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MINIMAL KULLBACK–LEIBLER DIVERGENCE FOR CONSTRAINED LÉVY–ITÔ PROCESSES*

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Abstract. Given an n-dimensional stochastic process $\boldsymbol{X}$ driven by $\mathbb{P}$-Brownian motions and Poisson random measures, we search for a probability measure $\mathbb{Q}$, with minimal relative entropy to $\mathbb{P}$, such that the $\mathbb{Q}$-expectations of some terminal and running costs are constrained. We prove existence and uniqueness of the optimal probability measure, derive the explicit form of the measure change, and characterize the optimal drift and compensator adjustments under the optimal measure. We provide an analytical solution for Value-at-Risk (quantile) constraints, discuss how to perturb a Brownian motion to have arbitrary variance, and show that pinned measures arise as a limiting case of optimal measures. The results are illustrated in a risk management setting—including an algorithm to simulate under the optimal measure—and explore an example where an agent seeks to answer the question what dynamics are induced by a perturbation of the Value-at-Risk and the average time spent below a barrier on the reference process?

Key words. controlled Lévy–Itô processes, Kullback–Leibler divergence, risk management, optimal probability measures

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1. Introduction.

We consider stochastic processes that follow Lévy–Itô dynamics under a reference probability measure $\mathbb{P}$ over a finite time horizon. The reference measure may arise in a data driven way and/or from modeling assumptions, however, it does not precisely capture all probabilistic beliefs of a modeler. In this work, misspecification under $\mathbb{P}$ is characterized via expected values of functions of the stochastic process at terminal time and expected running costs of the processes over the entire time horizon. To mitigate model error, we search over all absolutely continuous probability measures, under which the process satisfies these constraints, the one which is closest to the reference measure $\mathbb{P}$ in relative entropy, also called Kullback–Leibler (KL) divergence. Thus, the key contribution of this work is solving the following constrained optimization problem: Find the probability measure(s) that has minimal KL divergence subject to constraints that can be written as (i) expected values of functions applied to the stochastic process at terminal time, and (ii) expected running costs of the processes over the entire time horizon.

We proceed to solve the optimization problem by first considering a related optimization problem where we search over a subset of probability measures. Specifically, the subset consists of equivalent probability measures that arise from Doléans–Dade exponentials and we study this related problem using stochastic control techniques

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for Lévy–Itô processes. That is, we solve the dynamic programming equations and characterize a candidate solution (Proposition 2.5), prove that the candidate solution is indeed the value function associated with the optimization problem, and that the resulting controls (which induce the optimal measure change) are admissible (Theorems 2.6 and 2.9). Furthermore, we show that the optimal measure change can be written as the exponential of a collection of random variables corresponding to the constraints (Corollary 2.8 and Theorem 2.9) and can thus be seen as an Esscher-like transform. Esscher transforms have been used extensively in financial mathematics and insurance risk management; see, e.g., [18, 6] for early works and [20] for a comprehensive survey on Esscher martingale transforms in the context of minimal entropy martingale measures. Finally, we prove that if a solution to the subproblem (searching over the subset of equivalent measures) exists, then it is unique and, moreover, it is the unique solution to the original optimization problem, where we search over all absolutely continuous probability measures (Theorem 2.11).

We illustrate the dynamics of the stochastic process under the optimal measure using multiple examples. We consider the case of two Value-at-Risk (also known as quantile) constraints, provide an analytical expression for the optimal measure change, and contrast it to the solution of the static case. We further show how to optimally change the dynamics of a Brownian motion to have zero mean and arbitrary variance. We also discuss the connection of the solution to our constrained optimization problem to pinned measures; probability measures where the terminal value of the process lies almost surely within a Borel measurable set. While such measures are not equivalent with respect to $\mathbb{P}$ and thus do not fall into the set of admissible measures of our related problem, we derive them as a limiting case of solutions to our optimization problem. Finally, we provide an algorithm for solving and simulating from the optimal probability measure and illustrate the numerical results on a running cost constraint in a financial setting on real data.

Studying minimal relative entropy subject to constraints has a long history starting with the seminal paper of [11]. More recent applications to model risk assessment include [19] which uses the relative entropy to quantify worst-case model errors in a static setting. Similarly and also in a static setting, [5] proposes to quantify distributional model risk by considering alternative models that lie within a KL tolerance distance from a reference measure. The work in [24] investigates what happens in the limit of small KL tolerances. A related work is that of [7] which pursues model uncertainty in the context of stress testing using static minimization. Conceptually close to our work—though in a static setting—is [27] which considers a reference probability measure and finds the probability measure that satisfies risk measure constraints with minimal relative entropy to the reference measure. None of these works, however, consider stochastic processes and thus do not consider running cost constraints.

The KL divergence has many applications in financial mathematics. Starting with the influential work of [28], the vast majority of the literature on minimizing relative entropy focuses on its application for derivative pricing in incomplete markets. To avoid arbitrage, such questions require restricting oneself to martingale measures—a restriction which we relax. The early articles [3, 2], for example, consider a reference model and search over all equivalent martingale measures, in a simple diffusive setting, to ensure that a collection of prices of European contingent claims are matched correctly. The more recent article [10] extends [2] by using a compound Poisson process (with discrete jump sizes) as a reference model. The work in [22] studies the problem of finding martingale measures for exponential Lévy processes that minimize Rényi and KL divergences, and [12] uses convex regularization techniques, motivated by KL divergence as a regularizer, to calibrate local volatility models. The work [17] proves the existence of a minimal entropy martingale measure for geometric Lévy processes and provides explicit representations. There are similarities between their results and ours, namely, we also find closed-form representations that are of the Esscher-like family. Our results, however, establish a general framework to constrain terminal and running statistics of the state process under the resulting optimal measure which minimizes divergence from a reference measure; we study multidimensional Lévy–Ito processes and, hence, we are not restricted to geometric Lévy processes. The well-known work of [13] proves a duality result between (i) the problem of hedging a contingent claim by maximizing expected exponential utility, and (ii) the problem of maximizing price subject to an entropic penalty over a suitable class of local martingale measures. Connected with this duality result is the body of work on indifference pricing, where agents value (generally) nonhedgeable assets by finding strategies that optimize expected utility of terminal wealth with and without the presence of the asset and equate their value to determine price; see, e.g., the collection of papers in [8]. In this exposition, we consider a different problem in that we do not restrict ourselves to martingale measures but solve for the optimal dynamics of the process such that given constraints are fulfilled. In particular, in contrast to our setup, all of the above-mentioned literature work with risk-neutral measures (i.e., martingale measures). Moreover, the running cost constraint considered in this work is novel. A natural interpretation of the running cost constraint in mathematical finance is that of the average time spent below a barrier which we consider in the numerical example section.

Optimizing the (relative) entropy has a long tradition and many applications in physics. Article [21], for example, investigates the problem of specifying expectations of observables (random variables) and search over distributions (models) that match these expectations, and which maximize the Shannon entropy to obtain the model that best reflects the information contained in the expectations. This work has been extended in many directions and, for instance, [26] shows how relative entropy may inform about the arrow of time by looking at the relative entropy between the distribution of a process forward in time and its reversed version. As another example, [29] proposes a process for how a (physical) system may evolve to a state of minimal relative entropy, subject to an energy and mass constraint, based on the speed-gradient principle (see, e.g., [16]).

Calculations of the KL divergence of processes has been studied by [30], which establishes that f-divergences and, hence, the KL divergence, between two probability measures on path space may be approximated by focusing on their finite dimensional distributions. An application to uncertainty quantification in a dynamic setting is [15], which uses a variational representation of the Rényi divergence which encompasses the KL divergence to provide uncertainty quantification bounds for rare events.

This paper is structured as follows. Section 2.1 introduces the necessary notation and the two constraint optimization problems we consider. We present a formal derivation of a candidate solution and a verification theorem in section 2.2. Section 2.3 contains an alternative representation of the associated Radon–Nikodym derivatives and in section 2.4 we state the existence and uniqueness of the solutions to both optimization problems. Examples including analytical solution for Value-at-Risk (quantile) constraints, Brownian motion with arbitrary variance, and the connection of the solution to our optimization problem to pinned measures are discussed in section 3. Section 4 proposes an algorithm for calculating the dynamics of the process under the optimal measure, which we illustrate on a financial dataset and a running cost constraint.

2. Optimization problem and its solution.

2.1. Model setup and optimization problems. We work in a complete filtered probability space $(\Omega, \mathbb{P}, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]})$ with time horizon $T > 0$, and refer to $\mathbb{P}$ as the physical (or real-world) probability measure. In this space we introduce families of so-called Lévy–Itô processes. For aspects of the theory of such processes, see [1, 25, 4]. Here, we consider an m-dimensional $\mathbb{P}$-Brownian motion $\boldsymbol{W} = (W^1, \dots, W^m)^\intercal$ and l independent Poisson random measures $\boldsymbol{\mu}(\mathrm{d}t, \mathrm{d}\boldsymbol{z}) = (\mu_1(\mathrm{d}t, \mathrm{d}z_1), \dots, \mu_l(\mathrm{d}t, \mathrm{d}z_l))^\intercal$, $t \in [0,T]$, $\boldsymbol{z} = (z_1, \dots, z_l)^\intercal$, associated with l one-dimensional independent Lévy processes with finite second moments for all $t \in [0,T]$. Further, we denote by $\boldsymbol{\nu}(\mathrm{d}t, \mathrm{d}\boldsymbol{z}) = (\nu_1(\mathrm{d}t, \mathrm{d}z_1), \dots, \nu_l(\mathrm{d}t, \mathrm{d}z_l))^\intercal$ the compensator of $\boldsymbol{\mu}$ and by $\tilde{\boldsymbol{\mu}} = \boldsymbol{\mu} - \boldsymbol{\nu}$ the compensated measure.[1] That is, for any $i \in \mathcal{D} := \{1, \dots, l\}$, ν_i is the compensator associated with μ_i and $\tilde{\mu}_i = \mu_i - \nu_i$ the compensated random measure under $\mathbb{P}$.

We consider an n-dimensional stochastic process $\boldsymbol{X} := (\boldsymbol{X}_t)_{t\in[0,T]}$ starting at $\boldsymbol{X}_0 = \boldsymbol{x}_0 \in \mathbb{R}^n$ and which evolves according to the stochastic differential equation (SDE) under $\mathbb{P}$,

$$\mathrm{d}\boldsymbol{X}_t = \boldsymbol{\alpha}(t, \boldsymbol{X}_t)\,\mathrm{d}t + \boldsymbol{\sigma}(t, \boldsymbol{X}_t)\,\mathrm{d}\boldsymbol{W}_t + \int_{\mathbb{R}^l} \boldsymbol{\gamma}(t, \boldsymbol{X}_{t^-}, \boldsymbol{z})\,\tilde{\boldsymbol{\mu}}(\mathrm{d}t, \mathrm{d}\boldsymbol{z}), \tag{2.1}$$

where $\boldsymbol{\alpha} : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$, $\boldsymbol{\sigma} : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$, and $\boldsymbol{\gamma} : [0,T] \times \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^{n\times l}$ satisfy the standing Assumption 2.1 below. Equation (2.1) is the matrix notation meaning that the ith component of $(\boldsymbol{X}_t)_{t\in[0,T]}$ satisfies the SDE under $\mathbb{P}$,

$$\mathrm{d}X^i_t = \alpha_i(t, \boldsymbol{X}_t)\,\mathrm{d}t + \sum_{j=1}^m \sigma_{ij}(t, \boldsymbol{X}_t)\,\mathrm{d}W^j_t + \sum_{j\in\mathcal{D}} \int_{\mathbb{R}} \gamma_{ij}(t, \boldsymbol{X}_{t^-}, z_j)\,\tilde{\mu}_j(\mathrm{d}t, \mathrm{d}z_j)$$

with $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)^\intercal$, $\boldsymbol{\sigma} = [\sigma_{ij}]_{i,j}$, and $\boldsymbol{\gamma} = [\gamma_{ij}]_{i,j}$ for $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$. For $j \in \mathcal{D}$, we use the notation $\boldsymbol{\gamma}^{(j)} = (\gamma_{1,j}, \dots \gamma_{n,j})$ to refer to the jth column of $\boldsymbol{\gamma}$. Furthermore, we assume that each column $\boldsymbol{\gamma}^{(j)}$, $j \in \mathcal{D}$, depends on $\boldsymbol{z}$ only through z_j, i.e., $\boldsymbol{\gamma}^{(j)}(t, \boldsymbol{x}, \boldsymbol{z}) \equiv \boldsymbol{\gamma}^{(j)}(t, \boldsymbol{x}, z_j)$.

Throughout we use the following notation. For a function $\ell \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R}^n; \mathbb{R})$ we write $\nabla_{\boldsymbol{x}}\ell$ for the vector of its partial derivatives and $\nabla^2_{\boldsymbol{x}}\ell$ for the Hessian matrix of (mixed) second derivatives. We further define

$$\boldsymbol{\Delta}_{\boldsymbol{z}}\ell(t, \boldsymbol{x}) = \left(\Delta^1_{z_1}\ell(t, \boldsymbol{x}), \dots, \Delta^l_{z_l}\ell(t, \boldsymbol{x})\right), \quad \text{where} \tag{2.2a}$$

$$\Delta^j_{z_j}\ell(t, \boldsymbol{x}) := \ell\left(t, \boldsymbol{x} + \boldsymbol{\gamma}^{(j)}(t, \boldsymbol{x}, z_j)\right) - \ell(t, \boldsymbol{x}), \quad j \in \mathcal{D}. \tag{2.2b}$$

The next assumption guarantees that the stochastic process $(\boldsymbol{X}_t)_{t\in[0,T]}$ given in (2.1) is well-defined.

Assumption 2.1. The functions $\boldsymbol{\alpha} : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$, $\boldsymbol{\sigma} : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$, and $\boldsymbol{\gamma} : [0,T] \times \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^{n\times l}$ satisfy the usual linear growth and Lipschitz continuity conditions. That is, for all $t \in [0,T]$ and $\boldsymbol{x} \in \mathbb{R}^n$ there exists $C_1 < \infty$ such that

$$\|\boldsymbol{\sigma}(t, \boldsymbol{x})\|^2 + |\boldsymbol{\alpha}(t, \boldsymbol{x})|^2 + \int_{\mathbb{R}} \sum_{j\in\mathcal{D}} |\boldsymbol{\gamma}^{(j)}(t, \boldsymbol{x}, z_j)|^2 \nu_j(\mathrm{d}z_j) \le C_1\,(1 + |\boldsymbol{x}|^2),$$

[1] In the present framework $\boldsymbol{\nu}(\mathrm{d}t, \mathrm{d}\boldsymbol{z})$ can be written as $\boldsymbol{\nu}(\mathrm{d}\boldsymbol{z})\mathrm{d}t$ and we use them interchangeably.

where $|\cdot|$ is the Euclidean norm and $\|\boldsymbol{\sigma}\|^2 = \sum_{ij} \sigma_{ij}^2$ the Frobenius norm. Moreover, for all $t \in [0,T]$, $\boldsymbol{x} \in \mathbb{R}^n$, and $\boldsymbol{y} \in \mathbb{R}^n$ there exists $C_2 < \infty$ such that

$$\begin{aligned}&\|\boldsymbol{\sigma}(t,\boldsymbol{x}) - \boldsymbol{\sigma}(t,\boldsymbol{y})\|^2 + |\boldsymbol{\alpha}(t,\boldsymbol{x}) - \boldsymbol{\alpha}(t,\boldsymbol{y})|^2 \\ &\quad + \int_{\mathbb{R}} \sum_{j \in \mathcal{D}} |\boldsymbol{\gamma}^{(j)}(t,\boldsymbol{x},z_j) - \boldsymbol{\gamma}^{(j)}(t,\boldsymbol{y},z_j)|^2 \nu_j(\mathrm{d}z_j) \leq C_2\, |\boldsymbol{x} - \boldsymbol{y}|^2.\end{aligned}$$

As a consequence of Assumption 2.1 and by Theorem 1.19 in [25], there exists a unique càdlàg adapted process starting at $\boldsymbol{X}_0 = \boldsymbol{x}_0 \in \mathbb{R}^n$ that satisfies the SDE in (2.1); we refer to that process as this unique càdlàg solution $(\boldsymbol{X}_t)_{t\in[0,T]}$. Moreover, it holds for all $t \in [0,T]$ that $\mathbb{E}\left[|\boldsymbol{X}_t|^2\right] < \infty$.

