CURVED TWISTOR SPACES

by

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ABSTRACT.

This thesis is concerned with the problem of "coding" the information of various zero-rest-mass fields into the complex structure of "curved twistor spaces". Chapter 2 is devoted to various preliminaries: a brief outline of twistor theory; an introduction to vector bundles and sheaf cohomology and some of their applications in twistor theory; and a discussion of potentials for electromagnetic fields.

Chapter 3 deals with left-handed (i.e. anti-self-dual) electromagnetic fields and describes in some detail the associated curved twistor spaces. It is shown how holomorphic functions on the curved spaces give rise to "charged" zero-rest-mass fields on space-time. The first section of Chapter 4 gives the corresponding results for left-handed gravitational fields, using Penrose's "nonlinear graviton" construction. The rest of Chapter 4 is devoted to the concept of twistors relative to a hypersurface in a general curved space-time. In §4.2 the hypersurface is taken to be spacelike; the hypersurface twistors are described and the problem of using holomorphic hypersurface twistor functions to generate fields on the hypersurface and in space-time is discussed.

Next the hypersurface is taken to be null. The structure of the associated hypersuface twistor space and $\mathcal{H}$-space are described in some detail. The twistor space has a natural inner product and, if the hypersurface is shear-free, then it has a "fibred" structure as well. In §4.4 the hypersurface twistor language is used to show that the propagation of twistors through an analytic pp-wave is given by the unfolding of a canonical transformation. Chapter 5 extends the "electromagnetic" construction of Chapter 3 to non-Abelian gauge theories; left-handed gauage fields are described in terms of complex vector bundles over projective twistor space.

Acknowledgements.

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CONTENTS.

The numbers of equations and figures have the form (c.s.n), where c refers to the chapter and s to the section; but if an equation or figure in the same chapter is referred to, then the chapter number is omitted.

CHAPTER 1. INTRODUCTION.

One of the lessons which quantum mechanics teaches us is that complex numbers play an essential role in physics. And one of the features of the twistor programme [19], [21] is that this role is very strongly emphasized: twistor spaces are, first and foremost, complex manifolds. One of the most striking uses of the twistor complex structure lies in the fact that holomorphic twistor functions lead, via contour integration, to solutions of the (linear) zero-rest-mass free-field equations. This is sometimes called the passive description of zero-rest-mass fields. Recently it has emerged that there is a corresponding active description, in which space-time fields have the effect of deforming the complex structure of twistor spaces. The information of the fields is "coded" into the complex structure of the curved twistor spaces. The active description is in many ways more useful than the passive one: for one thing, non-linear theories fit more easily into the scheme; for another, it seems that interactions between fields can be handled in a particularly natural way.

The active description of zero-rest-mass fields forms the subject of this thesis. Chapter 2 is devoted to various preliminaries: a brief outline of basic twistor theory, an introduction to vector bundles and sheaf cohomology and the application of these notions in twistor theory, and a discussion of potentials for electromagnetic fields. Chapter 3 deals with left-handed electromagnetic fields, describing the associated curved twistor spaces and showing how solutions of the "charged" zero-rest-mass field equations may be generated using twistor functions. The first section of Chapter 4 contains the corresponding results for left-handed gravitational fields (using Penrose's "nonlinear graviton" construction [20]). The rest of Chapter 4 deals with the definition and properties of hypersurface twistors. Finally, in Chapter 5, the "electromagnetic" construction of Chapter 3 is generalized to other gauge theories.

Of course, zero-rest-mass fields are of limited physical interest, and the challenge to twistor theory is to come to grips with the problem of dealing with massive fields in a satisfactory way. The current idea in twistor theory is that mass is introduced by going from a 1-twistor to a 2- or more-twistor description [21]; this has the disadvantage that the theory becomes somewhat less natural, but there are compensating features. From the point of view of the active description, there appears to be no obstacle to generalizing the curved 1-twistor space constructions of this thesis to curved multi-twistor spaces (indeed, the "Yang-Mills" construction of chapter 5 seems to be leading us in that direction).

CHAPTER 2. PRELIMINARIES.

§2.1. SPINORS AND TWISTORS.

In this section we shall set up notation and conventions, and give a brief introduction to basic twistor theory.

First we list some definitions and some spinor formulae. Further details may be found in [17]. The abstract index notation will be used for tensors and spinors in space-time, according to the following scheme:

	Space-time 4-vector indices.	2-spinor indices.	
Abstract	a, b, c, ...	A, B, C, ...	A', B', C',...
Numerical	$\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}, \ldots$	$\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{C}, \ldots$	$\underset{\sim}{A'}, \underset{\sim}{B'}, \underset{\sim}{C'}, \ldots$
Running over the values	0,1,2,3.	0,1.	0',1'.

Abstract 4-vector indices will be identified with the corresponding pair of spinor indices (for example, $v^a = v^{AA'}$).

By a real <u>space-time</u> will be meant a pair (M, g_{ab}), where M is a connected open subset of $\mathbb{R}^4$ and g_{ab} is a real analytic metric (with signature +---) defined on M. On each space-time we set up an analytic spin-frame

$$\varepsilon_{\underset{\sim}{A}}{}^{A} = (o^A,\ \iota^A),$$

$$\varepsilon_{\underset{\sim}{A'}}{}^{A'} = (o^{A'}, \iota^{A'}), \text{ with } o^A = \overline{o^{A'}}, \text{ etc.,}$$

and an associated null tetrad $\{l^a, n^a, m^a, \bar{m}^a\}$. The spin coefficients $\gamma_{\underset{\sim}{ABCC'}}$ and $\bar{\gamma}_{\underset{\sim}{A'B'CC'}}$ are defined by

$$\gamma_{\underset{\sim}{ABCC'}} := \varepsilon_{\underset{\sim}{B}}{}^{A} \nabla_{\underset{\sim}{CC'}} \varepsilon_{\underset{\sim}{A}A},$$

$$\bar{\gamma}_{\underset{\sim}{A'B'CC'}} := \varepsilon_{\underset{\sim}{B'}}{}^{A'} \nabla_{\underset{\sim}{CC'}} \varepsilon_{\underset{\sim}{A'}A'} . \tag{1.1}$$

For later reference we write down the formula

$$\underset{\sim}{\varepsilon}_{A'}{}^{A'} \; \underset{\sim\sim}{\nabla}_{CC'} \; \theta_{A'} = \underset{\sim\sim}{\nabla}_{CC'} \; \underset{\sim}{\theta}_{A'} - \bar{\gamma}^{D'}{}_{\underset{\sim\sim\sim\sim}{A'CC'}} \; \theta_{D'} \; . \qquad (1.2)$$

In Minkowski space-time, the standard constant spin-frame will be used, in which a 4-vector v^a has the spinor representation

$$v^{\underset{\sim\sim}{AA'}} = \begin{bmatrix} v^{00'} & v^{01'} \\ v^{10'} & v^{11'} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} v^0 + v^1 & v^2 + iv^3 \\ v^2 - iv^3 & v^0 - v^1 \end{bmatrix} .$$

The distinction between abstract and numerical indices is not crucial in Minkowski space-time, and will usually be dropped.

The curvature tensor (with sign conventions $2\nabla_{[a}\nabla_{b}v_{d]} = R_{abcd}v^c$, $R_{ab} = R^c{}_{acb}$)* has the spinor decomposition

$$\begin{aligned} R_{abcd} = {} & \psi_{ABCD}\,\varepsilon_{A'B'}\,\varepsilon_{C'D'} + \bar{\psi}_{A'B'C'D'}\,\varepsilon_{AB}\,\varepsilon_{CD} \\ & + \varepsilon_{AB}\,\Phi_{CDA'B'}\,\varepsilon_{C'D'} + \varepsilon_{CD}\,\Phi_{ABC'D'}\,\varepsilon_{A'B'} \\ & + 2\Lambda\,(\varepsilon_{AC}\,\varepsilon_{BD}\,\varepsilon_{A'C'}\,\varepsilon_{B'D'} - \varepsilon_{AD}\,\varepsilon_{BC}\,\varepsilon_{A'D'}\,\varepsilon_{B'C'}) \; . \end{aligned}$$

Here $\psi_{ABCD} = \psi_{(ABCD)}$, $\Phi_{ABC'D'} = \Phi_{(AB)(C'D')} = \bar{\Phi}_{ABC'D'}$, $\bar{\Lambda} = \Lambda$.

The Weyl conformal curvature tensor is

$$C_{abcd} = C^-_{abcd} + C^+_{abcd},$$

where $\qquad C^+_{abcd} = \bar{\psi}_{A'B'C'D'}\,\varepsilon_{AB}\,\varepsilon_{CD}$

and $\qquad C^-_{abcd} = \psi_{ABCD}\,\varepsilon_{A'B'}\,\varepsilon_{C'D'} \qquad (1.3)$

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* Square brackets enclosing indices denote skew-symmetrization, round brackets denote symmetrization.

are respectively the self-dual and anti-self-dual parts of C_{abcd}. Einstein's vacuum equations $R_{ab} = 0$ are equivalent to

$$\Phi_{ABC'D'} = 0 , \quad \Lambda = 0. \qquad (1.4)$$

The alternating tensor $e_{abcd} = e_{[abcd]}$, $e_{0123} = 1$, has as its spinor equivalent

$$e_{abcd} = i \, \varepsilon_{AC} \, \varepsilon_{BD} \, \varepsilon_{A'D'} \, \varepsilon_{B'C'} - i \, \varepsilon_{AD} \, \varepsilon_{BC} \, \varepsilon_{A'C'} \, \varepsilon_{B'D'}.$$

The commutator $\nabla_{[a} \nabla_{b]}$ has the spinor decomposition

$$2 \nabla_{[a} \nabla_{b]} = \varepsilon_{A'B'} \Box_{AB} + \varepsilon_{AB} \Box_{A'B'},$$

where

$$\Box_{AB} := \nabla_{A'(A} \nabla^{A'}_{B)} ,$$

$$\Box_{A'B'} := \nabla_{A(A'} \nabla^{A}_{B')} .$$

If ϕ is a scalar on M, then

$$\Box_{AB}\phi = 0 = \Box_{A'B'}\phi, \qquad (1.5)$$

while

$$\varepsilon_{D\underset{\sim}{D}} \Box_{AB} \varepsilon_{C}{}^{\underset{\sim}{D}} = \psi_{ABCD} - 2 \Lambda \varepsilon_{D(A} \varepsilon_{B)C} ,$$

$$(1.6)$$

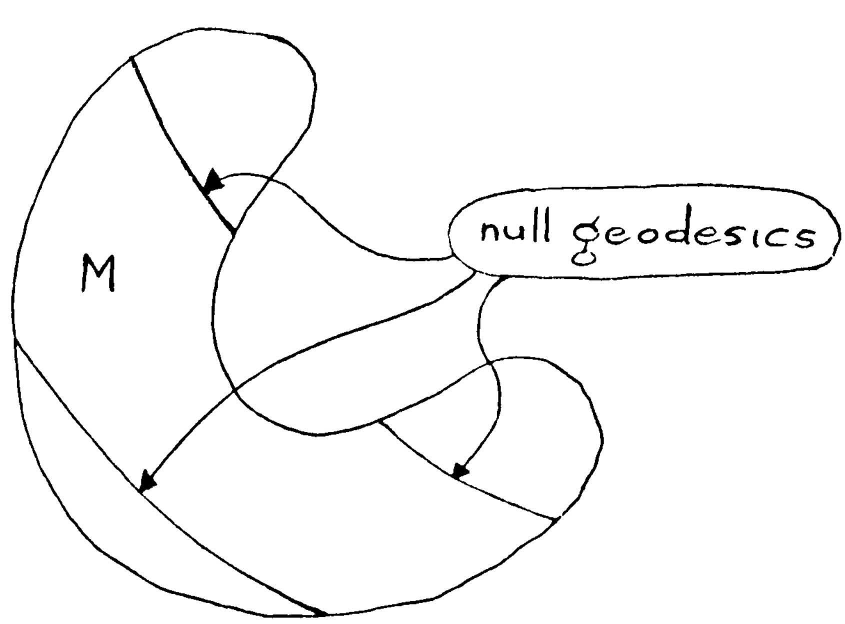
$$\varepsilon_{D\underset{\sim}{D}} \Box_{A'B'} \varepsilon_{C}{}^{\underset{\sim}{D}} = \Phi_{CDA'B'} .$$

We turn now to the notion of complexification. A complex space-time is a pair (N, g_{ab}), where N is a connected open subset of $\mathbb{C}^4$ and g_{ab} is a holomorphic nonsingular metric on N. Given a real space-time (M, g_{ab}), we can imbed it in a complex space-time $(\mathbb{C}M, g_{ab})$ by using the following procedure. If $x^{\underset{\sim}{a}}$ are real coordinates on M, simply allow the $x^{\underset{\sim}{a}}$ to take on complex values and extend the real-analytic metric g_{ab} analytically to these complex values. The complex space-time ($\mathbb{C}M, g_{ab}$) constructed in this way is said to be a <u>complexification</u> of (M, g_{ab}). In general, $\mathbb{C}M$ will just be a slight "thickening-out" of M : if we try to extend it too far into the complex, the metric will run into singularities.

The spin-frame and null tetrad can also be analytically extended to $\mathbb{C}M$, with the rule that complex conjugation bars are to be replaced by tildes. For example, $\bar{\psi}_{A'B'C'D'}$ becomes $\tilde{\psi}_{A'B'C'D'}$, which is now <u>independent</u> of ψ_{ABCD} ; and the real quantity Λ becomes a complex quantity.

By a <u>right-conformally-flat</u> space-time is meant a complex space-time which has $\tilde{\psi}_{A'B'C'D'} = 0$. A <u>right-flat</u> space-time is one which is both right-conformally-flat and Ricci-flat : i.e. $\tilde{\psi}_{A'B'C'D'} = 0$, $R_{ab} = 0$. Left-conformally-flat and left-flat space-times have $\psi_{ABCD} = 0$ and $\psi_{ABCD} = 0$, $R_{ab} = 0$, respectively.

It will be useful to have a notion of "convexity" of a space-time. The object of this is to rule out the type of situation illustrated in figure 1.1, where the space of null geodesics in M may be non-Hausdorff.* So let us say that a real or complex space-time (M, g_{ab}) is <u>convex</u> if the space of real or complex null geodesics in M is Hausdorff.



<u>Figure 1.1.</u>

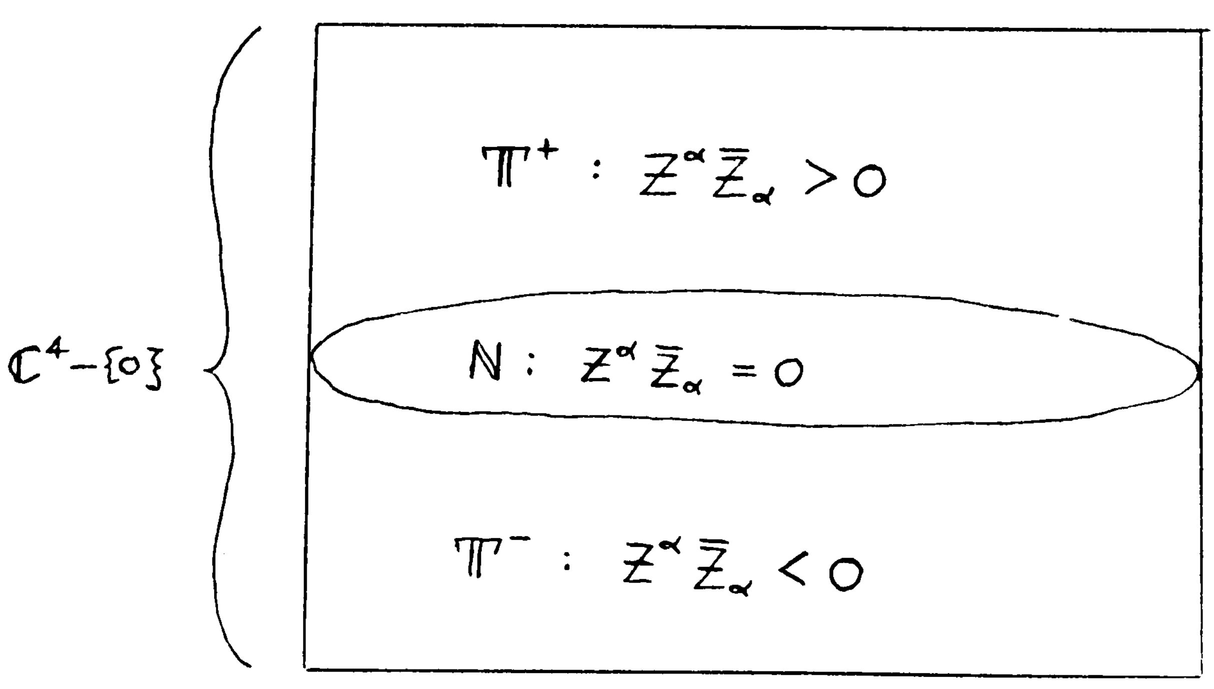
* By "geodesic in M" we mean "maximally extended geodesic in M".

The concept of a real or complex space-time being "close to Minkowski" will occasionally crop up. The statement that "a space-time has property P if it sufficiently close to Minkowski" means that "for every 1-parameter family of space-times $\{(M_\lambda, g_{ab}(\lambda)) \mid \lambda \in \mathbb{R} \text{ or } \mathbb{C}\}$ such that $(M_0, g_{ab}(0))$ is real or complex Minkowski, there exists $\varepsilon > 0$ such that $(M_\lambda, g_{ab}(\lambda))$ has P for $0 \leq |\lambda| < \varepsilon$ ".

We now move on to give a brief outline of twistor theory. For more details, the reader is referred to [24], [19], [8].

The basic non-projective twistor space is $\mathbb{C}^4 - \{0\}$, with coordinates Z^α (the index α runs over 0,1,2,3). The corresponding projective twistor space is the complex projective 3-space $\mathbb{CP}_3$; the coordinates Z^α serve as homogeneous coordinates on $\mathbb{CP}_3$. Define complex conjugate coordinates on $\mathbb{C}^4 - \{0\}$ by $\bar{Z}_0 = \overline{Z^2}$, $\bar{Z}_1 = \overline{Z^3}$, $\bar{Z}_2 = \overline{Z^0}$, $\bar{Z}_3 = \overline{Z^1}$. Twistor space is partitioned into three subsets $\mathbb{T}^+$, $\mathbb{N}$ and $\mathbb{T}^-$ according to whether the twistor norm $Z^\alpha \bar{Z}_\alpha$ is positive, zero or negative. See figure 1.2. Similarly, $\mathbb{CP}_3$ is partitioned into $\mathbb{PT}^+$, $\mathbb{PN}$ and $\mathbb{PT}^-$.

Figure 1.2



There is a useful geometric interpretation of twistors in complex Minkowski space-time, or, more generally, in some convex subregion $\mathbb{C}M$ of complex Minkowski space-time. A twistor determines two spinors ω^A and $\pi_{A'}$, according to the scheme

$$Z^\alpha = (\omega^A, \pi_{A'}).$$

Let Z denote the locus of points x^a in $\mathbb{C}M$ for which

$$\omega^A = i\, x^{AA'} \pi_{A'}. \tag{1.7}$$

Then Z is a complex 2-plane in M which is totally null, since each vector tangent to Z has the form $\lambda^A \pi^{A'}$, for some spinor λ^A, and is therefore null. We shall refer to such a set Z as a totally null 2-plane of primed type, abbreviated to TN2P. A complex conjugate twistor determines, via the complex conjugate version of equation (1.7), a 2-plane whose tangent vectors have the form $\eta^A \lambda^{A'}$, with η^A fixed and $\lambda^{A'}$ varying. Such a 2-plane will be referred to as a totally null 2-plane of unprimed type , abbreviated to $\overline{\text{TN2P}}$.

Since equation (1.7) is homogenous in (ω^A, $\pi_{A'}$), it is the projective twistor which determines the TN2P. Conversely, every TN2P is given by an equation of the form (1.7), and so a TN2P determines the pair $(\omega^A, \pi_{A'})$ up to proportionality, i.e. determines a projective twistor. In other words, we can regard

$$\mathbb{PT} := \{Z \mid Z \text{ is a TN2P in } \mathbb{C}M\}$$

as a subset of $\mathbb{CP}_3$. $\mathbb{PT}$ is the space of projective twistors in $\mathbb{C}$M.*

The space $\mathbb{T}$ of non-projective twistors in $\mathbb{C}$M can be pictured as follows*. As noted above, a TN2P Z determines a spinor $\pi_{A'}$ up to proportionality. Knowing a non-projective twistor corresponds to knowing $\pi_{A'}$ exactly (not just its direction). In other words, the space $\mathbb{T}$ may be thought of as the space of pairs (Z, $\pi_{A'}$), where $Z \in \mathbb{PT}$ and where the tangent vectors to Z have the form $\lambda^A \pi^{A'}$.

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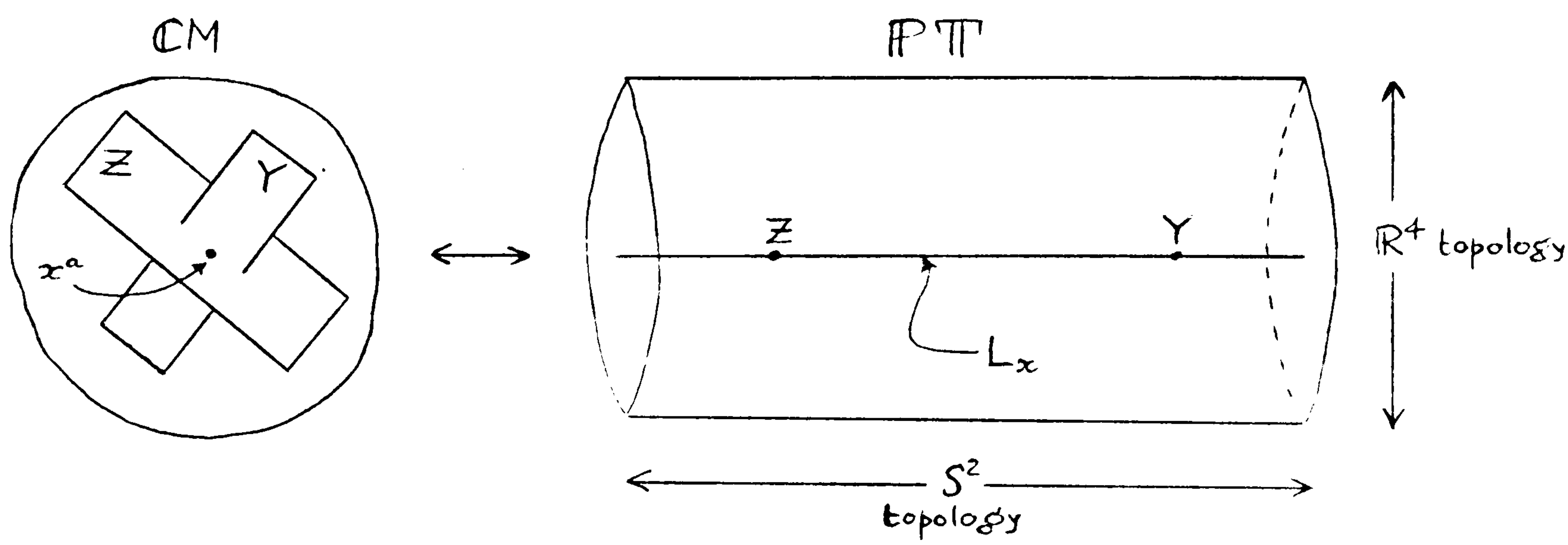
* We emphasize that $\mathbb{PT}$ and $\mathbb{T}$ refer not to the whole of $\mathbb{CP}_3$ and $\mathbb{C}^4-\{0\}$ (as is customary in the literature), but to those subsets of $\mathbb{CP}_3$ and $\mathbb{C}^4-\{0\}$ which correspond to the space-time region $\mathbb{C}$M under consideration.

A point x^a in $\mathbb{C}M$ can be represented by the collection of TN2P's passing through it. There is a sphere's worth of such TN2P's, and so x^a is represented in $\mathbb{P}\mathbb{T}$ by a compact holomorphic curve (topology S^2), denoted L_x (see figure 1.3).

Figure 1.3.



We now discuss twistor functions. If $f = f(Z^\alpha) = f(\omega^A, \pi_{A'})$ is a holomorphic twistor function (possibly with singularities), we can use equation (1.7) to define a function $F = F(x^a, \pi_{A'})$:

$$F(x^a, \pi_{A'}) \quad := f(i\, x^{AA'}\, \pi_{A'},\, \pi_{A'}). \qquad (1.3)$$

The function $F(x^a, \pi_{A'})$ is defined on the primed spin-bundle over $\mathbb{C}M$. From (1.8) it follows that

$$\nabla_{AA'}\, F(x^b, \pi_{B'}) = i\, \pi_{A'} \frac{\partial}{\partial \omega^A} f(i\, x^{BB'}\, \pi_{B'},\, \pi_{B'}), \qquad (1.9a)$$

$$\pi^{A'}\, \nabla_{AA'}\, F(x^b, \pi_{B'}) = 0. \qquad (1.9b)$$

To say that $f(Z^\alpha)$ is homogeneous of degree n in Z^α means the same as saying that $F(x^a, \pi_{A'})$ is homogeneous of degree n in $\pi_{A'}$.

One can use twistor functions to generate solutions of the zerorsst-mass free-field equations

$$\nabla^{AA'} \phi_{A'\ldots C'} = 0, \tag{1.10a}$$

$$\nabla^{AA'} \phi_{A\ldots C} = 0, \tag{1.10b}$$

where $\phi_{A'\ldots C'}$ and $\phi_{A\ldots C}$ are symmetric spinor fields on $\mathbb{C}M$. Let $f(Z^{\alpha})$ be holomorphic and homogeneous of degree n.

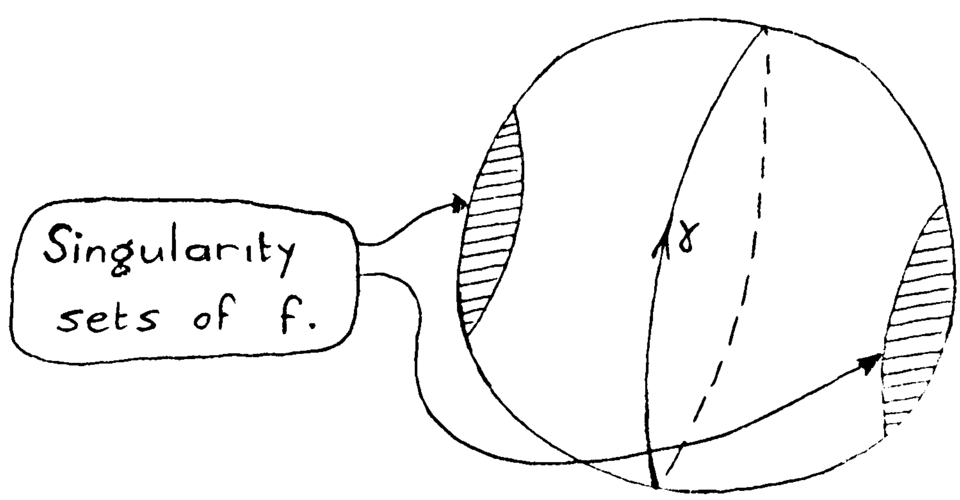
If $n \leqslant -2$, put $\phi_{A'\ldots C'}(x) = \frac{1}{2\pi i} \oint \underbrace{\pi_{A'}\ldots\pi_{C'}}_{-n-2} f(ix^{DD'}\pi_{D'}, \pi_{D'}) \; \Delta\pi.$ (1.11a)

If $n > -2$, put $\phi_{A\ldots C}(x) = \frac{1}{2\pi i} \oint \underbrace{\frac{\partial}{\partial \omega^{A}}\ldots\frac{\partial}{\partial \omega^{C}}}_{n-2} f(ix^{DD'}\pi_{D'}, \pi_{D'})\Delta\pi ,$ (1.11b)

where $\Delta\pi := \pi_{A'}\, d\pi^{A'}$.

Then (1.11) are solutions of (1.10). The integrals are performed on the Riemann sphere of projective π-spinors, over a contour γ which is a circle separating the singularities of f (see figure 1.4).

Figure 1.4.



