Simplified derivation of the gravitational wave stress tensor from the linearized Einstein field equations

Steven A. Balbus[a,1]

[a]Astrophysics, Department of Physics, University of Oxford, Oxford OX1 3RH, United Kingdom

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A conserved stress energy tensor for weak field gravitational waves propagating in vacuum is derived directly from the linearized general relativistic wave equation alone, for an arbitrary gauge. In any harmonic gauge, the form of the tensor leads directly to the classical expression for the outgoing wave energy. The method described here, however, is a much simpler, shorter, and more physically motivated approach than is the customary procedure, which involves a lengthy and cumbersome second-order (in wave-amplitude) calculation starting with the Einstein tensor. Our method has the added advantage of exhibiting the direct coupling between the outgoing wave energy flux and the work done by the gravitational field on the sources. For nonharmonic gauges, the directly derived wave stress tensor has an apparent index asymmetry. This coordinate artifact may be straightforwardly removed, and the symmetrized (still gauge-invariant) tensor then takes on its widely used form. Angular momentum conservation follows immediately. For any harmonic gauge, however, the stress tensor found is manifestly symmetric from the start, and its derivation depends, in its entirety, on the structure of the linearized wave equation.

gravitational radiation | general relativity | theoretical astrophysics

The recent detection of gravitational radiation (1) has greatly heightened interest in this subject. Deriving an expression for the correct form of the energy flux carried off in the form of gravitational waves is a famously difficult undertaking at both the conceptual and technical levels. The heart of the difficulty is that the stress energy of the gravitational field is neither a unique nor a localizable quantity, because local coordinates can be found for which the field can be made to vanish by the equivalence principle. It is not a source of spacetime curvature; it is part of the curvature itself, which manifests globally. Indeed, for many years, debate abounded as to whether there was any true energy propagated by gravitational radiation. We know now of course that there is, but the hunt for a suitable stress energy is a burdensome demand for those approaching the subject either as nonspecialists or newcomers. The currently generally adopted textbook approach (2) is to first write the metric tensor $g_{\mu\nu}$ as the following sum*:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad [1]$$

where $\eta_{\mu\nu}$ is the usual Minkowski metric and $h_{\mu\nu}$ is the departure therefrom, and then to treat the latter as a small quantity. We work throughout in quasi-Cartesian coordinates that differ only infinitesimally in linear order from strictly Cartesian coordinates, so that $\eta_{\mu\nu}$ is a constant tensor.* The Einstein tensor,

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{g_{\mu\nu}R}{2}, \qquad [2]$$

where $R_{\mu\nu}$ is the Ricci tensor and R is its R^{μ}_{μ} trace, is then expanded in powers of the amplitudes of $h_{\mu\nu}$ in its various forms. With the material stress energy tensor denoted by $T_{\mu\nu}$, the Newtonian gravitational constant by G, and the speed of light set to unity, the Einstein field equation is the following:

$$G_{\mu\nu} = -8\pi G T_{\mu\nu}, \qquad [3]$$

which upon expansion in $h_{\mu\nu}$ may be rewritten as follows:

$$G^{(1)}_{\mu\nu} = -8\pi G\left(T_{\mu\nu} + t_{\mu\nu}\right). \qquad [4]$$

Here, $G^{(1)}_{\mu\nu}$ consists of the terms in $G_{\mu\nu}$ that are linear in $h_{\mu\nu}$, and

$$t_{\mu\nu} = \frac{1}{8\pi G}\left(G^{(2)}_{\mu\nu} + \ldots\right), \qquad [5]$$

where $G^{(2)}_{\mu\nu}$ represents the Einstein tensor terms quadratic in $h_{\mu\nu}$, and so forth. Following standard practice, we refer to $t_{\mu\nu}$ as a "pseudotensor," because it is Lorentz covariant but not a true tensor, unlike $T^{\mu\nu}$, under full coordinate transformations. To leading nonvanishing order, the pseudotensor $t_{\mu\nu}$ is then interpreted as the stress energy of the gravitational radiation itself. The sum $T_{\mu\nu} + t_{\mu\nu}$ is often referred to as the energy-momentum pseudotensor; a yet more general version of the pseudotensor, using the full metric $g_{\mu\nu}$, is presented in the textbook of Landau and Lifschitz (3). There are by now many routes that lead to a suitable definition of an appropriate stress tensor for gravitational radiation without the use of a pseudotensor formalism. We make no pretense of doing anywhere near full justice to this elegant and sophisticated literature here; this is not the intent of this article. Our purpose, rather, is to show how to obtain a

Significance

Gravitational radiation provides a probe of unprecedented power with which to elucidate important astrophysical processes that are otherwise completely dark (e.g., black hole mergers) or impenetrable (e.g., supernova and early universe dynamics). Historically, the gap between propagating fluctuations in the spacetime metric and classical dynamical concepts such as energy and angular momentum conservation has bedeviled this subject. By now, there is a vast literature on this topic, and there are many powerful methods available. Because of their mathematical sophistication, however, they are not used in introductory texts, which are forced instead to a follow a much more cumbersome path. We present here a derivation of the most widely used form of the stress energy tensor of gravitational radiation, using elementary methods only.

