The Rödl Nibble

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1   Introduction

Given a set E of n elements and positive integers k, l, with $l \leq k \leq n$, the *covering number* $M(n, k, l)$ is the minimal size of a set $\mathcal{K}$ of k-tuples (subsets of E containing k elements each) such that every l-tuple of E is contained in at least one k-tuple of $\mathcal{K}$. Since each k-tuple contains exactly $\binom{k}{l}$ l-tuples and there is a total of $\binom{n}{l}$ l-tuples in E that need to be covered, then $M(n, k, l)$ must be at least $\binom{n}{l} / \binom{k}{l}$ with equality when each l-tuple is contained in exactly one k-tuple. In this case, the covering is called a (n, k, l) *tactical configuration* (see Figure 1 for an example).

The existence of tactical configurations is a central question in Combinatorial Analysis, and one in which probabilistic methods are not often encountered [5, 1]. In this essay, we will be concerned with tactical configurations in an asymptotic sense through the following conjecture. In 1963, Erdös and Hanani conjectured in [3] that for fixed k, l,

$$\lim_{n \to \infty} \frac{M(n, k, l)}{\binom{n}{l} / \binom{k}{l}} = 1. \tag{1.0.1}$$

Roughly, this states that one can get asymptotically close to a tactical configuration. Erdös and Hanani showed in [3] that (1.0.1) holds for $l = 2$ and some cases of $l = 3$. It is only about twenty years later, in 1985, that Rödl proved the full conjecture using a probabilistic argument.[1] During the same year, Frankl and Rödl realised that the probabilistic method Rödl used in [7], which became later known as the 'Rödl Nibble', could in fact be applied to a more general setting dealing with the covering of hypergraphs. In this essay, we explore a proof of the Erdös-Hanani conjecture formulated as a hypergraph covering problem.

2   Preliminaries

We begin with some notation and definitions which will be used repeatedly throughout the essay. For any positive integer n, $[n]$ denotes the set $\{1, 2, \ldots, n\}$. A *hypergraph H* is an ordered pair of sets (V, E) where V is a finite set and E a collection of subsets of V (i.e. a subset of the power set of V). An element of V is a *vertex* and an element of E is an *edge*. An edge e is said to be *incident* with a vertex x if it contains it. A hypergraph is said to be *r-uniform* if each of its edges contains exactly r vertices. We will sometimes use the notation $V(H)$ and $E(H)$ when referring to the vertex set and edge set, respectively, of a given hypergraph H. For any two vertices $x, y \in V(H)$, the degree of x in H, $d(x)$, denotes the number of edges containing x and we denote by $d(x, y)$ the number of edges in H containing both x and y. A hypergraph is said to be *regular* if all its vertices have the same degree. We will assume in this essay that $d(x) > 0$ for all $x \in V$ (i.e. that there are no isolated vertices). A *covering* or *cover* of a hypergraph H is a set of edges in $E(H)$ whose union contains all the vertices in $V(H)$.[2] For $W \subseteq V$, $H - W$ is the hypergraph obtained by deleting

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[1] In fact, Rödl's argument can be formulated in terms of covering number and packing number, where the packing number $m(n, k, l)$ is defined as the maximal size of a set $\mathcal{K}$ of k-tuples of $[n]$ such that every l-tuple is in at most one k-tuple of $\mathcal{K}$. However, it can be shown that $\lim_{n \to \infty} \frac{M(n,k,l)}{\binom{n}{l}/\binom{k}{l}} = 1 \Leftrightarrow \lim_{n \to \infty} \frac{m(n,k,l)}{\binom{n}{l}/\binom{k}{l}} = 1$, and thus it is sufficient to prove the conjecture for one [3, 6].

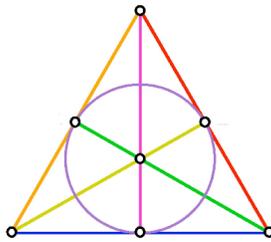[2] $d(x) > 0$ for all x in V ensures that a covering exists.

Figure 1: The *Fano plane*, a $(7, 3, 2)$ tactical configuration

the vertices in W and all edges containing them. The resulting hypergraph is called the *induced hypergraph* on $V - W$, the set of vertices in V which are not in W.

We will also be using some $o - O$ notation. For two functions $f(n)$ and $g(n)$, where n is a positive integer and $g(n) > 0$, we write $f = O(g)$ if there exists $C \in \mathbb{R}^+$ such that $|f(n)| \leq Cg(n)$ for all n. We write $f = o(g)$ if $\lim_{n \to \infty} f(n)/g(n) = 0$ and $f \sim g$ when $\lim_{n \to \infty} f(n)/g(n) = 1$ (i.e. if $f(n) = (1 + o(1))g(n)$).

With these definitions and notations in mind, we can begin investigating the Erdös-Hanani conjecture as a hypergraph covering problem.