If h is a twistor function which is holomorphic in a region on which γ is homologous to zero (for example, the region containing γ and the hemisphere "to the left of γ"), then the integral of h vanishes by Cauchy's theorem. So we can add such a function h to f without changing the result of the integrals (1.11), and we are really only interested in f modulo such functions h. As we shall see in section 2.3, this kind of "gauge freedom" can be neatly handled by regarding f as an element of a certain sheaf cohomology group.

Finally we remark on the concept of positive frequency of fields on Minkowski space-time. Let $\mathbb{C}M^+$ denote the subset of complex Minkowski space-time whose points have the form $x^a - i\, y^a$, where x^a and y^a are real and y^a is timelike future-pointing. A field on Minkowski space-time is positive frequency if it can be extended holomorphically to the whole of $\mathbb{C}M^+$ [19], [20]. The region of twistor space corresponding to $\mathbb{C}M^+$ is exactly $\mathbb{T}^+$ (cf. figure 1.2) [24], [20]. So the fields defined in equations (1.11) will be positive frequency provided the function f has a singularity structure on $\mathbb{T}^+$ which enables one to define its contour integral in a satisfactory way.

§2.2. Vector Bundles, Line Bundles and Twistor Functions.

In this section we define the concept of a holomorphic vector bundle (and, in particular, of a holomorphic line bundle) and show how homogeneous twistor functions may be regarded as cross-sections of certain line bundles over $\mathbb{PT}$. For further details on the subject of vector bundles, the reader is referred to [12], [10].

Let X be a complex manifold; assume it to be Hausdorff and paracompact. A holomorphic vector bundle of rank N over X is a complex manifold L and a holomorphic map p from L onto X such that, for a sufficiently fine locally finite open covering $\{U_j\}$ of X

(i) there exist biholomorphic maps

$g_j : p^{-1}(U_j) \longrightarrow U_j \times \mathbb{C}^N$ such that

$$\begin{array}{ccc} p^{-1}(U_j) & \xrightarrow{\quad g_j \quad} & U_j \times \mathbb{C}^N \\ & {\scriptstyle p}\searrow \qquad \swarrow{\scriptstyle p_j} & \\ & U_j & \end{array}$$

commutes, where $p_j(z, \zeta) = z$, $z \in U_j$, $\zeta \in \mathbb{C}^N$;

(ii) if $(z, \zeta_j) \in U_j \times \mathbb{C}^N$ and $(z, \zeta_k) \in U_k \times \mathbb{C}^N$, then $g_j \circ g_k^{-1}(z, \zeta_k) = (z, g_{jk}(z)\, \zeta_k)$, where $g_{jk}(z)$ is a non-singular N x N matrix whose elements are holomorphic functions on $U_j \cap U_k$ (regarding ζ_k as a 1 x N column vector : $\zeta_k = \begin{bmatrix} \zeta_{k1} \\ \vdots \\ \zeta_{kN} \end{bmatrix}$).

The condition (ii) means that if $(z, \zeta_j) \in U_j \times \mathbb{C}^N$ and $(z, \zeta_k) \in U_k \times \mathbb{C}^N$ represent the <u>same</u> point of L, then ζ_j and ζ_k are related by the linear transformation

$$\zeta_j = g_{jk}(z)\,\zeta_k \tag{2.1}$$

on the overlap region $(U_j \cap U_k) \times \mathbb{C}^N$. In matrix form, equation (2.1) is

$$\begin{bmatrix} \zeta_{j1} \\ \vdots \\ \zeta_{jN} \end{bmatrix} = \begin{bmatrix} g^1_{jk1} & \cdots & g^N_{jk1} \\ \vdots & & \vdots \\ g^1_{jkN} & \cdots & g^N_{jkN} \end{bmatrix} \begin{bmatrix} \zeta_{k1} \\ \vdots \\ \zeta_{kN} \end{bmatrix} .$$

The matrix-valued functions $g_{jk}(z)$ determine the vector bundle, since they describe how the pieces $U_j \times \mathbb{C}^N$ are "patched together". If z is a point of X, then $p^{-1}(z)$ is called the <u>fibre</u> over z : it is isomorphic to $\mathbb{C}^N$, as an N-dimensional complex vector space.

Two vector bundles L and L$'$ over X are said to be <u>equivalent</u> if there exists a biholomorphic map $H : L \longrightarrow L'$ such that

(1) the diagram $\begin{array}{ccc} L & \xrightarrow{H} & L' \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p'} & \\ & X & \end{array}$ commutes;

(2) H is linear on each fibre, i.e. there exist holomorphic matrix-valued functions $h_j(z)$ on U_j such that $g'_j \circ H \circ g_j^{-1} = h_j$ on U_j.

The condition (2) means that if $z \in U_j$, then the fibre $p^{-1}(z)$ (with coordinates ζ_j) is mapped to the fibre $(p')^{-1}(z)$ (with coordinates ζ'_j) by the linear transformation

$$\zeta'_j = h_j(z)\,\zeta_j. \tag{2.2}$$

Let us consider now the special case N = 1, i.e. vector bundles of rank one. Such vector bundles are called <u>line bundles</u>. In this case, the $g_{jk}(z)$ are non-singular 1 x 1 matrices, i.e. <u>nowhere zero holomorphic functions</u> on $U_j \cap U_k$. Given a line bundle L over X, determined by the transition functions $\{g_{jk}\}$, we define the <u>inverse</u> of L (denoted L^{-1}) to be the line bundle determined by the transition functions $\{g_{jk}^{-1}\}$. If L' (determined by $\{g'_{jk}\}$) is another line bundle over X, the <u>tensor product</u> of L and L' (denoted $L \otimes L'$) is defined to be the line bundle determined by the transition functions $\{g_{jk}\, g'_{jk}\}$. Define $L^2 := L \otimes L$, $L^3 := L \otimes L \otimes L$, etc. The line bundle determined by $g_{jk} = 1$ is called the <u>trivial bundle</u>; the trivial bundle over X is just the product $X \times \mathbb{C}$.

If $L_1, L_2, \ldots L_N$ are N line bundles over X (with transition functions $g_{(1)jk}, g_{(2)jk}, \ldots g_{(N)jk}$ respectively), then the <u>direct sum</u> of $L_1, L_2, \ldots, L_N$ (denoted $L_1 \oplus L_2 \oplus \ldots \oplus L_N$) is the vector bundle of rank N determined by the transition matrix

$$g_{jk} = \begin{bmatrix} g_{(1)jk} & & 0 \\ & g_{(2)jk} & \\ & & \ddots \\ 0 & & g_{(N)jk} \end{bmatrix} .$$

A holomorphic <u>cross-section</u> of a line bundle L over X is a holomorphic map $\sigma : X \to L$ such that $p \circ \sigma = id_X$. If $\{U_j\}$ is a sufficiently fine covering of X, then σ can be represented locally as a collection of functions $S_j : U_j \to \mathbb{C}$, where S_j is holomorphic on U_j, and where (cf.(2.1))

$$S_j(z) = g_{jk}(z) \quad S_k(z) \quad \text{on} \quad U_j \cap U_k . \qquad (2.3)$$

We now apply these definitions to the case when X is the projective twistor space $\mathbb{PT}$ corresponding to some convex subregion of $\mathbb{C}M^+$ (i.e. $\mathbb{PT}\subseteq\mathbb{PT}^+$). As coordinates on $\mathbb{PT}$, use the homogeneous coordinates Z^α. Let A and B be two disjoint planes in $\mathbb{PT}$, given by the homogeneous equations $A_\alpha Z^\alpha = 0$ and $B_\alpha Z^\alpha = 0$ respectively. Cover $\mathbb{PT}$ by the two open patches

$$\mathbb{P}U := \mathbb{PT} - A$$

and

$$\mathbb{P}\hat{U} := \mathbb{PT} - B .$$

The planes A and B are chosen so that $\mathbb{P}U$ and $\mathbb{P}\hat{U}$ are topologically trivial (the picture is as depicted in figure 2.1).

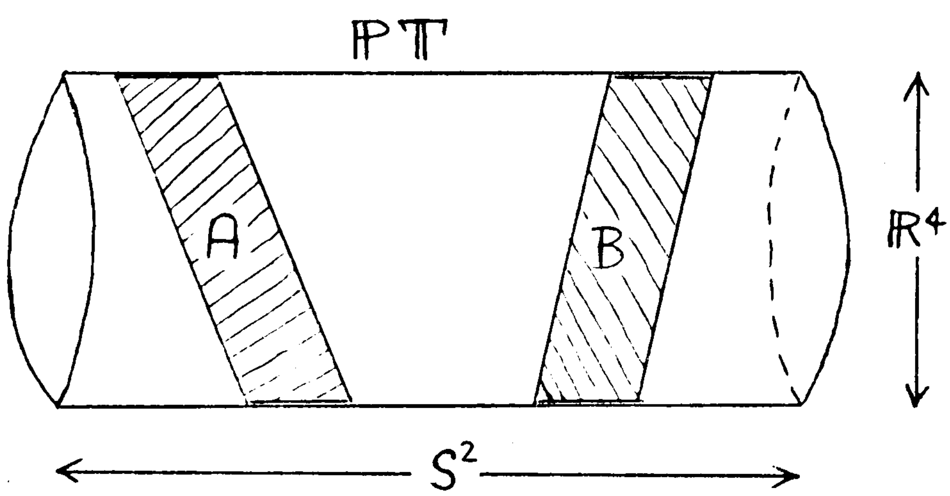


Figure 2.1

To build a line bundle over $\mathbb{PT}$, we take the two pieces $\mathbb{P}U \times \mathbb{C}$ (with coordinates (Z^α, ζ)) and $\mathbb{P}\hat{U} \times \mathbb{C}$ (with coordinates $(Z^\alpha,\hat{\zeta})$), and patch them together by

$$\hat{\zeta} = g(Z^\alpha)\, \zeta \tag{2.4}$$

(cf.(2.1)), where $g(Z^\alpha)$ is homogeneous of degree 0, holomorphic and nowhere zero on $\mathbb{P}U \cap \mathbb{P}\hat{U}$.

For example, we may take $g(Z^\alpha) = \dfrac{B_\alpha Z^\alpha}{A_\beta Z^\beta}$. (2.5)

In this case, we can define new coordinates X^α on $\mathbb{P}U\times\mathbb{C}$ and $\hat{X}^\alpha$ on $\mathbb{P}\hat{U}\times\mathbb{C}$ by

$$\left.\begin{aligned} X^\alpha &= \zeta\, \frac{Z^\alpha}{A_\beta Z^\beta} , \\ \hat{X}^\alpha &= \hat{\zeta}\, \frac{Z^\alpha}{B_\beta Z^\beta} . \end{aligned}\right\}$$

Then $\hat{X}^{\alpha} = \hat{\zeta} \dfrac{Z^{\alpha}}{B_{\beta} Z^{\beta}} = \dfrac{B_{\gamma} Z^{\gamma}}{A_{\delta} Z^{\delta}} \cdot \zeta \cdot \dfrac{Z^{\alpha}}{B_{\beta} Z^{\beta}} = X^{\alpha}$,

using (2.4) and (2.5), so that the line bundle is just the non-projective twistor space $\mathbb{T} \subset \mathbb{C}^4 - \{0\}$. Denote this line bundle by L(-1). By inverting L(-1), taking its tensor product with itself, etc., we obtain other bundles: define $L(n) := [L(-1)]^{-n}$. So L(n) is determined by the transition function

$$g(Z^{\alpha}) = \left(\frac{B_{\alpha} Z^{\alpha}}{A_{\beta} Z^{\beta}}\right)^{-n} .$$

Remarks: (1) The space $\mathbb{T}$, as a bundle over $\mathbb{PT}$, is really a principal fibre bundle with group $\mathbb{C}-\{0\}$ [11], rather than a line bundle. The point is that each fibre in $\mathbb{T}$ is $\mathbb{C}-\{0\}$ and not the vector space $\mathbb{C}$. But one can always add in the zero section if desired, so this distinction will be ignored.

Finally we consider what it means to have a cross-section of L(n). By definition, a cross-section corresponds to a pair of functions $S(Z^{\alpha})$ on $\mathbb{P}U$ and $\hat{S}(Z^{\alpha})$ on $\mathbb{P}\hat{U}$ such that

$$\hat{S}(Z^{\alpha}) = \left(\frac{B_{\beta} Z^{\beta}}{A_{\gamma} Z^{\gamma}}\right)^{-n} S(Z^{\alpha}) \tag{2.6}$$

on $\mathbb{P}U \cap \mathbb{P}\hat{U}$ (cf.(2.3)). Define two functions f and $\hat{f}$ by

$$\left.\begin{aligned} f(Z^{\alpha}) &= (A_{\beta} Z^{\beta})^{n} \; S(Z^{\alpha}), \\ \hat{f}(Z^{\alpha}) &= (B_{\beta} Z^{\beta})^{n} \; \hat{S}(Z^{\alpha}). \end{aligned}\right\}$$

Then

$$\begin{aligned} \hat{f}(Z^{\alpha}) &= (B_{\beta} Z^{\beta})^{n} \left(\frac{B_{\gamma} Z^{\gamma}}{A_{\delta} Z^{\delta}}\right)^{-n} S(Z^{\alpha}) \\ &= f(Z^{\alpha}) \end{aligned}$$

on the overlap region. What we have shown, therefore, is that cross-sections of L(n) correspond to functions on $\mathbb{T}$, homogeneous of degree n.

The same kind of result holds if we restrict our attention to a single line in $\mathbb{PT}$, i.e. to a Riemann sphere ($\mathbb{CP}_1$). We again obtain line bundles L(n), labelled by the integer n, and such that sections of L(n) correspond to functions of two variables homogeneous of degree n (the two variables being homogeneous coordinates for the sphere).

§2.3. Sheaf Cohomology.

This section is devoted to a brief discussion of sheaf cohomology theory and to some of its applications.

A rigorous definition of what a sheaf is may be found in [6] or [12]. For our purposes, it is sufficient to say what the local cross-sections of various sheaves are. Let X be a complex manifold and U an open subset of X. Let L be a line bundle over X. Then we define

$\Gamma(U,\mathcal{O})$:= Group of holomorphic functions on U;

$\Gamma(U,\mathcal{O}^*)$:= Multiplicative group of nowhere zero holomorphic functions on U;

$\Gamma(U,\mathcal{O}(L))$:= Group of holomorphic sections of L over U.

We now move on to discuss sheaf cohomology. Let X be a complex manifold, $\mathcal{U} = \{U_j\}_{j\in J}$ a covering of X, and $\mathcal{S}$ a sheaf over X. The group of p-cochains (denoted $\mathcal{C}^p(\mathcal{U},\mathcal{S})$) is defined as follows:

a o-cochain $\mathfrak{F}^0$ is a collection $\{f_j\}_{j\in J}$, where $f_j \in \Gamma(U_j,\mathcal{S})$;

a 1-cochain $\mathfrak{F}^1$ is a collection $\{f_{jk}\}_{j,k\in J}$, where $f_{jk} = -f_{kj}$ and $f_{jk} \in \Gamma(U_j \cap U_k, \mathcal{S})$;

a 2-cochain $\mathfrak{F}^2$ is a collection $\{f_{jkl}\}_{j,k,l\in J}$, where $f_{jkl} = f_{[jkl]}$ and $f_{jkl} \in \Gamma(U_j \cap U_k \cap U_l, \mathcal{S})$;

etc.

The coboundary operator $\delta : \mathcal{C}^p(\mathcal{U},\mathcal{S}) \to \mathcal{C}^{p+1}(\mathcal{U},\mathcal{S})$ is defined as follows. Suppose $\mathfrak{F}^p = \{f_{jk\ldots m}\} \in \mathcal{C}^p(\mathcal{U},\mathcal{S})$ and $\mathfrak{g}^{p+1} = \{g_{jk\ldots n}\} \in \mathcal{C}^{p+1}(\mathcal{U},\mathcal{S})$.

Then

$$\mathfrak{g}^1 = \delta\mathfrak{F}^0 \iff g_{jk} = f_k - f_j ;$$

$$\mathfrak{g}^2 = \delta\mathfrak{F}^1 \iff g_{jkl} = f_{kl} - f_{jl} + f_{jk} ;$$

$$\mathfrak{g}^3 = \delta\mathfrak{F}^2 \iff g_{jklm} = f_{klm} - f_{jlm} + f_{jkm} - f_{jkl} ;$$

etc.

A p-cochain $\mathcal{F}^p$ is said to be a <u>p-cocycle</u> if $\delta\mathcal{F}^p = 0$, and $\mathcal{F}^p$ is said to be a <u>p-coboundary</u> if there exists a $\mathcal{G}^{p-1} \in \mathcal{C}^{p-1}(\mathcal{U},\mathcal{S})$ such that $\delta\mathcal{G}^{p-1} = \mathcal{F}^p$. Denote the group of p-cocycles by $\mathcal{Z}^p(\mathcal{U},\mathcal{S})$ and the group of p-coboundaries by $\mathcal{B}^p(\mathcal{U},\mathcal{S})$.

Define $H^p(\mathcal{U},\mathcal{S}) := \mathcal{Z}^p(\mathcal{U},\mathcal{S})/\mathcal{B}^p(\mathcal{U},\mathcal{S})$ (it's not difficult to check that $\mathcal{B}^p$ is a subgroup of $\mathcal{Z}^p$, so the quotient group H^p is well-defined). The <u>p-th cohomology group of X with coefficients in $\mathcal{S}$</u>, denoted $H^p(X,\mathcal{S})$ is the direct limit of the groups $H^p(\mathcal{U},\mathcal{S})$ as the coverings $\mathcal{U} = \{U_j\}$ become finer. In practice one can generally find a "sufficiently fine" covering $\mathcal{U}$ such that $H^p(X,\mathcal{S}) = H^p(\mathcal{U},\mathcal{S})$.

Our first application of sheaf cohomology in twistor theory has to do with the twistor integrals discussed at the end of §2.1. The type of twistor function $f(Z^\alpha)$ considered there was homogeneous of degree n (and therefore a cross-section of the line bundle L(n) over $\mathbb{PT}$: cf. §2.2) and had singularities on two disconnected sets (say D and $\hat{D}$) in $\mathbb{PT}$. Define a cover $\mathcal{U} = \{\mathbb{P}U, \mathbb{P}\hat{U}\}$ of $\mathbb{PT}$ by

$$\mathbb{P}U := \mathbb{PT} - D,$$
$$\mathbb{P}\hat{U} := \mathbb{PT} - \hat{D}.$$

For the sake of brevity, denote the sheaf $\mathcal{O}(L(n))$ by $\mathcal{O}(n)$. The function $f(Z^\alpha)$ is a holomorphic cross-section of L(n) over $\mathbb{P}U \cap \mathbb{P}\hat{U}$, and so f represents a 1-cocycle in $\mathcal{Z}^1(\mathcal{U}, \mathcal{O}(n))$. [Strictly speaking, the cocycle is the set {f on $\mathbb{P}U \cap \mathbb{P}\hat{U}$, -f on $\mathbb{P}\hat{U} \cap \mathbb{P}U$, 0 on $\mathbb{P}U \cap \mathbb{P}U$, 0 on $\mathbb{P}\hat{U} \cap \mathbb{P}\hat{U}$ }.]

A 1-coboundary in $\mathcal{B}^1(\mathcal{U}, \mathcal{O}(n))$ has the form $h-\hat{h}$, where h and $\hat{h}$ are holomorphic cross-sections of L(n) over $\mathbb{P}U$ and $\mathbb{P}\hat{U}$ respectively. As was remarked in §2.1, the integrals of h and $\hat{h}$ vanish, and we are only interested in f modulo such functions.

In other words, we are interested in the quotient group $H^1(\mathcal{U}, \mathcal{O}(n)) = \mathcal{Z}^1(\mathcal{U}, \mathcal{O}(n))/\mathcal{B}^1(\mathcal{U}, \mathcal{O}(n))$. The function f may therefore be regarded as a representative cocycle of an element of $H^1(\mathcal{U}, \mathcal{O}(n))$, or, more generally, of

$H^1(\mathbb{PT}, \mathcal{O}(n))$. For further details, see [21], [22].

The second application of sheaf cohomology is to line bundles:

Theorem. The space of equivalence classes of equivalent line bundles over a complex manifold X, forms a group which is isomorphic to $H^1(X, \mathcal{O}^*)$.

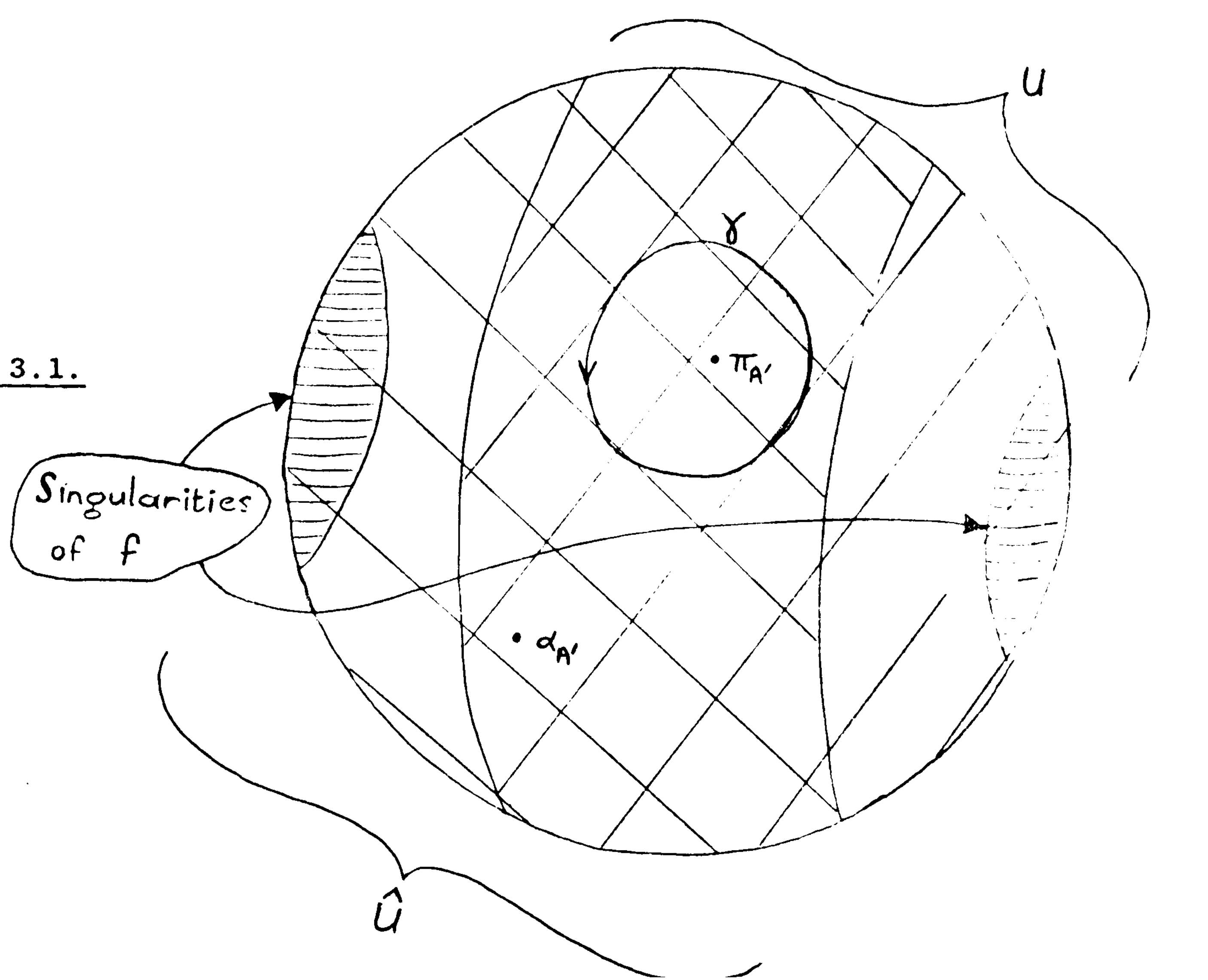
A proof may be found in [12], p.63. The group operation on the space of line bundles is tensor multiplication, and the group identity element is the trivial bundle.

Finally, we discuss a "splitting formula", originally suggested by G. A. J. Sparling [26].

Suppose that the Riemann sphere $\mathbb{CP}_1$ is covered by $\mathcal{U} = \{U, \hat{U}\}$, where U and $\hat{U}$ are two "hemispherical" patches (see figure 3.1). As homogeneous coordinates on $\mathbb{CP}_1$, use (the two components of) a primed spinor $\pi_{A'}$.

We shall show that $H^1(\mathcal{U}, \mathcal{O}(n)) = 0$ for $n \geq -1$, or in other words that every 1-cocycle is a 1-coboundary. In fact, $H^1(\mathbb{CP}_1, \mathcal{O}(n)) = H^1(\mathcal{U}, \mathcal{O}(n))$ ([12], p.35), so it will follow that $H^1(\mathbb{CP}_1, \mathcal{O}(n)) = 0$ for $n \geq -1$.

Figure 3.1.



A representative cocycle in $H^1(\mathcal{U}, \mathcal{O}(n))$ is essentially given by a function $f(\pi_{A'})$, homogeneous of degree n and holomorphic on $U \cap \hat{U}$. Let $\pi_{A'}$ and $\alpha_{A'}$ represent two distinct points in $U \cap \hat{U}$ and let γ be a closed contour surrounding $\pi_{A'}$ such that

$$f(\pi_{D'}) = \oint_\gamma \Omega , \tag{3.1a}$$

where Ω is the 1- form

$$\Omega := \frac{1}{2\pi i} \frac{(\alpha^{A'}\pi_{A'})^{n+1}}{(\alpha^{B'}\rho_{B'})^{n+1}} \frac{f(\rho_{D'})}{(\rho^{C'}\pi_{C'})} \rho_{E'} \, d\rho^{E'} \tag{3.1b}$$

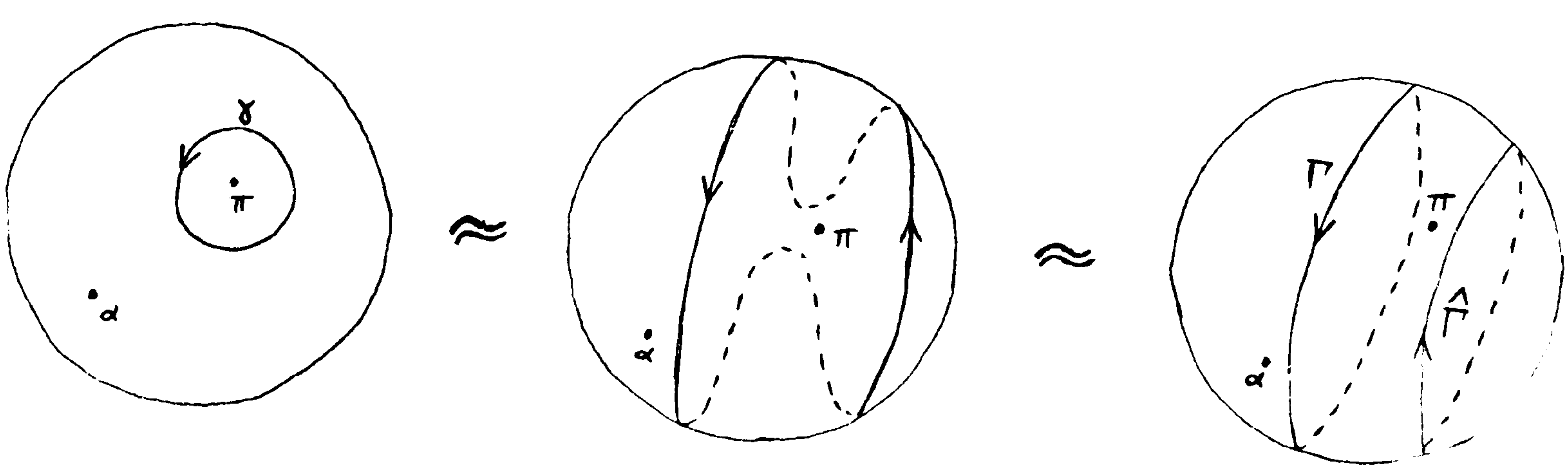
(This is essentially Cauchy's integral formula, as can be seen by putting $\rho_{A'} = \pi_{A'} + \zeta \alpha_{A'}$, where ζ is a complex variable; the integral in equation (3.1) then becomes

$$\frac{1}{\partial \pi i} \oint_\gamma \zeta^{-1} f(\pi_{A'} + \zeta \alpha_{A'}) \, d\zeta ,$$

where γ surrounds $\zeta = 0$.)