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[1]Email: Steven.Balbus@physics.ox.ac.uk.

*In this paper, Greek indices indicate spatiotemporal dimensions; Roman indices indicate spatial dimensions. We use the sign convention $[-+++]$ for the diagonal Minkowski metric $\eta_{\mu\nu}$. Indices of $h_{\mu\nu}$ $[h^{\mu\nu}]$ are raised [lowered] with $\eta^{\mu\nu}[=\eta_{\mu\nu}]$. For inline equations, we use the notation $\partial^{\mu} \equiv \partial/\partial x_{\mu}$, $\partial_{\mu} \equiv \partial/\partial x^{\mu}$.

widely used form of the stress energy tensor for gravitational radiation, making use only of elementary methods and conserved fluxes emerging from linear wave theory.

The calculation of the energy from the pseudotensor is sufficiently cumbersome that it is rarely done explicitly in textbooks (merely summarized), although the final answer is not unduly involved. In an arbitrary gauge and a background Minkowski spacetime (4),

$$t_{\mu\nu}=\frac{1}{32\pi G}\left[\left\langle\frac{\partial\bar{h}_{\kappa\lambda}}{\partial x^{\mu}}\frac{\partial\bar{h}^{\kappa\lambda}}{\partial x^{\nu}}\right\rangle-\left\langle\frac{\partial\bar{h}^{\lambda\kappa}}{\partial x^{\lambda}}\frac{\partial\bar{h}_{\kappa\mu}}{\partial x^{\nu}}\right\rangle-\left\langle\frac{\partial\bar{h}^{\lambda\kappa}}{\partial x^{\lambda}}\frac{\partial\bar{h}_{\kappa\nu}}{\partial x^{\mu}}\right\rangle\right.$$
$$\left.-\frac{1}{2}\left\langle\frac{\partial\bar{h}}{\partial x^{\mu}}\frac{\partial\bar{h}}{\partial x^{\nu}}\right\rangle\right],\qquad [6]$$

where

$$\bar{h}_{\mu\nu}=h_{\mu\nu}-\frac{\eta_{\mu\nu}}{2}h,\quad h\equiv h^{\mu}_{\mu},\quad \bar{h}\equiv\bar{h}^{\mu}_{\mu}=-h.$$

As explained in standard textbooks, the use of Eq. **6** as a stress tensor makes sense only if an average over many wavelengths is performed, so that oscillatory cross products do not contribute. This averaging is indicated by the angle bracket $\langle\rangle$ notation. Moreover, although the expression **[6]** is gauge invariant, in solving explicitly for $h_{\mu\nu}$, a choice of gauge must be made. The "harmonic gauge" is a convenient choice for the study of gravitational waves, as it greatly simplifies the mathematics. If $h_{\mu\nu}$ depends on its coordinates as a plane wave of the form $\exp(ik_{\mu}x^{\mu})$, a harmonic gauge is actually required if k_{μ} is a null vector, $k_{\mu}k^{\mu}=0$. All physical, curvature-inducing radiation (as opposed to oscillating coordinate transformations) has this property (4). The harmonic gauge is defined by the following condition:

$$\frac{\partial\bar{h}_{\mu\nu}}{\partial x_{\mu}}=0\quad\text{(harmonic gauge condition)}.\qquad [7]$$

That it is always possible to find such a gauge is well known (4); the proof is similar to that of being able to choose the Lorenz gauge condition in electrodynamics. In the "transverse traceless" (TT) gauge, there is the additional constraint $h=0$, which leads to the following simple result:

$$t_{\mu\nu}=\frac{1}{32\pi G}\left\langle\frac{\partial\bar{h}_{\kappa\lambda}}{\partial x^{\mu}}\frac{\partial\bar{h}^{\kappa\lambda}}{\partial x^{\nu}}\right\rangle\quad\text{(TT gauge)}.\qquad [8]$$

For linear gravitational plane waves propagating in vacuum [although not more generally (4)], a transformation to the TT gauge can always be found without departing from the harmonic constraint; there is also a precise electrodynamic counterpart.