We use the KL divergence also called relative entropy to quantify the distance between probability measures. Recall that the KL divergence of a probability measure $\mathbb{Q}$ with respect to $\mathbb{P}$ is given by

$$D_{KL}(\mathbb{Q} \mid\mid \mathbb{P}) = \begin{cases} \mathbb{E}\left[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \log \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right] & \text{if} \quad \mathbb{Q} \ll \mathbb{P}, \\ \infty & \text{otherwise,} \end{cases}$$

where we use the convention that $0\log 0 = 0$. For a probability measures $\mathbb{Q}$ we write $\mathbb{E}^{\mathbb{Q}}[\cdot]$ when we consider the $\mathbb{Q}$-expectation and for notational simplicity set $\mathbb{E}[\cdot] := \mathbb{E}^{\mathbb{P}}[\cdot]$.

Now we are ready to formally introduce the optimization problem which we will solve in the subsequent sections.

Optimization 2.2. *For functions $f_j, g_i : \mathbb{R}^n \to \mathbb{R}$ and constants $c_j, d_i \in \mathbb{R}$ with $j \in \mathcal{R}_1 := \{1,2,\ldots,r_1\}$, $i \in \mathcal{R}_2 := \{1,2,\ldots,r_2\}$, we consider the optimization problem*

$$(P) \qquad \begin{aligned}&\inf_{\mathbb{Q}\ll\mathbb{P}} D_{KL}(\mathbb{Q} \mid\mid \mathbb{P}) \quad \textit{subject to} \quad \mathbb{E}^{\mathbb{Q}}[f_j(\boldsymbol{X}_T)] = c_j \ \forall\, j \in \mathcal{R}_1, \quad \textit{and} \\ &\mathbb{E}^{\mathbb{Q}}\left[\int_0^T g_i(\boldsymbol{X}_s)\,\mathrm{d}s\right] = d_i \ \forall\, i \in \mathcal{R}_2,\end{aligned}$$

where the infimum is taken over probability measures that are absolutely continuous with respect to $\mathbb{P}$.

For $j \in \mathcal{R}_1$ and $i \in \mathcal{R}_2$, we call the equations $\mathbb{E}^{\mathbb{Q}}[f_j(\boldsymbol{X}_T)] = c_j$ and $\mathbb{E}^{\mathbb{Q}}[\int_0^T g_i(\boldsymbol{X}_s)\,\mathrm{d}s] = d_i$ constraints, and f_j and g_i constraint functions.

Before solving the optimization problem (P) we study the following closely related problem. Specifically, we consider optimization problem (P) where we only search over a subset of equivalent probability measures—the set of equivalent probability measures characterized by Doléans–Dade exponentials. For this, we define the following sets of stochastic processes:

$$\mathcal{P}_2([0,T]) := \left\{ \boldsymbol{\lambda} \;\middle|\; \boldsymbol{\lambda} := (\boldsymbol{\lambda}_t)_{t\in[0,T]} \text{ is } \mathbb{R}^m\text{-valued } \mathcal{F}\text{-adapted and } \mathbb{E}\left[\int_0^T |\boldsymbol{\lambda}_t|^2\,\mathrm{d}t\right] < \infty \right\}$$

and

$$\mathcal{P}_2([0,T]\times\mathbb{R}^l;\boldsymbol{\nu}) := \Bigg\{ \boldsymbol{h} \;\Bigg|\; \boldsymbol{h} := (\boldsymbol{h}_t(\boldsymbol{z}))_{t\in[0,T]} \text{ is } \mathbb{R}^l\text{-valued, predictable,}$$

$$\boldsymbol{h}_t(\boldsymbol{z}) \leq 1, \quad \text{and} \quad \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^l} (\boldsymbol{h}_t(\boldsymbol{z}) \odot \boldsymbol{h}_t(\boldsymbol{z}))\, \boldsymbol{\nu}(\mathrm{d}\boldsymbol{z})\mathrm{d}t\right] < \infty \Bigg\}.$$

Here $\odot$ stands for the Hadamard product for vectors, which is defined for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ by $\boldsymbol{x} \odot \boldsymbol{y} = (x_1\, y_1, \ldots, x_n\, y_n)$, and the inequality $\boldsymbol{h}_t(\boldsymbol{z}) \leq 1$, where $\boldsymbol{h}_t(\boldsymbol{z}) = (h_t^1(z_1), \ldots, h_t^l(z_l))$, is to be understood componentwise. For $\boldsymbol{\lambda} \in \mathcal{P}_2([0,T])$ and $\boldsymbol{h} \in \mathcal{P}_2([0,T] \times \mathbb{R}^l; \nu)$, we define the process $Z^{\boldsymbol{\lambda},\boldsymbol{h}} = (Z_t^{\boldsymbol{\lambda},\boldsymbol{h}})_{t\in[0,T]}$, given for $t \in [0,T]$ by

$$
\begin{aligned}
Z_t^{\boldsymbol{\lambda},\boldsymbol{h}} := \exp\Bigg(&- \int_0^t \boldsymbol{\lambda}_s \, \mathrm{d}\boldsymbol{W}_s - \frac{1}{2}\int_0^t |\boldsymbol{\lambda}_s|^2 \, \mathrm{d}s + \int_0^t \int_{\mathbb{R}^l} \mathbf{log}(\mathbf{1} - \boldsymbol{h}_s(\boldsymbol{z}))\, \tilde{\boldsymbol{\mu}}(\mathrm{d}s, \mathrm{d}\boldsymbol{z}) \\
&+ \int_0^t \int_{\mathbb{R}^l} \{\mathbf{log}(\mathbf{1} - \boldsymbol{h}_s(\boldsymbol{z})) + \boldsymbol{h}_s(\boldsymbol{z})\}\, \boldsymbol{\nu}(\mathrm{d}\boldsymbol{z})\, \mathrm{d}s \Bigg),
\end{aligned} \tag{2.3}
$$

where $\mathbf{log}(\mathbf{1} - \boldsymbol{h}_s(\boldsymbol{z})) := (\log(1 - h_s^1(z_1)), \ldots, \log(1 - h_s^l(z_l)))$. This process is a Doléans–Dade exponential and with, e.g., the Novikov assumption, defines a Radon–Nikodym (RN) derivative. We recall the Novikov's condition on $Z^{\boldsymbol{\lambda},\boldsymbol{h}}$ which is

$$
\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T |\boldsymbol{\lambda}_t|^2 \, \mathrm{d}t + \int_0^T \int_{\mathbb{R}^l} \boldsymbol{h}_t(\boldsymbol{z}) \odot \boldsymbol{h}_t(\boldsymbol{z})\, \boldsymbol{\mu}(\mathrm{d}t, \mathrm{d}\boldsymbol{z})\right)\right] < \infty, \tag{2.4}
$$

and which establishes sufficient conditions on $\boldsymbol{\lambda}$ and $\boldsymbol{h}$ such that $\mathbb{E}[Z_T^{\boldsymbol{\lambda},\boldsymbol{h}}] = 1$ and $(Z_t^{\boldsymbol{\lambda},\boldsymbol{h}})_{0\leq t\leq T}$ is a martingale. Thus, the measure $\mathbb{Q}_{\boldsymbol{\lambda},\boldsymbol{h}}$ characterized by the RN derivative

$$
\frac{\mathrm{d}\mathbb{Q}_{\boldsymbol{\lambda},\boldsymbol{h}}}{\mathrm{d}\mathbb{P}} = Z_T^{\boldsymbol{\lambda},\boldsymbol{h}}
$$

is a probability measure that is absolutely continuous with respect to $\mathbb{P}$; see Theorem 1.36 in [25]. We denote the probability measure $\mathbb{Q}_{\boldsymbol{\lambda},\boldsymbol{h}}$ by subscripts to indicate that it arises from an $\mathbb{R}^m$-valued process $\boldsymbol{\lambda}$ and an $\mathbb{R}^l$-valued random field $\boldsymbol{h}$.

With the above definitions we are ready to introduce a subset of absolutely continuous probability measures with respect to $\mathbb{P}$ given in (P), that are characterized by RN densities $Z^{\boldsymbol{\lambda},\boldsymbol{h}}$ with $\boldsymbol{\lambda} \in \mathcal{P}_2([0,T])$ and $\boldsymbol{h} \in \mathcal{P}_2([0,T] \times \mathbb{R}^l; \nu)$:

$$
\begin{aligned}
\boldsymbol{\mathcal{Q}} := \Bigg\{ \mathbb{Q}_{\boldsymbol{\lambda},\boldsymbol{h}} \;\Big|\; &\mathrm{d}\mathbb{Q}_{\boldsymbol{\lambda},\boldsymbol{h}} = Z_T^{\boldsymbol{\lambda},\boldsymbol{h}}\, \mathrm{d}\mathbb{P} \;\text{ s.t. }\; \mathbb{E}\left[Z_T^{\boldsymbol{\lambda},\boldsymbol{h}}\right] = 1,\; \mathbb{E}\left[\left|Z_T^{\boldsymbol{\lambda},\boldsymbol{h}} \log Z_T^{\boldsymbol{\lambda},\boldsymbol{h}}\right|\right] < \infty, \\
&\mathbb{E}^{\mathbb{Q}_{\boldsymbol{\lambda},\boldsymbol{h}}}\left[|f_j(\boldsymbol{X}_T)|\right] < \infty \;\; \forall\, j \in \mathcal{R}_1, \quad \mathbb{E}^{\mathbb{Q}_{\boldsymbol{\lambda},\boldsymbol{h}}}\left[\int_0^T |g_i(\boldsymbol{X}_s)|\, \mathrm{d}s\right] < \infty \;\; \forall\, i \in \mathcal{R}_2 \\
&\mathbb{E}^{\mathbb{Q}_{\boldsymbol{\lambda},\boldsymbol{h}}}\left[\sup_{t\in[0,T]} |\boldsymbol{X}_t|^2\right] < \infty, \text{ and } \boldsymbol{\lambda} \in \mathcal{P}_2\left([0,T]\right), \boldsymbol{h} \in \mathcal{P}_2\left([0,T] \times \mathbb{R}^l; \nu\right) \Bigg\}.
\end{aligned} \tag{2.5}
$$

Note that we do not assume Novikov's condition in (2.5).

Using the above class of equivalent probability measures, we consider the following optimization problem, which is optimization problem (P) but where we search over the subset of probability measures $\boldsymbol{\mathcal{Q}}$.

OPTIMIZATION 2.3. *For functions $f_j, g_i : \mathbb{R}^n \to \mathbb{R}$ and constants $c_j, d_i \in \mathbb{R}$ with $j \in \mathcal{R}_1$, $i \in \mathcal{R}_2$, we consider the optimization problem*

$$
\begin{aligned}
\inf_{\mathbb{Q}_{\boldsymbol{\lambda},\boldsymbol{h}} \in \boldsymbol{\mathcal{Q}}} \mathbb{E}\left[Z_T^{\boldsymbol{\lambda},\boldsymbol{h}} \log Z_T^{\boldsymbol{\lambda},\boldsymbol{h}}\right] \quad \textit{subject to} \quad &\mathbb{E}^{\mathbb{Q}_{\boldsymbol{\lambda},\boldsymbol{h}}}\left[f_j(\boldsymbol{X}_T)\right] = c_j \;\; \forall\, j \in \mathcal{R}_1, \quad \textit{and} \\
&\mathbb{E}^{\mathbb{Q}_{\boldsymbol{\lambda},\boldsymbol{h}}}\left[\int_0^T g_i(\boldsymbol{X}_s)\, \mathrm{d}s\right] = d_i \;\; \forall\, i \in \mathcal{R}_2.
\end{aligned} \tag{P'}
$$

For $\mathbb{Q}_{\boldsymbol{\lambda},\boldsymbol{h}} \in \boldsymbol{\mathcal{Q}}$ and, as a consequence of Girsanov's heorem, $\boldsymbol{W}^{\boldsymbol{\lambda}}$ defined by

$$\boldsymbol{W}^{\boldsymbol{\lambda}}{}_t := \int_0^t \boldsymbol{\lambda}_s^\intercal \,\mathrm{d}s + \boldsymbol{W}_t$$

is an m-dimensional $\mathbb{Q}_{\boldsymbol{\lambda},\boldsymbol{h}}$-Brownian motion and the $\mathbb{Q}_{\boldsymbol{\lambda},\boldsymbol{h}}$-compensator of $\boldsymbol{\mu}$ is

$$\boldsymbol{\nu}^{\boldsymbol{h}}(\mathrm{d}t, \mathrm{d}\boldsymbol{z}) := (\mathbf{1} - \boldsymbol{h}_t(\boldsymbol{z}))^\intercal \odot \boldsymbol{\nu}(\mathrm{d}\boldsymbol{z})\,\mathrm{d}t.$$

For notational simplicity we write $\boldsymbol{W}^{\boldsymbol{\lambda}}$ and $\boldsymbol{\nu}^{\boldsymbol{h}}$ as they only explicitly depend on $\boldsymbol{\lambda}$ and $\boldsymbol{h}$, respectively.

Using the above results, the KL divergence from $\mathbb{Q}_{\boldsymbol{\lambda},\boldsymbol{h}}$ to $\mathbb{P}$ becomes

$$\begin{aligned} &D_{KL}\left(\mathbb{Q}_{\boldsymbol{\lambda},\boldsymbol{h}} \;||\; \mathbb{P}\right) \\ &= \mathbb{E}^{\mathbb{Q}_{\boldsymbol{\lambda},\boldsymbol{h}}}\left[\frac{1}{2}\int_0^T |\boldsymbol{\lambda}_t|^2\,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^l} \left[\mathbf{log}(\mathbf{1} - \boldsymbol{h}_t(\boldsymbol{z})) \odot (\mathbf{1} - \boldsymbol{h}_t(\boldsymbol{z})) + \boldsymbol{h}_t(\boldsymbol{z})\right] \boldsymbol{\nu}(\mathrm{d}\boldsymbol{z})\mathrm{d}t\right]. \end{aligned}$$

Next, we discuss assumptions needed for the existence and uniqueness of the Lagrangian associated with optimization problem (P'). For this we first define the moment generating function (mgf) and the cumulant generating function (cgf) for random vectors. For a random vector $\boldsymbol{Y} = (Y_1, \ldots, Y_k)$, $k \in \mathbb{R}$, we define the set

$$D_{\boldsymbol{Y}} := \left\{\boldsymbol{a} \in \mathbb{R}^k \;\middle|\; \mathbb{E}\left[\exp(\boldsymbol{a} \cdot \boldsymbol{Y})\right] < \infty\right\}^\circ,$$

where $\{\}^\circ$ denotes the interior of a set. We note that $D_{\boldsymbol{Y}}$ is the interior of a convex set. If $D_{\boldsymbol{Y}} \neq \emptyset$, then the mgf $M_{\boldsymbol{Y}}$ and cgf $K_{\boldsymbol{Y}}$ of $\boldsymbol{Y}$ at $\boldsymbol{a} \in D_{\boldsymbol{Y}}$ exist and are, respectively, given by

$$M_{\boldsymbol{Y}}(\boldsymbol{a}) = \mathbb{E}[\exp(\boldsymbol{a} \cdot \boldsymbol{Y})] \qquad \text{and} \qquad K_{\boldsymbol{Y}}(\boldsymbol{a}) = \log M_{\boldsymbol{Y}}(\boldsymbol{a}).$$

Assumption 2.4. Let $\mathfrak{X}$ denote the $(r_1 + r_2)$-dimensional random vector given by

$$\mathfrak{X} := \left(\boldsymbol{f}(\boldsymbol{X}_T) - \boldsymbol{c}, \int_0^T \boldsymbol{g}(\boldsymbol{X}_s)\,\mathrm{d}s - \boldsymbol{d}\right),$$

where $\boldsymbol{f}(\boldsymbol{X}_T) := (f_1(\boldsymbol{X}_T), \ldots, f_{r_1}(\boldsymbol{X}_T))$, $\boldsymbol{g}(\boldsymbol{X}_T) := (g_1(\boldsymbol{X}_T), \ldots, g_{r_2}(\boldsymbol{X}_T))$, and constants $\boldsymbol{c} := (c_1, \ldots, c_{r_1})$, and $\boldsymbol{d} := (d_1, \ldots, d_{r_2})$. Here the integral in $\int_0^T \boldsymbol{g}(\boldsymbol{X}_s)\mathrm{d}s$ is understood to be applied componentwise. We assume that $D_{\mathfrak{X}} \neq \emptyset$ and that there exists $\boldsymbol{a}$ such that

$$\nabla_{\boldsymbol{a}} K_{\mathfrak{X}}(-\boldsymbol{a}) = \mathbf{0}. \tag{2.6}$$

In the next sections we first solve optimization problem (P') and then show that its solution, if it exists, is also the solution to optimization problem (P). To solve optimization problem (P') we next proceed by presenting a formal derivation of a candidate solution and a verification theorem.