The next step is to express the contour γ as the sum of two closed contours Γ and $\hat{\Gamma}$ lying in $U \cap \hat{U}$: see figure 3.2.

Figure 3.2.



So now if we define

$$g(\pi_{D'}) := \oint_\Gamma \Omega , \tag{3.2}$$

$$\hat{g}(\pi_{D'}) := - \oint_{\hat{\Gamma}} \Omega , \tag{3.3}$$

then equation (3.1) can be re-expressed as

$$f(\pi_{D'}) = g(\pi_{D'}) - \hat{g}(\pi_{D'}). \tag{3.4}$$

<u>Claim</u>: g is holomorphic on U. For the contour integral in (3.2) is well-defined for all $\pi_{A'}$ in U. It is only when $\pi_{A'}$ enters $\hat{U}$ that the contour Γ is in danger of being "pinched" between $\pi_{A'}$ and the singularities of f. Similarly, $\hat{g}$ is holomorphic on $\hat{U}$.

So we have succeeded in expressing the 1-cocycle f as a coboundary $g - \hat{g}$. For n> -1, these functions g and $\hat{g}$ are not unique (for one thing, they depend on the choice of the spinor $\alpha_{A'}$). The freedom in g and $\hat{g}$ (preserving their domains of holomorphicity and the difference $g - \hat{g}$) is

$$\left.\begin{aligned} g &\longmapsto g + h, \\ \hat{g} &\longmapsto \hat{g} + h, \end{aligned}\right\} \tag{3.5}$$

where $h(\pi_{A'})$ is holomorphic on $U \cup \hat{U}$ (i.e. on the whole Riemann sphere) and homogeneous of degree n. Thus h must have the form [6]

$$h(\pi_{A'}) = \begin{cases} 0, \text{ if } n = -1, & \\ K, \text{ if } n = 0 & (K \in \mathbb{C}), \\ K^{A' \ldots P'} \underbrace{\pi_{A'} \ldots \pi_{P'}}_{n}, \text{ if } n > 0 & (K^{A' \ldots P'} \text{ a constant symmetric } n\text{-spinor}). \end{cases}$$

§2.4. Hertz Potentials for Maxwell Fields.

Let M be a region of Minkowski space-time and let $F_{ab} = F_{[ab]}$ be a (possibly complex) 2-form on M representing an electromagnetic field. We have the spinor decomposition

$$F_{ab} = \phi_{AB}\ \varepsilon_{A'B'} + \tilde{\phi}_{A'B'}\ \varepsilon_{AB},$$

where ϕ_{AB} and $\tilde{\phi}_{A'B'}$ are symmetric (see, for example, [20]), and where F_{ab} real $\Leftrightarrow \tilde{\phi}_{A'B'} = \bar{\phi}_{A'B'}$. The field is said to be left-handed if $\tilde{\phi}_{A'B'} = 0$, and right-handed if $\phi_{AB} = 0$. The self-dual part of F_{ab} is exactly $\tilde{\phi}_{A'B'}\ \varepsilon_{AB}$, so

F_{ab} is self-dual $\Leftrightarrow F_{ab}$ is right-handed.

Maxwell's equations $\nabla^a F_{ab} = 0$, $\nabla_{[a} F_{bc]} = 0$ are equivalent to

$$\begin{cases} \nabla^{AA'}\ \phi_{AB} = 0, & (4.1a) \\ \nabla^{AA'}\ \tilde{\phi}_{A'B'} = 0\ . & (4.1b) \end{cases}$$

It is proved in [15] that if M is topologically trivial (i.e. homeomorphic to $\mathbb{R}^4$), if ϕ_{AB} satisfies (4.1a), and if $\alpha^{A'}$ is some constant spinor, then there exists a spinor field ψ_A such that

$$\nabla_{AA'}\ \psi^A = 0, \tag{4.2a}$$

$$\phi_{AB} = \alpha^{A'} \nabla_{AA'} \psi_B. \tag{4.2b}$$

Define a 1-form Φ_a by $\Phi_{AA'} := \psi_A\ \alpha_{A'}$. Then equations (4.2) can be rewritten as

$$\nabla_{AA'}\ \Phi^{A}{}_{B'} = 0, \tag{4.3a}$$

$$\nabla_{AA'}\ \Phi_{B}{}^{A'} = \phi_{AB}\ . \tag{4.3b}$$

Notice that the Lorenz gauge condition $\nabla_a \Phi^a = 0$ follows automatically from (4.3a).

The equations (4.3) are not conformally invariant, but the weaker equations

$$\nabla_{A(A'}\Phi^{A}_{B')} = 0, \tag{4.4a}$$

$$\phi_{AB} = \nabla_{A'(A}\Phi^{A'}_{B)} \tag{4.4b}$$

are conformally invariant, as may easily be checked by using the conformal transformation formulae in [24]. The essential difference between (4.3) and (4.4) is that we have dropped the Lorenz gauge condition. The equations (4.4) are in fact gauge-invariant, since if $\Phi_a \longmapsto \Phi_a + \nabla_a\lambda$ (with λ some scalar on M), then $\nabla_{A(A'}\Phi^{A}_{B')} \longmapsto \nabla_{A(A'}\Phi^{A}_{B')} + \nabla_{A(A'}\nabla^{A}_{B')}\lambda = \nabla_{A(A'}\Phi^{A}_{B')}$

(using equation (1.5)), and similarly

$$\nabla_{A'(A}\Phi^{A'}_{B)} \longmapsto \nabla_{A'(A}\Phi^{A'}_{B)}.$$

Theorem. If a 1-form Φ_a satisfies (4.4a), then the field ϕ_{AB}, defined by (4.4b), satisfies (4.1a).

Proof. (4.4a) $\Rightarrow 0 = 2\nabla^{A'}_{B}\left(\nabla_{A(A'}\Phi^{A}_{B')}\right)$

$$= \tfrac{1}{2}\varepsilon_{AB}\Box\Phi^{A}_{B'} + \nabla^{A'}_{B}\nabla_{AB'}\Phi^{A}_{A'}$$

$$= \tfrac{1}{2}\Box\Phi_{BB'} + \nabla^{A'}_{B}\nabla_{AB'}\Phi^{A}_{A'}, \tag{4.5}$$

where $\Box := \nabla_a\nabla^a$.

Now $2\nabla^{A}_{B'}\phi_{AB} = 2\nabla^{A}_{B'}\left(\nabla_{A'(A}\Phi^{A'}_{B)}\right)$

$$= \tfrac{1}{2}\varepsilon_{A'B'}\Box\Phi^{A'}_{B} + \nabla^{A}_{B'}\nabla_{A'B}\Phi^{A'}_{A}$$

$$= \tfrac{1}{2}\Box\Phi_{BB'} + \nabla^{A'}_{B}\nabla_{AB'}\Phi^{A}_{A'}$$

We say that Φ_a is <u>left-handed</u> if it satisfies equation (4.4a). If Φ_a is left-handed, then its exterior derivative

$$F_{ab} := 2\nabla_{[a}\Phi_{b]}$$
$$= \nabla_{C'(A}\Phi^{C'}_{B)}\,\varepsilon_{A'B'} + \nabla_{C(A'}\Phi^{C}_{B')}\,\varepsilon_{AB}$$
$$= \phi_{AB}\,\varepsilon_{A'B'}$$

is a left-handed solution of Maxwell's equations.
Similarly, if Ψ_a is right-handed, then $2\nabla_{[a}\Psi_{b]}$ is a right-handed solution of Maxwell's equations. Conversely, if F_{ab} satisfies Maxwell's equations in the topologically trivial region M, then there exist left-handed Φ_a and right-handed Ψ_a such that

$$F_{ab} = \nabla_a(\Phi_b + \Psi_b) - \nabla_b(\Phi_a + \Psi_b) = 2\nabla_{[a}\Phi_{b]} + 2\nabla_{[a}\Psi_{b]}.$$

All the above results are also valid in a region of <u>complexified</u> Minkowski space-time. In particular we could work in the region $\mathbb{C}M^+$ (cf. §2.1). A left-handed Maxwell field on $\mathbb{C}M^+$ represents the wave function (not necessarily normalizable) of a positive frequency photon, in an eigenstate of helicity with eigenvalue $-\hbar$ [20].

CHAPTER 3. ELECTROMAGNETIC FIELDS.

The object of this Chapter is to study two closely related problems:

(a) How may the "flat" twistor theory be modified so as to take account of an external electromagnetic field?

(b) How may the information relating to an electromagnetic field be coded into the complex structure of a (deformed) twistor space?

Throughout the Chapter the underlying space-time will be taken to be Minkowski. So the concept of a TN2P (which depends only on the conformal geometry of the space-time) and hence of a projective twistor, remains unchanged. One might expect, therefore, that the effect of an electromagnetic field will be to deform the non-projective twistor space, while preserving the projective twistor space. This turns out to be the case .

§3.1. Left-Handed Maxwell Fields and the Deformed Twistor Space.

In this section we define and give an explicit construction of the deformed twistor space $\mathcal{J}$ appropriate to a left-handed electromagnetic field.

Let $\mathbb{C}M$ be a convex and topologically trivial region of complex Minkowski space-time contained in the region $\mathbb{C}M^+$, and let $\mathbb{PT}$ be the corresponding projective twistor space. Let $F_{ab} = F_{[ab]}$ be a 2 - form on $\mathbb{C}M$ and let Φ_a be a 1 - form such that

$$F_{ab} = 2\nabla_{[a}\Phi_{b]} \qquad \text{(cf. §2.4).}$$

Recall from §2.1 that a non-projective twistor may be regarded as a pair $(Z, \pi_{A'})$, where Z is a TN2P whose tangent vectors all have the form $\lambda^A \pi^{A'}$. The spinor $\pi_{A'}$ may be viewed as a spinor field on Z, satisfying the (parallel propagation) equation

$$\lambda^A \pi^{A'} \nabla_{AA'} \pi_{B'} = 0 \qquad \text{for all } \lambda^A ,$$

$$\text{i.e.} \qquad \pi^{A'} \nabla_{AA'} \pi_{B'} = 0 \text{ on } Z. \tag{1.1}$$

A method of "minimally coupling" [5] the twistor to the electromagnetic field F_{ab} now suggests itself: replace (1.1) by

$$\pi^{A'}(\nabla_{AA'} - ie\,\Phi_{AA'})\,\pi_{B'} = 0, \tag{1.2}$$

where e is some complex number (at this stage arbitrary), which may be thought of as the "charge" of the twistor.

So let us define the deformed twistor space $\mathcal{T}$ as the space of pairs $(Z, \pi_{A'})$, where

(a) Z is a TN2P in $\mathbb{C}M$, with tangent vectors of the form $\lambda^A \pi^{A'}$;

(b) $\pi_{A'}$ is a spinor field on Z, satisfying (1.2).

We now find the condition under which equation (1.2) is integrable. If the field $\pi_{A'}$ has the value $\overset{o}{\pi}_{A'}$ at $\overset{o}{x}{}^a$ on Z, then it follows from (1.2) that the value of $\pi_{A'}$ at some other point x^a on Z is

$$\pi_{A'}(x^a) = \overset{o}{\pi}_{A'} \exp\{-i\,eF(x^b, \pi_{B'})\} , \tag{1.3}$$

$$\text{where } F(x^b, \pi_{B'}) := -\int_\Gamma \Phi_a(y^b)\, dy^a, \tag{1.4}$$

Γ being some contour from $\overset{o}{x}{}^a$ to x^a and lying in Z (see figure 1.1).

In order that equation (1.2) be integrable over Z, we require that the function F be independent of the choice of contour Γ.

So let Γ and Γ' be two contours in the TN2P $\mathcal{Z}$, from $\overset{o}{x}{}^a$ to x^a. We require that

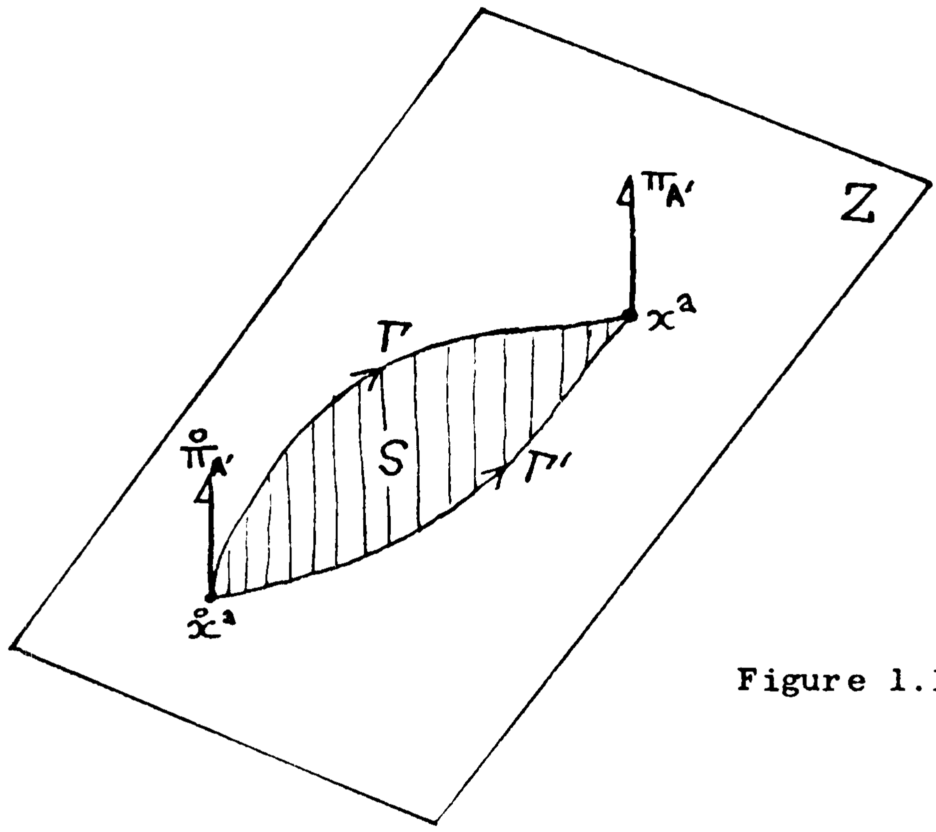


Figure 1.1.

$$\int_\Gamma \Phi_a dy^a = \int_{\Gamma'} \Phi_a dy^a$$

$$\Leftrightarrow \int_{\Gamma-\Gamma'} \Phi_a dy^a = 0$$

$$\Leftrightarrow \int_S d(\Phi_a dy^a) = 0 \quad \text{by Stokes' theorem,}$$

S being the 2-surface in $\mathcal{Z}$ bounded by the closed curve $\Gamma-\Gamma'$.

Now the points $y^a \in Z$ can be coordinatized by a spinor λ^A, according to

$$y^a = \overset{o}{x}{}^a + \lambda^A \pi^{A'}.$$

Hence $d(\Phi_a \, dy^a) = (\nabla_b \, \Phi_a) \, dy^b \wedge dy^a$

$$= (\nabla_{BB'} \, \Phi_{AA'}) \, \pi^{B'} \pi^{A'} \, d\lambda^B \wedge d\lambda^A.$$

But $d\lambda^B \wedge d\lambda^A = d\lambda^{[B} \wedge d\lambda^{A]} = \frac{1}{2}\varepsilon^{BA} \, d\lambda_C \wedge d\lambda^C$,

so $\int_S d(\Phi_a dy^a) = \frac{1}{2} \int_S \pi^{B'} \pi^{A'} (\nabla_{BB'} \Phi^B_{A'}) \, d\lambda_C \wedge d\lambda^C$,

which vanishes for arbitrary S iff

$$\nabla_{B(B'} \, \Phi^B_{A')} = 0 \qquad (1.5)$$

(cf. equation (2.4.4a)).

Thus $F(x^b,\pi_{B'})$ is contour-independent, provided the electromagnetic field is left-handed: we shall assume this to be the case in the rest of this section and in §3.2. The function $F(x^b,\pi_{B'})$ is homogeneous of degree zero in $\pi_{B'}$, since the integral in equation (1.4) is just the line integral of a 1-form and is therefore invariantly defined.

The remainder of this section is devoted to demonstrating, by explicit construction, that $\mathcal{J}$ has the structure of a holomorphic line bundle.*

Without loss of generality, we can assume that the origin O of the Minkowski coordinate system lies in the region $\mathbb{C}M$. Let $\{o^{A'},\iota^{A'}\}$ be the usual constant spin-frame (cf. §2.1). Define two twistors P^α and Q^α by

$$P^\alpha = (0,o_{A'}),$$

$$Q^\alpha = (0,\iota_{A'}).$$

Denote the corresponding TN2P's in $\mathbb{C}M$ by P and Q respectively (see fig. 1.2). Note that they both pass through the origin O.

Let $Z^\alpha = (\omega^A,\pi_{A'})$ represent some twistor with $\pi_{o'}\neq 0$, and denote its TN2P by Z. The point of intersection of P and Z is

$$p^{AA'} = \frac{1}{i\pi_{o'}}\,\omega^A o^{A'}$$

this being the unique solution of

$$\left.\begin{aligned} 0 &= i\,p^{AA'}o_{A'}, \\ \omega^A &= i\,p^{AA'}\pi_{A'}. \end{aligned}\right\}$$

* This construction was first described by G.A.J.Sparling [26].

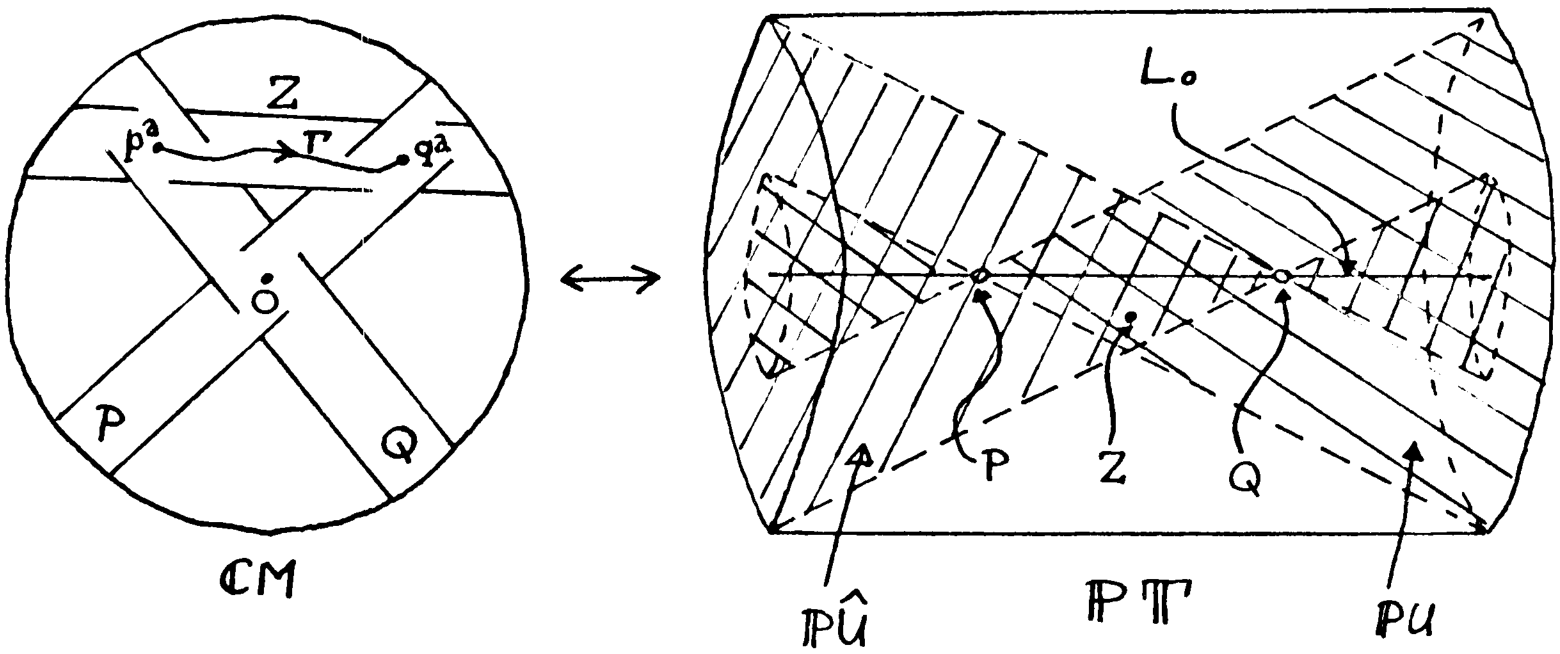


Figure 1.2.

Define a subset U of $\mathbb{T}$ by $U = \{Z^\alpha = (\omega^A, \pi_{A'}) \mid \pi_{0'} \neq 0$ and $p^a \in \mathbb{C}M\}$. Let $\mathbb{P}U$ be the corresponding projective space; in other words, the points of $\mathbb{P}U$ correspond to TN2P's which are not parallel to P, and whose intersection with P lies in $\mathbb{C}M$.

Similarly, define $\hat{U} \subset \mathcal{T}$ by $Z^\alpha \in \hat{U}$ iff $\pi_{1'} \neq 0$ and $q^a := \frac{1}{i\pi_{1'}} \omega^A \iota^{A'} \in \mathbb{C}M$.

It follows immediately from the definitions that (1) U and $\hat{U}$ are open subsets of $\mathbb{T}$;

(2) $\mathbb{P}U$ and $\mathbb{P}\hat{U}$ are open subsets of $\mathbb{P}\mathbb{T}$;

(3) the line L_0 is contained in $\mathbb{P}U \cup \mathbb{P}\hat{U}$ (see figure 1.2).

We are now in a position to construct the space $\mathcal{T}$. A point of $\mathcal{T}$ is a pair $(Z, \pi_{a'})$, where Z is a TN2P and $\pi_{A'}$ is a spinor field on Z. If $Z \in \mathbb{P}U$, we can label $(Z, \pi_{A'})$ by the coordinates $Z^\alpha = (\omega^A, \pi_{A'})$: the ratios between these coordinates determine Z, and $\pi_{A'}$ refers to the value of the spinor field $\pi_{A'}$ at the intersection point p^a.

Similarly, if $Z \in \mathbb{P}\hat{U}$, we can label $(Z, \pi_{A'})$ by the coordinates $\hat{Z}^\alpha = (\hat{\omega}^A, \hat{\pi}_{A'})$. In other words, the space $U \cup \hat{U}$ is covered by two patches, namely U(with coordinates Z^α) and $\hat{U}$ (with coordinates $\hat{Z}^\alpha$). On the overlap region $U \cap \hat{U}$, the coordinates are related by

$$\hat{Z}^{\alpha} = Z^{\alpha} \exp\{-ief(Z)\}, \tag{1.6}$$

where

$$f(Z) := -\int_{p}^{q} \Phi_a(x)\, dx^a \tag{1.7}$$

(cf. equations (1.3) and (1.4)).

Equation (1.6) tells us that $U \cup \hat{U}$ has the structure of a holomorphic line bundle over $\mathbb{P}U \cup \mathbb{P}\hat{U}$ (cf. §2.2). It is a deformation of the bundle L(-1) over $\mathbb{P}U \cup \mathbb{P}\hat{U}$ (recall that the bundle L(-1) was characterized by the patching relation $\hat{Z}^{\alpha} = Z^{\alpha}$). Our original claim that $\mathcal{J}$ would be shown to be a line bundle was a bit premature: in fact, it may not be possible to extend the line bundle structure from $U \cup \hat{U}$ to $\mathcal{J}$. The problem is that the field Φ_a might become singular "just outside" $\mathbb{C}M$. Since we are in any event only working locally in the space-time, we can get over this problem as follows. Let $\mathbb{C}M'$ be a subregion of $\mathbb{C}M$ and let $\mathbb{PT}'$ be the corresponding projective twistor space, such that $\mathbb{PT}' \subset \mathbb{P}U \cup \mathbb{P}\hat{U}$ (such a $\mathbb{C}M'$ exists, since $\mathbb{P}U \cup \mathbb{P}\hat{U}$ is a neighbourhood of the line L_0). Then the line bundle structure over $\mathbb{P}U \cup \mathbb{P}\hat{U}$ can be restricted to $\mathbb{PT}'$, and $\mathcal{J}'$ is a holomorphic line bundle over $\mathbb{PT}'$.

§3.2. Further Discussion.

This section is devoted to making a number of remarks about the construction of the previous section.

(a) We consider first the effect on f(Z) of a gauge transformation

$$\Phi_a \longmapsto \Phi_a + \nabla_a \lambda ,$$

where $\lambda = \lambda(x^a)$ is a scalar on $\mathbb{C}M$. Equation (1.7) gives

$$f(Z) \longmapsto -\int_{\Gamma} \{\Phi_a(x^b) + \nabla_a \lambda(x^b)\}\, dx^a$$

$$= f(Z) - \lambda(q^a) + \lambda(p^a).$$

Now $\lambda(q^a)$ represents a twistor function $\hat{g}(Z)$ which is holomorphic on $\mathbb{P}\hat{U}$; in fact, $\hat{g}$ is given by $\hat{g}(\omega^A, \pi_{A'}) := \lambda(\frac{1}{i\pi_{1'}} \omega^A \iota^{A'})$.

Similarly, $\lambda(p^a)$ represents a twistor function holomorphic on $\mathbb{P}U$. So $\lambda(q^a) - \lambda(p^a)$ represents a coboundary (cf. §2.3) and F(Z), considered as an element of $H^1(\mathbb{P}U \cup \mathbb{P}\hat{U}, \mathcal{O})$, is gauge-invariant.

(b) Next we wish to show that the relationship between f(Z) and the electromagnetic field F_{ab} is compatible with the contour integral formula (2.1.11b), i.e. that

$$F_{ab} = \phi_{AB}\, \varepsilon_{A'B'} = \varepsilon_{A'B'} \frac{1}{2\pi i} \oint \frac{\partial}{\partial \omega^A} \frac{\partial}{\partial \omega^B} f(ix^{DD'}\pi_{D'}, \pi_{D'})\, \Delta\pi$$

where $\Delta\pi := \pi_{A'}\, d\pi^{A'}$.

Let x^a be a point of $\mathbb{C}M$ corresponding to a line L_x in $\mathbb{P}U \cup \mathbb{P}\hat{U}$. The line has the structure of a Riemann sphere, and it is covered by the two patches

$$W_x := \mathbb{P}U \cap L_x \quad \text{and} \quad \hat{W}_x := \mathbb{P}\hat{U} \cap L_x .$$

Restrict the function f(Z) to L_x by putting

$$F(x^a; \pi_{A'}) := f(ix^{AA'}\pi_{A'}, \pi_{A'})$$

(cf. (2.1.8)). For each fixed value of x^a, $F(x^a;\pi_{A'})$ is a holomorphic function on $W_x \cap \hat{W}_x$, homogeneous of degree zero. Applying the "splitting" technique of §2.3, we get

$$F(x^a;\pi_{A'}) = g(x^a;\pi_{A'}) - \hat{g}(x^a;\pi_{A'}), \tag{2.1}$$

$$\text{where } g(x^a;\pi_{A'}) := \frac{1}{2\pi i}\oint \frac{(\alpha^{A'}\pi_{A'})\ F(x^d;\rho_{D'})}{(\alpha^{B'}\rho_{B'})\ (\rho^{C'}\pi_{C'})}\ \Delta\rho \tag{2.2}$$

is holomorphic on W_x; and similarly for $\hat{g}$.

But there is a more direct way of splitting F. This involves using the definition (1.7), which can be re-written

$$F(x^b;\pi_{A'}) = -\int_{p^b}^{q^b} \Phi_a(y^b)\ dy^a .$$

Put

$$\left.\begin{aligned} g'(x^b;\pi_{A'}) &:= -\int_{p^b}^{x^b} \Phi_a(y^b)\ dy^a \\ \text{and}\quad \hat{g}'(x^b;\pi_{A'}) &:= -\int_{q^b}^{x^b} \Phi_a(y^b) dy^a . \end{aligned}\right\} \tag{2.3}$$

$$\text{Then clearly } F(x^b;\pi_{A'}) = g'(x^b;\pi_{A'}) - \hat{g}'(x^b;\pi_{A'}) . \tag{2.4}$$

<u>Claim</u>: g' is holomorphic on W_X. For the line integral in (2.3) is well-defined provided the point p^a lies in $\mathbb{C}M$, i.e. provided $\pi_{A'}$ lies in W_X.