3   Erdös-Hanani Conjecture as a Hypergraph Covering Problem

Let H be an r-uniform hypergraph. As defined earlier, a covering of a hypergraph H is a set of edges in $E(H)$ whose union gives $V(H)$. This means that each vertex of H must be contained in at least one edge of the covering. As argued in the introduction for the covering number, since there is a total of $n = |V(H)|$ vertices and each edge contains exactly r vertices, then the hypergraph covering must contain at least n/r edges with equality if each vertex is covered exactly once. The following theorem is due to Frankl and Rödl (although this corresponds to a simpler formulation given by Pippenger and Spencer in [6]), who were the first to generalize Rödl's method in [7] to a hypergraph setting [4].

Theorem 3.1. *(**Frankl-Rödl**) For every integer $r \geq 2$ and reals $K \geq 1$ and $a > 0$, there exist $\gamma = \gamma(r, K, a)$ and $m = m(r, K, a)$ such that for every $n \geq D \geq m$ the following holds.*

Every r-uniform hypergraph $H = (V, E)$ on a set V of n vertices in which all vertices have positive degrees and which satisfies the following conditions:

  (i) For all but at most γn vertices $x \in V$, $d(x) = (1 \pm \gamma)D$,

  (ii) For all $x \in V, d(x) < KD$,

  (iii) For any two distinct $x, y \in V, d(x, y) < \gamma D$

contains a cover of at most $(1 + a)\frac{n}{r}$ edges,

where $\pm\gamma$ is a real number in the interval $(-\gamma, \gamma)$. Rephrasing the result using some asymptotic notation, the theorem roughly states that if, for some D, an r-uniform hypergraph H satisfies (i) $d(x) = (1 \pm o(1))D$ for almost all x, (ii) $d(x) < KD$ for all x and (iii) $d(x, y) = o(D)$ for all x, y, then there exists a cover of H with $(1 + o(1))\frac{n}{r}$ edges. Before commenting on the general idea of the proof, we show how this theorem implies the Erdös-Hanani conjecture.

Theorem 3.2. *(**Rödl**) For k, l, n fixed positive integers with $k \leq l \leq n$*

$$M(n, k, l) \leq (1 + o(1))\frac{\binom{n}{l}}{\binom{k}{l}} \tag{3.0.2}$$

where $o(1)$ tends to zero as n tends to infinity.

Proof. Let $r = \binom{k}{l}$ and consider the r-uniform hypergraph $H = (V, E)$ with vertices all the l-tuples of $[n]$ (so that $|V| = \binom{n}{l}$) and each edge the set of l-tuples found in one of the k-tuples of $[n]$ (so that $|e| = \binom{k}{l}$ and $|E| = \binom{n}{k}$)[3]. The degree of each vertex is the number of k-tuples it can be a part of, and since each vertex is an l-tuple we have $D := d(x) = \binom{n-l}{k-l}$. For two distinct vertices x and y, $d(x, y)$ is the number of k-tuples both vertices can be a part of (which is largest for maximal overlap). Since they can at most have $(l-1)$ elements in common (they need to be distinct), $d(x, y)$ is at most the number of k-tuples $(l-1) + 2$ elements can belong to, i.e. $d(x, y) \leq \binom{n-l-1}{k-l-l} = o(D)$. It follows by Theorem 3.1 that H has a cover of size at most $(1 + o(1))\frac{\binom{n}{l}}{\binom{k}{l}}$. Viewing the edges of H as the k-tuples they are constructed from, we get a set of k-tuples of size $(1 + o(1))\frac{\binom{n}{l}}{\binom{k}{l}}$ whose union contains all l-tuples of $[n]$, yielding the desired result. $\square$

Now that we have shown how the Erdös-Hanani conjecture follows directly from Theorem 3.1, we can explore the probabilistic proof technique behind the Frankl-Rödl theorem. Before going through the details, we give some intuition illustrating the main ideas of the proof. We start by randomly selecting, for small $\epsilon > 0$, a set of roughly $\epsilon n/r$ edges from an r-uniform hypergraph H satisfying conditions (i), (ii) and (iii) of Theorem 3.1. This set of edges contains a maximum of ϵn distinct vertices, since each edge contains r distinct vertices. We then show that with high probability, these edges will constitute a cover for at least $\epsilon n - O(\epsilon^2 n)$ vertices, covering only some $O(\epsilon^2 n)$ vertices more than once. By deleting the covered vertices, we can then repeat the same process on the hypergraph induced on the remaining vertices. The reason this works is because the induced hypergraph will also satisfy (i), (ii) and (iii) for new values of n, γ, K and D. We can therefore choose, in the same manner, a random set of edges which will cover roughly an ϵ-fraction of its vertices, and proceed this way until we are left with only a few (dependent on ϵ) vertices, which we can then cover one by one by taking an arbitrary edge containing it. As noted in [1], although this step is inefficient, it is tolerable for small ϵ. Putting all these partial covers together will then yield the desired covering of H.

