from these medals. For example, Cyrus H. McCormick, winner of a number of medals and founder of the future Harvester International, used these awards in his advertising [McKinsey & Company, 2009, p 31]. However, from the perspective of motivating the model of this chapter, as it relates to the introduction of new entrants in a market with an incumbent quality leader, there is a quote the authors reference [Brunt et al., 2012, p. 665] that is highly illuminating, especially in its broader context. *Newton's London Journal of Arts and Sciences* (August 1, 1860, p. 66) records that:

> ... a memorial signed by all the regular implement exhibitors at the annual shows, with the exception of two dissentients, represented to the Council the

> general feeling of the manufacturers to be opposed to the existing system of money prizes for competition. "We object," says the memorial, "to this system, on the ground that it operates as an undue stimulus to competiton..."
>
> ...

In other words, the incumbents object that these prizes are causing new entrants to develop innovations that allow them to compete more directly. Fortunately, the (welfare maximising?) prize sponsors at the RASE continued to award the prizes.

The NASA Astronaut Glove Challenge (2007)

The Astronaut Glove Challenge is one of eleven inducement prize contests (seven completed, four ongoing) offered by NASA for privately funded research into specific areas. The aim of the prizes is to discover innovative solutions from diverse and non-traditional sources. One of the stated objectives is to "Enable Challenge competitors to develop and/or expand business models and business base" [NASA, Space Technnology Mission Directorate website, Accessed 04/2013]. It is therefore clear that the potential future commercial enterprises of participating contestants form part of the design problem for NASA, and moreover that the assistance of these aims is part of the reason for establishing a prize in the first place. Two glove competitions have been run so far, one in 2007 and another in 2009. The emphasis here is not on any dynamic considerations in this example (see Chapter 2), so we focus on an examination of the 2007 prize.

In 2007, contestants had to produce the inner-most layer of a space suit glove, which

acts to restrain pressure during space walks. The main $200,000 prize was offered for the best glove that exceeded certain minimum technical specifications, which were based on the existing glove procured by NASA. That is, the entrants had to simply beat the incumbent quality level. The competition was managed on behalf of NASA by a non-profit educational venture, Volanz Aerospace, and the competition took place on the 2nd and 3rd of May.

The winner, an unemployed engineer named Peter Homer, went on to found Flagsuit LLC to manufacture gloves for the emerging commercial spaceflight market, and other commercial applications of the winning technology. Here is a clear example of a prize being offered, not just for research, but specifically for manufactured prototype designs. Moreover, it succeeded in incentivising research that allowed both higher quality and more competition to exist in the market.

The Cheap Access To Space prize (2000)

Another interesting example is the Cheap Access to Space (CATS) Prize. $250,000 was offered from November 1997 for the first private team to send a 2kg payload into space, 200km or higher. Several private launch companies participated, as well as a number of individuals, and several test launches were carried out. These launches broke records: the amateur altitude record; and the non-government liquid fuelled in-flight burn time. However, no team was able to reach the goal before the deadline of November 2000, and the prize expired unclaimed. Despite this seeming failure, the contestants continue to use

the research generated in the competition and some now compete in the launch-services market. The degree of quality of these launches varies greatly: Interorbital Systems aims to offer the first fully orbital tourist package and to commercially sell moondust, whereas JP Aerospace has as one of its programmes a free-to-user service that allows students to launch ping-pong balls into space. This example will be illustrative in later discussion about if having the prize won is always the best outcome from a welfare perspective.

Other verification prizes

There are numerous example of prizes where the research was developed into a commercial product. A few particularly pertinent cases are listed here for the interested reader: the Global Security Challenge, where winners often secure large contracts; the Alkali prize, which revolutionised the French chemical industry, and the Turbine Prize by the French Society for the Encouragement of Industry, which displaced existing turbine designs throughout Europe.

3.1.4 When and how should verification prizes be used?

By verifying the quality of a contestant's research, a prize can offer the potential for research to result in more market power than would otherwise be the case. This raises the ex-ante incentive for research. However, it also influences the ex-post market equilibrium. Prize design, particularly in cases where the potential value of the market for the research

is far larger than the size of the prizes offered, should take account of the informational content of winning a prize. The informational value of a prize is not necessarily purely psychic for the competitor, but may represent a real increase in his ability to monetize the research. In particular, the difficulty of winning the prize is a key determinant of the informational content, and therefore the welfare effects. Some of the key questions this chapter addresses are:

1. Can a purely informational technology prize raise welfare?
2. How does the existence of such a prize influence research decisions?
3. How does the existence of such a prize influence the equilibrium offered qualities?
4. How difficult should such a prize be to win?

3.1.5 Related work

The model itself is based on a simplified version of the vertical quality differentiation game of Shaked and Sutton [1982], which provides the basis of the post-prize product market. Although that paper looks to answer largely different questions than those posed in this chapter, it is useful to briefly see, in particular, what simplifications have been made. From a theoretical perspective, this paper largely builds on work about quality disclosure. Most other approaches to this question have looked at the incentives (or lack of incentives) for firms to reveal information they know, or can discover, privately. Moreover, this information usually affects the strategic decisions of consumers, rather than other firms. However, it is useful to review some key ideas in this literature.

In this chapter we assume that verifiable quality disclosure is impossible (perhaps because this would necessarily compromise trade secrets or proprietary knowledge), and that disclosure requires a third party intermediary. However, it has clear theoretical parallels with the mandatory disclosure literature, since the model effectively forces the entrant to submit its innovation to public assessment.

Shaked and Sutton [1982]

This paper is the foundation of the model in this chapter, although here it is simplified and changed in important ways, for focus, tractability, and for analysis of the particular trade-offs involved in the design of a stylised verification prize. The Shaked and Sutton [1982] paper is an analysis of competition over two variables: product specification and price, and is in three stages, where all firms act simultaneously in each. First, an entry decision, with a fixed cost of doing so. Then, qualities are set. Finally, the market equilibrium is determined via simultaneous price-setting. The main results are that only two firms will enter the market; and these will produce differentiated products, that cover the whole market; and therefore equilibrium prices result in positive profits. In fact, the model of the consumer market is itself taken from two earlier papers: Jaskold Gabszewicz and Thisse [1979] and Jaskold Gabszewicz and Thisse [1980].

A number of their results are assumed or made trivial by the analysis of this chapter. Only two firms exist, and the utility function has a large enough fixed utility from purchase to ensure the market is covered. The utility function is also simplified, so instead

of post-purchase income multiplying product quality, here there is a direct preference for quality. A very similar treatment can be found in Belleflamme and Peitz [2010, pp. 120-122]. But the most important changes are strategically important and are in the first two stages of the game. Regarding entry, one firm is already in the market, and the other has the chance to pay a fixed research cost to enter, but with uncertainty quality. In modelling the quality decision, the two most important differences exist. First, the incumbent firm also acts as a quality leader. This assumption is more in line with the idea of using prizes to stimulate additional competition in an existing market (as in the RASE example). Since the provision of prizes is in some way an intervention, it also helps to create the prior possibility of market failure in some sense, which the prize can act to correct. Second, the two firms are not symmetric in the quality they can provide. In Shaked and Sutton [1982], firms can provide any quality, limited by an exogenously given *common* upper bound. Here, the two firms differ in the maximum quality they can provide, and there is *ex ante* uncertainty about the entrant's maximum. This provides a further necessary element for sub-optimal welfare, because now the identity of the firm offering the higher quality matters, and therefore the move order introduces further inefficiency. This is an attempt to capture the idea that prizes are focused at the research boundary, with the possibility of previously impossible quality levels becoming available as the result of innovative investments by firms.

Board [2009]

This paper also uses a vertically differentiated market to explore the impact of quality information. It is motivated by the observation that there exists legislation that mandates the disclosure of product quality. It notes that the need for this legislation is unclear, since it seems unnecessary in the context of the standard verifiable disclosure results, which predict that non-disclosure will be interpreted in the least favourable way possible. Here, again a role for a third party in facilitating the disclosure of quality information is developed. Although the uninformed agents are consumers, there are parallels with the model of this chapter, since disclosure affects the perceived quality placement in the market, and therefore the strategic decisions of other firms.

A brief survey of the various theories as to why the unravelling result fails to hold is presented in the paper, but the main model is of particular interest in this context, because it focuses on the intensity of competition, and how this is influenced by quality level disclosure. The disclosure that products are close competitors has the strategic effect of raising price competition, which may be more harmful to a firm's profits than allowing consumers to perceive a product as potentially low quality. The characteristics of the equilbria that emerge are not dissimilar, with the expectation of increased competition among high-quality products, but less among low quality products.

The reason that this paper is a key reference here is that it considers the competitive effects of quality disclosure in a market with multiple firms. This is in contrast to the majority of quality disclosure papers, which posit the existence of a distribution of

hypothetical quality types, only one of which actually exists (see, for example, Grossman and Hart [1980] and Jovanovic [1982]).

Levin et al. [2009]

This is another paper which looks at the role of competition in determining firm's disclosure. This paper is particularly similar to the model presented in this chapter in that it assumes that, in a duopoly, a firm's characteristics are not observed by its rival. However, it differs by assuming a verifiable disclosure means of information transfer, and by modelling competition in a horizontally differentiated environment.

By comparing separate ownership to joint ownership, the main results are that more disclosure occurs under joint-ownership than under separate ownership. The former produces higher welfare if the disclosure costs is low, whereas the latter does if such costs are high. In either case the degree of disclosure is socially excessive. This occurs only because disclosure is costly, and there is a threshold value below which firms will not reveal their quality (as in Board [2009] above).

This is in contrast to the model presented here, which assumes that no disclosure can be made, except through a prize competition, and shows that the optimal such prize can improve on no disclosure. Here, a welfare-maximiser should intervene to raise the level of information in the market.

3.2 A model of prizes as quality verification mechanisms

The model is based on a simplified vertical quality differentiation game of Shaked and Sutton [1982], which provides the basis of the post-prize product market. Specific research and informational assumptions are used to generate insight into the design of prizes.

3.2.1 Assumptions

Players

The market consists of a set of consumers, a prize designer D and two firms: an incumbent I and a potential entrant E. Incumbency here relates to two distinct concepts: ex-ante capability to produce products of non-zero quality *and* to leadership in quality choice in the product market. Entry similarly relates to the potential to research and supply such a product offering *and* to being a quality follower.

Payoffs

Consumers each purchase a single unit to maximise utility $u = v + \theta s - p$, where p is the price paid, s is the quality of the product purchased, θ is a taste parameter and v is set high enough that all consumers purchase. There exists a unit mass of consumers, with θ uniform on [0,1].

I and E have constant and symmetric marginal costs of production c, and maximise

expected profits in the final product market. D acts to maximise total welfare.

Note that we abstract away from any monetary prize value to focus on how a prize can leverage existing demand to induce beneficial research.

Timing

1. Incumbent I is endowed with a publicly observable maximum quality level $\bar{s}_I \in (0, \infty)$; the existing state-of-the-art. Entrant E is endowed with a publicly observable fixed cost of research $F \in (0, 2/9]$; the opportunity to improve that technology.

2. Designer D publicly chooses $\hat{s}$ and commits to verify any entrant maximal quality $\bar{s}_E$ with a report of U (Unsuccessful) if $\bar{s}_E < \hat{s}$ or of S (Successful) if $\bar{s}_E \geq \hat{s}$.

3. E makes a public binary research decision $e \in \{R, N\}$. If he performs research (R), he privately draws an innovation $\bar{s}_E$ from a uniform distribution on $[0, 1]$ at cost F, which determines the maximum quality product he can offer. The distribution of the innovation is public knowledge. D observes $\bar{s}_E$ and reports U or S. If there is no research (N), then $\bar{s}_E = 0$. That is, even without research he can offer products of zero quality.

4. I publicly chooses his quality level $s_I \in [0, \bar{s}_I]$.

5. E publicly chooses his quality level $s_E \in [0, \bar{s}_E]$.

6. Bertrand price competition and payoffs.

3.2.2 Interpretations

The interpretation of the model should follow from the particular examples presented above. An incumbent is producing an established product of a given quality. This incumbent also acts as a quality leader, due to his market dominance. Another firm exists in the market, capable of producing a basic version of the product, but considering undertaking a costly and risky research project to allow the production of a higher-quality product, potentially of higher quality than the incumbent's.

Such an entrant firm, considering the cost of the project, and the leadership advantage of the incumbent, might well opt not to undertake the research. The major problem is that, even given successful research, there is no way to credibly communicate his new quality advantage to the incumbent without revealing proprietary information. That is, any information about the innovation sufficiently detailed to be credible prior to production is also necessarily sufficient to replicate it. Since this prevents any sort of strategic accommodation in the quality space, the likely degree of price competition after research is too severe to make it worthwhile.

In such a situation, with an un-innovative incumbent unchallenged by close competition, a welfare-maximiser might consider offering to act as an independent verifier of new product developments, hoping in this way to give potential innovators the possibility that successful research will achieve an advantageous position in the quality spectrum. But the designer of the prize may then be faced with a range of prizes that cause this socially beneficial research, and has to determine how good the innovation must be to

achieve the prize. The advantages of replacing one high quality firm with another similar are perhaps slim. He must tradeoff the need to incentivise research with the need to achieve a competitive and high-quality final product duopoly.

3.3 The final product market

When the final product market is reached, there is no uncertainty. Each firm has publicly chosen its quality, and there is a unique Nash equilibrium in the subgame. Let $s_1 < s_2$ be the two qualities offered by the firms, now labelled 1 and 2. Therefore, given corresponding prices p_1, p_2, the indifferent consumer is θ_i, such that $\theta_i s_1 - p_1 = \theta_i s_2 - p_2$. Therefore quantities demanded are $Q_1 = \theta_i = \frac{p_2 - p_1}{s_2 - s_1}$ and $Q_2 = 1 - \theta_i = 1 - \frac{p_2 - p_1}{s_2 - s_1}$. The profit functions are:

$$\pi_1(p_1, p_2, s_1, s_2) = (p_1 - c)\frac{p_2 - p_1}{s_2 - s_1},$$
$$\pi_2(p_1, p_2, s_1, s_2) = (p_2 - c)\left[1 - \frac{p_2 - p_1}{s_2 - s_1}\right].$$

Then the respective equilibrium Bertrand prices are $p_1^* = c + \frac{1}{3}(s_2 - s_1)$ and $p_2^* = c + \frac{2}{3}(s_2 - s_1)$. $\theta_i = \frac{1}{3}$, so consumers with $\theta < \frac{1}{3}$ purchase the lower quality, and those with $\theta \geq \frac{1}{3}$ the higher. This equilibrium is illustrated in Figure 3.1

In subsequent analysis, the following characteristics of the product market equilibrium will be useful:

- Profits are $\pi_1^* = \frac{1}{9}(s_2 - s_1)$ and $\pi_2^* = \frac{4}{9}(s_2 - s_1)$..
- Consumer surplus is $C^* = v - c + \frac{1}{18}(9s_1 - 2(s_2 - s_1))$.

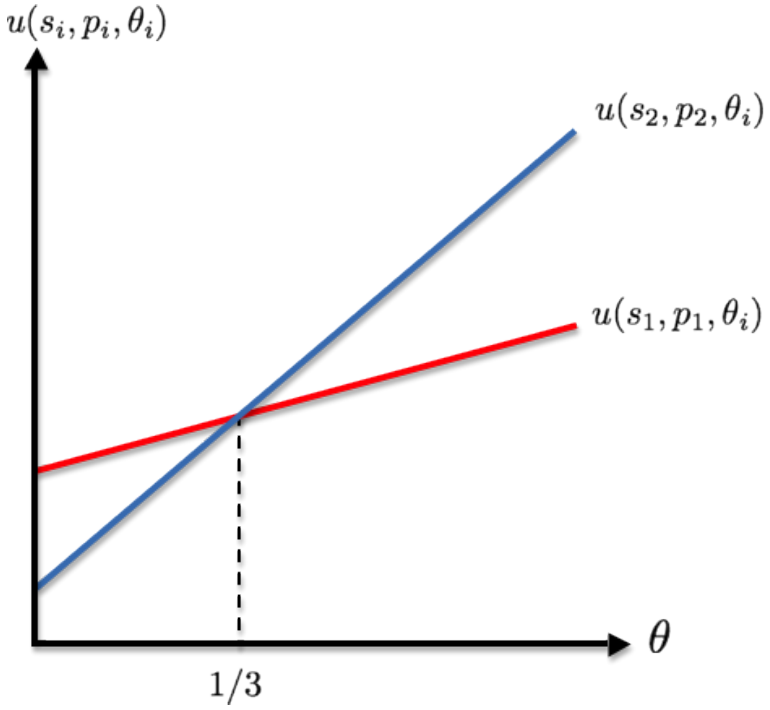


Figure 3.1: Equilibrium in the final product market

- Welfare $W^* = v - c + \frac{1}{18}(9s_1 + 8(s_2 - s_1))$.

Note that this implies that D's objective is to maximise $(s_1 + 8s_2)$, net of any research costs. That is, a weighted sum of the offered qualities.

3.4 Quality choices

Prior to the final product market, sequential public quality choices are made, with I leading and E following:

3.4.1 *E*'s quality choice

E only has a non-trivial choice in quality if he has earlier played R and performed research. If he played N, he trivially chooses to offer a zero-quality product, i.e. $s_E^* = 0$. E has already observed s_I, I's quality choice, and faces the following problem:

$$\max_{s_E \in [0, \bar{s}_E]} \pi_E(s_E; s_I^*),$$

$$\text{where } \pi_E(s_E; s_I^*) = \begin{cases} \frac{4}{9}(s_E - s_I^*) & \text{if } s_E \geq s_I^* \\ \\ \frac{1}{9}(s_I^* - s_E) & \text{if } s_E < s_I^* \end{cases}.$$

So E's profit falls linearly in own-quality, to zero when $s_E = s_I$, and then increases linearly thereafter. Evidently, $s_E = 0$ is the best of all $s_E \leq s_I$, and $s_E = \bar{s}_E$ is the best of all $s_E > s_I$. Therefore E's optimal quality choice is a cut-off strategy in $\bar{s}_E$, so E plays his maximum possible quality if it is sufficiently greater that I's offered quality. Otherwise he plays a zero quality. The solution is:

$$s_E^* = \begin{cases} \bar{s}_E & \text{if } \bar{s}_E \geq \frac{5}{4} s_I^* \\ \\ 0 & \text{if } \bar{s}_E < \frac{5}{4} s_I^* \end{cases}. \tag{3.1}$$

See Figure 3.2 for an illustration.

3.4.2 *I*'s quality choice

The effect of a prize is to give some information to I regarding the strength of E's innovation. Therefore, his product quality decision depends both on the difficulty of

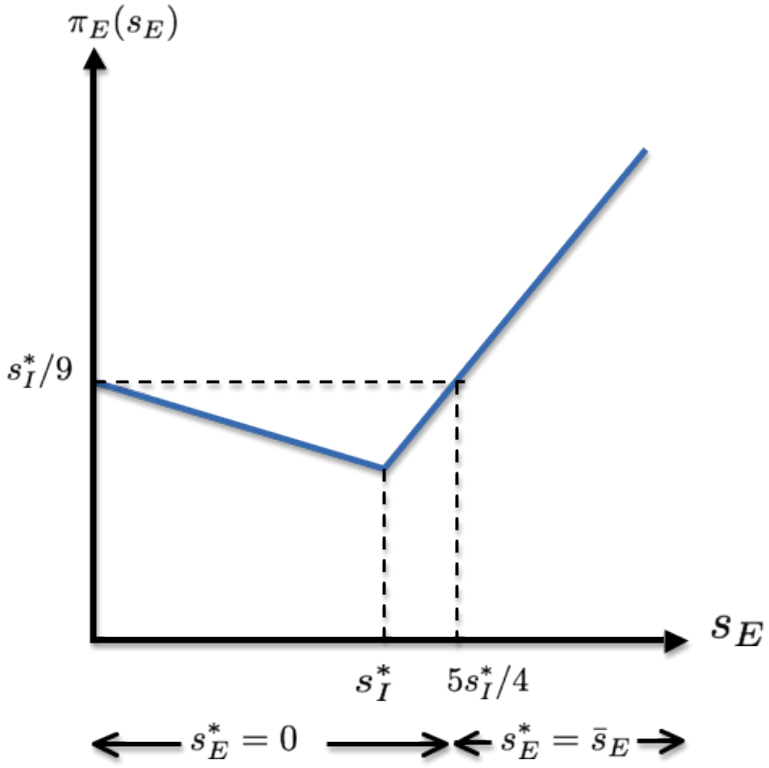


Figure 3.2: E's quality choice

achieving a prize, $\hat{s}$, and on the resulting signal $\{U, S\}$.

There are three subgames:

1. (N): No research by E.

2. (R, U): Research by E that does not win the prize.

3. (R, S): Research by E that wins the prize.

Subgame 1 N

If there is no research (N), so $s_E^* = 0$, trivially $s_I^* = \bar{s_I}$.

Subgame 2 - I's quality choice following R, U

I chooses his product quality to maximise expected profit, based on the fact that E has performed research that failed to win a prize of difficulty $\hat{s}$:

$$\max_{s_I \in [0, \bar{s}_I]} E[\pi_I(s_I, \hat{s}, U)].$$

Based on U and $\hat{s}$, I's posterior beliefs about E's quality are $\bar{s}_E \sim U[0, \hat{s})$. If it were the case that $\hat{s} \leq \frac{5}{4}\bar{s}_I$, this would remove all of I's strategic uncertainty, since E failed to win an easy prize, relative to the existing technology belonging to I. The fact that E lost means that I's producing a quality of $\bar{s}_I$ is certain to result in $s_E^* = 0$, since there are no longer any possible E qualities that would be sufficient to allow him to prefer to enter the market as the higher-quality producer. I knows he will be the high quality producer, and his profit is increasing linearly in his own quality, so $s_I^* = \bar{s}_I$.

But if $\hat{s} > \frac{5}{4}\bar{s}_I$, the failure of E to win the prize does not resolve all of I's uncertainty. He knows that E will play a zero quality if $\bar{s}_E \leq \frac{5}{4}s_I$, but $\bar{s}_E$ otherwise, and not all such sufficiently high qualities have been ruled out by the signal. In the former case, E has the lower quality, and in the latter, E has the higher quality. By choosing to produce a high quality, I runs the risk that he will end up as the lower-quality producer anyway, but in a less differentiated market, and therefore regret his decision not to offer the zero quality.