2.2. Candidate solution and verification. We proceed with a formal derivation of a candidate for the value function associated with the constrained optimization problem (P'). After, we provide a verification theorem that allows us to conclude that the candidate solution is indeed the value function.

Let $(\boldsymbol{\eta}_1, \boldsymbol{\eta}_2) \in D_{-\mathfrak{X}}$ with $\boldsymbol{\eta}_1 := (\eta_1, \ldots, \eta_{r_1}) \in \mathbb{R}^{r_1}$ and $\boldsymbol{\eta}_2 := (\eta_{r_1+1}, \ldots, \eta_{r_1+r_2}) \in \mathbb{R}^{r_2}$, then the Lagrangian of the constrained problem (P') with Lagrange multipliers $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ is given by

$$L^{\boldsymbol{\lambda},\boldsymbol{h}} := \mathbb{E}^{\mathbb{Q}_{\boldsymbol{\lambda},\boldsymbol{h}}}\left[\log Z_T^{\boldsymbol{\lambda},\boldsymbol{h}} + \boldsymbol{\eta}_1 \cdot (\boldsymbol{f}(\boldsymbol{X}_T) - \boldsymbol{c}) + \boldsymbol{\eta}_2 \cdot \left(\int_0^T \boldsymbol{g}(\boldsymbol{X}_s)\mathrm{d}s - \boldsymbol{d}\right)\right],$$

where $\cdot$ denotes the dot product. We define for a fixed control pair $(\boldsymbol{\lambda}, \boldsymbol{h})$, the value $J^{\boldsymbol{\lambda},\boldsymbol{h}} : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ associated with the Lagrangian $L^{\boldsymbol{\lambda},\boldsymbol{h}}$ by

$$\begin{aligned}
&J^{\boldsymbol{\lambda},\boldsymbol{h}}(t,\boldsymbol{x}) \\
&\quad := \mathbb{E}_{t,\boldsymbol{x}}^{\mathbb{Q}_{\boldsymbol{\lambda},\boldsymbol{h}}}\Bigg[\frac{1}{2}\int_t^T |\boldsymbol{\lambda}_t|^2\,\mathrm{d}t + \int_t^T\!\!\int_{\mathbb{R}^l}\big[\mathbf{log}(\mathbf{1} - \boldsymbol{h}_t(\boldsymbol{z})) \odot (\mathbf{1} - \boldsymbol{h}_t(\boldsymbol{z})) + \boldsymbol{h}_t(\boldsymbol{z})\big]\,\boldsymbol{\nu}(\mathrm{d}\boldsymbol{z})\mathrm{d}t \\
&\qquad + \boldsymbol{\eta}_1 \cdot (\boldsymbol{f}(\boldsymbol{X}_T) - \boldsymbol{c}) + \boldsymbol{\eta}_2 \cdot \left(\int_t^T \boldsymbol{g}(\boldsymbol{X}_s)\mathrm{d}s - \boldsymbol{d}\right)\Bigg],
\end{aligned} \tag{2.7}$$

where $\mathbb{E}_{t,\boldsymbol{x}}^{\mathbb{Q}_{\boldsymbol{\lambda},\boldsymbol{h}}}[\cdot]$ denotes the $\mathbb{Q}_{\boldsymbol{\lambda},\boldsymbol{h}}$-expectation conditioned on the event $\boldsymbol{X}_t = \boldsymbol{x}$. Observe that the expectation in (2.7) is finite because of the definition of $\boldsymbol{\mathcal{Q}}$—recall that $D_{KL}(\mathbb{Q}_{\boldsymbol{\lambda},\boldsymbol{h}} \,||\, \mathbb{P}) = \mathbb{E}[Z_T^{\boldsymbol{\lambda},\boldsymbol{h}} \log Z_T^{\boldsymbol{\lambda},\boldsymbol{h}}] < \infty$. We further define the optimal value function, which we often just refer to as the value function, by

$$J(t,\boldsymbol{x}) := \inf_{\substack{\boldsymbol{\lambda},\boldsymbol{h},\, s.t. \\ \mathbb{Q}_{\boldsymbol{\lambda},\boldsymbol{h}} \in \boldsymbol{\mathcal{Q}}}} J^{\boldsymbol{\lambda},\boldsymbol{h}}(t,\boldsymbol{x}). \tag{2.8}$$

For the purposes of the formal derivation we assume that the infimum in (2.8) is finite. We observe that as a consequence of the dynamic programming principle and Itô's formula—under the assumption that $J \in \mathcal{C}^{1,2}([0,T) \times \mathbb{R}^n; \mathbb{R}) \cap \mathcal{C}^0([0,T] \times \mathbb{R}^n; \mathbb{R})$—we have the following dynamic programming equation (DPE)

$$\begin{aligned}
&\partial_t J(t,\boldsymbol{x}) + \inf_{\boldsymbol{\lambda},\boldsymbol{h}}\Bigg\{\mathcal{L}^{\boldsymbol{\lambda},\boldsymbol{h}} J(t,\boldsymbol{x}) + \frac{1}{2}|\boldsymbol{\lambda}|^2 \\
&\qquad + \int_{\mathbb{R}^l}\big[\mathbf{log}(\mathbf{1} - \boldsymbol{h}_t(\boldsymbol{z})) \odot (\mathbf{1} - \boldsymbol{h}_t(\boldsymbol{z})) + \boldsymbol{h}_t(\boldsymbol{z})\big]\,\boldsymbol{\nu}(\mathrm{d}\boldsymbol{z}) + \boldsymbol{\eta}_2 \cdot \boldsymbol{g}(\boldsymbol{x})\Bigg\} = 0, \\
&J(T,\boldsymbol{x}) = \boldsymbol{\eta}_1 \cdot (\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{c}) - \boldsymbol{\eta}_2 \cdot \boldsymbol{d},
\end{aligned} \tag{2.9}$$

where the linear operator $\mathcal{L}^{\boldsymbol{\lambda},\boldsymbol{h}}$ is the $\mathbb{Q}_{\boldsymbol{\lambda},\boldsymbol{h}}$-generator of $\boldsymbol{X}$, and acts on functions as follows:

$$\begin{aligned}
\mathcal{L}^{\lambda,h} J(t,\boldsymbol{x}) &= (\boldsymbol{\alpha}(t,\boldsymbol{x}) - \boldsymbol{\sigma}(t,\boldsymbol{x})\,\boldsymbol{\lambda}^\intercal) \cdot \nabla_{\boldsymbol{x}} J + \frac{1}{2}\operatorname{Tr}\big(\boldsymbol{\sigma}(t,\boldsymbol{x})\,\boldsymbol{\sigma}(t,\boldsymbol{x})^\intercal\,\nabla_{\boldsymbol{x}}^2 J\big) \\
&\quad + \int_{\mathbb{R}^l} \boldsymbol{\Delta}_{\boldsymbol{z}} J(t,\boldsymbol{x})\,\boldsymbol{\nu}^{\boldsymbol{h}}(\mathrm{d}\boldsymbol{z}) - \int_{\mathbb{R}^l} (\nabla_{\boldsymbol{x}} J)^\intercal\,\gamma(t,\boldsymbol{x},\boldsymbol{z})\,\boldsymbol{\nu}(\mathrm{d}\boldsymbol{z}),
\end{aligned}$$

where $\boldsymbol{\Delta}_{\boldsymbol{z}} J(t,\boldsymbol{x})$ is defined in (2.2), and $\operatorname{Tr}(\cdot)$ denotes the trace of a matrix. The specific form of the DPE follows from writing (2.1) in terms of $\boldsymbol{W}^{\boldsymbol{\lambda}}$ and $\tilde{\boldsymbol{\mu}}^{\boldsymbol{h}} = \boldsymbol{\mu} - \boldsymbol{\nu}^{\boldsymbol{h}}$, so that

$$\begin{aligned}
\mathrm{d}\boldsymbol{X}_t &= \left(\boldsymbol{\alpha}(t,\boldsymbol{X}_t) - \boldsymbol{\sigma}(t,\boldsymbol{X}_t)\,\boldsymbol{\lambda}_t^\intercal - \int_{\mathbb{R}^l} \gamma(t,\boldsymbol{X}_{t^-},\boldsymbol{z})\,[\boldsymbol{h}_t^\intercal(\boldsymbol{z}) \odot \boldsymbol{\nu}(\mathrm{d}\boldsymbol{z})]\right)\mathrm{d}t \\
&\quad + \boldsymbol{\sigma}(t,\boldsymbol{X}_t)\,\mathrm{d}\boldsymbol{W^{\lambda}}_t + \int_{\mathbb{R}^l} \gamma(t,\boldsymbol{X}_{t^-},\boldsymbol{z})\,\tilde{\boldsymbol{\mu}}^{\boldsymbol{h}}(\mathrm{d}t,\mathrm{d}\boldsymbol{z}).
\end{aligned}$$

The measurable global minimizers $\boldsymbol{\lambda}^\dagger$ and $\boldsymbol{h}^\dagger$ (in feedback form) of the infimum in (2.9) are given by

$$\boldsymbol{\lambda}^\dagger(t,\boldsymbol{x}) = \left(\nabla_{\boldsymbol{x}} J(t,\boldsymbol{x})\right)\boldsymbol{\sigma}(t,\boldsymbol{x}) \quad \text{and} \quad \boldsymbol{h}^\dagger(t,\boldsymbol{x},\boldsymbol{z}) = \mathbf{1} - \boldsymbol{e}^{-\boldsymbol{\Delta}_{\boldsymbol{z}} J(t,\boldsymbol{x})}, \tag{2.10}$$

where $\mathbf{1} - \boldsymbol{e}^{-\boldsymbol{\Delta}_{\boldsymbol{z}} J(t,\boldsymbol{x})}$ stands for $(1 - e^{-\Delta^1_{z_1} J(t,\boldsymbol{x})}, \ldots, 1 - e^{-\Delta^l_{z_l} J(t,\boldsymbol{x})})$.

It follows immediately that $\boldsymbol{h}^\dagger$ is componentwise bounded from above by unity. Inserting the optimal controls $\boldsymbol{\lambda}^\dagger$ and $\boldsymbol{h}^\dagger$ in feedback form back into the DPE (2.9) (omitting the arguments $(t,\boldsymbol{x})$ when possible) we obtain that

$$\begin{aligned} &\partial_t J - \frac{1}{2}\left|\nabla_{\boldsymbol{x}} J\, \boldsymbol{\sigma}\right|^2 + \boldsymbol{\alpha}\cdot\nabla_{\boldsymbol{x}} J - \int_{\mathbb{R}^l} (\nabla_{\boldsymbol{x}} J)^\intercal\, \boldsymbol{\gamma}(t,\boldsymbol{x},\boldsymbol{z})\, \boldsymbol{\nu}(\mathrm{d}\boldsymbol{z}) \\ &\quad + \frac{1}{2}\operatorname{Tr}\left(\boldsymbol{\sigma}\,\boldsymbol{\sigma}^\intercal\, \nabla^2_{\boldsymbol{x}} J\right) + \int_{\mathbb{R}^l}\left(\mathbf{1} - \boldsymbol{e}^{-\boldsymbol{\Delta}_{\boldsymbol{z}} J}\right)\boldsymbol{\nu}(\mathrm{d}\boldsymbol{z}) + \boldsymbol{\eta}_2\cdot\boldsymbol{g} = 0 \end{aligned} \tag{2.11}$$

together with the terminal condition $J(T,\boldsymbol{x}) = \sum_{j=1}^{r_1} \eta_j\,(f_j(\boldsymbol{x}) - c_j) - \sum_{i=r_1+1}^{r_2} \eta_i\, d_i$. We observe that (2.11) can be written as

$$\begin{aligned} &\partial_t J + \mathcal{L}^c J - \frac{1}{2}\left|\nabla_{\boldsymbol{x}} J\, \boldsymbol{\sigma}\right|^2 - \int_{\mathbb{R}^l} (\nabla_{\boldsymbol{x}} J)^\intercal\, \boldsymbol{\gamma}(t,\boldsymbol{x},\boldsymbol{z})\, \boldsymbol{\nu}(\mathrm{d}\boldsymbol{z}) \\ &\quad + \int_{\mathbb{R}^l}\left(\mathbf{1} - \boldsymbol{e}^{-\boldsymbol{\Delta}_{\boldsymbol{z}} J}\right)\boldsymbol{\nu}(\mathrm{d}\boldsymbol{z}) + \boldsymbol{\eta}_2\cdot\boldsymbol{g} = 0, \end{aligned} \tag{2.12}$$

where the linear operator $\mathcal{L}^c$ is the $\mathbb{P}$-generator of the continuous part[2] of $\boldsymbol{X}$ and acts on functions as follows:

$$\mathcal{L}^c J = \boldsymbol{\alpha}\cdot\nabla_{\boldsymbol{x}} J + \frac{1}{2}\operatorname{Tr}\left(\boldsymbol{\sigma}\,\boldsymbol{\sigma}^\intercal\,\nabla^2_{\boldsymbol{x}} J\right).$$

Next, we construct a candidate for the solution to (2.12) by introducing the change of variables $J(t,\boldsymbol{x}) = -\log\omega(t,\boldsymbol{x})$. Hence,

$$\partial_t J = -\frac{\partial_t\omega}{\omega}, \quad \nabla_{\boldsymbol{x}} J = -\frac{1}{\omega}\nabla_{\boldsymbol{x}}\omega, \quad \nabla^2_{\boldsymbol{x}} J = -\frac{1}{\omega}\nabla^2_{\boldsymbol{x}}\omega + \frac{1}{\omega^2}(\nabla_{\boldsymbol{x}}\omega)^\intercal\,\nabla_{\boldsymbol{x}}\omega,$$

and, furthermore, $\Delta^j_{z_j} J(t,\boldsymbol{x}) = \log\left(w(t,\boldsymbol{x})/w(t,\boldsymbol{x}+\boldsymbol{\gamma}^{(j)}(t,\boldsymbol{x},z_j))\right)$. Equation (2.12) thus becomes

$$\begin{aligned} &-\frac{1}{\omega}\left\{\partial_t\omega + \boldsymbol{\alpha}\cdot\nabla_{\boldsymbol{x}}\omega + \frac{1}{2}\operatorname{Tr}\left(\boldsymbol{\sigma}\,\boldsymbol{\sigma}^\intercal\,\nabla^2_{\boldsymbol{x}}\omega\right) - \int_{\mathbb{R}^l}(\nabla_{\boldsymbol{x}}\omega)^\intercal\,\boldsymbol{\gamma}\,\boldsymbol{\nu}(\mathrm{d}\boldsymbol{z}) + \int_{\mathbb{R}^l}\boldsymbol{\Delta}_{\boldsymbol{z}}\omega\,\boldsymbol{\nu}(\mathrm{d}\boldsymbol{z})\right\} \\ &\quad + \frac{1}{2\,\omega^2}\operatorname{Tr}\left(\boldsymbol{\sigma}\,\boldsymbol{\sigma}^\intercal\,(\nabla_{\boldsymbol{x}}\omega)^\intercal\,\nabla_{\boldsymbol{x}}\omega\right) - \frac{1}{2\,\omega^2}\left|\nabla_{\boldsymbol{x}}\omega\,\boldsymbol{\sigma}\right|^2 + \boldsymbol{\eta}_2\cdot\boldsymbol{g} = 0 \end{aligned} \tag{2.13}$$

with $\omega(T,\boldsymbol{x}) = \exp(-\boldsymbol{\eta}_1\cdot(\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{c}) + \boldsymbol{\eta}_2\cdot\boldsymbol{d})$. Multiplying (2.13) by $-\omega(t,\boldsymbol{x})$, we have that

$$\begin{aligned} &\partial_t\omega + \boldsymbol{\alpha}\cdot\nabla_{\boldsymbol{x}}\omega + \frac{1}{2}\operatorname{Tr}\left(\boldsymbol{\sigma}\,\boldsymbol{\sigma}^\intercal\,\nabla^2_{\boldsymbol{x}}\omega\right) - \int_{\mathbb{R}^l}(\nabla_{\boldsymbol{x}}\omega)^\intercal\,\boldsymbol{\gamma}\,\boldsymbol{\nu}(\mathrm{d}\boldsymbol{z}) \\ &\quad + \int_{\mathbb{R}^l}\boldsymbol{\Delta}_{\boldsymbol{z}}\omega\,\boldsymbol{\nu}(\mathrm{d}\boldsymbol{z}) - \boldsymbol{\eta}_2\cdot\boldsymbol{g}\,\omega = 0, \end{aligned}$$

[2]The continuous part of $\boldsymbol{X}$ is defined by $\boldsymbol{X}^c_t := \boldsymbol{X}_t - \sum_{0\le s\le t}\Delta\boldsymbol{X}_s$, $\Delta\boldsymbol{X}_t := \boldsymbol{X}_t - \boldsymbol{X}_{t^-}$, where $\boldsymbol{X}_{t^-} := \lim_{s\uparrow t}\boldsymbol{X}_s$.

where we use the fact that

$$\frac{1}{2\,\omega^2}\,\mathrm{Tr}\left(\boldsymbol{\sigma}\,\boldsymbol{\sigma}^\intercal\,(\nabla_{\boldsymbol{x}}\omega)^\intercal\,\nabla_{\boldsymbol{x}}\omega\right)-\frac{1}{2\,\omega^2}\left|\nabla_{\boldsymbol{x}}\omega\,\boldsymbol{\sigma}\right|^2=0.$$

Then, one can write a Feynman–Kac representation for $\omega(t,\boldsymbol{x})$ given that the logarithmic transformation linearized the HJB equation.