In order to formalize this, we need to describe how the edges are selected and show that this random selection process will yield a partial cover of the desired size and an induced hypergraph on the

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[3]Note that the elements of an edge e are l-tuples and the elements of E are sets of l-tuples.

desired fraction of vertices satisfying properties (i), (ii) and (iii) at each iteration, for new values of n, γ, K and D. This is shown via a crucial lemma which requires repeated use of Chebyshev's Inequality.

4   Proof of the Frankl-Rödl Theorem

4.1   Chebyshev Inequality and the Second Moment Method

In the following section, we state some of the results we will use repeatedly throughout the proof of the lemma presented in the next section. A proof of these results can be found in [1].

Theorem 4.1. (**Chebyshev's Inequality**) *Let X be a random variable and $t \geq 0$. Then*

$$\mathbb{P}(|X - \mathbb{E}[X]| \geq t) \leq \frac{\mathrm{Var}[X]}{t^2}.$$

The use of Chebyshev's Inequality is called the *Second Moment Method*. The following corollary follows by applying Chebyshev's Inequality with $t = \epsilon \mathbb{E}[X_n]$ for $\epsilon > 0$.

Corollary 4.2. *Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of random variables with $\mathrm{Var}[X_n] = o(\mathbb{E}[X_n]^2)$. Then*

$$\lim_{n \to \infty} \mathbb{P}(X_n \sim \mathbb{E}[X_n]) = 1.$$

We define the *indicator function I_A* of an event A as the random variable which takes the value 1 when A holds and 0 when A does not hold. We will often deal with random variables X_n which are the sums of indicator functions I_i of events $A_i, i \in [k]$. Letting $X_n = \sum_{i=1}^{k} I_i$, where I_i is the indicator function of event A_i (which depends on n), it can be shown that

$$\mathbb{E}[X_n] = \sum_{i=1}^{k} \mathbb{P}(A_i) \qquad \text{and} \qquad \mathbb{E}[X_n^2] = \sum_{i=1}^{k}\sum_{j=1}^{k} \mathbb{P}(A_i \cap A_j). \tag{4.1.1}$$

Noting that $\mathrm{Var}[X_n] = \mathbb{E}[X_n^2] - (\mathbb{E}[X_n])^2$, this yields

$$
\begin{aligned}
\mathrm{Var}[X_n] &= \sum_{i=1}^{k}\sum_{j=1}^{k} \left(\mathbb{P}(A_i \cap A_j) - \mathbb{P}(A_i)\mathbb{P}(A_j) \right) \\
&= \sum_{i=1}^{k} \mathbb{P}(A_i) - (\mathbb{P}(A_i))^2 + \sum_{i=1}^{k}\sum_{j \sim i} \left(\mathbb{P}(A_i \cap A_j) - \mathbb{P}(A_i)\mathbb{P}(A_j) \right) \tag{4.1.2} \\
&\leq \mathbb{E}[X_n] + \sum_{i=1}^{k}\sum_{j \sim i} \mathbb{P}(A_i \cap A_j), \tag{4.1.3}
\end{aligned}
$$

where $j \sim i$ means that $i \neq j$ and A_i and A_j are dependent. The next corollary can be deduced by applying Chebyshev's Inequality with $t = \epsilon \mathbb{E}[X_n]$ as above and noting from the inequality in (4.1.3) that

$$\frac{\mathrm{Var}[X_n]}{\epsilon^2 \mathbb{E}[X_n]^2} \leq \frac{\mathbb{E}[X_n] + \Delta}{\epsilon^2 \mathbb{E}[X_n]^2},$$

where $\Delta = \sum_{i=1}^{k} \sum_{j \sim i} \mathbb{P}(A_i \cap A_j)$.

Corollary 4.3. *Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of random variables. If $\lim_{n \to \infty} \mathbb{E}[X_n] = \infty$ and $\Delta = \sum_{i=1}^{k} \sum_{j \sim i} \mathbb{P}(A_i \cap A_j) = o(\mathbb{E}[X_n]^2)$ then*

$$\lim_{n \to \infty} \mathbb{P}(X_n \sim \mathbb{E}[X_n]) = 1.$$

We can now state the lemma which constitutes the main step in the proof of Theorem 3.1, and whose proof will require the repeated use of corollary 4.3.

4.2   Crucial Lemma

Following the formulation in [1, 5], we state the crucial lemma from which Theorem 3.1 will follow.[4]

Lemma 4.4. *For every integer $r \geq 2$ and reals $K \geq 1$ and $\epsilon > 0$, and for every real $\delta' > 0$ there are $\delta = \delta(r, K, \epsilon, \delta')$ and $d_0 = d_0(r, K, \epsilon, \delta')$ such that for every $n \geq D \geq d_0$ the following holds.*

Every r-uniform hypergraph $H = (V, E)$ on a set V of n vertices which satisfies the following conditions:

 (i) *For all but at most δn vertices $x \in V$, $d(x) = (1 \pm \delta)D$,*

 (ii) *For all $x \in V$, $d(x) < KD$,*

 (iii) *For any two distinct $x, y \in V, d(x, y) < \delta D$*

contains a set E' of edges with the following properties:

