

# The Avrunin and Scott theorem and a truncated polynomial algebra

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## Abstract

We prove an analogue of a theorem of Avrunin and Scott for truncated polynomial algebras  $\Lambda_m := k[X_1, \dots, X_m]/(X_i^2)$  over an algebraically closed field of arbitrary characteristic. The Avrunin and Scott theorem relates the support variety for a finite-dimensional  $kE$ -module to its rank variety (where  $\text{char}(k) = p$  and  $E$  is an elementary abelian  $p$ -group). The analogue of the Avrunin and Scott theorem relates the support variety for a finite-dimensional  $\Lambda_m$ -module (using Hochschild cohomology) to its rank variety (developed in [K. Erdmann, M. Holloway, Rank varieties and projectivity for a class of local algebras, *Math. Z.* 247 (2004) 441–460] using Clifford algebras). Along the way to proving our main result we provide a new proof of the Avrunin and Scott theorem for elementary abelian  $p$ -group algebras which we are then able to generalise to the setting of  $\Lambda_m$ -algebras.

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## 1. Introduction

The theories of support and rank varieties for  $kG$ -modules (with  $G$  a finite group and  $k$  an algebraically closed field of characteristic  $p > 0$  dividing  $|G|$ ) have had a significant impact on modular representation theory. Inspired by this success similar theories have been developed in other contexts. In particular for:  $p$ -restricted Lie algebras (see [14,19]) and subsequently for the

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more general set up of finite group schemes (see [3–5,15,16]), Steenrod algebras (see [20,22]), quantum groups (see [17,21,23,24]) and complete intersections (see [1]). In addition the theory of support varieties has been developed for (well behaved) finite-dimensional selfinjective  $k$ -algebras using the Hochschild cohomology ring as a replacement for the group cohomology ring (see [13,25]).

One of the main results in the support variety theory for groups is a theorem of Avrunin and Scott (see [2]) extending Quillen's stratification theorem for  $V_G = V_G(k)$  (the support variety of the trivial module) to a stratification of  $V_G(M)$  the support variety of  $M$  an arbitrary finite-dimensional  $kG$ -module. A crucial part of their result was a theorem proving what was then Carlson's conjecture. This conjecture stated that for elementary abelian  $p$ -groups the support variety of a finite-dimensional module is the same as the rank variety for the module (the rank variety having been introduced by Carlson, the definition making no reference to cohomology). It is worth noting that subsequently Carlson gave another proof of his conjecture (see [10]) but his approach does not seem to generalise to our setting. Further details on the background and history of the Avrunin and Scott theorem and Carlson's conjecture can be found in [2].

It is this theorem of Avrunin and Scott establishing Carlson's conjecture and its obvious analogue for a certain truncated polynomial algebra that is the subject and motivation for this paper and for clarity we should say that throughout this paper when we refer to the Avrunin and Scott theorem we have in mind the theorem establishing Carlson's conjecture. In Section 4 we present a new proof of the Avrunin and Scott theorem by providing a more representation theoretic description of the support and rank varieties (we call these stable map descriptions). We then turn to a certain class truncated polynomial algebras,  $\Lambda_m$ , which possess a rank variety theory (see [12]). Moreover, these algebras satisfy appropriated finiteness conditions allowing the support variety theory (developed in [25] and further extended in [13]) to be applicable. We consider the analogue of the Avrunin and Scott theorem for these algebras and, with our more representation theoretic perspective, prove the corresponding analogue (Theorem 8.2).

The paper is organised as follows. In Section 2 we recall the definition of the support variety for a  $kG$ -module, and in Section 2.3 give our stable map description of these varieties. In Section 3 we review the definition of the rank variety for a  $kE$ -module and provide a stable map description of these varieties. In Section 4 we state the Avrunin and Scott theorem and give a proof using our stable map description of the varieties involved. The rest of the paper is concerned with the class of algebras  $\Lambda_m$ , which are defined in Section 5. Section 6 briefly recalls some of the definitions and results of the support variety theory for general selfinjective  $k$ -algebras, while Section 7 does the same for the rank variety theory for the algebras  $\Lambda_m$ . Finally, in Section 8, motivated by the arguments used in our proof of the Avrunin and Scott theorem, we prove the analogue for the algebras  $\Lambda_m$ . Appendix A contains some background details; in Appendix A.1 on the varieties considered in the paper and in Appendix A.2 on the Bockstein map in group cohomology. Appendix B describes, for the interested reader, some the technicalities involved with other proofs of the Avrunin and Scott theorem.

Throughout the paper  $k$  will always denote an algebraically closed field and where  $\text{char}(k) = p \geq 0$  becomes important we will specify it explicitly.  $G$  will denote an arbitrary finite group, with  $p > 0$  dividing  $|G|$ , and  $E$  will always be an elementary abelian  $p$ -group of rank  $n$ . All algebras considered will be finite-dimensional  $k$ -algebras and given such an algebra  $\Lambda$  we will denote by  $J(\Lambda)$  (or simply  $J$  if the context makes it clear which  $\Lambda$  is being referred to) the Jacobson radical of  $\Lambda$ . All modules considered will be finite-dimensional  $\Lambda$ -modules and the category of left (respectively right)  $\Lambda$ -modules will be denoted by  $\Lambda\text{-mod}$  (respectively  $\text{mod-}\Lambda$ ). The full subcategory of projective  $\Lambda$ -modules will be denoted by  $\Lambda\text{-proj}$  (respec-

tively  $\text{proj-}\Lambda$ ). We will denote by  $\Lambda\text{-mod}$  the stable module category and by  $\Omega$  the Heller shift. We adopt the following notation for dualities  $D- := \text{Hom}_k(-, k) : \Lambda\text{-mod} \rightarrow \text{mod-}\Lambda$  and  $-^\vee := \text{Hom}_\Lambda(-, \Lambda) : \Lambda\text{-proj} \rightarrow \text{proj-}\Lambda$ . For symmetric algebras, such as group algebras, it is well known that  $D- \cong -^\vee$  and we simply denote this common duality by  $-^*$ . Recall also that for a symmetric algebra  $\Lambda$ , if  $M$  is a  $\Lambda$ -bimodule which is projective as a left or right  $\Lambda$ -module then  $M^* \otimes_\Lambda -$  is both left and right adjoint to  $M \otimes_\Lambda -$ . Given a finite-dimensional  $k$ -algebra  $\Lambda$ ,  $\Lambda^e := \Lambda \otimes_k \Lambda^{\text{op}}$  denotes the enveloping algebra and we will freely view  $\Lambda$ -bimodules as equivalent to left  $\Lambda^e$ -modules. Given a graded ring  $R = R^*$ , the degree of a homogeneous element  $z \in R$  will be written  $\deg(z)$ .

## 2. The support variety for group algebras

In this section we recall the definition and some of the properties of support varieties for group algebras that do not depend upon the Avrunin and Scott theorem. A good reference for the results quoted here (and much more) can be found in the two volumes [6,7]. We then re-interpret some of them to provide what we call a stable map description of the support variety for a module.

Recall that for  $M, M' \in kG\text{-mod}$ ,  $\zeta \in \text{Ext}_{kG}^n(k, k)$  and  $\eta \in \text{Ext}_{kG}^m(M, M')$  we have the cup product  $\zeta \cup \eta \in \text{Ext}_{kG}^{m+n}(M, M')$  which turns  $\text{Ext}_{kG}^*(M, M')$  into a graded  $H^*(G, k)$ -module. We denote the ideal that is the annihilator of this module by  $I(M, M')$ . In the particular case when  $M = M'$  and  $\eta$  is the identity map  $1_M \in \text{Hom}_{kG}(M, M)$  then we have the ring homomorphism

$$H^*(G, k) \xrightarrow{- \otimes_k M} \text{Ext}_{kG}^*(M, M),$$

and we will write  $I(M) := \ker(- \otimes_k M)$ . The cup product factorises as  $\zeta \cup \eta = (\zeta \otimes_k M') \circ \eta = \pm \eta \circ (\zeta \otimes_k M)$ , where  $\circ$  denotes Yoneda composition, so that

$$I(M', M) \supseteq I(M) + I(M'). \quad (2.1)$$

Thanks to the Evens, Golod and Venkov theorem we know that  $H^*(G, k)$  is an affine  $k$ -algebra and that all the above modules (over  $H^*(G, k)$ ) are finitely generated. Hence we can define the affine variety  $V_G := \text{MaxSpec-}H^*(G, k)$ , and the support variety associated to the pair of modules  $M, M'$  to be the affine subvariety defined by the ideal  $I(M, M')$ , that is

$$V_G(M, M') := V(I(M, M')) = \{\mathfrak{m} \in V_G \mid I(M, M') \subseteq \mathfrak{m}\}.$$

The variety associated to a single module  $M$  is then simply defined as  $V_G(M) := V_G(M, M)$  (the variety associated to the ideal  $I(M)$ ). It follows from (2.1) that  $V_G(M, M') \subseteq V_G(M) \cap V_G(M')$ .

We summarise further properties of these support varieties, which have proofs independent of the Avrunin and Scott theorem. (For parts (i)–(iv) see [7]; for part (vi) see [13], and (v) follows from (vi).) The first proof of (vi) was given by Avrunin and Scott [2].

**Proposition 2.1.** *If  $M, M' \in kG\text{-mod}$  then the following hold:*

- (i)  $V_G(M) = V_G(M^*) = V_G(\Omega M)$ .
- (ii)  $V_G(M \oplus M') = V_G(M) \cup V_G(M')$ .
- (iii)  $V_G(M) = \bigcup_S V_G(S, M)$ , where  $S$  runs over the composition factors of  $M$ .

- (iv)  $V_G(M) = \{0\} \Leftrightarrow M$  is projective.  
 (v)  $V_G(M) \cap V_G(M') = \{0\} \Rightarrow \text{Ext}_{kG}^n(M, M') = 0$ , for  $n > 0$ .

In addition we have the tensor intersection rule:

- (vi)  $V_G(M \otimes_k M') = V_G(M) \cap V_G(M')$ .

### 2.1. The $L_\zeta$ -modules

Here we recall the construction of an important class of modules,  $L_\zeta$ , defined by homogeneous elements  $\zeta \in H^*(G, k)$ . Given  $\zeta \in H^n(G, k)$  we can represent  $\zeta$  as a map  $\zeta : \Omega^n k \rightarrow k$ , and  $L_\zeta$  is defined to be the kernel of this map. In fact, with  $P$  an injective hull for  $\Omega^n k$ , and  $M_\zeta := \Omega^{-1} L_\zeta$  we have the following diagram.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & L_\zeta & \xlongequal{\quad} & L_\zeta & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \Omega^n k & \longrightarrow & P & \longrightarrow & \Omega^{n-1} k \longrightarrow 0 \\
 & & \downarrow \zeta & & \downarrow & & \parallel \\
 0 & \longrightarrow & k & \longrightarrow & M_\zeta & \longrightarrow & \Omega^{n-1} k \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 
 \end{array}$$

What this diagram is really saying is that we have the following triangle in  $kG\text{-mod}$ :

$$\Omega^n k \xrightarrow{\zeta} k \rightarrow M_\zeta \rightarrow \Omega^{-1}(\Omega^n k).$$

Rotating the triangle we have

$$k \rightarrow M_\zeta \rightarrow \Omega^{n-1} k \xrightarrow{-\Omega^{-1}(\zeta)} \Omega^{-1} k. \quad (2.2)$$

We will need the following important fact about these modules, which can be proved without recourse to the Avrunin and Scott theorem, for example essentially the same proof as that used in [13] will do (but compare with [7]).

#### Theorem 2.2.

$$V_G(M_\zeta) = V_G(L_\zeta) = V(\langle \zeta \rangle).$$

**Remark 2.3.**  $V(\langle \zeta \rangle)$  is the variety associated to the ideal generated by  $\zeta$  and is therefore the hypersurface determined by the element  $\zeta$ .

## 2.2. Elementary abelian $p$ -groups

Let  $E$  denote an elementary abelian  $p$ -group of rank  $n$ . The cohomology ring of  $E$  (see [6, Corollary 3.5.7]) is given by following theorem.

### Theorem 2.4.

$$H^*(E, k) \cong \begin{cases} k[y_1, \dots, y_n] & \text{with } \deg(y_i) = 1 \text{ for } p = 2, \\ k[z_1, \dots, z_n] \otimes \Lambda(y_1, \dots, y_n) & \text{with } \deg(y_i) = 1, \deg(z_i) = 2 \text{ for } p > 2. \end{cases}$$

Here  $\Lambda(y_1, \dots, y_n)$  is the exterior algebra. In the case  $p > 2$  a Noether normalisation for  $H^*(E, k)$  is given by the polynomial subalgebra  $k[z_1, \dots, z_n]$ . Clearly in this case a choice of  $k$ -basis for the vector subspace of  $H^2(E, k)$  spanned by the  $\{z_i\}$  determines an identification  $V_E \cong \mathbb{A}^n$ . This vector subspace is the image of the Bockstein map  $\beta: H^1(E, k) \rightarrow H^2(E, k)$  (see Appendix A.2). The case  $p = 2$  is simpler with  $H^*(E, k)$  already a Noether normalisation and choice of system of parameters amounting to a choice of  $k$ -basis for  $H^1(E, k) = \text{Ext}_{kE}^1(k, k)$ .

For the case  $G = E$ , we can take, for  $p > 2$ , the generators  $\{z_i\}$  of the Noether normalisation as  $z_i = \beta(y_i)$ . If  $p = 2$ , then  $\beta(y_i) = y_i^2$ .

