

# A Quantum Witt Construction

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Given a quantum double and two suitably paired modules it is possible to construct a new quantum double, in a manner analogous to Witt's construction of simple Lie algebras. This construction generalises a standard construction of quantum groups, and also supergroups, but it also provides alternative constructions for some quantum groups, including the quantum exceptional group  $e_8$ .

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## 1. THE WITT CONSTRUCTION

Unlike ordinary Lie algebras which allow complete freedom to pick bases, the definition of the quantum groups always tends to drive one back to the standard generators. In this paper, motivated by the observation that it is closely related to the Witt construction for ordinary Lie algebras, we shall generalise the standard construction of quantum groups as quantum doubles to provide some alternative descriptions of quantum groups. We shall then show how this facilitates direct comparisons between different quantum unitary and orthogonal groups, and we shall give a construction of the quantum enveloping algebra of the exceptional Lie algebra  $e_8$ .

The ingredients for the standard Witt construction [11] are a semisimple Lie algebra  $\mathfrak{h}$ , with Killing form  $\phi$ , and an  $\mathfrak{h}$ -module  $M$ , equipped with a symmetric  $\mathfrak{h}$ -invariant bilinear form  $\psi$ . (The construction can easily be generalised; see the final section.) From these one constructs a Lie bracket on  $\mathfrak{h} \oplus M$  in the following way. For  $X$  and  $Y$  in  $\mathfrak{h}$ , and  $m$  and  $n$  in  $M$ , we set

$$[X \oplus m, Y \oplus n] = ([X, Y] + [m, n]) \oplus (X.n - Y.m),$$

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where the first term on the right-hand side denotes the Lie bracket on  $\mathfrak{h}$ , and the second term is the unique element of  $\mathfrak{h}$  which satisfies

$$\phi(X, [m, n]) = \psi(m, X.n).$$

The bracket is obviously antisymmetric, and the Jacobi identity holds for triples of elements in  $\mathfrak{h} \oplus M$ , of which at least one is in  $\mathfrak{h}$ , and for all triples when  $\dim(M) < 4$ . If the quadratic Casimir element of  $\mathfrak{h}$  has a single eigenvalue  $C_M$  on  $M$  and takes the value  $C_{\mathfrak{h}}$  in the adjoint representation, then (see the Appendix) the Jacobi identity holds for all triples precisely when  $M$  is trivial ( $C_M = 0$ ), or

$$\frac{\dim(M)}{\dim(\mathfrak{h})} + \frac{C_{\mathfrak{h}}}{C_M} = 2.$$

(The Killing form on  $\mathfrak{h} \oplus M$  is  $\phi \oplus C_M \psi$ , so that when  $M$  is trivial the resulting Lie algebra is certainly not semisimple.)

In the case of quantum enveloping algebras, a comultiplication is also needed. We generalise the Killing form on  $\mathfrak{h}$  to suppose that a Hopf algebra  $A$  has a nondegenerate pairing with another Hopf algebra  $B$ . We shall start by assuming, in a rather lavish way, that  $B$  has left and right (commuting) actions,  $\lambda$  and  $\rho$ , respectively, on a vector space  $N$ , which is itself paired with another space  $M$ . (All such modules will be assumed finite-dimensional unless the converse is explicitly stated.) The  $B$  actions give rise to unique dual maps  $\Delta_\lambda: N^* \rightarrow B^* \otimes N^*$  and  $\Delta_\rho: N^* \rightarrow N^* \otimes B^*$ . Under suitable conditions the pairings will allow us to identify  $B^*$  with  $A$  and  $N^*$  with  $M$ , to obtain

$$\Delta_\lambda: M \rightarrow A \otimes M, \quad \Delta_\rho: M \rightarrow M \otimes A,$$

which suggests generalising the usual comultiplication by setting

$$\Delta = \Delta_\lambda + \Delta_\rho: M \rightarrow A \otimes M + M \otimes A.$$

This will be coassociative provided that

$$\begin{aligned} & (\text{id} \otimes \Delta_\lambda)\Delta_\lambda + (\text{id} \otimes \Delta_A)\Delta_\rho + (\text{id} \otimes \Delta_\rho)\Delta_\lambda \\ &= (\Delta_A \otimes \text{id})\Delta_\lambda + (\Delta_\rho \otimes \text{id})\Delta_\rho + (\Delta_\lambda \otimes \text{id})\Delta_\rho. \end{aligned}$$

Now the condition that  $\lambda$  and  $\rho$  commute gives dually  $(\text{id} \otimes \Delta_\rho)\Delta_\lambda = (\Delta_\lambda \otimes \text{id})\Delta_\rho$ , so that it would suffice to show that  $(\text{id} \otimes \Delta_\lambda)\Delta_\lambda = (\Delta_A \otimes \text{id})\Delta_\lambda$  and a similar identity for  $\rho$ . There are two important situations in which these identities are automatic. The first of these, that of ordinary nondeformed groups, occurs when  $\lambda = e \otimes \text{id}$  and  $\rho = \text{id} \otimes e$  are trivial ( $e$

being the counit), and the identities are easily verified. The other, which occurs in the case of quantum enveloping algebras, is when  $\Delta_A$  is the dual of the multiplication on  $B$ , because then the identity just dualises the fact that  $\lambda$  and  $\rho$  define actions on  $N$ . The original convention for quantum groups, still almost universal in the physics literature, was to take  $\rho = \lambda S$ , the contragredient of  $\lambda$ , but we shall follow the algebraists' convention and take  $\rho$  trivial. (It is worth remarking that in quantum groups the adjoint action of  $B$  is just  $\lambda \times (\rho \circ S^{-1})$ , and so related to the comultiplication. This means that the undeformed theory, in which there is no such relationship, is not a special case of the quantum group theory.)

The doubling of the number of algebras and modules, and the pairings which make comultiplication the dual of multiplication all find their place in Drinfel'd's notion of a quantum double [3], though we shall follow the account in [5], which is better suited to our purpose. (We shall also use Joseph's simplification of Sweedler's convention for coproducts and write  $\Delta(a) = a_1 \otimes a_2$  with only an implied summation). The opposite comultiplication and antipode are denoted by  $\Delta^o$  and  $S^o$ , respectively, so that  $\Delta^o(a) = a_2 \otimes a_1$  and  $S^o = S^{-1}$ , with the implicit assumption that  $S$  is invertible. The counit is denoted by  $e$ . We shall occasionally denote the multiplication by  $\mu$ , but more often simply by juxtaposition. Let  $A$  and  $B$  be Hopf algebras over a field  $\mathbb{F}$  (which we shall generally assume to be  $\mathbb{R}$  or  $\mathbb{C}$ ). A pairing  $\phi: A \times B \rightarrow \mathbb{F}$  is said to be a skew Hopf pairing if  $\mu_A$ ,  $1_A$ ,  $\Delta_A$ ,  $e_A$ , and  $S_A$  are the transposes of  $\Delta_B^o$ ,  $e_B$ ,  $\mu_B$ ,  $1_B$ , and  $S_B^o$  with respect to  $\phi$ ; that is, they satisfy

- (i)  $\phi(1_A, b) = e_B(b)$  and  $\phi(a, 1_B) = e_A(a)$ ,
- (ii)  $\phi^2(\Delta_A(a), b \otimes b') = \phi(a, bb')$ ,
- (iii)  $\phi^2(a \otimes a', \Delta_B^o(b)) = \phi(aa', b)$ ,
- (iv)  $\phi(S_A(a), b) = \phi(a, S_B^o(b))$ ,

for all  $a, a' \in A$  and  $b, b' \in B$ , where  $\phi^2 = \phi \otimes \phi$ , and the suffices indicate to which algebra the operations refer.

The Drinfel'd quantum double  $D = D(A, B, \phi)$  is then the tensor product  $A \otimes B$  endowed with the obvious tensor product comultiplication, counit and antipode, and with the multiplication defined so that

- (i)  $a \mapsto a \otimes 1$  and  $b \mapsto 1 \otimes b$  are algebra homomorphisms,
- (ii)  $(a \otimes 1)(1 \otimes b) = (a \otimes b)$ ,
- (iii)  $(1 \otimes b)(a \otimes 1) = \phi(a_1, S_B(b_1))(a_2 \otimes b_2)\phi(a_3, b_3)$ ,

where  $\Delta_A^2(a) = (\Delta_A \otimes 1)\Delta_A(a) = a_1 \otimes a_2 \otimes a_3$  and  $\Delta_B^2(b) = b_1 \otimes b_2 \otimes b_3$ .

The standard quantum groups are obtained by taking the quotient by the radicals of the pairing, that is, the ideals of  $B$  or  $A$  which have

vanishing pairings with all elements of  $A$  or  $B$ , respectively, and also by a certain central ideal. (We shall write  $A^+$  and  $B^+$  for the quotients of the images of  $A$  and  $B$  in  $D(A, B, \phi)$ .)