 (iv) *$|E'| = \frac{\epsilon n}{r}(1 \pm \delta')$*

 (v) *The set $V' = V - \bigcup_{e \in E'} e$ is of cardinality $|V'| = ne^{-\epsilon}(1 \pm \delta')$,*

 (vi) *For all but at most $\delta'|V'|$ vertices $x \in V'$, the degree $d'(x)$ of x in H', the hypergraph induced by H on V', satisfies $d'(x) = De^{-\epsilon(r-1)}(1 \pm \delta')$.[5]*

Loosely speaking, the above lemma states that for an r-uniform hypergraph verifying conditions (i), (ii), (iii) of Theorem 3.1 for sufficiently small δ, there exists a random set of edges which will (iv) roughly have size $\frac{\epsilon n}{r}$, (v) roughly cover $n - ne^{-\epsilon} = n\epsilon - O(n\epsilon^2)$ vertices and (vi) the induced hypergraph on the set of remaining vertices will also be regular within tolerance $(1 \pm \delta')$. As described in [4], we can think of the δ's in the above lemma as the regularity tolerance: if we want the induced hypergraph on V', as defined above, to be regular within tolerance $1 \pm \delta'$ then it suffices

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[4]All the random variables in the proof depend on n but the subscript n will be suppressed for ease of writing.

[5]Although the notation is a little misleading, we emphasize that V' is the vertex set of H' but E' is not the edge set of H'.

that H is regular within tolerance $1 \pm \delta$ for appropriately small δ. This allows us to have control over how the probabilistic space changes after deleting vertices and edges from the hypergraph H. Basing ourselves on the proof given in [1, 5], we will now go over the main proof techniques used to prove Lemma 4.4. As the proof involves many steps, we present it more in the form of a discussion.

Proof. Throughout the proof, we will assume that D and (consequently) n are sufficiently large (i.e. $n > D > d_0(r, K, \epsilon, \delta')$); and we denote by $\delta_1, \delta_2, \dots$ positive constants that tend to 0 when δ tends to 0 and D tends to infinity (for fixed r, K, ϵ). Therefore, by choosing δ and d_0 appropriately we can ensure that each of these will be smaller than δ' (so that any $\pm \delta_i$ can be replaced by $\pm \delta'$).

We define a random subset E' of E by independently choosing each edge of E to be a member of E' with probability $p = \frac{\epsilon}{D}$, that is

$$\mathbb{P}(e \in E') = \frac{\epsilon}{D}, \quad \forall e \in E. \tag{4.2.1}$$

This way of selecting the edges is due to Rödl in [7] and is at the heart of the proof. We will now show that the properties (iv), (v) and (vi) hold with positive probability for such an edge set,[6] and hence that there exists a set of edges with these properties.

We start by showing that (iv) holds with positive probability. We know by (i) that H has at least $n - \delta n$ vertices with degrees at least $(1 - \delta)D$ and so the number of edges in $E(H)$ is at least

$$\frac{(n - \delta n)(1 - \delta)D}{r} = \frac{(1 - \delta)^2 n D}{r}.$$

Similarly, by (i) and (ii) the number of edges is at most

$$\frac{(1 + \delta)D(n - \delta n) + \max(KD, (1 + \delta)D)\delta n}{r} \le (1 + \delta)\frac{Dn}{r} + K\delta\frac{Dn}{r} = \frac{Dn}{r}[1 + \delta(1 + K)].$$

This implies that

$$\frac{Dn}{r}[1 - (2\delta - \delta^2)] \le \ |E| \ \le \frac{Dn}{r}[1 + \delta(1 + K)]$$
$$\Rightarrow \frac{Dn}{r}[1 - \delta_1] \le \ |E| \ \le \frac{Dn}{r}[1 + \delta_1], \tag{4.2.2}$$

where $\delta_1 = \max(2\delta - \delta^2, \delta(1 + K))$.[7] Now, we arbitrarily number the edges of E from 1 to $|E| < \infty$, and denote by A_i the event that the ith edge of E is in E' and by I_i the indicator function of A_i. We have that

$$\mathbb{E}\left[|E'|\right] = \mathbb{E}\left[\sum_{i=1}^{|E|} I_i\right] = \sum_{i=1}^{|E|} \mathbb{P}(A_i) = |E|p = (1 \pm \delta_1)\frac{\epsilon n}{r},$$

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[6]In fact, we will show that the properties hold with very high probility. For the purposes of proving Theorem 3.1, a positive probability would suffice, but the fact that it is very close to 1 makes the method described in the proof an efficient algorithm for constructing hypergraph coverings.