## 2.3. The stable map description

For  $M \in kG\text{-mod}$  we know that  $V_G(M)$  is a homogeneous affine variety. Given a point  $\alpha \in V_G$ , we denote by  $\ell_\alpha$  the line in  $V_G$  through  $\alpha$  (see Appendix A.1 for the meaning of lines). In the case  $G$  is a  $p$ -group (primarily we will be interested in  $G = E$  being elementary abelian) we will re-interpret some well-known results about the support variety to provide a stable map criterion for determining which lines  $\ell_\alpha$  are contained in  $V_G(M)$ . In order to do this we will need to construct modules which have varieties exactly  $\ell_\alpha$ . That this is possible follows from more general facts about support varieties for  $G$  an arbitrary finite group.

**Theorem 2.5.** (Carlson [9]) *Given any closed homogeneous subvariety  $V \subseteq V_G$  there exists a module  $M \in kG\text{-mod}$  such that  $V_G(M) = V$ .*

Note for  $V = \ell_\alpha$  every indecomposable non-projective submodule of the module given also has variety  $\ell_\alpha$ . The following result describes modules whose varieties are lines.

**Theorem 2.6.** (Carlson [8]) *If  $M \in kG\text{-mod}$  is indecomposable and has variety a line, then  $M$  is a periodic module and the period must divide the degree of one of the generators of a Noether normalisation of  $H^*(G, k)$ .*

There is also a converse result [7, Corollary 5.10.3] which says

**Theorem 2.7.** *If  $M \in kG\text{-mod}$  is indecomposable and periodic, then  $V_G(M)$  is a single line through the origin in  $V_G$ .*

For us the upshot of these theorems is contained in the following proposition.

**Proposition 2.8.** *If  $0 \neq \alpha \in V_G$  then there exists  $T_\alpha \in kG\text{-mod}$  such that*

$$\Omega T_\alpha \cong T_\alpha \quad \text{and} \quad V_G(T_\alpha) = \ell_\alpha.$$

**Proof.** By Theorem 2.5 there exists  $M \in kG\text{-mod}$  such that  $V_G(M) = \ell_\alpha$ . Choose  $M'$  to be any indecomposable non-projective summand of  $M$  (it has a non-projective summand as otherwise  $V_G(M) = \{0\}$ ). By Theorem 2.6  $M'$  is periodic of period  $r$  say. Now  $T_\alpha := M' \oplus \cdots \oplus \Omega^{r-1} M'$  clearly has the desired properties.  $\square$

**Remarks 2.9.**

- (1) Given  $\alpha \in V_G$  the  $T_\alpha$  given by Proposition 2.8 is not unique. In the sequel we will write  $T_\alpha$  to mean any module given by Proposition 2.8.
- (2) In the particular case when  $G = E$  is an elementary abelian group we will use the identification  $V_E \cong \mathbb{A}^n$ , with  $\alpha \in V_E$  corresponding to  $\lambda \in k^n$ , so that we have  $T_\lambda \in kE\text{-mod}$ , with  $V_E(T_\lambda) = \ell_\lambda$  (note in this case that the  $r$  appearing in the proof of Proposition 2.8 will be  $r \leq 2$ ).

For  $p$ -groups the modules  $T_\alpha$  have exactly the properties needed to give a stable map description of the support variety. To emphasise that it is the properties of the  $T_\alpha$ -modules (rather than their particular construction) that are important we have the following theorem.

**Theorem 2.10.** *Let  $G$  be a  $p$ -group,  $0 \neq \alpha \in V_G$  and  $X_\alpha \in kG\text{-mod}$  be such that  $V_G(X_\alpha) = \ell_\alpha$ , then, for  $M \in kG\text{-mod}$ ,*

$$\ell_\alpha \subseteq V_G(M) \quad \Leftrightarrow \quad \underline{\text{Hom}}_{kG}(X_\alpha, M) \neq 0.$$

**Proof.** By the tensor intersection property,  $V_G(X_\alpha \otimes_k M^*) = V_G(X_\alpha) \cap V_G(M^*) = V_G(X_\alpha) \cap V_G(M)$ . But  $V_G(X_\alpha) = \ell_\alpha$  and so

$$\ell_\alpha \subseteq V_G(M) \quad \Leftrightarrow \quad V_G(X_\alpha \otimes_k M^*) \neq \{0\}.$$

Moreover, by 2.1(iv) we know  $V_G(X_\alpha \otimes_k M^*) \neq 0$  if and only if  $X_\alpha \otimes_k M^*$  is not projective. Since  $G$  is a  $p$ -group, this is equivalent with  $\underline{\text{Hom}}_{kG}(X_\alpha \otimes_k M^*, k) \neq 0$ , that is  $\underline{\text{Hom}}_{kG}(X_\alpha, M) \neq 0$ .  $\square$

**Corollary 2.11.** *Let  $0 \neq \lambda \in k^n$  and  $M \in kE\text{-mod}$  then*

$$\ell_\lambda \subseteq V_E(M) \quad \Leftrightarrow \quad \underline{\text{Hom}}_{kE}(T_\lambda, M) \neq 0.$$

**Proof.** Immediate from Theorem 2.10 upon setting  $X_\alpha = T_\lambda$ , if  $\alpha \in V_E$  corresponds to  $\lambda$ , and using Proposition 2.8.  $\square$

### 3. The rank variety for elementary abelian $p$ -group algebras

In this section we recall Carlson's definition of the rank variety for an elementary abelian  $p$ -group. We then re-interpret the definition to provide a stable map description of these varieties analogous to that given in Theorem 2.10.

### 3.1. Definitions

Given  $E$ , an elementary abelian  $p$ -group of rank  $n$ , the rank variety assigns a homogeneous affine variety (or projective variety), denoted by  $V_E^r(M)$  to  $M \in kE\text{-mod}$ . In order to define this variety we will need to fix a coordinate system for the underlying affine space  $V_E^r := \mathbb{A}^n$  and this means choosing a  $k$ -basis for the  $n$ -dimensional vector space  $J(kE)/J^2(kE)$ . If  $E$  is presented as  $E := \langle g_1, \dots, g_n \mid g_i^p, [g_i, g_j] \rangle$  then an obvious basis for  $J(kE)/J^2(kE)$  is given by the cosets of  $\{x_i := g_i - 1\}$ , for  $1 \leq i \leq n$ . Clearly  $x_i^p = 0$  and we have

$$kE = k\langle x_1, \dots, x_n \rangle \cong k[X_1, \dots, X_n]/(X_i^p).$$

Before recalling the definition of the rank variety, note that a choice of  $k$ -basis for  $J(kE)/J^2(kE)$  is equivalent, via duality, to a choice of basis for the vector space

$$\text{Hom}_k(J(kE)/J^2(kE), k) \cong \text{Hom}_{kE}(J(kE)/J^2(kE), k).$$

But we have a natural isomorphism

$$\text{Hom}_{kE}(J(kE)/J^2(kE), k) \cong \text{Ext}_{kE}^1(k, k).$$

To summarise, a choice of identification  $V_E^r \cong \mathbb{A}^n$  is equivalent to fixing a  $k$ -basis for the vector space  $\text{Ext}_{kE}^1(k, k)$ .

The definition of the rank variety rests upon the notion of a shifted cyclic subgroup. Given  $0 \neq \lambda = (\lambda_1, \dots, \lambda_n) \in k^n$  define

$$u_\lambda := \sum_{i=1}^n \lambda_i x_i \in kE$$

and note that  $u_\lambda^p = 0$ , so that  $\langle 1 + u_\lambda \rangle$  is a cyclic  $p$ -group inside  $kE$ ; a so-called shifted subgroup.

**Definition 3.1.** Given  $M \in kE\text{-mod}$ , then the rank variety of  $M$  is defined by

$$V_E^r(M) := \{0\} \cup \{0 \neq \lambda \in V_E^r \mid \text{rank}(u_\lambda) < ((p-1)/p) \dim M\},$$

where  $u_\lambda : M \rightarrow M$  is multiplication by  $u_\lambda$ .

### Remarks 3.2.

- (i) This is clearly a homogeneous affine variety.
- (ii) The rank condition is equivalent to the condition that  $M \downarrow_{\langle 1+u_\lambda \rangle}$  is not a free  $k\langle 1+u_\lambda \rangle$ -module. This follows easily by considering the structure of indecomposable modules for a cyclic  $p$ -group. The same considerations also show that the rank condition is equivalent to the condition

$$\text{nullity}(u_\lambda) - \dim M + \text{nullity}(u_\lambda^{p-1}) > 0.$$

In order to provide a stable map description of the rank variety we will need to examine some properties of the cyclic modules  $kEu_\lambda$ .

**Proposition 3.3.** *Let  $0 \neq \lambda \in k^n$ . The cyclic module  $kEu_\lambda \in kE\text{-mod}$  has the following properties:*

- (1)  $kEu_\lambda$  is indecomposable,  $\Omega(kEu_\lambda) \cong kEu_\lambda^{p-1}$  and  $\Omega^2(kEu_\lambda) \cong kEu_\lambda$ . So, in particular, it is a periodic module of period 2 if  $p > 2$  and period 1 if  $p = 2$ .
- (2)  $V_E^r(kEu_\lambda) = \ell_\lambda$ .

**Proof.** Property (1) is easy to see, for we may choose a basis of  $J(kE)/J^2(kE)$  starting with  $u_\lambda$ . That is we may assume  $u_\lambda = x_1$  (so  $\lambda = (1, 0, \dots, 0)$ ) and now (1) is obvious. For property (2) we can assume  $n \geq 2$  since it is trivially true for  $n = 1$ . Keeping the assumption  $u_\lambda = x_1$ , and considering the map  $u_\mu : kEu_\lambda \rightarrow kEu_\lambda$ , we have to show that

$$\begin{aligned} \text{nullity}(u_\mu) &\leq p^{n-2}(p-1) && \text{if } \mu_i \neq 0, \text{ for some } i > 1, \\ \text{nullity}(u_\mu) &> p^{n-2}(p-1) && \text{if } \mu = (1, 0, \dots, 0). \end{aligned}$$

If some  $\mu_i \neq 0$  (for  $i > 1$ ) then we may take this as our second basis vector in a basis of  $J(kE)/J^2(kE)$ . That is we may assume  $x_2 = u_\mu$  and so we need to examine  $\ker(u_\mu)$ . But it is clear that

$$\left\{ \prod_{i=3}^n x_i^{s_i} x_2^{p-1} x_1^r \mid 1 \leq r \leq p-1 \text{ and } 0 \leq s_i \leq p-1 \text{ for } i > 2 \right\}$$

is a basis of  $\ker(u_\mu)$  therefore  $\text{nullity}(u_\mu) = p^{n-2}(p-1)$  and hence  $\mu \notin V_E^r(kEx_1)$ , if some  $\mu_i \neq 0$  (for  $i > 1$ ). The only possibility left is  $\mu_i = 0$  for  $i > 1$ , which means  $u_\mu = \mu_1 x_1$ , but in this case it is easy to see that

$$\left\{ \prod_{i=2}^n x_i^{s_i} x_1^{p-1} \mid 0 \leq s_i \leq p-1 \text{ for } i > 1 \right\}$$

is a basis for  $\ker(u_\mu)$ , so  $\text{nullity}(u_\mu) = p^{n-1} > p^{n-2}(p-1)$ , and therefore  $V_E^r(kEu_\lambda) = \ell_\lambda$ .  $\square$

### 3.2. The stable map description

The elements  $u_\lambda \in kE$ , for  $0 \neq \lambda \in k^n$ , can be used to give the following, stable map, description of the rank variety. The proof of this is essentially contained in [12, Lemma 3.7], but for completeness we include a proof here.

**Theorem 3.4.** *If  $M \in kE\text{-mod}$  and  $0 \neq \lambda \in k^n$  then*

$$\ell_\lambda \subseteq V_E^r(M) \iff \underline{\text{Hom}}_{kE}(kEu_\lambda, M) \neq 0 \iff \underline{\text{Hom}}_{kE}(kEu_\lambda^{p-1}, M) \neq 0.$$



**Proof.** Applying  $\text{Hom}_{kE}(-, M)$  to the exact sequence

$$kE \xrightarrow{u_\lambda^{p-1}} kE \rightarrow kEu_\lambda \rightarrow 0,$$

and using the identification  $\text{Hom}_{kE}(kE, M) \cong M$ , it is easy to see that  $\text{Hom}_{kE}(kEu_\lambda, M) \cong \ker(u_\lambda^{p-1} : M \rightarrow M)$ . Similarly  $\text{Hom}_{kE}(kEu_\lambda^{p-1}, M) \cong \ker(u_\lambda : M \rightarrow M)$ .

