

Mathematical Models of the Carding Process



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To
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Abstract

Carding is an essential pre-spinning process whereby masses of dirty tufted fibres are cleaned, disentangled and refined into a smooth coherent web. Research and development in this “low-technology” industry have hitherto depended on empirical evidence. In collaboration with the School of Textile Industries at the University of Leeds, a mathematical theory has been developed that describes the passage of fibres through the carding machine. The fibre dynamics in the carding machine are posed, modelled and simulated by three distinct physical problems: the journey of a single fibre, the extraction of fibres from a tuft or tufts and many interconnecting, entangled fibres.

A description of the life of a single fibre is given as it is transported through the carding machine. Many fibres are sparsely distributed across machine surfaces, therefore interactions with other neighbouring fibres, either hydrodynamically or by frictional contact points, can be neglected. The aerodynamic forces overwhelm the fibre’s ability to retain its crimp or natural curvature, and so the fibre is treated as an inextensible string. Two machine topologies are studied in detail, thin annular regions with hooked surfaces and the nip region between two rotating drums. The theoretical simulations suggest that fibres do not transfer between carding surfaces in annular machine geometries. In contrast to current carding theories, which are speculative, a novel explanation is developed for fibre transfer between the rotating drums. The mathematical simulations describe two distinct mechanisms: strong transferral forces between the taker-in and cylinder and a weaker mechanism between cylinder and doffer.

Most fibres enter the carding machine connected to and entangled with other fibres. Fibres are teased from their neighbours and in the case where their neighbours form a tuft, which is a cohesive and resistive fibre structure, a model has been developed to understand how a tuft is opened and broken down during the carding process. Hook-fibre-tuft competitions

are modelled in detail: a single fibre extracted from a tuft by a hook and diverging hook-entrained tufts with many interconnecting fibres. Consequently, for each scenario once fibres have been completely or partially extracted, estimates can be made as to the degree to which a tuft has been opened-up.

Finally, a continuum approach is used to simulate many interconnected, entangled fibre-tuft populations, focusing in particular on their deformations. A novel approach describes this medium by density, velocity, directionality, alignment and entanglement. The material responds to stress as an isotropic or transversely isotropic medium dependent on the degree of alignment. Additionally, the material's response to stress is a function of the degree of entanglement which we describe by using braid theory. Analytical solutions are found for elongational and shearing flows, and these compare very well with experiments for certain parameter regimes.

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Chapter 1

Introduction

A few years ago, a group from the School of Textile Industries in Leeds approached the Oxford Centre for Industrial and Applied Mathematics with the long-standing problem of understanding fibre dynamics in carding. Although the machines involved have undergone numerous modifications, predominantly fuelled by the advances in mechanical manufacturing technology, the rudiments behind the process have not changed for centuries. Furthermore, research and development in this “low-technology” industry have hitherto depended on empirical evidence. Within a multi-disciplinary group, including industrialists and experimentalists, we have endeavoured to shed light on this age-old and fundamental process.

This thesis presents the theoretical aspects of the work accomplished within this multi-disciplinary research framework. Bespoke mathematical models have been derived for the fibres on three different length scales. The first of the scales models the motion of a single fibre as it travels through the carding machine. The intermediate scale focuses on the interplay between fibres and a tuft in order to understand how fibres are extracted or teased out of tufts. Finally, we consider large volumes of interacting fibres as a continuum, and consider their evolution throughout the process. These models as a whole give us access to a theoretical simulation that caters for all areas of the machine at least to a first approximation. We describe the process in more detail and review the theoretical work done so far and then give a more detailed overview of the thesis.

1.1 Textile Manufacture

The fundamental operations when manufacturing yarns are carding, drawing, and twisting. Carding is the process of separating fibres from one another and combing them to form an ordered even web. Drawing consists of the attenuation of a loose

rope of carded fibres into a thinner rope until its thickness is suitable for the insertion of a twist. Twisting then gives the yarn coherence and some structural stability. The production processes of different yarns have a great deal in common, in contrast to the preparation of various textiles for yarn manufacture which is highly dependent on the chosen material.

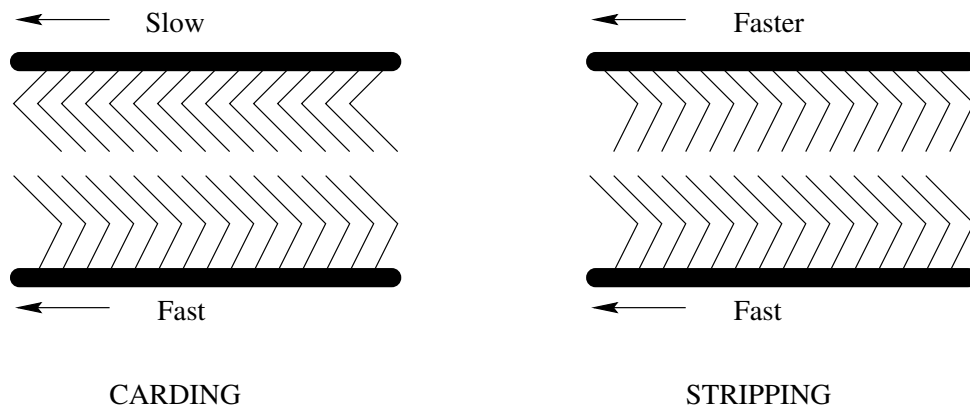


Figure 1.1: The two simplified carding process mechanisms: carding and stripping.

Most textile materials come to the spinning mill in masses of entangled fibres which can be described as tufts; the masses may be reduced by various breaking machines, but before the fibres are spun they must be disentangled and arranged in a smooth coherent web with uniform density and thickness. The process also incorporates the elimination of unwanted particles such as vegetation from cotton plants, unwanted short fibres and soils from wool. One of the negative outcomes of carding is that a group of fibres may evolve into tightly bound knots called “neps”, which to the yarn manufacturer are difficult to eliminate and consequently produce discrepancies in the yarn as they will include noticeable density irregularities.

The manufacturers of textile instruments attempt to create carding tools that optimise the speed and quality of the process. Modern machines usually consist of a number of revolving cylinders covered with fine hooks, and in some cases the cylinders are placed in surroundings that are also covered with hooks. The rudiments of current methods can be simplified to carding and stripping, see figure 1.1. In the diagram the fibres are attached to the bottom hooked surface and will interact with the hooks on the adjacent surface. Carding helps break down tufts by teasing fibres out of their neighbouring entanglements. Stripping moves fibres from one cylinder surface to another until it is ready for the next stage of the yarn production process. Although all the models we shall consider could be applied to all machines in the

carding genre, we give a detailed account of a revolving-flat short-staple machine, which is used to card fibres that are about half a centimetre in length and a micron in diameter such as cotton, polyester and short wool.

1.1.1 The Crosrol Revolving-Flats Carding Machine

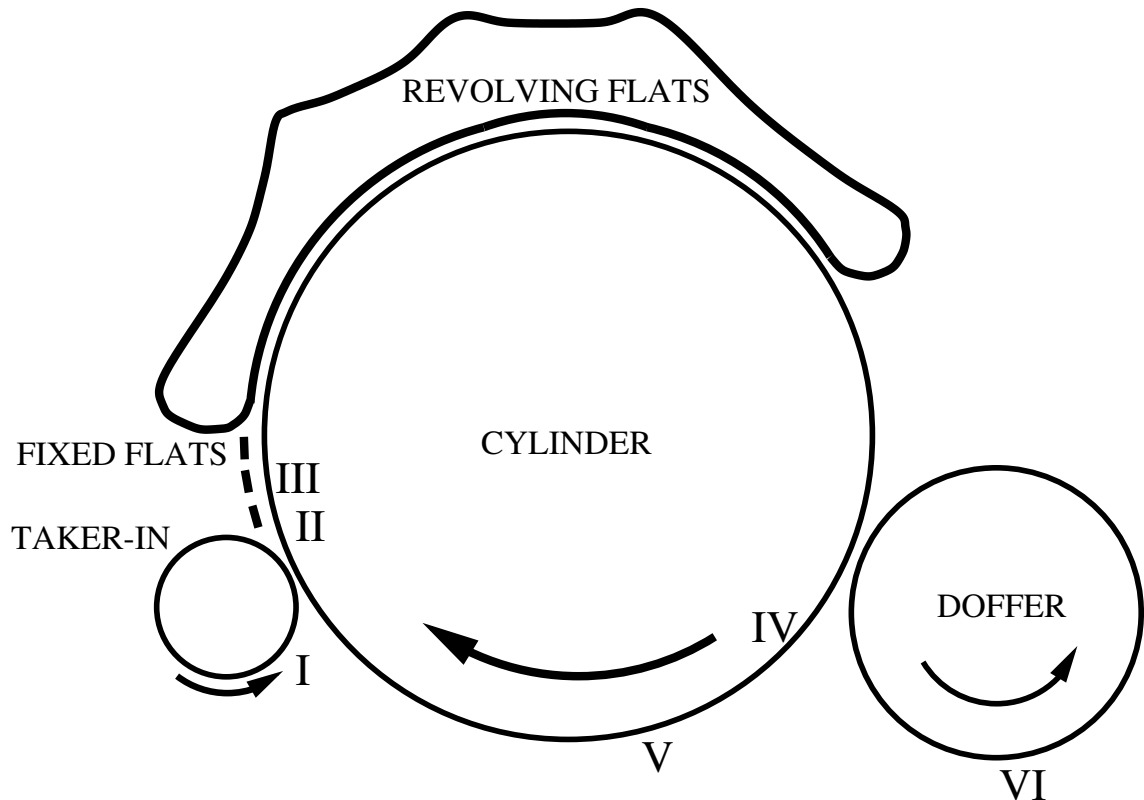


Figure 1.2: A diagram illustrating the points at which high speed photography was used to examine fibre orientations in the carding machine.

We begin with a qualitative overview of how a fibre travels through a short-staple carding machine. A machine is illustrated in figure 1.3. Once the raw materials have been prepared for yarn production, the fibres are then arranged in a densely packed entangled array which is called the “lap”. The “taker-in”, sometimes called the “licker-in”, is the first cylinder that the fibres encounter. When the lap is fed into the machine, the hooks on the rotating taker-in grab the fibres and carry them towards a larger drum covered in smaller hooks called the cylinder. All the fibres are transferred or stripped onto the cylinder. The fibres are then carried by the cylinder hooks into the carding region. The revolving-flats comb and disentangle the fibres and once they have been carded they travel towards the final drum called the “doffer”. The transfer

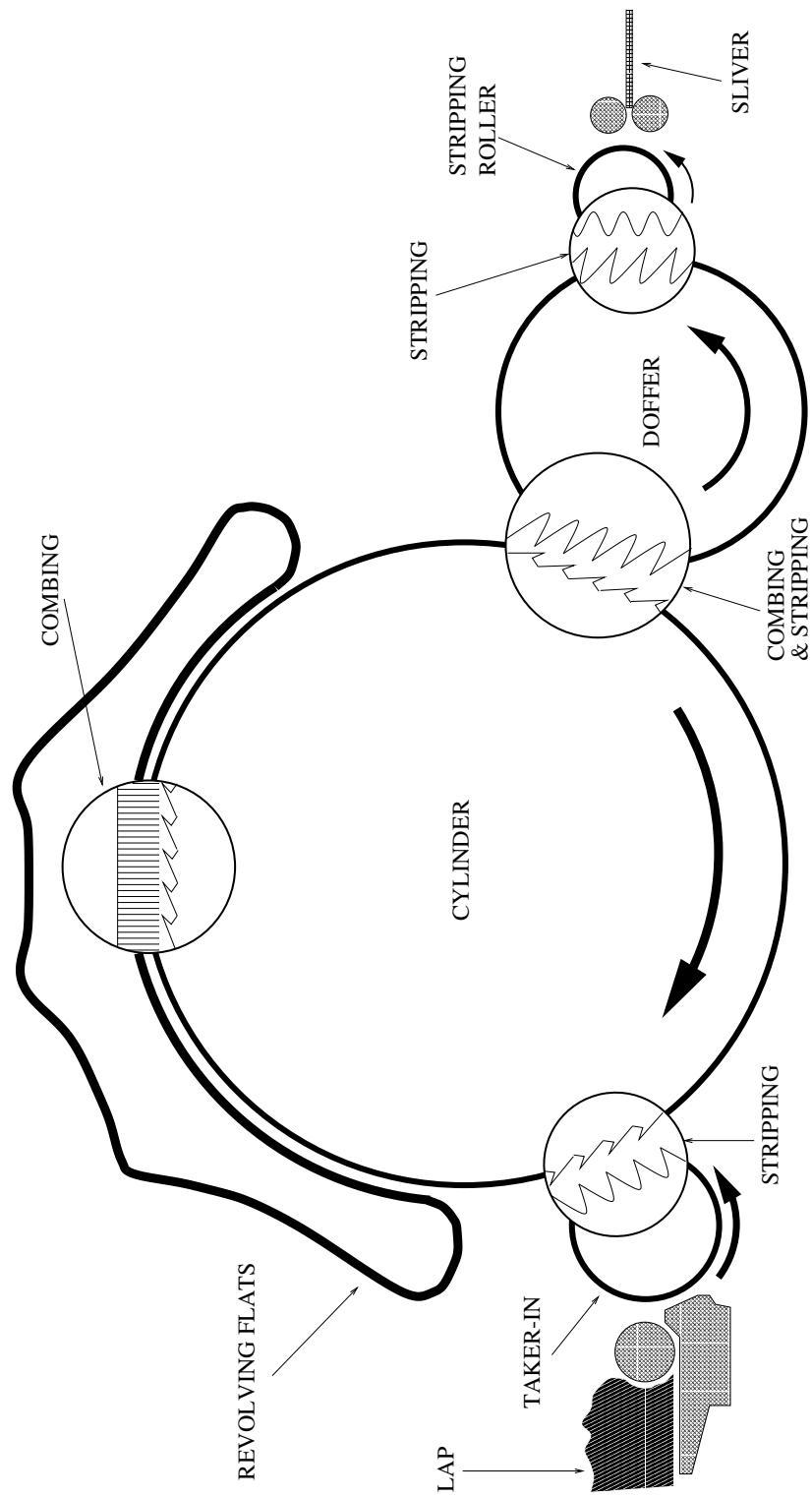


Figure 1.3: A diagram of the revolving-flat carding machine.

of fibres from the cylinder at this point only occurs for a fraction presented to the doffer hooks. The fibres that have transferred are carried around to a stripping tool and then exit the machine in the form of a lightly packed and ordered array called the “sliver”. The other fibres left on the cylinder revolve on its surface until they reach the doffer transfer region again, and this process is repeated until each fibre finally transfers onto the doffer. Now we turn to a quantitative and detailed account of fibres as they travel through a short-staple carding machine.

1.1.1.1 From Feeder-in to Taker-in

Once prepared for yarn production, the entangled fibres or tufts form the disordered and entangled lap, and for cotton this will typically have around 12,500 fibres per square centimetre of the lap (plan view). A small roller then continuously feeds the textile onto the first set of rotating wire hooks, see the left hand side of figure 1.3 and figure 1.4. The orientation of the fibres on the taker-in depends on the level of entanglement, which form locally connected structures, within the lap and the hook’s ability to retain fibres as it hits the lap surface. Typically the taker-in hooks or wires have a front angle 90° and a height of 3.93 mm, and their density on the taker-in surface is 6.5 per square centimetre, see figure 1.4. There is a millimetre clearance between the feeder and taker-in hooks and the number of fibres carried onto the taker-in surface is 18 per square centimetre. On average this means that there are 2-3 fibres per hook, and additionally not a homogeneous spread.

1.1.1.2 On the Taker-in

Figure 1.4 shows the mechanism by which a tuft is teased out of the lap by the taker-in hooks. The photographs in figure 1.5 are taken just before the fibres reach the taker-in to cylinder transfer region, point “I” on figure 1.2. We call the cohesive structures that are composed of fibres, “tuftlets”, and how they are broken down into individual fibres forms a key part of understanding the carding machine. Dehghani et al. (2000) give experimental information on the evolution of this population of structures for some parts of the machine. On the taker-in most tuftlets have a length-to-width aspect ratio of 1.5, are aligned in the direction of the hooks to within 5° , and have a mean length of 15 millimetres. At this stage of the carding machine, on the taker-in, when the fibres can be described as individuals or as part of tuftlets, it is estimated by Dehghani et al. (2000) that about half the fibres are independent of these structures. It is not clear how to relate the tuftlet-fibre medium on the taker-in with the fibre-structures inside the lap; for example, individuals could have been teased out of a

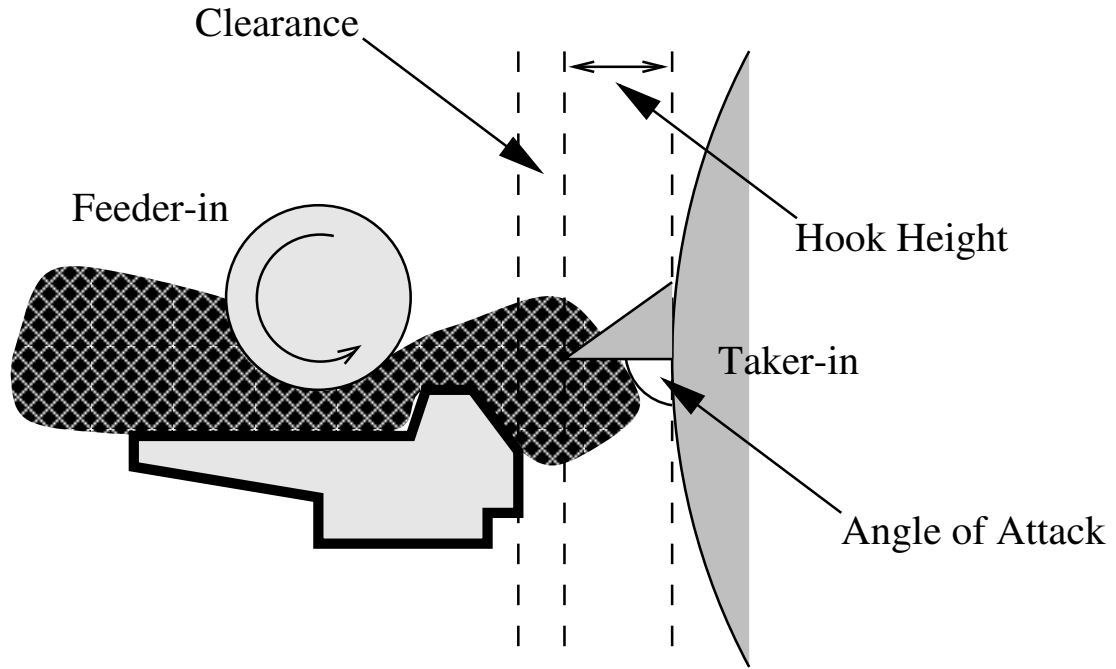


Figure 1.4: Profile view of a taker-in hook grabbing fibres from the lap.

tuftlet in the lap or it could have been an individual in the lap. Dehghani et al. (2000) were unable to photograph the transfer region due to the constraints of the machine casing.

1.1.1.3 From Taker-in to Cylinder

The movement of fibres from the taker-in to the cylinder is called drafting and the amount transferred is traditionally related to the ratio of surface speeds between the two drums, which is about 2.7 in favour of the cylinder. The cylinder hooks travel between 35 and 45 metres per second, they are 0.55 millimetres high and have a 63° angle of attack. The hooks on either drum are set with a 0.18 millimetre clearance. Therefore, the tuftlets are lifted off the taker-in and there is definite elongation caused by the disparity in surface speed. Figure 1.6 displays pictures taken from point “II” in figure 1.2, and shows a more elongated or “opened” tuftlet compared to that in figure 1.5. Dehghani et al. (2000) statistically validate this claim by showing that tuftlet lengths, aspect ratios and alignment to hook movement all significantly increase during this transfer region.

The photographic pictures in 1.5 and 1.6 do not aid in understanding how fibres transfer from taker-in to cylinder, as the transfer regions were difficult to photograph. Some textiles engineers (Varga and Cripps (1996)) postulate that it is the tail end

of the fibre being dragged by the taker-in hook which makes first contact with the hooks on the cylinder and it is this action that evokes the transfer mechanism. This claim will be debated in chapter 2.

1.1.1.4 On the Cylinder

Fibres on the cylinder live a complicated life, but we can consider the most important facets. If we assume that there is a perfect draft from the taker-in this leaves an average of 7 fibres per square centimetre on the cylinder. As the fibre-tuft medium is being transported on the surface of this drum at speeds of $O(10)$ metres per second, it firstly encounters the carding region, which is composed of three fixed flats and then the revolving-flats, which actually move at the near stationary speed of 10 meters per hour (relative to the cylinder surface speed), see figure 1.3. The fixed flats have just under 15 hooks per square centimetre and the revolving flats have around 60. The wires on the carding surfaces are about 5 millimetres in length and attack the tufts at 75° . At point “III” in figure 1.2, after one fixed flat, figure 1.7 shows that tufts still exist at this point but have undergone substantial changes. In fact Dehghani et al. (2000) suggest that about 90% of tufts are now aligned to within 5° hook motion, which is 20% more than the tuftlets on the taker-in. The aspect ratio and the length of tufts also significantly increase after just one fixed flat. In fact, the deformations in tuftlet structure may be greater than those measured in the experiment because there is normally a cloth of fibres on the cylinder that would encourage newly transferred material to protrude further into the carding hooks, and these were eliminated so as to minimise unwanted noise in data. Given the substantial changes produced by just one flat we expect the structures within the tuftlets to be broken down by the end of the carding region, resulting in a fairly disperse array of fibres.

The other key aspect of the life of a fibre on the cylinder, and in fact the whole process, is the transfer of fibres onto the doffer. This mechanism is paramount because the fibrous cloth laid onto the doffer is then stripped and becomes the final product or slivers that is then transported to the next phase of yarn production. There is little consensus on the ratio of fibres that transfer and on the mechanism itself. Figure 1.8 displays three photographs: the first shows the region just after the transfer, the second is a plan view of the doffer and what when lifted off the doffer becomes the sliver, and the final picture is a plan view of the cylinder after the transfer region. These photos are taken at point “IV”, “V” and “VI” in figure 1.2.

1.1.1.5 The Doffer

Industrialists (Varga and Cripps, 1996) assume that the motion of fibres on the doffer are unimportant. Fibres are carried around towards the stripping roller which then extracts all the fibres from the doffer, without significantly changing the structure of the fibres as they were on the doffer. The fibres as they exit the machine look like candy floss, predominantly uniform in density and they are certainly less tangled and more aligned than when they entered the process.

1.1.1.6 Summary

Although there is some useful experimental work, much of the fundamental mechanisms within the machine are still open to debate. How the transfer from one cylinder to another takes place is one topic of considerable importance, not only for individual fibres but also for tuftlets. Another vital area of study is the evolution of fibre structures throughout the machine. Therefore in this thesis we use mathematical modelling to illuminate our rudimentary knowledge of the carding machine. We produce novel explanations of the physical mechanisms that include key fibre properties, machine geometries and hook dimension.

1.1.2 Literature review

There are numerous articles that deal with aspects of the carding machine which are reviewed in Lawrence et al. (2000) and these are predominantly experimental. Indeed, there is only a small collection of articles that have attempted to analyse the machine from a mathematical point of view. A review is given of models that attempt to simulate the behaviour of fibres during this pre-spinning process and these fall into two categories, classical mechanics and probabilistic.

Potentially promising work by Roberts Jr (1996) claims to simulate fibre processing to the extent that it is near virtual prototyping. The model and the cases illustrated involved fibre networks that are extruded and transported by a turbulent free jet and are then electro-statically deposited through a turbulent jet boundary layer onto a moving conveyer belt. The primary reason for its inadequate description of the carding process is the fact that hooks are excluded from the problem and the fibre ensembles are linked together by springs. In section 1.1.1 we see the paramount importance of hooks and that within the machine there are numerous teasing processes that break down fibre ensembles in a non-elastic manner. It is even dubious whether this model can be applied to other pre-carding opening processes.

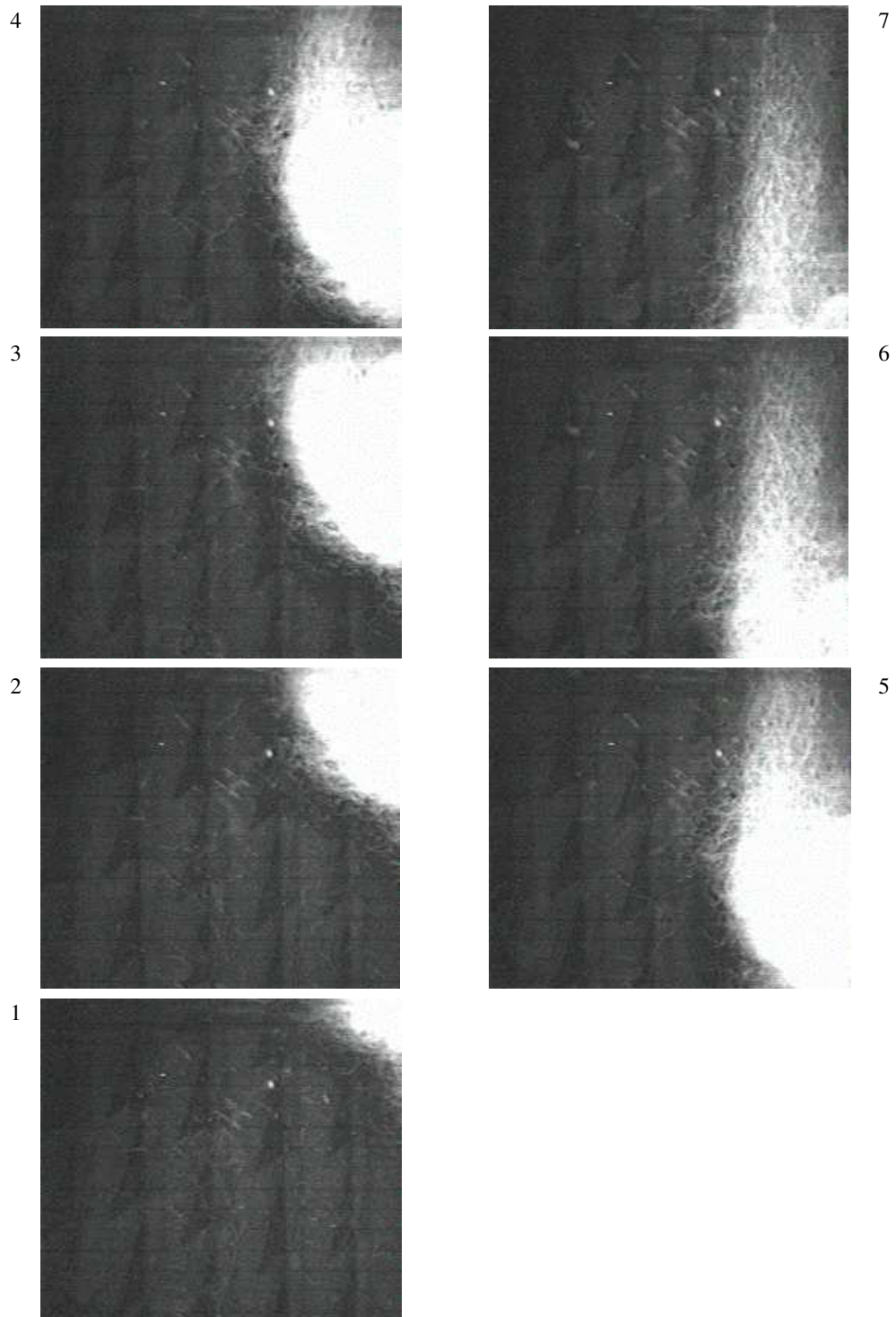


Figure 1.5: Plan view of the rotating taker-in hooked surface : snap-shots taken in numerical order from point I in figure 1.3 of a tuft on the taker-in (Dehghani et al., 2000).

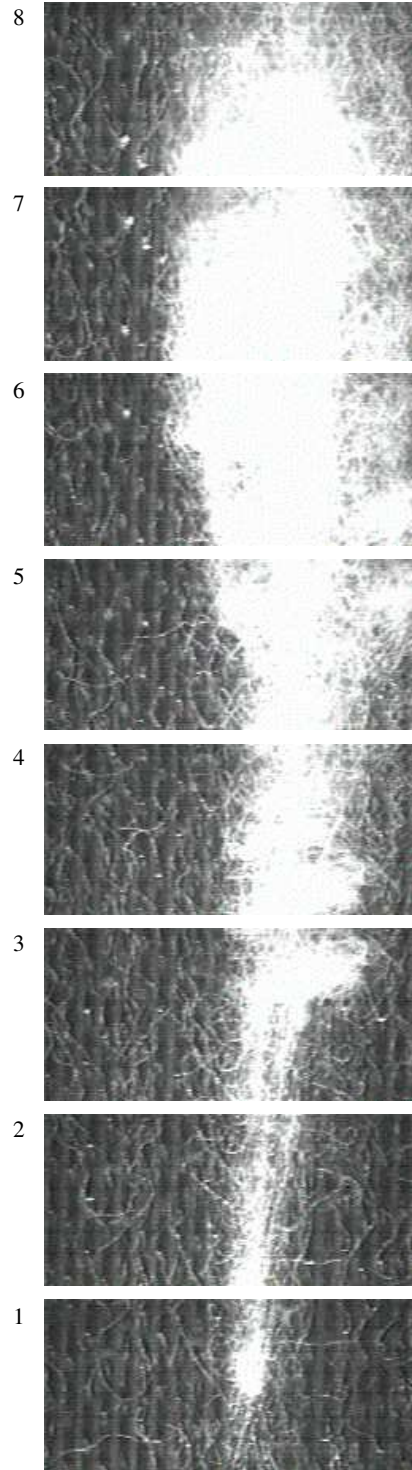


Figure 1.6: Plan view of the rotating cylinder surface just before the fixed and revolving-flats: snapshots taken in numerical order from point II in figure 1.3, of a tuft on the cylinder before the fixed flats (Dehghani et al., 2000).

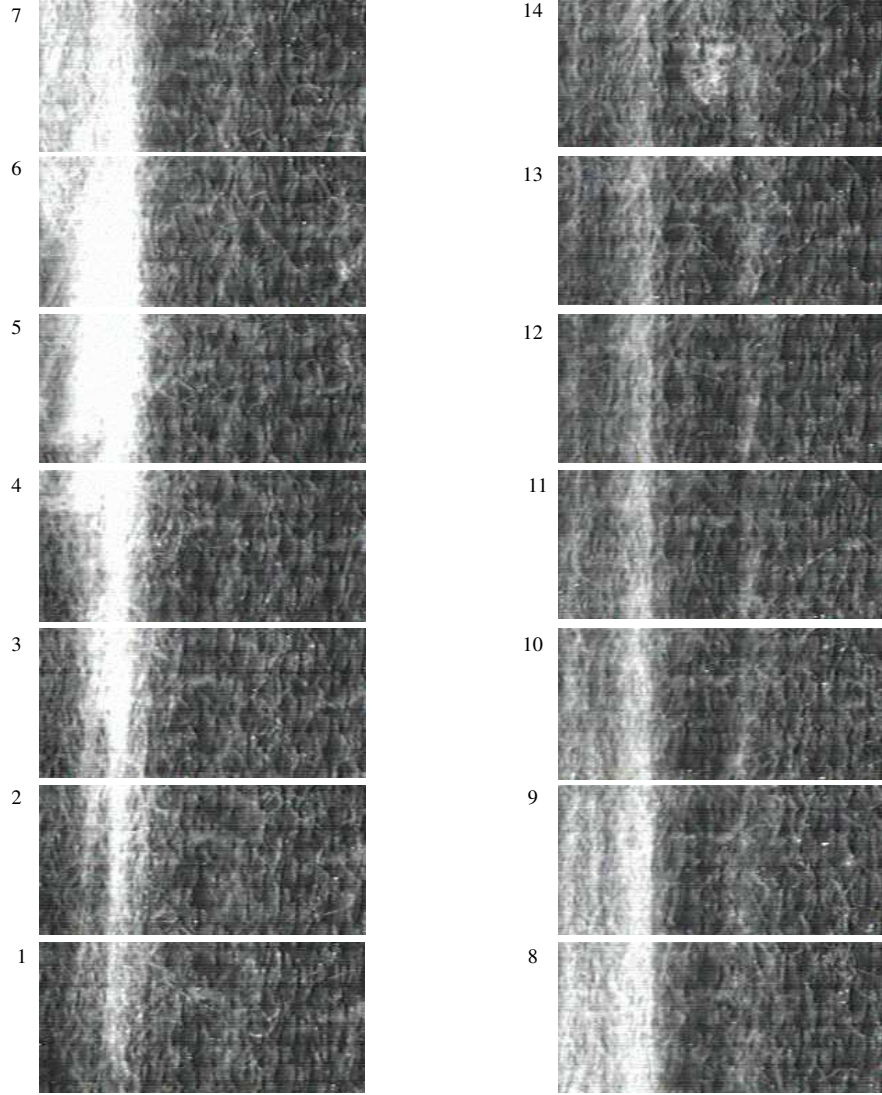
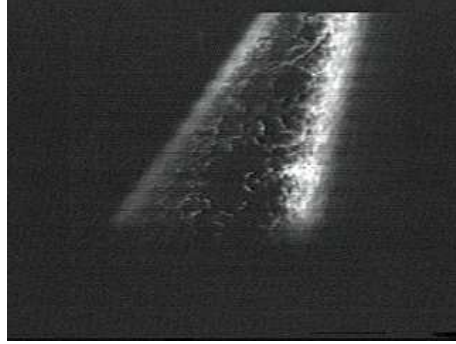
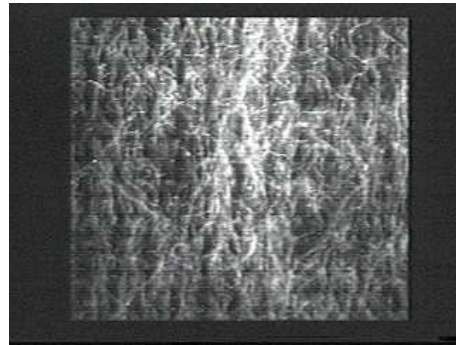


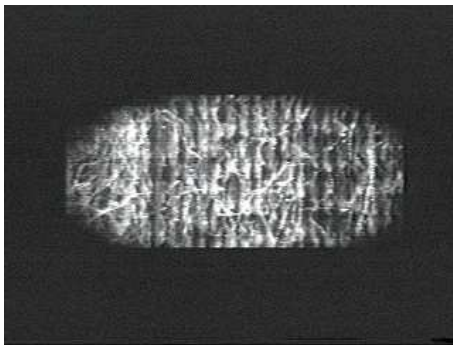
Figure 1.7: Plan view of the rotating cylinder surface just after one fixed flat but before the revolving-flats: snapshots taken in numerical order, from point III in figure 1.3, of a tuft on the cylinder after one fixed flat (Dehghani et al., 2000).



Cylinder and Doffer



Doffer



Cylinder

Figure 1.8: Snap-shots from point IV, V and VI in figure 1.3, of the regions just after the point of transfer between the cylinder to the doffer (Dehghani et al., 2000).

Smith and Roberts Jr. (1994) and Kong and Platfoot (1997) produce computational simulations of fibres which are being transported through converging channels where the forces acting on the fibres are purely aerodynamic. Although converging channel geometries are important, the aforementioned authors neglect hooks and this, in conjunction with one of our criticisms of Roberts Jr (1996), we believe to be naive. Furthermore, fibres within the carding machine are predominantly tethered by hooks attached to a rotating drum and this has been neglected. Aerodynamic transport may be important, but they have not explained how fibres transfer from one surface to another.

Probabilistic methods are employed by Abhiraman and George (1973), Cherkassky (1994, 1995), Gutierrez et al. (1995), Rust and Gutierrez (1994), Wibberly and Roberts Jr. (1997), all of whom consider the carding machine as a macroscopic process. The fibrous textile material is characterised by density, and the carding action and transfer regions are replaced by simple functions; density is assumed to decay exponentially during carding regions and a fraction of the fibres transfer from cylinder to doffer. The two primary modelling simplifications are to assume that fibre transfer and carding will redistribute mass at given fixed rates. Each of the aforementioned authors progress from this basic model in different ways and not surprisingly produce quite believable results. The main problem with this approach pertains to relating the parameters in the mathematical models with the machine variables. This means that the probabilistic approaches adopted by the aforementioned authors are very limited in terms of their predictive powers when considering advances in machine design and they do not explain the fundamental mechanisms within the machine.

The mathematical work applied specifically to the carding machine hitherto does not enlighten a reader who wishes to understand the evolution of fibres and fibre networks throughout the machine or aerodynamic forces on fibres. Applications of classical mechanics have been very limited; and the fundamental concepts of fibre-fibre, fibre-hook interactions and aerodynamic forces acting on fibres have not been considered. Probabilistic approaches look from a macroscopic view point but they do not address the crucial issue of how small-scale interactions produce global effects. Therefore this thesis attempts to outline the underlying physical mechanisms that govern the evolution of entangled and single fibres within the carding machine.

1.2 Thesis Overview

There is a multitude of physics in the carding machine that one could attempt to model mathematically, and posing relevant and tractable problems is not trivial. What is clear from the literature review is that there are no obvious starting points. After many discussions with carding practitioners and textile engineers, we decided to focus on three distinct areas: single fibres, tufts and fibres, and many fibres. It is quite conceivable that the whole thesis could have been dedicated to any one of these three areas. We chose these areas as they were the most pertinent to carding manufacturers and also posed interesting mathematical problems.

The simplest physical scenario examines how single fibres progress as they are carried through the carding machine. This assumes that the volume fraction of fibres is small at least locally. Such a model can be applied in part to the early stages of the fibre-tuft medium found on the taker-in, since up to half of the fibres can be found as non-interacting individuals. As the fibres are continually teased out of the tufts during the carding process, the single fibre approximation becomes more relevant. We neglect fibre-fibre interactions, and we model the fibre as an inextensible string with small bending stiffness compared to the external body forces. This follows on from the seminal work of Taylor (1952), Cox (1970) and Batchelor (1970), who consider hydrodynamic drag acting on bodies with slender geometries. We also consider rotational and internal forces. There are two distinct geometries that a fibre travels through, thin annular channels and adjacent rotating drums, and we find the appropriate respective fluid flows. Consequently we can determine the displacement of a fibre throughout the process and therefore study the effect of hook geometries using static friction laws.

Our work suggests that tethered individual fibres that do not interact with neighbouring fibres, either physically or aerodynamically, should remain close to drum surface (from which it is tethered) when in thin annular geometries. When individual fibres approach a neighbouring rotating drum, then we found mechanisms for transfer onto the next drum. We argue that fibres on the taker-in do not transfer by their tails first as suggested in Varga and Cripps (1996), and also explain the difference between taker-in to cylinder and cylinder to doffer transfer mechanisms.

The rest of the work examines how densely packed arrays of fibres are broken down into single fibres. The first part of this work considers how fibres are teased out of tufts composed of a uniformly entangled group of fibres and also how tufts can be broken down. Two cases are described in detail; a single fibre being extracted from a

tuft and two separating tufts with interconnecting fibres. The motion of a fibre being withdrawn from the tuft is assumed not to affect the mechanical integrity of the tuft. The basic model is derived and a one-parameter family of solutions are found. These are tested against experimental data from which we can attain values for the respective parameter. Our conclusion indicates the optimum conditions for extraction, fibre breakage conditions, and tuft distortion under both shearing, carding and pulling. This analysis is particularly relevant to modelling tufts and fibre behaviour at the feeder-in to taker-in, taker-in to cylinder and to a lesser extent the cylinder and revolving-flats region. Qualitative comparison with experiment, using the model for two tufts with many interconnecting fibres, are not satisfactory. Therefore, we go onto consider a model that describes many interacting fibres, where structural changes in the fibres intrinsically change physical properties of the mass of fibres.

In contrast to the work on single fibres and fibres and tufts, where classical mechanics is applied in novel settings, we derive a continuum model that describes the structure of entangled fibres and their evolution in the carding machine when fibre-fibre interactions are the dominant forces acting. Motivated partly by the work on fluid suspension (Hinch and Leal, 1975, Spencer, 1972, Toll and Månson, 1994), we characterise the material with velocity, density, directionality, and introduce the notion of alignment and entanglement. The model is created specifically for fibres under tension, and the conjectured governing equations provide a good basis for modelling evolution in the structure of fibres throughout the carding process. Comparisons with experiment are promising.

Chapter 2

Aerodynamics and Single Fibres

2.1 Introduction

Understanding the life of a single fibre subject to mechanical forces and aerodynamic drag during the carding process plays an important role in understanding the dominant mechanisms within the machine. We see from photographic evidence in section 1.1 that individual fibres are found throughout the process, where interaction with other neighbouring fibres either hydrodynamically or through frictional contact points can be ignored. Our aim is to predict how a single fibre moves through the machine, therefore it is important to understand the fundamental mechanisms governing its motion.

We shall work through every aspect of fibre motion that occurs inside the carding machine, and these can be placed under two fairly general headings, thin channels and transfer regions. Although the airflow within the machine travels through complicated geometries, and moreover is turbulent in many regions, we use quite crude mean flow approximations that enable us to highlight important mechanisms. One criticism of some mathematical models in the literature review of section 1.1.2, not only macroscopic probabilistic models but also those that consider forces acting on fibres in the carding machine, such as those of Roberts Jr (1996), Wibberly and Roberts Jr. (1997) and Rust and Gutierrez (1994), is the lack of insight the models and their respective solutions give. Our analysis will incorporate hook and machine geometries, physical fibre properties and machine sensitive aerodynamics.

There are a number of models that will cater for textile fibres that are attached to the hooks on a rotating drum. By finding the dominant external forces, and balancing these with the fibre's internal resistive forces, we find the appropriate governing equations. This results in the quasi-steady inextensible string equations, driven by

either high or low Reynolds number drag. We begin our study by introducing aspects of the problems that are central to modelling a single fibre in the carding machine.

We find that the geometries within the carding machine play an important role in calculating the body forces acting on a fibre. There are parts of the machine which intuitively seem as though they would produce similar forces on a fibre but in fact produce completely different fibre motion. For example, a fibre will probably remain on the cylinder whilst in the cylinder-revolving-flats region but may transfer onto the doffer when presented by the cylinder hooks. Physically, both the aforementioned scenarios involve similar dynamics and geometries but the fibres behave in very different ways. Such phenomena in the carding machine have not been explained. We present a model of a general fibre in the carding machine, and we give an aerodynamic explanation to the aforementioned paradox.

2.2 A Fibre in the Carding Machine

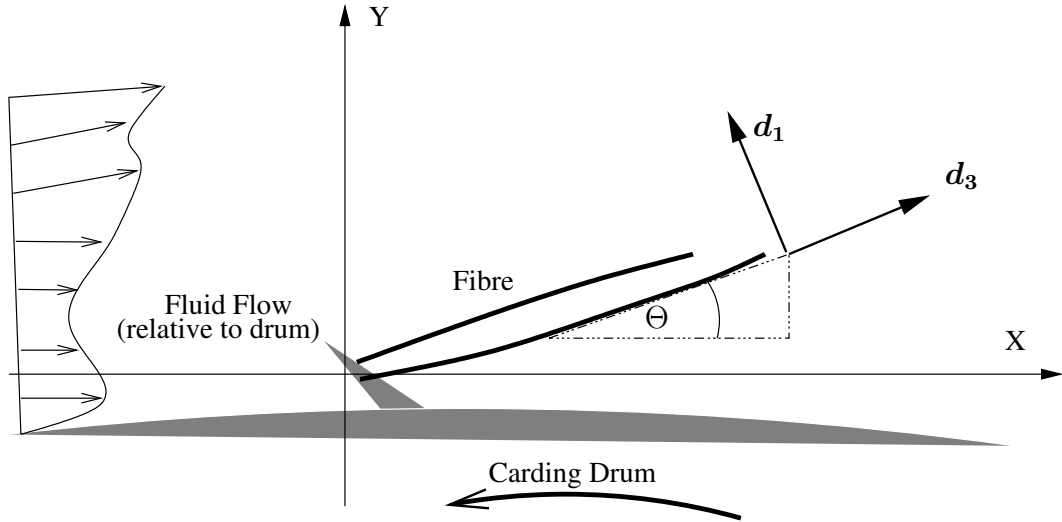


Figure 2.1: A diagram of a single fibre on a rotating drum.

A single fibre moves through the carding machine as a result of being tethered and dragged by hooks on the rotating drums, see figure 2.1. We can expect the scenario shown in figure 2.1, on each of the three major drums in the carding machine; the taker-in, cylinder and doffer. Rotational and aerodynamic forces are bound to affect a fibre's displacement, but the relative importance of these effects will depend on the drag acting on the fibre and the drum's size and speed. By considering the friction between the hook-surface and fibre, we may also measure the likelihood that

a fibre will slip off the hook from which it is tethered. The neighbouring machinery may also affect the fibre's motion throughout the carding process but we shall begin our calculations by assuming that a fibre is fixed to a hook at a single point and incorporate machine geometries into the fluid dynamics.

2.3 A Mathematical Model for a Single Fibre

Short wool, polyester and cotton are typical textiles that are carded in short staple machines. In appendix A.2 we observe that they share a number of physical characteristics, the most obvious being that they all have small aspect ratios, $\epsilon = \frac{a}{l}$, where a is the average diameter and l the average length of a fibre. Therefore, we can represent a point on the surface of a fibre as

$$\mathbf{R}(s, t) = (X(s, t), Y(s, t), Z(s, t))^T + a\mathbf{h}(s, t), \quad (2.1)$$

where (X, Y, Z) represents a point on the centre-line, using s for arclength along the fibre, t for time and $|\mathbf{h}| \sim O(1)$ represents cross sectional variances. The cross sectional variances are considered to be small compared to the length of the fibre, $0 < \epsilon \ll 1$. For our purposes, $\mathbf{R}$ will describe position in a rotating frame of reference for which $\mathbf{R}(0, t)$ is the point at which the fibre is tethered. There are a number of important fibre attributes: yield and breaking criterion when extended under tension, the roughness of fibre surface which affects the drag induced by the external fluid flow and the bending stiffness of the fibre which enables a fibre to keep its natural curvature or crimp.

2.3.1 Drag on a Fibre

We can take into account varying fibre surfaces in calculating the drag on a fibre. Cotton and polyester are good examples of the disparity within the micro-structure of textile fibres; the former is a natural material with a rough hairy surface and the latter is man-made with a smooth finish. Taylor (1952) obtained experimental evidence that shows how the texture of a slender cylindrical surface noticeably affects aerodynamic drag. This empirical work considered incompressible, unidirectional flow with Reynolds numbers between 20 and 10^6 based on the radius of a fibre, a , and the magnitude of fluid velocity, U . In practice the Reynolds number for flow around a fibre in the carding machine varies between 0.5 and 100. If we consider each rotating drum within short-staple carding machines separately, the steady state fluid flow is only dependent on the cylindrical polar radial variable r , the distance from the centre

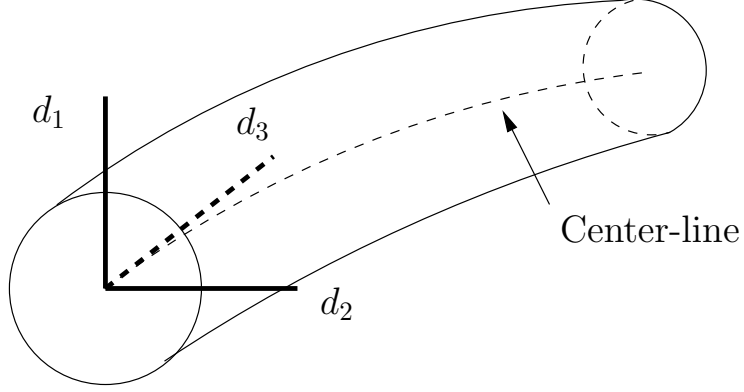


Figure 2.2: Orthonormal triad of vectors for a fibre's centre-line.

of the drum, and is independent of the azimuthal and axial directions of the cylinder. We neglect the width of a fibre and define an orthonormal triad for each point of a fibre with respect to the centre line to be $(\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3)$, where $\mathbf{d}_1$ and $\mathbf{d}_2$ are the principal normals in the cross sectional plane of the cylinder and $\mathbf{d}_3$ is the tangent, see figure 2.2. We make the assumption that the fibre is aligned in the plane of the unidirectional flow $\mathbf{U}(r)$, $\mathbf{d}_2 \cdot \mathbf{U} = 0$. Therefore, we consider a two-dimensional model for a fibre that will feel drag in the local tangential and normal directions to its centre-line. Using $\Theta(s, t)$ to represent the angle between the fibre's tangent and the incoming unidirectional flow, the drag term $\mathbf{D}$ per unit length for a smooth fibre when $Re \gg 1$ is given by Taylor (1952) as

$$\mathbf{D} = \frac{\rho_{air} a U^2}{2} \left[\left(\sin^2 \Theta + \frac{4}{\sqrt{Re}} \sin^{\frac{3}{2}} \Theta \right) \mathbf{d}_1 + \frac{5.2}{\sqrt{Re}} \cos \Theta \sqrt{\sin \Theta} \mathbf{d}_3 \right], \quad (2.2)$$

where ρ_{air} is the density of air and based on our assumption that the velocity of the fluid flow described in a global Cartesian framework $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ as $\mathbf{U} = (U, 0, 0)$ has no component in the $\mathbf{d}_2$ direction. Equation (2.2) suggests that a straight polyester fibre, modelled as a smooth cylinder, would not feel any drag when it is aligned parallel to the direction of the oncoming uniform flow. If we were to consider a natural fibre which has an approximately cylindrical but uneven surface, the Taylor expression for drag is

$$\mathbf{D} = \frac{\rho_{air} a U^2}{2} \left[\left(\sin^2 \Theta + \frac{4}{\sqrt{Re}} \sin \Theta \right) \mathbf{d}_1 + \cos \Theta \mathbf{d}_3 \right], \quad (2.3)$$

Alternatively if the fibre has a hairy surface, then

$$\mathbf{D} = \frac{\rho_{air} a U^2}{2} \left[\frac{4}{\sqrt{Re}} \sin \Theta \mathbf{d}_1 + \cos \Theta \mathbf{d}_3 \right]. \quad (2.4)$$

We note that for the aforementioned Taylor drags, $U \geq 0$ and $\theta \in [0, \frac{\pi}{2})$. Highlighting the differences in aerodynamic drag for varying fibre surfaces, we observe that for both cases (2.3) and (2.4) when compared to (2.2) the salient difference is the non-zero drag acting on a fibre that is aligned with the fluid flow.

When the fluid flow around a fibre is “slow”, $Re < 1$, it would be inconsistent to use the Taylor approximations for drag. Instead, we use an analytical expression derived from the Stokes flow approximation for slow viscous fluid dynamics (Keller and Rubinow, 1976) where the drag to leading order for a smooth fibre is

$$\mathbf{D} = \frac{8\pi\rho_{air}U^2a}{Re \log \frac{1}{\epsilon}} (2 \sin \Theta \mathbf{d}_1 + \cos \Theta \mathbf{d}_3). \quad (2.5)$$

Batchelor (1970) pays attention to small surface variations but this does not affect the first order terms given in equation (2.5). Other effects such as hydrodynamic interaction with other fibres and solid boundaries (Cox, 1970, Khayat and Cox, 1989) also give higher order modifications. Therefore, for drag induced by medium and high Reynolds numbers we can use the empirical approximation given by (2.3) or (2.4) and for low Reynolds numbers fluid flow we can use the leading order approximation for Stokes drag given by (2.5).

2.3.2 Internal Fibre Forces

Most fibres are not straight and polyester is even deliberately crimped in order to produce more cohesive yarns. We need to consider how a rod which has a surface described by (2.1) resists external body forces. The resulting equations are derived from a force balance between external and internal forces. In particular we see that all the aforementioned drag terms are dependent on the orientation of the fibre Θ , and it is therefore important to understand the interplay between drag and internal forces.

Although most textile fibres have bending moduli of the same order of magnitude, see appendix A.2, their natural shapes can vary considerably. For natural fibres a distribution of shapes occur due to the conditions during their formation but the uniform crimp in man-made fibres is regulated by machinery and variances could be considered to be statistically insignificant. Therefore each fibre will have natural curvature and due to the fact that textile fibres share low yield and breaking extensions, we begin by treating our cylinder as an inextensible, elastic rod.