[7]In the rest of the proof, although δ_i can be computed explicitly for all i, we will often just write δ_i without estimating it for ease of writing and to keep the focus on the proof technique rather than the technical details.

where the second equality follows from (4.1.1) and the last from (4.2.2). To prove (iv) it then suffices to show that $|E'| \sim \mathbb{E}\left[|E'|\right]$, which we do via Corollary 4.3. Noting that the A_i's are independent, it follows from (4.1.2) that

$$\text{Var}\left[|E'|\right] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \sum_{i=1}^{|E|} \mathbb{P}(A_i) - \mathbb{P}(A_i)^2 = |E|(p - p^2) \leq \mathbb{E}\left[|E'|\right],$$

and thus by corollary 4.3 that

$$\lim_{n \to \infty} \mathbb{P}\left(|E'| \sim \mathbb{E}\left[|E'|\right]\right) = 1,$$

that is

$$\mathbb{P}\left(|E'| = (1 \pm \delta_2)\frac{\epsilon n}{r}\right) > 0.99,$$

say, for appropriately chosen δ_2.

We now show that (v) holds with positive probability. Noting that $V' = V - \bigcup_{e \in E'} e$ is the set of remaining vertices after deleting those that were covered by the edges in E', (v) states that the number of covered vertices is

$$|V| - |V'| = n - ne^{-\epsilon}(1 \pm \delta') = (n\epsilon - O(\epsilon^2 n))(1 \pm \delta'),$$

and thus roughly $n\epsilon$ with only some $O(\epsilon^2 n)$ covered more than once.[8] The method of proof for (v) is the same as above, with the computation of the expectation and variance of vertices not covered by E' being slightly more involved. For each vertex $x_i \in V, i \in [n]$, define the event A_i to be '$x_i \notin \bigcup_{e \in E'} e$', that is, 'none of the edges incident with x_i are in E''. Let I_i be the indicator function of event A_i, so that so that $\sum_{i=1}^n I_i$ is the number of vertices not covered by E'. Call a vertex x_i *good* if $d(x_i) = (1 \pm \delta)$ and *bad* otherwise (recall there are at most δn bad vertices). If x_i is good, then we have that

$$\mathbb{E}(I_i) = \mathbb{P}(A_i) = (1 - p)^{d(x)} = \left(1 - \frac{\epsilon}{D}\right)^{(1 \pm \delta)D} = e^{-\epsilon}(1 \pm \delta_3), \qquad (4.2.3)$$

since as $\delta \to 0$ and $D \to \infty$, $\left(1 - \frac{\epsilon}{D}\right)^{(1 \pm \delta)D} \to e^{-\epsilon}$. If x_i is a bad vertex then

$$\mathbb{E}(I_i) = (1 - p)^{d(x)} \in [0, 1], \quad 0 < d(x) < KD. \qquad (4.2.4)$$

We now estimate $\mathbb{E}(|V'|)$, which from linearity of expectation is given by

$$\mathbb{E}(|V'|) = \mathbb{E}\left(\sum_{i=1}^n I_i\right) = \sum_{i=1}^n \mathbb{E}(I_i).$$

Using equations (4.2.3), (4.2.4) and the fact that there are at most δn bad vertices we can then write

$$(n - \delta n)e^{-\epsilon}(1 \pm \delta_3) \leq \mathbb{E}(|V'|) \leq ne^{-\epsilon}(1 \pm \delta_3) + \delta n$$

$$\Rightarrow \quad ne^{-\epsilon}(1 \pm \delta_3 - \delta \pm \delta\delta_3) \leq \mathbb{E}(|V'|) \leq ne^{-\epsilon}(1 \pm \delta_3 + \delta e^{\epsilon})$$

$$\Rightarrow \quad ne^{-\epsilon}(1 \pm \delta_4) \leq \mathbb{E}(|V'|) \leq ne^{-\epsilon}(1 \pm \delta_5),$$

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[8]A rigorous proof of this not assuming (v) and using Chebyshev's Inequality can be found in [8].

and thus

$$\mathbb{E}(|V'|) = ne^{-\epsilon}(1 \pm \delta_6). \tag{4.2.5}$$

We next compute the variance of $|V'|$, which by (4.1.2) satisfies

$$\mathrm{Var}[|V'|] \le \sum_{i=1}^{n} \mathbb{E}(I_i) + \sum_{i=1}^{k} \sum_{j \sim i} \left(\mathbb{P}(A_i \cap A_j) - \mathbb{P}(A_i)\mathbb{P}(A_j) \right).$$