Now apply  $\text{Hom}_{kE}(-, M)$  to the short exact sequence

$$0 \rightarrow kEu_\lambda^{p-1} \rightarrow kE \xrightarrow{u_\lambda} kEu_\lambda \rightarrow 0.$$

This gives the exact sequence

$$0 \rightarrow \text{Hom}_{kE}(kEu_\lambda, M) \rightarrow \text{Hom}_{kE}(kE, M) \rightarrow \text{Hom}_{kE}(kEu_\lambda^{p-1}, M) \rightarrow \text{Ext}_{kE}^1(kEu_\lambda, M) \rightarrow 0.$$

Using the identifications above together with  $\text{Ext}_{kE}^1(kEu_\lambda, M) \cong \underline{\text{Hom}}_{kE}(\Omega kEu_\lambda, M) \cong \underline{\text{Hom}}_{kE}(kEu_\lambda^{p-1}, M)$ , then on taking the alternating sum of the dimensions we arrive at

$$\text{nullity}(u_\lambda) - \dim M + \text{nullity}(u_\lambda^{p-1}) = \dim \underline{\text{Hom}}_{kE}(kEu_\lambda^{p-1}, M).$$

Interchanging the role of  $u_\lambda$  and  $u_\lambda^{p-1}$  we get

$$\text{nullity}(u_\lambda^{p-1}) - \dim M + \text{nullity}(u_\lambda) = \dim \underline{\text{Hom}}_{kE}(kEu_\lambda, M).$$

These two formulae together finish the proof by Remarks 3.2(ii).  $\square$

#### 4. The Avrunin and Scott theorem

In this section we prove the Avrunin and Scott theorem (Theorem 4.2) using the stable map descriptions of both the rank and support varieties described earlier and refer the interested reader to Appendix B for a discussion of some of the technicalities involved with earlier proofs of the Avrunin and Scott theorem. Given  $M \in kE\text{-mod}$  we have two different varieties associated to  $M$ : the support variety  $V_E(M)$ , and the rank variety  $V_E^r(M)$  and the Avrunin and Scott theorem says that these two varieties agree. To be more precise there exists a morphism of the underlying affine varieties  $F : V_E^r \rightarrow V_E$  which maps the subvariety  $V_E^r(M)$  onto  $V_E(M)$ . To define the morphism  $F$  we must choose a system of coordinates for  $V_E^r$  and  $V_E$  (equivalently identifications  $V_E^r \cong \mathbb{A}^n$  and  $V_E \cong \mathbb{A}^n$ ). In Section 2.2 a system of coordinates for  $V_E$  was shown to amount to a choice of  $k$ -basis for the  $k$ -vector space  $\text{Ext}_{kE}^1(k, k)$ , while from Section 3.1 we saw that a system of coordinates for  $V_E^r$  was also equivalent (via duality) to a choice of  $k$ -basis for  $\text{Ext}_{kE}^1(k, k)$ . Clearly we should make the same choice of  $k$ -basis for  $\text{Ext}_{kE}^1(k, k)$  and once this has been done we can define the morphism  $F$  as follows.

**Definition 4.1.** Given a fixed  $k$ -basis for  $\text{Ext}_{kE}^1(k, k)$ , and hence identifications  $V_E^r \cong \mathbb{A}^n \cong V_E$ . Define the morphism  $F : V_E^r \rightarrow V_E$  by, for  $\lambda = (\lambda_1, \dots, \lambda_n) \in V_E^r$ ,

$$F(\lambda) = \begin{cases} (\lambda_1, \dots, \lambda_n) & \text{if } p = 2, \\ (\lambda_1^p, \dots, \lambda_n^p) & \text{if } p > 2. \end{cases}$$

For  $p > 2$  this is essentially the Bockstein map  $\beta: H^1(E, k) \rightarrow H^2(E, k)$  (see Appendix A.2) whose image  $W$  is the subspace of  $\text{Ext}_{kE}^2(k, k)$  spanned by the generators of a Noether normalisation. Moreover, in the case  $p > 2$ , the map  $F$  is a morphism of varieties and a bijection of sets, but clearly there is no inverse morphism to  $F$  in this case.

**Theorem 4.2.** (Avrunin and Scott [2]) *Given  $M \in kE\text{-mod}$  then*

$$F(V_E^r(M)) = V_E(M).$$

To prove Theorem 4.2 we will use the stable map descriptions of the support variety (Theorem 2.10) and the rank variety (Theorem 3.4). The main fact to establish in proving Theorem 4.2 is the following proposition.

**Proposition 4.3.** *If  $0 \neq \lambda \in k^n$  then  $V_E(kEu_\lambda) = \ell_{F(\lambda)}$ .*

Assuming this proposition for the moment we can prove Theorem 4.2 as follows.

**Proof of Theorem 4.2.** Define

$$X_{F(\lambda)} := kEu_\lambda.$$

By Proposition 4.3,  $X_{F(\lambda)}$  clearly satisfies the conditions of Theorem 2.10 and so

$$\ell_{F(\lambda)} \subseteq V_E(M) \quad \Leftrightarrow \quad \underline{\text{Hom}}_{kE}(X_{F(\lambda)}, M) \neq 0.$$

The stable map description of the rank variety, Theorem 3.4, implies

$$\ell_\lambda \subseteq V_E^r(M) \quad \Leftrightarrow \quad \underline{\text{Hom}}_{kE}(X_{F(\lambda)}, M) \neq 0.$$

Taken together they clearly finish the proof of 4.2.  $\square$

Recall from Theorem 2.4 that the  $\{y_i\}$  appearing in Theorem 2.4 are a  $k$ -basis for  $\text{Ext}_{kE}^1(k, k)$ , and that a Noether normalisation for  $H^*(E, k)$  can be generated by the  $\{y_i\}$  if  $p = 2$  and the  $\{z_i = \beta(y_i)\}$  if  $p > 2$ . If we choose a presentation of  $kE = k\langle x_1, \dots, x_n \rangle$  as described in Section 3 (so in particular the  $\{x_i\}$  are a  $k$ -basis for  $J(kE)/J^2(kE)$ ) then the dual space is  $\text{Ext}_{kE}^1(k, k)$ . Hence to define  $F$  as in Definition 4.1 we must make the same choice of  $k$ -basis for  $\text{Ext}_{kE}^1(k, k)$  and so we will take, for  $1 \leq i \leq n$ ,  $y_i = x_i^\vee$  to be the dual basis vectors. For the rest of this section we assume that this has been done and we keep this notation.

To prove Proposition 4.3, we will need to understand how the elements

$$\begin{aligned} y_i &\in \text{Ext}_{kE}^1(k, k) \cong \underline{\text{Hom}}_{kE}(\Omega k, k) \quad \text{for } p = 2, \quad \text{and} \\ z_i = \beta(y_i) &\in \text{Ext}_{kE}^2(k, k) \cong \underline{\text{Hom}}_{kE}(\Omega^2 k, k) \quad \text{for } p > 2 \end{aligned}$$

act when interpreted as maps in the stable module category. In what follows we will take as a representative of  $\Omega k \in kE\text{-}\underline{\text{mod}}$ , the module  $J(kE)$ .

**Lemma 4.4.** For  $1 \leq i \leq n$  and  $p > 2$  the map  $\Omega^{-1}z_i \in \underline{\text{Hom}}_{kE}(\Omega k, \Omega^{-1}k)$  factorises as  $\Omega^{-1}z_i = \pi \eta_i$  where  $\pi : kE \rightarrow kE/\text{soc } kE \cong \Omega^{-1}k$  is the natural quotient map and  $\eta_i : \Omega k \rightarrow kE$  is the linear map defined by

$$\eta_i|_{\text{rad } \Omega k} = 0 \quad \text{and} \quad \eta_i(x_j) = \begin{cases} 0 & \text{if } i \neq j, \\ c_i := \prod_{\{s|s \neq i\}} x_s^{p-1} & \text{if } i = j. \end{cases}$$

For  $p = 2$  the map  $\Omega^{-1}y_i \in \underline{\text{Hom}}_{kE}(k, \Omega^{-1}k)$  factorises as  $\Omega^{-1}y_i = \pi \eta_i$  with  $\pi$  as before and  $\eta_i : k \rightarrow kE$  defined by

$$\eta_i(1) = c_i := \prod_{\{s|s \neq i\}} x_s.$$

**Proof.** We will only prove the case  $p > 2$  (the argument in the case  $p = 2$  is similar and somewhat simpler). We can use the fact that

$$kE = k\langle x_1, \dots, x_n \rangle \cong k[X_1, \dots, X_n]/(X_i^p) \cong k[X_1]/(X_1^p) \otimes_k \cdots \otimes_k k[X_n]/(X_n^p)$$

to see that the cohomology of  $kE$  can be obtained, by the Künneth theorem, from tensoring together the cohomologies of the various  $k[X_i]/(X_i^p)$ . In particular it is clear that  $z_i = \beta(y_i) \in \text{Ext}_{kE}^2(k, k)$  is given by  $1 \otimes_k \cdots \otimes_k z'_i \otimes_k \cdots \otimes_k 1$ , where  $z'_i$  is defined by the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & kX_i^{p-1} & \longrightarrow & k[X_i]/(X_i^p) & \xrightarrow{X_i} & k[X_i]/(X_i^p) \\ & & \downarrow z'_i & & & & \\ & & k & & & & \end{array}$$

with  $z'_i$  mapping  $X_i^{p-1}$  to 1. Forming the tensor product it is clear that  $z_i$  comes from the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega^2(k) & \longrightarrow & \bigoplus^n kE & \xrightarrow{m=(x_1, \dots, x_n)} & kE \\ & & \downarrow z_i & & & & \\ & & k & & & & \end{array}$$

The module  $\Omega^2(k)$  has generators  $(0, \dots, x_i^{p-1}, 0, \dots)$  with the non-zero entry in the  $i$ th coordinate, and  $(0, \dots, x_i, 0, \dots, -x_j, 0, \dots)$ , for all  $1 \leq i < j \leq n$ . In the above diagram,  $z_i$  maps  $(0, \dots, 0, x_i^{p-1}, 0, \dots)$  to 1 and all other generators to zero. To calculate  $\Omega^{-1}(z_i)$  we must calculate a map  $\alpha_i$  in the following (commutative) diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^2(k) & \longrightarrow & \bigoplus^n kE & \longrightarrow & \text{im}(m) = \Omega(k) & \longrightarrow & 0 \\ & & \downarrow z_i & & \downarrow \alpha_i & & \downarrow \Omega^{-1}(z_i) & & \\ 0 & \longrightarrow & k & \longrightarrow & kE & \xrightarrow{\pi} & kE/\text{soc}(kE) & \longrightarrow & 0 \end{array}$$

It is clear from the above that one can take  $\alpha_i$  sending the 1 in the  $i$ th summand of  $\bigoplus^n kE$  to  $c_i = \prod_{\{s|s \neq i\}} x_s^{p-1}$ , and sending other summands to 0. Hence  $\Omega^{-1}(z_i)$  factorises as desired.  $\square$

Given  $0 \neq \mu = (\mu_1, \dots, \mu_n) \in k^n$ , define

$$\zeta := \begin{cases} \sum_{i=1}^n \mu_i z_i \in \text{Ext}_{kE}^2(k, k) & \text{if } p > 2, \\ \sum_{i=1}^n \mu_i y_i \in \text{Ext}_{kE}^1(k, k) & \text{for } p = 2. \end{cases}$$

(Note, by Lemma 4.4,  $\Omega^{-1}\zeta = \pi\eta$  where  $\eta = \sum_{i=1}^n \mu_i \eta_i$ .) The module  $M_\zeta$  constructed in Section 2.1 has, by Theorem 2.2, support variety  $V_E(M_\zeta) = V(\langle \zeta \rangle)$  which is the hypersurface determined by the element  $\zeta$ . But clearly under our identification  $V_E \cong \mathbb{A}^n$ , this hypersurface is the hyperplane through the origin perpendicular to the line  $\ell_\mu$  and we will denote this hyperplane by  ${}^\perp\ell_\mu$  so that  $V_E(M_\zeta) = {}^\perp\ell_\mu$ . Keeping this notation the key fact we will need to establish Proposition 4.3 is the following lemma.

**Lemma 4.5.** *If  $0 \neq \lambda, \mu \in k^n$  are such that  $F(\lambda) \in {}^\perp\ell_\mu$  (i.e.  $\sum_{i=1}^n \lambda_i^p \mu_i = 0$  if  $p > 2$  and  $\sum_{i=1}^n \lambda_i \mu_i = 0$  if  $p = 2$ ) then there is a monomorphism  $kEu_\lambda \rightarrow M_\zeta$  which gives a non-zero map in  $\underline{\text{Hom}}_{kE}(kEu_\lambda, M_\zeta)$ .*

**Proof.** Assume  $p > 2$  and recall the diagram defining  $M_\zeta$ .

$$\begin{array}{ccccccc} 0 & \longrightarrow & k & \longrightarrow & M_\zeta & \longrightarrow & \Omega k \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow -\Omega^{-1}\zeta \\ 0 & \longrightarrow & k & \longrightarrow & kE & \xrightarrow{\pi} & \Omega^{-1}k \longrightarrow 0 \end{array}$$

It is sufficient to find a map  $f$  such that the following composition is zero:

$$kEu_\lambda \xrightarrow{f} kE \oplus \Omega k \xrightarrow{(\pi, \Omega^{-1}\zeta)} \Omega^{-1}k.$$

Since  $kEu_\lambda$  is a cyclic module we only have to determine the image of  $u_\lambda$ . Setting  $f(u_\lambda) = (\eta(u_\lambda), -u_\lambda)$  (recall  $\Omega^{-1}\zeta = \pi\eta$ ), we will have such a map, provided the annihilator of  $u_\lambda$  kills  $f(u_\lambda)$ . But the annihilator of  $u_\lambda$  is generated by  $u_\lambda^{p-1}$  and so we must check that  $u_\lambda^{p-1}\eta(u_\lambda) = 0$ . By Lemma 4.4  $\eta(u_\lambda) = \sum_{i=1}^n \lambda_i \mu_i c_i$ , and an elementary calculation shows  $u_\lambda^{p-1}c_i = \lambda_i^{p-1}c$ , where  $c = \prod_{i=1}^n x_i^{p-1}$  generates the socle of  $kE$ . Hence

$$u_\lambda^{p-1}\eta(u_\lambda) = \left( \sum_{i=1}^n \lambda_i^p \mu_i \right) c$$

and this is zero by assumption.