There has been much work on the Kirchhoff-Love theory of linearly elastic rods, sometimes known as elasticas. We give a brief outline of the equations concerned,

consequently allowing us to outline the important dimensionless parameters when considering a fibre in the carding machine. The basis of this theory (Antman, 1995, Love, 1927) is that extensional and shear deformations are small compared to bending and twisting and because the fibre is treated as a thin cylinder with a circular cross section we also neglect the effects of warping. We write down the geometrical relationship, that the fibre tangent is normal to the cross-sectional plane:

$$\frac{\partial \mathbf{R}}{\partial s} = \mathbf{d}_3. \quad (2.6)$$

We define the stress resultant vector $\mathbf{N}(s, t)$ and the couple resultant vector $\mathbf{M}(s, t)$ to be

$$\mathbf{N}(s, t) = \sum_{i=1}^3 N_i(s, t) \mathbf{d}_i(s, t), \quad (2.7)$$

$$\mathbf{M}(s, t) = \sum_{i=1}^3 M_i(s, t) \mathbf{d}_i(s, t), \quad (2.8)$$

where N_1 and N_2 are the shear forces and M_1 and M_2 are the bending moments along the principal axes of the cross-sectional plane, while N_3 is the tensile force and M_3 is the twisting moment. Balancing forces and couples at each cross section in an inertial frame gives:

$$\frac{\partial \mathbf{N}}{\partial s} + \mathbf{F} = \pi a^2 \rho_{fibre} \frac{\partial^2 \mathbf{R}}{\partial t^2}, \quad (2.9)$$

$$\frac{\partial \mathbf{M}}{\partial s} + \frac{\partial \mathbf{R}}{\partial s} \wedge \mathbf{N} = \rho_{fibre} I \sum_{i=1}^2 \frac{\partial^2 \mathbf{d}_i}{\partial t^2} \wedge \mathbf{d}_i, \quad (2.10)$$

where $\mathbf{F}$ is the external body force per unit length, $I = \frac{\pi a^4}{2}$ is the coefficient of inertia, and ρ_{fibre} is the density of the fibre. To complete the equations (2.9) and (2.10) we use the Euler-Bernoulli constitutive law, which relates twist linearly with components of curvature:

$$\mathbf{M} = EI(\kappa_1 \mathbf{d}_1 + \kappa_2 \mathbf{d}_2) + GJ\kappa_3 \mathbf{d}_3, \quad (2.11)$$

E is the Young's modulus, G is the shear modulus, J is the polar moment of inertia, and the κ_i 's are the components of curvature. Equation (2.11) is true for a naturally straight rod; the moments are proportional to curvature, so in the two limits as the curvature tends to infinity the moments become singular and when the fibre is straight there are no moments in the fibre. If a fibre is naturally curved then we would need

to alter (2.11) so that there would be no moments in the fibre when in its preferred natural form.

In order to capture the important terms we derive a dimensionless form to equations (2.9) – (2.11). Appropriate scalings for each term are:

$$\begin{aligned} s &= l\bar{s}, \quad \mathbf{R} = l\bar{\mathbf{R}}, \quad \kappa_i = \frac{\bar{\kappa}_i}{l}, \quad \mathbf{F} = F\bar{\mathbf{F}}, \\ \mathbf{N} &= lF\bar{\mathbf{N}}, \quad \mathbf{M} = \frac{EI}{l}\bar{\mathbf{N}}, \quad t = \frac{l}{U}\bar{t}, \end{aligned} \quad (2.12)$$

where the dimensionless terms denoted by an over-bar are $O(1)$. The typical body force F represents the size of the drag terms $\mathbf{D}$, which for Taylor drag will be $\frac{\rho_{air}aU^2}{2}$ as discussed in section 2.3.1 and for slow flow $\frac{8\pi U\mu}{\log \frac{1}{\epsilon}}$, where μ is the viscosity of air. The equations (2.9) and (2.10) become:

$$\frac{\partial \bar{\mathbf{N}}}{\partial \bar{s}} + \bar{\mathbf{F}} = \Lambda_1 \frac{\partial^2 \bar{\mathbf{R}}}{\partial \bar{t}^2}, \quad (2.13)$$

$$\frac{\partial \bar{\mathbf{M}}}{\partial \bar{s}} + \Lambda_2 \frac{\partial \bar{\mathbf{R}}}{\partial \bar{s}} \wedge \bar{\mathbf{N}} = \Lambda_3 \sum_{i=1}^2 \frac{\partial^2 \mathbf{d}_i}{\partial \bar{t}^2} \wedge \mathbf{d}_i, \quad (2.14)$$

respectively. This results in three important parameters:

$$\Lambda_1 = \frac{\rho_{fibre}\pi a^2 U^2}{lF}, \quad \Lambda_2 = \frac{l^3 F}{EI}, \quad \Lambda_3 = \frac{\rho_{fibre} U^2}{E}, \quad (2.15)$$

where Λ_1 represents fibre acceleration over drag, Λ_2 is drag over flexural rigidity and Λ_3 is the torque over elastic modulus; N.B. $\Lambda_3 = \frac{\Lambda_1 \Lambda_2}{2} \epsilon^2$.

2.3.3 Aerodynamics

As we have considered the primary forces that will affect the fibre, we now focus on the external forces, in particular aerodynamics within the machine. Due to the complexity of the aerodynamics in the carding machine we elect to describe fluid flow regimes for specific geometries that cover the key areas of process, but there are a few general features that can be outlined before any particular scenarios are considered. The air interacting with the fibres will be modelled as a homogeneous incompressible fluid, and this is accurate if the flow is sufficiently sub-sonic, for example the Mach number is less than 0.7, see Liepmann and Roshko (1957). The drag on the fibre depends on the Reynolds number Re , where the length scales of interaction will be comparable to the diameter of the fibre. The Reynolds number for the flow in general Re_{drum} , assuming that the hydrodynamic effect caused by fibre motion is negligible, will depend on the drum size length scales. Therefore as the rotating drums have

radii in the order of metres, see appendix A.1, and the single fibres have radii in the order of microns, see appendix A.2, the Reynolds numbers for flow regimes inside the carding machine will tend to be six orders of magnitude greater than flow around a single fibre, see appendix A.3. Consequently, we can expect turbulent or laminar boundary layers to play an important role in determining the external body force acting on a fibre.

2.3.4 Important Parameters

PARAMETERS	APPROPRIATE APPLICATION
$Re_{fibre} < 1$	Stokes Drag (approximately smooth surfaces only)
$Re_{fibre} > 1$	Taylor Drag (dependent on surface roughness)
$\Lambda_2 = O(1)$ and $\Lambda_1, \Lambda_3 \ll 1$	quasi steady elastica
$\Lambda_2 \gg 1$ and $\Lambda_1, \Lambda_3 = O(1)$	unsteady string with no bending stiffness
$\Lambda_2 \gg 1$ and $\Lambda_1, \Lambda_3 \ll 1$	quasi steady string with no bending stiffness

Table 2.1: A table of parameter regimes for a fibre in the carding machine.

This leaves us with a number of parameters that we calculate *a priori* and in some regimes, before any serious computation, this will indicate the dominant mechanisms of fibre transport and will also allow us to simplify the models accordingly. We give examples of such regimes in table 2.1. The Reynolds numbers around a fibre suggest the appropriate drag approximations and the Λ_i 's which may indicate that bending stiffness, accelerations and torques are negligible. Notice in table 2.1 that we have not included all possible scenarios, in particular, $\Lambda_2 \ll 1$ is not relevant in practice and we also neglected significant differences in Λ_1 and Λ_3 as they would produce less physically relevant scenarios. We did not include the effect of rotational accelerations but these can be included in the body forces $\mathbf{F}$ in equation (2.9) and (2.13). Now we are in a good position to begin formulating problems specifically for different areas of the machine and thence we can write down the respective leading order governing equations.

We begin the dimensionless evaluation by writing down the approximate magnitudes of dimensional quantities, see table 2.2. Then we place the dimensional quantities into the formulae given in (2.15), see table 2.3. The Reynolds number Re , is given in appendix A.3 for each drum and with the Λ_i 's we suggest the appropriate models. Table 2.3 suggests that a fibre in the carding machine can be modelled as a

QUANTITY	VALUE	UNITS
ρ_{air}	1.1	gcm^{-3}
μ	1.7	$gcm^{-1}s^{-1}$
U	0.5-40	ms^{-1}
l	2.5-4	cm
a	0.5-11	$micron$
E	10	$kNmm^{-2}$
$I = \frac{\pi a^4}{2}$	150	$micron^4$

Table 2.2: A table of the dimensional numbers for a fibre and air, see appendix A.2.

quasi-steady string. It should be noted that in some cases bending stiffness will be important, for example near the point of contact with a hook.

DRUM	Λ_1	Λ_2	Λ_3	Re	MODEL
Taker-In	10^{-3}	10^4	10^{-14}	10-26	steady string with Taylor drag
Main Cylinder	10^{-3}	10^4	10^{-14}	34-52	steady string with Taylor drag
Doffer	10^{-5}	10^3	10^{-9}	0.8 - 3.4	steady string with Stokes drag

Table 2.3: A table of the dimensionless parameters and the suitable models for Taker-in, Cylinder and Doffer.

2.4 Fibres on the Cylinder and Taker-in

Using the table 2.3, assuming that $\Lambda_1, \Lambda_3 \ll 1$ and $\Lambda_2 \gg 1$, the approximate governing equations (2.13) and (2.14), where we now remove the over-bars, reduce to

$$\frac{\partial \mathbf{N}}{\partial s} + \mathbf{F} = 0, \quad (2.16)$$

$$\frac{\partial \mathbf{R}}{\partial s} \wedge \mathbf{N} = \mathbf{d}_3 \wedge \mathbf{N} = 0. \quad (2.17)$$

Equation (2.17), due to the fact that the fibre is to leading order a cylindrical surface, and that we have neglected warping, implies that the stress resultant only has components in the direction of the tangent, which is the tension and we write $N_3 = T$ so that $\mathbf{N} = T(s, t)\mathbf{d}_3$. The Bernoulli-Euler equations (2.11) and the force (2.9) and resultant (2.10) balances for an elastica are derived with the implicit assumption that the fibre is inextensible (Antman, 1995, Love, 1927). As equation (2.11) is no longer needed, we use an inextensibility constraint to close the equation (2.16):

$$\mathbf{d}_3 \cdot \mathbf{d}_3 = 1. \quad (2.18)$$

Before we write down the components of equations (2.16), there are two other aspects that need further consideration in order to determine $\mathbf{F}$: rotational effects caused by the motion of the cylinder and the fluid flow in the machine.

2.4.1 Rotational Forces

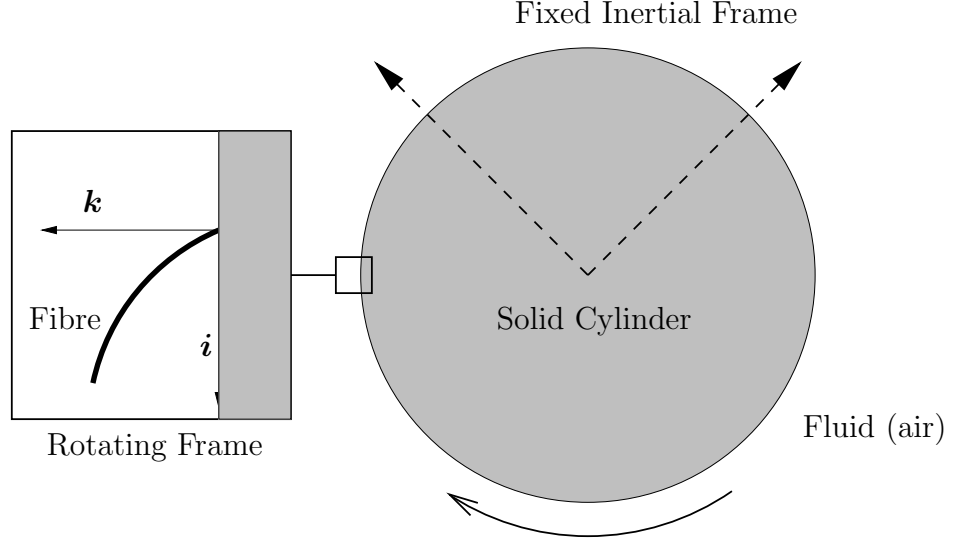


Figure 2.3: A rotating cylinder and the local frame of reference for a fibre

Now that we have established the fibre equations as being quasi-steady, we suppose the fibre is fixed in a frame rotating with the drum. Within the carding machine, the angular velocities of each drum stays constant, see figure 2.3. In general, the force on a mass m at position $\mathbf{X}$ in a frame rotating with angular velocity $\mathbf{\Omega} = \Omega \mathbf{e}_\theta$ is

$$m(\ddot{\mathbf{X}} + \dot{\mathbf{\Omega}} \wedge \mathbf{X} + 2\mathbf{\Omega} \wedge \dot{\mathbf{X}} + \mathbf{\Omega} \wedge (\mathbf{\Omega} \wedge \mathbf{X})), \quad (2.19)$$

where a dot means the derivative with respect to time in the rotating frame of reference. Since the fibre is fixed in the rotating frame, the rotational body force acting on the fibre is the centrifugal force per unit length

$$\rho_{fibre} \pi a^2 A \Omega^2 \left[\mathbf{e}_\theta \wedge (\mathbf{e}_\theta \wedge \mathbf{e}_r) + O\left(\frac{l}{A}\right) \right], \quad (2.20)$$

to leading order, as $\mathbf{R} = O(l)$, where we have written $\mathbf{X} = A\mathbf{e}_r + \mathbf{R}$ and A is the radius of the cylinder. More precisely, ρ_{fibre} should be the difference between the fibre density and the density of air, but we can take the latter to be negligible compared to the former. The centrifugal force produces a body force that encourages the fibre to move away from the drum surface.

2.4.2 Fluid Dynamics

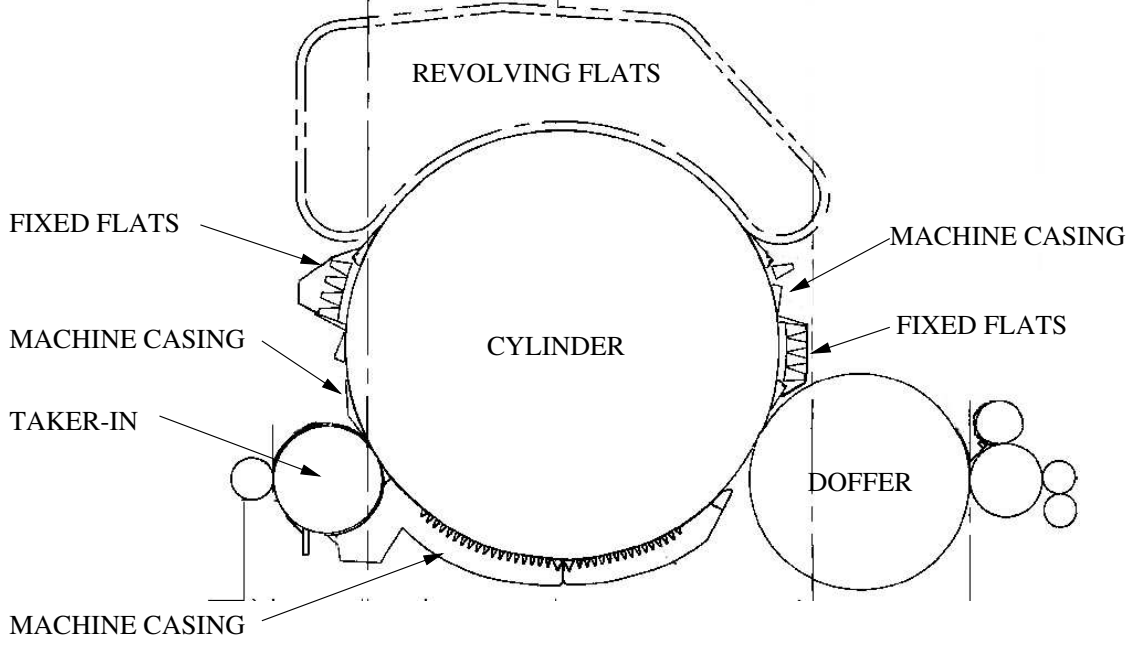


Figure 2.4: A diagram of the mark V Crosrol Revolving-Flats Carding Machine.

In the preliminary investigations we stated that the air can be treated as an homogeneous, incompressible, viscous fluid. Newtonian flow is governed by the Navier-Stokes and continuity equations (Ockendon and Ockendon, 1995)

$$\rho_{air} \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{f}, \quad (2.21)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (2.22)$$

where $\mathbf{u}(\mathbf{x}, t)$ is the velocity, $p(\mathbf{x}, t)$ pressure, $\mathbf{f}(\mathbf{x}, t)$ is body force, ρ_{air} density of air and μ viscosity. There is a strong possibility that the fluid flow will be turbulent as the Reynolds numbers based on drum radii are $O(10^6)$, see appendix A.3, and the hooks may act as vortex generators inducing turbulent boundary layers (Smith, 1994). However, a good starting point is to consider a laminar flow. This could characterise the mean flow of the turbulent fluid as long as Reynolds stresses are negligible, i.e. variations $\mathbf{u}'$ about the mean velocity produce stresses $(\mathbf{u}' \cdot \nabla) \mathbf{u}'$ which must be small.

A diagram of a carding machine is given in figure 2.4. We can see that for most parts of the machine a fibre will travel through thin geometries. The only part of the machine where this is not true is on the doffer. Therefore, we begin the modelling process by approximating the carding machine as three rotating drums encased by

a solid boundary, see figure 2.5. To simplify the problem we shall consider annular flow.

2.4.2.1 Annular Flow without Hooks

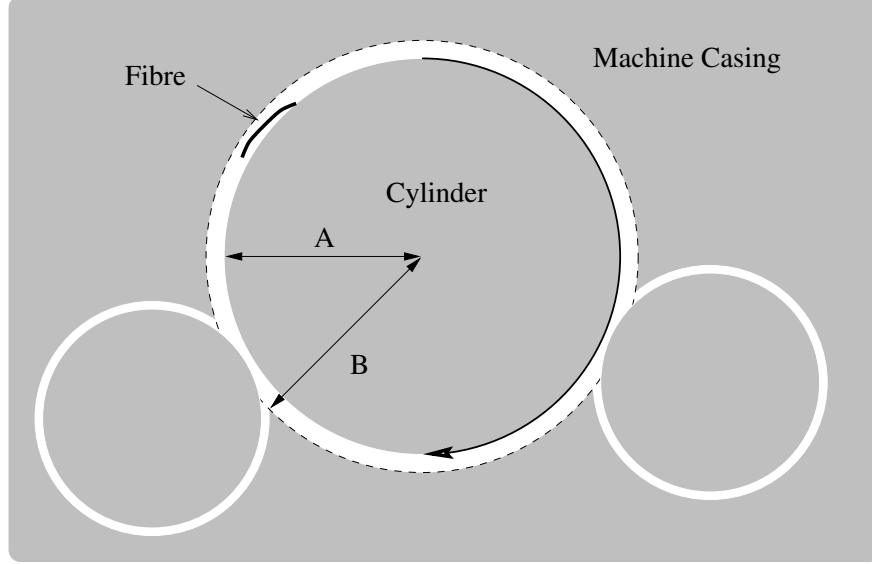


Figure 2.5: A diagram illustrating the regions of annular flow.

If we ignore the resistive forces produced by the hooks, we can find a simple flow encased by two coaxial cylinders driven by the inner drum's rotation, see figure 2.5. Using polar coordinates $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$, the unidirectional velocity is prescribed as $\mathbf{u} = u_\theta(r, \theta, z, t)\mathbf{e}_\theta$, and we look for a steady state ($\frac{\partial}{\partial t} = 0$), axially symmetric flow ($\frac{\partial}{\partial z} = \frac{\partial}{\partial \theta} = 0$), and so the governing equations for the fluid flow (2.22) and (2.21) in component form become

$$-\frac{\rho_{air}u_\theta^2}{r} = \frac{dp}{dr}, \quad (2.23)$$

$$\frac{d^2u_\theta}{dr^2} + \frac{1}{r}\frac{du_\theta}{dr} - \frac{u_\theta}{r^2} = 0. \quad (2.24)$$

To accompany the equations (2.23) and (2.24) we impose no-slip conditions on the inner drum ($r = A$) with angular velocity Ω and outer ($r = B$) stationary cylindrical surface:

$$u_\theta = 0 \quad \text{at} \quad r = B \quad \text{and} \quad u_\theta = A\Omega \quad \text{at} \quad r = A, \quad (2.25)$$

and note that the imposition of no-penetration is not required as this condition is satisfied implicitly through the prescribed azimuthal velocity. The fluid flow is

$$u_\theta = \frac{\Omega A^2 B^2}{B^2 - A^2} \left(\frac{1}{r} - \frac{r}{B^2} \right), \quad (2.26)$$

and the pressure can be found accordingly by integrating (2.23). For the components in the carding machine $B = A + \delta$, where $A = 0.5m$ and $\delta = 10^{-2}m$, which means that the velocity profile is approximately Couette flow.

2.4.2.2 Annular Flow with Hooks

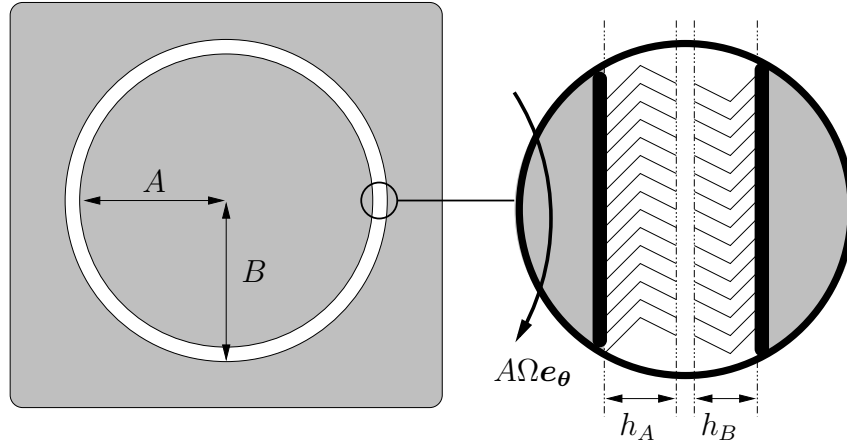


Figure 2.6: A diagram illustrating annular flow with hooks.

To progress onto the next level of sophistication we incorporate a body force on the fluid that is due to the interaction of the hooks on both the rotating drum and the outer drum casing. We can use an array of cylinders to describe the hooks, with radius a_{hook} spaced distance d apart, and that move through the fluid at velocity $\mathbf{U}$. Based on high porosity

$$\frac{a_{hook}}{d} \ll 1, \quad (2.27)$$

for fluid moving around the hooks, this produces the body force (Ockendon and Terrill, 1993, Terrill, 1990):

$$\mathbf{f} = -\frac{4\mu\varrho}{a_{hook}^2} N(\varrho) (\mathbf{u} - \mathbf{U}), \quad (2.28)$$

where $\varrho = \frac{\pi a_{hook}^2}{4d^2} \ll 1$ is the volume fraction of hooks in the fluid and N is a diagonal matrix, which to $O(\varrho)$ is

$$N_{33}(\varrho) = \frac{1 - \varrho}{\log \frac{1}{\varrho} - 1.476 + 2\varrho}, \quad (2.29)$$

$$N_{11} = N_{22} = \frac{2}{(1 - \varrho)(\log \frac{1}{\varrho} - 1.476 + 2\varrho)}, \quad (2.30)$$

and this assumes that the hooks are perpendicular to the flow. Notice that the high porosity condition (2.27) translates to small volume fraction $\varrho \ll 1$. In fact the volume fraction of the hooks on the rotating drums and the fixed flats vary between $\varrho = \frac{1}{7}$ and $\frac{1}{10}$ but for the hooks on the revolving-flats the volume fraction is $\varrho = \frac{1}{100}$. Here the hooks on the outer casing $r = B$ are stationary and protrude a height of h_B and those on the moving inner drum have velocity $A\Omega\mathbf{e}_\theta$ and height h_A , see figure 2.6. This produces the regional body forces:

$$\mathbf{f} = \begin{cases} -\frac{4\mu\varrho}{a_{hook}^2}N(\varrho)(\mathbf{u} - A\Omega\mathbf{e}_\theta) & \text{when } A < r < A + h_A \\ 0 & \text{when } A + h_A < r < B - h_B \\ -\frac{4\mu\varrho}{a_{hook}^2}N(\varrho)\mathbf{u} & \text{when } B - h_B < r < B \end{cases}, \quad (2.31)$$

where $A + h_A < B - h_B$. The dimensionless form of the Navier-Stokes, with scalings $\mathbf{u} = A\Omega\bar{\mathbf{u}}$, $t = \frac{1}{\Omega}\bar{t}$, $p = \rho_{air}A^2\Omega^2\bar{p}$, $\nabla = \frac{1}{A}\bar{\nabla}$ and $\mathbf{f} = \frac{4\mu\varrho N_{11}(\varrho)}{a_{hook}^2}\bar{\mathbf{f}}$, for a steady state become:

$$\frac{\partial \bar{\mathbf{u}}}{\partial \bar{t}} + (\bar{\mathbf{u}} \cdot \bar{\nabla})\bar{\mathbf{u}} = -\bar{\nabla}\bar{p} + \frac{1}{Re_{drum}}\bar{\nabla}^2\bar{\mathbf{u}} + \Upsilon\bar{\mathbf{f}}, \quad (2.32)$$

where Reynolds number is $Re_{drum} = \frac{\rho_{air}A^2\Omega}{\mu}$ and the hook body force over inertia is $\Upsilon = \frac{4\mu\varrho N_{11}}{a_{hook}^2\rho_{air}A\Omega^2}$. Typical values for Υ are 10^2 and as the Reynolds number is high this means that the effect of the hooks dominate, and therefore we can expect the flow to be entrained with the carding surfaces. Fluid in between the two regions will be very similar to the flow in an annulus where the solid boundaries are theoretically closer. In equation (2.26), the drum parameters A and B can be simply changed to $A + h_A$ and $B - h_B$ respectively. So we can model the airflow with a mixture of shear and plug flow when incorporating hooks into the aerodynamics as shown in figure 2.7, or more simply we can consider just shear flow. There is no machine casing around the doffer so a shear flow may be more suitable in this case.

To summarise, we have equations (2.16) and (2.17) for a quasi-steady inextensible string. The forces that affect a fibre which are produced in the carding machine environment are aerodynamic and rotational and these can now be included in the

governing equations (2.13) and (2.14). Rotation is approximated by a centrifugal force (2.20) and the drag of the fibre will be given by one of the following equations (2.3) – (2.5). The drag will also depend on the fluid flow and that will in turn depend on the machine geometry and dynamics, but the full governing flow equations that incorporate hooks are given in (2.32).

2.4.3 The Equations

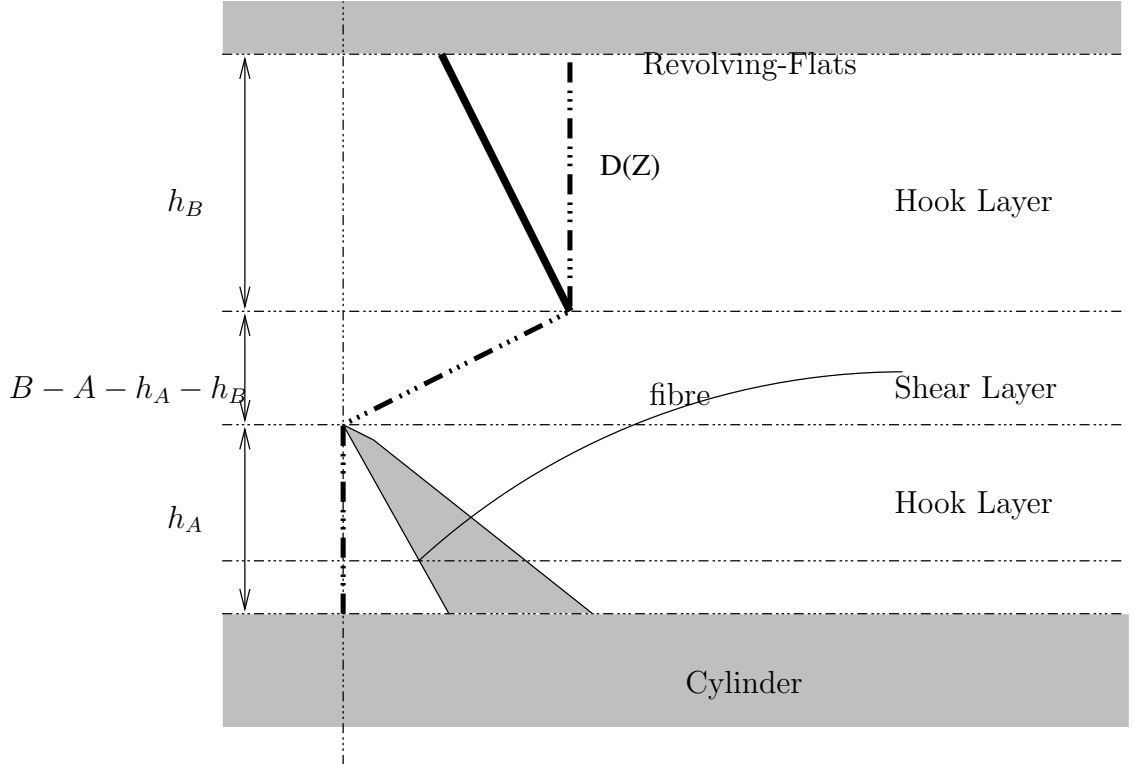


Figure 2.7: A diagram for annular flow local to a fibre, where both surfaces are covered in hooks.

Channels with hooked surfaces can be found in between the cylinder and the revolving and fixed flats, between the taker-in and machine casing and in between the cylinder and machine casing, see figure 2.4. Without being too concerned with the regions near transfer between the cylinders, we model the airflow in the aforementioned channels as above. For the region between the cylinder and fixed or moving flats both surfaces have hooks, but in the other regions the machine casing is smooth.

From section 2.3, the Taylor drag approximation holds for a fibre on the cylinder and taker-in and to begin with we choose to model the fibre as cotton, i.e. with a hairy surface. Although the analysis allows for quasi-steady states for the fibre we are

only interested in steady states in the fluid, and this is still consistent with Λ_1 and Λ_3 being small compared to Λ_2 . We find solutions that will give the displacement of a fibre tethered by a hook and its respective tension. Consequently we can estimate the amount a fibre will protrude from the rotating drum's surface and also the forces acting on the hook-fibre contact point by a static-friction analysis. We find parameter regimes that will allow fibres to slip off hooks whilst travelling in annular geometries.

Once the drums have attained a steady-state, we can consider the cross planar forces to be negligible as there is no component of airflow or rotational forces across the rotating drums. Due to the fact that the bending stiffness is small, we can consider a two dimensional model. The aerodynamic modelling in this section has been based on a unidirectional assumption, so we can write $\mathbf{U} = U(Z)\mathbf{i}$ where the global position vector relative to the centre of the cylinder $r\mathbf{e}_r = (A + Z)\mathbf{k}$, where A is the radius of the cylinder and Z corresponds to the radial component of the position vector from the surface of the inner drum; we can revert to a Cartesian frame of reference as the curvature of the drum $\frac{1}{A}$ is small compared with $\frac{1}{l}$, making the region near the fibre to be approximately planar. $U(Z)$ is the approximation discussed in section 2.4.2 and is shown in figure 2.7,

$$U(Z) = \begin{cases} \Omega A & B - A - h_B < Z < B \\ \Omega A \frac{Z - h_A}{B - A - h_A - h_B} & h_A < Z < B - A - h_B \\ 0 & 0 < Z < h_A \end{cases} \quad (2.33)$$

Now we can write the equations (2.18) governing the position of the string, where the forces can now be given explicitly:

$$\frac{\partial}{\partial s}(T\mathbf{d}_3) + \Omega^2 \rho_{fibre} \pi a^2 A \mathbf{k} + \mathbf{D}(U) = 0 \quad (2.34)$$

where $\mathbf{i}$ and $\mathbf{k}$ are global unit vectors, $\mathbf{d}_3$ is the tangent vector which can be written as $(\cos \Theta, \sin \Theta)$, and $\mathbf{D}$ can be taken from (2.2), (2.3), or (2.4) depending on the fibre's surface roughness. Using the scalings (2.12), where tension is scaled with the product of fibre length and aerodynamic drag, i.e. $\frac{\rho_{air} a A^2 \Omega^2}{2}$, equation (2.4) for drag on a hairy fibre and the inextensibility constraint (2.18) we find that

$$\frac{d\bar{T}}{d\bar{s}} + D(\bar{Z}(\bar{s})) \cos \Theta + \varsigma \sin \Theta = 0 \quad (2.35)$$

$$\frac{d\Theta}{d\bar{s}} \bar{T} - 4D(\bar{Z}(\bar{s})) \frac{\sin \Theta}{\sqrt{Re}} + \varsigma \cos \Theta = 0, \quad (2.36)$$

where over-bars denotes dimensionless variable, $D(\bar{Z})$ is the dimensionless magnitude of the drag and $\varsigma = \frac{2\pi a \rho_{fibre}}{A \rho_{air}}$ is a dimensionless parameter. ς is the ratio of centrifugal

force over drag and the values are given in appendix A.3.2. From equation (2.4) the drag imposed is the square of velocity (2.33), so that

$$D(\bar{Z}(\bar{s})) = \begin{cases} 1 & \frac{B-A-h_B}{l} < \bar{Z}(\bar{s}) < \frac{B-A}{l} \\ \left(\frac{l\bar{Z}(\bar{s})-h_A}{B-A-h_B-h_A} \right)^2 & \frac{h_A}{l} < \bar{Z}(\bar{s}) < \frac{B-A-h_B}{l} \\ 0 & 0 < \bar{Z}(\bar{s}) < \frac{h_A}{l} \end{cases}. \quad (2.37)$$

If we are to consider the shear flow scenario when $h_A = h_B = 0$, the dimensional velocity function is

$$U(Z) = \frac{Z\Omega A}{B-A}, \quad (2.38)$$

non-dimensionalising so that ς will in this case be $\frac{2\pi a(B-A)^2 \rho_{fibre}}{Al^2 \rho_{air}}$, where $U \sim \frac{\Omega Al}{B-A}$. We have adopted this different description for ς in the case of shear flow alone as it allows us to simplify the find a numerical computation, and actual values are given in appendix A.3.2. Consequently, the drag function becomes

$$D(\bar{Z}(\bar{s})) = \bar{Z}(\bar{s})^2 \quad 0 < \bar{Z}(\bar{s}) < \frac{B-A}{l}. \quad (2.39)$$

For consistency we have scaled Z with l although we could have re-scaled with $B-A$.

To close the system (2.35)–(2.36) we relate the height of the fibre from the inner drum surface $Z(s)$ to the fibre angle $\Theta(s)$, by simply writing the geometrical constraint:

$$\frac{d\bar{Z}}{d\bar{s}} = \sin \Theta. \quad (2.40)$$

2.4.3.1 Boundary Conditions

The boundary conditions for the problem with hooks can be posed in a similar way to the case where there are no hooks and use D as given by (2.39). We assume we know the point at which the fibre is held to the hook and choose this to be the origin of the position vector $\mathbf{R}$ and the tension at the free end must be zero in order to keep the applied external forces non-singular. There is one more condition if we are to pin-point a particular solution and this comes from ensuring that Θ is regular, and from equation (2.36) at the free end $s = 1$, we find the appropriate condition. These three conditions are

$$\bar{T}(1) = 0, \quad \Theta(1) = \operatorname{arccot} \frac{4\bar{Z}(1)^2}{\varsigma \sqrt{Re}}, \quad \bar{Z}(0) = 0. \quad (2.41)$$

Imposing a zero tension at the free-end of the fibre will mean that equation (2.36) becomes singular during numerical computations and so a Taylor expansion from the end point $s = 1$ is used to find approximate values close to that end of the fibre:

$$\bar{T}(1 - \varepsilon) = \varepsilon \frac{4h^4 + \sqrt{Re} \varsigma^2}{\sqrt{16h^4 + Re \varsigma^2}}, \quad \Theta(1) = \operatorname{arccot} \frac{4Z(1)^2}{\varsigma \sqrt{Re}}, \quad \bar{Z}(0) = 0 \quad (2.42)$$

when $\bar{Z}(1) = h$ and where $0 < \varepsilon \ll 1$. Although h is unknown the approximation is fine for $h \gg \varepsilon$. We now solve the equations asymptotically and numerically and will also consider how to solve the problem with the drag that includes the hooks (2.37).

2.4.4 Asymptotic Solution

There are a number of methodologies that can be applied to find solutions to this problem. Asymptotics can be applied for small or large parameter regimes and numerical computations for all values of ς . We can plot the relationship between the height of the free-end from the drum surface h against ς . This will indicate the parameter regimes for which a fibre will interact with other carding surfaces.

2.4.4.1 Annular Flow without Hooks

For airflow without hooks we use equations (2.35), (2.36), (2.39) and (2.40). Asymptotic analysis can be employed for small ς which is relevant for the carding machine. Using a regular perturbation expansion in powers of $\varsigma^{\frac{1}{3}}$ for tension T , height Z and tangent angle Θ , we find that to leading order the fibre is flat, moreover

$$\Theta = Z = O(\varsigma^{\frac{1}{3}}), \quad \text{and} \quad T = O(\varsigma^{\frac{2}{3}}). \quad (2.43)$$

The next order in the asymptotic expansion requires numerical work, but we can deduce that the maximum height of the fibre is $O(\varsigma^{\frac{1}{3}})$. This will also allow us to assess the gripping powers of a hook if we use a static friction analysis between fibre tangent at $s = 0$ and the hook. We have chosen not to document the numerical calculations for the small ς asymptotic expansions as they are very similar to the full numerical solutions, see below in section 2.4.5.

2.4.4.2 Annular Flow with Hooks

To incorporate hooks we use the equations (2.35), (2.36), (2.37) and (2.40). We could think of solving the problem for each of the three regions where different drag forces are applied by equation (2.37). We would need to prescribe conditions at the interfaces $\bar{Z} = \frac{h_A}{l}$ and $\bar{Z} = \frac{B-A-h_B}{l}$, these would be continuity of tension and tangent angle

at the interfaces. We shall consider this in more detail when we solve the equations numerically.

We assume that ς is small and pursue an asymptotic solution in powers of $\varsigma^{\frac{1}{3}}$, and find that the leading order solution has constant tangent angle Θ in the lower entrained region. We then impose continuity of tension and tangent angle at the interface between the shear layer and the lower hook-entrained region, $Z = \frac{h_A}{l}$. For the first order problem, the solution cannot have a continuous tangent-angle Θ at the transition point and so we conclude that the transition point is at the end of the fibre. A straight line is the first order solution to the fibre's position from the point at which it is tethered to the shear-flow interface, and so we write

$$\sin \Theta = \frac{h_A}{l^2}. \quad (2.44)$$

Going to the next term in the expansion we see that only a small fraction of the fibre, of length $O(\varsigma^{\frac{1}{3}})$, will be in the shear flow and this will require numerical work. A static friction analysis will produce a basic calculation for preferred location of the hook-fibre interface and this follows in section 2.4.5.3.

2.4.5 Numerical Computations

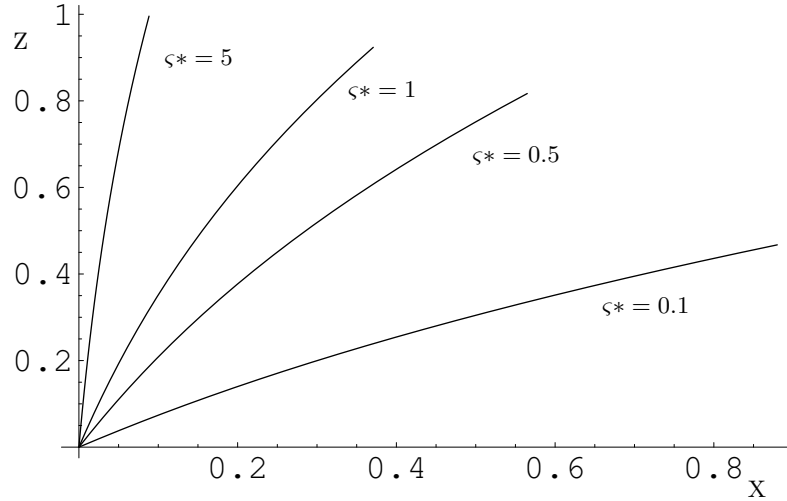


Figure 2.8: Fibre displacements with a Taylor drag that is induced by shear flow, where $\varsigma = h^2\varsigma^*$.

2.4.5.1 Annular Flow without Hooks

Now we look to solve the ordinary differential equations (2.35) and (2.36) numerically with equation (2.39) for the drag term. We could solve the boundary value problem

by using a shooting method. This method reposes the boundary value problem as an initial value problem, and guesses what the new initial value is, then iterates to find the initial condition that produces the original boundary condition. Guessing the initial conditions can often be difficult. It is usually easier, particularly for nonlinear equations, to solve initial value problems, and using a subtle rescaling we can repose the problem in order to eliminate the boundary condition at $s = 0$ without having to employ a shooting method.

We consider the height of the end point to be a dummy variable $Z(1) = h$. Now, the following rescaling will allow us to prescribe conditions at one end of the fibre only. We write

$$\Theta(\bar{s}) = \theta(z), \quad \bar{Z}(\bar{s}) = hz(s), \quad D(\bar{Z}) = h^2 D(\bar{z}) \quad T(\bar{s}) = h^3 t(s), \quad (2.45)$$

and the dimensionless parameters are re-scaled as follows

$$\varsigma = h^2 \varsigma^*, \quad \varepsilon = \delta h^2. \quad (2.46)$$

Using the autonomy of equations (2.35), (2.36) and (2.40) they can be re-written as

$$\frac{dt}{dz} = -(\varsigma^* + D(z) \cot \Theta), \quad (2.47)$$

$$\frac{d\Theta}{dz} = \frac{1}{t} \left(\frac{4D(z)}{\sqrt{Re}} - \varsigma^* \cot \Theta \right), \quad (2.48)$$

and the free-end conditions are now at the fixed prescribed position $z = 1$

$$t(1 - \delta) = \delta \frac{4 + \varsigma^{*2} \sqrt{Re}}{\sqrt{16 + \varsigma^{*2} Re}}, \quad \theta(1) = \text{arccot} \left(\frac{4}{\varsigma^* \sqrt{Re}} \right). \quad (2.49)$$

The solution is found parametrically by firstly solving the problem for a given ς^* , and then the “dummy variable” h can be found by

$$h = \frac{1}{\int_0^1 \text{cosec}[\theta(z')] dz'}, \quad (2.50)$$

which will then give ς . Now choosing the scale $X = hx$,

$$\frac{dx}{dz} = \cot \theta(z) \quad (2.51)$$

and so

$$X = h \int_0^z \cot[\theta(z')] dz', \quad (2.52)$$

and now we can go on to find the the position of the fibre. The main problem in using this rescaling is that we can solve the equations, using simpler methods but one cannot prescribe the value of the dimensionless variable ς as we do not know the value of h *a priori*. This can be overcome by using an iterative technique, such as a secant method, that will find the appropriate value of ς^* for a given ς . A fourth order Runge-Kutta method is used to solve the differential equation and a secant algorithm is employed to find the scaling h . Examples are given in figure 2.8.

2.4.5.2 Annular Flow with Hooks

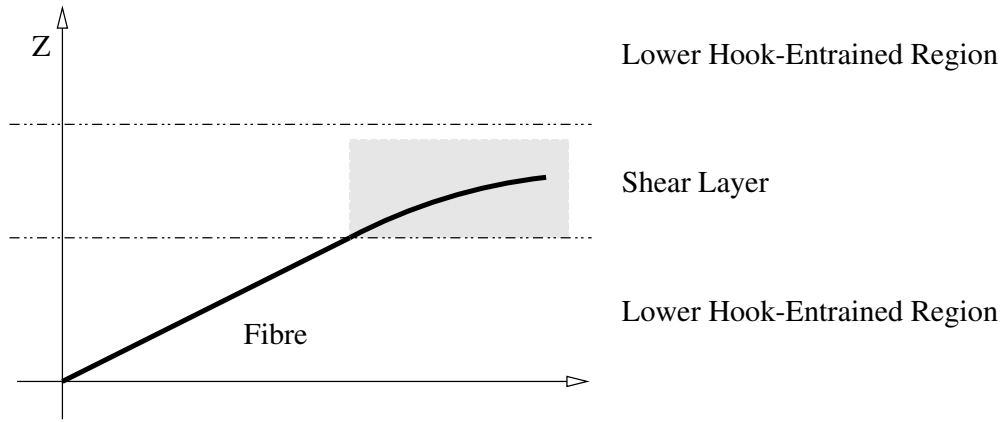


Figure 2.9: A diagram of a fibre in hook entrained flow. The shaded region is very similar to the problem where there is just shear flow.

For the aerodynamics that incorporate hooks, we have our “triple-layer” of zero, shear and plug flow, and so we can break down the calculations for fibre displacement into three regions, see figure 2.9. Our plight is made significantly easier by the fact that when $\varsigma \ll 1$ the fibre does not intrude significantly into the upper plug flow region as the aerodynamic body forces are so dominant. For the shear region, immediately above the lower hook-entrained region, this produces a problem that is almost exactly the same as the case for just the shear layer, although this is a free boundary problem in arc-length s , as we do not know where the interface between the shear and lower hook-entrained region is.

We can write the equations (2.35), (2.36), (2.37) and (2.40) explicitly for each region, removing over-bars, as

$$\left. \begin{aligned} T'_{low} + \varsigma \sin \Theta_{low} &= 0 \\ \Theta'_{low} T_{low} + \varsigma \cos \Theta_{low} &= 0 \end{aligned} \right\} \quad 0 < \bar{Z} < \frac{h_A}{l}, \quad (2.53)$$

$$\left. \begin{aligned} T' + D(Z) \cos \Theta + \varsigma \sin \Theta &= 0 \\ \Theta' T - 4D(Z) \frac{\sin \Theta}{\sqrt{Re}} + \varsigma \cos \Theta &= 0 \end{aligned} \right\} \quad \frac{h_A}{l} < \bar{Z} < \frac{B-A-h_B}{l}, \quad (2.54)$$

where T_{low} and Θ_{low} are tension and tangent-angle respectively for the part of the fibre in the lower hook entrained region $0 < \bar{Z} < \frac{h_A}{l}$. The interface conditions are simply continuity as there is no mechanism to support a jump in either Θ or T and so we add

$$\Theta_{low} = \Theta \quad \text{and} \quad T_{low} = T \quad \text{at} \quad Z = \frac{h_A}{l} \quad (2.55)$$

to the boundary conditions (2.42). We note that $Z = \frac{h_A}{l}$ corresponds to the fibre arclength point $s = S_A$, which is not known *a priori*.

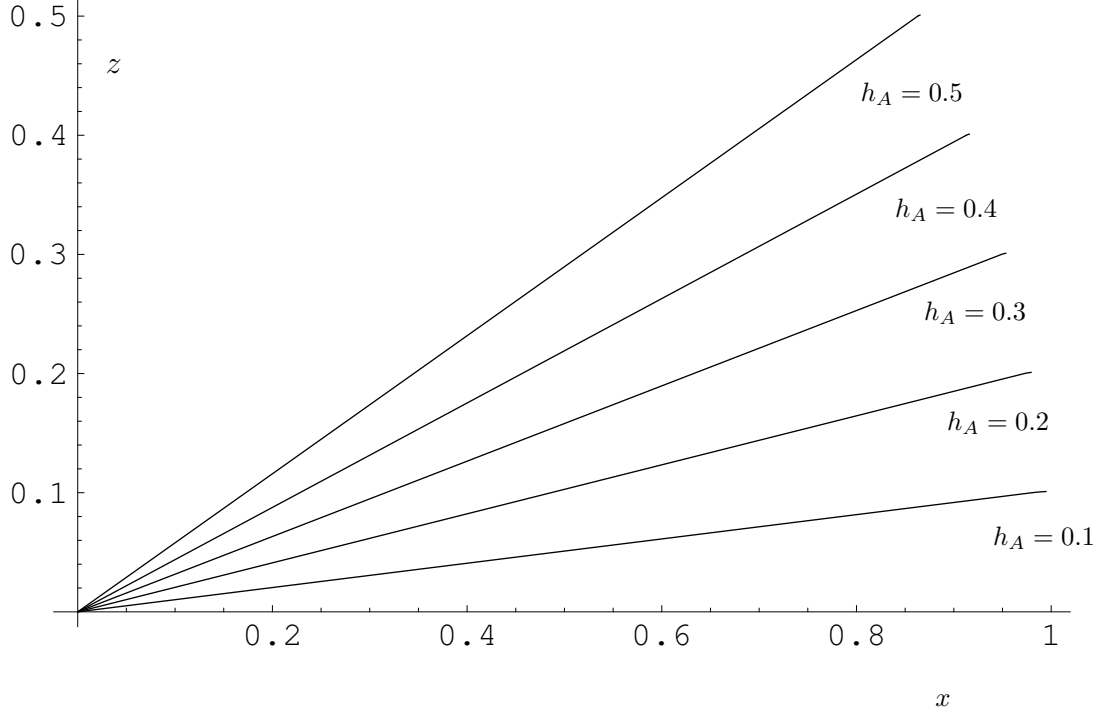


Figure 2.10: The displacement of a fibre with Taylor drag induced by the “triple-layer” airflow; $\varsigma^* = 0.0005$

As we have discussed already, the shear layer problem for $\frac{h_A}{l} < \bar{Z} < \frac{B-A-h_B}{l}$, is very similar to the problem that considers Couette flow with (2.39), and we can find a solution the the shear flow problem replacing ς with $\varsigma_{dum} = \frac{2\pi a(B-A)^2 \rho_{fibre}}{Al_{dum}^2 \rho_{air}}$, where l_{dum} is the length of the fibre in the shear layer. Therefore, one way to solve this problem is to use the numerical simulations obtained by solving equations (2.47) and (2.48). Then we use the fact that the dimensionless length of the fibre is one, and

so we try to find ς_{dum} such that the length of the fibre in the shear layer and the length of the fibre in the lower entrained region add up to one. Using the interface continuity condition for Θ we know the angle of the fibre in the lower region as any uniform θ is a solution for the fibre in the lower hook-entrained region. Therefore, we can find ς_{dum} such that

$$\frac{h_A}{l \sin \Theta(0)} + \sqrt{2\pi a(B-A)^2 \frac{\rho_{fibre}}{A\rho_{air}\varsigma_{dum}}} = 1, \quad (2.56)$$

for which we can employ the secant method. As we have used the same re-scalings of (2.45) and (2.46), when we solve the equations we find the solution for $\varsigma_{dum}^* = h^2 \varsigma_{dum}$. To find the corresponding ς_{dum} that satisfies (2.56) involves a further root finding iterative scheme. Examples of fibre profiles are given in figure 2.10 for varying h_A . We see that the fibre barely enters the shear layer. The fibre's deformation in the shear layer are the same as those in figure 2.8. We plot the relationship between the height of the fibres from the drum surface and the dimensionless parameter ς in figure 2.11.