By way of contrast, in classical wave problems, finding a conserved wave energy flux is much more straightforward. Consider the simplest example of a wave equation for a quantity f,

$$\frac{\partial^2 f}{\partial t^2}-\frac{\partial^2 f}{\partial x^2}=0.\qquad [9]$$

Start by looking for a conserved flux. If we multiply by $\partial f/\partial t\equiv\dot{f}$, integrate the second term $-\dot{f}\partial^2 f/\partial x^2$ by parts, and regroup, this leads to the following:

$$\frac{\partial}{\partial t}\left[\frac{\dot{f}^2}{2}+\frac{(f')^2}{2}\right]-\frac{\partial}{\partial x}(\dot{f}f')=0,\qquad [10]$$

where $f'\equiv\partial f/\partial x$. This readily lends itself to the interpretation of an energy density $[\dot{f}^2+f'^2]/2$ and an energy flux $-\dot{f}f'$, although with an uncertain overall normalization factor that must be determined by such considerations as the work done on the wave sources. Note in particular that the equation for the second-order energy flux is entirely determined by a linear-in-f wave equation.

A linear scalar wave equation is yet more revealing, and only slightly more complicated. With Φ the effective potential, and ρ the source density, consider the wave equation of scalar gravity,

$$\Box\Phi\equiv-\frac{\partial^2\Phi}{\partial t^2}+\nabla^2\Phi=4\pi G\rho.\qquad [11]$$

(Here, $\Box$ and ∇^2 are the usual d'Alembertian and Laplacian operators, respectively.) Then, if we multiply by $(1/4\pi G)\partial_t\Phi$, integrate $(\partial_t\Phi)\nabla^2\Phi$ by parts, and regroup, this leads to the following:

$$-\frac{1}{8\pi G}\frac{\partial}{\partial t}\left[\left(\frac{\partial\Phi}{\partial t}\right)^2+|\nabla\Phi|^2\right]+\nabla\cdot\left(\frac{1}{4\pi G}\frac{\partial\Phi}{\partial t}\nabla\Phi\right)=\rho\frac{\partial\Phi}{\partial t}.\qquad [12]$$

However,

$$\rho\frac{\partial\Phi}{\partial t}=\frac{\partial(\rho\Phi)}{\partial t}-\Phi\frac{\partial\rho}{\partial t}$$
$$=\frac{\partial(\rho\Phi)}{\partial t}+\Phi\nabla\cdot(\rho\boldsymbol{v})\qquad [13]$$

$$=\frac{\partial(\rho\Phi)}{\partial t}+\nabla\cdot(\rho\boldsymbol{v}\Phi)-\rho\boldsymbol{v}\cdot\nabla\Phi,\qquad [14]$$

where $\boldsymbol{v}$ is the velocity and the usual mass conservation equation has been used in the second equality. A simple rearrangement then leads to the following:

$$\frac{\partial\mathcal{E}}{\partial t}+\nabla\cdot\boldsymbol{F}=\rho\boldsymbol{v}\cdot\nabla\Phi,\qquad [15]$$

where

$$\mathcal{E}=\rho\Phi+\frac{1}{8\pi G}\left[\left(\frac{\partial\Phi}{\partial t}\right)^2+|\nabla\Phi|^2\right],\quad \boldsymbol{F}=\rho\boldsymbol{v}\Phi-\frac{1}{4\pi G}\frac{\partial\Phi}{\partial t}\nabla\Phi.\qquad [16]$$

The right side of **[15]** is minus the volumetric rate at which work is being done on the sources. For the usual case of compact sources, the left side may then be interpreted as a far-field wave energy density of $[(\partial_t\Phi)^2+|\nabla\Phi|^2]/8\pi G$ and a wave energy flux of $-(\partial_t\Phi)\nabla\Phi/4\pi G$. The question we raise here is whether an analogous formal "direct method" might be used to shed some light on the origin of Eq. **6**, including, very importantly, a means of extracting the overall normalization factor.

There does indeed seem to be such a formulation, which we now discuss.

Analysis

Conserved Densities and Fluxes. Begin with the standard, gauge-invariant general weak field linearized wave equation (2, 4):

$$\Box\bar{h}_{\mu\nu}-\frac{\partial^2\bar{h}^{\lambda}_{\mu}}{\partial x^{\nu}\partial x^{\lambda}}-\frac{\partial^2\bar{h}^{\lambda}_{\nu}}{\partial x^{\mu}\partial x^{\lambda}}+\eta_{\mu\nu}\frac{\partial^2\bar{h}^{\lambda\rho}}{\partial x^{\lambda}\partial x^{\rho}}=-\kappa T_{\mu\nu},\qquad [17]$$

where $\kappa=16\pi G$. We restrict our attention throughout this work to the case of a small metric disturbance $h_{\mu\nu}$ on a background Minkowski spacetime. The material stress tensor $T_{\mu\nu}$ is treated as completely Newtonian.

