Essays on competitive and strategic bidding in multi-object auctions



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To my mother

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Abstract

This thesis addresses four important issues in multi-object auction markets, focusing on a theory of strategic bidding and corresponding empirical evidence, on questions of equilibrium existence with distributional preferences, and on the equivalence of market objectives.

The product-mix auction (PMA) designed by Paul Klemperer is used by the Bank of England to allocate loans to commercial banks at least monthly. If only a few bidders were thought to be participating, strategic behaviour may harm efficiency. I develop a theoretical model of strategic bidding in the PMA that enables the comparison with other auction formats commonly used for the sale of differentiated substitutes. While the theory predicts that the considered auctions perform (nearly) identically in bidder surplus, revenue, and efficiency, I also show contrasting empirical evidence. In a virtual lab experiment, the PMA achieves highest bidder surplus and efficiency. When bidders are symmetric, payoffs in the PMA are even between 90% and 156% higher and efficiency is 12% higher than in other auctions.

The product-mix auction was developed for substitute preferences, but similar concepts may be useful in markets with substitutes and complements. Extending the product-mix preferences and bidding language, I consider an even richer market with substitutes, complements, and *distributional* preferences. In this market, the seller can favour, e.g., a more concentrated allocation of products among buyers, or an even distribution of products. This type of preference, while common in many real world applications, is novel in the literature. I develop a graph-based bidding language and derive new results on the existence of competitive (Walrasian) equilibria, which can be computed via linear programming. The auction design I develop guarantees fair and transparent pricing.

In the context of substitutes, it is often practical for buyers to state their preferences with a budget. This may be spent on several products, while trade-offs between those products are stated as prices. Without budget constraints, it is known that revenue and welfare optimum may not coincide. We show, however, that with budget constraints, the market objectives of maximising revenue and maximising welfare result in the same, unique, market clearing prices and the same allocation. Our proof of this result also constructs an algorithm to identify these prices.

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1 Introduction

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1.1 Background

The allocation of a fixed set of resources among different interested parties is an ubiquitous problem. Various stakeholders take part in the allocation mechanism, for example the recipients of the resources, the distributor or seller, or a social planner like the government. Some of their objectives may align, but others compete; therefore the choice of allocation procedure is rarely straightforward. Of special interest and notoriously difficult to analyse are markets in which multiple units of distinct objects are allocated or sold. Such markets often appear in the context of licensing; for example when national (or local) governments sell telecommunication spectrum (Bichler et al., 2017), (Bichler and Goeree, 2017), allocate transportation routes Cantillon and Pesendorfer (2005), or manage the use of their natural resources (Cramton, 2009; Stoneham et al., 2003; Teytelboym, 2019). Licences can be designed flexibly to fit the requirements of the user and the issuer; hence this implies, in many cases, product differentiation. As most licences are indivisible objects, complications with the existence of ex-post desirable allocations and prices may arise. In other contexts, products may be seen as (almost) infinitely divisible, as is the case with various financial instruments, e.g. on

the stock exchange, in treasury auctions (Friederich and Payne, 2007; Hasbrouck et al., 1993; Hortaçsu et al., 2018; Hortaçsu and McAdams, 2010), or in the sale of online advertisement slots (ad auctions) (Conitzer et al. (2022),Conitzer et al. (2021),Varian (2009)). Other buyer characteristics, however, such as budget constraints may complicate the market design.

A crucial ingredient in the design of a market are the agents' preferences. Products may be substitutes for buyers, for example two ready-made cakes like apple pie and carrot cake. When the baker is shopping for ingredients, however, flour, sugar, and eggs will be complementary as there will be no cake unless all ingredients are included. Substitute preferences are, with respect to the existence of competitive equilibria, easier to handle, and this has been famously leveraged for the allocation of liquidity through central banks (Klemperer, 2008, 2010, 2018). More complex preferences may occur in procurement auctions, which have also received much attention in the field of operations research (Bichler et al., 2011; Fan et al., 2003; Gartner, 2008; Giunipero et al., 2009).

But even with relatively straightforward preferences, concerning only few goods but involving both substitutes *and* complements, it may be more difficult for markets to deliver satisfying results. A famous concept of a 'satisfying result' is competitive equilibrium: at given prices, all buyers and sellers involved in the market buy or sell exactly those amounts of each product which they find optimal to buy or sell. While there are very general existence theorems for 'unimodular' preferences (Baldwin and Klemperer, 2019*b*), which include many natural substitute preferences, existence results with complements are notoriously difficult to derive (Candogan et al., 2016, 2015; Sun and Yang, 2006, 2014; Teytelboym, 2014). The computation and existence of competitive equilibrium is indeed an economic problem that has not only fascinated researchers for decades (Arrow and Debreu, 1954; Ausubel, 2004, 2006; Bikhchandani and Mamer, 1997; Bikhchandani and Ostroy, 2002; Devanur et al., 2008; Nisan et al., 2007; Paes Leme and Wong, 2017), but also bears immediate practical implications. Once fundamental existence properties of a market environment are understood, an allocation mechanism can be designed. Without, a mechanism will still produce an distribution of objects, but it may be significantly harder to justify such distribution to the participants in the market.

Other inherently practical but also mathematically challenging issues to consider are, first, the conception of a language for the buyers and sellers to communicate with the market owner or its delegate (often the auctioneer); and second, the means to facilitate the computation of the allocation and prices, according to a set of rules or optimising some objective. Languages to communicate preferences should be

rich enough for the agents to express any relevant preference relations in the given application, that is, for example, substitute relations between different kinds of cake, or, instead of bids for two separate products, a single bid for the bundle containing a bag of flour and a bag of sugar. Such and much more complex 'bidding languages' have been studied, most prominently, by Klemperer (2008, 2018); Lehmann et al. (2006, 2002); Milgrom (2009); Nisan (2000). Bidding languages not only have to be adapted to the agents' preferences, they also may be required to have structural properties that facilitate the computation of the desired market outcomes, prices and allocations. In practice, this means being able to compute market outcomes *within a reasonable time* even when a large number of buyers participates in the auction, and even when a large number of products and bundles of products is sold together (e.g. in procurement or auctions for electricity). To achieve this, the algorithms used for the computation must often be *efficient*, that is, their running time must be a polynomial function of the number of objects or bundles, and the number of submitted bids. Economists and computer scientist alike have dedicated much work to the search for such efficient algorithms (Ausubel, 2004, 2006; Baldwin et al., 2022, 2019; Bichler et al., 2021; Fujishima et al., 1999; Garg and Kapoor, 2006; Guruswami et al., 2005; Nisan, 2000; Orlin, 2010; Paes Leme, 2017; Sandholm, 2002).

If we imagine a 'well-behaved' auction market, in which buyers and sellers have tractable preferences, competitive equilibrium with ex-post desirable allocation and prices exist, and the auctioneer has efficient algorithms and/or the necessary computing power to calculate market outcome, then, unfortunately, things may still go wrong. Most agents involved in the market will want to obtain products at the cheapest possible price, regardless of how much the products are actually worth to them. To achieve this, the agents will behave strategically, i.e. bid to maximise their own payoff in some way. This is usually no easy task, as the bidding environment contains a large amount of uncertainty, e.g. with respect to the other bidders (and their preferences) involved, the number of objects being sold, or information pertinent to the market that arises after the auction. But in extremely high-stake auctions like those for telecommunication spectrum, participants go as far as hiring teams of economists and mathematicians to devise optimal bidding strategies. On the other hand, perhaps aligned with the interest of many social planners, academics have put significant effort into developing incentive-compatible auctions and mechanisms. In an incentive-compatible auction, any incentive to behave strategically is non-existent in the sense that it is in every participant's best interest to simply report their true preferences for the objects that are to be allocated. Most famously, this quest started with Vickrey (1961) and has been

followed by several other ingenious mechanisms for multiple goods (Ausubel, 2006; Fujishige and Yang, 2020; Sun and Yang, 2014) but constraining preferences for example. Often, incentive-compatible mechanisms are not feasible in practice or come with other drawbacks such as low revenue. Therefore, it is indispensable to understand potential strategies in the mechanisms employed in practice, e.g. in electricity markets (Borenstein et al., 2002; Hortaçsu and Puller, 2008; Wolfram, 1998) or treasury auctions (Hortaçsu et al., 2018; Hortaçsu and McAdams, 2010). Once strategies are understood, and perhaps even empirically proven, the market designer can draw the necessary consequences with respect to the auction design.

Finally, of vital importance to the existence of competitive equilibria, to computability, and to bidders' potential desire to behave strategically, is the objective the auctioneer sets for the allocation mechanism. For instance, a social planner purely motivated by achieving the most productive allocation of objects in the market, may choose an allocation that maximises social welfare (efficiency). But even a government may care about revenue, as in the spectrum auctions in the early 2000s (Cramton, 2001; Klemperer, 2002*a*,*b*), and balancing competing objectives often adds additional complexity in designing a good auction. While efficiency and revenue have been perhaps the most relevant performance criteria, considerations of fairness and post-auction market outcomes also come into focus (Engelmann et al., 2018; Ilvento et al., 2020; Kasberger, 2020).

1.2 Contributions

This thesis addresses several important questions in the design of auctions, exploring strategic behaviour in the context of substitutes (Chapter 2), providing empirical evidence on this strategic behaviour (Chapter 3), deriving new existence results in the context of distributional preferences (Chapter 4), and providing a novel equivalence of design objectives in budget-constrained auctions (Chapter 5). The focus is always on practically relevant settings, each with a specific focus on existing or potential applications.

Chapter 2 provides the first tractable model for the comparison of strategic bidding in product-mix auctions with standard non-combinatorial auction formats. The product-mix auction (PMA) was designed by Paul Klemperer for the Bank of England for the allocation of liquidity among commercial banks (Klemperer, 2008, 2010, 2018). As one of the important novelties of this auction, bidders can express trade-offs between multiple substitute goods. The Bank of England

still uses this mechanism at least monthly to auction loans. However, if bidders thought that the number of participants were small, they might attempt to bid strategically and harm efficiency.

My theoretical model encompasses markets with any number of substitutes and a representative buyer competing against several competitive fringes. I derive results on equilibrium bidding in a product-mix, a sequential, and a simultaneous auction, and I compare bidder surplus, revenue, and welfare. Numerical results show that in the risk-neutral Bayes-Nash equilibrium with uniform priors, the product-mix auction and the sequential auction are equivalent in terms of outcomes, and the simultaneous auction performs only slightly worse and is near-equivalent. These results hold under the first-price and the second-price payment rule. The model I analyse can be adapted to various extensions. For example, the representative buyer's priors can vary subject to very weak regularity assumptions and I provide some robustness checks for the first-price auction formats. I also provide some uniqueness results on equilibrium bidding for a setup with only two goods. My findings may be applied to financial markets, online advertisement, or electricity markets, where buyers decide between substitute products. My findings imply that in a simultaneous sale, allowing product-mix bids gives a small advantage to buyers and the seller. A sequential sale has no drawbacks compared to the product-mix auction.

Chapter 3 studies a special case of the model developed in Chapter 2 in a virtual lab experiment. I provide the first experimental evidence on bidding in product-mix, sequential, and simultaneous auctions that is accompanied by a theoretical model. The experimental study covers three treatments modelled precisely according to the theory. Moreover, it also studies three treatments in a closely related setting with symmetric bidders. The difficulty of strategic behaviour in the context of multiple objects is that bid functions are inherently multi-variate, which renders the standard computation of Bayes-Nash equilibrium bid functions in single-good models infeasible. To date, no theoretical model can predict Bayes-Nash equilibrium with symmetric buyers in a multi-object model. My experimental design with symmetric bidders, however, is comparable in many ways to the design that follows the theory precisely, and may thus be much more informative than, e.g. a multi-object market modelled after an application.

The study is conducted with a large sample with over 22,000 observations.[1] To the best of my knowledge, my study is first to document over-bidding compared to

[1]At the date of thesis submission, the data collection was not yet concluded, so the results should be seen as somewhat preliminary, with about one third of the desired sample size being available.

Bayes-Nash equilibrium in a setting with multiple, heterogeneous goods. Contrasting with the theory, the experimental results indicate that the product-mix format dominates the sequential format in terms of bidder surplus, revenue, and welfare. It dominates the simultaneous format in bidder surplus and welfare, but not in revenue. The simultaneous auction, while performing significantly worse in bidder surplus, achieves the highest revenue and somewhat higher revenue than the sequential auction.

Interestingly, in the setup with symmetric bidders, the product-mix format outperforms the sequential in terms of average bidder payoffs by 90% and the simultaneous auction even by 156%. The simultaneous auction still delivers the highest revenue, but the product-mix format delivers 12% higher efficiency than both other auction formats.

The experimental study therefore provides the first empirical evidence that, in the context of heterogeneous substitutes, the product-mix auction may be the superior auction design. This applies to many high-stake industries such as central bank loan management, online advertisement, electricity generation, but also consumer goods auctions. Significant gains in bidder surplus may attract additional buyers, with a knock-on effect on already higher social welfare.

Chapter 4 develops a theory of distributional preferences, precisely, preferences of the seller over the partitioning of supply in the market. For example, a manufacturer holding a procurement auction may prefer obtaining multiple products from the same supplier, e.g. to save on transaction costs. For a market with multiple, indivisible objects, I prove two main results on the existence of competitive equilibrium: First, I provide necessary and sufficient conditions for existence in the most general model. In this model, buyers and the seller can express substitute *and* complements preferences over multiple copies of multiple different objects. In addition, the seller can state preferences over allocations in the market through the means of a novel cost function graph. The graph structure provides flexibility to express a rich set of preferences while maintaining analytical tractability. The equilibrium considered involves prices that are 'linear in packages', ensuring fairness and transparency in pricing.

Second, in a restricted environment with only-complements and any number of objects, but only one copy per object for sale, I derive sufficient conditions for the existence of a competitive equilibrium. The theory of distributional preferences in this single-copy environment also relates to a more traditional approach of defining reserve prices for the seller (Sun and Yang, 2014). I derive a useful duality between superadditive and 'set-cover submodular' functions, connecting reserve price functions to the cost functions I introduce.

The results I derive connect also to some of earliest literature on existence of competitive equilibrium (Bikhchandani and Mamer, 1997; Bikhchandani and Ostroy, 2002), introducing non-additive marginal costs in the form of distributional preferences. In practice, these kind of preferences are very common: if the bundling of different products from/for different sellers/buyers leads to cost savings or additional costs for the seller/buyer, we have, in fact, distributional preferences. Applications may be in procurement, in land auctions with subsidies, auctions for biodiversity conservation, spectrum auctions, or transport licensing.

Chapter 5 proves a novel result on market objectives in a setting with multiple, divisible substitute goods and budget-constrained bidders: the equivalence of the (envy-free) revenue-maximising and the welfare-maximising market outcome. It is shown that, at the market-clearing price vector, the seller is able to gain the maximum of buyers' available budgets, while buyers are content purchasing their assigned allocation. Clearing prices are shown to be buyer-optimal among all revenue-maximising prices in the sense that they induce maximum quantities to be allocated to buyers. Earlier work (Chen et al., 2007) implies that efficient inner-point methods can determine the revenue-maximising prices and allocation.

The geometric structure introduced, called the 'feasible region' of prices, may be of independent interest. The region forms a lower semi-lattice, and hence contains a unique, element-wise minimal price vector, which is proven to clear the market. While the feasible region is non-convex, it can be traversed by a price-scaling procedure that bears similarities with a simplex-like algorithm by Adsul et al. (2012).

Moreover, the considered market is a generalisation of standard Fisher markets where buyers' utilities are quasi-linear. The generalised setting has been also been labelled a 'quasi-Fisher' market and has been studied from a perspective of competitive equilibrium and Nash social welfare (Chen et al., 2007; Cole et al., 2017; Murray et al., 2020). With budget constraints, however, maximising revenue may be an important concern for the seller as well.

This work provides a clear recommendation for the use of this budget-constrained allocation procedure, when the two objectives of optimising for revenue or efficiency need to be balanced. Prominent applications of budget-constrained settings with multiple goods include the exchange of financial instruments for one another, debt restructuring, the (re-)division of firms between shareholders, as well as many ad auctions in which advertising companies bid to have their ads placed on websites at different publishers.

2

Strategic bidding in multi-object auctions

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2.1 Introduction

If buyers are interested in two different goods, but want to buy at most one of them, they face a difficult strategic situation in standard simultaneous auction: they do not know which auction to participate in. Holding auctions sequentially mitigates this difficulty, and the product-mix auction design resolves it altogether. Following the financial crisis in 2007, the product-mix auction (PMA) was designed for the Bank of England to allocate loans to commercial banks against different collateral (Klemperer (2008, 2010, 2018)) and has been in use ever since. In a single-round procedure, bidders can express trade-offs between different varieties of goods. The auction implements the efficient allocation and if bidders act as price takers it determines a competitive equilibrium.

The PMA allows a bidder with the aforementioned preferences to submit bids on both goods, under the constraint to win at most one. It therefore greatly simplifies the bidder's bidding decision, and it is expected to result in a more efficient allocation if the number of bidder is sufficiently large. However, this auction has not been studied in game-theoretic models: if the number of bidders in the auction is small, bidders are have a strong incentive to bid strategically in order to influence auction prices and maximise their surplus. My contribution is, to the best of my knowledge, the first comparison of equilibrium bidding in PMAs with standard simultaneous and sequential auctions. I consider first-price and second-price payment rules, where the standard PMA with uniform prices falls under the second-price rule. The first-price PMA I discuss is an instance of the menu auction by Bernheim and Whinston (1986), equipped with the product-mix bidding language.

The PMA has further applications in finance and various other industries, e.g. in online advertisement. Advertisers bid for web space to display their ads on publishers' websites. If two websites have different audiences, an advertiser with say only one ad they want to display has a trade-off between publishing on either (but not both) of the two websites. Another important application are electricity markets. Capacity-constrained suppliers, e.g. operators of pumped-storage hydro power facilities, face a trade-off between bidding for adjacent time slots. Suppose they can supply either in the morning or the afternoon, but not both. The product-mix format would allow them to express a trade-off between the morning-product and the afternoon-product, without taking on the risk of having to supply both. Consumer good auctions provide many other examples for possible applications of the PMA.[1]

In my analysis, simultaneous auctions perform worst in terms of efficiency, revenue, and bidder surplus. The performance of PMAs and sequential auctions is similar, with a slight advantage for sequential auctions in terms of efficiency and bidder surplus. When bidders are restricted to bid for at most one unit, the standard PMA is outcome-equivalent to the general VCG mechanism[2,3] and thus sets the benchmark for efficiency. The first-price PMA achieves the highest revenue, closely followed by the sequential auction. The flexible bidder prefers the PMA under the second-price rule, and she prefers a sequential auction under the first-price rule. Among first-price auctions, efficiency is highest in the PMA, close to a

[1]Vintage cars are an obvious example. I have also run several product-mix auctions among students at Oxford, to sell licences for rental of community-owned exercise equipment. Rowing machines of different quality are rented to students, each of whom (most likely) does not want more than one.

[2]Ausubel and Milgrom (2005)

[3]A proof for this can be found e.g. in Leonard (1991). When bidders are restricted to bid for one unit, the PMA is equivalent to his formulation of an assignment problem of individuals to positions. Demange (1982) proves a closely related result in a two-sided market setup.

sequential format. I obtain most of my results assuming bidders' values are uniformly distributed; but varying the distributions does not significantly change the results.

In a simple version of my model, the auctioneer sells two differentiated goods, one unit of each. All bidders have constant marginal values for one or two goods and unit demand. There is one flexible bidder who values both goods, up to one unit overall; and there are two groups of inflexible bidders who are interested in only either one of the goods, respectively. Inflexible bidders are modelled as a competitive fringe: in equilibrium, it is optimal for them to bid their true value. This allows us to focus on the flexible bidder's behaviour. She has to balance the trade-off between winning one good or the other and not too often winning both goods at the same time. Under the first-price rule, there is an additional incentive to shade bids. In product-mix auctions, the former trade-off is eliminated because the bidding format allows to specify bids (so called "paired" bids[4]) that prevent the flexible bidder from winning both goods at once. Similarly, in sequential auctions the flexible bidder can choose when to expose herself to potentially winning more than one unit.

I find that differences between auction formats are more extreme for high values of the flexible bidder.[5] I also vary the inflexible bidders' value distribution in a way that is straightforwardly interpreted as varying the degree of competition the flexible bidder faces. I find that outcomes in the first-price PMA and the first-price sequential auction formats do not differ by much, irrespective of the degree of competition. However, in the comparison of the first-price PMA/sequential auction versus the first-price simultaneous auction, differences in bidder surplus increase considerably with increased competition, while differences in efficiency and revenue decline with increased competition.

This research is among the first to study strategic bidding in PMAs, and it is the first to compare relevant auction formats for the sale of heterogeneous goods with respect to equilibrium bidding, efficiency, revenue, and bidder surplus. Holmberg et al. (2018) extend Klemperer and Meyer (1989)'s supply function equilibrium to two goods. Their characterisation of linear equilibria under demand uncertainty can be used to describe equilibrium bidding in two-variety PMAs with a number of symmetric bidders holding private information. However, the authors provide no information on how these equilibria compare to other mechanisms used in practice.

Our setting builds on Krishna and Rosenthal (1996)'s model with "global" and "local" bidders. In contrast to their work, I study the sale of heterogeneous goods

[4]In the original design of the PMA, these bids are called paired bids (Klemperer (2010)); in the computer science literature, they are better known as XOR-bids.

[5]I only compare mechanisms with the same payment rule, as comparisons between payment rules are inherently flawed by the setup of my model: inflexible bidders are comparatively stronger under the first-price rule.

and my flexible (global) bidder has *substitutes* preferences: if she wins both goods she can only use one of them, whereas in Krishna and Rosenthal (1996) the flexible bidder has increasing returns.[6] Albano et al. (2001) study a similar framework and draw comparisons with the VCG mechanism, but they also restrict to super-additive valuations.[7] Vickrey (1961) discussed simultaneous and "successive" auctions with unit-demand bidders, however, with identical objects.

Simultaneous auctions have been also studied by Gerding et al. (2008), in a setup similar to ours. For identical goods and identical second-price auctions, they show that a flexible bidder participates in each auction with a strictly positive bid. The setup is extended to non-identical auctions, but identical goods. My model differs in that I assume identical distributions for the inflexible bidders' values across all auctions, but I allow for the goods to be non-identical to the flexible bidder. In addition, I provide a result on the uniqueness of equilibria with two goods, and also consider first-price payments. Simultaneous first-price auctions under full information have been studied by Palfrey (1980), and a related optimal bidding problem is discussed in Rothkopf (1977). Both papers consider a setup with constraints on the sum of all amounts bid for.

A related approach to modelling multi-unit auctions with large and small bidders is taken by Baisa and Burkett (2017). In their model, the small bidders each demand only an infinitesimal amount. They bid truthfully in the uniform-price auction, but shade their bids under discriminatory payments. Their model is less distorting than ours under the first-price rule; however, I found it not to be tractable for product-mix auctions.

This chapter is structured as follows. I present a simple version of the model in section 2.2 and provide results on equilibrium behaviour. In section 2.3, I describe the general model. I present my numerical analysis in section 2.4, assuming different distributions for the inflexible bidders' values. Section 5.6 concludes.

2.2 A simple two-good model

The flexible bidder's decision problem is best illustrated in a simple model with just two differentiated goods. I describe a model where two indivisible goods A and B are for sale, one unit of each, but all my results hold if the goods are

[6] In their paper, the authors establish equilibria for simultaneous second-price auctions and, in an example, compare equilibrium bidding and revenue to sequential second-price and combinatorial auctions.

[7] Albano et al. (2001) provide efficiency and revenue comparisons between a variant of the 1994 FCC simultaneous ascending auction, simultaneous and sequential second-price auctions, using the VCG mechanism as a benchmark.

divisible as well. Three types of bidders compete for the goods: a flexible bidder with unit demand, bidder F (which I refer to as "she") values one unit of good A at v_A or one unit of good B at v_B and has no value for more than one unit in total. The two goods are differentiated for bidder F with $v_B \geq v_A > 0$.[8] All bidders are risk neutral. Formally, I have

$$u^F(p_A, p_B) = \begin{cases} v_A - p_A & \text{if A is won} \\ v_B - p_B & \text{if B is won} \\ v_B - p_B - p_A & \text{if A and B are won} \\ 0 & \text{otherwise} \end{cases}$$

where p_A and p_B denote the prices of good A and B. There is a group G_A of $n_A \geq 2$ inflexible bidders, each of which is interested only in good A, and there is another group G_B of $n_B \geq 2$ inflexible bidders, each of which is interested only in good B. It is without loss of generality to assume that $n_A = n_B = n$; the size of each competitive fringe does not matter. Bidders of group G_K value one unit of good $K = A, B$ at $k = a, b, k \geq 0$, that is

$$u^K(p_A, p_B) = \begin{cases} k - p_K & \text{if good } K \text{ is won} \\ 0 & \text{otherwise} \end{cases}$$

The auctioneer ("he") has one unit of good A and one unit of good B for sale. Both goods are identical to him at a zero reserve price. Each bidder knows their value privately, but the distribution of the inflexible bidders' values is common knowledge. The inflexible bidders also know that they have at least one competitor with an identical value. The inflexible bidders' values a and b are random variables drawn from an absolutely continuous probability distribution G with probability density function g and support $[0, \overline{v}]$ where $\overline{v} \in \mathbb{R} \cup \{\infty\}$. I make the following assumption:

Assumption 1. The reverse hazard rate $\frac{g(x)}{G(x)}$ is weakly decreasing on $[0, \overline{v}]$.

For most of my results, I assume $0 < v_A \leq v_B < \overline{v}$.[9] I now describe the auction formats I consider.

Product-mix auctions. In describing the PMA for my setup, I define (i) the bidding language, (ii) the allocation rule, and (iii) the payment rule.

(i) The bidding language allows single and paired bids.[10] The flexible bidder submits a bid (w_A, w_B) to state demand for 1 unit of good A *or* good B, but not

[8] I will sometimes note illustrative results for the corner case $v_A = 0$, or $v_A = v_B = 0$.

[9] I note some corner solutions for $v_B \geq \overline{v}$, but this is not the focus of this study.

[10] Paired bids are an instance of XOR bids (see, e.g., Sandholm (2002)) where each component specifies the quantity and price of exactly one good. Let p_A and p_B denote the auction prices. A paired bid (w_A, w_B) for good A and good B respectively expresses the following preference: if $w_A - p_A > w_B - p_B$, then the bidder would like to receive up to one unit of good A. If the inequality sign is switched, she would like to receive up to one unit of good B. If the inequality sign is replaced by an equality, the bidder is indifferent between receiving good A and B, assuming indivisibility of goods.

both, at prices of up to w_A or w_B respectively. The inflexible bidders of group G_K submit a single bid each for good $K = A, B$. Call the bid value of this bid y_K.

(ii) The auctioneer is given bids $(w_A, w_B), (y_A, 0), (0, y_B)$. In the standard PMA, he then determines the efficient allocation given the reported values, i.e. the flexible bidder obtains good A if and only if $w_A - y_A > w_B - y_B$ and $w_A \geq y_A$, and she wins good B if and only if $w_B - y_B \geq w_A - y_A$ and $w_B \geq y_B$.[11] In the first-price PMA, the same allocation rule applies. Together with the pricing rule, this is equivalent to choosing the revenue-maximising allocation, as in the menu auction by Bernheim and Whinston (1986).

(iii) In the standard PMA, the auctioneer determines the lowest competitive equilibrium prices. In my model, prices will always be set by the inflexible bidders' bids y_A and y_B.[12] In the first-price PMA, each winning bidder pays their bid price for the unit won.

Simultaneous auction. Bidders can participate in either one or both of two simultaneous single-unit auctions. The auction price is set by the highest losing bid, respectively, under the second-price rule. A winner pays her bid under the first-price rule.[13]

Sequential auction. Two single-unit auctions for good A and B, respectively, are held sequentially. After the first auction, the auctioneer announces the winning allocation and the clearing price. Bidders then choose their bid in the second auction. They are allowed to participate in either or both auctions. Payment rules are as in the simultaneous auction. I distinguish the case where good A is sold first and the case where good B is sold first.

Remark: In my setup the standard PMA (second-price rule) is equivalent to the VCG mechanism. Bidders pay precisely their social externality from participating in the auction. This also holds more generally for the standard PMA when bidders are restricted to bid for only one unit (possibly with a paired bid).

Throughout the chapter, I use subscripts to denote the auction format. The subscripts are "$P2$" (standard PMA), "$Sim2$" (simultaneous second-price), "$SeqK2$" (sequential second-price with good K sold first), "$P1$" (first-price PMA), "$Sim1$" (simultaneous first-price), and "$SeqK1$" (sequential first-price with good K sold first).

[11] Ties are broken in favour of good B in case of indifference between the components of the paired bid, and in favour of the flexible bidder between her and the inflexible bidders; but this is not important for my analysis.

[12] In general, prices are determined in a linear programme. A detailed description of the price setting concept would go beyond the scope of my study and can be found in Baldwin and Klemperer (2019*a*).

[13] Ties are broken in favour of the flexible bidder, but again this is not important in my analysis.

2.2.1 Equilibrium bidding

First, I formalise that inflexible bidders bid indeed truthfully as claimed above. All proofs are relegated to the appendix.