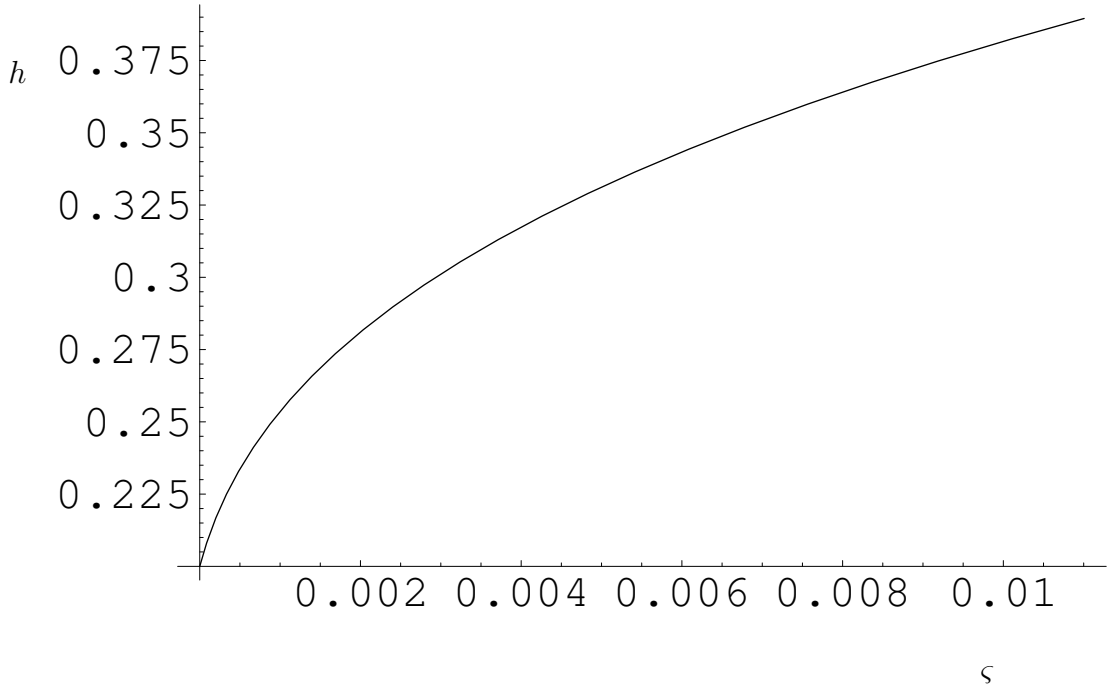


Figure 2.11: A plot of the height of the trailing end of the fibre h plotted against varying ς , for a fibre with Taylor drag that is induced by shear flow.

2.4.5.3 Friction between and fibre and a hook

A fault with the aforementioned model is the presumption that the fibre will not fly off the hook, and is therefore fixed at one end point. One way to overcome this is to consider the forces acting on the hook fibre contact point, and we apply static friction to estimate whether a fibre will move given the fibre's displacement. Using a classical static force balance, we can find the conditions for the fibre to slip and this is illustrated in figure 2.12. In terms of Cartesian components the tension force is

$$T(0) \begin{pmatrix} \cos[\theta(0)] \\ \sin[\theta(0)] \end{pmatrix}.$$

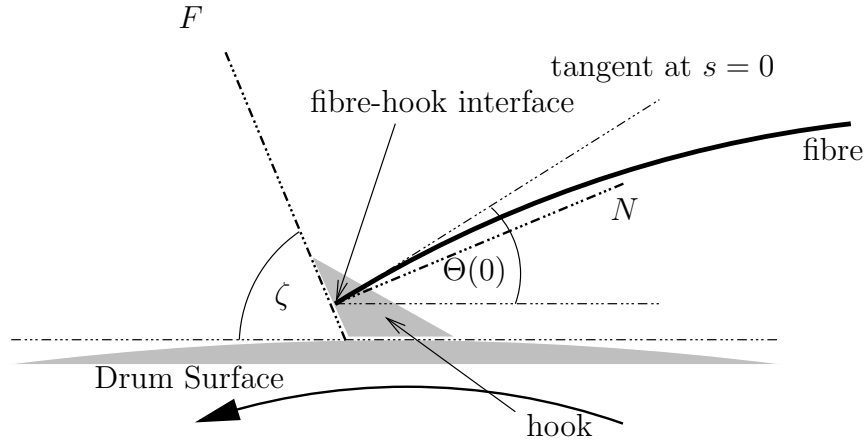


Figure 2.12: A diagram of the frictional forces between a hook and fibre on a rotating drum.

If the hook has an angle of attack ζ as in figure 2.12 then in equilibrium the friction force F and the normal force N are defined as follows,

$$F = T(0) \cos[\theta(0) + \zeta], \quad (2.57)$$

$$N = T(0) \sin[\theta(0) + \zeta]. \quad (2.58)$$

The fibre will slip if

$$\left| \frac{F}{N} \right| > \mu_s, \quad (2.59)$$

where μ_s is the friction coefficient between the hook and the fibre. Then given frictional coefficient μ_s , we can evaluate the range at which the fibre may be attached to the hook:

$$|\cot[\theta(0)]| > \mu_s. \quad (2.60)$$

This gives us good insight into how the hooks should be designed for the purposes of keeping the fibre on the hook in the annular regions.

2.4.6 The Doffer

The drum surface speed is an order of magnitude slower on the doffer when compared to the surface speeds of the cylinder and taker-in. We have previously suggested in section 2.3 that a quasi-steady inextensible string with Stokes drag (2.5) would be a suitable model and this approximation is sufficient for a fibre on the doffer. Unlike the cylinder, the doffer is not surrounded by a casing of hooks or any other mechanical objects so the flow can be represented by shear flow near the drum surface. It isn't clear whether or not the disentangled fibres actually interact with one another here, but it seems likely from the photographic evidence given by Dehghani et al. (2000).

The fibres once transferred onto the doffer move around the drum surface and are then stripped off and transported out of the machine. Due to the cylinder moving an order of magnitude faster than the doffer, the global airflow near the transfer region will be driven by the fluid entrained by the cylinder. The Reynolds number based on the doffer's radius will be $O(10^5)$ and the high speed photography in section 1.1 suggests that there will be an area of re-circulation just after the transfer region. For the flow around a fibre, apart from the aforementioned area near the cylinder, fluid flow near the doffer can be approximated by a cylinder rotating in an infinite fluid which has a Reynolds number, based on the fibre diameter, of $O(10^{-1})$. The velocity field is simply $\frac{\Omega_d A_d^2}{r}$, where $A_d \Omega_d$ is the surface speed of the doffer and close to the doffer this gives a shear flow approximation for flow relative to the cylinder.

For the fibre itself we derive governing equations in the same way as in section 2.4.3, but now we use the drag formula (2.5) since the Reynolds number for fluid local to a fibre on the doffer is smaller than unity and this has equations

$$\frac{dT}{ds} + \kappa Z \cos \Theta + \sin \Theta = 0, \quad (2.61)$$

$$T \frac{d\Theta}{ds} - 2\kappa Z \sin \Theta + \cos \Theta = 0, \quad (2.62)$$

$$\frac{dZ}{ds} - \sin \theta = 0, \quad (2.63)$$

where $\kappa = \frac{8A_d \rho_{air}}{a Re \rho_{fibre} \log \frac{1}{\epsilon}}$, which is similar to $\frac{1}{\varsigma}$ but the drag is now based on a Stokes flow approximation, see appendix A.3.1 for values. These can be solved using the same asymptotic and numerical approaches used in sections 2.4.4 and 2.4.5 respectively, see figure 2.13. This predicts the fibre to protrude from the drum surface by the amount

$\frac{1}{\kappa^2}$ and this is usually an order of magnitude greater than ς , which is the amount the fibre will protrude on the cylinder and taker-in.

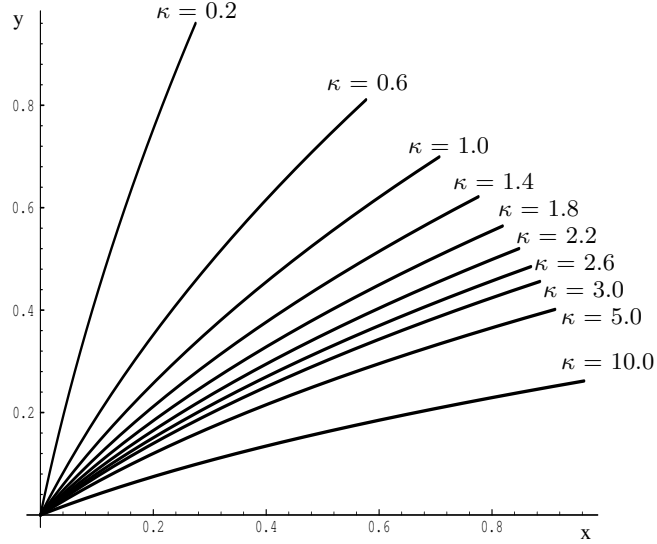


Figure 2.13: Displacements for a fibre with Stokes Drag that is induced by a shear flow, for varying κ

2.4.7 Summary

All these models, whether the fibre is approximated by shear or triple layered flow, give evidence that a single fibre will not interact with other machine surfaces, including hooks on the revolving-flats and any other neighbouring parts of the carding machine. Although the rotational forces acting on a fibre will project the fibre away from the surface from which it is tethered, it will not be significant enough so that the fibre will make contact with other machine parts. This result is still important to those in the carding industry as there is no consensus on how fibres behave in the annular regions. The linear stability analysis in appendix C, about a plug flow with velocity comparable to carding speeds, shows that a fibre will not fluctuate uncontrollably under the prescribed airflow and rotational forces found in the carding machine. Of course we have ignored important physics such as the effect of the neighbouring fibres on both aerodynamics and the fibre itself, and we consider fibre-fibre interactions in chapters 3 and 4. Therefore, to complete our understanding of the life of a single fibre in the carding machine we need to postulate a theory for fibre transfer between the three main drums; taker-in, doffer and cylinder. The dominant body force acting on

a fibre is due to aerodynamic drag and we will now consider fluid flow in non-annular geometries to recognise how fibres transfer between drums.

2.5 Transfer Mechanisms

We have already considered external forces for a fibre in laminar flow in annular geometries and found that they do not give a transfer mechanism. In section 2.3 we ascertained the importance of the aerodynamic forces, so we now turn our attention to the aerodynamics in non-annular geometries. Due to the fluid flow being characterised by high Reynolds numbers when considering scales of motion comparable with the machine, if the flow is considered to be laminar we can expect predominantly inviscid flow, $Re = \infty$. Near the machine surfaces we expect boundary layers of thickness $O\left(\frac{1}{\sqrt{Re}}\right)$. We consider the inviscid case which gives rise to a “strong” and “weak” transfer mechanism for a fibre travelling in between two rotating drums. The strong transfer mechanism occurs between the taker-in and cylinder, and the weak mechanism models the transfer from cylinder to doffer. This coincides with observation and we find that transfers are not surprisingly dependent on the holding properties of the hooks, machine geometries, and drum speeds.

2.5.1 Aerodynamics

We are interested in the region between two cylinders of radius r_C and r_D rotating in proximity to each other. For simplicity, we consider aerodynamic analysis and exclude the effects of the hooks and the fibres inside the machine. We consider two-dimensional incompressible inviscid flow. Then from the continuity equation (2.22), assuming the flow is irrotational, we introduce a stream function which satisfies Laplace’s equation:

$$\nabla^2 \psi = 0. \tag{2.64}$$

This can be solved to give the flow streamlines when applying the appropriate boundary conditions. We prescribe a no-penetration condition which translates to the stream-function being constant on the given boundary, and this is applied to the surface of the cylinders’ boundaries. Another interpretation, due to lack of a no-slip condition, is that any streamline can be a solid boundary and so this means that we wish to find two neighbouring circular streamlines.

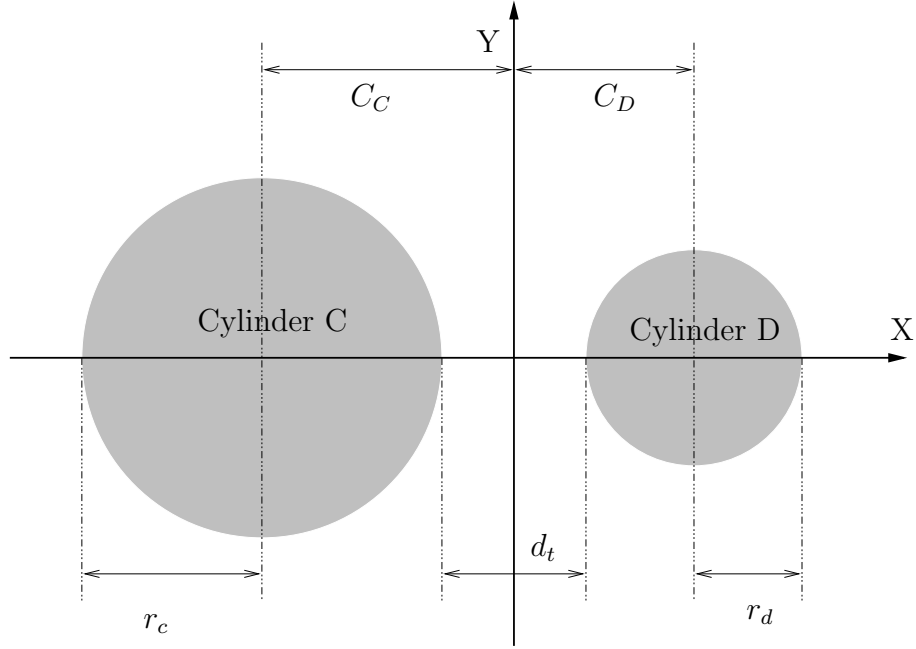


Figure 2.14: A diagram for fluid flow between two rotating cylinders.

Of course two rotating drums in truly inviscid fluid would not produce any motion, but because the viscous effects will cause the fluid to rotate we prescribe circulations around each drum C and D such that:

$$\begin{aligned}\Gamma_C &= \oint_C \mathbf{u} \cdot d\mathbf{r}, \\ \Gamma_D &= \oint_D \mathbf{u} \cdot d\mathbf{r},\end{aligned}\tag{2.65}$$

respectively. This is equivalent to placing a point vortex within each cylinder. The circulation should then provide a matching parameter for the full viscous problem. Note that a cylinder, of radius r , rotating in an infinite fluid at any Reynolds number flow will eventually create a circulation $\Gamma = 2\pi Ur$ around itself when rotating at angular velocity U and thus $\Gamma_C = 2\pi u_C r_C$ and $\Gamma_D = 2\pi u_D r_D$ are plausible values to take.

There are a number of possible scenarios that can occur as a consequence of variations in the circulations Γ_C and Γ_D . There are four possible flow topologies, two with stagnation points on drum surfaces and two without, an example of each is illustrated in figures 2.15. We are interested when flow stagnation points can be found on one of the two drums, as this invokes the transport of fluid from one drum surface to another, and consequently this may produce the desired transfer of fibres.

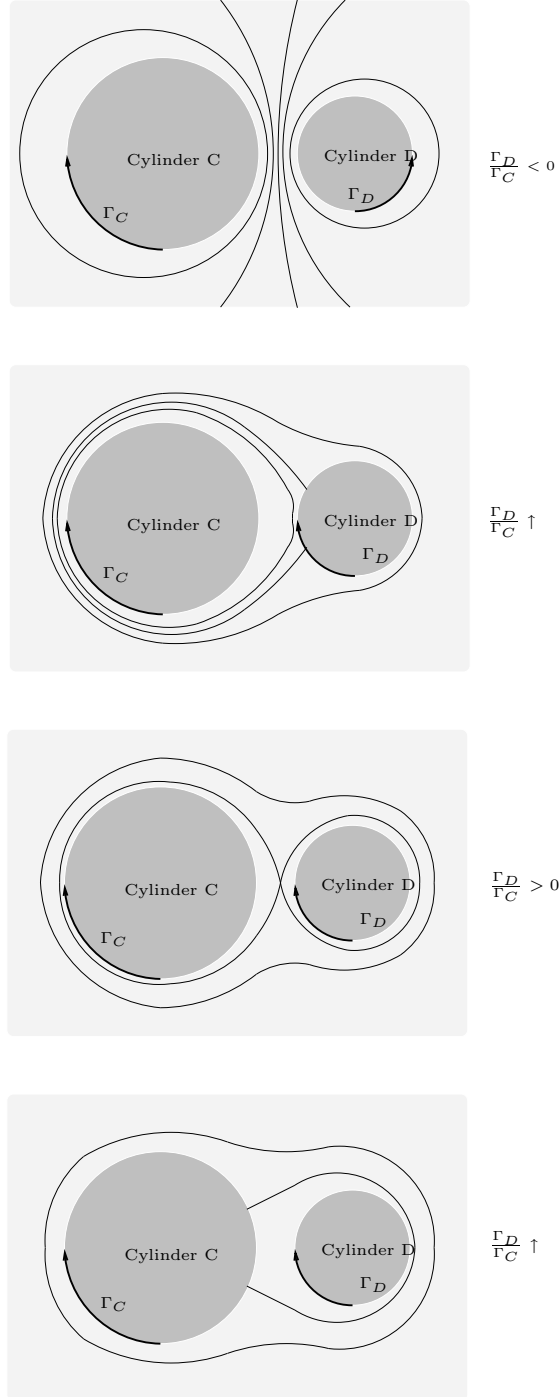


Figure 2.15: Possible fluid flow topologies between two rotating cylinders. The diagrams are in order of increasing $\frac{\Gamma_D}{\Gamma_C}$.

When we consider the circulations to be proportional to the speed of the surfaces of the carding drums we can expect the magnitude of Γ_C to be much greater than the magnitude of Γ_D , where C is the cylinder and D can be either the doffer or taker-in and this makes the likelihood of stagnation points to be found on D to be quite probable. Unfortunately it is not really so simple to determine circulations around each drum, the machine casing and the hook densities on each respective cylinder will affect the flow topology. Nonetheless, we stay with this simple model and we begin by outlining the conditions for the four types of fluid flow.

Using a conformal map, Jones (2000a) found that the four flow topologies could be characterised by the ratio of the circulations

$$\frac{\Gamma_D}{\Gamma_C}, \quad (2.66)$$

see figure 2.15. Starting with smallest ratio (2.66), a negative one, the circulations will be in opposite directions and the flow driven around by each drum will not interact with its respective neighbouring drum surface. Then as we increase the ratios so that the circulations are in the same direction, the flow will be driven by cylinder C . As fluid is circulated around C towards the nip region (with drum D) some of the fluid will go through the nip and some will go around drum D , and in this case there will be two stagnation points. As we increase the ratio $\frac{\Gamma_D}{\Gamma_C}$ the two stagnation points on drum D move close together until there is only one. Then as we further increase (2.66), drum D dominates the flow, and the stagnation point moves towards drum C and once on the surface of the drum will form two stagnation points.

The resulting velocities from this simple inviscid analysis can be used to determine the external body forces on a fibre, but we can consider much simpler geometries that still encapsulate the behaviour of fluid flow near the points of transfer. For the “strong” transfer we consider flow around the cylinder and taker-in where two stagnation points are on the taker-in. The “weak” transfer consists of flow around the cylinder and doffer where there are two stagnation points on the doffer. Both the transfer mechanisms correspond to the second picture in figure 2.15.

2.5.1.1 From Taker-In to Cylinder (Strong Transfer)

We consider the flow near a fibre on the taker-in, where fluid entrained by the cylinder is driven onto the adjacent drums. We assume that the two stagnation points are on the taker-in, shown in the second diagram in figure 2.5.1.1, and we can represent the

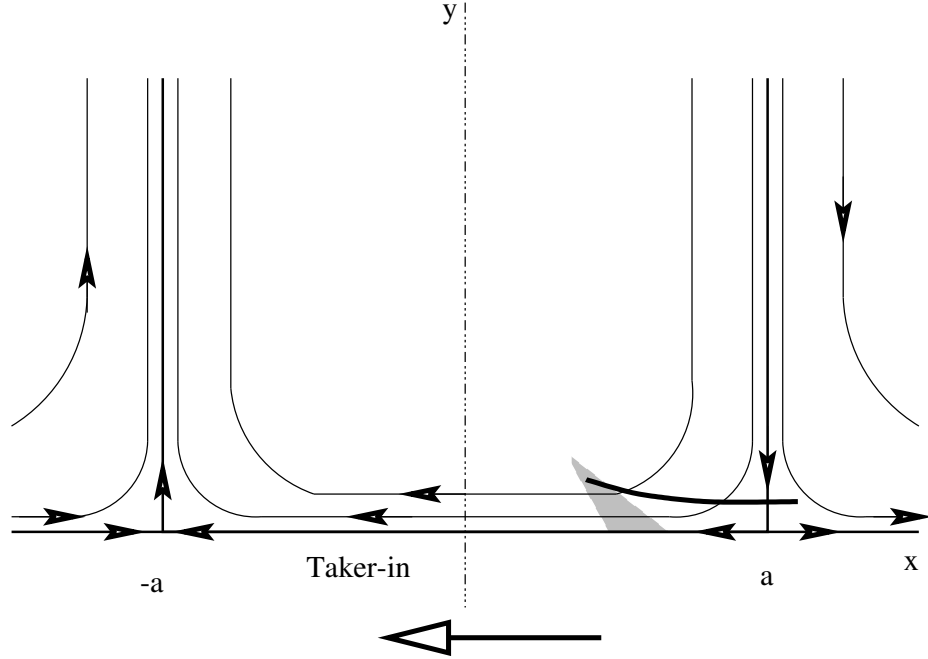


Figure 2.16: Fluid flow, local to the taker-in, near the point of transfer with the cylinder.

flow by considering a complex potential within an infinite half plane. If the stagnation points are at $x = a$ and $x = -a$ on an infinite plate, the complex potential is

$$w(z) = A_0 \left(\frac{z^3}{3} - za^2 \right), \quad (2.67)$$

where z is a complex variable. Thence the velocity is given by

$$U = A_0(x^2 + y^2 - a^2), \quad V = 2A_0xy. \quad (2.68)$$

We are not interested in the far field flow, as this is unlikely to be accurate but locally this description is adequate to illustrate our theory, and obviously a lot simpler than the stream function for flow around two adjacent rotating cylinders. This velocity profile is used to compute fibre trajectories in the next section 2.5.2.

2.5.1.2 From Cylinder to Doffer (Weak Transfer)

The transfer to the doffer is different as shown in our simple inviscid model in section 2.5.1. If we were to consider three rotating drums together as in figure 2.5, the cylinder drives flow around the carding machine and the entrained air hits both doffer and taker-in, and so we could envisage an almost symmetric fluid flow about the cylinder. The crucial difference for a fibre is that as the fibre moves around the taker-in, the

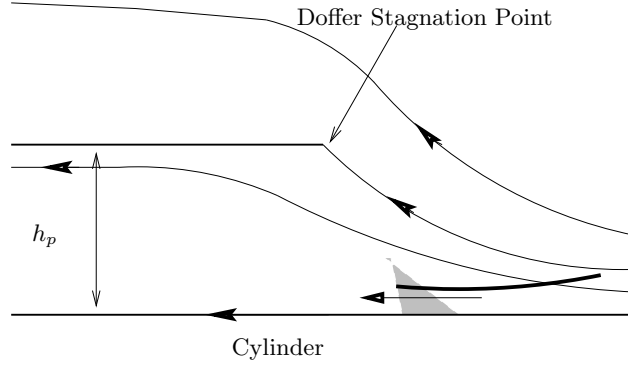


Figure 2.17: Fluid flow, local to the cylinder, near the point of transfer onto the doffer.

surface from which it is being tethered intersects a stagnation point. Once the fibre is on the cylinder the fibre is dragged around the cylinders axis and then towards the doffer but in this transfer region the stagnation points are on the neighbouring drum surface.

A fibre tethered by hooks on the cylinder moves towards the doffer on which two stagnation points reside. There are several inviscid flow-fields that we could use to approximate such a fluid flow, for example a block adjacent to a semi-infinite plane, but we choose a fixed plate which is a height h_p from the cylinder surface, see diagram 2.17. In the high Reynolds number laminar analysis, the leading edge region is predominantly governed by Euler's equations that matches onto thin boundary layers, and we will use this outer-inviscid solution for the drag acting on the fibre given by Jones (2000b).

2.5.2 Motion of a fibre at a transfer point

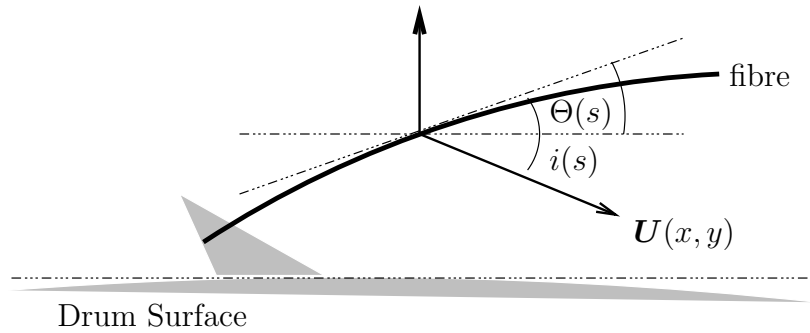


Figure 2.18: A diagram illustrating a two dimensional fluid velocity acting on a fibre.

From section 2.3, we can treat the equations for dynamic fibre motion as quasi-steady. Unlike section 2.4, here it would be inconsistent to incorporate the rotational forces as they are smaller than accelerations in an inertial frame. The equations we solve are still (2.16) and (2.17) but the only variant is that the velocity is no longer unidirectional and therefore a little care is needed to re-write the equations as we have two components in the velocity, see figure 2.18. From section 2.3, we will use the Taylor drag given for a hairy surface given by (2.4), as opposed to Stokes drag, as the fibre will be most affected by the fluid driven by the cylinder, since Re based on the cylinder speed is $O(10)$.

The fibre equations now incorporate a general fluid velocity $\mathbf{U}(x, y) = (U, V)$. Taylor drag approximations are based on uniform flow around cylindrical shaped bodies. The fibres can be arranged so that the flow hits the fibre at oblique angles, which due to the simplicity of the flow on previous occasion we related simply with the fibre angle $\Theta(s)$ but now as the angle varies we write the angle of attack as

$$i(s) = \Theta(s) - \arctan \frac{U(x(s), y(s))}{V(x(s), y(s))}, \quad (2.69)$$

see figure 2.18. The dimensionless governing equations now read:

$$\frac{dT}{ds} + (U(X, Z)^2 + V(X, Z)^2) \cos i = 0, \quad (2.70)$$

$$T \frac{d\Theta}{ds} - \frac{4}{\sqrt{Re}} \text{sgn}(i) (U(X, Z)^2 + V(X, Z)^2) \sin i = 0, \quad (2.71)$$

$$\frac{dX}{ds} - \cos \Theta = 0, \quad (2.72)$$

$$\frac{dZ}{ds} - \sin \Theta = 0. \quad (2.73)$$

It remains to apply boundary conditions and these are very similar in essence to (2.41), i.e. no tension at the free end, and spatial conditions for the end of the fibre that is being tethered,

$$T(1 - \varepsilon) \sim (U^2 + V^2)\varepsilon, \quad (2.74)$$

$$X(0) = X_{hook}, \quad (2.75)$$

$$Z(0) = Z_{hook}, \quad (2.76)$$

but the angle of the free end $\Theta(1)$ needs to be found by solving the equation

$$\sin i(1) = 0. \quad (2.77)$$

$$(2.78)$$

This leaves us with a fourth order system of coupled ordinary equations (2.70)-(2.73) with four boundary conditions.

2.5.2.1 Solutions

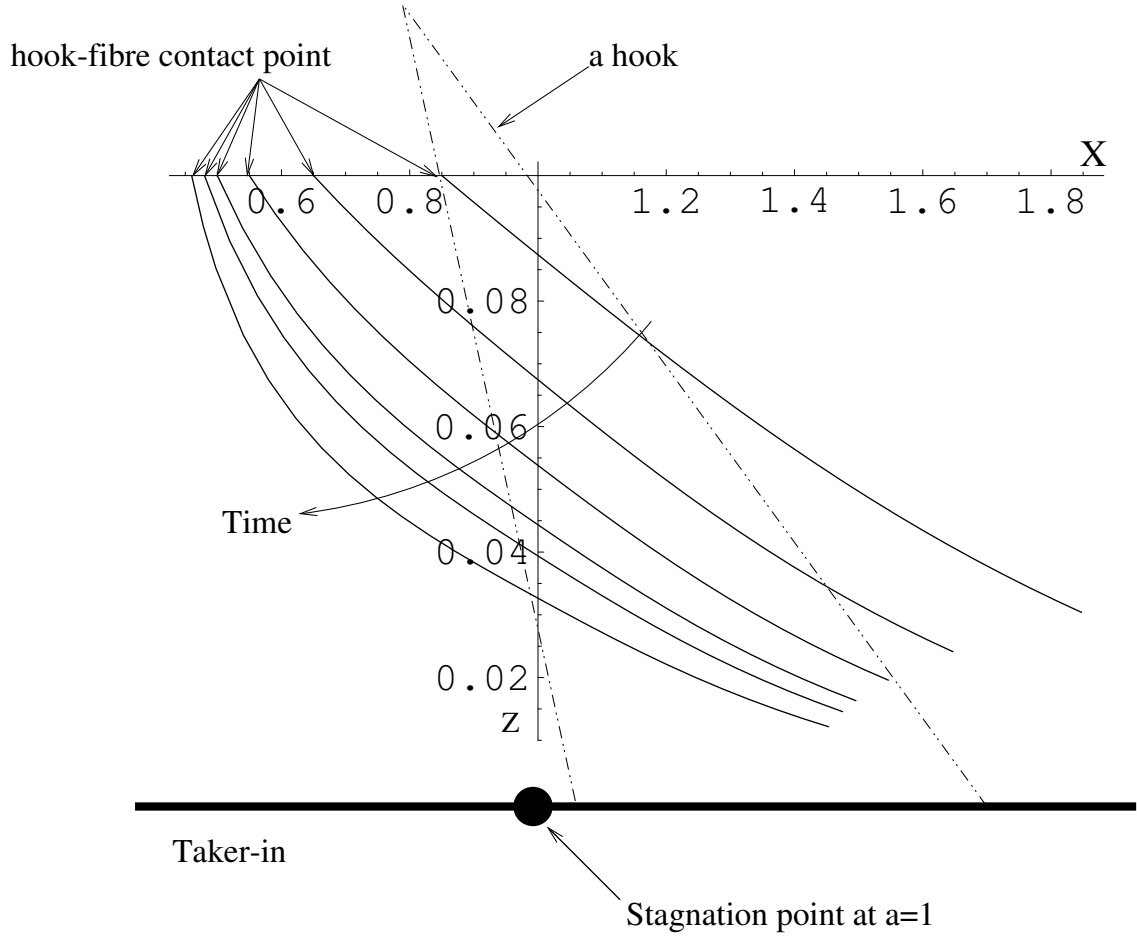
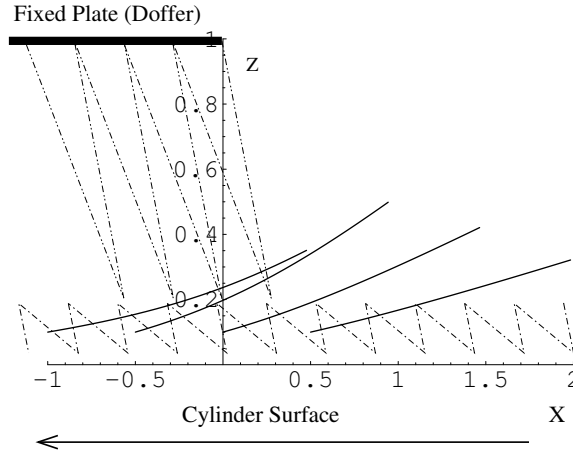


Figure 2.19: Fibre displacement on the taker-in moving past the first stagnation point in the transfer region, see figure 2.5.1.1. As time increases the angle between the hook and fibre contact point decreases, and the fibre will slip off the hook.

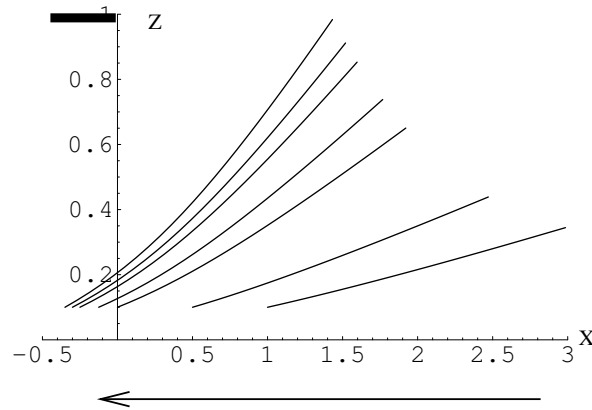
In formulating the numerical solution we cannot use the same rescaling as we did in section 2.4.5, as this relies not only on the autonomy of the differential equations but also on one spatial variable, in that case Z . Now that the velocity depends on X and Z a rescaling is not so simple nor is it as intuitive. Instead, we use a shooting method, guessing two conditions at one end of the fibre where the other two remaining conditions are given *a priori*. We choose the spatial conditions

$$X(1) = X_{end}, \quad Z(1) = Z_{end} \quad (2.79)$$

to be the shooting parameters. Then a Runge-Kutta discretisation is used to compute the solution to the given system of ordinary differential equations.



(a) fibre of dimensionless length 1



(b) fibre of dimensionless length 2

Figure 2.20: Fibre displacement near the cylinder-doffer transfer region.

For the taker-in to cylinder region, where airflow in section 2.5.1 predicts that there will be stagnation points on the taker-in, the fibre as it reaches the first stagnation point may slip down the fibre as air from the cylinder flows onto the taker-in, see figure 2.5.1.1. On either side of the first stagnation point, the fluid moves in opposite directions. When the fibre moves past the stagnation point drag on the fibre, either side of the stagnation streamline, will compete against each other and this will probably keep the fibre tethered to the hook. When approximately half the fibre is either side of the first stagnation point the airflow drags the textile off the hook, see figure 2.19. The fibre, now free, aligns itself to the flow Hinch (1976). Then the fluid transports the fibre towards the second stagnation point from which the fibre

is carried away from the surface of the taker-in onto the cylinder. It is difficult to envisage a scenario where a fibre can remain on the taker-in, even regardless of hook dimensions and holding properties.

When we consider a plate above an infinite half plane, this describes local behaviour in the cylinder to doffer transfer region. In figure 2.20, we see quasi-steady animations of a fibre that moves underneath the stagnation point. We note that the hooks on the doffer will typically be found in the region $0.2 < Z < 1$ on the plots in figure 2.20, and so in both cases shown there will be a good chance of interaction with the doffer-hooks. These hooks are the largest in every dimension of all the hooks found in the carding machine, and as the drum hooks are made of similar materials the frictional holding forces will be greater.

In the case where the fibre is double the length keeping all other parameters fixed, plotted in (b) of figure 2.20, the tail end rises to touch the surface of the fixed semi-infinite plate or the doffer, which may encourage slip at the cylinder-hook-fibre contact point, and this is due to aerodynamic forces alone. Alternatively the rising fibre of longer length may attach itself more convincingly to a hook on the upper surface. To complete the study of the transfer mechanics we need to understand the inter-play between the dimension of the hooks involved and the frictional resistance caused by hook-fibre contact points.

2.5.3 Frictional Contact Points

There are three possible ways in which a fibre may slip off the hook from which it is being tethered. The first we consider is when a fibre is held by two hooks and slips off the end of one. We can apply a fairly simple frictional analysis (Baturin, 1964) that indicates which hook will retain the fibre, see figure 2.21, picture (a). Assuming the fibre is approximately parallel to the drum surfaces, a condition for slipping is

$$\cos \Theta_1 + \mu_c \sin \Theta_1 > \cos \Theta_2 + \mu_d \sin \Theta_2, \quad (2.80)$$

where $\mu_{c,d}$ are the frictional coefficients between the fibre and hooks on the cylinder and doffer. The fibre will stay on the bottom cylinder and when (2.21) is not satisfied the fibre will transfer onto the doffer. There are two other possibilities for a fibre to transfer from cylinder to doffer. The first is when a fibre is removed from a hook purely by aerodynamic forces as described in section 2.4.5.3 and we can use the same inequality (2.59)

$$\left| \frac{F}{N} \right| > \mu_s.$$

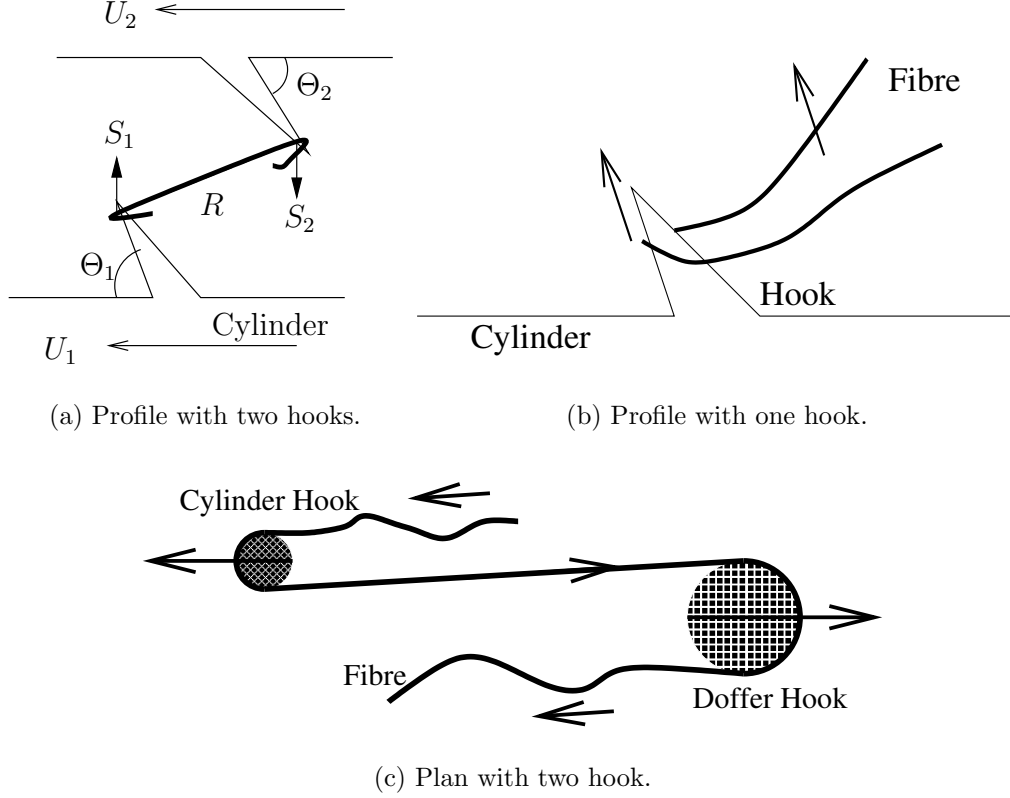


Figure 2.21: Friction forces acting on a fibre connected to a hook or a couple of hooks.

The second is when one part of the fibre is dragged around and off one hook as shown in picture (c) of figure 2.21. For this case, we can assume the fibre held between the two hooks will stay on the cylinder if the following inequality is satisfied,

$$\mu_c a_c^2 > \mu_d a_d^2, \quad (2.81)$$

where a_c and a_d are the radii of the cylinder and doffer hooks respectively. Inequality (2.81) is a rather simple approach that approximated the hook cross sections to be circular, but even if this is not sufficient for some, the reader can envisage, certainly for this case, that the relationship will depend on hook architecture and in particular the surface contact. We have three conditions from the inequalities (2.59), (2.80) and (2.81), and these are not necessarily mutually exclusive.

2.6 Conclusion

We began the work on a single fibre by considering what internal and external forces are important, as well as how machine geometries may affect the modelling process.

Subsequently we found that a fibre that is dragged through air travelling at speeds of 30 metres per second, in a rotating frame of reference, would approximately behave as a quasi-steady string; the fibres bending rigidity and the centrifugal forces were dwarfed by aerodynamic drag.

It was clear that when considering flow in thin annular geometries, a fibre would remain close to the body of the rotating drum. It was also shown that a fibre is linearly stable in a plug flow. Therefore, we predict that single fibres tethered by hooks on any of the carding drums do not interact with neighbouring machine surfaces, such as the flats. This however did not enlighten us on how a fibre moves between cylinders, as the reach of the hooks on adjacent drums or in other carding machinery such as the revolving flats are similar. Fluid flow was the dominant external body force and for the velocities in the channels we studied there was no normal component to the flow. This led us onto consider different flow geometries.

An inviscid analysis allowed us to consider flow regimes between adjacent drums. We concluded that stagnation flows may well give rise to a transfer mechanism. The flow in the carding machine will be predominantly driven by the air entrained by the cylinder. This suggested that there are stagnation points on both doffer and taker-in. We discovered that if a fibre is tethered to the drum on which the stagnation points are this will invoke a “strong” transfer mechanism, forcing all fibres to migrate onto the neighbouring drum. When the stagnation points are on the surface of the other neighbouring drum then this will invoke a “weak” transfer mechanism. In the latter case, “weak” transfer, the fibres will not necessarily transfer as it will depend on whether significant parts of the fibre will make contact with the hooks on the neighbouring surface. If the fibre connects well with a hook, we outlined three possible contests that neighbouring hooks would undertake in order to retain or obtain the respective fibre.

For the transfer mechanisms, the next stage would be to include viscous fluid mechanics as these may affect the possible flow topologies found near the respective transfer regions. Another additional level of sophistication, that can be applied to both models, would be to consider a semi-dilute suspension where fibres interact with each other and this is considered in the next two chapter. Nonetheless we have postulated a sensible theory to this age-old mystery of how a fibre behaves in the carding machine and hopefully dispelled the myth that fibres from the taker-in transfer onto the cylinder by their trailing ends.

Chapter 3

Tufts and Fibres

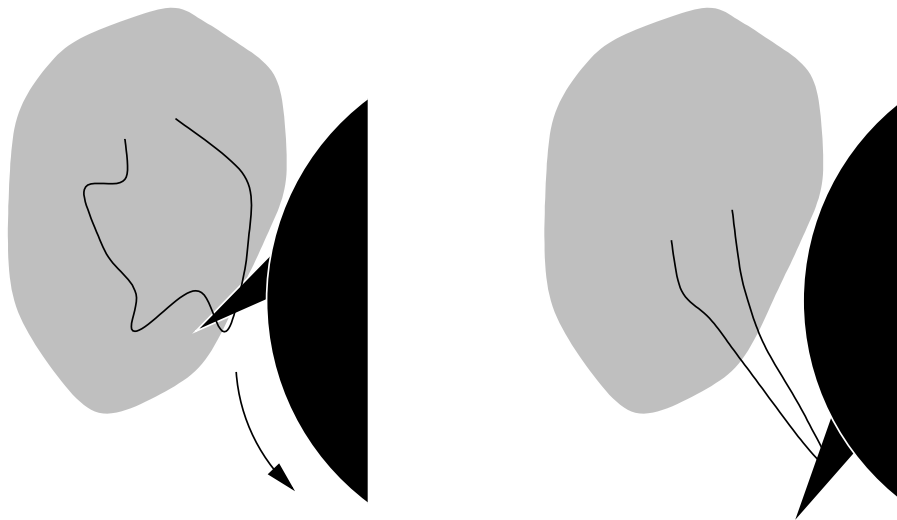


Figure 3.1: A diagram of a hook attaching itself to a fibre in a tuft.

3.1 Introduction

The fibre withdrawal problems presented in this chapter are motivated by the study of tufts within the carding machine. Photographic evidence, given in section 1.1 from work by Dehghani et al. (2000) supports the hypothesis that tufts can be found on the taker-in and cylinder. More specifically, it is found that about 50% of the fibres on the taker-in and on the cylinder before interaction with revolving and fixed flats are in tufts and the other 50% are laid down as individual fibres. During the carding process, tufts are broken down into individual fibres by the time they arrive at the cylinder-doffer transfer region. This is predominantly a consequence of the carding

action between the cylinder hooks and the revolving-flats, but part of the teasing process occurs when taker-in hooks attack the entangled fibres entering the machine via the feeder-in, see figure 3.1, and again at transfer from taker-in to cylinder. From the point of entry into the carding machine, fibres are being continually teased away from a cluster of neighbouring fibres or tuft, therefore in this chapter we examine the inter-play between fibre and tuft.

Within the carding machine, when a hook moves through a tuft there are a few possible outcomes. The hook may:

- move through the tuft without changing the orientations of the fibres a great deal.
- attach itself to a single fibre, either extracting the fibre completely or leaving part of the fibre extruding out of the original body of fibres, see figure 3.1.
- attach itself to many interconnected fibres, breaking down the original tuft into two tufts, see figure 3.2.

We model each case with the exception of the unaffected tuft. These scenarios are likely to occur when the taker-in hooks grab fibres from the lap in the transfer region between feeder-in and taker-in, taker-in and cylinder, and to a lesser extent between cylinder hooks and revolving-flats' hooks, see figures 3.1 and 3.2.

Firstly, we describe the model for a single fibre that is withdrawn from a tuft. Although this is not directly applicable to the carding machine it does correspond to an experiment completed by the School of Textile Industries which allows us to validate the modelling assumptions and estimate some of the parameters. A natural extension to the extraction of a single fibre from a tuft is to consider a hook that attaches itself to a fibre where both ends of the fibre are in the material, see figure 3.1. We go onto consider how tufts are broken down and we do this by modelling a single tuft, that is being extended, and evolves into two discrete entities connected by n individual fibres, see figure 3.2.

Therefore we address the key fibre-tuft problems that are relevant to the carding machine. Once we find the tension, because our fibres are treated as unbreakable and therefore no breaking stress is prescribed *a priori*, we can approximate the point in the simulation where breakage occurs. The control of breakage is paramount for the production of good quality yarns. The idea behind all our applications in this chapter is based on the extraction of an inextensible single thread from a material that is on average a uniform continuous media.

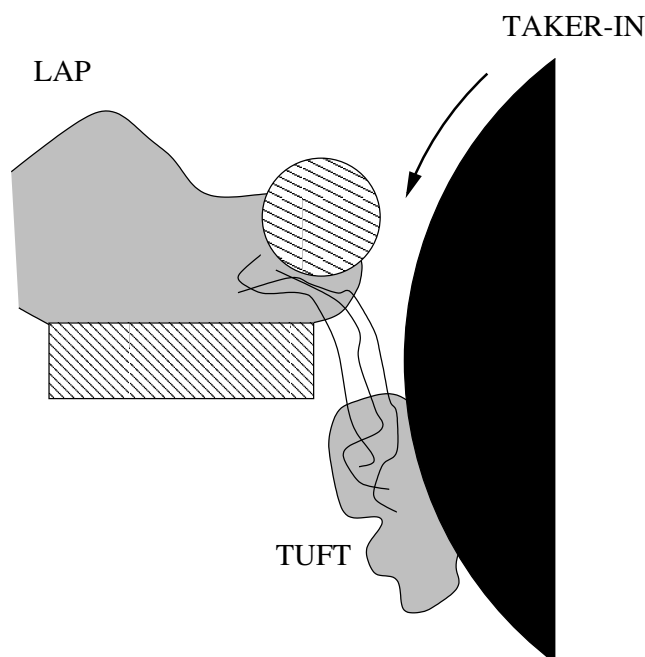


Figure 3.2: A diagram of taker-in hooks grabbing a tuft from the lap with interconnecting fibres.

3.2 The Fibres

The fibres found in the textiles industry have fairly low breaking extensions, for example 7% for polynosic viscose and high tenacity polyester, and 5%–10% for cotton. For the vast majority of textile fibres, yield strains occur when fibres are extended approximately 1%; that is, a fibre will deform elastically up to this point. Due to the low yield strain and breaking extensions for the materials concerned we will focus on the stresses acting on a naturally curved or crimped, inextensible fibre with bending rigidity.

Man-made fibres such as polyester are uniformly crimped, for example a fibre that is 3 centimeters long may have around 10 uniformly distributed turning points. Many natural fibres such as cotton have non-uniform curvature, and there may be a few scales on which turning points occur. A simple way to view the extraction of a fibre from a tuft is to consider a straight line between the two ends, see figure 3.3, and model the fibre as a spring lying along this line. Note that the fibre-centre line, due to crimp or natural curvature will be two to three times shorter than the length of the fibre itself and this means that the straight line representing the fibre will be extensible although the fibre itself is inextensible, see figure 3.5. Work by Cooke (2000) examined models for a crimped fibre under tension and showed that

the tension-extension relations were linear for small tension and tended to a fixed maximum extension as the tension increased. We examine this in more detail when we derive the equations in section 3.3.

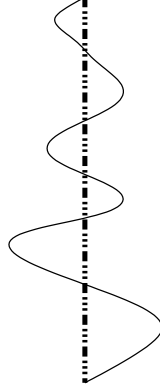


Figure 3.3: A diagram of a naturally curved fibre and its centre line.

Tufts are composed of many fibres that are entangled, forming a cohesive structure due to their intrinsic topological arrangement. When a body force is applied to the tuft, resistance is due to friction generated by the fibre-fibre contact points. If we apply a force to the end of a fibre, extracting it from the tuft, the motion of the fibre will be constrained by the geometry of its neighbours. This is similar to the concept of reptation in the theory of polymers for dense arrays of long-chain molecules. Although the topologies and densities may change throughout the tuft, we treat it as a homogeneous material, which is an adequate assumption for dense materials. Consequently we focus on the stress in the fibre that is being extracted and assume there are no deformations in the tuft from the deformations in the fibre. The typical dimensions of the tuft that we consider should be greater than the length of a fibre's centre-line. Finally we assume that when a fibre is extracted from the tuft, this does not significantly affect the physical properties of the tuft itself.

3.3 The Withdrawal of a Single Fibre

Figure 3.4 illustrates the geometry of our model. The point at $y = 0$ is the interface between the tuft boundary and the air. The vertical line at $x = 0$ represents the centre-line of the fibre and $y \in (0, D(t))$ is the part of the fibre extending out of the tuft and $y \in (-h(t), 0)$ is part of the fibre embedded in the tuft. Movement in the boundary of the tuft is assumed to be negligible as we do not expect the movement of

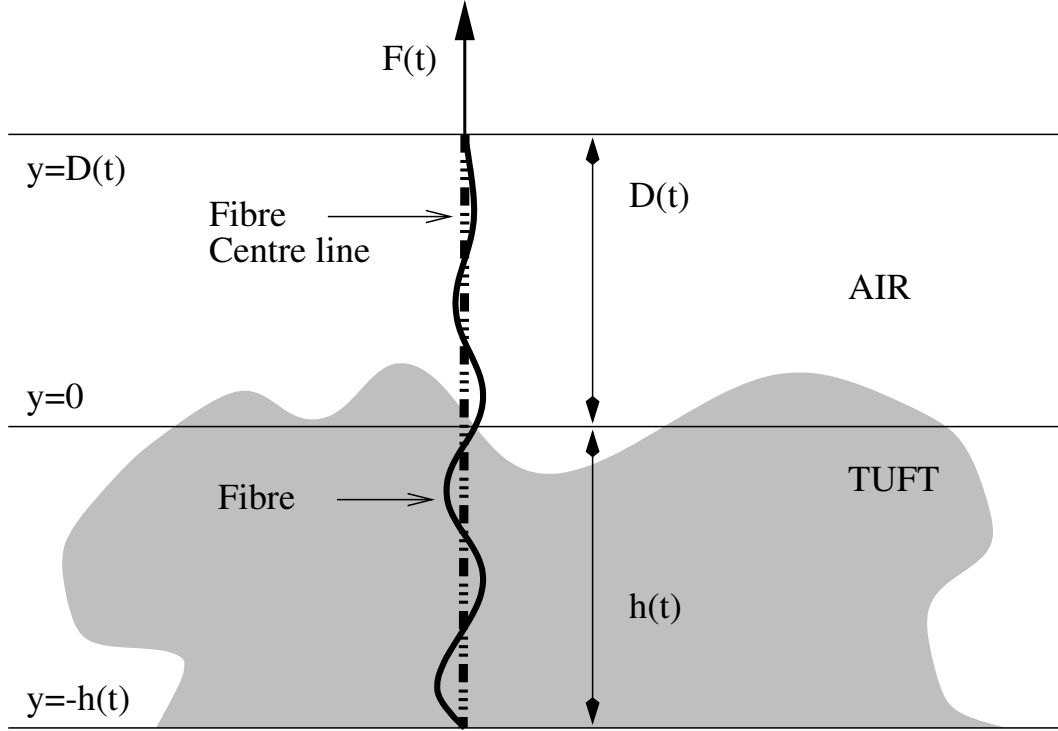


Figure 3.4: A diagram of a single fibre being withdrawn from a tuft.

a single fibre to significantly affect the orientation of the tuft. For the part of the fibre in the tuft there is a force acting on the surface of the fibre that is due to dynamic friction. Outside the tuft we ignore the effect of drag caused by the air. Gravity is ignored throughout the problem.