As shown in the analysis of the final product market, the form of I's profit function varies in the two cases, depending on the identity of the high quality producer. Given

$\hat{s} \leq \frac{5}{4}\bar{s}_I$, expected profit is:

$$E[\pi_I(s_I, \hat{s}, U)] = \Pr\left(\bar{s}_E \leq \frac{5}{4}s_I\right)\frac{4}{9}s_I + \Pr\left(\bar{s}_E > \frac{5}{4}s_I\right)\frac{1}{9}E\left[\bar{s}_E - s_I \middle| \bar{s}_E > \frac{5}{4}s_I\right],$$
$$= \frac{175s_I^2 - 32s_I\hat{s} + 16\hat{s}^2}{288\hat{s}}.$$

So I's expected profit is quadratic in own-quality, with a minimum at $\bar{s}_I = \frac{16\hat{s}}{175}$. In a similar way to the analysis of E's quality choice, this implies a cut-off strategy, with the two possible optima being $s_I^* = 0$ or $s_I^* = \bar{s}_I$:

$$E[\pi_I(0; \hat{s})] = \frac{\hat{s}}{18},$$
$$\implies E[\pi_I(0)] \leq E[\pi_I(s_I)] \iff s_I \geq \frac{32}{175}\hat{s}.$$

Note that this cut-off is less than $\frac{4}{5}\hat{s}$ and therefore in the domain of the expected profit function, so I's optimal quality strategy after U is:

$$s_I^*(U) = \begin{cases} \bar{s}_I & \text{if } \hat{s} \leq \frac{175}{32}\bar{s}_I \\ 0 & \text{if } \hat{s} > \frac{175}{32}\bar{s}_I \end{cases}.$$

A summary of these results, for all $\hat{s} \in [0, 1]$ is shown in Figure 3.3.

Subgame 3 - I's quality choice following R, S

If E wins the prize, I's posterior beliefs are $\bar{s}_E \sim U[\hat{s}, 1]$. Following S, it is in the case that $\hat{s} \geq \frac{5}{4}\bar{s}_I$ that all strategic uncertainty is removed - that is, if the entrant wins a particularly difficult prize, relative to the prevailing technology. In this case, I is

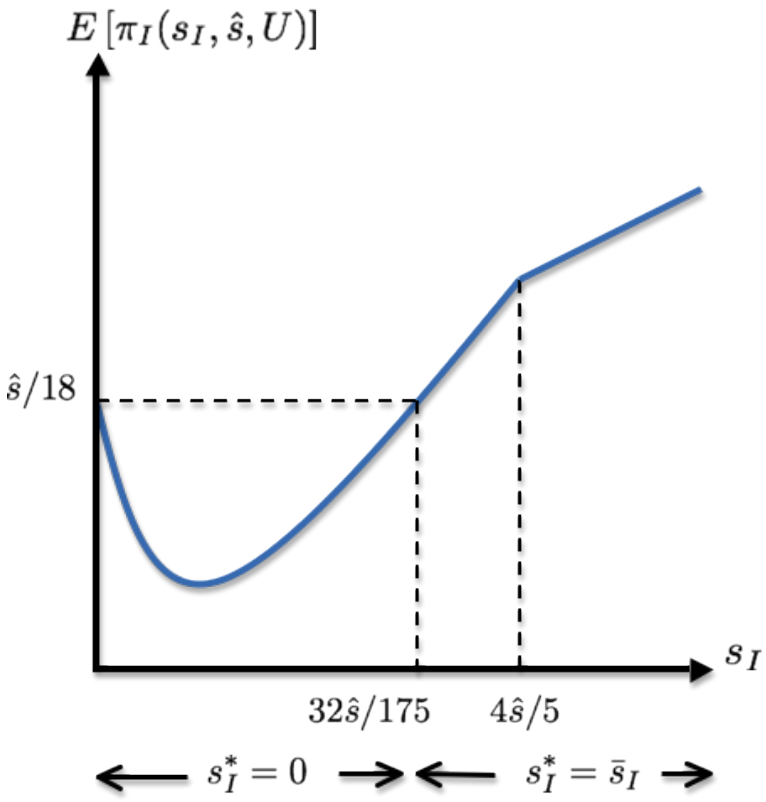


Figure 3.3: I's quality choice following (R, U)

sure that E's innovation is of a sufficiently high quality that E will produce $\bar{s}_E > \bar{s}_I$, regardless of I's quality choice. In this case, $s_I^* = 0$.

If $\hat{s} < \frac{5}{4}\bar{s}_I$, then there remains some uncertainty for I as to whether he can still establish a place as the higher-quality producer. Although E's innovation was good enough to win the prize, this does not in this case mean it is sufficiently good for it to be optimal for E to improve on a high quality product offered by I. The expected profit is:

$$\begin{aligned} E[\pi_I(s_I;\hat{s},H)] &= \Pr\left(\bar{s}_E \leq \frac{5s_I}{4}\right)\frac{4s_I}{9} + \Pr\left(\bar{s}_E > \frac{5s_I}{4}\right) E\left[\frac{\bar{s}_E - s_I}{9}\middle|\bar{s}_E > \frac{5}{4}s_I\right], \\ &= \frac{175s_I^2 - 128s_I\hat{s} - 32s_I + 16}{288(1-\hat{s})}. \end{aligned}$$

Again, expected profit is quadratic in own-quality, with a minimum at $s_I = \frac{16}{175}(4\hat{s}+1)$. If I can produce a sufficiently high quality, this is preferred. Otherwise, a zero quality is optimal. See Figure 3.4.

$$\begin{aligned} E[\pi_I(0;\hat{s})] &= \frac{1}{18}\left(\hat{s}+1\right), \\ \implies E[\pi_I(0;\hat{s})] \leq E[\pi_I(0;\hat{s})] &\iff \hat{s} \leq 4s_I - \frac{\sqrt{s_I}}{4}\sqrt{81s_I+32}. \end{aligned}$$

Optimal play is therefore a cut-off strategy. Call the cut-off point in this case $s^C(\bar{s}_I)$:

$$s_I^*(\hat{s}) = \begin{cases} \bar{s}_I & \text{if } \hat{s} < 4\bar{s}_I - \frac{\sqrt{\bar{s}_I}}{4}\sqrt{81\bar{s}_I+32} = s^C(\bar{s}_I) \\ 0 & \text{if } \hat{s} \geq 4\bar{s}_I - \frac{\sqrt{\bar{s}_I}}{4}\sqrt{81\bar{s}_I+32} = s^C(\bar{s}_I) \end{cases}.$$

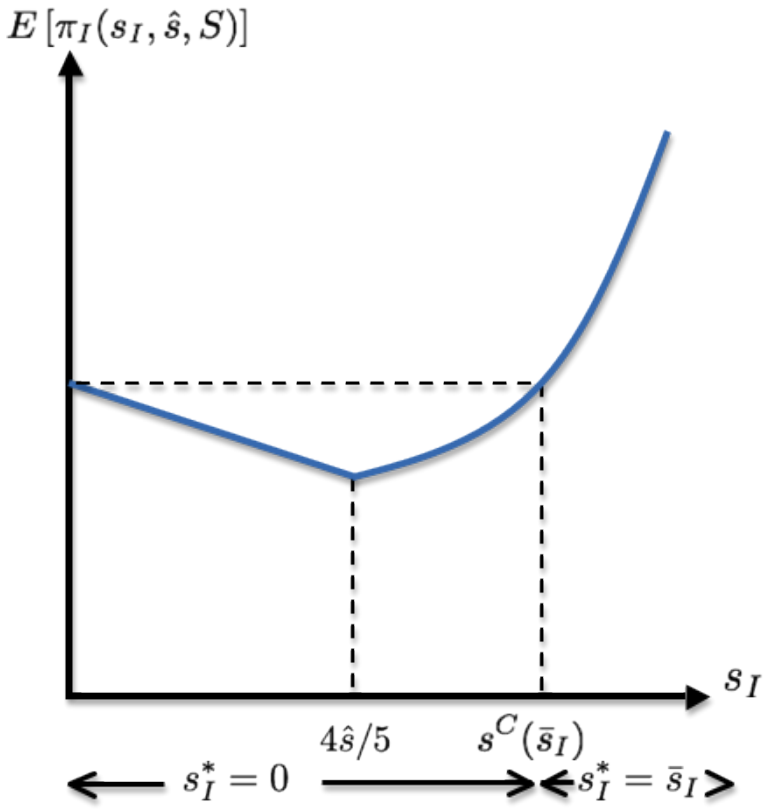


Figure 3.4: I's quality choice following (R, S)

3.4.3 I's quality strategies

Since I's optimal play following N is always the same, and the only two possible optimal qualities are 0 and $\bar{s}_I$, we can label his strategy based on the play following R in the following way:

- Aggressive (A) strategy if he plays $\bar{s}_I$ after both S and U;
- Passive (P) strategy if he plays 0 after both S and U;
- Contingent (C) strategy if he plays $\bar{s}_I$ after U but 0 after S.

Therefore a firm strategy profile (i.e. given $\hat{s}$, D's strategic decision) is in general $\{i, e\}$, where i represents play in the case when $e = R$.

Using this notation, we can summarise I's best responses to R as follows:

$$A \quad \text{if } \bar{s}_I > 32/175 \text{ and } \hat{s} < s^C(\bar{s}_I);$$

$$P \quad \text{if } \bar{s}_I < 32/175 \text{ and } \hat{s} > \tfrac{175}{32}\bar{s}_I;$$

$$C \quad \text{otherwise.}$$

These regions are shown in Figure 3.5. Note that D is not unconstrained in the strategy

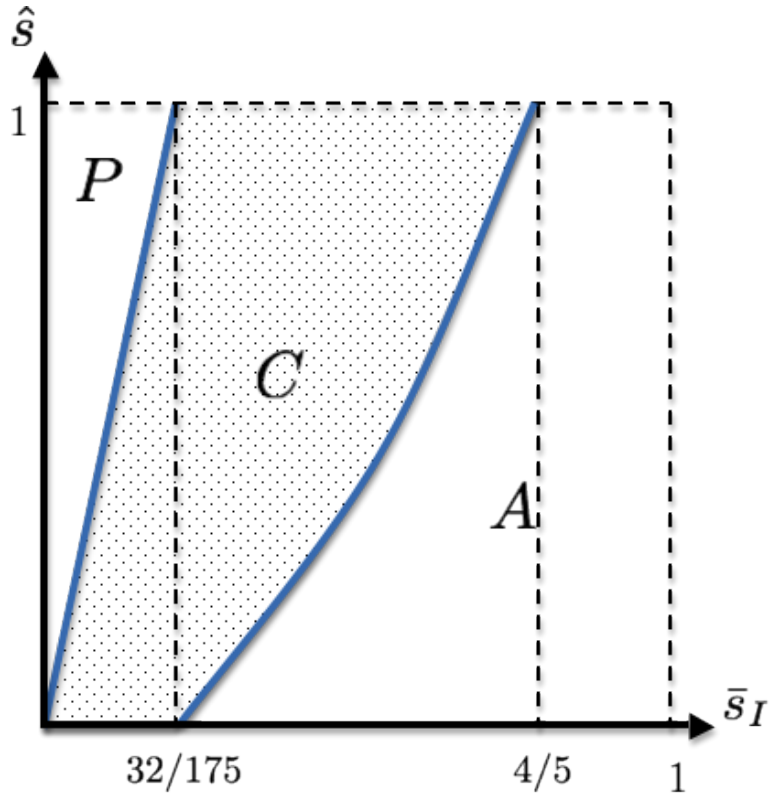


Figure 3.5: I's optimal quality strategies

choice responses his signals can produce in I. The value of s_I relative to 32/175 determines the feasible responses that D can achieve. Note that this is I's strategy cut-off

point in the no information benchmark; that is, if $\hat{s} = 0$ or $\hat{s} = 1$. Such prizes are completely uninformative given the publicly known support of E's innovation distribution. Therefore, in the absence of a prize, and following R, I plays $\bar{s}_I$ if $\bar{s}_I \geq \frac{32}{175}$, and plays 0 otherwise.

For a general prize, when $s_I < 32/175$, I will either play L, or C if the U signal is sufficiently convincing about the poor quality of E's innovation. Conversely, when $s_I \geq 32/175$ I will either play H, or will play C if S is sufficiently convincing of the high quality of E's innovation.

A corollary is that there no $\hat{s}$ that causes both the response to S and to U to be conditional on $\bar{s}_I$. The critical value is $\hat{s} = \frac{32}{175}$, for intuitive reasons: If no information causes I to enter with a zero quality, report S can never induce a higher quality choice. Similarly, if no information causes I to enter with his maximal quality, report U can never induce him to produce a lower quality.

3.5 The research decision

When I plays C, E's expected profit function depends on if there is positive probability that both will enter with non-zero quality. In fact, in such a case, both will enter with their maximal quality. This occurs if $\bar{s}_E \in [\frac{5}{4}\bar{s}, \hat{s}]$ (where $\frac{5}{4}\bar{s} < \hat{s}$). In such cases, E's innovation is not sufficiently good to win the prize, so I plays $\bar{s}_I$, but is good enough to make it profitable for the entrant to enter the market with a higher quality product afterwards anyway.

- $\frac{5}{4}\bar{s}_I < \hat{s}$: with positive probability, both enter with non-zero qualities:
  - If $\bar{s}_E \in [0, \frac{5}{4}\bar{s}_I]$, then U and $\{s_I^*, s_E^*\} = \{\bar{s}_I, 0\}$,
  - If $\bar{s}_E \in (\frac{5}{4}\bar{s}_I, \hat{s}]$, then U and $\{s_I^*, s_E^*\} = \{\bar{s}_I, \bar{s}_I\}$,
  - If $\bar{s}_E \in (\hat{s}, 1]$, then S and $\{s_I^*, s_E^*\} = \{0, \bar{s}_E\}$.
- $\frac{5}{4}\bar{s} \geq \hat{s}$: There is only ever one firm with non-zero quality:
  - If $\bar{s}_E \in [0, \hat{s}]$ then U and $\{s_I^*, s_E^*\} = \{\bar{s}_I, 0\}$,
  - If $\bar{s}_E \in (\hat{s}, 1]$ the S and $\{s_I^*, s_E^*\} = \{0, \bar{s}_E\}$.

The following functions are E's expected profit, given $\hat{s}$, with the quality strategy of I held constant, but all other choices (E's quality, and prices) responding optimally.

If no research is performed, N, the strategy of I following R is irrelevant for expected profit:

$$\pi_E^*(\hat{s}, \bar{s}_I, e, N) = \frac{1}{9}\bar{s}_I \qquad \forall e \in \{A, P, C\}.$$

If research is performed, R, the profit function varies with I's strategy:

$$E\left[\pi_E^*(\hat{s}, \bar{s}_I, A, R)\right] = \frac{1}{72}(25\bar{s}_I^2 - 32\bar{s}_I + 16) - F,$$
$$E\left[\pi_E^*(\hat{s}, \bar{s}_I, P, R)\right] = \frac{2}{9} - F,$$
$$E\left[\pi_E^*(\hat{s}, \bar{s}_I, C, R)\right] = \begin{cases} \frac{1}{9}(\bar{s}_I\hat{s} - 2\hat{s}^2 + 2) - F & \text{if } \hat{s} < \frac{5}{4}\bar{s}_I \\ \frac{1}{72}(25\bar{s}_I^2 - 32\bar{s}_I\hat{s} + 16) - F & \text{if } \hat{s} \geq \frac{5}{4}\bar{s}_I \end{cases}.$$

Note that the research payoffs when facing A or P are also those that exist in the no prize (i.e. no information) case, because these are equivalent to the strategies that are

played by I in that case: either $s_I^* = \bar{s}_I$ for certain or $s_I^* = 0$ for certain. Since I's play is not dependent on the outcome of the prize, the difficulty of the prize has no bearing. Thus if A is played, E plays R if $F < \frac{1}{72}(4 - 5\bar{s}_I)^2$. If P is played, E plays R if $F < \frac{1}{9}(2 - \bar{s}_I)$.

Note also that, given strategies, only for (C, R) is the profit dependent on $\hat{s}$. E's profit when I plays C is continuous, with a unique maximum at $\hat{s} = \frac{\bar{s}_I}{4}$, that results in profit of $\frac{2}{9} + \frac{\bar{s}_I^2}{72} - F$. Note that the maximum profit available is *increasing* in the existing rival technology. This is because the cases where E's research is poor are improved for E, even though these cases occur more often.

Profit is increasing below this value, and decreasing above. Although the form of the function changes, it is continuous. In fact, the linear part lies tangent to the curve at the point where the change occurs. Conversely, profit is fixed if N is played. Therefore, there exist a lower and an upper bound on $\hat{s}$ if the designer wants to induce E to play R when I is playing C. Denote these bounds respectively $\{\underline{s}, \bar{s}\}$, regardless of the specific part of the profit function they exist on, or if they exist on the 0 or 1 boundaries. The functions determining these values are given below, and Figure 3.6 provides an example.

$$\underline{s} = \begin{cases} \frac{1}{4}(\bar{s}_I - \sqrt{\bar{s}_I^2 - 8\bar{s}_I - 72F + 16}) & \text{if } F \geq \frac{1}{9}(2 - \bar{s}_I) \\ 0 & \text{if } F < \frac{1}{9}(2 - \bar{s}_I) \end{cases}$$

$$\bar{s} = \begin{cases} \frac{1}{4}(\bar{s}_I + \sqrt{\bar{s}_I^2 - 8\bar{s}_I - 72F + 16}) & \text{if } F \geq -\frac{5\bar{s}_I^2}{24} - \frac{\bar{s}_I}{9} + \frac{2}{9} \\ \frac{25\bar{s}_I^2 - 8\bar{s} + 16 - 72F}{32\bar{s}_I} & \text{if } F < -\frac{5\bar{s}_I^2}{24} - \frac{\bar{s}_I}{9} + \frac{2}{9} \end{cases}$$

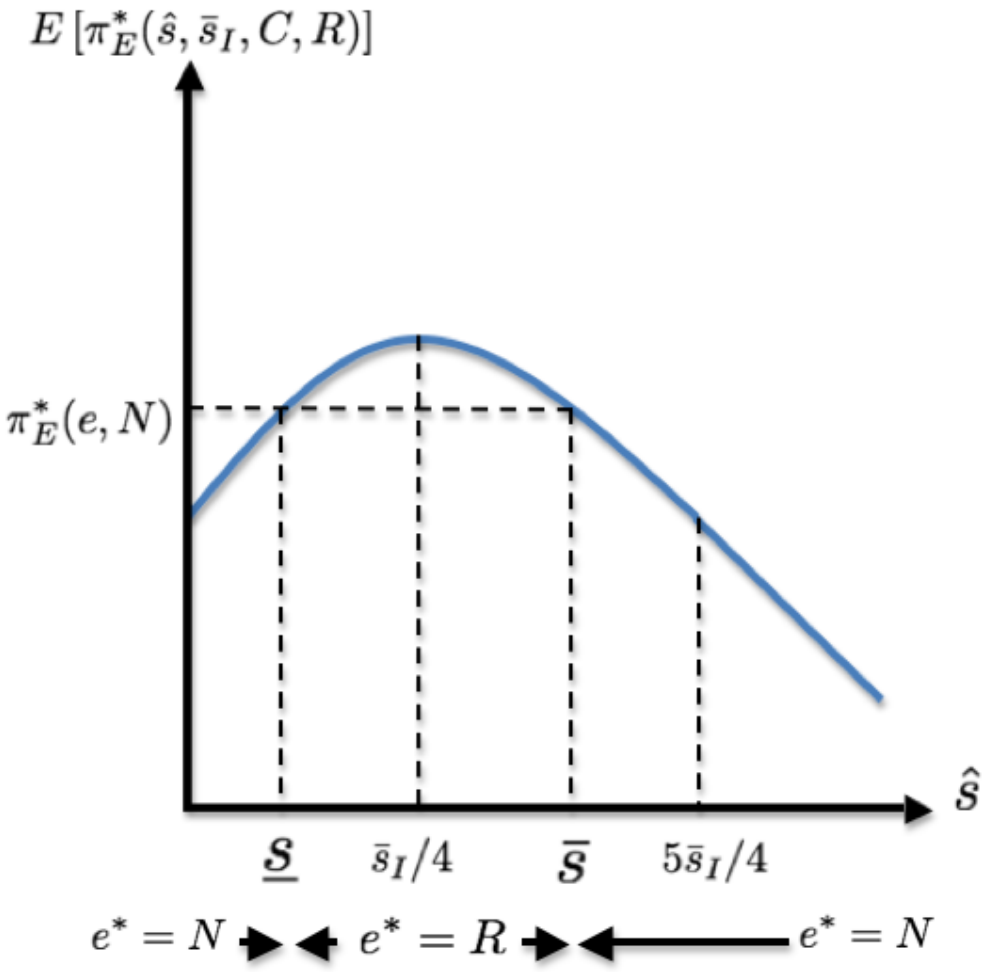


Figure 3.6: E's research decision, N versus R, when I plays C

Note that $\underline{s}$ is greater than zero only in the case where there is no research if P. C with a zero prize and P are equivalent: in each case, E ends up playing his maximum quality and I plays zero. If this is not the case, E gains profit by having the prize made marginally harder; so that in the event that he produces a very poor innovation, he can avoid crowding out I's better innovation and thereby reducing differentiation.

Figure 3.7 extends Figure 3.5 to include the research decision for $F = 0.05$. Note that, in this example, $\underline{s} = 0$, and that the research response to A depends on $\bar{s}_I$, whereas the response to P does not. Neither of these is generally true, but is a feature of the particular choice of fixed cost level.