Next, establish an identity by contracting Eq. **17** on $\mu\nu$:

$$\Box\bar{h}+2\frac{\partial^2\bar{h}^{\lambda\rho}}{\partial x^\lambda\partial x^\rho}=-\kappa T^\mu_\mu\equiv-\kappa T.$$

Hence:

$$\frac{\partial^2\bar{h}^{\lambda\rho}}{\partial x^\lambda\partial x^\rho}=-\frac{1}{2}\Box\bar{h}-\frac{\kappa T}{2},\qquad [18]$$

and we rewrite Eq. **17** as follows:

$$\Box\bar{h}_{\mu\nu}-\frac{\partial^2\bar{h}^\lambda_\mu}{\partial x^\nu\partial x^\lambda}-\frac{\partial^2\bar{h}^\lambda_\nu}{\partial x^\mu\partial x^\lambda}-\frac{\eta_{\mu\nu}}{2}\Box\bar{h}=-\kappa S_{\mu\nu},\qquad [19]$$

where the source function $S_{\mu\nu}$ is the following:

$$S_{\mu\nu}=T_{\mu\nu}-\frac{\eta_{\mu\nu}T}{2}.\qquad [20]$$

We seek an energy-like conservation equation from the wave equation in the form displayed in Eq. **19**. Toward that end, multiply by $\partial_\sigma\bar{h}^{\mu\nu}$, summing over $\mu\nu$ as usual and leaving σ free. The first term on the left side of **[19]** is then the following:

$$\begin{aligned}\frac{\partial^2\bar{h}_{\mu\nu}}{\partial x^\rho\partial x_\rho}\frac{\partial\bar{h}^{\mu\nu}}{\partial x^\sigma}&=\frac{\partial}{\partial x_\rho}\left(\frac{\partial\bar{h}_{\mu\nu}}{\partial x^\rho}\frac{\partial\bar{h}^{\mu\nu}}{\partial x^\sigma}\right)-\frac{\partial\bar{h}_{\mu\nu}}{\partial x^\rho}\frac{\partial^2\bar{h}^{\mu\nu}}{\partial x_\rho\partial x^\sigma},\\&=\frac{\partial}{\partial x_\rho}\left(\frac{\partial\bar{h}_{\mu\nu}}{\partial x^\rho}\frac{\partial\bar{h}^{\mu\nu}}{\partial x^\sigma}\right)-\frac{\partial\bar{h}_{\mu\nu}}{\partial x^\rho}\frac{\partial}{\partial x^\sigma}\frac{\partial\bar{h}^{\mu\nu}}{\partial x_\rho},\\&=\frac{\partial}{\partial x_\rho}\left(\frac{\partial\bar{h}_{\mu\nu}}{\partial x^\rho}\frac{\partial\bar{h}^{\mu\nu}}{\partial x^\sigma}\right)-\frac{1}{2}\frac{\partial}{\partial x^\sigma}\left(\frac{\partial\bar{h}_{\mu\nu}}{\partial x^\rho}\frac{\partial\bar{h}^{\mu\nu}}{\partial x_\rho}\right).\end{aligned}\qquad [21]$$

The second term on the left is handled similarly. Juggling indices,

$$-\frac{\partial^2\bar{h}^\lambda_\mu}{\partial x^\nu\partial x^\lambda}\frac{\partial\bar{h}^{\mu\nu}}{\partial x^\sigma}=-\frac{\partial^2\bar{h}^{\lambda\mu}}{\partial x_\nu\partial x^\lambda}\frac{\partial\bar{h}_{\mu\nu}}{\partial x^\sigma},\qquad [22]$$

leads to the following:

$$-\frac{\partial^2\bar{h}^\lambda_\mu}{\partial x^\nu\partial x^\lambda}\frac{\partial\bar{h}^{\mu\nu}}{\partial x^\sigma}=-\frac{\partial}{\partial x_\nu}\left(\frac{\partial\bar{h}^{\lambda\mu}}{\partial x^\lambda}\frac{\partial\bar{h}_{\mu\nu}}{\partial x^\sigma}\right)+\frac{\partial\bar{h}^{\lambda\mu}}{\partial x^\lambda}\frac{\partial^2\bar{h}_{\mu\nu}}{\partial x^\sigma\partial x_\nu},\qquad [23]$$

or equivalently,

$$-\frac{\partial^2\bar{h}^\lambda_\mu}{\partial x^\nu\partial x^\lambda}\frac{\partial\bar{h}^{\mu\nu}}{\partial x^\sigma}=-\frac{\partial}{\partial x_\rho}\left(\frac{\partial\bar{h}^{\lambda\mu}}{\partial x^\lambda}\frac{\partial\bar{h}_{\mu\rho}}{\partial x^\sigma}\right)+\frac{1}{2}\frac{\partial}{\partial x^\sigma}\left(\frac{\partial\bar{h}^{\lambda\mu}}{\partial x^\lambda}\frac{\partial\bar{h}_{\mu\nu}}{\partial x_\nu}\right).\qquad [24]$$