We will apply a withdrawal condition at the end point and this consists of either prescribing the velocity, $\dot{D}(t)$, or the force, $F(t)$. In our case it is useful to prescribe velocity as we wish to compare mathematical simulation with experiments. Furthermore in all carding machine scenarios, a hook will move approximately at a constant velocity through a tuft. Using an Eulerian description, denoting T as tension and y as the fibre length variable, we find the global force balance:

$$T(y, t) = F(t) \quad \text{for} \quad 0 < y \leq D(t), \quad (3.1)$$

$$\frac{\partial}{\partial y} T(y, t) = f \quad \text{for} \quad -h(t) < y < 0, \quad (3.2)$$

where f is the resistance per unit length to motion caused by the inter-fibre contact points. As there are no forces acting on the fibre when $y \in (0, D(t))$ we consequently have the simple form for tension (3.1). It remains to impose the boundary conditions and also the constitutive equations for the fibre. At the end of the fibre in the tuft

there will be no tension, at the tuft-air interface the tension is continuous, and initially before the fibre is withdrawn from the tuft there will be no tension in the fibre, and so these conditions correspond to

$$T(y, t) = 0 \quad \text{at} \quad y = -h(t), \quad (3.3)$$

$$T(y, t) = F(t) \quad \text{at} \quad y = 0, \quad (3.4)$$

$$T(y, t) = 0 \quad \text{when} \quad t = 0, \quad (3.5)$$

respectively. At the moment we choose to prescribe $D(t)$, the rate of withdrawal, but this means that $F(t)$ the tension in the string in the air, and $-h(t)$ the depth of the fibre inside tuft, are unknown.

To complete the system we need to relate strain with stress in the fibre but before we go on to explain the possible constitutive relationships we change the coordinate system to a more intuitive framework, namely Lagrangian variables ξ and t . In a Lagrangian description we follow elements of the fibre centre-line and this is a natural system to work in, particularly when considering elastic deformations. We define $\xi = y + h(0)$ at $t = 0$, where for a fibre of length l , $\xi = l$ is the end from which the fibre is withdrawn from the material, and $\xi = 0$ is the trailing end inside the tuft. Our spatial coordinate system is now bound by the fibre end points 0 and l . The tuft-air interface moves relative to the fibre elements and we define this point as $\xi = \xi_0(t)$. The Eulerian coordinates transform to the Lagrangian framework as follows:

$$y = D(t) \quad \mapsto \quad \xi = l, \quad (3.6)$$

$$y = 0 \quad \mapsto \quad \xi = \xi_0(t), \quad (3.7)$$

$$y = -h(t) \quad \mapsto \quad \xi = 0. \quad (3.8)$$

The position y can now be written as a function of time and element, i.e. $Y(t, \xi)$, and then the velocity is $\frac{\partial Y}{\partial t}$ and $Y(0, \xi) = \xi - h(0)$. To complete the Lagrangian transformation we must map the field equations from (3.1) and (3.2) to

$$T(\xi, t) = F(t) \quad \text{for} \quad \xi_0(t) < \xi \leq l, \quad (3.9)$$

$$\frac{\partial \xi}{\partial Y} \frac{\partial}{\partial \xi} T(\xi, t) = f \quad \text{for} \quad 0 < y < \xi_0(t), \quad (3.10)$$

where $\frac{\partial \xi}{\partial Y} = \frac{1}{\frac{\partial Y}{\partial \xi}}$. The boundary and initial conditions, (3.3)–(3.5) become

$$T(\xi, t) = 0 \quad \text{at} \quad \xi = 0, \quad (3.11)$$

$$T(\xi, t) = F(t) \quad \text{at} \quad \xi = \xi_0(t), \quad (3.12)$$

$$T(\xi, t) = 0 \quad \text{when} \quad t = 0, \quad (3.13)$$

respectively.

3.3.1 Constitutive Law

With the introduction of the Lagrangian variable, we are in a good position to define a plausible constitutive relationship:

$$\frac{\partial}{\partial \xi} Y(\xi, t) = \Phi \left(\frac{T(\xi, t)}{k} \right) \quad (3.14)$$

where k is the elastic modulus or spring constant. For Φ , we expect linear elasticity to hold where one applies Hooke's Law for small stresses. When the string is extended fully at tension greater than T_c then Φ is constant, which means the fibre has been extended fully. Thus a simple choice for Φ is

$$\Phi(T) = \begin{cases} 1 + T & 0 < T < T_c \\ 1 + T_c & T_c < T \end{cases} . \quad (3.15)$$

Alternatively for a smoother transition near the point where the fibre is fully extended we could use $\Phi \left(\frac{T}{k} \right) = 1 + (T_c - 1) \tanh \frac{T}{k}$, see figure 3.5. We note that regardless of

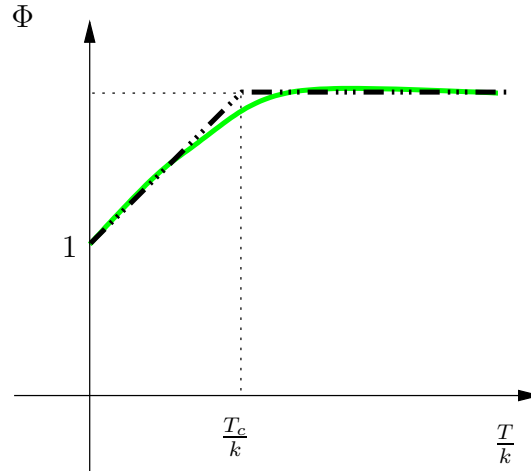


Figure 3.5: Possible constitutive relations for tension and strain for a spring or a crimped fibre.

the choice of Φ , the condition $\Phi(0) = 1$ must be satisfied, which simply ensures that when there is no tension in the string there is no extension and the fibre remains in its natural state.

3.3.2 Friction

The discrete contact points in a dense tuft are approximated by a continuous surface force on the fibre. The classical laws attributed to Coulomb and Amonton state that static friction between two objects is independent of the surface area of contact and is proportional to the normal force between them. Coulomb pointed out that this is not necessarily true when considering kinetic friction, the force resisting motion when two bodies in contact slide over one another. Much of the study of textile fibre friction focuses on departures from the classical laws, the reasons for such departures and their consequences (Morton and Hearle, 1975). An example of such study is the directional dependence of the frictional behaviour of wool, which is due to microscopic structure of the fibre. Of the two types of friction, kinetic friction is certainly the least understood, but a general rule (Morton and Hearle, 1975) for textile fibres is that as speed increases so does frictional resistance. There have been some experiments that measure the relationship between velocity and friction, and these tend to be nonlinear and dependent on the material. For simplicity, we shall assume that the many contact points produce a linear relationship between friction and velocity, $V = \frac{\partial}{\partial t} Y(\xi, t)$ for a stationary tuft. So we write

$$f = \lambda V \quad (3.16)$$

where f is the force per unit length exerted on the fibre's centre-line, V is the velocity of the fibre relative to the tuft and λ is a coefficient of friction which is assumed constant. However λ will depend on the properties of the tuft, such as density and fibre material.

3.3.3 The Equations

All the principal components of our model, from the geometry (3.6) and (3.8), the field equations (3.9), (3.10), the constitutive laws (3.14), and boundary conditions (3.11) – (3.13) have been specified. We are left with the task of finding three unknowns $T(\xi, t)$, $F(t)$ and $\xi_0(t)$. The area of interest is in the partial differential equation (3.10) for the region $0 < \xi < \xi_0(t)$, because once we solve this equation we know the tension throughout the string from (3.12).

Firstly we can re-write our field equation (3.10) for the region inside the tuft, $0 < \xi < \xi_0(t)$, by using the constitutive law (3.14),

$$\frac{\partial T}{\partial \xi} = f \Phi\left(\frac{T}{k}\right). \quad (3.17)$$

Differentiating the equation for linear dynamic friction (3.16) with respect to ξ and again using the constitutive law (3.14) we get:

$$\frac{\partial f}{\partial \xi} = \frac{\lambda}{k} \frac{\partial T}{\partial t} \Phi' \left(\frac{T}{k} \right), \quad (3.18)$$

where dash denotes full derivative with respect to the argument. Now eliminating $\frac{\partial f}{\partial \xi}$ from equations (3.18) and by differentiating (3.17) with respect to ξ we obtain an equation for T :

$$\frac{\partial}{\partial \xi} \left[\frac{1}{\Phi \left(\frac{T}{k} \right)} \frac{\partial T}{\partial \xi} \right] = \frac{\lambda}{k} \Phi' \left(\frac{T}{k} \right) \frac{\partial T}{\partial t}. \quad (3.19)$$

Equation (3.19) is a nonlinear parabolic partial differential equation for T and boundary conditions (3.11) – (3.13) would be sufficient for the partial differential equation (3.19) to be well posed if $F(t)$ and $\xi_0(t)$ were known.

Now we are left with solving equations that are similar to a nonlinear Stefan problem and for completeness we need to define extra conditions at the free boundary. For the region outside the tuft, $l > \xi > \xi_0(t)$, we use the condition that the tension in the fibre is uniform for any given time. Integrating the constitutive law (3.14), we find

$$Y(\xi, t) = \Phi \left(\frac{F(t)}{k} \right) (\xi - \xi_0(t)), \quad (3.20)$$

and imposing the geometrical condition (3.6) we get the equation

$$\xi_0(t) = l - \frac{D(t)}{\Phi \left(\frac{F(t)}{k} \right)}, \quad (3.21)$$

which means that once $F(t)$ is found, we will know the location of the free boundary. By differentiating $Y(\xi_0(t), t) = 0$ with respect to time we find that

$$\frac{\partial Y}{\partial \xi} \frac{d\xi_0(t)}{dt} + \frac{\partial Y}{\partial t} = 0 \quad \text{at} \quad \xi = \xi_0(t). \quad (3.22)$$

Using the frictional law (3.16) with (3.17), the constitutive law (3.14) becomes:

$$f = \frac{\partial T}{\partial \xi} \frac{1}{\Phi \left(\frac{T}{k} \right)} = -\lambda \Phi \left(\frac{T}{k} \right) \dot{\xi}_0, \quad (3.23)$$

and differentiating (3.21) with respect to time and then substituting into (3.23) we obtain:

$$\frac{\partial T}{\partial \xi} = \lambda \left[\Phi \left(\frac{T}{k} \right) \dot{D} - \frac{D(t) \Phi' \left(\frac{T}{k} \right) \dot{F}}{k} \right] \quad \text{at} \quad \xi = \xi_0, \quad (3.24)$$

which is the final condition and will consequently give us F .

So we have a nonlinear diffusion problem (3.19) with boundary and initial conditions (3.11), (3.12), and (3.13), a condition for the location of the free boundary (3.21) and an additional condition that ultimately gives the tension required to withdraw a fibre at constant velocity (3.24). Now we will look for solutions but we begin the analysis by finding a dimensionless form for the aforementioned equations.

3.3.3.1 Dimensionless Equations

By prescribing a constant velocity of the free end in the air, giving $D(t) = Ut$, we scale velocity with U , ξ with the length of fibre's centre-line l , time scales with the amount of time needed to travel length l at speed U , and we scale tension with the elastic constant k , therefore,

$$\xi = l\xi', \quad D = lD', \quad T = kT' \quad F = kF', \quad t = \frac{l}{U}t'. \quad (3.25)$$

Now dropping the primes, the dimensionless field equation becomes:

$$\beta \Phi'(T) \frac{\partial T}{\partial t} = \frac{\partial}{\partial \xi} \left[\frac{1}{\Phi(T)} \frac{\partial T}{\partial \xi} \right]. \quad (3.26)$$

The boundary and initial conditions become:

$$T(\xi, t) = 0 \quad \text{at} \quad \xi = 0 \quad (3.27)$$

$$T(\xi, t) = F(t) \quad \text{at} \quad \xi = \xi_0 \quad (3.28)$$

$$T(\xi, t) = 0 \quad \text{at} \quad t = 0 \quad (3.29)$$

$$\frac{\partial T}{\partial \xi} = \beta [\Phi(T) D'(t) - D(t) \Phi'(T) F'(t)] \quad \text{at} \quad \xi = \xi_0 \quad (3.30)$$

with

$$\xi_0(t) = 1 - \frac{D(t)}{\Phi(F(t))}. \quad (3.31)$$

The dimensionless number β is like a Stefan number:

$$\beta = \frac{\lambda U l}{k} \quad (3.32)$$

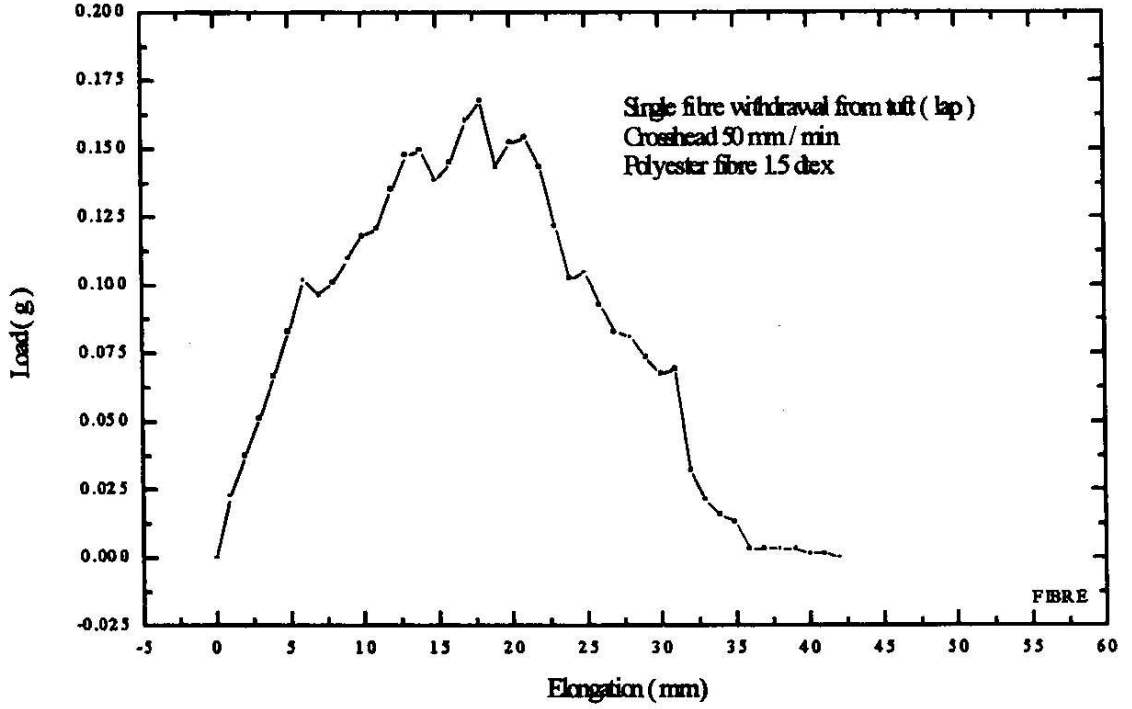


Figure 3.6: Experimental results of a single fibre being withdrawn from a tuft

where U is the withdrawal speed, l is the fibre length, λ is the dynamic friction coefficient and k is the linear elastic constant.

An experiment was conducted by the School of Textile Industries at the University of Leeds, where the force exerted on a fibre, as it was being withdrawn at a uniform velocity, was measured. The average measurements are plotted in figure 3.6, where the maximum force measured is approximately 1.7×10^{-3} Newtons and we expect the force $F \sim k$, the elastic coefficient. The average length of a fibre is 10^{-2} m and the withdrawal speed is 10^{-3} m s^{-1} , therefore $\beta \sim [\lambda] \times 10^{-2}$, where $[\lambda]$ is the value of λ in msKg^{-1} . Due to the qualitative shape of the solutions we go onto compute in section 3.3.5, we expect $\beta \sim 10^{-1}$.

3.3.4 Asymptotic Solutions

We consider the solutions for small β and these could be considered as experiments with “slow” withdrawal speeds. The asymptotic analysis provides a useful test for the numerical computations. We will also need a small time solution to begin numerical simulations as there is a sudden jump in tension as we begin to withdraw the fibre, near the tuft-air interface. We begin with the small β asymptotics.

3.3.4.1 Small β Asymptotics

We consider the case where $0 < \beta \ll 1$ for a fibre that is withdrawn at a constant speed so that $D = t$. This can apply to the case of short fibres, small resistive friction caused by neighbouring fibres, slow withdrawal speeds or large spring constant k . From the dimensionless problem (3.26) – (3.31), tension is re-scaled with the small parameter β , and then we obtain the problem for $T = \beta\tilde{T}$ and $F = \beta\tilde{F}$:

$$\beta\Phi'(\beta\tilde{T})\frac{\partial\tilde{T}}{\partial t} = \frac{\partial}{\partial\xi} \left[\frac{1}{\Phi(\beta\tilde{T})} \frac{\partial\tilde{T}}{\partial\xi} \right], \quad (3.33)$$

$$\tilde{T} = 0 \quad \text{at} \quad \xi = 0, \quad (3.34)$$

$$\tilde{T} = \tilde{F}(t) \quad \text{at} \quad \xi = \xi_0, \quad (3.35)$$

$$\frac{\partial\tilde{T}}{\partial\xi} = \Phi(\beta\tilde{T}) - t\Phi'(\beta\tilde{T})\beta\tilde{F} \quad \text{at} \quad \xi = \xi_0, \quad (3.36)$$

$$\xi_0 = 1 - \frac{t}{\phi(\beta\tilde{F})}. \quad (3.37)$$

Now letting $\beta \rightarrow 0$, and using $\phi(0) = 1$ and by choosing k appropriately so that $\phi'(0) = 1$, the leading-order problem is denoted with subscript zero,

$$\frac{\partial^2\tilde{T}_0}{\partial\xi^2} = 0 \quad \text{when} \quad 0 < \xi < \xi_{0,0}(t), \quad (3.38)$$

$$\tilde{T}_0 = 0 \quad \text{at} \quad \xi = 0, \quad (3.39)$$

$$\left. \begin{array}{l} \frac{\partial\tilde{T}_0}{\partial\xi} = 1 \\ \tilde{T}_0 = \tilde{F}_0 \end{array} \right\} \quad \text{at} \quad \xi = \xi_{0,0} \quad (3.40)$$

and

$$\xi_{0,0} = 1 - t. \quad (3.41)$$

The solution is simply

$$\tilde{T}_0 = \xi, \quad (3.42)$$

$$\tilde{F}_0 = 1 - t \quad (3.43)$$

i.e. a linear decrease in the force as the fibre is pulled out. The equations (3.42) and (3.43) do not satisfy the initial condition $\tilde{T} = \tilde{F} = 0$ at $t = 0$. We use a singular perturbation analysis, in particular a boundary layer near the initial time $t = 0$ and so (3.38)–(3.41) becomes the “outer” problem to leading order. For the “inner” problem

we re-scale time with β , $t = \beta\tau$, and the leading order problem for this case is

$$\frac{\partial \tilde{T}_{in,0}}{\partial \tau} = \frac{\partial^2 \tilde{T}_{in,0}}{\partial \xi^2} \quad \text{when } 0 < \xi < \xi_{0,0}(t) \quad (3.44)$$

$$\tilde{T}_{in,0} = 0 \quad \text{at } \xi = 0 \quad (3.45)$$

$$\left. \begin{aligned} \tilde{T}_{in,0} &= \tilde{F}_{in,0} \\ \frac{\partial \tilde{T}_{in,0}}{\partial \xi} &= 1 \end{aligned} \right\} \quad \text{at } \xi = \xi_{in,0}(t) \quad (3.46)$$

$$\left. \begin{aligned} \xi_{in,0} &= 1 \\ \tilde{T}_{in,0} &= 0 \end{aligned} \right\} \quad \text{at } \tau = 0. \quad (3.47)$$

Using the method of separation of variables for the linear parabolic partial differential equation (3.44) the leading-order solution is thus

$$\tilde{T}_{in,0} = \xi + \frac{8}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(1+2n)^2} \exp \left[- \left(n + \frac{1}{2} \right)^2 \pi^2 \tau \right] \sin \left[\left(n + \frac{1}{2} \right) \pi \xi \right], \quad (3.48)$$

$$\tilde{F}_{in,0} = 1 - \frac{8}{\pi^2} \sum_{n=0}^{\infty} \frac{\exp \left[- \left(n + \frac{1}{2} \right)^2 \pi^2 \tau \right]}{(1+2n)^2}, \quad (3.49)$$

$$\xi_{in,0} = 1. \quad (3.50)$$

We note that the inner solution naturally satisfies the leading order matching condition, as $\tau \rightarrow \infty$ for $F_{in,0} \rightarrow 1$ and when $t \rightarrow 0$ then $\tilde{F}_0 \rightarrow 1$. Similarly for the tension inside the tuft, (3.42) matches (3.48), when $t \rightarrow 0$ and $\tau \rightarrow \infty$ respectively. Thus the composite leading-order expansion is

$$T \sim \xi + \frac{8}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(1+2n)^2} \exp \left[- \left(n + \frac{1}{2} \right)^2 \pi^2 \frac{t}{\beta} \right] \sin \left[\left(n + \frac{1}{2} \right) \pi \xi \right] \quad (3.51)$$

$$F \sim 1 - t - \frac{8}{\pi^2} \sum_{n=0}^{\infty} \frac{\exp \left\{ - \left(n + \frac{1}{2} \right)^2 \pi^2 \frac{t}{\beta} \right\}}{(1+2n)^2}. \quad (3.52)$$

The results in figure 3.7 show the linear force profile except for a small region of size β near $t = 0$. The boundary layer represents the period when the tension rapidly changes along the fibre in the tuft from the tuft-air interface. The tension in the fibre in the tuft will then evolve towards a linear profile as information propagates down the fibre. From $t \sim \beta$, the withdrawal occurs at a steady rate, see equation (3.41), and then tension in the fibre is linear (3.42). If we withdrew the fibre at greater speeds, keeping all other parameters fixed, then the diffusion of tension down the fibre from the tuft air interface will be much slower as β will be larger and $\frac{1}{\beta}$ is the diffusion coefficient. In this case we would need to consider a numerical solution.

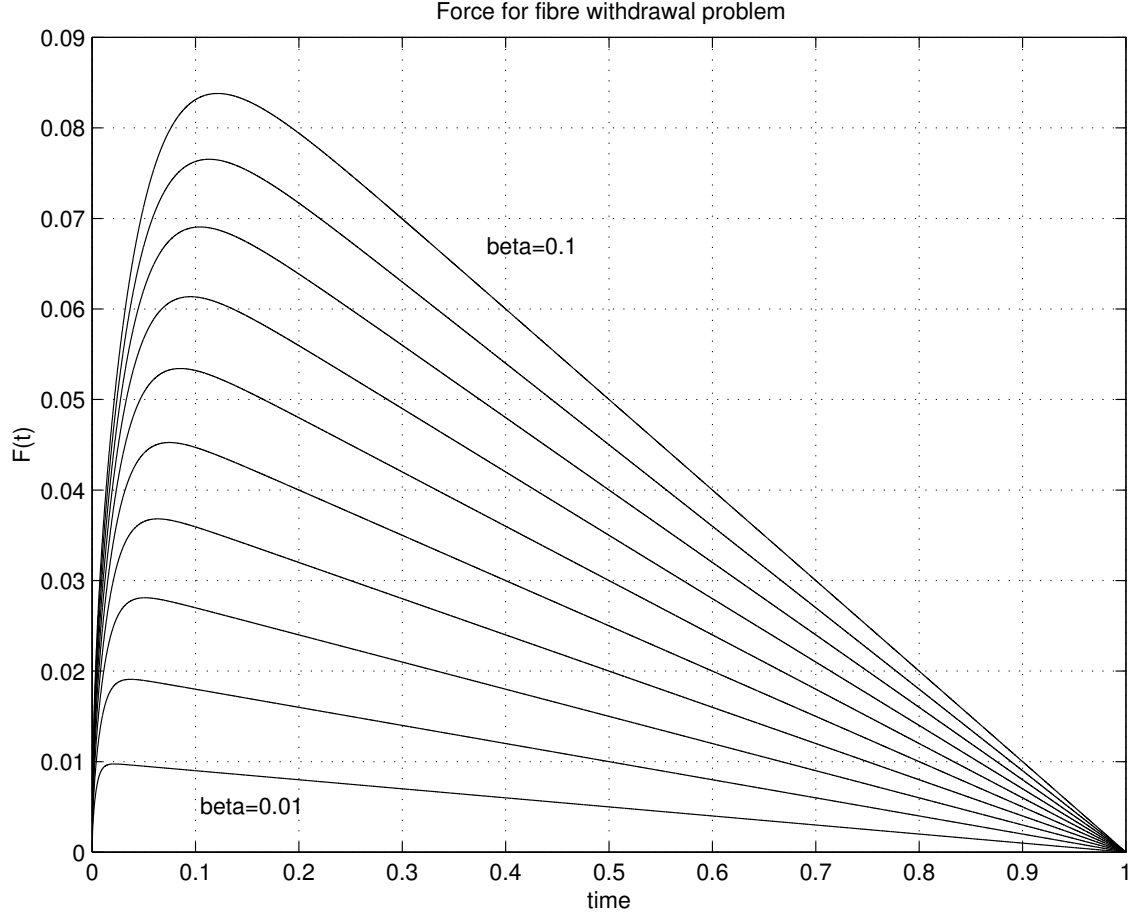


Figure 3.7: Plot of the force acting on a single fibre being withdrawn from a tuft. The small β asymptotic solution, where β ranges from 0.01 to 0.1.

3.3.4.2 Small Time Solution for $\beta = O(1)$

As there is a region of rapid change in the fibre's tension as $t \rightarrow 0$, which is invoked by the sudden withdrawal of the fibre, this will be difficult to compute numerically and so we consider a small time solution. For the small time where $t = \epsilon \bar{t}$ and $0 < \epsilon \ll 1$, we substitute the new time scaling into the equations (3.26) – (3.31) and the asymptotic expansions in ϵ

$$T(\xi, t) = T_0(\xi, t) + \epsilon T_1(\xi, t) + \epsilon^2 T_2(\xi, t) + \dots, \quad (3.53)$$

$$F(t) = F_0(t) + \epsilon F_1(t) + \epsilon^2 F_2(t) + \dots, \quad (3.54)$$

$$\xi_0(t) = \xi_{0,0}(t) + \epsilon \xi_{0,1}(t) + \dots, \quad (3.55)$$

and with the constitutive law for Φ given by (3.15) this gives the leading order problem:

$$\frac{\beta}{\epsilon} \frac{\partial T_0}{\partial t} = \frac{\partial}{\partial \xi} \left\{ \frac{1}{1 + T_0} \frac{\partial T_0}{\partial \xi} \right\}, \quad (3.56)$$

with boundary conditions (3.27) – (3.30) becoming

$$T_0(\xi, \bar{t}) = 0 \quad \text{at} \quad \xi = 0, \quad (3.57)$$

$$T_0(\xi, \bar{t}) = F_0(\bar{t}) \quad \text{at} \quad \xi = 1 - \epsilon \frac{\bar{t}}{1 + F_0(\bar{t})}, \quad (3.58)$$

$$\frac{\partial T_0}{\partial \xi} = \beta [\Phi(T_0) - \bar{t} \Phi'(T_0) F_0'(\bar{t})] \quad \text{at} \quad \xi = 1 - \epsilon \frac{\bar{t}}{1 + F_0(\bar{t})}, \quad (3.59)$$

$$T_0 = 0 \quad \text{at} \quad \bar{t} = 0. \quad (3.60)$$

For these equations (3.56) – (3.60), it is assumed that the tension is bounded by 0 and T_c . Now, the leading order behaviour is simply

$$T_0(\xi, t) = C(\xi). \quad (3.61)$$

This equation (3.61) cannot satisfy both the boundary conditions (3.58) and (3.60). The only way to avoid a solution that depends only on ξ , equation (3.61), is to introduce a boundary layer of width $O(\epsilon^{\frac{1}{2}})$ in ξ , which will be near $\xi = \xi_0$ or $\xi = 0$, since $t = O(\epsilon)$. Therefore this problem (3.56) – (3.60) requires a singular perturbation analysis. The tension will be small for most of the string for small time except for a small region near the free boundary $\xi = \xi_0(t)$. The outer region is composed of most of the fibre, which is embedded in the tuft and is not yet in motion, where the outer solution for tension is $T_0 = 0$. There is a neighbourhood near the free boundary $\xi_0(t)$, which is defined as the inner region or boundary layer.

From equations (3.56)–(3.60), for the inner problem we use the scalings: $F = \epsilon^{\frac{1}{2}} \bar{F}$, $T = \epsilon^{\frac{1}{2}} \bar{T}$, and $\xi = 1 - \epsilon^{\frac{1}{2}} \eta$. Using an asymptotic expansion for $\bar{F}$ and $\bar{T}$ in powers of $\epsilon^{\frac{1}{2}}$ we find the leading order inner problem to be:

$$\beta \frac{\partial \bar{T}_0}{\partial \bar{t}} = \frac{\partial^2 \bar{T}_0}{\partial \eta^2}, \quad (3.62)$$

$$\frac{\partial \bar{T}_0}{\partial \eta} = -\beta \text{ at } \eta = 0, \quad \bar{T}_0 = 0 \text{ at } \bar{t} = 0. \quad (3.63)$$

The additional condition that we need comes from the Van Dyke matching principle and in this case we will consider leading order matching:

$$\bar{T}_0(\eta, t) \rightarrow 0 \quad \text{as} \quad \eta \rightarrow \infty. \quad (3.64)$$

Solving equations (3.62) – (3.63) will give $\bar{F}_0$, which is just $\bar{T}_0$ evaluated at $\eta = 0$. Using a similarity solution of the form $\bar{T}_0(\eta, t) = \bar{t}^{\frac{1}{2}} g\left(\frac{\eta}{\bar{t}^{\frac{1}{2}}}\right)$, we rewrite equations (3.62) and (3.63) as

$$2\frac{d^2g}{dx^2} + \beta x \frac{dg}{dx} - \beta g = 0, \quad (3.65)$$

$$\frac{dg}{dx}(x=0) = -\beta, \quad (3.66)$$

respectively, where $x = \frac{\eta}{\bar{t}^{\frac{1}{2}}}$. Notice that $x \rightarrow \infty$ when either $\eta \rightarrow \infty$ or $\bar{t} \rightarrow 0$, but as long as g is non-singular at $x \rightarrow \infty$ then it naturally satisfies the initial condition $\bar{T}_0 = 0$ at $\bar{t} = 0$. So the remaining spatial boundary condition is dealt with during the matching process. Now we use the substitution $g(x) = xQ(x)$ in equation (3.65),

$$2x\frac{d^2Q}{dx^2} + \frac{dQ}{dx}(4 + \beta x^2) = 0, \quad (3.67)$$

and then integrate using the boundary condition at $x = 0$ from equation (3.66) and this gives the leading order term:

$$\bar{T}_0 = C_0 \bar{t}^{\frac{1}{2}} \left(\exp\left\{-\frac{\eta^2 \beta}{4\bar{t}}\right\} - \frac{\eta}{2} \sqrt{\frac{\pi \beta}{\bar{t}}} \operatorname{erf}\left\{\frac{\eta}{2} \sqrt{\frac{\beta}{\bar{t}}}\right\} - \beta \frac{\eta}{\bar{t}^{\frac{1}{2}}} \right). \quad (3.68)$$

Applying the leading order matching condition (3.64) determines C_0 as $-2\sqrt{\frac{\beta}{\pi}}$ and then the solution in terms of ξ and t is

$$T \sim 2\sqrt{\frac{\beta t}{\pi}} \exp\left\{-\frac{\beta(\xi-1)^2}{4t}\right\} + \beta(\xi-1) \operatorname{erfc}\left\{\frac{1-\xi}{2} \sqrt{\frac{\beta}{t}}\right\} \quad (3.69)$$

$$F \sim 2\sqrt{\frac{\beta t}{\pi}} \quad (3.70)$$

$$\xi_0 \sim 1. \quad (3.71)$$

Figure 3.8 shows boundary layer behaviour of thickness $O(\sqrt{\epsilon})$ near the free boundary which is at $\xi = 1 + O(\epsilon)$.

3.3.5 Numerical Computations

We need to solve a nonlinear parabolic partial differential equation (3.26), with boundary conditions (3.27) – (3.31). For the general β case this requires numerical simulations. Using the linear form for the constitutive law (3.15), the field equation (3.26) becomes

$$\beta \frac{\partial T}{\partial t} = \frac{\partial}{\partial \xi} \left[\frac{1}{1+T} \frac{\partial T}{\partial \xi} \right], \quad (3.72)$$

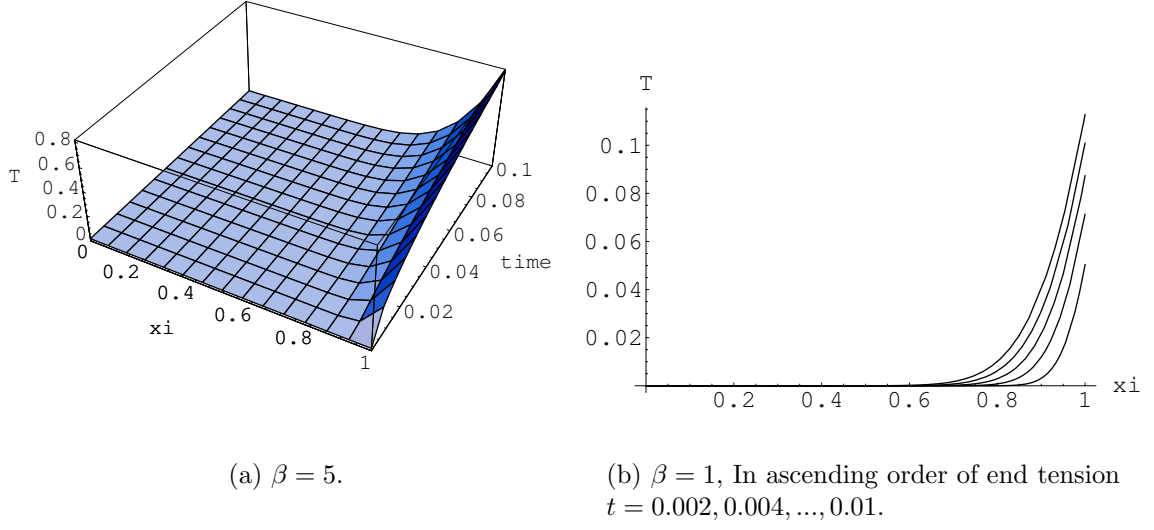


Figure 3.8: A single fibre being withdrawn from a tuft: asymptotic solutions for small time.

with the boundary and initial conditions (3.27) – (3.31) becoming:

$$T = 0 \quad \text{at} \quad \xi = 0, t = 0 \quad (3.73)$$

$$T = F \quad \text{at} \quad \xi = \xi_0, \quad (3.74)$$

$$\frac{\partial T(t, \xi)}{\partial \xi} = \beta \left[1 + F(t) - t \frac{dF(t)}{dt} \right] \quad \text{at} \quad \xi = \xi_0. \quad (3.75)$$

and

$$\xi_0(t) = 1 - \frac{t}{1 + F(t)} \quad (3.76)$$

To compute the solution of (3.72) with (3.73) – (3.76) we use a finite difference scheme based on the “method of lines”.

Before we go on to explain the numerical scheme in more detail, we need to find a suitable substitution for time t because, from equation (3.70), the behaviour of $F \sim \sqrt{t}$ for small time. This means that a finite difference scheme would have large truncation errors for small time. A suitable substitution would be to write $s = \sqrt{t}$ and so the governing equation (3.72) and boundary condition (3.75) become:

$$\frac{\beta}{2s} \frac{\partial T(\xi, s)}{\partial s} = \frac{\partial}{\partial \xi} \left[\frac{1}{1 + T(\xi, s)} \frac{\partial T(\xi, s)}{\partial \xi} \right], \quad (3.77)$$

$$\frac{s}{2} \frac{dF(s)}{ds} = 1 + F(s) - \frac{1}{\beta} \frac{\partial T(\xi, s)}{\partial \xi} \quad \text{at} \quad \xi = \xi_0(s), \quad (3.78)$$

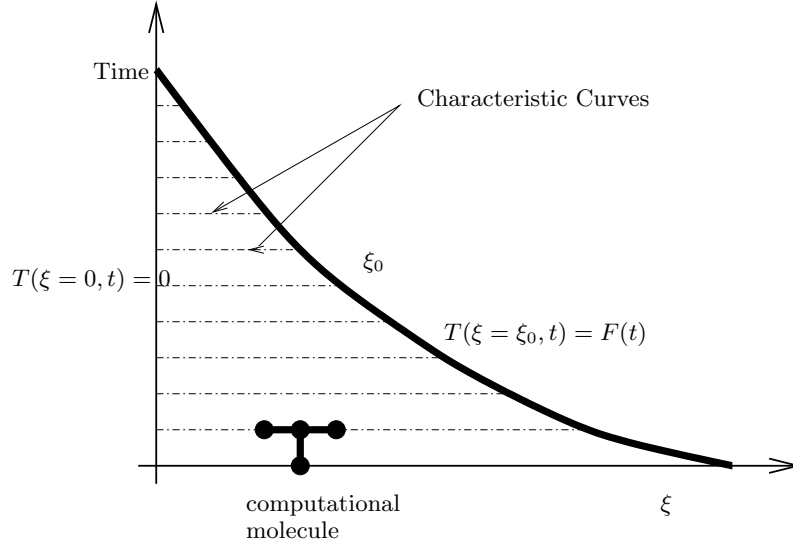


Figure 3.9: The computational grid and molecule for a parabolic partial differential with a free boundary $\xi = \xi_0$.

respectively.

The method of lines is based on the fact that information within the system, for a quasi-linear parabolic partial differential equation, travels along the one-parameter family of characteristic curves, in this case $t = \text{constant}$, and these lines are illustrated in figure 3.9. Thus, as we wish to compute exactly how information travels along these lines, we discretise equation (3.77) in s and what remains is an ordinary differential equation in fibre centre-line arclength ξ . So at each time step we evaluate the tension, T^{n+1} , but due to the time step will include the previous time-step's tension, T^n ,

$$\frac{\partial^2 T^{n+1}}{\partial \xi^2} = \frac{1}{1 + T^{n+1}} \left(\frac{\partial T^{n+1}}{\partial \xi} \right)^2 - \frac{\beta(T^{n+1} - T^n)(1 + T^{n+1})}{2(n+1)(\Delta s)^2}. \quad (3.79)$$

Equation (3.79) has two boundary conditions from (3.73) and (3.73):

$$T^{n+1}(\xi = 0) = 0 \quad \text{and} \quad T^{n+1}(\xi = \xi_0) = F^{n+1}(s). \quad (3.80)$$

To solve the ordinary differential equation for each time-step, we shall use an implicit method because, for small time, there is a boundary layer near the tuft-air interface, $\xi_0(t)$, as shown in section 3.3.4. We are now in a position to compute a solution for each time-step, but we need to find the appropriate boundary conditions.

It remains to find the location of the free boundary and the force applied there. We use equation (3.78), which is a first-order ordinary differential equation for the evolution of the free boundary force F^n . Once we are aware of F at the n -th time

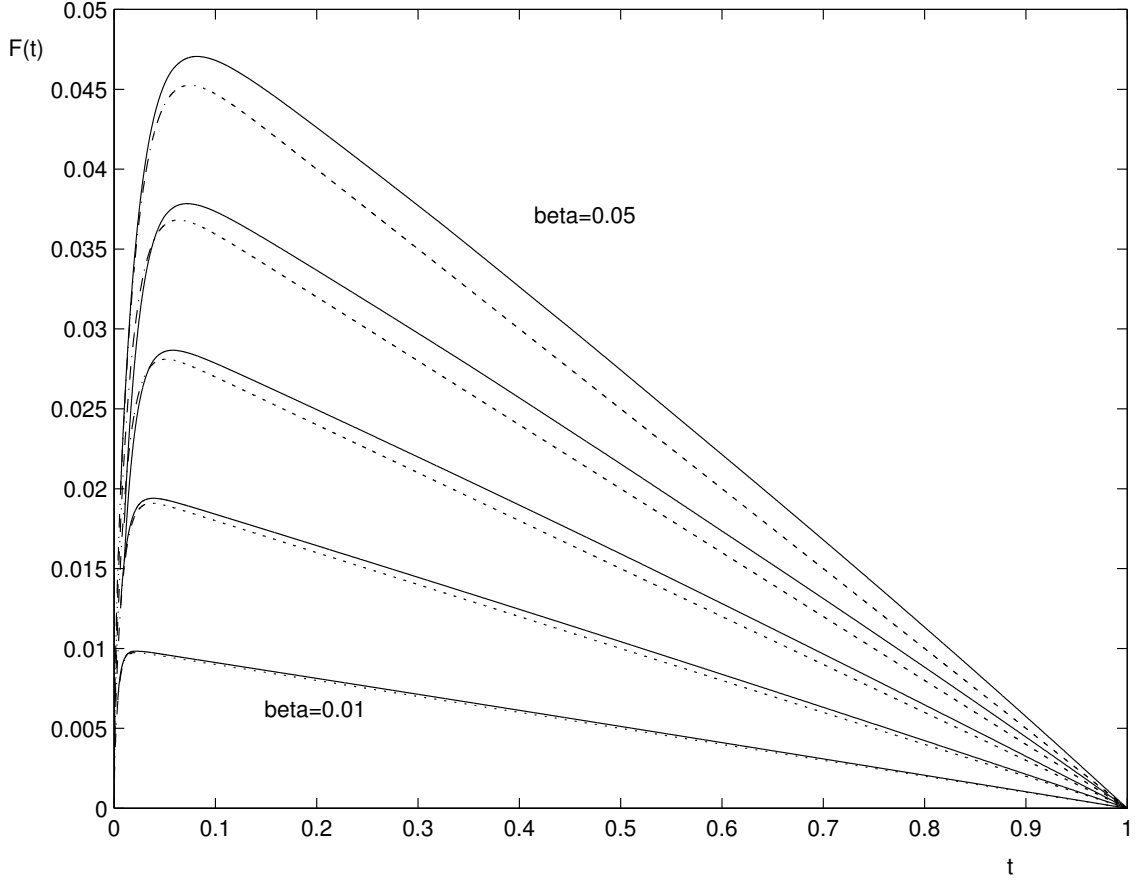


Figure 3.10: The withdrawal force on of a single fibre: dotted lines plot asymptotic solutions and the solid lines plot the numerical computations with $\beta = 0.01, 0.02, 0.03, 0.04, 0.05$ in ascending order for both sets of results.

step then we can find $\xi_0^n(t)$ from (3.76). A solution to (3.78) can be obtained using a linear multi-step predictor-corrector method, namely Adams-Bashforth and Adams-Moulton. The predictor step is

$$\begin{aligned}
 F^{n+1} = & F^n + \frac{1}{12} \left[\frac{55}{n} \left\{ 1 + F^n - \frac{1}{\beta} \frac{\partial T^n}{\partial \xi} (\xi_0^n) \right\} \right. \\
 & + \frac{59}{n-1} \left\{ 1 + F^{n-1} - \frac{1}{\beta} \frac{\partial T^{n-1}}{\partial \xi} (\xi_0^{n-1}) \right\} \\
 & - \frac{37}{n-2} \left\{ 1 + F^{n-2} - \frac{1}{\beta} \frac{\partial T^{n-2}}{\partial \xi} (\xi_0^{n-2}) \right\} \\
 & \left. - \frac{9}{n-3} \left\{ 1 + F^{n-3} - \frac{1}{\beta} \frac{\partial T^{n-3}}{\partial \xi} (\xi_0^{n-3}) \right\} \right], \quad (3.81)
 \end{aligned}$$

and the corrector step is

$$\begin{aligned}
F^{n+1} = & F^n + \frac{1}{12} \left[\frac{9}{n+1} \left\{ 1 + F^{n+1} - \frac{1}{\beta} \frac{\partial T^{n+1}}{\partial \xi} (\xi_0^{n+1}) \right\} \right. \\
& + \frac{19}{n} \left\{ 1 + F^n - \frac{1}{\beta} \frac{\partial T^n}{\partial \xi} (\xi_0^n) \right\} \\
& - \frac{5}{n-1} \left\{ 1 + F^{n-1} - \frac{1}{\beta} \frac{\partial T^{n-1}}{\partial \xi} (\xi_0^{n-1}) \right\} \\
& \left. - \frac{9}{n-2} \left\{ 1 + F^{n-2} - \frac{1}{\beta} \frac{\partial T^{n-2}}{\partial \xi} (\xi_0^{n-2}) \right\} \right]. \tag{3.82}
\end{aligned}$$

We then simply approximate ξ_0^{n+1} with the algorithm

$$\xi_0^{n+1} = 1 - \frac{(n+1)\Delta t}{1 + F^{n+1}}. \tag{3.83}$$

The explicit Adams-Bashforth method (3.81) finds a solution, F^{n+1} and ξ_0^{n+1} , for the partial differential equation at the $n+1$ step. Then this solution is used to evaluate the right-hand-side of the implicit Adams-Moulton method (3.82). The correction step then computes F^{n+1} by using (3.82) based on the evaluation step. Once this is done the evaluation step can be done again for the newly corrected F^{n+1} , then the correction computation can be made again. Repetition in the evaluation and correction step were carried out until an *a priori* convergence was achieved. Now we have for the $n+1$ step F^{n+1} and ξ_0^{n+1} and so we can solve the ODE at the $n+1$ step with boundary conditions at $\xi = 0, \xi_0^{n+1}$ being $T = 0, F^{n+1}$ respectively.

Due to the singular nature of the problem in the limit as time tends to zero, we begin the numerical computation with the small time asymptotic solution given by (3.69) – (3.71). The linear multi-step methods also use the asymptotic analysis for the first four steps in (3.82) and (3.81). We see in figure 3.10 that the difference in magnitude between asymptotic and numerical solution is at most order β which gives us confidence in the numerical algorithm. In figure 3.11 we plot the numerical findings for larger values of β . The computations failed when the fibre is just about to exit the tuft, i.e. we expect $\frac{dF}{ds} \rightarrow \infty$ as $F \rightarrow t-1$. The magnitude of the gradient steepens near the point in which the numerical code fails, see figure 3.11.

3.4 Teasing out Fibres with a Hook

We can now address some industrially relevant scenarios that occur when a hook attacks a group of entangled fibres, see figure 3.1. In particular, we explain the case where a hook attaches itself to a single fibre and attempts to extract it from a tuft.

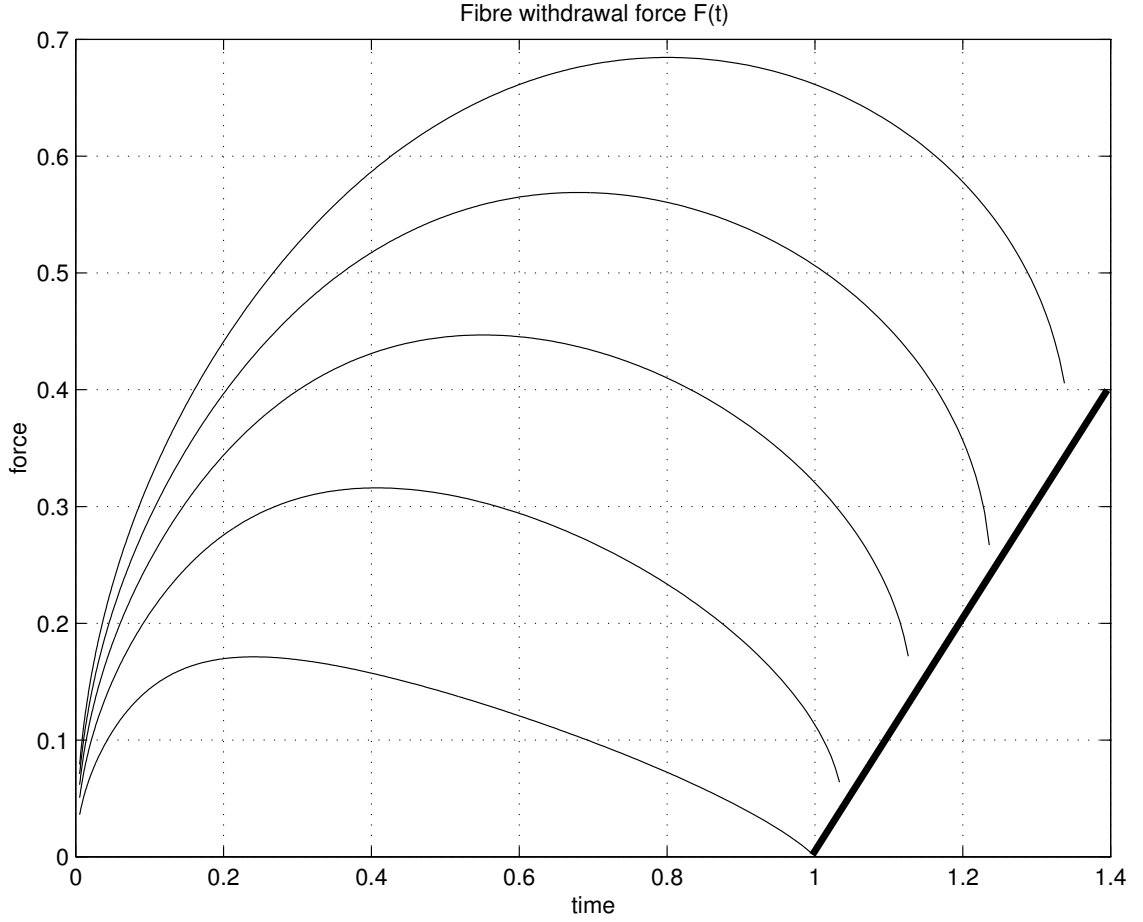


Figure 3.11: The withdrawal forces on a single fibre. Numerical computations of force for $\beta = 0.2, 0.4, 0.6, 0.8, 1.0$. Plots ascending with respect to β .

There is a competition between the two fibre ends, that are embedded in the tuft, to remain in the tuft. A simplified drawing of this is given in figure 3.12.

For simplicity, we assume that the hook is round. Then for a fibre to overcome the static friction generated by the contact with the hook, it must satisfy the Amonton's law:

$$\frac{T_2}{T_1} \geq e^{\mu\theta} \quad \text{or} \quad \frac{T_2}{T_1} \leq e^{-\mu\theta}, \quad (3.84)$$

where T_1 and T_2 are the leaving and incoming tensions respectively, θ is the angle of contact and μ is the coefficient of friction, see figure 3.12.

The easiest problem to solve mathematically is when there is no slip between fibre and hook, as this means the fibre is extracted with uniform velocity, and allows us to decouple the solutions in the fibre either side of the hook. This would either mean

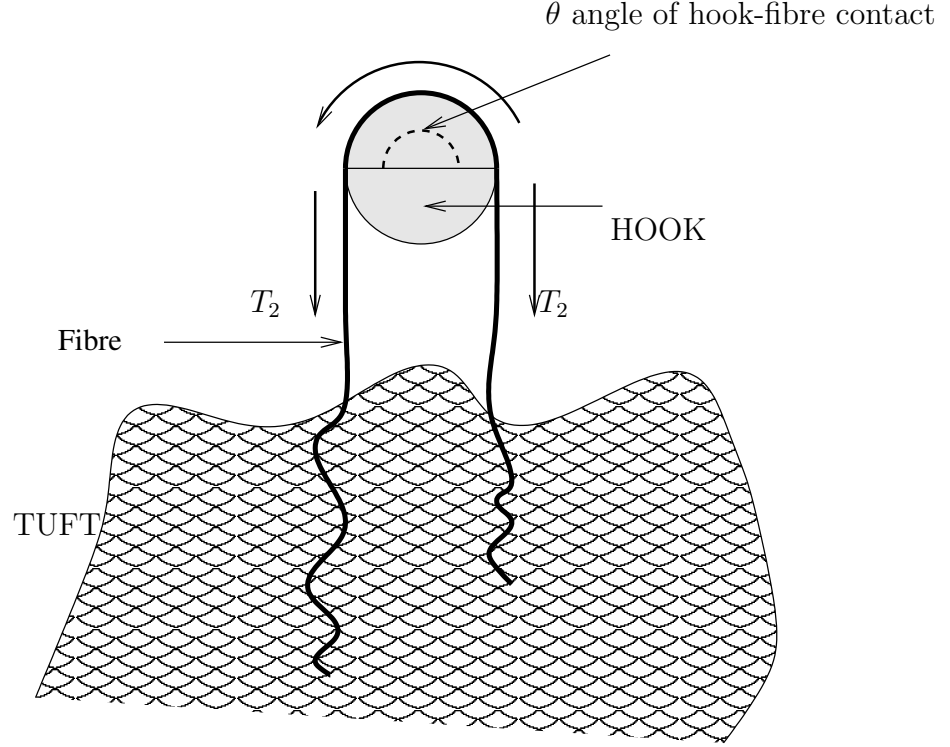


Figure 3.12: A diagram of a hook teasing a fibre from a tuft; θa_{hook} is the length of contact between fibre and hook, where a_{hook} is the radius of the hook.

that the coefficient of friction is large or that the lengths of fibre embedded in the tuft are nearly equal. For this situation we can apply the work done in the previous section 3.3. For fibres with uneven lengths embedded, either side of a hook, in a tuft but still with significant fibre-hook friction we are left with a similar problem. In this case one fibre end will be freed from the tuft and then the loose end will slip around the hook, leaving the fibre partly extracted from the tuft. Using the withdrawal tension F from section 3.3, for different β , we can find T_1 and T_2 . In the next section we will deal with a fibre that is affected by forces due to dynamic friction in two tufts.