Lemma 1. *In the auction formats introduced above, at least two inflexible bidders in each group bid their true values in any equilibrium.*

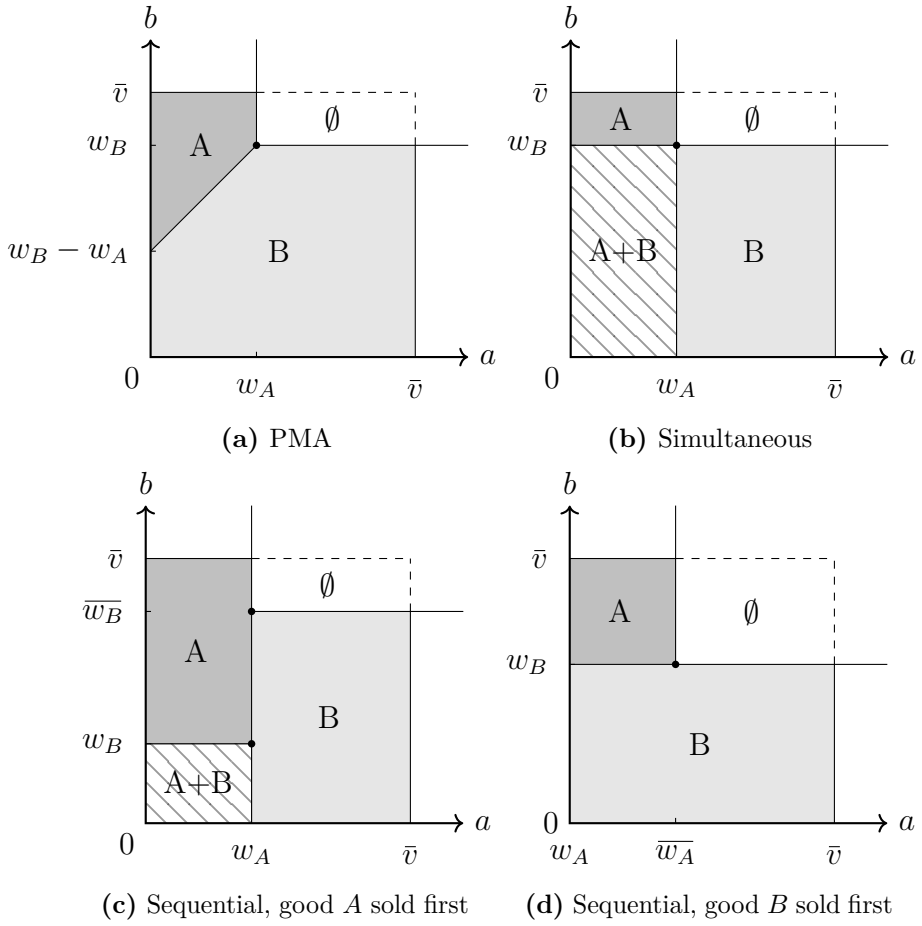
Under the second-price rule, truthful bidding is a dominant strategy for the inflexible bidders in all auction formats. With first-price payments, a simple Bertrand-type argument can be made to establish the lemma. I show this for the first-price PMA in the appendix, and the arguments for the other first-price auctions are analogous. The flexible bidder's choice set can be restricted by eliminating weakly dominated strategies: she never bids above her true values. Knowing the inflexible bidders' equilibrium strategy, I can illustrate the allocation to the flexible bidder, as an outcome of each mechanism's allocation rule, graphically in a-b-space in Figures 3.1(a) - 3.1(d). The dark shaded region marked with "A" corresponds to bidder F winning good A, the light shaded region to winning good B, and the striped region to winning both goods. Note that w_A and w_B, assuming $w_A \leq w_B$,[14] are not drawn at the optimal values, although proportions approximately resemble equilibrium bidding.[15]

Product-mix auctions.
The product-mix auction caters to the unit demand preference of the flexible bidder. The format alleviates the exposure to win more than one unit, maintaining the facility to win either of the goods. Given the flexible bidder's preference, it is without loss of generality to assume that she makes one flat paired bid.[16] Denote this bid (w_A, w_B). In the same way, each inflexible bidder of group G_A or G_B makes a bid $(y_A, 0) = (a, 0)$ and $(0, y_B) = (0, b)$, respectively. As stated above, the flexible bidder obtains good A if and only if $w_A - a > w_B - b$ and $w_A \geq a$, i.e. north of the 45 degree line in Figure 3.1(a). She wins good B if and only if $w_B - b \geq w_A - a$ and $w_B \geq b$. Ties such that $w_A = a$, $w_B = b$, or $w_A + b = w_B + a$ occur with zero probability in equilibrium.

[14] The Figure corresponding to $w_A > w_B$ is symmetric.

[15] The only precisely optimal value drawn is w_A in the sequential auction with good B sold first.

[16] This is argued in many related models, see for example Baisa and Burkett (2017). Note that, when the good is divisible, this only holds if the flexible bidder knows that her bid (or a part of it) will never win *and* set the auction price at the same time. For example, if she knew that the inflexible bidders might demand only half a unit of some good, it might be her bid which ends up setting the price. In this case, the flexible bidder would have an incentive to reduce demand on the second half of the unit she is bidding for.



(a) PMA **(b)** Simultaneous

(c) Sequential, good A sold first **(d)** Sequential, good B sold first

Figure 2.1: Allocations to the flexible bidder

Proposition 1. *In the standard PMA, there exists a unique equilibrium in which the flexible bidder bids truthfully.*

The standard PMA achieves full efficiency in my model. Indeed, it is equivalent to the VCG-mechanism. It is straightforward to calculate bidder surplus, revenue, and efficiency, which can be found in the appendix and will be needed for my numerical analysis later.

The product-mix auction can also be implemented with first-price payments, i.e. each winner pays their respective bid for the goods won. Naturally, the flexible bidder then has an incentive to shade her bids. Again, she makes a flat paired bid (w_A, w_B), given the inflexible bidders' flat, truthful bid. Clearly, any decreasing bid would leave her with an opportunity to improve her payoff ex-post. I denote by $P_A(w_A, w_B)$ and $P_B(w_A, w_B)$ denote the probability of bidder F winning good A and good B, respectively. The allocation rule remains unchanged from the standard PMA; or, looking at it differently, the auctioneer now chooses the

revenue-maximising allocation. To study equilibrium play, I write out the flexible bidder's expected payoff function. We simply have

$$\Pi^F_{P1|v_A,v_B}(w_A, w_B) = P_A(w_A, w_B)\,[v_A - w_A] + P_B(w_A, w_B)\,[v_B - w_B] \tag{2.1}$$

To determine the probabilities of winning good A or B, respectively, I note the following lemma.

Lemma 2. *In any BNE, bids must be such that $w_A \leq w_B$.*

The argument is simple: $w_A > w_B$ cannot be part of an equilibrium because simply switching the bid amounts between good A and B would guarantee the bidder a strictly better payoff. Using Figure 3.1(a), it is not hard to see that

$$P_A(w_A, w_B) = \int_0^{w_A} \int_{w_B - w_A + a}^{\bar{v}} \mathrm{d}G(b)\,\mathrm{d}G(a) \text{ and} \tag{2.2}$$

$$P_B(w_A, w_B) = \int_0^{w_A} \int_0^{w_B - w_A + a} \mathrm{d}G(b)\,\mathrm{d}G(a) + \int_{w_A}^{\bar{v}} \int_0^{w_B} \mathrm{d}G(b)\,\mathrm{d}G(a) \tag{2.3}$$

So the flexible bidder's maximisation problem is

$$\max_{w_A, w_B} \Pi^F_{P1|v_A,v_B}(w_A, w_B)$$
$$\text{s.t. } 0 \leq w_K \leq v_K,\ K = A, B, \text{ and (2.2) and (2.3) hold}$$

Proposition 2. *In the first-price product-mix auction, there exists an equilibrium, in which the flexible bidder makes a strictly positive bid $({w_A}^*, {w_B}^*)$, characterised by equations (A.7) and (A.8) (in the appendix).*

The fact that bidder F makes strictly positive bids on both goods in equilibrium may seem intuitive, but in fact is not obvious considering carefully the probabilities of winning. Notice that by increasing her bid on one of the goods, while increasing the probability of winning on that respective good, but also the bid price, she decreases the probability of winning the respective other good. A priori, it is not clear in what direction the trade-off between these three effects goes. This is illustrated in Figure A.3 in the appendix. I demonstrate that any point on the boundary of $[0, v_A] \times [0, v_B]$ cannot be optimal and therefore the global optimum must be interior and coincide with a stationary point; i.e., first-order conditions fully characterise the global optimum of the flexible bidder's maximisation problem.

Sequential auctions.

A standard alternative to sell good A and B, without creating exposure to win

both goods at once, is to sell them sequentially. The flexible bidder learns the outcome of the first auction before she decides how to bid in the second auction. Let ${w_K}^*$ denote the equilibrium bid after winning the first auction, and let $\overline{w_K}^*$ denote the equilibrium bid after losing the first auction, for $K = A, B$, where good $J \neq K$ is sold prior to good K. The flexible bidder's optimal strategy is found by straightforward backwards-induction. Because of the asymmetry in the flexible bidder's values the order in which good A and B are sold matters; I first present the case where good A is sold first.

Proposition 3. *In the sequential second-price auction, where good A is sold prior to good B, there exists a unique equilibrium, in which*

(i) in the first auction, the flexible bidder submits a strictly positive bid, and

(ii) in the second auction, the flexible bidder always submits a strictly positive bid.

All characterisations of optimal bids can be found in the appendix. Intuitively, the second stage adjusts the flexible bidder's true first-stage value for good A; it is the difference in the expected second-stage payoffs conditional on winning or losing the first auction. If she won the first auction, she will make a strictly positive bid in the auction for good B again, but only so high that her potential additional payment is entirely hedged: she bids exactly the difference in value between good B and good A. This is different, of course, when good B is sold first; there is no incentive to bid again in the second auction if the first auction and the higher-value good is won.

Proposition 4. *In the sequential second-price auction, where good B is sold prior to good A, there exists a unique equilibrium, in which*

(i) in the first auction, the flexible bidder submits a strictly positive bid, and

(ii) in the second auction, the flexible bidder bids always submits a strictly positive, truthful bid after losing the first auction, and a zero bid after losing the first auction.

Suppose now that good A and B are sold in two sequential first-price auctions. I first discuss the case where good A is prior to good B. Fix v_B and let $\widehat{v_A} := G(\overline{w_B}^*(\widehat{v_A}, v_B))(v_B - \overline{w_B}^*(\widehat{v_A}, v_B)) - G({w_B}^*(\widehat{v_A}, v_B))(v_B - {w_B}^*(\widehat{v_A}, v_B) - v_A)$.

Proposition 5. *In the sequential first-price auction, where good A is sold prior to good B, there exists a unique equilibrium, in which*

(i) in the first auction, the flexible bidder submits a strictly positive bid, iff $v_A < \widehat{v_A}$ *for given* v_B*, and he submits a zero bid iff* $v_A \geq \widehat{v_A}$ *for given* v_B*, and*

(ii) in the second auction, the flexible bidder bids always submits a strictly positive bid.

Intuitively, if the expected additional gain from winning the second auction after the first auction is won already, is too small relative to v_A, then it is not worth bidding in the first auction to begin with. Instead, a higher expected gain can be achieved by forgoing the bid in the first auction, and making an optimal standard first-price auction bid in the subsequent auction. The proof proceeds by backwards-induction. After having won the first auction, bidder F occurs a sunk cost of w_A. To determine her bid in the second auction, she updates her value for good B and makes a correspondingly optimal bid. Again, the incremental gain from winning good B is $v_B - v_A$, but has to be adjusted for optimal shading due to the first-price rule. Now consider the sequential auction where good B is sold prior to good A.

Proposition 6. *In the sequential first-price auction, where good B is sold prior to good A, there exists a unique equilibrium, in which*

(i) in the first auction, the flexible bidder always submits a strictly positive bid, and

(ii) in the second auction, the flexible bidder bids always submits a strictly positive bid after losing the first auction, and a zero bid after winning the first auction.

Naturally, after winning the higher value good in the first auction, there is nothing to be gained from participating in the second auction. If the first auction is lost, the flexible bidder can bid again in the second auction. The difference in expected payoff between the case where she wins or loses the first auction determines her true value.

Simultaneous auctions.

Suppose now the auctioneer sells good A and good B in two separate auctions which take place simultaneously. Bidder F can participate either in both auctions or bid only in either the A-auction or the B-auction. Participating in both auctions comes with the risk of potentially winning in both auction, and having to pay for

a good that brings no additional value. In reality, this exposure may prevent the flexible bidder from bidding optimally. Her maximisation problem is

$$\begin{aligned}\max_{w_A,w_B} \Pi^F_{Sim2|v_A,v_B}(w_A,w_B) &= G(w_A)\left(1-G(w_B)\right)v_A + G(w_B)v_B \\ &\quad - \int_0^{w_A} a\,\mathrm{d}G(a) - \int_0^{w_B} b\,\mathrm{d}G(b) \\ \text{s.t. } 0 \le w_K \le v_K, K &= A,B\end{aligned}$$

Proposition 7. *In the simultaneous second-price auction, there exists an equilibrium, in which the flexible bidder makes a strictly positive bid* $({w_A}^*, {w_B}^*)$*, characterised by equations (A.16) and (A.17) (in the appendix).*

In the appendix, I show that the two equations fully characterise the global optimum of bidder F's profit maximisation problem. This global optimum is always interior, i.e. despite the risk of having to pay for both goods, the flexible bidder participates in both auctions, if she bids optimally. This result also holds under first-price payments. Proceeding similarly, I write bidder F's maximisation problem as

$$\begin{aligned}\max_{w_A,w_B} \Pi^F_{Sim1|v_A,v_B}(w_A,w_B) &= G(w_A)\left(1-G(w_B)\right)v_A - G(w_A)w_A + G(w_B)\left(v_B - w_B\right) \\ \text{s.t. } 0 \le w_K \le v_K,\ K &= A,B\end{aligned}$$

Proposition 8. *In the simultaneous first-price auction, there exists an equilibrium, in which the flexible bidder makes a strictly positive bid* $({w_A}^*, {w_B}^*)$*, characterised by equations (A.24) and (A.25) (in the appendix).*

In addition to the standard trade-off in a single good first-price auction, increasing the probability to win the respective other good decreases the expected payoff on the first good, because of the looming possibility to have to overpay for the higher-value good. Through the equilibrium characterisation I also obtain the following result, which will be useful for numerical analysis.

Proposition 9. *In the simultaneous auction (first-price or second-price), if* $G(x)$ *is convex* $\forall x \in [0,\overline{v}]$*, or if* $G(x)$ *is concave* $\forall x \in [0,\overline{v}]$*, the equilibrium is unique.*

2.3 General model

There are M indivisible goods sold in an auction, one unit of each good. $M+1$ types of bidders compete for the goods. There is one flexible bidder F ("she") who values one unit of good j at v_j, $j \in \mathcal{J} := \{1, ..., M\}$, but at most one unit in total. I assume $v_M \geq v_{M-1} \geq ... \geq v_1 > 0$. Let $\mathbf{v} = (v_1, ..., v_M)$, let $q_j \in \{0, 1\}$ denote the quantity of good j allocated to the flexible bidder at price p_j, and let $\mathbf{q}$ and $\mathbf{p}$ denote the corresponding M-vectors. Furthermore, let j^{max} denote the good with the highest index she is allocated, i.e. $j^{max} := \max\{j \in \mathcal{J} : q_j = 1\}$. $\langle x, y \rangle$ denotes the dot product of two vectors x and y. The flexible bidder's utility is then

$$u^F(\mathbf{p}, \mathbf{q}) = v_{j^{max}} - \langle \mathbf{p}, \mathbf{q} \rangle$$

There is a group G_j of $n_j \geq 2$ inflexible bidders, each of which is interested only in good j; there are M groups in total. Each such group can be seen as a competitive fringe for a specific good. It is without loss of generality to assume that $n_j = n \ \forall j$: the size of each competitive fringe does not matter. Bidders of group G_j value one unit of good $j \in \mathcal{J}$ at $x_j \geq 0$. Letting $\mathbf{q}$ denote the vector of quantities allocated, we have

$$u^j(\mathbf{p}, \mathbf{q}) = x_j q_j - \langle \mathbf{p}, \mathbf{q} \rangle$$

All M goods are identical to the auctioneer at a zero reserve price. Each bidder knows their value privately, but the distribution of the inflexible bidders' values is common knowledge. The inflexible bidders also know that they have at least one competitor with an identical value. All x_j are random variables drawn from an absolutely continuous probability distribution G with probability density function g and support $[0, \overline{v}]$, where $\overline{v} \in \mathbb{R} \cup \{\infty\}$. Assumption 1 still holds. For most of my results, I assume $v_M \leq \overline{v}$. The description of the auction formats is easily extended to the case with M goods.

Product-mix auctions. I define (i) the bidding language, (ii) the allocation rule, and (iii) the payment rule.

(i) The bidding language allows single and paired bids.[17] The flexible bidder submits a bid $\mathbf{w} = (w_j)_{j=1,...,M}$ to state demand for 1 unit of good 1 *or* good 2 *or*

[17]Paired bids are an instance of XOR bids (see, e.g., Sandholm (2002)) where each component specifies the quantity and price of exactly one good. Let $\mathbf{p}$ denote the auction prices. A paired bid $\mathbf{w} = (w_j)_{j=1,...,M}$ expresses the following preference: if $b_k - p_k > b_j - p_j$ for all $j \neq k$, then the bidder would like to receive up to one unit of good k. If the inequality sign is replaced by an equality for some good j, the bidder is indifferent between receiving good k and j, assuming indivisibility of goods.

... *or* good M, but not more than one in total, at prices of up to w_j for good j, respectively. The inflexible bidders of group G_j submit a single bid each for good j.

(ii) The auctioneer is given a bid list $\mathbf{w}, (y_1, 0, ..., 0), ..., (0, ..., y_M)$. In the standard PMA, he then determines the efficient allocation given the reported values, i.e. the flexible bidder obtains good k if and only if $w_k - y_k > w_j - y_j$ for all $j \neq k$, and $w_k - y_k \geq 0$.[18] In the first-price PMA, the same allocation rule applies. Together with the pricing rule, this is equivalent to choosing the revenue-maximising allocation, as in the menu auction by Bernheim and Whinston (1986).

(iii) In the standard PMA, the auctioneer determines the lowest competitive equilibrium prices. In my model, prices will always be set by the inflexible bidders' bids $y_j, j \in \mathcal{J}$.[19] In the first-price PMA, each winning bidder pays their bid price for the unit won.

Simultaneous auctions. Bidders can participate in any number of M simultaneous single-unit auctions. In particular, a bidder could choose to bid only in one auction, or in all of them. The auction price is set by the highest losing bid, respectively, under the second-price rule. A winner pays her bid under the first-price rule.

Sequential auctions. M single-unit auctions for one of the goods, respectively, are held sequentially. Bidders can participate in any number of the M auctions. After each auction, the auctioneer announces the winning allocation and the clearing price. Bidders then choose their bid in the subsequent auction. Payment rules are as in the simultaneous auction. There are $M!$ ways of ordering the sale. We are only interested in two orderings: the case where the goods are ordered according to the flexible bidder's values from lowest to highest, and reverse.

Remark: Again, the standard PMA (second-price rule) is equivalent to the VCG mechanism (see section 2.2).

2.3.1 Equilibrium bidding

As in the case with two goods, I make use of the following preliminary lemma. The proof from the two good case trivially extends to this model, and again, the flexible bidder never chooses a weakly dominated strategy, i.e. bids not more than her value on any good.

[18] Ties are broken in favour of higher value goods in case of indifference between the components of the paired bid, and in favour of the flexible bidder between her and the inflexible bidders; but this is not important for my analysis.

[19] In general, prices are determined in a linear programme. A detailed description of the price setting concept would go beyond the scope of my study and can be found in Baldwin and Klemperer (2019*a*).

Lemma 3. *In the auction formats introduced above, at least two inflexible bidders in each group bid their true values in any equilibrium.*

Product-mix auctions. Again, it is without loss of generality to assume that the flexible bidder makes one flat paired bid. Denote this bid $\mathbf{w} = (w_1, ..., w_M)$. In the same way, each bidder of group G_j makes a single bid $(0, ...0, x_j, 0, ...0), j \in \mathcal{J}$. Then, the efficient allocation is such that bidder F wins good k if and only if $w_k \geq x_k$ and $w_k - x_k = \max_{j \in \mathcal{J}} w_j - x_j$.[20] Given the equilibrium behaviour of all inflexible bidders, it is easy to argue the flexible bidder's optimal strategy, which straightforwardly extends the two-good model.

Proposition 10. *In the PMA with the second-price rule, there exists an equilibrium in which the flexible bidder bids truthfully. If $v_M \geq \bar{v}$, there exists a continuum of equilibria in which $w_M \geq \bar{v}$ and $w_j = \max\{w_M + v_j - v_M, 0\}$ $\forall j \in \mathcal{J}, j \neq M$.*

The flexible bidder's payoff function is not straightforward to construct with M goods. I skip this step and discuss the first-price PMA instead. The payoff function in the first-price PMA can easily be translated to the standard PMA.

In the first-price PMA, by straightforward extension of the two-good case, it is without loss of generality to assume that she makes a flat paired bid $\mathbf{w} = (w_1, ..., w_M)$. The inflexible bidders bid their value in equilibrium. Let $P_k(\mathbf{w})$ denote the probability of bidder F winning good k. She wins good k if and only if $w_k \geq x_k$ and $w_k - x_k = \max_{j \in \mathcal{J}} w_j - x_j$. So the probability of winning good k is given by

$$P_k(\mathbf{w}) = Prob(w_k \geq x_k \text{ and } w_k - x_k = \max_{j \in \mathcal{J}} w_j - x_j) \tag{2.4}$$

As a first step, I write bidder F's expected payoff function as

$$\Pi^F_{P1|\mathbf{v}}(\mathbf{w}) = \sum_{k=1}^{M} P_k(\mathbf{w}) \left[v_k - w_k\right] \tag{2.5}$$

To find a closed form expression for $P_k(\mathbf{w})$, I first attempt to find some structure in the flexible bidder's equilibrium bids. Given her values $v_M \geq v_{M-1} \geq ... \geq v_1 > 0$, I can show that $w_{k'} > w_k$ for $k' < k$ cannot be part of an equilibrium, because switching the bid prices between good k and k' would guarantee a strictly higher profit.

Lemma 4. *In equilibrium, the flexible bidder's bids must be such that $w_M \geq w_{M-1} \geq ... \geq w_1$.*

[20] The events $w_j = x_j$, $w_k = x_k$, and $w_j + x_k = w_k + x_j$ for $k \neq j \in \mathcal{J}$ occur with zero probability.

Using Lemma 4 and the fact that all inflexible bidders bid truthfully, we can now write down an expression for $P_k(\mathbf{w})$. Let k denote the good that the flexible bidder wins. Let $\{l_1, l_2, ..., l_P\} =: \mathcal{F} \subseteq \mathcal{J}$ denote any possible subset of goods (including the empty set), which the flexible bidder obviously loses to the inflexible bidders, that is good $l \in \mathcal{F} \Leftrightarrow w_l < x_l$. Let σ denote the permutations of goods that the flexible bidder does *not obviously* lose to the inflexible bidders, i.e.

$$\sigma = \begin{pmatrix} j_1 & j_2 & ... & j_{M-P-1} \\ \sigma(j_1) & \sigma(j_2) & ... & \sigma(j_{M-P-1}) \end{pmatrix}$$

where $\{j_1, ..., j_{M-P-1}\} = \mathcal{J} \setminus \mathcal{F}$. A permutation σ may be interpreted as follows: given a realisation $\mathbf{x}$ and a bid $\mathbf{w}$, the order of σ corresponds to the "order of efficiency" among the goods the flexible bidder does not obviously lose to the inflexible bidders. That is, the permutation σ is such that $w_{\sigma(j_1)} - x_{\sigma(j_1)} \geq w_{\sigma(j_2)} - x_{\sigma(j_2)} \geq ... \geq w_{\sigma(j_{M-P-1})} - x_{\sigma(j_{M-P-1})}$. I also define $\sigma(j_0) := k$.

Further, let $\overline{x}_{\sigma(j)}$ denote the upper bound of the integral corresponding to integration over $x_{\sigma(j)}$, and let $\underline{x}_{\sigma(j)}$ denote the lower bound of the integral corresponding to integration over $x_{\sigma(j)}$.

Then, we have

$$\begin{aligned} P_k(\mathbf{w}) = \sum_{\mathcal{F} \subseteq \mathcal{J} \setminus \{k\}} \sum_{\substack{\sigma(j): \\ j \in \mathcal{J} \setminus \{\mathcal{F} \cup \{k\}\}}} & \int_{w_{l_1}}^{\overline{v}} ... \int_{w_{l_P}}^{\overline{v}} \\ & \int_{\max\{\underline{x}_{\sigma(j_{M-P-2})} - \overline{x}_{\sigma(j_{M-P-2})} + x_{\sigma(j_{M-P-1})}, 0\}}^{w_{\sigma(j_{M-P-1})}} \int_{\max\{\underline{x}_{\sigma(j_{M-P-3})} - \overline{x}_{\sigma(j_{M-P-3})} + x_{\sigma(j_{M-P-2})}, 0\}}^{w_{\sigma(j_{M-P-2})} - w_{\sigma(j_{M-P-1})} + x_{\sigma(j_{M-P-1})}} ... \\ & \int_{\max\{\underline{x}_{\sigma(j_0)} - \overline{x}_{\sigma(j_0)} + x_{\sigma(j_1)}, 0\}}^{w_{\sigma(j_1)} - w_{\sigma(j_2)} + x_{\sigma(j_2)}} \int_0^{w_{\sigma(j_0)} - w_{\sigma(j_1)} + x_{\sigma(j_1)}} \\ & \mathrm{d}G(x_{\sigma(j_0)}) \, \mathrm{d}G(x_{\sigma(j_1)}) \, ... \, \mathrm{d}G(x_{\sigma(j_{M-P-2})}) \, \mathrm{d}G(x_{\sigma(j_{M-P-1})}) \, \mathrm{d}G(x_{l_P}) \, ... \, \mathrm{d}G(x_{l_1}) \end{aligned} \tag{2.6}$$

We are adding $\sum_{l=0}^{M-1} \binom{M-1}{l} (M-1-l)!$ integral terms.

Using equations (2.5) and the expression for P_k, bidder F's maximisation problem is

$$\max_{\mathbf{w}} \quad \Pi^F_{P1|\mathbf{v}}(\mathbf{w}) \quad \text{s.t. } 0 \leq \mathbf{w} \leq \mathbf{v}, \text{ and (2.5) and (2.6) hold}$$

The choice set $\mathcal{C}_M := [0, v_1] \times ... \times [0, v_M]$ is an M-dimensional parallelepiped. I show that the function $\Pi^F(\mathbf{w})$ cannot attain its maximum on the boundary of $\mathcal{C}_M$. This immediately implies that the global optimum must be interior and coincide with a stationary point. Hence, the first-order conditions fully characterise the solution to bidder F's optimal bidding problem.

Proposition 11. *In the first-price PMA, there exists an equilibrium, in which the flexible bidder makes a strictly positive bid* $({w_k}^*)_{k=1,...,M}$*, which is implicitly defined by the first-order condition* $\boldsymbol{\nabla}\Pi^F_{P1|\mathbf{v}}(\mathbf{w}^*) = 0$.

The intuition behind this proposition is similar to the two-good case: the decrease in probability of winning a more favourable good is of a different (lower) order than the increase in probability of winning some good in events where otherwise the bidder would have won nothing. For small bids, the order effect outweighs that a higher value could be potentially achieved from winning more favourable goods, i.e. it is better to win more often overall, but potentially a lower-value good.

Simultaneous auctions. In a model with M simultaneous second-price auctions, I show that the flexible bidder always bids a strictly positive bid in each of the M auctions. Note that Gerding et al. (2008) have developed a similar result, when the goods are all identical to the flexible bidder.[21] However, we are interested in a model of differentiated goods, i.e. we maintain different values for different goods for the flexible bidder. Of course, dominated strategies will never be played, so bidder F's reduced choice set is again $\mathcal{C}_M := [0, v_1] \times ... \times [0, v_M]$. Because each inflexible bidder of group G_k bids their value x_k in equilibrium, the flexible bidder's expected payoff function is

$$\Pi^F_{Sim2|\mathbf{v}}(\mathbf{w}) = \sum_{k=1}^{M} \left[v_k G(w_k) \prod_{l=k+1}^{M} (1 - G(w_l)) - \int_0^{w_k} x g(x)\, \mathrm{d}x \right]$$

and she solves the maximisation problem

$$\max_{\mathbf{w}} \quad \Pi^F(\mathbf{w}) \quad \text{s.t. } 0 \leq \mathbf{w} \leq \mathbf{v}$$

The first-order condition for w_j of the corresponding unconstrained problem is

$$w_j = -\sum_{k=1}^{j-1} v_k G(w_k) \prod_{l=k+1, l\neq j}^{M} (1 - G(w_l)) + v_j \prod_{l=j+1}^{M} (1 - G(w_l)) \tag{2.7}$$

Proposition 12. *In the simultaneous second-price auctions, there exists an equilibrium, in which the flexible bidder makes a strictly positive bid* $({w_k}^*)_{k=1,...,M}$*, which is implicitly defined by the first-order condition* $\boldsymbol{\nabla}\Pi^F_{Sim2|\mathbf{v}}(\mathbf{w}^*) = 0$.

[21] For certain distributions of the inflexible bidders' values, they provide additional results on equilibrium characterisation.

Example 1. Note that symmetric values of the flexible bidder do not necessarily imply symmetric bids in equilibrium. To demonstrate this, let $M = 2$ and $v_1 = v_2 = v < 1$. I assume a probability distribution with support $[0, 1]$ and a uniform spike around $x \in [0, 1]$, where $x < v$. Let $h := \frac{1-\epsilon+2\epsilon^2}{2\epsilon}$ and $\epsilon < x$. Formally, the probability density function is

$$g(t) = \begin{cases} \epsilon & \text{if} \quad t < x - \epsilon \\ h & \text{if} \quad t \in [x - \epsilon, x + \epsilon) \\ \epsilon & \text{if} \quad t \geq x + \epsilon \end{cases}$$

Now let $\epsilon = 0.1$ (hence $h = 4.6$), $v = 0.7$, and $x = 0.5$. In the appendix, I show that the global maximum is at $({w_A}^*, {w_B}^*) \approx (0.02, 0.70)$. Depending on how much probability mass concentrates around the spike, we may also obtain a symmetric equilibrium. For example, let $\epsilon = 0.4$ (hence $h = 1.15$), $v = 0.7$, and $x = 0.5$. The global maximum here is $({w_A}^*, {w_B}^*) \approx (0.42, 0.42)$.