If  $p = 2$ , the corresponding diagram for  $M_\zeta$  is

$$\begin{array}{ccccccc} 0 & \longrightarrow & k & \longrightarrow & M_\zeta & \longrightarrow & k \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow -\Omega^{-1}\zeta \\ 0 & \longrightarrow & k & \longrightarrow & kE & \xrightarrow{\pi} & \Omega^{-1}k \longrightarrow 0 \end{array}$$

Again it suffices to find a map  $f$  for which the following diagram is exact:

$$kEu_\lambda \xrightarrow{f} kE \oplus k \xrightarrow{(\pi, \Omega^{-1}\zeta)} \Omega^{-1}k.$$

Setting  $f(u_\lambda) = (\eta(1), -1)$  will work ( $\Omega^{-1}\zeta = \pi\eta$ ) provided, as before, the annihilator of  $u_\lambda$  kills  $f(u_\lambda)$ . But the annihilator of  $u_\lambda$  is generated by  $u_\lambda$  itself and clearly kills the trivial module  $k$ , so we have to check that  $u_\lambda\eta(1) = 0$ . By Lemma 4.4  $\eta(1) = \sum_{i=1}^n \mu_i c_i$ , and we have

$$u_\lambda \left( \sum_{i=1}^n \mu_i c_i \right) = \sum_{i,j} \lambda_j \mu_i x_j c_i = \left( \sum_{i=1}^n \lambda_i \mu_i \right) c$$

where  $c = \prod_{i=1}^n x_i$  generates the socle of  $kE$ . So we require  $\sum_{i=1}^n \lambda_i \mu_i = 0$ , and this is our assumption.

In both constructions it is clear that  $f$  is a monomorphism and it is also easy to see that  $f$  does not factor through a projective module and hence is non-zero in  $\underline{\text{Hom}}_{kE}(kEu_\lambda, M_\zeta)$ .  $\square$

**Proof of Proposition 4.3.** Proposition 3.3 and Theorem 2.7 together imply  $V_E(kEu_\lambda) = \ell_\gamma$ , for some  $0 \neq \gamma \in k^n$ . Now consider  $0 \neq \mu \in k^n$  and

$$\zeta := \begin{cases} \sum_{i=1}^n \mu_i z_i \in \text{Ext}_{kE}^2(k, k) & \text{if } p > 2, \\ \sum_{i=1}^n \mu_i y_i \in \text{Ext}_{kE}^1(k, k) & \text{for } p = 2. \end{cases}$$

Then, as noted before Lemma 4.5, we have  $V_E(M_\zeta) = {}^\perp \ell_\mu$ . If we now define

$$X_\gamma := kEu_\lambda$$

so that  $V_E(X_\gamma) = \ell_\gamma$  then, by Theorem 2.10, we have

$$\ell_\gamma \subseteq {}^\perp \ell_\mu = V_E(M_\zeta) \Leftrightarrow \underline{\text{Hom}}_{kE}(X_\gamma, M_\zeta) \neq 0.$$

But by Lemma 4.5  $\underline{\text{Hom}}_{kE}(X_\gamma, M_\zeta) \neq 0$  if  $F(\gamma) \in {}^\perp \ell_\mu$ . Hence  $\ell_\gamma$  lies in every hyperplane that contains  $\ell_{F(\gamma)}$ . The intersection of all hyperplanes containing a given line is precisely that line, hence we have  $\ell_\gamma = \ell_{F(\gamma)}$  as required.  $\square$

## 5. The algebras $\Lambda_m$

For the remainder of this paper we will be concerned with looking at the analogous situation for a certain class of finite-dimensional symmetric  $k$ -algebras  $\Lambda_m$ . In particular we will use our fresh perspective on the Avrunin and Scott theorem to prove an analog for the algebras  $\Lambda_m$ . We begin with the definition of our class of algebras.

**Definition 5.1.** Let  $k$  be an algebraically closed field and  $m$  be a positive integer. The finite-dimensional  $k$ -algebra  $\Lambda_m$  is defined by

$$\Lambda_m := k[X_1, \dots, X_m]/(X_i^2).$$

### Remarks 5.2.

- (1) Note that no restriction on the characteristic of the field  $k$  is made.
- (2) If  $\text{char}(k) = 2$  then  $\Lambda_m$  is just the group algebra of an elementary abelian 2-group of rank  $m$ .
- (3) If  $\text{char}(k) \neq 2$  then  $\Lambda_m$  will not be a group algebra nor even a Hopf algebra in any obvious way. In particular we cannot just naively copy the support variety set up for group algebras to give a support variety theory for  $\Lambda_m$ . Nor can we just copy the rank variety theory for elementary abelian 2-groups to give a rank variety theory for  $\Lambda_m$  (because  $(\sum_{i=1}^m \lambda_i X_i)^2$  is not necessarily zero in  $\Lambda_m$ ).

In the light of Remarks 5.2(3) we will need to develop a support and rank variety theory for the algebra  $\Lambda_m$ . Fortunately there does exist a rank variety theory for the algebra  $\Lambda_m$  (see [12]) and we recall the relevant facts in Section 7. We are also fortunate that, in the context of well-behaved finite-dimensional selfinjective  $k$ -algebras a theory of support varieties has been developed in analogy to that for the group algebra (see [13,25]). In this theory, the role played by the group cohomology ring is taken up by the Hochschild cohomology ring. In Section 6 we will recall the necessary details of the support varieties, focusing on how they apply to the algebra  $\Lambda_m$ .

But first it is worth recording the structure of the Hochschild cohomology ring,  $\text{HH}^*(\Lambda_m^n)$ , of  $\Lambda_m^n$ . More generally for the truncated polynomial algebra  $\Lambda_m^n := k[X_1, \dots, X_m]/(X_i^n)$ , we have the following result.

**Theorem 5.3.** (Holm [18]) Let  $\text{char}(k) = p \geq 0$ . Then

$$\text{HH}^*(\Lambda_m^n) \cong k[x_1, \dots, x_m, y_1, \dots, y_m, z_1, \dots, z_m]/I$$

where  $\deg(x_i) = 0$ ,  $\deg(y_i) = 1$  and  $\deg(z_i) = 2$  for  $1 \leq i \leq m$  and the ideal  $I$  is defined by

$$I := \begin{cases} (x_i^n, y_i^2) & \text{if } p \mid n \text{ and } p \neq 2 \text{ or } p = 2 \text{ and } n \equiv 0 \pmod{4}, \\ (x_i^n, y_i^2 - x_i^{n-2}z_i) & \text{if } p = 2, p \mid n \text{ and } n \equiv 2 \pmod{4}, \\ (x_i^n, nx_i^{n-1}z_i, y_ix_i^{n-1}, y_i^2) & \text{if } p \nmid n. \end{cases}$$

**Remark 5.4.** The Hochschild cohomology ring  $\text{HH}(\Lambda_m^n)$  is a graded commutative ring, so that for homogeneous elements  $\alpha, \beta \in \text{HH}(\Lambda_m^n)$ ,  $\alpha\beta = (-1)^{\deg(\alpha)\deg(\beta)}\beta\alpha$ . Such commutativity relations (up to a possible sign) are implicitly understood and are therefore not explicit in the ideal  $I$  of Theorem 5.3.

**Corollary 5.5.** *Let  $\text{char}(k) = p \geq 0$ . Then*

$$\text{HH}^*(\Lambda_m) \cong k[x_1, \dots, x_m, y_1, \dots, y_m, z_1, \dots, z_m]/I$$

where  $\deg(x_i) = 0$ ,  $\deg(y_i) = 1$  and  $\deg(z_i) = 2$  for  $1 \leq i \leq m$  and the ideal  $I$  is defined by

$$I := \begin{cases} (x_i^2, y_i^2 - z_i) & \text{if } p = 2, \\ (x_i^2, x_i z_i, y_i x_i, y_i^2) & \text{if } p \neq 2. \end{cases}$$

## 6. Support varieties for finite-dimensional selfinjective $k$ -algebras

In this section we recall some of the main results from [13,25] which detail the theory of support varieties for finite-dimensional selfinjective  $k$ -algebras. Our aim is to record sufficient results so as to allow us to provide a stable map description of these varieties, for the algebra  $\Lambda_m$ , in Section 6.2.

The theory of support varieties for group algebras rests upon the foundation stone of the Evens, Golod and Venkov theorem (see, for example, [7]) which says that the cohomology ring,  $H^*(G, k)$  is an affine ring and that cohomology of a finitely generated module is itself a finite module over  $H^*(G, k)$ . In order to develop a similar theory for a finite-dimensional selfinjective  $k$ -algebra  $\Lambda$ , one needs to obtain a (graded) ring,  $R^*$  say, dependent on  $\Lambda$  such that:

- (1)  $R^*$  is an affine  $k$ -algebra.
- (2) Given  $M, N \in \Lambda\text{-mod}$  then  $\text{Ext}_\Lambda^*(M, N)$  is a (graded)  $R^*$ -module in a natural way.
- (3)  $\text{Ext}_\Lambda^*(M, N)$  is a finite  $R^*$ -module.

Addressing the existence of an  $R^*$  and (2) first, the starting point in [25] is to consider  $R^* = \text{HH}^*(\Lambda)$  (or suitable graded subalgebras). This is because  $\text{HH}^*(\Lambda)$  is a graded commutative  $k$ -algebra, and given  $M \in \Lambda\text{-mod}$  we do have an action of  $\text{HH}^*(\Lambda)$  on  $\text{Ext}_\Lambda^*(M, M)$  given by the map

$$\text{HH}^*(\Lambda) \xrightarrow{-\otimes_\Lambda M} \text{Ext}_\Lambda^*(M, M)$$

and more generally, for  $M, N \in \Lambda\text{-mod}$ , an action of  $\text{HH}^*(\Lambda)$  on  $\text{Ext}_\Lambda^*(M, N)$  given by the composite map

$$\text{HH}^*(\Lambda) \xrightarrow{-\otimes_\Lambda M} \text{Ext}_\Lambda^*(M, M) \xrightarrow{\text{Yoneda composition}} \text{Ext}_\Lambda^*(M, N).$$

It is worth recalling Theorem 1.1 in [25] which shows that this composite map is also (gradedly) equal to composite map

$$\text{HH}^*(\Lambda) \xrightarrow{-\otimes_\Lambda N} \text{Ext}_\Lambda^*(N, N) \xrightarrow{\text{Yoneda composition}} \text{Ext}_\Lambda^*(M, N).$$

This clearly satisfies requirement (2). In the absence of an Evens, Golod, Venkov result for  $\Lambda$  we address requirements (1) and (3) above by making them assumptions. More precisely we recall the finite generation assumptions on  $\Lambda$  that form the basis for the support variety in [13].

**Assumption 1.** There exists a graded subalgebra  $H = H^* \subseteq \mathrm{HH}^*(\Lambda)$  such that:

- (i)  $H$  is a commutative affine ring,
- (ii)  $H^0 = \mathrm{HH}^0(\Lambda) = Z(\Lambda)$ ,
- (iii)  $\mathrm{Ext}_\Lambda^*(\Lambda/J, \Lambda/J)$  is a finitely generated  $H$ -module.

**Remarks 6.1.**

- (1) These are assumptions **Fg1** and **Fg2** in [13].
- (2) Clearly property (1) and (2) hold with  $R^* = H^*$ . That property (3) also holds easily follows from (iii) (this is [13, Proposition 2.4]).

For the rest of this section  $\Lambda$  is an algebra for which Assumption 1 holds, and recall (see [13,25]) some definitions and basic properties of support varieties.

The underlying affine variety,  $V_H$  is defined to be

$$V_H := \mathrm{MaxSpec}\text{-}H.$$

Given  $M, N \in \Lambda\text{-mod}$  then  $\mathrm{Ext}_\Lambda^*(M, N)$  is a finite  $H$ -module. Let its annihilator be  $I(M, N)$  and define the support variety for the pair of modules  $(M, N)$  to be the variety associated to the ideal  $I(M, N)$ , that is

$$V_H(M, N) := \{\mathfrak{m} \in V_H \mid I(M, N) \subseteq \mathfrak{m}\}.$$

The support variety for  $M \in \Lambda\text{-mod}$  is then defined by

$$V_H(M) := V_H(M, M).$$

The obvious modifications to these definitions are made for  $M \in \mathrm{mod}\text{-}\Lambda$ . We also have  $V_H = V_H(\Lambda/J)$  [25, Proposition 4.4] and for  $M, N \in \Lambda\text{-mod}$  we have [25, Proposition 3.1]

$$V_H(M, N) \subseteq V_H(M) \cap V_H(N).$$

These support varieties are homogeneous affine varieties and satisfy analogues of the elementary properties (i)–(iv) in Proposition 2.1 (see [13,25] for details).

**Proposition 6.2.** *Given  $M, M' \in \Lambda\text{-mod}$  then the following hold.*

- (i)  $V_H(M) = V_H(DM) = V_H(\Omega M)$ .
- (ii)  $V_H(M \oplus M') = V_H(M) \cup V_H(M')$ .
- (iii)  $V_H(M) = V_H(M, \Lambda/J) = V_H(\Lambda/J, M)$ .
- (iv)  $V_H(M) = \{0\} \Leftrightarrow M$  is projective.
- (v)  $V_H(M) \cap V_H(M') = \{0\} \Rightarrow \mathrm{Ext}_\Lambda^n(M, M') = 0$ , for  $n \gg 0$ .

**Remark 6.3.** Because  $\Lambda$  is not necessarily a Hopf algebra it does not make sense for  $M \otimes_k M'$  to be a  $\Lambda$ -module, so we cannot directly have the analog of Proposition 2.1(vi). Nevertheless there is a sort of replacement, involving analogues of the  $L_\zeta$ -modules of Section 2.1, which will now be described in Section 6.1.



### 6.1. The $\Lambda$ -bimodules $M_\zeta$

We will briefly recall the construction of the  $\Lambda$ -bimodules  $M_\zeta$  (for  $\zeta \in H^*$  a homogeneous element), given in [13]. These bimodules are the analogues of the  $kG$ -modules  $M_\zeta = \Omega^{-1}(L_\zeta)$  considered in Section 2.1.