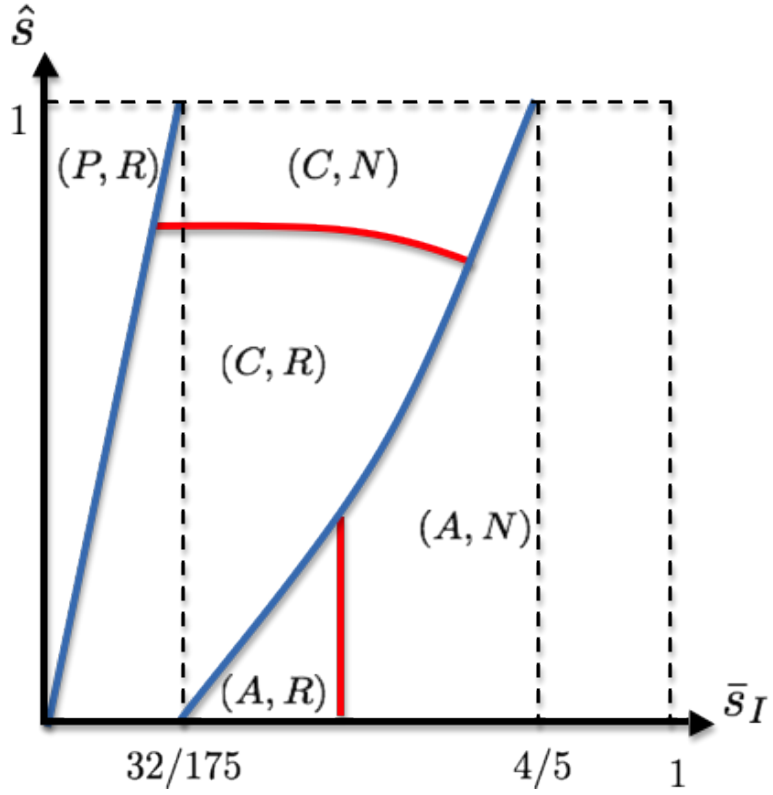


Figure 3.7: E's research decision (red), given I's quality strategy (blue); for $F = 0.05$

3.6 Prize difficulty

We are now in a position to analyse the first strategic choice of the game: the criterion for awarding a prize. We are interested in expected welfare from D's perspective at

the outset of the game. First, the welfare properties of $\hat{s}$ conditional on (the relevant part of) the strategies of I and E are analysed. Second, the feasibility of inducing these strategies will be examined.

Before the analysis, it is worth providing an interpretation of the situation in which the designer cannot improve the outcome with any prize, compared to if no prize were possible. For example, both $\hat{s} = 0$ and $\hat{s} = 1$ convey no information and result in identical equilbria. Such prizes do not improve on the no-prize situation. There will be occasions in which intervals of $\hat{s}$ may convey information in some strict sense, but that information is strategically irrelevant; that is, for no draw of $\bar{s}_E$ is the equilibrium outcome altered from the no-information case. This also requires that the research decision is unchanged. Such prizes are ineffective, and have the same welfare as $\hat{s} = 1$. In such cases we will say that the designer offers *no prize*, as an intuitive shorthand for these types of optima.

3.6.1 Welfare, given strategies

If N is played, welfare is independent of I's quality strategy. If R is played, welfare depends on i.

$$W(\bar{s}_I, \hat{s}, i, N) = v - c + \frac{4}{9}\bar{s}_I \qquad \forall i \in \{C, H, L\}, \tag{3.2}$$

$$W(\bar{s}_I, \hat{s}, A, R) = v - c + \frac{1}{36}(5\bar{s}_I^2 + 2\bar{s}_I + 8) - F, \tag{3.3}$$

$$W(\bar{s}_I, \hat{s}, P, R) = v - c + \frac{2}{9} - F, \tag{3.4}$$

$$W(\bar{s}_I, \hat{s}, C, R) = \begin{cases} v - c + \frac{2}{9}(2\bar{s}_I\hat{s} - \hat{s}^2 + 1) - F & \text{if } \hat{s} \leq \frac{5}{4}\bar{s}_I \\ v - c + \frac{1}{36}(5\bar{s}_I^2 + 2\bar{s}_I\hat{s} + 8) - F & \text{if } \hat{s} > \frac{5}{4}\bar{s}_I \end{cases}. \tag{3.5}$$

Note that, as for E's profit, welfare only depends on $\hat{s}$ if (C, R) is played. From now on, discussion of welfare levels takes place relative to the common level of $v - c$. A sketch of this relationship is shown in Figure 3.8.

The intuition behind this shape is straightforward. Recall that either a single firm offers their maximum quality and the other a zero quality, or both produce their maximum quality. Welfare is determined by the ranges of $\bar{s}_E$ for which each of these cases occurs, and the identity of the zero-quality firm.

For $\hat{s} < \frac{5}{4}\bar{s}_I$, the situation in which both offer their maximum quality is excluded, for the reasons explained above. Given this, welfare is maximised by having the firm with the greater maximum quality provide it, and the other provide zero quality. Since we assume that C is being played, and I is the zero-quality provider if and only if E wins the prize, welfare is maximised in this region by allowing E to win only if he is in fact

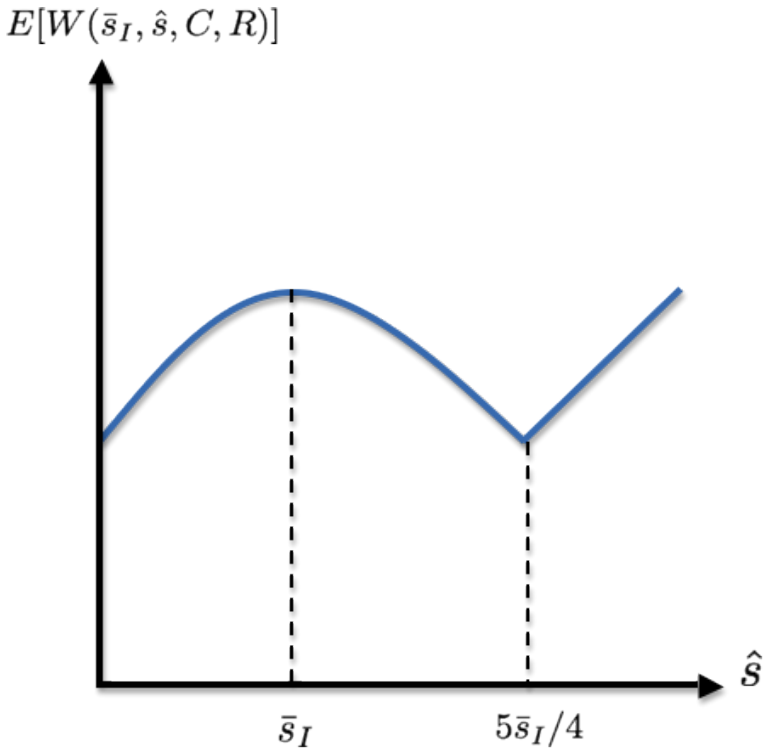


Figure 3.8: Expected welfare when (C, R) is played

of a higher quality than I.

Below this point, for $\hat{s} < \bar{s}_I$ there is a possibility that E will get to provide the non-zero quality, even if he is not the highest-quality. Above this point, $\bar{s}_I < \hat{s} < \frac{5}{4}\bar{s}_I$ there is a possibility that I will provide the only non-zero quality, even though E's innovation is better. As $\hat{s}$ increases above the local maximum, we are replacing equilibrium outcomes $\{0, \bar{s}_E\}$ with $\{\bar{s}_I, 0\}$, with $\bar{s}_E > \bar{s}_I$, which is clearly welfare reducing. Welfare is therefore falling in $\hat{s}$, until $\hat{s} = \frac{5}{4}\bar{s}_I$. At this point, the additional s_E levels which result in U and therefore $s_I^* = \bar{s}_I$, no longer have E play a zero quality. Rather, they result in the best possible outcome: two non-zero qualities in equilibrium, where there are higher qualities

and lower prices. This arrests the fall in welfare and begins to reverse it. Beyond this point, increases in the difficulty of the prize replace equilibrium quality outcomes $\{0, \bar{s}_E\}$ with $\{\bar{s}_I, \bar{s}_E\}$, which is evidently a welfare improvement.

3.6.2 Welfare with high incumbent quality; $\bar{s}_I > \frac{4}{5}$

For such $\bar{s}_I$, I always plays quality strategy A (see Figure 3.5), regardless of the prize difficulty. Therefore, no prize can affect the equilibrium research decision, or the equilibrium qualities. All prizes are equally ineffective, and the designer offers no prize

3.6.3 Welfare with medium incumbent quality; $\bar{s}_I \in [\frac{32}{175}, \frac{4}{5}]$

The analysis in this section and the next determines the optimal $\hat{s}$, via a series of results (all written under the assumption that $\bar{s}_I \geq \frac{32}{175}$). First, a region where $\hat{s} = 0$ is optimal is derived. Next, it is shown that in the remaining region, (C, R) is a feasible outcome. Then, the possible optima within and without (C, R) are characterised and compared. Finally, it is shown that (C, R) is only optimal within the region where it is feasible. For consistency, propositions are used when the result directly establishes the optimality of some prize difficulty, whereas lemmas are used for intermediate results, regardless of the simplicity of the analysis.

In this part of the parameter space, I plays either A or C, depending on the value of $\hat{s}$ relative to $s^C(\bar{s}_I)$. If A is played, then the research decision is independent of $\hat{s}$. This is the case when there is no prize, and all prizes that result in A are therefore strategically

and welfare equivalent to the no prize situation

If C is played, then $e = R$ only if $\hat{s} \in [\underline{s}, \bar{s}]$. The feasible set of strategy profiles that can be achieved by choice of $\hat{s}$ therefore depends only on the relative sizes of these three strategic threshold values, $s^C(\bar{s}_I), \underline{s}, \bar{s}$.

Proposition 3.1. *When* $F > \frac{\bar{s}_I^2}{72} - \frac{\bar{s}_I}{9} + \frac{2}{9}$*, no prize is optimal.*

Proof. Note that the maximum profit that can be given to E after R is $\frac{1}{9}(\frac{\bar{s}_I^2}{8} + 2) - F$, by the choice of $\hat{s} = \frac{\bar{s}_I}{4}$. Therefore, if this profit is less than the N profit, there can be no research, regardless of the prize. Therefore, research is feasible if and only if $F \leq \frac{\bar{s}_I^2}{72} - \frac{\bar{s}_I}{9} + \frac{2}{9}$ Otherwise, all prizes result in the no-prize outcome. □

Lemma 3.1. $F \leq \frac{\bar{s}_I^2}{72} - \frac{\bar{s}_I}{9} + \frac{2}{9}$ *is sufficient to permit the designer to select* (C, R) *by an appropriate choice of* $\hat{s}$ *in the interval* $[s^C, \bar{s}]$.

Proof. First, $s^C \geq 0$, whereas $\underline{s} = 0$ if $F < \frac{1}{9}(2 - \bar{s}_I)$. Therefore, for sufficiently low existing technology compared to the level of research costs, the lower bound of those $\hat{s}$ that cause (C, R) is not $\underline{s}$, E's research boundary, but rather s^C, I's A, C strategic boundary.

Second, note that E's (C, R) profit is not less than $\frac{1}{9}(\bar{s}_I\hat{s} - 2\hat{s}^2 + 2) - F$. Looking at the two line segments that make up the profit function, note that this quadratic element is always below the linear, for any given $\bar{s}_I$. Therefore, $\bar{s}$, the highest $\hat{s}$ that induces R when C is not less than $\frac{1}{4}(\bar{s}_I + \sqrt{\bar{s}_I^2 - 8\bar{s}_I - 72F + 16})$.

Therefore, comparing this to s^C, we can see that $F \leq \frac{\bar{s}_I^2}{72} - \frac{\bar{s}_I}{9} + \frac{2}{9}$ is sufficient to ensure

$\hat{s}$	$[0, s^C(\bar{s}_I))$	$[s^C(\bar{s}_I), \bar{s})$	$[\bar{s}, 1]$
$i^*(R)$	A	C	C
$e^*(i)$	N/R	R	N

Table 3.1: Optimal strategies for medium I quality, when $F \leq \frac{\bar{s}_I^2}{72} - \frac{\bar{s}_I}{9} + \frac{2}{9}$, by prize difficulty. N/R indicates that the optimal choice depends on other parameters.

that $s^C < \bar{s}_I$. This is a stricter condition than the first, and the result follows. □

The optimal strategies of each firm when $F \leq \frac{\bar{s}_I^2}{72} - \frac{\bar{s}_I}{9} + \frac{2}{9}$ are summarised in Table 3.1. Note that the research decision in the A case is determined independently of the prize difficulty, and is the same as if there were no prize. Recall that I's strategy i is the response if R were to occur.

Lemma 3.2. *There are four possible welfare-maximising $\hat{s}$:* $\{0, s^C, \bar{s}_I, \bar{s}\}$.

Proof. $s^C, \bar{s}_I, \bar{s}$ are all possible maxima by inspection of the welfare function given (C, R), given the interval of prizes that result in this strategy profile, established in Lemma 3.1.

$\hat{s} = 0$ is weakly the best of the options outside the range that induce (C, R). To see this, first note that all prizes harder than the interval of $\hat{s}$ that induce (C, R), that is $\hat{s} > \bar{s}$ are welfare-equivalent and all result in N, no research. All prizes easier than the interval of $\hat{s}$ that induce (C, R) are independent (from a welfare perspective) of $\hat{s}$ and result in either (A, R) or (A, N) depending on $(\bar{s}_I, F)$. Comparing the relevant welfare functions, (3.2) and (3.3), (A, R) is welfare-preferred to N when $F \leq \frac{1}{36}(5\bar{s}_I^2 - 14\bar{s}_I + 8)$. But R is the optimal response to A only when $F < \frac{1}{72}(25\bar{s}_I^2 - 40\bar{s}_I + 16)$, a stricter condition.

So, if (A, R) is feasible $\hat{s} = 0$ is preferred to all non-(C, R) equilibria. If (A, R) is not feasible, all non-(C, R) equilibria are equally preferred from a welfare perspective, including $\hat{s} = 0$. □

Proposition 3.2. *$\hat{s} = \bar{s}_I$ is never welfare-optimal .*

Proof. First, there is another $\hat{s}$ on the $W(C, R)$ line that gives the same welfare as the local maximum $\bar{s}_I$, call it $\hat{s}^\dagger$, defined as: $\frac{1}{36}(5\bar{s}_I + 2\bar{s}_I\hat{s}^\dagger + 8) = \frac{2}{9}(\bar{s}_I^2 + 1)$ This gives that $\hat{s}^\dagger = \frac{3}{2}\bar{s}_I$.

Therefore, since this point is necessarily on an upward-sloping part of the welfare function for (C, R), the condition $\bar{s} > \frac{3}{2}\bar{s}_I$ is sufficient to show that $\hat{s} = \bar{s}_I$ is not optimal in that region.

We have already seen that $\bar{s}$ is not less than $\frac{1}{4}(\bar{s}_I + \sqrt{\bar{s}_I^2 - 8\bar{s}_I - 72F + 16})$. Therefore, the stricter condition $\frac{1}{4}(\bar{s}_I + \sqrt{\bar{s}_I^2 - 8\bar{s}_I - 72F + 16}) > \frac{3}{2}\bar{s}_I \iff F < \frac{1}{9}(2 - 3\bar{s}_I^2 - \bar{s}_I)$ is also sufficient for this.

Second, for $\hat{s} = \bar{s}_I$ to be optimal, it has to improve on the no prize case: $\hat{s} = 0$. This case can lead to research or no research, but if R is played (i.e. it is privately optimal) then it is also socially preferred to N, as shown in Lemma 3.2. Thus, for $\bar{s}_I$ to be optimal, it necessarily must give a better welfare than the no research. And conversely, if the N case produces greater welfare, then $\bar{s}_I$ is necessarily non-optimal: $\frac{2}{9}(\bar{s}_I^2 + 1) - F < \frac{4}{9}\bar{s} \iff F > \frac{2}{9}(\bar{s}_I^2 - 2\bar{s}_I + 1)$.

Therefore we have two regions of the parameter space in which $\bar{s}_I$ is non-optimal as a

prize level. Combining the two, we can see that a necessary condition for $\hat{s} = \bar{s}_I$ to be the optimal prize level is that $\bar{s}_I > \frac{3}{5}$.

However, for $\bar{s}_I$ to be feasible as a means of inducing (C, R) in the first place, we require that $\bar{s}_I \geq s^C \iff \bar{s}_I \leq \frac{32}{63}$. And therefore, $\hat{s} = \bar{s}_I$ is never optimal when feasible. □

To summarise the position so far: if the Lemma 3.1 condition holds, D optimally chooses $\hat{s}$ from one of $\{s^C, \bar{s}, 0\}$ (by Lemma 3.2 and Proposition 3.2), where the first two result in (C, R) and the other one results in the no-prize outcome, either (A, N) or (A, R).

Here again it assists with the interpretation of the model to introduce some further descriptive terminology. We already know that $\hat{s} = 0$ corresponds to a no-prize situation, and stands in for all such ineffective prizes, or the absence of prizes, in the notation. The two other possible optimal prize difficulties can be profitably described by the mechanics of how they result in high welfare. $\hat{s} = s^C$ is the closest that the designer can get to the local welfare maximum $\bar{s}_I$. That is, it the (C,R) prize that most often selects the highest quality firm to produce the non-zero quality. Therefore, it will be referred to as the *selection* prize. In contrast, $\hat{s} = \bar{s}$, is an attempt to induce research without providing market power to the entrant in an excessive number of cases, and only rewards the entrant with a win (and thereby, with a zero-quality rival) in the smallest possible number of cases; where $\bar{s}_E$ is particularly high. This type of prize can therefore be termed an *excellence* prize.

The final stage of the analysis compares these possible outcomes to find the regions in the parameter space, if any, where these choices are optimal.

Proposition 3.3. *The "selective" prize s^C is preferred to the "excellence" prize $\bar{s}$ in the union of the regions $F > -\frac{8}{3}\sqrt{81\bar{s}_I + 32}\bar{s}_I^{\frac{3}{2}} + \frac{1777}{72}\bar{s}_I^2 + \frac{31}{9}\bar{s}_I + \frac{2}{9}$ and $F > -\frac{5}{24}\bar{s}_I^2 - \frac{\bar{s}_I}{9} + \frac{2}{9}$.*

Proof. Recall that these prizes are the two remaining possible optima that result in (C, R), and are the two extreme values of $\hat{s}$ that do so. Note that s^C is never greater than $\frac{5}{4}\bar{s}_I$ in the region of interest (i.e. $\bar{s}_I < \frac{4}{5}$, where prizes can alter the outcome). Therefore, the interval of prizes that result in (C, R) is never completely on the upward-sloping part of the welfare function. Note also, from the definition of $\bar{s}$, that it is on the downward-sloping part of the welfare function if $F > -\frac{5}{24}\bar{s}_I^2 - \frac{\bar{s}_I}{9} + \frac{2}{9}$. In such cases, s^C is necessarily preferred.

When this condition does not hold, the remaining case occurs, where the interval of $\hat{s}$ that result in (C, R) includes $\frac{5}{4}\bar{s}_I$. Combining the welfare function with s^C and the appropriate form of $\bar{s}$, we can see that s^C is preferred to $\bar{s}$ if and only if $F > -\frac{8}{3}\sqrt{81\bar{s}_I + 32}\bar{s}_I^{\frac{3}{2}} + \frac{1777}{72}\bar{s}_I^2 + \frac{31}{9}\bar{s}_I + \frac{2}{9}$. The union of these two regions gives the result. □

Proposition 3.4. *If (A, R) is feasible, it is also optimal.*

Proof. Using the welfare functions 3.3 and 3.5 (A, R) gives greater welfare than $\bar{s}$ if and only if $F < \frac{25}{72}\bar{s}_I^2 - \frac{5}{9}\bar{s}_I + \frac{2}{9}$. Since this region is contained within the region in which $\bar{s}$ is preferred to s^C, by Proposition 3.3, it follows that this is sufficient for the optimality of (A, R). □

Note that this is exactly the condition established in the no-prize benchmark for research

to occur in the absence of a prize. So if research occurs when $\bar{s} = 0$, then the designer should optimally offer no prize.

Although it appears natural that the existence of research with a prize being offered eliminates the need for a prize, this proposition is more substantial than it might seem. Incentivising research is not the only effect of a prize: it also alters the quality selections. In particular, we might imagine that the welfare gains from better selection, that is, allowing the higher-quality firm to more often produce the high-quality good, might outweigh the associated welfare losses arising from greater strategic accommodation by the firms. However, this is not the case under this model's specification. Therefore, incentivising research is a necessary element of a prize being useful.

It is also important for an understanding of the model to note how the optimal prize changes at this boundary. At the boundary, $\bar{s} = 1$. This is because it is the boundary at which a no prize situation gives just sufficient profit to research to make it worthwhile, even though A is being played. Therefore C with a difficulty of $\hat{s} = 1$ is equivalent.

Increasing F or $\bar{s}_I$ means that research has to made more appealing; there has to be a region in which $\bar{s}_I^* = 0$. So therefore the critical upper difficulty, given a play of C, $\bar{s}$ has to fall.

Proposition 3.5. *The "selective" prize s^C is preferred to no research if $F < \frac{1}{3}\sqrt{81\bar{s}_I + 32}\bar{s}_I^{\frac{3}{2}} - \frac{209}{72}\bar{s}_I^2 - \frac{8}{9}\bar{s}_I + \frac{2}{9}$. The "excellence" prize $\bar{s}$ is preferred to no research if $F < \frac{35}{216}\bar{s}_I^2 - \frac{11}{27}\bar{s}_I + \frac{2}{9}$.*

Proof. These follow immediately from the welfare functions. □

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The results to this point are sufficient to determine the optimum prize difficulty for all F and all $\bar{s}_I > \frac{32}{175}$. The relevant comparisons are summarised in Table 3.2, and their combination illustrated in Figure 3.9

$W(s^C(\bar{s}_I))<W(\bar{s})$	$F<-\frac{8}{3}\sqrt{81\bar{s}_I+32}\bar{s}_I^{\frac{3}{2}}+\frac{1777}{72}\bar{s}_I^2+\frac{31}{9}\bar{s}_I+\frac{2}{9}$ and $F<-\frac{5}{24}\bar{s}_I^2-\frac{\bar{s}_I}{9}+\frac{2}{9}$
$W(N)<W(s^C(\bar{s}_I))$	$F<\frac{1}{3}\sqrt{81\bar{s}_I+32}\bar{s}_I^{\frac{3}{2}}-\frac{209}{72}\bar{s}_I^2-\frac{8}{9}\bar{s}_I+\frac{2}{9}$
$W(N)<W(\bar{s})$	$F<\frac{35}{216}\bar{s}_I^2-\frac{11}{27}\bar{s}_I+\frac{2}{9}$
$W(\bar{s})<W(0,R)$	$F<\frac{25}{72}\bar{s}_I^2-\frac{5}{9}\bar{s}_I+\frac{2}{9}$

Table 3.2: Summary of the comparisons that determine the optimal prize level

Notice that there is a single point at which the three possible optima $s^C, \bar{s}, 0$ are all equally preferred. This is because $\hat{s}=0$ results in N, an outcome where the welfare is independent of F. The point where all three boundaries intersect is simply that unique value of F where the common welfare produced by s^C and $\bar{s}$ when they are equally preferred is equal to $\frac{4}{9}\bar{s}_I$.