The third term is identical to the second upon summation over μ and ν. The fourth and final term of the left side of Eq. **19** is as follows:

$$-\frac{1}{2}\frac{\partial^2\bar{h}}{\partial x^\rho\partial x_\rho}\frac{\partial\bar{h}}{\partial x^\sigma}=-\frac{1}{2}\frac{\partial}{\partial x_\rho}\left(\frac{\partial\bar{h}}{\partial x^\rho}\frac{\partial\bar{h}}{\partial x^\sigma}\right)+\frac{1}{4}\frac{\partial}{\partial x^\sigma}\left(\frac{\partial\bar{h}}{\partial x^\rho}\frac{\partial\bar{h}}{\partial x_\rho}\right).\qquad [25]$$

Thus, after dividing by 2κ, Eq. **19** takes the form of

$$\frac{\partial\mathcal{S}}{\partial x^\sigma}+\frac{\partial\mathcal{T}_{\rho\sigma}}{\partial x_\rho}=-\frac{1}{2}S_{\mu\nu}\frac{\partial\bar{h}^{\mu\nu}}{\partial x^\sigma},\qquad [26]$$

where $\mathcal{S}$ is a scalar density:

$$\mathcal{S}=-\frac{1}{4\kappa}\left(\frac{\partial\bar{h}_{\mu\nu}}{\partial x^\rho}\frac{\partial\bar{h}^{\mu\nu}}{\partial x_\rho}\right)+\frac{1}{2\kappa}\left(\frac{\partial\bar{h}^{\lambda\mu}}{\partial x^\lambda}\frac{\partial\bar{h}_{\mu\nu}}{\partial x_\nu}\right)+\frac{1}{8\kappa}\left(\frac{\partial\bar{h}}{\partial x^\rho}\frac{\partial\bar{h}}{\partial x_\rho}\right),\qquad [27]$$

and $\mathcal{T}_{\rho\sigma}$ is a flux tensor:

$$\mathcal{T}_{\rho\sigma}=\frac{1}{2\kappa}\left(\frac{\partial\bar{h}_{\mu\nu}}{\partial x^\rho}\frac{\partial\bar{h}^{\mu\nu}}{\partial x^\sigma}\right)-\frac{1}{\kappa}\left(\frac{\partial\bar{h}^{\lambda\mu}}{\partial x^\lambda}\frac{\partial\bar{h}_{\mu\rho}}{\partial x^\sigma}\right)-\frac{1}{4\kappa}\left(\frac{\partial\bar{h}}{\partial x^\rho}\frac{\partial\bar{h}}{\partial x^\sigma}\right).\qquad [28]$$

Due to its second term, $\mathcal{T}_{\rho\sigma}$ is not symmetric in its indices. Index asymmetry also arises in the development of the stress tensor of electromagnetic theory, and there are methods to correct this deficiency (5). Similar techniques may be brought to bear on the current problem, as we discuss below. For the moment, we may note that, in a harmonic gauge $\partial_\mu\bar{h}^{\mu\nu}=0$ (not necessarily traceless), the asymmetry vanishes and the tensor becomes manifestly symmetric in $\rho\sigma$. Notice that the wave stress tensor **[6]** is simply a symmetrized version of **[28]**.

Rather than work with $\mathcal{S}$ and $\mathcal{T}_{\rho\sigma}$ each on its own, it is more natural to form the composite tensor $\mathcal{U}_{\rho\sigma}$,

$$\mathcal{U}_{\rho\sigma}\equiv\mathcal{T}_{\rho\sigma}+\eta_{\rho\sigma}\mathcal{S}.\qquad [29]$$

The left side of Eq. **26** may then be written more compactly as a 4-divergence:

$$\frac{\partial\mathcal{U}_{\rho\sigma}}{\partial x_\rho}=-\frac{1}{2}S_{\mu\nu}\frac{\partial\bar{h}^{\mu\nu}}{\partial x^\sigma}.\qquad [30]$$