Let us now suppose that  $M$  and  $N$  are modules for the quantum double  $D = D(A, B, \phi)$  and that  $\psi: M \times N \rightarrow \mathbb{F}$  is a  $D$ -invariant pairing, that is,  $\psi(S(d).m, n) = \psi(m, d.n)$  for all  $d \in D$ ,  $m \in M$ , and  $n \in N$ . (It is usually convenient to take  $\psi$  to be nondegenerate, but it is not strictly necessary, because any degeneracy is removed when one factors out the radicals.) Most of our construction involves only the action of the subalgebras  $A$  on  $M$  and  $B$  on  $N$ . Let  $\mathcal{M}$  and  $\mathcal{N}$  denote the free unital algebras generated by  $M$  and  $N$ , that is, the finite linear combinations of units and formal products of elements in  $M$  or  $N$ . (These are essentially the tensor algebras, but to avoid confusion with other uses of tensor products we shall write them simply as products. When dealing with quantum groups at roots of unity, it is better to take the quotient by proper Frobenius twistings, or else the Serre relations become more complicated. However, we shall suppress this detail in the interests of simplicity.) By defining  $a.1 = e(a)1$  and  $a.(wx) = (a_1.w)(a_2.x)$ , we obtain an action of  $A$  on  $\mathcal{M}$ , and similarly of  $B$  on  $\mathcal{N}$ , which allows us to introduce the smash product algebras  $\mathcal{A} = \mathcal{M} \# A$  and  $\mathcal{B} = \mathcal{N} \# B$  (the latter using the opposite coproduct on  $B$ ). (We recall that the smash product is defined so that the given action of  $A$  on  $\mathcal{M}$  coincides with its adjoint action, defined by  $\text{ad}(a)x = a_1 x S(a_2)$ .) Our main theorem shows that, under suitable technical conditions, we may define comultiplications, counits, and antipodes on  $\mathcal{A}$  and  $\mathcal{B}$  and to extend the pairings  $\phi$  and  $\psi$  to a skew Hopf pairing  $\varphi: \mathcal{A} \times \mathcal{B} \rightarrow \mathbb{F}$ , so that we can construct a new quantum double.

**THEOREM 1.1.** *Let  $D = D(A, B, \phi)$  be a quantum double of Hopf algebras, with modules  $M$  and  $N$ , and  $\psi: M \times N \rightarrow \mathbb{F}$  is a  $D$ -invariant pairing, such that for every  $m \in M$  and  $n \in N$  there are elements  $a_{mn} \in A$  and  $b_{mn} \in B$  such that each matrix element  $\psi(m, b.n)$  can be expressed as  $\phi(a_{mn}, b)$  for all  $b \in B$  and  $\psi(a.m, n)$  as  $\phi(a, b_{mn})$  for all  $a \in A$ . Then there are Hopf structures on  $\mathcal{A} = \mathcal{M} \# A$  and  $\mathcal{B} = \mathcal{N} \# B$  unique modulo the radicals of the pairings, and a unique skew Hopf pairing  $\varphi: \mathcal{A} \times \mathcal{B} \rightarrow \mathbb{F}$  such that*

- (a)  $\Delta_{\mathcal{A}}(m) - m \otimes 1 \in A \otimes M$  for all  $m \in M$ ;
- (b)  $\Delta_{\mathcal{B}}(n) - n \otimes 1 \in B \otimes N$  for all  $n \in N$ ;
- (c)  $\varphi$  vanishes on  $A \times N$  and  $M \times B$ ;
- (d)  $\varphi(a, b) = \phi(a, b)$  for all  $a \in A$ ,  $b \in B$ , and  $\varphi(m, n) = \psi(m, n)$  for all  $m \in M$  and  $n \in N$ .

This result is essentially similar to Proposition 3.1.10 in [5], although the starting point is rather different and somewhat more general. Its proof will

be accomplished in a series of lemmata which will occupy the next two sections. In Section 4 we shall discuss the multiplication of the double algebra. The result will then be applied to a commutative Hopf algebra in Section 5, to show how the standard Hopf algebras arise in this context. Section 6 proves some results which enable us to recognize the double structure in known Hopf algebras. This is followed, in Section 7, by a fairly detailed account of the relationship between the quantum enveloping algebras  $U_q(\mathfrak{su}(n))$  and  $U_q(\mathfrak{su}(n+1))$ , and then, in the next three sections, by shorter discussions of quantum orthogonal algebras, orthosymplectic superalgebras, and a construction of  $U_q(e_8)$  from quantum  $\mathfrak{so}(14) \times \mathfrak{so}(2)$  and two spin modules. The main references for any standard unexplained notation are [2], [5], [6], and [8].

## 2. THE UNIQUENESS OF THE DOUBLE

We shall start by showing that the conditions of the theorem uniquely determine the Hopf structure and the extension of the pairing to  $\mathcal{A} \times \mathcal{B}$ . In this section and the next, unless explicitly countermanded, we shall use the notation  $a \in A$ ,  $b \in B$ ,  $x \in \mathcal{M}$ ,  $y \in \mathcal{N}$ ,  $\xi, \omega \in \mathcal{A}$ ,  $\eta, \zeta \in \mathcal{B}$ , and with these conventions we may drop the suffices on  $\Delta$ ,  $e$ , and  $S$ . The action of  $A$  on  $M$  is written  $a.m$  to distinguish it from the product  $am$  in  $\mathcal{A}$ , and similarly for  $B$  on  $N$ . We note that  $\mathcal{M}$  (and  $\mathcal{N}$ ) are graded by the number of factors from  $M$  (or  $N$ , respectively) occurring in each product. The actions of  $A$  and  $B$  respect this grading so that it can be extended to  $\mathcal{A}$  and  $\mathcal{B}$  and most of the proofs use induction on the degree of terms. We start by proving the uniqueness of the Hopf structure, and before doing so we note one useful consequence of our hypotheses.

**LEMMA 2.1.** *Under conditions (a)–(d) of Theorem 1.1 we have  $\varphi(xa, yb) = \varphi(x, y)\phi(a, b)$  for all  $a \in A$ ,  $b \in B$ ,  $x \in \mathcal{M}$ , and  $y \in \mathcal{N}$ .*

*Proof.* Using the skew Hopf property and induction on the degree of its arguments, it is easy to show that  $\varphi$  vanishes unless its arguments have the same degree. The skew Hopf property gives

$$\varphi(xa, yb) = \varphi(x, y_2 b_2) \varphi(a, y_1 b_1),$$

so bearing in mind the form of the coproducts, the only nonvanishing term is

$$\begin{aligned} \varphi(x, y b_2) \varphi(a, b_1) &= \varphi(x_1, y) \varphi(x_2, b_2) \varphi(a, b_1) \\ &= \varphi(x, y) \phi(1, b_2) \phi(a, b_1), \end{aligned}$$

and the identity  $b = e(b_2)b_1$  tells us that this is  $\varphi(x, y)\phi(a, b)$ , as asserted. ■

COROLLARY 2.2. *Under conditions (a)–(d) of Theorem 1.1, for any  $m \in M$ ,  $n \in N$ ,  $a \in A$ , and  $b \in B$  we have*

$$\varphi(m, bn) = \psi(m, b.n), \quad \varphi(am, n) = \psi(a.m, n).$$

*Proof.* We have

$$\begin{aligned} \varphi(m, bn) &= \varphi(m, (b_1.n)b_2) = \varphi(m, (b_1.n))\phi(1, b_2) \\ &= \psi(m, (b_1.n))e(b_2) = \psi(m, b.n), \end{aligned}$$

and similarly for the other identity. ■

LEMMA 2.3. *The algebras  $\mathcal{A}$  and  $\mathcal{B}$  each admit at most one Hopf structure consistent with conditions (a)–(d) of Theorem 1.1.*

*Proof.* We shall use only weak forms of the conditions of the theorem. In particular, the duality between multiplication and comultiplication with respect to the pairing will be used only in degree  $\leq 1$ , where  $\varphi$  is already completely determined by  $\phi$  and  $\psi$ , and the duality of the antipodes will not be used at all. Since the comultiplication on  $\mathcal{A}$  is an algebra homomorphism it is completely determined by its values on the generating set consisting of  $A$ , where it is already known, and  $M$ . By Corollary 2.2 we have

$$\varphi^2(\Delta(m), b \otimes n) = \varphi(m, bn) = \psi(m, b.n).$$

Since we wish to write  $\Delta(m) = m \otimes 1 + m'_1 \otimes m'_2$ , with  $m'_1 \in A$  and  $m'_2 \in M$ , the above identity gives  $\phi(m'_1, b)\psi(m'_2, n) = \psi(m, b.n) = \psi(S(b).m, n)$ . This suggests that  $S(b).m = \phi(m'_1, b)m'_2$ , and as the matrix elements of  $S(b)$  are the transpose of those of  $b$ , they are certainly expressible in this form, thus determining  $m'_1 \otimes m'_2$  to within elements of the radicals of  $\phi$  and  $\psi$ . (We also see that  $a_{mn} = \psi(m'_2, n)m'_1$ , so that when  $\Delta(m)$  has the required form the matrix elements can be represented.) The same applies to  $\Delta^o(n)$ , where we have the identities

$$\varphi^2(a \otimes m, \Delta^o(n)) = \varphi(am, n) = \psi(a.m, n).$$

We extend  $\Delta$  and  $\Delta^o$  to  $\mathcal{A}$  and  $\mathcal{B}$  as homomorphisms, which are coassociative for the reasons discussed in the introduction. Although the comultiplications are defined to be algebra homomorphisms, we need to check

that this is consistent with the adjoint action. We first note that

$$\begin{aligned}
 \Delta(a.m) &= \Delta(a_1 m S(a_2)) \\
 &= (a_1 \otimes a_2)(m_1 \otimes m_2)(S(a_4) \otimes S(a_3)) \\
 &= (a_1 m_1 S(a_4)) \otimes (a_2 m_2 S(a_3)) \\
 &= (a_1 m_1 S(a_3)) \otimes (a_2 . m_2) \\
 &= (a_1 m S(a_3)) \otimes (a_2 . 1) + (a_1 m'_1 S(a_3)) \otimes (a_2 . m'_2) \\
 &= (a_1 m S(a_2)) \otimes 1 + (a_1 m'_1 S(a_3)) \otimes (a_2 . m'_2) \\
 &= a.m \otimes 1 + (a_1 m'_1 S(a_3)) \otimes (a_2 . m'_2).
 \end{aligned}$$

The first term is of the required form and the other is determined by

$$\begin{aligned}
 \varphi^2(\Delta(a.m), S(b) \otimes n) \\
 &= \phi(a_1 m'_1 S(a_3), S(b)) \psi(a_2 . m'_2, n) \\
 &= \phi(a_1, S(b_1)) \phi(m'_1, S(b_2)) \phi(S(a_3), S(b_3)) \psi(a_2 . m'_2, n).
 \end{aligned}$$

On the other hand, using the formulae for the coproduct, and the multiplication in  $D$ , this can be rewritten as

$$\begin{aligned}
 &\phi(a_1, S(b_1)) \phi(a_3, b_3) \varphi^2(\Delta m, S(b_2) \otimes S^o(a_2).n) \\
 &= \varphi^2(\Delta(m), S^o(a) \otimes S(b).n) \\
 &= \psi(a.m, S(b).n),
 \end{aligned}$$

so that  $\varphi^2(\Delta(a.m), S(b) \otimes n) = \psi(a.m, S(b).n)$ , exactly as needed for consistency with the usual definition.