Let  $\zeta \in H^n$  be a homogeneous element, of degree  $n$  say, which we choose to be represented as a  $\Lambda$ -bimodule map  $\zeta : \Omega_{\Lambda^e}^n(\Lambda) \rightarrow \Lambda$ . The  $\Lambda^e$ -module  $M_\zeta$  is then defined by the following pushout diagram.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega_{\Lambda^e}^n(\Lambda) & \longrightarrow & P^{n-1} & \longrightarrow & \Omega_{\Lambda^e}^{n-1}(\Lambda) \longrightarrow 0 \\ & & \downarrow \zeta & & \downarrow & & \parallel \\ 0 & \longrightarrow & \Lambda & \longrightarrow & M_\zeta & \longrightarrow & \Omega_{\Lambda^e}^{n-1}(\Lambda) \longrightarrow 0 \end{array}$$

Here  $P^{n-1}$  is a projective  $\Lambda$ -bimodule. The following result is the replacement to Proposition 2.1(vi) alluded to in Remark 6.3, it may also be viewed as a sort of analogue of Theorem 2.2.

**Theorem 6.4.** [13, Proposition 4.3] *Let  $\zeta \in H$  be a homogeneous element, and  $M \in \Lambda\text{-mod}$ . Then*

$$V_H(M_\zeta \otimes_\Lambda M) = V_H(\langle \zeta \rangle) \cap V_H(M).$$

### Remarks 6.5.

- (1)  $V_H(\langle \zeta \rangle)$  denotes the hypersurface in  $V_H$  determined by the homogeneous element  $\zeta$ .
- (2) If we define  $L_\zeta := M_\zeta \otimes_\Lambda \Lambda/J \in \Lambda\text{-mod}$  then  $V_H(L_\zeta) = V_H(\langle \zeta \rangle)$ .
- (3) The  $\Lambda$ -bimodule  $M_\zeta$  is projective when viewed as a left or right  $\Lambda$ -module.

Just as in the situation for group algebras, the modules  $M_\zeta$  can be used to construct modules with arbitrary homogeneous closed varieties. In particular the following result is the analogue to Theorem 2.5.

**Theorem 6.6.** [13, Theorem 4.4] *Let  $V$  be a closed homogeneous subvariety of  $V_H$ . Then there exists a module  $M \in \Lambda\text{-mod}$  such that  $V_H(M) = V$ .*

Finally, before turning to  $\Lambda_m$  and a stable map description of its support varieties in Section 6.2, we recall the analogous results to Theorems 2.6 and 2.7.

**Theorem 6.7.** [13, Theorem 4.3] *If  $M \in \Lambda\text{-mod}$  is indecomposable and has variety a line, then  $M$  is a periodic module. Moreover, the period must divide the degree of one of the homogeneous generators of  $H$ .*

**Theorem 6.8.** [13, Proposition 4.2] *If  $M \in \Lambda\text{-mod}$  is an indecomposable periodic module then  $V_H(M)$  is a line.*

## 6.2. Support varieties and stable map description for $\Lambda_m$

Here we want to specialise to the case  $\Lambda = \Lambda_m$  and use the above results to provide a stable map description of support varieties for  $\Lambda_m$ -modules analogous to that given in Corollary 2.11 for  $kE$ -modules. However before going further we must address the fact that all the results (even definitions!) quoted above for support varieties for a general finite-dimensional selfinjective  $k$ -algebra  $\Lambda$  were based upon Assumption 1 holding. From now on we will only consider the case  $\Lambda = \Lambda_m$  and we define

$$H = H^* := \begin{cases} HH^{\text{even}}(\Lambda_m) & \text{if } p \neq 2, \\ HH^*(\Lambda_m) & \text{if } p = 2. \end{cases}$$

It follows from Corollary 5.5 that Assumption 1(i) and (ii) hold and the following result immediately implies that (iii) also holds so that Assumption 1 is satisfied for  $\Lambda_m$ .

**Lemma 6.9.** *As a graded ring (under Yoneda composition as product) we have*

$$\text{Ext}_{\Lambda_m}^*(k, k) \cong k\langle \alpha_1, \dots, \alpha_m \rangle / (\alpha_i \alpha_j + \alpha_j \alpha_i, \forall i \neq j)$$

where  $\deg(\alpha_i) = 1$ , for  $1 \leq i \leq m$ . The ring map  $\phi := - \otimes_{\Lambda_m} k|_H : H \rightarrow \text{Ext}_{\Lambda_m}^*(k, k)$  is as follows. For  $1 \leq i \leq m$ ,

$$\begin{aligned} \phi(z_i) &= \alpha_i^2, & \phi(x_i) &= 0 & \text{if } p \neq 2, \\ \phi(y_i) &= \alpha_i, & \phi(x_i) &= 0 & \text{if } p = 2. \end{aligned}$$

**Proof.** The algebra  $\Lambda_m$  is easily seen to be a Koszul algebra and general structure theory shows that  $\text{Ext}_{\Lambda_m}^*(k, k)$  is of the form as stated. The final part concerning  $\text{im}(\phi)$  is straightforward and follows directly from considering the case  $m = 1$ . We have  $\Omega_{\Lambda_1}^2(\Lambda_1) \cong \Lambda_1$  as a bimodule and  $z_1$  can be represented by the identity map of  $\Lambda_1$ ; and then  $z_1 \otimes_{\Lambda_1} k$  obviously represents  $\alpha_1^2$ .  $\square$

Corollary 5.5 shows that  $V_H \cong \mathbb{A}^m$  with  $\mu \in k^m$  corresponding to a point in  $V_H$ . Given  $0 \neq \mu = (\mu_1, \dots, \mu_m) \in k^m$  define  $\zeta \in H$  as follows:

$$\zeta = \begin{cases} \sum_{i=1}^m \mu_i z_i \in HH^2(\Lambda_m) & \text{if } p \neq 2, \\ \sum_{i=1}^m \mu_i y_i \in HH^2(\Lambda_m) & \text{if } p = 2. \end{cases}$$

Then the hypersurface given by the variety associated to the ideal  $\langle \zeta \rangle$  is simply the hyperplane  ${}^\perp \ell_\mu$  (recall this is the hyperplane perpendicular to the line  $\ell_\mu$ ). Now given a line  $\ell_\lambda \subseteq \mathbb{A}^m$ , let  ${}^\perp \ell_{\gamma_1}, \dots, {}^\perp \ell_{\gamma_{m-1}}$  be  $m - 1$  hyperplanes such that

$$\bigcap_{i=1}^{m-1} {}^\perp \ell_{\gamma_i} = \ell_\lambda.$$

If now  $\zeta_i \in H$  is such that  $V_H(\langle \zeta_i \rangle) = {}^\perp \ell_{\gamma_i}$  for  $1 \leq i \leq m - 1$  then it follows from Theorem 6.4 that if we define  $M_{\underline{\zeta}} := M_{\zeta_1} \otimes_{\Lambda_m} \dots \otimes_{\Lambda_m} M_{\zeta_{m-1}}$  then

$$V_H(M_{\underline{\zeta}} \otimes_{\Lambda_m} k) = \ell_\lambda. \quad (6.1)$$

Keeping this notation we have the following result which follows from repeated application of Theorem 6.4.

**Corollary 6.10.** *Let  $0 \neq \lambda \in \mathbb{A}^m$  and let  $M \in \Lambda_m\text{-mod}$ . Then*

$$\ell_\lambda \subseteq V_H(M) \quad \Leftrightarrow \quad V_H(M_\zeta \otimes_{\Lambda_m} M) \neq 0.$$

We will also need to consider the dual bimodules,  $M_\zeta^*$ . In particular we need the following result.

**Theorem 6.11.** *Let  $\zeta \in H$  be a homogeneous element and let  $M \in \Lambda_m\text{-mod}$ , then*

$$V_H(M_\zeta^* \otimes_{\Lambda_m} M) = V_H(\langle \zeta \rangle) \cap V_H(M).$$

**Proof.** We would like to argue as follows. Using Proposition 6.2(iii) and the fact that

$$\text{Ext}_{\Lambda_m}^*(M_\zeta^* \otimes_{\Lambda_m} M, k) \cong \text{Ext}_{\Lambda_m}^*(M, M_\zeta \otimes_{\Lambda_m} k)$$

by adjointness of the functors  $M_\zeta \otimes_{\Lambda_m} -, M_\zeta^* \otimes_{\Lambda_m} -$ , we have

$$V_H(M_\zeta^* \otimes_{\Lambda_m} M) = V_H(M_\zeta^* \otimes_{\Lambda_m} M, k) = V_H(M, M_\zeta \otimes_{\Lambda_m} k).$$

If  $M = k$  then we are done by Proposition 6.2(iii) and Remarks 6.5(2), but in general all we can say without further analysis is (by the observations before Proposition 6.2)  $V_H(M, M_\zeta \otimes_{\Lambda_m} k) \subseteq V_H(M) \cap V_H(\langle \zeta \rangle)$ . To argue further we must examine the  $\Lambda_m$ -bimodule  $M_\zeta^*$  in more detail.

Suppose  $\deg(\zeta) = n$  and consider the dual diagram to the pushout diagram defining  $M_\zeta$ . The top row of this dual diagram is

$$0 \rightarrow (\Omega_{\Lambda_m^e}^{n-1}(\Lambda_m))^* \rightarrow M_\zeta^* \rightarrow \Lambda_m^* \rightarrow 0.$$

Using the fact that  $(\Omega^k(-))^* \cong \Omega^{-k}(-^*)$  together with  $\Lambda_m^* \cong \Lambda_m$  as  $\Lambda_m$ -bimodules (because  $\Lambda_m$  is a symmetric  $k$ -algebra) we can apply  $\Omega^{n-1}(-)$  to the above sequence and obtain the short exact sequence of  $\Lambda_m$ -bimodules

$$0 \rightarrow \Lambda \rightarrow \Omega_{\Lambda_m^e}^{n-1}(M_\zeta^*) \oplus \text{proj} \rightarrow \Omega_{\Lambda_m^e}^{n-1}(\Lambda_m) \rightarrow 0.$$

By considering a projective  $\Lambda_m$ -bimodule resolution of  $\Lambda_m$  we must have some  $\eta \in \text{HH}^n(\Lambda_m)$  giving us the following diagram (where  $P^{n-1}$  is some projective  $\Lambda_m^e$ -module).

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega_{\Lambda_m^e}^n(\Lambda_m) & \longrightarrow & P^{n-1} & \longrightarrow & \Omega_{\Lambda_m^e}^{n-1}(\Lambda_m) \longrightarrow 0 \\ & & \downarrow \eta & & \downarrow & & \parallel \\ 0 & \longrightarrow & \Lambda_m & \longrightarrow & \Omega_{\Lambda_m^e}^{n-1}(M_\zeta^*) & \longrightarrow & \Omega_{\Lambda_m^e}^{n-1}(\Lambda_m) \longrightarrow 0 \end{array}$$

Note  $\deg(\eta) = \deg(\zeta) = n$  so that in fact  $\eta \in H^n$  and by construction of  $M_\eta$  we must have the following isomorphism of  $\Lambda_m$ -bimodules:

$$M_\eta \cong \Omega_{\Lambda_m^e}^{n-1}(M_\zeta^*) \oplus \text{proj}.$$

We can now use the fact that

$$\Omega_{\Lambda_m^e}^{n-1}(M_\zeta^*) \otimes_{\Lambda_m} M \cong \Omega_{\Lambda_m}^{n-1}(M_\zeta^* \otimes_{\Lambda_m} M) \oplus \text{proj}$$

together with Proposition 6.2 (i) and (iv) to get

$$V_H(M_\eta \otimes_{\Lambda_m} M) = V_H(\Omega_{\Lambda_m^e}^{n-1}(M_\zeta^*) \otimes_{\Lambda_m} M) = V_H(M_\zeta^* \otimes_{\Lambda_m} M).$$

Theorem 6.4 finishes the proof once we know that  $V_H(\langle \eta \rangle) = V_H(\langle \zeta \rangle)$ , but this follows because we already know the desired result in the case  $M = k$ .  $\square$

We are now in a position to provide a stable map description of the support variety for  $\Lambda_m$ -modules analogous to that given in Corollary 2.11 for  $kE$ -modules. Before doing so we need to define (for  $0 \neq \lambda \in k^m$ ) a  $T_\lambda \in \Lambda_m\text{-mod}$  to play the analogous role to its  $kE$ -module namesake. Given  $0 \neq \lambda \in k^m$  by the construction following Lemma 6.9 we have a sequence  $\underline{\zeta} = (\zeta_i)$  (with  $\zeta_i \in H^2$  for  $1 \leq i \leq m-1$ ) and a  $\Lambda_m$ -bimodule  $M_{\underline{\zeta}} := M_{\zeta_1} \otimes_{\Lambda_m} \cdots \otimes_{\Lambda_m} M_{\zeta_{m-1}}$  such that  $V_H(M_{\underline{\zeta}} \otimes_{\Lambda_m} k) = \ell_\lambda$ . Define  $T_\lambda$  to be the non-projective  $\Lambda_m$ -module summand of  $M_{\underline{\zeta}} \otimes_{\Lambda_m} k$  so that  $V_H(T_\lambda) = \ell_\lambda$ . Moreover, by Theorem 6.7, each indecomposable summand of  $T_\lambda$  is periodic of period at most 2 and hence  $T_\lambda$  itself is also periodic of period at most 2. With this construction and notation we can now state the stable map description of  $\Lambda_m$ -support varieties.