Our results under the second-price rule also hold under first-price payments. Although obvious that truthful bidding cannot be optimal in any one auction, it is not clear that the flexible bidder still makes a strictly positive bid in each auction. The flexible bidder's expected payoff function is

$$\Pi^F_{Sim1|\mathbf{v}}(\mathbf{w}) = \sum_{k=1}^{M} \left[v_k G(w_k) \prod_{l=k+1}^{M} (1 - G(w_l)) - G(w_k) w_k \right]$$

with the first-order conditions

$$w_j = -\sum_{k=1}^{j-1} v_k G(w_k) \prod_{l=k+1, l\neq j}^{M} (1 - G(w_l)) + v_j \prod_{l=j+1}^{M} (1 - G(w_l)) - \frac{G(w_j)}{g(w_j)} \qquad (2.8)$$

In addition to the standard trade-off between payment and probability of winning in a single good first-price auction, increasing the probability to win one good decreases the expected payoff on any higher-value good.

Proposition 13. *In the simultaneous first-price auctions, there exists an equilibrium, in which the flexible bidder makes a strictly positive bid* $({w_k}^*)_{k=1,...,M}$*, which is implicitly defined by the first-order condition* $\boldsymbol{\nabla} \Pi^F_{Sim1|\mathbf{v}}(\mathbf{w}^*) = 0$.

Sequential auctions. When M auctions are held sequentially, the history of past auction outcomes at each stage is relevant for bidder F's bidding decision. There are $M!$ different orders in which the goods can be sold. I focus on the two most natural orders, which are of course ordering the goods by value from high to low, and from low to high. Let us assume first that goods are sold in the order $1, 2, ..., M$.

In each auction, the flexible bidder knows which past auctions she has won, and at what price. Prices paid in the past are sunk costs, however, and do not matter in the backwards-induction. It only matters to the bidder by how much she could potentially improve her current payoff. Denote by H_j the history known in the auction of good $j+1$, i.e. auction outcomes of auctions $1, ..., j$. Let v_{j-1}^{max} denote the value of the highest-value object bidder F obtained in the history H_{j-1}.

Proposition 14. *In the sequential second-price auction, in which goods* $1, ..., M$ *are sold ordered from the lowest-value to the highest-value good, there exists an equilibrium, in which the flexible bidder's bid in the auction for good* j *is characterised by equation (A.32).*

When the order of sale is reversed, i.e. goods are sold in the order $M, ..., 1$, bidder F's strategy simplifies. She submits a strictly positive bid in each auction until she wins for the first time; thereafter she submits zero bids in all subsequent auctions. Details can be found in the appendix.

A similar result as above can be obtained for M sequential first-price auctions.

Proposition 15. *In the sequential first-price auction, in which goods* $1, ..., M$ *are sold ordered from the lowest-value to the highest-value good, there exists an equilibrium, in which the flexible bidder's bid in the auction for good* j *is characterised by equation (A.33).*

The case with the reverse sales order is again similar to the second-price auction and omitted.

2.3.2 Identical values

Equilibrium bidding can be illustrated further if we consider identical values for the flexible bidder, i.e. $v_j = v \ \forall j \in \mathcal{J}$. In the product-mix auction and in the simultaneous auctions, I restrict my analysis to symmetric equilibria.[22] In sequential auctions, equilibria are inherently asymmetric. I characterise equilibrium bidding, the flexible bidder's payoff function and the auctioneer's revenue for first-price auctions in appendix A.2.7. Revenue in second-price auctions is identical for all auction formats, because auction prices are always given by the inflexible bidders' bids. I use the equilibrium characterisation in the example below.

[22]Symmetric equilibria may not exist for more extreme distributions of the inflexible bidders' values; see Example 1.

Example 2. More insightful are some numerical computations for different numbers of goods M. In this example, let the flexible bidder's value $v = 0.9$ and let the inflexible bidders' values be uniformly distributed on $[0, 1]$. The flexible bidder's equilibrium bids and profits are shown in Figures 2.2 - 2.5. Note that each dot of the sequential auction, corresponding to M goods, represents the equilibrium bid in the first of M auctions. Because the equilibrium is subgame-perfect, the bids up to good M also depict the sequence of equilibrium bids when M goods are sold.

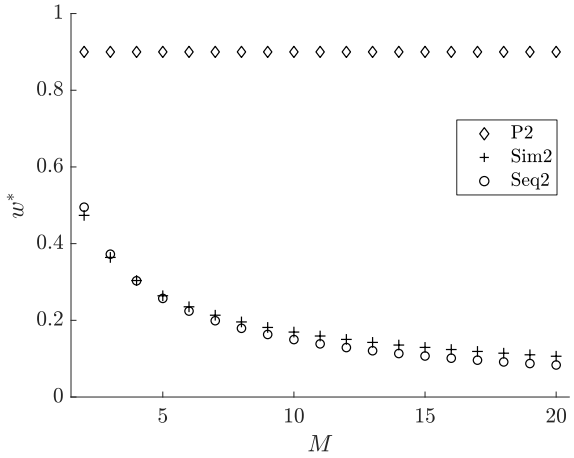


Figure 2.2: Equilibrium bids in second-price auctions

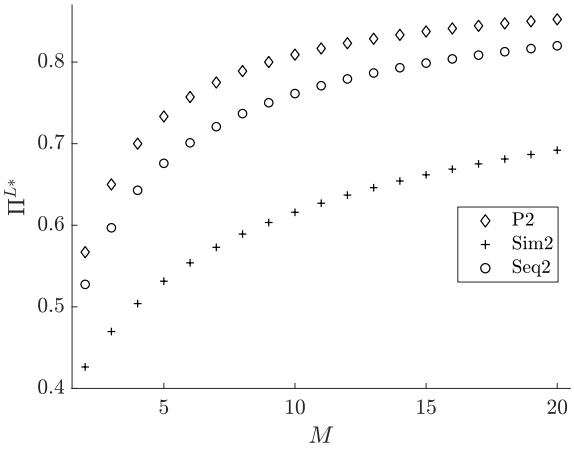


Figure 2.3: Bidder surplus in second-price auctions

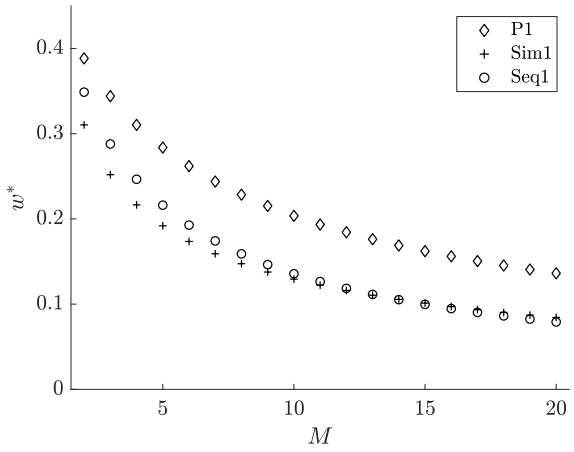


Figure 2.4: Equilibrium bids in first-price auctions

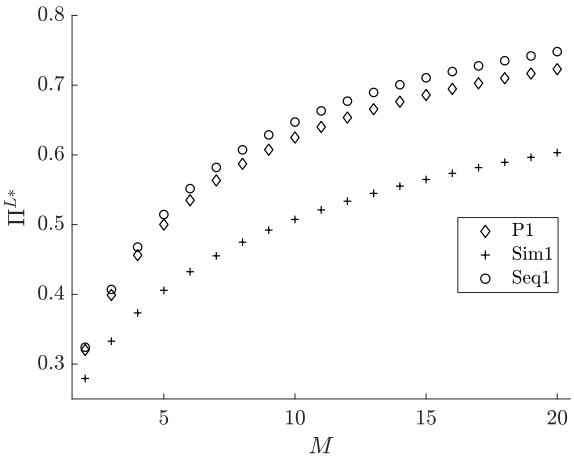


Figure 2.5: Bidder surplus in first-price auctions

Under second-price payments, as the number of goods increases, the flexible bidder increasingly shades her bids in the first of the sequential auctions. As evident from Figure 2.2, with 20 goods, the optimal bid in the first of 20 sequential auctions

is even lower than her optimal bid in 20 simultaneous auctions.[23] This translates directly into the flexible bidder's payoffs: differences between the sequential and the simultaneous auction are increasing in the number of goods. In both auction formats, the flexible bidder is more likely to win at least one of the goods with more goods for sale. However, in the sequential auction, she can better avoid the risk of winning more than one good. Interestingly, the differences in payoffs between the PMA and the sequential auction are first increasing and then decreasing slightly in the number of goods.

Under first-price payments, the effects observed in the flexible bidder's optimal bidding are similar to those under second-price payments. The optimal bids in the sequential and simultaneous auction move closer together as the sequential auction offers a higher probability to win exactly one good. The optimal bids in the first-price PMA compared to sequential and simultaneous auctions are moving further apart: the more goods are for sale, the less bid shading is necessary in the PMA relative to the other formats. The flexible bidder increasingly benefits from stronger bid shading in the sequential auction as the number of goods increases, comparatively to the PMA. The simultaneous auction does worse in terms of payoffs as more goods are sold: the risk of winning more than one good paired with the first-price disadvantage outweighs the benefit of a higher probability of winning at least one good.

Note that for approximately $M \geq 10$, the differences in payoffs stagnate for all auction formats. This suggests some kind of saturation in how much the flexible bidder can exploit the benefits of a specific auction format, as the number of goods for sale grows larger.

2.4 Numerical results

In this section, I derive and compute results on bidder surplus, revenue, and efficiency for the two-good model. First, I focus on the example where a and b are uniformly distributed on $[0, 1]$, and I provide a detailed discussion. Then, I also present results for different distributions of a and b, varying the strength of the competitive fringes (symmetrically). Many of my results turn out to be robust to changes in the prior on a and b. Bidder surplus is denoted Π_Y^F, revenue R_Y, and efficiency/welfare W_Y, where Y is substituted by the abbreviation for the respective auction format introduced in section 2.2.

[23] I am assuming symmetric equilibria in simultaneous auctions. Asymmetric equilibria may exist for distributions of the inflexible bidders' values with more concentrated probability mass than in this uniform example.

2.4.1 Uniform distributions of a and b

First, I compare the bidder surplus, revenue, and efficiency given bidder F's values v_A and v_B (I call these "interim" outcomes). I restrict my attention to $(v_A, v_B) \in [0,1]^2$. For the comparison with sequential auctions, I introduce a straightforward metric which combines the respective outcomes from the sequential format with good A sold first, and the format with good B sold first: $\Pi^F_{Seq2|v_A,v_B} = \frac{1}{2}\Pi^F_{SeqA2|v_A,v_B} + \frac{1}{2}\Pi^F_{SeqB2|v_A,v_B}$, and analogously for the first-price sequential auction.[24]

For most auction formats, a closed form solution can be derived when a and b are uniform on $[0,1]$ (closed form solutions are stated in the appendix). The first-price PMA is an exception because the equilibrium bids are characterised implicitly, and comparisons with this auction are done numerically (implemented in *Matlab*; to compare closed form solutions I used *Mathematica*).

Interim bidder surplus. First I note the obvious: inflexible bidders make zero surplus in equilibrium in all auction types. Thus, the flexible bidder's equilibrium surplus equals overall bidder surplus. I state my results for $0 < v_A < v_B < 1$. When $v_A = 0$, $v_B = 1$, or $v_A = v_B$ (or a combination of these conditions), the flexible bidder may be indifferent between some of the auction types.[25] We find that

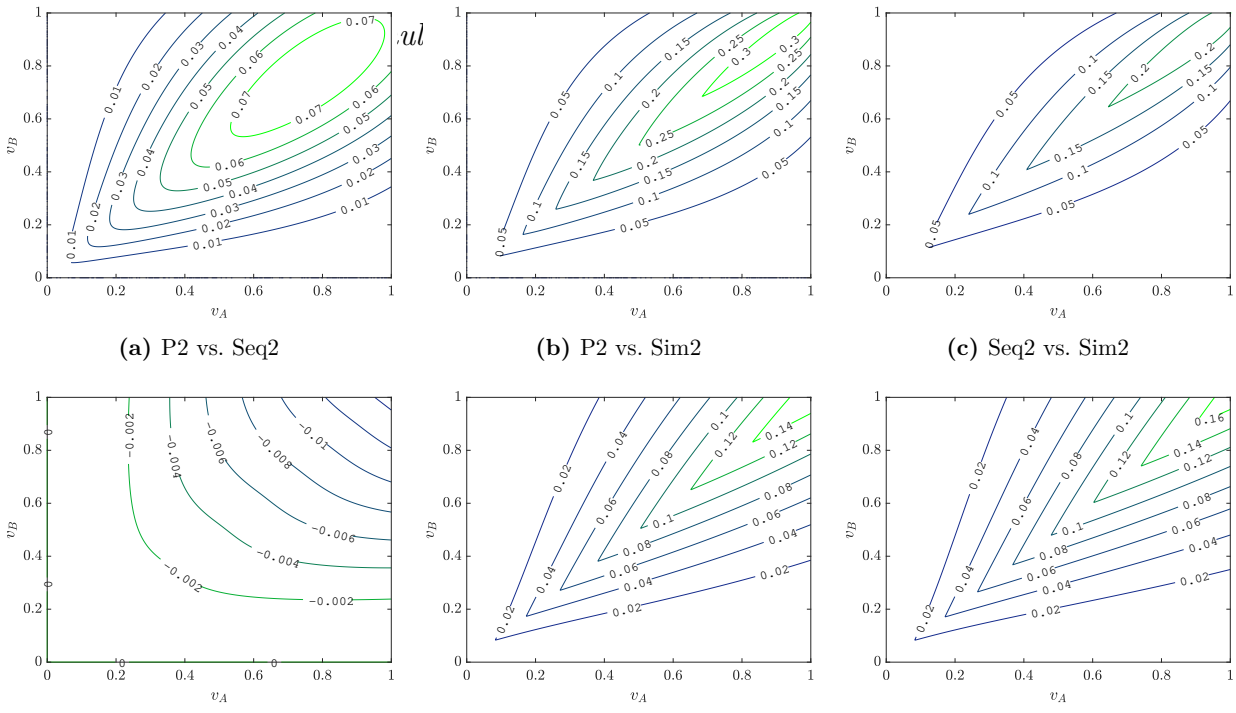
$$\Pi^F_{P2|v_A,v_B} > \Pi^F_{Seq2|v_A,v_B} > \Pi^F_{Sim2|v_A,v_B} \text{ and } \Pi^F_{Seq1|v_A,v_B} > \Pi^F_{P1|v_A,v_B} > \Pi^F_{Sim1|v_A,v_B}$$

The magnitude of the flexible bidder's preference for one auction format relative to another is illustrated in Figure 2.6. I plot the relative deviation of her equilibrium payoff in the PMA from her payoff in the sequential and simultaneous format under the corresponding payment rule as a contour plot, i.e. the 2D-projection of given levels of relative deviations as a function of v_A and v_B. For example, the left-most line in Figure 2.6(a) describes all combinations of v_A and v_B for which the standard PMA results in a 1% higher payoff for the flexible bidder than the sequential second-price auction. The contour lines are mirrored at the 45-degree line because equilibria are symmetric for valuations $v_A \geq v_B$.

Interim revenue. I state my results for $0 < v_A < v_B \leq 1$. When $v_A = 0$, or $v_A = v_B$ (or a combination of these conditions), revenue may be identical for some

[24] This metric is due to the fact that sometimes we may not know which of two differentiated objects will be higher value for the buyer, e.g. a red car or a blue car, because it may depend more on taste than an objective difference in quality. Being unaware of the true ordering in value, I simply assume that each ordering happens with equal possibility.

[25] When $v_A = v_B$, the sales order in sequential auctions is obviously irrelevant. When $v_B = 1$, the sales order in the sequential second-price auction is irrelevant. When $v_A = 0$, all auction types under the same pricing rule are strategically equivalent.



(a) P2 vs. Seq2 **(b)** P2 vs. Sim2 **(c)** Seq2 vs. Sim2

(d) P1 vs. Seq1 **(e)** P1 vs. Sim1 **(f)** Seq1 vs. Sim1

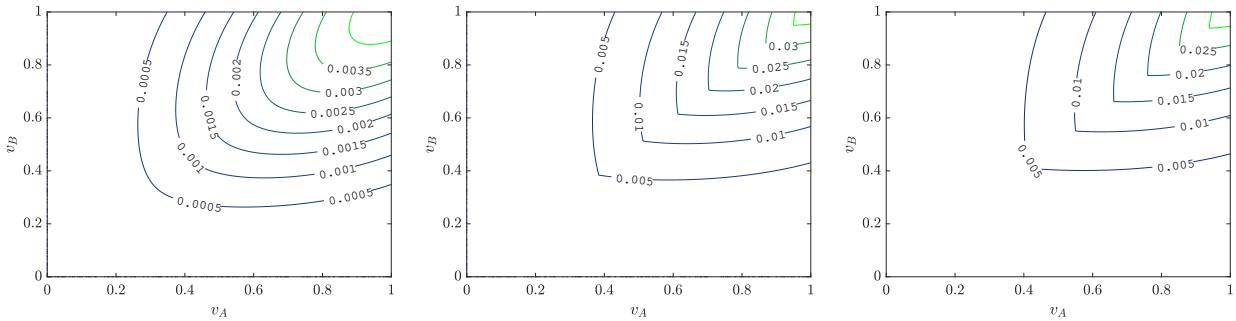
Figure 2.6: The flexible bidder's interim payoffs: "P2 vs. Seq2" shows the relative gain of mechanism $P2$ over mechanism $Seq2$, i.e. $(\Pi^F_{P2|v_A,v_B} - \Pi^F_{Seq2|v_A,v_B})/\Pi^F_{Seq2|v_A,v_B})$, etc.

of the auction types.[26]

Under the second-price rule, prices are always determined by the inflexible bidders' bids. We have

$$R_{Y|v_A,v_B} = E_{a,b}\left[a+b \mid v_A, v_B\right] = 1 \text{ and } R_{P1|v_A,v_B} > R_{Seq1|v_A,v_B} > R_{Sim1|v_A,v_B}$$

with $Y \in \{P2, Sim2, Seq2\}$. I illustrate the magnitude of the relative difference in revenues between one auction format and another in Figure 2.7.



(a) P1 vs. Seq1 **(b)** P1 vs. Sim1 **(c)** Seq1 vs. Sim1

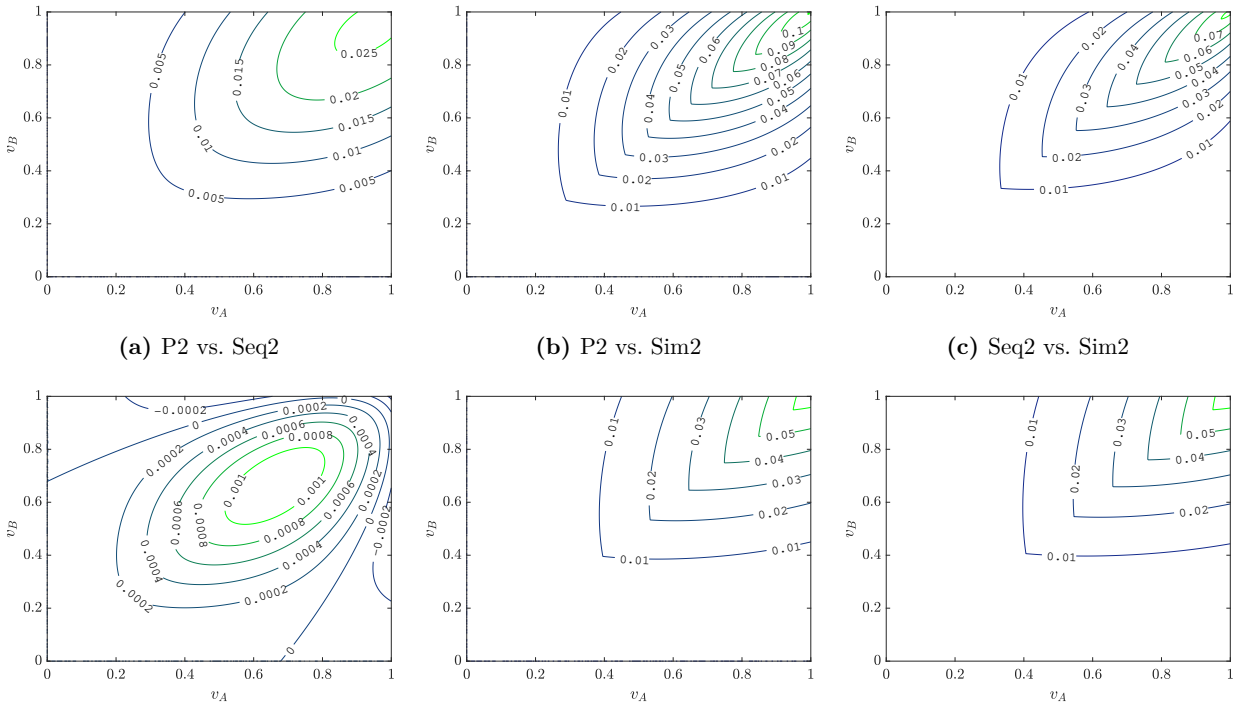
Figure 2.7: Interim revenue: "P1 vs. Seq1" shows the relative gain of mechanism $P1$ over mechanism $Seq1$, i.e. $(R_{P1|v_A,v_B} - R_{Seq1|v_A,v_B})/R_{Seq1|v_A,v_B})$, etc.

[26] Again, when $v_A = v_B$, the sales order in sequential auctions is irrelevant. When $v_A = 0$, all auction types under the same pricing rule are strategically equivalent, hence revenue is also the same.

Interim efficiency. To compare efficiency between the auction formats, I compute welfare as the sum of the flexible bidder's payoff and revenue (closed form solutions are listed in the appendix). I state my results for $0 < v_A < v_B < 1$, and by composition of the previous comparisons, indifference may hold for $v_A = 0$ or $v_A = v_B$. We find that

$$\mathcal{W}_{P2|v_A,v_B} > \mathcal{W}_{Seq2|v_A,v_B} > \mathcal{W}_{Sim2|v_A,v_B} \text{ and } \mathcal{W}_{P1|v_A,v_B} > \mathcal{W}_{Sim1|v_A,v_B}$$

The relation between (P1) and (Seq1) is ambiguous, depending on the values of v_A and v_B. The magnitudes of deviations are illustrated in Figure 2.8.



(a) P2 vs. Seq2 **(b)** P2 vs. Sim2 **(c)** Seq2 vs. Sim2

(d) P1 vs. Seq1 **(e)** P1 vs. Sim1 **(f)** Seq1 vs. Sim1

Figure 2.8: Interim efficiency: "P2 vs. Seq2" shows the relative gain of mechanism $P2$ over mechanism $Seq2$, i.e. $(W_{P2|v_A,v_B} - W_{Seq2|v_A,v_B})/W_{Seq2|v_A,v_B})$, etc.

Ex-ante outcomes. I also derive average outcomes across all possible realisations of v_A and v_B, and I call these "ex-ante outcomes". Table 2.1 shows the ex-ante bidder surplus, revenue, and efficiency rounded to 3 decimals. Note these are absolute values. They are computed assuming each value pair $(v_A, v_B) \in \mathbb{R}^2 | 0 < v_A < 1, v_A < v_B < 1$ occurring with equal probability.[27] Differences are very small; so in order to get a full and meaningful comparison, I distinguish the two sequential auction types in this table. The ordering of interim outcomes largely translates into ex-ante outcomes,

[27] If v_A and v_B were truly continuous, they would be drawn from a uniform distribution on $\{\mathbb{R}^2 | 0 < v_A < 1, v_A < v_B < 1\}$ with density function $g(v_A, v_B) = 2$. I compute the average for approx. 2M value pairs on the above domain.

Table 2.1: Ex-ante bidder surplus, revenue, and efficiency

$M =$	(P2)	(SeqA2)	(SeqB2)	(Sim2)	(P1)	(SeqA1)	(SeqB1)	(Sim1)
Π_M^L	0.283	0.272	0.275	0.258	0.149	0.149	0.151	0.142
R_M	1.000	1.000	1.000	1.000	1.069	1.067	1.068	1.062
W_M	1.283	1.272	1.275	1.258	1.218	1.215	1.219	1.204

despite the disaggregation of the two sequential auctions. Under the second-price rule, the ordering product-mix, Sequential, Simultaneous holds, with the sequential auction with the higher-value good sold first performing better than the other sequential format. Under the first-price rule, the sequential format with the lower-value good sold first performs identical to product-mix in bidder surplus and revenue, but falls back in efficiency. In contrast, the sequential format with the higher-value good sold first outperforms the product-mix auction in bidder surplus and efficiency, but not in revenue. The first-price PMA yields the highest revenues across all auction formats, and, naturally, the standard PMA achieves the highest efficiency overall.

2.4.2 Discussion

In Table A.1 I present several examples of equilibrium bids and interim outcomes for specific valuations v_A and v_B, which are helpful for understanding the intuition behind equilibrium outcomes. Note that the multi-column of the sequential auction formats displays the second-stage bid conditional on winning or losing the first stage, when applicable. Relating to payment rules and auction formats, I give some intuition for bidding behaviour and outcomes with the help of Figures 2.6, 2.7, and 2.8.

First-price and second-price rule. First, I note a significant difference between first-price and second-price formats. The structure of my model means that first-price auctions are unusually bad for the flexible bidder because her competitors also bid truthfully under the first-price rule. This effect is even more pronounced when her values are small. To see this, consider the sale of a single unit. The flexible bidder always bids half of her value in a first-price auction,[28] whether her valuation is 0.1 or 1. In a second-price auction, she always bids truthfully. Now consider the sale of 2 units, one of each good. Then, in the standard PMA, she still bids truthfully for any values. In the first-price auction (e.g. the sequential), with small values, the flexible bidder simply repeats her strategy (approximately)

[28] Bidding half of her value maximises her expected payoff given the trade-off between maximising the probability of winning and minimising her payment.

from the single good case and bids (approximately) half her valuation, because the likelihood of winning one unit (or both units) is negligible. With high values, things are less bad for her: because she needs to win only one unit, she can still significantly shade her bids. This implies that, comparing payment rules, the difference in payoffs is large especially for small valuations.

Bidder surplus. Because of the structural difference between pricing rules, my focus is on comparing different auction formats under the same pricing rule. The fact that winning two units is unlikely with low valuations also has a straightforward implication for those comparisons. There is little difference between the flexible bidder's payoffs in different second-price or first-price auction formats for small valuations, and similarly for strongly asymmetric valuations, because the flexible bidder focuses her bidding mainly on the high value good.[29]

Sequential auctions. An important aspect in sequential auctions is the "scope for gambling" for the good sold in the first auction, i.e. bidding for it at a very low price with a low probability of success. The second auction offers the possibility to rebid in case of losing the first auction, or even to improve on one's payoff when the lower value good was sold first (i.e. the flexible bidder can make up for a "too low" bid in the first auction). Three related effects are worth noting.

(i) "Commitment effect." The "scope for gambling" is higher in second-price auctions than in first-price auctions, because paying her own bid diminishes the opportunity to make up for losing the first auction. In other words, under the first-price rule the flexible bidder is more nervous about entering the second auction. Hence it is optimal for her to commit more to her first-stage bid, that is to shade her bid less relative to her value.[30]

(ii) "Reversed bid effect." Another effect can be observed in the sequential auction where the higher value good is sold first: the ratio of bid price to value is always higher in the second auction. When both values are high, the flexible bidder may even choose to make a higher bid for the low value good than for the high value good.

(iii) "Preferred gamble." Keeping the pricing rule fixed, the sequential format is better for the flexible bidder when the higher value good is sold first, compared to when the lower value good is sold first. In the former case, she has a higher chance of winning her preferred good at a cheap price compared to the latter case, because "gambling" in the first auction may strike her a good bargain.

[29] See Figure 2.6(a)-(c), (e),(f).

[30] This is illustrated for example in Table A.1 for values $v_A = 0.5$ and $v_B = 1$: comparing the equilibrium bids between (SeqA2) and (SeqA1), $w^*_{A^{SeqA2}} = 0.125$ and $w^*_{A^{SeqA1}} = 0.156$, the commitment effect in the first auction is obvious.

The sequential first-price auction does, on average, better for the flexible bidder than the first-price PMA. For low valuations the differences are negligible, and similarly for strongly asymmetric valuations: bidding behaviour from a single-unit sale is (approximately) replicated, and the "scope for gambling" is very low. This results in the contour lines in Figure 2.6 (d). If values are high, in the sequential auction where the high value good is sold first, we have the "reverse bid effect". Reversing her bid order relative to her values is, intuitively, a bad bidding strategy in a simultaneous or product-mix format. In the sequential decision, however, this reverse bid ordering can make the flexible bidder better off ex-ante.[31,32] Note that the "commitment" effect is beneficial to the flexible bidder. Under the first-price rule, the sequential auction with the higher-value good sold first is, on average, indeed better for her. Stronger "gambling" behaviour (although optimal ex-ante) makes her worse off comparatively.

Simultaneous auctions. Under the second-price rule, it is also true that the flexible bidder's payoff is always weakly higher in the PMA than in the simultaneous auction. Any action the flexible bidder takes in the simultaneous auction is also available to her in the PMA (two separate bid instead of one paired bid would allowed), yet she chooses to bid differently and must therefore be better off. This holds for general distributions of a and b.

Under the first-price rule, my numerical results show that with uniform distributions, on average, the flexible bidder is better off in the PMA compared to the simultaneous auction (see section 2.4.1). Generally, however, there may be states where the flexible bidder is better off in the simultaneous auction.[33,34]

[31] See Table A.1 for $v_A = 0.9$ and $v_B = 1$: $\overline{w_A}^*_{SeqB1} = 0.450$ and $w_{B}^*_{SeqB1} = 0.399$, and payoffs are larger in the sequential auction than the first-price PMA.