3.6.4 Welfare with low incumbent quality $\bar{s}_I < \frac{32}{175}$

The analysis here is similar, in that there are three critical values that determine the strategic outcome of a prize $\underline{s}, \bar{s}, \frac{175}{32}\bar{s}_I$, but different in that the critical value for I's strategic decision has C below, rather than above. Therefore the latter two prizes are both types of excellence prizes if optimal.

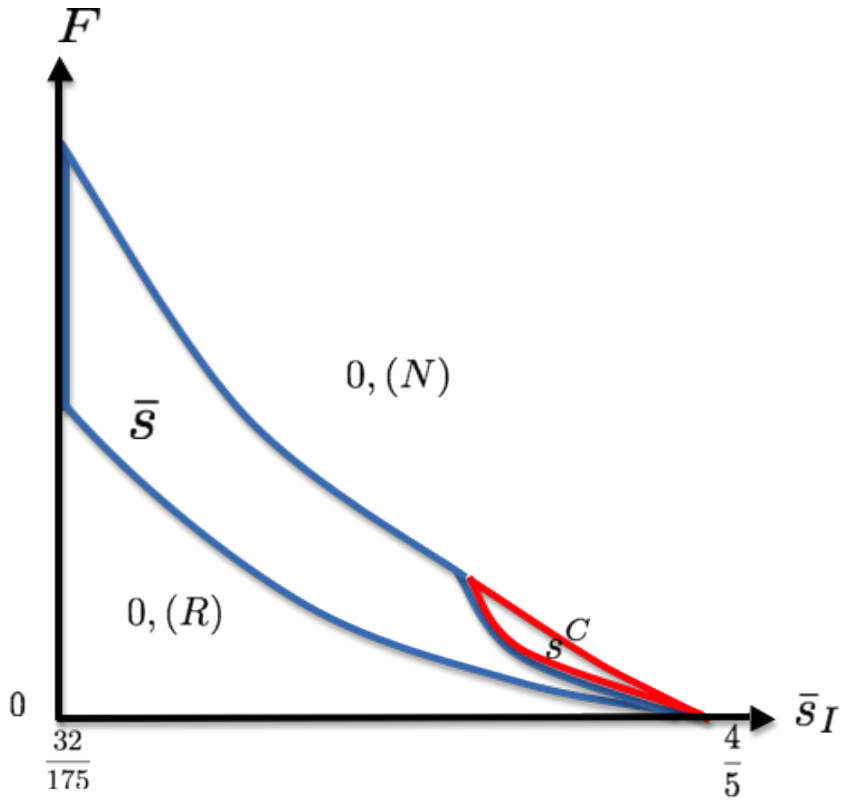


Figure 3.9: Optimal prize types with medium incumbent technology: the excellence prize (blue region) and the selection prize (red region). Regions in which no prize is optimal are denoted 0 and labelled by R if research occurs or N if not.

Lemma 3.3. *For low I tech, the (C,R) interval of prizes is bounded below by $\underline{s}$, and above by $\frac{175}{32}\bar{s}_I$ if $F \leq -\frac{25}{12}\bar{s}_I^2 - \frac{1}{9}\bar{s}_I + \frac{2}{9}$, or by $\bar{s}$ otherwise.*

Proof. Note that $\underline{s}$, the lower profit bound on the (C,R) entrant research decision is lower than the other two critical values; $\bar{s}$ by definition, and $\frac{175}{32}\bar{s}_I$ because this is greater than $\frac{\bar{s}_I}{4}$, the highest possible value of $\underline{s}$.

The upper bound on (C,R) is the lower of $\bar{s}$, $\frac{175}{32}\bar{s}_I$, since the prize difficulty needs to be easier than both these boundaries to induce (C,R), respectively. Note that $\bar{s}$ can only be greater than $\frac{175}{32}\bar{s}_I$ if $\bar{s} \geq \frac{5}{4}\bar{s}_I$. Therefore, we know the form of $\bar{s}$ in such a case. So $\bar{s} > \frac{175}{32}\bar{s}_I$ if and only if $F \leq -\frac{25}{12}\bar{s}_I - \frac{1}{9}\bar{s}_I + \frac{2}{9}$ □

Note that if $\frac{175}{32}\bar{s}_I$ is the upper bound on (C,R), then necessarily $F < \frac{1}{9}(2 - \bar{s}_I)$ and therefore both $\underline{s} = 0$ and $e^*(P) = R$.

The implication of Lemma 3.3 for optimal firm strategies can be summarised in Tables 3.3 and 3.4:

$\hat{s}$	$[0, \frac{175}{32}\bar{s}_I)$	$[\frac{175}{32}\bar{s}_I, 1]$
$i^*(R)$	C	P
$e^*(i)$	R	R

Table 3.3: Optimal strategies for medium I quality, when $F < -\frac{25}{12}\bar{s}_I^2 - \frac{1}{9}\bar{s}_I + \frac{2}{9}$, by prize difficulty

Note that in Table 3.4, the research condition binds, and the designer has the option of removing the incentive to invest in an additional region. A final element of prize

$\hat{s}$	$[0, \underline{s})$	$[\underline{s}, \bar{s}]$	$(\bar{s}, \frac{175}{32}\bar{s}_I]$	$(\frac{175}{32}\bar{s}_I, 1]$
$i^*(R)$	C	C	C	P
$e^*(i)$	N	R	N	R/N

Table 3.4: Optimal strategies for medium I quality, when $F \geq -\frac{25}{12}\bar{s}_I^2 - \frac{1}{9}\bar{s}_I + \frac{2}{9}$, by prize difficulty

terminology is now introduced. Call any of the welfare-equivalent $\hat{s} \in (\bar{s}, \frac{175}{32}\bar{s}_I)$ a *blocking* prize, if this interval exists, and represent any such by s^B. This is because such a prize blocks research from occurring, even if it would have done in the absence of a prize.

Now that the feasible outcomes in each have been established, candidate welfare maxima can be identified and compared.

Proposition 3.6. *If N is not a feasible outcome, then $\frac{175}{32}\bar{s}_I$ is the optimal prize difficulty.*

Proof. The condition that N is not a feasible outcome is equivalent to saying that $\frac{175}{32}\bar{s}_I$ is the upper bound on prizes that induce (C, R). That is, the strategic situation in Table 3.3 prevails. In such a case, there are three possible welfare optima. In the (C, R) interval, the two possible maxima are $\bar{s}_I$ and $\frac{175}{32}\bar{s}_I$. Outside this, the (P, R) outcome is independent of $\hat{s}$, but is equivalent to the no-prize outcome, represented by $\hat{s} = 0$.

$\hat{s} = \frac{175}{32}\bar{s}_I$ gives expected welfare of $\frac{2}{9} + \frac{85}{192}\bar{s}_I^2 - F$. By inspection, this is preferred to the (P, R) outcome $\frac{2}{9} - F$ and to the other possible (C, R) maximum, $\bar{s}_I$, which gives $\frac{2}{9}(1 + \bar{s}_I^2) - F$. □

Lemma 3.4. *$\hat{s} = \bar{s}_I$ is never optimal when N is feasible.*

Proof. If $\bar{s}_I$ is a feasible prize generating (C, R), then so is $\bar{s}$. The welfare generated by $\bar{s}_I$ is $\frac{2}{9}(1+\bar{s}_I^2)-F$. By the same reasoning as before, the welfare generated by $\bar{s}$ is greater than or equal to $\frac{1}{4} - \frac{\bar{s}_I}{72} + \frac{35}{192}\bar{s}_I^2 - \frac{9}{8}F$. Therefore a region in which $\bar{s}_I$ is non-optimal is $F < \frac{1}{72}(-23\bar{s}_I^2 - 8\bar{s}_I + 16)$. Similarly, the welfare of N is $\frac{4}{9}\bar{s}_I$, and therefore a further region of non-optimality is $F > \frac{2}{9}(1 - \bar{s}_I)^2$. These two regions cover the low technology parameter space. □

Proposition 3.7. *If N is feasible, it is optimal if $F > \frac{1}{216}(35\bar{s}_I^2 - 88\bar{s}_I + 48)$; either via a blocking prize, or when it is the no prize outcome. Otherwise, the excellence prize $\bar{s}$ is optimal.*

Proof. N is a feasible outcome whenever s^B exists, as illustrated in Table 3.4. It is the no prize outcome if $F > \frac{1}{9}(2 - \bar{s}_I)$, (in which case, a too-low prize also prevents R), otherwise a blocking prize is necessary.

$\bar{s}$ is preferred to N if $F > \frac{1}{216}(35\bar{s}_I^2 - 88\bar{s}_I + 48)$, which intersects the region in which N is feasible only through use of a blocking prize. In this region, $\bar{s}$ does indeed result in (C, R). □

These two propositions together give the optimal prize difficulty for the low quality incumbent case, illustrated in Figure 3.10, and therefore completes the solution of the designer's problem.

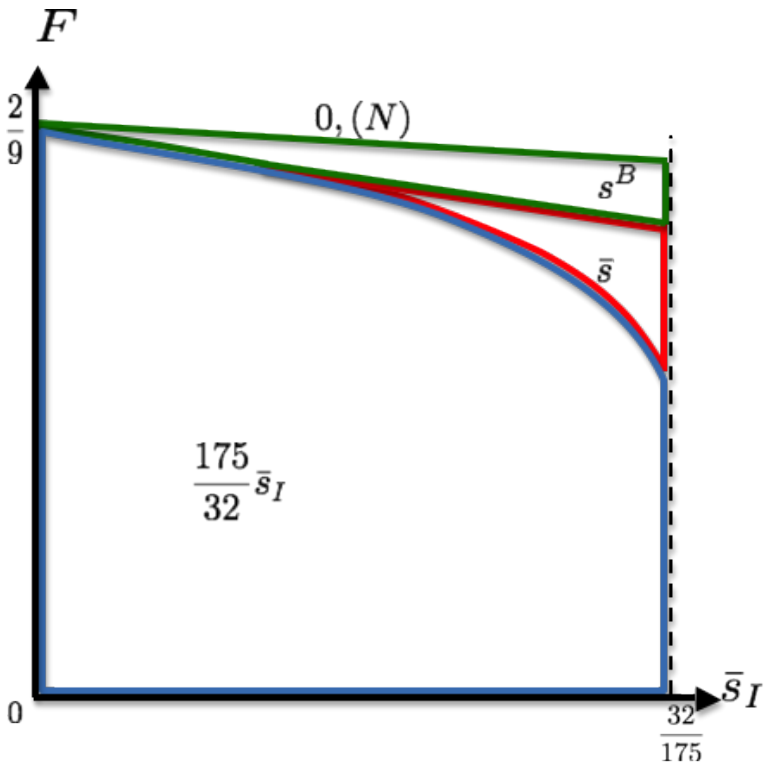


Figure 3.10: Optimal prize types with low incumbent technology: the excellence prize (blue region) and the selection prize (red region), and the blocking prize (green region). The region in which no prize is optimal is denoted 0 and labelled N since no research occurs

3.7 Further analysis and generalisations

Before proceeding to ask what, if any, interesting lessons can be drawn from this model, it is important to address some outstanding issues with the model specification. A number of reasonable possibilities for how agents might act in the real world were removed by assumption. For example, the designer was not permitted to run full-information verification prizes, where E's quality was revealed precisely, and the contestants were not allowed to communicate via cheap talk. This section relaxes these restrictions, and others, made for the sake of conciseness, to demonstrate the robustness of the use of binary prizes.

3.7.1 Comparison with full information prizes

The prize assumptions of this model are theoretically restrictive, if intuitively natural and commonly observed. For example, the CATS prize criterion was the launch of a 2kg payload 200km *or higher*. But team attempts would not necessarily have had their exact achievements publicly released, whether they won or lost. Another natural prize type to consider if that where the attributes of all attempts are made public. However, this is not appealing to a designer under the assumptions of this model: The model is robust to permitting D to offer a full information prize.

Allowing D to offer full information prizes does not change this analysis, since there is a binary information prize that is equivalent. Note that if E plays R and I receives perfect information about $\bar{s}_E$, play is:

$$(s_I^*, s_E^*) = \begin{cases} (0, \bar{s}_E) & \text{if } \bar{s}_E \geq \frac{5}{4}\bar{s}_I \\ (\bar{s}_I, 0) & \text{if } \bar{s}_E < \frac{5}{4}\bar{s}_I \end{cases}.$$

But such a situation can be mimicked in all respects by a binary prize with $\hat{s} = \frac{5}{4}\bar{s}_I$. Therefore, if we give D the option of offering a full information prize, any time $\hat{s} = \frac{5}{4}\bar{s}_I$ is part of the welfare-maximising set of prizes, the full information prize is also. If not, there is a binary prize design that is preferred to the full information prize. In fact, this prize level is a local welfare minimum when (C, R), and since $\frac{5}{4}\bar{s}_I \geq s^C$ it can therefore only be optimal when it results in N.

3.7.2 Comparison with cheap talk

This model rules out verifiable disclosure, by assuming that the nature of technology is such that verification requires the release of sufficient information to permit a rival to replicate the quality. In this way, contestants will not disclose their abilities, except to a third party, such as the contest designer.

It is almost impossible, however, to reasonably rule out the possibility of cheap talk communication from E to I (except by assuming that E's research decision is made before I's quality decision, but that the outcome of the research is not known until after). However, allowing for this possibility *instead* of the no information status quo, after the research decision and before I's quality decision, does not alter the outcome of the model: All cheap-talk equilibria are improved upon by the optimal prize.

Posit a cheap-talk equilibrium, a general partition of the $\bar{s}_E$ space, where E tells I which interval his innovation falls in. Assume truth-telling by E for the moment, and consider I's optimal responses to these messages. From the analysis of I's quality decision above, it is clear that only the report of the interval that covers $\frac{4}{5}\bar{s}_I$ leaves any strategic uncertainty. If $\bar{s}_E$ is certainly known to be below $\frac{4}{5}\bar{s}_I$, then $s_I^* = \bar{s}_I$. Conversely, if $\bar{s}_E$ is certainly known to be above $\frac{4}{5}\bar{s}_I$, then $s_I^* = 0$. So only the single report interval that does not generate either case, call it $[a, b]$, requires any analysis. As in the prize analysis above, the expected profit of I as a function of s_I^* is quadratic, with a minimum, so that this report also results in one of $s_I^* = \bar{s}_I$ or $s_I^* = 0$, depending on the end-points of the interval. Therefore, all of E's reports result in one of these two values.

When is E willing to tell the truth in such a situation? If the partition is such that I's response is independent of the report (e.g. a binary cheap talk equilibrium partitioned at a level that, if instituted as a prize, would cause A or P), then truth-telling is weakly optimal, but strategically uninformative. It is the equivalent of the no prize situation. On the other hand, suppose the various reports result in different I responses. E wants I to produce the higher quality if and only if $\bar{s}_E < \frac{\bar{s}_I}{4}$. Thus, any report that indicates that this is the case will be believed. What about reports that the innovation is of a higher quality? I will believe a report that indicates just this information, since he knows that if it was not true, it would not be optimal. Such a report of $\bar{s}_E > \frac{\bar{s}_I}{4}$ is sufficient to induce C if $\bar{s} < \frac{2}{9}$. In this case, this is equivalent to a prize of $\hat{s} = \frac{\bar{s}_I}{4}$. If this is not the case, then a binary cheap talk partition is strategically uninformative, and results in a response A.

But a finer partition above $\frac{\bar{s}_I}{4}$ that does induce a strategic response from I (that is, sometimes causes zero quality and other times not) is not possible, since in these cases E always wants to be the high quality producer. Therefore he will always deviate and report one of the intervals that results in I playing zero.

Therefore, a cheap talk equilibrium is, at its most informative, the equivalent of some binary signal prize. It is subject to the same incentive constraints as the prize, plus an additional truth-telling constraint. It should be obvious that removing the additional constraint that the signal is ex-post optimal for E allows for better welfare outcomes if the signal threshold is controlled by a welfare-maximiser.

3.7.3 Allowing E to opt out of prize; or allowing I to compete for the prize

It should be clear that if E does not enter his innovation for the prize, I will assume that the signal would have been U if he had.

For clarity and analytic convenience, this chapter modelled a prize as a binary signal on E's research, ruling out the option of I competing for the prize. This is perhaps not unreasonable, since it can often be observed that, for example, government agencies are precluded from competing for the prize (there is discussion of the reasonableness of such a government-as-incumbent interpretation in the conclusion). However, it should be clear that, whatever the particular rules relating to awarding the prize in the case when both exceed the difficulty, it is never optimal for D to institute a prize that can

be won with the existing technology. Nowhere in the analysis above was the optimal prize so easily won, and if I won the prize, he would learn nothing about E's quality: the no-prize situation again.

3.7.4 Adding a monetary prize

The original formulation of this model excluded any direct monetary reward from winning the prize. This was to focus on the potential benefits of the post-research market position, and to show that informative prizes were valuable, even if the prize money was negligible compared to the size of the market. However, it is straightforward to extend the model to include a monetary prize and make a few broad comments about the effect of this on the conclusion.

Suppose that D has prize funds X, such that he can commit to making an award $x \leq X$ to E, contingent on a research outcome of S; that is, that E wins the prize. Note that the award or non-award of the prize makes no direct difference to overall welfare, since it is just a transfer payment. Similarly, it makes no difference to the optimal quality or price decision for either firm, since its award is purely contingent on the random outcome of the research process. It does, however, influence the initial decision of whether to undertake research or not.

Effectively, the introduction of a prize purse x, at difficulty $\hat{s}$, causes E to act as if he had a fixed research cost of $F-(1-\hat{s})x$, since after paying F to undertake research, with probability $(1-\hat{s})$ he wins back x. The remainder of the profit function is unchanged.

Thus, the region $[\underline{s}, \bar{s}]$ is expanded in both directions.

Since $\underline{s}$ never binds in the solution, this simply has the effect of raising the relative welfare of the excellence prize (when it is $\bar{s}$), versus both the selection prize, the no prize case and the discouragement prize. The maximum prize $x = X$ should be offered, since it acts only to relax E's incentive constraint and raise the interval of prizes where research occurs. The only exception to this are the cases where N remains optimal compared to the enhanced excellence prize: in this case, a too-large x will remove the option of s^B for the designer.

3.7.5 Adding horizontal differentiation

In a model driven largely by vertical product differentiation, it is natural to ask what effect the introduction of horizontal differentiation might have. In the model as specified above, this addition reduces analytic tractability and intuition to a significant degree. The purpose of this subsection is to show that the ideas underlying the analysis of the original model are robust to some limited transportation costs, even if a full analytic solution to the generalised model cannot be presented.

The model remains identical, except now the unit mass of consumers is differentiated along two dimensions. One dimension is the degree of preference for quality, exactly as before. The other is a linear transport cost, based on location d. The two firms are exogenously located a unit distance apart, and without loss of generality, suppose the firm offering the lower quality is located at zero and named Firm 1. Thus, the new

consumer utility function is: $u = v + \theta s_1 - td - p_1$ if purchasing from Firm 1, and $u = v + \theta s_2 - t(1-d) - p_2$ if purchasing from Firm 2. Note that everywhere in this analysis the original model can be obtained as a special case with $t = 0$.

Therefore, the indifferent consumer θ_i is now given by:

$$\theta_i(d) = \frac{p_2 - p_1 + t - 2td}{s_2 - s_1}$$

This boundary that determines the relative market sizes that the two firm serve. However, the exact form of market size varies depending on who serves the corners. To avoid excess complexity arising from these possibilities, and to ensure a pure-strategy equilibrium, make the assumption that the transportation cost is always small relative to the extent of vertical differentiation. That is, the lower quality firm serves all consumers with $\theta = 0$, and the higher quality firm serves all consumers with $\theta = 1$, so the market boundary line intersects the $d = 0$ and $d = 1$ lines. This case nests the case of a zero transport cost, and essentially asks if the results of the model are robust to small degrees of horizontal product differentiation. The exact nature of the required smallness will be derived during the analysis.

Since $\theta(0) = \frac{p_2 - p_1 + t}{s_2 - s_1}$ and $\theta(1) = \frac{p_2 - p_1 - t}{s_2 - s_1}$, the respective profit functions are:

$$\pi_1 = (p_1 - c)\frac{p_2 - p_1}{s_2 - s_1}$$

$$, \pi_2 = (p_2 - c)\left[1 - \frac{p_2 - p_1}{s_2 - s_1}\right].$$

But these are independent of t, and are identical to those derived in the original game, and the equilibrium is identical. Here, increased transport costs redistribute the available consumers between the firms, but makes no difference to the size of the market served by each firm.

To show that this equilibrium does indeed exist, it should be confirmed that the prices derived do indeed cause each firm to serve the boundaries of quality preference as assumed above. Since $p_2 - p_1 = \frac{1}{3}(s_2 - s_1)$, the indifferent consumers are:

$$\theta_i(d) = \frac{s_2 - s_1 + 3t - 6td}{3(s_2 - s_1)}$$

For the original assumption about the location of this boundary to be true, it must be that $\theta_i(0) \in [0, 1]$ and $\theta_i(0) \in [0, 1]$, which are the conditions $3t \leq 2(s_2 - s_1)$ and $3t \leq (s_2 - s_1)$ respectively, with the latter being sufficient alone. Clearly, the existence of this equilibrium depends on the realised degree of product differentiation. However, in the sequential quality choice game, it was shown that the minimum degree of differentiation occurs when the leader produces $\bar{s}_I$ and the follower is just good enough to also have a non-zero quality: $\bar{s}_E = \frac{5}{4}\bar{s}_I$, which means that the lowest degree of product differentiation that occurs is $\frac{\bar{s}_I}{4}$, and therefore the condition that $12t \leq \bar{s}_I$ ensures that this equilibrium is available regardless of the realised quality of E's innovation.