The above formal calculations provide the following candidate solution for the value function.

PROPOSITION 2.5. *A candidate solution to the value function* (2.8) *is given by*

$$J(t,\boldsymbol{x})=-\log\mathbb{E}_{t,\boldsymbol{x}}\left[\exp\left(-\boldsymbol{\eta}_1\cdot(\boldsymbol{f}(\boldsymbol{X}_T)-\boldsymbol{c})-\boldsymbol{\eta}_2\cdot\left(\int_t^T\boldsymbol{g}(\boldsymbol{X}_u)\mathrm{d}u-\boldsymbol{d}\right)\right)\right]$$

with the optimal Markovian controls given by (2.10) *in terms of* J *defined above.*

Next, we prove that, under certain conditions and for fixed Lagrange multipliers, this candidate solution does indeed coincide with the value function.

THEOREM 2.6 (verification). *Under Assumption* 2.1*, let* $D_{-\mathfrak{X}}\neq\emptyset$ *and* $(\boldsymbol{\eta}_1,\boldsymbol{\eta}_2)\in D_{-\mathfrak{X}}$*. Define*

$$J^\dagger(t,\boldsymbol{x}):=-\log\omega^\dagger(t,\boldsymbol{x}),$$

where

$$\omega^\dagger(t,\boldsymbol{x})=\mathbb{E}_{t,\boldsymbol{x}}\left[\exp\left(-\boldsymbol{\eta}_1\cdot(\boldsymbol{f}(\boldsymbol{X}_T)-\boldsymbol{c})-\boldsymbol{\eta}_2\cdot\left(\int_t^T\boldsymbol{g}(\boldsymbol{X}_u)\mathrm{d}u-\boldsymbol{d}\right)\right)\right] \tag{2.14}$$

and suppose that $J^\dagger\in\mathcal{C}^{1,2}([0,T)\times\mathbb{R}^n;\mathbb{R})\cap\mathcal{C}^0([0,T]\times\mathbb{R}^n;\mathbb{R})$ *with* $J^\dagger$ *having at most quadratic growth, i.e., there is* $C_3\in\mathbb{R}^+$ *such that* $\left|J^\dagger(t,\boldsymbol{x})\right|\le C_3(1+|\boldsymbol{x}|^2)$*. Let*

$$\boldsymbol{\lambda}^\dagger_t:=-\frac{\nabla_{\boldsymbol{x}}\omega^\dagger(t,\boldsymbol{X}_t)}{\omega^\dagger(t,\boldsymbol{X}_t)}\,\boldsymbol{\sigma}(t,\boldsymbol{X}_t)\qquad\text{and}\qquad\boldsymbol{h}^\dagger_t(\boldsymbol{z}):=-\frac{\boldsymbol{\Delta}_{\boldsymbol{z}}\omega^\dagger(t,\boldsymbol{X}_{t^-})}{\omega^\dagger(t,\boldsymbol{X}_{t^-})}, \tag{2.15}$$

and assume that $\boldsymbol{\lambda}^\dagger$ *and* $\boldsymbol{h}^\dagger$ *satisfy Novikov's condition* (2.4) *and are such that* $\mathbb{Q}_{\boldsymbol{\lambda}^\dagger,\boldsymbol{h}^\dagger}\in\boldsymbol{\mathcal{Q}}$*. Then,* $\boldsymbol{\lambda}^\dagger$ *and* $\boldsymbol{h}^\dagger$ *are admissible controls and* $J^\dagger=J$*.*

Proof. Note that Novikov's condition and $\mathbb{Q}_{\boldsymbol{\lambda}^\dagger,\boldsymbol{h}^\dagger}\in\boldsymbol{\mathcal{Q}}$ are sufficient to guarantee that $\boldsymbol{\lambda}^\dagger$ and $\boldsymbol{h}^\dagger$ are admissible and that they induce a measure $\mathbb{Q}_{\boldsymbol{\lambda}^\dagger,\boldsymbol{h}^\dagger}$ that is well-defined. Next, we observe that

$$\begin{aligned}&\partial_t J^\dagger(t,\boldsymbol{x})+\inf_{\boldsymbol{\lambda},\boldsymbol{h}}\left\{\mathcal{L}^{\boldsymbol{\lambda},\boldsymbol{h}}J^\dagger(t,\boldsymbol{x})+\frac{1}{2}|\boldsymbol{\lambda}|^2+\mathfrak{h}_t+\boldsymbol{\eta}_2\cdot\boldsymbol{g}(\boldsymbol{x})\right\}\\&\qquad=\partial_t J^\dagger(t,\boldsymbol{x})+\mathcal{L}^{\boldsymbol{\lambda}^\dagger,\boldsymbol{h}^\dagger}J^\dagger(t,\boldsymbol{x})+\frac{1}{2}|\boldsymbol{\lambda}^\dagger|^2+\mathfrak{h}^\dagger_t+\boldsymbol{\eta}_2\cdot\boldsymbol{g}(\boldsymbol{x})=0,\end{aligned}$$

where we set $\mathfrak{h}_t:=\int_{\mathbb{R}^l}\left[\log(\mathbf{1}-\boldsymbol{h}_t(\boldsymbol{z}))\odot(\mathbf{1}-\boldsymbol{h}_t(\boldsymbol{z}))+\boldsymbol{h}_t(\boldsymbol{z})\right]\boldsymbol{\nu}(\mathrm{d}\boldsymbol{z})$ and $\mathfrak{h}^\dagger_t:=\int_{\mathbb{R}^l}\left[\log(\mathbf{1}-\boldsymbol{h}^\dagger_t(\boldsymbol{z}))\odot(\mathbf{1}-\boldsymbol{h}^\dagger_t(\boldsymbol{z}))+\boldsymbol{h}^\dagger_t(\boldsymbol{z})\right]\boldsymbol{\nu}(\mathrm{d}\boldsymbol{z})$. Then, for arbitrary $(\boldsymbol{\lambda},\boldsymbol{h})$ s.t. $\mathbb{Q}_{\boldsymbol{\lambda},\boldsymbol{h}}\in\boldsymbol{\mathcal{Q}}$, $(t,\boldsymbol{x})\in[0,T)\times\mathbb{R}^n$, $s\in[t,T)$, and a stopping time $\tau_n=\inf\{u\ge t:|\boldsymbol{X}_u|>n\}$ for $n\in\mathbb{Z}_+,n<\infty$, we use Dynkin's formula to obtain

$$\begin{aligned}&\mathbb{E}^{\mathbb{Q}_{\boldsymbol{\lambda},\boldsymbol{h}}}_{t,\boldsymbol{x}}\left[J^\dagger(s\wedge\tau_n,\boldsymbol{X}_{s\wedge\tau_n})\right]\\&\qquad=J^\dagger(t,\boldsymbol{x})+\mathbb{E}^{\mathbb{Q}_{\boldsymbol{\lambda},\boldsymbol{h}}}_{t,\boldsymbol{x}}\left[\int_t^{s\wedge\tau_n}\partial_t J^\dagger(u,\boldsymbol{X}_u)+\mathcal{L}^{\boldsymbol{\lambda},\boldsymbol{h}}J^\dagger(u,\boldsymbol{X}_u)\,\mathrm{d}u\right],\end{aligned}$$

and as

$$\partial_t J^\dagger(t,\boldsymbol{x}) + \mathcal{L}^{\boldsymbol{\lambda},\boldsymbol{h}} J^\dagger(t,\boldsymbol{x}) + \frac{1}{2}|\boldsymbol{\lambda}|^2 + \mathfrak{h}_t + \boldsymbol{\eta}_2 \cdot \boldsymbol{g}(\boldsymbol{x}) \geq 0,$$

we conclude that

$$\mathbb{E}_{t,\boldsymbol{x}}^{\mathbb{Q}_{\boldsymbol{\lambda},\boldsymbol{h}}}\left[J^\dagger(s\wedge\tau_n, \boldsymbol{X}_{s\wedge\tau_n})\right] \geq J^\dagger(t,\boldsymbol{x}) - \mathbb{E}_{t,\boldsymbol{x}}^{\mathbb{Q}_{\boldsymbol{\lambda},\boldsymbol{h}}}\left[\int_t^{s\wedge\tau_n} \frac{1}{2}|\boldsymbol{\lambda}_u|^2 + \boldsymbol{\eta}_2\cdot\boldsymbol{g}(\boldsymbol{X}_u) + \mathfrak{h}_u \,\mathrm{d}u\right].$$

Since $\mathbb{Q}_{\boldsymbol{\lambda},\boldsymbol{h}} \in \boldsymbol{\mathcal{Q}}$ we have that

$$\left|\mathbb{E}_{t,\boldsymbol{x}}^{\mathbb{Q}_{\boldsymbol{\lambda},\boldsymbol{h}}}\left[\int_t^{s\wedge\tau_n} \frac{1}{2}|\boldsymbol{\lambda}_u|^2 + \mathfrak{h}_u\,\mathrm{d}u\right]\right| \leq \mathbb{E}_{t,\boldsymbol{x}}^{\mathbb{Q}_{\boldsymbol{\lambda},\boldsymbol{h}}}\left[\int_0^T \frac{1}{2}|\boldsymbol{\lambda}_u|^2 + \mathfrak{h}_u\,\mathrm{d}u\right] < \infty,$$

where we used that $\log(1-y)(1-y)+y \geq 0$ for $y \leq 1$. Similarly, since $\mathbb{Q}_{\boldsymbol{\lambda},\boldsymbol{h}} \in \boldsymbol{\mathcal{Q}}$

$$\left|\mathbb{E}_{t,\boldsymbol{x}}^{\mathbb{Q}_{\boldsymbol{\lambda},\boldsymbol{h}}}\left[\int_t^{s\wedge\tau_n} \boldsymbol{\eta}_2\cdot\boldsymbol{g}(\boldsymbol{X}_u)\mathrm{d}u\right]\right| \leq |\boldsymbol{\eta}_2| \cdot \mathbb{E}_{t,\boldsymbol{x}}^{\mathbb{Q}_{\boldsymbol{\lambda},\boldsymbol{h}}}\left[\int_0^T |\boldsymbol{g}(\boldsymbol{X}_u)|\,\mathrm{d}u\right] < \infty.$$

Finally, using the quadratic growth condition imposed on $J^\dagger$, we have

$$\left|J^\dagger(s\wedge\tau_n, \boldsymbol{X}_{s\wedge\tau_n})\right| \leq C_3\left(1 + \sup_{u\in[t,T]}|\boldsymbol{X}_u|^2\right),$$

and the right-hand side of the inequality is integrable with respect to $\mathbb{Q}_{\boldsymbol{\lambda},\boldsymbol{h}}$ because $\mathbb{Q}_{\boldsymbol{\lambda},\boldsymbol{h}} \in \boldsymbol{\mathcal{Q}}$. Thus, as a consequence of the dominated convergence theorem, we can take the limit when $n\to\infty$ to obtain for all $s\in[t,T)$

$$\mathbb{E}_{t,\boldsymbol{x}}^{\mathbb{Q}_{\boldsymbol{\lambda},\boldsymbol{h}}}\left[J^\dagger(s,\boldsymbol{X}_s)\right] \geq J^\dagger(t,\boldsymbol{x}) - \mathbb{E}_{t,\boldsymbol{x}}^{\mathbb{Q}_{\boldsymbol{\lambda},\boldsymbol{h}}}\left[\int_t^s \frac{1}{2}|\boldsymbol{\lambda}_u|^2 + \boldsymbol{\eta}_2\cdot\boldsymbol{g}(\boldsymbol{X}_u) + \mathfrak{h}_u\,\mathrm{d}u\right],$$

and by continuity of $J^\dagger$, as we send $s\nearrow T$, we obtain

$$\begin{aligned}&\mathbb{E}_{t,\boldsymbol{x}}^{\mathbb{Q}_{\boldsymbol{\lambda},\boldsymbol{h}}}\Big[\boldsymbol{\eta}_1\cdot(\boldsymbol{f}(\boldsymbol{X}_T)-\boldsymbol{c}) - \boldsymbol{\eta}_2\cdot\boldsymbol{d}\Big]\\ &\quad\geq J^\dagger(t,\boldsymbol{x}) - \mathbb{E}_{t,\boldsymbol{x}}^{\mathbb{Q}_{\boldsymbol{\lambda},\boldsymbol{h}}}\left[\int_t^T \frac{1}{2}|\boldsymbol{\lambda}_u|^2 + \boldsymbol{\eta}_2\cdot\boldsymbol{g}(\boldsymbol{X}_u) + \mathfrak{h}_u\,\mathrm{d}u\right].\end{aligned}$$

After rearranging the above equation we have that $J^\dagger \leq J^{\boldsymbol{\lambda},\boldsymbol{h}}$, and as a consequence of the arbitrariness of $\boldsymbol{\lambda}$ and $\boldsymbol{h}$, we obtain $J^\dagger \leq J$.

Finally, using a similar localization technique as above, this time with $(\boldsymbol{\lambda}^\dagger, \boldsymbol{h}^\dagger)$ and corresponding measure $\mathbb{Q}_{\boldsymbol{\lambda}^\dagger,\boldsymbol{h}^\dagger} \in \boldsymbol{\mathcal{Q}}$, and since it holds by construction that

$$\partial_t J^\dagger(t,\boldsymbol{x}) + \mathcal{L}^{\boldsymbol{\lambda}^\dagger,\boldsymbol{h}^\dagger} J^\dagger(t,\boldsymbol{x}) + \frac{1}{2}|\boldsymbol{\lambda}^\dagger|^2 + \mathfrak{h}_t^\dagger + \boldsymbol{\eta}_2\cdot\boldsymbol{g}(\boldsymbol{x}) = 0,$$

we have that

$$\begin{aligned}&\mathbb{E}_{t,\boldsymbol{x}}^{\mathbb{Q}_{\boldsymbol{\lambda}^\dagger,\boldsymbol{h}^\dagger}}\left[\boldsymbol{\eta}_1\cdot(\boldsymbol{f}(\boldsymbol{X}_T)-\boldsymbol{c}) - \boldsymbol{\eta}_2\cdot\boldsymbol{d}\right]\\ &\quad= J^\dagger(t,\boldsymbol{x}) - \mathbb{E}_{t,\boldsymbol{x}}^{\mathbb{Q}_{\boldsymbol{\lambda}^\dagger,\boldsymbol{h}^\dagger}}\left[\int_t^T \frac{1}{2}|\boldsymbol{\lambda}_u^\dagger|^2 + \boldsymbol{\eta}_2\cdot\boldsymbol{g}(\boldsymbol{X}_u) + \mathfrak{h}_u^\dagger\,\mathrm{d}u\right].\end{aligned}$$

Therefore, after rearranging, we have that $J \leq J^\dagger$. Combining both inequalities we obtain that $J = J^\dagger$ which concludes the proof. ∎

We introduce the notation $\mathbb{Q}^\dagger := \mathbb{Q}_{\boldsymbol{\lambda}^\dagger,\boldsymbol{h}^\dagger}$ to refer to the measure change induced by choosing $\boldsymbol{\lambda}^\dagger, \boldsymbol{h}^\dagger$ as in (2.15).