Being another algebra homomorphism, the counit is determined by the fact that it is already known on  $\mathcal{A}$  and must vanish on  $M$ , where  $e(m) = \varphi(m, 1) = 0$ . Taking  $b = 1$  in the above definition of the comultiplication, it is easily checked that these are compatible with the requirement that

$$m = (e \otimes \text{id})\Delta(m) = e(m) + e(m'_1)m'_2.$$

The antipode, an antihomomorphism which is also known on  $\mathcal{A}$ , is similarly determined on  $M$  by the condition

$$0 = e(m) = (S \otimes \text{id})\Delta(m) = S(m)1 + S(m'_1)m'_2,$$

which gives  $S(m) = -S(m'_1)m'_2$ , and then, as an antiautomorphism  $S$  is determined on  $\mathcal{A}$ . The same applies on  $\mathcal{B}$ . (One may also check that the second condition on  $S$  is automatic.) ■

We can also extract one useful corollary of the method:

**COROLLARY 2.4.** *For all  $m \in M$  and  $a \in A$ , if  $\Delta(m) = m \otimes 1 + m'_1 \otimes m'_2$ , then*

$$\Delta(a.m) = a.m \otimes 1 + (a_1 m'_1 S(a_3)) \otimes (a_2.m'_2).$$

**LEMMA 2.5.** *There is a unique pairing  $\varphi$  determined by the following conditions:*

- (a) *for all  $\xi, \omega \in \mathcal{A}$ ,  $\varphi(\xi\omega, \eta) = \varphi^2(\xi \otimes \omega, \Delta^o\eta)$ ;*
- (b) *for all  $\eta, \zeta \in \mathcal{B}$  and  $\xi \in \mathcal{A}$  of degree  $\leq 1$ ,  $\varphi(\xi, \eta\zeta) = \varphi^2(\Delta\xi, \eta \otimes \zeta)$ ;*
- (c)  *$\varphi$  vanishes on  $A \times N$  and  $M \times B$ ;*
- (d) *for all  $m \in M$  and  $n \in N$ ,  $\varphi(m, n) = \psi(m, n)$ ;*
- (e) *for all  $a \in A$  and  $b \in B$ ,  $\varphi(a, b) = \phi(a, b)$ .*

*Proof.* The first condition, which may be written as

$$\varphi(\xi\omega, \eta) = \varphi(\xi, \eta_2)\varphi(\omega, \eta_1),$$

means that  $\varphi$  is determined by its values on elements of degree 0 and 1 in  $\mathcal{A}$ . The second condition

$$\varphi(\xi, \eta\zeta) = \varphi(\xi_1, \eta)\varphi(\xi_2, \zeta)$$

then allows us to reduce to the case when the right-hand argument also has degree at most 1. When both arguments have degree 0 then (d) tells us that  $\varphi$  is just  $\phi$ . When one has degree 0 and the other 1, then  $\varphi$  vanishes (by (c)), and in the remaining case, when both arguments have degree 1,  $\varphi(ma, nb) = \psi(m, n)\phi(a, b)$ , by Lemma 2.1. ■

### 3. THE SKEW HOPF PROPERTY

This section will be devoted to showing that the uniquely defined structure identified above does indeed meet all the requirements of a double. We first note that the Hopf structure and pairing have been constructed to satisfy the four conditions of Theorem 1.1.



LEMMA 3.1. *The unique pairing  $\varphi$  constructed in the last section is a skew Hopf pairing.*

*Proof.* We shall sketch the arguments only and, where the cases of  $\mathcal{A}$  and  $\mathcal{B}$  are similar, prove just one of them. The first condition of Lemma 2.5 tells us that  $\mu_{\mathcal{A}}$  is dual to  $\Delta_{\mathcal{B}}$ , while the second condition gives the duality of  $\Delta_{\mathcal{A}}$  and  $\mu_{\mathcal{B}}$  for elements of  $\mathcal{A}$  having degree at most 1. We may prove the general case by induction on the degree of the element in  $\mathcal{A}$ , for we have

$$\varphi(\xi\omega, \eta\zeta) = \varphi(\xi, \eta_2\zeta_2)\varphi(\omega, \eta_1\zeta_1),$$

and using the inductive hypothesis, if each of  $\xi$  and  $\omega$  has degree at least 1, this reduces to

$$\begin{aligned} \varphi(\xi_1, \eta_2)\varphi(\xi_2, \zeta_2)\varphi(\omega_1, \eta_1)\varphi(\omega_2, \zeta_1) &= \varphi(\xi_1\omega_1, \eta)\varphi(\xi_2\omega_2, \zeta) \\ &= \varphi^2(\Delta(\xi\omega), \eta \otimes \zeta), \end{aligned}$$

as required. It is now easy to check that  $e_{\mathcal{B}}(\eta) = \varphi(1, \eta)$ . Lemma 2.5(e) tells us that this vanishes unless  $\eta \in B$ , when it follows from the properties of  $\phi$ . Finally, the duality of the two antipodes can be proved by induction on the degree of their arguments. The inductive step is supplied by the calculation

$$\begin{aligned} \varphi(S(\xi\omega), \eta) &= \varphi(S(\xi), \eta_1)\varphi(S(\omega), \eta_2) \\ &= \varphi(\xi, S^o(\eta_1))\varphi(\omega, S^o(\eta_2)) = \varphi(\xi\omega, S^o(\eta)). \end{aligned}$$

To establish the validity of the result in degree 1 we note that, dropping terms which vanish,

$$\varphi(S(m), n) = -\varphi(S(m'_1)m'_2, n) = -\varphi(S(m'_1), n'_2)\varphi(m'_2, n'_1).$$

Since  $S$  appears only in the degree zero term where the result is true by hypothesis this completes the proof. ■

#### 4. MULTIPLICATION ON THE DOUBLE ALGEBRA

From this data we may construct a new Hopf algebra  $\mathcal{D}(\mathcal{A}, \mathcal{B}, \varphi)$ . For convenience, we shall write

$$\Delta^2(m) = m \otimes 1 \otimes 1 + m'_1 \otimes m'_2 \otimes 1 + m''_0 \otimes m'_1 \otimes m'_2,$$

where  $\Delta(m'_1) = m''_0 \otimes m'_1$ , and similarly

$$\Delta^2(n) = 1 \otimes 1 \otimes n + 1 \otimes n'_1 \otimes n'_2 + n'_1 \otimes n'_2 \otimes n''_3.$$

The following proposition shows a clear analogy to the usual Witt construction.

**PROPOSITION 4.1.** *For  $m \in M$ ,  $n \in N$ ,  $a \in A$ , and  $b \in B$  we have*

- (a)  $nm - mn = [n, m]_A + [n, m]_B$ , where  $[n, m]_A \in A$  and  $[n, m]_B \in B$  satisfy  $\phi([n, m]_A, b) = \psi(m, b.n)$  and  $\phi(a, [n, m]_B) = -\psi(a.m, n)$ ;
- (b)  $\text{ad}(S^{-1}(b))m = \phi(m'_1, b)m'_2$ ;
- (c)  $\text{ad}(S^{-1}(a))n = n'_1\phi(a, n'_2)$ ;
- (d)  $\varphi(\text{ad}(a)m, n) = \varphi(m, \text{ad}(S^{-1}(a)).n)$ .

*Proof.* The multiplication rule tells us that within the double

$$\begin{aligned} nm &= \varphi(m_1, S(n_1))m_2n_2\varphi(m_3, n_3) \\ &= \varphi(m, S(n'_1))n''_2\varphi(1, n''_3) + \varphi(m'_1, 1)m'_2n'_1\varphi(1, n'_2) \\ &\quad + \varphi(m''_0, 1)m'_1\varphi(m'_2, n). \end{aligned}$$

Using the standard Hopf algebra identities, these terms reduce to

$$\varphi(m, S(n'_1))n'_2 + mn + m'_1\varphi(m'_2, n),$$

so that

$$\begin{aligned} nm - mn &= \varphi(m, S(n'_1))n'_2 + m'_1\varphi(m'_2, n) \\ &= \psi(m, S(n'_1))n'_2 + \psi(m'_2, n)m'_1. \end{aligned}$$

This is clearly in  $A + B$  and its  $A$  component satisfies  $\phi([n, m]_A, b) = \psi(m, b.n)$ . The  $B$  component follows similarly. (These are directly comparable with the Witt formulae for the Lie bracket.)

For elements  $b \in B$ ,  $\phi(m'_1, S(b))m'_2b$  is the only nonzero element of  $bm$ , so that

$$\begin{aligned} \text{ad}(b)m &= \phi(m'_1, S(b_3))m'_2b_2bS(b_1) \\ &= \phi(m'_1, S(b_2))m'_2e(b_1) = \phi(m'_1, S(b))m'_2. \end{aligned}$$

This proves (b) and also means that  $\text{ad}(S^{-1}(b))$  is the same as the dual map  $b'$  introduced earlier. The third assertion is proved similarly, and the final part follows since

$$\varphi(\text{ad}(a)m, n) = \varphi(am, n) = \varphi(m, \delta_a(n)) = \varphi(m, \text{ad}(S^{-1}(a)).n).$$

■

## 5. THE RADICAL OF THE PAIRING

In ordinary quantum groups the radicals of  $\varphi$  in  $\mathcal{A}$  and  $\mathcal{B}$  are generated by the Serre relations. We cannot expect to obtain any such precise description in the general situation, but we can give some useful criteria. To be precise we define the radical  $\mathcal{R}_{\mathcal{A}}$  in  $\mathcal{A}$  to be

$$\mathcal{R}_{\mathcal{A}} = \{ \alpha \in \mathcal{A}: \varphi(\alpha, \eta) = 0, \eta \in \mathcal{B} \},$$

and similarly for  $\mathcal{R}_{\mathcal{B}}$  in  $\mathcal{B}$ .