[32] The relationship between the first-price PMA and the sequential auction with the lower value good sold first is ambiguous, in line with the flexible bidder's "preferred gamble" described above. For sufficiently dissimilar valuations the flexible bidder slightly prefers the first-price PMA, because the incentive to "gamble" for the higher value good introduces a loss in expectations compared to the flexibility between goods that the PMA offers. When values are similar and sufficiently high, however, the disadvantage from bidding for the lower value good first becomes less significant, and the flexible bidder can better utilise her market power from the information update.

[33] Such states must be such that she wins good A in the PMA because the distance between her bid and the inflexible bidders' bid is smaller on good A, but wins both goods in the simultaneous auction, and is still better off despite having to pay for both goods. For such scenario to arise, it would have to be true that $v_A - w_A^{*P1} < v_B - w_B^{*Sim1} - w_A^{*Sim1}$. Due to the complexity of the first-order conditions, I cannot show this to be true or false for general G. Note that under second-price payments this condition becomes $v_A - a < v_B - b - a$, which contradicts the flexible bidder having won good A in the PMA.

[34] Under the second-price rule, the inflexible bidders do not respond to the flexible bidder's behaviour. Under the first-price rule, if the inflexible bidders do not behave competitively among themselves, their behaviour has to take into account the flexible bidder's optimal shading, and vice versa. In this case, it is still true that the flexible bidder's strategy space expands in the PMA, but her bidding would have to account for the inflexible bidders' shading too.

Revenue. My model will naturally tend to generate higher prices for first-price than second-price payment rules. To see this, consider again the single unit case. Assume the flexible bidder has value v and competes against a number of inflexible bidders who always bid their value, which is uniform on $[0, 1]$. Then, in the first-price auction the flexible bidder bids $\frac{v}{2}$, and in the second-price auction she bids her value v. The auctioneer's revenue is higher under the first-price rule: $R_{FP} = \int_0^{v/2} \frac{v}{2}\,\mathrm{d}x + \int_{v/2}^1 x\,\mathrm{d}x = \frac{1}{2} + \frac{v^2}{8} > \frac{1}{2} = \int_0^1 x\,\mathrm{d}x = R_{SP}$. In the two-good case, the same effect - the inflexible bidders bid the same under the first-price and the second-price rule - means that first-price auctions outperform second-price auctions in terms of revenue. Intuitively, and obvious from the single-unit case, this effect is stronger for high valuations of the flexible bidder.

Due to the "commitment effect" in sequential first-price auctions described above, the flexible bidder's bid prices in the first-price PMA and sequential first-price auctions differ only slightly. Committing to a higher bid in the first auction is beneficial for the flexible bidder, whereas the "gambling" behaviour (although optimal ex-ante) makes her worse off comparatively. Obviously, this "gambling" is not only bad for the flexible bidder but also for the auctioneer: when values are high, the PMA's advantage becomes slightly more pronounced. The relationship between revenues in the sequential and the simultaneous auction is similar to "P1 vs. Sim" as described below (figure 2.7 (c)).

It is intuitive that prices in the first-price PMA are always higher than in the simultaneous first-price auction because expected losses from winning both goods are anticipated through higher bid shading. As explained above, differences for low and strongly asymmetric values of the flexible bidder are negligible, whereas high and more similar values correspond to higher anticipated losses and hence more extreme differences between the two auction formats (illustrated in Figure 2.7(d)). The difference in bid shading between the first-price PMA and the simultaneous first-price auction is *greater on the low value good*, and thus the minimum of the two values predominantly determines the magnitude of the deviation in revenue ("Leontief-like" contour lines).[35]

Efficiency. It is clear that the standard PMA is efficient when each bidder's total demand does not exceed one unit. All other auction formats I consider do not achieve full efficiency due to the exposure to win more than one unit, or the first-price payment rule. Keeping the auction format fixed, the welfare increment in second-price auctions relative to first-price auctions, gained from a more efficient allocation to the flexible bidder, outweighs the welfare loss in terms of revenue

[35] See for example Table A.1 for $v_A = 0.5, v_B = 1$.

due to her exercising some market power. Thus, second-price payments lead to higher welfare in each auction format considered.[36]

Due to the invariance of revenues in second-price formats in my model, differences in efficiency between those auction formats are entirely determined by the flexible bidder's profits. This naturally implies that Figures 2.8(a) - (c) are just a reshaped version of the deviations in payoffs.[37] Comparing auction formats within the first-price rule, variations in the flexible bidder's payoffs are slightly stronger (most of the time), but revenues are much higher in absolute magnitudes. Hence, the *relative* difference in efficiency between auction formats is similar to the difference in revenues, when comparing simultaneous auctions to product-mix and sequential auctions. Comparing PMA and sequential auctions, the opposing effects of bidder surplus and revenue nearly cancel each other out.

2.4.3 Varying the competitive fringes

In this section, I vary the strength of the competitive fringes the flexible bidder is competing against. Again, I consider the model with two goods A and B for sale, and two competitive fringes. Each variation considered is identical for the two fringes. I present the analysis for first-price auctions only (but may extend this to second-price auctions in the future). In the previous section, I discussed the case where a and b are uniformly distributed on $[0, 1]$. Here, I examine distributions with probability density functions $g_k(x) = kx^{k-1}$ and $\widetilde{g}_k(x) = k(1-x)^{k-1}$, for $k \in \mathbb{N}$, and support $[0, 1]$. The corresponding distribution functions are $G_k(x) = x^k$ and $\widetilde{G}_k(x) = 1 - (1-x)^k$.

Notice that these distributions have a straightforward interpretation in terms of increased or decreased competition. g_k corresponds to k independent groups, where each group is composed of at least two identical bidders in perfect competition. All k groups are interested in the same good. Values across groups are drawn independently from a uniform distribution on $[0, 1]$. The flexible bidder faces the maximum of k values, i.e. increased competition, represented by g_k. Similarly, $\widetilde{g}_k$ also corresponds to k independent groups, where each group is composed of at least two identical bidders in perfect competition. Again, values across groups are drawn independently from a uniform distribution on $[0, 1]$; however, the group with the lowest value among the k groups will be selected as the competition the flexible bidder faces.[38] The distributions are depicted in Figure 2.9.

[36] A trivial observation from Table A.1 is that absolute differences between the flexible bidder's payoffs under the first-price and the second-price rule are higher than absolute differences between

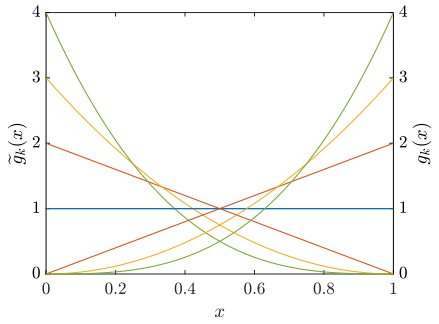


Figure 2.9: Probability density functions on $[0, 1]$

In each of the Tables A.2 - A.4 in appendix 2.4.3, I compare two auction formats. I choose two different metrics: (i) the relative difference with (approximate) maximum absolute value on $[0, v_A] \times [0, v_B]$, denoted "max diff", and (ii) the relative difference of average outcomes across $[0, v_A] \times [0, v_B]$, denoted "avg diff". For example, in the second line in Table A.2 we read that, for the uniform distribution, efficiency in the PMA is maximally 6.5% higher than in the simultaneous auction, at values $(v_A, v_B) = (1, 1)$ of the flexible bidder. In line 9, we read that, on average, for the uniform distribution, efficiency in the PMA is 1.2% higher than in the simultaneous auction.

I calculated outcomes for 1326 values pairs $(v_A, v_B) \in \{(x, y)|0 \leq x, y \leq 1, x \leq y\}$.[39] In each comparison of auction formats, for each outcome dimension and for every distribution of a and b, the value pair with the maximum absolute value relative difference is selected among those pairs, and the average across those pairs is computed.[40]

In Table A.2, I compare the first-price PMA with the first-price simultaneous auction. Note that maximum differences in efficiency and revenue decrease as competition for the flexible bidder increases, but the maximum difference in bidder surplus increases first slightly, and then rather dramatically from 15.5% to 43.4%.

revenues under the first-price and the second-price rule.

[37]Because I use relative deviations, the magnitudes of deviations in profits and revenues do not simply add up to magnitudes in efficiency deviations.

[38]One interpretation might be, that there are exactly two bidders in each group, and $2k-1$ units of one respective good are for sale, i.e. decreased competition. Important here is the modelling: the flexible bidder competes only against the weakest of k groups.

[39]For distributions of a and b other than the uniform distribution, I cannot obtain closed form solutions. Moreover, we only know for simultaneous and sequential auctions that the equilibrium is unique. In the first-price PMA, multiple equilibria, and multiple stationary points may exists. My propositions guarantee interior equilibria; therefore, I simply search among all stationary points.

[40]I chose approx. 2M value pairs for calculating averages for the uniform distribution in section 2.4.1. With different distributions, the model is computationally much more demanding; therefore I need to dispense with much fewer value pairs. However, varying the number of pairs convinced us to achieve more than sufficient accuracy with 1326 pairs.

Average values change in the same direction, but not with the same magnitude. In fact, differences in the average are nearly invariant to increased or decreased competition for efficiency and revenue. The difference in averages rises from 4.3% to 7.2% for bidder surplus, but by far not as much as the maximum difference. With high values and against strong competition, the flexible bidder has an incentive to bid much higher in the simultaneous auction; therefore, the exposure to win two units instead of one is much more pronounced compared to the PMA.

Comparing the first-price PMA with the first-price sequential auction with good A sold first in Table A.3, I find virtually no difference in efficiency and revenue, for maximum differences and differences in average. The flexible bidder does between 1.4% and 2.3% worse in the PMA, but this is also largely invariant to the strength of her competition. Differences in average bidder surplus disappear entirely. In Table A.4, I compare differences in outcomes between the first-price PMA and the first-price sequential auction with good B sold first. Varying the flexible bidder's competition does not impact the differences by much. Simply comparing absolute values across different distributions, we naturally find that efficiency and revenue are improving significantly, whereas bidder surplus drops by a factor of 10 going from low to high competition (see Tables A.5-A.7).

To summarise, varying the degree of competition impacts the differences between simultaneous and PMA/sequential auctions. Maximum differences in efficiency and revenue decrease as competition increases. The maximum difference in bidder surplus increases with increasing competition. Average difference do not respond as much to varying competition, and comparisons between other auction formats are also largely invariant to the degree of competition.

2.5 Conclusion

This chapter compares equilibrium bidding in product-mix, sequential, and simultaneous auctions. Considering first-price and second-price payment rules, I characterise unique equilibria in most auction formats under some regularity assumptions on value distributions. For the example of uniform distributions, equilibria are unique in all auction formats. My results are derived for a model of indivisible goods, but all results go through for the divisible good case.

For a broad class of value distributions and for any number of goods, I demonstrate that the flexible bidder faces an exposure problem in simultaneous auctions. Because it is ex-ante optimal to participate in every auction with a strictly positive bid, she runs the risk of winning both goods, when she can use only one (assuming

no possibility for resale). I compare alternative formats that alleviate this exposure problem. The answer is inconclusive: the product-mix and the sequential auction perform similarly in terms of efficiency, revenue, and bidder surplus; but subtle differences can be observed.

For uniform value distributions, the performance comparisons depend on the variation in the flexible bidder's values. Relative differences across auction formats, keeping the payment rule fixed, are most extreme for high values of the flexible bidder. The reason is that for small values the flexible bidder's probability of winning one (or both) goods is very small; her optimal strategy is to approximately repeat her optimal bid from the single-unit case twice. Averaging over the flexible bidder's values, I find that the first-price PMA performs best compared to other first-price auctions, in terms of revenue and efficiency. The flexible bidder, on the other hand, prefers the sequential format (on average) to the first-price PMA. Among second-price auctions, the sequential auction with the higher-value good sold first is the runner-up to the standard PMA, from the flexible bidder's and an efficiency perspective.

For first-price mechanisms, I vary the inflexible bidders' distribution to evaluate the robustness of my findings. I also obtain comparative statics with respect to the competition the flexible bidder is faced with. I select distributions that may be interpreted as varying the strength of the two competitive fringes. Differences between the first-price product-mix and first-price sequential formats are very subtle, and nearly invariant to changing the strength of competition. As competition becomes more fierce, the flexible bidder does increasingly worse in the simultaneous auction compared to an auction format eliminating the exposure to win both goods. In contrast, differences in efficiency and revenue between the PMA and the simultaneous auction become less pronounced with increased competition.

It remains an open problem to characterise equilibrium bidding under the first-price rule if inflexible bidders did not have identical competitors, or if two or more flexible bidders participated.

3

Strategic bidding for multiple objects: experimental evidence

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3.1 Introduction

Simultaneous and sequential sealed-bid auctions for multiple objects are common in electricity markets, for the allocation of online advertisement slots, for consumer goods, and many other products. In these auctions, buyers may often find it difficult to work out which objects to bid for, or which auction to participate in. Suppose a buyer wants either of two distinct objects, but no more than one in total. In which of two simultaneous or sequential auctions should they participate? How should they bid? Product-mix auctions, and combinatorial auctions more generally, offer a simple solution: the buyer can submit a bid for either object but is guaranteed to win at most one of them. If objects are ordinary or strong substitutes for bidders, they can perfectly represent their preferences. In practice, this auction format has been used by the Bank of England to allocate central bank loans (Klemperer (2008, 2010)).

An important open question is how well a product-mix auction performs when bidders have informational advantages and strategically influence the market price and allocation. The crux in the analysis of strategic behaviour in multi-object auctions is that standard game-theoretical tools like the Bayes-Nash equilibrium cannot be easily applied. Even in very simple settings involving bidders with valuations for multiple distinct objects, bid functions become multivariate functions which are much harder to work with. A long-standing tool to establish some understanding of more difficult multi-unit and multi-object markets are lab experiments Alsemgeest et al. (1998); Goeree et al. (2006); Kagel and Levin (2001); Ledyard et al. (1997). However, especially if not connected to a theory, it can be difficult to derive fundamental results and easy to run into issues with external validity.

My study is, to the best of my knowledge, the first to analyse strategic behaviour in a setting with multiple, distinct objects while simultaneously testing the theory with experimental results. I study the outcomes of a product-mix, a sequential, and a simultaneous auction under the *pay-as-bid pricing rule*, which has been widely used in practice for auctions of homogeneous goods,[1] and is also common in multi-unit markets that exhibit product differentiation.[2] Building on a theoretical model, I compare the performance of all three auction mechanisms with numerical simulations and I test these predictions in a virtual lab experiment. Three of the

[1] In auctions for homogeneous goods, Pycia and Woodward (2021) show that pay-as-bid pricing is revenue dominant and possibly welfare dominant over uniform pricing.

[2] Pay-as-bid pricing is also common in electricity markets, e.g. in England&Wales, Mexico, Peru, Panama, and Chile. It is also used for online advertisement auctions, conducted on platforms such as Google's AdSense (Xandr and OpenX, e.g., use a mix of first-price and second-price auctions).

treatments (one for each auction format) are modelled precisely after the theoretical model, in a market setting called the *computerised environment*. The other three treatments, while corresponding to the same auction formats, study a closely related market setting with symmetric bidders, the *human environment*. My theory predicts that, if bidders bid optimally, the product-mix auction performs nearly identically to the sequential auction with respect to the bidders' surplus, the auctioneer's revenue, and efficiency. Similarly, the simultaneous auction performs only slightly less well with a mere ∼5% (7%, 0.6%, 1%) difference in bidder surplus (flexible bidder payment, revenue, welfare). The experimental data, however, show substantial and significant differences between all three auction formats. The product-mix auction generates the highest bidder surplus and welfare. This effect holds for both the computerised and the human environment, but is much stronger with symmetric human bidders: in the product-mix auction payoffs are 90% - 156% higher and efficiency is 12% higher than in the other two auction formats. The simultaneous auction performs best in terms of revenue. All pairwise equivalence tests of outcomes (following the predictions of the theory) are non-significant, with the exception of the efficiency comparison between the sequential and the simultaneous auction. I also discuss bidding behaviour and to analyse to what extent bidders shade their bids. I find that bidders overbid significantly in all three considered auction formats, a phenomenon that has been documented and studied in many other auction experiments (Breitmoser, 2019; Cooper and Fang, 2008; Dorsey and Razzolini, 2003; Georganas et al., 2017; Goeree et al., 2002).

In my setup, two differentiated objects A and B are sold, and two markets, or environments, are considered. In the first environment, a single bidder (called the 'flexible bidder') who wishes to buy either object, but not more than one overall, competes against two competitive fringes, represented by a computerised agent who bids truthfully. This is a special case of the theoretical model set up in Chapter 2. However, in the experiment, I also study the more general environment in which three flexible bidders compete against each other for the two objects. This environment is closely related: the flexible bidder may win either object or nothing, but, of course, the prior about the highest competing bids are different. While currently there does not exist any theoretical model that could predict bidding behaviour is the three-bidder environment, my study sheds some light on how it may connect to a simpler, but solvable model with competitive fringes. In the experiment, in the computerised environment, each subject assumes the role of a flexible bidder and plays against computerised agents. In the human environment, subjects compete against other human bidders with the same type of preference and

symmetric priors in groups of three. My study is set up as a between-subject design: each participant is assigned to one auction mechanism in one given environment; hence, I consider six treatments. Each subject plays several auction rounds, bidding for one or both of the two objects, with round-independent induced values.

3.1.1 Related literature

Strategic behaviour in multi-unit markets has received much attention in the literature on auctions. For one, the stakes in many public and private sector auctions are extremely high: e.g., in wholesale electricity markets, in auctions for spectrum licensing, pollution permits, treasury bonds, government loans, and other financial markets. Moreover, market power often plays an important role in practice, as shown for examples for treasury auctions in various countries by Armantier and Sbaï (2006) (France), Malvey and Archibald (1998), Hortaçsu et al. (2018) (US), Hortaçsu and McAdams (2010) (Turkey). In electricity auctions, evidence of market power was found, e.g., by Borenstein et al. (2002) (California), Wolfram (1998) (UK), and Hortaçsu and Puller (2008) (Texas). Understanding and potentially improving the auction design in place was and still is in the interest of many decision-makers.

When the FCC began auctioning spectrum licences in 1994, first experiments with multi-object auctions were conducted not long after by Ledyard et al. (1997), testing the FCC's simultaneous procedure against sequential and combinatorial auctions. The auction formats, however, were all of ascending nature, and the considered market settings involved a greater number of players, rendering a theoretical analysis infeasible. In the more complex setups they study, the combinatorial auction outperforms the simultaneous. Similarly, Alsemgeest et al. (1998) study an ascending auction and a sealed-bid auction with uniform pricing, finding evidence of strategic demand reduction, and Gillen et al. (2016) and Katok and Roth (2004) investigate a descending-price auction for multiple, homogeneous units.[3] The question of uniform vs. discriminatory pricing has received great interest especially in the context of treasury auctions, as well as electricity auctions. In an experimental context, Abbink et al. (2006) find evidence that the uniform and a hybrid format used by the Bank of Spain raise higher revenue than discriminatory multi-unit auctions.

Other important experimental (and theoretical) papers set their focus e.g. on anticipated loser regret in first-price auctions (Filiz-Ozbay and Ozbay, 2007), collusion in ascending auctions (Sherstyuk and Dulatre, 2008), closing rules (Sherstyuk, 2009), resale (Filiz-Ozbay et al., 2015), or the comparison of supply function equilibrium

[3]Katok and Roth (2004) compare the Dutch auction also to an ascending uniform-price auction.

to a standard multi-unit auction model (Brandts et al., 2013). Special attention has also been given to multi-unit auctions in which truth-telling is an equilibrium, namely the Vickrey and Ausubel's 'clinching auction' (Engelmann and Grimm, 2009; Kagel and Levin, 2009)[4] and mechanisms by Leonard (1991), Demange and Gale (1985) and Demange et al. (1986) in experiments by Andersson et al. (2013), with respect to performance and truthful reporting.

An experimental study closely related to my research was conducted by Goeree et al. (2006), also comparing different first-price auctions for heterogeneous objects. However, in their setting every bidder is restricted to winning at most one object in all considered auction formats. This is only the case for the first-price product-mix auction I consider, a format also studied in Goeree et al. (2006). Importantly, however, I provide not only an experimental analysis but also a theoretical benchmark. Contrasting with Goeree et al. (2006)'s work, I am especially interested in the strategic incentives that arise if bidders are exposed to winning more than one object.

The remainder of the chapter is organised as follows. In Section 3.2, I briefly present the results and predictions from the theoretical model (an in-depth analysis of the more general model is given in Chapter 2). The experimental design is detailed in Section 3.3. The empirical model and the hypotheses tested in this study are set out in Section 3.5. In Section 3.6, I present the results of the experiment and the outcomes of the hypothesis tests, and Section 5.6 concludes.

3.2 Theoretical results

For the first bidding environment, in which subjects play against computerised agents, theoretical and numerical results can be derived. The market model is of the type 'local-local-flexible' (LLF).[5] The 'flexible' bidder has substitutes preferences across different objects. She is competing against two groups of local bidders. Each subject in the experiment assumes the role of a flexible bidder, whereas a group of local bidders is represented by a computerised agent. In this section, I briefly describe the model and summarise the theoretical results. A more detailed analysis and discussion was given in 2.

Two indivisible objects A and B are for sale, at zero cost. There are two types of buyers in the market, a flexible bidder and two groups of local bidders. All bidders

[4]Engelmann and Grimm (2009)test all of five different multi-unit auction format, an open, ascending-price and a sealed-bid uniform price auctions, a vickrey, an Ausubel, a discriminatory sealed-bid auction.

[5]This is similar to LLG models, in which a global bidder competes against local bidders. A global bidders usually has weakly complementary preferences between objects, i.e. is interested in both and there may be synergies (cf. Krishna and Rosenthal (1996)).

have quasi-linear utility. The *flexible bidder* wishes to buy object A or object B, but at most one. Her values for object A and B are given by $v_A > 0$ and $v_B > 0$, respectively, and at fixed prices p_A and p_B, her utility is

$$u(p_A, p_B) = \begin{cases} v_A - p_A & \text{if A is won} \\ v_B - p_B & \text{if B is won} \\ v_B - p_B - p_A & \text{if A and B are won} \\ 0 & \text{otherwise} \end{cases}$$

One group of at least two local bidders is interested only in object A, and another group of at least two local bidders is interested only in object B. Each local bidder of the first group values object A at a (and object B at zero), and each bidder of the second group values object B at b (and object A at zero). All values are privately known and the flexible bidder has a simple prior over the distribution of the local bidders' values: a and b are drawn from a uniform distribution on $(0, \bar{v})$. For the flexible bidder's values I assume that $v_A, v_B \in (v_{min}, v_{max})$ for some $v_{min} > 0, v_{max} \leq \bar{v}$. For my theoretical results, I set w.l.o.g $\bar{v} = 1$. For simulations and in the experiment, I scale the distribution and values such that $v_{min} = 0$ and $\bar{v} = v_{max} = 100$.

Local bidders know that, within their group, bidders have identical values. Therefore, at least two local bidders in each group aim to outbid or undercut each other in bid prices, and hence bid their true value in any Bayes-Nash equilibrium (a formal proof is given in **?**). That is, the highest bids the flexible bidder is competing against, denoted by s_A and s_B, are also drawn uniformly from $(0, 1)$. As a point of reference, I note that if restricted to bidding in a standard first-price auction for only one object, say B, against a group of local bidders, the flexible bidder's optimal bid is $\frac{v_B}{2}$. In the following, I define the considered auction mechanisms. All auction mechanisms are pay-as-bid (first-price) mechanism, i.e. a winning bidder pays their own bid prices for the objects they won.

Product-mix auction. The flexible bidder submits a 'paired bid' (w_A, w_B) to state demand for object A *or* object B, but not both, at prices of up to w_A or w_B respectively.[6] The local bidders' highest bid on object A is $(s_A, 0)$ and $(0, s_B)$ on object B. Given the bids $(w_A, w_B), (s_A, 0), (0, s_B)$, the auctioneer allocates

[6]Paired bids are an instance of XOR bids (see, e.g., Sandholm (2002)) where each component of an XOR bid specifies the quantity and price of exactly one object. Let p_A and p_B denote the auction prices. A paired bid (w_A, w_B) for object A and object B respectively expresses the following preferences: if $w_A - p_A > w_B - p_B$ and $w_A > p_A$, then the bidder wants up to one unit of object A. If the first inequality is reversed and $w_B > p_B$, she wants up to one unit of object B (equalities corresponds to indifference).

objects A and B in order to maximise efficiency with respect to the reported preferences.[7] The nature of the flexible bidder's bid does not allow her winning object A and B. Hence the 'valid combinations' of potential winning bids are $C := \{(w_A, s_B), (s_A, w_B), (s_A, s_B)\}$. The winning bid x_A on A and the winning bid x_B on B are those that maximise the sum of the valid combinations of submitted bids, i.e. $(x_A, x_B) \in \arg\max_{(x_A,x_B)\in C} \{x_A + x_B\}$.[8,9]

Simultaneous auction. Two single-unit first-price auctions, one for object A and one for object B, are held simultaneously. Bidders can participate in either one or both of the two auctions. The flexible bidder submits a bid w_A for object A and a bid w_B for object B (either of which may be zero). The local bidders' highest bids are s_A and s_B for the two objects, respectively. In each of the two simultaneous auctions, the respective highest bid wins.

Sequential auction. Two single-unit first-price auctions, one for object A and one for object B are held sequentially. The (for the flexible bidder) less valuable object A is sold first. After the first auction, the winner is announced. Bidders then choose their bid in the second auction. The flexible bidder submits a bid w_A for object A and a bid w_B for object B in case she won A, and a bid $\overline{w}_B$ for object B in case she did not win A. The local bidders' highest bids are s_A and s_B. In each of the two sequential auctions, the respective highest bid wins.

The abbreviations '*pma*', '*seq*', and '*sim*' indicate the product-mix, the sequential, and the simultaneous mechanism and $\mathcal{M} := \{pma, seq, sim\}$ denotes the set of all considered mechanisms.

3.2.1 Equilibrium bidding

In all three auction mechanisms, I uniquely characterise and, in some cases numerically, compute the optimal bids of the flexible bidder. As noted above, the local bidders' highest, optimal bids truthfully reveal their values and therefore equilibrium bidding boils down to the task of determining the flexible bidder's optimal bids. Fix an auction mechanism $m \in \mathcal{M}$. Then I let w_A^m and w_B^m denote the flexible bidder's bid for object A and B. In the sequential auction, let w_B^{seq} denote the flexible bidder's bid for object B conditional on winning object A, and let $\overline{w}_B^{seq}$ denote the flexible bidder's bid for object B conditional on *not* winning object A. When the context is clear, I omit the superscript m for legibility.

[7] In the product-mix auction with uniform pricing, assuming all bidders bid truthfully, actual efficiency is maximised. However, with pay-as-bid pricing, truthful bidding cannot be expected.

[8] Ties are broken by randomly (with equal probability) choosing one of the (sets of) winners, but this is not important for the theoretical analysis.

[9] The allocation and pricing rule are equivalent to choosing the revenue-maximising allocation, as defined for the 'menu auction' by Bernheim and Whinston (1986).

In the product-mix auction and the simultaneous auction, i.e. for $m \in \{pma, sim\}$, the flexible bidder's expected utility is given by

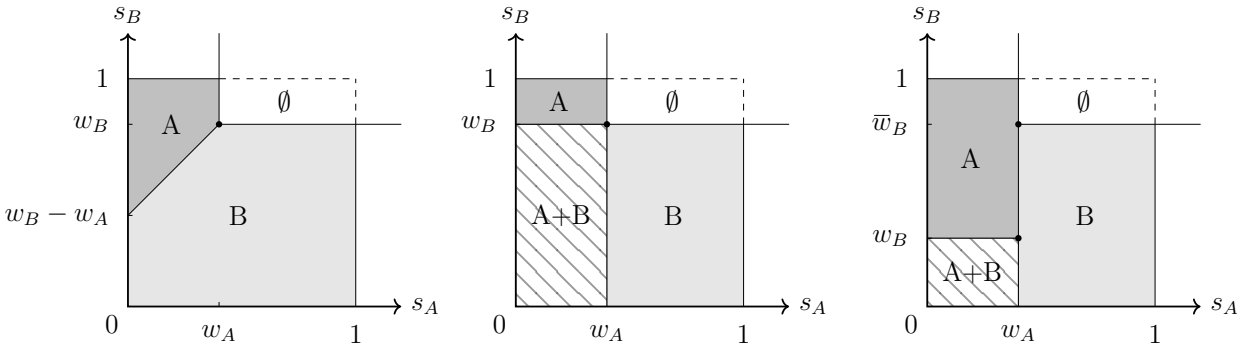
$$\begin{aligned}\mathcal{U}^m(w_A, w_B) = P_A^m(w_A, w_B)[v_A - w_A] + P_B^m(w_A, w_B)[v_B - w_B] \\ + P_{AB}^m(w_A, w_B)[v_B - w_A - w_B]\end{aligned}$$

where P_A^m, P_B^m, and P_{AB}^m denote the flexible bidder's probability of winning object A only, object B only, and objects A and B together in mechanism m. In the sequential auction, the flexible bidder's expected utility is given by

$$\begin{aligned}\mathcal{U}^{seq}(w_A, w_B) = P_A^{seq}(w_A, w_B)[v_A - w_A] + P_B^{seq}(w_A, \overline{w}_B)[v_B - \overline{w}_B] \\ + P_{AB}^{seq}(w_A, w_B)[v_B - w_A - w_B]\end{aligned}$$

To determine the probabilities of winning in the individual auction mechanisms, I illustrate the allocation to the flexible bidder as a function of the local bidders' highest bids s_A and s_B in Fig. 3.1. Note that in the product-mix auction, the allocation rule implies that the flexible bidder obtains object A if and only if $w_A - s_A > w_B - s_B$ and $w_A > s_A$, and she wins object B if and only if $w_B - s_B > w_A - s_A$ and $w_B > s_B$.

The dark shaded region marked with 'A' indicates the flexible bidder winning object A, the light shaded region marked with 'B' indicates her winning object B, and the striped region marked with 'A+B' indicates her winning both objects. The figures are drawn assuming w_A and w_B ($\overline{w}_B$ in the sequential auction) to be the same across auction mechanisms, and assuming $w_A \leq w_B$. The optimal bids will differ between mechanisms. Recall that s_A and s_B are distributed uniformly on $(0, 1)$. Hence, the marked regions immediately correspond to the probabilities of winning object A, object B, and objects A and B, respectively.