2.3. Representation of RN density. The next result shows that for $\mathbb{Q}_{\boldsymbol{\lambda},\boldsymbol{h}} \in \boldsymbol{\mathcal{Q}}$ with carefully chosen $\boldsymbol{\lambda}$ and $\boldsymbol{h}$, the corresponding RN density $Z_T^{\boldsymbol{\lambda},\boldsymbol{h}}$ of $\mathbb{Q}_{\boldsymbol{\lambda},\boldsymbol{h}}$ has an alternative representation. This leads to an Esscher-like representation of the RN density that characterizes a solution to (P'); see Corollary 2.8.

THEOREM 2.7 (representation of RN density). *Let Assumption* 2.1 *be fulfilled. Let* $Z^{\boldsymbol{\lambda},\boldsymbol{h}}$ *be given in* (2.3) *with* $\boldsymbol{\lambda},\boldsymbol{h}$ *specifically chosen to be*

$$\boldsymbol{\lambda}_t = -\frac{\nabla_{\boldsymbol{x}} w(t,\boldsymbol{X}_t)}{w(t,\boldsymbol{X}_t)}\,\boldsymbol{\sigma}(t,\boldsymbol{X}_t) \qquad \text{and} \qquad \boldsymbol{h}_t(\boldsymbol{z}) = -\frac{\boldsymbol{\Delta}_{\boldsymbol{z}} w(t,\boldsymbol{X}_{t^-})}{w(t,\boldsymbol{X}_{t^-})},$$

where $w:[0,T]\times\mathbb{R}^n \to \mathbb{R}$ *is*

$$w(t,\boldsymbol{x}) := \mathbb{E}_{t,\boldsymbol{x}}\left[H\left(\boldsymbol{X}_T\right) G\left(\int_t^T \ell(\boldsymbol{X}_s)\,\mathrm{d}s\right)\right]$$

with $\ell\colon \mathbb{R}^n \to \mathbb{R}$ *and for some* $H:\mathbb{R}^n \to \mathbb{R}^+\setminus\{0\}$ *such that* $\mathbb{E}[H(X_T)]<\infty$*, and* $G:\mathbb{R}\to\mathbb{R}^+\setminus\{0\}$ *is* $\mathcal{C}^1$ *and such that* $G(a)\,G(b) = G(a+b)$ *for* $a,b\in\mathbb{R}$.[3] *We assume that* H *and* G *are such that* $w\in\mathcal{C}^{1,2}([0,T]\times\mathbb{R}^n)$ *and that Novikov's condition* (2.4) *holds for the above choice of* $\boldsymbol{\lambda}$ *and* $\boldsymbol{h}$*. Then, we have, for all* $t\in[0,T]$*, that*

$$Z_t^{\boldsymbol{\lambda},\boldsymbol{h}} = \frac{w(t,\boldsymbol{X}_t)\,G\left(\int_0^t \ell(\boldsymbol{X}_s)\,\mathrm{d}s\right)}{\mathbb{E}\left[w(T,\boldsymbol{X}_T)\,G\left(\int_0^T \ell(\boldsymbol{X}_s)\,\mathrm{d}s\right)\right]}.$$

Note that $w(T,\boldsymbol{X}_T) = H(\boldsymbol{X}_T)\,G(0)$ *and* $G(0)=1$.

Proof. The proof is conceptually known, but for completeness we provide the details below. For simplicity we drop the superscripts of $Z^{\boldsymbol{\lambda},\boldsymbol{h}}$ and just write Z. By Novikov's condition we have that $(Z_t)_{t\in[0,T]}$ is a martingale and thus $Z_t = \mathbb{E}_t[Z_T]$ for $0\le t\le T$, where $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot|\mathcal{F}_t]$. As Z_t is a stochastic exponential, from Itô's lemma, we have that

$$\mathrm{d}Z_t = \frac{Z_{t^-}}{w}\,\nabla_{\boldsymbol{x}} w\,\boldsymbol{\sigma}\,\mathrm{d}\boldsymbol{W}_t + \frac{Z_{t^-}}{w}\int_{\mathbb{R}^l} \boldsymbol{\Delta}_{\boldsymbol{z}} w(t,\boldsymbol{X}_{t^-})\,\tilde{\boldsymbol{\mu}}(\mathrm{d}t,\mathrm{d}\boldsymbol{z}). \tag{2.16}$$

Next, we observe that the process

$$\tilde{w}_t := G_t\, w(t,\boldsymbol{X}_t) = \mathbb{E}_{t,\boldsymbol{X}_t}\left[H\left(\boldsymbol{X}_T\right)\, G_T\right]$$

is a martingale, where $G_t := G(\int_0^t \ell(\boldsymbol{X}_s)\mathrm{d}s)$. Given that $w\in\mathcal{C}^{1,2}([0,T)\times\mathbb{R}^n)$, we derive the SDE satisfied by $\tilde{w}$, and since $\tilde{w}$ is a martingale, the dt-term vanishes and we obtain an identity that we use to obtain the following: the process $V_t := 1/\tilde{w}_t$, which can be written as $V_t = v(t,\boldsymbol{X}_t)\,/G_t$, where $v(t,\boldsymbol{x}) := 1/w(t,\boldsymbol{x})$ satisfies

(2.17)

$$\begin{aligned}
\mathrm{d}V_t = {} & \frac{1}{G_t\,w^2}\int_{\mathbb{R}^l} \boldsymbol{\Delta}_{\boldsymbol{z}} w(t,\boldsymbol{X}_t)\,\boldsymbol{\nu}(\mathrm{d}t,\mathrm{d}\boldsymbol{z}) + \frac{1}{w\,G_t^2}\,G'\left(\int_0^t \ell(\boldsymbol{X}_s)\,\mathrm{d}s\right)\ell(\boldsymbol{X}_t)\,\mathrm{d}t \\
& - \frac{1}{G_t\,w^2}(\nabla_{\boldsymbol{x}} w)\,\boldsymbol{\sigma}\,\mathrm{d}\boldsymbol{W}_t + \frac{1}{G_t\,w^3}\,\left|(\nabla_{\boldsymbol{x}} w)\,\boldsymbol{\sigma}\right|^2\mathrm{d}t - G'\left(-\int_0^t \ell(\boldsymbol{X}_s)\,\mathrm{d}s\right)\ell(\boldsymbol{X}_t)\,\frac{1}{w}\,\mathrm{d}t \\
& + \frac{1}{G_t}\sum_{j\in\mathcal{D}}\int_{\mathbb{R}}\left(\frac{1}{w(t,\boldsymbol{X}_{t^-}+\boldsymbol{\gamma}^{(j)})} - \frac{1}{w(t,\boldsymbol{X}_{t^-})}\right)\mu^j(\mathrm{d}t,\mathrm{d}z).
\end{aligned}$$

[3]From this property, it follows that $G(0)=1$ and $G(-a) = 1/G(a)$. In fact, a simple calculation shows that $G(x)$ is of the form $\exp(\kappa\,x)$ for some constant κ.

Next, define the process $\phi_t := Z_t V_t$ for $t \in [0,T]$. We claim that $\phi_t = c$ for all $t \in [0,T]$ for some constant $c \in \mathbb{R}$. To see this, note that

$$\mathrm{d}\phi_t = Z_{t^-}\,\mathrm{d}V_t + V_{t^-}\,\mathrm{d}Z_t + \mathrm{d}[Z,V]_t,$$

which after direct substitution, using (2.17), (2.16), and the formula for $\mathrm{d}[Z,V]_t$, we have

$$\begin{aligned}
\mathrm{d}\phi_t &= \frac{Z_{t^-}}{G_t\,w^2}\int_{\mathbb{R}^l} \boldsymbol{\Delta_z} w(t,\boldsymbol{X}_{t^-})\,\boldsymbol{\nu}(\mathrm{d}t,\mathrm{d}\boldsymbol{z}) + \frac{Z_{t^-}}{w\,G_t^2}\,G'\left(\int_0^t \ell(\boldsymbol{X}_s)\,\mathrm{d}s\right)\ell(\boldsymbol{X}_t)\,\mathrm{d}t \\
&\quad \underbrace{-\frac{Z_{t^-}}{G_t\,w^2}(\nabla_{\boldsymbol{x}} w)\,\boldsymbol{\sigma}\,\mathrm{d}\boldsymbol{W}_t}_{(a)} + \underbrace{\frac{Z_{t^-}}{G_t\,w^3}\,|(\nabla_{\boldsymbol{x}} w)\,\boldsymbol{\sigma}|^2\,\mathrm{d}t}_{(b)} \\
&\quad + \frac{Z_{t^-}}{G_t}\sum_{j\in\mathcal{D}}\int_{\mathbb{R}}\left(\frac{1}{w(t,\boldsymbol{X}_{t^-}+\boldsymbol{\gamma}^{(j)})} - \frac{1}{w(t,\boldsymbol{X}_{t^-})}\right)\mu^j(\mathrm{d}t,\mathrm{d}z) \\
&\quad - Z_{t^-}\,G'\left(-\int_0^t \ell(\boldsymbol{X}_s)\,\mathrm{d}s\right)\ell(\boldsymbol{X}_t)\,\frac{1}{w}\,\mathrm{d}t + \underbrace{V_{t^-}\,Z_{t^-}\,(\nabla_{\boldsymbol{x}}\log w)\,\boldsymbol{\sigma}\,\mathrm{d}\boldsymbol{W}_t}_{(c)} \\
&\quad + \frac{V_{t^-}\,Z_{t^-}}{w}\int_{\mathbb{R}^l}\boldsymbol{\Delta_z} w(t,\boldsymbol{X}_{t^-})\,\tilde{\boldsymbol{\mu}}(\mathrm{d}t,\mathrm{d}\boldsymbol{z}) \underbrace{-\frac{Z_{t^-}}{G_t\,w^2}\mathrm{Tr}\Big(\boldsymbol{\sigma}\,\boldsymbol{\sigma}^{\intercal}\,(\nabla_{\boldsymbol{x}}\log w)^{\intercal}\,\nabla_{\boldsymbol{x}} w\Big)\mathrm{d}t}_{(d)} \\
&\quad - \frac{Z_{t^-}}{G_t}\sum_{j\in\mathcal{D}}\int_{\mathbb{R}}\left(\frac{1}{w(t,\boldsymbol{X}_{t^-}+\boldsymbol{\gamma}^{(j)})} - \frac{1}{w(t,\boldsymbol{X}_{t^-})}\right)\left(1-\frac{w(t,\boldsymbol{X}_{t^-}+\boldsymbol{\gamma}^{(j)})}{w(t,\boldsymbol{X}_{t^-})}\right)\mu^j(\mathrm{d}t,\mathrm{d}z).
\end{aligned}$$

As $\nabla_{\boldsymbol{x}}\log w = \frac{\nabla_{\boldsymbol{x}} w}{w}$ and $V_t = 1/(w\,G_t)$, after a short calculation we find that (a) cancels with (c). Similarly, (b) cancels with (d) by factoring $1/w$ out of the $\mathrm{Tr}(\cdot)$ operator. Then, it follows that $\mathrm{d}\phi_t$ reduces to

(2.18)
$$\begin{aligned}
\mathrm{d}\phi_t &= \frac{Z_{t^-}}{G_t\,w^2}\int_{\mathbb{R}^l} \boldsymbol{\Delta_z} w(t,\boldsymbol{X}_{t^-})\,\boldsymbol{\nu}(\mathrm{d}t,\mathrm{d}\boldsymbol{z}) + \underbrace{\frac{Z_{t^-}}{w\,G_t^2}\,G'\left(\int_0^t \ell(\boldsymbol{X}_s)\,\mathrm{d}s\right)\ell(\boldsymbol{X}_t)\,\mathrm{d}t}_{(e)} \\
&\quad + \frac{Z_{t^-}}{G_t}\sum_{j\in\mathcal{D}}\int_{\mathbb{R}}\left(\frac{1}{w(t,\boldsymbol{X}_{t^-}+\boldsymbol{\gamma}^{(j)})} - \frac{1}{w(t,\boldsymbol{X}_{t^-})}\right)\mu^j(\mathrm{d}t,\mathrm{d}z) \\
&\quad \underbrace{-Z_{t^-}\,G'\left(-\int_0^t \ell(\boldsymbol{X}_s)\,\mathrm{d}s\right)\ell(\boldsymbol{X}_t)\,\frac{1}{w}\,\mathrm{d}t}_{(f)} + \frac{V_{t^-}\,Z_{t^-}}{w}\int_{\mathbb{R}^l}\boldsymbol{\Delta_z} w(t,\boldsymbol{X}_{t^-})\,\tilde{\boldsymbol{\mu}}(\mathrm{d}t,\mathrm{d}\boldsymbol{z}) \\
&\quad - \frac{Z_{t^-}}{G_t}\sum_{j\in\mathcal{D}}\int_{\mathbb{R}}\left(\frac{1}{w(t,\boldsymbol{X}_{t^-}+\boldsymbol{\gamma}^{(j)})} - \frac{1}{w(t,\boldsymbol{X}_{t^-})}\right) \\
&\quad \times\left(1-\frac{w(t,\boldsymbol{X}_{t^-}+\boldsymbol{\gamma}^{(j)})}{w(t,\boldsymbol{X}_{t^-})}\right)\mu^j(\mathrm{d}t,\mathrm{d}z).
\end{aligned}$$

Note that $G(-x) = 1/G(x)$ (as $G(x-x) = G(x)G(-x)$ and $G(0)=1$), hence $G'(-x) = \frac{G'(x)}{G^2(x)}$, and thus $G'(-\int_0^t \ell(\boldsymbol{X}_s)\,\mathrm{d}s) = G'(\int_0^t \ell(\boldsymbol{X}_s)\,\mathrm{d}s)/G^2(\int_0^t \ell(\boldsymbol{X}_s)\,\mathrm{d}s)$. Using this relationship, the (e) and (f) terms in (2.18) cancel, in which case we have

$$
\begin{aligned}
\mathrm{d}\phi_t = & \underbrace{\frac{Z_{t^-}}{G_t\, w^2} \int_{\mathbb{R}^l} \boldsymbol{\Delta_z} w(t, \boldsymbol{X}_{t^-})\, \boldsymbol{\nu}(\mathrm{d}t, \mathrm{d}\boldsymbol{z})}_{(g)} \\
& + \frac{Z_{t^-}}{G_t} \sum_{j\in\mathcal{D}} \int_{\mathbb{R}} \left(\frac{1}{w(t, \boldsymbol{X}_{t^-} + \boldsymbol{\gamma}^{(j)})} - \frac{1}{w(t, \boldsymbol{X}_{t^-})} \right) \mu^j(\mathrm{d}t, \mathrm{d}z) \\
& + \frac{V_{t^-}\, Z_{t^-}}{w} \int_{\mathbb{R}^l} \boldsymbol{\Delta_z} w(t, \boldsymbol{X}_{t^-})\, \boldsymbol{\mu}(\mathrm{d}t, \mathrm{d}\boldsymbol{z}) \underbrace{- \frac{V_{t^-}\, Z_{t^-}}{w} \int_{\mathbb{R}^l} \boldsymbol{\Delta_z} w(t, \boldsymbol{X}_{t^-})\, \boldsymbol{\nu}(\mathrm{d}t, \mathrm{d}\boldsymbol{z})}_{(h)} \\
& - \frac{Z_{t^-}}{G_t} \sum_{j\in\mathcal{D}} \int_{\mathbb{R}} \left(\frac{1}{w(t, \boldsymbol{X}_{t^-} + \boldsymbol{\gamma}^{(j)})} - \frac{1}{w(t, \boldsymbol{X}_{t^-})} \right) \\
& \times \left(1 - \frac{w(t, \boldsymbol{X}_{t^-} + \boldsymbol{\gamma}^{(j)})}{w(t, \boldsymbol{X}_{t^-})} \right) \mu^j(\mathrm{d}t, \mathrm{d}z).
\end{aligned}
$$

From the definition of V_t, we see that (g) and (h) cancel. Finally, the remaining terms cancel by collecting like terms and we obtain $\mathrm{d}\phi_t = 0$. Thus, $\phi_t = c$ for all $t \in [0, T]$, and some $c \in \mathbb{R}$. As $Z_t = c\, w(t, \boldsymbol{X}_t)\, G_t$ and $\mathbb{E}[Z_T] = 1$, it follows that

$$
c = \mathbb{E}[w(T, \boldsymbol{X}_T)\, G_T]^{-1} = \mathbb{E}\left[w(T, \boldsymbol{X}_T)\, G \left(\int_0^T \ell(\boldsymbol{X}_s)\, \mathrm{d}s \right) \right]^{-1},
$$

from which we obtain the required results

$$
Z_t = \frac{w(t, \boldsymbol{X}_t)\, G_t}{\mathbb{E}[w(T, \boldsymbol{X}_T)\, G_T]}, \qquad t \in [0, T]. \qquad \square
$$

The next result specifies that the optimal RN density can be written as an Esscher-like transform of a function depending only on the terminal value of the processes and the running costs. This reflects similarities to the static case as well as indifference pricing, where the optimal RN density also admits an Esscher-like transform; indicatively see [11, 13, 17, 8, 19, 27].