**THEOREM 5.1.** *The radical  $\mathcal{R}_{\mathcal{A}}$  is an ideal and a coideal in  $\mathcal{A}$ . Elements of degree greater than 1 for which  $\Delta\alpha \in \alpha \otimes \mathcal{A} + \mathcal{A} \otimes \alpha$  are in the radical.*

*Proof.* If  $\alpha \in \mathcal{R}_{\mathcal{A}}$  then  $\varphi(a\alpha, \eta) = \varphi(a, \eta_1)\varphi(\alpha, \eta_2) = 0$ , so that  $a\alpha \in \mathcal{R}_{\mathcal{A}}$ , and similarly  $\alpha a \in \mathcal{R}_{\mathcal{A}}$ . We also have

$$\varphi^2(\Delta\alpha, \eta \otimes \tilde{\eta}) = \varphi(\alpha, \eta\tilde{\eta}) = 0,$$

from which it follows that  $\mathcal{R}_{\mathcal{A}}$  is a coideal. This means that elements of the radical do satisfy

$$\Delta\alpha \in \mathcal{R}_{\mathcal{A}} \otimes \mathcal{A} + \mathcal{A} \otimes \mathcal{R}_{\mathcal{A}}.$$

Suppose now that if  $\Delta\alpha = a_1 \otimes \alpha + \alpha \otimes a_2$ , so that

$$\varphi(\alpha, \eta\tilde{\eta}) = \varphi^2(\Delta\alpha, \eta \otimes \tilde{\eta}) = \varphi(a_1, \eta)\varphi(\alpha, \tilde{\eta}) + \varphi(\alpha, \eta)\varphi(a_2, \tilde{\eta}).$$

Elements of  $\mathcal{A}$  can only have nonvanishing pairings with elements of  $\mathcal{B}$  having the same degree, so that it is sufficient to check the pairing when the sum of the degrees of  $\eta$  and  $\tilde{\eta}$  is the degree of  $\alpha$ . Since that degree is greater than 1 we may choose both  $\eta$  and  $\tilde{\eta}$  having positive degree. Then both pairings with  $\alpha$  vanish, so that  $\alpha$  is in the radical. ■

It is useful to note that, when dealing with quantum doubles constructed from modules, the relations for  $A$  and  $B$  may be assumed already to be known as part of the structure of the original quantum double. Similarly, the relations linking  $A$  and  $M$  (or  $B$  and  $N$ ) in which the module elements appear linearly are already encoded in the module structure, so that it is only the nonlinear relations on  $M$  (and  $N$ ) which are new.

## 6. THE STANDARD QUANTUM ENVELOPING ALGEBRAS

In this section and in later examples unexplained notation will generally follow the conventions of [8] or of [2].

Let  $C$  denote a commutative Hopf algebra which is generated by grouplike elements, and on which there is a nondegenerate symmetric bilinear form  $\phi$  with the properties assumed above when we identify both  $A$  and  $B$  with  $C$ . Let  $M$  and  $N$  be semisimple  $C$ -modules and  $\psi$  a  $C$ -invariant pairing between them, satisfying our conditions. Being semisimple,  $N$  has a basis of  $C$ -eigenvectors  $F_j$ , such that for some linear functional  $\lambda_j \in C^*$  we have  $b.F_j = \lambda_j(b)F_j$  for all  $b \in C$ . Since matrix elements are always expressible through  $\phi$ , there must exist an element  $L_j \in C$  such that  $\phi(L_j, b) = \lambda_j(b)$  for all  $b \in C$ , and giving  $b.F_j = \phi(L_j, b)F_j$ . Clearly we must have

$$\phi(L_j, a)\phi(L_j, b) = \phi(L_j, ab) = \phi^2(\Delta(L_j), a \otimes b),$$

so that, modulo the radicals,  $\Delta(L_j) = L_j \otimes L_j$  and the element  $L_j$  is grouplike. Taking a dual basis  $E_j$  for  $M$ , and using Corollary 2.2, we know that the comultiplication on  $M$  satisfies

$$\begin{aligned} \varphi(\Delta E_j, b \otimes F_k) &= \psi(E_j, b.F_k) = \varphi(L_k, b)\psi(E_j, F_k) \\ &= \varphi^2(L_k \otimes E_j, b \otimes F_k), \end{aligned}$$

and since this vanishes unless  $j = k$ , we conclude that  $\Delta(E_j) = E_j \otimes 1 + L_j \otimes E_j$ . By the  $C$ -invariance of  $\psi$ , we also see that

$$\psi(a.E_j, F_k) = \psi(E_j, S^o(a).F_k) = \phi(L_k, S^o(a))\psi(E_j, F_k),$$

so that  $a.E_j = \phi(L_k, S^o(a)E_j) = \phi(S(L_k), a)E_j$ . By the assumed symmetry of  $\phi$ , this can also be written as  $a.E_j = \phi(a, S(L_k))E_j$ , from which we deduce that  $\Delta^o(F_k) = F_k \otimes 1 + S(L_k) \otimes F_k$ . Finally, in the quantum double, we have the commutation relation

$$\begin{aligned} F_k E_j - E_j F_k &= \psi(E_j, S(F'_{k1}))F'_{k2} + \psi(E'_{j2}, F_k)E'_{j1} \\ &= \psi(E_j, -F_k)S(L_k) + \psi(E_j, F_k)L_j \\ &= (L_j - S(L_j))\delta_{jk}. \end{aligned}$$

By writing  $L_j = \tilde{K}_j$ , we see that these agree with the usual formulae for the standard quantum groups apart from the normalisation of  $E_j$  and  $F_k$ . (We have followed Lusztig in writing  $\tilde{K}_j$  because in the case of non-simply laced groups it is these elements which appear, rather than the  $K_j$ . The normalisation can be dealt with by replacing  $F_k$  by  $q_k(S^2 - 1)F_k$ .) In other words any such example will share many of the features of the standard quantum group enveloping algebras. The most obvious difference, that usually  $C$  is assumed to be freely generated by the elements  $\tilde{K}_j$ , is largely

illusory. If the number of  $\tilde{K}_j$  is insufficient to generate  $C$  then the ideal  $C_0$  of elements of  $C$  which are  $\phi$ -orthogonal to all the  $\tilde{K}_j$  will lie in the centre of the quantum double, and the quotient by this still gives the usual quantum enveloping algebra. At the opposite extreme there may be relations between the  $\tilde{K}_j$ , but then the algebra obtained is a quotient of that without relations.

To make the link with the standard case still firmer, we note that, when  $C$  is freely generated by the  $L_j$  and their inverses, the form  $\phi$  is determined by the values  $q_{jk} = \phi(L_j, L_k)$ , since we already know that  $\phi$  is a bicharacter on grouplike elements. Its image will form a multiplicative subgroup of  $\mathbb{F}$ . If this is a lattice, we define  $q$  to be the generator, and then we must be able to write

$$\phi(L_j, L_k) = q^{-\alpha_{jk}}$$

for suitable integer  $\alpha_{jk}$ . If  $\alpha_{jj}$  is positive and even and  $\alpha_{jk} = 2\alpha_{jk}/\alpha_{jj}$  is nonpositive then we are back in the usual situation, otherwise we can have a sort of “ergodic” version of the usual quantum groups.

When one works over the real numbers with positive  $q$  and a symmetric Cartan matrix, then  $\phi$  defines an inner product on  $C = A = B$ . It is then natural to complete  $C$  to a Hilbert space, in which the Riesz theorem ensures that the matrix elements are represented in the required way.

It is also worth considering the adjoint action of the elements  $E_j$ .

**LEMMA 6.1.** *The adjoint action is given by  $\text{ad}(E_j)Z = E_jZ - (\text{ad}(L_j)Z)E_j$ , and satisfies the skew derivation property*

$$\text{ad}(E_j)(ZW) = (\text{ad}(E_j)Z)W + (\text{ad}(L_j)Z)(\text{ad}(E_j)W).$$

*Proof.* Writing  $\mu_L$  and  $\mu_R$  for left and right multiplication respectively, we have

$$\begin{aligned} \text{ad}(E_j)Z &= (\mu_L \otimes \mu_R)(\text{id} \otimes S)\Delta(E_j)Z \\ &= (\mu_L \otimes \mu_R)(\text{id} \otimes S)(E_j \otimes 1 + L_j \otimes E_j)Z \\ &= (\mu_L \otimes \mu_R)(E_j \otimes 1 - L_j \otimes S(L_j)E_j)Z \\ &= E_jZ - L_jZS(L_j)E_j. \end{aligned}$$

(We note, in particular,

$$\text{ad}(E_j)E_k = E_jE_k - q_{jk}E_kE_j,$$

which shows that  $\text{ad}(E_j)E_k \neq \text{ad}(E_k)E_j$ , and that  $\text{ad}(E_j)E_j \neq 0$ .) In the ordinary situation factoring out the radical of the inner product  $\varphi$  is equivalent to imposing the Serre relations, and on  $M$  these are just

$$\text{ad}(E_j)^{1-a_{jk}}E_k = 0,$$

while those for the  $N$  may be similarly interpreted. The skew derivation property is easily checked. ■

## 7. THE CONSTRUCTION OF KNOWN QUANTUM GROUPS

The procedure which we have given can be used to construct quantum groups directly, but often one is looking for an alternative description of a known quantum double, and then some of the conditions needed for the construction will automatically hold. In this section we shall give some conditions sufficient to ensure that the method described above can be used to reconstruct a given quantum double.