Comparing (2.1) and (2.4), and using eqn.(2.3.5), we must have

$$g'(x^b;\ \pi_{A'}) = g\ (x^b;\pi_{A'}) + h\ (x^b),$$

for some function $h(x^b)$. In other words,

$$\int_{p^b}^{x^b} \Phi_a(y^b)dy^a = -\frac{1}{2\pi i}\oint \frac{(\alpha^{A'}\pi_{A'})\;F(x^d;\rho_{D'})}{(\alpha^{B'}\rho_{B'})\;(\rho^{C'}\pi_{C'})}\Delta\rho + h(x^b). \tag{2.5}$$

Operating on (2.5) with $\pi^{A'}\nabla_{AA'}$, and using equation (2.1.9a), yields

$$\Phi_{AA'}(x^b)\,\pi^{A'} = \frac{1}{2\pi i}\oint\frac{(\alpha^{A'}\pi_{A'})}{(\alpha^{B'}\rho_{B'})}\;\frac{\partial}{\partial\omega^A}\;f(ix^{DD'}\rho_{D'},\rho_{D'})\,\Delta\rho$$

$$+\pi^{A'}\nabla_{AA'}\;h\,(x^b). \tag{2.6}$$

Thus $\phi_{AB}(x) := \varepsilon^{B'A'}\,\nabla_{B'(B}\Phi_{A)A'}$ (cf. equation (2.4.4.b))

$$= \left(\pi_{C'}\eta^{C'}\right)^{-1}\left(\pi^{B'}\eta^{A'} - \pi^{A'}\eta^{B'}\right)\nabla_{B'(B}\Phi_{A)A'}$$

$$= -\left(\pi_{C'}\eta^{C'}\right)^{-1}\frac{1}{2\pi i}\oint\frac{(\alpha^{A'}\eta_{A'})(\pi^{B'}\rho_{B'})-(\alpha^{A'}\pi_{A'})(\eta^{B'}\rho_{B'})}{(\alpha^{D'}\rho_{D'})}$$

$$\cdot\frac{\partial}{\partial\omega^B}\frac{\partial}{\partial\omega^A}\;f(i\,x^{DD'}\rho_{D'},\rho_{D'})\Delta\rho\;, \tag{2.7}$$

using equations (2.1.9a) and (2.1.5).

But $(\alpha^{A'}\eta_{A'})(\pi^{B'}\rho_{B'}) - (\alpha^{A'}\pi_{A'})(\eta^{B'}\rho_{B'})$

$$= -(\pi_{C'}\rho^{C'})\,\alpha_{A'}\,\rho_{B'}\,\varepsilon^{B'A'}$$

$$= -(\pi_{C'}\rho^{C'})\,(\alpha^{A'}\rho_{A'})\;. \tag{2.8}$$

Substituting (2.8) in (2.7) and then replacing $\rho_{A'}$ by $\pi_{A'}$ gives

$$\phi_{AB}(x^d) = \frac{1}{2\pi i}\oint \frac{\partial}{\partial\omega^A}\ \frac{\partial}{\partial\omega^B}\ f\,(ix^{DD'}\ \pi_{D'},\ \pi_{D'})\ \Delta\pi$$

as required.

(c) The function $f(Z)$ depends on the choice of spin-frame $\{o^{A'}, \iota^{A'}\}$, since $o^{A'}$ and $\iota^{A'}$ appear in the definition of the two twistors P^α and Q^α. A change in, say, $o^{A'}$ will change the endpoint p^a in the line integral (1.7), and hence f will change by having added into it a function which is holomorphic all over $\mathbb{P}U$. It follows that $f(Z)$, regarded as an element of $H^1(\mathbb{P}U \cup \mathbb{P}\hat{U}, \mathcal{O})$, is <u>independent</u> of the choice of spin-frame.

(d) If the region of Minkowski space-time under consideration is the region $\mathbb{C}M^+$, then the deformed twistor space $\mathcal{J}$ will be a deformation of $\mathbb{T}^+$ (cf.§2.1) and will correspond to the wave function of a positive frequency, negative helicity photon (cf. §2.4.).

(e) Let ϕ_{AB} and ψ_{AB} be two left-handed Maxwell fields, and $f(Z)$ and $g(Z)$ the corresponding twistor functions. The linearity of the contour integral formula (2.1.11b) shows that $f(Z) + g(Z)$ corresponds to $\phi_{AB} + \psi_{AB}$; the appropriate twistor space "patching" is

$$\hat{Z}^\alpha = Z^\alpha \exp\{-ie(f(Z) + g(Z))\}$$

$$= Z^\alpha \exp\{-ief(Z)\}\ .\ \exp\{-ieg(Z)\}.$$

But <u>multiplication</u> of transition functions means taking the <u>tensor product</u> of the corresponding line bundles. In other words, adding left-handed Maxwell fields corresponds to taking the tensor product of the associated line bundles.

(f) We have seen how to describe a left-handed electromagnetic field in terms of a deformation of the "flat" twistor line bundle $\mathbb{T}$. To deal with right-handed fields, one can either

(i) deform the complex conjugate twistor space $\widetilde{\mathbb{T}}$, using the complex conjugate version of the construction of §3.1;or

(ii) deform the twistor space $\mathbb{T}$ in a different way: Penrose [23] has recently described a construction which seems to have the desired features.

The general solution of Maxwell's equations can (at least locally) be written as the sum of a left-handed and a right-handed field (cf. §2.4). So the general Maxwell field (and in particular a real Maxwell field on real Minkowski space-time) may be described by

(i) a deformation of the product space $\mathbb{T} \times \widetilde{\mathbb{T}}$: $\mathbb{T}$ is deformed by the left-handed part of the field, and $\widetilde{\mathbb{T}}$ by the right-handed part;

(ii) combining the construction of §3.1 with Penrose's [23] construction: unfortunately this seems to have the feature that the right-handed part of the field interacts with the left-handed part;

(iii) employing the notion of hypersurface twistors. These will be discussed in Chapter 4 in the context of gravitation, but they can also be defined in the presence of electromagnetism. The main defect of the hypersurface twistor description is its non-global nature, resulting from its being "tied" to a particular hypersurface in space-time.

§3.3. Charged Functions and Fields.

In §§3.1 and 3.2 we discussed the problem of coding the information of an electromagnetic field into the complex structure of twistor space. Part of the idea behind doing all this is that interactions between the electromagnetic field and other fields would be described by working in the deformed twistor space $\mathcal{J}$ rather than in the flat twistor space $\mathbb{T}$. There are some indications that this programme can be carried out: in particular, we shall show in this section that cross-sections of deformed line bundles correspond to charged zero-rest-mass fields on space-time. Of course, charged zero-rest-mass fields are not observed in nature. But the idea is that the procedure should generalize to the case of massive fields at the 2- or 3-twistor level [21].

So let $\phi_{AB} = \nabla_{A'(A}\Phi^{A'}_{B)}$ be a left-handed Maxwell field and let $\mathcal{J}$ be the associated twistor space, regarded as a line bundle over $\mathbb{PT}$. If $s>o$ and q are integers, define a line bundle $L(s,q)$ over $\mathbb{PT}$ by

$$L(s,q) := \mathcal{J}^{q} \otimes \mathbb{T}^{2s+2-q} . \tag{3.1}$$

Here $\mathbb{T}$ is being regarded as the standard line bundle over $\mathbb{PT}$ (i.e. $\mathbb{T}$ is the bundle $L(-1)$: cf. §2.2).

First we need to find the "patching function" for $L(s,q)$. Suppose that $\mathbb{PT}$ is covered by the two patches $\mathbb{P}U$ and $\mathbb{P}\hat{U}$ and (as in §2.2) that A and B are planes in $\mathbb{P}U$ and $\mathbb{P}\hat{U}$ respectively. Recall from §2.2 that the patching for $\mathbb{T}$ goes as follows:

the patch $\mathbb{P}U \times \mathbb{C}$ (with coordinates (Z^{α},ζ)) and the patch $\mathbb{P}\hat{U} \times \mathbb{C}$ (with coordinates $(Z^{\alpha},\hat{\zeta})$) are patched together by

$$\hat{\zeta} = \frac{B_{\alpha} Z^{\alpha}}{A_{\beta} Z^{\beta}} \zeta ,$$

which amounts to the same thing as

$$\hat{Z}^{\alpha} = Z^{\alpha} .$$

The patching for $\mathfrak{I}$ is given by

$$\hat{\zeta} = \frac{B_\alpha Z^\alpha}{A_\beta Z^\beta} \quad \exp\{-ief(Z)\}\ \zeta$$

which means the same as (cf. equation (1.6))

$$\hat{Z}^\alpha = \exp\{-ief(Z)\}Z^\alpha .$$

So the patching for $L(s,q) = \mathfrak{I}^q \otimes \mathbb{T}^{2s+2-q}$

is
$$\hat{\zeta} = \left(\frac{B_\alpha Z^\alpha}{A_\beta Z^\beta}\right)^{2s+2} \exp\{-iqef(Z)\}\ \zeta . \tag{3.2}$$

Now what is meant by a holomorphic cross-section of $L(s,q)$ over $\mathbb{P}U \cap \mathbb{P}\hat{U}$?
By definition (cf.§2.2) this is a pair of holomorphic functions $\{h(Z), \hat{h}(Z)\}$
on $\mathbb{P}U \cap \mathbb{P}\hat{U}$ such that (cf. equation (3.2))

$$\hat{h}(Z) = \left(\frac{B_\alpha Z^\alpha}{A_\beta Z^\beta}\right)^{2s+2} \exp\{-iqef(Z)\}\ h(Z) . \tag{3.3}$$

Define $H(Z^\alpha) \ := (A_\alpha Z^\alpha)^{-2s-2}\ h(Z^\beta)$

and $\hat{H}(Z^\alpha) := (B_\alpha Z^\alpha)^{-2s-2}\ \hat{h}(Z^\beta)$.

Then H and $\hat{H}$ are homogeneous of degree $-2s-2$ and satisfy

$$\hat{H}(Z^\alpha) = \exp\{-iqef(Z)\}\ H(Z^\alpha) . \tag{3.4}$$

What we want to do is obtain from the pair $\{H, \hat{H}\}$ a space-time field $\Psi_{A'\ldots L'}$ in an unambiguous way. This can be achieved as follows. Split the function f according to equation (2.3) :

$$f(ix^{AA'}\pi_{A'}, \pi_{A'}) = g'(x^a, \pi_{A'}) - \hat{g}'(x^a,\pi_{A'}),$$

where
$$g'(x^b,\pi_{B'}) := -\int_{p^b}^{x^b} \Phi_a(y^b)\, dy^a . \tag{3.5}$$

Put $\quad H'(x^a,\pi_{A'}) := H(ix^{AA'}\pi_{A'}, \pi_{A'}) \exp\{-iqeg'(x^a,\pi_{A'})\}, \qquad (3\cdot 6\text{ a})$

$$\hat{H}'(x^a,\pi_{A'}) := \hat{H}(ix^{AA'}\pi_{A'},\ \pi_{A'})\ \exp\{-iqe\hat{g}'(x^a,\pi_{A'})\}\ . \qquad (3.6b)$$

Then the relation (3.4) becomes

$$\hat{H}' = H'\ . \qquad (3.7)$$

Define a spinor field $\Psi_{A'\ldots L'}(x^b)$ by

$$\underbrace{\Psi_{A'\ldots L'}}_{2s}(x^b) := \frac{1}{2\pi i}\oint \underbrace{\pi_{A'}\ldots\pi_{L'}}_{2s}\ H'(x^b,\pi_{B'})\ \Delta\pi\ . \qquad (3.8)$$

Then $\nabla^{AA'}\Psi_{A'\ldots L'} = \frac{1}{2\pi i}\oint \nabla^{AA'}\pi_{A'}\ldots\pi_{L'}\ H'\ \Delta\pi$

$$= iqe\ \Phi^{AA'}\ \frac{1}{2\pi i}\oint \pi_{A'}\ldots\pi_{L'}\ H'\Delta\pi,$$

using (3.5) and (3.6). Thus

$$(\nabla^{AA'} - iqe\Phi^{AA'})\ \Psi_{A'\ldots L'} = 0.$$

In other words, the zero-rest-mass field $\Psi_{A'\ldots L}$ is minimally coupled to the left-handed electromagnetic field [5]. The charge of $\Psi_{A'\ldots L'}$ is qe (recall that q was an integer, and e was the "charge" of the twistor).

Notice that we could have used $\hat{H}'$ instead of H' in (3.8): by virtue of equation (3.7), this would make no difference to the answer. This was what was meant when we said that we wanted to obtain $\Psi_{A'\ldots L'}$ in an unambiguous way. To sum up: holomorphic cross-sections of the line bundle L(s,q) correspond to zero-rest-mass fields with charge qe, where e is the charge of the twistor.

CHAPTER 4. GRAVITATIONAL FIELDS

The problem of defining twistors in curved space-times has long been one of the central issues in twistor theory. The difficulty involved may be illustrated by considering the Kerr theorem [16]. This states essentially that a shear-free congruence of null geodesics in Minkowski space-time corresponds to a holomorphic 2-surface in twistor space $\mathbb{PT}$. In other words, the equations of "shear-free-ness" in space-time correspond to the Cauchy-Riemann equations in $\mathbb{PT}$. But conformal curvature is something which creates shear [17] : a shear-free congruence of null geodesics will pick up shear (in general) if it passes through a conformally curved region of space-time. Thus conformal curvature would seem to destroy the complex structure of twistor space.

A number of different approaches to the problem have been proposed; more details on these may be found in [24], [19].

(i) Local twistors. The idea here is to erect a flat twistor space at each point of space-time. This provides a conformally invariant algebra on the space-time, but from our point of view is not much use : the local twistor spaces contain no information about the gravitational field.

(ii) Global twistors. Here we start with the real 5-dimensional space of unscaled null geodesics in space-time and extend it to a 6 - dimensional manifold : the space of global twistors. But this twistor space has no natural complex structure, only a (weaker) symplectic structure.

(iii) Hypersurface twistors. See §§4.2, 4.3. The space of twistors relative to a hypersurface has a complex structure and contains some information about the gravitational field. But it suffers from the defect of being "tied" to a hypersurface in space-time. The special case of asymptotic twistors in asymptotically flat space-times is particularly useful.

(iv) The Nonlinear Graviton [20]. See §4.1. This is the most satisfactory approach so far : the description is global and holomorphic, and all the information about the gravitational field is contained in the twistor space structure. Unfortunately it applies only to right- or left- flat fields - what is really needed is a more general construction which works for general solutions of Einstein's vacuum equations, but at present no such construction is known.

§4.1. Zero-Rest-Mass Free Fields on Right-Flat Space-Times.

This section is concerned with right-flat space-times, i.e. complex space-times which have $\tilde{\Psi}_{A'B'C'D'} = 0$ and $R_{ab} = 0$ (these are what Plebanski [25] calls "strong heavens"). A construction due to Penrose (the "nonlinear graviton" [20] has shown that right-flat space-times are closely associated with curved twistor spaces. We shall begin with a brief description of this association and then go on to describe how it may be used in dealing with zero-rest-mass free fields on a right-flat background.

The central concept is that of a totally null 2-surface (of primed type), which will be referred to as a TN2S. A TN2S is a complex 2-surface Z, such that the tangent space at each point of Z is spanned by vectors of the form $\lambda^A \pi^{A'}$, with $\pi^{A'}$ fixed and λ^A varying. A necessary and sufficient condition for a complex space-time to admit a three-complex-parameter family of TN2S's is [20]

$$\tilde{\Psi}_{A'B'C'D'} = 0.$$

So if (M, g_{ab}) is a right-flat space-time, we define the associated projective twistor space $\mathbb{P}\mathcal{T}$ to be the space of TN2S's in M (see figure 2.1.3.). Provided (M, g_{ab}) is convex and sufficiently close to Minkowski (cf. §2.1), $\mathbb{P}\mathcal{T}$ will have the structure of a 3-dimensional complex manifold.

As was the case in Minkowski space-time, each point $x \in M$ corresponds to a compact holomorphic curve L_x in $\mathbb{P}\mathcal{T}$. With every TN2S Z, there is associated a primed spinor field $\pi_{A'}$ on Z, up to proportionality. Knowing $\pi_{A'}$ exactly (as a covariantly constant spinor field on Z)gives us a non-projective twistor. More precisely, the non-projective twistor space $\mathcal{T}$ is the space of pairs $(Z, \pi_{A'})$, where Z is a TN2S in M and $\pi_{A'}$ is a constant spinor field on Z, such that the tangent vectors to Z have the form $\lambda^A \pi^{A'}$.

The fact that M is right-flat enables one to choose a primed spin-frame which is covariantly constant on M [25]:

$$\nabla_b \varepsilon_{\underset{\sim}{A}'}{}^{A'} = 0. \qquad (1.1a)$$

Equation (1.1a) means that the primed spin coefficients vanish:

$$\tilde{\gamma}_{\underset{\sim}{A}'\underset{\sim}{B}'\underset{\sim}{C}\underset{\sim}{C}'} = 0. \qquad (1.1b)$$

The practical implication of (1.1) is that primed spinor indices "commute with the covariant derivative operator." Let $\mathcal{B}$ denote the primed spin-bundle over M; as coordinates on $\mathcal{B}$ use $(x^{\underset{\sim}{a}}, \pi_{\underset{\sim}{A}'})$. The following lemma describes a representation of twistor functions as functions on $\mathcal{B}$ (cf. equations (2.1.8) and (2.1.9)).

Lemma. Holomorphic functions on $\mathcal{T}$, homogeneous of degree n, correspond to holomorphic functions $F = F(x^{\underset{\sim}{a}}, \pi_{\underset{\sim}{A}'})$ on $\mathcal{B}$, homogeneous of degree n in $\pi_{\underset{\sim}{A}'}$ and satisfying

$$\pi^{A'}_{\sim} \nabla_{\underset{\sim}{A}\underset{\sim}{A}'} F(x^b_{\sim}, \pi_{\underset{\sim}{B}'}) = 0. \tag{1.2}$$

<u>Proof</u>. There is a natural map $\sigma : \mathcal{B} \to \mathcal{T}$, defined as follows:

$$(x^a_{\sim}, \pi_{\underset{\sim}{A}'}) \xmapsto{\sigma} (Z, \pi_{A'}),$$

where Z is that TN2S which passes through the point $x^a_{\sim}$ and which has tangent vectors of the form $\lambda^A \pi^{A'}$. Clearly, if $x^a_{\sim}$ and $y^a_{\sim}$ are two infinitesimally separated points and the vector from $x^a_{\sim}$ to $y^a_{\sim}$ has the form $\lambda^A \pi^{A'}$, then $(x^a_{\sim}, \pi_{\underset{\sim}{A}'})$ and $(y^a_{\sim}, \pi_{\underset{\sim}{A}'})$ are mapped by σ to the <u>same</u> $(Z, \pi_{A'})$.

Now, given $f : \mathcal{T} \to \mathbb{C}$, we define $F = F(x^a_{\sim}, \pi_{\underset{\sim}{A}'})$ on $\mathcal{B}$ by $F = f \circ \sigma$. The above comment implies that $\lambda^A \pi^{A'} \nabla_{AA'} F(x^b_{\sim}, \pi_{\underset{\sim}{B}'}) = 0$, for all λ^A, or in other words that $\pi^{A'} \nabla_{AA'} F(x^b_{\sim}, \pi_{\underset{\sim}{B}'}) = 0$. □

We move on to a discussion of zero-rest-mass fields on M, dealing first with scalar and right-handed fields and then with left-handed fields.

<u>Theorem</u>. Suppose that $F = F(x^a_{\sim}, \pi_{\underset{\sim}{A}'})$ satisfies (1.2).

(i) If F is homogeneous of degree $-n-2$ (n a positive integer), then the field $\phi_{A'\ldots C'}$ on M, defined by

$$\phi_{\underset{\sim}{A}'\underset{\sim}{B}'\ldots\underset{\sim}{D}'} = \frac{1}{2\pi i} \oint \overbrace{\pi_{A'} \pi_{B'} \cdots \pi_{D'}}^{n} F(x^e_{\sim}, \pi_{\underset{\sim}{E}'}) \, \Delta\pi$$

is a solution of $\nabla^{AA'} \phi_{A'B'\ldots D'} = 0$.

(ii) If F is homogeneous of degree -2, then the field

$$\phi = \frac{1}{2\pi i} \oint F(x^d_{\sim}, \pi_{\underset{\sim}{D}'}) \, \Delta\pi$$

is a solution of $\Box\phi = 0$ on M.

<u>Proof</u>. (i) $\varepsilon_A{}^{\underset{\sim}{A}} \varepsilon_{\underset{\sim}{B}'}{}^{B'} \cdots \varepsilon_{\underset{\sim}{D}'}{}^{D'} \nabla^{AA'} \phi_{A'B'\cdots D'}$

$$= \nabla^{\underset{\sim}{A}\underset{\sim}{A}'} \phi_{\underset{\sim}{A}'\underset{\sim}{B}'\ldots\underset{\sim}{D}'} \qquad \text{by (1.1b)}$$

$$= \frac{1}{2\pi i} \oint \pi_{\underset{\sim}{A}'} \pi_{\underset{\sim}{B}'} \ldots \pi_{\underset{\sim}{D}'} \nabla^{\underset{\sim}{A}\underset{\sim}{A}'} F(x^e, \pi_{\underset{\sim}{E}'}) \quad \Delta\pi$$

$= 0$ by (1.2) .

(ii) Equation (1.2) implies that there exists a spinor field ξ_A such that

$$\nabla_{AA'} F = \xi_A \pi_{A'} . \tag{1.3}$$

Now $\pi_{A'} \pi^{B'} \nabla_{BB'} \xi_A = \pi^{B'} \nabla_{BB'} (\pi_{A'} \xi_A)$

$$= \pi^{B'} \nabla_{BB'} \nabla_{AA'} F$$

$$= \nabla_{AA'} (\pi^{B'} \nabla_{BB'} F)$$

$$= 0 \quad \text{by (1.2).}$$

Thus $\pi^{B'} \nabla_{BB'} \xi_A = 0$, so there exists a spinor field ξ_{AB} such that

$$\nabla_{BB'} \xi_A = \xi_{AB} \pi_{B'} . \tag{1.4}$$

Combining (1.3) and (1.4) gives

$$\nabla_{BB'} \nabla_{AA'} F = \xi_{AB} \pi_{A'} \pi_{B'} , \tag{1.5}$$

from which it follows that ξ_{AB} is symmetric in AB.

It also follows that

$$\Box \left\{ \frac{1}{2\pi i} \oint F(x^{\underset{\sim}{d}}, \pi_{\underset{\sim}{D}'}) \; \Delta\pi \right\} = \frac{1}{2\pi i} \oint \nabla_{AA'} \nabla^{AA'} F(x^{\underset{\sim}{d}}, \pi_{\underset{\sim}{D}'}) \quad \Delta\pi$$

$$= 0,$$

which completes the proof of the theorem. $\Box$

Remarks.

(a) Define $\partial_A F := i\,\xi_A$ and $\partial_A \partial_B F := -\xi_{AB}$. (1.6)

Then $\partial_A F$ and $\partial_A \partial_B F$ reduce, in the case of flat-space twistor theory, to $\frac{\partial F}{\partial \omega^A}$ and $\frac{\partial^2 F}{\partial \omega^A \partial \omega^B}$ respectively (cf. equation (2.1.9a)).

(b) Solutions of the equation $\nabla^{AA'}\phi_{\underbrace{A'\ldots C'}_{2s}} = 0$ in curved space-time are (for $s \geqslant \frac{3}{2}$) subject to the "Buchdahl conditions" [17]

$$\tilde{\Psi}^{A'B'C'}{}_{(D'}\phi_{E'\ldots L')A'B'C'} = 0.$$

In the present case $\tilde{\Psi}_{A'B'C'D'} = 0$, so the conditions are void.

But for left-handed fields, these conditions have to be taken into account:

solutions of $\nabla^{AA'}\phi_{\underbrace{A\ldots C}_{2s}} = 0$, with $s \geqslant \frac{3}{2}$, are subject to

$$\Psi^{ABC}{}_{(D}\phi_{E\ldots L)ABC} = 0. \tag{1.7}$$

For $S = \frac{1}{2}$ or 1, solutions can be generated using twistor functions:

Theorem. Let $F = F(x^{\underset{\sim}{a}}, \pi_{\underset{\sim}{A'}})$ satisfy (1.2).

(i) If F is homogeneous of degree -1, then

$$\phi_A := \frac{1}{2\pi i}\oint \partial_A F(x^{\underset{\sim}{b}}, \pi_{\underset{\sim}{B'}})\ \Delta\pi$$

satisfies $\nabla^{AA'}\phi_A = 0$.

(ii) If F is homogeneous of degree 0, then

$$\phi_{AB} := \frac{1}{2\pi i}\oint \partial_A \partial_B F(x^{\underset{\sim}{c}}, \pi_{\underset{\sim}{C'}})\ \Delta\pi$$

satisfies $\nabla^{AA'}\phi_{AB} = 0$.

<u>Proof</u>. (i) $\nabla^{AA'}\phi_A = \frac{1}{2\pi i}\oint \nabla^{AA'}(\partial_A F)\ \Delta\pi$

$= \frac{1}{2\pi i}\oint (-i\pi^{A'})\ \xi^A{}_A\ \ \Delta\pi$ by (1.6), (1.4)

$= 0$ by symmetry of ξ_{AB}.

(ii) By (1.6), $\nabla_{BC'}(\partial^B\partial_A F)\ \pi_{B'}\ \pi_{A'} = -\nabla_{BC'}\ \nabla^B{}_{B'}\ \nabla_{AA'}\ F$. (1.8)

Now $\nabla_{B(C'}\nabla^B{}_{B')}\nabla_{AA'}F = \Box_{C'B'}(\nabla_{AA'}F) = 0$ (cf. (2.1.6)).

So the right-side of (1.8) is skew in C'B', while the left-hand side is clearly symmetric in B'A'. Thus both sides vanish and

$$\nabla_{BC'}(\partial^B\partial_A F) = 0. \tag{1.9}$$

The desired result follows immediately:

$\nabla^{AA'}\phi_{AB} = \frac{1}{2\pi i}\oint \nabla^{AA'}(\partial_A\partial_B F)\ \Delta\pi$

$= 0$ by (1.9). □

There appears to be no natural generalization of this kind of procedure to left-handed fields of higher spin (in view of the Buchdahl conditions (1.7), this is not surprising).

To sum up, the space $\mathcal{T}$ has the information about the right-flat gravitational field coded into it, and holomorphic functions on $\mathcal{T}$ correspond to zero-rest-mass fields which are correctly coupled to the gravitational field.

§4.2. Twistors Relative to a Spacelike Hypersurface.

This section is devoted to discussing the concept of twistors relative to a spacelike hypersurface in a general space-time, and to verifying certain conjectures made by Penrose ([19], p.388) concerning the space-time fields derived from functions of such twistors via contour integration.

As was noted in the previous section, the existence of TN2S's in a space-time imposes restrictions on its conformal curvature: in a general curved space-time, no TN2S's will exist. To get around this problem, we note that in Minkowski space-time a TN2P can be represented by its intersection with a given spacelike hypersurface : this intersection will be a complex curve (see [19] for details). This observation motivates the definition that follows.

Let M be a convex space-time and let $\mathcal{S}$ be an analytic spacelike hypersurface in M. Suppose that $\mathcal{S}$ is given by the equation $S(x^a) = 0$, where S is an analytic function on M.
Now "thicken" M and $\mathcal{S}$ into the complex, giving a complex space-time $\mathbb{C}M$ and a complex hypersurface $\mathbb{C}\mathcal{S}$ embedded in it. Define

$$t^a := \left[(\nabla_b S)(\nabla^b S)\right]^{-\frac{1}{2}} \nabla^a S.$$

Then t^a is a holomorphic unit vector on $\mathbb{C}\mathcal{S}$, provided we don't "thicken out" too far (note that $(\nabla_b S)(\nabla^b S) > 0$ on M).