Recalling that A_i is the event 'none of the edges incident with x_i are in E'' and thus that $A_i \cap A_j$ is the event that 'none of the edges incident with either x_i or x_j are in E'', it follows that

$$
\begin{aligned}
&\mathbb{P}(A_i \cap A_j) - \mathbb{P}(A_i)\mathbb{P}(A_j) \\
={}& (1-p)^{d(x_i)+d(x_j)-d(x_i,x_j)} - (1-p)^{d(x_i)+d(x_j)} \\
={}& (1-p)^{d(x_i)+d(x_j)} [(1-p)^{-d(x_i,x_j)} - 1] \\
\le{}& (1-p)^{-d(x_i,x_j)} - 1 \le \left(1 - \frac{\epsilon}{D}\right)^{-\delta D} - 1 \le e^{+\delta\epsilon} - 1 \le \delta_7,
\end{aligned}
\tag{4.2.6}
$$

where we have used the fact that $\lim_{n\to\infty} \left(\left(1 - \frac{\epsilon}{D}\right)^D \right)^{-\delta} = e^{+\delta\epsilon}$ and $\lim_{\delta\to 0} e^{+\delta\epsilon} - 1 = 0$. Combining the bounds in (4.2.5) and (4.2.6) yields

$$\mathrm{Var}(|V'|) \le \sum_{i=1}^{n} \mathbb{E}(I_i) + \sum_{i=1}^{k} \sum_{j \sim i} \left(\mathbb{P}(A_i \cap A_j) - \mathbb{P}(A_i)\mathbb{P}(A_j) \right) \le \mathbb{E}(|V'|) + n^2 \delta_7,$$

and thus by corollary 4.3 we have that

$$\lim_{n\to\infty} \mathbb{P}(|V'| \sim \mathbb{E}(|V'|)) \to 1,$$

that is,

$$\mathbb{P}\left(|V'| = ne^{-\epsilon}(1 \pm \delta_8)\right) > 0.99,$$

say, for appropriately chosen δ_8.

Property (vi) is a key part of the proof as it ensures that the induced hypergraph will have similar properties to H and thus that the process can be iterated. We will focus on the ideas of the proof of (vi) rather than the technical details (these can be found in [1]).

The intuition behind (vi) is to show that the induced subgraph on the set of vertices which are not covered by E' is regular within tolerance $(1 \pm \delta')$ (i.e. verifies (i) for some D), to ensure that the process of selecting a random set of edges as in (4.2.1) can be reiterated. Consider a vertex x in V which has not been covered by E' and an edge e containing x. We look at whether e belongs to H', the hypergraph induced by H on the vertices which are not covered by E'. For e not to belong H' (and thus for the degree of x to change), a vertex in e different from x (since we assume x is not covered) would need to be covered by E' so that e would then constitute an edge incident with this vertex and be eliminated from the induced hypergraph. In other words, the edge e will not

belong to H' if e intersects an edge in E'. Equivalently, for the edge e to belong to the induced hypergraph H', none of the edges which intersect e not at x can be in E'. A natural starting point is then to estimate the size of precisely this set

$$F_{n,e} = |\{f \in E : x \notin f, f \cap e \neq \emptyset\}|.$$

In particular, we will show that for all but at most $\delta_9 n$ vertices x, the following two properties hold:

(A) $d(x) = (1 \pm \delta)D$, and

(B) for all but at most $\delta_{10} D$ edges $e \in E$ with $x \in e$

$$|F_{n,e}| = (1 \pm \delta_{12})(r - 1)D. \tag{4.2.7}$$

Property (A) follows directly from (i) by taking $\delta \leq \delta_9$. Now, as earlier, we call a vertex x good if $d(x) = (1 \pm \delta)$ and bad otherwise. Since there are at most δn bad vertices x, and these all have $d(x) < KD$, the number of edges containing bad vertices is at most $\delta n K D$ and thus the number of vertices (good and bad) contained in more than $\delta_{10} D$ such edges is at most $(\delta n K D / \delta_{10} D) r < \delta_9 n$, for appropriate δ_9, δ_{10}.[9] This means that for all but at most $\delta_9 n$ vertices x, all but at most $\delta_{10} D$ of the edges incident with x will contain only good vertices. It then remains to show that for an edge containing only good vertices, (4.2.7) holds.

Let e be an edge with only good vertices and $x \in e$. We want to find a lower and upper bound for the number of edges that intersect e at a point other than x. To do this we will need to take into account the number of edges two vertices y, z have in common, $d(y, z)$, which by (iii) verifies $d(y, z) < \delta n$ for all y, z. The maximum number of edges not containing x which can intersect e is for minimal $d(y, z)$ for all y, z in e. Assuming e is their only common edge and noting that each vertex in e must be incident with $(1 \pm \delta)D - 1$ edges other than e (since each vertex in e has degree $(1 \pm \delta)D$), $|F_{n,e}|$ satisfies

$$|F_{n,e}| \leq (r - 1)[(1 \pm \delta)D - 1] < (1 \pm \delta)(r - 1)D. \tag{4.2.8}$$

Now, the minimum number of edges not containing x which can intersect e is for maximal $d(y, z)$ between all pairs of vertices in e, that is

$$
\begin{aligned}
|F_{n,e}| \quad \geq \quad & (r-1)[(1 \pm \delta)D - 1] - \underbrace{(r-1)(\delta D - 1)}_{d(x,y)\forall y} - \underbrace{\binom{r-1}{2}(\delta D - 1)}_{d(y,z)\forall y,z} \\
= \quad & (r-1)(1 \pm \delta)D - (r-1)\delta D - \frac{(r-1)(r-2)}{2}(\delta D - 1) \\
\geq \quad & (r-1)D\left[(1 \pm \delta) - \delta(1 + (r-2)/2)\right] \\
= \quad & (r-1)D(1 \pm \delta_{11}).
\end{aligned}
\tag{4.2.9}
$$