**Theorem 6.12.** *Let  $0 \neq \lambda \in k^m$  and  $M \in \Lambda_m\text{-mod}$ . Then*

$$\ell_\lambda \subseteq V_H(M) \iff \underline{\text{Hom}}_{\Lambda_m}(T_\lambda \oplus \Omega T_\lambda, M) \neq 0.$$

**Proof.** By repeated use of Theorem 6.11 and construction of  $\underline{\zeta}$  we have

$$V_H(M_{\zeta_{m-1}}^* \otimes_{\Lambda_m} \cdots \otimes_{\Lambda_m} M_{\zeta_1}^* \otimes_{\Lambda_m} M) = \ell_\lambda \cap V_H(M).$$

It follows from this and Proposition 6.2 that

$$\begin{aligned} \ell_\lambda \subseteq V_H(M) &\iff V_H(k, M_{\zeta_{m-1}}^* \otimes_{\Lambda_m} \cdots \otimes_{\Lambda_m} M_{\zeta_1}^* \otimes_{\Lambda_m} M) \neq 0 \\ &\iff \text{Ext}_{\Lambda_m}^i(k, M_{\zeta_{m-1}}^* \otimes_{\Lambda_m} \cdots \otimes_{\Lambda_m} M_{\zeta_1}^* \otimes_{\Lambda_m} M) \neq 0 \quad \text{for } i \gg 0. \end{aligned}$$

But by repeated use of adjointness we have

$$\text{Ext}_{\Lambda_m}^i(k, M_{\zeta_{m-1}}^* \otimes_{\Lambda_m} \cdots \otimes_{\Lambda_m} M_{\zeta_1}^* \otimes_{\Lambda_m} M) = \text{Ext}_{\Lambda_m}^i(M_{\underline{\zeta}} \otimes_{\Lambda_m} k, M).$$

Finally by construction, for  $i > 0$ ,  $\text{Ext}_{\Lambda_m}^i(M_{\underline{\zeta}} \otimes_{\Lambda_m} k, M) \cong \text{Ext}_{\Lambda_m}^i(T_\lambda, M)$  and since  $\Omega^i T_\lambda \cong T_\lambda$  (respectively  $\Omega T_\lambda$ ) if  $i$  is even (respectively  $i$  odd) the result follows.  $\square$

**Remark 6.13.** This result is weaker than the corresponding result for  $kE$ -modules (Corollary 2.11). This is because we cannot just rely upon the properties of the  $T_\lambda$ -modules (i.e.  $V_H(T_\lambda) = \ell_\lambda$  and  $T_\lambda$  periodic of period at most 2) but also have to use the fact that they are the non-projective summands of the  $M_\zeta$ -bimodules. This reflects the fact that in the group case Corollary 2.11 made essential use of the tensor intersection property whereas for  $\Lambda_m$ -modules the closest we have is Theorem 6.4.

## 7. Rank varieties for $\Lambda_m$ -modules

Having established a stable map description of the support variety for a  $\Lambda_m$ -module, we turn in this section to doing the same for the rank variety of a  $\Lambda_m$ -module. As noted in Remarks 5.2(3), in the case  $\text{char}(k) \neq 2$  it is far from obvious that  $\Lambda_m$ -modules have rank varieties. The existence of suitable rank varieties for  $\Lambda_m$ -modules was established in [12] and in this section we wish to recall those details of the construction and results from [12] that will be needed in Section 8 when we come to establishing the analogue of the Avrunin and Scott theorem for  $\Lambda_m$ .

In [12] the problems identified in Remarks 5.2(3) are circumvented by use of Clifford algebras. Recall that the  $m$ -generated Clifford algebra (over  $k$ )  $C_m$  is defined as follows.

### Definition 7.1.

$$C_m := k\langle e_1, \dots, e_m \rangle / (e_i^2 + 1, e_i e_j + e_j e_i \text{ for } i \neq j).$$

### Remarks 7.2.

- (1) If  $\text{char}(k) = 2$  then  $C_m$  is the group algebras of an elementary abelian 2-group of rank  $m$ .
- (2) If  $\text{char}(k) \neq 2$  then  $C_m$  is a semisimple  $k$ -algebra. Up to isomorphism it has only 1 (respectively 2) simple module if  $m$  is even (respectively if  $m$  odd).

To define a rank variety for  $\Lambda_m$ -modules we need a replacement to the module map multiplication by  $u_\lambda$  in Section 3.1. To do this fix an irreducible representation  $\rho: C_m \rightarrow GL(W)$ . We have the following definition from [12].

**Definition 7.3.** If  $0 \neq \lambda \in k^m$  and  $M \in \Lambda_m\text{-mod}$  let

$$\sigma_M(\lambda) := \sum_{j=1}^m \lambda_j \rho(e_j) \otimes_k X_j: W \otimes_k M \rightarrow W \otimes_k M.$$

### Remarks 7.4.

- (1) We identify  $X_j \in \Lambda_m$ , for  $1 \leq j \leq m$ , with the map  $X_j: M \rightarrow M$  that is multiplication by  $X_j$ .
- (2) By construction  $\sigma_M(\lambda)^2 = 0$ .
- (3) If  $\text{char}(k) = 2$  then  $\sigma_M(\lambda)$  is the map multiplication by  $u_\lambda$ .

We can now define the rank variety for  $\Lambda_m$ -modules [12, Definition 4.1].

**Definition 7.5.** If  $M \in \Lambda_m\text{-mod}$  define the rank variety  $V^r(M)$  by

$$V^r(M) := \{0\} \cup \{0 \neq \lambda \in k^m \mid \text{rank}(\sigma_M(\lambda)) < (1/2) \dim(W) \dim(M)\}.$$

**Remarks 7.6.**

- (1) In [12]  $V^r(M)$  is defined to be a projective variety, but for consistency with the other varieties considered here we view it as an affine variety. It is of course a homogeneous affine variety.
- (2) In the case  $\text{char}(k) \neq 2$  and  $m$  odd the choice of irreducible representation  $W$  is unimportant; another choice only changes the signs of the  $\lambda_i$ , but  $V^r(M)$  is invariant under all changes of signs of the  $\lambda_i$ .

We recall some elementary properties of the rank variety.

**Proposition 7.7.** Given  $M_i \in \Lambda_m\text{-mod}$ , for  $1 \leq i \leq 3$ , then:

- (1)  $V^r(M_1 \oplus M_2) = V^r(M_1) \cup V^r(M_2)$ .
- (2)  $V^r(\Omega M_1) = V^r(M_1)$ .
- (3) If  $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$  is a short exact sequence then  $V^r(M_i) \subseteq V^r(M_j) \cup V^r(M_s)$  for  $\{i, j, s\} = \{1, 2, 3\}$ .

We will also need one of the main results from [12] showing that the rank variety detects projectivity (in the case of rank varieties for elementary abelian  $p$ -groups this result goes back to Dade's lemma; see [11, Lemma 11]).

**Theorem 7.8.** [12, Theorem 4.3] If  $M \in \Lambda_m\text{-mod}$  then  $V^r(M) = \{0\} \Leftrightarrow M \in \Lambda_m\text{-proj}$ .

To provide a stable map description for these rank varieties we must recall the definition and properties of the modules  $V(\lambda)$  defined in [12] which play the role of the modules  $kEu_\lambda$  in Section 3.2.

**Definition 7.9.** If  $0 \neq \lambda \in k^m$  let  $V(\lambda) \in \Lambda_m\text{-mod}$  be defined by  $V(\lambda) := \text{im}(\sigma_{\Lambda_m}(\lambda))$ .

Some of the properties of the modules  $V(\lambda)$  are given in the following result.

**Proposition 7.10.** If  $0 \neq \lambda \in k^m$  then  $V(\lambda) \in \Lambda_m\text{-mod}$  is such that:

- (1)  $\dim(V(\lambda)) = (1/2) \dim(W) \dim(\Lambda_m)$ .
- (2) A minimal projective resolution for  $V(\lambda)$  is

$$\cdots \rightarrow W \otimes_k \Lambda_m \xrightarrow{\sigma_{\Lambda_m}} W \otimes_k \Lambda_m \xrightarrow{\sigma_{\Lambda_m}} W \otimes_k \Lambda_m.$$

- (3)  $\Omega(V(\lambda)) \cong V(\lambda)$ .

(4) Let  $t := |\{i: \lambda_i = 0\}|$  then

$$V(\lambda) \cong \begin{cases} \oplus^{2^{\lfloor t/2 \rfloor}} V & \text{if } m \text{ is odd,} \\ \oplus^{2^{\lfloor (t+1)/2 \rfloor}} V & \text{if } m \text{ is even,} \end{cases}$$

with  $V$  an indecomposable  $\Lambda_m$ -module.

**Proof.** Properties (1), (2) and (3) are fairly easy results whose proofs can be found in Proposition 3.4 in [12] (indeed property (3) is immediate from property (2)). For property (4), the result follows from Lemma 3.5 in [12] provided one can establish it for the case  $t = m$ . That is we need to show that if  $t = m$  then  $V(\lambda)$  is indecomposable. To do this we use Proposition 3.4 in [12] which says that  $V(\lambda)$  restricted to  $\Lambda_{m-1}$  is free of rank  $\dim(W)$ ; for the rest of this proof we identify  $\Lambda_{m-1}$  with the subalgebra of  $\Lambda_m$  generated by  $\{X_i \mid 1 \leq i \leq m-1\}$ . Since  $\sigma_{\Lambda_m}(\lambda)$  annihilates  $V(\lambda)$  the action of  $X_m$  on  $V(\lambda)$ , viewed as a free  $\Lambda_{m-1}$ -module of rank  $\dim(W)$ , is easily seen to be given by

$$-(\lambda_m \rho(e_m))^{-1} \sum_{i=1}^{m-1} \lambda_i \rho(e_i) \otimes_k X_i. \quad (7.1)$$

It is clear that  $\text{End}_k(V(\lambda) \downarrow_{\Lambda_{m-1}}) \cong \rho(C_m) \otimes_k \Lambda_{m-1}$  and therefore the endomorphism ring  $\text{End}_{\Lambda_m}(V(\lambda))$  can be identified with the centraliser of (7.1) in  $\rho(C_m) \otimes_k \Lambda_{m-1}$ . Because  $\lambda_i \neq 0$  for  $1 \leq i \leq m-1$  it now follows that any element in this centraliser is a sum of an invertible element and a nilpotent element (the non-nilpotent part must lie in  $\rho(C_m) \otimes_k 1$  and centralise each  $\rho(e_i) \otimes_k X_i$  and therefore actually lie in  $Z(\rho(C_m)) \otimes_k 1$  and so is invertible). Hence  $\text{End}_{\Lambda_m}(V(\lambda))$  is a local ring and therefore  $V(\lambda)$  is indecomposable.  $\square$

We can now state the stable map description of the rank variety for a  $\Lambda_m$ -module.

**Theorem 7.11.** [12, Lemma 3.7] Let  $0 \neq \lambda \in k^m$  and  $M \in \Lambda_m\text{-mod}$  then

$$\ell_\lambda \subseteq V^r(M) \iff \underline{\text{Hom}}_{\Lambda_m}(V(\lambda), M) \neq 0.$$

We finish this subsection with an easy observation that will prove useful.

**Lemma 7.12.** Let  $\zeta \in H$  be some homogeneous and  $M \in \Lambda_m\text{-mod}$ . Then

$$V^r(M_\zeta \otimes_{\Lambda_m} M) \subseteq V^r(M).$$

**Proof.** Suppose  $\deg(\zeta) = n$  and consider the second row in the pushout diagram of  $\Lambda_m$ -bimodules that defines  $M_\zeta$ . Since all modules in this row are projective as right  $\Lambda_m$ -modules, upon applying the functor  $-\otimes_{\Lambda_m} M$  to this row we have the short exact sequence of left  $\Lambda_m$ -modules

$$0 \rightarrow M \rightarrow M_\zeta \otimes_{\Lambda_m} M \rightarrow \Omega_{\Lambda_m^e}^{n-1}(\Lambda_m) \otimes_{\Lambda_m} M \rightarrow 0.$$

Now use the fact that  $\Omega_{\Lambda_m^e}^{n-1}(\Lambda_m) \otimes_{\Lambda_m} M \cong \Omega^{n-1}(M) \oplus \text{proj}$  together with Proposition 7.7 and Theorem 7.8 to finish the proof.  $\square$

## 8. Relating the rank and support varieties for $\Lambda_m$ -modules

Now that we have at our disposal both support and rank varieties for  $\Lambda_m$ -modules it is natural to ask whether an analogue of the Avrunin and Scott theorem (Theorem 4.2) for  $kE$ -modules, also holds for  $\Lambda_m$ -modules. In this section we show (Theorem 8.2) that this is indeed the case. Before we can proceed we need to define the morphism  $F$  between our two varieties and that means fixing a system of parameters for the underlying affine  $m$ -space  $\mathbb{A}^m$ . However in the case  $\text{char}(k) \neq 2$  the situation is simpler than in the group case,  $kE$ , because the  $\{X_i\}$  in Definition 5.1 are a distinguished set of generators for the algebra  $\Lambda_m$  (as noted in Remarks 5.2(3) a linear combination cannot be a generator, i.e. square to 0). The rank variety was defined in terms of these generators and for the support variety the generators  $\{z_i\}$  in  $H^2$  are again distinguished (being determined by the  $\{X_i\}$ ). In the case  $\text{char}(k) = 2$  then, as in Section 3.1, an identification  $V^r \cong \mathbb{A}^m$  amounts to a choice of  $k$ -basis for  $J(\Lambda_m)/J(\Lambda_m)^2$ , whilst an identification  $V_H \cong \mathbb{A}^m$  amounts to a choice of generators  $\{y_i\}$  which is a  $k$ -basis of  $H^1 = \text{HH}^1(\Lambda_m)$ . But in this case  $\text{HH}^1(\Lambda_m)$  is the  $k$ -space of derivations of  $\Lambda_m$  (any inner derivation is zero as  $\Lambda_m$  is commutative) and a choice of  $k$ -basis is determined by a choice of  $k$ -basis for the dual space  $J/J^2$ . Hence in this, as we did in the group case, we choose the dual basis. With these identifications assumed we can now define the morphism  $F: V^r \rightarrow V_H$  as follows.