Definition. A hypersurface twistor (with respect to the spacelike hypersurface $\mathcal{S}$) is a pair $(\Lambda, \pi_{A'})$, where Λ is a complex curve (regarded as a point set) in $\mathbb{C}\mathcal{S}$ and where $\pi_{A'}$ is a spinor defined along Λ, such that

(a) the vector $t^{AB'}\pi_{B'}\pi^{A'}$ is tangent to Λ; (2.1a)

(b) $\pi_{A'}$ is parallelly propagated along Λ , i.e.

$$t^{AB'}\pi_{B'}\,\pi^{A'}\,\nabla_{AA'}\,\pi_{C'} = 0. \qquad (2.1b)$$

Let $\mathcal{J}(\mathcal{S})$ denote the space of hypersurface twistors with respect to $\mathcal{S}$, and for brevity write

$$v^{a} := t^{AB'}\,\pi_{B'}\pi^{A'}.$$

There is another way of visualising hypersurface twistors, namely as follows. Let $\mathcal{B}(\mathcal{S})$ be the bundle of primed spinors over $\mathbb{C}\mathcal{S}$; so $\mathcal{B}(\mathcal{S})$ is a five-dimensional complex manifold on which we can use the coordinates $(x^{\underset{\sim}{a}}, \pi_{\underset{\sim}{A}'})$ with $x^{\underset{\sim}{a}} \in \mathbb{C}\mathcal{S}$.

Equation (2.1b) implies that

$$0 = \varepsilon_{\underset{\sim}{C}'}{}^{C'}\, v^{AA'}\nabla_{AA'}\pi_{C'} = v^{\underset{\sim}{AA}'}\,\nabla_{\underset{\sim}{AA}'}\pi_{\underset{\sim}{C}'} + \tilde{\gamma}_{\underset{\sim}{C}'\underset{\sim}{D}'\underset{\sim}{AA}'}\, v^{\underset{\sim}{AA}'}\pi^{\underset{\sim}{D}'}$$

(cf. (2.1.2)). So let us define a vector field V on $\mathcal{B}(\mathcal{S})$ by

$$V := v^{\underset{\sim}{AA}'}\left(\nabla_{\underset{\sim}{AA}'} - \tilde{\gamma}_{\underset{\sim}{C}'\underset{\sim}{D}'\underset{\sim}{AA}'}\,\pi^{\underset{\sim}{D}'}\frac{\partial}{\partial \pi_{\underset{\sim}{C}'}}\right) \tag{2.2}$$

acting on scalars. Then the integral curves of V in $\mathcal{B}(\mathcal{S})$ are in one-to-one correspondence with hypersurface twistors. In fact, each integral curve of V projects down to a hypersurface twistor curve Λ in $\mathbb{C}\mathcal{S}$ (see figure 2.1).

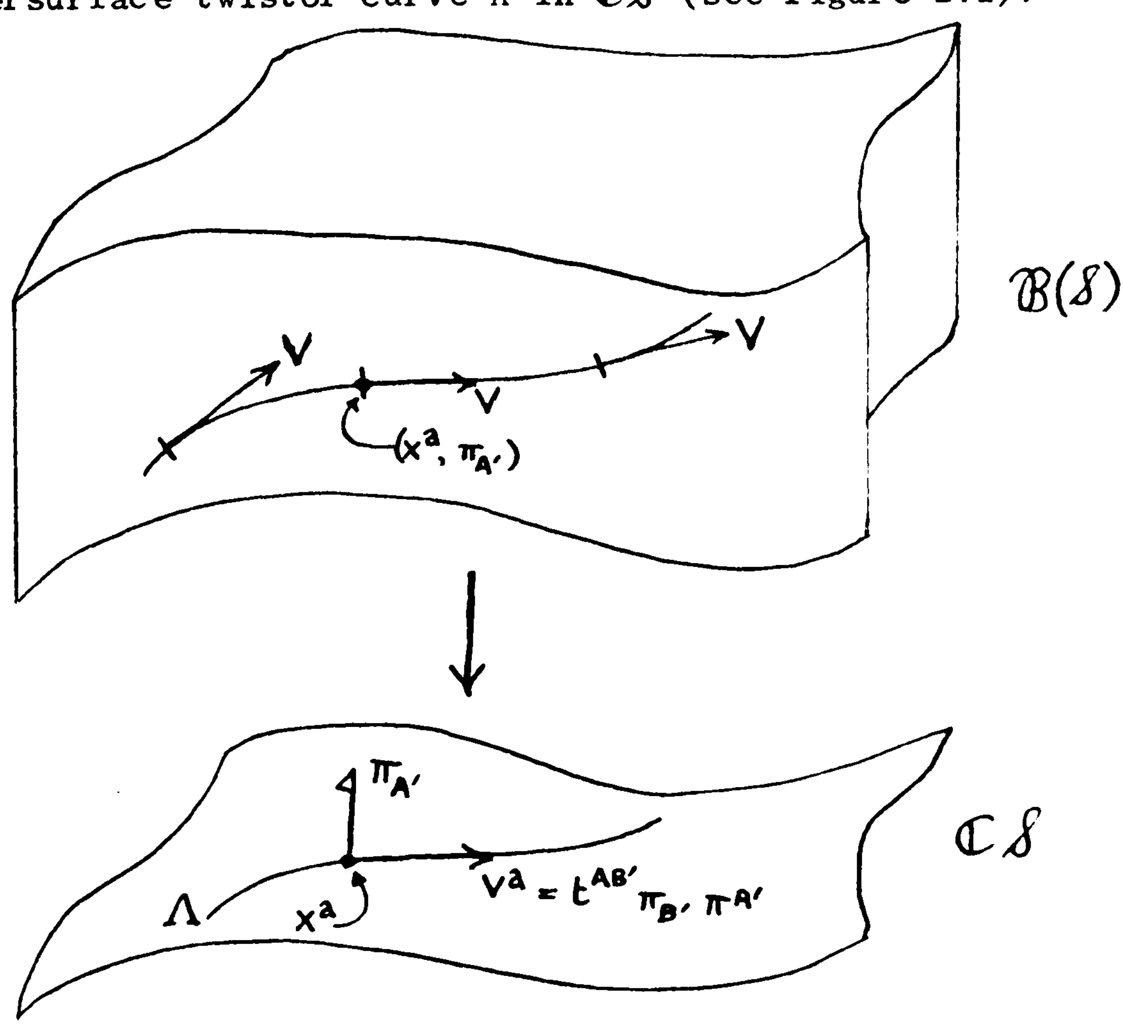


Figure 2.1.

Since the conditions (4.1) are homogeneous in $\pi_{A'}$, it makes sense also to talk about projective hypersurface twistors. We say that two hypersurface twistors ($\Lambda, \pi_{A'}$) and $(\hat{\Lambda}, \hat{\pi}_{A'})$ are projectively equivalent iff

$$\Lambda = \hat{\Lambda} \text{ and } \pi_{A'} = C\hat{\pi}_{A'}$$

for some nonzero complex number C.
A projective hypersurface twistor is an equivalence class of projectively equivalent hypersurface twistors. Denote the space of projective hypersurface twistors by $\mathbb{P}\mathcal{J}(\mathcal{S})$.

In exactly similar fashion one can define the space $\tilde{\mathcal{J}}(\mathcal{S})$ of conjugate hypersurface twistors with respect to the spacelike hypersurface $\mathcal{S}$. An element of $\tilde{\mathcal{J}}(\mathcal{S})$ is a pair $(\tilde{\Lambda}, \eta_A)$ such that

(a) $t^{A'B} \eta_B \eta^A$ is tangent to the curve $\tilde{\Lambda}$, (2.3a)

(b) $t^{A'B} \eta_B \eta^A \nabla_{AA'} \eta_C = 0$ along $\tilde{\Lambda}$. (2.3b)

If ($\Lambda, \pi_{A'}$) is a hypersurface twistor, then its complex conjugate is defined to be the conjugate hypersurface twistor $(\bar{\Lambda}, \bar{\pi}_A)$.

An element $(\Lambda, \pi_{A'})$ of $\mathcal{J}(\mathcal{S})$ and an element $(\tilde{\Lambda}, \eta_A)$ of $\tilde{\mathcal{J}}(\mathcal{S})$ are said to be orthogonal to each other iff $\Lambda \cap \tilde{\Lambda} \neq \phi$. This "conformal scalar product" seems to be all that remains of the flat-space twistor scalar product $Z^\alpha \tilde{Z}_\alpha$ (i.e. the "complexified" version of the twistor norm $Z^\alpha \bar{Z}_\alpha$).

We now turn to consideration of functions on $\mathcal{J}(\mathcal{S})$, and of their contour integrals.

A holomorphic function on $\mathcal{J}(\mathcal{S})$, homogeneous of degree n, may be thought of as a holomorphic function $f = f(x^{\underset{\sim}{a}}, \pi_{\underset{\sim}{A'}})$ on $\mathcal{B}(\mathcal{S})$, homogeneous of degree n in $\pi_{A'}$, and constant along the integral curves of V, i.e.

$$Vf = v^{\underset{\sim}{A}\underset{\sim}{A}'}(\nabla_{\underset{\sim}{A}\underset{\sim}{A}'} - \tilde{\gamma}_{\underset{\sim}{C}'\underset{\sim}{D}'\underset{\sim}{A}\underset{\sim}{A}'}\,\pi^{\underset{\sim}{C}'}\frac{\partial}{\partial\pi_{\underset{\sim}{D}'}})\,f = 0. \tag{2.4}$$

[Note: the partial differentiation $\nabla_{AA'}$ is carried out while keeping the components $\pi_{A'}$ of the π- spinor fixed.]

If $n \leq -2$ and if f has the appropriate singularity structure, we can integrate out the π-dependence of f to produce a spinor field $\phi_{A'\ldots C'}(x^d)$ on $\mathbb{C}\mathcal{S}$: its components are

$$\underbrace{\phi_{\underset{\sim}{A}'\ldots\underset{\sim}{C}'}}_{-n-2} = \frac{1}{2\pi i}\oint \underbrace{\pi_{\underset{\sim}{A}'}\ldots\pi_{\underset{\sim}{C}'}}_{-n-2}\, f(x^{\underset{\sim}{d}}, \pi_{\underset{\sim}{D}'})\quad \Delta\pi\ . \tag{2.5}$$

The obvious question that now arises is : is $\phi_{A'\ldots C'}$ the restriction to $\mathbb{C}\mathcal{S}$ of a zero-rest-mass free field in $\mathbb{C}M$?

(i) The case of spin $\geq \frac{3}{2}$ (i.e. $n \leq -5$) can be ruled out immediately, since the Buchdahl conditions (cf. §4.1) show that there is an "obstruction" to the existence of zero-rest-mass fields in curved space-time, for $s \geq 3/2$.

(ii) for spin 0 (scalar field: n = -2) and spin $\frac{1}{2}$ (neutrino field: n = -3), there are no differential constraints on the fields restricted to a hypersurface. Thus the integral expression (2.5) serves to describe scalar or neutrino fields restricted to $\mathbb{C}\mathcal{S}$.

(iii) The most interesting case is that of spin 1 (electromagnetic field: n = - 4). An electromagnetic free-field restricted to a hypersurface has to satisfy certain constraint equations.

In 3-vector notation, these are the well-known $\mathrm{div}\,\vec{E} = 0 = \mathrm{div}\,\vec{B}$. The following lemma gives their spinorial version.

Lemma. Let $\phi_{A'B'}$ be a spinor field on $\mathbb{C}\mathcal{S}$. Then $\phi_{A'B'}$ is the restriction to $\mathbb{C}\mathcal{S}$ of a zero-rest-mass free field in $\mathbb{C}M$ iff

$$t^{A}{}_{B'} \nabla_{AA'} \phi^{A'B'} = 0. \tag{2.6}$$

Proof. Let D_a be the intrinsic covariant derivative in $\mathbb{C}\mathcal{S}$ [9]. If $F_{ab} = F_{[ab]}$ is a source-free electromagnetic field in $\mathbb{C}M$, then the electric and magnetic field vectors E^a and B^a are defined by

$$\left.\begin{aligned} E^a &:= F^{ab}\, t_b, \\ B^a &:= F^{*ab}\, t_b, \end{aligned}\right\} \tag{2.7}$$

where $F^{*ab} := \frac{1}{2} e^{abcd} F_{cd}$ is the dual of F_{ab}.

The constraint equations div $E^a = 0 =$ div B^a can be restated as

$$D_a E^a = 0 = D_a B^a . \tag{2.8}$$

Now if we take

$$F_{ab} = \phi_{A'B'}\, \varepsilon_{AB},$$

then
$$\begin{aligned} F^*_{ab} &= i\, \phi_{A'B'}\, \varepsilon_{AB}, \\ E^a &= t^{A}{}_{B'}\, \phi^{A'B'}, \\ B^a &= i\, t^{A}{}_{B'}\, \phi^{A'B'}, \end{aligned}$$

and the constraint equations (2.8) can be rewritten as

$$D_{AA'}(t^{A}{}_{B'}\, \phi^{A'B'}) = 0 . \tag{2.9}$$

By definition of D_a,

$$D_{AA'}(t^A_{B'}\, \phi^{A'B'}) = (\varepsilon_A{}^C\, \varepsilon_{A'}{}^{C'} - t_{AA'}\, t^{CC'})\, \nabla_{CC'}\, (t^A_{B'}\, \phi^{A'B'})$$

$$= t^A_{B'}\, \nabla_{AA'}\, \phi^{A'B'} + \phi^{A'B'} \left[\nabla_{AA'}\, t^A_{B'} - t_{AA'}\, t^{CC'}\, \nabla_{CC'} t^A_{B'}\right]$$

$$-(t_{AA'} t^A_{B'})(t^{CC'}\, \nabla_{CC'}\, \phi^{A'B'}). \qquad (2.10)$$

The last term on the right-hand side of (2.10) vanishes because $t_{AA'} t^A_{B'} = \frac{1}{2}\, \varepsilon_{A'B'}$ is skew and $\phi^{A'B'}$ is symmetric. Similarly, the second last term vanishes because the quantity in square brackets is skew in A'B' : to see this, we use the facts that t^a is hypersurface-orthogonal and has unit length. These imply that

$$0 = 3\, t^c t_{[c}\, \nabla_a\, t_{b]} = \nabla_{[a} t_{b]} - t^c\, t_{[a}\, \nabla_c\, t_{b]}$$

$$\Rightarrow 0 = \nabla_{A(A'}\, t^A_{B')} - t^{CC'}\, t_{A(A'}\, \nabla_{|CC'|}\, t^A_{B')}\, .$$

Thus $D_{AA'}(\phi^{A'B'}\, t^A_{B'}) = t^A_{B'}\, \nabla_{AA'}\, \phi^{A'B'}$ and the constraint equation (2.9) is equivalent to (2.6). □

We are now in a position to verify the following result:

Theorem. Let f be a holomorphic function satisfying (2.4) and homogeneous of degree -4. Then the spinor field $\phi_{A'B'}$ on $\mathbb{C}\mathcal{S}$, defined by

$$\phi_{\underset{\sim}{A}'\underset{\sim}{B}'} = \frac{1}{2\pi i} \oint \pi_{\underset{\sim}{A}'}\, \pi_{\underset{\sim}{B}'}\, f(x^{\underset{\sim}{c}}, \pi_{\underset{\sim}{C}'})\, \Delta\pi\, ,$$

satisfies the constraint equation (2.6).

Proof.

$$t^A_{B'}\nabla_{AA'}\phi^{A'B'} = t^{\tilde{A}}_{\tilde{B}'}(\nabla_{\underset{\sim}{AA}'}\phi^{\underset{\sim}{A'B'}} + \tilde{\gamma}^{\tilde{A}'}{}_{\underset{\sim}{C'AA}'}\phi^{\underset{\sim}{C'B'}} + \tilde{\gamma}^{\tilde{B}'}{}_{\underset{\sim}{C'AA}'}\phi^{\underset{\sim}{A'C'}})$$

$$= \frac{1}{2\pi i}\oint t^{\tilde{A}}_{\tilde{B}'}\Big\{\pi^{\tilde{A}'}\pi^{\tilde{B}'}\nabla_{\underset{\sim}{AA}'} f + \tilde{\gamma}^{\tilde{A}'}{}_{\underset{\sim}{C'AA}'}\pi^{\tilde{C}'}\pi^{\tilde{B}'} f + \tilde{\gamma}^{\tilde{B}'}{}_{\underset{\sim}{C'AA}'}\pi^{\tilde{A}'}\pi^{\tilde{C}'} f\Big\}\Delta\pi$$

$$= \frac{1}{2\pi i}\oint t^{\tilde{A}}_{\tilde{B}'}\Big\{\tilde{\gamma}_{\underset{\sim}{C'D'AA}'}\pi^{\tilde{C}'}\pi^{\tilde{A}'}\pi^{\tilde{B}'}\frac{\partial f}{\partial \pi_{\tilde{D}'}} + \tilde{\gamma}^{\tilde{A}'}{}_{\underset{\sim}{C'AA}'}\pi^{\tilde{C}'}\pi^{\tilde{B}'} f + \tilde{\gamma}^{\tilde{B}'}{}_{\underset{\sim}{C'AA}'}\pi^{\tilde{A}'}\pi^{\tilde{C}'} f\Big\}\Delta\pi,$$

using (2.4).

The expression in braces is easily seen to be equal to

$$\frac{\partial}{\partial \pi_{\tilde{D}'}}\Big[\tilde{\gamma}_{\underset{\sim}{C'D'AA}'}\pi^{\tilde{C}'}\pi^{\tilde{A}'}\pi^{\tilde{B}'} f\Big],$$

and so its integral over a closed contour vanishes. Thus $t^A_{B'}\nabla_{AA'}\phi^{A'B'} = 0$. □

The final matter to be discussed in this section is that of finding solutions of the equation

$$\nabla^{AA'}\Phi_{A\ldots CA'\ldots D'} = 0 \quad . \qquad (2.11)$$

in $\mathbb{C}M$.

Consider the product space $\mathcal{J}(\mathcal{S}) \times \tilde{\mathcal{J}}(\mathcal{S})$ and let $\Delta(\mathcal{S})$ be the "diagonal" subspace

$$\Delta(\mathcal{S}) = \{((\Lambda,\pi_{A'}),\ (\tilde{\Lambda},\eta_A)) \in \mathcal{J}(\mathcal{S}) \times \tilde{\mathcal{J}}(\mathcal{S})\ |\ \Lambda\cap\tilde{\Lambda} \neq \phi\}\ .$$

In other words, an element of $\Delta(\mathcal{S})$ consists of an orthogonal pair $(\Lambda, \pi_{A'})$ and $(\tilde{\Lambda}, \eta_A)$. The vector $\eta^A \pi^{A'}$ at the intersection of Λ and $\tilde{\Lambda}$ determines a null geodesic in $\mathbb{C}M$ (see figure 2.2). Conversely, a null geodesic in $\mathbb{C}M$ determines an element of $\Delta(\mathcal{S})$ in the obvious way. In order to generate solutions of equation (2.11), we shall use holomorphic functions on the space of null geodesics. Such functions may essentially be thought of as twistor functions defined on the space $\Delta(\mathcal{S}) \subset \mathcal{J}(\mathcal{S}) \times \mathcal{J}(\mathcal{S})$.

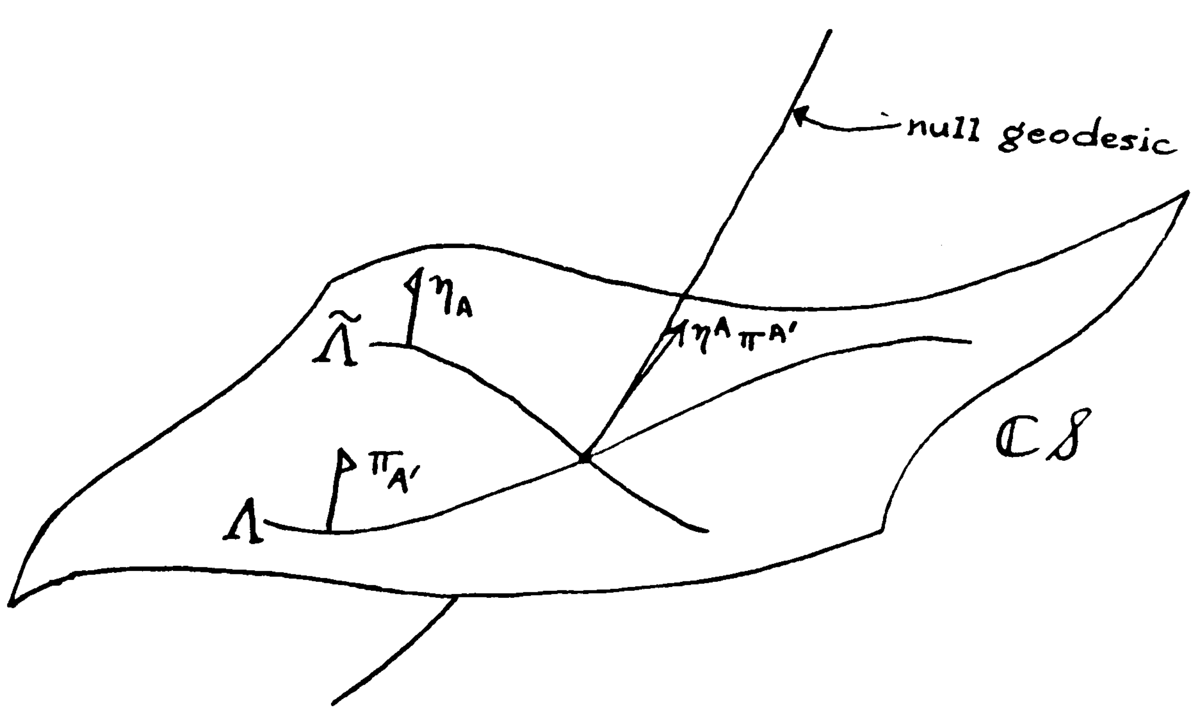
A function f on the space of null geodesics in $\mathbb{C}M$ may be written as

$$f = f(x^{\underset{\sim}{a}}, \eta_{\underset{\sim}{A}}, \pi_{\underset{\sim}{A}'}) ,$$

where
$$\eta^{\underset{\sim}{A}} \pi^{\underset{\sim}{A}'} (\nabla_{\underset{\sim}{AA'}} - \gamma_{\underset{\sim}{BCAA'}} \eta^{\underset{\sim}{B}} \frac{\partial}{\partial \eta_{\underset{\sim}{C}}} - \tilde{\gamma}_{\underset{\sim}{B'C'AA'}} \pi^{\underset{\sim}{B'}} \frac{\partial}{\partial \pi_{\underset{\sim}{C'}}}) f = 0. \qquad (2.12)$$

Equation (2.12) simply says that f is constant up the null geodesic with tangent vector $\eta^A \pi^{A'}$. The presence of the γ and $\tilde{\gamma}$ terms express the fact that η^A and $\pi^{A'}$ move up the null geodesic by parallel propagation.

Figure 2.2.



Define a field $\Phi_{A\ldots CA'\ldots D'}$ on $\mathbb{C}M$ by

$$\Phi_{\underbrace{\underset{\sim}{A}..\underset{\sim}{C}}_{n}\underbrace{\underset{\sim}{A}'..\underset{\sim}{D}'}_{m}}(x^{\underset{\sim}{e}}) = \frac{1}{(2\pi i)^2}\oint \underbrace{\eta_{\underset{\sim}{A}}\cdots\eta_{\underset{\sim}{C}}}_{n}\underbrace{\pi_{\underset{\sim}{A}'}\cdots\pi_{\underset{\sim}{D}'}}_{m}\, f(x^{\underset{\sim}{e}},\eta_{\underset{\sim}{E}},\eta_{\underset{\sim}{E}'})\,\Delta\eta_{\wedge}\Delta\pi \,, \qquad (2.13)$$

where f is required to be homogeneous of degree -n-2 in η_E, -m-2 in $\pi_{E'}$.

<u>Theorem</u>. If f satisfies (2.12), then the field $\Phi_{A..CA'..D'}$ on $\mathbb{C}M$, defined by (2.13), satisfies

$$\nabla^{AA'}\Phi_{A\ldots CA'\ldots D'} = 0.$$

<u>Proof</u>. For brevity let us assume that m = n = 2 ; the proof for other values of m and n is essentially the same.

$$\varepsilon_{\underset{\sim}{B}}{}^{B}\,\varepsilon_{\underset{\sim}{B}'}{}^{B'}\,\nabla^{AA'}\,\Phi_{ABA'B'} = \nabla^{\underset{\sim}{A}\underset{\sim}{A}'}\Phi_{\underset{\sim}{A}\underset{\sim}{B}\underset{\sim}{A}'\underset{\sim}{B}'} - \gamma^{\underset{\sim}{C}}{}_{\underset{\sim}{A}}{}^{\underset{\sim}{A}\underset{\sim}{A}'}\Phi_{\underset{\sim}{C}\underset{\sim}{B}\underset{\sim}{A}'\underset{\sim}{B}'} - \gamma^{\underset{\sim}{C}}{}_{\underset{\sim}{B}}{}^{\underset{\sim}{A}\underset{\sim}{A}'}\,\Phi_{\underset{\sim}{A}\underset{\sim}{C}\underset{\sim}{A}'\underset{\sim}{B}'}$$

$$-\tilde{\gamma}^{\underset{\sim}{C}'}{}_{\underset{\sim}{A}'}{}^{\underset{\sim}{A}\underset{\sim}{A}'}\,\Phi_{\underset{\sim}{A}\underset{\sim}{B}\underset{\sim}{C}'\underset{\sim}{B}'} - \tilde{\gamma}^{\underset{\sim}{C}'}{}_{\underset{\sim}{B}'}{}^{\underset{\sim}{A}\underset{\sim}{A}'}\,\Phi_{\underset{\sim}{A}\underset{\sim}{B}\underset{\sim}{A}'\underset{\sim}{C}'}$$

$$= \frac{1}{(2\pi i)^2}\oint \Big\{ \eta_{\underset{\sim}{A}}\eta_{\underset{\sim}{B}}\pi_{\underset{\sim}{A}'}\pi_{\underset{\sim}{B}'}\,\nabla^{\underset{\sim}{A}\underset{\sim}{A}'} f \;-\; \gamma^{\underset{\sim}{C}}{}_{\underset{\sim}{A}}{}^{\underset{\sim}{A}\underset{\sim}{A}'}\eta_{\underset{\sim}{C}}\eta_{\underset{\sim}{B}}\pi_{\underset{\sim}{A}'}\pi_{\underset{\sim}{B}'}\, f$$

$$-\gamma^{\underset{\sim}{C}}{}_{\underset{\sim}{B}}{}^{\underset{\sim}{A}\underset{\sim}{A}'}\eta_{\underset{\sim}{A}}\eta_{\underset{\sim}{C}}\pi_{\underset{\sim}{A}'}\pi_{\underset{\sim}{B}'}\, f \;-\; \tilde{\gamma}^{\underset{\sim}{C}'}{}_{\underset{\sim}{A}'}{}^{\underset{\sim}{A}\underset{\sim}{A}'}\eta_{\underset{\sim}{A}}\eta_{\underset{\sim}{B}}\pi_{\underset{\sim}{C}'}\pi_{\underset{\sim}{B}'}\, f$$

$$-\tilde{\gamma}^{\underset{\sim}{C}'}{}_{\underset{\sim}{B}'}{}^{\underset{\sim}{A}\underset{\sim}{A}'}\eta_{\underset{\sim}{A}}\eta_{\underset{\sim}{B}}\pi_{\underset{\sim}{A}'}\pi_{\underset{\sim}{C}'}\, f \Big\}\,\Delta\eta_{\wedge}\,\Delta\pi \,.$$

Now using (2.12), the expression in braces can be written as

$$\frac{\partial}{\partial\eta_{\underset{\sim}{D}}}\Big[\eta_{\underset{\sim}{A}}\,\eta_{\underset{\sim}{B}}\,\eta^{\underset{\sim}{C}}\,\pi_{\underset{\sim}{A}'}\,\pi_{\underset{\sim}{B}'}\,\gamma_{\underset{\sim}{C}\underset{\sim}{D}}{}^{\underset{\sim}{A}\underset{\sim}{A}'} f\Big] + \frac{\partial}{\partial\pi_{\underset{\sim}{D}'}}\Big[\eta_{\underset{\sim}{A}}\,\eta_{\underset{\sim}{B}}\,\pi_{\underset{\sim}{A}'}\,\pi_{\underset{\sim}{B}'}\,\pi^{\underset{\sim}{C}'}\tilde{\gamma}_{\underset{\sim}{C}'\underset{\sim}{D}'}{}^{\underset{\sim}{A}\underset{\sim}{A}'}\, f\Big],$$

and thus its integral (over a closed contour) vanishes. The result follows. □

§4.3 Twistors Relative to a Null Hypersurface.

The concept of a hypersurface twistor introduced in the previous section carries over to the case when the hypersurface $\mathcal{S}$ is null, but some extra features arise. These features are well-known in the case when the space-time is asymptotically flat [18] and $\mathcal{S} = \mathcal{J}^+$: we then obtain the additional structure of asymptotic twistor space and $\mathcal{H}$- space [13], [24], [19]. In this section we will show that the additional structure arises in more general situations as well.