---

[9]In a r-uniform hypergraph, $(\delta n K D / \delta_{10} D) r$ is the largest number of vertices that can be repeated $\delta_{10} D$ times in a set of $\delta n K D$ edges. This simplifies to $(r \delta K / \delta_{10}) n$, and so by choosing δ_9, δ_{10} such that $\delta_{10} \delta_9 > \delta K r$ we can ensure that $(\delta n K D / \delta_{10} D) r < \delta_9 n$ holds.

It follows from (4.2.8) and (4.2.9) that $|F_{n,e}| = (1 \pm \delta_{12})(r-1)D$, for all but at most $\delta_9 n$ vertices x in all but at most $\delta_{10}D$ of their edges.

The rest of the proof follows the same technique as the one used to prove (v) and (vi), and consists in showing, with high probability, that the degree in H' of most vertices satisfying (A) and (B) is $e^{-\epsilon(r-1)}D(1 \pm \delta')$. We will outline the steps of the proof but will not go over most of the technical details. We will call an edge good if it contains only good vertices and bad otherwise (recall all but $\delta_{10}D$ of the edges incident with x are good). As mentioned earlier, for all but at most $\delta_9 n$ vertices x and conditioning on $x \in V'$, the probability that a good edge e containing x is in H' is the probability that none of the edges in $F_{n,e}$ are in E', which yields

$$\mathbb{P}(e \subset V') = (1-p)^{(1 \pm \delta_{12})(r-1)D}$$

by (4.2.7). It follows that the expected value of $d'(x)$ is

$$\mathbb{E}[d'(x)] = (1 \pm \delta_{10} \pm \delta)D(1-p)^{(1 \pm \delta_{12})(r-1)D} \pm \delta_{10}D = e^{-\epsilon(r-1)}D(1 \pm \delta_{13}).$$

One can also show, using the usual upper bounds on the variance given in (4.1.2) that

$$[d'(x)] \leq \mathbb{E}[d'(x)] + \delta_{14}D^2$$

for all but at most $\delta_{15}n$ and thus by corollary 4.3 we have that

$$\lim_{n \to \infty} \mathbb{P}(d'(x) \sim \mathbb{E}[d'(x)]) \to 1,$$

that is,

$$\mathbb{P}(d'(x) = De^{-\epsilon(r-1)}(1 \pm \delta_{16}) > 0.99,$$

say, for appropriately chosen δ_{16}. This ends the proof of Lemma 4.4.   □

4.3   Hypergraph Covering from Lemma

By selecting a random subset E' such that $Pr(e \in E') = \epsilon/D$ for all $e \in E$, we have shown in the above lemma that for a hypergraph H verifying properties (i), (ii), (iii) of Theorem 3.1 with sufficiently small regularity tolerance, there exists a set of roughly $n\epsilon$ edges that cover $n\epsilon - O(n\epsilon^2)$ of the vertices in H; and the induced hypergraph on the remaining vertices will also be regular within a specified tolerance. To end the proof of the Frankl-Rödl theorem given in Theorem 3.1, basing ourselves on [1], we give details on how the process can be iterated (i.e. on how the parameters are updated at each iteration) and show how to construct the hypergraph covering from the partial covers obtained at each iteration.

Proof. Theorem 3.1 from lemma 4.4. Fix $\epsilon > 0$ such that

$$\frac{\epsilon}{1 - e^{-\epsilon}} + r\epsilon < 1 + a, \tag{4.3.1}$$

and fix $\delta > 0$ such that

$$(1 + 2\delta)\frac{\epsilon}{1 - e^{-\epsilon}} + r\epsilon < 1 + a. \tag{4.3.2}$$

Fix an integer t such that $e^{-\epsilon t} < \epsilon$. We prove Theorem 3.1 by applying Lemma 4.4 t times for an appropriate choice of variables. Since the regularity tolerance of the induced hypergraph depends on that of the hypergraph the edges are selected from, we work backwards to determine the δ's. Put $\delta = \delta_t$, and define by reverse induction $\delta_t > \delta_{t-1} > \ldots > \delta_0$ such that $\delta_{i-1} \leq \delta_i e^{-\epsilon(r-1)}, i \in [t]$ and $\prod_{i=0}^{t}(1 + \delta_i) = (1 + \delta)\prod_{i=0}^{t-1}(1 + \delta_i) < 1 + 2\delta.$[10] For r fixed by Theorem 3.1 and ϵ as in (4.3.1), the first iteration of the lemma is on V, D, K, n as specified in Theorem 3.1, and for $\delta' = \delta_1, \delta = \delta_0$, with $n \geq D \geq d_0.$[11] Applying the lemma t times will yield a decreasing sequence of sets of vertices $V = V_0 \supset V_1 \supset \ldots \supset V_{t-1} \supset V_t$ and a sequence of sets of edges $E_1, E_2, \ldots, E_t$, where E_i is the set of edges E' obtained in the application of the lemma to the hypergraph induced on V_{i-1}. Furthermore, the lemma is applied at iteration $i \geq 1$ with