However, although firm strategies are identical, the designer has an additional consideration. Welfare is given by:

$$W = \frac{1}{18}(8s_2 + s_1) - \frac{t}{2} + \frac{t^2}{6(s_2 - s_1)}.$$

There is an extra incentive to create strong price competition (have less vertically differentiated products). This is because vertical differentiation now carries an additional cost, where consumers have to pay an increased transport cost to purchase from the firm that is not nearest them.

The expected welfare functions under (C, R) then become:

$$\begin{cases} v - c + t^2 \left[\frac{\hat{s}}{6\bar{s}_I} + \frac{1}{3(1+\hat{s})} - \frac{\hat{s}}{3(1+\hat{s})} \right] - \frac{t}{2} + \frac{2}{9}(2\bar{s}_I\hat{s} - \hat{s}^2 + 1) - F & \text{if } \hat{s} \leq \frac{5}{4}\bar{s}_I \\ v - c + t^2 \left[\frac{2\bar{s}_I}{9\bar{s}_I - 12\hat{s}} + \frac{2}{3(\hat{s}+1)} + \frac{5}{24} \right] - \frac{t}{2} + \frac{1}{36}(5\bar{s}_I^2 + 2\bar{s}_I\hat{s} + 8) - F & \text{if } \hat{s} > \frac{5}{4}\bar{s}_I \end{cases} . \quad (3.6)$$

The first order conditions in $\hat{s}$ are cubic, and the solutions are not simple. However, a plot of this function for a range of t suffices to illustrate that the designer faces the same essential dilemma: there is a local maximum in the region $\hat{s} \in [0, \frac{5\bar{s}_I}{4}]$, and the function is increasing above this region. Of course, the exact trade-off is not identical. Compared with the $t = 0$ case, greater weight is places on minimising product differentiation, and therefore the local maximum is not perfect selection.

3.8 Conclusions, limitations and future directions

This section examines what, if any, insights can be achieved from the solution of this model. First, it looks at the solution and interprets it in the context of real prize design. Then there is a discussion about the validity and generality of the assumptions of the model. Finally, there is a brief discussion about what issues remain outstanding or unclear.

3.8.1 Implications for prize design

There are two main results of this model. The first is that welfare-improving research can be incentivised by the introduction of a prize that potentially improves product positioning for the entrant. The second is that the difficulty of such prizes is determined by a trade-off between the degree of selection accuracy and the desire to avoid giving excessive market power to the entrant. Of course, the specific areas of the model's parameter space in which the various types of prizes are optimal are a consequence of the particular functional forms and distributions assumed. However, the underlying mechanics governing the types of trade-off, and the strategic boundaries that determine the optimum, can be understood intuitively.

There were in general two types of prize that resulted in research. It was noted above that the prize that resolved all of the incumbent's strategic uncertainty in every case ($\hat{s} = \frac{5}{4}\bar{s}_I$) about the entrant was a local welfare minimum. The two types of research prize were movements in different directions away from this minimum. The excellence prizes were more difficult to win. If there was a low-technology state, it was optimal if it was the incumbent's strategic bound - more difficult and a loss would not be strategically meaningful. If it was a medium-technology state, it was the entrant's research condition - more difficult and the entrant did not expect to win often enough to make research worthwhile. Evidently, it is simply a case of selecting the most difficult prize that gives sufficient profit to the entrant, and is sufficiently informative to change the incumbent's behaviour.

In either case, the force causing welfare to increase in prize difficulty is via an improved chance that the entrant will in fact fail to win the progress prize, but not by too much. This illustrates that a failure to be won is not necessarily indicative of poor prize design, since research is uncertain in outcome, nor in fact of a poor outcome from the designer's perspective. This is obvious, but often ignored. There is a seeming bias in the literature to reference and study prizes that were won. Undoubtedly, they are in the nature of a better and more interesting story. However, it is clear from this model that, when an excellence prize is the optimal structure, welfare is not increasing strictly in the quality of the innovation achieved. A marginal fail, with the resultant strong competition, is preferred to a marginal success, which results in market supremacy for the entrant.

The CATS example is illustrative of this. It was prize that was never won, but incentivised a wave of research, with contestants continuing to compete today. If the prize had been won, it might have been to the detriment of the long-term competitiveness of the industry. The point of the excellence prize is that, only in cases where the innovation is a significant improvement on the status quo is the entrant rewarded with market dominance.

In the real world of prize design, excellence prizes are prizes offered for radical breakthroughs. They are probably not likely to be won. If they are won, it is likely that the winner will have some large quality advantage and enjoy substantial market power, having forced the incumbent to give way. They exist to provide an additional incentive to research via the credible verification of what might be an unexpected breakthrough.

The second type of research prize was the selection prize. This moved away from the local welfare minimum by making the prize easier. In this case, the lower boundary was not always reached, because of the existence of the intuitive selection local welfare maximum at $\hat{s} = \bar{s}_I$. The lower boundary, if it exists, is the incumbent's strategic boundary - any easier, and the winning the prize is essentially uninformative - in the case of medium incumbent technology. The lower boundary that is the lower bound on entrant's profit can never matter, because welfare is maximised at a difficulty above profit. The selection prize is more intuitive in how it maximises welfare. Simply, as often as possible, given the strategic constraints, the higher-quality firm produces the higher quality. There is no attempt to force firms into close competition with minimally differentiated products. Note that in the low-technology state, there were no selection prizes offered.

These types of prizes are more incremental in their goals. Welfare is only improved if they are won. The criteria for winning may be defined as any minor improvement on existing technology levels, as in the NASA glove challenge. (See also, The Netflix Prize, the M-Prize). They exist to allow higher-quality entrants to replace existing incumbents as the quality provider in the market.

There was also a third type of prize that arose in the solution of the game: the blocking prize. The results from the fact that in the low-technology, no information case, the entrant performs research that is privately optimal, but reduces welfare. This is because any research is sufficient to cause the incumbent to accommodate, even if the existing technology is not very low. This is a feature of the model that is not likely to be observed

in practice. It is hard to imagine a philanthropist establishing a difficult prize with the express intention of discouraging research. This feature arises because research is a binary decision. However, it perhaps is particularly illustrative of another point regarding the effective design of prizes. It is possible for these prizes to act as discouragements to research (even, in the model, when it is not optimal to do so). Therefore, designers should be wary of this potential pitfall: an attempt to establish, say, an excellence prize, with poor information, could result in the opposite of the intended consequence.

This leads directly to another point, about the information the designer needs in order to design an optimal prize in the context of this model. The existing quality in the market should not pose a problem, since the designer necessarily is able to assess the quality of these products. However, the cost of research is an important variable that could potentially be known only to the firms themselves. An understanding of the technological circumstance of the innovative process is necessary for a well-designed prize. (Although one of the great advantages of the use of prizes is that they are not paid in the event that they are made too difficult.)

More broadly, this chapter contextualises prize-winning research within a differentiated product market. The final market for the research is important. Much modelling of innovation considers additional research that results in more than one agent making a discovery to be wasteful. However, this assumes that the breakthrough becomes public property. In the case where the ability to produce is private or protected knowledge, research duplication, or the discovery of alternative means of producing the same outcome,

has potential benefits in terms of price competition.

3.8.2 Discussion of modelling choices

The assumptions of this model are quite specific regarding the market structure and the timing of player movement. The purpose of this section is to give some explanation and justification as to why these particular choices were made.

The first assumption of this nature regards the nature of incumbency. As noted in the introduction, incumbency here relates to two quite separate features. One is the existence of a producer of known quality. An alternative approach would have been to consider a number of potential entrants considering research, with no firm already in the market. A similar trade-off would likely have emerged; namely that better sorting of contestants' maximal quality in a contest raises the number of firms that perform research by raising expected profits via more informed product differentiation. On the other hand this would naturally result in the loss of price competition in the final market. However, the approach of the model in this chapter allows direct parameterisation of the existing technology level, relative to the support of research outcomes. It seems that this variable is likely to be a key factor in determining the interest of a prize-giver.

The second feature of incumbency is quality leadership. This assumption has the dual advantage of increasing tractability and generating a market inefficiency, in combination with the non-verifiable nature of the entrant's research. Such an assumption has not been uncommon in models of differentiation, for example Hay [1976] and Lane [1980].

The use of a vertically differentiated market environment is perhaps the most natural of the assumptions made. It seems to follow readily from the use of a prize, which are typically characterised by the use of some objective scale of measurement, or weighted sum of such, to determine contestant performance. In some contexts, where the prize is largely focused on research for its own sake, it might seem unrealistic to assume that the designer's definition of quality is identical to the consumers'. However, in the model presented here, the designer's concern is exactly with the distribution of quality in the final consumer market. Therefore, as near as possible, he will want to measure exactly that parameter which enters the consumers' utility functions. The assumption is then simply that perfect measurement of this is possible.

The possibility of the incumbent winning the prize was ruled out by assumption. The designer simply assesses the quality of the entrant's innovation. Real prizes do sometimes prevent competitors with substantial resources or existing expertise from competing. For example, one of the prizes offered in the Global Security Challenge, is open only to new start-ups. Often, governmental organisations are excluded from competing in IPCs (e.g. the Ansari X Prize), although the government-as-incumbent interpretation has difficulty explaining the assumed price competition of the final round, unless there is actually an ex ante state-owned monopoly provider of the product.

However, even if there is no reason to assume that the incumbent would be excluded in the rules, it has been shown that the optimal prize is never one that the incumbent can win. Clearly there is no information content to an incumbent winning a prize, so

extending the game to include this possibility would complicate without clarifying. This leads directly to another, related, assumption, which is that the incumbent is unable to conduct similar research. This is justified by considering that prizes are usually offer in circumstances where the existing market has failed to innovate, and external approaches or expertise need to be introduced. However, this assumption is not trivial and is important in determining the validity of the model's insights.

3.8.3 Future directions and alternative answers

What else determines the difficulty of prizes? Most obviously, where the realised quality of the innovation is a function of the amount spent on research, the difficulty of the prize might be designed to elicit different levels of research spending. This is probably the most natural explanation.

In other circumstances, even when a continuous scale of quality exists, the social payoff may be discontinuous at some point. For example, imagine a prize offered for a non-polluting variant of an existing technology. The goal of the designer is to completely displace the current technology by making its adoption individually rational for users. In order for this to happen, the new variant must be as cheap or cheaper than the existing technology level. Further decreases in cost (increases in quality) are second-order effects. The cost-level of the prevailing state of technology again offers itself as a natural benchmark.

Chapter 4

Prizes, subsidies and grants in Tullock contests

4.1 Introduction and motivation

Perhaps the most fundamental decision facing a principal looking to sponsor research into a particular technological problem is whether to establish a prize, or to use the same money to dispense a series of grants. In the former, contestants raise money in pursuit of a prize, whereas in the latter, they spend what they are allocated. A middle way between these two extremes is to offer a prize, and also provide matching funds to contestants, to increase their spending. This paper generalises the Tullock contest so that prizes (with and without matching funds) and grants arise endogenously. This generates insight into the factors that influence when to offer IPCs in the first place.

Many papers consider what the optimal contest success function (CSF) for a particular contest might be. Designers are modelled as varying the accuracy or the contest, or the treatment of contestants based on their characteristics. However, the alteration of the CSF is taken to be costless. In this model, the CSF is taken as given, and an alternative, costly, contest change is motivated and analysed. This is possible because of a different interpretation of the nature of costs, efforts and prizes. In particular, the cash nature of IPCs is used to justify some one-to-one relationships that are not necessarily reasonable in other contest contexts.

This chapter makes two contributions: it endogenises the asymmetric treatment of contestants by adding a cost; and it captures within a single model the competing attractions of prizes versus grants.

4.1.1 Outline of the chapter

The chapter begins by considering the contexts in which IPCs are usually offered, and contrasts them with the use of grants. The key question of contestant access to funding is looked at. Two strands of related literature are briefly reviewed: those that deal with tax and/or subsidy in contest design, and those that look at the optimal contest success function.

A simple generalisation of the Tullock contest is presented in answer to the questions raised. A designer chooses levels of matching funds for contestant spending in the contest. Matching funds made available in this way reduce the prize available in the contest. Two

firms with varying costs of raising funds then compete in the contest, with the designer aiming to maximise the amount spent in pursuit of the prize.

The version with a lottery CSF is solved first, and observations about the relative merits of prizes and grants are considered, as well as the use of subsidies. Many results are then generalised to the Tullock CSF with accuracy $r < 1$. Finally, there are a number of limitations to the model that need to be considered.

4.1.2 Features of prizes, grants, and hybrid designs

IPCs, although growing in popularity, certainly account for a minority of the research undertaken. Traditional funding has involved the provision of a grant to be spent on research, with no or only marginal incentives to spend additional funds at the researcher's own expense. Archetypal prizes reverse this formula - they require contestants to fund their own spending, but provide the potential of a monetary reward in order to incentivise this activity.

One significant problem with IPCs is that the very research they purport to incentivise, that is, highly innovative research drawn from outside mainstream or established channels, is exactly the type of venture that has difficultly raising funding. Although the problem of insufficient designer information, and the associated difficulty of "picking winners" is avoided by sponsoring a prize, this may simply relocate the problem of informational asymmetry if those able to perform the research do not have substantial endowments. There may exist informational asymmetry between the researcher and

potential investors. This in turn causes two problems in the prize design: first, high funding costs reduce the amount that can be spent by competitors; and second, asymmetric funding costs blunt the competitive process.

In the McKinsey & Company [2009] report into effective prize design, the authors recognised these and other considerations that affect the decision to use prizes over other philanthropic instruments: specifically, hybrid prize-grant schemes; fees or contracts; grants; or infrastructure investment. They consider three factors in choosing between these five options: the clarity of the goal, the number of agents, and the risk-aversion of those agents. Although there is not an exact parallel to the questions of this chapter, their approach nevertheless shows the importance of this decision and the relevance of these issues. Our focus is on contestant cost of funding.

When contestant spending is a key consideration, the obvious trade-off is between providing funds and providing incentives to raise funds. However, questions of symmetry are also important in contests. The following examples provide some insight into how real incentive schemes were designed, and where they fell on the prize-grant spectrum. This chapter focuses on three types: pure prizes; subsidised prizes; and grants.

4.1.3 Examples

The Ansari X prize; a pure prize

The Ansari X prize is probably the most celebrated contemporary IPC. Established in 1996, and itself inspired by the Orteig Prize, it offered a $10m prize for the first non-

government organisation to successfully complete two spaceflights in two weeks with a single manned spacecraft. Twenty six teams participated in the competition. The teams were extremely heterogeneous, with some participating teams essentially amateurs, while the winning team attracted early investment from Microsoft co-founder Paul Allen. The prize was won in October 2004, and the team continues, in partnership with Virgin Galactic, to move towards the commercial provision of spaceflight. The prize sponsors and designers have since endorsed their model by launching a number of further prizes in fields ranging from automotive engineering to genome sequencing.

In observing the design of this highly successful prize the key thing to note is that, although the contestant field was evidently extremely asymmetric, all contestants were treated equally. This seems entirely natural, perhaps more than anything else because it seems fair. But it is theoretically puzzling. If such contests are well modelled by Tullock-type tournaments, by giving some assistance or preference to the weaker (if not the weakest) contestants, the designer should be able to increase the effort of the best as well.

DARPA Grand Challenge 2007; a prize with funding

The Defense Advanced Research Projects Agency (DARPA) is one of the research bodies within the US Department of Defense. Their Grand Challenges are a series of technological contests aimed at developing innovative new military hardware. The first three, in 2004, 2005, and 2007, were contests for driverless ground vehicles. The time taken to

complete a course of obstacles is the assessment criterion. In 2004, no team completed the off-road course, and no prize was awarded. The next year, five teams completed the course. After that, the 2007 challenge was extended to include the navigation of urban environments. But there was another modification to the contest as well. Based on their proposal applications, the teams were split into two tracks: Track A and Track B. All these teams competed against each other on the same course for the same prize, but the eleven Track A teams were given $1million of advance funding for the development of their entry. The means by which the teams were sorted into tracks remains somewhat opaque. However, DARPA clearly felt that there were additional gains to be made in the outcome of the contest by applying funds directly to competitors. This decision is particularly revealing, because the prize schedule was only $2million for first place and $1million for second place. That is, with the same funds, they could have more than quadrupled the prize schedule. However, the designer preferred to raise the spending of a certain number of teams by $1million each.

Why would this be the case? One possible explanation is that the teams were simply unable to mobilise significant resources. Even if a quadrupling of prizes led to a quadrupling of contestant spending, if this was from a low base, it might not be particularly appealing. By splitting the funds between prizes and grants, the designer was able to achieve a balance between an incentive to compete and the means to do so effectively.

The Biomedical Catalyst; grants without prizes

Direct grants are so ubiquitous that an example is scarcely necessary. This example is presented because it is focused on creating the type of technological innovations that a prize might be expected to adequately incentivise. The Biomedical Catalyst Fund is distributed under the auspices of the Technology Strategy Board, the UK's state innovation agency, and the Medical Research Council, another UK quango. The £180million programme "offers funding to to innovative small and medium-sized enterprises (SMEs) and academics looking to work either individually or in collaboration to develop solutions to healthcare challenges" [The Technology Strategy Board and The Medical Research Council, 2012, p 2]. The solutions of these challenges could just as easily be made the subject of a prize, but this grant based design has been chosen in preference. Perhaps a indicator as to why is that the programme is targeted at academics and SMEs - precisely those agents that would struggle to cultivate sufficient funding to carry out a risky large-scale research project.

Specific questions arising from motivation and examples

These examples were chosen to illustrate the diversity with which a principal can apply his money in order to produce research outcomes. One crucial difference that arises is the source of the money spent on research. It can be provided by the designer, or it can be sourced by the contestant. By applying this distinction to a standard Tullock lottery model, this chapter aims to generate some insight into the following questions:

1. Why has the use of IPCs relative to grants increased in recent times? (See the introduction to this thesis for an overview of this trend)
2. When should prizes be used in preference to grants?
3. When can offering matching funds raise output?
4. Why do so many prizes *not* take account of contestant asymmetry?

4.1.4 Related work: Tax and subsidy in contests

There are two strands of related literature. First, and most directly, there are those papers that consider the impact of taxation or subsidy in contests.

Secondly, there are those papers that discuss the optimal form of contest success functions.

Fu et al. [2012]

The treatment of subsidy in contests that is closest to the one considered in this chapter is the recent paper of Fu et al. [2012]. As in my treatment, a budget-constrained designer can choose between "subsidies" and "prizes" in a Tullock - equivalent contest.

There are important differences however. The type of subsidy assumed by Fu et al. [2012] is multiplicative on output, and has the intuitive interpretation of contestant-specific capital, and therefore increases the principal's payoff even if there is no strategic effect on effort. That is, it directly enters the principal's objective function. In contrast,

the subsidy in this paper reduces the marginal cost of effort for contestants, but in the absence of any change in behaviour by the contestants, does not change the principal's payoff. It has the interpretation of matching funds. Relatedly, the cost of the subsidy to the designer is independent of effort in the Fu et al. [2012] paper, but increases proportionally with cost to contestant in this analysis.

They find that preferentially subsidising the weaker contestant is optimal if and only if R&D effort is sufficiently effective. This is in contrast to the usual result that more even contests are preferred (which is also the result of this chapter) and that preferring the weaker contestant is always optimal. This is a result of the direct effect of subsidies noted above. The strategic effect acts in the usual direction.

Glazer and Konrad [1999]

This paper considers taxes on contestants when they are engaged in rent-seeking contests. The taxes can be on contest winnings or on contest expenditures. The prize can be awarded to either in an all-pay auction, or through a Tullock lottery contest. They find the conditions under which taxes do not reduce rent-seeking effort, generate tax revenue with no tax burden, and cause no reduction in the profits of contestants. For example, a proportional and symmetric tax on spending in an all-pay auction changes neither the total spending (including taxes) of any firm, nor their equilibrium probability of winning, and therefore, nor their expected profit. The insight of the paper is that, because rent-seeking activity imposes a bilateral externality, a tax can raise revenue

without any deadweight loss.

This paper is, in a sense, the reverse of the one analysed here. The use of taxation to increase the endowment of the designer has been excluded in this chapter's model, mainly because it does not sit well as an assumption given the IPC context. Instead, resources are given to firms to encourage additional spending. This is because effort in an IPC is not wasteful rent-seeking, but rather valuable research.

Matros and Armanios [2009]

In this paper, the symmetric contestants are reimbursed a proportion of their effort. The winner is reimbursed a certain proportion, and the loser is reimbursed a potentially different proportion. This is similar to the model of this paper, in that a proportion of effort cost can be reduced by the designer. However, in this paper, the proportion is dependent on the identity of the contestant, and independent of who wins or loses.

Another similarity is that the provision of reimbursements may have a cost to the designer, depending on his objective function. Three alternatives are considered: maximisation of net total spending (which has the interpretation of a designer collecting the most contributions); minimisation of net total spending (interpreted as a designer deterring rent-seeking); or maximisation of total spending. This last case is of the most interest in the context of this chapter, since it most directly relates to a designer who cares about the aggregate effort of the contestants, and not his own net resource position at the end of the contest. The loser-reimbursed contest generates greater total spend-

ing than the winner-reimbursed contest, which has some parallels with the model here, which finds that it is more beneficial to subsidise that contestant who is least likely to win in equilibrium. Excessive comparison of the results is not too helpful, however, since it is precisely in the case of total-spending maximisation that the return of spending is not costly for the designer. In the model here, distributing resources to contestants is costly because it results in a reduced prize, and therefore in lower incentives to the effort the designer cares about.