COROLLARY 2.8. *Let Assumptions* 2.1 *be fulfilled, $D_{-\mathfrak{x}} \neq \emptyset$, and $(\boldsymbol{\eta}_1, \boldsymbol{\eta}_2) \in D_{-\mathfrak{x}}$. Further let $\boldsymbol{\lambda}^\dagger$, $\boldsymbol{h}^\dagger$, and $\omega^\dagger(t, \boldsymbol{x})$ be as in Theorem* 2.6 *and satisfying its assumptions. Then, the probability measure $\mathbb{Q}^\dagger$ has RN density*

$$
\frac{\mathrm{d}\mathbb{Q}^\dagger}{\mathrm{d}\mathbb{P}} = Z_T^{\boldsymbol{\lambda}^\dagger, \boldsymbol{h}^\dagger} = \frac{\exp\left(-\boldsymbol{\eta}_1 \cdot \boldsymbol{f}(\boldsymbol{X}_T) - \boldsymbol{\eta}_2 \cdot \int_0^T \boldsymbol{g}(\boldsymbol{X}_u)\, \mathrm{d}u \right)}{\mathbb{E}\left[\exp\left(-\boldsymbol{\eta}_1 \cdot \boldsymbol{f}(\boldsymbol{X}_T) - \boldsymbol{\eta}_2 \cdot \int_0^T \boldsymbol{g}(\boldsymbol{X}_u)\, \mathrm{d}u \right) \right]}. \tag{2.19}
$$

Proof. By the definition of $\omega^\dagger(t, x)$ in (2.14) we have

$$
\omega^\dagger(t, \boldsymbol{x}) = \mathbb{E}_{t,\boldsymbol{x}} \left[H(\boldsymbol{X}_T)\, G \left(\int_t^T \ell(\boldsymbol{X}_u) \mathrm{d}u \right) \right],
$$

where we set $H(\boldsymbol{x}) := \exp(-\boldsymbol{\eta}_1 \cdot (\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{c}))$, $\ell(\boldsymbol{x}) = \boldsymbol{\eta}_2 \cdot \boldsymbol{g}(\boldsymbol{x})$, and $G(x) := \exp(-x + \boldsymbol{\eta}_2 \cdot \boldsymbol{d})$. Applying Theorem 2.7, the RN density becomes

$$
Z_T^{\boldsymbol{\lambda}^\dagger, \boldsymbol{h}^\dagger} = \frac{\exp\left(-\boldsymbol{\eta}_1 \cdot (\boldsymbol{f}(\boldsymbol{X}_T) - \boldsymbol{c}) - \boldsymbol{\eta}_2 \cdot \left(\int_0^T \boldsymbol{g}(\boldsymbol{X}_u)\, \mathrm{d}u - \boldsymbol{d} \right) \right)}{\mathbb{E}\left[\exp\left(-\boldsymbol{\eta}_1 \cdot (\boldsymbol{f}(\boldsymbol{X}_T) - \boldsymbol{c}) - \boldsymbol{\eta}_2 \cdot \left(\int_0^T \boldsymbol{g}(\boldsymbol{X}_u)\, \mathrm{d}u - \boldsymbol{d} \right) \right) \right]}, \tag{2.20}
$$

which after simplification concludes the proof. $\square$

Note that the above corollary states that the RN density is a function only of the terminal value of the processes $\boldsymbol{X}_T$ and the running costs $\int_0^T \boldsymbol{g}(\boldsymbol{X}_s)\mathrm{d}s$. Thus, even though the RN density $Z_T^{\boldsymbol{\lambda}^\dagger,\boldsymbol{h}^\dagger}$ was characterized by the stochastic process $\boldsymbol{\lambda}^\dagger$ and the random vector field $\boldsymbol{h}^\dagger$, it has a representation where it does not (explicitly) depend on them. As we show in the next subsection, an optimal RN density which attains the infimum in the optimization problem (P') will be of the form (2.19) for some $\boldsymbol{\lambda}$ and $\boldsymbol{h}$ and, moreover, it will indirectly depend on them through the constraints.

2.4. Solution to optimization problems (P') and (*P*). In this section, we present the solution to the control problem (P') and show that, if the solution exists, it is also the unique solution to the constrained optimization problem (P). The next result states the solution to the optimization problem (P').

THEOREM 2.9 (solution to (P')). *Let Assumptions* 2.1 *and* 2.4 *be fulfilled and suppose that* $\boldsymbol{\lambda}^*$, $\boldsymbol{h}^*$ *are as in Theorem* 2.6*, with Lagrange multipliers* $(\boldsymbol{\eta}_1^*,\boldsymbol{\eta}_2^*)$ *solving* (2.6)*, and satisfying its assumptions. Then, there exists a solution* $\mathbb{Q}_{\boldsymbol{\lambda}^*,\boldsymbol{h}^*}$ *to* (P') *which is given in Corollary* 2.8 *with optimal Lagrange multipliers* $(\boldsymbol{\eta}_1^*,\boldsymbol{\eta}_2^*)$ *and where* $\boldsymbol{\lambda}^*$, $\boldsymbol{h}^*$ *generate the measure change.*

Proof. For fixed $(\boldsymbol{\eta}_1,\boldsymbol{\eta}_2)\in D_{-\mathfrak{X}}$, we take $\omega^\dagger(t,\boldsymbol{x})$, $\boldsymbol{\lambda}^\dagger$, and $\boldsymbol{h}^\dagger$ as given in (2.14) and (2.15). Denote the corresponding measure by $\mathbb{Q}^\dagger := \mathbb{Q}_{\boldsymbol{\lambda}^\dagger,\boldsymbol{h}^\dagger}$. Recall that $\frac{\mathrm{d}\mathbb{Q}^\dagger}{\mathrm{d}\mathbb{P}} = Z_T^{\boldsymbol{\lambda}^\dagger,\boldsymbol{h}^\dagger}$. Then we may rewrite the constraints as

$$\begin{aligned} &\mathbb{E}\left[Z_T^{\boldsymbol{\lambda}^\dagger,\boldsymbol{h}^\dagger}\left(f_j(\boldsymbol{X}_T)-c_j\right)\right]=0 \qquad \text{for } j\in\mathcal{R}_1, \\ \text{and}\quad &\mathbb{E}\left[Z_T^{\boldsymbol{\lambda}^\dagger,\boldsymbol{h}^\dagger}\left(\int_0^T g_i(\boldsymbol{X}_s)\,\mathrm{d}s-d_i\right)\right]=0 \qquad \text{for } i\in\mathcal{R}_2. \end{aligned} \tag{2.21}$$

By Corollary 2.8 we further have that $Z_T^{\boldsymbol{\lambda}^\dagger,\boldsymbol{h}^\dagger}$ is given in (2.20), which allows us to rewrite the set of equations (2.21) as

$$-\partial_{\eta_k}\log\mathbb{E}\left[e^{-\boldsymbol{\eta}_1\cdot(\boldsymbol{f}(\boldsymbol{X}_T)-\boldsymbol{c})-\boldsymbol{\eta}_2\cdot\left(\int_0^T\boldsymbol{g}(\boldsymbol{X}_u)\,\mathrm{d}u-\boldsymbol{d}\right)}\right]=0 \quad \forall\, k\in\mathcal{R}_1\cup\mathcal{R}_2.$$

The above set of equations can be compactly written as the system of equations

$$\nabla_{\boldsymbol{a}}K_{\mathfrak{X}}(-\boldsymbol{a})=\boldsymbol{0},$$

which, by Assumption 2.4, has a solution, denoted here by $(\boldsymbol{\eta}_1^*,\boldsymbol{\eta}_2^*)$. Further, if for this choice of Lagrange multipliers, the assumptions in Theorem 2.6 are satisfied, then, by Theorem 2.6, the corresponding optimal controls $\boldsymbol{\lambda}^*,\boldsymbol{h}^*$ are attainable and generate the required measure change $\mathbb{Q}_{\boldsymbol{\lambda}^*,\boldsymbol{h}^*}$. ∎

PROPOSITION 2.10 (SDE under $\mathbb{Q}^*$). *Let the conditions of Theorem* 2.9 *be fulfilled and* $\mathbb{Q}^*=\mathbb{Q}_{\boldsymbol{\lambda}^*,\boldsymbol{h}^*}$ *given in Theorem* 2.9*. Then,* $\boldsymbol{X}$ *satisfies the following SDE in terms of* $\mathbb{Q}^*$*-martingales,*

$$\begin{aligned}\mathrm{d}\boldsymbol{X}_t=&\left(\boldsymbol{\alpha}(t,\boldsymbol{X}_t)-\boldsymbol{\sigma}(t,\boldsymbol{X}_t)\,\boldsymbol{\lambda}_t^{*\intercal}-\int_{\mathbb{R}^l}\gamma(t,\boldsymbol{X}_{t^-},\boldsymbol{z})\left[\boldsymbol{h}_t^{*\intercal}(\boldsymbol{z})\odot\boldsymbol{\nu}(\mathrm{d}\boldsymbol{z})\right]\right)\mathrm{d}t\\ &+\boldsymbol{\sigma}(t,\boldsymbol{X}_t)\,\mathrm{d}\boldsymbol{W}_t^{\boldsymbol{\lambda}^*}+\int_{\mathbb{R}^l}\gamma(t,\boldsymbol{X}_{t^-},\boldsymbol{z})\,\tilde{\boldsymbol{\mu}}^{\boldsymbol{h}^*}(\mathrm{d}t,\mathrm{d}\boldsymbol{z}),\end{aligned}$$

where $\tilde{\boldsymbol{\mu}}^{\boldsymbol{h}^*}:=\boldsymbol{\mu}-\boldsymbol{\nu}^{\boldsymbol{h}^*}$, $\boldsymbol{W}^{\boldsymbol{\lambda}^*}$ *is a* $\mathbb{Q}^*$*-Brownian motion, and* $\boldsymbol{\nu}^{\boldsymbol{h}^*}(\mathrm{d}t,\mathrm{d}\boldsymbol{z})=(\boldsymbol{1}-\boldsymbol{h}_t^*(\boldsymbol{z}))^\intercal\odot\boldsymbol{\nu}(\mathrm{d}\boldsymbol{z})\,\mathrm{d}t$ *is the* $\mathbb{Q}^*$*-compensator of* $\boldsymbol{\mu}$*.*

Proof. This follows immediately from Girsanov's theorem and by writing (2.1) in terms of $\boldsymbol{W}^{\boldsymbol{\lambda}^*}$ and $\tilde{\boldsymbol{\mu}}^{\boldsymbol{h}^*}$. ◻

Next, we prove that the optimal probability measure of optimization problem (P') is also a solution to optimization problem (P). Thus, restricting ourselvs to the class of equivalent probability measures provides a way of solving and characterizing the solution to optimization problem (P).

THEOREM 2.11 (Solution to (P)). *If the optimization problem* (P') *has a solution, then it is unique, and moreover it is the unique solution to optimization problem* (P).

Proof. By Theorem 2.9 a solution to optimization problem (P') is $\mathbb{Q}^* = \mathbb{Q}_{\boldsymbol{\lambda}^*,\boldsymbol{h}^*}$, where $\boldsymbol{\lambda}^*$, $\boldsymbol{h}^*$, and $\omega^*(t,\boldsymbol{x})$ are as in Theorem 2.6 with Lagrange multipliers $(\boldsymbol{\eta}_1^*, \boldsymbol{\eta}_2^*)$ solving (2.6). By Corollary 2.8 (multiplying and dividing by the constants), we have that

$$\frac{\mathrm{d}\mathbb{Q}^*}{\mathrm{d}\mathbb{P}} = \frac{\exp\left(-\boldsymbol{\eta}_1^* \cdot (\boldsymbol{f}(\boldsymbol{X}_T) - \boldsymbol{c}) - \boldsymbol{\eta}_2^* \cdot \left(\int_0^T \boldsymbol{g}(\boldsymbol{X}_u)\,\mathrm{d}u - \boldsymbol{d}\right)\right)}{\mathbb{E}\left[\exp\left(-\boldsymbol{\eta}_1^* \cdot (\boldsymbol{f}(\boldsymbol{X}_T) - \boldsymbol{c}) - \boldsymbol{\eta}_2^* \cdot \left(\int_0^T \boldsymbol{g}(\boldsymbol{X}_u)\,\mathrm{d}u - \boldsymbol{d}\right)\right)\right]}.$$

Next, let $\tilde{\mathbb{Q}}$ be any probability measure that is absolutely continuous with respect to $\mathbb{P}$ and under which the constraints are fulfilled. Then, observe that

$$\begin{aligned}
&\mathbb{E}\left[\left(\frac{\mathrm{d}\mathbb{Q}^*}{\mathrm{d}\mathbb{P}} - \frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\right)\log\frac{\mathrm{d}\mathbb{Q}^*}{\mathrm{d}\mathbb{P}}\right] \\
&\quad = \mathbb{E}\left[\left(\frac{\mathrm{d}\mathbb{Q}^*}{\mathrm{d}\mathbb{P}} - \frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\right)\left\{-\boldsymbol{\eta}_1^* \cdot (\boldsymbol{f}(\boldsymbol{X}_T) - \boldsymbol{c}) - \boldsymbol{\eta}_2^* \cdot \left(\int_0^T \boldsymbol{g}(\boldsymbol{X}_u)\,\mathrm{d}u - \boldsymbol{d}\right)\right.\right. \\
&\qquad \left.\left. -\ \log\mathbb{E}\left[\exp\left(-\boldsymbol{\eta}_1^* \cdot (\boldsymbol{f}(\boldsymbol{X}_T) - \boldsymbol{c}) - \boldsymbol{\eta}_2^* \cdot \left(\int_0^T \boldsymbol{g}(\boldsymbol{X}_u)\,\mathrm{d}u - \boldsymbol{d}\right)\right)\right]\right\}\right] = 0.
\end{aligned}$$

Using the above equality, the KL divergence from $\tilde{\mathbb{Q}}$ to $\mathbb{P}$ can be bounded below as follows:

$$\begin{aligned}
D_{KL}\big(\tilde{\mathbb{Q}} \;||\; \mathbb{P}\big) &= \mathbb{E}\left[\frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\log\frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\right] + \mathbb{E}\left[\left(\frac{\mathrm{d}\mathbb{Q}^*}{\mathrm{d}\mathbb{P}} - \frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\right)\log\frac{\mathrm{d}\mathbb{Q}^*}{\mathrm{d}\mathbb{P}}\right] \\
&= \mathbb{E}\left[\frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\left(\log\frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{P}} + \log\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}^*}\right)\right] + D_{KL}\big(\mathbb{Q}^* \;||\; \mathbb{P}\big) \\
&= \mathbb{E}^{\tilde{\mathbb{Q}}}\left[\log\frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}^*}\right] + D_{KL}\big(\mathbb{Q}^* \;||\; \mathbb{P}\big) \\
&= D_{KL}\big(\tilde{\mathbb{Q}} \;||\; \mathbb{Q}^*\big) + D_{KL}\left(\mathbb{Q}^* \;||\; \mathbb{P}\right) \geq D_{KL}\left(\mathbb{Q}^* \;||\; \mathbb{P}\right).
\end{aligned}$$

Thus, $\mathbb{Q}^*$ is indeed a solution to (P). Uniqueness of the solution to (P) follows by strict convexity of the KL divergence, which implies uniqueness of (P'). ◻

3. Analytically tractable examples. Here, we provide analytically tractable examples to provide insight on how the dynamics of processes change when moving from $\mathbb{P}$ to $\mathbb{Q}^*$. As far as we are aware, the results are all new and illustrate the ease of use and applicability of our main results. In section 4, we develop an algorithm to simulate the processes under $\mathbb{Q}^*$, when analytical solutions are not available. As a first example, we discuss the sign of the optimal Lagrange multiplier under one single constraint. Second, we consider how the solution to the optimization problem (P) is connected to pinned measures. Third, we provide explicit expressions for the Lagrange multipliers and the optimal RN density under two Value-at-Risk (VaR) constraints. Fourth, we consider a constraint on the mean when the underlying process has independent increments. Finally, we study how a Brownian motion is perturbed when we keep its mean equal to 0 but alter its standard deviation.