3.5 Tufts held together by a single fibre

We continue modelling carding machine scenarios with the breakup of a tuft, by considering the case where a tuft is broken into two discrete bodies which are held together by, to begin with, only one interconnecting fibre. This case can then be extended to consider many inter-connecting fibres. Within the carding machine this would mimic the case where the two tuft entities are attached to different moving

parts of the carding machine, such as the lap-to-taker-in and taker-in-to-cylinder regions, see figure 3.2.

The rudiments of the problem are similar to the single fibre withdrawal problem given in section 3.3. Again the crimped or naturally curved fibre is modelled as a spring or an extensible, linearly elastic rod. Then we derive the respective Lagrangian field equations for the fibre inside and between the two tufts. In contrast to the single free boundary in the single tuft scenario we will now have two tufts and therefore two unknown boundaries or tuft-air interfaces.

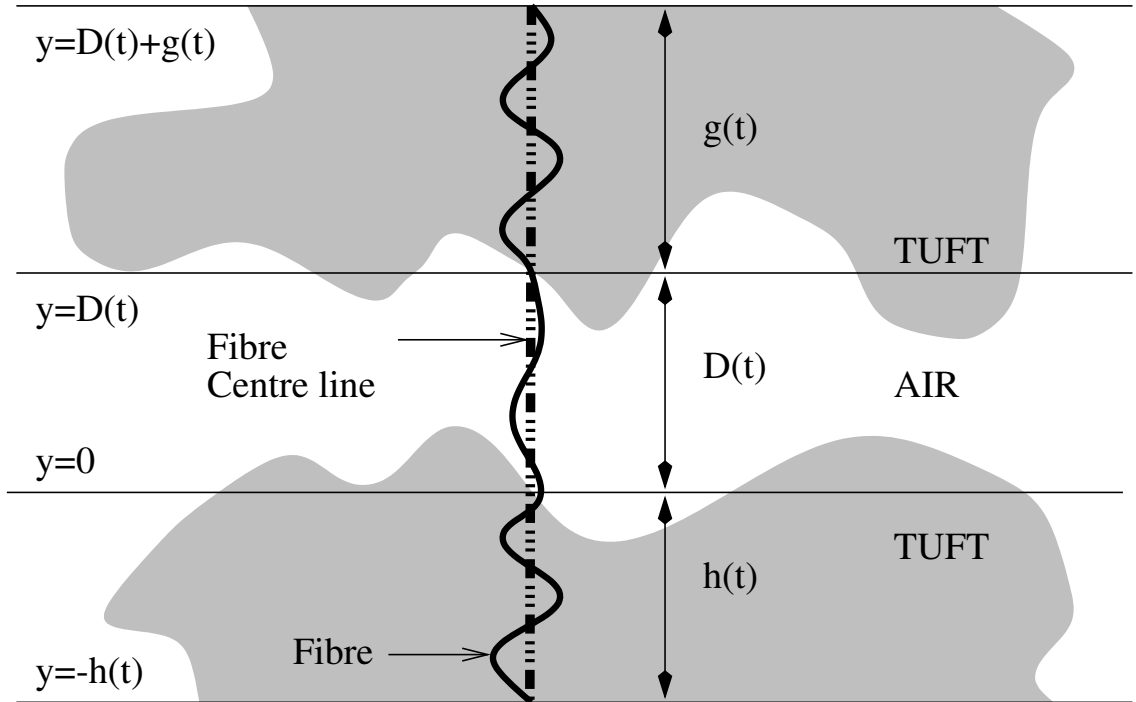


Figure 3.13: A diagram of two tufts with one inter-connecting fibre.

The double tuft fibre withdrawal problem is illustrated in figure 3.13. We choose the top boundary of the bottom tuft, which is always fixed, to be described by the Eulerian coordinate $y = 0$ and the end of the fibre in the bottom tuft to be at $y = -h(t)$. The bottom boundary of the top tuft, which moves away at a prescribed speed is at the point $y = D(t)$, and the end of the fibre in the top tuft is $y = g(t) + D(t)$. The initial state, $D(t = 0) = 0$, means that the boundary of both tufts are in the same place at the beginning of the experiment. Furthermore, $h(0)$ and $g(0)$ are the amounts of fibre in the bottom and top tufts respectively.

We use the friction law (3.16) and remember that V is now the velocity of the fibre relative to the tuft for which the direction of damping is reversed in the upper

tuft. As the upper tuft moves at speed $\frac{dD}{dt}$, this gives us the field equations inside the tuft to be

$$\frac{\partial T}{\partial y} = \lambda v \quad \text{for} \quad -h(t) < y < 0, \quad (3.85)$$

$$\frac{\partial T}{\partial y} = \lambda \left(v - \frac{dD}{dt} \right) \quad \text{for} \quad D(t) < y < D(t) + g(t), \quad (3.86)$$

where v is the actual velocity of the fibre. In equations (3.85) and (3.86), we have chosen the same frictional coefficient in the two respective regions as we assume that the whole tuft, before it is broken down into two, can be approximated as a uniform media. As we are considering just one interconnecting fibre then the resulting interaction with the other fibres caused by its extraction will not significantly change the physical properties of either tuft.

To complete the Eulerian description of the problem, (3.85) and (3.86), we need to find a constitutive equation and some physical constraints, and we begin with the latter. There is no tension at the fibre end-points otherwise there would need to be a singularity in the frictional force. We impose continuity in tension at the tuft-air interfaces and no tension in the fibre when the tufts have touching boundaries. The aforementioned physical conditions translate into the following boundary and initial conditions:

$$T(y, t) = 0 \quad \text{at} \quad y = -h(t) \text{ and } y = D(t) + g(t), \quad (3.87)$$

$$T(y, t) = F(t) \quad \text{at} \quad y = 0 \text{ and } y = D(t), \quad (3.88)$$

$$T(y, t) = 0 \quad \text{when} \quad t = 0. \quad (3.89)$$

Now we are left with the problem of finding a constitutive equation relating the tension to extension.

As in the case for the single tuft problem in section 3.3, due to the fact that we are considering the centre line of the fibre to behave as a spring or a linearly elastic extensible beam, we convert the coordinate system from Eulerian to Lagrangian variables, thus writing

$$y = g(t) + D(t) \quad \mapsto \quad \xi = l, \quad (3.90)$$

$$y = D(t) \quad \mapsto \quad \xi = \xi_1(t), \quad (3.91)$$

$$y = 0 \quad \mapsto \quad \xi = \xi_0(t), \quad (3.92)$$

$$y = -h(t) \quad \mapsto \quad \xi = 0, \quad (3.93)$$

where $y(x, 0) = \xi$ and $Y(\xi, t) = y(x, t)$. The transformations (3.90) – (3.93) give three regions to consider, the bottom tuft $0 < \xi < \xi_0(t)$, the volume in between the two tufts $\xi_0(t) < \xi < \xi_1(t)$ and the top tuft $\xi_1(t) < \xi < l$. The constitutive equation (3.14) still holds throughout the tuft. We note that position $Y(\xi, t)$ is expected to be a monotonic increasing function with respect to element ξ and time t .

3.5.1 The Equations

The field equations, (3.85) and (3.86), can be transformed into a Lagrangian framework, by using the constitutive law (3.14); and this relation holds throughout the tuft as we are assuming a fibre which has uniform properties throughout its length. Thence we find,

$$\frac{\partial T}{\partial \xi} = \Phi \lambda \frac{\partial Y}{\partial t} \quad \text{when} \quad 0 < \xi < \xi_0, \quad (3.94)$$

$$\frac{\partial T}{\partial \xi} = 0 \quad \text{when} \quad \xi_0 < \xi < \xi_1, \quad (3.95)$$

$$\frac{\partial T}{\partial \xi} = \Phi \lambda \left(\frac{\partial Y}{\partial t} - \frac{dD}{dt} \right) \quad \text{when} \quad \xi_1 < \xi < l, \quad (3.96)$$

where $\Phi \left(\frac{T}{k} \right)$ can be described by the equation (3.15), which is a piecewise linear function of tension, or alternatively we can prescribe a nonlinear relationship.

In order to find an equation for the tension in both tufts, by differentiating (3.94) and (3.96) with respect to particular element ξ , and using the constitutive law (3.14) we get

$$\frac{\partial f}{\partial \xi} = \lambda \frac{\partial^2 Y}{\partial \xi \partial t} = \frac{\lambda \Phi'}{k} \frac{\partial T}{\partial t} \quad \text{when} \quad \begin{cases} 0 < \xi < \xi_0(t) \\ \xi_1(t) < \xi < l \end{cases}. \quad (3.97)$$

Now we eliminate $\frac{\partial f}{\partial \xi}$ from (3.97) by differentiating the equations (3.94) with respect to ξ ; this results in the field equations for the double tuft problem to be:

$$\frac{\partial}{\partial \xi} \left[\frac{1}{\Phi \left(\frac{T}{k} \right)} \frac{\partial T}{\partial \xi} \right] = \frac{\lambda}{k} \Phi' \left(\frac{T}{k} \right) \frac{\partial T}{\partial t} \quad \text{when} \quad \begin{cases} 0 < \xi < \xi_0(t) \\ \xi_1(t) < \xi < l \end{cases}, \quad (3.98)$$

and this is exactly the same as the tuft field equation (3.19).

Finally we re-write the boundary and initial conditions for tension (3.87)–(3.89) as:

$$T(\xi, t) = 0 \quad \text{at} \quad \xi = 0, l. \quad (3.99)$$

$$T(\xi, t) = F(t) \quad \text{at} \quad \xi = \xi_0(t), \xi_1(t), \quad (3.100)$$

$$T(\xi, t) = 0 \quad \text{at} \quad t = 0. \quad (3.101)$$

The initial length of fibre in top and bottom tufts are given by $l - \xi_0(0)$ and $\xi_0(0)$ respectively and the tension at $\xi_0(t = 0)$ is simply $F = 0$. The ratio of fibre length in each tuft will be a critical in the problem. Let us write

$$\xi_0(0) = \xi_1(0) = \frac{\theta}{l}, \quad (3.102)$$

where l is the length of the fibre. We are left with solving two coupled parabolic partial differential equations (3.98) with sufficient boundary and initial conditions (3.99)–(3.101) in order to find a unique solution if we know $\xi_0(t)$, $\xi_1(t)$ and $F(t)$. We close the aforementioned system of equations by finding three more conditions that ultimately give the location of the two tuft-air interfaces, $\xi_0(t)$ and $\xi_1(t)$, and the tension $F(t)$ of the fibre in between the tufts, $\xi_0(t) < \xi < \xi_1(t)$.

The first condition is found by integrating the constitutive equations (3.14) in the region outside both tufts, where the tension is uniform with respect to space $T(\xi, t) = F(t)$, then applying the geometrical conditions, (3.91) and (3.92), and this gives

$$D(t) = \Phi \left(\frac{F(t)}{k} \right) [\xi_1(t) - \xi_0(t)]. \quad (3.103)$$

Similar to the derivation in section 3.3, we use the geometrical conditions $Y(\xi_0(t), t) = 0$ and $Y(\xi_1(t), t) = D(t)$ for the tuft-air interfaces and consequently their respective velocities in order to find the final additional equations so that we can close the system of equations. We find

$$\frac{\partial}{\partial \xi} Y(\xi, t) \frac{d}{dt} \xi_0(t) + \frac{\partial}{\partial t} Y(\xi, t) = 0, \quad (3.104)$$

$$\frac{\partial}{\partial \xi} Y(\xi, t) \frac{d}{dt} \xi_1(t) + \frac{\partial}{\partial t} Y(\xi, t) = \frac{d}{dt} D(t). \quad (3.105)$$

Then substituting these equations, (3.104) and (3.105), and the constitutive laws given by (3.14) into the respective frictional laws (3.85) and (3.86) for the two tufts, we find the free boundary conditions

$$\frac{\partial T}{\partial \xi} = -\lambda \Phi \left(\frac{F}{k} \right)^2 \frac{d}{dt} \xi_0(t) \quad \text{at} \quad \xi = \xi_0(t), \quad (3.106)$$

$$\frac{\partial T}{\partial \xi} = -\lambda \Phi \left(\frac{F}{k} \right)^2 \frac{d}{dt} \xi_1(t) \quad \text{at} \quad \xi = \xi_1(t). \quad (3.107)$$

Now we have a complete set of equations that are composed of two coupled parabolic partial differential equations (3.98), with free boundary conditions (3.103), (3.106) and (3.107) and the initial and boundary conditions (3.99) – (3.101).

3.5.1.1 Dimensionless Equations

Now we need to consider the scalings for our equations and similar to those prescribed in the single tuft problem by the equations (3.25), for the upper tuft we write down:

$$\xi = l_U \xi', \quad \xi_0(0) = l - l_U \quad D = l_U D', \quad T = k T' \quad F = k F', \quad t = \frac{l_U}{U} t', \quad (3.108)$$

where subscript U is short hand for the upper tuft, and thence

$$\beta_U \Phi'(T) \frac{\partial T}{\partial t} = \frac{\partial}{\partial \xi} \left\{ \frac{1}{\Phi(T)} \frac{\partial T}{\partial \xi} \right\} \quad \text{when} \quad \xi_1(t) < \xi < \frac{l}{l_U} = \theta + 1, \quad (3.109)$$

where $\beta_U = \frac{\lambda U l_U}{k}$ and $\theta = \frac{l - l_U}{l_U}$ is the ratio of the length of the fibre originally in the bottom tuft to that originally in the top tuft. For the lower tuft we use the same scalings. The spring constants are assumed to be the same in both tufts, and this is a reasonable proposition as the fibre is then considered to have uniform properties throughout its length. We drop the primes in the notation and get the dimensionless equations:

$$\beta_U \Phi'(T) \frac{\partial T}{\partial t} = \frac{\partial}{\partial \xi} \left\{ \frac{1}{\Phi(T)} \frac{\partial T}{\partial \xi} \right\} \quad \text{when } 0 < \xi < \xi_0(t) \text{ and } \xi_1(0) < \xi < 1 + \theta \quad (3.110)$$

The fixed initial and boundary conditions are

$$T(\xi, t) = 0 \quad \text{at} \quad t = 0, \quad (3.111)$$

$$T(\xi, t) = 0 \quad \text{at} \quad \xi = 0, \theta + 1, \quad (3.112)$$

$$T(\xi, t) = F(t) \quad \text{at} \quad \xi = \xi_0(t), \xi_1(t), \quad (3.113)$$

$$\xi_0(0) = \xi_1(0) = \theta. \quad (3.114)$$

The free boundary conditions are given from (3.106) and (3.107) by

$$\frac{\partial T}{\partial \xi} = -\beta_U \Phi(F)^2 \frac{d}{dt} \xi_0(t) \quad \text{at} \quad \xi = \xi_0(t), \quad (3.115)$$

$$\frac{\partial T}{\partial \xi} = -\beta_U \Phi(F)^2 \frac{d}{dt} \xi_1(t) \quad \text{at} \quad \xi = \xi_1(t). \quad (3.116)$$

Finally the tension in the fibre between the tufts is given by

$$D(t) = \Phi(F) [\xi_1(t) - \xi_0(t)] \quad (3.117)$$

3.5.2 Asymptotic Solutions

The system of equations we are looking to solve are two coupled free boundary problems for nonlinear parabolic partial differential equations. There are two types of problems that we could consider analytically, $0 < \beta_U \ll 1$ and $\beta_U \gg 1$, and these relate to slow and fast withdrawal speeds respectively. For each case there is a further set of situations that can be modelled, and the various cases will depend on the ratio θ of the two fibre lengths within each tuft. We will focus on the slow tuft separation problem, i.e. $\beta_U \ll 1$.

3.5.2.1 Small β Asymptotics

We begin the analysis by prescribing the dimensionless initial velocity $D = t$, and using the linear form for Φ from equation (3.15). This means we can write the dimensionless system of equations (3.110), (3.115) and (3.116) as

$$\beta_U \frac{\partial T}{\partial t} = \frac{\partial}{\partial \xi} \left\{ \frac{1}{1+T} \frac{\partial T}{\partial \xi} \right\} \quad \text{when} \quad \xi_1(t) < \xi < \theta + 1, \quad (3.118)$$

$$\beta_U \frac{\partial T}{\partial t} = \frac{\partial}{\partial \xi} \left\{ \frac{1}{1+T} \frac{\partial T}{\partial \xi} \right\} \quad \text{when} \quad 0 < \xi < \xi_0(t), \quad (3.119)$$

$$\frac{\partial T}{\partial \xi} = -\beta_U(1+F)^2 \frac{d}{dt} \xi_0(t) \quad \text{at} \quad \xi = \xi_0(t), \quad (3.120)$$

$$\frac{\partial T}{\partial \xi} = -\beta_U(1+F)^2 \frac{d}{dt} \xi_1(t) \quad \text{at} \quad \xi = \xi_1(t), \quad (3.121)$$

and

$$\frac{t}{1+F} = [\xi_1(t) - \xi_0(t)]. \quad (3.122)$$

The initial and fixed boundary conditions remain unchanged from (3.111)–(3.113).

For the situation where the initial lengths of fibre in each tuft are comparable, $\theta = O(1)$ and $\beta_U \ll 1$ we employ an asymptotic expansion. In a similar fashion to section 3.3.4.1 and from equations (3.118)–(3.122), this suggests that we use the following:

$$T^U(\xi, t) = \beta_U T_0^U(\xi, t) + \beta_U^2 T_1^U(\xi, t) + \dots \quad (3.123)$$

$$T^B(\xi, t) = \beta_U T_0^B(\xi, t) + \beta_U^2 T_1^B(\xi, t) + \dots \quad (3.124)$$

$$F(t) = \beta_U F_0(t) + \beta_U^2 F_1(t) + \dots \quad (3.125)$$

$$\xi_1(t) = \xi_{1,0}(t) + \beta_U \xi_{1,1}(t) + \dots \quad (3.126)$$

$$\xi_0(t) = \xi_{0,0}(t) + \beta_U \xi_{0,1}(t) + \dots \quad (3.127)$$

and this leaves us with the following leading order equations:

$$\frac{\partial^2 T_0^U}{\partial \xi^2} = \frac{\partial^2 T_0^B}{\partial \xi^2} = 0 \quad (3.128)$$

$$T_0^U(\theta + 1, t) = T_0^B(0, t) = T_0^U(\xi, 0) = T_0^B(\xi, 0) = 0 \quad (3.129)$$

$$T_0^U(\xi_{1,0}, t) = T_0^B(\xi_{0,0}, t) = F_0 \quad (3.130)$$

$$\frac{\partial T_0^B}{\partial \xi} = -\frac{d\xi_{0,0}}{dt} \quad \text{at} \quad \xi = \xi_{0,0} \quad (3.131)$$

$$\frac{\partial T_0^U}{\partial \xi} = -\frac{d\xi_{1,0}}{dt} \quad \text{at} \quad \xi = \xi_{1,0} \quad (3.132)$$

and

$$t = \xi_{1,0}(t) - \xi_{0,0}(t), \quad (3.133)$$

where

$$\xi_{0,0}(0) = \xi_{1,0}(0) = \theta. \quad (3.134)$$

Integrating the field equation (3.128) and applying the no tension condition at the end points from (3.129), we are left with finding the remaining integrating constants, $c_1(t)$ and $c_2(t)$ from

$$T_0^U = c_1(t)(\xi - 1 - \theta) \quad (3.135)$$

$$T_0^B = c_2(t)\xi \quad (3.136)$$

The respective free boundary conditions (3.131) and (3.132) are then used to relate c_1 and c_2 with the free boundary velocities which are then eliminated by using (3.130) – (3.132). The leading order solution is then given by:

$$\xi_{0,0}(t) = \frac{2(1+\theta)(\theta+t) + t^2}{2(1+\theta) - 2t} \quad (3.137)$$

$$\xi_{1,0}(t) = \frac{2\theta(1+\theta) - t^2}{2(\theta+1) - 2t} \quad (3.138)$$

$$T_0^B(\xi, t) = \frac{t^2 - 2t(1+\theta) + 2(1+\theta)}{2(1-t+\theta)^2} \xi \quad (3.139)$$

$$T_0^U(\xi, t) = \frac{t^2 + 2\theta(1+\theta) - 2t(1+\theta)}{2(1-t+\theta)^2} (\theta + 1 - \xi) \quad (3.140)$$

$$F_0(t; \theta) = \frac{(t^2 + 2(1+\theta) - 2t(1+\theta))(t^2 - 2t(1+\theta) + 2\theta(1+\theta))}{4(\theta + 1 - t)^3} \quad (3.141)$$

We illustrate the tension in the string F , from equation (3.141), that lies between

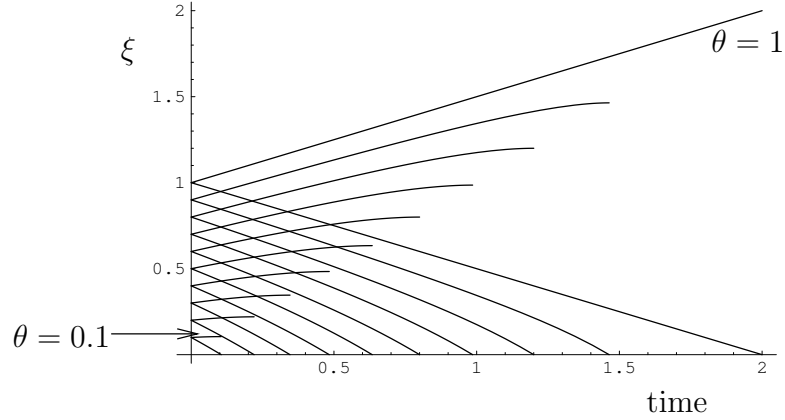


Figure 3.14: The position of the free boundaries, ξ_0 and ξ_1 , for two tufts with an interconnecting fibre; with varying $\theta \in [0.1, 1]$ in steps of 0.1.

the two tufts in figure 3.15 and also the position of the free boundary in figure 3.14. The free boundaries have turning points in time but the solution is no longer valid when these occur as the other end of the fibre is free by then.

The solution when there is an equal length of fibre in the two separating tufts, $\theta = 1$, is consistent with solutions for the outer problem of the single fibre withdrawal from a tuft (3.42) and (3.43). We expect this result as the solution is now symmetric. Notice that when $\xi_{0,0} = 0$ the fibre leaves the bottom tuft and when $\xi_{1,0} = 1 + \theta$ it leaves the top tuft, and when either part of the fibre leaves its respective tuft $F = 0$. Furthermore, for $\xi_{0,0} = 0$ then $\theta \leq 0$ and for $\xi_{1,0} = 1 + \theta$ then $\theta \geq 0$. Finally we observe that the force in $F_0(t) = T_0^B(\xi_0, t) = T_0^U(\xi_1, t)$ plotted in figure 3.15 does not satisfy the initial condition that $F(t = 0) = 0$.

We have deliberately omitted the imposition of the initial conditions as we expected a small region near $t = 0$ of size β_U for which the terms with time derivatives become important. Rescaling time $t = \beta_U \tau$ and using the asymptotic expansion,

$$T^U(\xi, t) = \beta_U T_0^{U,in}(\xi, t) + \beta_U^2 T_1^{U,in}(\xi, t) + \dots \quad (3.142)$$

$$T^B(\xi, t) = \beta_U T_0^{B,in}(\xi, t) + \beta_U^2 T_1^{B,in}(\xi, t) + \dots \quad (3.143)$$

$$F(t) = \beta_U F_0^{in}(t) + \beta_U^2 F_1^{in}(t) + \dots \quad (3.144)$$

$$\xi_1(t) = \xi_{1,0}^{in}(t) + \beta_U \xi_{1,1}^{in}(t) + \dots \quad (3.145)$$

$$\xi_0(t) = \xi_{0,0}^{in}(t) + \beta_U \xi_{0,1}^{in}(t) + \dots \quad (3.146)$$

for the small time region, the leading order equations are

$$\frac{\partial T_0^{U,in}}{\partial t} = \frac{\partial^2 T_0^{U,in}}{\partial \xi^2} \quad (3.147)$$

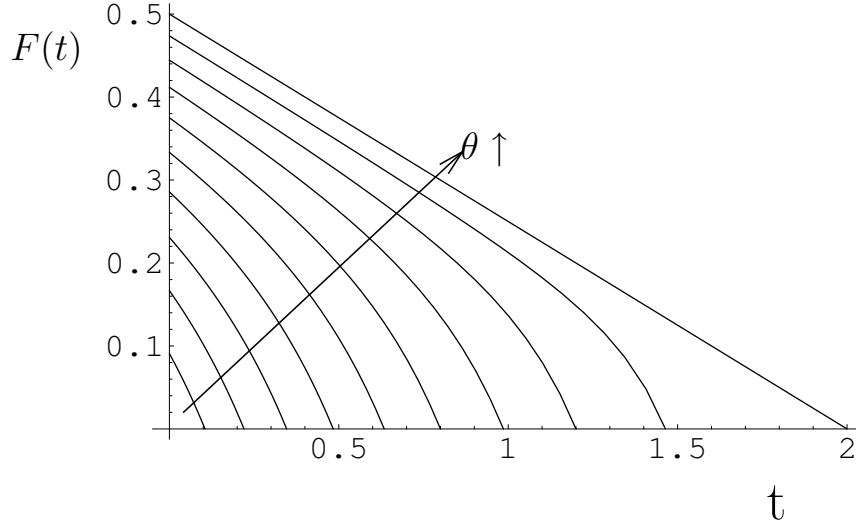


Figure 3.15: Tension in a fibre between the tufts: the small β problem. The ratio of length varies in steps of 0.1 in the interval $\theta \in [0.1, 1]$.

$$\frac{\partial T_0^{B,in}}{\partial t} = \frac{\partial^2 T_0^{B,in}}{\partial \xi^2} \quad (3.148)$$

$$T_0^{U,in}(\theta + 1, t) = T_0^{B,in}(0, t) = T_0^{U,in}(\xi, 0) = T_0^{B,in}(\xi, 0) = 0. \quad (3.149)$$

The moving boundaries to leading order can be approximated by their respective initial condition $\xi_{0,0}^{in} = \xi_{1,0}^{in} = \theta$, and the conditions imposed here simplify to

$$T_0^{U,in}(\theta, t) = T_0^{B,in}(\theta, t) \quad (3.150)$$

$$\frac{\partial T_0^{U,in}}{\partial \xi} - \frac{\partial T_0^{B,in}}{\partial \xi} = -1. \quad (3.151)$$

By reposing the problem with $T_0^{B,in} = u(\xi) - U(\xi, \tau)$ and $T_0^{U,in} = b(\xi) - B(\xi, \tau)$, we can find a solution using either Laplace transforms or Fourier analysis. As the diffusion problem (3.130) – (3.131) is valid for $t \sim O(\beta_U)$ we can match this to the steady-state or outer solution (3.137) – (3.140). To apply the leading order matching principle we simply define $u = T_0^U$ and $b = T_0^B$. The equations governing U and B are very similar to (3.130) – (3.151), the only differences are the initial conditions:

$$\frac{\partial B}{\partial \tau} = \frac{\partial^2 B}{\partial \xi^2} \quad \text{when } \xi < \theta, \quad (3.152)$$

$$\frac{\partial U}{\partial \tau} = \frac{\partial^2 U}{\partial \xi^2} \quad \text{when } \xi > \theta, \quad (3.153)$$

$$U(\theta + 1, \tau) = B(0, \tau) = 0, \quad (3.154)$$

$$U(\theta, \tau) = B(\theta, \tau), \quad (3.155)$$

$$\frac{\partial U}{\partial \xi} = \frac{\partial B}{\partial \xi}, \quad \text{at } \xi = \theta \quad (3.156)$$

$$U(\xi, 0) = \frac{\theta}{1+\theta}(\theta + 1 - \xi) \quad \text{when } \xi < \theta, \quad (3.157)$$

$$B(\xi, 0) = \frac{\xi}{1+\theta} \quad \text{when } \xi > \theta. \quad (3.158)$$

The conditions (3.155) and (3.156) allow us to consider a solution $J(\xi, \tau)$ which is applicable for the whole interval $(0, \theta + 1)$.

$$J = \sum_{n=1}^{\infty} C_n \exp \left\{ -\frac{n^2 \pi^2}{(\theta + 1)^2} \tau \right\} \sin \frac{n\pi \xi}{\theta + 1}, \quad (3.159)$$

$$C_n^U = \frac{\theta}{n^2 \pi^2} \left\{ n\pi \cos \left[\frac{n\pi \theta}{1+\theta} \right] + (1+\theta) \left(\sin \left[\frac{n\pi \theta}{1+\theta} \right] - \sin[n\pi] \right) \right\}, \quad (3.160)$$

$$C_n^B = \frac{\theta n\pi \cos \left[\frac{n\pi \theta}{1+\theta} \right] + (1+\theta) \sin \left[\frac{n\pi \theta}{1+\theta} \right]}{n^2 \pi^2}. \quad (3.161)$$

This gives rise to the composite solution:

$$T^B(\xi, t) = \frac{\xi}{1+\theta} - \sum_{n=1}^{\infty} C_n^B \exp \left\{ -\frac{n^2 \pi^2 t}{\beta_U (\theta + 1)^2} \right\} \sin \frac{n\pi \xi}{\theta + 1} \quad (3.162)$$

$$T^U(\xi, t) = \frac{\theta(\theta + 1 - \xi)}{\theta + 1} - \sum_{n=1}^{\infty} C_n^U \exp \left\{ -\frac{n^2 \pi^2 t}{\beta_U (\theta + 1)^2} \right\} \sin \frac{n\pi \xi}{\theta + 1} \quad (3.163)$$

where C_n^U and C_n^B are defined from (3.160) and (3.161) respectively. The equations (3.162) and (3.163) are the composite asymptotic solutions for small β_U and we have a complete solution valid for all time.

3.6 Tuft breaking

This section is really an extension to the asymptotic solution of 3.5.2. We have found a small β_U solution for a single tuft held by two separating tufts. Now we proceed to consider many fibres that are being teased out of two neighbouring tufts, see figure 3.2. We make the simplifying assumptions that the withdrawing fibres do not affect the physical properties of the tufts and the speed at which the tufts are separated is slow i.e. $0 < \beta^U \ll 1$. We do expect for large n that the assumption decoupling fibre withdrawal with the physical properties of the tuft fibres will break down.

For n fibres we wish to find the force F_n required to break a tuft into two. Let us introduce a probability density function $P(\theta)$, which gives the likelihood that a fibre connecting the two tufts has the ratio lengths θ and 1 in each tuft. By using

symmetry, we can consider $\theta \in [0, 1]$, which translates to $l_B \in [0, \frac{l}{2}]$, which covers half the possibilities of fibre ratios in the bottom tuft. Therefore, we require that

$$\int_0^1 P(\theta') d\theta' = \frac{1}{2}. \quad (3.164)$$

Examples of suitable functions that could represent the distribution of n -fibres connecting the two tufts are

$$P(\theta) = \frac{1}{2}, \quad (3.165)$$

$$P(\theta) = 1 - \theta, \quad (3.166)$$

$$P(\theta) = \theta, \quad (3.167)$$

see figure 3.2. The first equation (3.165) represents the scenario that for the n fibres it is equally likely to find any ratio of lengths. The second example (3.166) considers the case where the lengths of each fibre, which connect the two separating tufts, are predominantly in one or the other. Finally equation (3.167) could be a simulation of fibres that are predominantly equally distributed in the two tufts.

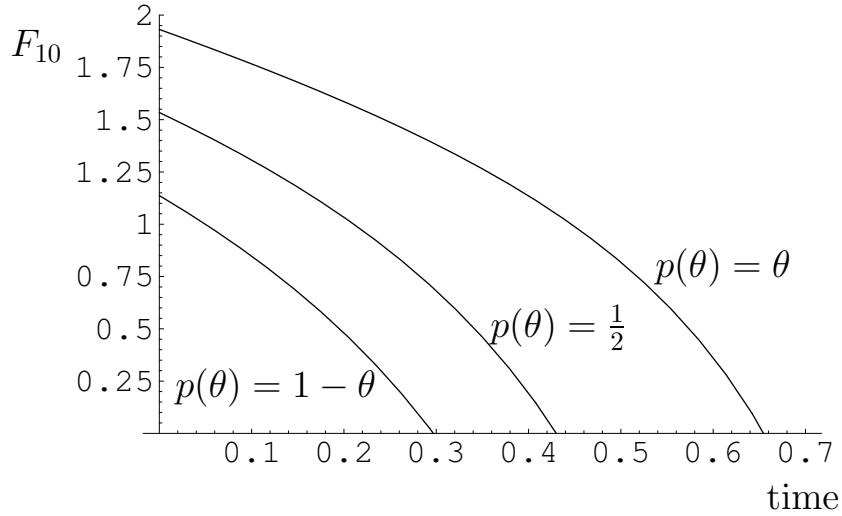


Figure 3.16: The force required to pull two tufts apart held by 10 fibres

To find the force resisting the breaking of a tuft which is being continually elongated, we need to consider the force required to pull these apart. If we consider n fibres that contribute individually with force $F(t; \theta)$, then the expected total force F_n can be found with the use of the length ratio distribution $P(\theta)$:

$$F_n = 2n \int_0^1 P(\theta) F(t, \theta) d\theta, \quad \text{whenever } F > 0. \quad (3.168)$$

For a slow withdrawal, $0 < \beta^U \ll 1$, we use F_0 from equation (3.141) to be $F(t, \theta)$ and find the following solutions illustrated in figure 3.16. The situation where the fibres tended to have equal lengths in each fibre (3.167) produced the largest resistance to breakage. The case where there were originally just fractions of each fibre in either tuft, (3.166), produced the least resistance.

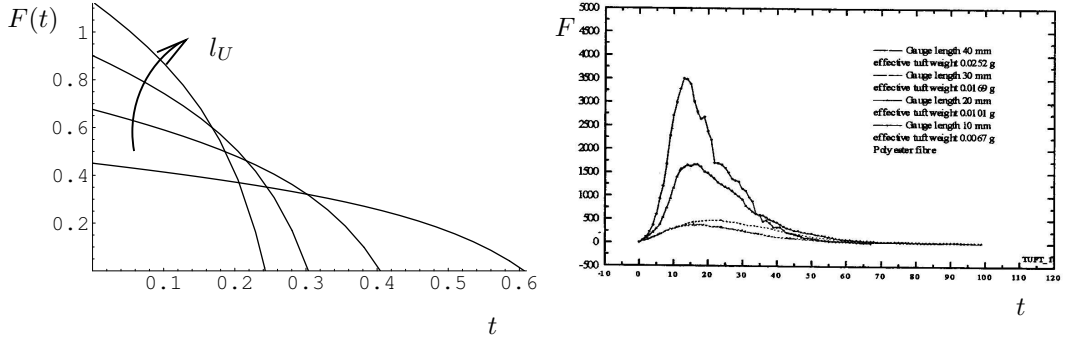


Figure 3.17: Tuft breaking for varying initial gauge lengths: a comparison with experiment; $l_U = 1, 1.5, 2, 2.5$

The plots in figure 3.16 are sensible results, as the case where the fibres produce the least resistance is when θ is small or large for most of the fibres. We can also test these results with experiments of fibre elongation. From the analysis in section 3.5.2.1, we note that our plots in 3.16 exclude a small region near $t = 0+$ of size $O(\beta^U)$ where the force increases exponentially from zero. This would make our simulation resemble the tuft elongation experiment in figure 3.17. We will go into more detail with regards to the experiment in chapter 4. The solutions are qualitatively similar for different example distributions P , unfortunately none of them compare well with the experiment. By varying the velocity, we see in figure 3.17 that the comparison is not very good, as one is convex and the other is concave. This suggests that assumptions are not valid anymore, and as the model worked well with a single fibre the problem in this model must be due to the fact that we have decoupled the withdrawal of many fibres with the tuft. Something more dramatic must occur with the overall structure of the tuft fibres and we examine this in chapter 4.

3.7 Conclusion

We started with a model for a single fibre being withdrawn from a tuft. The tuft was assumed to be homogeneous throughout, where the withdrawal did not affect the frictional properties of the tuft. Textile fibres, which are always naturally curved, are

approximated by a linearly elastic centre-line. After deriving a model that consisted of a nonlinear parabolic partial differential equation with a free boundary condition, we validated the model by testing the solutions against experiment. We then went on to consider two industrially relevant applications of the simpler single fibre withdrawal problem. The first scenario was a hook that moved through a tuft and grabbed a fibre, and the point of interest was whether or not the fibre would be withdrawn completely or partially. The second scenario went on to consider the case when a hook or a few hooks grab part of a dense tuft. Then the elongational forces break the tuft into two, where there are interconnecting fibres. Although we start with just one interconnecting fibre, we progress to consider many interconnecting fibres. The results are not completely satisfactory and this leads us on to consider how entangled fibres are broken down into individuals in the next chapter.

Chapter 4

Continuum Models for Interacting Fibres



Figure 4.1: A picture of the lap consisting of polyester fibres.

4.1 Introduction

We investigate a model for deformations in large populations of inter-connected, approximately inextensible man-made and natural fibres. In chapter 3, we assumed that the fibre extraction did not affect the tuft structure. The next step is to focus on how forces applied to a entangled body of fibres result in the disordered fibre structures being broken down either into individual fibres or into tufts in which the fibres are in a more ordered state. Teasing many fibres simultaneously or combing tufts which cause non-recoverable changes is fundamental to the carding process, and therefore understanding the transport of fibres relative to one another is crucial if one is to engage in good carding practice.

Much work on fibre structures consider elastic or visco-elastic deformations in fibre structures (Buckley, 1980, Pheonix, 1999, Phoenix, 1979). A material composed of coherent textile fibres will initially respond elastically to stress but our main focus is on a process that destroys implicit structures that allow for elastic recovery. On the other hand, visco-elastic models such as those by Maxwell and Jeffreys (Fowler, 1997) have creeping and relaxation time scales, and are not relevant in the carding machine as the relaxation time scale is negligible. Therefore we consider a novel approach that models large populations of inter-connected tufts and fibres under tension, which on a macroscopic scale will behave as a viscous material.

We begin with a simple approach that considers fibre density and velocity as the variables in a continuum model. By solving conservation equations for mass and momentum we find solutions for an elongational flow. After comparing simulations with experiment we find that except for the initial part of the experiment when the fibres are re-orientating, the forces in the model were in good agreement with experiment. Even though the model compares reasonably well with experiment for the majority of time, we look to include other variables so that we can capture the behaviour of the re-orientating “alignment phase” and this also allows us to predict how the internal structure of the lap evolves during the process. The initial “alignment phase” does not significantly change the density of the tuft or fibre structure, but it can be thought of as placing the fibres in a state of tension.

We use a vector field to describe average-fibre directionality, and to avoid ambiguity in what directionality describes in terms of fibre arrangements, we introduce an associated scalar field that measures the microscopic alignment to this direction. In the scientific theories of liquid crystals, there is a particular method to describe

the aforementioned directionality and alignment (de Gennes, 1974, Doi and Edwards, 1986, Leslie, 1968) and we gratefully adopt their approach and terminology.

The evolution of each of the new variables requires the introduction of their respective governing equations, and we also need to consider how directionality and alignment affect the material's response to stress. There are a number of theories of mathematics and theoretical physics that attempt to incorporate anisotropies, for example nematic liquid crystals or fibres in a suspension of fluid. One such example is the theory of polymers that encompasses a number of interesting modelling concepts, which may be applicable to a fibrous assembly. A possible modelling assumption we could employ is polymer reptation, which occurs when a molecular chain in a concentrated polymer solution is constrained to move strictly in a geometry that is prescribed by its neighbouring molecules. This may be relevant when extracting a single fibre but does not apply in the current situation when the whole mass of fibres is distorted by the applied forces.

Forest et al. (1997) consider how to incorporate directionality and degree of alignment into a stress tensor for a polymeric liquid crystal; although the physical properties of the material may be different there are some fundamental similarities in design. Closer to the study of fibrous materials are the mathematics applied in fluid suspensions (Hinch and Leal, 1975, 1976), fibre-reinforced fluids (Spencer, 1972, 1997) and planar constrained dense fibre arrays (Toll and Månson, 1994, 1995) and using similar ideology we conjecture a bulk stress tensor. Although there are a number of analogies with many-body fluid suspensions, there are clearly some major differences in the physics involved, such as the absence of pressure and advection in a fibre only continuum. We derive a set of coupled Eulerian field equations, which are composed of conservation laws, a kinematic condition, and a constitutive evolution equation. Comparison with experimental data shows that our model with directionality and alignment to be a good starting point for modelling entangled fibres in the carding machine. Although the qualitative behaviour is captured in our simulations we find that there are counter-intuitive problems due to the exclusion of the concept of entanglement. This leads us onto consider inter-fibre topologies within the continuous medium.

The final piece of the jigsaw is a scalar function that measures the degree of inter-fibre entanglement. Entanglement is a subject that has seen little coverage in the scientific press, and we define a method of quantifying the degree of entanglement by analysing the topology of braids. The description of braids based on group theory is somewhat sophisticated for our purposes, but it does give a complete methodology

that allows us to classify the degree of entanglement of tufts found in the carding machine. We use equivalence classes of braids and the Artin braid relations, and this allows us to use a one-to-one mapping to define entanglement as a scalar field. Incorporating the entanglement into the governing equations results in modifying the stress tensor and introducing an empirical law for its evolution. The revised theoretical simulations now compare very well with experiments, both for extensional and shearing scenarios. We also consider how to include the effects of hooks as they are dragged through the fibres. Thus we establish a promising continuum model that describes a many-fibre medium.

4.2 Experiments

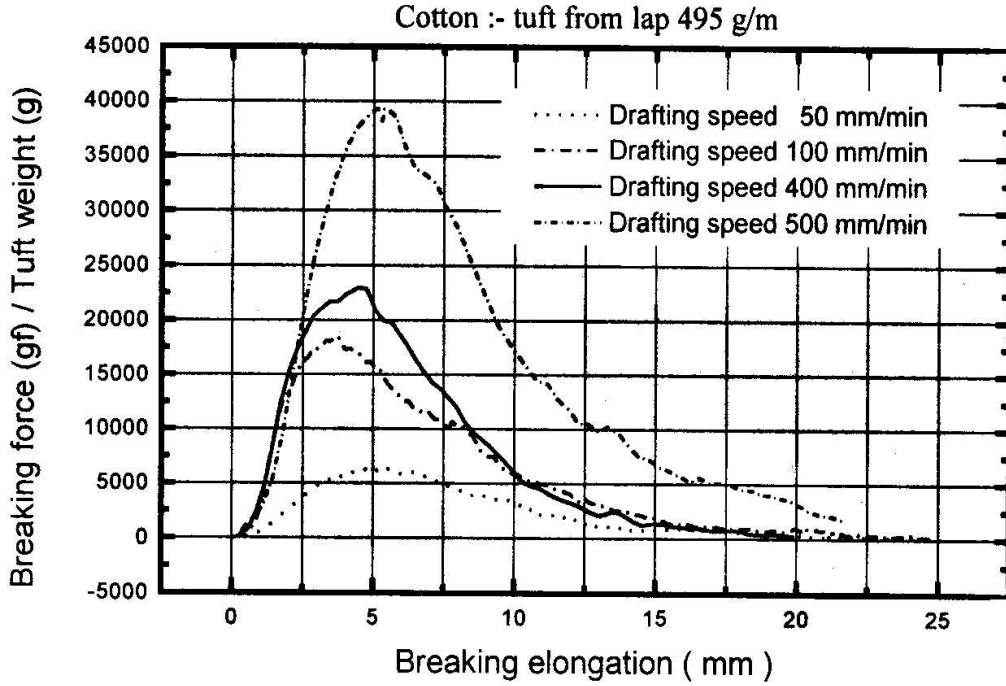
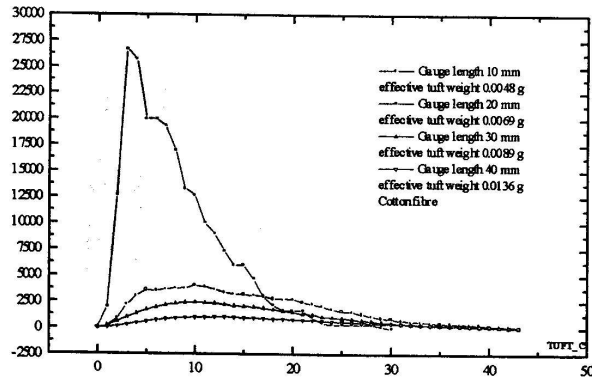
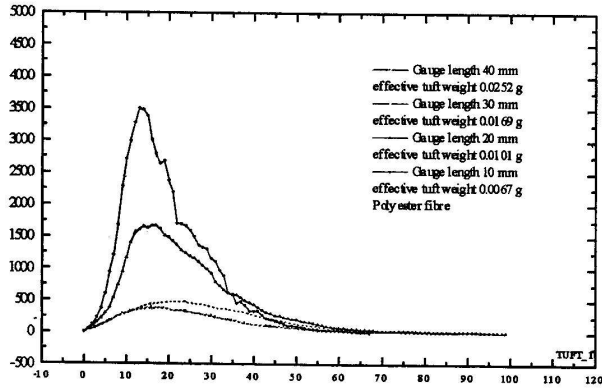


Figure 4.2: Graphs of the tuft breaking force experiment for cotton with variable elongation velocities. The initial gauge length is 20 mm.

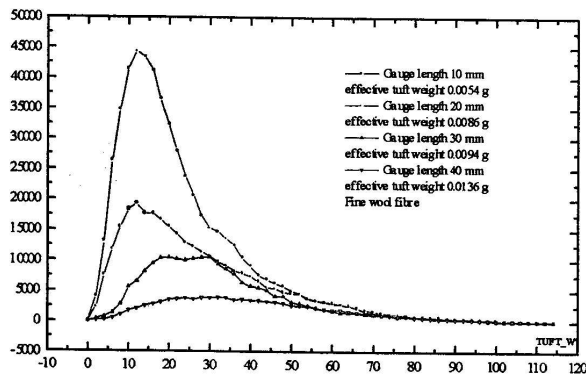
We have been collaborating with experimentalists at the School of Textile Industries, University of Leeds. In order to aid the analysis and test the validity of our models we asked the Leeds group to measure the forces required to pull apart a group of fibres, loosely labelled as a tuft. They kindly agreed to run two experiments titled “tuft breaking force” and “tuft shear force”.



(a) Cotton



(b) Polyester



(c) Wool

Figure 4.3: Graphs of the tuft breaking force experiment with variable initial tuft lengths: $\frac{force}{weight}$ plotted against extension (mm). Elongation speed of 50 mm/min.

For the first of the aforementioned experiments polyester, cotton and short fine wool tufts were tested on the Instron tensile tester at one of four different gauge lengths (10, 20, 30 and 40 mm) depending on tuft size. Care was taken to ensure that a fibre would not be attached to both clamps holding the tuft in place during the experiment. The elongational speed is 50 mm/min. The experimentalists considered the tuft to be broken once the force measured on the tensile tester had reached its maximum. Working from this definition the magnitude of the force at which tufts are broken down increase with a decrease in initial tuft length. See figure 4.3 for the results of this experiment. After further discussion this experiment was also carried out at four speeds 50, 100, 400 and 500 mm/min, see figure 4.2.

In the second experiment, tuft shear force, a tuft was placed between two wired surfaces where one traversed the other. The distance between the hooks were varied (0.007, 0.01 and 0.013 inches), as well as the relative velocity of surfaces (50, 100 and 200 mm/min). On the moving surface a tensile tester measured the forces during the motion. The shear force experiment in some senses mimics the environment of the carding machine although in reality the disparity in surface speed is usually between $O(10^4)$ and $O(10^5)$ mm/min. As in the tuft breaking experiment polyester, cotton and wool fibre tufts were used, see figures B.1, B.2, and B.3 in appendix B.

The consistency of the graphs can be called into question. We highlight two possible anomalies, one in the tuft breaking force problem with variable speeds, see figure 4.2, and the other is in the tuft shear force experiment, see B.3 in appendix B. In the tuft breaking experiment it is difficult to explain why the resistive forces for elongational speeds 100 mm/min and 400 mm/min are similar and then we observe a force that approximately doubles when the speed is increased by just 100 mm/min. For the tuft shear force experiment, there is a dramatic increase in the force and in the location of the maximum, for speeds of 100 and 200 mm/min, when the distance between surfaces decrease from 0.01 inches to 0.007. We can also see differences in the measured forces in the two similar but independent cotton experiments of figures 4.2 and 4.3, which display the measurements for cotton tufts of gauge length 20 mm elongated at speed 50 mm/min. It is difficult to replicate any experiment involving tufts that are taken from the lap in a consistent way as the tufts used must differ in terms of initial entanglement and orientation.

4.3 A Simple Viscous Model

There is considerable complexity if one attempts to create a model that incorporates every single interaction between all fibres in the lap. We have chosen a macroscopic model in an attempt to get a usable and tractable model. Alternatively we could use a microscopic approach, whereby probability density functions for microscopic scales are integrated and averaged to give macroscopic variables. To significantly improve on a continuum model, we would need to consider discrete interactions but this will dramatically increase complexity and the mathematics quickly becomes impenetrable. Furthermore, evolution of many-fibre topologies, see figure 4.7, is difficult to encapsulate succinctly consequently making the microscopic method cumbersome. For the industrialist, the process of disentangling and ordering fibres is only really important from a macroscopic point of view, as it is the sliver and its intrinsic qualities that interests the textile manufacturer. So we build a continuum model that encapsulates the essentials of the carding material without having to account for every single physical interaction. We begin the modelling process by considering a continuum model and begin with the simplest case first.

We define the Eulerian scalar function $\rho(\mathbf{x}, t)$ to be the average fibre density or volume fraction,

$$\rho = \frac{N_V \pi a^2 l}{V}, \quad (4.1)$$

where N_V is the number of fibres in volume V , a and l are average fibre radius and length respectively. The vector function $\mathbf{u}(\mathbf{x}, t)$ is the average velocity of the fibres. We make the rather crude assumption that the global behaviour of the microstructure, consisting of many interacting fibres, is that of an isotropic medium. The two factors that give rise to internal forces and structural rigidity are individual contact points between fibres and the orientation of the fibre network. We neglect aerodynamic forces, electro-static forces and gravity. These assumptions are tested by juxtaposing simulation with experiment.

4.3.1 Problem Formulation

Using conservation laws for mass and momentum (Batchelor, 1967) we write down the governing field equations, in an arbitrary domain V , for a tuft population:

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial u_k}{\partial x_k} + u_l \frac{\partial \rho}{\partial x_l} = 0, \quad (4.2)$$

$$\rho \left(\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} \right) = \frac{\partial \sigma_{il}}{\partial x_l} + f_i, \quad (4.3)$$

where we employ the summation convention, σ_{ij} is the stress tensor and f_i are the external body forces. It is necessary to impose an initial state, and boundary conditions at ∂V , if we wish to find a solution and an example of these are:

$$\left. \begin{array}{l} \mathbf{u}(\mathbf{x}, t) = \mathbf{u}_0(\mathbf{x}) \\ \rho(\mathbf{x}, t) = \rho_0(\mathbf{x}) \end{array} \right\} \text{ at } t = 0 \quad \text{with} \quad \mathbf{u}(\mathbf{x}, t) \text{ or } \sigma_{ij}n_j \text{ given on } \partial V, \quad (4.4)$$

where $\mathbf{x} = (x_1, x_2, x_3)^T$ and $\mathbf{u} = (u_1, u_2, u_3)^T$. To close the system of equations (4.2) and (4.3) it remains to relate stress with velocity, or rate-of-strain, using a constitutive assumption. The simplest approach is to use a Newtonian viscous stress tensor (Batchelor, 1967), which suggests that stress is linearly proportional to the rate-of-strain:

$$\sigma_{ij} = \frac{\mu}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \lambda \frac{\partial u_k}{\partial x_k} \delta_{ij} \quad (4.5)$$

where δ_{ij} is the Kronecker delta and μ and λ are effective viscosities which will depend on frictional forces generated by fibre-fibre contact points.