The authors make use of an example, referencing the prototype competition between Boeing and Lockheed Martin for a DoD Joint Strike Fighter contract. The winner receives a contract for future jets, and each firm also received a large sum to demonstrate their concepts. The contract was interpreted as being the prize, and the commission of jets the reimbursement to the winner, while the direct payments were considered loser reimbursements. A case can instead be made that the contract and attendant purchases (which, after all, did not necessarily vary with the costs of design) are a prize, and the moneys distributed a subsidy. That is, the type of contest design problem considered in this paper is very similar to the motivation here, and the model built is a different interpretation of similar facts.

Konrad [2000]

In this paper, subsidies take the form of additional payments that are made by governments in the event of winning an export contract. The governments are strategically

competing agents. In the non-cooperative equilibrium, one country subsidises its domestic contestant, to be paid if they win they contract, the other country chooses an export tax, reducing the value of the contract. Both exporting countries are better off than without strategic trade policy.

This is a contest with a subsidy, but it is very different it its motivation, and therefore in its assumptions. The subsidy is a lump sum. Moreover, since the prize is an export contract, the subsidy is not a subsidy to contest effort, but rather an increase in the effective prize level. Although we use the same terminology, the instrument examined by Konrad is, in the context of an IPC, more like a contestant-specific prize.

4.1.5 Related work: Optimal contest success functions

Although these are more tangential, since this model assumes a fixed contest success function, the matching funds instrument is equivalent to a form of effort-weighting from the perspective of the contestants. As opposed to the costless change available to the principal in these models, the change in this paper is costly. Nonetheless, the intuition of what determines the optimal costless change will be important in understanding the underlying mechanics of this paper.

Nti [2004]

This paper begins with the observation that total efforts are in general reduced when there are asymmetries between contestants. It looks at different classes of contest success

functions and identifies the optimal member of each class from the perspective of an effort-maximising designer. Of particular interest here is the result relating to contests where effort is transformed by a contest technology into output, and the ratio of an individual's output relative to the total output determines the probability of winning, since this is a generalisation of the Tullock form used in this chapter.

In particular, the paper shows that the optimal contest technologies weight each contestant's effort by his opponent's valuation in the contest success function. By the standard Tullock results, this is equivalent to weighting each by his own marginal cost, in a contest where prizes are identical and costs vary. This is relevant because this chapter allows contest designers to reduce marginal cost, but in a costly way.

This result of this paper provides one of the key questions this chapter offers insight on; since it is optimal to treat contestants differently, except in the case of perfect symmetry, why is this quite rarely seen?

Runkel [2006] generalises Nti [2004] to situations where the designer also cares about "competitive balance" - the difference in the equilibrium winning probabilities, and also achieves the value-weighted CSF in certain cases.

Wang [2010]

This paper considers designer discretion over r in the Tullock contest success function. This is another kind of modification that the designer could perform to induce more effort. Again, such a change is considered costless for the designer. The main results

are that, for sufficiently heterogeneous contestants, there is an optimal r, and that this optimum is falling in heterogeneity. This adjustment of the contest accuracy is an alternative way of "subsidising" contestants when they are heterogeneous. Less accurate contests act preferentially towards the weaker contestants, since for any given disparity in effort, they are more likely to win. However, more accurate contests increase overall incentives to effort, for all contestants.

4.2 A model of subsidies in Tullock contests

4.2.1 Assumptions

The model is an extensive form game, a generalisation of Tullock [1980].

Players

There are three players: a contest designer D, and two contestants labelled 1 and 2. These contestants differ only in their marginal cost of spending, $\{c_1, c_2\} \in (0, \infty)^2$.

Payoffs

Contestant i receives:

$$A\frac{x_i^r}{x_i^r + x_j^r} - c_i(1 - \alpha_i)x_i$$

That is, the risk-neutral contestant spends x_i at constant marginal cost c_i in a Tullock contest for a prize A, but only pays $(1 - \alpha_i)$ of the cost.

D receives a payoff of $x_1 + x_2$, the sum of contestant spend on research, subject to his budget constraint being satisfied: $A + \alpha_1 x_1 + \alpha_2 x_2 \leq 1$. If the budget constraint is violated, he gets a large negative payoff.

See below for a more thorough interpretation.

Timing

1. D chooses $\{A, \alpha_1, \alpha_2\} \in [0, \infty)^3$.
2. 1 and 2 choose efforts x_1 and x_2 respectively.
3. Payoffs are distributed to all three players, as described immediately above.

4.2.2 Interpretation

A monetary view of research has been used to motivate the model. Contestants spend money x_i in order to compete for a cash prize A. The designer cares about the magnitude of x_i - the direct spend on research. This spending is also what matters in the contest success function.

However, the costliness of spend to the contestant is not x_i, but rather $c_i x_i$. We interpret the parameter c_i as being the contestant's cost of funds. Thus, if $c_i > 1$, perhaps the contestant has additional costs to collecting those funds; i.e. financing costs. For example, if the contestant borrows £100 to build a prototype for the contest, perhaps it spends £10 on interest. Only the £100 matters in the CSF, or to the designer, but

the contestant has $c_i = 1.1$ and faces a higher cost. Similarly, for $c_i < 1$, the contestant gains efficiencies from the spend. Perhaps 10% of the cash spent was earmarked for such research in the absence of a prize. Then the contestant has $c_i = 0.9$. Another interpretation might be an expected commercial return on a portion of the research undertaken.

The designer's strategic choice is to supply some matching funds to that spend which directly contributes to success in the contest. Since we consider both the prize and the research spend to be cash, the designer's budget constraint follows. Suppose, as above, for every £1 spent on research, the contestant incurs a 10% financing cost (i.e. $c_i = 1.1$). If the designer offers to match research spend one-to-one, $\alpha_i = 0.5$. For the total x_i that is spent on research by the contestant, only $(1 - \alpha_i)x_i$ is provided directly by the contestant and so his total costs are $c_i(1 - \alpha_i)x_i$, whereas D spends $\alpha_i x_i$.

It is important to note that the contestant-specific costs are related to the generation of the funds *provided by the contestant itself.* That is, not the money provided by the designer. The crucial point is that the contestant-specific marginal cost is incurred collecting non-subsidy research funds, rather than at the later stage of spending them. Otherwise the designer's contribution would be subject to the additional cost in the same way, and c_i would appear in the budget constraint. For example, if every £1 spent on research required an additional 10% in overhead costs, this model would not be appropriate. See the conclusion for a brief examination of the alternative formulation.

In this way, the direct relationship between subsidy, spending and prize moneys can be

motivated.

4.3 Optimal design in the lottery contest

In this section, we consider the case where $r = 1$, the Tullock lottery contest. The solution is by backwards induction, and is structured in four parts: the first provides some structure to the designer's problem and eliminates some non-optimal regions from the strategy space; the second examines the properties of the remaining regions; the third ties together this analysis to solve the designer's problem and discuss the solution; and the fourth uses various benchmarks or additional constraints to gain further insight from the model.

4.3.1 Analysis of limiting cases and interior solutions

This subsection eliminates, via a series of lemmas, a part of D's choice set as always suboptimal or infeasible:

Lemma 4.1. *The designer's optimal contest design is in the unit cube:* $\{A^*, \alpha_1^*, \alpha_2^*\} \in [0,1]^3$

Proof. First, no $\alpha_i > 1$ can form part of the solution. Note that we restrict the designer to offering only non-negative prizes, $A \geq 0$. Then, by inspection of the contestants' payoffs, note that if $\alpha_i > 1$, that is, if the marginal cost of effort is negative, Contestant i optimally increases his effort without limit to infinity. This will necessarily violate D's

budget constraint. Therefore, $\alpha_i^* \in [0, 1]$.

Note that the same argument does not necessarily apply to $\alpha_i = 1$, since if $A = 0$ optimal effort is indeterminate. We will return to this point shortly.

Since resources cannot be extracted from contestants (as we assume $\alpha_i \geq 0$, so taxation of effort is impossible) D cannot offer a prize greater than his total funds. Thus $A^* \in [0, 1]$. □

Ignoring the case of $\alpha_i = 1$ for the moment, we have the standard Tullock case, and the following standard results apply:

$$x_i^* = A\frac{c_j(1-\alpha_j)}{[c_i(1-\alpha_i)+c_j(1-\alpha_j)]^2} \tag{4.1}$$

$$, \implies x_1^* + x_2^* = A\frac{1}{c_i(1-\alpha_i)+c_j(1-\alpha_j)}. \tag{4.2}$$

Before proceeding to consider the D's problem directly, we need to consider the case where one or both of α_1 and α_2 equal 1. As specified so far, contestant efforts, and therefore the objective function, are undefined at certain borders of the (reduced) designer strategy space. If $\alpha_i = 1$ then the marginal cost of effort of effort for Firm i is zero. If $\alpha_j < 1$, then all $x_i \geq X$ cause Firm j to play zero, and all such are optimal choices, giving equal payoff to Firm i, but not to the designer. It seems natural to assume that $x_i^*(\alpha_i = 1) = \lim_{\alpha_i \to 1^{(+)}} x_i^*(\alpha_i) = \frac{1}{(1-\alpha_j)c_j}$. That is, equal to the one-sided limit, approaching from the region in which the designer's budget constraint can be satisfied.

This can be justified by giving the firm lexicographic preferences over profit and spending: for all choices that give equal profit, the one with least spending is preferred. This avoids the problem of maximising over an open interval, and keeps the designer's payoff continuous. (The derivative is not defined, but in all cases where the derivative is used in the analysis, simply note that the spending function is increasing). This assumption means that for $A > 0$, as long as one subsidy is less than 1, the designer receives a finite payoff. Therefore 4.1 and 4.2 are both valid for $\alpha_i = 1$ or $\alpha_j = 1$.

When both subsidies are one, a similar problem emerges. Here, however, we cannot appeal to the fact that one firm's spending is eventually pushed to zero - each can costlessly add spending. Of course, such a situation will violate the designer's budget constraint unless $A = 0$. In such a case π_D has the indeterminate form $\frac{0}{0}$. But we can use similar reasoning to justify using the limit as the outcome, since the designer can get arbitrarily close to it. This limit will depend on the relative rates at which the various instruments approach their limits, which is constrained by the budget constraint, but is also determined in part by the designer. The exact outcome, which is finite and non-zero, will be determined later, once more about the designer's optimisation is known. For now, we simply accept that this point in the strategy space is not undefined.

However, it does seem appropriate at this point to make a comment on the interpretation of $A = 0$ and $\alpha_i = 1$. If $\alpha_i = 1$, so the subsidy has been increased to the point where all funds are provided by the designer, this has the natural interpretation of a research grant. As shown above, this can be coupled with a non-zero prize.

However, zero prizes also form part of the designer's strategy space. Such a limiting case does not seem immediately relevant to a study of contests, of which prizes are a defining feature. However, part of the aim of this paper is to study exactly when the competitive advantages of a prize are useful, and when other forms of incentive might be more appropriate. $A = 0$ might occur when both contestants are given grants $\alpha_i = \alpha_j = 1$. Then contestants face only infinitesimal costs and incentives (recalling that we consider these extremes to be limiting situations, which the designer can be arbitrarily close to); exactly as we usually consider a grant system.

The analysis now begins to identify potentially optimal contest designs:

Lemma 4.2. *The solution is on the budget constraint:* $A^* + \alpha_1^* x_1^* + \alpha_2^* x_2^* = 1$.

Proof. The designer's payoff is increasing in all three arguments in the unit cube, where we know the solution to exist. Even where the derivative is undefined, it is easy to see that the limit is an upper bound. □

These standard Tullock results for equilibrium x_i, together with Lemmas 1 and 2, allow us to rewrite D's problem:

$$\max_{\{A,\alpha_1,\alpha_2\}\in[0,1]^3} \frac{A}{c_i(1-\alpha_i)+c_j(1-\alpha_j)}$$
$$\text{subject to} \qquad A + A\frac{c_i(1-\alpha_i)\alpha_j + c_j(1-\alpha_j)\alpha_i}{[c_i(1-\alpha_i)+c_j(1-\alpha_j)]^2} = 1.$$

Lemma 4.3. *The solution is never strictly inside the unit cube:* $\{A^*, \alpha_1^*, \alpha_2^*\} \notin (0,1)^3$, *unless* $c_1 + c_2 = 4c_1c_2$, *when all contest designs on the budget constraint with* $(1-c_1)\alpha_1^* = (1-c_2)\alpha_2^*$ *are optimal.*

Proof. Suppose such an interior optimum exists. Call it $\{\hat{A}, \hat{\alpha_1}, \hat{\alpha_2}\}$. By the Kuhn-Tucker Theorem, for such an optimum, the ratio between the derivative of the objective function and the derivative of the constraint for each of α_1 and α_2 must be equal. Call this common ratio λ_1.

$$\frac{\partial \pi_D}{\partial \alpha_1} = 0 \implies c_1 = \lambda_1 \frac{(1-\alpha_2)^2 c_2^2 + (1-\alpha_1) c_1^2 \alpha_2 + (1-\alpha_2) c_2 c_1 (\alpha_1 - \alpha_2 + 1)}{(1-\alpha_1) c_1 + (1-\alpha_2) c_2},$$
$$\frac{\partial \pi_D}{\partial \alpha_2} = 0 \implies c_2 = \lambda_1 \frac{(1-\alpha_1)^2 c_1^2 + (1-\alpha_2) c_2^2 \alpha_1 + (1-\alpha_1) c_1 c_2 (\alpha_2 - \alpha_1 + 1)}{(1-\alpha_1) c_1 + (1-\alpha_2) c_2},$$
$$(1-\alpha_1) c_1 + (1-\alpha_2) c_2 \neq 0.$$

Combining these two first order conditions to eliminate λ_1 gives a condition relating $\hat{\alpha_1}$ and $\hat{\alpha_2}$ that must be satisfied at any optimum not on one or more boundaries of the designer's strategy space:

$$\frac{c_1}{c_2} = \frac{(1-\hat{\alpha_2})^2 c_2^2 + (1-\hat{\alpha_1}) c_1^2 \hat{\alpha_2} + (1-\hat{\alpha_2}) c_2 c_1 (\hat{\alpha_1} - \hat{\alpha_2} + 1)}{(1-\hat{\alpha_1})^2 c_1^2 + (1-\hat{\alpha_2}) c_2^2 \hat{\alpha_1} + (1-\hat{\alpha_1}) c_1 c_2 (\hat{\alpha_2} - \hat{\alpha_1} + 1)},$$
$$0 = \frac{(c_1 + c_2)((\hat{\alpha_2} - 1)^2 c_2^2 - (\hat{\alpha_1} - 1)^2 c_1^2)}{(c_2(-(\hat{\alpha_1} - 1) c_1 c_2 (\hat{\alpha_1} - \hat{\alpha_2} - 1) + \hat{\alpha_1}(\hat{\alpha_2} - 1) c_2^2 - (\hat{\alpha_1} - 1)^2 c_1^2)},$$
$$\implies (1-\hat{\alpha_2}) c_2 = (1-\hat{\alpha_1}) c_1.$$

Therefore, in any such interior optimum, the effective marginal costs of the contestants must be equalised. Note that the budget constraint does not treat $c_1(1-\alpha_1)$ and $c_2(1-\alpha_2)$ symmetrically, so the intuition underlying this result is not as straightforward as might be imagined.

Therefore we can see that $\{\hat{A}, \hat{\alpha_1}, \hat{\alpha_2}\}$ must be on the line created by the intersection of this plane and the budget constraint. That is, we can remove two variables, say $\hat{\alpha_2}$ and $\hat{A}$, as determined and solve a univariate problem:

$$\hat{A} = \frac{4c_1(1-\hat{\alpha_1})}{4c_1(1-\hat{\alpha_1}) + \hat{\alpha_1} + \left(1 - \frac{c_1(1-\hat{\alpha_1})}{c_2}\right)},$$

$$\implies \max_{\hat{\alpha_1} \in [0,1]} \pi_D = \frac{2}{4c_1(1-\hat{\alpha_1}) + (1 - \frac{c_1}{c_2}(1-\hat{\alpha_1}) + \hat{\alpha_1})}.$$

Then the denominator is a linear function of $\hat{\alpha_1}$ and the maximum is therefore obtained by an extreme value of $\hat{\alpha_1}$, contradicting the assertion that $\hat{\alpha_1}$ was an interior value. The only exception to this is when $c_1 + c_2 = 4c_1c_2$, when the expression is independent of $\hat{\alpha_1}$. In this case, all contests that exhaust D's budget and where the post-subsidy costs are equal give an equal payoff to D, equal to one. For symmetric contestants, this is equivalent to requiring $c_1 = c_2 = \frac{1}{2}$. □

Therefore, the solution is necessarily at the intersection of the budget constraint and one or more edges of the unit cube. We can therefore search through the four points and the four lines in turn:

4.3.2 Identifying some potential optima and non-optima

This section analyses the remaining four curves of possibly optimal contest designs via a series of lemma to determine their payoffs. Some are eliminated as non-optimal.

Lemma 4.4. *If* $\{A^*, \alpha_1^*, \alpha_2^*\} = \{1, 0, 0\}$*, so the designer offers a prize but no subsidies,* $\pi_D = \frac{1}{c_1+c_2}$

Proof. This is the standard Tullock lottery. □

Lemma 4.5. *If the designer offers no prize, and only subsidies,* $\{A^*, \alpha_1^*, \alpha_2^*\} \to \{0, 1, 1\}$, *then* $\pi_D = 1$, *regardless of* $\{c_1, c_2\}$.

Proof. The optimal way to offer full subsidy and no prize can be derived from the analysis of Lemma 4.3.

Consider moving along the budget constraint, increasing α_1 and α_2. It was shown that the optimal way for the designer to do this is to equate contestant marginal costs: $c_1(1-\alpha_1) = c_2(1-\alpha_2)$. Otherwise, there is a way to reduce one subsidy and use the resulting increase in funds to increase the other in a way that spends no more on subsidies in equilibrium but results in greater contestant spending in total.

Substituting this into the budget constraint implies again:

$$\pi_D = \frac{2}{4c_1(1-\alpha_1) + (1 - \frac{c_1}{c_2}(1-\alpha_1)) + \alpha_1)}.$$

And thus the limiting value of π_D as both subsidies tend to 1 in the optimal ratio is 1. □

This is the result that was postponed earlier when considering the limits when both contestants were offered grants. Lemma 4.5 says that the full subsidy outcome gives the designer a payoff of 1, independent of cost levels. This independence is intuitive, since if the full cost of the spend is being supplied by the designer, the ability of the contestant to raise other funds becomes irrelevant. The level of the payoff is also intuitive. Since

there is effectively no prize, no additional funds are raised or spent in pursuit of it. Therefore the total spend is just the amount given as grants to the contestants - which equals the total funds available to the designer.

For convenience we will say $\pi_D(0,1,1) = 1$, without necessarily referencing the limiting nature of the argument that underlies this result

Lemma 4.6. *The contests with $\alpha_i = 1$ and $\alpha_j \in [0,1)$ are never optimal.*

Proof. We can show that such a contest is always improved upon by the full subsidy outcome. That is, it always gives a payoff less than one.

Recall that we used a limit argument to justify $x_i^* = \frac{A}{c_j(1-\alpha_j)}, x_j^* = 0$ when $\alpha_i \to 1, \alpha_j \in [0,1)$. Therefore, the budget constraint gives that $A = \frac{c_j(1-\alpha_j)}{1+c_j(1-\alpha_j)}$ and this implies $\pi_D = \frac{1}{1+c_j(1-\alpha_j)}$. This holds for all $\alpha_j \in [0,1)$, and it is clear that this is always less than one. That is, if one contestant is fully subsidised, it is always beneficial to increase the other subsidy (and decrease the prize as necessary). □

This is because, if one and only one contestant is fully subsidised, there is an extreme asymmetry in marginal costs. In this case, the strategic effect of increasing the other subsidy certainly increases the payoff to the designer.

The two remaining cases to consider are $\alpha_1 = 0, \alpha_2 > 0$ and $\alpha_1 > 0, \alpha_2 = 0$. These can then be compared to the two other potentially optimal contests $\{1,0,0\}$ and $\{0,1,1\}$.

Lemma 4.7. *With α_i constrained to equal zero, there is at most one stationary point in π_D when varying $\alpha_j \in (0,1)$ along the budget constraint. The stationary point exists*

if $c_j \in [\frac{1}{2}(\sqrt{c_i(c_i+4)}-c_i),\infty]$ *if* $c_i \leq 1$ *or* $c_j \in [\frac{1}{2}(\sqrt{c_i(c_i+4)}-c_i),\frac{c_i}{c_1-1}]$ *if* $c_i > 1$ *and is then at* $\alpha_j = 1+\frac{c_i}{c_j}-\sqrt{\frac{c_i(c_i+c_j)}{c_j^3}}$ *causing* $\pi_D = \frac{c_j}{2\sqrt{c_i c_j}\sqrt{c_i+c_j}-c_i}$.