$$D_i = D_{i-1}e^{-\epsilon(r-1)} = De^{-\epsilon i(r-1)} \quad \text{and} \quad K_i = K_{i-1}e^{\epsilon(r-1)} = Ke^{\epsilon i(r-1)},$$

and $\delta' = \delta_i$, $\delta = \delta_{i-1}$ for $n_i = |V_i| \geq D_i \geq d_i$. The lemma can be applied on V_i since the first condition concerning near-regularity will hold by (vi) of the previous iteration, and the second two conditions will hold since

(ii) For all $x \in V_i, d_{H_i}(x) < d_H(x) < KD = Ke^{\epsilon i(r-1)}De^{-\epsilon i(r-1)} = K_iD_i,$

(iii) For any two distinct $x, y \in V_i, d_{H_i}(x, y) < d_H(x, y) < \delta_0 D \leq e^{-\epsilon i(r-1)}\delta_i D = \delta_i D_i,$

where $d_{H_i}(x)$ is the degree of x in the hypergraph induced by H on V_i.

Now, consider the assertion of Theorem 3.1 with $\gamma = \delta_0$ and $m = \max_{[t]\cup\{0\}} d_i$. We will show that applying the lemma t times yields the desired cover for H. By (iv) and (v) of Lemma 4.4, we have for all $i \in [t]$

$$|V_i| = |V_{i-1}|e^{-\epsilon}(1 \pm \delta_i) = |V_0|e^{-i\epsilon}\prod_{k=0}^{i}(1 \pm \delta_k) \leq ne^{-i\epsilon}(1 \pm 2\delta) \tag{4.3.3}$$

$$|E_i| = \frac{\epsilon|V_{i-1}|}{r}(1 \pm \delta_i) = \frac{\epsilon|V_0|e^{-(i-1)\epsilon}}{r}\prod_{k=0}^{i}(1 \pm \delta_k) \leq \frac{\epsilon n}{r}e^{-(i-1)\epsilon}(1 \pm 2\delta), \tag{4.3.4}$$

where the last inequality follows from (4.3.2), and where V_i and E_i are as defined earlier. Suppose we group together all the edges from all sets E_i and cover each vertex in V_t, the vertices that remain after t iterations of the lemma, by an edge containing it. This will give a set of edges that covers

---

[10]This is possible since $\delta_i, i \in [t]$ can be made arbitrarily small.

[11]Recall that, as mentioned at the start of the proof of Lemma 4.4, δ_0 and d_0 must be chosen such that all the δ_is as defined in the proof (not to be confused with δ_is in this section) are smaller than δ'.

all vertices of H and is of size at most

$$\sum_{i=0}^{t-1} |E_i| + |V_t| \quad \leq \quad (1 \pm 2\delta)\frac{\epsilon n}{r}\sum_{i=0}^{t-1} e^{-(i-1)\epsilon} + n\underbrace{e^{-t\epsilon}}_{<\epsilon}(1 \pm 2\delta) \tag{4.3.5}$$

$$\leq \quad (1 \pm 2\delta)\left[\frac{\epsilon n}{r}\frac{1}{1-e^{-\epsilon}} + n\epsilon\right] \tag{4.3.6}$$

$$= \quad (1 \pm 2\delta)\frac{n}{r}\left[\frac{\epsilon}{1-e^{-\epsilon}} + r\epsilon\right] \tag{4.3.7}$$

$$< \quad \frac{n}{r}(1+a), \tag{4.3.8}$$

where we have used the inequality in (4.3.1) and the fact that $e^{-\epsilon t} < \epsilon$. This ends the proof of Theorem 3.1. $\qquad\square$

5   Conclusion

In this essay, we presented a proof of the Erdös-Hanani conjecture on tactical configurations reformulated as a hypergraph covering problem, using the probabilistic techniques introduced by Rödl in [7] and the Second Moment Method. The idea that it is possible to get asymptotically close to a tactical configuration for large enough n as well as the probabilistic techniques introduced by Rödl in [7] have had significant impact on both theoretical and computational fronts [5, 9]. They have lead to some results on the asymptotic behavior of the chromatic index for hypergraphs, the use of more modern proof techniques such as branching processes to prove the Erdös-Hanani conjecture, as well as the development of efficient hypergraph coloring and hypergraph covering algorithms [6, 8, 2]. Furthermore, Füredi states in his survey that "almost all combinatorial questions can be reformulated as either a matching or covering problem of a hypergraph", and gives in [5] many more important combinatorial results that have also been studied in the setting of a hypergraph covering problem.

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