4.3.1.1 Fibre Contact Points

To complete the equations we would like to incorporate the effect of friction between fibres. We can now explore the relationship between the viscous coefficient μ in equation (4.5) and space $\mathbf{x}$, velocity $\mathbf{u}$ and density ρ . By considering two extreme situations we can gain some idea of how we can implement this into our model. When we take the limit as fibre density tends to zero we expect $\mu, \lambda \rightarrow 0$ and conversely when the density of fibres tends to unity we postulate that $\mu, \lambda \rightarrow \infty$. Assuming that a greater density implies a greater number of contact points per unit volume, then one would expect the viscous coefficient μ to depend on density (4.1). We could use a number of power laws based on intuition, but we can refine this assumption by using a probabilistic method based on the work of Komori and Makishima (1977).

Let us define a region bounded by parallel planes that intersect the centre lines of two straight fibres A and B ; the fibres have polar angles (θ, ϕ) and (θ', ϕ') , and their respective surfaces are in contact at one point. By connecting the corresponding vertices with straight line segments, we have a parallelepiped, the bases have lengths l , the average length of a fibre, see figure 4.4. The volume of the parallelepiped is

$$v_1(\theta, \phi; \theta', \phi) = 2al^2 \sin \chi, \quad (4.6)$$

where χ is the angle between two adjacent sides of the base or equivalently the angle between the axes of the fibres A and B , see figure 4.4, and this is explicitly:

$$\cos \chi = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi'). \quad (4.7)$$

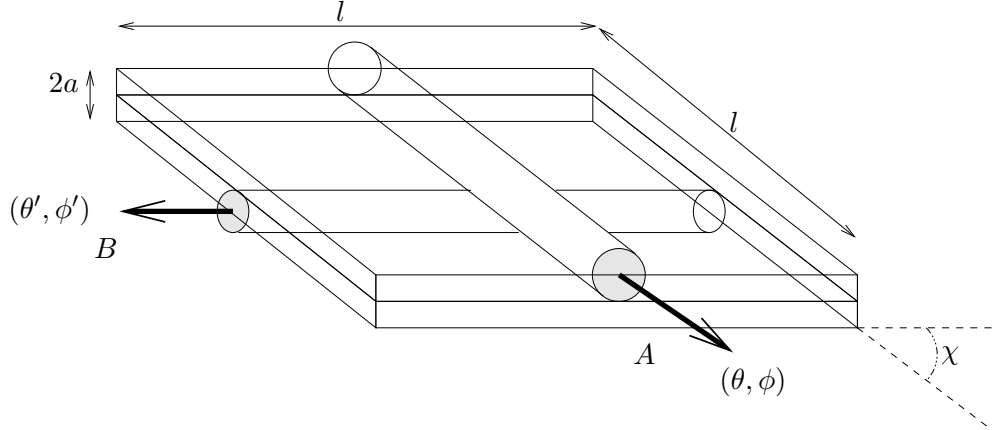


Figure 4.4: A diagram illustrating the likelihood of contact between a couple of fibres.

Whenever the centre of mass of a straight fibre with orientation (θ', ϕ') can be found in the parallelepiped described in figure 4.4 it will necessarily make contact with fibre A . Equally this would also be true if we reflected the volume through the plane intersecting A ; this means that we are concerned with a larger parallelepiped of depth $4a$, centred on A and this volume is labelled as $v = 2v_1$. We assume that the distribution of the centre of mass, of B , is random throughout the material concerned, and we also neglect the frictional forces due to the fibre ends, which is acceptable for fibres with small aspect ratio $\frac{a}{l} \ll 1$. The probability that a fibre B comes into contact with A is equivalent to the probability that the centre of mass of B can be found in the volume v :

$$P(C \subset V) = P(\theta, \phi; \theta', \phi') = \frac{v}{V} = \frac{4al^2}{V} \sin \chi, \quad (4.8)$$

where V is an arbitrary volume that contains v . When N fibres are in V in addition to fibre A , the average number of of fibre contact points or crossings on A is

$$n(\theta, \phi) = N \int_0^\pi \int_0^\pi P(\theta, \phi; \theta', \phi') \Omega(\theta', \phi') \sin \theta' d\theta' d\phi' \quad (4.9)$$

$$= \frac{4al^2 N}{V} \int_0^\pi \int_0^\pi \Omega(\theta', \phi') \sin \chi \sin \theta' d\theta' d\phi', \quad (4.10)$$

where Ω is defined such that the probability of finding a fibre with orientation $(\theta + \Delta\theta, \phi + \Delta\phi)$ as $\Delta\theta$ and $\Delta\phi \rightarrow 0$ is $\Omega(\theta, \phi) \sin \theta d\theta d\phi$. The average number of contact points for an arbitrary fibre of length l is

$$\bar{n} = \frac{4aNl^2}{V} I \quad (4.11)$$

where

$$I = \int_0^\pi \int_0^\pi \int_0^\pi \int_0^\pi \Omega(\theta', \phi') \Omega(\theta, \phi) \sin \chi \sin \theta d\theta' d\phi' d\theta d\phi. \quad (4.12)$$

As we have not incorporated a quantity that represents directionality or alignment in this model then I in equation (4.12) is constant. We do, however, go on to introduce such variables and these will be closely related to I . This means that by considering the average length of fibre in a unit volume we can predict the number of contact points per volume V , and this can be written in terms of the density as follows:

$$\nu = \frac{2aN^2l^2I}{V} = \frac{2I}{\pi^2a^3}\rho^2. \quad (4.13)$$

This result that ν is proportional to ρ^2 coincides with work by Toll (1993) and Toll and Månson (1994) and can also be generalised for non straight fibres (Komori and Makishima, 1977). We could evaluate I in a number of orientations, for example either totally random $\Omega(\theta, \phi) = \frac{1}{2\pi}$ or aligned $\Omega(0, 0) = \delta(\theta)\delta(\phi)$. Therefore we begin by modelling the viscous coefficient in (4.5) by:

$$\mu = \nu_1\rho^2 \quad \text{and} \quad \lambda = \nu_2\rho^2, \quad (4.14)$$

where ν_1 and ν_2 are constant. We note that for highly heterogeneous densities, the implicit assumption that the centre of masses of fibres in V being uniformly distributed is no longer valid, but for our purposes this assumption is adequate. Now the model is complete, though parameters ν_1 and ν_2 are still unknown.

The governing equations (4.2) - (4.3) represent a continuum of fibres and to verify the modelling assumptions made in this section we test theoretical experiment with reality. The simplest case for both experiment and mathematical solution leads us to consider the quasi one-dimensional problem of a tuft or entangled fibre-mass being elongated.

4.3.2 Dimensionless Equations

Before we go on to solve the problem (4.2), (4.3), (4.5) and (4.14), in order to give greater understanding into the interplay between their respective terms we find their dimensionless form. The velocity is scaled with a representative value of $u(\mathbf{x}_0, t)$ from the boundary conditions (4.4), U , and similarly for the density a suitable quantity from $\rho_0(\mathbf{x})$ or $\rho(\mathbf{x}_0, t)$, which we call $\tilde{\rho}$. Scaling distance with the length-scale l , the

initial size of the tuft, and time with $\frac{l}{U}$ means the equations (4.2), (4.3), (4.5) and (4.14) can be re-written as

$$\frac{\partial \bar{\rho}}{\partial t} + \bar{\rho} \frac{\partial \bar{u}_k}{\partial \bar{x}_k} + \bar{u}_l \frac{\partial \bar{\rho}}{\partial \bar{x}_l} = 0, \quad (4.15)$$

$$\bar{\rho} \left(\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_k \frac{\partial \bar{u}_i}{\partial \bar{x}_k} \right) = \Lambda \frac{\partial \bar{\sigma}_{il}}{\partial \bar{x}_l}, \quad (4.16)$$

where

$$\bar{\sigma}_{ij} = \bar{\rho}^2 \left(\bar{e}_{ij} + \frac{\nu_2}{\nu_1} \frac{\partial \bar{u}_k}{\partial \bar{x}_k} \delta_{ij} \right), \quad (4.17)$$

$$\bar{e}_{ij} = \frac{1}{2} \left(\frac{\partial \bar{u}_i}{\partial \bar{x}_j} + \frac{\partial \bar{u}_j}{\partial \bar{x}_i} \right), \quad (4.18)$$

$\bar{e}_{ij}$ is the linear rate-of-strain tensor in dimensionless form and we have assumed that there are no external body forces $f_i = 0$. The dimensionless scalar $\Lambda = \frac{\nu_1 \bar{\rho}}{U l}$ is analogous to the inverse of the Reynolds number.

To find an estimate for the size of the parameter Λ , we use the experimental data given in section 4.2, in figure 4.3. The load for cotton, with initial gauge length of 10 mm, is 132.4 grams for the point at which the extensional force is at its maximum, and the corresponding force of 1.29752 Newtons, see figure 4.3 and appendix B. When we compare this to the force generated by the fibre contact points in unidirectional extension, approximated by $F = \nu_1 \rho^2 \frac{\partial u}{\partial x} \sim \nu_1 \rho^2 \frac{U}{l}$ where density, velocity and length scale with $\rho \sim 10^2 \text{ kg m}^{-3}$, $U \sim 10^{-3} \text{ m s}^{-1}$, and length $l \sim 10^{-3} \text{ m}$ respectively, then $\nu_1 \sim 10^{-2}$ with dimensions $\text{m}^3 \text{ kg}^{-1} \text{ s}^{-1}$. This means that Λ is $O(10^5)$ for the experiment and in the carding machine itself, basing the length on the distance between cylinder surfaces $O(10^{-2})$ m/s and typical velocity as $O(1)$ m/s, is $O(10^2)$. Therefore we shall consider Λ to be large, and this means that the viscous terms dominate over the inertia terms in equation (4.16).

It must be noted that we never expect a scenario for the motion of entangled fibres to display noticeable inertia, due to the internal forces generated by fibre-fibre contact points. For the textile fibres concerned, a low density fibrous media could be modelled as hydro-dynamically dilute or semi-dilute suspension, and the model may include inertia but it would then be driven by the fluid dynamics.

4.3.3 The Extensional Simulation

A group of entangled fibres, that form a tuft, can be represented by a simply connected region in a field. The tuft, unlike a fluid, will retain its initial structural integrity

when ambient body forces are applied, such as gravity. Our microscopic structures that form the continuum are on the scale of microns, unlike conventional fluid flows that are fundamentally based on molecular interactions. We shall neglect any effects due to the tuft boundaries, such as surface tension due to the lack of any meniscus forces. Alternatively one could view this as the tuft being of infinite in width. A tuft

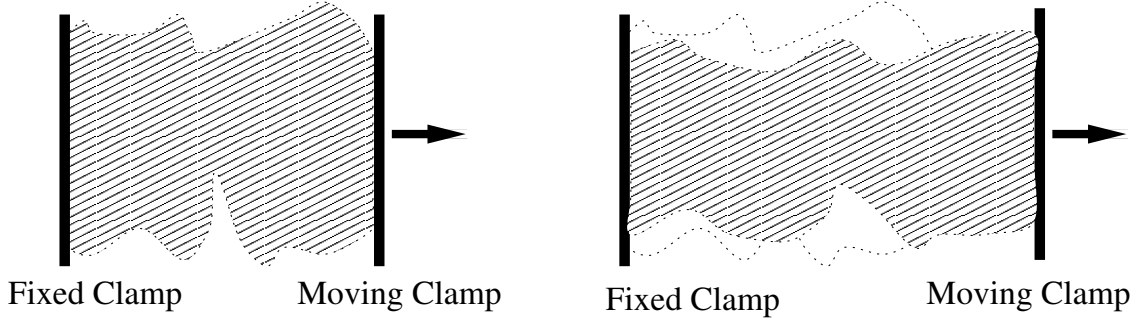


Figure 4.5: A diagram of unidirectional elongation.

of length $l(t)$ is pulled apart with force $F(t)$ and speed $U(t)$, see figure 4.5, where either the force or the rate of extension may be specified on one boundary and the other boundary is held fixed.

For the experiments the elongation speed $U(t) = U_0$ is constant throughout and the tuft length varies but has initial length l . We treat the body as quasi one-dimensional and work with one spatial dimension x and time t , which means that $\mathbf{u} = u(x, t)\mathbf{i}$ and $\rho = \rho(x, t)$. For the governing field equations we explicitly specify the scalings for the dimensionless form: distance with l , viz. $x = lx'$, velocity U with U_0 and density $\tilde{\rho}$ with the maximum initial density $\rho(x, t = 0)$. Equations (4.15) – (4.17) with $\Lambda \gg 1$, become

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0, \quad (4.19)$$

$$\frac{\partial}{\partial x} \left(\rho^2 \frac{\partial u}{\partial x} \right) = 0. \quad (4.20)$$

With the aforementioned scalings, the moving boundary has speed unity, and the initial and boundary conditions become

$$\rho = \rho_0(x) \text{ at } t = 0, \text{ and } u = 0 \text{ at } x = 0, \quad (4.21)$$

$$\rho^2 \frac{\partial u}{\partial x} = F(t) \text{ and } u = 1 \text{ at } x = 1 + t. \quad (4.22)$$

The form of (4.19) suggests that a Lagrangian formulation would be appropriate if we wish to find analytical solutions. By letting ξ represent a particular material element

throughout the motion, position is now a function of element and time, $x = X(\xi, t)$. Velocity is therefore defined as $\left. \frac{\partial X(\xi, t)}{\partial t} \right|_{\xi} = u(x, t)$. Then $X = 1 + t$ at $\xi = 1$, which is the point of the fibre continuum that is dragged away by the moving boundary. Under the transformation of variables we find that the Lagrangian formulation of the field equations (4.19) – (4.20) become:

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} = 0, \quad (4.23)$$

$$\frac{\partial \xi}{\partial x} \frac{\partial}{\partial \xi} \left(\rho^2 \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} \right) = 0. \quad (4.24)$$

The boundary conditions (4.21) and (4.22) are now:

$$\rho = \rho_0(\xi) \text{ at } t = 0, \text{ and } u = 0 \text{ at } \xi = 0, \quad (4.25)$$

$$\sigma_{11} = F(t) \text{ and } u = 1 \text{ at } \xi = 1, \quad (4.26)$$

where $\xi[X(\xi, t), t] = \xi$ and $\frac{\partial \xi}{\partial x} \frac{\partial x}{\partial \xi} = 1$. Substituting (4.23) and also using the first equation in (4.26), equation (4.24) becomes:

$$-\rho(\xi, t) \frac{\partial \rho(\xi, t)}{\partial t} = F(t). \quad (4.27)$$

A parametric solution can be found by integrating equation (4.27), and imposing the initial condition in (4.25) the solution is:

$$\rho(\xi, t) = \sqrt{\rho_0(\xi)^2 - 2 \int_0^t F(s) ds}, \quad X(\xi, t) = \int_0^\xi \frac{\rho_0(p) dp}{\sqrt{\rho_0(p)^2 - 2 \int_0^t F(s) ds}}. \quad (4.28)$$

Applying the boundary condition at $\xi = 1$ so that $X(1, t) = 1 + t$ gives the force at the moving clamp, $F(t)$ from (4.28). The case when we have a uniform initial density, $\rho_0(\xi) = 1$ is particularly straightforward to solve, and in this case:

$$u = \xi, \quad F(t) = \frac{1}{(1+t)^3} \quad \text{and} \quad \rho(t) = \frac{1}{1+t}. \quad (4.29)$$

This gives monotonic decreasing functions in time for both the withdrawal force and the density, which is similar to the analogous hydrodynamic simulation.

The dimensional force from (4.29) is $F_{dim}(t) = \frac{\nu_1 U \rho_0^2 l^2}{(l+Ut)^3}$. This agrees well with the experiment but only when the tuft breaking force has reached its maximum, see figure 4.6. If we visualise the experiment, the fibres orientate themselves to the direction of extension in the initial stage and then are pulled apart in the latter stage. We have assumed that the fibre structures have not fractured, but even if they don't fracture, as $\rho \rightarrow 0$ the continuum assumption will break down.

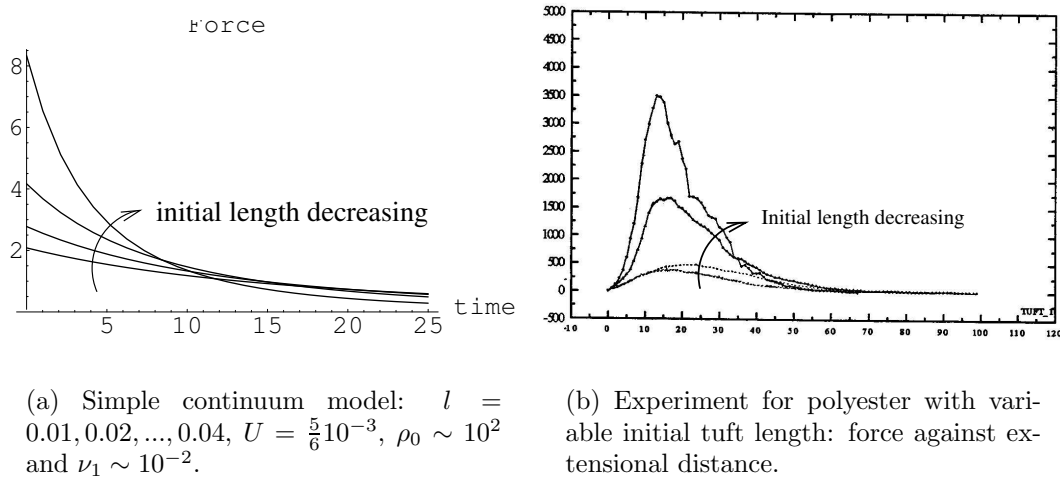


Figure 4.6: Solutions for the elongation of a tuft: comparison between experiment and the simple continuum model.

So this simple model would be sufficient to model aligned fibre arrays within the constraints of one spatial dimension. This assumption is fine because we are only considering a unidirectional flow for fibres aligned with the direction on elongation. However we need to include other mechanisms into the model that would cater for the non-monotonicity observed in the force when a tuft is pulled apart at constant velocity. We therefore review some of the areas where the effects of the directionality and anisotropy in materials are relevant, and introduce and incorporate new variables.

4.4 A Continuum Model with Direction and Alignment

The simple viscous model, see section 4.3, failed due to absence of orientation in the model. Actually, not only does directionality and the degree of alignment allow us to create a more sophisticated model but it also gives important information on the quality of carding. We consider a well combed bunch of fibres and a group of fibres orientated isotropically, and the obvious disparity in these two orientations will allow us to consider how the material will respond to stress. Therefore, in addition to the variables, density and velocity, used in the simple viscous model (section 4.3) we add two more properties from the theory of nematic Liquid Crystals (de Gennes, 1974, Leslie, 1968), namely the order parameter and the director. Although the concept of characterising groups of long slender bodies are similar, in this case there are different physics that govern the evolution of their quantities.

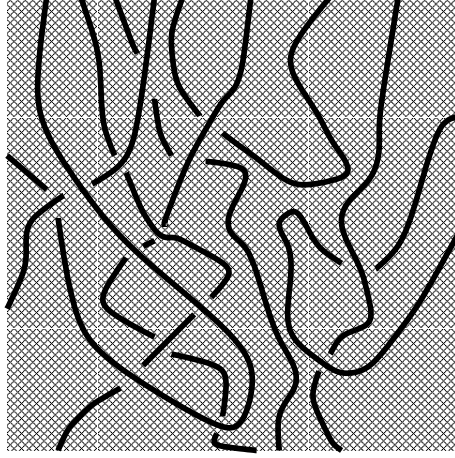


Figure 4.7: A plan view of the fibre arrangement as they enter the carding machine.

The director $\mathbf{a}$, is a unit vector and ϕ is the associated order. The director represents the macroscopic-average direction of the fibres local to a given point. A group of fibres that have the same average direction can vary in terms of the degree to which the individual fibres are aligned to the average direction. In figure 4.8, we see that a 3-fibre bundle can have the same director but the orientation is different, and so this ambiguity is eliminated by using an order parameter ϕ .

At each point we measure the fibre orientation with a given directionality. If a typical fibre-tangent lies at an angle θ relative to a given unit vector $\mathbf{a}$, see figure 4.9, the order parameter is related to the average angle θ between all the neighbouring fibres and the director, more specifically

$$\phi = \mathbf{E} \left[\frac{3 \cos^2 \theta - 1}{2} \right]. \quad (4.30)$$

where $\mathbf{E}$ is the average over all fibres in a neighbouring region. For any arbitrary unit vector $\mathbf{a}$, we can specify an order parameter for a given group of neighbouring fibres. So we specify a unique macroscopic directionality $\mathbf{a}$ to be the unit vector for which the order ϕ attains its maximum value. The limits of the order parameter suggest three distinct states: “nematic” $\phi = 1$, “isotropic” $\phi = 0$ and “planar” $\phi = -\frac{1}{2}$. An illustration of these are given in figure 4.10. The “planar” case, in figure 4.10, which is a viable form in liquid crystals but is physically unlikely in our fibre model, particularly when the material is dilated, sheared or elongated. Therefore, we could assume that the order parameter will always lie between the isotropic to random phases inclusively, i.e. $\phi \in [0, 1]$. Furthermore in this chapter we shall only consider two dimensional cases, which make the planar isotropy $\phi = -\frac{1}{2}$ redundant. We note

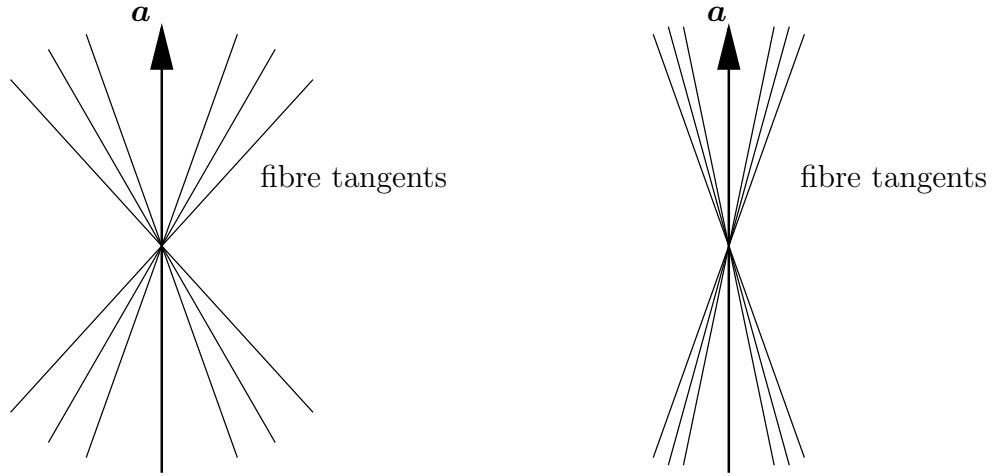


Figure 4.8: A comparison of a couple of 3-fibre bundles with the same average directionality.

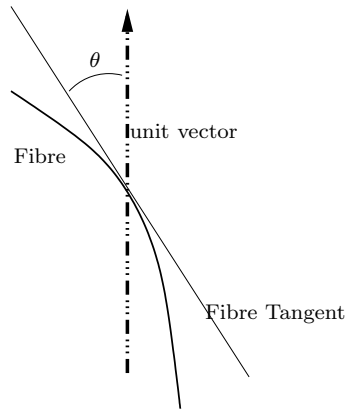


Figure 4.9: An illustration of the angle averaged for the order parameter.

that ϕ is closely related to I defined in 4.12, but it is different because the the average direction $\mathbf{a}$ is implicit in the definition of ϕ , whereas I measures the general degree of alignment.

4.4.1 The Governing Equations

At the moment, to describe the fibre medium, we have an order parameter ϕ , a director $\mathbf{a}$, density ρ and components of velocity u_i . Conservation laws for mass and momentum (4.2) and (4.3) respectively are given in the simple viscous model of section 4.3 and these are still valid in this more sophisticated model.

For the “simple viscous” model in section 4.3 we neglected inertia by assuming

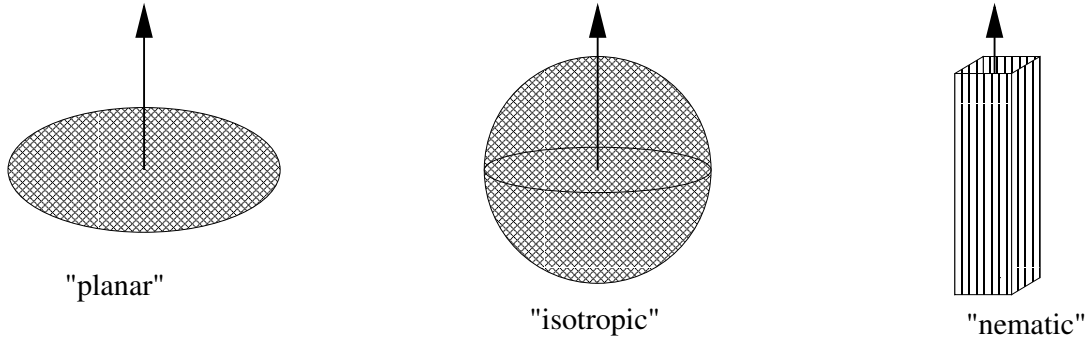


Figure 4.10: Three distinct states for liquid crystals that can be represented by the order parameter: $\phi = -\frac{1}{2}$, $\phi = 0$ and $\phi = 1$ respectively.

that $\Lambda \gg 1$ and this assumption, that the frictional forces induced by the contact points dominate over the advective properties of the material, is still valid. So with negligible inertia, conservation of momentum (4.3) becomes

$$\frac{\partial \sigma_{ik}}{\partial x_k} + f_i = 0, \quad (4.31)$$

where f_i are the components of the external body forces and σ_{ij} is the stress tensor. However we need to incorporate the effects of the new characteristic quantities, directionality and degree of alignment, into the stress-rate-of-strain relationship, and this forms a key component in our model.

4.4.1.1 The Stress Tensor

Much of the work on fibres in fluids produce a bulk stress tensor that includes a component from the fluid and a component from the fibre interactions. Usually the fluids concerned are treated as Newtonian (Batchelor, 1971, Hinch and Leal, 1976, Spencer, 1972, Toll and Månson, 1995) where the bulk stress tensor varies depending on the physical effects of the fibres concerned. Hinch and Leal (1975, 1976) derive governing equations for a material where “micro-scale” interactions are averaged into a “macro-scale” continuous media. The authors focus on particles in a Newtonian fluid suspension described by a single scalar-type function and a direction-function, which are similar to our order parameter and director. A volume average of the micro-structural state described by a probability density function N governed by the Fokker-Planck diffusion equations is related to the bulk stress consisting of a normal viscous component and a contribution due to the fibres,

$$\sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij} + 2\mu\Phi \{2A\langle a_i a_j a_k a_l \rangle e_{kl} +$$

$$2B [\langle a_i a_k e_{jk} + a_j a_k e_{ki} \rangle] + C e_{ij} + F \langle a_i a_j \rangle D \}. \quad (4.32)$$

$\langle \cdot \rangle$ denotes the average with respect to the probability density function N and A, B, C , and F are shape factors, Φ is a small volume fraction of fibres and μ the Newtonian viscosity. The terms multiplied by the volume fraction in equation (4.32), with the exception of the last term which is multiplied by the shape factor F , are the same as the stress tensor for an incompressible viscous, transversely isotropic medium. We go on to look at transverse isotropy when we derive our stress tensor and illustrate this point more clearly.

Toll and Månson (1994, 1995) are the only authors that focus on viscous deformations in fibre suspensions where the fibre-fibre contact points make a significant contribution to the bulk stress, although there are numerous articles that study elastic and visco-elastic deformations such as (Phoenix, 1999, Phoenix, 1979) and (Buckley, 1980) respectively. An analytical form of stress is found where linear friction dominates over hydrodynamic effects and the cases of nearly random orientation, shearing and extensional flows are given. This work is the closest study to that of the carding machine, but the body of fibres we are interested in do not have any internal or external forces keeping an array of fibres approximately planar. Axial fibre rotations under planar compression form the distinctive part of this model.

Spencer (1972) examines fibre-reinforced materials. Similar to the work by Hinch and Leal (1975, 1976) the stress tensor has two components, a reaction stress due to the constraints and a remainder stress which is dependent on the chosen material. In general, dynamic cases involve a reaction stress that is dependent on strain-rates and for elastic fibres a remainder stress that is proportional to strain. None of the cases studied focussed on the effect of interacting particles which play an important role in the carding process, but we do adopt a similar ethos in creating our model, with bulk stress consisting of isotropic and anisotropic components and the use of transverse isotropic symmetries in our continuum model.

The stress tensor for an anisotropic material needs special consideration. We write the bulk stress tensor as two components, and these are derived from considering the two distinct fibre orientations, random and nematic. If the material is in the nematic phase, see figure 4.11, where all the fibres are theoretically aligned, we expect a line of symmetry about the direction of alignment, and the stress tensor should also be independent of a reflection through a plane transverse to this line. This physical phenomena is called transverse isotropy and it is well documented for linear elasticity (Green and Zerna, 1968, Love, 1927, Spencer, 1972). When the fibres are randomly

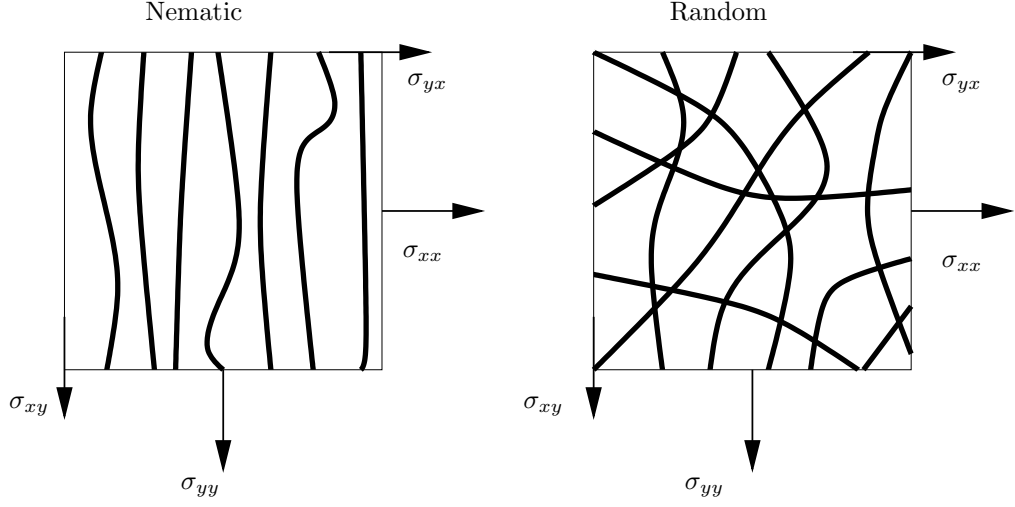


Figure 4.11: A diagram of the stresses acting on a nematic body of fibres and a randomly orientated body of fibres.

arranged, the medium will respond as an isotropic medium, and this has already been described in section 4.3.

To derive a transversely-isotropic stress tensor σ^N , with axial symmetry about $\mathbf{a}$, we begin by using a similar methodology illustrated in Love (1927) where the director is treated as constant, $\mathbf{a} = (0, 0, 1)^T$, but for a viscous material, which assumes that stresses are related linearly to strain-rates. The stress tensor is then generalised for an arbitrary directionality.

By representing stress and strain rates with six-vectors, three components for dilation and three for torsion, we can write:

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{pmatrix} = \mathbf{C} \begin{pmatrix} u_x \\ v_y \\ w_z \\ v_z + w_y \\ w_x + u_z \\ u_y + v_x \end{pmatrix} \quad (4.33)$$

where $\mathbf{C}$ is a square symmetric six by six matrix. To deduce the form of $\mathbf{C}$ we need to ensure that the strain-rate energy function W , which has the form

$$\begin{aligned} 2W = & c_{11}e_{xx}^2 + c_{12}e_{xx}e_{yy} + c_{13}e_{xx}e_{zz} + c_{14}e_{xx}e_{yz} + c_{15}e_{xx}e_{zx} + c_{16}e_{xx}e_{xy} \\ & + c_{22}e_{yy}^2 + c_{23}e_{yy}e_{zz} + c_{24}e_{yy}e_{yz} + c_{25}e_{yy}e_{zx} + c_{26}e_{yy}e_{xy} \\ & + c_{33}e_{zz}^2 + c_{34}e_{zz}e_{yz} + c_{35}e_{zz}e_{zx} + c_{36}e_{zz}e_{xy} \\ & + c_{44}e_{yz}^2 + c_{45}e_{yz}e_{zx} + c_{46}e_{yz}e_{xy} \end{aligned}$$

$$\begin{aligned}
& + c_{55}e_{zx}^2 + c_{56}e_{zx}e_{xy} \\
& + c_{66}e_{xy}^2, \quad (4.34)
\end{aligned}$$

remains unchanged under rotations within and reflections through the transverse plane perpendicular to $\mathbf{a} = (0, 0, 1)^T$. In (4.34), e_{ij} is the rate-of-strain tensor given in (4.18) and c_{ij} are components of the matrix $\mathbf{C}$. The reflection through the plane perpendicular to $\mathbf{a}$ takes the form

$$x' = x; \quad y' = y \quad \text{and} \quad z' = -z. \quad (4.35)$$

This will leave the energy function (4.34) unchanged only if

$$c_{14} = c_{15} = c_{24} = c_{25} = c_{34} = c_{35} = c_{46} = c_{56} = 0. \quad (4.36)$$

Similarly if we rotate about z by an angle ψ then we write

$$x' = x \cos \psi + y \sin \psi; \quad y' = -x \sin \psi + y \cos \psi; \quad z' = z. \quad (4.37)$$

By using the chain rule one can write the new rate-of-strain terms:

$$e_{x'x'} = e_{xx} \cos^2 \psi + e_{yy} \sin^2 \psi + e_{xy} \sin \psi \cos \psi, \quad (4.38)$$

$$e_{y'y'} = e_{xx} \sin^2 \psi + e_{yy} \cos^2 \psi - e_{xy} \sin \psi \cos \psi, \quad (4.39)$$

$$e_{z'z'} = e_{zz}, \quad (4.40)$$

$$e_{y'z'} = e_{yz} \cos \psi - e_{zx} \sin \psi, \quad (4.41)$$

$$e_{z'x'} = e_{zx} \cos \psi + e_{yz} \sin \psi, \quad (4.42)$$

$$e_{x'y'} = e_{xy}(\cos^2 \psi - \sin^2 \psi) - 2 \cos \psi \sin \psi (e_{xx} - e_{yy}). \quad (4.43)$$

The equations (4.38) – (4.43) can be used to express the old strain-rates in vector form, $\mathbf{e}$, in terms of the new strain-rates $\mathbf{e}' = \mathbf{R}_e \mathbf{e}$, where $\mathbf{R}_e$ is a six by six matrix. Similarly, the rotation also means that the stress vector $\boldsymbol{\sigma}$ can be related linearly with the new variable, $\boldsymbol{\sigma}' = \mathbf{R}_s \boldsymbol{\sigma}$, where $\mathbf{R}_s$ is a six by six matrix. After the rotation we expect $\boldsymbol{\sigma}' = \mathbf{C} \mathbf{e}'$ as it would be inconsistent if the coefficients in $\mathbf{C}$ changed, therefore we find that

$$\mathbf{C}' = \mathbf{R}_s^{-1} \mathbf{C} \mathbf{R}_e. \quad (4.44)$$

This gives six dependent equations for $c_{11}, c_{12}, c_{13}, c_{16}, c_{22}, c_{23}, c_{26}, c_{36}, c_{44}, c_{45}, c_{55}$ and c_{66} . Further algebraic manipulations give the following simplifications:

$$c_{11} = c_{22}, \quad c_{66} = \frac{c_{11} - c_{12}}{2}, \quad c_{13} = c_{23}, \quad c_{44} = c_{55} \quad \text{and} \quad c_{33} \neq 0 \quad (4.45)$$

$$c_{14} = c_{15} = c_{16} = c_{24} = c_{25} = c_{26} = c_{34} = c_{35} = c_{36} = c_{45} = c_{46} = c_{56} = 0. \quad (4.46)$$

An alternative to this method has been given by Spencer (1972), which again considers an elastic medium, but uses linear algebra, namely the Cayley-Hamilton Theorem. Both methods produce the same result, that is five non-zero coefficients, in a relatively sparse block-diagonal matrix:

$$\mathbf{C} = \begin{pmatrix} A & A-2N & 2F & 0 & 0 & 0 \\ A-2N & A & 2F & 0 & 0 & 0 \\ 2F & 2F & C & 0 & 0 & 0 \\ 0 & 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & N \end{pmatrix}, \quad (4.47)$$

where $c_{11} = A$, $c_{13} = 2F$, $c_{33} = C$, $c_{44} = L$, and $c_{66} = N$.

We need to generalise this for any director $\mathbf{a}$, and do this using a tensor method. The stress vector $\boldsymbol{\sigma}$ and strain-rate vector $\mathbf{e}$ can be re-written using tensor notation σ_{ij} and e_{ij} respectively and the relating coefficients found in $\mathbf{C}$ from (4.47) can now be found in a fourth order tensor B_{ijrs} , and this is written as

$$\sigma_{ij} = B_{ijrs}e_{rs}. \quad (4.48)$$

As both stress and rate-of-strain are symmetric, and this imposes the conditions,

$$B_{ijrs} = B_{jirs} = B_{ijsr} = B_{jisr}. \quad (4.49)$$

For transverse isotropy, most of the elements of the four-tensor, B_{ijrs} , are zero but the non-zero elements equate to the non-zeros in the matrix equation (4.47) as follows:

$$\left. \begin{aligned} B_{1111} &= \frac{A}{2}, & B_{1122} &= \frac{A-2N}{2}, & B_{1133} &= F, \\ B_{2211} &= \frac{A-2N}{2}, & B_{2222} &= \frac{A}{2}, & B_{2233} &= F, \\ B_{3311} &= F, & B_{3322} &= F, & B_{3333} &= \frac{C}{2}, \\ B_{1212} &= N, & B_{1313} &= L, & B_{2323} &= L. \end{aligned} \right\} \quad (4.50)$$

With a little foresight we rearrange the constants as follows

$$\begin{aligned} B_{2211} &= B_{1122} = \frac{\lambda_2}{2}, & B_{1212} &= \lambda_1, \\ B_{3311} &= B_{3322} = B_{1133} = B_{2233} = \frac{\lambda_2}{2} + \lambda_4, & B_{1313} &= B_{2323} = \lambda_1 + \lambda_3, \\ B_{3333} &= \lambda_1 + \frac{\lambda_2}{2} + 2\lambda_3 + 2\lambda_4 + \lambda_5, & B_{1111} &= B_{2222} = \lambda_1 + \frac{\lambda_2}{2} \end{aligned} \quad (4.51)$$

and so this gives the form

$$\begin{aligned} B_{ijrs} = & \lambda_1 \frac{\delta_{ir}\delta_{js} + \delta_{is}\delta_{jr}}{2} + \lambda_2 \frac{\delta_{ij}\delta_{rs}}{2} \\ & + \lambda_3 \frac{(\delta_{j3}\delta_{r3}\delta_{is} + \delta_{i3}\delta_{s3}\delta_{jr} + \delta_{i3}\delta_{r3}\delta_{js} + \delta_{j3}\delta_{s3}\delta_{ir})}{2} \\ & + \lambda_4(\delta_{i3}\delta_{j3}\delta_{rs} + \delta_{ij}\delta_{r3}\delta_{s3}) + \lambda_5\delta_{i3}\delta_{j3}\delta_{r3}\delta_{s3}. \end{aligned} \quad (4.52)$$

So far our analysis is valid when $\mathbf{a} = (0, 0, 1)^T$, and now we proceed to find the formula for a general transverse isotropic media with director $\mathbf{a} = (a_1, a_2, a_3)$. We rotate the specified director $(0, 0, 1)$ with an orthonormal matrix t_{ij} so $a_i = t_{ij}(0, 0, 1) = t_{i3}$, then

$$\sigma'_{\alpha\beta} = t_{\alpha i} t_{\beta j} t_{kr} t_{ls} B_{ijrs} e'_{kl}. \quad (4.53)$$

Now for the third term in (4.52) we find that

$$\begin{aligned} & t_{\alpha i} t_{\beta j} t_{kr} t_{ls} (\delta_{j3} \delta_{r3} \delta_{is} + \delta_{i3} \delta_{s3} \delta_{jr} + \delta_{i3} \delta_{r3} \delta_{js} + \delta_{j3} \delta_{s3} \delta_{ir}) \\ &= a_k a_\beta \delta_{\alpha l} + a_\alpha a_l \delta_{\beta k} + a_\alpha a_k \delta_{\beta l} + a_\beta a_l \delta_{kl}, \end{aligned} \quad (4.54)$$

for the fourth term

$$t_{\alpha i} t_{\beta j} t_{kr} t_{ls} (\delta_{i3} \delta_{j3} \delta_{rs} + \delta_{ij} \delta_{r3} \delta_{s3}) = a_\alpha a_\beta \delta_{kl} + a_k a_l \delta_{\alpha\beta}, \quad (4.55)$$

and the final term becomes

$$t_{\alpha i} t_{\beta j} t_{kr} t_{ls} \delta_{i3} \delta_{j3} \delta_{r3} \delta_{s3} = a_\alpha a_\beta a_k a_l. \quad (4.56)$$

Thence dropping dashes we find the stress tensor for a general transversely isotropic material

$$\begin{aligned} \sigma_{\alpha\beta}^N = & \lambda_1 e_{\alpha\beta} + \lambda_2 \frac{e_{kk}}{2} \delta_{\alpha\beta} + \lambda_3 (a_\alpha a_i e_{\beta i} + a_\beta a_i e_{i\alpha}) \\ & \lambda_4 (a_\alpha a_\beta e_{kk} + \delta_{\alpha\beta} a_k a_l e_{kl}) + \lambda_5 a_\alpha a_\beta a_k a_l e_{kl}, \end{aligned} \quad (4.57)$$

where we have renamed the stress σ which has transverse isotropic symmetries as σ_{ij}^N .

If the material is completely aligned, as in the “nematic phase” in figure 4.11, the stress tensor should have the form given in (4.57). Notice that if we were to solve the elongational simulation in section 4.3.3 with the stress tensor (4.57) instead of equation (4.5), for $\mathbf{a} = (1, 0, 0)^T$, the results would be very similar.

The other distinct orientation of the fibre material is the random phase $\phi \sim 0$, and we expect a different response such as that of an isotropic medium.

$$\sigma_{\alpha\beta}^I = \mu_1 e_{\alpha\beta} + \mu_2 e_{kk} \delta_{\alpha\beta}, \quad (4.58)$$

and this is the same as (4.5).

Adopting a similar approach to Hinch and Leal (1975, 1976) and Spencer (1972), we break down the stress into two components, which we label nematic and isotropic. The bulk stress should behave like (4.57) when $\phi \rightarrow 1$ and when the material is in

the random phase $\phi = 0$ we expect (4.58) to dominate. A general form for bulk stress can be written as

$$\boldsymbol{\sigma} = f(\phi)\boldsymbol{\sigma}_{ij}^N + g(\phi)\boldsymbol{\sigma}_{ij}^I, \quad (4.59)$$

where g and f can take a number of forms. A simple definition that reflects the difference between the cases $\phi = 1$ and $\phi = 0$ is

$$g(\phi) = (1 - \phi)^m \quad \text{and} \quad f(\phi) = \phi^n. \quad (4.60)$$

We will begin with the simplest case first and use a linear relation, $m = n = 1$.

Finally we need to approximate how the viscous coefficients are dependent on the frictional contact points. As in the case for the simple viscous model of section 4.3 we assume that the coefficients are dependent on the square of density. We write each coefficient as follows:

$$\mu_i = \rho^p \bar{\mu}_i \quad \text{and} \quad \lambda_i = \rho^q \bar{\lambda}_i, \quad (4.61)$$

where p and q are positive constants, and again for simplicity we start our analysis with $p = q = 2$.

The stress tensor for a dilute fibre suspension will have the form (4.32) whereas our fibre continuum has the form (4.59). There are bound to be differences as the presence of the fluid is significant, where one would expect to include pressure which is due to the fluid. Spencer (1997) and Hinch and Leal (1975, 1976) both incorporate pressure into the representative bulk stress. In our stress tensor, unlike (4.32), $\mathbf{u}$ is the velocity of the fibres not the fluid. Although our fibres are surrounded by air we assume that the aerodynamic contribution to the stress tensor is negligible, our stress is dominated by fibre-fibre interactions. There are two other differences in the stress given in (4.59) and (4.32), and they arise from the fluid velocity field being incompressible whereas our fibrous material is compressible and the fibrous viscous coefficients being dependent on the number of contact points.

Now we have completed the conservation of momentum equation by defining the stress (4.59) with viscous coefficients defined by (4.60) and (4.61). We are left with the task of finding evolution equations for the director and order. These equations are found in the form of a kinematic condition and an empirical law.

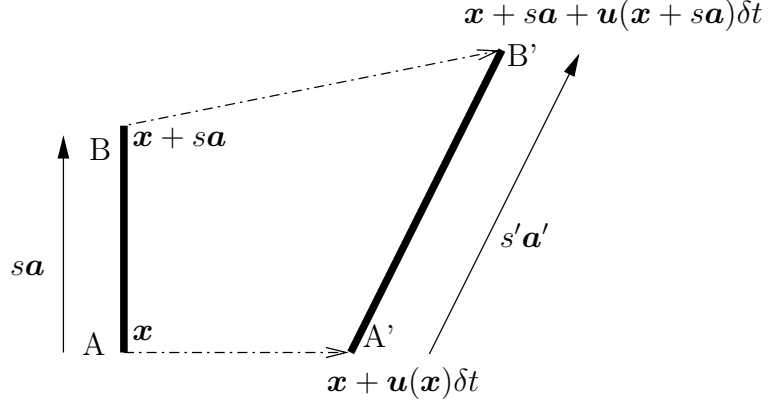


Figure 4.12: A diagram illustrating the evolution of the director using kinematics.

4.4.2 Kinematic Condition

The motion of the fibres will govern the evolution of their directionality, and therefore we derive a governing equation for the director $\mathbf{a}$. Consider an arbitrary element evolving in time where $A \rightarrow A'$ and $B \rightarrow B'$, illustrated in figure 4.12, where δt is a small time and s is a small length scale along $\mathbf{a}$. Applying elementary kinematics we have the following equation,

$$s'\mathbf{a}' = s\mathbf{a} + s\delta t(\mathbf{a} \cdot \nabla)\mathbf{u} + O(s^2). \quad (4.62)$$

Now we take the limit as $\delta t \rightarrow 0$ and ignoring $O(s)$ terms, we find

$$\frac{\partial \mathbf{a}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{a} + \frac{\dot{s}}{s}\mathbf{a} = (\mathbf{a} \cdot \nabla)\mathbf{u}. \quad (4.63)$$

where $\dot{}$ is full derivative with respect to time. In equation (4.62), we have introduced a new variable s which represents the length of the element in direction $\mathbf{a}$, and $\frac{\dot{s}}{s}$ which represents the rate of extension of a material element. The concept of extensible fibres and the inclusion of compressibility make this model unique amongst those that model viscous deformations in fibre suspensions. In the model of fibre reinforced fluids by Spencer (1972), the fibres are treated as inextensible, and they could represent say hairs moving in the wind. Our many-fibres' extensibility will be represented by the director $\mathbf{a}$ and length s , and when the fibres are moved by a non-uniform velocity field they may be susceptible to viscous deformations, where fibres slide over one another. This, then, is the crux of our model: fibres slipping over one another. As the macroscopic continuous medium represents the average of many fibres, this produces the effect that the macroscopic material elements are extensible. The actual

director can be represented by a unit vector

$$a_k a_k = 1, \quad (4.64)$$

as s takes care of the vector length and $\dot{s}$ its respective evolution. Consequently we can eliminate the new variables s if we so desire by using (4.64).

We can contrast the kinematic condition (4.63) with a director for a dilute fibre-fluid suspension that is driven by the fluid flow (Hinch and Leal, 1975, 1976). When placed in a time-dependent linear flow $\mathbf{u}(\mathbf{x}, t) = \mathbf{\Gamma}(t) \cdot \mathbf{x}$ with $\mathbf{\Gamma} = \mathbf{E} + \mathbf{\Omega}$, $\mathbf{E} = \mathbf{E}^T$ and $\mathbf{\Omega} = -\mathbf{\Omega}^T$, symmetric and anti-symmetric tensors, a particle aligned in the direction of the unit vector $\mathbf{a}$ rotates according to

$$\dot{\mathbf{a}} = \mathbf{\Omega} \cdot \mathbf{a} + G [\mathbf{E} \cdot \mathbf{a} - \mathbf{a} (\mathbf{a} \cdot \mathbf{E} \cdot \mathbf{a})]. \quad (4.65)$$

G represents a shape factor modulus. This means that the evolution of the fibres are driven purely by the external fluid flow, whereas our director evolves according to the average fibre velocity (4.63).

4.4.3 Empirical Law for the Order Parameter

Finally we need to find one more evolution equation for the order parameter to close the system given by (4.2), (4.31), (4.59), (4.60), (4.61), (4.63) and (4.64). There are a number of scenarios that are possible but we will focus on the effect of density reducing, elongational and extensional flows. We attempt to encapsulate the leading order behaviour to get a usable equation, which is based on simple empirical evidence.

If one strains a bunch of entangled fibres, we can see two phases of interaction: the first involves the orientation of fibres in the direction of the applied force and in the second phase the fibres begin to slip over one another. This observation cannot be made categorically for all fibre types and length, nor do we claim that the two phases are distinct and exclusive processes, but just the dominant qualitative behaviour. So we conjecture the evolution of the order parameter to be a function of the rate of extension of the material element $\frac{\dot{s}}{s}$, and to begin with we use a linear relationship. Constraining the order parameter to be between zero and unity, where for extensional or shearing flows we expect ϕ to be monotonic increasing, we postulate:

$$\frac{\partial \phi}{\partial t} + (\mathbf{u} \cdot \nabla) \phi = \beta(1 - \phi) \frac{1}{s} \left[\frac{\partial s}{\partial t} + (\mathbf{u} \cdot \nabla) s \right], \quad (4.66)$$

where β is the rate at which the order increases compared to the rate of extension, $\frac{\dot{s}}{s}$. We could have used other polynomial functions of ϕ on the right hand side of equation

4.66, but we found when comparing the experimental results of section 4.2 with other polynomials, $1 - \phi$ is a good starting point. We re-write (4.66) by eliminating s by using (4.63) with (4.64)

$$\frac{\partial \phi}{\partial t} + u_k \frac{\partial \phi}{\partial x_k} = \beta(1 - \phi) a_k a_l \frac{\partial u_k}{\partial x_l}. \quad (4.67)$$

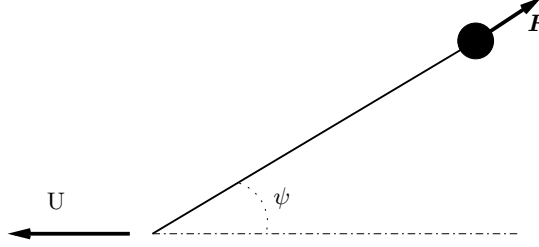


Figure 4.13: A diagram that illustrates the evolution of the order parameter with a linearly damped rod.

To support the conjectured evolution equation (4.66), we consider a simple case of a string, see figure 4.13. A string at one end moves at a constant velocity U , and the end of the string is constrained by linear damping. Without loss of generality we consider planar motion, and so it suffices to define the order parameter as $\phi = \cos \psi$ instead of using (4.30). On neglecting inertia the order then evolves as $\frac{d\phi}{dt} = \frac{U}{l}(1 - \phi^2)$, where l is the length of the string. This bears a fair resemblance to (4.66) in terms of dimensions.