It should be noted that the content of Eq. **30** is exactly the same as that of the wave equation **[17]**: no more, no less. At this stage, note that we have not done any spatial averaging. In the TT gauge, Eq. **29** leads directly to the following:

$$\mathcal{U}_{00}=\frac{1}{4\kappa}\left(\frac{\partial\bar{h}_{\mu\nu}}{\partial x^i}\frac{\partial\bar{h}^{\mu\nu}}{\partial x^i}+\frac{\partial\bar{h}_{\mu\nu}}{\partial t}\frac{\partial\bar{h}^{\mu\nu}}{\partial t}\right),\qquad [31]$$

$$\mathcal{U}_{0i}=\frac{1}{2\kappa}\left(\frac{\partial\bar{h}_{\mu\nu}}{\partial t}\frac{\partial\bar{h}^{\mu\nu}}{\partial x^i}\right),\qquad [32]$$

$$\mathcal{U}_{ij}=\frac{1}{2\kappa}\left(\frac{\partial\bar{h}_{\mu\nu}}{\partial x^i}\frac{\partial\bar{h}^{\mu\nu}}{\partial x^j}-\frac{\delta_{ij}}{2}\frac{\partial\bar{h}_{\mu\nu}}{\partial x^\rho}\frac{\partial\bar{h}^{\mu\nu}}{\partial x_\rho}\right).\qquad [33]$$

By these canonical forms, the component $\mathcal{U}_{00}$ is readily interpreted as a wave energy density, $\mathcal{U}_{0i}$ as a wave energy flux, and $\mathcal{U}_{ij}$ as a wave momentum stress. However, in fact, the combination $(\partial^\rho\bar{h}_{\mu\nu})(\partial_\rho\bar{h}^{\mu\nu})$ (and $[\partial^\rho\bar{h}][\partial_\rho\bar{h}]$ in a more general harmonic gauge) must vanish when averaged over many wavelengths, because the Fourier wave vector components satisfy the null constraint $k^\rho k_\rho=0$. In the end, there emerges the very simple results:

$$\begin{aligned}\mathcal{U}_{\rho\sigma}=t_{\rho\sigma}&=\frac{1}{2\kappa}\left\langle\frac{\partial\bar{h}_{\mu\nu}}{\partial x^\rho}\frac{\partial\bar{h}^{\mu\nu}}{\partial x^\sigma}-\frac{1}{2}\frac{\partial\bar{h}}{\partial x^\rho}\frac{\partial\bar{h}}{\partial x^\sigma}\right\rangle\text{(harmonic)},\\&=\frac{1}{2\kappa}\left\langle\frac{\partial\bar{h}_{\mu\nu}}{\partial x^\rho}\frac{\partial\bar{h}^{\mu\nu}}{\partial x^\sigma}\right\rangle\qquad\text{(TT)}.\end{aligned}\qquad [34]$$

Direct Energy Loss. In going from the wave equation **[19]** to an energy equation **[30]**, we divided by 2κ. How do we know that this particular normalization is the proper one for producing a true energy flux? It is the right side of Eq. **26** that tells this story. This is as follows:

$$-\frac{1}{2}S_{\mu\nu}\frac{\partial\bar{h}^{\mu\nu}}{\partial x^{\sigma}} = -\frac{1}{2}\left(T_{\mu\nu}-\frac{\eta_{\mu\nu}}{2}T\right)\left(\frac{\partial h^{\mu\nu}}{\partial x^{\sigma}}-\frac{\eta^{\mu\nu}}{2}\frac{\partial h}{\partial x^{\sigma}}\right)$$
$$= -\frac{1}{2}T_{\mu\nu}\frac{\partial h^{\mu\nu}}{\partial x^{\sigma}}. \quad [35]$$

We now set $\sigma=0$, picking out the time component, and work in the Newtonian limit $h^{00}\simeq-2\Phi$, where Φ is the gravitational potential. We are then dominated by the 00 components of $h^{\mu\nu}$ and $T_{\mu\nu}$. Using the right arrow $\rightarrow$ to mean integrate by parts and ignore the pure derivatives (as inconsequential for wave losses), and recalling the mass–energy conservation relation $\partial_\mu T^{0\mu}=0$, we perform the following manipulations:

$$-\frac{1}{2}T_{00}\frac{\partial h^{00}}{\partial x^0}\rightarrow\frac{1}{2}\frac{\partial T_{00}}{\partial x^0}h^{00}=\frac{1}{2}\frac{\partial T^{00}}{\partial x^0}h^{00}$$
$$=-\frac{1}{2}\frac{\partial T^{0i}}{\partial x^i}h^{00}$$
$$\rightarrow\frac{1}{2}T^{0i}\frac{\partial h^{00}}{\partial x^i}$$
$$\simeq-\rho\boldsymbol{v}\cdot\nabla\Phi, \quad [36]$$

which is the rate at which the effective Newtonian potential Φ does net work on the matter. (Here, ρ is the Newtonian mass density and $\boldsymbol{v}$ is the normal kinetic velocity. Averaging is understood; the $\langle\rangle$ notation has been suppressed in **[36]** for ease of presentation.) This is negative if the force is oppositely directed to the velocity, so that the source is losing energy by generating outgoing waves. Our TT gauge expression **[8]** for $\mathcal{T}_{0i}$ is also negative for an outward flowing wave of argument $(r-t)$, r being spherical radius and t time. (By contrast, $\mathcal{T}^{0i}$ would be positive.)