The section consists of three parts. First, the null hypersurface twistor space is defined and its structure described. Second, $\mathcal{H}$- space is introduced. Finally, we discuss the special case when $\mathcal{S}$ is shear-free.

We shall be considering two different cases simultaneously : the "purely local" case in which $\mathcal{S}$ has topology $\mathbb{R}^3$, and the "null cone" case in which $\mathcal{S}$ has topology $\mathbb{R}^1 \times S^2$. Let (M, g_{ab}) be a real space-time and $\mathcal{S}$ a null hypersurface in M. As coordinates on M, use $(u, r, \zeta, \bar{\zeta})$, where

(i) u and r are real, ζ is complex;

(ii) $\mathcal{S}$ is given by $u = o$;

(iii) r is an affine parameter up the null generators of $\mathcal{S}$;

(iv) ζ and $\bar{\zeta}$ label the null generators of $\mathcal{S}$.

In the "null cone" case, ζ is allowed to take on the value ∞; i.e. ζ varies over the Riemann sphere. Choose a null tetrad $\{l^a, n^a, m^a, \bar{m}^a\}$ as in [3] : the vectors l^a and m^a have the form

$$\left.\begin{aligned} l^a\nabla_a &= D = \frac{\partial}{\partial r}, \quad l_a = \nabla_a u, \\ m^a\nabla_a &= \delta = \xi\frac{\partial}{\partial\zeta} + \tilde{\xi}\frac{\partial}{\partial\bar{\zeta}} \ ; \end{aligned}\right\} \tag{3.1a}$$

and the spin coefficients satisfy

$$\left.\begin{aligned} &\kappa = \bar{\kappa} = \epsilon = \bar{\epsilon} = \rho - \bar{\rho} = \mu - \bar{\mu} = 0, \\ &\tau = \bar{\pi} = \beta + \bar{\alpha}, \\ &\bar{\tau} = \pi = \bar{\beta} + \alpha . \end{aligned}\right\} \tag{3.1b}$$

Let $\mathbb{C}M$ and $\mathbb{C}\mathscr{S}$ be complexifications of M and $\mathscr{S}$. Extend the null tetrad and spin coefficients to $\mathbb{C}M$ in the usual way (cf. §2.1).

The vector normal to $\mathbb{C}\mathscr{S}$ is $l^a = o^A o^{A'}$, so the vector $t^{AB'}\pi_{B'}\pi^{A'}$ of the previous section should be replaced by

$$o^A o^{B'} \pi_{B'} \pi^{A'} = (\pi_{B'} o^{B'}) \; o^A \pi^{A'}.$$

This suggests the following

Definition.

A hypersurface twistor (with respect to the null hypersurface $\mathscr{S}$) is a complex curve Λ in $\mathbb{C}\mathscr{S}$, together with a spinor $\pi_{A'}$ on Λ, such that

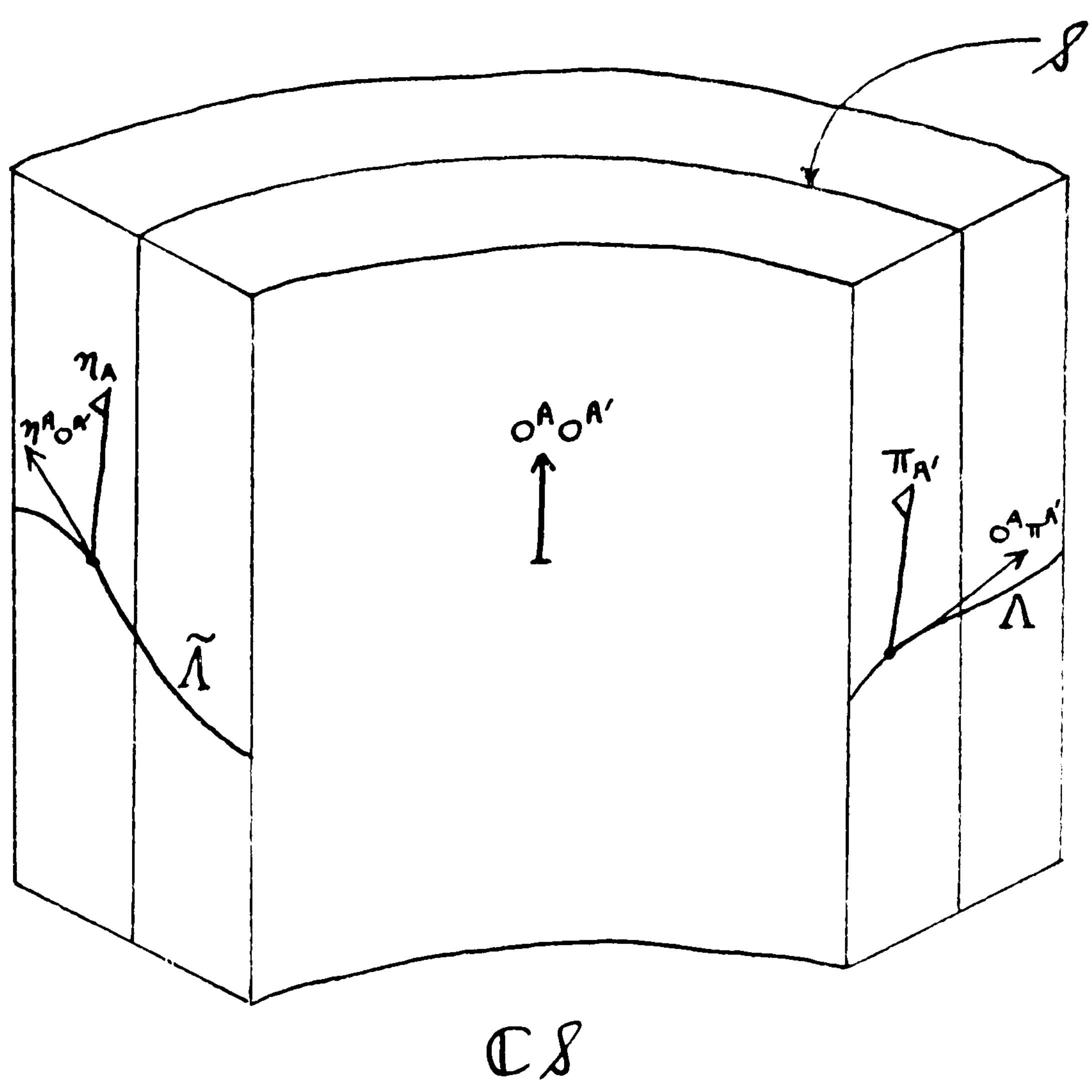
(a) the vector $o^A\pi^{A'}$ is tangent to Λ;

(b) $\pi_{A'}$ is parallelly propagated along Λ, i.e. $o^A\pi^{A'}\nabla_{AA'}\pi_{B'} = 0$; (3.2)

(c) $\pi_{0'} \neq 0$ on Λ

(see figure 3.1).

Figure 3.1.



Denote by $\mathcal{J}(\mathcal{S})$ the space of hypersurface twistors with respect to $\mathcal{S}$.

There is an alternative way of stating what $\mathcal{J}(\mathcal{S})$ is, namely as follows. Let $\mathcal{B}(\mathcal{S})$ be the primed spin-bundle over $\mathbb{C}\mathcal{S}$, with spinors proportional to $o_{A'}$ deleted; i.e. $\mathcal{B}(\mathcal{S}) = \{(x^{\underset{\sim}{a}}, \pi_{\underset{\sim}{B}'})\ x^{\underset{\sim}{a}} \in \mathbb{C}\mathcal{S},\ \pi_{0'} \neq 0\}$. Define a vector field V on $\mathcal{B}(\mathcal{S})$ by

$$V = o^{\underset{\sim}{A}}\pi^{\underset{\sim}{A}'}\left(\nabla_{\underset{\sim}{AA}'} - \tilde{\gamma}_{\underset{\sim}{B'C'AA'}}\ \pi^{\underset{\sim}{B}'}\ \frac{\partial}{\partial \pi_{\underset{\sim}{C}'}}\right). \qquad (3.3)$$

Then $\mathcal{J}(\mathcal{S})$ is the space of integral curves of V in $\mathcal{B}(\mathcal{S})$: we may regard $\mathcal{J}(\mathcal{S})$ as the quotient space $\mathcal{B}(\mathcal{S})/V$.

The condition $\pi_{0'} \neq 0$ is added for technical reasons. If $\pi_{A'}$ were allowed to be proportional to $o_{A'}$, then the corresponding hypersurface twistor curve Λ would be a (complex) generator of $\mathbb{C}\mathcal{S}$: such a curve may be called a "blown up twistor entirely on $\mathcal{S}$" (cf. [24]).

We now claim (but will not give an explicit proof here) that $\mathcal{J}(\mathcal{S})$ has the structure of a complex manifold, provided that $\mathbb{C}$M is convex and sufficiently close to Minkowski.

In similar fashion, an element of the space $\tilde{\mathcal{J}}(\mathcal{S})$ of conjugate hypersurface twistors (with respect to the null hypersurface $\mathcal{S}$) is defined to be a curve $\tilde{\Lambda}$ and a spinor η_A on $\tilde{\Lambda}$, satisfying

$$\left.\begin{array}{ll}\text{(a)} & \eta^A o^{A'} \text{ is tangent to } \tilde{\Lambda};\\ \text{(b)} & \eta^A o^{A'}\ \nabla_{AA'}\ \eta_B = 0 \text{ on } \tilde{\Lambda};\\ \text{(c)} & \eta_0 \neq 0 \text{ on } \Lambda.\end{array}\right\}$$

In addition to its complex structure, the space $\mathcal{J}(\mathcal{S})$ possesses the following three structures.

(a) A Projective structure. This is defined in exactly the same way as in the spacelike hypersurface case(cf. §4.2).

Another way of looking at this projective structure is to define a vector field Υ on $\mathcal{B}(\mathcal{S})$ by

$$\Upsilon := \pi_{\underset{\sim}{A}'} \frac{\partial}{\partial \pi_{\underset{\sim}{A}'}} .$$

Then $[\Upsilon, V] = V$, which implies that Υ projects down to a vector field $\hat{\Upsilon}$ on the quotient space $\mathcal{J}(\mathcal{S}) = \mathcal{B}(\mathcal{S})/V$ [2]. In flat twistor space, $\hat{\Upsilon}$ would just be the Euler operator $Z^\alpha \partial/\partial Z^\alpha$. The integral curves of $\hat{\Upsilon}$ in $\mathcal{J}(\mathcal{S})$ are the points of $\mathbb{P}\mathcal{J}(\mathcal{S})$.

(b) A scalar product. This is a straightforward generalization of the scalar product for asymptotic twistors [24]. The definition goes as follows. Let $(\Lambda, \pi_{A'}) \in \mathcal{J}(\mathcal{S})$ and $(\tilde{\Lambda}, \eta_A) \in \tilde{\mathcal{J}}(\mathcal{S})$. There is at most one generator γ of $\mathbb{C}\mathcal{S}$ which meets both Λ and $\tilde{\Lambda}$: suppose that such a γ exists and let p^a and q^a be the points where γ intersects Λ and $\tilde{\Lambda}$ respectively. (If such a γ does not exist, then the scalar product is not defined. For example, see figure 3.1 : in this picture, $\mathbb{C}\mathcal{S}$ does not extend far enough into the complex for γ to exist.)

We now employ the language of local twistors [19]. The hypersurface twistor $(\Lambda, \pi_{A'})$ is represented by the local twistor $(\omega^A, \pi_{A'}) = (0, \pi_{A'})$ at p^a. This local twistor is then propagated up γ from p^a to q^a by local twistor transport :

$$\left.\begin{aligned} o^A o^{A'} \nabla_{AA'} \omega^B(x) &= -i \pi_{0'}(x) o^B, \\ o^A o^{A'} \nabla_{AA'} \pi_{B'}(x) &= -i P_{00'BB'}(x) \omega^B(x), \end{aligned}\right\} \qquad (3.4)$$

where $P_{AA'BB'} := \Phi_{ABA'B'} - \Lambda\varepsilon_{AB}\varepsilon_{A'B'}$. The conjugate hypersurface twistor $(\tilde{\Lambda}, \eta_A)$ is represented by the conjugate local twistor $(\eta_A, \xi^{A'}) = (\eta_A, 0)$ at q^a. The scalar product of $(\Lambda, \pi_{A'})$ and $(\tilde{\Lambda}, \eta_A)$ is defined to be the scalar product of the corresponding local twistors at q^a, namely

$$\omega^A(q)\ \eta_A(q) + \pi_{A'}(q)\ \xi^{A'}(q) = \omega^A(q)\ \eta_A(q).$$

Remarks. (i) We could also propagate the conjugate local twistor $(\eta_A, \xi^{A'})$ down γ from q^a to p^a by

$$\left.\begin{aligned} o^A o^{A'} \nabla_{AA'}\ \xi^{B'}(x) &= i\,\eta_o(x)\ o^{B'}, \\ o^A o^{A'} \nabla_{AA'}\ \eta_B(x) &= i\ P_{CO'BB'}(x)\ \xi^{B'}, \end{aligned}\right\} \qquad (3.5)$$

and then take the scalar product at p^a. This gives the same value for the scalar product, since (3.4) and (3.5) imply that

$$\omega^A(x)\ \eta_A(x) + \pi_{A'}(x)\ \xi^{A'}(x)$$

is constant along γ.

(ii) In the case when (M, g_{ab}) is Minkowski space-time, the above scalar product agrees with the usual flat-space scalar product $Z^\alpha \tilde{Z}_\alpha$. To see this, notice that in flat space the solution of (3.4) is

$$\left.\begin{aligned} \pi_{B'} &= \text{constant}, \\ \omega^B(x) &= i(p^{BB'} - x^{BB'})\ \pi_{B'}. \end{aligned}\right\}$$

So $\omega^B(q)\ \eta_B(q) = i(p^{BB'} - q^{BB'})\pi_{B'}\ \eta_B$

$$= (i\ p^{BB'}\pi_{B'})\ \eta_B + (-i\ q^{BB'}\eta_B)\pi_{B'}$$

$$= Z^\alpha \tilde{Z}_\alpha,$$

using equation (2.1.7) and its complex conjugate version.

(iii) The scalar product of $(\Lambda, \pi_{A'})$ and $(\tilde{\Lambda}, \eta_A)$ is zero iff $\Lambda \cap \tilde{\Lambda} \neq \phi$. To show this, it is sufficient to show that

$$\omega^B(q)\ \eta_B(q) = 0 \Leftrightarrow p^a = q^a.$$

The $\Leftarrow$ part follows immediately from the definition. Conversely, (3.4) implies that $\omega^B(q) = k o^B(q)$ for some scalar k, so $\omega^B(q)\ \eta_B(q) = 0 \Rightarrow \omega^B(q) = 0$ (since $\eta^B o_B \neq 0$). It follows that $q^a = p^a$, as required.

(c) A Kähler Structure. The scalar product serves as a Kähler scalar on $\mathcal{T}(\mathcal{S})$ [12]. If Z^α are local complex coordinates on $\mathcal{T}(\mathcal{S})$ and the scalar product is $g(Z^\alpha, \overline{Z^\alpha})$, then the Kahler 2 - form is

$$\frac{\partial^2 g(Z^\beta, \overline{Z^\beta})}{\partial Z^\beta\ \partial \overline{Z^\beta}}\ dZ^\beta \wedge d\overline{Z^\beta}\ .$$

Remark. The hypersurface twistor space structure is conformally invariant, in the following sense:

Theorem.

The structure of $\mathcal{T}(\mathcal{S})$ is invariant under a conformal rescaling of the space-time metric :

$$g_{ab} \longmapsto \hat{g}_{ab} = \Omega^2\ g_{ab}.$$

Proof.

Take $\pi_{A'}$ and o_A to be invariant under the rescaling: i.e. $\hat{\pi}_{A'} = \pi_{A'}$ and $\hat{o}_A = o_A$. Then the conditions (3.2) are preserved under the rescaling; for example (using the transformation formulae of [24])

$$\hat{o}^A\ \hat{\pi}^{A'}\ \hat{\nabla}_{AA'}\ \hat{\pi}_{B'} = \Omega^{-2}\ o^A\ \pi^{A'}\{\nabla_{AA'}\ \pi_{B'} - \Omega^{-1}\ \pi_{A'}\ \nabla_{AB'}\ \Omega\}$$

$$= \Omega^{-2}\ o^A\ \pi^{A'}\ \nabla_{AA'}\ \pi_{B'}\ ,$$

so $\hat{o}^A\hat{\pi}^{A'}\ \hat{\nabla}_{AA'}\ \hat{\pi}_{B'} = 0 \Leftrightarrow o^A\ \pi^{A'}\ \nabla_{AA'}\pi_{B'} = 0.$

The scalar product is also invariant; this follows from the conformal invariance of local twistor transport and of the local twistor scalar product [19]. □

We come now to the concept of $\mathcal{H}$- space. For the remainder of this section, suppose that $\mathscr{I}$ is of the "null-cone" type (i.e. with topology $\mathbb{R}^1 \times S^2$). The space $\mathcal{H}$ is defined to be the space of compact holomorphic curves (with topology S^2) in $\mathbb{P}\mathcal{J}(\mathscr{I})$, belonging to a certain homology class. To specify what this homology class is, it is sufficient to consider the case when $\mathbb{P}\mathcal{J}(\mathscr{I}) = \mathbb{PT}$, i.e. flat twistor space (this is because homology is preserved under sufficiently small deformations [12]). The homology class is taken to be that one to which the projective straight lines in $\mathbb{PT}$ belong. For further details, see [20].

[20] also contains an existence theorem which can be stated as follows.

Theorem. Provided $\mathbb{P}\mathcal{J}(\mathscr{I})$ is a sufficiently small deformation of $\mathbb{PT}$ (i.e. provided M is sufficiently close to Minkowski), the space $\mathcal{H}$ is a 4-dimensional complex manifold.

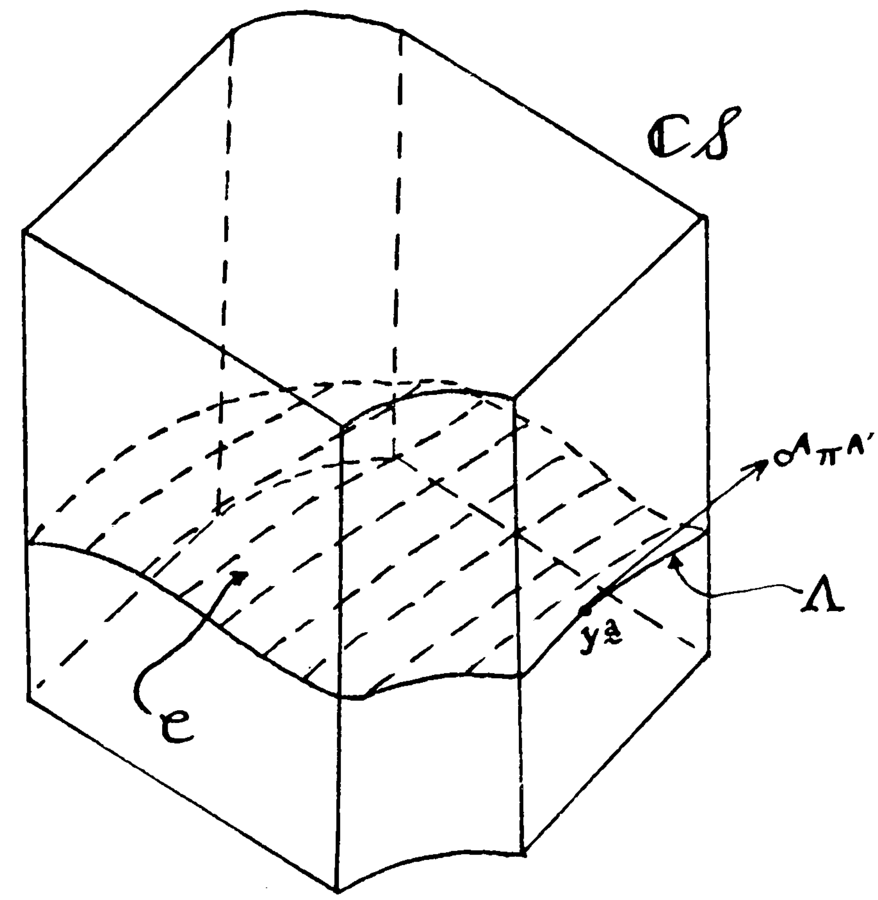
The points of $\mathcal{H}$ have an interpretation in the space-time picture, namely as cross-sections of $\mathbb{C}\mathscr{I}$. By a cross-section of $\mathbb{C}\mathscr{I}$ we mean a complex 2-surface $\mathcal{C}$ in $\mathbb{C}\mathscr{I}$ given by an equation of the form $\mathfrak{r} = f(\zeta, \tilde{\zeta})$, where $f(\zeta,\tilde{\zeta})$ is holomorphic in some neighbourhood of $\tilde{\zeta} = \bar{\zeta}$. In other words, $\mathcal{C}$ intersects each of the generators of $\mathbb{C}\mathscr{I}$ at most once. A point of $\mathcal{H}$ is an S^2's worth of projective twistors, each of which corresponds to a curve Λ in $\mathbb{C}\mathscr{I}$. So we have an S^2's worth of curves in $\mathbb{C}\mathscr{I}$ making up a cross-section which is ruled by hypersurface twistor curves (see figure 3.2) Such a cross-section will be called a good cut of $\mathbb{C}\mathscr{I}$.

Theorem. The cross-section $\mathcal{C}$ of $\mathbb{C}\mathscr{I}$ given by $\mathfrak{r} = f(\zeta,\tilde{\zeta})$ is a good cut iff

$$\delta^2 f + 2\tilde{\rho}(\delta f)^2 + (\tilde{\pi} + 2\tilde{\alpha})(\delta f) + \sigma(\delta f)(\tilde{\delta} f) + \tilde{\lambda} = 0$$

on $\mathfrak{r} = f(\zeta,\tilde{\zeta})$.

Figure 3.2.



Proof. Write $F(x^a) := f(\zeta,\tilde{\zeta}) - r$, so that $\mathcal{C}$ is given by $F(x^a) = 0$. Let y^a be a point on $\mathcal{C}$ and let $(\Lambda, \pi_{A'})$ be a hypersurface twistor such that Λ intersects, and is tangent to, $\mathcal{C}$ at y^a (see figure 3.2). The condition for Λ to be tangent to $\mathcal{C}$ at y^a is

$$o^A \pi^A \nabla_{AA'} F(x^b) \Big|_{x^b = y^b} = 0 . \qquad (3.6)$$

The condition that Λ should lie entirely in $\mathcal{C}$ is that equation (3.6) should be preserved along Λ, i.e. that

$$0 = o^B \pi^B \nabla_{BB'} (o^A \pi^{A'} \nabla_{AA'} F)$$

$$= \pi^{B'} \pi^{A'} \nabla_{OB'} \nabla_{OA'} F \qquad \text{using (3.2)} \qquad (3.7)$$

$$= \pi^{B'} \pi^{A'} \nabla_{OB'} \nabla_{OA'} F - \gamma^{C'}{}_{A'OB'} \pi^{B'} \pi^{A'} \nabla_{OC'} F,$$

using (2.1.2).

Now equation (3.6) implies that $\pi_{A'}$ is proportional to

$$o^A \nabla_{AA'} F = (o_{A'} \delta - \iota_{A'} D)(f(\zeta,\tilde{\zeta}) - r)$$

$$= o_{A'} (\delta f) + \iota_{A'} \text{ at } y^a. \qquad (3.8)$$

Since (3.7) is preserved under multiplication by some nonzero scalar, we can make the substitution $\pi_{\underset{\sim}{A}'} \longmapsto k\, o_{\underset{\sim}{A}'} + \iota_{\underset{\sim}{A}'}$ (3.9)

(where $k := f(y^{\underset{\sim}{a}})$) in (3.7), giving

$$0 = (k\, o^{\underset{\sim}{A}'} + \iota^{\underset{\sim}{A}'})(kD + \delta)\left[(\delta f) o_{\underset{\sim}{A}'} + \iota_{\underset{\sim}{A}'}\right]$$

$$+ \tilde{\gamma}_{\underset{\sim}{C}'\underset{\sim}{A}'\underset{\sim}{B}'}(k\, o^{\underset{\sim}{B}'} + \iota^{\underset{\sim}{B}'})(k\, o^{\underset{\sim}{A}'} + \iota^{\underset{\sim}{A}'})(k\, o^{\underset{\sim}{C}'} + \iota^{\underset{\sim}{C}'})$$

$$= \delta^2 f + k\, D\delta f + \tilde{\rho}\, k^2 + (\tilde{\pi} + 2\tilde{\alpha})\, k + \tilde{\lambda} \qquad (3.10)$$

Now $D\delta f = (D\delta - \delta D) f = (\sigma\tilde{\delta} + \tilde{\rho}\,\delta) f$ (see for example, [14])

$$= \sigma\,\tilde{\delta} f + \tilde{\rho}\,\delta f,$$

so that equation (3.10) becomes

$$\delta^2 f + 2\tilde{\rho}(\delta f)^2 + (\tilde{\pi} + 2\tilde{\alpha})(\delta f) + \sigma(\delta f)(\tilde{\delta} f) + \tilde{\lambda} = 0 \qquad (3.11)$$

at the point $y^{\underset{\sim}{a}}$. Thus a necessary and sufficient condition for $\mathcal{C}$ to be ruled by twistor curves is that (3.11) should hold on $\mathcal{C}$. □

Equation (3.11) is a generalization of the "good cut" equation of Newman ([13], equation 11), which applies to the particular case when $\mathcal{S} = \mathcal{I}^+$. Newman defines a good cut to be a cross-section which is shear-free, in the following sense.

Definition. A cross-section of $\mathbb{C}\mathcal{S}$ given by $F(x^{\underset{\sim}{a}}) = f(\zeta,\tilde{\zeta}) - r = 0$ is said to be shear-free if F can be extended off $\mathbb{C}\mathcal{S}$ in such a way that the vector field $\xi^a := -\nabla^a F$ is null and shear-free (i.e. $\xi_{AA'} = \xi_A \xi_{A'}$, $o^A \xi^{A'} \xi^{B'} \nabla_{AA'}\, \xi_{B'} = 0$: cf. [17]).

But Newman's definition is consistent with ours:

Theorem. A cross-section of $\mathbb{C}\mathcal{S}$ is shear-free iff it is a good cut.