(a) Product-mix **(b)** Simultaneous **(c)** Sequential

Figure 3.1: Allocations to the flexible bidder

First, I make trivial observation: in any auction mechanism, bidding more than the true value v_A or v_B on the corresponding object can never bid optimal. Now consider the product-mix auction. To write down the flexible bidder's payoff function, I first ascertain that, in equilibrium, bids are such that $w_A \leq w_B$. A simple proof is given in Chapter 2: suppose $w_A > w_B$ in equilibrium, then simply switching the bids between the two objects would make the flexible bidder strictly better off. From the allocation rule (and illustrated in Fig. 3.1(a)), it is also clear that the flexible bidder cannot win both objects at once, i.e. $P_{AB}^{pma} = 0$. I obtain

$$P_A^{pma}(w_A, w_B) = \int_0^{w_A} \int_{w_B - w_A + s_A}^{1} \mathrm{d}s_B \, \mathrm{d}s_A \tag{3.1}$$

$$P_B^{pma}(w_A, w_B) = \int_0^{w_A} \int_0^{w_B - w_A + s_A} \mathrm{d}s_B \, \mathrm{d}s_A + (1 - w_A) w_B \tag{3.2}$$

In the simultaneous auction, if the flexible bidder submits strictly positive bids w_A and w_B, there is a non-zero probability of winning both objects. The probabilities of winning are easily determined as $P_A^{sim}(w_A, w_B) = (1 - w_B)w_A$, $P_B^{sim}(w_A, w_B) = (1 - w_A)w_B$, and $P_{AB}^{sim}(w_A, w_B) = w_A w_B$.

In the sequential auction, the flexible bidder can condition her bid on object B on (not) having won object A. The probabilities of winning are $P_A^{seq}(w_A, w_B) = (1 - w_B)w_A$, $P_B^{seq}(w_A, \overline{w}_B) = (1 - w_A)\overline{w}_B$, and $P_{AB}^{seq}(w_A, w_B) = w_A w_B$. Given these characterisations of winning probabilities, I can provide the following characterisation of the flexible bidder's optimal bids:

Proposition 16. *In the first-price product-mix auction, the flexible bidder's optimal (equilibrium) bids are uniquely characterised on* $(0, v_A) \times (0, v_B)$ *by equations*

$$-\frac{3}{2}{w_A}^2 + w_A\left(-2 + 3w_B + v_A - v_B\right) + v_A(1 - w_B) = 0 \text{ and} \tag{3.3}$$

$$\frac{3}{2}{w_A}^2 + v_B - 2w_B - v_A w_A = 0. \tag{3.4}$$

The proof is given in Appendix B.1. In Chapter 2, I show that the optimal bid must be interior, i.e. on $(0, v_A) \times (0, v_B)$, and therefore must be characterised by first-order conditions. Furthermore, Eqs. (3.3) and (3.4) admit a unique, real solution on this interval. The flexible bidder making a strictly positive bid on object A and object B (as opposed to one zero bid) is not entirely obvious, as increasing the bid on one object, while increasing the probability of winning that object, also decreases the probability of winning the respective other object. However, at an arbitrarily small (but positive) price, winning the less valuable object with some positive probability is always profitable in expectation.

Proposition 17. *In the first-price sequential auction, the flexible bidder's optimal bid is uniquely characterised by* $w_A = \frac{1}{4}\left(\frac{{v_A}^2}{2} + v_A(2 - v_B)\right)$, $w_B = \frac{1}{2}(v_B - v_A)$, *and* $\overline{w}_B = \frac{v_B}{2}$.

The flexible bidder's optimal bid for object B, in case she won object A, adjusts for the potential incremental gain in value. Because at most $v_B - v_A$ can be gained by winning object B in addition to object A, the bidder's willingness to pay exactly half of this potential gain. The price paid for A is treated as a sunk cost and not relevant for the subsequent bid. Trivially, in case she did not win object A, the flexible bidder's optimal bid is that of a standard first-price auction in my setup. Her optimal bid in the first auction takes the trade-offs of subsequently winning/not winning object B into consideration.[10] A more formal proof is given in Appendix B.1.

Proposition 18. *In the first-price simultaneous auction, the flexible bidder's optimal bid is uniquely characterised by* $w_A = \frac{v_A(2-v_B)}{4-{v_A}^2}$ *and* $w_B = \frac{2v_B - {v_A}^2}{4-{v_A}^2}$.

The simultaneous auction is intuitively the most disadvantageous auction mechanism for the flexible bidder. She has to balance the trade-off between not winning two objects at a loss too often, but bidding high enough so as to win often enough, in expectation. It is indeed optimal to bid such that she might win both objects only at prices that still guarantee a positive surplus, i.e. such that $w_A + w_B < v_B$. Just as in the product-mix auction, it is not obvious that two non-zero bids are optimal, but, similarly, it can be shown that the solution must indeed be interior on $(0, v_A) \times (0, v_B)$ (see Chapter 2). The formal proof of the proposition is given in Appendix B.1.

I note two more properties of optimal bids that hold across all three mechanisms: First, the flexible bidder bids less than what she would bid for a single object, that is $w_A^m < \frac{v_A}{2}$ and $w_B^m < \frac{v_B}{2}$ for all mechanisms $m \in \mathcal{M}$. Second, the comparative statics of optimal bids are such that $\frac{\partial w_A^m}{\partial v_A} > 0$, $\frac{\partial w_A^m}{\partial v_B} < 0$, $\frac{\partial w_B^m}{\partial v_A} < 0$, and $\frac{\partial w_B^m}{\partial v_B} > 0$ for all m. Hence, all auction formats considered allow the flexible bidder to express the substitutability between A and B to some, but varying, degree.

3.2.2 Profits, payments, revenue, welfare

The three auction formats are evaluated with respect to the flexible bidder's utility $\mathcal{U}^m$ and the flexible bidder's payment $\mathcal{P}^m$, the auctioneer's revenue $\mathcal{R}^m$, and total welfare $\mathcal{W}^m$. All outcomes are computed in expected values across

[10] For general distributions of the local bidders' bids, a zero bid in the first auction may be optimal even for $v_A \neq 0$.

the local bidders' bids. In the simulations in Section 3.4, I also average over the flexible bidder's values.

In the product-mix auction and the simultaneous auction, the auctioneer's expected revenue is given by

$$\begin{aligned}\mathcal{R}^m(w_A, w_B) = {} & P_A^m(w_A, w_B)\mathbb{E}_{s_B}\left[w_A + s_B \mid \text{flexible bidder wins } A\right] \\ & + P_B^m(w_A, w_B)\mathbb{E}_{s_A}\left[s_A + w_B \mid \text{flexible bidder wins } B\right] \\ & + P_{AB}^m(w_A, w_B)\left[w_A + w_B\right] \\ & + P_\emptyset^m(w_A, w_B)\mathbb{E}_{s_A, s_B}\left[s_A + s_B \mid \text{flexible bidder wins nothing}\right]\end{aligned}$$

In the sequential auction, the auctioneer's expected revenue is given by

$$\begin{aligned}\mathcal{R}^{seq}(w_A, w_B, \overline{w}_B) = {} & P_A^{seq}(w_A, w_B)\mathbb{E}_{s_B}\left[w_A + s_B \mid \text{flexible bidder wins } A\right] \\ & + P_B^{seq}(w_A, \overline{w}_B)\mathbb{E}_{s_A}\left[s_A + \overline{w}_B \mid \text{flexible bidder wins } B\right] \\ & + P_{AB}^{seq}(w_A, w_B)[w_A + w_B] \\ & + P_\emptyset^m(w_A, w_B)\mathbb{E}_{s_A, s_B}\left[s_A + s_B \mid \text{flexible bidder wins nothing}\right]\end{aligned}$$

The flexible bidder's payment in the product-mix and the simultaneous auctions is given by

$$\mathcal{P}^m(w_A, w_B) = P_A^m(w_A, w_B)w_A + P_B^m(w_A, w_B)w_B + P_{AB}^m(w_A, w_B)\left[w_A + w_B\right]$$

In the sequential auction, the flexible bidder's payment is given by

$$\mathcal{P}^{seq}(w_A, w_B, \overline{w}_B) = P_A^{seq}(w_A, w_B)w_A + P_B^{seq}(w_A, \overline{w}_B)\overline{w}_B + P_{AB}^{seq}(w_A, w_B)[w_A + w_B]$$

Welfare is given by $\mathcal{W}^m = \mathcal{U}^m + \mathcal{R}^m$.

3.3 Experimental design

Each auction mechanism in a given environment corresponds to one treatment. I run a between-subject design, in which each subject is randomly assigned to one of six treatments. In the *computerised environment*, each subject competes against two computerised bidders for the objects A and B. The computerised bidders represent the two groups of local bidders interested in object A and object B, respectively. In the *human environment*, subjects compete in groups of three for the two objects. In each treatment, subject play a number of bidding rounds. The outcome of each round is shown to the subject at the end of each round and payments are disclosed at the end of the experiment. The study is implemented in oTree (Chen et al., 2016).

3.3.1 Notation

Each subject $i \in [i] = \{1, ..., I\}$ is assigned to one auction mechanism, denoted by $m \in \mathcal{M}$, and to one environments $e \in [e] = \{1, 2\}$, where $e = 1$ represents the *computerised environment* and $e = 2$ represents *human environment*. Treatments are denoted by $\tau \in \mathcal{T} := \mathcal{M} \times [e]$. Each subject plays R auction rounds, denoted by $r \in [r] = \{1, ..., R\}$. In round r, subject i draws values v_{ir}^A, v_{ir}^B and submits bids w_{ir}^A and w_{ir}^B. The subject's *expected* utility (profit or loss) is denoted by $\mathcal{U}_{ir}$. In each corresponding auction, the expected flexible bidder's payments, the auctioneer's revenue, and welfare are denoted by $\mathcal{P}_{ir}$, $\mathcal{R}_{ir}$, and $\mathcal{W}_{ir}$. In environment $e = 1$, computerised bidders draw values s_{ir}^A, s_{ir}^B. For each subject i, I also observe a vector of covariates c_i. The covariates are established in a survey after the auction rounds. Note that v_{ir}^A and v_{ir}^B are not ordered in the experiment, i.e. a subject's value for object B may be higher or lower than its value for object A, unlike in the theoretical model.[11]

3.3.2 Treatments

Product-mix Auction. In the product-mix auction, subject i submits a bid w_{ir}^A for object A and a bid w_{ir}^B for object B, and is guaranteed to win at most one of the two objects. The winning bid on A and the winning bid on B are those that maximise the sum of the valid combinations of submitted bids. In environment $e = 1$, the flexible bidder's bids can be combined with either of the computerised bidders' bids, i.e. the valid combinations of potential winning bids are $C_1 := \{(w_i^A, s^B), (s^A, w_i^B)\}$. In environment $e = 2$, let $(w_i^A, w_i^B)_{i=1,2,3}$ denote the bids of three subjects competing against each other. All subjects are flexible bidders and the nature of their bids allows winning at most one object. Thus, the valid combinations of potential winning bids are $C_2 := \{(w_i^A, w_j^B)\}_{i,j \in 1,2,3, i \neq j}$.

Sequential Auction. Subject i participates in two sequential first-price auctions. Bids are submitted using the strategy method, whereby the subject makes three simultaneous choices: a bid w_{ir}^J for object J in the first auction, a bid w_{ir}^K for object K in the second auction conditional on having won in the first auction, and a bid $\overline{w}_{ir}^K$ for K conditional on *not* having won in the first auction.[12] In environment

[11] For the analysis, objects may be relabelled such that $v_{ir}^A \leq v_{ir}^B$.

[12] The strategy method (due to Selten (1967)) has been shown to induce no different behaviour than exposing a subject to the actual move in the game (Brandts and Charness (2011)). Using this method for bid elicitation allows me to collect all relevant bids to compute expected outcomes. Moreover, eliciting bids this way mitigates confounding effects from the two-stage nature of the sequential auction (e.g. wealth effects, emotions).

$e = 1$, the less valuable object is always sold first, i.e. the two sequential auctions are ordered such that $J = j, K = k$ if values are drawn such that $v_{ir}^j \leq v_{ir}^k$, $j, k \in \{A, B\}$. In environment $e = 2$, the object to be sold first is chosen at random.[13]

Simultaneous Auction. Subject i participates in two simultaneous fist-price auctions, submitting a bid w_{ir}^A for object A and a bid w_{ir}^B for object B. The subject can win in both auctions and would then have to pay for both objects. In each of the two simultaneous auctions, the highest submitted bid wins.

Treatment group sizes are balanced in each session between the product-mix and the simultaneous auction treatment. The two environments may be run in separate sessions for grouping purposes.

3.3.2.1 Environments

'Subject vs. computerised bidders' ($e = 1$). Each subject plays against two computerised agents. One of the computerised agents is only interested in object A, the other agent is only interested in object B. The computerised agents' exact bids are unknown to the subject, but the subject learns the distribution from which these bids are drawn.

'Groups of 3 subjects' ($e = 2$). In each round, subjects within the same treatments are randomly assigned to groups of three. Each subject can be matched with any other two subjects within the same treatment, including previously matched players, and subjects are made aware of this. The three subjects compete for the two object A and B in the auction mechanism corresponding to their treatment. Each subject knows that it is playing against two opponents with preferences just like themselves, i.e. that each opponent is interested in either of the two objects but not both. To each subject, their opponents' exact values are unknown, but they know the distribution of their opponents' values. In the sequential auction mechanism, one object will be randomly chosen to be sold first, i.e. some bidders will bid for their more valuable object first.

[13] In environment $e = 2$, I am only interested data points from subjects whose values are such that $v_{i2r}^A \leq v_{i2r}^B$. To account for discarded data points in $e = 2$, I may recruit additional subjects for the sequential treatment (but the overall sample size will be guided by power calculations for environment $e = 1$).

3.3.3 Rounds

In every bidding round, subjects are assigned values for object A and B. In environment $e = 1$, computerised bidders are also assigned values for object A and B. Subjects' and computerised bidders' values are drawn uniformly (with equal probability) from integer values between 0 and 100 (inclusive), approximating a continuous uniform distribution on $[0, 100]$. The draws are independent between objects, subjects, and rounds. After drawing their values, each subject submits bids corresponding to their treatment, and winners are determined using the respective treatment's allocation rule. Three practice rounds are played first, which are not payoff-relevant. The practice rounds are followed by 20 'actual' auction rounds. Any of those rounds can be payoff-relevant (more detail below).

3.3.4 Survey

After the auction rounds, each subject is asked to complete a survey and several additional tasks, including a Holt and Laury (2002) risk aversion (probability weighting) test, a risk aversion test following Drichoutis and Lusk (2016), an ambiguity aversion test Gneezy et al. (2015), the adaptive Berlin numeracy test by Cokely et al. (2012)), and a questionnaire on bidding experience, demographics, and self-reported risk attitude (Dohmen et al., 2011). Subjects are also asked to give a brief explanation of their rationale in choosing their bids.

3.3.5 Payoffs

The auction tasks and three multiple price lists are incentivised with monetary payments. Among the 20 actual rounds played, 3 rounds will be randomly selected to determine payoffs. Note that the average payoffs in each auction round (across all possible induced values) are 15.1 points in the product-mix format, 15.0 points in the sequential format, and 14.4 points in the simultaneous format (cf. Table 3.1).[14] The payoffs from the 3 randomly selected rounds are added together. If this sum is non-negative, it will be added to the subject's payoff tally. If the sum is negative, the payoff tally remains at zero.

One list among the three multiple price lists is randomly selected (with equal probability). From the decision sheet of this list, one decision is selected to be payoff-relevant. The payoff from this decision is added to the subject's payoff tally. The average payment from the three multiple price lists is 15.28 points.

[14]The computerised environment is taken as a reference point for both environments, as theoretical predictions and simulations are only feasible in this environment.

The subject's final tally will be converted to GBP, at a rate of 10 points = 1 GBP for the earnings from the auction task in all three treatments, and at a rate of 10 points = 1 GBP for the earnings from the multiple price lists. Thus, subjects have identical perceived incentives across treatments. Average payoffs are slightly higher in the product-mix and the sequential auction format, but subjects have an equal chance of being in any treatment, hence are treated ex-ante the same.

Based on a small, technical pilot the study is expected to last approximately 30 minutes for treatments in the computerised environment, and 45 minutes for treatments in the human environment. The increased duration in the human environment is due to additional waiting time for the arrival of all participants (within treatment) to form subgroups of 3 players.

The variable average payments for the study are therefore {£12.16/hour, £12.14/hour, £11.72/hour} for the {product-mix, sequential, simultaneous} treatments in the computerised environment, and {£8.11/hour, £8.09/hour, £7.81/hour} for the {product-mix, sequential, simultaneous} treatments in the human environments. In addition, all participants who complete the study receive a completion fee of £3. Total hourly average payoffs are higher in the computerised environments, but subjects have an ex-ante equal chance of being in either environment.

In the product-mix format, subjects cannot make any losses. In the sequential and simultaneous format, however, subjects can make a loss if they do not bid wisely. Although it is optimal to bid such that one would never make a loss, i.e. $w^A + w^B \leq v^B$, it is possible that subjects overbid. The probabilities to win objects A and B are very low, on average 1.5 (3.6)%, if one bids optimally (Table B.22).[15] Even if one were to overbid 'in the worst possible way', i.e. to bids one's induced values[16] the average probabilities of winning both objects are only at 25.7% (Table B.23). Overbidding in this way results loss of -13.80 points in the sequential and the simultaneous auction (Table B.24).[17] Of course, payments are bounded from below at zero. However, because I select multiple, payoff-relevant rounds to calculate payments, I believe the incentives in the sequential and simultaneous format are still intact.[18]

[15]The probabilities are averaged across all possible (uniformly distributed) induced values.

[16]Subjects are constrained to bid at most their value on each object.

[17]Overbidding this way always leads to payoffs of at most zero, but in those cases where the subject wins both objects, it will result in a loss amounting to the price paid for the lower value object. The losses are averaged across all possible (uniformly distributed) induced values.

[18]Endowments would provide more guarantee that the lower bound is not hit. However, endowments come with several drawbacks as well: they would diminish incentives to 'try hard' in each auction round, and it is difficult to determine an appropriate amount. In the product-

3.3.6 Further details

Comprehension quiz. After reading the instructions, subjects have to pass a comprehension quiz to be admitted for participation in the experiment. To pass, they must answer at least four questions of different types correctly, and two questions of different subtypes. Each type corresponds to a specific allocation to the 'participant' in the question.[19] In one subtype, subjects are asked to determine the winning bid for one of the two objects, and in the other subtype, subjects are asked to calculate earnings based on given values, allocations, and prices. Subjects have an unlimited number of attempts. After a wrong attempt, the correct answer is shown and an additional question of the same type is added to the end of the quiz. The questions alternate in subtypes while at most one question of each subtype is answered correctly. The initial order of the questions is randomised.

There is no time limit to answer any one question, but there is a (generous) overall time limit for passing the quiz, after which subjects would time out and be excluded from the study.

Attention checks. Throughout the study, subjects will be asked to complete attention checks, confirming their continued participation in the study. To do so, they must submit the page within a time limit of 1 minute. Failure to complete the attention check results in exclusion from the study without payment.

Timeouts. Subjects will have a fixed time of 45 seconds for each auction round in the experiment. A timer appears when 20 seconds are left to alert subjects to the remaining time. Subjects also have a fixed time of 4 minutes for each of the incentivised, additional tasks (timer appears when 90 seconds are left) and a fixed time of 8 minutes for the survey (timer appears when 2 minutes are left). For reading the instructions, subjects have 8 minutes (timer appears when 1 minute are left). For passing the comprehension quiz, subjects have 10 minutes (timer appears when 2 minutes are left).

On most pages, timing out does not result in exclusion from the study, but subjects are automatically advanced to the next page. In every auction round, in the comprehension quiz, in the survey, and in the additional, incentivised tasks (multiple price lists), subjects must advance within the time limit; otherwise, they

mix auction, endowments are not necessary, and introducing treatment-specific endowments also distorts incentives between treatments. Therefore, I choose not to provide subjects with endowments.

[19] There are only three different types in the PMA treatments.

are excluded from the study (and they are specifically made aware of it). Participants who fail to complete the comprehension quiz receive £2 for their participation. All other participants only receive payment if they complete the study.

Dropouts. A subject may qualify as a dropout if they manually exit the session, their browser or internet connection is interrupted for any reason, or if they time out during the quiz, the auction rounds, the survey, or the additional, incentivised tasks. If possible, the subject will be redirected to the end of the experiment. If the subject qualifies as a dropout, they do not receive any payments (except if they fail to complete the comprehension quiz) and this is emphasised in the instructions.

3.4 Simulations

I simulate bidding in the computerised environment building on the theoretical model set out in Section 3.2, assuming that flexible bidders are risk-neutral quasi-linear utility maximisers. From $10,000$ independent, uniform integer draws of $v_A, v_B \in [0, 100]$, I calculate optimal bids and corresponding expected utility of the flexible bidder, expected revenue, and expected welfare. Average outcomes, standard deviations, medians and quartiles Q1 and Q3 are presented in Table 3.1. The results are striking: the product-mix and the sequential format perform nearly

		Mean	SD	Median	Q1	Q3
$\mathcal{U}$	pma	15.18	8.73	15.17	7.86	22.08
	seq	15.14	8.73	15.05	7.85	21.99
	sim	14.44	8.06	14.56	7.63	21.17
$\mathcal{R}$	pma	106.97	3.79	107.11	103.79	110.17
	seq	106.78	3.67	106.91	103.71	109.89
	sim	106.29	3.33	106.46	103.54	109.17
$\mathcal{W}$	pma	122.16	12.5	122.28	111.7	132.29
	seq	121.92	12.37	122.06	111.54	131.93
	sim	120.73	11.32	121.09	111.26	130.33
$\mathcal{P}$	pma	13.51	7.23	13.85	7.47	19.71
	seq	13.56	7.34	13.81	7.42	19.77
	sim	12.57	6.67	12.91	7.08	18.34

Table 3.1: Average outcomes and standard deviations across auction mechanisms

identically on average. For the flexible bidder, these two formats could be thought of as more advantageous than the simultaneous auction, but it is surprising that these

two mechanisms perform nearly *identically* with respect to all outcome variables. The simultaneous auction performs only slightly worse than the other two auction formats: approximately 5% in utility, 7% in flexible bidder payments, 0.6% in revenue, and 1% in welfare.

3.5 Model and evaluation

Let y_{ir} denote an outcome variable in $\{\mathcal{U}_{ir}, \mathcal{P}_{ir}, \mathcal{R}_{ir}, \mathcal{W}_{ir}\}$. Recall that each subject draws values v_{ir}^A, v_{ir}^B and submits bids w_{ir}^A, w_{ir}^B. I denote by t_i^τ, $\tau \in \mathcal{T}$ a binary treatment variable which equals one if subject i is in treatment τ and zero otherwise. The corresponding coefficients are given by θ^τ, $\tau \in \mathcal{T}$. I evaluate each environment separately and compare mechanisms pairwise. The main hypotheses are tested using equivalence tests (e.g. with R-package 'TOSTER' by Lakens (2017)). For robustness checks, I use regressions. For all treatments $(j, e) \neq (k, e) \in \mathcal{T}$, $e = 1, 2$, the model is given by

$$y_{ir} = \theta^{(j,e)} + \theta^{(k,e)} t_i^{(k,e)} + \epsilon_{ir}$$

where ϵ_{ir} is an error term orthogonal to the treatment variables and $\epsilon_{ir} \sim \mathcal{N}(0, \sigma_\epsilon)$. Due to the randomisation of the treatments and values in each round, population orthogonality and the rank condition are satisfied, so $\theta^\tau, \tau \in \mathcal{T}$ can be consistently estimated using least squares.

Additional robustness checks include covariates c_i and drawn values v_{ir}^A and v_{ir}^B (and potentially their interaction). To account for any additional subject-specific variation I also estimate a model with random subject-specific intercepts, given by

$$y_{ir} = \theta^{(j,e)} + \theta^{(k,e)} t_i^{(k,e)} + z_i + u_{ir}$$

where the error term is decomposed in a residual between-subject error $u_{ir} \sim \mathcal{N}(0, \sigma_u)$ and the random effect $z_i \sim \mathcal{N}(0, \sigma_z)$. The standard assumptions $E(u_{ir} \mid t_i^{(k,e)}, z_i) = 0$ and $E(z_i \mid t_i^{(k,e)}) = E(z_i) = 0$ clearly hold because of the randomisation into treatments and the random assignment of values in each round; hence z_i is independent of $t_i^{(k,e)}, k \in \mathcal{M}$. I may also include additional round-specific random intercepts. The random effect model can be estimated, e.g., with the R-package 'lme4' (Bates et al., 2015).

Hypotheses. My hypotheses are formulated for absolute differences between outcomes.[20] I test for pairwise equivalence of outcomes between all auction formats. My hypotheses are formulated below.

(1) $\theta^{(pma,e)}$ and $\theta^{(seq,e)}$ are equivalent up to ±0.2SD equivalence bounds (Null hypothesis $\theta^{(pma,e)} \neq \theta^{(seq,e)}$)

(2) $\theta^{(pma,e)}$ and $\theta^{(sim,e)}$ are equivalent up to ±0.2SD equivalence bounds (Null hypothesis $\theta^{(pma,e)} \neq \theta^{(sim,e)}$)

(3) $\theta^{(seq,e)}$ and $\theta^{(sim,e)}$ are equivalent up to ±0.2SD equivalence bounds (Null hypothesis $\theta^{(seq,e)} \neq \theta^{(sim,e)}$)

The hypotheses for $e = 1$ follow immediately from the simulations building on the theoretical model, and the hypotheses for $e = 2$ use this as a reference point. The standard deviation for computing the equivalence bounds will be taken from the simulated data also for the equivalence tests of the real data.[21] Hypotheses (1), (2), and (3) are powered for $e = 1$, and the analysis for $e = 2$ should be seen as exploratory. Further analysis will consider within-treatment effects across environments, aiming to understand, e.g., which bidding environment is more competitive for the flexible bidder, with its implications for revenue and welfare. From a theoretical angle, it is not feasible to predict if a competitive fringe or a multi-product oligopoly market are easier to compete against.

3.5.1 Power analysis

I power the analysis with respect to hypotheses (1), (2), and (3). Effects between the computerised and the human environment will be studied from an exploratory perspective. I conduct the power analysis with the R package 'TOSTER' (Lakens, 2017). I also correct for serial correlation within observations from a single subject between rounds (cf. Diggle et al. (2002); Liu and Wu (2005); Potvin and Schutz (2000)), by setting $\tilde{n} = (1 + (R-1)\rho) \cdot n/R$ as the serial-correlation-adjusted sample size, where R is the number of observations per subject and ρ is the correlation between observations. Setting ρ a priori is not trivial. I follow the example of Drichoutis et al. (2015) who demonstrate that their study is powered for a variety

[20] A previous version set out to compare relative differences. After a few sessions, however, it was evident that a non-trivial number of subjects obtained negative or zero payoffs even in expectation. Therefore, a log-transformation is not feasible.

[21] The subject data may have an even greater variance, such that basing equivalence bounds the corresponding standard deviation may not be meaningful.

of parameter constellations, including the values $\rho \in \{0.1, 0.3, 0.5\}$.[22] The pooled standard deviation of the expected payoff for the comparison of the PMA and the SEQ treatment, for example, is given by $SD = sqrt((8.73^2 + 8.73^2)/2) = 8.73$. The hypothesised effect size is $\delta = 0$ and I choose a significance level of $\alpha = 0.05$ and equivalence bounds $\pm\Delta = \pm 0.2SD$, where the pooled standard deviation SD depends on the outcome and the two treatments that are tested for equivalence. For testing equivalence of expected payoffs between PMA and SEQ, e.g., the bounds are given by $\pm 0.2 * 8.73$. Note that the required sample size will be identical for all outcomes, as the bounds vary with the corresponding standard deviation. I present power $1 - \beta$ and sample sizes for various scenarios of within-subject correlation ρ in Table 3.2. I choose a sample size of 185 subjects per treatment arm of the computerised treatments. This powers my hypotheses conservatively for a serial correlation between rounds of up to $\rho = 0.4$. I expect $\rho = 0.4$ to be a strict upper bound (cf. Footnote 22). The sample size for the human treatments may be chosen somewhat lower as they serve an exploratory analysis.

Table 3.2: Sample sizes (per treatment arm) needed to power study for different serial correlation. Numbers in bold indicate that the required sample size is lower than the planned sample of 185 subjects per treatment arm.

		$b = \pm 0.2SD$	$b = \pm 0.15SD$
$p = 0.8$	$\rho = 0.1$	**63**	**111**
	$\rho = 0.2$	**103**	**183**
	$\rho = 0.3$	**144**	256
	$\rho = 0.4$	**185**	328
	$\rho = 0.5$	226	401
$p = 0.9$	$\rho = 0.1$	**79**	**140**
	$\rho = 0.2$	**131**	232
	$\rho = 0.3$	**182**	323
	$\rho = 0.4$	233	414
	$\rho = 0.5$	285	506

[22] Drichoutis et al. (2015) base their results on computations for previous auction experiments. In their experiment, they have 24 rounds with 3 blocks of 8 randomly chosen, induced values, from a range of only 8 values to choose from in total. Because in my setup bidders play 20 rounds with completely independent values for *two* goods instead of one, their chosen correlation may serve as an upper bound for my study.