3.1. Sign of Lagrange multiplier for single constraint. For the case when there is only one constraint, i.e., $r_1 = 1$, $r_2 = 0$ or $r_1 = 0$, $r_2 = 1$, we can specify the sign of the Lagrange multiplier. From Corollary 2.8 we observe that the sign of the Lagrange multiplier determines how the RN density distorts the distribution of $\boldsymbol{X}_T$, e.g., in the one-dimensional case and if $f(x) = x$, then a positive (negative) Lagrange multiplier increases (decreases) the probability of large values of X_T. For simplicity we assume that $r_2 = 0$; however, the following proposition also holds for one running cost constraint.

PROPOSITION 3.1 (sign of Lagrange multiplier). *Let Assumptions* 2.1 *and* 2.4 *be fulfilled and denote by* η^* *the unique solution to* (2.6). *Consider problem* (P) *with one constraint, i.e.* $r_1 = 1$, $r_2 = 0$, *which we write as* $\mathbb{E}^{\mathbb{Q}}[f(\boldsymbol{X}_T)] = c$. *Then, the sign of the optimal Lagrange multiplier* η^* *is given by* $\operatorname{sgn}(\eta^*) = \operatorname{sgn}(\mathbb{E}[f(\boldsymbol{X}_T)] - c)$.

Proof. Using the optimal RN density given in Corollary 2.8, the optimal Lagrange multiplier η^* fulfills

$$
\begin{aligned}
0 &= \mathbb{E}\left[\frac{e^{-\eta^*(f(\boldsymbol{X}_T)-c)}}{\mathbb{E}[e^{-\eta^*(f(\boldsymbol{X}_T)-c)}]}\,(f(\boldsymbol{X}_T)-c)\right] = \frac{\mathrm{d}}{\mathrm{d}\eta}\log\left(\mathbb{E}\left[e^{\eta(f(\boldsymbol{X}_T)-c)}\right]\right)\bigg|_{\eta=-\eta^*} \\
&= \frac{\mathrm{d}}{\mathrm{d}\eta}K_{f(\boldsymbol{X}_T)-c}\,(\eta)\bigg|_{\eta=-\eta^*}.
\end{aligned}
$$

As the derivative of a cgf of a random variable Y, $\frac{d}{da}K_Y(a)$, is strictly increasing in its argument a, we have that $\frac{\mathrm{d}}{\mathrm{d}\eta}K_{f(\boldsymbol{X}_T)-c}(\eta)|_{\eta=-\eta^*}$ is strictly decreasing in η^*. Moreover,

$$
\frac{\mathrm{d}}{\mathrm{d}\eta}K_{f(\boldsymbol{X}_T)-c}\,(\eta)\,\Big|_{\eta=0} = \mathbb{E}[f(\boldsymbol{X}_T)] - c.
$$

Clearly, if the right-hand side vanishes, then $\eta^* = 0$. Further, if $\mathbb{E}[f(\boldsymbol{X}_T)] - c > 0$, we must have $\eta^* > 0$. Similarly, if $\mathbb{E}[f(\boldsymbol{X}_T)] - c < 0$, we must have $\eta^* < 0$. ∎

The above proposition states that a constraint $\mathbb{E}^{\mathbb{Q}^*}[f(\boldsymbol{X}_T)] = c > \mathbb{E}[f(\boldsymbol{X}_T)]$, i.e., an increase in the expected value of $f(\boldsymbol{X}_T)$ from $\mathbb{P}$ to $\mathbb{Q}^*$ corresponds to a negative optimal Lagrange multiplier. Similarly, if the expected value of $f(\boldsymbol{X}_T)$ is decreased from $\mathbb{P}$ to $\mathbb{Q}^*$, i.e., $\mathbb{E}^{\mathbb{Q}^*}[f(\boldsymbol{X}_T)] = c < \mathbb{E}[f(\boldsymbol{X}_T)]$, then η^* is positive.

3.2. Pinned measures. Pinned measures have a long history and are connected to the classical Doob's h-transform [14]. In this example, we construct a sequence of probability measures, each being a solution of an optimization problem (P'), that converges to a pinned measure. Note, that the sequence of probability measures is equivalent to $\mathbb{P}$ while the limiting measure—the pinned measure—is not. The resulting sequence of measures provide approximations of a pinned measure, where each one is equivalent to the original measure. For this consider a Borel measurable set $B \in \mathcal{B}(\mathbb{R}^n)$ and the constraint function $f(\boldsymbol{x}) = \mathbb{1}_{\{\boldsymbol{x}\in B\}}$ which results in the constraint

$\mathbb{Q}(\boldsymbol{X}_T \in B) = q_k$, where we choose $q_k := 1 - \frac{1}{k}$, $k \in \mathbb{Z}_+ \setminus \{0\}$. For each k, the optimal probability measure is

$$\frac{\mathrm{d}\mathbb{Q}_k^*}{\mathrm{d}\mathbb{P}} = \frac{e^{-\eta_k^*} \mathbb{1}_{\{\boldsymbol{X}_T \in B\}} + \mathbb{1}_{\{\boldsymbol{X}_T \notin B\}}}{e^{-\eta_k^*} p_B + (1 - p_B)}, \tag{3.1}$$

where $p_B := \mathbb{P}(\boldsymbol{X}_T \in B)$ and η_k^* is such that the constraint is binding. We include the subscript index on $\mathbb{Q}_k^*$ as we aim to consider the limiting measure for $k \uparrow \infty$. By enforcing the constraint, we have that

$$\mathbb{E}\left[\frac{e^{-\eta_k^*} \mathbb{1}_{\{\boldsymbol{X}_T \in B\}} + \mathbb{1}_{\{\boldsymbol{X}_T \notin B\}}}{e^{-\eta_k^*} p_B + (1 - p_B)} \mathbb{1}_{\{\boldsymbol{X}_T \in B\}}\right] = q_k,$$

which gives

$$\eta_k^* = -\log\left(\frac{p_B}{q_k} \frac{1 - q_k}{1 - p_B}\right). \tag{3.2}$$

Substituting (3.2) into (3.1), the RN density becomes

$$\frac{\mathrm{d}\mathbb{Q}_k^*}{\mathrm{d}\mathbb{P}} = \left(\frac{1 - \frac{1}{k}}{p_B}\right) \mathbb{1}_{\{\boldsymbol{X}_T \in B\}} + \left(\frac{1}{k(1 - p_B)}\right) \mathbb{1}_{\{\boldsymbol{X}_T \notin B\}}. \tag{3.3}$$

The limiting measure induced by the RN density $\frac{\mathrm{d}\mathbb{Q}^*}{\mathrm{d}\mathbb{P}} := \lim_{k \to \infty} \frac{\mathrm{d}\mathbb{Q}_k^*}{\mathrm{d}\mathbb{P}} = \frac{\mathbb{1}_{\{\boldsymbol{X}_T \in B\}}}{\mathbb{P}(\boldsymbol{X}_T \in B)}$ coincides with the so-called pinned measures. Pinned measures are those for which the terminal value of the process $\boldsymbol{X}$ must lie within the set B. Note this limiting measure is not equivalent to $\mathbb{P}$, but absolutely continuous $\mathbb{Q} \ll \mathbb{P}$.

3.3. VaR constraints. In a static and risk management setting, a single quantile constraint has been considered in [27]. Perhaps surprisingly, for a single quantile constraint at terminal time of a stochastic process, the optimal RN density coincides with the optimal RN density of the static problem; see (3.5) and [27, Prop. 3.2]. Here, we consider the case of two quantile constraints. That is, the constraint functions are $f_1(\boldsymbol{x}) = \mathbb{1}_{\{x^j \le q_1\}}$ and $f_2(\boldsymbol{x}) = \mathbb{1}_{\{x^j \le q_2\}}$ with $q_1, q_2 \in \mathbb{R}$ such that $\operatorname{essinf} X_T^j < q_1 < q_2 < \operatorname{esssup} X_T^j$ and $j \in \{1, 2, \dots, n\}$. The corresponding constraints are $\mathbb{Q}(X_T^j \le q_i) = \beta_i$, $0 < \beta_1 < \beta_2 < 1$, $i = 1, 2$, which are VaR[4] constraints at levels β_i if the $\mathbb{P}$-distribution of X_T^j is continuous. Next, we rewrite the second constraint as $\mathbb{Q}(q_1 < X_T^j \le q_2) = \beta_2 - \beta_1$ and by Corollary 2.8 the RN density for fixed Lagrange multipliers η_1, η_2 becomes

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \frac{e^{-\eta_1} \mathbb{1}_{\{X_T^j \le q_1\}} + e^{-\eta_2} \mathbb{1}_{\{q_1 < X_T^j \le q_2\}} + \mathbb{1}_{\{X_T^j > q_2\}}}{C(\eta_1, \eta_2)}, \tag{3.4}$$

where $C(\eta_1, \eta_2) := e^{-\eta_1} \mathbb{P}(X_T^j \le q_1) + e^{-\eta_2} \mathbb{P}(q_1 < X_T^j \le q_2) + \mathbb{P}(X_T^j > q_2)$ is the normalizing constant. Further, the optimal Lagrange multipliers η_1^* and η_2^* satisfy

$$\beta_1 = \frac{e^{-\eta_1^*} \mathbb{P}(X_T^j \le q_1)}{C(\eta_1^*, \eta_2^*)} \quad \text{and} \quad \beta_2 - \beta_1 = \frac{e^{-\eta_2^*} \mathbb{P}(q_1 < X_T^j \le q_2)}{C(\eta_1^*, \eta_2^*)}.$$

[4]For a univariate random variable Y and a probability measure $\mathbb{Q}$, the $\mathbb{Q}$-VaR at level $\beta \in (0,1)$ is defined as $\mathrm{VaR}_\beta^{\mathbb{Q}}(Y) := \inf\{y \in \mathbb{R} \mid \mathbb{Q}(Y \le y) \ge \beta\}$. For simplicity of notation we write $\mathrm{VaR}_\beta(Y) := \mathrm{VaR}_\beta^{\mathbb{P}}(Y)$.

Inserting this into (3.4) the optimal RN density is

(3.5)
$$\frac{\mathrm{d}\mathbb{Q}^*}{\mathrm{d}\mathbb{P}} = \frac{\beta_1}{\mathbb{P}(X_T^j \le q_1)} \mathbb{1}_{\{X_T^j \le q_1\}} + \frac{\beta_2 - \beta_1}{\mathbb{P}(q_1 < X_T^j \le q_2)} \mathbb{1}_{\{q_1 < X_T^j \le q_2\}} + \frac{1-\beta_2}{\mathbb{P}(X_T^j > q_2)} \mathbb{1}_{\{X_T^j > q_2\}}.$$

Moreover, the optimal Lagrange multipliers are given by

$$\eta_1^* = \log\left(\frac{\mathbb{P}\left(X_T^j \le q_1\right)}{\beta_1} \frac{1-\beta_2}{\mathbb{P}\left(X_T^j > q_2\right)} \right) \quad \text{and}$$
$$\eta_2^* = \log\left(\frac{\mathbb{P}\left(q_1 < X_T^j \le q_2\right)}{\beta_2 - \beta_1} \frac{1-\beta_2}{\mathbb{P}\left(X_T^j > q_2\right)} \right),$$

and the normalizing constant simplifies to $C(\eta_1^*, \eta_2^*) = \frac{\mathbb{P}\left(X_T^j > q_2\right)}{1-\beta_2}$. Hence, the RN density is piecewise constant as a function of X_T^j with jumps at the constrained quantile levels.

Note that the explicit formulas for the Lagrange multipliers and the RN density hold for any Lévy–Itô process $(\boldsymbol{X}_t)_{t\in[0,T]}$. Moreover, the assumptions on q_1, q_2, β_1, and β_2 are enough to guarantee existence of the solution to (P) with VaR constraints.

3.4. Linear constraint function for process with independent increments. Next, we consider what happens when a processes' expected terminal value is altered. For example, if the process represents the price of an asset, and a trader believes the expected value is higher than that of the estimated model, they may use this approach to perturb their model. For simplicity we consider a one-dimensional process and a linear constraint function. That is, we let $(X_t)_{t\in[0,T]}$ be the solution to the SDE under $\mathbb{P}$,

$$\mathrm{d}X_t = \mu(t)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}W_t + \int_{\mathbb{R}} z\,\tilde{\mu}(\mathrm{d}t, \mathrm{d}z),$$

where W is a one-dimensional $\mathbb{P}$-Brownian motion, $\mu(\mathrm{d}t, \mathrm{d}z)$ the Poisson random measure describing Poisson arrivals of independent and identically distributed marks, and $X_0 = x_0 \in \mathbb{R}$. We consider optimization problem (P) with a constraint on the expected value of X_T, i.e., $\mathbb{E}^{\mathbb{Q}}[X_T] = c$, $c \in \mathbb{R}$. Note that this constraint encompasses linear constraint functions $f(x) = a_1 x + a_2$ with $a_1, a_2 \in \mathbb{R}$, $a_1 \neq 0$, as, for this choice of f, the constraint $\mathbb{E}^{\mathbb{Q}}[f(X_T)] = c$ is equivalent to $\mathbb{E}^{\mathbb{Q}}[X_T] = (c - a_2)/a_1$.

From Theorem 2.6 for fixed Lagrange multiplier $\eta \in D_{-(X_T - c)}$, and as X has independent increments, we have

$$\omega(t,x) = \mathbb{E}_{t,x}\left[e^{-\eta\,(X_T - c)}\right] = e^{-\eta\,(x-c)}\,\mathbb{E}_{t,x}\left[e^{-\eta\,(X_T - X_t)}\right],$$
$$\lambda(t,x) = \eta\,\sigma(t), \quad \text{and} \quad h_t(z) = 1 - e^{-\eta\, z}.$$

We note that $\omega \in \mathcal{C}^{1,2}([0,T]\times\mathbb{R})$ and that λ, h induce $\mathbb{Q}_{\lambda,h} \in \boldsymbol{\mathcal{Q}}$. Hence, the perturbed compensator is exponentially tilted relative to the original, and the drift adjustment is linear in the volatility.

If we further assume that $\nu(\mathrm{d}z, \mathrm{d}t) = \ell\,\Phi_{a,b}(\mathrm{d}z)\,\mathrm{d}t$, where $\ell > 0$ is the rate parameter of the Poisson process and $\Phi_{a,b}(z) := \Phi((z-a)/b)$ is the distribution function of the marks—here Φ is the distribution function of a standard normal random variable.