**THEOREM 7.1.** *Suppose that  $\mathcal{A}$  and  $\mathcal{B}$  define a quantum double, and that  $A$  and  $B$  are Hopf subalgebras of  $\mathcal{A}$  and  $\mathcal{B}$ , respectively, and that  $M \subset \mathcal{A}$  and  $N \subset \mathcal{B}$  are subspaces satisfying the following conditions:*

- (a) *with respect to the pairing of  $\mathcal{A}$  and  $\mathcal{B}$ ,  $M$  annihilates  $B$  and  $N$  annihilates  $A$ ;*
- (b)  *$\text{ad}(A)M \subseteq M$  and  $\text{ad}(B)N \subseteq N$ ;*
- (c) *for  $m \in M$ , we have  $\Delta(m) - m \otimes 1 \in A \otimes M$  and for  $n \in N$ ,  $\Delta(n) - 1 \otimes n \in N \otimes B$ ;*
- (d)  *$A$  and  $M$  generate  $\mathcal{A}$ , and  $B$  and  $N$  generate  $\mathcal{B}$ .*

*Then the double generated by  $\mathcal{A}$  and  $\mathcal{B}$  using  $\varphi$  is isomorphic to that constructed from  $A, B, M, N, \phi = \varphi_{A \times B}$ , and  $\psi = \varphi|_{M \times N}$ .*

*Proof.* We first note that the action of  $B$  on  $N$  can be transposed to a dual action on  $M$ , enabling us to regard  $M$  as a  $D$ -module, and similarly for  $N$ . Since  $\mathcal{A}$  and  $\mathcal{B}$  form a quantum double with a pairing  $\varphi$ , we may define  $\phi = \varphi|_{A \times B}$  and  $\psi = \varphi|_{M \times N}$ . From the comultiplication we may represent the matrix elements of the action as noted in the proof of Lemma 2.3. If the conditions of Theorem 1.1 are satisfied then by uniqueness we recover the double defined by  $\mathcal{A}$  and  $\mathcal{B}$  by applying the construction to  $A, B, M$ , and  $N$ , and we have already noted that not all the conditions are needed. Condition (d) is true by definition and the first

two conditions are covered by our third assumption, and we already know that the pairing has the skew Hopf property. The first assumption gives Theorem 1.1(c). ■

Bearing in mind that the usual description of a quantum group uses only generators, we expect that it should be sufficient to specify  $M$  and  $N$  by giving the vectors generating their cyclic submodules. The following result describes what happens when  $M$  and  $N$  are themselves cyclic.

**THEOREM 7.2.** *Let  $A$  and  $B$  be Hopf subalgebras of  $\mathcal{A}$  and  $\mathcal{B}$ , let  $\mu$  be an element of  $\mathcal{A}$  which is in the annihilator of  $B$ , and let  $\nu$  be an element of the annihilator of  $A$  in  $\mathcal{B}$ , which satisfy the following conditions.*

- (a)  $A$  and  $\mu$  generate  $\mathcal{A}$ , and  $\nu$  and  $B$  generate  $\mathcal{B}$ ;
- (b)  $\Delta(\mu) - \mu \otimes 1 \in A \otimes \text{ad}(A)\mu$  and  $\Delta(\nu) - 1 \otimes \nu \in \text{ad}(B)\nu \otimes B$ ;

*Then  $A$  and  $B$  together with  $M = \text{ad}(A)\mu$  and  $N = \text{ad}(B)\nu$  satisfy the conditions of the preceding theorem.*

*Proof.* It is easy to check that  $\varphi(\text{ad}(a)\mu, b) = \varphi(a_1, b_3) \cdot \varphi(\mu, b_2)\varphi(S(a_2), b_1)$ , from which it follows that  $\text{ad}(A)\mu$  annihilates  $B$ , and  $\text{ad}(B)\nu$  similarly annihilates  $A$ . The first condition of the previous theorem therefore holds. It is clear from the definitions that  $\text{ad}(A)M = \text{ad}(A)\mu = M$ , and similarly  $\text{ad}(B)\nu = N$ , so that the second condition of the last theorem holds, and also the first assumption above tells us that  $\mathcal{A}$  and  $\mathcal{B}$  are generated as required in the fourth condition. Finally, from Corollary 2.4 we have

$$\Delta(\text{ad}(a)\mu) - \text{ad}(a)\mu \otimes 1 = a_1 \mu'_1 S(a_3) \otimes \text{ad}(a_2)\mu'_2.$$

Writing  $\mu'_2 = \text{ad}(a')\mu$  leaves  $a_1 \mu'_1 S(a_3) \otimes \text{ad}(a_2 a')\mu$ , which is clearly in  $A \otimes \text{ad}(A)\mu$ , as required. Since  $\Delta(\text{ad}(b)\nu)$  may be treated similarly, this tells us that the comultiplications satisfy the third condition of the preceding theorem. ■

The assumption that  $\Delta(\mu) - \mu \otimes 1 \in A \otimes \text{ad}(A)\mu$  amounts to the requirement that  $\mu'_1 \in A$  and  $\mu'_2 \in \text{ad}(A)\mu$ . We recall from the last section that  $\text{ad}(S^{-1}(b))\mu = \varphi(\mu'_1, b)\mu'_2$ , which shows that the second requirement is equivalent to  $\text{ad}(S^{-1}(b))\mu \in \text{ad}(A)\mu$ . In some circumstances this is automatic, as when we are factoring out central elements of the form  $S^{-1}(b)a - 1$ .

**THEOREM 7.3.** *Let  $\mathcal{D}$  be a simple quantum enveloping algebra constructed as a quantum double from  $\mathcal{A} = \langle K_1^{\pm 1}, \dots, K_n^{\pm 1}, E_1, \dots, E_n \rangle$  and  $\mathcal{B} = \langle K_1^{\pm 1}, \dots, K_n^{\pm 1}, F_1, \dots, F_n \rangle$ . Then  $\mathcal{D}$  can also be constructed from the*

quantum double  $A = \langle K_1^{\pm 1}, \dots, K_n^{\pm 1}, E_1, \dots, E_{n-1} \rangle$ ,  $B = \langle K_1^{\pm 1}, \dots, K_n^{\pm 1}, F_1, \dots, F_{n-1} \rangle$ , and the two cyclic modules generated by  $\mu = E_n$  and  $\nu = F_n$ , using the above constructions.

*Proof.* We already know that  $\mathcal{A} = \langle K_1^{\pm 1}, \dots, K_n^{\pm 1}, E_1, \dots, E_n \rangle = \langle A, E_n \rangle$ , and similarly  $\mathcal{B} = \langle B, F_n \rangle$ , so that the first condition of Theorem 7.2 holds. We also have  $\Delta E_n = E_n \otimes 1 + \tilde{K}_n \otimes E_n$  and a similar expression for  $\Delta F_n$  which prove the second condition of the theorem, so that  $\mathcal{D}$  can also be constructed from  $A$ ,  $B$ ,  $\mu$ , and  $\nu$ . ■

Although this gives a fairly uniform approach to the construction of various simple quantum groups, the details will depend on the Cartan matrix and associated Serre relations. We have already noted in Section 5, that the only Serre relations other than those of  $A$  and  $B$  and those encoded in the module structure arise from the nonlinear relations in  $\mu = E_n$  and  $\nu = F_n$ . The linear relation  $\text{ad}(E_j)^{(1-a_{jn})}E_n = 0$  can be rewritten in terms of the module structure, as  $E_j^{1-a_{jn}}.E_n = 0$ . Whenever the entry  $a_{jn}$  in the Cartan matrix vanishes we have  $\text{ad}(E_j)E_n = E_j.E_n - E_n.E_j = -\text{ad}(E_n)E_j$ , so that this covers two sets of Serre relation. The nonlinear relations can also be handled quite easily, as the following example shows.

LEMMA 7.4. *Suppose that  $a_{jn} = -1$ . Then the Serre relation  $\text{ad}(E_n)^2 E_j = 0$  is equivalent to the relation*

$$qE_n(E_j.E_n) = (E_j.E_n)E_n.$$

*Proof.* The standard Serre relation  $\text{ad}(E_n)^2 E_j = 0$  can be expressed as

$$\begin{aligned} 0 &= E_n^2 E_j - (q + q^{-1})E_n E_j E_n + E_j E_n^2 \\ &= -qE_n(E_j E_n - q^{-1}E_n E_j) + (E_j E_n - q^{-1}E_n E_j)E_n \\ &= qE_n(\text{ad}(E_j)E_n) - (\text{ad}(E_j)E_n)E_n, \end{aligned}$$

from which the result follows. ■

Higher order Serre relations can be handled similarly. They can usually also be derived directly using the criterion in Theorem 5.1.

Theorem 7.3 includes a number of interesting examples to which we shall give more attention in the next few sections. Most obviously it covers  $U_q(\mathfrak{su}(n+1))$ , where the double generated by  $A$  and  $B$  is just  $U_q(\mathfrak{su}(n) \times \mathfrak{u}(1))$ , with the extra copy of  $\mathfrak{u}(1)$  generated by  $K_n$ . We shall often think of this as  $U_q(\mathfrak{u}(n))$ . As we shall show in more detail in the next section the module  $M$  is the natural  $n$ -dimensional module and  $N$  its dual.