3.6 Results

3.6.1 Data collection

Subjects participated in this study in a virtual lab environment on their own device. CESS at Nuffield College, Oxford, provided the virtual lab environment and collected the data. At the time of writing, the data collection is not yet complete and two thirds of the desired sample size remain to be collected. However, the existing data already reveals interesting, albeit preliminary results. The data collection took place in September and October 2022 with subjects between 18 and 65 years of age from all over the UK. Subjects interacted only anonymously and their non-anonymous data is protected by the experimental laboratory CESS, from whom I obtained completely anonymised and unidentifiable datasets. Data was collected on ten different days, where on most days two sessions were run,[23] one with the computerised environment, and one with the human environment. The order of environments was randomised for each days. For each session, between 50 and 75 subjects were recruited with approximately 15% no-shows. The sessions on the first two days served as technical pilot sessions and were helpful in making improvements to the design, e.g. inclusion of attention checks, spreading the presentation of examples, and simplifying the comprehension test.

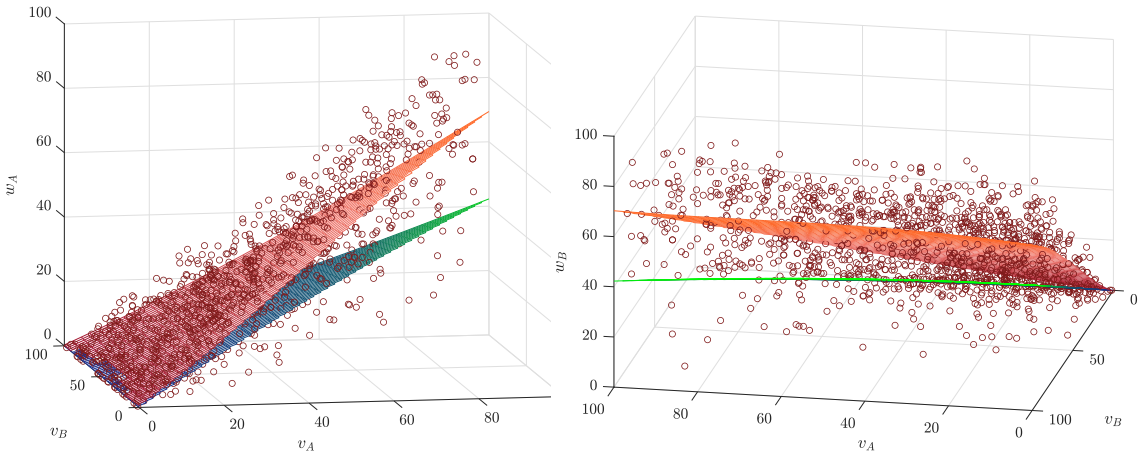
Across the 24 sessions, 370 subjects completed the study. About 11% did not pass the comprehension test and were therefore excluded from the study. Further details are given in Table 3.3. Among those who completed the study, the average and median time spent on the study were between 40 and 41 minutes. The quickest participant finished in 15 minutes, and the slowest participant took 64 minutes.

Table 3.3: Overview participants

	PMA/c	SEQ/c	SIM/c	PMA/h	SEQ/h	SIM/h	All
failed quiz	5	11	7	14	12	9	58
passed quiz	78	75	78	76	75	88	470
dropout after quiz	2	12	6	9	9	5	43
with dropout	0	0	0	9	12	4	25
unmatched	0	0	0	11	11	10	32
completed study	76	63	72	47	43	69	370

Of the 370 subjects, each played 20 (non-practice) auction rounds. In the following, I provide a graphical analysis of bidding behaviour as well as descriptive summaries of outcomes.

[23] One one day, two computerised and two human environment sessions were run.



(a) Bids for object A **(b)** Bids of object B

Figure 3.2: Product-mix format computerised: observed bids, fitted polynomial surface (red), and optimal bid function (blue/green)

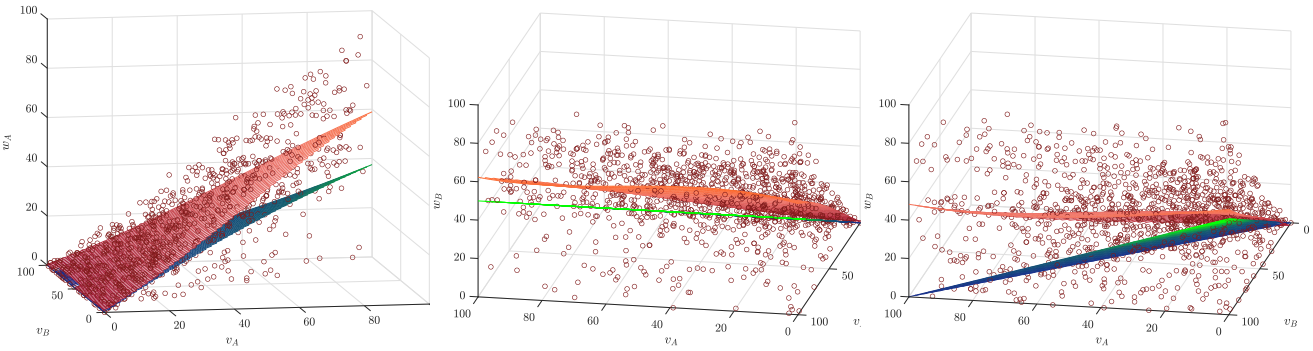
3.6.2 Bidding behaviour

Subject draw independent value for object A and object B in each round. In the experiment, either object could be the less valuable one, but for the analysis I relabelled objects such that object A is always worth less object B.[24] For the computerised environments I calculate, given the subjects' drawn values, their optimal bids. In Figs. 3.2 to 3.4 I show the observed bids in the computerised environments as scatter plots, together with fitted polynomial surfaces and the corresponding optimal bid surfaces. In the computerised treatments, we observe significant overbidding compared to the risk-neutral Bayes-Nash equilibrium. Relative overbidding appears to be strongest in the simultaneous auction, and in bids for object B conditional on winning A in the sequential auction. In Figs. 3.5 to 3.7, I plot the observed bids in the human environments, together with fitted polynomial surfaces and the fitted polynomials from the corresponding computerised treatment. In most auction formats and bids, the fitted surfaces are very close, with an exception of the sequential auction, where bids for the more valuable object B are higher in the human than the computerised environment.

3.6.3 Outcomes

In all treatments, payoffs are observed as direct outcomes of the experiment and compute revenue and welfare. For the computerised treatments, I also consider the

[24]Note that in the sequential auction in the human environment, this is not possible. To keep priors about the distribution of values independent, it could not be the case that the cheaper object is sold first for every bidder. Therefore, I have to discard about half of the data in the SEQ/h treatment.

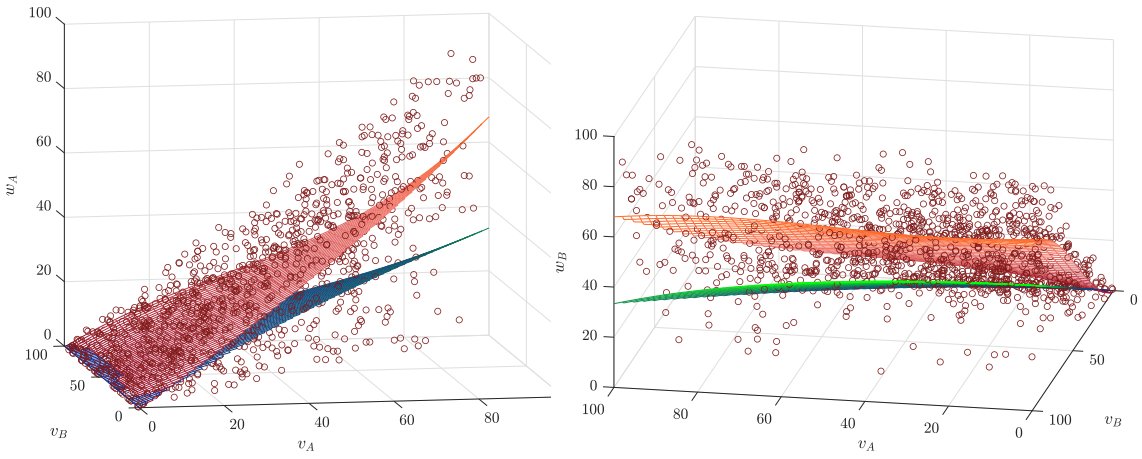


(a) Bids for object A

(b) Bids for object B if lost A

(c) Bids for object B if won A

Figure 3.3: Sequential format computerised: observed bids, fitted polynomial surface (red), and optimal bid function (blue/green)



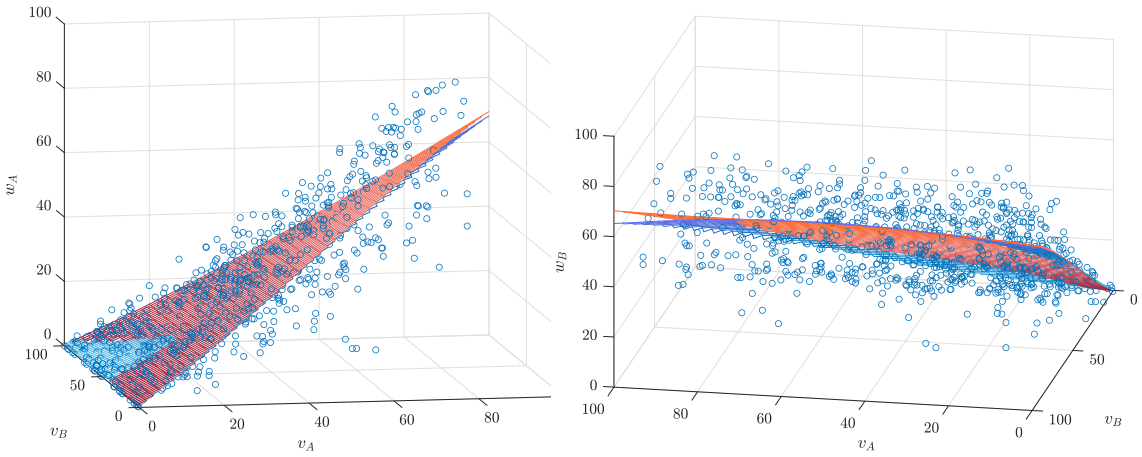
(a) Bids for object A

(b) Bids for object B

Figure 3.4: Simultaneous format computerised: observed bids, fitted polynomial surface (red), and optimal bid function (blue/green)

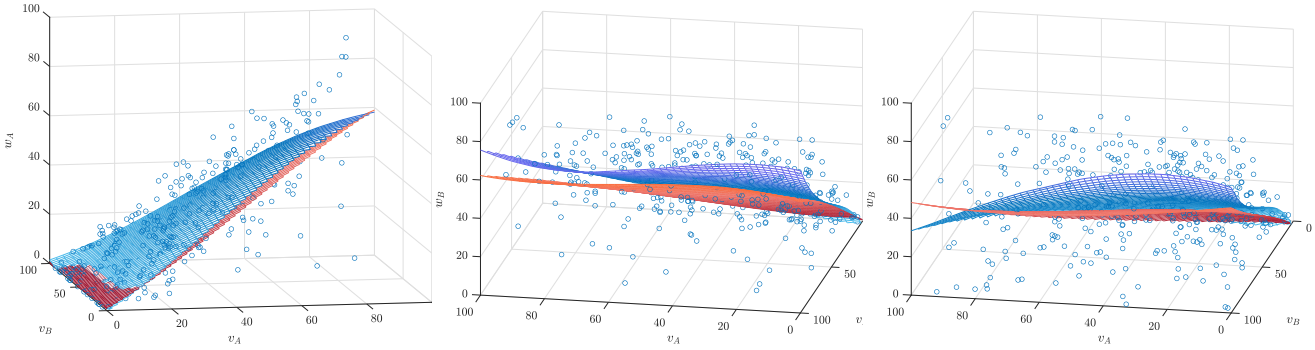
payments of the flexible bidder. In computerised treatments, I compute expected outcomes based on the observed bids and values.[25] The main goal of this section is to compare expected outcomes in the computerised treatments with expected outcomes based on optimal bids (cf. simulations in Section 3.4). In human environments, i.e. auction sessions in which subjects played against other human participants in groups of three, I cannot compute expected or optimal outcomes. However, I compare realised outcomes of the human treatments to the realised outcomes of the

[25]For a very large number of observations, the expected outcomes are, on average, equal to average realised outcomes, as the opponents' bids replicate the density which they are drawn from. However, the distribution of expected outcomes and realised outcomes will still differ. For this reason and to avoid any small sample bias, I chose to work with expected outcomes.



(a) Bids for object A **(b)** Bids for object B

Figure 3.5: Product-mix format: observed bids and fitted polynomial surface (blue) for human environment, fitted polynomial surface computerised (red)

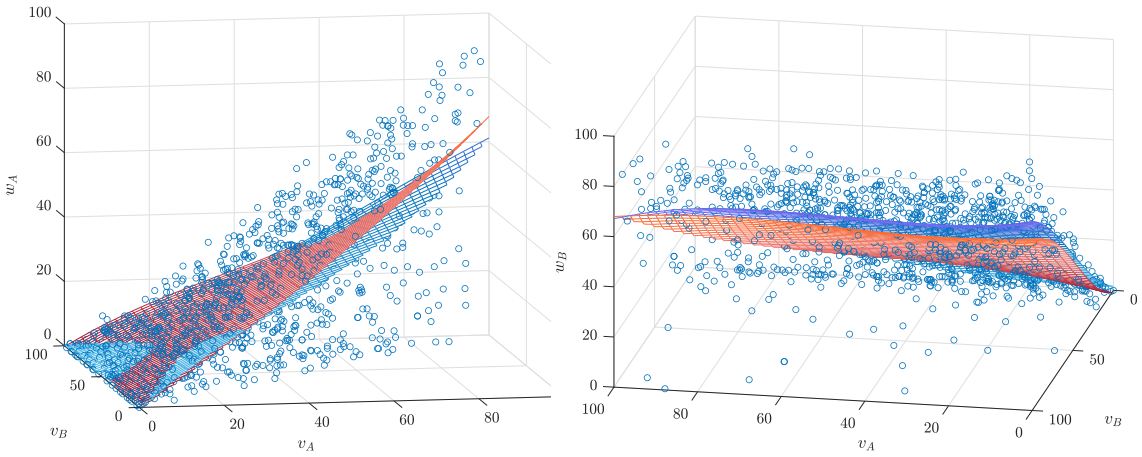


(a) Bids for object A **(b)** Bids for object B if lost A **(c)** Bids for object B if won A

Figure 3.6: Sequential format: observed bids and fitted polynomial surface (blue) for human environment, fitted polynomial surface computerised (red)

corresponding computerised treatments. As the two environments are very alike in the market setting, except for the type and strength of opponents a subject is facing, I obtain some results on the robustness of the theoretical model with competitive fringes.

In Figs. 3.8 and 3.9 I plot the densities of all considered outcomes, and the outcomes based on optimal bidding in the computerised version as a benchmark. We observe that realised payoffs have a spike at zero, which is to be expected as the realised payoff is zero whenever the subject does not win anything. Expected payoffs, however, still have a spike close to zero, indicating that a substantial number of subjects may have overbid by a significant amount, even close to their value,

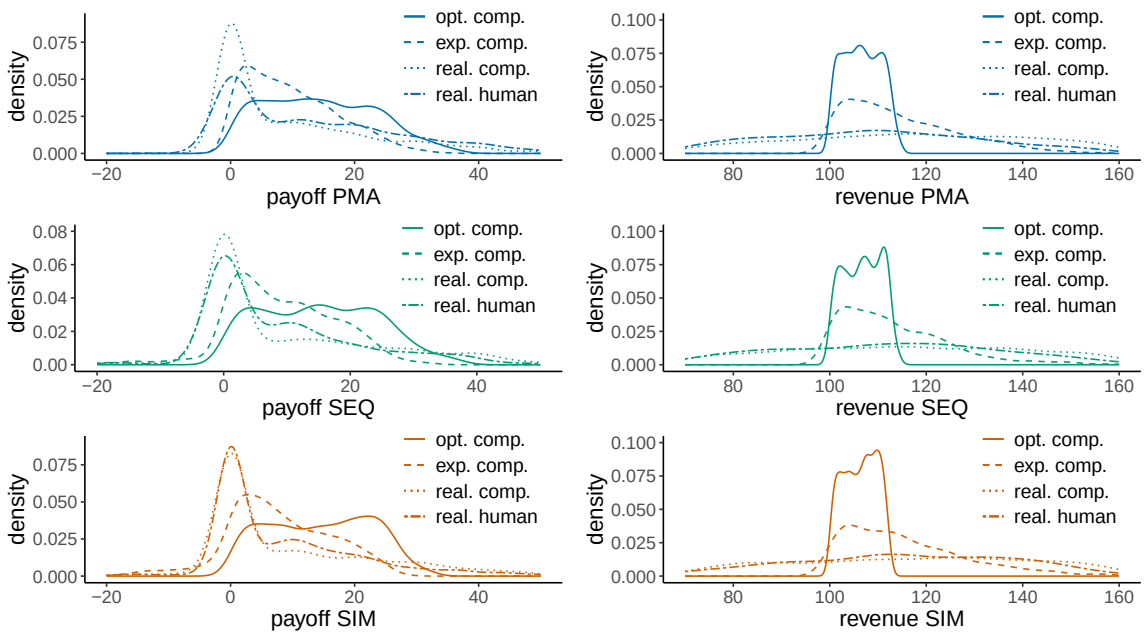


(a) Bids for object A **(b)** Bids for object B

Figure 3.7: Simultaneous format: observed bids and fitted polynomial surface (blue) for human environment, fitted polynomial surface computerised (red)

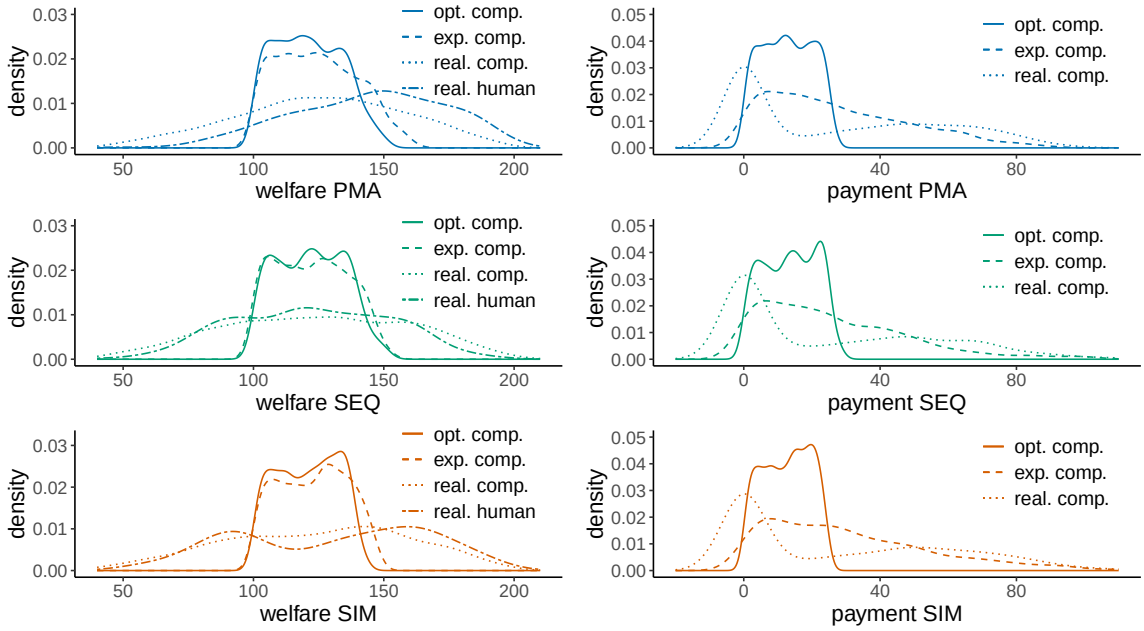
and this is confirmed by the data. These results hold across treatments, as well as the fact that realised and expected payoff distributions are skewed to the left and are tailing more than optimal payoffs. Revenues are consequently also much more tailed, and, at least in expectation, some probability mass is shifted to the right: the bidders' loss in payoff is often the auctioneer's gain in revenue. Surprisingly, the distributions of expected and optimal welfare very close. This immediately suggest that my theoretical model may be doing well in predicting efficiency of the considered auction formats. However, the distribution of welfare between the auctioneer and bidders is skewed towards the auctioneer in the experimental data. In the PMA treatment, realised welfare is shifted to the right compared to optimal and expected outcomes. Payments of the flexible bidder (the subjects) have, as expected from the preceding discussion, a spike at zero and a long tail to the right compared to payments under optimal bidding.

In Fig. 3.10, I plot the densities of expected outcomes, comparing the three computerised treatments in each outcome. It is striking how similar the distributions are, encouraging the detailed equivalence analysis I set out to do in the pre-analysis (see Section 3.6.5). The density plots of outcomes in the three human treatments (Fig. 3.11), however, exhibit a different pattern: while revenues across all treatments and other outcomes between SEQ and SIM are very similarly distributed, the density of payoffs, efficiency, and bidder surplus in the product-mix format is shifted significantly to the right, compared to SEQ and SIM. Note that the flexible bidder payments equal revenue in the human environment, whereas bidder surplus equals



(a) payoff **(b)** revenue

Figure 3.8: Density plots of outcomes payoff and revenue



(a) welfare **(b)** payments flexible bidder

Figure 3.9: Density plots of outcomes welfare and flexible bidder payments

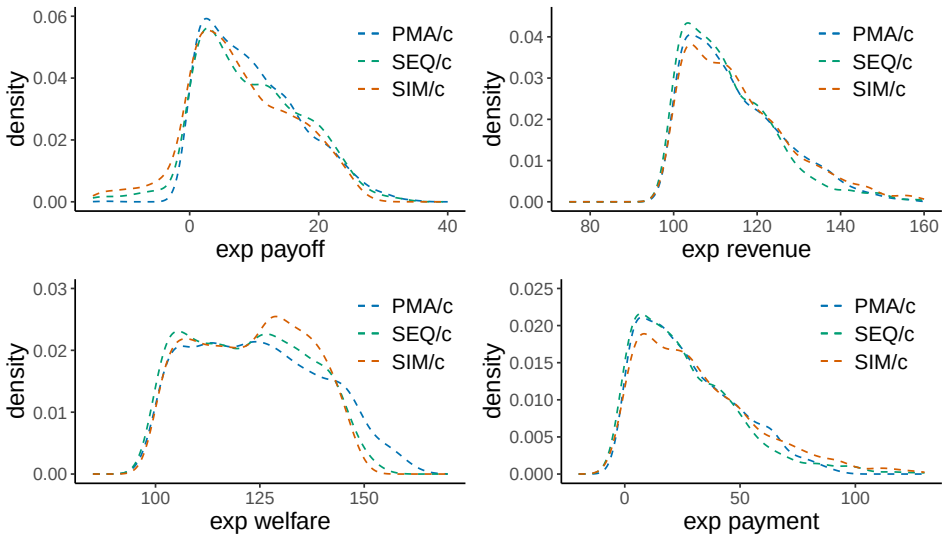


Figure 3.10: Density plots of expected outcomes comparing computerised treatments PMA, SEQ, SIM

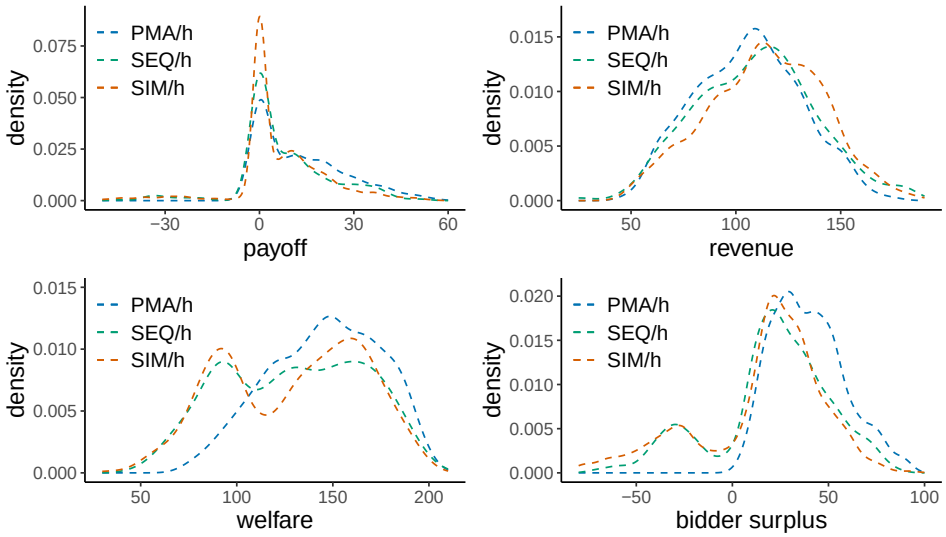


Figure 3.11: Density plots of outcomes comparing human treatments PMA, SEQ, SIM

payoff in the computerised environment (computerised agents make zero profit).

I briefly present some summary statistic of outcomes. Expected outcomes are shown in Table 3.4. Comparing to the theoretical predictions (Table 3.1), average expected payoffs $\mathcal{U}$ are between 35% and 47% lower than predicted. Correspondingly, payments of the flexible bidder $\mathcal{P}$ are between double or triple of the predicted amount, with a large standard deviation, and revenue is also somewhat, between 6% and 9% higher. Surprisingly, the disadvantage of the flexible bidder does not come at the cost of efficiency.

When looking at the realised outcomes (Table 3.5 and Table 3.6), I first note that the outcomes in the computerised treatments are, on average, identical to the expected outcomes, meaning the number of observations is large enough to account for the whole distribution of the opponents' values. Naturally, the variability in realised outcomes is higher.

The comparison of outcomes between the computerised and human environments needs some qualification. Note that the environments were chosen to be as similar as possible, while transforming the competitive fringe into a less competitive kind of oligopoly. Each flexible bidder makes a bid in structure identical to the computerised environment, and they may win either object A, object B, neither object, or, in treatments SEQ and SIM, both objects. However, their guess about the opponents' highest competing bid on each good, is much more difficult, as it depends on their values and strategies. In the sequential and simultaneous auctions, this resulted in lower average payoffs for each bidder. However, in the product-mix format, bidders did remarkably well, comparing to the payoffs against competitive fringes and, even more, against the other auction formats in the oligopoly environment: the average payoff is 90% higher than in the sequential and 156% higher than in the simultaneous auction.

Although the auctioneer's revenue is lower in the PMA than other auctions, the higher surplus translate also into stronger performance in terms of efficiency: the PMA is 12% more efficient than the other auction formats. I note that for the computation of average payoffs in the sequential treatment, I only use subjects for whom the cheaper object was sold first, in order to not distort the comparison further.[26] However, revenue, welfare, and bidder surplus can only sensibly reported for 'complete markets', i.e. groups of three bidders, and therefore all subjects in SEQ/h are included.

3.6.4 Covariates

For robustness checks, I also collected a number of covariates after the auction task in the study. I show contingency tables of the means of expected outcomes for all computerised environments in Appendix B.2.1. The most obvious trend is suggested by Table B.7, where higher scores show higher average expected payoffs in all three treatments. Otherwise, there are no obvious trends in covariates. The tables

[26]The difference to including all subjects in SEQ/h is small.

Table 3.4: Expected outcomes summary statistics

		Mean	SD	Median	Q1	Q3
$\mathcal{U}$	pma	9.84	7.33	8.58	3.74	14.79
	seq	8.60	9.82	7.94	2.87	15.01
	sim	6.80	10.68	6.59	1.79	13.69
$\mathcal{R}$	pma	114.73	11.58	112.39	105.44	121.71
	seq	113.65	11.97	111.05	104.55	119.73
	sim	116.22	13.51	113.00	105.63	123.13
$\mathcal{W}$	pma	124.56	15.25	123.65	111.86	136.67
	seq	122.25	13.85	122.40	109.95	133.63
	sim	123.02	13.12	123.76	111.59	133.83
$\mathcal{P}$	pma	27.65	20.80	23.93	10.57	42.25
	seq	27.30	23.93	22.10	9.11	39.47
	sim	32.44	27.02	26.00	11.25	46.25

Table 3.5: Outcomes summary statistics computerised treatments

		Mean	SD	Median	Q1	Q3
$\mathcal{U}$	pma	9.34	12.66	2.00	0.00	15.00
	seq	8.78	17.62	0.00	0.00	18.00
	sim	6.63	17.99	0.00	0.00	15.00
$\mathcal{R}$	pma	115.27	34.39	117.00	92.00	140.00
	seq	113.90	36.06	115.00	90.00	140.00
	sim	117.09	35.35	119.00	91.00	143.25
$\mathcal{W}$	pma	124.61	33.56	125.00	102.00	149.00
	seq	122.68	36.81	124.00	96.00	152.25
	sim	123.72	36.37	127.00	97.00	151.00
$\mathcal{P}$	pma	27.31	29.27	20.00	0.00	52.25
	seq	27.77	34.07	10.00	0.00	50.00
	sim	32.77	37.55	22.50	0.00	60.00

Table 3.6: Outcomes summary statistics human treatments

		Mean	SD	Median	Q1	Q3
$\mathcal{U}$	pma	13.20	13.92	10.00	0.00	22.00
	seq	6.96	16.64	1.50	0.00	15.00
	sim	5.16	16.98	0.00	0.00	13.00
$\mathcal{R}$	pma	105.98	24.98	106.00	87.00	124.00
	seq	109.42	28.51	110.00	89.00	129.00
	sim	113.73	27.86	115.00	95.00	134.00
$\mathcal{W}$	pma	145.57	28.61	148.00	124.00	169.00
	seq	129.71	36.13	131.00	97.00	160.00
	sim	129.22	36.78	135.00	95.00	160.00
$\mathcal{S}$	pma	39.59	18.63	38.00	25.00	51.00
	seq	20.29	31.81	23.00	10.00	40.00
	sim	15.49	32.87	22.00	4.00	36.00

suggest, however, that some variation across covariate outcomes does exist. For example, there may be a difference in payoffs between female and male participants in SEQ and SIM, but not PMA.[27] Payoffs also vary somewhat with age, ethnicity, and socio-economic status.[28] We see payoff variation in income, e.g. subjects with doctoral degrees have the highest payoffs in SIM but lowest payoffs in SEQ. Subjects with intermediate self-reported risk aversion appear to have higher payoffs than those with extreme risk attitudes. The multiple-price-list tests after Drichoutis et al.