4.4.4 The Two Dimensional Equations

Now there are nine unknown quantities: three components of directionality $\mathbf{a}$, an order parameter ϕ , three components of velocity $\mathbf{u}$, average density ρ , and the extension of directional elements s . There are also nine equations: (4.2), (4.31), (4.63), (4.64), and (4.66). We eliminate the extension s , by using the fact that $\mathbf{a}$ is a unit vector, from (4.64), and write down the two dimensional equations:

$$\rho_t + (\rho u)_x + (\rho v)_y = 0, \quad (4.68)$$

$$\theta_t + u\theta_x + v\theta_y = \cos^2 \theta v_x - \sin^2 \theta u_y + \cos \theta \sin \theta (v_y - u_x), \quad (4.69)$$

$$\phi_t + u\phi_x + v\phi_y = \beta(1 - \phi) [\cos^2 \theta u_x + \sin^2 \theta v_y + \sin \theta \cos \theta (v_x + u_y)], \quad (4.70)$$

$$\sigma_{11,x} + \sigma_{12,y} = 0, \quad (4.71)$$

$$\sigma_{21,x} + \sigma_{22,y} = 0, \quad (4.72)$$

where subscripts x , y and t represent partial derivatives $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial t}$ respectively and

$$\sigma_{\alpha\beta} = \sigma_{\alpha\beta}^I + \sigma_{\alpha\beta}^N, \quad (4.73)$$

$$\frac{\sigma_{\alpha\beta}^I}{\rho^2(1-\phi)} = \bar{\mu}_1 e_{\alpha\beta} + \frac{\bar{\mu}_2}{2} e_{kk} \delta_{\alpha\beta}, \quad (4.74)$$

$$\begin{aligned} \frac{\sigma_{\alpha\beta}^N}{\rho^2\phi} = & \bar{\lambda}_1 e_{\alpha\beta} + \bar{\lambda}_2 \frac{e_{kk}}{2} \delta_{\alpha\beta} + \bar{\lambda}_3 (a_\alpha a_i e_{\beta i} + a_\beta a_i e_{i\alpha}) \\ & + \bar{\lambda}_4 (a_\alpha a_\beta e_{kk} + \delta_{\alpha\beta} a_k a_l e_{kl}) + \bar{\lambda}_5 a_\alpha a_\beta a_k a_l e_{kl}, \end{aligned} \quad (4.75)$$

with $a_1 = \cos \theta$ and $a_2 = \sin \theta$, and u, v are the components of the velocity $\mathbf{u}$. Certainly for the radial or azimuthal flow geometries found in carding machines these equations are perfectly adequate.

Now we look to test the proposed continuum model against experiment. There are two relevant experiments described in section 4.2: tuft breakage force and tuft shear force. Up till now we have been rather vague with regard to prescribing boundary conditions for the field equations, but now they will play an essential part in modelling the physics of the problem, in particular, the tuft shear force comparison where two possible mathematical approaches are tested. Firstly, we will begin by testing the model against the tuft breakage experiment as it is in some senses the simplest case before we can progress onto more complicated scenarios. We shall actually find that the description we have so far is not quite adequate.

4.4.5 Elongation of a Fibrous Mass

We expect the initial tuft, or population of tufts, to be approximately randomly orientated. In terms of the order parameter this means that $\phi \sim 0$. We set this problem up in exactly the same way as in section 4.3.3, where there are two parallel clamps holding the material that is separated initially by a prescribed distance. We measure the force as one clamp moves away from the other at a uniform speed, see figure 4.5.

4.4.5.1 The Governing Equations

We note that the velocities in the experiment are considerably slower than carding machine speeds, but still we expect the elastic response of the model to be a secondary effect. The problem can be simplified by constraining the motion to be in one direction, so that $v = 0$ and the functions depend on x and t only.

Before continuing we can introduce a further constraint by considering the two equations for stress and the divergence of the stress tensor, (4.73)–(4.75),

$$\sigma_{11} = \rho^2 \frac{\partial u}{\partial x} [A(1 - \phi) + \phi (B + 2C \cos^2 \theta + D \cos^4 \theta)], \quad (4.76)$$

$$\sigma_{12} = \rho^2 \phi \frac{\partial u}{\partial x} [C \cos \theta \sin \theta + D \cos^3 \theta \sin \theta], \quad (4.77)$$

$$\frac{\partial \sigma_{11}}{\partial x} = \frac{\partial \sigma_{12}}{\partial x} = 0, \quad (4.78)$$

where $A = \bar{\mu}_1 + \bar{\mu}_2$, $B = \bar{\lambda}_1 + \bar{\lambda}_2$, $C = \bar{\lambda}_3 + \bar{\lambda}_4$ and $D = \bar{\lambda}_5$. There is nothing to constrain the motion transverse to the direction of elongation so we need to ensure that $\sigma_{12} = 0$. In fact if we consider (4.77) the only sensible problem would be to choose

$$\theta \equiv 0. \quad (4.79)$$

For the case where $\theta \equiv \frac{\pi}{2}$, the macroscopic direction of the fibres should eventually align themselves to the direction of the applied force and if we consider (4.69), a unidirectional solution would allow the appropriate evolution in $\mathbf{a}$. As long as the initial order of the media is approximately isotropic, imposing (4.79) does not negate the applicability of the model.

From the governing two dimensional equations, (4.68) – (4.75), conservation of mass and the evolution of the order parameter simplify dramatically to give:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0, \quad (4.80)$$

$$\frac{\partial \phi}{\partial t} + u \frac{\partial \phi}{\partial x} = \beta(1 - \phi) \frac{\partial u}{\partial x}, \quad (4.81)$$

and two equations from the divergence of the stress tensor reduce to

$$\sigma_{11} = \rho^2 \frac{\partial u}{\partial x} [A(1 - \phi) + E\phi], \quad (4.82)$$

$$\frac{\partial \sigma_{11}}{\partial x} = 0, \quad (4.83)$$

where $A = \bar{\mu}_1 + \bar{\mu}_2$ and $E = \bar{\lambda}_1 + \bar{\lambda}_2 + 2\bar{\lambda}_3 + 2\bar{\lambda}_4 + \bar{\lambda}_5$.

4.4.5.2 Boundary Conditions

At the two solid boundaries we impose no-slip velocity conditions, where one boundary, $x = l(t)$, moves at a prescribed speed and the other $x = 0$ is fixed, so

$$u = 0 \text{ at } x = 0, \quad (4.84)$$

$$\text{and } u = U \text{ at } x = l(t). \quad (4.85)$$

Initial conditions are then set for density and order:

$$\rho(x, t = 0) = \rho_0(x) \quad \text{and} \quad \phi(x, t = 0) = \phi_0(x). \quad (4.86)$$

4.4.5.3 Dimensionless Lagrangian Formulation

We apply the same methodology for solving this problem as we did in the simple viscous model of section 4.3, as we have similar equations. So we reformulate the problem using Lagrangian variables. This means that position can now be written as a function of time and particular element, where $X(\xi, 0) = x$. We create dimensionless equations using the scalings $t = \frac{l(0)}{U}\bar{t}$, $x = l(0)\bar{x}$, $\xi = l(0)\bar{\xi}$, $\rho = \tilde{\rho}\bar{\rho}$, and $u = U\bar{u}$, where dimensionless variables denoted with over-bar are $O(1)$. Furthermore we remind ourselves that the order ϕ is already normalised. Now transforming the derivatives, using

$$\left. \frac{\partial}{\partial t} \right|_{x \text{ fixed}} + \left. \frac{\partial x}{\partial t} \right|_{\xi \text{ fixed}} \frac{\partial}{\partial x} = \left. \frac{\partial}{\partial t} \right|_{\xi \text{ fixed}}, \quad (4.87)$$

we can re-write equations (4.80), (4.81) and (4.82) as

$$\frac{\partial \bar{\rho}}{\partial \bar{t}} = -\bar{\rho} \frac{\partial \bar{u}}{\partial \bar{\xi}} \frac{\partial \bar{\xi}}{\partial \bar{x}}, \quad (4.88)$$

$$\frac{\partial \phi}{\partial \bar{t}} = \tilde{\beta}(1 - \phi) \frac{\partial \bar{u}}{\partial \bar{\xi}} \frac{\partial \bar{\xi}}{\partial \bar{x}}, \quad (4.89)$$

$$\frac{\partial \sigma_{11}}{\partial \bar{\xi}} = 0 \quad (4.90)$$

$$\text{where } \sigma_{11} = \gamma \bar{\rho}^2 \frac{\partial \bar{u}}{\partial \bar{\xi}} \frac{\partial \bar{\xi}}{\partial \bar{x}} [\nu(1 - \phi) + \phi], \quad (4.91)$$

respectively, where $\gamma = \frac{\tilde{\rho}^2 U B}{l}$ and $\nu = \frac{A}{E}$. Due to the fact that we ignore inertia in the conservation of momentum equation (4.90), we can eliminate γ from the problem. The boundary conditions become

$$\bar{u}(0, \bar{t}) = 0, \quad \bar{u}(1, \bar{t}) = 1, \quad \bar{\rho}(\bar{\xi}, 0) = \bar{\rho}_0(\bar{\xi}), \quad \text{and} \quad \phi(\bar{\xi}, 0) = \phi_0(\bar{\xi}). \quad (4.92)$$

Now we drop the dimensionless notation, removing the over-bars. We can now find relationships between two of the three unknowns by eliminating $\frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x}$ in equations (4.88)–(4.90), and integrate to get

$$\phi(\xi, t) = 1 + (\phi_0(\xi) - 1) \left(\frac{\rho(\xi, t)}{\rho_0(\xi)} \right)^\beta. \quad (4.93)$$

To find the solution of the differential equations (4.88)–(4.90) with boundary conditions (4.92) we need to integrate equation (4.90)

$$- \left[1 + (1 - \nu)(\phi_0(\xi) - 1) \left(\frac{\rho(\xi, t)}{\rho_0(\xi)} \right)^\beta \right] \rho \frac{\partial \rho(\xi, t)}{\partial t} = F(t), \quad (4.94)$$

where $F(t)$ is the non-dimensional force prescribed at one end of the tuft, i.e. $\xi = 1$. Integrating this with respect to time leaves an inversion to get density as a function of time which is not tractable.

4.4.5.4 The Solution for a Uniformly Dense Tuft

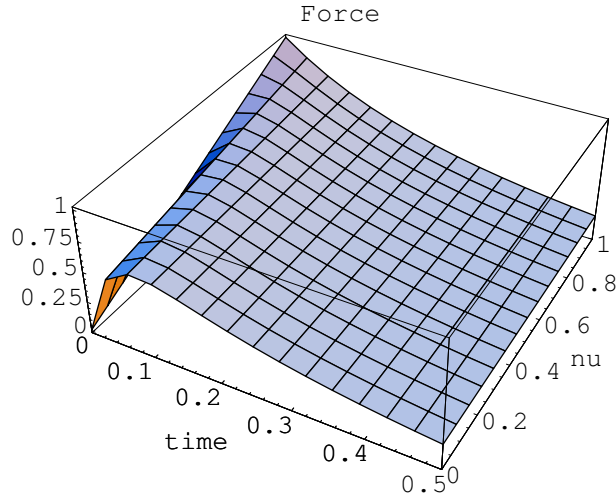


Figure 4.14: The dimensionless force required to elongate the fibre continuum at uniform velocity; $\beta = 10$ and $\phi_0 = 0$.

A much simpler form can be deduced if we assume a constant initial density and order. This means that in the Lagrangian framework, all the characteristic variables except velocity will depend on time only. So from conservation of mass (4.88) we find that

$$\frac{\partial^2 X}{\partial \tau \partial \xi} \frac{\partial \xi}{\partial X} = -\frac{1}{\rho} \frac{\partial \rho}{\partial \tau}. \quad (4.95)$$

Integrating (4.95) we find that

$$X(\xi, t) = \int_0^\xi \frac{\rho_0(\xi')}{\rho(\xi', t)} d\xi', \quad (4.96)$$

and if $\rho_0(\xi) = 1$ as we have made $\rho(\xi, t)$ dimensionless with respect to ρ_0 , we get the relationship

$$X(\xi, t) = \frac{1}{\rho(t)}\xi + C_0. \quad (4.97)$$

From the boundary condition (4.92), we can deduce that $X(\xi, t) = 0$ at $\xi = 0$ and at $\xi = 1$ we have the condition $X(\xi, t) = 1 + t$, thence giving the explicit solutions:

$$\rho(t) = \frac{1}{1+t}, \quad (4.98)$$

$$\phi(t) = 1 + \frac{\phi_0 - 1}{(1+t)^\beta}, \quad (4.99)$$

$$F(t) = \frac{1}{(t+1)^3} \left[\frac{(\nu-1)(1-\phi_0)}{(1+t)^\beta} + 1 \right]. \quad (4.100)$$

The dimensionless density from equation (4.98) tends monotonically to zero as time tends to infinity. The order tends to one and depends linearly on its initial condition. Again as $\rho \rightarrow 0$, we expect the continuum model to be no longer adequate, as either the fibre structures will fracture or be broken down into non-interacting fibres.

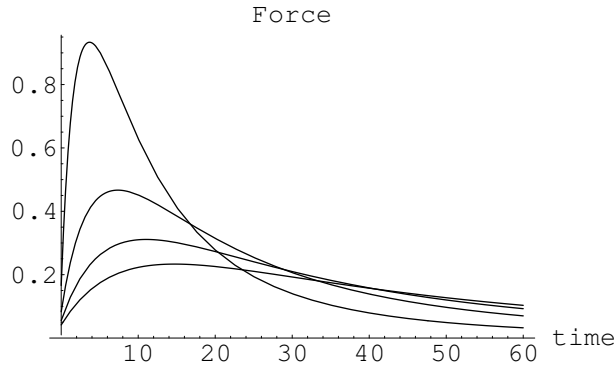


Figure 4.15: The dimensionless force required to elongate the fibre continuum; $u = \frac{5}{6}10^{-3}$, $\phi_0 = 0$, $\beta = 1$ and $\nu = 0.01$. The function with the highest maximum corresponds to length 0.01 and for increasing gauge lengths 0.02, 0.03 and 0.04, the respective maximum decreases.

The results from the simple viscous model in section 4.3 showed that the theoretical simulations did not compare well in the initial stage of experimental data. Qualitatively the force function lacked a maximum. The new model (4.100) allows for turning points for $t > 0$ when two conditions are satisfied:

$$(\nu - 1)(\phi_0 - 1)(\beta + 3) > 3, \quad (4.101)$$

$$\nu < 1. \quad (4.102)$$

Since $\nu = \frac{A}{E}$ is the ratio of the isotropic and nematic viscosities, this means that the anisotropic stresses must dominate over the isotropic. In fact the greater the disparity in viscous coefficients ($\nu \rightarrow 0$) the later the turning point will occur. We also note that when the fibres are initially aligned so that $\phi_0 = 1$ there is no turning point. If we assume that the initial order is approximately random $\phi \sim 0$ then we can approximate values for ν and β based on qualitative comparisons with experimental results. Finally we mention the fact that the initial force is not zero, which is what was observed in the experiments, section 4.2. In fact our initial force is

$$F(0) = \nu(1 + \phi_0) + \phi_0, \quad (4.103)$$

and with the maximum condition (4.102) and the assumption that the initial tuft is disordered, we expect $F(0)$ to be small. When fibres are at rest there should be no force measured, but we have neglected the tuft population's elastic properties and so our model breaks down when considering small stresses and is only appropriate for a tuft population under tension.

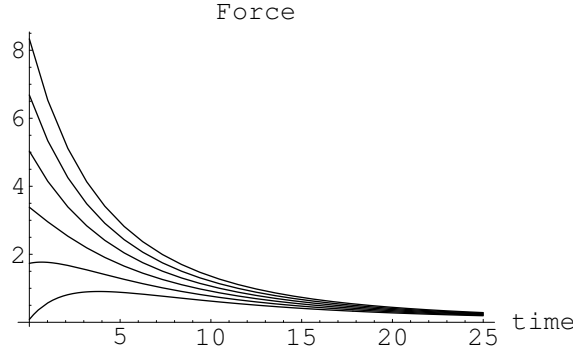


Figure 4.16: The dimensionless force required to elongate the fibre continuum; $u = \frac{5}{6}10^{-3}$, $\beta = 1$ and $\nu = 0.01$. The largest force corresponds to $\phi_0 = 1$ and decreases with respect to the order parameter $\phi_0 = 0.8, 0.6, 0.2, 0$.

What is still not correct though, is the counter-intuitive behaviour in (4.100), where initially random orientations are easier to pull apart than aligned fibres of the same density, as it can be seen in figure 4.16. On the contrary we would expect that, under the action of carding wire, when fibres are aligned they will tend to be more susceptible to viscous deformations. As the number of contact points may in fact decrease when fibres become more aligned, this suggests that density is not sufficient in describing the number of contact points that contribute to the bulk stress. The concept that a number of neighbouring fibres may either be entangled around one

another or juxtaposed has been ignored so far, but now we expect this to significantly effect the bulk stress tensor.

4.5 Continuum Model with Entanglement

The issue of how the bulk stress depends on contact points needs to be addressed in more detail. Although we introduced anisotropies in the continuum with the director and order parameter, we found that choosing viscous coefficients to be proportional to the square of density lacked the required information on fibre topologies. Our fibre assembly model also distinguishes itself from fibre-suspensions or liquid crystals with the inclusion of the degree of entanglements. Clearly the more tangled a tuft the harder it will be to tease apart, and clearly this is paramount to the carding process as the machines should order and disentangle. With this in mind we start by formulating a measure of the entanglement of a bunch of fibres.

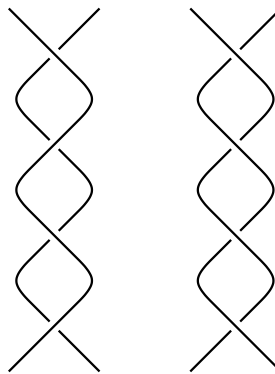


Figure 4.17: A comparison of two quasi-planar braids with the same order and directionality.

The topology of a braid will play an important role in identifying structural robustness. The problem of using just the order parameter and director in the previous section is illustrated in figure 4.17. A simple example is pairs of fibres that have the same alignment and direction, but the resistance to motion when one fibre moves away from the other is very different. In fact the pair of fibres on the left in figure 4.17 would produce little resistance when the fibres pulled apart. The pair of fibres on the right of figure 4.7 have a fair degree of entanglement and the resistance would be far greater than its adjacent counterpart. We quantify and model the degree of entanglement. We review a systematic way of classifying braid topology, based on the seminal work of Artin (1965). We go on to define an entanglement function and

then a corresponding evolution equation for it. Finally we test the new model against experiment

4.5.1 Degree of Entanglement and Braid Theory

We begin the process of quantifying the entanglement of a group of fibres by classifying their braid topology. If we consider a bundle of n fibres, a braid is a three-dimensional object that can be described topologically by a planar diagram. In this plane there are two parallel lines where on each there lies a set of points $M_i = (i, 0, 0)$ and $N_i = (i, 0, 1)$, a fibre will begin from a point M_k and end on the point N_l where the fibres do not intersect on the two parallel lines. Between the two parallel lines the fibres go over and under each other, see figure 4.18. In the planar braid diagram, one always assumes that a strand descends without any regions of ascent from one parallel line containing M_i 's to the other parallel line containing N_i 's. For each diagram we can construct an infinite number of topologically equivalent three dimensional objects. Artin (1965) began studying braids and showed that braids under the aforementioned description form a group. We will use this theory as the basis for our description of the inter-fibre knottedness.

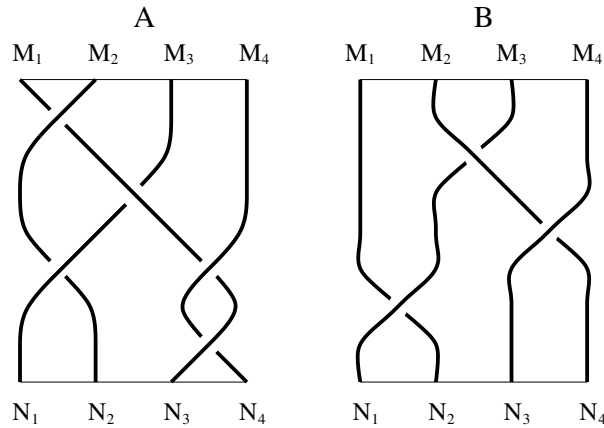


Figure 4.18: A couple of braid diagrams.

For a braid, the group operation is the product of two braids A and B . For an n -braid group, B_n , each braid has n fibres or strands. In the planar diagram of a three-dimensional braid we prescribe the beginning and end points to be equidistant. The associative binary operation, the product, is the result of placing the top of B at the bottom of A , see figure 4.19. The inverse of a braid, A , is simply the reflection

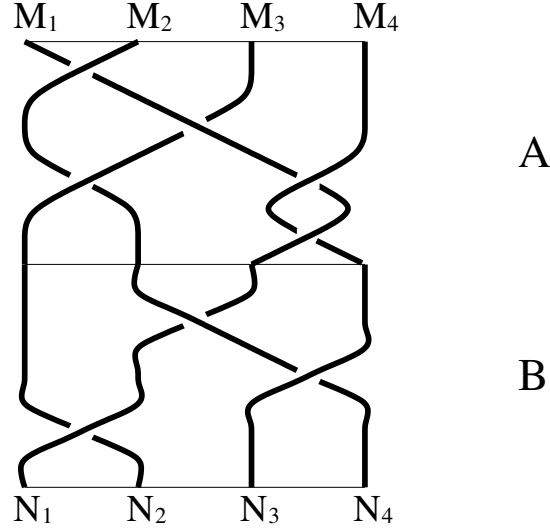


Figure 4.19: A diagram illustrating the product of two braids given in figure 4.18

at either one of the bounding parallel lines in the braid diagram. The unit element is the braid where there are no cross-overs.

It is important to mention the equivalence class of braids. If there are two points on a strand, bounded by $\mathbf{x}_1$ and $\mathbf{x}_2$, one can choose a third point, $\mathbf{x}_3$, that is not on the chosen strand and change the original strand element from the original bounded by $\mathbf{x}_1$ and $\mathbf{x}_2$, by using a polygonal link from $\mathbf{x}_1$ to $\mathbf{x}_3$ and then from $\mathbf{x}_3$ to $\mathbf{x}_2$. When the polygonal links, $\mathbf{x}_1 \rightarrow \mathbf{x}_3$ and $\mathbf{x}_3 \rightarrow \mathbf{x}_2$, are always descending and if the surface bounded by the polygonal links $\mathbf{x}_1 \rightarrow \mathbf{x}_2$, $\mathbf{x}_1 \rightarrow \mathbf{x}_3$ and $\mathbf{x}_3 \rightarrow \mathbf{x}_2$ do not intersect any other strand, we call this geometric deviation an *elementary move*. Two braids are equivalent if one braid can be transformed into another braid by using elementary moves. A corollary of this is that the product operation of three braids is associative.

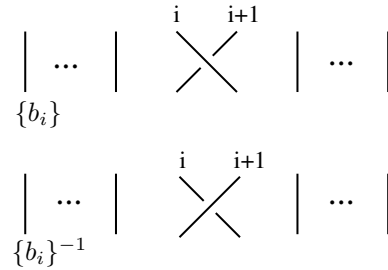


Figure 4.20: A i -th braid generator and its inverse.

Now we progress on to developing the final piece of the braid topology jigsaw, braid generators. We define a simple set of braid generators where there is only

one cross-over between two fibres, $\{b_1, \dots, b_{n-1}\}$, see figure 4.20. Any braid in B_n can be constructed using the generators as long as the representative diagram in the projection plane is composed of cross-overs at different altitudes. When this latter condition is violated we can find a representative case in the equivalence class that will satisfy this condition. Now we consider some illustrations in order to find three relationships between braid generators that allows us to migrate between equivalent braids. This should also eliminate any ambiguity caused by the choice of projection plane for the braid diagram.

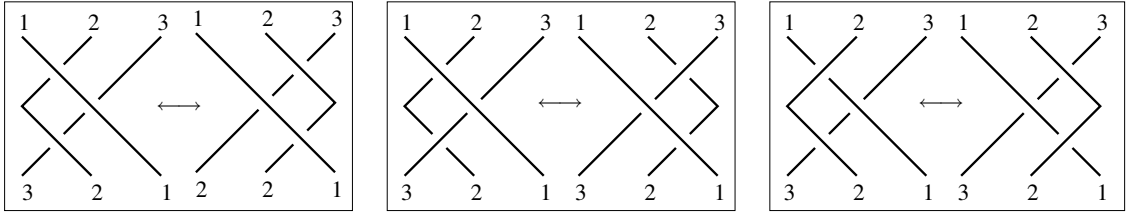


Figure 4.21: Three couples of braids illustrating transformations that yield the braid relation.

The first of three braid generator relationships is the simplest: a generator followed by its inverse is equivalent to the identity braid. Topologically, they are simply fibres standing side-by-side. A simple illustration is the braid on the left-hand-side in figure 4.17, and this gives the result

$$b_i b_i^{-1} = 1. \quad (4.104)$$

Now we consider three pairs of braids that illustrate the *braid relation*, see figure 4.21. Each pair is equivalent under elementary moves. The two equivalent braids on the left of figure 4.21 give the relation

$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}. \quad (4.105)$$

The other two transformations in figure 4.21 produce similar relational equalities $b_i b_{i+1} b_i^{-1} = b_{i+1}^{-1} b_i b_{i+1}$ and $b_i^{-1} b_{i+1} b_i = b_{i+1} b_i b_{i+1}$ respectively, and by using equation (4.104) these equations are equivalent to (4.105). There remains just one more braid generator relation to complete the picture. The property is called *far commutativity*. We can see that the cross-overs in figure 4.22 can be moved up or down relative to each other by using elementary moves, and so again are equivalent. This yields the relationship:

$$b_i b_j = b_j b_i \quad \text{whenever } |i - j| \geq 2. \quad (4.106)$$

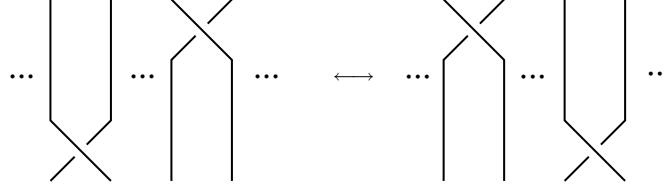


Figure 4.22: Two couples of braids illustrating transformations that yield the far commutativity relationship.

The *braid relation* and *far commutativity* are also known as *Artin's Braid Relations*. Prasolov and Sossinsky (1997) show that the braid group B_n is isomorphic to the abstract group generated by the braid generators $b_1, \dots, b_{n-1}$ that satisfy Artin's Braid Relations, (4.105) and (4.106). This gives a method of writing down the entanglement of a braid.

Within the equivalence class of a braid, the member with the least number of braid generators may be found and this we label as a *canonical braid*, and the set of these braids we call ω . The canonical braid is not unique but equivalent canonical braids will be composed of the same number of generators. We define entanglement, κ , to be the number of generators found in the canonical braid per elementary volume. A function, Ω can be defined, that operates on the set of canonical braids, ω , varying on the interval $\Omega : \omega \rightarrow [0, a]$. Furthermore, Ω is strictly monotonically increasing with respect to the number of braid generators, such that for the identity element, i say, $\Omega(i) = 0$ and for the braid with the most generators, m say, $\Omega(m) = a$. We write:

$$\kappa = \frac{\Omega(B)}{a}, \quad (4.107)$$

where a is a positive real number. One method of calculating Ω would be to choose a spheroid centred on a point, then compose a braid with the fibres in the spheroid. Another method for finding the braid that will define entanglement at a chosen point, would be to include all fibres that are topologically tangled with fibres in the spheroid.

4.5.2 Governing Equations

The equations for conservation of mass (4.2) and momentum (4.31), kinematic condition (4.63), evolution of order parameter (4.66), still hold. In addition to these we modify the viscous coefficients (4.61) of the stress tensor (4.59) and introduce a new equation for the evolution of the entanglement.

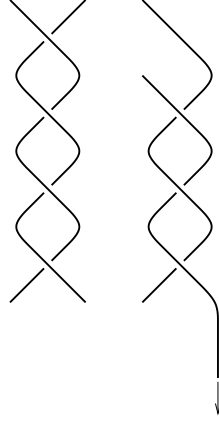


Figure 4.23: A couple of braids illustrating how extension of an element will intuitively reduce entanglement.

4.5.2.1 Empirical Law for Entanglement

We ignore the effect of extreme fibre clustering, what is known in the textiles industry as “nep” formation, as this would coincide with spikes in the density and entanglement functions. In the equation for directionality (4.63), we used the concept of a group of fibres being moved relative to one another, where extensions were quantified by a length factor s . As this action occurs we also expect fibres to disentangle, certainly if we ignore nep formation. Therefore we assume that the concentration of entanglements will decrease, when material elements are stretched. The length of fibres concerned will influence the rate at which the decrease occurs. We begin by assuming a simple relationship and ensuring that the entanglement concentration remains positive, we write:

$$\frac{\partial \kappa}{\partial t} + (\mathbf{u} \cdot \nabla) \kappa = -\alpha \kappa \frac{1}{s} \left[\frac{\partial s}{\partial t} + (\mathbf{u} \cdot \nabla) s \right]. \quad (4.108)$$

An illustration of the concept behind this equation (4.108) can be seen in figure 4.23. We reiterate that this is not always the case as some braids will need more than just simple local elongations in the direction $\mathbf{a}$, in order to become less or completely disentangled.

4.5.2.2 The Stress Tensor

For the stress tensor (4.59) we look to incorporate the additional information of the fibres entanglement. When the fibres are highly tangled then they will resist stress

better than when fibres are only loosely connected. A simple polynomial relationship is:

$$\mu_i = \kappa^p \rho^2 \bar{\mu}_i \text{ and } \lambda_i = \kappa^q \rho^2 \bar{\lambda}_i, \quad (4.109)$$

and a good starting point for this heuristic power law is $p = q = 1$. Notice that the degree of entanglement is completely independent of the director and order, and also braid topology is independent of density. Therefore our viscous coefficients in (4.109) now include the important physical characteristics that describe a body of fibres. Again we look to test the proposed continuum model against experiments from section 4.2, tuft breakage force and tuft shear force.

4.5.3 Elongation of a Fibrous Mass

As in the case for the simple viscous model in section 4.3 and the model with order and director 4.4.5, there are two parallel clamps holding the material that are separated initially by a prescribed distance and we measure the force required to elongate the material at a uniform velocity.

4.5.3.1 The Governing Equations

As in section 4.4.5 the problem can be simplified by constraining the motion to be in one direction, so that $v = 0$ and the functions depend only on x and t . The governing two dimensional equations, (4.68)–(4.75), then simplify to give:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0, \quad (4.110)$$

$$\frac{\partial \theta}{\partial t} + u \frac{\partial \theta}{\partial x} = -\cos \theta \sin \theta \frac{\partial u}{\partial x}, \quad (4.111)$$

$$\frac{\partial \phi}{\partial t} + u \frac{\partial \phi}{\partial x} = \beta(1 - \phi) \cos^2 \theta \frac{\partial u}{\partial x}, \quad (4.112)$$

$$\frac{\partial \kappa}{\partial t} + u \frac{\partial \kappa}{\partial x} = -\alpha \kappa \cos^2 \theta \frac{\partial u}{\partial x}. \quad (4.113)$$

As before in section 4.4.5, we constrain motion transverse to the direction of elongation by imposing (4.79).

4.5.3.2 Boundary Conditions

At the two solid boundaries we impose no-slip velocity conditions, where one boundary, $x = l(t)$, moves at a prescribed speed and the other at $x = 0$ is fixed, so

$$u = 0 \text{ at } x = 0, \quad (4.114)$$

$$\text{and } u = U \text{ at } x = l(t). \quad (4.115)$$

Initial conditions are then set for density, directionality and order:

$$\rho(x, t = 0) = \rho_0(x), \quad \kappa(\xi, t) = \kappa_0(\xi) \quad \text{and} \quad \phi(x, t = 0) = \phi_0(x). \quad (4.116)$$

4.5.3.3 Dimensionless Lagrangian Formulation

We can solve this problem by using the same method as that used in section 4.3 and 4.4. Reformulation of the problem using Lagrangian variables and the creation of dimensionless equations using the scalings of section 4.4, equations (4.2), (4.31), (4.63), (4.66), (4.109) and (4.59) simplifying to

$$\frac{\partial \bar{\rho}}{\partial \bar{t}} = -\bar{\rho} \frac{\partial \bar{u}}{\partial \bar{\xi}} \frac{\partial \bar{\xi}}{\partial \bar{x}}, \quad (4.117)$$

$$\frac{\partial \phi}{\partial \bar{t}} = \tilde{\beta}(1 - \phi) \frac{\partial \bar{u}}{\partial \bar{\xi}} \frac{\partial \bar{\xi}}{\partial \bar{x}}, \quad (4.118)$$

$$\frac{\partial \kappa}{\partial \bar{t}} = -\tilde{\alpha} \kappa \frac{\partial \bar{u}}{\partial \bar{\xi}} \frac{\partial \bar{\xi}}{\partial \bar{x}}, \quad (4.119)$$

$$\frac{\partial \sigma_{11}}{\partial \bar{\xi}} = 0 \quad (4.120)$$

$$\text{where } \sigma_{11} = \gamma \kappa \bar{\rho}^2 \frac{\partial \bar{u}}{\partial \bar{\xi}} \frac{\partial \bar{\xi}}{\partial \bar{x}} [\nu(1 - \phi) + \phi], \quad (4.121)$$

respectively, where $\gamma = \frac{\bar{\rho}^2 U}{l}$ and $\nu = \frac{A}{B+2C+D}$. We note that the entanglement is a dimensionless variable already. The boundary conditions become

$$\bar{u}(0, \bar{t}) = 0, \bar{u}(1, \bar{t}) = 1, \quad (4.122)$$

$$\bar{\rho}(\bar{\xi}, 0) = \bar{\rho}_0(\bar{\xi}), \kappa(\bar{\xi}, 0) = \kappa_0(\bar{\xi}) \text{ and } \phi(\bar{\xi}, 0) = \phi_0(\bar{\xi}). \quad (4.123)$$

Now we drop the dimensionless notation, removing the over-bars. We can now find relationships between three of the four unknowns by eliminating $\frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x}$ in equations (4.117)–(4.119) and integrating, to get

$$\phi(\xi, t) = 1 + (\phi_0(\xi) - 1) \left(\frac{\rho(\xi, t)}{\rho_0(\xi)} \right)^\beta \quad (4.124)$$

$$= 1 + (\phi_0(\xi) - 1) \left(\frac{\kappa(\xi, t)}{\kappa_0(\xi)} \right)^{\frac{\beta}{\alpha}}. \quad (4.125)$$

To find the solution of the differential equations (4.117)–(4.120) with boundary conditions (4.122) we need to integrate equation (4.120) and use equation (4.117):

$$-\frac{\rho(\xi, t)^{\alpha+1}}{\rho_0(\xi)^\alpha} \left[1 + (1 - \nu)(\phi_0(\xi) - 1) \left(\frac{\rho(\xi, t)}{\rho_0(\xi)} \right)^\beta \right] \frac{\partial \rho(\xi, t)}{\partial t} = F(t) \quad (4.126)$$

where $F(t)$ is the force prescribed at one end of the tuft, i.e. $\xi = l$.

4.5.3.4 The Solution For a Uniformly Dense Tuft

Similar to the problem solved in section 4.4, a much simpler form can be deduced if we assume a constant initial density, order ϕ_0 and entanglement κ_0 . This means that in the Lagrangian framework, all the characteristic variables except velocity will depend on time only. Using the boundary condition (4.122) and by writing $X(\xi, t) = 1 + t$, the explicit solutions are:

$$\rho(t) = \frac{1}{1+t}, \quad (4.127)$$

$$\phi(t) = 1 + \frac{\phi_0 - 1}{(1+t)^\beta}, \quad (4.128)$$

$$\kappa(t) = \frac{\kappa_0}{(1+t)^\alpha}, \quad (4.129)$$

$$F(t) = \frac{\kappa_0}{(1+t)^{3+\alpha}} \left[1 + \frac{(1-\nu)(\phi_0 - 1)}{(1+t)^\beta} \right]. \quad (4.130)$$

The dimensionless density from equation (4.127), still tends monotonically to zero as time tends to infinity. The order and entanglement both depend on one parameter, and their respective initial conditions. Order tends to one and entanglement tends to zero as time progresses, but we have now introduced κ_0 and the exponent α which allows more flexibility in matching $F(t)$ with experiment.

Finally the force produced at the moving boundary as a consequence of the motion is given in equation (4.130). A turning point for F occurs at

$$t_{max} = \left(\frac{(1-\nu)(1-\phi_0)(3+\alpha+\beta)}{3+\alpha} \right)^{\frac{1}{\beta}} - 1. \quad (4.131)$$

One condition for a maximum to exist, when $t_{max} \geq 0$, is $\nu < 1$, and this means that the nematic component of the stress tensor should dominate over the isotropic part in terms of their respective contributions to the bulk stress. Moreover we require:

$$(1-\nu)(1-\phi_0) > \frac{3+\alpha}{3+\alpha+\beta}, \quad (4.132)$$

and as the disparity in the inequality increases (i.e. ν gets smaller), t_{max} moves away from the origin. For the same material where α , β and ν are the same then the initial order of the material is the only parameter that influences the position of the maximum, whereas the initial entanglement effects the magnitude of the resistive force linearly. The height of the maximum is:

$$\frac{F_{max}}{\kappa_0} = \left[\frac{3+\alpha}{(1-\nu)(1-\phi_0)(3+\alpha+\beta)} \right]^{\frac{3+\alpha}{\beta}} \left(\frac{\beta}{3+\alpha+\beta} \right). \quad (4.133)$$

Local analysis near the turning point, indicates that the behaviour will always be a maximum provided $\alpha, \beta \geq 0$. The equation (4.132) suggests that if the rate at which the order parameter evolves is large ($\beta \gg 3 + \alpha$), then the later the maximum occurs in the experiment and also the smaller its the magnitude. We note that if the maximum point doesn't exist, $t_{max} < 0$, then the behaviour is similar to the “simple” viscous model given in section 4.3.

The results for density, order, entanglement and force given by (4.127)–(4.130), have been plotted, see figures 4.24 and 4.25 for the results. The dimensional force from (4.130) is

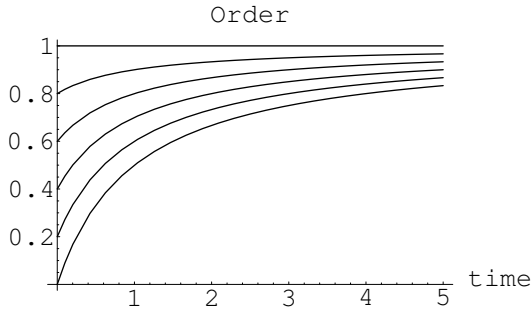
$$F_{dim}(t) = \frac{\kappa_0(B + 2C + D)\rho_0^2 l^{\alpha+2} U}{(l + Ut)^{3+\alpha}} \left[1 - \frac{(1 - \nu)(1 - \phi_0)l^\beta}{(l + Ut)^\beta} \right]. \quad (4.134)$$

We note that the initial force for the experiment that includes entanglement (4.130) is the same as (4.103) except that it is multiplied by the initial knottedness κ_0 . This implies that for the same entanglement, the initial force required to elongate a tuft will increase linearly with ϕ_0 . Now we have the ability to differentiate between a combed group of straight fibres and an entangled group of straight fibres, where the latter produces greater resistance to elongation. Moreover any group of fibres that are highly tangled $\kappa \sim 1$ will now produce greater resistance to viscous deformations than a group of fibres that are well combed ($\kappa \sim 0$).

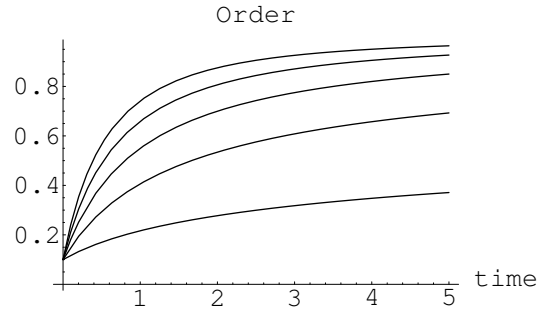
4.5.3.5 Comparison with Experiment

When the initial tuft size length varies in the tuft breaking force experiment, see figure 4.25 plots (a) and (b), the qualitative comparison between experiment and mathematical simulation can be very good by choosing α and β appropriately. As our analysis shows we can change numerous parameters, for example α and β , that will depend on the chosen material, and we have chosen our results to fit the experiment for polyester.

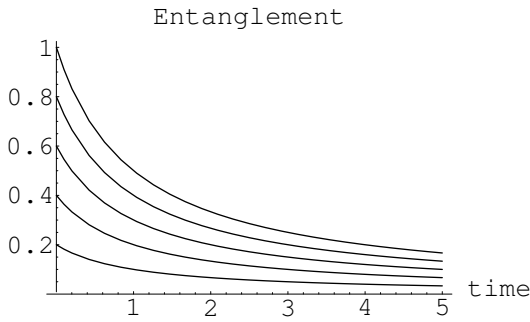
For the variable speed tuft breaking experiment, see figure 4.25 plots (c) and (d), the comparison is less good. As mentioned in section 4.2, it is quite likely that the results recorded include noise produced by the difficulty in obtaining equivalent tufts for each test. In terms of the model, for the same material, the initial order, entanglement and director may differ. All these parameters would also significantly change the mathematical simulation. Regardless of the possibly spurious comparative data the qualitative shape of the curves are similar in the aforementioned plots.



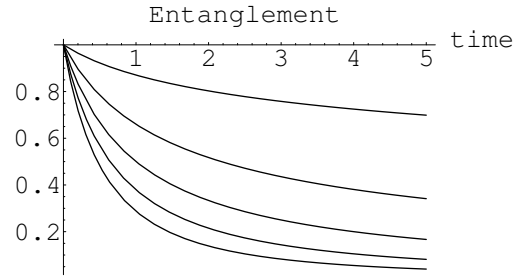
(a) $\beta = 1$ and $\phi_0 = 0, 0.2, \dots, 1$.



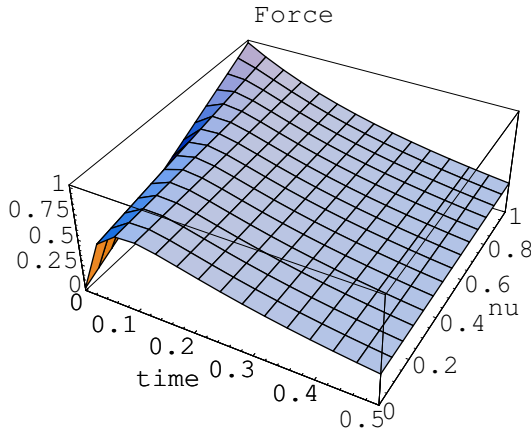
(b) $\beta = 0.2, 0.6, \dots, 1.8$ and $\phi_0 = 0.1$



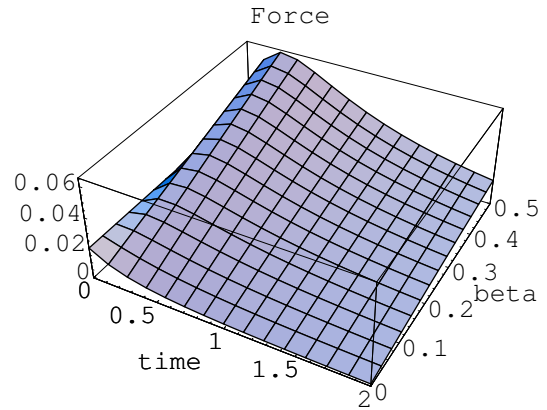
(c) $\alpha = 1$ and $\kappa_0 = 0.2, 0.4, \dots, 1$.



(d) $\alpha = 1.8, 1.4, \dots, 0.2$ and $\kappa_0 = 1$.

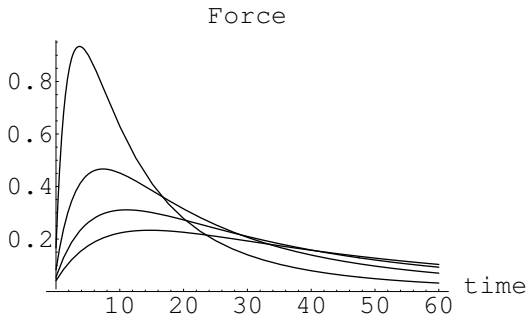


(e) $\kappa_0 = 1$, $\phi_0 = 0.01$, $\alpha = 1$, $\beta = 25$.

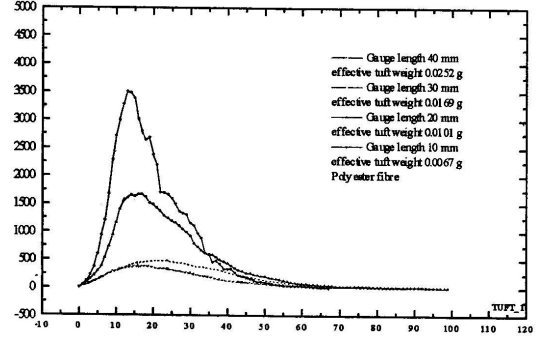


(f) $\kappa_0 = \nu = 0.01$ and $\phi_0 = \kappa_0 = 1$.

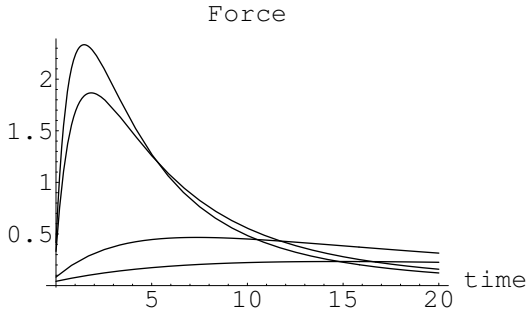
Figure 4.24: Results of the dimensionless elongation problem for the continuum model: (a) and (b) are plots of the order parameter, (c) and (d) are plots of the entanglement, and (e) and (f) are plots of the force. The graphs with multiple functions correlate to the given values beneath the graph in ascending order.



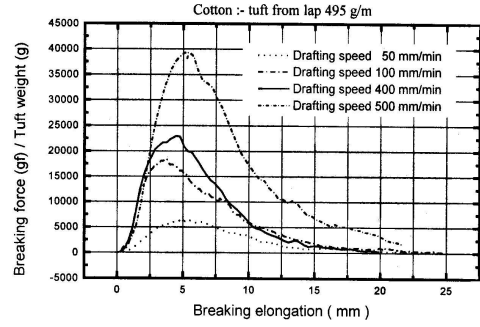
(a) $\kappa_0 = \alpha = \beta = 1$, $U = \frac{5}{6}10^{-3}\text{m/s}$, $\phi_0 = \nu = 0.01$, $l = 0.04, 0.03, 0.02, 0.01\text{m}$.



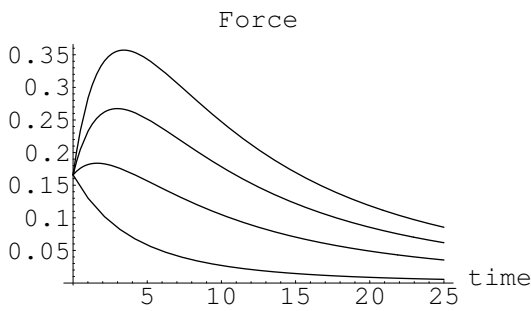
(b) Experiment for polyester with variable initial lengths: force against extension.



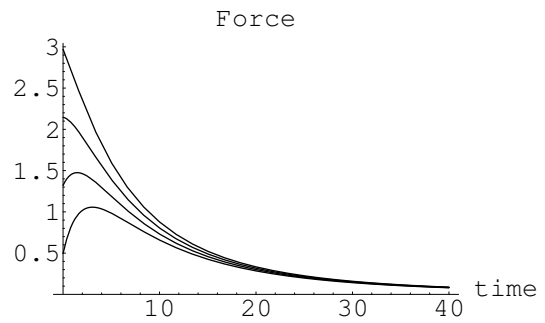
(c) $\kappa_0 = \alpha = \beta = 1$, $l = 0.04\text{m}$, $\phi_0 = \nu = 0.01$, $U = \frac{5}{6}10^{-3}, \frac{10}{6}10^{-3}, \frac{40}{6}10^{-3}, \frac{50}{6}10^{-3}\text{m/s}$.



(d) Experiment for polyester with variable elongation velocity.



(e) $\phi_0 = 0.01$, $\kappa_0 = \alpha = 1$, $l = 0.01\text{m}$, $U = \frac{5}{6}10^{-3}\text{m/s}$, $\beta = 0, 0.1, 0.2, 0.3$.



(f) $\kappa_0 = \alpha = \beta = 1$, $\phi_0 = 0.01$, $l = 0.01\text{m}$, $U = \frac{5}{6}10^{-3}\text{m/s}$, $\nu = 0.05, 0.15, 0.25, 0.35$.

Figure 4.25: Results of the dimensional elongation problem for the continuum model: (a) and (b) compares experiment with mathematical simulation for varying gauge lengths, (c) and (d) is a comparison for varying velocity. The functions on each graph correlate to the given values below where the respective plots in ascending order.

4.5.4 A Simple Shearing Problem

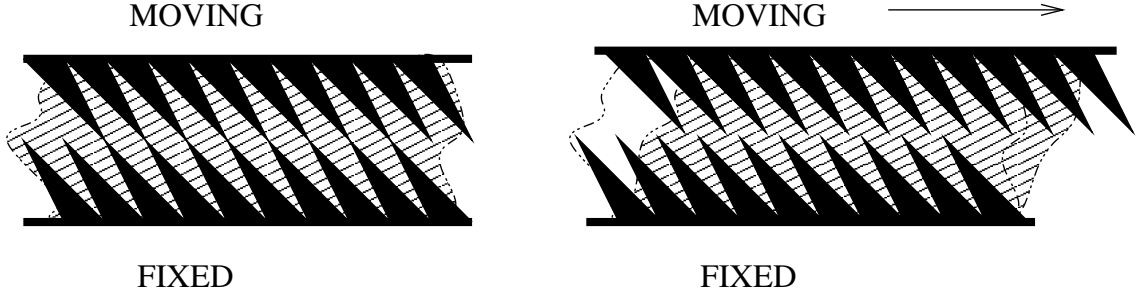


Figure 4.26: A diagram of the shearing problem.

We continue testing this mathematical model by considering the shearing problem, see figure 4.26. The material is now sheared by two parallel solid boundaries. If the two plates bounding the material are smooth, there would no change during the motion or very little non-recoverable change. One should witness constant slipping between the fibres and the solid surface, unless the surface is rough or sticky. Small hooks would produce the desired effect of shearing the material. A mathematical simulation, where we attach the fibres to the solid boundary, could mimic events in the carding machine or the experiment discussed in section 4.2. The effectiveness of the model will depend on the hooks used in the experiments, moreover their ability to hold onto fibres and in the model below we take the boundary condition to be a no-slip on the wall, i.e. we neglect the effect of the hooks within the fibres.

4.5.4.1 The Governing Equations

As in paragraph 4.5.3 we constrain the motion to be in one direction, and we assume that the initial state is uniform in the x -direction, allowing only for variations in time and in y , and assuming $v = 0$. The director is written as $\mathbf{a} = (\cos \theta, \sin \theta, 0)^T$. The governing equations from (4.68) – (4.75), become

$$\frac{\partial \rho}{\partial t} = 0, \quad (4.135)$$

$$\frac{\partial \theta}{\partial t} = -\sin^2 \theta \frac{\partial u}{\partial y}, \quad (4.136)$$

$$\frac{\partial \phi}{\partial t} = \beta(1 - \phi) \sin \theta \cos \theta \frac{\partial u}{\partial y}, \quad (4.137)$$

$$\frac{\partial \kappa}{\partial t} = -\alpha \kappa \sin \theta \cos \theta \frac{\partial u}{\partial y}. \quad (4.138)$$

This means that the density does not evolve in time but could depend on the spatial variable y . Equations (4.71) and (4.72) become:

$$\frac{\partial}{\partial y} \left\{ \rho^2 \kappa \frac{\partial u}{\partial y} \left[\mu_1(1 - \phi) + \phi \left(\lambda_1 + \frac{\lambda_3}{2} + \lambda_5 \sin^3 \theta \cos \theta \right) \right] \right\} = 0, \quad (4.139)$$

$$\frac{\partial}{\partial y} \left\{ \rho^2 \kappa \phi \frac{\partial u}{\partial y} [(\lambda_1 + \lambda_2) \sin \theta \cos \theta + \lambda_3 \sin^3 \theta \cos \theta] \right\} = 0. \quad (4.140)$$

Unlike the extensional problem we do not have to impose a condition on the director because in this problem the fibre material is bounded by two solid “walls” and there is no need to ensure that there is no transverse forces. The boundary and initial conditions are simply no slip at the solid boundaries and initial conditions for density, order and the director:

$$u = 0 \text{ at } y = 0, \quad u = U \text{ at } y = h, \quad (4.141)$$

$$\kappa(y, t) = \kappa_0(y), \quad \theta(y, t) = \theta_0(y), \quad \phi(y, t) = \phi_0(y) \quad \text{at } t = 0. \quad (4.142)$$

Now we can progress to writing the equation in dimensionless form.