Proof. If $\alpha_i = 0$, then the prize that exhausts the budget is:

$$A = \frac{(c_i + c_j(1-\alpha_j))^2}{(c_i + c_j(1-\alpha_j))^2 + c_i\alpha_j}.$$

The resulting payoff to the designer is:

$$\pi_D = \frac{c_i + c_j(1-\alpha_j)}{(c_i + c_j(1-\alpha_j))^2 + c_i\alpha_j}.$$

Then the first order condition for a stationary point is:

$$\frac{(-c_ic_j(2(\alpha_j-1)c_j+1)+(\alpha_j-1)^2c_j^3+c_i^2(c_j-1)}{(\alpha_j^2c_j^2+\alpha_j(-2c_ic_j+c_i-2c_j^2)+(c_i+c_j))^2} = 0,$$

$$\implies (-c_ic_j(2(\alpha_j-1)c_j+1)+(\alpha_j-1)^2c_j^3+c_i^2(c_j-1) = 0.$$

Note denominator is never zero for $\alpha_j \in [0,1]$. This is quadratic in α_j, and the solutions for α_j are:

$$\alpha_j = \frac{\pm\sqrt{c_ic_j^3(c_i+c_j)}+c_ic_j^2+c_j^3}{c_j^3}.$$

Since marginal costs are non-negative these stationary points always exist, and can be expressed as:

$$\alpha_j = \pm\sqrt{\frac{c_i(c_i+c_j)}{c_j^3}}+\frac{c_i}{c_j}+1. \tag{4.3}$$

Evidently, the only candidate stationary point in the interval [0,1] has a negative sign on the square root term. Call the subsidy level at this stationary point $\alpha_j^\dagger$. But this stationary point is only in this interval if two conditions are met. First, that $\alpha_j^\dagger \geq 0$

$$1+\frac{c_i}{c_j}-\sqrt{\frac{c_i(c_i+c_j)}{c_j^3}} \geq 0.$$

Which has the simpler form:

$$c_j^2 + c_i c_j - c_i \geq 0.$$

Or as a lower bound on c_j:

$$c_j \geq \frac{1}{2}(\sqrt{c_i(c_i+4)} - c_i). \tag{4.4}$$

The boundary can also be solved for c_i:

$$c_i = \frac{c_j^2}{1-c_j}.$$

Or, the inequality is:

$$c_i \leq \begin{cases} \frac{c_j^2}{1-c_j} & \text{if } c_j < 1 \\ \infty & \text{if } c_j \geq 1 \end{cases}.$$

Second, that $\alpha_j^\dagger \leq 1$

$$1 + \frac{c_i}{c_j} - \sqrt{\frac{c_i(c_i+c_j)}{c_j^3}} \leq 1.$$

This simply requires that the sum of marginal costs is greater than the product:

$$c_i + c_j - c_i c_j \geq 0.$$

Or as an upper bound on c_j:

$$c_j \leq \begin{cases} \frac{c_i}{c_i-1} & \text{if } c_i > 1 \\ \infty & \text{if } c_i \leq 1 \end{cases}. \tag{4.5}$$

Since the condition is symmetric, the condition on c_i has the same form:

$$c_i \leq \begin{cases} \frac{c_j}{c_j-1} & \text{if } c_j > 1 \\ \infty & \text{if } c_j \leq 1 \end{cases}.$$

These then are necessary upper and lower bounds on c_j (or on c_i) for such an optimum to be possible, otherwise the maximum is necessarily at $\alpha_j = 0$ or $\alpha_j = 1$, which are the full prize or single full subsidy situations considered above.

Designer payoff when $\alpha_j = \alpha_j^\dagger$ is:

$$\pi_D^\dagger = \frac{c_j}{2\sqrt{c_i c_j}\sqrt{c_i + c_j} - c_i}.$$

□

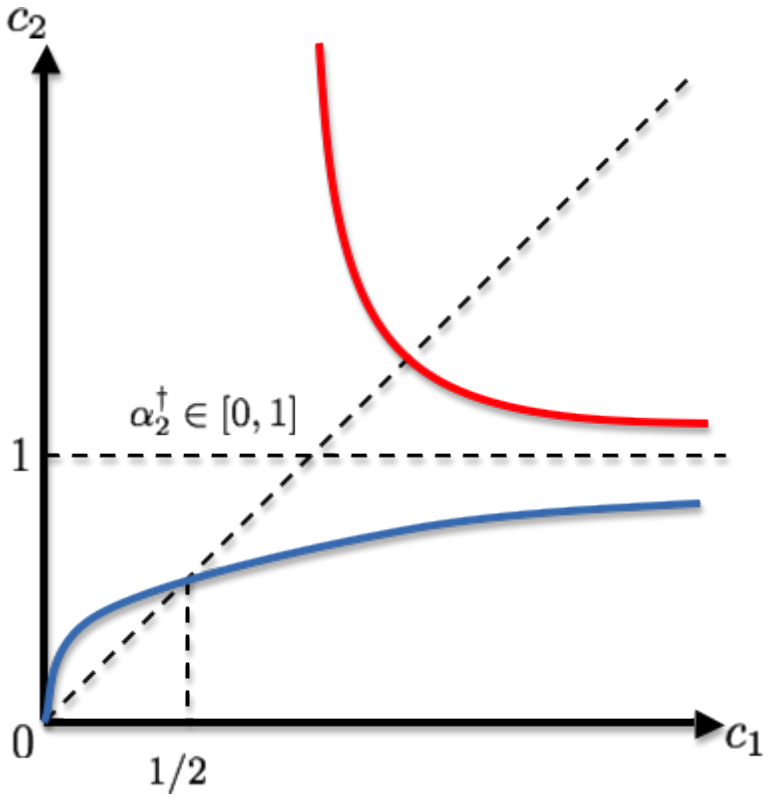


Figure 4.1: The existence of stationary point $\alpha_2^\dagger$: the upper bound on c_2 (red); the lower bound on c_2 (blue), together with the line $c_1 = c_2$ and the common asymptote

From now on, define $A_j^\dagger$ as the level of prize which exhausts the budget constraint when $(\alpha_i, \alpha_j) = (0, \alpha_j^\dagger)$

4.3.3 The optimal contest design

After Lemmas 3, 6 and 7, the only remaining possible optimal contest designs are the grand prize $(1, 0, 0)$, the grant scheme $(0, 1, 1)$, and the two single subsidy cases: $(A_1^\dagger, 0, \alpha_1^\dagger)$ or $(A_2^\dagger, 0, \alpha_2^\dagger)$, and the designer payoffs and existence conditions for each of these cases is also known. This subsection solves the designer's problem by direct comparisons of these in a series of three propositions. Once the solution is known, some comment is made, relating the results to the motivation.

Proposition 4.1. *The contest design that is the best possible single subsidy of contestant* j, $(A_j^\dagger, 0, \alpha_j^\dagger)$ *is optimal if* $c_i + c_j > 4c_ic_j$ *and* $c_j^2 + c_ic_j - c_i \geq 0$.

Proof. The optimal single subsidy of any contestant is preferred to the grant scheme when $\pi_D^\dagger \geq 1$:

$$\frac{c_j}{2\sqrt{c_ic_j}\sqrt{c_i + c_j} - c_i} \geq 1, \tag{4.6}$$

$$c_i + c_j \geq 4c_ic_j. \tag{4.7}$$

This constraint is stricter than the upper bound condition on c_j (4.5) required for the stationary point to be in [0,1].

Similarly, to see when the optimal subsidy of the contestant is preferred to simply offering a prize, compare $\pi_D^\dagger > \frac{1}{c_i+c_j}$. It is not sufficient to note that the full prize contest is on

the same line, since we have not yet shown the stationary point to be a maximum.

$$\frac{c_j}{2\sqrt{c_i c_j}\sqrt{c_i + c_j} - c_i} \geq \frac{1}{c_i + c_j}.$$

Cross-multiplying and expanding, the condition can then be factorised as:

$$(c_j^2 + c_i c_j - c_i)^2 \geq 0,$$

which is always true. This implies incidentally that the unique stationary point in this interval is always a maximum. □

Corollary 1. *Subsidy of the stronger contestant is never optimal.*

Proof. Moreover, note that, if j were the stronger firm, $c_j \leq c_i$ and $c_j^2 + c_i c_j - c_i \geq 0$ together imply that $c_i + c_j \leq 4c_i c_j$ - that is, if the stationary point exists for subsidising the stronger firm, it is necessarily preferable to award two grants instead. □

Proposition 4.2. *Awarding grants to both contestants* $(0, 1, 1)$ *is the optimal contest design if* $c_i + c_j \leq 4c_i c_j$.

Proof. Proposition 4.1 has already shown that in the region $c_i + c_j \leq 4c_i c_j$, the provision of grants to both contestants is preferred to a single subsidy. It just remains to note this is also preferred to full prize if $1 > \frac{1}{c_i + c_j}$, which is a weaker condition. □

Proposition 4.3. *Full prize is optimal if both* $c_1^2 + c_1 c_2 - c_1 \leq 0$ *and* $c_2^2 + c_1 c_2 - c_2 \leq 0$.

Proof. First note again that $c_i + c_j \leq 1$ is necessary for full prize to be preferred to a grant scheme.

Second, from Proposition 4.1 the single-subsidy stationary points $\alpha_i^\dagger, \alpha_j^\dagger$ for each contestant must lie below zero (i.e. below the full prize contest) for it not to be preferred. That is, $c_1^2 + c_1c_2 - c_2 \leq 0$ and $c_2^2 + c_1c_2 - c_1 \leq 0$. These two conditions together are stronger than the first. □

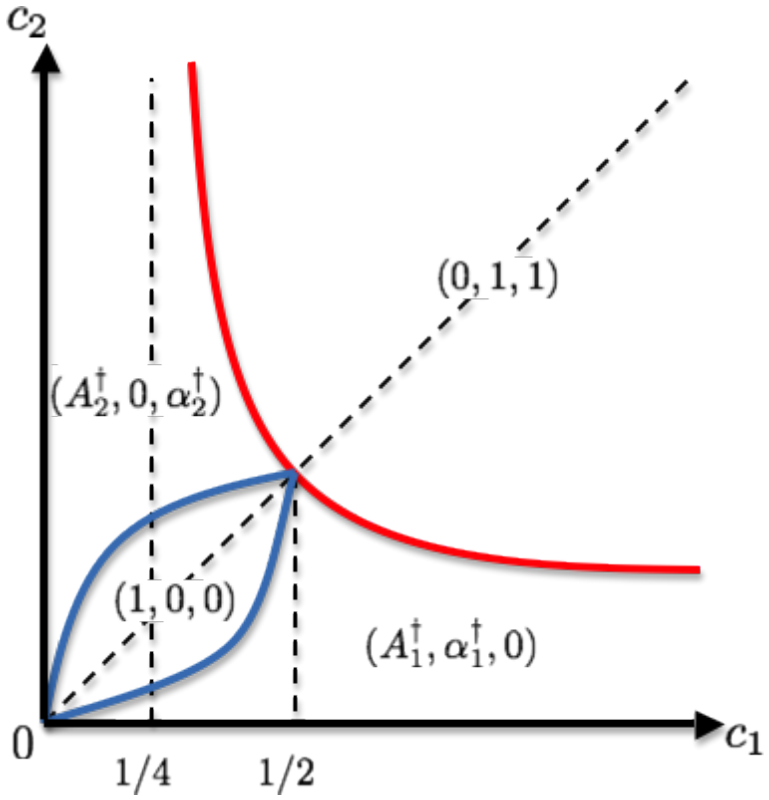


Figure 4.2: The optimal use of designer funds

Propositions 1, 2 and 3 together solve the designer's contest design problem. A few comments can be made before some more formal analysis. Note that the absolute level of contestants cost of funds is important. This is because the designer's cost of funds is set at one, since the assumption is that the principal owns the prize fund. In the symmetric case $c_1 = c_2$ the movement from optimal prize to optimal subsidy occurs

when $c_1 = c_2 \geq \frac{1}{2}$; that is, when the combined cost of funds exceeds the designer's. Notice as well that the motivation for asymmetric subsidy seems to come entirely from contestant asymmetry - contestant's costs of funds have to diverge sufficiently from each other for a single prize or symmetric grant scheme to become non-optimal. This may help explain why we observe contest designers ignoring information. Offering a subsidy to one contestant reduces the funds available to incentivise both simultaneously. Only when the strategic effect from increased symmetry is particularly great is this worthwhile.

However, the strategic effect is not the whole story. By considering the designer's optimal choice under some additional constraints, further insight can be gained into the strategic mechanics underlying the model.

4.3.4 Some further constraints and analysis of the solution

This section generates further insight into the meaning of the solution by highlighting particular features, and commenting on their meaning. It also uses the results above to consider optimal contest design in more restrictive environments, with discussion as to if these generate additional insight, or explain any observed behaviours. In contrast to the preceding subsection, each result stands alone (some are derived using incompatible constraints), and each is commented on in turn.

Proposition 4.4. *If a single subsidy is optimal, the subsidised contestant is still at an effective cost disadvantage*

Proof. Rearranging (4.3), if only a single contestant is subsidised, we can obtain an

expression for the subsidised contestant's effective marginal cost $c_j^\dagger$ at the optimal level of subsidy $\alpha_j^\dagger$:

$$c_j^\dagger = c_j(1-\alpha_j^\dagger) = \sqrt{\frac{c_i(c_i+c_j)}{c_j}} - c_i.$$

When is this greater than the unsubsidised contestant's marginal cost?

$$\sqrt{\frac{c_i(c_i+c_j)}{c_j}} - c_i \geq c_i \iff c_i + c_j \geq 4c_ic_j.$$

That is, if the stationary point is preferred to a grant scheme, it necessarily means that the optimal single subsidy is insufficient to cause the weaker contestant to have a lower effective marginal cost. The limiting case, where $c_i + c_j = 4c_ic_j$, has the single subsidy which equates effective marginal costs giving a payoff of 1, the same as a grant scheme. That is, the optimal single subsidy is one of those contest designs which equate marginal costs and exhaust the budget. From Lemma 4.3, we know that all such designs are equally preferred in such a case. The line created by the intersection of the budget surface and the plane $c_i(1-\alpha_i) = c_j(1-\alpha_j)$ runs from $(A^\dagger, 0, \alpha_j^\dagger)$ to $(0,1,1)$, and all the points on that line give a payoff of one. □

Proposition 4.5. *If the designer is constrained to subsidise at most one contestant, the region where the use of a grand prize is optimal is unchanged. Where both contestants have single-subsidy stationary points, it is preferable to subsidise the weaker. For $c_i+c_j < c_ic_j$, a single grant to the weaker contestant is optimal.*

Proof. Here, the possibility of the symmetric grant scheme (0,1,1) has been excluded. This means that both stationary point existence conditions (4.5) and (4.4) are now

relevant. The designer can subsidise neither (1,0,0), subsidise one partially, as both, or provide a grant to one contestant. Lemma 6 gives the payoff for this last case.

It was already shown in Proposition 1 that the stationary point is a maximum. Therefore, if it exists in the interval [0,1], since there are no other stationary points in that interval, it is preferred to the endpoints of the line. These end points are the grand prize and single grant designs. If both $\alpha_i^\dagger \in [0,1]$ and $\alpha_j^\dagger \in [0,1]$, then it is optimal to subsidise the contestant with the higher marginal cost of funding. This is clear from the form of $\pi_D^\dagger$.

In the cases where the stationary point does not exist on the line, a comparison of three outcomes is necessary: $\frac{1}{c_1+c_2}$ for a grand prize, $\frac{1}{1+c_1}$ for a grant to firm 2, and $\frac{1}{1+c_2}$ for a grant to firm 1. Of the grants, the grant to the weaker firm is preferred. Grants are preferred if (4.5) is violated, whereas a grand prize is preferred if (4.4) is violated □

Proposition 4.6. *The optimal single subsidy level $\alpha_j^\dagger$ for a contestant is falling in c_i and increasing in c_j.*

Proof. It is slightly clearer to work with the derivative of effective marginal cost. If effective marginal cost is increasing, the optimal level of subsidy must be falling and vice versa.

$$c_j^\dagger = c_j(1-\alpha_j^\dagger) = \sqrt{\frac{c_i(c_i+c_j)}{c_j}} - c_i,$$
$$\frac{\partial c_j^\dagger}{\partial c_i} \geq 0 \iff 2c_i + c_j \geq \sqrt{c_i c_j (c_i + c_j)}.$$

In the region of interest ($\alpha_i^\dagger \in [0,1]$) it must be that $c_i + c_j \geq c_i c_j$ (4.5). Therefore, a

stricter condition is:

$$2c_i + c_j \geq \sqrt{(c_i + c_i)(c_i + c_j)}$$

Which is clearly satisfied. So the post-subsidy marginal cost is increasing as the rival's cost increases, and therefore the optimal subsidy level is falling in rival cost.

$$\frac{\partial c_j^\dagger}{\partial c_j} = -\frac{1}{2}\sqrt{\frac{c_j}{c_i(c_i + c_j)}}\frac{c_i^2}{c_j^2} < 0.$$

So the optimal level of subsidy is increasing in own cost. □

Note that this result is derived without reference to the strength of the contestant, and is true of the optimal subsidy of the stronger contestant. This gives more detail to the response of the optimum contest design. Suppose initially c_1 is marginally greater than zero, and that c_2 is fixed at a value in $(\frac{1}{4}, \frac{1}{2})$. The initial optimum is a subsidy to c_2, at a level close to 1. As c_1 increases, the optimal subsidy falls. At the point where the optimal subsidy is zero, where the asymmetry between the firms exists but is not too great, the region where a full prize is optimal is entered. This remains optimal as c_1 increases past c_2, until the asymmetry is particularly great again. Then a subsidy is again optimal, beginning from zero, and increasing with c_2 until the effective marginal costs of the contestants are equalised. Further increases in c_2 then mean that contestants are jointly so bad at raising funds that it is preferred for the designer to provide all the funds, and the dual grant system is then optimal. This is shown in Figure 4.3.

Note that, because the level of costs is important, not all such progression move through

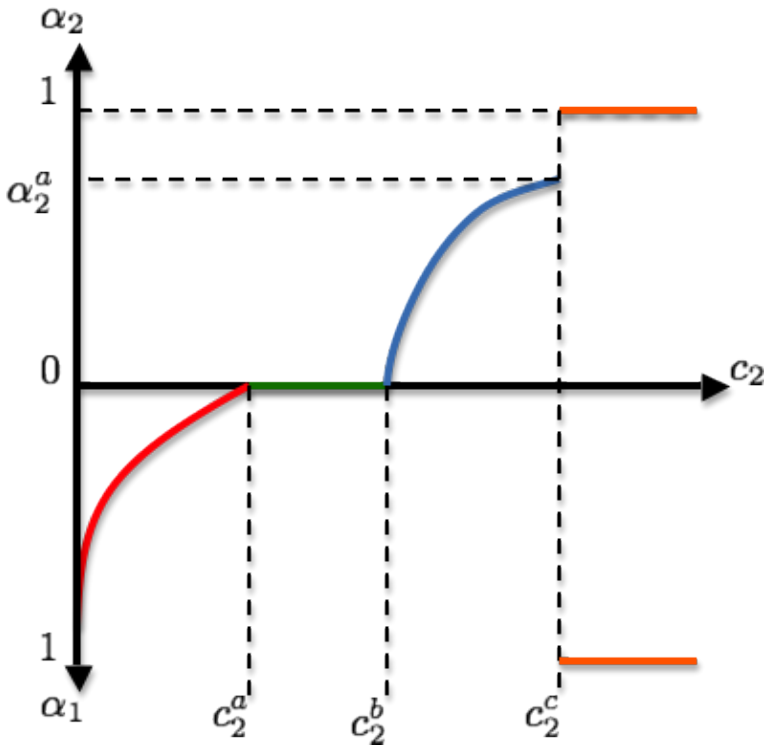


Figure 4.3: Optimal subsidy level as a function of c_2 when $c_1 \in (1/4, 1/2)$: subsidy of 1 (red); no subsidy (green); subsidy of 2 (blue); full subsidy of both (orange). Threshold values are in Table 4.3.4

c_2^a	$\frac{c_1^2}{1-c_1}$
c_2^b	$\frac{1}{2}\left(\sqrt{c_1(c_1+4)}-c_1\right)$
c_2^c	$\frac{c_1}{1-c_1}$
α_i^a	$1-\frac{c_j}{c_i}$

Table 4.1: Threshold values for Figures 4.3 and 4.4

all the regions. If $c_2 \leq \frac{1}{4}$, then no matter how big c_1 gets, it is never optimal to offer joint grants, rather the optimal subsidy of contestant 1 tends to 1 ($c_2^c = \infty$). This is because the funding costs of contestant 2 are so low as to preclude the optimality of directly offering funds to them.

If instead, $c_2 \geq \frac{1}{2}$, a prize is never optimal when one of the contestant is this inefficient at spending. Either they alone are subsidised, or, if the other is sufficiently inefficient, both are given grants. This situation is shown in Figure 4.4.

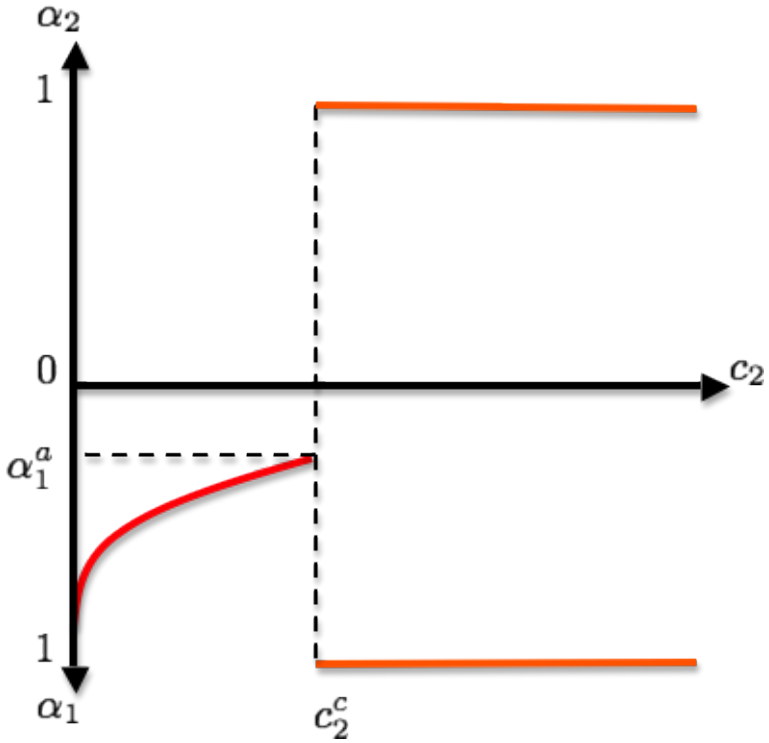


Figure 4.4: Optimal subsidy level as a function of c_2 when $c_1 > 1/2$: subsidy of 1 (red); full subsidy of both (orange).

Proposition 4.7. *If the designer is constrained not to subsidise the weaker contestant,*

he can still improve on the full prize outcome by subsidising the stronger, i, if $\alpha_i^\dagger \in (0,1)$; or from awarding a grant, if $c_i + c_j - c_i c_j \leq 0$.

Proof. The only possibilities here are full prize, subsidy of the stronger contestant, or a grant to the stronger contestant. Since $\alpha_j = 0$ by constraint, these are respectively the $\alpha_i = 0$, $\alpha_i \in (0,1)$, and $\alpha_i = 1$ cases.