## 8. QUANTUM UNITARY GROUPS

We have already noted that  $U_q(\mathfrak{su}(n+1))$  can be constructed by our methods using the quantum double  $D = U_q(\mathfrak{u}(n))$  with  $M$  the module generated by  $E_k$  and  $N$  its dual. In this case it is possible to give a very detailed description of the structure. It will be useful in the discussion which follows to define iteratively, for all  $k = 1, \dots, n$  and  $j = 2, \dots, k$ ,

$$e_k^k = 1, \quad e_k^{k-1} = (q - q^{-1})E_k, \quad e_k^{k-j} = \text{ad}(E_{k-j+1})e_k^{k-j+1}.$$

Except in the case of  $j = k - 1$ , the Serre relations give  $\text{ad}(E_j)E_k = 0$ , so that this definition covers all the nontrivial terms. We shall also use the notation

$$u_k^k = 1, \quad u_k^{k-j} = K_k K_{k-1} \cdots K_{k-j+1}.$$

The adjoint action of the  $K_i$  on these elements is easily seen to be

$$\text{ad}(K_i)e_k^{k-j} = \phi(K_i, u_k^{k-j})e_k^{k-j}.$$

In order to determine the full adjoint action of  $U_q(\mathfrak{u}(n))$  on these elements we use the following result.

**LEMMA 8.1.** *If  $i_1 < i_r - 1$  and  $i_r < k$  for all  $r = 2, \dots, s$ , then*

$$\text{ad}(E_{i_1}E_{i_2} \cdots E_{i_s})e_k^{k-1} = 0.$$

*For all values of  $r \neq k - j$  one has  $\text{ad}(E_r)e_k^{k-j} = 0$ .*

*Proof.* According to the Serre relations,  $E_{i_1}$  commutes with all the other  $E_{i_j}$ , so that we have

$$\text{ad}(E_{i_1}E_{i_2} \cdots E_{i_s})e_k^{k-1} = \text{ad}(E_{i_2}E_{i_3} \cdots E_{i_s}E_{i_1})e_k^{k-1},$$

and the first result follows since  $i_1 < i_r - 1 < k - 1$  forces  $\text{ad}(E_{i_1})E_k = 0$ . If  $r < k - j$  then  $\text{ad}(E_r)e_k^{k-j} = \text{ad}(E_rE_k \cdots E_{n-j})e_k^{k-j}$  vanishes by direct use of the lemma. If  $r > k - j$  then commuting  $E_r$  through the product as far as possible, we have

$$\begin{aligned} \text{ad}(E_r)e_k^{k-j} &= \text{ad}(E_rE_{k-j+1} \cdots E_{k-1})e_k^{k-1} \\ &= \text{ad}(E_{k-j+1} \cdots E_{r-2})\text{ad}(E_rE_{r-1}E_r \cdots E_{n-1})e_k^{k-1}. \end{aligned}$$

Now the Serre relations give

$$\begin{aligned} E_r E_{r-1} E_r E_{r+1} &= (q + q^{-1})^{-1} (E_r^2 E_{r-1} + E_{r-1} E_r^2) E_{r+1} \\ &= (q + q^{-1})^{-1} (E_r^2 E_{r-1} E_{r+1} + E_{r-1} ((q + q^{-1}) E_r E_{r+1} E_r \\ &\quad - E_{r+1} E_r^2)). \end{aligned}$$

Each of the terms therefore gives rise to a sequence of  $E_i$  in which the suffix jumps by two at either  $r$  or  $r - 1$ , and so the lemma tells us that the result vanishes. ■

It therefore follows that  $M$  is the module spanned by the vectors  $m_j = e_n^{j-1}$ , for  $j = 1, \dots, n$ , with the action

$$K_j.m_k = q^{ajk} m_k, \quad E_j.m_k = \delta_{j, k-1} m_{k-1},$$

and  $M$  is the restriction to  $A$  of the natural module of  $U_q(\mathfrak{sl}(n))$ .

Now consider the comultiplication. Starting with

$$\begin{aligned} \Delta e_k^{k-1} &= (q - q^{-1}) \Delta E_k = (q - q^{-1}) (E_k \otimes 1 + K_k \otimes E_k) \\ &= e_k^{k-1} \otimes 1 + u_k^{k-1} \otimes e_k^{k-1}, \end{aligned}$$

and using induction on  $j$  it is easy to show that

**THEOREM 8.2.** *With  $u_k^j$  and  $e_j^k$  defined as above we have*

$$\Delta(e_k^{k-j}) = \sum_{r=0}^j u_k^{k-r} e_{k-r}^{k-j} \otimes e_k^{k-r}.$$

Taking  $k = n$  and noting that  $u_k^{k-r} e_{k-r}^{k-j} \in A$  this confirms that the comultiplication has the required form for elements of  $M = \text{span}(m_1, \dots, m_n)$ . Using this it is easy to derive the above relations directly, since

$$\begin{aligned} \Delta(qm_1 m_2 - m_2 m_1) &= (qm_1 m_2 - m_2 m_1) \otimes 1 \\ &\quad + K_n^2 K_{n-1} \otimes (qm_1 m_2 - m_2 m_1), \end{aligned}$$

which shows that  $(qm_1 m_2 - m_2 m_1)$  is in the radical, by Theorem 5.1. This can also be shown directly from Lemma 7.4:

**THEOREM 8.3.** *The elements  $m_r = e_n^{n-r}$  satisfy the relations*

$$m_r m_s = q m_s m_r$$

for  $r > s$ .

*Proof.* It follows from Theorem 7.4 that  $qm_1m_2 = m_2m_1$ . We now work by induction, first assuming that we have the identity  $qm_1m_j = m_jm_1$  for  $j > 1$ . Applying  $\text{ad}(E_{j-1})$  to each side, and using the fact that  $\text{ad}(E_{j-1})m_1$  vanishes, we get  $qm_1m_{j-1} = m_{j-1}m_1$ . Similarly applying  $\text{ad}(E_{k-1})$  to  $qm_km_j = m_jm_k$  gives  $qm_{k-1}m_j = m_jm_{k-1}$ , enabling us to prove the general result. ■

We already know that the comultiplication is the dual of multiplication in  $B$ , with respect to the pairing. We shall now show how that enables us to reconstruct the pairing itself. We first set  $\text{ad}^o(n) = (\mu_L \otimes \mu_R)(\text{id} \otimes S^o)\Delta^o(n)$ , so that  $\text{ad}^o(F_j)y = F_jy - (\text{ad}(K_j^{-1})y)F_j$ , and then define

$$f_n^n = 1, \quad f_n^{n-1} = (q - q^{-1})F_n, \quad f_n^{n-r} = \text{ad}^o(F_{n-r+1})f_n^{n-r+1}.$$

We may now define  $N$  to be the span of  $f_n^0, \dots, f_n^{n-1}$ . It remains only to find the pairing between  $M$  and  $N$  in a more direct form than simply as the restriction of the pairing between  $\mathcal{A}$  and  $\mathcal{B}$ .

**THEOREM 8.4.** *Let  $e_n^{n-r}$  and  $f_n^{n-r}$  be defined as above. Then we have*

$$\varphi(e_n^{n-r}, f_n^{n-s}) = \delta_{rs}(1 - q^{-2})^{r+1}q^2.$$

*Proof.* This can be proved by a double induction on  $r$  and  $s$ . For  $s = 1$  the product vanishes unless  $r = 1$  when we have

$$\varphi(e_n^{n-r}, f_n^{n-s}) = (q - q^{-1})^2 \varphi(E_n, F_n) = (1 - q^{-2})^2 q^2,$$

in line with the assertion. We therefore turn our attention to the case of  $s > 1$ . From the definitions we have

$$\begin{aligned} \varphi(e_n^{n-r}, f_n^{n-s}) &= \varphi^2(\Delta(e_n^{n-r}), F_{n-s+1} \otimes f_n^{n-s+1} \\ &\quad - \text{ad}(K_{n-s+1})f_n^{n-s+1} \otimes F_{n-s+1}). \end{aligned}$$

Using our previous formulae we know that

$$\begin{aligned} &\varphi^2(\Delta(e_n^{n-r}), F_{n-s+1} \otimes f_n^{n-s+1}) \\ &= \sum \varphi(u_n^{n-j}e_n^{n-r}, F_{n-s+1})\varphi(e_n^{n-j}, f_n^{n-s+1}). \end{aligned}$$

The first factor in the summand clearly vanishes unless  $j = r - 1 = s - 1$ , so that, using the inductive hypothesis the sum reduces to

$$\begin{aligned} &\delta_{rs} \varphi((q - q^{-1})u_n^{n-r+1}E_{n-r+1}, F_{n-r+1})\varphi(e_n^{n-r+1}, f_n^{n-s+1}) \\ &= \delta_{rs}(q - q^{-1})(1 - q^{-2})^r q^2 \varphi(u_n^{n-r+1}E_{n-r+1}, F_{n-r+1}). \end{aligned}$$

Now we can conclude that

$$\varphi(u_n^{n-r+1}E_{n-r+1}, F_{n-r+1}) = \varphi(q^{-1}E_{n-r+1}, F_{n-r+1}) = q^{-1},$$

so that we get

$$\delta_{rs}(1 - q^{-2})^{r+1}q^2.$$

The second term is

$$\begin{aligned} & \varphi^2(\Delta(e_n^{n-r}), \text{ad}(K_{n-s+1})(f_n^{n-s+1}) \otimes F_{n-s+1}) \\ &= \sum \varphi(u_n^{n-j}e_{n-j}^{n-r}, \text{ad}(K_{n-s+1})(f_n^{n-s+1}))\varphi(e_n^{n-j}, F_{n-s+1}), \end{aligned}$$

and this vanishes because  $s > 1$ , completing the proof. ■

## 9. SOME QUANTUM ORTHOGONAL AND PSEUDO-ORTHOGONAL GROUPS

The quantum orthogonal algebras  $U_q(\mathfrak{so}(P+1))$  can also be constructed by the method of Theorem 7.3. In this case the quantum double  $D = U_q(\mathfrak{so}(P-1) \times \mathfrak{so}(2))$  (thus in Cartan's notation  $\mathfrak{h}_P$  is constructed from  $\mathfrak{h}_{P-1}$  and  $\mathfrak{d}_P$  from  $\mathfrak{d}_{P-1}$ ). The modules  $M$  and  $N$  are each  $(P-1)$ -dimensional.