Then, for fixed η and $\mathbb{Q}_{\lambda,h} \in \boldsymbol{\mathcal{Q}}$ the constraint equation $\mathbb{E}^{\mathbb{Q}_{\lambda,h}}[X_T] = c$ becomes (after some calculations)

$$x_0 + A - \eta\,\Sigma^2 - \ell\,T\,\eta\,b^2\,e^{-a\,\eta + \frac{1}{2}\eta^2\,b^2} - \ell\,a\,T = c, \tag{3.6}$$

where $A := \int_0^T \alpha(t)\,\mathrm{d}t$ and $\Sigma^2 := \int_0^T \sigma^2(t)\mathrm{d}t$. The optimal Lagrange multiplier η^* that binds the constraint exists since the left-hand side of (3.6) is continuous in η and diverges to $-\infty$ for $\eta \to \infty$ and diverges to ∞ for $\eta \to -\infty$. Uniqueness of η^* follows by uniqueness of the solution to (P).

3.5. Brownian motion with arbitrary variance. Finally, we investigate how a Brownian motion is transformed when we modify the variance while keeping the mean fixed. This provides an interesting (and, as far as we know, new) connection between a Brownian motion and an Ornstein–Uhlenbeck (OU) process. Let $X_t = W_t$, $t \in [0,T]$, be a one-dimensional $\mathbb{P}$-Brownian motion and consider the constraints $\mathbb{E}^{\mathbb{Q}}[X_T] = 0$ and $\mathbb{E}^{\mathbb{Q}}[X_T^2] = \kappa T$ for $\kappa > 0$, $\kappa \neq 1$. Note that since $\mathbb{E}[X_T] = 0$ and $\mathbb{E}[X_T^2] = T$ the constraints result that under the optimal probability measure, the mean of X_T is kept fixed to its $\mathbb{P}$ value while the variance is scaled by κ. For Lagrange multipliers $\eta_1, \eta_2 \in D_{-\mathfrak{X}}$ we have by Theorem 2.6 that

$$\omega(t,x) = \mathbb{E}_{t,x}\left[e^{-\eta_1\,X_T - \eta_2\,(X_T^2 - \kappa\,T)}\right] \quad \text{and}$$
$$\lambda(t,x) = \frac{2\,\eta_2}{2\,\eta_2(T-t)+1}\,x + \frac{\eta_1}{2\,\eta_2(T-t)+1}.$$

Therefore, $(X_t)_{t\in[0,T]}$ satisfies the following SDE under $\mathbb{Q}_\lambda$, $\mathrm{d}X_t = a_t\,(\beta - X_t)\,\mathrm{d}t + \mathrm{d}W_t^\lambda$, where W^λ is a $\mathbb{Q}_\lambda$-Brownian motion, $a_t := 2\,\eta_2/(2\,\eta_2(T-t)+1)$, and $\beta := -\eta_1/(2\,\eta_2)$ with η_1, η_2 to be determined to bind the constraints. By employing Itô's formula on the process

$$Y_t := X_t\,e^{\int_0^t a_s\,\mathrm{d}s}, \quad t \geq 0,$$

we obtain that

$$X_t = -\eta_1 \int_0^t \frac{2\,\eta_2\,(T-t)+1}{(2\,\eta_2\,(T-s)+1)^2}\,\mathrm{d}s + \int_0^t \frac{2\,\eta_2\,(T-t)+1}{2\,\eta_2\,(T-s)+1}\,\mathrm{d}W_s^\lambda.$$

As the coefficient of the Itô integral is deterministic, X_T is normally distributed, and as the Itô integral has zero $\mathbb{Q}_\lambda$-mean, we see that $\eta_1^* = 0$ to enforce the mean constraint. Using Itô's isometry we find that $\eta_2^* = (1-\kappa)/(2\,\kappa\,T)$ is required to enforce the variance constraint. Note that if $\kappa \in (0,1)$—a reduction of the variance under $\mathbb{Q}^*$—then $\eta_2^* > 0$, which implies that the process $(X_t)_{t\in[0,T]}$ mean-reverts around zero to reduce the variance; similarly, if $\kappa > 1$—an increase of the variance under $\mathbb{Q}^*$—, then $\eta_2^* < 0$, which implies that the process $(X_t)_{t\in[0,T]}$ is mean-avoiding to increase the variance. Finally, under the optimal measure $\mathbb{Q}^*$ the process $(X_t)_{t\in[0,T]}$ satisfies the SDE

$$\mathrm{d}X_t = -\left(\frac{T}{1-\kappa} - t\right)^{-1} X_t\,\mathrm{d}t + \mathrm{d}W_t^*, \tag{3.7}$$

where W^* is a $\mathbb{Q}^*$-Brownian motion. The above SDE shows that X is an OU process. Note the coefficient of the drift remains finite for all $t \in [0,T]$.

4. Numerical example. In this section, we illustrate how our methodology may be applied in practice. In particular, for simplicity of exposition, we assume that X is a one-dimensional Itô process, which more specifically satisfies the SDE under $\mathbb{P}$,

$$\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t, \tag{4.1}$$

where $\mu(x)$ and $\sigma(x)$ are parameterized by artificial neural networks. We estimate μ and σ using maximum likelihood estimation based on an Euler discretization of the SDE (4.1); for details on this procedure and other refinements, see, e.g., [9].

The data are from an automatic marker making (AMM) pool known as sushi-swap for the USDC-WETH cryptocurrency pair, and X represents the amount of WETH received for one USDC. We normalize prices by shifting and scaling them using the mean and standard deviation of the sample path. The data considered are for the week of December 10 to December 17, 2020, and Figure 1 shows the normalized data and the estimated drift and volatility functions (time is measured in hours). The estimation of μ and σ seen in the figure illustrate that the drift/volatility have a general upward/downward trend with a bump at $x = 0.7$.

A trader may have a specific view on, e.g., the expected return in the cryptocurrency pool and/or the expected time that prices spend below some level. Using the methodology we developed in this paper, the trader then wishes to update the model estimated on historical data to reflect their beliefs. Hence, with μ and σ estimated (under $\mathbb{P}$), and a specified set of constraints, the trader proceeds to estimate the optimal measure using the algorithmic steps shown in Algorithm 4.1. We use the Crank–Nicholson scheme for solving the PDEs in the algorithm with $N = 500$ time steps and $\Delta x = \sqrt{3\,\Delta t}\,\bar{\sigma}$, where $\bar{\sigma} = \frac{1}{0.4}\int_{-0.2}^{0.2}\sigma(x)\,dx$. The optimal η^* are obtained using the `scipy.optimize.fsolve` function (which is a wrapper for MINPACK's "hybrd" algorithm) to enforce the constraints. The algorithm appears to converge quickly in our experiments (within a dozen iterations) for a variety of different constraints.

We consider two numerical examples: (i) we increase $\mathrm{VaR}_{0.9}(X_T)$ by 20% and decrease $\mathrm{VaR}_{0.5}(X_T)$ by 10%, and (ii) we increase the $\mathrm{VaR}_{0.9}(X_T)$ by 20% and reduce the average time spent below the barrier $X_t = -0.05$ by 50%; all percentages are relative to their values under the reference measure $\mathbb{P}$. The VaR_α constraints are induced by constraint functions $f(x) = \mathbb{1}_{\{x<q_\alpha\}}$ with constraint constants α, i.e., $\mathbb{Q}(X_T \leq q_\alpha) = \alpha$, and where q_α are the VaR values under $\mathbb{Q}$. The average time spent

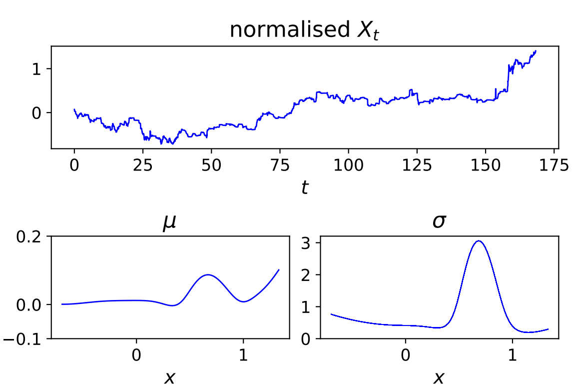

Fig. 1. *Top panel: Normalized data. Bottom panels: $\mu(x)$ and $\sigma(x)$ estimated from USDC-WETH cryptocurrency AMM pool using data from December* 10 *to December* 17, 2020. *Note: color appears only in the online article.*

Algorithm 4.1. Optimal measure computation.

Input: drift μ, volatility σ functions, constraint functions $\boldsymbol{f}$ and $\boldsymbol{g}$, and targets $\boldsymbol{c}$ and $\boldsymbol{d}$;

Initialize $\boldsymbol{\eta}_1 = \boldsymbol{0}$, $\boldsymbol{\eta}_2 = \boldsymbol{0}$, and $n = 0$;

do

increment $n \leftarrow n + 1$

Solve for $\omega^{(n)}(t,x)$ satisfying (2.14) by solving the PDE

$$\left(\partial_t + \mu(x)\,\partial_x + \tfrac{1}{2}\sigma^2(x)\,\partial_{xx} - \boldsymbol{\eta}_2 \cdot \boldsymbol{g}(x)\right)\omega^{(n)}(t,x) = 0,$$

s.t. $\omega^{(n)}(T,x) = e^{-\boldsymbol{\eta}_1 \cdot (\boldsymbol{f}(x) - \boldsymbol{c})}$ using finite-difference (FD) methods;

Compute (using (2.15)) $\lambda^{(n)}(t,x) = -\sigma(x)\,\partial_x \log \omega^{(n)}(t,x)$ with FD;

Set $\boldsymbol{k}^{(n)}(t,x) := \mathbb{E}^{\mathbb{Q}^{(n)}}[(\boldsymbol{f}(X_T) - \boldsymbol{c})|X_t = x]$. $\boldsymbol{k}^{(n)}(0,x_0)$ is the terminal constraint errors in (P');

Solve

$$\left(\partial_t + (\mu(x) - \sigma(x)\,\lambda^{(n)}(t,x))\,\partial_x + \sigma^2(x)\,\partial_{xx}\right)\boldsymbol{k}^{(n)} = \boldsymbol{0}$$

s.t. $\boldsymbol{k}^{(n)}(T,x) = \boldsymbol{f}(x) - \boldsymbol{c}$ using FD;

Set $\boldsymbol{\ell}^{(n)}(t,x) := \mathbb{E}^{\mathbb{Q}^{(n)}}[\int_t^T \boldsymbol{g}(X_s)\,\mathrm{d}s|X_t = x]$. $\boldsymbol{\ell}^{(n)}(0,x_0)$ is the running constraint errors in (P');

Solve

$$\left(\partial_t - \sigma(x)\,\lambda^{\dagger}(t,x)\,\partial_x + \sigma^2(x)\,\partial_{xx} + \boldsymbol{g}(x)\right)\boldsymbol{\ell}^{(n)}(t,x) = \boldsymbol{0}$$

s.t. $\boldsymbol{\ell}^{(n)}(T,x) = \boldsymbol{0}$ using FD;

Update $\boldsymbol{\eta}_1$, $\boldsymbol{\eta}_2$ using an optimization engine

while $|\boldsymbol{k}^{(n)}(0,x_0)|, |\boldsymbol{\ell}^{(n)}(0,x_0)| > tol$;

Output: $\lambda^{(n)}(t,x)$ which is the estimate of $\lambda^*(t,x)$;

below a barrier is achieved by imposing a running cost constraint. To this end, define $\tau = \int_0^T g(X_s)\,\mathrm{d}s$ with $g(x) = \mathbb{1}_{\{x \le -0.05\}}$. Thus, the constraint $\mathbb{E}^{\mathbb{Q}}[\int_0^T g(X_t)\,\mathrm{d}t] = c$ corresponds to constraining the average time spent below the barrier -0.05 to be equal to c. We also refer the reader to [23] for further applications to risk measure constraints and Poisson processes.

We first investigate the case of the two VaR constraints, example (i). In this case, Algorithm 4.1 converged to a tolerance of 10^{-8} after 9 iterations. The histogram of X_T under the reference measure $\mathbb{P}$ and the optimal measure $\mathbb{Q}^*$ is shown in the top panel of Figure 2. From the figure, we observe, when comparing the distribution of X_T under $\mathbb{P}$ with $\mathbb{Q}^*$, that probability mass from the center of the distribution is pushed into the left and right tails to ensure that the median is reduced and the 90%-quantile is increased. The bottom panel of Figure 2 shows the drift under $\mathbb{Q}^*$. Recall that under $\mathbb{P}$ the drift is a function of the process X only (see (4.1)); under $\mathbb{Q}^*$, however, the drift now depends on the value of the process and on time. From the bottom panel of Figure 2, we observe that the probability mass transport seen in the histograms of X_T in the left panel, is achieved by having excess positive/negative drift to the right/left of the original median value. Moreover, we see upward/downward spikes at the locations of the new quantiles whose intensity increases as the terminal time approaches.

Next, we investigate the case of a 10% increase in the 90% quantile and a 50% reduction in the average time spent below the barrier, i.e., example (ii). In this case, Algorithm 4.1 converged to a tolerance of 10^{-8} after 11 iterations. The middle panel of Figure 3 displays the histogram of τ under the reference $\mathbb{P}$ and optimal measure $\mathbb{Q}^*$. The figure shows that under $\mathbb{Q}^*$, the amount of time spent below the barrier is more concentrated towards zero than it is under the reference measure. The left panel shows the histogram of X_T, and while under $\mathbb{Q}^*$ the 90% quantile is increased, which

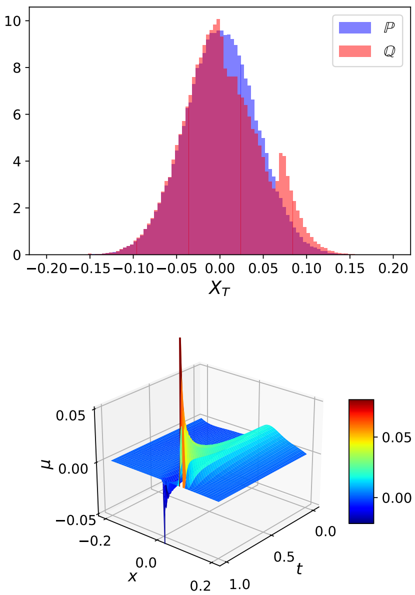

FIG. 2. *Under $\mathbb{Q}^*$, VaR at levels $\alpha = 0.5$ and $\alpha = 0.9$ are decreased by 20% and increased by 20%, respectively. Percentage changes are relative to those under $\mathbb{P}$. Top panel: histogram of X_T under $\mathbb{P}$ (blue) and under $\mathbb{Q}^*$ (red). Bottom panel: drift of X under $\mathbb{Q}^*$. Note: color appears only in the online article.*

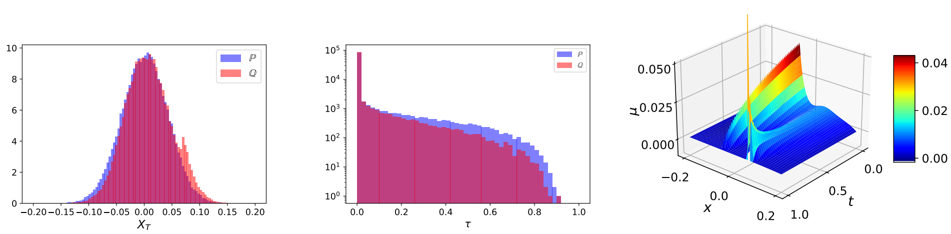

FIG. 3. *Under $\mathbb{Q}^*$, the VaR at level $\alpha = 0.9$ is increased by 20% and the average time spent below the barrier level -0.05 is decreased by 50%. Percentage changes are relative to those under $\mathbb{P}$. Under the $\mathbb{P}$-measure, the process spends about 4.3% of the time below the level -0.05. Left panel: histogram of X_T under $\mathbb{P}$ (blue) and under $\mathbb{Q}^*$ (red). Middle panel: histogram of the time spent below the barrier τ under $\mathbb{P}$ (blue) and under $\mathbb{Q}^*$ (red). Right panel: drift of X under $\mathbb{Q}^*$. Note: color appears only in the online article.*

is seen by the additional mass in the right tail, the running cost constraint moves mass away from the left tail. The right panel of the figure shows the drift under $\mathbb{Q}^*$. We observe that as the process crosses to negative values, it receives a positive drift which prevents the process from spending additional time below the barrier. The process also receives a drift if it approaches the target quantile whose intensity increases as the terminal time approaches.

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