**Definition 8.1.** With the identifications  $V^r \cong \mathbb{A}^m$  and  $V_H \cong \mathbb{A}^m$  already fixed in the case  $\text{char}(k) \neq 2$  and chosen compatibly (by fixed choice of  $k$ -basis for  $J/J^2$ ) in the case  $\text{char}(k) = 2$ , we define the morphism  $F: V^r \rightarrow V_H$  by, for  $\lambda = (\lambda_1, \dots, \lambda_m) \in k^m$ ,

$$F(\lambda) = \begin{cases} (\lambda_1, \dots, \lambda_m) & \text{if } \text{char}(k) = 2, \\ (\lambda_1^2, \dots, \lambda_m^2) & \text{if } \text{char}(k) \neq 2. \end{cases}$$

We can now state the analogue of the Avrunin and Scott theorem for  $\Lambda_m$ -modules.

**Theorem 8.2.** *Given  $M \in \Lambda_m\text{-mod}$  then*

$$F(V^r(M)) = V_H(M).$$

To prove this theorem we will imitate our proof of the Avrunin and Scott theorem in Section 4. In particular we will make use of the stable map descriptions of both the support and rank variety described in Sections 6.2 and 7. As in the group case the main task will be to establish the following proposition.

**Proposition 8.3.** *If  $0 \neq \lambda \in k^m$  then  $V_H(V(\lambda)) = \ell_{F(\lambda)}$ .*

Assuming this proposition the proof of Theorem 8.2 proceeds as follows.

**Proof of Theorem 8.2.** The stable map description of the rank variety, Theorem 7.11, says

$$\ell_\lambda \subseteq V^r(M) \iff \underline{\text{Hom}}_{\Lambda_m}(V(\lambda), M) \neq 0.$$

Because  $V(\lambda)$  is periodic, with period 1, we have

$$\underline{\text{Hom}}_{\Lambda_m}(V(\lambda), M) \neq 0 \iff \text{Ext}_{\Lambda_m}^i(V(\lambda), M) \neq 0 \quad (\text{for } i \gg 0).$$



But by Proposition 6.2 we have

$$\mathrm{Ext}_{\Lambda_m}^i(V(\lambda), M) \neq 0 \quad (\text{for } i \gg 0) \Rightarrow V_H(V(\lambda)) \cap V_H(M) \neq \{0\}.$$

By Proposition 8.3, this is equivalent to  $\ell_{F(\lambda)} \subseteq V_H(M)$  and hence we have the inclusion  $F(V^r(M)) \subseteq V_H(M)$ . To obtain the reverse inclusion let  $0 \neq \mu \in k^m$  be such that  $\ell_\mu \subseteq V_H(M)$  and let  $M_\zeta$  be the  $\Lambda_m$ -bimodule constructed in Section 6.2 such that  $V_H(M_\zeta \otimes_{\Lambda_m} k) = \ell_\mu$ . Then  $V_H(M_\zeta \otimes_{\Lambda_m} M) = \ell_\mu$  by Corollary 6.10. Now consider  $F(V^r(M_\zeta \otimes_{\Lambda_m} M))$ . We have already established that

$$F(V^r(M_\zeta \otimes_{\Lambda_m} M)) \subseteq V_H(M_\zeta \otimes_{\Lambda_m} V(\lambda)) = \ell_\mu.$$

Suppose  $F(V^r(M_\zeta \otimes_{\Lambda_m} M)) = \{0\}$  so that  $V^r(M_\zeta \otimes_{\Lambda_m} V(\lambda)) = \{0\}$ , then this would mean  $M_\zeta \otimes_{\Lambda_m} M$  is projective, by Theorem 7.8 and hence  $V_H(M_\zeta \otimes_{\Lambda_m} M) = \{0\}$ . Since this is not the case and  $V^r(M)$  is a homogeneous affine variety, we must have  $V^r(M_\zeta \otimes_{\Lambda_m} M) = \{\ell_\lambda \mid F(\lambda) = \mu\}$  (see Remarks 7.6(2)). To finish we simply use Lemma 7.12 to see that  $F^{-1}(\ell_\mu) = \{\ell_\lambda \mid F(\lambda) = \mu\} \subseteq V^r(M)$ .  $\square$

The proof of Proposition 8.3 will be, with suitable modifications, a variation on that given for Proposition 4.3 in the group case. We begin by giving the suitable modification of Lemma 4.4. Here we need to understand how the elements

$$\begin{aligned} y_i \otimes_{\Lambda_m} k &\in \mathrm{Ext}_{\Lambda_m}^1(k, k) \cong \underline{\mathrm{Hom}}_{\Lambda_m}(\Omega k, k) \quad \text{for } \mathrm{char}(k) = 2, \quad \text{and} \\ z_i \otimes_{\Lambda_m} k &\in \mathrm{Ext}_{\Lambda_m}^2(k, k) \cong \underline{\mathrm{Hom}}_{\Lambda_m}(\Omega^2 k, k) \quad \text{for } \mathrm{char}(k) > 2 \end{aligned}$$

act when interpreted as maps in the stable module category. In what follows we will take as a representative of  $\Omega k \in \Lambda_m\text{-mod}$ , the module  $J(\Lambda_m)$ .

**Lemma 8.4.** *For  $1 \leq i \leq m$  and  $\mathrm{char}(k) > 2$  the map  $\Omega^{-1}(z_i \otimes_{\Lambda_m} k) \in \underline{\mathrm{Hom}}_{\Lambda_m}(\Omega k, \Omega^{-1}k)$  factorises as  $\Omega^{-1}(z_i \otimes_{\Lambda_m} k) = \pi \eta_i$  where  $\pi : \Lambda_m \rightarrow \Lambda_m / \mathrm{soc} \Lambda_m \cong \Omega^{-1}k$  is the natural quotient map and  $\eta_i : \Omega k \rightarrow \Lambda_m$  is the linear map defined by*

$$\eta_i|_{\mathrm{rad} \Omega k} = 0 \quad \text{and} \quad \eta_i(X_j) = \begin{cases} 0 & \text{if } i \neq j, \\ c_i := \prod_{\{s \mid s \neq i\}} X_s & \text{if } i = j. \end{cases}$$

*For  $\mathrm{char}(k) = 2$  the map  $\Omega^{-1}(y_i \otimes_{\Lambda_m} k) \in \underline{\mathrm{Hom}}_{\Lambda_m}(k, \Omega^{-1}k)$  factorises as  $\Omega^{-1}(y_i \otimes_{\Lambda_m} k) = \pi \eta_i$  with  $\pi$  as before and  $\eta_i : k \rightarrow \Lambda_m$  defined by*

$$\eta_i(1) = c_i := \prod_{\{s \mid s \neq i\}} X_s.$$

**Proof.** Exactly the same as that for Lemma 4.4.  $\square$

We also need a modification of Lemma 4.5. Given  $0 \neq \mu = (\mu_1, \dots, \mu_m) \in k^m$ , define

$$\zeta := \begin{cases} \sum_{i=1}^m \mu_i z_i \in H^2 & \text{if } \mathrm{char}(k) > 2, \\ \sum_{i=1}^m \mu_i y_i \in H^1 & \text{if } \mathrm{char}(k) = 2. \end{cases}$$

Then  $\Omega^{-1}(\zeta \otimes_{\Lambda_m} k) = \pi \eta$ , by Lemma 8.4, where  $\eta = \sum_{i=1}^m \mu_i \eta_i$ . Recall that the  $\Lambda_m$ -bimodule,  $M_\zeta$ , constructed in Section 6.1 is such that  $V_H(M_\zeta \otimes_{\Lambda_m} k) = V(\langle \zeta \rangle)$ , which is the hypersurface determined by the element  $\zeta$  and under the identification  $V_H \cong \mathbb{A}^m$ , this is the hyperplane  ${}^\perp \ell_\mu$ . With this notation we have the following analogue to Lemma 4.5.

**Lemma 8.5.** *If  $0 \neq \lambda, \mu \in k^m$  are such that  $F(\lambda) \in {}^\perp \ell_\mu$  (i.e.  $\sum_{i=1}^m \lambda_i^2 \mu_i = 0$  if  $\text{char}(k) > 2$  and  $\sum_{i=1}^m \lambda_i \mu_i = 0$  if  $\text{char}(k) = 2$ ) then there is a monomorphism  $V(\lambda) \rightarrow M_\zeta \otimes_{\Lambda_m} k$  which gives a non-zero element in  $\underline{\text{Hom}}_{\Lambda_m}(V(\lambda), M_\zeta \otimes_{\Lambda_m} k)$ .*

**Proof.** It would suffice to construct a map from  $V(\lambda)$  to a finite direct sum of  $M_\zeta \otimes_{\Lambda_m} k$  which has the stated properties and in fact we will construct a map to  $W \otimes_k (M_\zeta \otimes_{\Lambda_m} k)$ . With this aim in mind, let  $\sigma := \sigma_{\Lambda_m}(\lambda): W \otimes_k \Lambda_m \rightarrow W \otimes_k \Lambda_m$ , so that  $V(\lambda) = \text{im}(\sigma)$  and let  $\hat{\zeta} := \Omega^{-1}(\zeta \otimes_{\Lambda_m} k) = \pi \eta: \Omega k \rightarrow \Omega^{-1}k$  be as above. We can now use the construction of  $M_\zeta \otimes_{\Lambda_m} k$  as the pull-back in the diagram

$$\begin{array}{ccc} M_\zeta \otimes_{\Lambda_m} k & \longrightarrow & \Omega k \\ \downarrow & & \downarrow \hat{\zeta} \\ \Lambda_m & \xrightarrow{\pi} & \Omega^{-1}k \end{array}$$

to see that it suffices to find a map  $f$  such that the following composition:

$$V(\lambda) \xrightarrow{f} W \otimes_k \Lambda_m \oplus W \otimes_k \Omega k \xrightarrow{(W \otimes_k \pi, W \otimes_k \hat{\zeta})} W \otimes_k \Omega^{-1}k$$

is zero. To define  $f$  we will actually construct a map  $g$  such that the following composition:

$$W \otimes_k \Lambda_m \xrightarrow{g=(g_1, g_2)} W \otimes_k \Lambda_m \oplus W \otimes_k \Omega k \xrightarrow{(W \otimes_k \pi, W \otimes_k \hat{\zeta})} W \otimes_k \Omega^{-1}k$$

is zero. Simply take  $g_2 := -W \otimes_k \sigma$  and, because  $W \otimes_k \Lambda_m$  is a free module, to describe  $g_1$  it suffices to describe the images of a basis for  $W \otimes_k \Lambda_m$ . We define  $g_1$  by mapping basis elements to their images under the linear map  $(W \otimes_k \eta) \circ \sigma$ . It is clear that the resulting map  $g$  satisfies our requirements.

We can now use the exact sequence

$$V(\lambda) \hookrightarrow W \otimes_k \Lambda_m \rightarrow W \otimes_k \Lambda_m / V(\lambda) \cong V(\lambda)$$

derived from property (2) of Proposition 7.10, to see that  $g$  will induce our map  $f$  (by factoring through the cokernel) if  $g$  restricts to zero on the submodule  $V(\lambda)$ . This is immediate for  $g_2$  and for  $g_1$  this will be the case if the images of basis elements are annihilated by  $\sigma$  and so we require  $\sigma \circ (W \otimes_k \eta) \circ \sigma = 0$ . But

$$\begin{aligned} \sigma(W \otimes_k \eta)\sigma &= \sigma\left[\sum \lambda_j \rho(e_j) \otimes_k \mu_j c_j\right] = \left(\sum \lambda_i \rho(e_i) \otimes_k X_i\right)\left(\sum \lambda_j \rho(e_j) \otimes_k \mu_j c_j\right) \\ &= \left(\sum \lambda_i^2 \mu_i\right)c \end{aligned}$$

where  $c = \prod_{i=1}^m X_i$  and  $\sum \lambda_i^2 \mu_i$  is zero by assumption. Again it is relatively simple to see that the map  $f$  does not factor through a projective module and hence we have our non-zero element in  $\underline{\text{Hom}}_{\Lambda_m}(V(\lambda), M_\zeta \otimes_{\Lambda_m} k)$  as required.  $\square$

Before we prove Proposition 8.3 we need one further lemma.