A subtle but important point: can one be sure that such a potential actually exists? An ordinary Newtonian potential would conserve mechanical energy over the course of the system's evolution. That a gauge does exist in which an appropriate effective potential function emerges is shown in standard texts (2, 4). This is the Burke–Thorne potential (6, 7), which is proportional to the leading order "radiation reaction" term in an expansion of h^{00}. The effective potential must emerge as part of the radiation reaction terms in $\bar{h}_{\mu\nu}$ if it is to deplete mechanical energy. This time-dependent potential may be precisely defined in a suitable "Newtonian gauge." Although we make no explicit use of it here, it is given by the following (4):

$$\Phi=-\frac{1}{2}h_{00}=\frac{G}{5}I^{(\mathrm{V})}_{jk}x_jx_k,$$

where I_{jk} is the traceless moment of inertia tensor, $I^{(\mathrm{V})}$ refers to its fifth time derivative, and x_j is a spatial Cartesian coordinate. This justifies our overall normalization factor of $1/2\kappa$. Our final energy equation in an arbitrary gauge thus takes the following form:

$$\frac{\partial\mathcal{U}_{\rho\sigma}}{\partial x_\rho}=-\frac{1}{2}T_{\mu\nu}\frac{\partial h^{\mu\nu}}{\partial x^\sigma}. \quad [37]$$

The fact that the Newtonian gauge is not harmonic may have contributed to this rather basic (work done) $\leftrightarrow$ (wave flux) conservation equation, the analog of our scalar prototype introductory example, not being highlighted previously in the literature. If, for example, we follow custom and go directly to an harmonic gauge straight from Eq. **17**, one obtains the following familiar result:

$$\Box\bar{h}_{\mu\nu}=-\kappa T_{\mu\nu}\text{(harmonic gauge)}. \quad [38]$$

This is certainly useful as a means to solve for $\bar{h}_{\mu\nu}$, but if we now multiply by $\partial_\sigma\bar{h}^{\mu\nu}$, and regroup as before, we find the following:

$$\frac{\partial}{\partial x_\rho}\left(\frac{1}{2\kappa}\frac{\partial\bar{h}_{\mu\nu}}{\partial x^\rho}\frac{\partial\bar{h}^{\mu\nu}}{\partial x^\sigma}\right)=-\frac{1}{2}\left(\frac{\partial\bar{h}^{\mu\nu}}{\partial x^\sigma}\right)T_{\mu\nu}\text{(harmonic gauge)}. \quad [39]$$

The difficulty now is that the right side has no obvious physical interpretation, and we are gauge-bound. It is only if we follow the path of Eq. **37**, retaining full gauge freedom, that we may simultaneously formulate a conserved flux on one side of the equation with the readily interpretable "work done" combination $-(T_{\mu\nu}/2)\partial_\sigma h^{\mu\nu}$ on the other. Within the same equation, the radiated waves and the effective Newtonian potential are best understood each in their own gauge. Gauge selection (as an aid to interpretation) is the last, not the first, step of the analysis. Very different gauges in very different regions are illustrative of the nonlocal character of this problem.

Index Symmetry and Angular Momentum Conservation. The tensor $\mathcal{U}_{\rho\sigma}$ lacks index symmetry for nonharmonic gauges. This is awkward for angular momentum conservation. To address this problem, begin in Eq. **37** by setting $\sigma=k$, a spatial index. Next, multiply the equation by $\epsilon_{ijk}x_j$, adhering to the usual summation convention for repeated spatial indices but not distinguishing between their covariant and contravariant placement. An integration by parts then gives the following:

$$\frac{\partial\mathcal{J}'_{i\rho}}{\partial x^\rho}-\epsilon_{ijk}\mathcal{U}_{jk}=-\frac{1}{2}\epsilon_{ijk}x_jT_{\mu\nu}\frac{\partial h^{\mu\nu}}{\partial x_k}, \quad [40]$$

where we have introduced a provisional angular momentum flux of gravitational waves,

$$\mathcal{J}'_{i\rho}\equiv\epsilon_{ijk}x_j\mathcal{U}_{\rho k}. \quad [41]$$

Were $\mathcal{U}_{\rho\sigma}$ a symmetric tensor, $\epsilon_{ijk}\mathcal{U}_{jk}$ would vanish identically, and an equation of strict angular momentum conservation would emerge. This suggests that the asymmetry is not truly fundamental.