Proof. Let $F(x^{a}) = f(\zeta,\tilde{\zeta}) - r = 0$ be a cross-section of $\mathbb{C}\mathcal{S}$. Then

$$\xi_a := -\nabla_a F$$

$$= - g_a{}^{b}\nabla_b F$$

$$= - (\ell_a n^b + n_a \ell^b - m_a \tilde{m}^b - \tilde{m}_a m^b)\nabla_b F$$

$$= -\ell_a \Delta F + n_a + m_a \tilde{\delta} F + \tilde{m}_a \delta f. \qquad (3.12)$$

The quantity ΔF is only defined if we know what F is off $\mathbb{C}\mathcal{S}$; if we extend F off $\mathbb{C}\mathcal{S}$ in such a way that $\Delta F = -(\delta f)(\tilde{\delta} f)$, then ξ_a will be null, since (3.12) implies that

$$\xi_a \xi^a = - 2\left[\Delta F + (\delta f)(\tilde{\delta} f)\right].$$

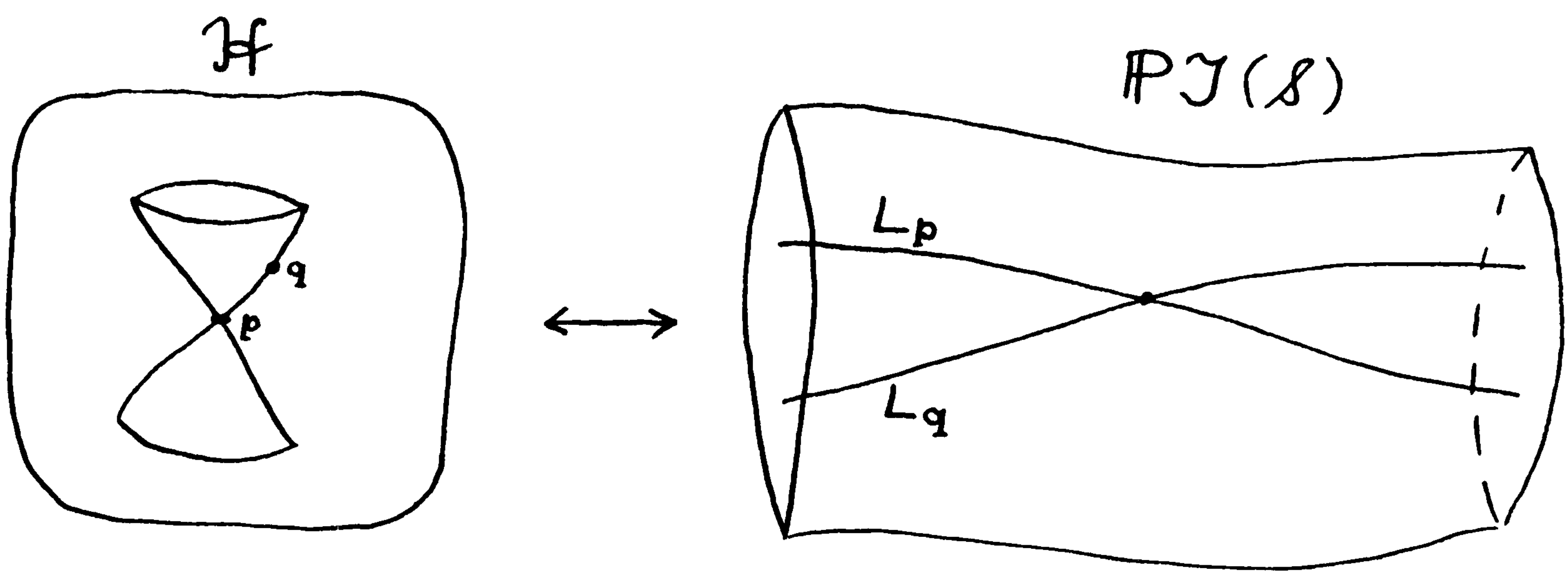
It follows that ξ_a has the form $\xi_A \xi_{A'}$, where

$$\xi_{A'} = (\delta f)\, o_{A'} + \iota_{A'}. \qquad (3.13)$$

Now, combining equations (3.7), (3.8), (3.9) and (3.13) we see that the shear-free condition $o^A \xi^{A'} \xi^{B'} \nabla_{AA'} \xi_{B'} = 0$ is equivalent to the good cut condition (3.11). □

The space $\mathcal{H}$ has a natural conformal structure, defined as follows. Let p and q be two neighbouring points in $\mathcal{H}$ and let L_p and L_q be the corresponding curves in $\mathbb{P}\mathcal{T}(\mathcal{S})$ (see figure 3.3).
Then p and q are defined to have a null separation if and only if the curves L_p and L_q intersect. It is proved in [20] that this concept of null separation is in fact a quadratic (i.e. Riemannian) conformal structure.

Figure 3.3.



Theorem. The conformal structure of $\mathcal{H}$ is right-conformally-flat (i.e. $\tilde{\Psi}_{A'B'C'D'} = 0$: cf. §2.1).

Proof. Fix a point $Z \in \mathbb{P}\mathcal{T}(\mathcal{S})$, and consider all the compact holomorphic curves through Z : these form a 2-complex-parameter family. The corresponding object in $\mathcal{H}$ is a 2-surface, and this 2-surface is totally null, since each holomorphic curve intersects all the others in the family (namely at Z). So there exists a 3-parameter family of totally null 2-surfaces in $\mathcal{H}$ (one for each point of $\mathbb{P}\mathcal{T}(\mathcal{S})$).

By convention, we say that these totally null 2-surfaces are of primed type (cf. §2.1). The reason for making this choice is that the fixed twistor Z would, if (M, g_{ab}) were Minkowski, correspond to a totally null 2-plane of primed type in $\mathbb{C}M$. But the integrability condition for the existence of a 3-parameter family of TN2S's is $\tilde{\Psi}_{A'B'C'D'} = 0$ [20]. This proves the theorem. □

To sum up: associated with the null hypersurface $\mathcal{S}$ is a 4-dimensional complex manifold $\mathcal{H}$, the points of which may be regarded either as good cuts of $\mathbb{C}\mathcal{S}$, or as compact holomorphic curves in $\mathbb{P}\mathcal{T}(\mathcal{S})$. It has a natural conformal structure which is right-conformally-flat.

Finally, let us consider the special case of $\mathcal{S}$ being shear-free (i.e. $\sigma = \tilde{\sigma} = 0$). This condition is, for example, satisfied by the $\mathscr{I}^+$ of an asymptotically flat space-time [18].

The commutator equation between D and δ now becomes

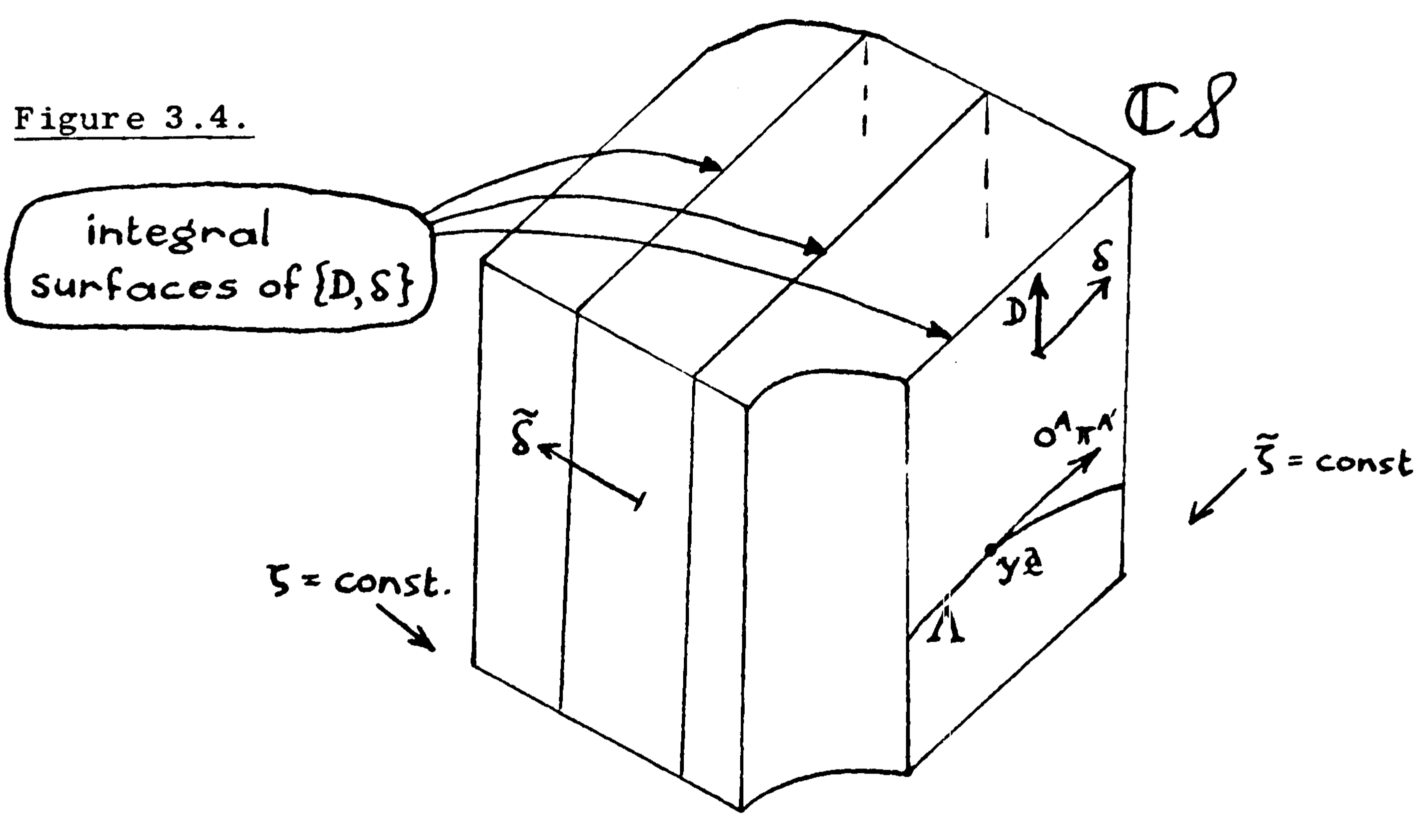
$$D\delta - D\delta = \tilde{\rho}\,\delta \;; \tag{3.14}$$

in other words, the pair of vector fields $\{D, \delta\}$ spans an involutive distribution [2]. Therefore, Frobenius' theorem implies that $\mathbb{C}\mathcal{S}$ is fibred into 2-surfaces, each of which has tangent vectors of the form

$$\begin{aligned} u^a &= b\ell^a + cm^a \\ &= o^A\,(bo^{A'} + c\iota^{A'}), \end{aligned} \tag{3.15}$$

where b and c are arbitrary complex numbers. We refer to such a surface as a totally null 2-surface of unprimed type, or $\overline{\text{TN2S}}$ (see figure 3.4). Without loss of generality, the coordinates ζ and $\tilde{\zeta}$ can be chosen so that the $\overline{\text{TN2S}}$'s are given by $\tilde{\zeta}$ = constant, and the (complex) generators of $\mathbb{C}\mathcal{S}$ by ζ = const., $\tilde{\zeta}$ = const.

Figure 3.4.



Theorem. Each hypersurface twistor curve Λ lies entirely in a $\overline{\text{TN2S}}$.

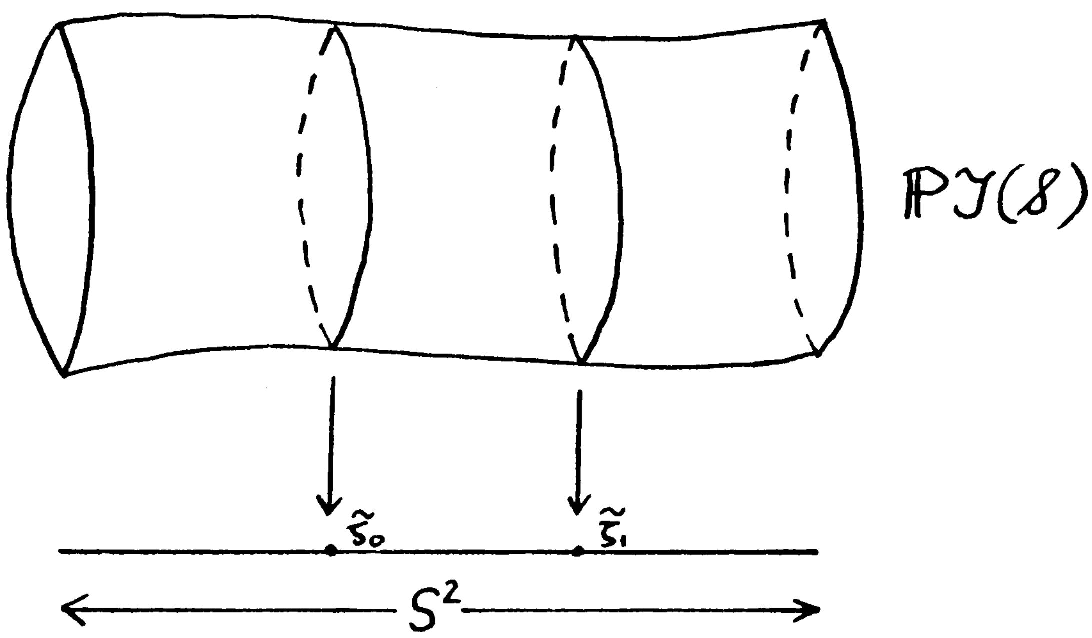
Proof. Let Λ be a hypersurface twistor curve and let $y^{\underset{\sim}{a}}$ be some point on Λ. Suppose that $y^{\underset{\sim}{a}}$ lies in the $\overline{\text{TN2S}}$ given by $\tilde{\zeta} = \tilde{\zeta}_0$. To prove the theorem, it suffices to prove that Λ is tangent to this $\overline{\text{TN2S}}$ at the arbitrary point $y^{\underset{\sim}{a}}$.

But this follows immediately from the definitions:

the tangent vector to Λ at $y^{\underset{\sim}{a}}$ has the form $o^A \pi^{A'}$, which is of the form (3.15). □

It follows that $\mathbb{P}\mathcal{J}(\mathcal{S})$ has a fibred structure: $\mathbb{P}\mathcal{J}(\mathcal{S})$ is a bundle over the space of $\overline{\text{TN2S}}$'s, i.e. over the $\tilde{\zeta}$-sphere (see figure 3.5).

Figure 3.5.



In the case $\mathcal{S} = \mathcal{I}^+$, this $\tilde{\zeta}$-sphere is called the (projective) asymptotic spin-space [24], and it is an important part of the asymptotic structure.

§4.4. Propagation of Hypersurface Twistors.

In [24] there is a discussion of how a twistor is "scattered" through a weak impulsive gravitational wave. The scattering turns out to have the form of a canonical transformation

$$\left.\begin{aligned} Z^{\alpha} &\longmapsto Z^{\alpha} - i \frac{\partial}{\partial \bar{Z}_{\alpha}} H(Z^{\beta}, \bar{Z}_{\beta}), \\ \bar{Z}_{\alpha} &\longmapsto \bar{Z}_{\alpha} + i \frac{\partial}{\partial Z^{\alpha}} H(Z^{\beta}, \bar{Z}_{\beta}), \end{aligned}\right\}$$

where H is a suitable real Hamiltonian. The transformation preserves the symplectic form $dZ^{\alpha} \wedge d\bar{Z}_{\alpha}$. In this section we consider the propagation of a twistor through an analytic plane-fronted wave with parallel rays (pp - wave for short), using the formalism of hypersurface twistors. It will turn out that the propagation is given by the continuous unfolding of a canonical transformation.

The metric of a real-analytic pp-wave has the form [4]

$$ds^2 = 2\,du\,dv - 2d\zeta d\bar{\zeta} + 2h(v,\zeta,\bar{\zeta})\,dv^2,$$

where (i) u and v are real coordinates;

(ii) ζ and $\bar{\zeta}$ are complex, ranging over the Riemann sphere;

(iii) h is real-analytic.

If $h = 0$, the space-time is Minkowski.

We set up a null tetrad as follows

$$D = \frac{\partial}{\partial u}, \quad \Delta = -h\frac{\partial}{\partial u} + \frac{\partial}{\partial v}, \quad \delta = \frac{\partial}{\partial \zeta}, \quad \bar{\delta} = \frac{\partial}{\partial \bar{\zeta}} \quad . \tag{4.1}$$

The only nonzero spin coefficients turn out to be $\nu = \bar{\delta}h$ and $\bar{\nu} = \delta h$; in other words,

$$\bar{\gamma}_{B'C'AA'} = (\delta h)\, o_{B'} o_{C'} o_A o_{A'} \; . \tag{4.2}$$

The Ricci tensor is

$$R_{ab} = -2\,(\delta\bar{\delta} h)\,\ell_a \ell_b \;,$$

so Einstein's vacuum equations are satisfied iff

$$\delta\tilde{\delta} h = \frac{\partial^2 h}{\partial\zeta\partial\tilde{\zeta}} = 0,$$

i.e. iff h has the form

$$h(v,\zeta,\bar{\zeta}) = h^{+}(v,\zeta) + h^{-}(v,\bar{\zeta}), \tag{4.3}$$

with $h^{+} = \overline{h^{-}}$.

Let $\mathscr{S}_o$ be the null hypersurface $v = v_o$, where v_o is a real constant. We shall consider hypersurface twistors with respect to $\mathscr{S}_o$. Complexify by allowing u to become complex, replacing $\bar{\zeta}$ by $\tilde{\zeta}$, etc. Recall from §4.3 that hypersurface twistors can be regarded as integral curves in $\mathscr{B}(\mathscr{S}_o)$ of the vector field

$$V = o^{A}_{\sim}\pi^{A'}_{\sim}\left(\nabla_{\underset{\sim}{AA'}} - \tilde{\gamma}_{\underset{\sim}{B'C'AA'}}\,\pi^{B'}_{\sim}\frac{\partial}{\partial\pi_{\underset{\sim}{C'}}}\right)$$

$$= o^{A}\pi^{A'}\nabla_{AA'} \qquad \text{using (4.2)}$$

$$= \pi^{o'} D + \pi^{1'}\delta$$

$$= \pi_{1'} D - \pi_{o'}\delta\ . \tag{4.4}$$

We see from equation (4.4) that V does not involve the function h, so we expect that the twistor space $\mathscr{T}(\mathscr{S}_o)$ will turn out to be <u>flat</u> (i.e. a subset of $\mathbb{C}^4 - \{0\}$). This is because V does not "notice" whether or not h is zero, and h = 0 corresponds to Minkowski space-time.

We shall now verify that $\mathscr{T}(\mathscr{S}_o)$ <u>is</u> flat, by defining natural coordinates $Z^{\alpha} = (\omega^{A}_{\sim}, \pi_{\underset{\sim}{A'}})$ on $\mathscr{T}(\mathscr{S}_o)$.

Let $\omega^{A}_{\sim}$ be the pair of functions on $\mathscr{B}(\mathscr{S}_o)$ defined by

$$\omega^{A}_{\sim} := i(u\pi_{o'} + \zeta\pi_{1'},\ \tilde{\zeta}\pi_{o'} + v_o\,\pi_{1'}). \tag{4.5}$$

The motivation for this definition is the equation $\omega^{A} = i\,x^{AA'}\pi_{A'}$.

Theorem. $Z^\alpha = (\omega^{\underset{\sim}{A}}, \pi_{\underset{\sim}{A}'})$ serve as global coordinates on $\mathcal{J}(\mathcal{S}_o)$.

Proof. First, notice that $V(\pi_{\underset{\sim}{A}'}) = 0$ and $V(\omega^{\underset{\sim}{A}}) = 0$, so the four functions Z^α are constant along the integral curves of V, i.e. for each hypersurface twistor they take on a constant value.

Conversely, given $(\omega^{\underset{\sim}{A}}, \pi_{\underset{\sim}{A}'})$ with $\pi_{0'} \neq 0$, we can employ (4.5) to obtain a curve Γ in $\mathcal{B}(\mathcal{S}_o)$: Γ is given by

$$\left.\begin{aligned} v &= v_o, \\ \tilde{\zeta} &= -\frac{i}{\pi_{0'}}(\omega^1 - i\, v_o\, \pi_{1'}), \\ u &= -\frac{i}{\pi_{0'}}(\omega^0 - i\, \zeta\, \pi_{1'}), \\ \zeta &= \zeta, \\ \pi_{A'} &= \text{constant}. \end{aligned}\right\} \qquad (4.6)$$

The tangent vector to Γ is

$$\frac{\partial x^{\underset{\sim}{a}}}{\partial \zeta}\frac{\partial}{\partial x^{\underset{\sim}{a}}} + \frac{\partial \pi_{\underset{\sim}{A}'}}{\partial \zeta}\frac{\partial}{\partial \pi_{\underset{\sim}{A}'}} = -\frac{\pi_{1'}}{\pi_{0'}}\frac{\partial}{\partial u} + \frac{\partial}{\partial \zeta} = -\frac{1}{\pi_{0'}} V,$$

so Γ is indeed an integral curve of V, i.e. a hypersurface twistor. □

In a similar way we can deal with the conjugate hypersurface twistor space $\tilde{\mathcal{J}}(\mathcal{S}_o)$. Elements of $\tilde{\mathcal{J}}(\mathcal{S}_o)$ correspond to integral curves of

$$\tilde{V} = \eta_1 D - \eta_0 \tilde{\delta}$$

and are labelled by coordinates $\tilde{Z}_\alpha = (\eta_{\underset{\sim}{A}}, \xi^{\underset{\sim}{A}'})$, where

$$\xi^{\underset{\sim}{A}'} := -i(u\eta_0 + \tilde{\zeta}\eta_1,\ \ \zeta\eta_0 + v_o\,\eta_1).$$

<u>Theorem</u>. Let $Z^\alpha = (\omega^{A}, \pi_{A'})$ be the coordinates of an element $(\Lambda, \pi_{A'})$ of $\mathcal{J}(\mathcal{S}_o)$. Put $\bar{Z}_\alpha = (\bar{\pi}_A, \bar{\omega}^{A'})$. Then Λ contains a real point iff $Z^\alpha \bar{Z}_\alpha = 0$, in which case $(\Lambda, \pi_{A'})$ is said to be <u>null</u>.

<u>Proof</u>. If Λ does have a real point, then this point has to be given by

$$\left.\begin{aligned} v &= v_o, \\ \bar{\zeta} &= -\frac{i}{\pi_{o'}}(\omega^1 - i\, v_o \pi_{1'}), \\ \zeta &= \frac{i}{\bar{\pi}_o}(\bar{\omega}^{1'} + i\, v_o\, \bar{\pi}_1), \\ u &= -\frac{i}{\pi_{o'}}\left[\omega^o + \frac{\pi_{1'}}{\bar{\pi}_o}(\bar{\omega}^{1'} + i\, v_o\, \bar{\pi}_1)\right], \end{aligned}\right\} \tag{4.7}$$

using equation (4.6). The condition for u to be real is

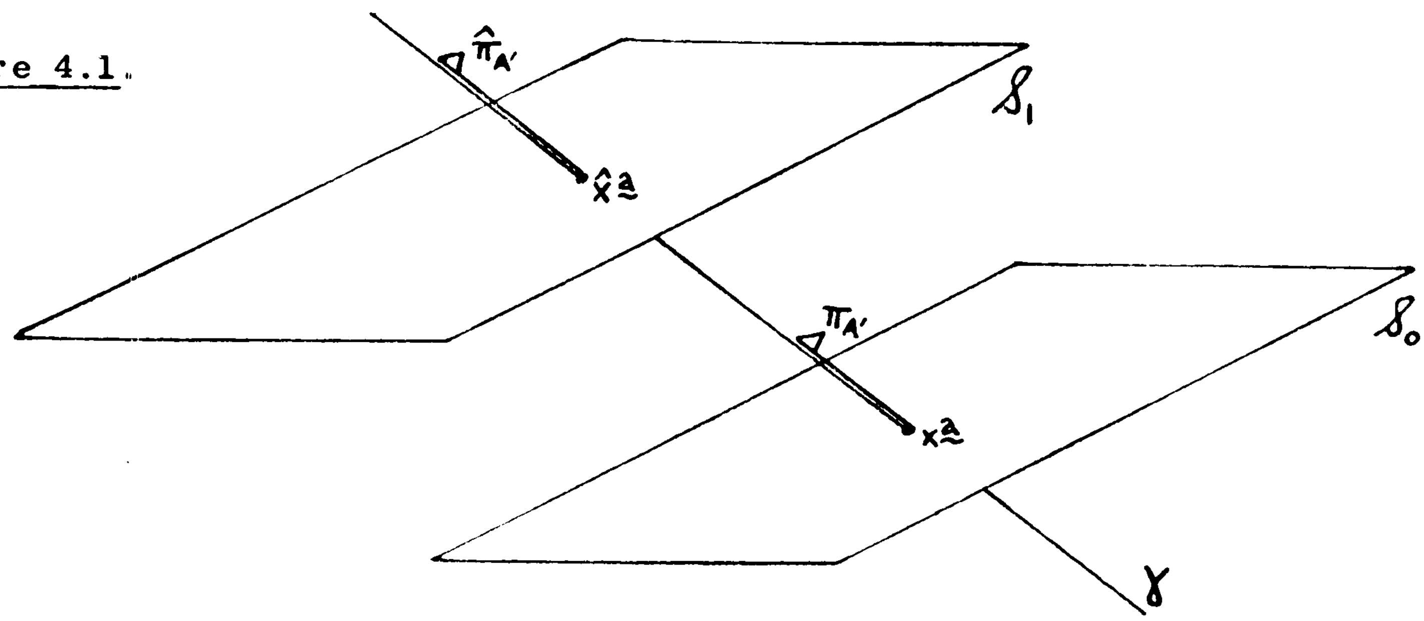
$$\begin{aligned} 0 &= u - \bar{u} \\ &= -\frac{i}{\bar{\pi}_o \pi_{o'}}\left[\omega^o\, \bar{\pi}_o + \bar{\omega}^{1'} \pi_{1'} + i\, v_o\, \bar{\pi}_1\, \pi_{1'} + \bar{\omega}^{o'}\, \pi_{o'} + \omega^1\, \bar{\pi}_1 - i\, v_o\, \bar{\pi}_1\, \pi_{1'}\right] \\ &= -\frac{i}{\bar{\pi}_o \pi_{o'}} Z^\alpha \bar{Z}_\alpha, \end{aligned}$$

so $Z^\alpha \bar{Z}_\alpha = 0 \iff \Lambda$ has a real point. □

We now discuss the <u>propagation</u> of a twistor from $\mathcal{S}_o$ to a neighbouring hypersurface $\mathcal{S}_1$, (given by $v = v_1$). The propagation law for <u>null</u> twistors is defined as follows. Given a null hypersurface twistor (with coordinates Z^α) in $\mathcal{J}(\mathcal{S}_o)$, let x^a denote the real point given by (4.7). Let γ be the real null geodesic through x^a with tangent vector $\bar{\pi}^A \pi^{A'}$ at x^a; suppose $\pi_{A'}$ to be parallelly propagated up γ. Let $\hat{x}^a$ be the point where γ intersects $\mathcal{S}_1$ and let $\hat{\pi}_{A'}$ denote the value of the spinor $\pi_{A'}$ at $\hat{x}^a$ (see figure 4.1).

Having $\hat{\pi}_{A'}$ at $\hat{x}^{\underset{\sim}{a}}$ determines an element $(\hat{\Lambda}, \hat{\pi}_{A'})$ of $\mathcal{T}(\mathcal{S}_1)$; suppose that the coordinates of this hypersurface twistor are $\hat{Z}^{\alpha}$. We want to find an expression for $\hat{Z}^{\alpha}$ in terms of Z^{α}. Let us consider first the infinitesimal case.

Figure 4.1.



Theorem. If $v_1 - v_o =: \varepsilon$ is infinitesimal, then

$$\hat{Z}^{\alpha} = Z^{\alpha} - i\varepsilon \frac{\partial}{\partial \bar{Z}_{\alpha}} H(v_o, Z^{\beta}, \bar{Z}_{\beta})$$

where $H(v, Z^{\beta}, \bar{Z}_{\beta}) := \bar{\pi}_o \; \pi_{o'} \; h\left(v, \frac{i\bar{\omega}^{1'} - v\bar{\pi}_1}{\bar{\pi}_o}, \frac{-i\omega^1 - v\pi_{1'}}{\pi_{o'}}\right)$.

Proof. Write $x^{\underset{\sim}{a}} = (u, v, \zeta, \bar{\zeta})$ (so these quantities are given by (4.7)) and $\hat{x}^{\underset{\sim}{a}} = (\hat{u}, \hat{v}, \hat{\zeta}, \hat{\bar{\zeta}})$.

The vector tangent to γ is $p^a := \bar{\pi}^A \pi^{A'}$, and

$$p^a \nabla_a = p_b \, g^{ab} \nabla_a$$

$$= p_b \, (\ell^a n^b + \ell^b n^a - m^a \bar{m}^b - m^b \bar{m}^a) \nabla_a$$

$$= \bar{\pi}_1 \; \pi_{1'} \frac{\partial}{\partial u} + \bar{\pi}_o \; \pi_{o'} \left(-h \frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)$$

$$- \bar{\pi}_1 \; \pi_{o'} \frac{\partial}{\partial \zeta} - \bar{\pi}_o \; \pi_{1'} \frac{\partial}{\partial \bar{\zeta}} \; .$$

Thus moving a distance λ along γ means that

$$\left.\begin{aligned} u &\longmapsto u + \lambda(\bar{\pi}_1\pi_{1'} - h\bar{\pi}_o\pi_{o'}), \\ v &\longmapsto v + \lambda\,\bar{\pi}_o\pi_{o'}, \\ \zeta &\longmapsto \zeta - \lambda\bar{\pi}_1\pi_{o'}, \\ \bar{\zeta} &\longmapsto \bar{\zeta} - \lambda\bar{\pi}_o\pi_{1'}. \end{aligned}\right\} \qquad (4.8)$$

We want $v \longmapsto \hat{v} = v + \varepsilon$, so $\lambda = \varepsilon/\bar{\pi}_o\pi_{o'}$.