4.5.4.2 Dimensionless Formulation

Velocity is scaled with the speed of the moving solid boundary U , so $u = U\bar{u}$. To scale density we use a constant ρ_0 characterising the prescribed initial condition, so that $\rho = \rho_0\bar{\rho}$. The entanglement and order are defined as normalised variables and the angle θ is order unity. The typical length-scale of motion is distance between the two parallel plates, so $y = h\bar{y}$ and finally we are left with scaling time conventionally as $t = \frac{h}{U}\bar{t}$. The dimensionless equations (4.135)–(4.138) are:

$$\frac{\partial \theta}{\partial \bar{t}} = -\sin^2 \theta \frac{\partial \bar{u}}{\partial \bar{y}}, \quad (4.143)$$

$$\frac{\partial \phi}{\partial \bar{t}} = \beta(1 - \phi) \sin \theta \cos \theta \frac{\partial \bar{u}}{\partial \bar{y}}, \quad (4.144)$$

$$\frac{\partial \kappa}{\partial \bar{t}} = -\alpha \kappa \sin \theta \cos \theta \frac{\partial \bar{u}}{\partial \bar{y}}, \quad (4.145)$$

and conservation of momentum equations (4.139) and (4.140) become:

$$\frac{\partial}{\partial \bar{y}} \left\{ \bar{\rho}^2 \kappa \frac{\partial \bar{u}}{\partial \bar{y}} [\nu_1(1 - \phi) + \phi(1 + \nu_2 + \nu_3 \sin^3 \theta \cos \theta)] \right\} = 0, \quad (4.146)$$

$$\frac{\partial}{\partial \bar{y}} \left\{ \bar{\rho}^2 \kappa \phi \frac{\partial \bar{u}}{\partial \bar{y}} [(1 + \nu_4) \sin \theta \cos \theta + 2\nu_2 \sin^3 \theta \cos \theta] \right\} = 0, \quad (4.147)$$

where $\nu_1 = \frac{\mu_1}{\lambda_1}$, $\nu_2 = \frac{\lambda_3}{2\lambda_1}$, $\nu_3 = \frac{\lambda_5}{\lambda_1}$, and $\nu_4 = \frac{\lambda_2}{\lambda_1}$. The boundary and initial conditions, from equations (4.141) become:

$$\bar{u} = 0 \text{ at } \bar{y} = 0, \quad \bar{u} = 1 \text{ at } \bar{y} = 1. \quad (4.148)$$

and equations (4.142) remain the same. We now drop the dimensionless notation for tidiness, i.e. do not write the over-bars any more.

4.5.4.3 The Solution

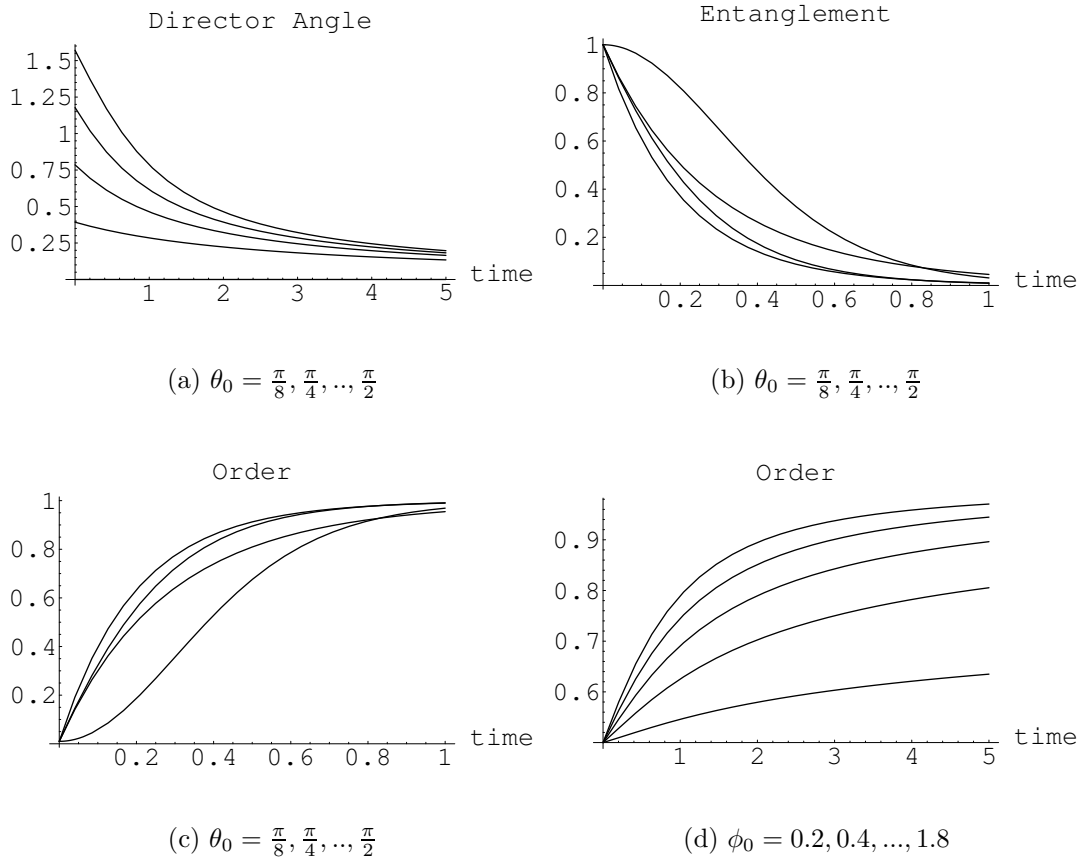


Figure 4.27: Results of the dimensionless shearing problem for the continuum model: (a) a plot of the director, (b) entanglement, (c) and (d) order. The plots on each graph correlate to the number written below each graph in ascending order.

By considering partial differential equations (4.143)–(4.145), we look for solutions in which κ , ϕ and θ are only time dependent. This allows us to integrate equations (4.146) and (4.147) with respect to y to give the velocity:

$$u = \frac{f_1(t)y}{\rho^2\kappa [\nu_1(1 - \phi) + \phi (1 + \nu_2 + \nu_3 \sin^3 \theta \cos \theta)]} + g_1(t) \quad (4.149)$$

$$u = \frac{f_2(t)y}{\rho^2 \kappa \phi [(1 + \nu_4) \sin \theta \cos \theta + \nu_2 \sin^3 \theta \cos \theta]} + g_2(t) \quad (4.150)$$

respectively. Applying the boundary conditions (4.148) gives a simple linear velocity profile, analogous to Couette Flow (see Batchelor (1967)),

$$u = y. \quad (4.151)$$

Now with equation (4.151), the partial differential equations in (4.143) – (4.145) become ordinary differential equations in time. The solutions are:

$$\theta = \operatorname{arccot}(t + \cot \theta_0), \quad (4.152)$$

$$\phi = 1 + (\phi_0 - 1) \left[\frac{1 + \cot^2 \theta_0}{1 + (\cot \theta_0 + t)^2} \right]^{\frac{\beta}{2}}, \quad (4.153)$$

$$\kappa = \kappa_0 \left(\frac{1 - \phi_0}{1 - \phi} \right)^{\frac{\alpha}{\beta}} = \kappa_0 \left[\frac{1 + \cot^2 \theta_0}{1 + (\cot \theta_0 + t)^2} \right]^{\frac{\alpha}{2}}, \quad (4.154)$$

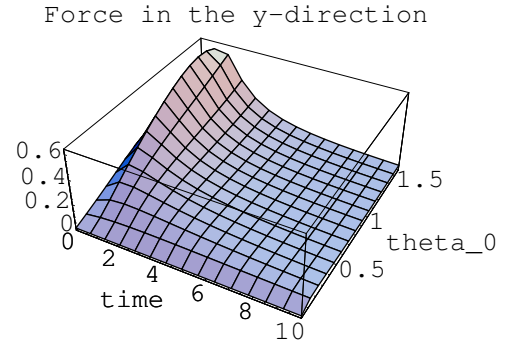
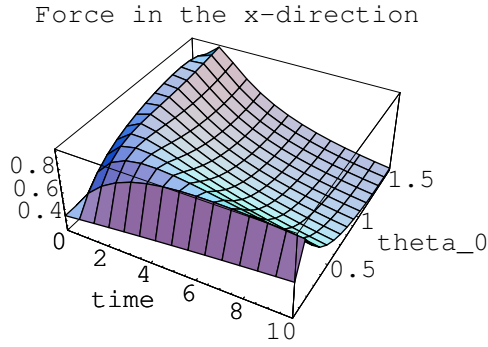
and we note that in this case that the density is constant. The director, regardless of initial condition, will always tend to align itself parallel to the shearing boundaries. Order evolves to the nematic phase, $\phi \rightarrow 1$, but depends on the initial direction of the fibres, θ_0 . Similarly for the entanglement, we see an evolution that eliminates knottedness but the rate at which this occurs depends on the initial director. For plots of the functions (4.152)–(4.154) see figure 4.27. Two components of force $(F, G)^T$ are found using the stress tensor,

$$\begin{aligned} F(t) = & \left(\frac{1 + \cot^2 \theta_0}{1 + (t + \cot \theta_0)^2} \right)^{\frac{\alpha}{2}} \left\{ \nu_1 (1 - \phi_0) \left(\frac{1 + \cot^2 \theta_0}{1 + (t + \cot \theta_0)^2} \right)^{\frac{\beta}{2}} \right. \\ & + \left[1 - (1 - \phi_0) \left(\frac{1 + \cot^2 \theta_0}{1 + (t + \cot \theta_0)^2} \right)^{\frac{\beta}{2}} \right] \\ & \times \left[1 + \nu_2 + \frac{\nu_3(t + \cot \theta_0)}{[(t + \cot \theta_0)^2 + 1]^2} \right] \Big\} \end{aligned} \quad (4.155)$$

and

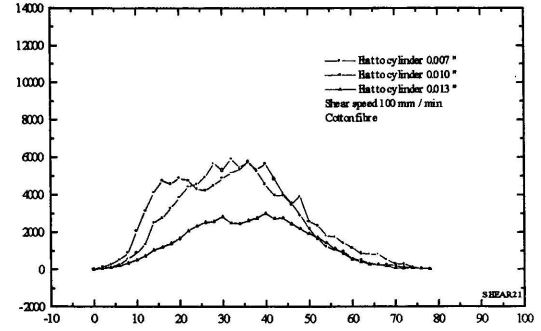
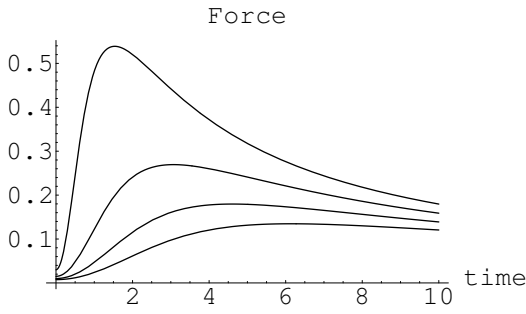
$$\begin{aligned} G(t) = & \left[\frac{1 + \cot^2 \theta_0}{1 + (t + \cot \theta_0)^2} \right]^{\frac{\alpha}{2}} \left[1 - (1 - \phi_0) \left(\frac{1 + \cot^2 \theta_0}{1 + (t + \cot \theta_0)^2} \right)^{\frac{\beta}{2}} \right] \\ & \times \left[\frac{(1 + \nu_4)(t + \cot \theta_0)}{1 + (t + \cot \theta_0)^2} + \frac{\nu_2(t + \cot \theta_0)}{[(t + \cot \theta_0)^2 + 1]^2} \right]. \end{aligned} \quad (4.156)$$

The dimensional forces are found by rescaling time with $\frac{U}{l}$ and multiplying the force by $\rho^2 B \frac{U}{l}$, see figure 4.28. Now it remains to compare the results with experimental data.



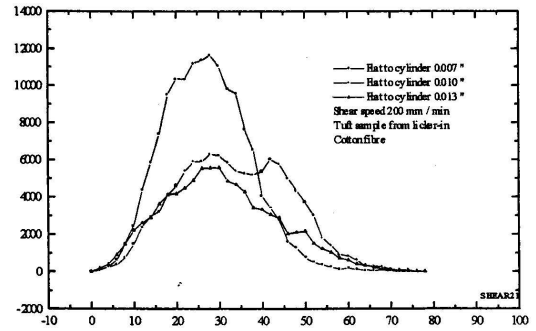
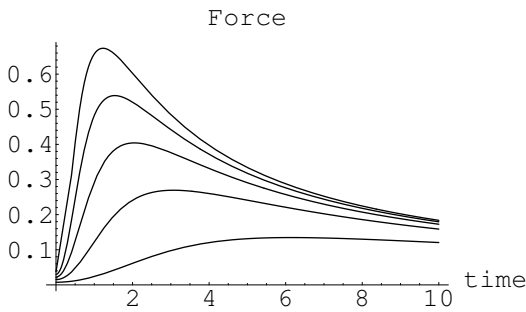
(a) $\phi_0 = \nu_1 = 0.1$, $\alpha = \beta = 1$, $\nu_2 = \nu_3 = 2$.

(b) $\phi_0 = 0.1$, $\alpha = \beta = 1$, $\nu_2 = \nu_4 = 2$.



(c) Model with varying hook distances: $l = 0.001, 0.002, \dots, 0.004\text{m}$, $U = 0.001\text{m/s}$.

(d) Shear force experiment for cotton with varying hook-to-hook distance and shear speed of 100mm/min.



(e) Model with varying shear speeds: $U = 0.001, 0.002, \dots, 0.005\text{m/s}$, $l = 0.004\text{m}$.

(f) Shear force experiment for cotton with varying hook-to-hook distance and shear speed of 200mm/min.

Figure 4.28: Force graphs for continuum model and the shearing experiment. The pairs juxtaposed in this figure are simulations and their corresponding experiment.

4.5.4.4 Comparison with Experiments

The comparisons juxtaposed in figure 4.28 are fairly good. The main discrepancy is that the experimental graphs show a greater decrease in the measured force after their respective maxima are reached. We can account for this, as the mathematical model is based on an infinite channel of fibrous material and the experiments are conducted with tufts, a cluster of fibres. This means that for the mathematical simulation the density remains constant throughout, which is not what we would expect in reality. It is also conceivable that as entanglement decreases the hooks' ability to hold the fibres will diminish, and this suggests that the no-slip condition may not be valid. The qualitative initial shape of the force plots is captured well with the model, except for the previously documented fact that the force never begins at zero, as explained in section 4.5.3.

4.5.5 An Array of Hooks

Another method of modelling the “shear force” experiment, is considering the hooks as a body force acting on the continuous medium. For a viscous Newtonian fluid, an array of fibres moving normal to the flow is modelled by Terrill (1990). In this work, the author incorporates a body force on the fluid due to the presence of a homogeneous uniform array of fibres, which are considered as a large number of long slender cylinders to leading order. We have already transposed this problem before by letting the cylinders represent hooks in section 2.4.

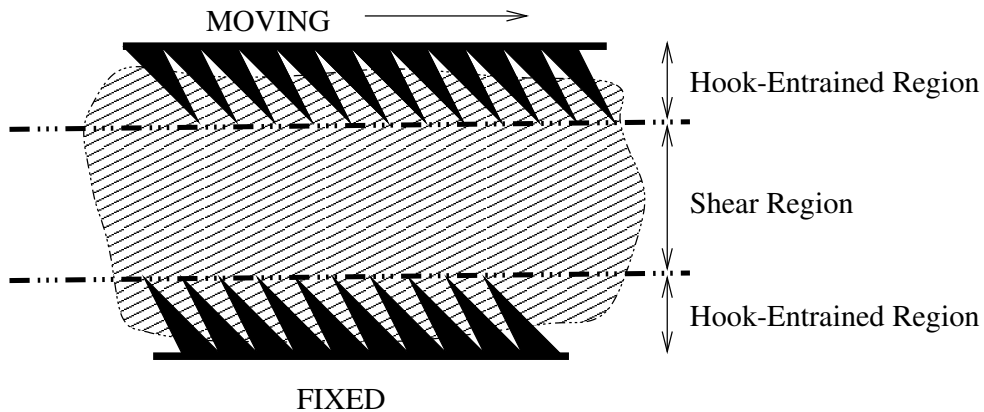


Figure 4.29: A diagram of the fibre continuum which is sheared by two arrays parallel hooks.

For our non-Newtonian viscous material, which is composed of fibres we can add a body force to the momentum equation (4.3) which is due to the movement of hooks

through the fibrous material. In the work by Terrill (1990), the approximation is for slow flow, and in our case the fibres behave in a similar way, i.e. $\Lambda \gg 1$. From our work in section 2.4 we could think of the approximation (2.31) as the sum of the forces produced by one hook in low Reynolds number flow:

$$f_i = H u_i \delta(\mathbf{x} - \mathbf{x}_{hook}), \quad (4.157)$$

where H is a viscous coefficient, u_i is relative velocity and $\mathbf{x}_{hook}$ is the position of the hook. The forces (4.157) can be added into the equation of conservation of momentum (4.31) to produce a body force on the fibre continuum proportional to the velocity of the fibres relative to the hooks. When the forces produced by the hooks dominate over inertia and viscous stresses, fibres will be entrained by the shearing surfaces, and there will be a region of shear in between the two hook layers. This means that our shearing calculations of section 4.5.2 would then be useful in the gap region. Consequently one approximation that includes hooks will produce exactly the same results as the shearing experiment in section 4.5.4, as the two entrained fibre regions create two no slip conditions at interfaces between the shear-region and the hook-entrained regions, see figure 4.29.

4.6 Conclusion

We have identified three potential areas of applied mathematics and theoretical physics that model “microscopic” nematic structures, namely liquid crystals, fibre reinforced fluids and polymers. Within all these fields there is one unifying methodology and that is the use of continuum models and the idea of a director. None of the theories and techniques migrated well enough to model fibre deformations in the carding machine so we designed a new continuum model where the dominant physical mechanism was friction between fibres. The key prescribed concepts were that the material’s dominant behaviour in the carding machine was viscous and that these deformations occurred under tension.

Using just velocity and density to describe a many-fibre body, we began to explore the possibilities of finding a tractable model. An isotropic bulk stress tensor was carefully derived using a probabilistic technique, where the viscous coefficients were proportional to the square of density. Comparisons with experiment and our model partially justified the viscous assumption as there was a fair degree of similarity, but the theoretical simulations failed to capture the behaviour of a fibre-medium

when forces are initially applied. We postulated that this was due to the lack of directionality and anisotropy in the model.

After defining two new continuum quantities, the order parameter and director, we defined a kinematic condition and an empirical evolution equation. A new bulk stress tensor incorporated both isotropic and anisotropic components, where the latter was based on transversely isotropic media. The comparison with experiments improved significantly as we captured, qualitatively, the behaviour of a many-fibre material as it was being elongated, in particular during the initial stages of the experiment. This result however, was not so conclusive when one looked closely at the relationship between initial order and the magnitude of the force produced. It was clear that neglecting how the fibres were interconnected led to counter-intuitive results.

Finally we introduced a function that defined the degree of entanglement. An explanation of the required braid theory was followed by the introduction of a new evolution equation for entanglement, and the viscous coefficients of the bulk stress were modified. The comparisons with experiments, for both elongational and shearing cases, were very promising. We have established a model for many-fibres, interconnected and entangled deformed under tension, that incorporates alignment, directionality and entanglement. All of these three characteristics are vital to textile engineers.

Chapter 5

Conclusions

We have created a number of models that provide insight into and understanding of the principal physical processes and mechanisms of the carding machine. We have done this by applying a number of areas of classical mechanics in novel settings and created an original model for a many-fibre-medium. Numerous methodologies were employed from perturbation theory to braid topology. There were three key areas of interest that arose from conversations with industrialists and experimentalists involved with textile manufacturing processes. These were: the behaviour of a single fibre in the carding machine, in particular their transfer between surfaces, the behaviour of fibre being teased out of tufts, and a macroscopic model for many interconnected fibres.

In chapter 2 we derived a model for a textile fibre in the carding machine, which included rotational and aerodynamic forces. We ignored any interaction with other fibres and focussed on fluid flow and hook-fibre friction. Working from the simplest scenario, where a fibre is tethered by a hook on a rotating drum, we examine the issue of fibre transfer between carding surfaces. The fibres would lie close to the hooked-drum surface from which it was being tethered, except when the fibre entered the nip regions between taker-in and cylinder and also cylinder and doffer. In these two nip regions, we found mechanisms for transfer were completely different.

The work in chapter 3 considered how a fibre is extracted from a tuft. We decoupled the internal forces within the tuft and the tension in the fibre. Then we went on to consider how a tuft, when sheared or teased can be modelled by two tufts with inter-connecting fibres. This work illuminates the forces acting on a fibre as it moves through a mass of fibres in a tuft.

The assumption, that fibre withdrawal does not affect the tuft structure, was not adequate when considering many interconnecting fibres and so in chapter 4, we create a continuum model for many interacting fibres. We introduced an order parameter, a

director and an entanglement function. The governing equations were derived using the mathematics from subjects such as transverse isotropic symmetry in continuum models and braid theory. The limitation to the model from the industrialist's point of view was that we could not incorporate the possibility of nep formation, singularities in the entanglement function. The mathematical model compared very well with experimental data, which included an elongational and shearing problem. Now we go on to describe how all the aforementioned models cumulatively describe the transport and deformation of fibres in the carding process.

5.1 The Life of Fibres in the Carding Machine

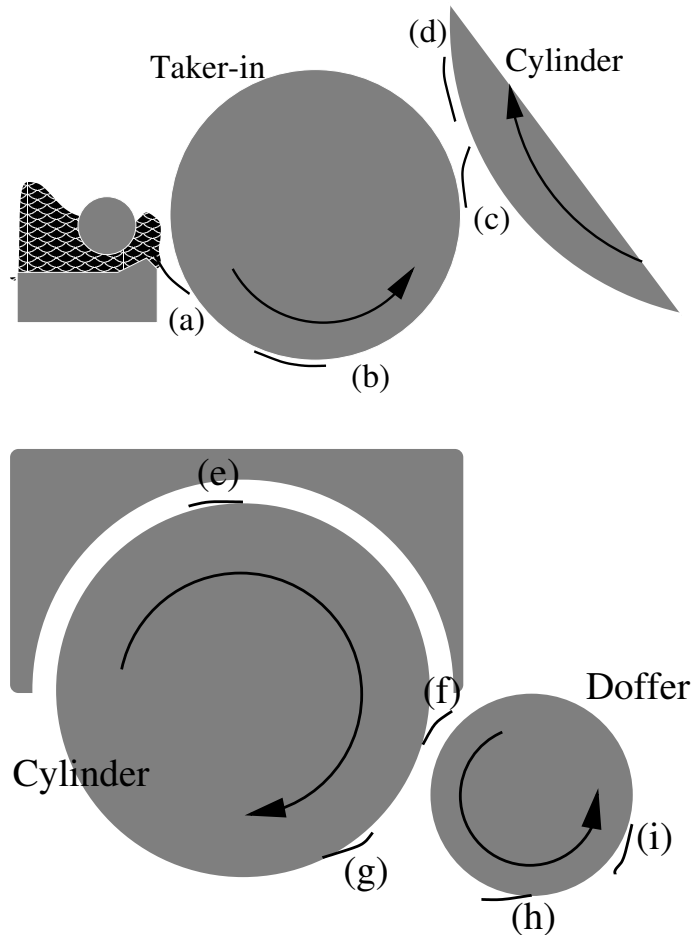


Figure 5.1: Diagrams illustrating the life of a single fibre in the carding machine.

5.1.1 The Taker-In

The taker-in teases out a single fibre from the lap on the feeder-in or it will grab a tuft, see (a) in figure 5.1 and (i) in figure 5.2. In both cases we can apply the models of chapter 3, that is the single fibre withdrawal problem from a tuft and two tufts with interconnecting fibres. From the solutions we can then predict fibre breakage based on the machine setting and the degree of entanglement in the lap. The continuum model of chapter 4 could also be applied to this region, because as the fibres are fed onto the taker-in, the rotation of the taker-in shears the fibre mass. Our continuum model can predict how the order in the fibre-medium evolves from this shearing.

It is found experimentally that half the fibres on the taker-in do not interact with other fibres. For the single fibres we can use the models of chapter 2, where we expect the fibres to remain close to the drum surface (b) until they reach the nip region between taker-in and doffer, the single fibre transfer onto the cylinder by the leading end first, see point (c) in figure 5.1. This mechanism is governed by the aerodynamics as described in section 2.5. As the tufts transfer onto the cylinder there may be slight changes in their respective structure which would be generated by the shearing motion in the nip region, see (ii) in figure 5.2. The reason for assuming that the changes are slight is that the hooks on both surfaces in the nip region all face the same direction, and consequently the resistance to motion relies on the ability of the taker-in hooks to hold onto a tuft, and this will be minimal.

5.1.2 The Cylinder

Tufts that enter the carding region have already been teased by their extraction from the lap with the taker-in hooks. The fixed and revolving flats now produce considerable resistance to tufts that are transported through this region, see (iii) in figure 5.2. Here we can apply the shearing problem of chapter 4 for a continuous media. We can use the machine parameters in our model to approximate how the tufts will evolve through this part of the process, although in order to find some of the parameters in the model, we will require experimental data. Single fibres, as they enter the region with the revolving and fixed flats, will remain close to the drum surface (see (e) in figure 5.1), and they should not interact with neighbouring machine surfaces. By the end of the revolving flats, from section 1.1, all the tufts should be broken down into individual fibres.

All the fibres now can be treated as single fibres, and modelled by the equations in chapter 2. The transfer mechanism from cylinder to doffer is also governed by

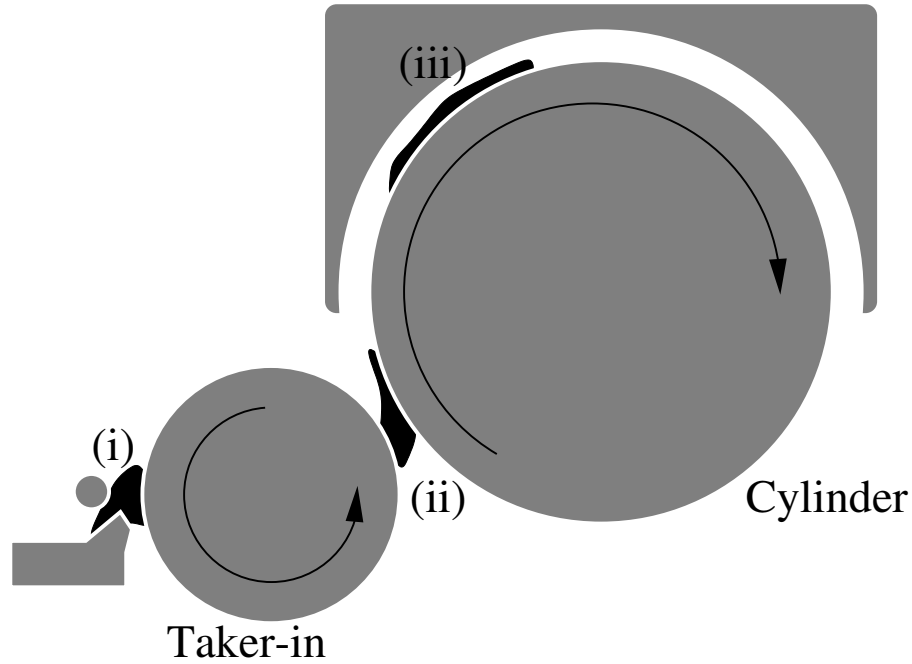


Figure 5.2: A diagram illustrating the regions where entangled tufts are teased into individual fibres or evolve into less entangle tufts.

the fluid dynamics, and the tails of the fibres are offered to the doffer hooks, see (f) in figure 5.1. Some fibres transfer onto the doffer and some remain on the cylinder. Those that remain on the cylinder, see (g) in figure 5.1 will always remain close to the cylinder surface until they return to the doffer nip region where they will be offered again for transfer.

5.1.3 The Doffer

Fibres that transfer onto the doffer are then transported to the stripper, which takes the fibres out of the machine in the form of the sliver, see (j) in figure 5.1. We modelled a single fibre on the doffer in chapter 2, see (h) in figure 5.1. All these models can help carding machine manufacturers, by firstly explaining how fibres evolve and move throughout the machine and then by using the theoretical simulations to suggest optimal hook design, machine settings and machine geometries.

5.1.4 Suggested Further Work

For the single fibre model, the main difficulty arose from finding the correct fluid flow. For both annular and transfer regions improvements could be made to our

aerodynamic approximations. In the case of annular flow, the incorporation of the hooks assumed that the hooks were cylindrical and not radially dependent. This is not the case for the cylinder, doffer and taker-in hooks. The hooks are usually thin triangular prisms. This may give a radial dependence for the amount of fluid entrained by the hooks, which will in turn affect the aerodynamic forces acting on the fibre. Other modifications could be made by considering the viscous model to flow around the two cylinders, and to quantify the level of local instability on a fibre due to turbulence.

There is much more work that could be done on the fibre withdrawal problems and the continuum model. The issue of modelling inter-fibre dynamic friction could be addressed in a number of ways, but the outcome of any such study will benefit both models in chapters 3 and 4. For the two tufts with many interconnecting fibres it is not clear whether or not a change in the body force would have made the solutions more realistic. We could also couple the fibre withdrawal problem with the fibre continuum, and this would hopefully explain the differences in the fibre elongation experiment and the breaking of two tufts with interconnecting fibres.

The main area that would benefit from more analysis is the continuum model of chapter 4. There are three important areas. The first would be to compute the size of the unknown parameters in the model and this could be achieved by constructing a number of experiments. This may require further simulations of the model which will require numerical computation. The second would be theoretical improvements to the empirical laws that govern the evolution of the order parameter and the entanglement scalar field. For the empirical laws, in particular the evolution of entanglement, one may be able to consider the braid topologies so that they understand structural evolution of a tuft under the various stresses.

Appendix A

Dimensional and Dimensionless Numbers

A.1 Drum Speeds

The specifications of the three main cylindrical drums inside the revolving-flat carding machine.

	Unit	Taker-In	Cylinder	Doffer
Diameter	metres	0.25	1.0	0.5
Revolutions	rpm	600 – 1500	500 – 750	25 – 100
Surface Speed	metres per minute	471.2 – 1178	1570 – 2356	39.27 – 157.1
	metres per second	7.854 – 19.63	26.18 – 39.27	0.6545 – 2.618

Table A.1: Drum specifications.

A.2 The Fibres

Two commonly used short-staple fibres are cotton and polyester, and a description of their properties are given in table A.2. The average typical mass flow of polyester through the carding machine is about $\frac{1}{2}$ kg/min.

A.3 Fluid Dynamics and Drag

We compute the Reynolds number, $Re = \frac{UL}{\nu}$, using the circumference of the cross section of the drum, the surface speed of the cylinder and the kinematic viscosity, $\nu = 0.150 \text{ cm}^2/\text{sec}$, for air at twenty degrees Celsius, found in Batchelor (1967) on

	Cotton	Polyester
Length	20.0–48.0 mm	
Average Length	28 mm	38 mm
Diameter	11.5–22.0 micron	
Average Diameter	17 micron	6.26 micron
weight	$6.832 \cdot 10^{-6}$ g	$6.46 \cdot 10^{-6}$ g
Average Density	$1.075 \cdot 10^6$ g/m ³	$5.784 \cdot 10^6$ g/m ³
Moment of Inertia $I = \int_A r^2 dA$	$8.2 \cdot 10^{-21}$ m ⁴	$1.508 \cdot 10^{-22}$ m ⁴
Young's Modulus E	$9 \cdot 10^9$ N/m ²	$7.7 \cdot 10^9$ N/m ²

Table A.2: The properties of cotton and polyester Fibres.

page 594. For the local Reynolds number near the fibre we consider the circumference of the fibre. A polyester fibre has an approximate diameter of 6.26 microns. We estimate the thickness of the boundary layer, in metres, on an infinite smooth cylinder to be $\delta = O(\frac{l}{\sqrt{Re}})$, where l is the typical length, see Rosenhead (1963).

Reynolds No.	Taker-In	Cylinder	Doffer
Cylinder	4.1×10^5 – 10^6	5.5×10^6 – 8.2×10^6	6.9×10^5 – 2.7×10^5
Single fibre	10–26	34–51	0.86–3.4
Boundary layer	7.8×10^{-4} – 1.2×10^{-3}	1.1×10^{-3} – 1.3×10^{-3}	3.0×10^{-3} – 6.0×10^{-3}

Table A.3: The Reynolds numbers of a drum in a homogeneous fluid.

A.3.1 Stokes Drag

The slow flow approximation for flow around a cylinder breaks down (Khayat and Cox, 1989), when

$$\left| \frac{\ln Re - \ln \varepsilon}{(\ln \varepsilon)^2} \right| \ll \left| \frac{1}{\ln \varepsilon} \right|. \quad (\text{A.1})$$

Considering a polyester fibre with aspect ratio $\varepsilon = \frac{a}{l} = 0.000824$, then $\ln \varepsilon = -2.497$ and we find that (A.1) is true only for the doffer.

In chapter 2, a single fibre in Stokes flow was discussed. The important dimensionless parameter $\kappa = \frac{8A\rho_{air}}{aRe\rho_{fibre} \log \frac{1}{\varepsilon}}$, see table A.4, is the ratio of centrifugal force over viscous drag, where A is the radius of the rotating drum, a is the radius of the fibre, Re is the Reynolds number for flow around the fibre, and ρ_{air} and ρ_{fibre} are the respective air and fibre densities.

	Doffer
κ	$6.22 \times 10^5 - 2.48 \times 10^6$

Table A.4: Dimensionless parameter κ for a single fibre with Stokes drag.

A.3.2 Taylor Drag

In section 2, a fibre on the cylinder and taker-in involved the dimensionless parameter ς , which was the ratio of Taylor drag over centrifugal force. The two definitions used are dependent on the local flow, “triple-layer” and shear flow, and these were $\varsigma = \frac{2\pi a(B-A)^2 \rho_{fibre}}{Al^2 \rho_{air}}$ and $\varsigma = \frac{2\pi a \rho_{fibre}}{A \rho_{air}}$ respectively, see table A.5.

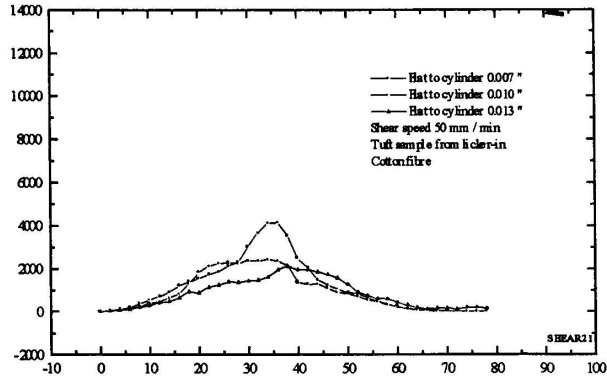
ς	Taker-In	Cylinder
“triple-layer” flow	1.27×10^{-4}	5.01×10^{-4}
shear flow	1.41×10^{-5}	5.57×10^{-5}

Table A.5: Dimensionless parameter ς for single polyester fibre with Taylor drag.

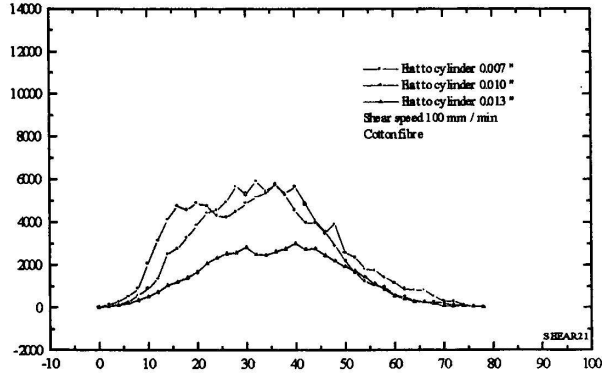
Appendix B

Shear Breaking Experiments on Tufts

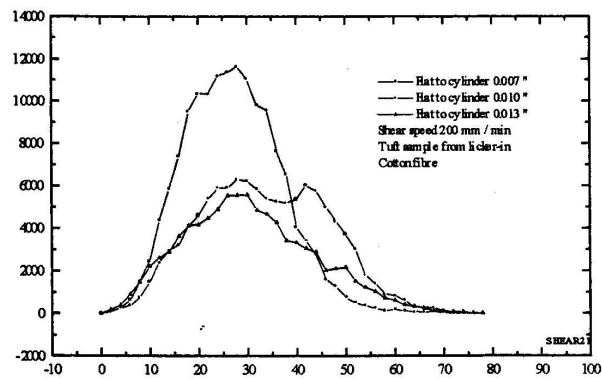
In section 4.2, the “tuft shear force” experiment was described. A tuft is placed between two wired surfaces where one traversed the other at a uniform velocity. The distance between the hooks were varied (0.007, 0.01 and 0.013 inches), as well as the relative velocity of surfaces (50, 100 and 200 mm/min). There are three sets of experiments for cotton, polyester and fine wool, see figures B.1, B.2 and B.3 which plot the force acting on the one of the moving wired surfaces. The two parameters that were changed for each material were the gap between the teeth of the shearing surfaces, namely the flat and cylinder, and the relative speed between the two surfaces. We note that in the experiments, the tufts were hand picked out of typical Laps that would be fed into the carding machine. This underlines the difficulty of the experiment and may go some way to explain, in some cases, the rather volatile findings.



(a) 50 mm/min

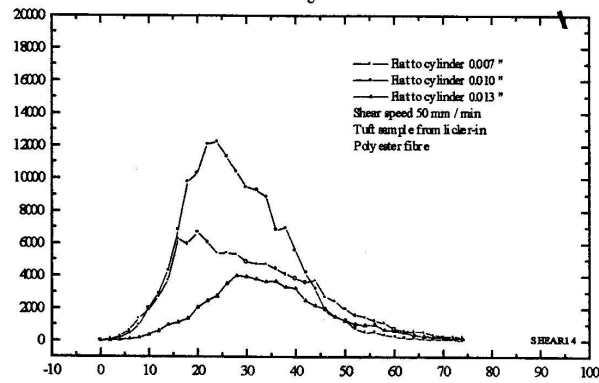


(b) 100 mm/min

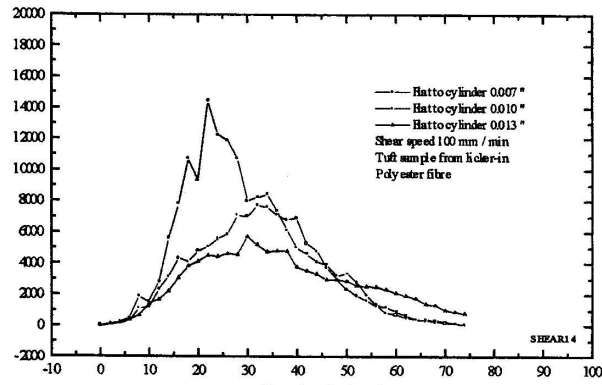


(c) 200 mm/min

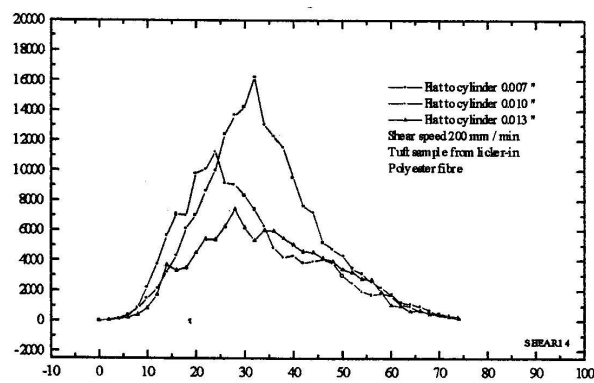
Figure B.1: Graphs of the tuft shear force experiment for cotton: $\frac{\text{force}}{\text{weight}}$ plotted against shearing distance (mm).



(a) 50 mm/min

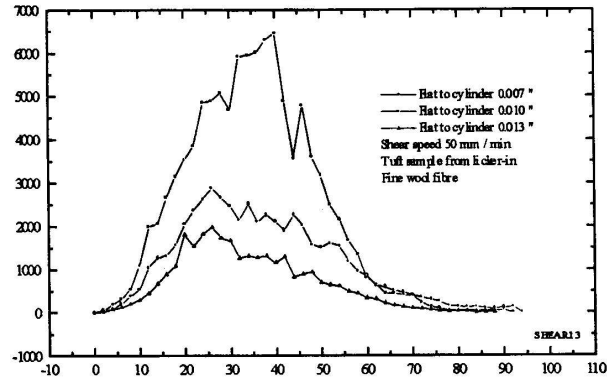


(b) 100 mm/min

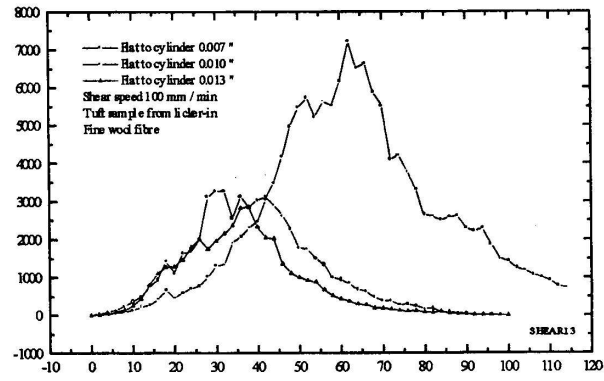


(c) 200 mm/min

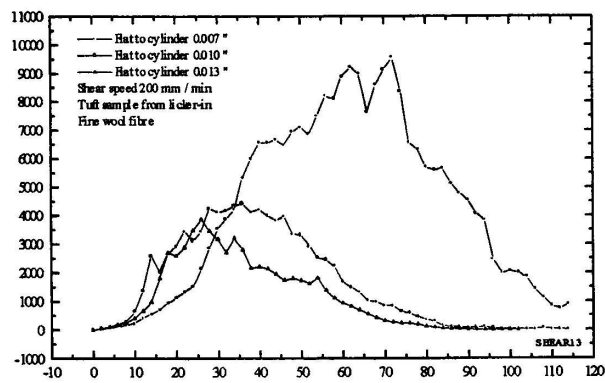
Figure B.2: Graphs of the tuft shear force experiment for polyester with variable speeds: $\frac{force}{weight}$ plotted against extension (mm).



(a) 50 mm/min



(b) 100 mm/min



(c) 200 mm/min

Figure B.3: Graphs of the tuft shear force experiment for fine wool: $\frac{\text{force}}{\text{weight}}$ plotted against extension (mm).

Appendix C

Stability Analysis of a Fibre in the Carding Machine

In chapter 2, we modelled a textile fibre as an inextensible elastica. Due to the dominant aerodynamic forces within the carding machine a fibre behaved approximately as a quasi-steady string. We remind ourselves of the governing dimensionless equations (2.13) and (2.14), from section 2.3.2, and the inextensibility condition:

$$\frac{\partial \mathbf{N}}{\partial s} + \mathbf{F} = \Lambda_1 \frac{\partial^2 \mathbf{R}}{\partial t^2}, \quad (\text{C.1})$$

$$\frac{\partial \mathbf{M}}{\partial s} + \Lambda_2 \frac{\partial \mathbf{R}}{\partial s} \wedge \mathbf{N} = \Lambda_1 \Lambda_2 \frac{\epsilon^2}{2} \sum_{i=1}^2 \frac{\partial^2 \mathbf{d}_i}{\partial t^2} \wedge \mathbf{d}_i, \quad (\text{C.2})$$

$$\frac{\partial \mathbf{R}}{\partial s} \cdot \frac{\partial \mathbf{R}}{\partial s} = 1, \quad (\text{C.3})$$

where the stress resultant vector is $\mathbf{N}(s, t)$, the couple resultant vector is $\mathbf{M}(s, t)$, the body force is $\mathbf{F}$, $\mathbf{R} = (X, Y, Z)$ describes position with respect to a global coordinate system, $(\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3)$ are a local orthonormal triad of principal normals and tangent with respect to the centre line of a fibre and (Λ_1, Λ_2) are dimensionless parameters. We assume as in section 2.3.4 that drag dominates flexural rigidity: $\Lambda_1 \ll \frac{1}{\epsilon^2}$, $\Lambda_2 \gg 1$, where $\epsilon = \frac{a}{l} \ll 1$ is the aspect ratio. In fact $\Lambda_1 \sim 10^{-3}$, $\Lambda_2 \sim 10^4$ and $\epsilon \sim 10^{-4}$, which means that from (C.2),

$$\frac{\partial \mathbf{R}}{\partial s} \wedge \mathbf{N} \approx \mathbf{0}, \quad (\text{C.4})$$

and so,

$$\mathbf{N} \approx T(s, t) \mathbf{d}_3, \quad (\text{C.5})$$

where T is the tension in the fibre and $\frac{\partial}{\partial s} \mathbf{R} = \mathbf{d}_3$. Therefore, it suffices to consider just (C.1), (C.3) and (C.5).

We begin the linear stability analysis by writing

$$\mathbf{R} = \bar{\mathbf{R}}(s) + \tilde{\mathbf{R}}(s, t), \quad (\text{C.6})$$

where bar and tilde denote steady and perturbed solutions respectively. Then from equation (C.5) we write

$$T = \bar{T}(s) + \tilde{T}(s, t). \quad (\text{C.7})$$

We note that for the steady state problem the global coordinates relate to the fibre's local frame of reference with

$$\frac{d\bar{\mathbf{R}}}{ds} = (\cos \theta(s), \sin \theta(s), 0) = \mathbf{d}_3(s), \quad (\text{C.8})$$

where θ is the angle between the global X -axis and the local $\mathbf{d}_3$ tangent vector.

The first step is to find an analytical steady state solution $\bar{\mathbf{R}}(s)$ and $\bar{T}(s)$. We choose to consider the problem of uniform flow and Stokes drag, see equation (2.61). Therefore the body force $\mathbf{F}$ in equation (C.1) is composed of Stokes drag and rotational force

$$\mathbf{F} = \kappa \left(2I - \frac{\partial \mathbf{R}}{\partial s} \frac{\partial \mathbf{R}^T}{\partial s} \right) \mathbf{i} + \mathbf{j}, \quad (\text{C.9})$$

where I is the identity matrix, κ is a dimensionless parameter of drag over rotational force, $(\mathbf{i}, \mathbf{j})$ are Cartesian orthonormal unit vectors and $\mathbf{j}$ represents the contribution due to centrifugal force. We could also compute a solution for Taylor drag.

The steady state problem consists of two equations from (C.1) and two geometrical conditions from (C.8):

$$\frac{d\bar{T}}{ds} + \kappa \cos \theta + \sin \theta = 0, \quad (\text{C.10})$$

$$\bar{T} \frac{d\theta}{ds} - 2\kappa \sin \theta + \cos \theta = 0, \quad (\text{C.11})$$

$$\frac{d\bar{Y}}{ds} - \sin \theta = 0, \quad (\text{C.12})$$

$$\frac{d\bar{X}}{ds} - \cos \theta = 0, \quad (\text{C.13})$$

where $\bar{Y}$ and $\bar{X}$ represent the approximate distance from and along the carding drum surface. The boundary conditions for (C.10) – (C.13) are

$$T = 0 \quad \text{at} \quad s = 1, \quad (\text{C.14})$$

$$\mathbf{R} = \mathbf{0} \quad \text{at} \quad s = 0, \quad (\text{C.15})$$

$$\theta \text{ is non-singular}, \quad (\text{C.16})$$

and these also apply to the unsteady problem (C.1), (C.3) and (C.5) for all time t . The steady state solution is a straight thread:

$$\theta = \operatorname{arccot} 2\kappa \quad \text{and} \quad \bar{T}(s) = \frac{1}{\sqrt{4\kappa^2 + 1}} [2\kappa^2 + 1] (1 - s), \quad (\text{C.17})$$

$$\bar{Y}(s) = \frac{s}{\sqrt{4\kappa^2 + 1}}, \quad \bar{X}(s) = \frac{2\kappa s}{\sqrt{4\kappa^2 + 1}}, \quad \bar{Y} = \frac{\bar{X}}{2\kappa}, \quad (\text{C.18})$$

where $\bar{R} = (\bar{X}, \bar{Y}, 0)$.

For linear stability analysis we now consider equations (C.1), (C.3) and (C.5) with (C.6) and (C.7), ignoring quadratic terms in $\tilde{T}$ and $\tilde{\mathbf{R}}$. We write the displacement in local variables $\tilde{\mathbf{R}} = (\mathbf{d}_3 f(s, t) + \mathbf{d}_1 g(s, t) + \mathbf{d}_2 h(s, t))$ to simplify the algebra. From the inextensibility constraint (C.3) we find that

$$\frac{\partial f}{\partial s} = 0, \quad (\text{C.19})$$

and with boundary condition (C.15) this simplifies to

$$f(s, t) = 0. \quad (\text{C.20})$$

The linear stability equations, after some algebraic manipulation, can be written as

$$\Lambda_1 \frac{\partial^2 g}{\partial t^2} = \frac{1}{\sqrt{4\kappa^2 + 1}} \left[(2\kappa^2 + 1)(1 - s) \frac{\partial^2 g}{\partial s^2} - (4\kappa^2 + 1) \frac{\partial g}{\partial s} \right], \quad (\text{C.21})$$

$$\Lambda_1 \frac{\partial^2 h}{\partial t^2} = \frac{1}{\sqrt{4\kappa^2 + 1}} \left[(2\kappa^2 + 1)(1 - s) \frac{\partial^2 h}{\partial s^2} - (4\kappa^2 + 1) \frac{\partial h}{\partial s} \right], \quad (\text{C.22})$$

$$\frac{\partial \tilde{T}}{\partial s} = -\frac{\kappa}{\sqrt{4\kappa^2 + 1}} \frac{\partial g}{\partial s}, \quad (\text{C.23})$$

with boundary conditions

$$h = g = 0 \quad \text{at} \quad s = 0, \quad (\text{C.24})$$

$$\tilde{T} = 0 \quad \text{at} \quad s = 1, \quad (\text{C.25})$$

$$h(s, t) \text{ and } g(s, t) \text{ are non-singular.} \quad (\text{C.26})$$

We note that both the orthogonal perturbations satisfy the same governing equation.

Looking for separable solutions for either g and h of the form $e^{i\frac{\omega t}{\sqrt{\Lambda_1}}} \Sigma(s)$ we find that either (C.21) or (C.22) becomes

$$a(1 - s) \frac{d^2 \Sigma}{ds^2} - b \frac{d \Sigma}{ds} + \omega^2 \Sigma = 0, \quad (\text{C.27})$$

where $a = \frac{2\kappa^2+1}{\sqrt{4\kappa^2+1}}$ and $b = \sqrt{4\kappa^2+1}$. The solution to equation (C.27) is composed of Bessel functions

$$\Sigma(s) = (1-s)^{\frac{a-b}{2a}} \left[C_0 J_{1-\frac{b}{a}} \left(2\omega \sqrt{\frac{1-s}{a}} \right) + C_1 J_{\frac{b}{a}-1} \left(2\omega \sqrt{\frac{1-s}{a}} \right) \right]. \quad (\text{C.28})$$

where $\frac{b}{a} = 2 - \frac{1}{2\kappa^2+1}$ which $\in (1, 2)$, $1 - \frac{b}{a} < 0$ and $\frac{b}{a} - 1 > 0$. At $s = 1$, $J_{1-\frac{b}{a}}$ becomes singular and to satisfy condition (C.26) we suppress the singular term by imposing $C_0 = 0$. Implementing the boundary condition (C.24) gives

$$C_1 J_{\frac{b}{a}-1} \left(\frac{2\omega}{\sqrt{a}} \right) = 0, \quad (\text{C.29})$$

which means that

$$\omega = \omega_n = \frac{\sqrt{a} j_n}{2}, \quad (\text{C.30})$$

where j_n is the n -th root of $J_{\frac{b}{a}-1}(\xi)$. Hence the set ω_n is a sequence of positive numbers. Since ω and Λ_1 are real, the perturbations oscillate neutrally, and the flow is linearly stable.

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