This construction also applies to pseudo-orthogonal groups such as the quantum de Sitter group  $U_q(\mathfrak{b}_2) = U_q(\mathfrak{sp}(2))$ , of [7], where we take for  $A$  and  $B$  the Hopf algebras  $\langle K_1^{\pm 1}, K_2^{\pm 1}, E_1 \rangle$  and  $\langle K_1^{\pm 1}, K_2^{\pm 1}, F_1 \rangle$ , respectively, and for  $M$  the three-dimensional  $A$ -module with cyclic vector  $E_2$ , and for  $N$  the three-dimensional  $B$ -module with cyclic vector  $F_2$ . We may then contract this as in [7] to obtain the quantum Poincaré group. The construction is, however, considerably simplified by the fact that the contraction simply involves a rescaling by a factor of  $R$  within a subspace of  $M \oplus N$  together with the introduction of a new deformation parameter  $p = q^R$ , and a new  $\tilde{K}_2 = K_2^R$ . When  $R$  goes to  $\infty$  the contraction is obtained.

## 10. A QUANTUM EXCEPTIONAL GROUP

One of the most useful applications of the ordinary Witt procedure is to construct the exceptional simple Lie algebra  $e_8$  from the orthogonal algebra  $\mathfrak{so}(16)$  and one of its 128-dimensional spin representations  $M$  [10, 11]. (The Casimir element takes values 28 on the adjoint representation

and 30 on the spin representation, and since  $128/120 + 28/30 = 2$  the criterion of the Appendix for the Jacobi identity is satisfied.) The obvious analogue of the methods already exploited for unitary and orthogonal groups is to set  $A = \langle K_1^{\pm 1}, \dots, K_n^{\pm 1}, E_1, \dots, E_{n-1} \rangle$ ,  $B = \langle K_1^{\pm 1}, \dots, K_n^{\pm 1}, F_1, \dots, F_{n-1} \rangle$ , with  $M$  and  $N$  the modules generated by the  $E_n$  and  $F_n$ , respectively. However, although closely related to Witt's construction this is subtly different.

The simple roots of  $\mathfrak{e}_8$  are as follows (cf. [1]):

$$\begin{array}{cccc} (\varepsilon_1 - \varepsilon_2) & (\varepsilon_2 - \varepsilon_3) & (\varepsilon_3 - \varepsilon_4) & (\varepsilon_4 - \varepsilon_5) \\ (\varepsilon_5 - \varepsilon_6) & (\varepsilon_6 - \varepsilon_7) & (-\delta + \varepsilon_7 + \varepsilon_8), & \\ (\varepsilon_6 + \varepsilon_7) & & & \end{array}$$

where  $\delta = \frac{1}{2} \sum_j \varepsilon_j$ . These are positive in the sense that their inner products with the vector  $(6, 5, 4, 3, 2, 1, 0, 23)$  are positive integers; in fact all are 1. Unfortunately this is not the same positivity structure that we use for  $\mathfrak{so}(16)$  whose system of simple roots is

$$\begin{array}{cccccc} (\varepsilon_1 - \varepsilon_2) & (\varepsilon_2 - \varepsilon_3) & (\varepsilon_3 - \varepsilon_4) & (\varepsilon_4 - \varepsilon_5) & (\varepsilon_5 - \varepsilon_6) & \\ (\varepsilon_6 - \varepsilon_7) & (\varepsilon_7 - \varepsilon_8), & & & & \\ (\varepsilon_7 + \varepsilon_8) & & & & & \end{array}$$

so that the obvious double construction of  $\mathfrak{so}(16)$  is not compatible with that of  $\mathfrak{e}_8$ . However, this problem is easily remedied by dropping a couple of dimensions to  $\mathfrak{so}(14) \times \mathfrak{so}(2)$  instead of  $\mathfrak{so}(16)$ . In fact  $\mathfrak{so}(14) \times \mathfrak{so}(2)$  is precisely the undeformed version of the double generated by the above choice of  $A$  and  $B$ . This has the following simple roots with respect to  $\mathfrak{so}(14)$ :

$$\begin{array}{cccccc} (\varepsilon_1 - \varepsilon_2) & (\varepsilon_2 - \varepsilon_3) & (\varepsilon_3 - \varepsilon_4) & (\varepsilon_4 - \varepsilon_5) & (\varepsilon_5 - \varepsilon_6) & (\varepsilon_6 - \varepsilon_7), \\ & & & & (\varepsilon_6 + \varepsilon_7) & \end{array}$$

which is obviously a part of the corresponding Dynkin diagram of  $\mathfrak{e}_8$ . Moreover the restriction of the  $\mathfrak{so}(16)$  spin module to  $\mathfrak{so}(14)$  naturally splits into two pieces,  $M = \mathcal{S}^+ \oplus \mathcal{S}^-$ , where we have chosen  $\mathcal{S}^+$  to be that which contains the root  $\delta - \varepsilon_1 - \varepsilon_2$ . By taking  $A$  and  $B$  to be the natural quantum analogues of the positive and negative root parts of the quantum double for  $\mathfrak{so}(14)$  and  $M = \mathcal{S}^+$  and  $N = \mathcal{S}^-$ , with  $\mathfrak{so}(2)$  acting with weight  $\varepsilon_7 + \varepsilon_8$  on  $M$  and  $-\varepsilon_7 - \varepsilon_8$  on  $N$ , we may perform the construction of the earlier sections. To understand the way in which this is related to Witt's construction it is useful to note that in the undeformed case one has the identities

$$S^2(\mathcal{S}^\pm) = \lambda^7 \oplus \lambda^3, \quad \Lambda^2(\mathcal{S}^\pm) = \lambda^5 \oplus \lambda^1,$$

where  $\lambda^j$  denotes the  $j$ th exterior power of the natural representation of  $\mathfrak{so}(14)$  on  $\mathbf{R}^{14}$ . When the algebras  $\mathcal{M}$  and  $\mathcal{N}$  are constructed each contains a copy of  $\lambda^1$ . However,  $\mathfrak{so}(16)$  can be constructed from  $\mathfrak{so}(14) \oplus \mathfrak{so}(2)$  by using just these modules, so that this repairs the deficit caused by using  $\mathfrak{so}(14)$  instead of  $\mathfrak{so}(16)$ . (Another way of thinking about this is to note that the Serre relation no longer takes quite so simple a form in this case, and allows room to define some extra elements corresponding to roots in  $\mathfrak{so}(16)$ .) In the undeformed case the two lowest components of the decomposition

$$\mathcal{S}^+ \otimes \mathcal{S}^- = \lambda^6 \oplus \lambda^4 \oplus \lambda^2 \oplus \lambda^0$$

correspond to the fact that commutators of elements in the module lie in  $\mathfrak{so}(14) \oplus \mathfrak{so}(2) \cong \lambda^2 \oplus \lambda^0$ .

## 11. SUPERALGEBRAS

The Witt construction can be generalized non-semisimple  $\mathfrak{h}$  containing an  $\mathfrak{h}$ -invariant subspace  $V$ , with an  $\mathfrak{h}$ -invariant nonsingular symmetric bilinear form  $g$ , and  $T$  is a tensor operator; that is, for each  $v \in V$ ,  $T(v)$  is a linear transformation on  $M$  and, for all  $X \in \mathfrak{h}$ ,

$$[X, T(v)] = T(X.v).$$

If  $T(v)$  is skew symmetric with respect to  $\psi$ , then

$$g(v, [m, n]) = \psi(m, T(v)n)$$

defines a Lie bracket provided that the Jacobi identity holds. When  $V = \mathfrak{h}$ ,  $g = \phi$  and  $T(v)$  is defined by the action of  $\mathfrak{h}$  on  $M$ ; this reduces to the previous case.

This version has a useful extension to superalgebras. For example, given a Lie algebra  $\mathfrak{h}$ ,  $V \subseteq \mathfrak{h}$ , and form  $g$ , and an  $\mathfrak{h}$ -module  $M$  and tensor operator  $T$ , as above, with an *antisymmetric*  $\mathfrak{h}$ -invariant bilinear form  $\psi$  on  $M$ , then we can construct a superalgebra  $\mathfrak{h} \oplus M$  whose odd part is  $M$ , by taking

$$g(v, [m, n]) = \psi(m, T(v)n).$$

Since we are assuming that  $T(v)$  is skew symmetric, the bracket  $[m, n]$  is now symmetric. One simple example arises when  $\mathfrak{h}$  is the Lie algebra of the Poincaré group,  $V$  the translation subalgebra, and  $g$  the Minkowski inner product. If we take for  $M$  the Dirac spinors, and  $T(v) = \gamma(v)$  the Clifford algebra element corresponding to  $v$ , then there is a natural

Lorentz-invariant symplectic form  $\psi$  on  $M$ . The Lie superalgebra  $\mathfrak{h} \oplus M$  is the super-Poincaré algebra. (In [4] this construction is used, but without reference to its relation to the usual Witt construction.)