Writing $\mu := \pi_{1'}/\pi_{o'}$, the transformations (4.8) become

$$\left.\begin{aligned} u &\longmapsto \hat{u} = u + \varepsilon(\mu\bar{\mu} - h), \\ v &\longmapsto \hat{v} = v + \varepsilon, \\ \zeta &\longmapsto \hat{\zeta} = \zeta - \varepsilon\bar{\mu}, \\ \bar{\zeta} &\longmapsto \hat{\bar{\zeta}} = \bar{\zeta} - \varepsilon\mu. \end{aligned}\right\} \qquad (4.9)$$

Now the parallel propagation of $\pi_{A'}$ means (using equation (2.1.2)) that

$$0 = \varepsilon_{\underset{\sim}{B'}}{}^{B'} \; \lambda p^a \, \nabla_a \; \pi_{B'}$$

$$= \lambda\, p^{AA'} \, (\nabla_{AA'} \pi_{\underset{\sim}{B'}} - \gamma_{\underset{\sim}{B'}}{}^{C'}{}_{AA'}\, \pi_{C'})$$

$$= (\hat{\pi}_{\underset{\sim}{B'}} - \pi_{\underset{\sim}{B'}}) + \frac{\varepsilon}{\bar{\pi}_o\pi_{o'}} \; \bar{\pi}^A \, \pi^{A'} \, \pi^{C'} \, \bar{\gamma}_{\underset{\sim}{B'}\,C'\,AA'}$$

$$= (\hat{\pi}_{\underset{\sim}{B'}} - \pi_{\underset{\sim}{B'}}) - \varepsilon\pi_{o'}(\delta h)\, o_{\underset{\sim}{B'}} \qquad \text{by equation (4.2)}.$$

Thus $\hat{\pi}_{B'} = \pi_{B'} + \varepsilon\pi_{O'}\,(\delta h)\, o_{B'}$. (4.10)

The "hatted" version of equation (4.5) gives

$$\hat{\omega}^{A}_{\sim} = \perp (\hat{u}\, \hat{\pi}_{o'} + \hat{\zeta}\, \hat{\pi}_{1'},\ \hat{\bar{\zeta}}\, \hat{\pi}_{o'} + \hat{v}\, \hat{\pi}_{1'})$$

$$= \omega^{A}_{\sim} + i\varepsilon((\mu\bar{\mu} - h)\, \pi_{o'} - \bar{\mu}\pi_{1'} + \zeta\, \delta h\, \pi_{o'} ,$$

$$- \mu\pi_{o'} + \pi_{1'} + v_o\, \delta h\, \pi_{o'})$$

$$= \omega^{A}_{\sim} - i\, \varepsilon\, \pi_{o'}(h - \zeta\, \delta h,\ - v_o\, \delta h), \qquad (4.11)$$

where use has been made of (4.9) and (4.10), and where terms of the second order in ε have been ignored.

Defining

$$H(v, Z^{\alpha}, \bar{Z}_{\alpha}) := \bar{\pi}_{o}\, \pi_{o'}\, h\left(v, \frac{i\bar{\omega}^{1'} - v\bar{\pi}_{1}}{\bar{\pi}_{o}}, \frac{-i\omega^{1} - v\pi_{1'}}{\pi_{o'}}\right) \qquad (4.12)$$

and using $\zeta = \frac{i}{\bar{\pi}_{o}} (\bar{\omega}^{1'} + i\, v_o\, \bar{\pi}_{1})$ (cf.(4.7)),

it is not hard to check that (4.10) and (4.11) are equivalent to

$$\hat{Z}^{\alpha} = Z^{\alpha} - i\varepsilon \frac{\partial}{\partial \bar{Z}_{\alpha}} H(v_o, Z^{\beta}, \bar{Z}_{\beta}), \qquad (4.13)$$

as required. □

The propagation equation (4.13) has been derived for null twistors. We define the propagation of non-null twistors to be that given by (4.13). To propagate a finite (i.e. non-infinitesimal) distance, one has to integrate (4.13); in other words, one exponentiates the Hamiltonian transformation.

Finally, let us consider the consequences of the vanishing of the Ricci tensor. Combining equations (4.3) and (4.12) leads to

$$H = H^{+} + \overline{H^{+}},$$

where $H^{+}(v, Z^{\gamma}, \bar{Z}_{\gamma}) := \bar{Z}_{\alpha}\, I^{\alpha\beta} \frac{\partial}{\partial Z^{\beta}}\, g(v, Z^{\gamma}),$

$$g(v, Z^{\alpha}) := i\pi_{o'}^{2} \left[\int_{o}^{\bar{\zeta}} h^{-}(v, \xi)\, d\xi \right]_{\bar{\zeta} = \frac{-i\omega^{1} - v\pi_{1'}}{\pi_{o'}}},$$

and where $I^{\alpha\beta}$ represents the infinity twistor:

$$I^{\alpha\beta} = \begin{bmatrix} \varepsilon^{AB} & 0 \\ & \\ 0 & 0 \end{bmatrix} .$$

In other words, the propagation or scattering of a twistor through a vacuum pp-wave is essentially described by the holomorphic twistor function $g(v, Z^{\alpha})$. This is one more illustration of the important role played by holomorphic functions in twistor theory.

Chapter 5. Yang-Mills Fields.

The object of this Chapter is to generalize the "left-handed Maxwell field" construction of Chapter 3 to a "left-handed Yang-Mills field" construction. We shall use the term "Yang-Mills" to encompass all gauge theories, not just the one which arises from the group SU(2).

§5.1. Mathematical Preliminaries.

In this section we shall give a brief review of the theory of Yang-Mills fields. For more details, the reader is referred to the review article by Abers & Lee [1].

Yang-Mills theory grew out of the observation that the theory of electromagnetism is invariant under the action of a gauge group U(1). If U(1) is replaced by some larger gauge group, one is led to the theory of Yang-Mills fields. The basic mathematical setup is as follows.

Let G be a compact semi-simple Lie group. Let $\{T_p \mid p = 1,2,\ldots, n\}$ be a matrix representation of the infinitesimal generators of G. In other words, the T_p are N x N complex matrices (for some integer N) and they span the Lie algebra of G. Suppose that the T_p satisfy the commutation relations*

$$[T_p, T_q] = i\, C_{pq}{}^{r}\, T_r \quad ; \qquad (1.1)$$

So the $C_{pq}{}^{r}$ are the structure constants of G [7].

---

* The indices p,q,r ... run from 1 to n.

Let M be a region of real or complexified Minkowski space-time. A Yang-Mills field on M is defined to be a field $F^p_{ab} = F^p_{[ab]}$ on M (this gives us a collection of n 2-forms on M, as p runs over the values 1,2,..., n); the field is assumed (i) to possess a potential Φ^p_a on M (i.e. a collection of n 1-forms), such that

$$F^p_{ab} = 2\nabla_{[a} \Phi^p_{b]} + g\ C_{qr}{}^p\ \Phi^q_a\ \Phi^r_b\ , \tag{1.2}$$

where g is some constant (the coupling constant of the field);

(ii) to satisfy the Yang-Mills equations

$$\nabla^a F^p_{ab} + g\ C_{qr}{}^p\ \Phi^{qa}\ F_{ab}{}^r = 0, \tag{1.3a}$$

$$\nabla^a F^*_{ab}{}^p + g\ C_{qr}{}^p\ \Phi^{qa}\ F^*_{ab}{}^r = 0, \tag{1.3b}$$

where $F^{*p}_{ab} := \frac{1}{2}\, e_{abcd}\, F^{pcd}$ is the dual of F^p_{ab}.

We are particularly interested in fields which are left-handed (or anti-self-dual). The field $F_{ab}{}^p$ is said to be left-handed if it has the form

$$F_{ab}{}^p = \phi^p_{AB}\ \varepsilon_{A'B'}, \tag{1.4}$$

where $\phi^p_{AB} = \phi^p_{(AB)}$. Using equation (1.4), the Yang-Mills equations (1.3) become

$$\nabla^{AA'} \phi^p_{AB} + g\ C_{qr}{}^p\ \Phi^{qAA'}\ \phi^r_{AB} = 0, \tag{1.5}$$

while equation (1.2) is equivalent to the two equations

$$\phi^p_{AB} = \nabla_{A'(A}\ \Phi^{pA'}_{B)} + \tfrac{1}{2}\, g\ C_{qr}{}^p\ \Phi^q_{A'(A}\ \Phi^{rA'}_{B)}\ , \tag{1.6}$$

$$0 = \nabla_{A(A'}\ \Phi^{pA}_{B')} + \tfrac{1}{2}\, g\ C_{qr}{}^p\ \Phi^q_{A(A'}\ \Phi^{rA}_{B')}\ . \tag{1.7}$$

A potential Φ^p_a satisfying (1.7) is said to be <u>left-handed</u>.

<u>Theorem</u>. Equations (1.7) and (1.6) imply equation (1.5). In other words, the field derived from a left-handed potential <u>automatically</u> satisfies the Yang-Mills equations.

<u>Proof</u>. First note that since $C_{qr}{}^p$ is skew in qr,

$$C_{qr}{}^p \, \Phi^q_{AA'} \, \Phi^{rA}_{B'} = C_{qr}{}^p \, \Phi^q_{A(A'} \, \Phi^{r\,A}_{B')} \quad \text{and}$$

$$C_{qr}{}^p \, \Phi^q_{A'A} \, \Phi^{rA'}_{B} = C_{qr}{}^p \, \Phi^q_{A'(A} \, \Phi^{rA'}_{B)} \; .$$

Define $\Psi^p := \frac{1}{2} \nabla_a \Phi^{pa}$, so that

$$\nabla_{A'A} \, \Phi^{pA'}_{B} = \nabla_{A'(A} \, \Phi^{pA'}_{B)} + \varepsilon_{AB} \, \Psi^p \; .$$

Then (1.6) and (1.7) can be rewritten as

$$\phi^p_{AB} = \nabla_{AA'} \, \Phi^{pA'}_{B} - \varepsilon_{AB} \, \Psi^p + \tfrac{1}{2} g \, C_{qr}{}^p \, \Phi^q_{AA'} \, \Phi^{rA'}_{B} \; , \tag{1.8}$$

$$0 = \nabla_{AA'} \, \Phi^{pA}_{B'} - \varepsilon_{A'B'} \, \Psi^p + \tfrac{1}{2} g \, C_{qr}{}^p \, \Phi^q_{AA'} \, \Phi^{rA}_{B'} \; . \tag{1.9}$$

Operating on (1.9) with $\nabla^{BB'}$ and using (1.8) yields

$$0 = \nabla_{AA'} \Big[- \phi^{pAB} + \varepsilon^{AB} \, \Psi^p + \tfrac{1}{2} g \, C_{qr}{}^p \, \Phi^{qA}_{B'} \, \Phi^{rB'B} \Big]$$

$$- \varepsilon_{A'B'} \, \nabla^{BB'} \, \Psi^p + \tfrac{1}{2} g \, C_{qr}{}^p \, (\nabla^{BB'} \, \Phi^q_{AA'}) \, \Phi^{rA}_{B'}$$

$$+ \tfrac{1}{2} g \, C_{qr}{}^p \, \Phi^q_{AA'} \Big[- \phi^{rAB} + \varepsilon^{AB} \, \Psi^r + \tfrac{1}{2} g \, C_{st}{}^r \, \Phi^{sA}_{B'} \, \Phi^{tBB'} \Big] \; . \tag{1.10}$$

Using

$$\nabla_A^{A'} \Phi_B^{pB'} = \nabla_{(A}^{(A} \Phi_{B)}^{B')p} + \tfrac{1}{2} \epsilon^{A'B'} (\phi_{AB}^{p} - \tfrac{1}{2} g C_{st}^{\ \ p} \Phi_{C'A}^{s} \Phi_B^{tC'})$$

$$+ \tfrac{1}{2} \varepsilon_{AB} (- \tfrac{1}{2} g C_{st}^{\ \ p} \Phi_C^{sA'} \Phi^{tCB'}) + \tfrac{1}{4} \varepsilon_{AB} \varepsilon^{A'B'} \psi^p ,$$

equation (1.10) reduces to

$$0 = - (\nabla_{AA'} \phi^{pAB} + g C_{qr}^{\ \ p} \Phi_{AA'}^{q} \phi^{rAB})$$

$$+ g^2 C_{qr}^{\ \ p} C_{st}^{\ \ q} \Gamma^{tsrB}_{\ \ \ \ A'}, \qquad (1.11)$$

where $\Gamma^{tsrB}_{\ \ \ \ A'} := \Phi^{tAB'} \Phi_{AA'}^{[s} \Phi_{B'}^{r]B}$.

Clearly $\Gamma^{t(sr)}{}_a = 0$, and it is easy to check that $\Gamma^{(ts)r}{}_a = 0$, so it follows that $\Gamma^{tsr}{}_a = \Gamma^{[tsr]}{}_a$.

But the Jacobi identity [7] states that

$$C_{q[r}^{\ \ \ p} C_{st]}^{\ \ \ q} = 0,$$

so that the last term in equation (1.11) vanishes, and we are left with equation (1.5), as required. □

Nothing has been said so far about the way in which G acts as a gauge group. This matter will not be entered into here, beyond saying that $\phi^p{}_{AB}$ and Φ_a^p transform in a definite way under gauge transformations and that equations (1.5), (1.6) and (1.7) are <u>gauge-invariant</u>. In addition, it is easy to see (using the conformal transformation formulae of [24]) that these equations are <u>conformally invariant</u>.

Finally, note if we define

$$F_{ab} := F^{p}{}_{ab} \; T_p \, ,$$

$$\phi_{AB} := \phi^{p} \; T_p \, ,$$

$$\Phi_a := \Phi^{p}{}_{a} \; T_p$$

(so Φ_a is an N x N matrix of 1-forms, etc.), then equations (1.5), (1.6) and (1.7) become

$$\nabla^{AA'} \phi_{AB} - i\, g\, (\Phi^{AA'} \phi_{AB} - \phi_{AB} \Phi^{AA'}), \qquad (1.12)$$

$$\phi_{AB} = \nabla_{A'(A} \Phi^{A'}{}_{B)} - i\, g\, \Phi_{A'(A} \Phi^{A'}{}_{B)} \, , \qquad (1.13)$$

$$0 = \nabla_{A(A'} \Phi^{A}{}_{B')} - i\, g \Phi_{A(A'} \Phi^{A}{}_{B')} \qquad (1.14)$$

respectively, where use has been made of equation (1.1).

§5.2. The Twistorial Construction.

Recall from Chapter 3 that in the case of electromagnetic fields, the space that was deformed was the non-projective twistor space $\mathbb{T}$, considered as a line bundle L(-1) over $\mathbb{PT}$ (cf. §2.2).
In the Yang-Mills case, the space we shall deform is a vector bundle K over $\mathbb{PT}$, where K is the direct sum of N copies of $\mathbb{T}$ = L(-1):

$$K := \underbrace{L(-1) \oplus --- \oplus L(-1)}_{N \text{ times}}.$$

The bundle K has rank N, i.e. each fibre is a copy of $\mathbb{C}^N$: recall that $\mathbb{C}^N$ is the representation space on which the matrices T_p act. K may be thought of as the space of N-tuples of non-projective twistors

$$Z^\alpha = \begin{bmatrix} Z^\alpha_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ Z^\alpha_N \end{bmatrix},$$

such that Z^α_1, Z^α_2, ... Z^α_N, are all proportional to one another.

Let $\mathbb{C}M$ be a convex region of complexified Minkowski space-time and let $\mathbb{PT}$ be the corresponding projective twistor space. So $\mathbb{PT}$ is the space of TN2P's in $\mathbb{C}M$. Let $F_{ab} = F^p_{ab}\ T_p$ be a Yang-Mills field in $\mathbb{C}M$, with potential $\Phi_a = \Phi^p_a\ T_p$, satisfying the Yang-Mills equations (1.3).

Recall from §3.1 that $\mathbb{T}$ could be regarded as the space of pairs $(Z, \pi_{A'})$ where Z is a TN2P in $\mathbb{C}M$ and $\pi_{A'}$ is a constant spinor field on Z which points along Z (meaning that the tangent vectors to Z have the form $\lambda^A \pi^{A'}$).
Analogously, the vector bundle K may be regarded as the space of pairs $(Z, \pi_{A'})$ where Z is a TN2P in $\mathbb{C}M$ and $\pi_{A'}$ is a 1 x N column vector of spinor fields on Z :

$$\pi_{A'} = \begin{bmatrix} \pi_{1A'} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \pi_{NA'} \end{bmatrix},$$

such that each of $\pi_{1A'}$, ---, $\pi_{NA'}$ points along Z. It follows that $\pi_{1A'}$,---, $\pi_{NA'}$ are all proportional to one another; let $\xi_{A'}$ denote some spinor in the proportionality class they determine. So the tangent vectors to Z have the form $\lambda^A \xi^{A'}$ for some λ^A, and the condition for $\pi_{A'}$ to be constant on Z is

$$\xi^{A'} \nabla_{AA'} \pi_{B'} = 0 \text{ on Z .} \tag{2.1}$$

A way of coupling this setup to the Yang-Mills field now suggests itself: replace (2.1) by

$$\xi^{A'}(\nabla_{AA'} -ig\, \Phi_{AA'}) \pi_{B'} = 0 \text{ on Z.} \tag{2.2}$$

In matrix form, equation (2.2) is

$$\xi^{A'}\nabla_{AA'} \begin{bmatrix} \pi_{1B'} \\ \vdots \\ \pi_{NB'} \end{bmatrix} - ig \begin{bmatrix} \xi^{A'} \Phi^{1}_{1\,AA'} & \text{-----} & \xi^{A'} \Phi^{N}_{1\,AA'} \\ \vdots & & \vdots \\ \xi^{A'} \Phi^{1}_{N\,AA'} & \text{-----} & \xi^{A'} \Phi^{N}_{N\,AA'} \end{bmatrix} \begin{bmatrix} \pi_{1B'} \\ \vdots \\ \pi_{NB'} \end{bmatrix} = 0.$$

To sum up, on each TN2P Z we have an N-tuple of proportional primed spinors, and equation (2.2)tells us how to propagate these spinors over Z. Equation (2.2) consists of two equations

$$\left.\begin{aligned} V_o\, \pi_{B'} &= 0, \\ V_1\, \pi_{B'} &= 0, \end{aligned}\right\} \tag{2.3}$$

where the linear operators V_o and V_1 are defined by

$$V_{\underset{\sim}{A}} := \xi^{A'}(\nabla_{\underset{\sim}{AA'}} - i\, g\, \Phi_{\underset{\sim}{AA'}}) .$$

The condition for the system of equations (2.3) to be <u>integrable</u> is that the commutator $[V_0, V_1]$ be a linear combination of V_0 and V_1. Integrability of (2.3) will mean that the propagation law is integrable, in the sense that if we propagate $\pi_{B'}$ around a closed loop in Z, then $\pi_{B'}$ returns to its original value.

Now

$$[V_0, V_1] = \xi^{A'} \xi^{B'}(\nabla_{0A'} - i\, g\, \Phi_{0A'})(\nabla_{1B'} - i\, g\Phi_{1B'}) - (0 \leftrightarrow 1)$$

$$= - i\, g\, \xi^{A'}\xi^{B'} (\nabla_{AA'}\, \Phi^{A}{}_{B'} - i\, g\, \Phi_{AA'}\, \Phi^{A}{}_{B'}) ;$$

and the only way that this can be a linear combination of V_0 and V_1 is for it to vanish. It follows that the integrability condition is

$$\nabla_{A(A'}\, \Phi^{A}{}_{B')} - i\, g\, \Phi_{A(A'}\, \Phi^{A}{}_{B')} = 0 ,$$

i.e. Φ_a is <u>left-handed</u> (cf. equation (1.14)). We shall assume for the rest of this section that Φ_a <u>is</u> left-handed.

Thus we can define a vector bundle $\mathcal{K}$ over $\mathbb{PT}$ to be the space of pairs $(Z, \pi_{A'})$, where $Z \in \mathbb{PT}$ and where $\pi_{A'}$ is an N-tuple of spinor fields on Z, each pointing along Z and satisfying equation (2.2). The space $\mathcal{K}$ is a deformation of the "flat" vector bundle K. A construction analogous to the one of §3.1 enables us to see how $\mathcal{K}$ may be patched together out of two patches U and $\hat{U}$. This construction goes as follows.

Let P and Q be two fixed TN2P's in $\mathbb{C}M$ and define two subsets $\mathbb{P}U$ and $\mathbb{P}\hat{U}$ of $\mathbb{PT}$ as in §3.1. Let U be the space of all

$$Z^{\alpha} = \begin{bmatrix} Z^{\alpha}_{1} \\ \vdots \\ Z^{\alpha}_{N} \end{bmatrix}$$

such that the Z_1^α, ---, Z_N^α all determine the <u>same</u> TN2P Z in $\mathbb{P}U$ (i.e. Z_1^α, ---, Z_N^α are all proportional to one another and their proportionality class lies in $\mathbb{P}U$). Define $\hat{U}$ similarly. The "flat" bundle K is given by patching U and $\hat{U}$ together according to

$$\hat{Z}^\alpha = Z^\alpha \quad \text{on } U \cap \hat{U}.$$

The deformed bundle $\mathcal{K}$ is determined by the patching

$$\hat{Z}^\alpha = f(Z)\, Z^\alpha \quad \text{on } U \cap \hat{U}, \tag{2.4}$$

where f(Z) is a nonsingular N x N matrix, the elements of which are holomorphic functions on $\mathbb{P}U \cap \mathbb{P}\hat{U}$. The matrix f(Z) arises as follows. If $Z \in \mathbb{P}U \cap \mathbb{P}\hat{U}$, let Γ be some path in Z from p^a (the intersection point of Z and P) to q^a (the intersection point of Z and Q) (see figure 3.1.2). Propagate $\pi_{A'}$ along Γ using the propagation law (2.2) : this leads to a relationship of the form

$$\pi_{A'}(q^b) = f(Z)\, \pi_{A'}(p^b), \tag{2.5}$$

and serves to define f(Z).

In other words, given a left-handed Yang-Mills potential Φ_a, we can build the space $\mathcal{K}$ according to (2.4). Conversely, <u>given</u> the matrix f(Z) which determines $\mathcal{K}$, how can we get back to Φ_a? One way of achieving this is set out below.

The first step is to fix a point $x^a \in \mathbb{C}M$ (i.e. fix a line L_x in $\mathbb{P}\mathbb{T}$) and to restrict f(Z) to L_x by putting

$$F(x^a;\ \xi_{A'}) := f(i\, x^{AA'}\xi_{A'}, \xi_{A'})$$

(cf. equation 2.1.8 and §3.2; recall that $\xi_{A'}$ was a constant projective spinor pointing along Z). Identify the line L_x with the sphere of projective spinors $\xi_{A'}$; this sphere is covered by the two patches

$$W_x := L_x \cap \mathbb{P}U\ ,$$
$$\hat{W}_x := L_x \cap \mathbb{P}\hat{U}\ .$$

Now "split" F into the quotient of two matrices G and $\hat{G}$:

$$F(x^a;\ \xi_{A'}) = \hat{G}(x^a;\ \xi_{A'})\ G^{-1}(x^a;\ \xi_{A'}) , \tag{2.6}$$

where $G(x^a;\ \xi_{A'})$ is holomorphic and nonsingular on W_x and $\hat{G}(x^a;\ \xi_{A'})$ on $\hat{W}_x$. In practice, it might be rather difficult to carry out this splitting (except in cases like electromagnetism where there is an explicit "splitting formula": cf. §3.2).

Operating on equation (2.6) with $\xi^{A'}\nabla_{AA'}$ yields

$$0 = (\xi^{A'}\nabla_{AA'}\hat{G})\ G^{-1} + \hat{G}(\xi^{A'}\nabla_{AA'}G^{-1})$$

$$= (\xi^{A'}\nabla_{AA'}\hat{G})\ G^{-1} - \hat{G}\ G^{-1}\ (\xi^{A'}\nabla_{AA'}G)\ G^{-1}$$

$$\Longrightarrow\ G^{-1}(\xi^{A'}\nabla_{AA'}G) = \hat{G}^{-1}(\xi^{A'}\nabla_{AA'}\hat{G}). \tag{2.7}$$

The left-hand side of (2.7) is holomorphic on W_x and the right-hand side on $\hat{W}_x$; so both sides must be globally holomorphic.
And both sides are homogeneous of degree one in $\xi^{A'}$ (G and $\hat{G}$ are homogeneous of degree zero since they are defined on the projective $\xi_{A'}$ - space). It follows that both sides of (2.7) must be linear in $\xi^{A'}$. In other words, there exists a unique N x N matrix $\Phi_a\ (x^b)$ of 1-forms on $\mathbb{C}M$, such that

$$G^{-1}(x^b;\ \xi_{B'})\ \xi^{A'}\ \nabla_{AA'}\ G(x^b;\ \xi_{B'}) =: -ig\ \xi^{A'}\ \Phi_{AA'}(x^b). \tag{2.8}$$

<u>Theorem</u>. Φ_a is left-handed (i.e. satisfies equation (1.14).

<u>Proof</u>. Using equations (2.8) and (2.1.5),

$$\xi^{A'}\ \xi^{B'}\ \nabla_{AA'}\ \Phi^{A}_{\ B'} = i\ g^{-1}\ (\xi^{A'}\ \nabla_{AA'}\ G^{-1})(\xi^{B'}\ \nabla^{A}_{B'}\ G)$$

$$= -\ ig^{-1}(G^{-1}\ \xi^{A'}\ \nabla_{AA'}G)(G^{-1}\ \xi^{B'}\ \nabla^{A}_{B'}\ G)$$

$$= ig\ \xi^{A'}\ \xi^{B'}\ \Phi_{AA'}\ \Phi^{A}_{\ B'} ,$$

from which equation (1.14) follows immediately. □

So we now have a procedure for getting from a left-handed Yang-Mills potential Φ_a to a vector bundle $\mathcal{K}$ over $\mathbb{PT}$ with patching function f(Z) :

$$\Phi_a \longmapsto f \ ;$$

and we also have a procedure for getting from a patching function f to a potential Φ_a:

$$f \longmapsto \Phi_a \ .$$

Are these two procedures compatible, in the sense that the composite procedure

$$f \longmapsto \Phi_a \longmapsto f' \qquad (2.9)$$

implies that f = f′ ? In fact, the answer to this question is no, in general. But what <u>is</u> true is that (2.9) implies that there exist matrices h(Z) (holomorphic on $\mathbb{P}U$) and $\hat{h}(Z)$ (holomorphic on $\mathbb{P}\hat{U}$) such that

$$f' = \hat{h} \ f \ h^{-1} . \qquad (2.10)$$

This means precisely that f and f′ determine <u>equivalent</u> vector bundles (cf.§2.2). So compatibility is ensured if we agree to identify equivalent vector bundles over $\mathbb{PT}$.

To verify that equation (2.10) holds, we can reason as follows. Given a patching function f(Z), "split" it according to (2.6) and define Φ_a by (2.8). Adopt the propagation law (2.2) for $\pi_{B'}$. Then (using (2.2) and (2.8))

$$\xi^{A'} \nabla_{AA'}(G \ \pi_{B'}) = 0.$$

It follows that

$$G(x^a; \xi_{A'}) \ \pi_{B'} (x^a) = G(p^a; \ \xi_{A'}) \ \pi_{B'}(p^a). \qquad (2.11)$$

Similarly,

$$\hat{G}(x^a; \xi_{A'}) \ \pi_{B'}(x^a) = \hat{G}(q^a; \ \xi_{A'}) \ \pi_{B'} \ (q^a) . \qquad (2.12)$$

Combining (2.11) and (2.12) yields

$$\pi_{B'}(q^a) = \hat{G}(q^a;\xi_{A'})\,\hat{G}(x^a;\ \xi_{A'})\ G^{-1}(x^a;\ \xi_{A'})\ G(p^a;\ \xi_{A'})\pi_{B'}(p^a)$$

$$= \hat{h}(Z)\ f(Z)\ h^{-1}(Z)\ \pi_{B'}(p^a), \tag{2.13}$$

using equation (2.6) and defining

$$\hat{h}(Z) := \hat{G}(q^a;\xi_{A'}),$$

$$h(Z) := G(p^a;\ \xi_{A'}).$$

Comparing (2.13) with (2.5) yields the desired result.

What we have shown in this Chapter is that the information of a left-handed Yang-Mills field can be coded into the complex structure of a vector bundle $\mathcal{K}$ over $\mathbb{PT}$. It remains to be seen whether the construction will provide a useful technique for finding explicit left-handed solutions of the Yang-Mills equations.

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