[27]Other gender categories have a too low number of subjects to report averages.

[28]Some variation like very low payoffs for subjects who prefer not to reveal their socio-economic status is puzzling.

(2015) and Holt and Laury (2002) confirm only that more risk-seeking subjects may have lower payoffs. Strongly ambiguity-seeking subjects also seem to have lower payoffs, but a trend is, as with risk-aversion, not apparent. In the PMA and SIM format, the perceived difficulty of bidding in the auction rounds seems to be correlated with expected payoffs, while in the sequential such trend is not apparent.

A more in-depth analysis is beyond the scope of this thesis; nonetheless, details (also on revenue, efficiency, and flexible bidder payments) are shown in Tables B.1 to B.7 and B.9 to B.12.

3.6.5 Hypothesis testing

In this section, I formally test the hypotheses of pairwise equivalence between the auction formats with respect to the outcomes of expected payoffs, expected revenue, expected efficiency, and expected payments of the flexible bidder. These equivalence tests are only done for the computerised environments. I report the results from two-one-sided-tests procedures (TOST) for payoffs in Table 3.7. Details are given in the appendix in Tables B.17 to B.19. The equivalence bounds for each outcome are set at $\pm 0.2 * SD$, where SD is the pooled standard deviation across the considered treatments. The significance level is $\alpha = 0.05$ and 0.9-confidence intervals are used. All equivalence tests are run with the R-package 'TOSTER' (cf. Lakens (2017)).

Table 3.7: Overview equivalence tests for expected outcomes in computerised environment

		equivalence test significant	t-test for difference significant
PMA/c vs. SEQ/c	payoff	-	yes
	revenue	-	yes
	welfare	-	yes
	payment	-	-
PMA/c vs. SIM/c	payoff	-	yes
	revenue	-	yes
	welfare	-	yes
	payment	-	yes
SEQ/c vs. SIM/c	payoff	-	yes
	revenue	-	yes
	welfare	yes	-
	payment	-	yes

Equivalence is rejected for almost all pairwise comparisons at the pre-selected bounds (the tests fail to reject the null hypothesis of equivalence). The exception is

equivalence in welfare between the sequential and the simultaneous auction, but other confidence intervals are also just beyond the set equivalence bounds: payoffs and flexible bidder payments in SEQ vs. SIM and welfare in PMA vs SIM. I plot the distribution of mean differences in Fig. 3.12. I also conduct robustness checks with the Wilcoxon TOST equivalence test (Lakens, 2017) and TOST equivalence test with bootstrapped standard errors (Lakens, 2017). The robustness checks confirm the findings of the standard equivalence tests, with the exception of the Wilcoxon test for the equivalence of payoffs between PMA and SEQ.[29]

3.6.6 Regression analysis

To check the difference in outcomes for robustness, I run the linear regressions (Table 3.8), also including covariates (Table B.21) and subject-specific random effects (Table B.20). This section only analyses the outcomes from computerised environments.[30]

For the inclusion of covariates, I run a step-wise search (backward direction) for the best combination of covariates, which outputs nearly all covariates for all comparisons. For simplicity, I therefore include all covariates in the analysis. In the standard linear model, I find, in line with Table 3.7, a significant difference in payoffs and revenue in all pairwise comparisons. The difference in welfare is not significant in the comparison of SEQ/c and SIM/c, and the difference in payment is not significant in the comparison of PMA and SEQ.

The random effect model confirms the difference between payoffs, but standard errors increase substantially for the revenue comparison; as a result, the difference PMA vs. SEQ and PMA vs. SIM is not significant anymore, and similarly for the comparison of payments between PMA and SEQ. The difference in welfare between SEQ and SIM is non-significant throughout; indeed the TOST procedure confirmed equivalence in welfare given the set bounds. Overall, the results of the random effect model are encouraging: the subject-specific variance is small relative to the residual variance for most estimates. This indicates that observations are largely independent between rounds. The variation in my estimated differences is predominantly (but of course not exclusively) by subject-independent.

Finally, I note that the inclusion of covariates naturally changes the estimates in size, but not in direction, which provides some robustness. Because, at the time

[29]The standard test for equivalence, the bootstrap version, and the test for difference provide, together, overwhelming evidence that equivalence does not hold.

[30]A regression analysis for the human environment is planned in future work, but beyond the scope of this thesis.

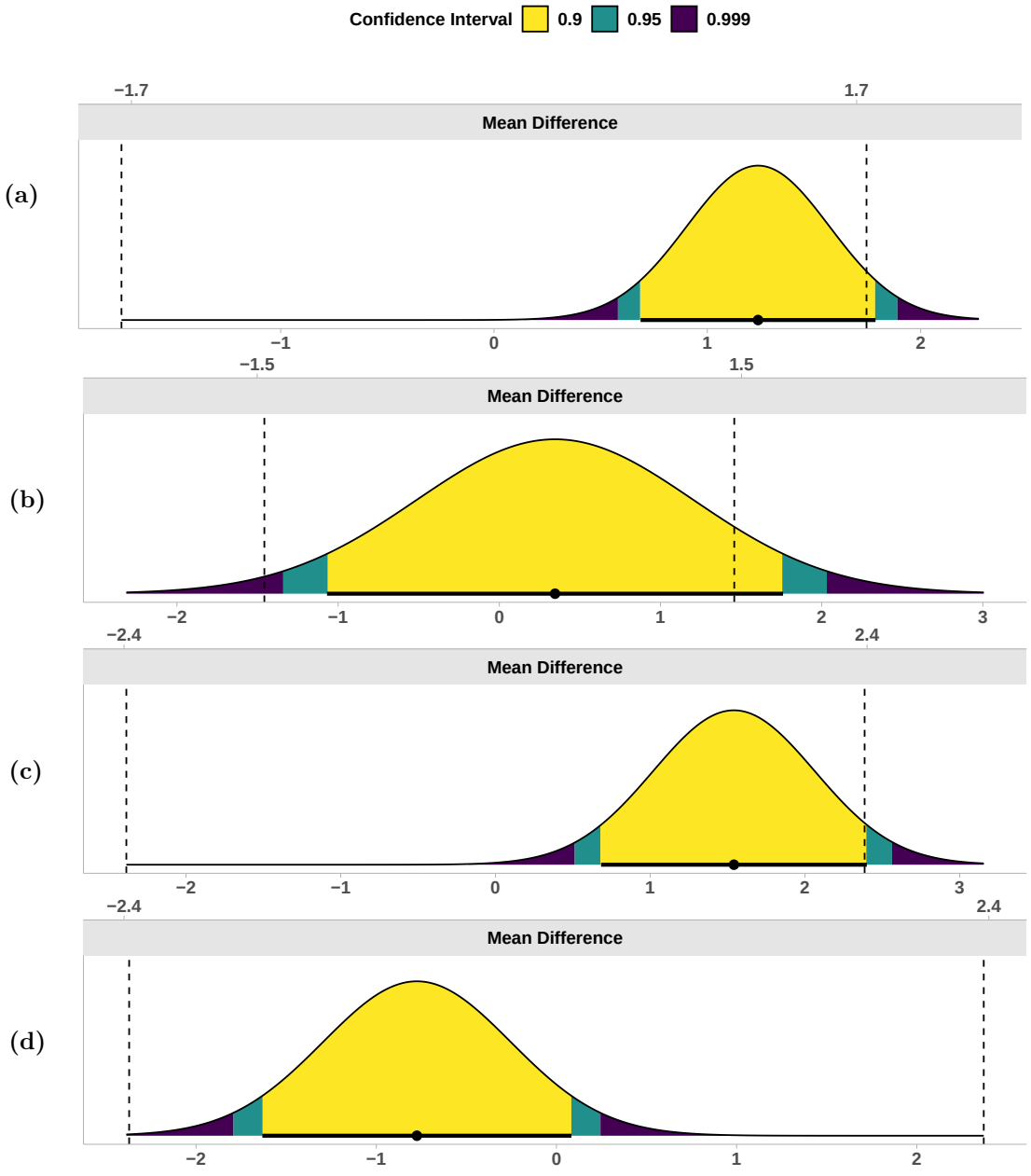


Figure 3.12: Distribution of mean differences in equivalence tests of expected outcomes, computerised env.: a) payoffs SEQ vs. SIM, b) payments SEQ vs. SIM, c) welfare PMA vs. SIM, d) welfare SEQ vs. SIM

Table 3.8: Linear models, testing for difference in expected outcomes, computerised environment

	Dependent variable:			
	Exp. payoff	Exp. revenue	Exp. welfare	Exp. payments
	(1)	(2)	(3)	(4)
PMA/c vs. SEQ/c	1.238***	1.078**	2.315***	0.346
	(0.326)	(0.448)	(0.557)	(0.849)
SEQ/c	8.597***	113.651***	122.248***	27.301***
	(0.241)	(0.331)	(0.412)	(0.628)
PMA/c vs. SIM/c	3.032***	−1.491***	1.541***	−4.791***
	(0.335)	(0.462)	(0.524)	(0.884)
SIM/c	6.804***	116.219***	123.023***	32.438***
	(0.240)	(0.331)	(0.376)	(0.633)
SEQ/c vs. SIM/c	1.794***	−2.568***	−0.775	−5.137***
	(0.397)	(0.494)	(0.519)	(0.988)
SIM/c	6.804***	116.219***	123.023***	32.438***
	(0.271)	(0.338)	(0.355)	(0.675)
Note:				*p<0.1; **p<0.05; ***p<0.01

of writing, the sample is not complete yet, the results from the regression with covariates are to be interpreted with caution. Some categories like the category 'Prefer not so say' in gender do not have enough observations to allow for a meaningful interpretation of the effect. Similarly, the number of auction experiments participated in appears to have some significant, but in size rather erratic effects, that may be only spurious. The subjects' numeracy score has a significant effect on welfare but not on payoffs. This may be due to some endogeneity, as Table B.7 shows an unambiguous, increasing trend in payoffs in all treatments. The number of errors subjects made in the comprehension quiz seems to be a significant predictor consistent across payoffs, payments, and revenue. However, a closer look at Table B.8 suggests that this may be driven mainly by the simultaneous auction treatment.

3.7 Conclusion

This experiment delivers the first evidence on the comparison of strategic bidding in the product-mix auction, a sequential, and a simultaneous auction. While the

results should still be considered preliminary, there is some clear indication that the product-mix format performs best in terms of bidder surplus and efficiency, whereas from the auctioneer's perspective, a simultaneous auction may be preferred as it achieves the highest revenue. These effects are even more pronounced in the treatments with symmetric, flexible bidders. These results contrast the theory developed alongside the experimental study: all pairwise equivalence tests fail to reject a hypothesised difference between outcomes, with the exception of efficiency in the sequential and the simultaneous auction.

Bidders with unit demand face a difficult strategic choice when several independent auctions offer the opportunity to win an object. Regardless if flexible bidders and competitive fringes, or several symmetric, flexible bidders participate in the auction, optimal bidding behaviour is non-trivial. The experimental results suggest that risk-neutral Bayes-Nash equilibrium cannot explain the subjects' strategies and that additional factors are at play. Indeed, bidding in the experiment was far from optimal; a brief analysis of bidding behaviour showed strong evidence of overbidding. Future work may attempt to shed light on why the different auction formats generates such different outcomes. For example, systematic differences in bid-shading across the auctions formats may be a possible channel. Nonetheless, the presented results already indicate that an auction format with combinatorial bids like the product-mix auction is the vastly more efficient solution in markets for heterogeneous substitutes.

4

Competitive equilibrium with allocation preferences

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4.1 Introduction

In many multi-object auctions, it is standard that buyers can submit bids on packages of indivisible objects. Submitting preferences over allocations in the

market, on the other hand, is typically not allowed, even though cost savings from specific allocations may arise naturally. In procurement auctions, e.g., a manufacturer may want to favour package allocations: by obtaining multiple products or services from a single supplier they can save transaction costs. In the literature on Walrasian equilibrium with indivisibilities to date, the preferences considered do not permit such costs either.

I study a competitive market setting for multiple, indivisible goods, in which the seller's preferences may depend on the *partitioning of supply* among buyers. The objective is to find a socially efficient allocation. I model the seller's preferences over allocations as *packaging costs*, where the key ingredient is a novel graph structure to express marginal cost functions. This *cost function graph* allows me to obtain anonymous, Walrasian equilibrium prices, which are linear in packages[1] and reflect the structure of marginal costs. The anonymity, package-linearity, and the structure itself guarantee fair and transparent pricing. The seller's marginal costs can reflect cost savings (or additional costs) arising from bundling certain objects together; at the same time, there is flexibility to sell individual objects separately if demand requires it. On the buyers' side, I also allow for a rich set of preferences admitting complementarities as well as some substitutabilities between objects. I characterise Walrasian equilibria and derive necessary and sufficient conditions for the existence of a Walrasian equilibrium. In addition, I provide sufficient conditions for the existence of a Walrasian equilibrium in terms of the agents' preferences.

In my model, a seller ("she") supplies multiple indivisible units of multiple varieties. Several bidders ("he") want to buy multiple units of each variety, and each bidder has a preference over combinations of these varieties. Similar settings, with the goal of characterising Walrasian equilibria, have been studied by Bikhchandani and Mamer (1997) and Bikhchandani and Ostroy (2002). However, they consider an exchange economy and a seller whose reservation utility (or marginal cost) is normalised to zero, respectively. Their model can be straightforwardly extended to marginal costs which are additive between objects, but these do not admit preferences over the partitioning of supply among buyers. For this partitioning problem I develop a tractable version of non-additive marginal costs, allowing me to provide insights into Walrasian equilibrium.

In the real world, non-additive marginal costs are present whenever the bundling of objects results in cost savings or additional expenses for the seller; I call these

[1]Under package-linear pricing, the same price applies to identical packages, and the price of a collection of several packages equals the sum of prices of the packages contained in the collection. A package-linear pricing function is non-linear in items, i.e. the price of a bundle may be different to the sum of prices of the items contained.

packaging costs. They can take the form of transaction costs (delivery, drawing up a contract), cost savings from realised synergies in the allocation, or even subsidies.

Many examples for realised synergies can be found in procurement. Consider a company holding a procurement auction for two input factors, which could be two different legal services for the same department. One legal team providing both services is likely more effective than two different legal teams, so the company would prefer to obtain the two services from the same law firm. At the same time, it wants to maintain flexibility to be supplied by two different firms if this is significantly cheaper. The two products may also be machines with a servicing contract. One supplier could provide maintenance and employee training for the machines more efficiently than two different suppliers. The literature in operations research has already taken an interest in related procurement issues. E.g. Bichler et al. (2011) propose a bidding language that accommodates various types of discounts a procurement manager may want to offer, although not in the context of Walrasian equilibrium. Some decision support systems used in practice[2] also allow for different types of discounts and sophisticated bids (Bichler et al. (2011), Giunipero et al. (2009), Gartner (2008)).

Subsidies are especially relevant in auctions held by the government. Consider the allocation of land plots for farming: a government with expert knowledge of land productivity may want to subsidise the allocation of complementary plots in order to achieve higher land productivity.[3] In auctions for biodiversity conservation contracts[4] certain allocations may also be favoured because different measures are more effective if implemented on the same piece of land. Applications may extend to transportation[5] , telecommunication licensing, and many others.

While my main results characterise Walrasian equilibria and their existence, my work also contributes to the literature on bidding languages, specifically the OR-of-XOR language (e.g. Nisan (2000) and Lehmann et al. (2006)).[6] A version of this language was also independently developed in the product-mix auction (PMA) by Klemperer (2008, 2010, 2018), which has been in use by the Bank of England

[2]Giunipero et al. (2009) interviewed the companies Ariba/Procuri, CombineNet, Emptoris, and Iasta.

[3]Market design approaches to land allocation have been studied, e.g., in Bryan et al. (2017).

[4]See, e.g., the well-known BushTender auctions in Victoria (Australia) studied by Stoneham et al. (2003) and others.

[5]A notable combinatorial auction in the transport sector is analysed in Cantillon and Pesendorfer (2005).

[6]XOR bids are standard in combinatorial auctions: e.g. the early package auction *i*Bundle Parkes (1999) (and many others) uses such bids. However, allowing high-dimensional XOR bids increases the computational complexity of the allocation problem. Using OR-of-XOR language when possible (see also Section 4.3.4) aims to reduce the computational complexity.

following the financial crisis in 2007 until today. In my model, each buyer has a valuation that is a relaxation of "OR-of-XOR" valuations. Such valuations can be represented in a finite list of "XOR-bids". I identify conditions under which a buyer can use this representation of preferences in the auction, under the relaxation that they may *repackage* their allocated collection of packages in any way desired. The possibility of repackaging a collection of bundles of items in the most profitable way seems natural in many settings.

Work by Baldwin and Klemperer (2019*b*) implies that OR-of-XOR preferences without allowing repackaging satisfy the strong substitutes (between packages) property,[7,8] and it is well known that Walrasian equilibrium exists if all buyers have strong substitutes valuations and the supply bundle is fixed (Danilov et al. (2001), Baldwin and Klemperer (2019*b*)). Consequently, a package-linear pricing Walrasian equilibrium exists if all buyers have strong substitutes valuations between packages and the supply bundle *of packages* is fixed. However, if the seller partitions a given supply bundle of different *objects* into packages *and* package-linear pricing functions are allowed, her preference cannot satisfy the "strong substitutes between packages" property (see Appendix C.2). The partitioning problem is significantly more difficult precisely because of this subtlety. Nevertheless, I derive a condition for the existence of Walrasian equilibrium, using linear programming techniques in the spirit of Bikhchandani and Mamer (1997). This approach has the additional advantage of providing a computationally tractable framework. Linear programming relaxations can be solved in polynomial time, and even for associated integer programmes duality theory provides avenues for practical computation (see, e.g., Vries and Vohra (2003)).

Moreover, I establish a second result on equilibrium existence that relies only on the agents' value and cost functions: a non-linear pricing Walrasian equilibrium always exists if buyers' values are superadditive, the seller's costs are subadditive, and only one unit per variety is for sale. This restricted environment is closely related to Sun and Yang (2014) (henceforth SY), but it is more general in that my seller's preferences distinguish between different partitionings of objects among buyers. In contrast, SY's seller has a reserve price for each bundle; once the reserve price is met, the bundle can be sold in any partition. I show that these reserve prices can be expressed as cost functions. Furthermore, I establish a new connection between superadditive (subadditive) set functions and their dual set

[7]See e.g. Milgrom and Strulovici (2009) or Baldwin and Klemperer (2019*b*) for a definition of strong substitutes.

[8]The attribute "between packages" signifies that packages, not objects, are priced individually. Thus, a buyer may substitute one package for another.

functions. This allows me to characterise properties of those cost functions that are equivalent to superadditive (subadditive) values.

I describe the general model in Section 4.2, and discuss the sealed-bid auction and the main result in Section 4.3. In Section 4.4, I present the second result on equilibrium existence and my results on the relationship between cost functions and reserve prices. Section 4.5 is a conclusion. All proofs are deferred to the appendix.

4.1.1 Walrasian equilibrium and packaging cost: an example

Consider a seller (e.g. the government) who has two distinct and indivisible land plots for sale, and there are two buyers in the market, Kate and Leon. Plot A is particularly suited to growing crop A(lfalfa), and plot B is suited to crop B(eans). Kate wants to buy either plot, but not both, and values plot A at 40 and plot B at 50. Leon, on the other hand, wants to buy both plots, but is not interested in a single plot. He values the combination of both plots AB at 70. All agents have transferable utility. The seller is denoted by index 0, Kate by K, and Leon by L. We are interested in a Walrasian equilibrium, i.e. an allocation $x_S^i \in \{0,1\}, i \in \{K,L,0\}, S \in \{\emptyset, A, B, AB\}$ and a pricing function $p : \{\emptyset, A, B, AB\} \to \mathbb{R}$ such that buyers demand their allocation (obtain their preferred bundle S) and the seller maximises her profit, at the given prices; and demand equals supply. A pricing function p satisfies by definition $p(\emptyset) = 0$ and it satisfies the standard notion of Walrasian equilibrium if competitive prices $p(A)$, $p(B)$, and $p(AB)$ are such that $p(AB) = p(A) + p(B)$. I call this a linear pricing Walrasian equilibrium. This chapter is concerned with a more general notion of Walrasian equilibrium, in which $p(AB) \neq p(A) + p(B)$. In the case where only a single unit of each of multiple distinct objects is sold, this is called a non-linear pricing Walrasian equilibrium.

First, assume the seller's marginal cost is zero for the two plots. It is not hard to verify that no linear pricing Walrasian equilibrium exists.[9] Surprisingly, even the more general non-linear pricing Walrasian equilibrium fails: there exists no solution in $p(A), p(B)$, and $p(AB) \neq p(A) + p(B)$.[10] Now I will modify the example.

[9]Suppose $p(A) + p(B) \leq 70$, then Leon demands AB. However, Kate also demands either A or B, unless $p(A) + p(B) > 90$, so markets cannot clear. Suppose $p(A) + p(B) > 70$, then Leon demands nothing. Kate demands either plot A or plot B (or nothing), but at the given prices, the seller wants to supply both (or the plot that is not demanded if one of p_A or p_B equals zero).

[10]Because no linear pricing equilibrium exists, it must be either (i) $p(A) + p(B) < p(AB)$ or (ii) $p(A) + p(B) > p(AB)$. If (i), the seller wants to sell the bundle AB. For Leon to demand AB, it must be that $p(AB) \leq 70$. For Kate to demand neither A nor B, it must be that $p(A) > 40$ and $p(B) > 50$, a contraction. If (ii), the seller prefers to sell A and B separately, but in this case markets do not clear.

Suppose that Kate does want to buy both plots, and she values the combination AB at 90 (and plot A at 40 and plot B at 50 as before). Suppose Leon also wants to buy plot A alone and values it at 50 (and the bundle AB at 70 as before). The agents' valuations v^i for this example are summarised in Table 4.1. Sun and Yang (2014) show that a non-linear pricing Walrasian equilibrium always exists, with any number of distinct land plots for sale, if all agents have superadditive values. Superadditivity of values in this example only requires $v^i(AB) \geq v^i(A) + v^i(B)$ for all $i \in \{K, L, 0\}$, which is now satisfied.[11] The seller also may have superadditive values for the bundles that she does not sell and retains for her own use instead. For example, the seller (seller 1) may have values $v^0(A) = 10$, $v^0(B) = 20$, and $v^0(AB) = 60$. The seller's profit is defined as the revenue she obtains from sold bundles plus the value she derives from unsold bundles.

	$\emptyset$	A	B	AB
Kate	0	40	50	90
Leon	0	50	0	70
Seller 1 (values)	0	10	20	60
Seller 2 (marg. cost)	0	40	50	60

Table 4.1: Agents' values and marginal costs over objects

In a different model, assume the seller (seller 2) prefers the two plots to be allocated to a single farmer.[12] The seller may not care if Kate or Leon obtains the bundle of AB as long as one farmer obtains both plots, i.e. the seller is indifferent about the buyer's identity (or legally required to be). Suppose the seller has a marginal cost function where $c^0(A) = 40$, $c^0(B) = 50$, and $c^0(AB) = 60$. Her profits are defined as revenue minus marginal cost of any bundle sold. Notice that this marginal cost function is not defined arbitrarily. It is the set function dual of v^0: if the value of retaining both plots A and B is 60, and the value of retaining only B is 20, the cost of selling A is 40. Indeed, there exists a transformation between a seller with values for retained items, and a seller who maximises revenue minus marginal cost (Proposition 22). However, the cost functions arising from such transformation can never accommodate preferences over the partitioning of supply (unlike the cost function defined here).

In a non-linear pricing Walrasian equilibrium, a seller with the value function v^0 for retained land plots sells plot A to Leon and plot B to Kate. On the contrary,

[11] Note that a linear pricing Walrasian equilibrium also exists in this example, supported by $p(A) = p(B) = 50$. In general, this is not the case as shown in Example 1 in Sun and Yang (2014).

[12] For example, a farm growing different crops may be less susceptible to weather extremes or demand fluctuations, and thus be more likely to thrive and yield higher long-term productivity.

equipping the seller with the marginal cost function c^0, the plot AB is allocated to Kate. This is precisely because in the presence of a seller with marginal cost function c^0, cost savings of 30 (40 + 50 - 60) are realised when allocating the bundle to a single buyer instead of allocating plot A and B separately. As is standard, a non-linear pricing (and the more general notion of package-linear pricing) Walrasian equilibrium implements an efficient allocation (Lemma 5). My Theorem 2 and Corollary 4 guarantee that a non-linear pricing Walrasian equilibrium always exists, with any number of distinct land plots for sale, if all buyers have superadditive values and the seller has subadditive marginal costs over the partitioning of supply.[13]

The results illustrated so far provide a baseline for Walrasian equilibrium with packaging costs and connect it to the previous literature. In reality, however, we want to go even further: a model of pure complements may still be unsatisfactory. Indeed, most of the time multiple units of distinct varieties need to be allocated, e.g. two land plots of type A, two land plots of type B, and three land plots of type C. Distinct or identical land plots may be complements for some buyers but substitutes for others. The seller may also have richer preferences over the partitioning, e.g., she may prefer allocations in which plots of type A and B are allocated to a single buyer, but plots of type B and C are each allocated to different buyers. She may prefer a more equal distributions of land among buyers, or she may prefer allocations where plots of the same type are allocated to a single buyer. Even more, she may not only favour a specific *type* of concentration (or equal distribution), she may also favour a certain *degree* of concentration: the cost savings from allocating A and B to a single buyer may be decreasing in the number of packages AB allocated. Finally, the marginal cost of allocating a plot of a given type may be increasing in the number of plots of that given type already allocated.

All these preferences, and many more dependencies between marginal costs of any bundles of land plots, are allowed in my general model; importantly, the preferences are independent of any buyer's identity. Precisely this requirement allows me to characterise a package-linear pricing Walrasian equilibrium with *anonymous* prices (Proposition 21 and Corollary 1), and I provide necessary and sufficient conditions for the existence of such equilibrium (Theorem 1).

[13]Note that introducing the seller's cost function to the example, a linear pricing Walrasian equilibrium does not exist: to implement the efficient allocation $x_{AB}^{Kate} = 1$ we must have $p(A) > 50$ so that Leon demands nothing, but Kate demands AB over B only if $90-p(A) \geq 50$, a contradiction.

4.2 A model of a competitive market with packaging costs

4.2.1 Preliminaries

I use basic multiset theory in my model and formally define multisets, feasible multisets, and basic operations on multisets.[14]

Definition 1 (Multiset)**.** Given an underlying finite set $A = \{a_1, \ldots, a_n\}$, a multiset is defined as a mapping $m : A \to \mathbb{N}_0$, and written as $\mathbf{m} := (m(a_1), \ldots, m(a_n))$.

Each element $a \in A$ is distinguishable, but occurrences of the same element are indistinguishable. $m(a)$ denotes the multiplicity of a, i.e. there are $m(a)$ occurrences of a in the multiset $\mathbf{m}$. For ease of notation, I denote $m(a)$ as m_a. There are n indivisible, distinguishable varieties in the economy, denoted $j \in N := \{1, \ldots, n\}$, with a maximum supply of Ω_j units per variety. I formulate my model such that packages of complementary objects (on the buyers' side) and packages associated with the seller's packaging costs only consist of distinct objects. Packages of identical objects will be treated as multisets of packages, or "collections of packages". Making this assumption simplifies the exposition of my multi-unit environment, and it is without loss of generality: relabelling appropriately I can translate any setting with complementarities or packaging costs between identical objects into mine, and I demonstrate this in Section 4.3.4.

Definition 2 (Package)**.** A package is defined as a set $S \subseteq N$.

A package contains at most one unit of each different variety, and there exist $2^n - 1$ distinct packages in the economy, not including the empty package. The powerset of N is denoted 2^N. If $|S| = 1$, the package may also be denoted by j, where j is the only variety contained in S. I write a multiset of varieties (with underlying set N) as $\mathbf{m} \in \mathbb{Z}_+^n$, a vector $(m_1, m_2, \ldots, m_n)$, where each m_j denotes the multiplicity of variety j. I write a multiset of packages (with underlying set 2^N) as $\mathbf{k} \in \mathbb{Z}_+^{|2^N|-1}$, a vector $(k_{S_1}, k_{S_2}, \ldots, k_{S_{2^n-1}})$, where each k_{S_x} denotes the multiplicity of package S_x. Given a set $A \subseteq 2^N$, $\mathbf{k}_{-A}$ denotes the vector $(k_{S_1}, k_{S_2}, \ldots, k_{S_{2^n-1}})$, where $k_{S_x} = 0$ for all $S_x \in A$. The feasibility of multisets in the economy is restricted by the maximum supply per variety. I denote the universe of all feasible multisets by $\mathbb{K}$. Formally,

$$\mathbb{K} := \left\{ \mathbf{k} \in \mathbb{Z}_+^{|2^N|-1} : \sum_{S \in 2^N, S \ni j} k_S \leq \Omega_j \quad \forall j \in N \right\}$$

[14] See for example Blizard (1989) for an extensive treatment of multisets.

If a multiset $\mathbf{k}$ contains only a single set S, i.e. $k_S = 1, k_{S'}, = 0$ for all $S' \neq S$, I denote $\mathbf{k}$ simply by S. For a function $f : \mathbb{Z}_+^{|2^N|-1} \to \mathbb{R}$, $f(S)$ is short for $f(\mathbf{k}$ with $k_S = 1, k'_S = 0 \; \forall S' \neq S)$. The maximum supply of all units is given by the multiset of varieties $\mathcal{N} := (\Omega_1, ..., \Omega_n)$. I define the following basic operations for multisets:

Definition 3. Let $\mathbf{k} = (k_S)_{S \in 2^N}, \mathbf{k}' = (k'_S)_{S \in 2^N} \in \mathbb{Z}_+^{|2^N|-1}$.

(i) Sum. $\mathbf{k} + \mathbf{k}' = (k_S + k'_S)_{S \in 2^N}$. The sum operator for multisets is denoted $\dotplus$.