It was shown that when $\alpha_i^\dagger \in (0,1)$, so when (4.4) and (4.5) are satisfied, it is a maximum, and so is optimal. Previously, these local maxima were ruled out in the general (unrestricted) solution as inferior to both optimal subsidy of the weaker contestant, and to a joint grant scheme.

Lemma 4.6 gives the outcome for a single grant on contestant i: $\pi_D = \frac{1}{1+c_j}$. When $c_i > 1$, this is evidently preferred by the designer to the full prize outcome, which gives him a payoff of $\frac{1}{c_i+c_j}$. That is, full subsidy of the stronger contestant is only preferred to a full prize design if the contestant has a higher cost of funds than the designer. □

This proposition reinforces the important point that the effect of the subsidy is not purely strategic. In fact, for certain cost levels, the designer is willing to increase the asymmetry of the contest in order to reduce the stronger contestant's cost of funding still further.

When the (0,1,1) solution is ruled out, the maximum moves from full prize, along the subsidy line, to full subsidy of the contestant.

Beyond this additional insight into the mechanics of the model, it is reasonable to ask

at this point why a designer would ever be prevented from subsidising those contestants with the higher cost of funds. This situation might prevail , for example, if only the strongest contestant can be identified, in a large population of unproductive rent-seekers and one weaker contestant. This might be the case, for example, if the stronger contestants are established or incumbent firms, and the weaker contestants are those "outsiders" whose unlikely proposals are indistinguishable from impossible proposals. Such a scenario would also explain the asymmetry in funding costs, where the outsiders have more difficulty raising funds, as well as being unable to attract matching funds from the designer. This is perhaps a helpful interpretation to place on the DARPA 2007 design example, where the strong contestants were placed in Track A.

4.4 Optimal design in general Tullock contests

This section generalises the above analysis to allow $r \in [0, 1]$. Although some tractability in the solution is lost by this generalisation, it serves to show that a number of the more surprising results are robust. Roughly the same analysis is performed: where the arguments are analogous and clear, they are omitted. The contest success functions with $r > 1$ are excluded because of the tractability problems they pose: namely, that only limited homogeneity is permitted if searching for a pure strategy equilibrium (see Nti for why, and Fu et al. [2012] for an identical assumption), and the degree of homogeneity is endogenous.

Lemma 4.8. *The designer's optimal contest design is in the unit cube:* $\{A^*, \alpha_1^*, \alpha_2^*\} \in$

$[0,1]^3$

Proof. This is shown in an entirely analogous way. □

Ignoring the case of $\alpha_i = 1$, we have the standard Tullock case, and the following standard results apply:

$$x_i^* = rA\frac{c_j^r(1-\alpha_j)^r c_i^{r-1}(1-\alpha_i)^{r-1}}{[c_i^r(1-\alpha_i)^r + c_j^r(1-\alpha_j)^r]^2}, \quad (4.8)$$

$$\implies x_1^* + x_2^* = rA\frac{(c_j^{r-1}(1-\alpha_j)^{r-1}c_i^{r-1}(1-\alpha_i)^{r-1})(c_i(1-\alpha_i)+c_j(1-\alpha_j))}{[c_i^r(1-\alpha_i)^r + c_j^r(1-\alpha_j)^r]^2}. \quad (4.9)$$

Again, the cases where a subsidy or subsidies are one are taken to be valued at the one-sided limit of this expression, as the subsidy approaches 1 from below. The precise value is derived later.

These standard Tullock results, together with the reduced strategy space, allow us to rewrite D's problem:

$$\underset{\{A,\alpha_1,\alpha_2\}\in[0,1]^3}{\arg\max} \; rA\frac{(c_j^{r-1}(1-\alpha_j)^{r-1}c_i^{r-1}(1-\alpha_i)^{r-1})(c_i(1-\alpha_i)+c_j(1-\alpha_j))}{[c_i^r(1-\alpha_i)^r + c_j^r(1-\alpha_j)^r]^2},$$

$$\text{subject to} \quad A + rA\frac{(c_j^{r-1}(1-\alpha_j)^{r-1}c_i^{r-1}(1-\alpha_i)^{r-1})(c_i(1-\alpha_i)\alpha_j + c_j(1-\alpha_j)\alpha_i)}{[c_i^r(1-\alpha_i)^r + c_j^r(1-\alpha_j)^r]^2} = 1.$$

Lemma 4.9. *The solution is on the budget constraint:* $A^* + \alpha_1^* x_1^* + \alpha_2^* x_2^* = 1$

Proof. This is shown in an analogous way □

This implies that designer payoff can be expressed as:

$$\pi_D = \frac{r(c_j^{r-1}(1-\alpha_j)^{r-1}c_i^{r-1}(1-\alpha_i)^{r-1})(c_i(1-\alpha_i)+c_j(1-\alpha_j))}{[c_i^r(1-\alpha_i)^r + c_j^r(1-\alpha_j)^r]^2 + r(c_j^{r-1}(1-\alpha_j)^{r-1}c_i^{r-1}(1-\alpha_i)^{r-1})(c_i(1-\alpha_i)\alpha_j + c_j(1-\alpha_j)\alpha_i)}.$$

Lemma 4.10. *The solution is never strictly inside the unit cube:* $\{A^*, \alpha_1^*, \alpha_2^*\} \notin (0,1)^3$, *unless* $c_1+c_2 = \frac{4}{r}c_1c_2$, *when all contest designs on the budget constraint with* $(1-c_1)\alpha_1^* = (1-c_2)\alpha_2^*$ *are optimal.*

Proof. Again, the Kuhn-Tucker first order conditions for an interior solution are used. Use the following definitions:

$$P = c_1(1-\alpha_1)c_2(1-\alpha_2),$$

$$C_r = c_1^r(1-\alpha_1)^r + c_2^r(1-\alpha_2)^r,$$

$$X = c_1(1-\alpha_1)\alpha_2 + c_2(1-\alpha_2)\alpha_2,$$

$$M = rA\frac{P^{r-1}}{C_r^2}.$$

And then use a subscript notation to denote first derivatives of these functions, so, for example:

$$M_i = \frac{\partial M}{\partial \alpha_i}$$

Then, introduce the following additional definition:

$$M_i = M\left(\frac{1-r}{1-\alpha_i} + 2\frac{c_1^r r(1-\alpha_i)^r}{C_r(1-\alpha_i)}\right) = MT_i.$$

And so the relevant first order Kuhn-Tucker conditions are

$$Mc_i + [c_1(1-\alpha_1) + c_2(1-\alpha_2)]\,MT_i = \lambda[MX_i + XMT_i] \qquad i = 1, 2.$$

Which together give the condition that:

$$\frac{\underline{c}T_1 - c_1}{\underline{c}T_2 - c_2} = \frac{(1-\alpha_2)c_2 - \alpha_2 c_1 + [c_1(1-\alpha_1)\alpha_2 + c_2(1-\alpha_2)\alpha_1]T_1}{(1-\alpha_1)c_1 - \alpha_1 c_2 + [c_1(1-\alpha_1)\alpha_2 + c_2(1-\alpha_2)\alpha_1]T_2}.$$

It is easy to verify that the solution of the lottery contest $c_1(1-\alpha_1) = c_2(1-\alpha_2)$ also solves the general case.

Call a hypothetic interior optimum $\{\hat{A}, \hat{\alpha_1}, \hat{\alpha_2}\}$. Since it is an interior optimum, $c_1(1-\hat{\alpha_1}) = c_2(1-\hat{\alpha_2})$, and the designer's problem reduces to:

$$\max_{\hat{\alpha_1}\in[0,1]} \pi_D = \frac{2}{c_1(\frac{4}{r} - \frac{1}{c_2})(1-\alpha_1) + (1+\alpha_1)}.$$

Which implies that the maximum is obtained by an extreme value of $\hat{\alpha_1}$, contradicting the assertion that $\hat{\alpha_1}$ was an interior value. The only exception to this is when $r(c_1+c_2) = 4c_1c_2$, when the expression is independent of $\hat{\alpha_1}$. In this case, all contests that exhaust D's budget and where the post-subsidy costs are equal give an equal payoff to D, equal to one. For symmetric contestants, this is equivalent to requiring $c_1 = c_2 = \frac{r}{2}$. □

Lemma 4.11. *If $\{A^*, \alpha_1^*, \alpha_2^*\} = \{1, 0, 0\}$, so the designer offers a prize but no subsidies, $\pi_D = \frac{rc_1^{r-1}c_2^{r-1}(c_1+c_2)}{(c_1^r+c_2^r)^2}$.*

Proof. This is the standard Tullock lottery. □

Lemma 4.12. *As any single subsidy tends to 1, the designer's payoff tends to one.*

Proof. This is slightly different. In fact, we can see here that the non-zero and finite limit on designer payoff as one effective cost goes to zero is unique to the $r = 1$ case.

Consider fixing one subsidy $\alpha_i \in [0, 1)$ and letting $\alpha_j \to 1$. Express designer payoff on the budget constraint as a function of the two subsidies:

$$\pi_D = \frac{1}{\frac{(c_i^r(1-\alpha_i)^r + c_j^r(1-\alpha_j)^r)^2}{rc_i^{r-1}(1-\alpha_i)^{r-1}c_j^{r-1}(1-\alpha_j)^{r-1}(c_1(1-\alpha_i)+c_j(1-\alpha_j))} + \frac{(c_1(1-\alpha_i)\alpha_j + c_j(1-\alpha_j)\alpha_i)}{(c_1(1-\alpha_i)+c_j(1-\alpha_j))}},$$

$$\lim_{\alpha_j \to 1} \pi_D = \begin{cases} 1 & \text{if } r < 1 \\ \frac{1}{1+c_i(1-\alpha_i)} & \text{if } r = 1 \\ 0 & \text{if } r > 1. \end{cases}$$

To consider the case where both subsidies tend to one, the argument relies on the analysis of Lemma 4.10 in an analogous way to $r = 1$. □

There is no analytic solution for the optimal subsidy of a single contestant for the general case. That is, there is no equivalent analytic form to $\alpha_i^\dagger$. However, this does not prevent us from establishing sufficient conditions for the optimality of such a prize:

Proposition 4.8. *If $r(c_i + c_j) > 4c_ic_j$ the optimal scheme does not involve a grant and has at least one zero subsidy. Otherwise, any of the equally preferred schemes where one or more grants is given is optimal.*

Proof. Notice that this generalises the condition under which the grant scheme is non-optimal in the $r = 1$ case.

Setting $\alpha_j = 0$ gives a condition under which there exists such a contest which is preferred to the full grant scheme; that is, where it gives a payoff greater than one:

$$\pi_D = \frac{rc_j^{r-1}c_i^{r-1}(1-\alpha_i)^{r-1}(c_i(1-\alpha_i)+c_j)}{[c_i^r(1-\alpha_i)^r + c_j^r]^2 + rc_j^{r-1}c_i^{r-1}(1-\alpha_i)^{r-1}c_j\alpha_i} > 1.$$

We need to check if there is an $\alpha_i \in (0,1)$ that makes this true; grouping terms gives:

$$\sqrt{rc_i^{r-1}c_j^{r-1}(c_i+c_j)} > c_i^r\sqrt{(1-\alpha_i)^r} + \frac{c_j^r}{\sqrt{(1-\alpha_i)^r}}$$

So such a contest is preferred if the RHS of this inequality (in the interval $(0,1)$) falls below the constant LHS at any point. We show that in this interval there is a unique global minimum of the LHS, and derive the conditions under which this minimum is valued less than the LHS.

$$\frac{\partial RHS}{\partial \alpha_i} = -\frac{r}{2}c_i^r(1-\alpha_i)^{\frac{r}{2}-1} + c_j^r\frac{r}{2}(1-\alpha_i)^{-\frac{r}{2}-1} \gtrless 0,$$

$$\frac{c_j}{c_i} \gtrless (1-\alpha_i).$$

Therefore $\alpha = 1 - \frac{c_j}{c_i}$ is the global minimum of this function - that is, where the effective marginal costs are equalised, and α_i is in (0,1) if $c_j < c_i$, so for any asymmetric contestants.

Using this minimum, therefore, the optimal contest has at least one zero subsidy iff:

$$r(c_i+c_j) > 4c_ic_j.$$

Since there is never an interior optimum, if this condition does not hold, the optimum must be one of the grant schemes that all result in a payoff of one. □

Proposition 4.9. *A sufficient condition for a single non-zero subsidy to be optimal is:* $r(c_i+c_j)(c_i(c_j^{2r}-c_i^{2r})+(c_ic_j)^r) > c_ic_j(c_i^r+c_j^r)^2$. *It is never optimal to subsidise the stronger contestant.*

Proof. If the first derivative of the α_i = profit function is positive at $\alpha_j = 0$, this is sufficient to show that some scheme with $\alpha_j > 0$ is preferred to the full prize scheme of $\alpha_j = 0$.

$$\frac{\partial \pi_D}{\partial \alpha_i}\Big|_{\alpha_i=0} > 0 \implies r(c_i + c_j)(c_i(c_j^{2r} - c_i^{2r}) + (c_i c_j)^r) > c_i c_j (c_i^r + c_j^r)^2.$$

Since the cost variables are always positive. □

Note that this generalises the conditions in proposition 3. This condition also implies that subsidy of the stronger contestant is never optimal, except in region where full subsidy is optimal.

4.5 A comment on robustness

Several of the results of this chapter are quite extreme. The solution in large regions of the parameter space is either $(1, 0, 0)$ or $(0, 1, 1)$, which are opposite extrema in the designer's feasible strategy space of the unit cube. Moreover, the transition between these designs is not smooth. Designer payoff changes continuously, if at all, but the optimal design changes discontinuously. (This is also the case for the movement from the optimal subsidy to the optimal grant scheme). For example, when considering the case of symmetric contestants, so there is no indirect incentive to consider subsidy, as soon as the common cost exceeds a certain threshold, $c > \frac{r}{2}$, the optimal costs moves from being a pure prize to pure grants.

It might reasonably be supposed that this lack of smoothness is related to the linear cost functions. In fact, this particular result, for symmetric contestants and $r = 1$, hold for any increasing cost function:

Proposition 4.10. *In the lottery contest with subsidies, as defined above, but with a general increasing common cost function $f(cx_i)$, so that c is a scale parameter, the only two possible optimal designs are (1,0,0) and (0,1,1).*

Proof. In a lottery contest with such a general cost function, and symmetric subsidy, contestant i's profit is:

$$\pi_i = A\frac{x_i}{x_i + x_j} - f(c(1-\alpha)x_i).$$

As throughout, x_i is spent by the researcher, but αx_i is provided directly by the designer. $(1-\alpha)x_i$ is raised by the researcher himself, at a total cost of $f(c(1-\alpha)x_i)$. The parameter c simply scales the cost function, and is without loss of generality.

Therefore, the first order condition for each contestant i is

$$\frac{x_i}{(x_1 + x_2)^2}A = c(1-\alpha)f'(c(1-\alpha)x_i).$$

Therefore, in the symmetric equilibrium, since the optimum is on the budget constraint, $A + 2\alpha x = 1$, the effective marginal cost of each contestant is:

$$c(1-\alpha)f'(c(1-\alpha)x^*(\alpha)) = \frac{1 - 2\alpha x^*(\alpha)}{4x^*(\alpha)}.$$

The designer's payoff is simply $2x$, so the designer's problem is:

$$\max_{\alpha\in[0,1]} \pi_D(\alpha) = \frac{1}{\alpha + 2c(1-\alpha)f'(c(1-\alpha)x^*(\alpha))}.$$

This generalises the problem as derived in Lemma 3. Note that the equilibrium value of effort in these expressions is obviously itself a function of the choice of subsidy level. To show that there is no $\alpha \in (0,1)$ that maximises this expression, it suffices to show that there is no stationary point in this interval in the denominator expression that improves on the $\alpha = 0$ and $\alpha = 1$ payoffs. Suppose that there were such a stationary point, call it $\tilde{\alpha}$. The condition for such a stationary point is:

$$-2(1-\tilde{\alpha})c^2\left[x^*(\tilde{\alpha}) - (1-\tilde{\alpha})x^{*\prime}(\tilde{\alpha})\right] f''((1-\tilde{\alpha})cx^*(\tilde{\alpha})) - 2cf'((1-\tilde{\alpha})cx^*(\tilde{\alpha})) + 1 = 0.$$

The following substitutions simplify the expression to an expression in terms of $x^*(\alpha)$: first, note that $x^{*\prime}(\tilde{\alpha}) = 0$ at an interior optimum; second, from the first order condition on effective marginal cost, $cf'((1-\tilde{\alpha})cx^*(\tilde{\alpha})) = \frac{1-2\tilde{\alpha}x^*(\tilde{\alpha})}{4x^*(\tilde{\alpha})(1-\tilde{\alpha})}$; and third, from this same condition, $f''((1-\tilde{\alpha})cx^*(\tilde{\alpha}))$ can also be calculated in terms of $x^*(\tilde{\alpha})$ (since $x^{*\prime}(\tilde{\alpha})$ equals zero at the hypothetical optimum) as $\frac{x^*(\tilde{\alpha}-2x^*(\tilde{\alpha})^2}{4(1-\tilde{\alpha})cx^*(\tilde{\alpha})}$. Therefore, at the stationary point, it must be that:

$$-2(1-\tilde{\alpha})c^2x^*(\tilde{\alpha})\frac{x^*(\tilde{\alpha}) - 2x^*(\tilde{\alpha})^2}{4(1-\tilde{\alpha})cx^*(\tilde{\alpha})} - 2\frac{1-2\tilde{\alpha}x^*(\tilde{\alpha})}{4x^*(\tilde{\alpha})(1-\tilde{\alpha})} + 1 = 0.$$

Factorisation gives that:

$$\frac{[2x^*(\tilde{\alpha}) - 1]\left[(1-\tilde{\alpha})cx^*(\tilde{\alpha})^2 + 1\right]}{2(1-\tilde{\alpha})x^*(\tilde{\alpha})} = 0.$$

But no such $\tilde{\alpha}$ exists. For it to be an optimum, it must at least improve on the payoff from $\alpha = 1$, so that $x^*(\tilde{\alpha}) > \frac{1}{2}$. But in this case, the expression in the left-hand square bracket is strictly positive. This, combined with the strictly positive expression in the right-hand square bracket means that there is no solution to this condition. Therefore,

no $\alpha \in (0,1)$ produces a stationary point in $\pi_D(\alpha)$ (and also necessarily in $x^*(\alpha)$), so the maximum must be at $\alpha = 1$ or $\alpha = 0$. Note that $\alpha = 1$, with $2x^*(1) = 1$ does in fact satisfy this equation. As in the linear cost model, this solution is a corner solution characterised by a tangency condition. □

4.6 Conclusion and discussion

4.6.1 Implications for prize design

This model has shown how variations in the level and symmetry of the cost of funds for researchers can explain some of the diversity in the design of programmes to incentivise innovation. The model suggests that prizes are preferable to grants when firms have cheap access to resources. There is a role for grants and subsidies when contestants are asymmetric in the cost of their access to funds, building on the well known result that more symmetric Tullock contests produce greater aggregate output. However, matching funds can profitably be applied to the more advantaged contestants, even though this necessarily has the effect of increasing asymmetry. That is, the role of offering funds to contestants is not purely strategic, but increases overall spending by lowering the effective marginal cost of that contestant.

In considering the trade-offs between prizes and grants, the designer also has to recognise that funds offered as a prize will incentivise all contestant simultaneously to spend more, whereas the use of matching funds or grants can only raise the output of a single

contestant. This effect will obviously become greater as we consider contests with a larger number of competitors. This may explain why so many prizes take no account of contestant asymmetry, even when it is particularly great.

It is often noted that real world prizes have the ability to generate more spending by contestants in aggregate than the value of the prize. In this model, this is in fact the only time when a prize should be offered, since the designer always has the option of giving the same money to a researcher to spend directly.

These insights may be able to explain some of the significant increase in the use of prizes (and particularly IPCs) from 1990 to 2007, as credit availability increased, and access to the most sophisticated technology (particularly computing power) became more widespread, so contestants could rely more on their own resources, and less on costly external sources. Moreover, there are more credible contestants, so the number is increasing and therefore the marginal value of prize funds is also increasing.

4.6.2 Discussion of modelling choices

On one level, only a single modelling element has been added to the standard, and much discussed, Tullock contest. (See Konrad [2009, pp 48-65] for a summary of the axiomatic, experimental, and evolutionary defences of this model.) The addition of a costly designer intervention in the contestant cost function is, as far as I know, novel. An interpretation in the context of IPCs was given above.

However, another natural choice of assumption also presents itself, and it is important

to emphasise how this decision has been made. Although both are equally compelling, only the assumption made here generates an interesting model or any real insight. The alternative is to assume that instead of subsidising spending in the contest, a designer could subsidise overall spending, including the cost of funds. In such a case, the cost function of the competitor is identical, since instead of paying a cost of funds multiplier of c_i on costs of $(1-\alpha_i)x_i$, now he pays $c_i x_i$, but only $(1-\alpha_i)$ of it is not reimbursed by the designer. Of course if such a scenario were to be consistent, the designer would pay $\alpha_i c_i x_i$ out of his prize funds, rather than $\alpha_i x_i$. This assumption, while seemingly similar to the one made in this chapter, in fact has strikingly different consequences, but generates little insight either alone or in combination with the existing model. In every case, the optimal solution for the designer is to offer full subsidies to both contestants.

Aside from this, by making the structure of the contest the strategic decision of a designer, important additional informational assumptions are made. Assuming that the contestants can accurately observe the designer's choices is natural, as is the assumption that the designer's offers are binding. An area that bears closer scrutiny, however, is the designer's knowledge of the value of the c_i variables. This assumption does not seem too strong when one considers that Tullock contestants are often assumed to have perfect knowledge of each other's cost functions. Moreover, in the cost of funding interpretation, the c_i is exactly the cost of funding that outside investors demand, based on public information. Thus, the designer does not have to directly observe this variable, but can infer it from public information and credit market conditions.

4.6.3 Future directions and alternative answers

There is one important question outstanding that is suppressed completely in the model, but would likely yield important insights. In the model above, it is assumed that each contestant is equally productive. That is, productivity of spending is independent of cost of funds. The differing costs of funding are assumed to be derived from other circumstances. In reality, more productive or promising contestants will be more able to access funds, and these two variables should be correlated.

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