Indeed, symmetry is easily restored. The method is to subtract off an appropriate "difference tensor" from $\mathcal{U}_{\rho\sigma}$, which leaves the fundamental conservation equations intact. Begin by rewriting a spatially averaged form of $\mathcal{U}_{\rho\sigma}$ as follows:

$$\mathcal{U}_{\rho\sigma}=t_{\rho\sigma}+\left\langle\eta_{\rho\sigma}\mathcal{S}+\frac{1}{2\kappa}\frac{\partial\bar{h}^{\lambda\mu}}{\partial x^\lambda}\left(\frac{\partial\bar{h}_{\mu\sigma}}{\partial x^\rho}-\frac{\partial\bar{h}_{\mu\rho}}{\partial x^\sigma}\right)\right\rangle$$
$$\equiv t_{\rho\sigma}+\mathcal{U}^D_{\rho\sigma}, \quad [42]$$

where $t_{\rho\sigma}$ is the standard symmetric wave tensor given by Eq. **6** and $\mathcal{U}^D_{\rho\sigma}$, the (spatially averaged) difference tensor, is defined by this equation. It is easy to verify that $\mathcal{U}^D_{\rho\sigma}$ vanishes for the TT gauge (we have already done so), but it is also a gauge-invariant quantity under the infinitesimal coordinate transformation $x^\lambda\rightarrow x^\lambda+\xi^\lambda$, with the following:

$$\bar{h}_{\mu\nu}\rightarrow\bar{h}_{\mu\nu}-\frac{\partial\xi_\mu}{\partial x^\nu}-\frac{\partial\xi_\nu}{\partial x^\mu}+\eta_{\mu\nu}\frac{\partial\xi^\lambda}{\partial x^\lambda}. \quad [43]$$

Here, ξ_μ is a well-behaved but otherwise arbitrary vector function. In fact, it is a straightforward exercise to show that $\mathcal{S}$ and the final $\rho\sigma$-antisymmetric term in **[42]** are each gauge invariant on their own. A standard textbook problem is to show that $t_{\rho\sigma}$ is a gauge-invariant quantity (4); here, we have done so indirectly, because $\mathcal{U}_{\rho\sigma}$ must be gauge invariant by virtue of its original construction. Thus, if we evaluate $\mathcal{U}^D_{\rho\sigma}$ in the TT gauge, in which it vanishes, and transform to any other gauge, the result must still vanish. We may therefore conclude that $\mathcal{U}_{\rho\sigma} = t_{\rho\sigma}$ quite generally.

Returning to the question of angular momentum conservation, we replace $\mathcal{U}_{\rho\sigma}$ with $t_{\rho\sigma}$, and define the symmetrized angular momentum flux tensor as follows:

$$\mathcal{J}_{i\rho} \equiv \epsilon_{ijk} x_j t_{\rho k}. \quad [44]$$

The precise statement of angular momentum conservation is then as follows:

$$\frac{\partial \mathcal{J}_{i\rho}}{\partial x^\rho} = -\frac{1}{2}\epsilon_{ijk} x_j T_{\mu\nu}\frac{\partial h^{\mu\nu}}{\partial x_k}. \quad [45]$$

The right side affords a direct method for computing angular momentum loss via the explicit Burke–Thorne potential.

Conclusion

The linear wave equation that emerges from the Einstein field equations, either in the form of **[17]** or **[19]**, contains in itself all of the ingredients needed for determining a conserved gravitational wave energy flux tensor, propagating in a background Minkowski spacetime and produced by slowly moving sources. The stress tensor that is calculated via a more lengthy and complex second-order analysis of the Einstein tensor is, for any harmonic gauge, identical to that which emerges from our first-order calculation, that is, $\mathcal{U}_{\rho\sigma}$ and $t_{\rho\sigma}$ are identical in this case. It is only for the construction of a symmetric wave stress tensor in nonharmonic gauges that an alteration of form is needed.

The presented calculation also illuminates the physical connection between the radiated gravitational waves and the effective Newtonian potential that serves to deplete mechanical energy from the matter source for these waves. The precise form of this "Burke–Thorne" potential does not itself play a role in our analysis. Merely the fact that it exists, and that in common with any Newtonian potential function it is associated with $-h^{00}/2$ to leading order, is sufficient to determine the normalization constant of the conserved flux tensor. Indeed, the entire calculation could be performed in a harmonic gauge, in which case $-(T_{\mu\nu}/2)\partial_\sigma h^{\mu\nu}$ must be the generic expression for the work done on the sources, even if its manifestation is less transparent than in the Newtonian gauge.

We have shown how to remove an apparent index asymmetry in $\mathcal{U}_{\rho\sigma}$, which in its symmetrized and spatially averaged form reverts to $t_{\rho\sigma}$. Angular momentum conservation readily follows.

The approach that is presented in this paper seems to be the simplest, the most concise, and ultimately the most physically transparent route to understanding the form of the stress energy tensor of gravitational radiation, especially in its most natural harmonic gauges, as embodied in Eq. **34**.

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