**Lemma 8.6.** *For  $0 \neq \mu, \lambda \in k^m$  let  $T_\lambda \in \Lambda_m\text{-mod}$  be such that  $V_H(T_\lambda) = \ell_\lambda$  then*

$$\underline{\text{Hom}}_{\Lambda_m}(T_\lambda, M_\zeta \otimes_{\Lambda_m} k) \neq 0 \quad \Leftrightarrow \quad \ell_\lambda \subseteq {}^\perp \ell_\mu.$$

**Proof.** We have by adjunction,

$$\underline{\text{Hom}}_{\Lambda_m}(T_\lambda, M_\zeta \otimes_{\Lambda_m} k) \neq 0 \quad \Leftrightarrow \quad \underline{\text{Hom}}_{\Lambda_m}(M_\zeta^* \otimes_{\Lambda_m} T_\lambda, k) \neq 0.$$

But

$$\underline{\text{Hom}}_{\Lambda_m}(M_\zeta^* \otimes_{\Lambda_m} T_\lambda, k) \neq 0 \quad \Leftrightarrow \quad V_H(M_\zeta^* \otimes_{\Lambda_m} T_\lambda) \neq \{0\},$$

and by Theorem 6.11 we have

$$V_H(M_\zeta^* \otimes_{\Lambda_m} T_\lambda) = V(\langle \zeta \rangle) \cap V_H(T_\lambda) = {}^\perp \ell_\mu \cap \ell_\lambda.$$

The result follows.  $\square$

**Proof of Proposition 8.3.** Thanks to Proposition 7.10 we know that  $V(\lambda)$  decomposes into a direct sum of copies of  $V$ , an indecomposable periodic module, of period 1. By Theorem 6.8 the support variety of  $V$  and hence that of  $V(\lambda)$  must be a line. That is,  $V_H(V(\lambda)) = \ell_\alpha$  say, for some  $0 \neq \alpha \in k^m$ . Now choose  $0 \neq \mu \in k^m$  and take  $\zeta \in H^2(\Lambda_m)$  such that  $V_H(M_\zeta \otimes_{\Lambda_m} k) = {}^\perp \ell_\mu$ . Since  $V_H(V(\lambda)) = \ell_\alpha$  Lemma 8.6 says

$$\ell_\alpha \subseteq {}^\perp \ell_\mu \quad \Leftrightarrow \quad \underline{\text{Hom}}_{\Lambda_m}(V(\lambda), M_\zeta \otimes_{\Lambda_m} k) \neq 0.$$

But we know from Lemma 8.5 that if we choose  $0 \neq \mu \in k^m$  so that  $F(\lambda) \in {}^\perp \ell_\mu$ , then  $\underline{\text{Hom}}_{\Lambda_m}(V(\lambda), M_\zeta \otimes_{\Lambda_m} k) \neq 0$ . The proof is now exactly the same as that in Proposition 4.3 and we conclude  $V_H(V(\lambda)) = \ell_{F(\lambda)}$  as required.  $\square$

## Appendix A. Background details

### A.1. The varieties considered

The support varieties associated to modules are subvarieties of a variety  $V$ , associated to a finitely generated graded local ring  $R = R^* = \bigoplus_{n \geq 0} R^n$  where we may assume  $k = R^0$  so that  $R$  is an affine  $k$ -algebra (typically we are concerned with  $\text{Ext}_{kG}^*(M, M)$  with  $M$  indecomposable and we implicitly take  $R$  to be the quotient of  $\text{Ext}_{kG}^*(M, M)$  by the nilpotent maximal ideal in  $\text{End}_{kG}(M) = \text{Ext}_{kG}^0(M, M)$ , which does not affect the varieties). In particular  $V$  is defined as the maximal ideal spectrum,  $V := \text{MaxSpec-}R$ . Now  $V$  has a distinguished point, viz the maximal ideal  $R^+ = \bigoplus_{i \geq 1} R^i$ . Moreover, the ideals  $I$ , that define the support subvarieties of  $V$ , will be

homogeneous ideals, and so these support varieties  $V(I)$  will be homogeneous affine varieties that contain the distinguished point.

A presentation of  $R$  as  $R = k[x_1, \dots, x_n]/J$  (with  $x_i$  homogeneous elements,  $\deg(x_i) = n_i > 0$  and  $J$  a homogeneous ideal) yields an embedding of  $V$  as a closed set in  $\mathbb{A}^n$ . Moreover,  $R^+$  then corresponds to the origin in  $\mathbb{A}^n$  under this embedding. Typically, because the varieties considered are homogeneous, one can view them as collections of lines through the origin. However one has to be careful, taking into account the grading, in defining the lines (see [7, Section 5.4]). To be more precise, given  $\lambda \in k$ , there is a ring homomorphism  $m_\lambda: R \rightarrow R$  which simply multiplies an element of degree  $r$  by  $\lambda^r$ . The induced map  $m_\lambda^*: V \rightarrow V$ , is called dilation by  $\lambda$  and given a point  $\alpha \in V$ , the set  $\{m_\lambda^* \alpha \mid \lambda \in k\}$  forms a homogeneous subvariety of  $V$  that is a line in  $V$ . For example, if  $R = k[x, y, z]$ , with  $\deg(x) = 1$ ,  $\deg(y) = 2$ ,  $\deg(z) = 3$ , then the line through  $(1, 2, 2) \in V$  would be defined by the homogeneous ideal  $(2x^2 - y, 2x^3 - z)$ . Because the varieties considered are invariant under dilation, it might be more efficient to consider the projective variety

$$\bar{V} = \text{Proj } R := \{\mathfrak{p} \subsetneq R^+, \mathfrak{p} \text{ a homogeneous prime ideal}\}.$$

However in this paper (in keeping with the existing literature) we will view the varieties considered as homogeneous affine varieties.

## A.2. The Bockstein map

The Bockstein homomorphism is a degree 1 graded map  $\beta: H^*(G, \mathbb{F}_p) \rightarrow H^{*+1}(G, \mathbb{F}_p)$ . It is defined to be the connecting homomorphism in the long exact sequence associated to the short exact sequence

$$0 \rightarrow \mathbb{F}_p \rightarrow \mathbb{Z}/p^2 \rightarrow \mathbb{F}_p \rightarrow 0.$$

Given a field extension  $k \supseteq \mathbb{F}_p$  then we have  $H^*(G, k) \cong k \otimes_{\mathbb{F}_p} H^*(G, \mathbb{F}_p)$ . The Bockstein map is extended semilinearly, through the Frobenius map on  $k$ , to a map  $\beta: H^*(G, k) \rightarrow H^{*+1}(G, k)$ . That is, given  $\sum_i \lambda_i x_i \in H^r(G, k)$ , with  $x_i \in H^r(G, \mathbb{F}_p)$ ,  $\lambda_i \in k$ , then

$$\beta\left(\sum_i \lambda_i x_i\right) = \sum_i \lambda_i^p \beta(x_i).$$

## Appendix B. Aspects of other proofs of the Avrunin and Scott theorem

We recall some of the aspects of the existing proofs of the Avrunin and Scott theorem (Theorem 4.2 in Section 4) for elementary abelian  $p$ -group algebras. For more details and background on the issues involved see [7].

One of the difficulties (as outlined in [7]) in proving Theorem 4.2, is that, for  $H_\alpha := \langle 1 + u_\alpha \rangle$  a cyclic shifted subgroup of  $kE$ , the following diagram does not necessarily commute.

$$\begin{array}{ccc} \mathrm{Ext}_{kE}^*(k, k) & \xrightarrow{-\otimes M} & \mathrm{Ext}_{kE}^*(M, M) \\ \downarrow \mathrm{res}_{E, H_\alpha} & & \downarrow \mathrm{res}_{E, H_\alpha} \\ \mathrm{Ext}_{kH_\alpha}^*(k, k) & \xrightarrow{-\otimes M} & \mathrm{Ext}_{kH_\alpha}^*(M, M) \end{array}$$

Let us assume  $p > 2$  and suppose that the above diagram did commute and then see how an argument proving Theorem 4.2 would proceed. Let  $J(M) = I(M) \cap k[x_1, \dots, x_n]$  be the ideal defining the variety  $V_E(M)$ , that is  $J(M)$  is the annihilator, in the Noether normalisation subring of  $H^*(E, k)$ , of  $1_M \in \mathrm{Hom}(M, M)$  the identity map. Let  $\zeta \in J(M)$  be a homogeneous element, say  $\zeta = f(x_1, \dots, x_n)$ , then to establish

$$F(V_E^r(M)) \subseteq V_E(M)$$

we would want to see that  $f(\alpha_1^p, \dots, \alpha_n^p) = 0$  for  $\alpha = (\alpha_1, \dots, \alpha_n) \in V_E^r(M)$ .

With the above diagram commuting we would have

$$\mathrm{res}_{E, H_\alpha}(\zeta)1_M = \mathrm{res}_{E, H_\alpha}(\zeta 1_M) = 0.$$

A result of Carlson [8, Proposition 2.20] says

$$\mathrm{res}_{E, H_\alpha}(\zeta) = f(\alpha_1^p, \dots, \alpha_n^p)\gamma$$

where  $\gamma$  is the canonical generator of  $\mathrm{Ext}_{kH_\alpha}^*(k, k)$  in degree  $2\deg(f)$ . By construction  $\alpha \in V_E^r(M)$  means  $M \downarrow_{H_\alpha}$  is not free and so  $\gamma 1_M \neq 0$ . Hence  $f(\alpha_1^p, \dots, \alpha_n^p) = 0$  as desired.

To obtain the reverse inclusion

$$V_E(M) \subseteq F(V_E^r(M)),$$

consider the corresponding inclusion of ideals. One would want to know that for  $\zeta$  homogeneous and arbitrary subject to  $\mathrm{res}_{E, H_\alpha}(\zeta) = 0$  for all  $\alpha \in V_E^r(M)$  (in other words  $\zeta$  is obtained from a polynomial defining  $V_E^r(M)$  by raising coefficients to the power  $p$ ), then  $\zeta \in I(V_E(M)) = \sqrt{J(M)}$ . Still assuming the commutativity of the diagram we have

$$0 = \mathrm{res}_{E, H_\alpha}(\zeta)1_M = \mathrm{res}_{E, H_\alpha}(\zeta 1_M).$$

Another theorem of Carlson [10, Theorem 3.1] says that  $\theta \in \mathrm{Ext}_{kE}^t(M, M)$  ( $t \geq 0$ ) is nilpotent if and only if  $\mathrm{res}_{E, H_\alpha}(\theta)$  is nilpotent for all  $H_\alpha$  such that  $\alpha \in V_E^r(M)$ . Hence  $\zeta 1_M$  is nilpotent, i.e.  $\zeta \in \sqrt{J(M)}$ , and we have the reverse inclusion.

In any event the above diagram does not necessarily commute, essentially because in restricting to a shifted subgroup we are changing the group action on the image of the functor  $-\otimes_k M$ . In [7] this problem is somewhat circumvented by noting that  $V_E(M) = V_E(k, M)$  (as  $E$  is a  $p$ -group) and that the action of  $H^*(E, k)$  on  $V_E(k, M)$  employs Yoneda composition, i.e. not the Hopf structure needed for tensor products over  $k$  (this same observation lies behind the stable

map description of support varieties for  $p$ -groups given in Theorem 2.10). However one still has to check that the morphism  $F$  (equivalently the Bockstein) is unaffected by changing to a shifted subgroup and this is what is done in [7].

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## References

- [1] L.L. Avramov, R.O. Buchweitz, Support varieties and cohomology over complete intersections, *Invent. Math.* 142 (2000) 285–318.
- [2] G.S. Avrunin, L.L. Scott, Quillen stratification for modules, *Invent. Math.* 66 (1982) 277–286.
- [3] C. Bendel, E. Friedlander, A. Suslin, Infinitesimal 1-parameter subgroups and cohomology, *J. Amer. Math. Soc.* 10 (1997) 693–728.
- [4] C. Bendel, E. Friedlander, A. Suslin, Support varieties for infinitesimal group schemes, *J. Amer. Math. Soc.* 10 (1997) 729–759.
- [5] C. Bendel, D.K. Nakano, Complexes and vanishing of cohomology for group schemes, *J. Algebra* 214 (1999) 668–713.
- [6] D.J. Benson, *Representations and Cohomology I: Basic Representation Theory of Finite Groups and Associative Algebras*, Cambridge Stud. Adv. Math., vol. 30, Cambridge Univ. Press, 1991. Reprinted in paperback, 1998.
- [7] D.J. Benson, *Representations and Cohomology II: Cohomology of Groups and Modules*, Cambridge Stud. Adv. Math., vol. 31, Cambridge Univ. Press, 1991, reprinted in paperback, 1998.
- [8] J.F. Carlson, The varieties and cohomology ring of a module, *J. Algebra* 85 (1983) 104–143.
- [9] J.F. Carlson, The variety of an indecomposable module is connected, *Invent. Math.* 77 (1984) 291–299.
- [10] J.F. Carlson, The cohomology ring of a module, *J. Pure Appl. Algebra* 36 (1985) 105–121.
- [11] E.C. Dade, Endo-permutation modules over  $p$ -groups, II, *Ann. of Math.* 108 (1978) 317–346.
- [12] K. Erdmann, M. Holloway, Rank varieties and projectivity for a class of local algebras, *Math. Z.* 247 (2004) 441–460.
- [13] K. Erdmann, M. Holloway, N. Snashall, Ø. Solberg, R. Taillefer, Support varieties for selfinjective algebras, *K-Theory* 33 (2004) 67–87.
- [14] E.M. Friedlander, B.J. Parshall, Support varieties for restricted Lie algebras, *Invent. Math.* 86 (1986) 553–562.
- [15] E.M. Friedlander, J. Pevtsova, Representation theoretic support spaces for finite group schemes, *Amer. J. Math.* 127 (2005) 379–420.
- [16] E.M. Friedlander, A. Suslin, Cohomology of finite group schemes over a field, *Invent. Math.* 127 (1997) 209–270.
- [17] V. Ginzburg, S. Kumar, Cohomology of quantum groups at roots of unity, *Duke Math. J.* 69 (1993) 179–198.
- [18] T. Holm, Hochschild cohomology rings of algebras  $k[X]/(f)$ , *Beiträge Algebra Geom.* 1 (2000) 291–301.
- [19] J.C. Jantzen, Kohomologie von  $p$ -Lie Algebren und nilpotente Elemente, *Abh. Math. Sem. Univ. Hamburg* 76 (1986) 191–219.
- [20] D.K. Nakano, J. Palmieri, Support varieties for the Steenrod algebra, *Math. Z.* 227 (1998) 663–684.
- [21] V. Ostrik, Cohomological supports for quantum groups, *Funct. Anal. Appl.* 32 (1999) 237–246 (translation from Russian).
- [22] J.H. Palmieri, Quillen stratification for the Steenrod algebra, *Ann. of Math.* 149 (1999) 421–449.
- [23] B. Parshall, J.P. Wang, Cohomology of infinitesimal quantum groups, I, *Tohoku Math. J.* 44 (1992) 395–423.
- [24] B. Parshall, J.P. Wang, Cohomology of infinitesimal quantum groups: The quantum dimension, *Canad. J. Math.* 45 (1993) 1276–1298.
- [25] N. Snashall, Ø. Solberg, Support varieties and Hochschild cohomology rings, *Proc. London Math. Soc.* 88 (2004) 705–732.