For quantum groups there is a very simple modification of our double construction which produces superalgebras: all that one needs to do is to use a  $\mathbf{Z}_2$ -graded multiplication, where the odd and even parts are decided on the basis of the  $\mathbf{Z}$  grading introduced earlier. Consider, for example, the case when the action of  $A = B$ , generated by the single grouplike element  $K$ , on the one-dimensional modules  $M$  spanned by  $v_+$  and  $N$  spanned by the dual basis vector  $v_-$ , is given by

$$K.v_{\pm} = q^{\pm 1}v_{\pm}.$$

As for the algebra  $U_q(\mathfrak{su}(2))$  one has the comultiplication

$$\Delta(v_+) = v_+ \otimes 1 + K \otimes v_+, \quad \Delta(v_-) = 1 \otimes v_- + v_- \otimes K^{-1}.$$

It is useful to introduce the elements  $E_{\pm} = (1 + q^{\pm 1})v_{\pm}^2$ , for which  $K.E_{\pm} = q^{\pm 2}E_{\pm}$  and (recalling that  $v_{\pm}$  has degree 1 in the graded product)

$$\begin{aligned} \Delta(E_+) &= (1 + q)\Delta(v_+)^2 \\ &= (1 + q)(v_+^2 \otimes 1 - Kv_+ \otimes v_+ + v_+K \otimes v_+ + K^2 \otimes v_+^2) \\ &= E_+ \otimes 1 + (1 - q^2)v_+K \otimes v_+ + K^2 \otimes E_+, \end{aligned}$$

with a similar formula for  $\Delta(E_-)$ . Clearly the even degree elements  $E_{\pm}$  commute with the corresponding  $v_{\pm}$ . The grading shows up again when one calculates the product of  $v_-$  and  $v_+$ , where one has, instead of the earlier formula

$$v_-v_+ = -(v_+v_- + \psi(v_+, S(v_-))K^{-1} + \psi(v_+, v_-)K),$$

so that

$$v_-v_+ + v_+v_- = \psi(v_+, v_-)K^{-1} - \psi(v_+, v_-)K = K^{-1} - K.$$

Apart from some obvious notation changes this is the quantum algebra  $U_q(\mathfrak{osp}(1|2))$  of [7].

We shall also briefly mention an example of a multiparameter deformation of  $\mathfrak{sl}(2|1)$  due to Zhang [12], where our approach breaks down. (This is hardly surprising, since it is not known whether this example admits a Hopf structure.) We take for  $A = B$  the commutative algebra generated by two grouplike elements  $K_1$  and  $K_2$ , with the pairing  $\phi(K_i, K_j) = q_i^{\delta_{ij}-1}$ . Take for  $M$  the two-dimensional module with eigenvector basis

$E_1, E_2$  such that  $K_j.E_k = \phi(K_j, K_k)E_k$ , and for  $N$  its dual with dual basis  $F_1, F_2$ , so that  $K_j.F_k = \phi(K_j, K_k)^{-1}F_k$ . We now try to form a  $\mathbf{Z}_2$ -graded superalgebra, from these. It is easily checked that our definition of the comultiplication gives  $\Delta^o F_k = F_k \otimes 1 + K_k \otimes F_k$ . This means, on taking account of the grading, that

$$\Delta^o(F_j^2) = F_j^2 \otimes 1 + (F_j K_j - K_j F_j) \otimes F_j + K_j^2 \otimes F_j^2,$$

and, since  $K_j.F_j = F_j$ , we have  $\Delta^o(F_j^2) = F_j^2 \otimes 1 + K_j^2 \otimes F_j^2$ . By Theorem 5.1 we see that  $F_j^2$  is in the radical of the pairing, in other words we have a relation  $F_j^2 = 0$ . Unfortunately we cannot generally use the same argument for  $\Delta E_j$ , because  $\phi(K_j, K_k)$  can only be expressed in the form  $\phi(K_k, L_j)$ , if the two parameters  $q_1$  and  $q_2$  are related. This means that our method breaks down. Nonetheless, it is interesting that when  $q_1 = q_2$ , the vanishing of  $F_j^2$  and in that case of  $E_k^2$  also, can be interpreted in terms of the radical, making them play the role of Serre relations.

## 12. APPENDIX

The Witt construction for ordinary algebras always yields a bracket, but this may not satisfy the Jacobi identity. Witt gave two conditions sufficient to ensure that it does, but one of these served only to normalise the Killing form and is unnecessary with the coordinate free definitions we have given. They may be replaced with the single condition stated in the introduction. This is probably well known, but, since I have been unable to find it elsewhere, I include it here.

**THEOREM 12.1.** *Let  $\mathfrak{h}$  be a semisimple Lie algebra with Killing form  $\phi$ , and let  $M$  be an  $\mathfrak{h}$ -module, equipped with a symmetric  $\mathfrak{h}$ -invariant bilinear form  $\psi$ . Define the bracket of  $X \oplus m, Y \oplus n \in \mathfrak{h} \oplus M$  by*

$$[X \oplus m, Y \oplus n] = ([X, Y] + [m, n]) \oplus (X.n - Y.m),$$

where  $\phi(X, [m, n]) = \psi(m, X.n)$ . This satisfies the Jacobi identity and so defines a Lie bracket on  $\mathfrak{h} \oplus M$  if  $\dim(M) < 4$ . If the quadratic Casimir element of  $\mathfrak{h}$  has a single eigenvalue  $C_M$  on  $M$  and takes the value  $C_{\mathfrak{h}}$  in the adjoint representation, then the Jacobi identity holds precisely when  $M$  is trivial or

$$\frac{\dim(M)}{\dim(\mathfrak{h})} + \frac{C_{\mathfrak{h}}}{C_M} = 2.$$

*Proof.* As already noted in the introduction, it is easy to check that the Jacobi identity automatically holds for any triple of elements of which at



least one is in  $\mathfrak{h}$ , so that we need only look at triples of elements  $m, n$ , and  $p$  in  $M$ . By choosing an orthonormal basis  $\{E_j\}$  for  $\mathfrak{h}$  we may explicitly write

$$[m, n] = \sum_j \phi(E_j, [m, n])E_j = \sum_j \psi(m, E_j \cdot n)E_j,$$

so that the Jacobi identity becomes

$$\sum_j \psi(m, E_j \cdot n)E_j \cdot p + \sum_j \psi(n, E_j \cdot p)E_j \cdot m + \sum_j \psi(p, E_j \cdot m)E_j \cdot n = 0.$$

We therefore want

$$\begin{aligned} & \sum_j \psi(m, E_j \cdot n)\psi(q, E_j \cdot p) + \sum_j \psi(n, E_j \cdot p)\psi(q, E_j \cdot m) \\ & + \sum_j \psi(p, E_j \cdot m)\psi(q, E_j \cdot n) \end{aligned}$$

(which we shall denote by  $b(q, m, n, p)$ ) to vanish for all  $q, p, m$ , and  $n$  in  $M$ . Since it is clearly totally antisymmetric in its arguments,  $b$  must vanish if  $\dim(M) < 4$ . Witt showed [11, 10] that the sum of terms

$$\sum_k (\psi(p, E_k \cdot q)\psi(n, E_k \cdot m))b(q, p, m, n) = \frac{1}{3}b(q, p, m, n)^2,$$

as  $p, q, m$ , and  $n$  run over an orthonormal basis is just

$$\sum_{jk} \operatorname{tr}(E_k E_j)^2 - 2 \sum_{jk} \operatorname{tr}((E_k E_j)^2).$$

If this vanishes then so do all the components of  $b$ , and conversely. This leads to Witt's condition

$$\sum_{jk} \operatorname{tr}(E_k E_j)^2 = 2 \sum_{jk} \operatorname{tr}((E_k E_j)^2)$$

for the Jacobi identity to hold. We now note that

$$\begin{aligned} \operatorname{tr}((E_k E_j)^2) &= \operatorname{tr}(E_k E_j E_j E_k) + \operatorname{tr}(E_k E_j [E_k, E_j]) \\ &= \operatorname{tr}(E_j E_j E_k E_k) + \frac{1}{2} \operatorname{tr}([E_k, E_j]^2). \end{aligned}$$

Writing  $C$  for the Casimir invariant  $\sum_j E_j^2$ , we have

$$\begin{aligned} \sum_{jk} \operatorname{tr}([E_k, E_j]^2) &= - \sum_j \operatorname{tr}\left(E_j \sum_k [E_k, [E_k, E_j]]\right) \\ &= - \sum_j \operatorname{tr}(E_j \operatorname{ad}(C) E_j). \end{aligned}$$

Since the adjoint representation of a simple group is irreducible,  $\operatorname{ad}(C)$  is just multiplication by a constant,  $C_{\mathfrak{h}}$ , and we obtain

$$\sum_{jk} \operatorname{tr}([E_k, E_j]^2) = -C_{\mathfrak{h}} \sum_j \operatorname{tr}(E_j E_j) = -C_{\mathfrak{h}} \operatorname{tr}(C),$$

which gives

$$\operatorname{tr}((E_k E_j)^2) = \operatorname{tr}(C^2) - \frac{1}{2} C_{\mathfrak{h}} \operatorname{tr}(C).$$

Since  $\operatorname{tr}(XY)$  is  $\operatorname{ad}(h)$ -invariant it is a multiple of the Killing form  $\phi(X, Y)$ , which means that  $\operatorname{tr}(E_j E_k)$  vanishes unless  $j = k$ , and in that case its value must be  $\operatorname{tr}(C)/\dim(\mathfrak{h})$ , so that we have

$$\sum_{jk} \operatorname{tr}(E_k E_j)^2 = \dim(\mathfrak{h})(\operatorname{tr}(C)/\dim(\mathfrak{h}))^2 = \operatorname{tr}(C)^2/\dim(\mathfrak{h}).$$

Thus Witt's criterion reduces to the requirement that

$$\operatorname{tr}(C)^2/\dim(\mathfrak{h}) = 2 \operatorname{tr}(C^2) - C_{\mathfrak{h}} \operatorname{tr}(C).$$

When there is only a single irreducible, or all components give the same value  $C_M$ , we have  $\operatorname{tr}(C) = C_M \dim(M)$ , and the condition becomes

$$C_M^2 \dim(M)^2/\dim(\mathfrak{h}) = 2C_M^2 \dim(M) - C_{\mathfrak{h}} C_M \dim(M).$$

This condition is automatically fulfilled if  $C_M = 0$ , in which case  $M$  is trivial, and otherwise we may simplify it to get

$$\frac{\dim(M)}{\dim(\mathfrak{h})} + \frac{C_{\mathfrak{h}}}{C_M} = 2,$$

as asserted. ■

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