(ii) Multiplication with a scalar. $\alpha\mathbf{k} = (\alpha k_S)_{S \in 2^N}$

(iii) Unpacking. $\mathbf{k}^* = (m_j)_{j \in N}$ with $m_j = \sum_{S \in 2^N, S \ni j} k_S \quad \forall j \in N$

(iv) Cardinality. $|\mathbf{k}| = \sum_{S \in 2^N} k_S$

The unpacking-operator * breaks a multiset of packages $\mathbf{k}$ down into a multiset of the individual varieties contained in $\mathbf{k}$.

4.2.2 Agents and preferences

There is one seller ("she") in the economy, denoted by 0, and a set of L buyers (bidders) denoted by $l \in \mathcal{L} := \{1, \ldots, L\}$. The set of all agents is denoted by $\mathcal{L}_0 := \mathcal{L} \cup \{0\}$.

4.2.2.1 Buyers

Each buyer ("he") has a preference over packages of goods, which can be summarised by a value function $V^l : \mathbb{Z}_+^{|2^N|-1} \to \mathbb{Z}_+$ with $V^l(\emptyset) = 0$. I add some structure on this value function such that preferences bear the characteristic of superadditivity between complementary objects, and, at the same time, the substitute characteristic of weakly decreasing marginal values between identical varieties and packages. In the literature, superadditivity is the most general concept of complementarity comprising supermodularity and gross complements (Samuelson (1974),Sun and Yang (2014)).

The preference structure is "OR-of-XOR" (see, e.g., Nisan (2000) or Lehmann et al. (2006)) with some additional requirements. Hence, each bidder's valuation can be mapped into a finite list of XOR-bids (and this will be shown formally later). However, I also introduce a new and natural feature into this preference: buyers may repackage an allocated collection of packages in any way they desire. Therefore, and to simplify the exposition of my results, I use a different definition of buyers' values.[15] Given a multiset of packages $\mathbf{k}$, a buyer's aggregate value of $\mathbf{k}$

[15] If repackaging is *not* allowed, my definition is equivalent to OR-of-XOR preferences.

is derived from an assignment of packages in the multiset to positions $\{1, \ldots, |\boldsymbol{k}|\}$ (where not all positions have to be filled), treating each copy as a separate package. The aggregate value of $\mathbf{k}$ will be defined by the value-maximal assignment of the copies of packages contained in $\mathbf{k}$.

Definition 4 (Marginal values)**.** A marginal value function is defined as $v^l : \mathbb{Z}_+ \times 2^N \to \mathbb{Z}_+$. $v^l(q, S)$ describes the marginal value bidder l derives from package $S \in 2^N$ if he assigns it the qth position in $\{1, \ldots, |\boldsymbol{k}|\}$. Interdependencies between valuations of different bidders are not allowed.

Definition 5 (Marginal value aggregation)**.** For any multiset $\mathbf{k}$, a bidder's overall value is defined as

$$V^l(\mathbf{k}) = \max_{\{x_{q,S}\}} \sum_{S \in 2^N} \sum_{q=1}^{\overline{M}} v^l(q, S) x_{q,S} \quad \text{s.t.} \sum_{q=1}^{\overline{M}} x_{q,S} \leq k_S \ \forall S, \sum_{S \in 2^N} x_{q,S} \leq 1 \ \forall q, x_{q,S} \in \{0, 1\} \ \forall S, q$$

where $\overline{M} := \max_{q,S}\{q : v^l(q, S) > 0\}$. $x_{q,S}$ is one if package S is assigned to the qth position in $\{1, \ldots, |\boldsymbol{k}|\}$, and zero otherwise.

If a bidder is given a multiset of packages, the bidder may not be able to repackage the varieties contained in the multiset of packages, as in Definition 5, due to physical or contractual constraints. I also consider bidders who *can* repackage the assigned varieties into those packages that are most valuable to them.

Definition 6 (Marginal value aggregation with repackaging)**.** For any multiset $\mathbf{k}$, a bidder's overall value is defined as

$$\widetilde{V}^l(\mathbf{k}) = \max_{\{x_{q,S}\}} \sum_{S \in 2^N} \sum_{q=1}^{\overline{M}} v^l(q, S) x_{q,S} \quad \text{s.t.} \sum_{S \ni j} \sum_{q=1}^{\overline{M}} x_{q,S} \leq \mathbf{k}^*_j \ \forall j, \sum_{S \in 2^N} x_{q,S} \leq 1 \ \forall q, x_{q,S} \in \{0, 1\} \ \forall S, q$$

where $\overline{M} := \max_{q,S}\{q : v^l(q, S) > 0\}$. Again, $x_{q,S}$ is one if package S is assigned to the qth position in $\{1, \ldots, |\boldsymbol{k}|\}$, and zero otherwise.

Example 6 in the appendix illustrates that the marginal value aggregation is useful and parsimonious when dealing with valuation for packages. It also illustrates the difference between the aggregating marginal values with and without repackaging. With this structure, it is now straightforward to define the properties of superadditivity between different varieties, and weakly decreasing marginal values between identical packages and varieties.

Definition 7 (Superadditivity)**.** A marginal value function $v^l : \mathbb{Z}_+ \times 2^N \to \mathbb{Z}_+$ is superadditive if for any disjoint sets $S_1, S_2 \in 2^N$ and for all $q \geq 1$ it holds that

$$v^l(q, S_1 \cup S_2) \quad \geq \quad v^l(q, S_1) + v^l(q, S_2).$$

Definition 8 (Decreasing marginal values)**.** A marginal value function $v^l : \mathbb{Z}_+ \times 2^N \to \mathbb{Z}_+$ is decreasing if for any package $S \in 2^N$, for all $q \geq 1$, it holds that

$$v^l(q, S) \quad \geq \quad v^l(q+1, S).$$

If a marginal value function $v^l : \mathbb{Z}_+ \times 2^N \to \mathbb{Z}_+$ satisfies Definitions 7 and 8, I call it *multi-unit-decreasing superadditive* (short MU-decreasing superadditive). I call the aggregate value function constructed from $v^l(q, S)$ according to Definition 5 or 6 *multi-unit concave superadditive* (short MU-concave superadditive) if $v^l(q, S)$ is MU-decreasing superadditive. I make the following assumptions on the bidders' values and utility:

Assumption 2 (Multi-unit-concave superadditivity)**.** Every bidders' value function V^l is MU-concave superadditive.

Assumption 3 (Quasi-linear utility)**.** Every bidder's utility is given by $u^l(\mathbf{k}, p) = V^l(\mathbf{k}) - p(\mathbf{k})$ when he receives a multiset of packages $\mathbf{k}$.

4.2.2.2 Seller

The seller's ("she") preferences are given by (i) an (incremental) marginal cost function for each individual variety sold, (ii) incremental marginal cost functions which may take effect when varieties are sold as a package, and (iii) a graph characterising cost relations, i.e. *which* incremental marginal cost functions take effect when a given package is sold. These three elements define the seller's cost functions I allow in my model. First, I define the graph characterising cost relations. This graph is called a cost function graph (CFG).

Definition 9 (Cost function graph)**.** A CFG is defined as a directed graph $G := (V, E)$, where the set of vertices is given by $V = 2^N$. The set of edges E defines cost relations subject to the conditions below.

Definition 10 (Path existence and length)**.**

(i) Whenever there exists a sequence of vertices $(S_1, S_2, ..., S_t)$ such that $(S_1, S_2), ..., (S_{t-1}, S_t) \in E$, I say there exists a path from S_1 to S_t and write $\exists(S_1...S_t)$. There always exists a path $(S, ..., S)$, i.e. from any package S to itself.

(ii) Given a path $H := (S_1, S_2, ..., S_t)$, $|H| := t - 1$, i.e. $|H|$ denotes the length of path H.

$(S_1, ..., S_t)$ may not be uniquely defined; but this is not important for my purpose. The CFG serves two purposes: First, it describes the cost relations between packages. Whenever $\exists(T...S)$, i.e. there exists a path from T to S, I say that "package T has a cost relation to package S". This means that a cost increment corresponding to S also contributes to the marginal cost of T. Second, the CFG facilitates the counting of varieties contained in packages, so that overall supply is never violated. Each node in the CFG also corresponds to an incremental marginal cost function. The incremental marginal cost function corresponding to a single variety j contributes to the overall cost of a multiset, whenever a copy of j is sold, if separately or as part of a package; hence, counting these contributions allows me to keep track of overall supply. To fulfil the two described purposes, the graph is required to have the following properties.

Assumption 4 (Permissible cost relations)**.**

(i) If $(T, S) \in E$, then $S \subset T$.

(ii) There exists a directed path from every package S to every single variety j that is contained in S, i.e. $\forall S,\ \forall j \in S,\ \exists(S...j)$.

Assumption 4(i) is intuitive in an economic sense: if a bundle has no "physical" relation to another package (in terms of being a superset of it), it also cannot have a cost relation to it. From Assumption 4(i) it follows that node N is always a source; but G may contain other sources as well. It also follows that G contains no cycles. From Assumption 4(i) and 4(ii) it follows that S is a sink if and only if S is a single variety. This is required because the seller's "physical" supply is given by the number of units available of each individual variety Ω_j. A marginal cost function corresponding to a sink will count the number of units of each single variety allocated.[16] Assumption 4(ii) ensures that whenever a unit of a given package S is allocated, the single varieties contained in that package are counted correctly. It also follows that G is weakly connected. If $\exists(T...S)$, I say that "package T has a cost relation to package S".

Each node of the CFG also corresponds to an incremental marginal cost function $\Delta c(\cdot, S), S \in 2^N$. Suppose one copy of package T is sold and nothing else. Then the

[16]Should the graph be such that S with $|S| > 1$ is a sink, S would have to be defined as a "new" single variety, available in quantity Ω_S.

marginal cost of T is obtained by adding all incremental marginal costs $\Delta c(\cdot, S)$ to which T has a cost relation, i.e. the marginal cost of T is $\sum_{S\in 2^N:\exists(T...S)} \Delta c(\cdot, S)$. Whenever a package with cost relation to S is sold, the incremental marginal cost $\Delta c(\cdot, S)$ is added to the overall cost of the supplied multiset. I allow the incremental cost to depend on q, the number of packages with cost relation to S sold.

Definition 11 (Incremental marginal cost)**.** Incremental marginal cost functions are defined as $\Delta c : \mathbb{Z}_+ \times 2^N \to \mathbb{Z}$, where $\Delta c(q, S)$ is the incremental marginal cost the seller incurs from selling a given copy of a package T due to its cost relation to package S, when she sells $q-1$ copies of other packages with cost relation to S, not including the original copy of T. For $q = 0$ and for all S, I define $\Delta c(0, S) := 0$. For all $j \in N$, and for all $q > \Omega_j$, I define $\Delta c(q, j) := \infty$.

Note that I permit negative incremental marginal costs.

Definition 12 (Marginal cost)**.** Given a CFG, and an incremental marginal cost function Δc, the marginal cost of a bundle $S \in 2^N$ is defined as a function $c^0 : \mathbb{Z}_+ \times 2^N \times \mathbb{Z}_+^{2^n-1} \to \mathbb{Z}_+$. Given $B \subseteq 2^N$, $c^0(k, S, \mathbf{k}_{-B})$ denotes the seller's marginal cost from selling the kth "unit" of package S, when she also sells the packages $\mathbf{k}_{-B}$. Let $S \in B$. Then

$$c^0(k, S, \mathbf{k}_{-B}) = \sum_{\substack{\gamma\in 2^N:\\ \exists(S...\gamma)}} \Delta c(q_\gamma, \gamma), \qquad \text{where } q_\gamma := \sum_{\substack{A\in 2^N\setminus B:\\ \exists(A...\gamma)}} k_A + k$$

Note that k_A counts the number of packages of type A supplied, whereas q_γ counts the number of packages of any type $A : \exists(A...\gamma)$ supplied, excluding the packages in B. For single varieties $j \in N$ we have $c^0(k, j, \mathbf{k}_{-j}) = \Delta c(k, j)$, so the marginal cost equals the incremental marginal cost of the kth unit of package j.

Definition 13 (Cost)**.** Given a CFG, and an incremental marginal cost function Δc, the cost of selling a number of packages given by $\mathbf{k} \in \mathbb{K}$ is defined as a function $C^0 : \mathbb{K} \to \mathbb{Z}_+$, and given by

$$C^0(\mathbf{k}) = \sum_{S\in 2^N} \sum_{y=1}^{q_S} \Delta c(y, S), \qquad \text{where } q_S := \sum_{\substack{\gamma\in 2^N:\\ \exists(\gamma...S)}} k_\gamma$$

The seller's overall cost of selling a multiset of packages is given by adding all incremental marginal costs that occur due to cost relations of the packages she supplies. This also takes the interaction of packages with overlapping subsets of single varieties into account; the cost of each additionally supplied package may

differ from the cost of a previously supplied, identical package. From Definition 11, it follows immediately that for any $\mathbf{k} = (k_S)_{S\in 2^N}$ for which $\exists\ j \in A \subseteq N : \sum_{S\in 2^N, S\ni j} k_S > \Omega_j$, we have $C^0(k, A, \mathbf{k}_{-A}) := \infty$, i.e. package A (and by extension the multiset $\mathbf{k}$) cannot be sold because the seller is constrained in the supply of at least one variety in A. Note that the seller cannot procure a bundle at the cost of the bundle and then split it up into subsets to be sold individually, or vice versa. I also make the following assumptions on the seller's preferences.

Assumption 5 (Increasing incremental marginal cost)**.** The seller's incremental marginal cost functions Δc are increasing, i.e., for any package $S \in 2^N$ and for all $q \geq 1$, it holds that

$$\Delta c(q, S) \quad \leq \quad \Delta c(q+1, S).$$

The seller incurs at least the cost of each individual package sold; if she sells two packages, the cost of selling both is weakly greater than the sum of the cost of each individual package, due to Assumption 5 and Definition 13.

Assumption 6 (Seller's quasi-linear profit)**.** The seller's profit is given by $u^0(\mathbf{k}, p) = p(\mathbf{k}) - C^0(\mathbf{k})$ when she sells the multiset of packages $\mathbf{k}$.

Definition 14 (Subadditive marginal cost)**.** Let $S_1, S_2 \in 2^N$ be two disjoint packages, and let $B := \{S_1 \cup S_2, S_1, S_2\}$. The seller's marginal cost is subadditive if it holds that, for all $\mathbf{k}_{-B} \in \mathbb{K}$, and for all $k \geq 1$,

$$c^0(k, S_1 \cup S_2, \mathbf{k}_{-B}) \quad \leq \quad c^0(k, S_1, \mathbf{k}_{-B}) + c^0(k, S_2, \mathbf{k}_{-B}).$$

For general cost function graphs, subadditivity of marginal cost is not obvious; one would have to compute the marginal cost for each possible multiset that could be supplied. For a special class of CFG, however, there is a straightforward criterion, which I describe in section 4.3.3.1.

4.2.3 Demand, supply, and equilibrium

The aim is to develop a framework in which equilibrium prices are *package-linear*.[17] Given the objects available for sale, the seller chooses a multiset $\mathbf{k}$ of packages to supply. $\mathbf{k}$ must be feasible, i.e. $\mathbf{k} \in \mathbb{K}$, or $\bigplus_{S\in 2^N} k_S S^* \subseteq \mathcal{N}$. An allocation of objects in $\mathcal{N}$ is defined as an assignment $\pi = (\pi(l), l \in \mathcal{L}_0)$ of these

[17] SY call a pricing function $p : 2^N \to \mathbb{R}$ *non-linear*, and the equilibrium concept *Non-linear* pricing Walrasian equilibrium (NPW). With only one unit per variety for sale, the non-linear equilibrium and the package-linear equilibrium are the same concept.

objects among the bidders and the seller, such that $\biguplus_{l\in\mathcal{L}} \pi(l) = \mathbf{k}$ and $\pi(0) = \mathcal{N} - \mathbf{k}^*$. $\pi(l)$ is the multiset of packages assigned to agent l under the allocation π, where $\pi(l)$ may be the empty set, and $\pi(0) \neq \emptyset$ means that the objects in $\pi(0)$ are not sold. Bidder l's demand correspondence and indirect utility are defined as

$$D^l(p) := \arg\max_{\mathbf{k}\in\mathbb{K}} u^l(\mathbf{k}, p) \text{ and } \mathcal{V}^l(p) := \max_{\mathbf{k}\in\mathbb{K}} u^l(\mathbf{k}, p).$$

I define the seller's supply correspondence $S(p)$ and profit functions as

$$S(p) := \arg\max_{\mathbf{k}\in\mathbb{K}} p(\mathbf{k}) - C^0(\mathbf{k}) \text{ and } \Pi(p) := \max_{\mathbf{k}\in\mathbb{K}} p(\mathbf{k}) - C^0(\mathbf{k}).$$

Definition 15. An allocation π is efficient if it holds for every allocation π' that

$$\sum_{l\in\mathcal{L}} V^l(\pi(l)) - C^0(\mathbf{k}) \geq \sum_{l\in\mathcal{L}} V^l(\pi'(l)) - C^0(\mathbf{k}')$$

where $\mathbf{k} = \biguplus_{l\in\mathcal{L}} \pi(l)$ and $\mathbf{k}' = \biguplus_{l\in\mathcal{L}} \pi'(l)$.

Given an efficient allocation π, the market value is defined as $V(\mathcal{N}) := \sum_{l\in\mathcal{L}} V^l(\pi(l)) - C^0(\mathbf{k})$. Formally, a pricing function is a function $p : \mathbb{Z}_+^{|2^N|-1} \to \mathbb{R}$ with $p(\emptyset) = 0$. This function is *nonlinear* in varieties $j \in N$, i.e. for any package $S \in 2^N$ we may have $p(S) \neq \sum_{j\in S} p(\{j\})$.

Definition 16. A pricing function $p : \mathbb{Z}_+^{|2^N|-1} \to \mathbb{R}$ with $p(\emptyset) = 0$ is *package-linear* if and only if, for all $\mathbf{k} \in \mathbb{Z}_+^{|2^N|-1}$, $p(\mathbf{k}) = \sum_{S\in 2^N} k_S p(S)$.

Thus, a package-linear pricing function can be represented as a mapping $p : 2^N \to \mathbb{R}$.

Definition 17. A *package-linear* pricing Walrasian equilibrium is given by a *package-linear* pricing function $p^* : 2^N \to \mathbb{R}$ and an allocation π^* such that $\biguplus_{l\in\mathcal{L}} \pi(l) \in S(p^*)$ and $\pi^*(l) \in D^l(p^*)$ for every bidder $l \in \mathcal{L}$.

If the pricing function is linear in varieties, the package-linear Walrasian equilibrium is simply the standard Walrasian equilibrium. Walrasian equilibrium and and efficient allocations are connected in the standard way.

Lemma 5. *If (p^*, π^*) is a package-linear Walrasian equilibrium, π^* is an efficient allocation. Furthermore, if π' is another efficient allocation, (p^*, π') is a package-linear Walrasian equilibrium as well.*

Note that I defined efficiency and Walrasian equilibrium assuming bidders have aggregate marginal values without repackaging. If bidders have the technology to repackage bundles they have been allocated in the auction, V^l needs to be substituted by $\widetilde{V}^l$ (in the definition of u^l).

4.3 The sealed-bid package auction and Walrasian equilibrium

In this section I describe how the agents can submit their preferences in the auction. The auction design is based on the assumption of agents submitting (approximately) truthful preferences. It will be shown later that this results in a Walrasian equilibrium allocation supported by competitive prices; but it is helpful to keep the assumption of truthful behaviour in mind in this section.

4.3.1 Buyers' bidding language

Each bidder l submits a list of bids to the auctioneer in a sealed envelope. The auctioneer treats each bid as independent from other bids, including the ones made by the same bidder, i.e. the auctioneer only cares about the aggregate list of bids. It is therefore convenient to simply enumerate the bids by $i \in \mathcal{I} := \{1, \ldots, m\}$. I also denote the list of bids submitted from bidder l by $\mathcal{I}^l$, i.e. $\bigdoteq_{l \in \mathcal{L}} \mathcal{I}^l = \mathcal{I}$.

Each bid submitted in the auction is a *package XOR-bid* and each bidder can submit any finite number of package XOR-bid.

Definition 18 (Package XOR-bid)**.** A package XOR-bid is a vector of length 2^n. It specifies a maximum price v_S^i for each available package in the auction (except the empty package) which the bidder is willing to pay, and an overall maximum quantity κ^i.

I denote by x_S^i the allocation of bundle S to bid i. I will show that each package XOR-bid is guaranteed to obtain only packages that maximise its surplus.

Package XOR-bids are the straightforward extension of so called "paired bids" used in the product-mix auction by Klemperer (2010) to an environment where packages are sold in addition to single varieties. An allocation $x_S^i, S \in 2^N, i \in \mathcal{I}^l$ maps into a multiset $\pi(l) = \mathbf{k}^l$, where $k_S^l = \sum_{i \in \mathcal{I}^l} x_S^i$. Naturally, this mapping is not one-to-one, as different allocations may map into the same multiset. I say that a value function V^l can be represented by a list of bids, if (i) there exists a mapping from V^l to the list of bids, and (ii) given an allocation to bidder l, the bidder's value V^l from this allocation is identical to the value entering the auctioneer's objective. Package XOR-bids allow each bidder to represent their preferences:

Proposition 19. *If the auctioneer maximises social welfare, every bidder's value function V^l satisfying Definitions 4 and 5 and Assumption 3 can be represented by a finite list of XOR-bids.*

The proposition above tells us that when bidders cannot repackage their received objects, preferences can be easily translated into the XOR-bid language. However, once bidders are allowed to repackage, the bidding language loses some of its power. Consider Example 6 in Appendix C.3. If bidder 1 gets assigned A and B as separate bundles and she can repackage, her value of this assignment is $v^1(1, AB) = 9$, not $v^1(1, B) + v^1(2, A) = 5 + 1$. Thus, the interpretation of the bids is ambiguous to the seller: because she cannot distinguish which bids were made by which bidder, she may assume that A and B are used as separate bundles, when in fact the bidder uses them as a package, and hence should also pay the package price p_{AB}, instead of $p_A + p_B$. If repackaging is possible, in general, an accurate representation of the *seller's* preferences is not possible. Given an allocation, the bidder is always weakly better off compared to the value the auctioneer assigns to his contribution to welfare. However, when the preferences of the seller and bidders align such that all varieties are complementary to one another, repackaging can be allowed.

Proposition 20. *Assume the auctioneer maximises social welfare, and let bidder l's preference satisfy Assumption 2 (MU-concave-superadditivity) and let the seller's preference satisfy subadditivity of marginal costs according to Definition 14. (Assume also that the standard Definitions 4, 5, 9, 11, 12, 13 hold and standard Assumptions 3, 4, 5 of my model are satisfied.) Then the bidder never has to repackage bundles to achieve the highest aggregate of marginal values, i.e. $V^l = \widetilde{V}^l$.*

Given the XOR-bid structure and Proposition 19, bidder l's indirect utility is

$$\mathcal{V}^l(p) := \max_{x_S^i, i \in \mathcal{I}^l} \sum_{S, i \in \mathcal{I}^l} (v_S^i - p_S) x_S^i \quad \text{s.t.} \ \sum_S x_S^i \leq \kappa^i \ \forall i \in \mathcal{I}^l \tag{4.1}$$

For three packages $\{A\}, \{B\}, \{AB\}$, a bid $(v_A, v_B, v_{AB}, 1)$ can be illustrated in price space $p_A \times p_B \times p_{AB}$ as shown in Figure C.6 in the appendix.

Note that, allowing repackaging of collections of bids, MU-concave superadditivity of bidders' values and subadditivity of the seller's marginal costs is required such that bidders can represent their preferences accurately in the mechanism. Generally, not allowing repackaging, any buyers' preference that can be represented as a list of XOR-bids is admissible in the auction. If a bidder were interested in at most one unit of each variety, a package XOR-bid would be fully expressive, i.e. allow the representation of *any* preference. Although somewhat more restrictive with multiple units per variety, bidders can express a rich set of preferences in this language.

It needs to be emphasised that the assumptions imposed on the agents' preferences are only to comply with the bidding language of the mechanism. Without the possibility to repackage, the sealed-bid auction still yields a Walrasian equilibrium (if it exists) if some varieties are substitutes.

4.3.2 Seller's bidding language

The seller can also perfectly represent her preferences in the auction. She submits her preferences through two components: (i) supply functions, and (ii) a graph describing the relations between supply functions. If she reports her preferences truthfully, the supply functions will correspond precisely to the incremental marginal cost functions, and the graph describing relations between supply functions will correspond to her cost function graph (CFG). The submitted CFG is called a supply function graph (SFG).

The seller submits $2^n - 1$ supply functions. Each is required to be a weakly increasing step function. Analogously to incremental marginal cost functions, a supply function f_j for an individual variety j describes the marginal cost of each unit supplied of variety j. A supply function f_S for package S with $|S| > 1$ describes the incremental marginal cost for every package $S' : \exists(S'...S)$, i.e. the additional cost, or the cost savings, inherited by S', which also contains the objects bundled together in S.

f_S is a step function, where the qth step has length l_S^q and height μ_S^q. Without loss of generality the length of each step can be normalised to one. I denote by y_S^q the amount allocated on step q of f_S. Each supply function consists of an infinite number of steps, and there exists a finite $\bar{q}_S$ such that $\mu_S^q = \infty$ for all $q > \bar{q}_S$. Thus, any step $q > \bar{q}_S$ will never be allocated. I define $\bar{q} := \max\{\bar{q}_S | S \in 2^N\}$.

I make the following two requirements on supply function graphs to ensure non-negative costs and weakly increasing incremental marginal costs.

Requirement 1 (Non-negative costs)**.** $\forall\, \mathbf{k} \in \mathbb{K} : \sum_{S \in 2^N} \sum_{y=1}^{q_S} \mu_S^y \geq 0,$ where $q_S := \sum_{\substack{\gamma \in 2^N: \\ \exists(\gamma...S)}} k_\gamma$

Requirement 2 (Weakly increasing marginal costs)**.** $\forall\, S, q : \mu_S^q \leq \mu_S^{q+1}$

Because supply function f_S describes the incremental marginal cost that is inherited by all packages S' with a cost relation to S, i.e. $\exists(S'...S)$, I define the allocation on supply function as follows.

Definition 19 (Supply step allocation)**.** Whenever a step on supply function f_S is allocated, a step on every supply function $f_{S'}$ is allocated, for all $S' : \exists(S...S')$.

The allocation on supply curve f_S is limited by the minimum number of steps with finite height among all supply curves $f_\gamma : \exists(S...\gamma)$. Note that this reflects precisely my definition of marginal costs: for $\mathbf{k} = (k_S)_{S \in 2^N}$ for which $\exists\, j \in A \subseteq N : \sum_{S \in 2^N, S \ni j} k_S > \Omega_j$, I defined $C^0(k, A, \mathbf{k}_{-A}) := \infty$. An important property of this supply function graph is that an allocation $(y_S^q)_{S \in 2^N, q \leq \bar{q}}$ uniquely identifies a multiset of packages:

Lemma 6. *Given a supply function graph $G = (V, E)$ satisfying Assumption 4, there exists a linear one-to-one mapping between a supply function allocation $(y_S^q)_{S\in 2^N, q\leq \bar{q}}$ and a corresponding multiset of packages* $\mathbf{k}$.

Now I can rewrite the seller's problem as follows.

$$\begin{aligned}
& \max_{\mathbf{k}\in\mathbb{K}} \left\{ p(\mathbf{k}) - C^0(\mathbf{k}) \right\} \\
\Leftrightarrow \quad & \max_{k_S} \left\{ \sum_{S\in 2^N} k_S p(S) - \sum_{S\in 2^N} \sum_{y=1}^{q_S} \Delta c(y, S) \right\} \qquad \text{where } q_S := \sum_{\substack{\gamma\in 2^N: \\ \exists(\gamma \ldots S)}} k_\gamma \\
\Leftrightarrow \quad & \max_{k_S, y_S^q} \left\{ \sum_{S\in 2^N} k_S p(S) - \sum_{S\in 2^N} \sum_{q=0}^{\bar{q}_S} \mu_S^q y_S^q \right\} \qquad \text{s.t. } y_S^q \leq l_S^q \ \forall S, q
\end{aligned}$$

Subtracting the seller's cost simply amounts to subtracting the amounts allocated on each step multiplied with the height of the corresponding step; Lemma 6 allowed me to substitute some k_γ with the y_γ^q, which are now additional choice variables.

4.3.3 Solving the auction

The auction input is a list of bids $\{(v_S^i)_{S\in 2^N}, \kappa^i\}_{i\in\mathcal{I}}$, a list $(\mu_S^q, l_S^q)_{S\in 2^N, q\leq\bar{q}}$, and a supply function graph given by a graph G_{SFG}. x_S^i denotes the allocation of bundle S to bid i, y_S^q denotes the allocation on step q of the supply function f_S, and Y_S denotes the total amount of bundle S allocated. The auctioneer's objective is to find the efficiency maximising partitioning of supply. That is, the objective is to find an allocation $\{x_S^i, y_S^q, Y_S\}_{S\in 2^N, i\in\mathcal{I}, q\leq\bar{q}}$ that maximises the sum of the buyers' surplus and of the seller's profit. The SFG determines the relations between the supply functions $(f_S)_{S\in 2^N}$. Lemma 6 tells us that given a SFG with the necessary properties, there exists a linear one-to-one mapping between $\{y_S^q\}_{S\in 2^N, q\leq\bar{q}}$ and Y_S, for all $S \in 2^N$. I denote this mapping by $\phi : \mathbb{Z}_+^{2^n\times\bar{q}} \to \mathbb{Z}^{2^n}$, where $\phi_S(\{y_S^q\}) = Y_S$. The auctioneer's problem can be written as an integer programme, which is given by

IP

$$\max_{\{x_S^i, y_S^q, Y_S\}} \left[\sum_{i,S} v_S^i x_S^i - \sum_{S,q} \mu_S^q y_S^q \right]$$

$$\text{s.t.} \begin{cases}
\sum_S x_S^i \leq \kappa^i & \forall i & \text{bid size constraint} \\
y_S^q \leq l_S^q & \forall q, \forall S & \text{step size constraint} \\
\sum_i x_S^i \leq Y_S & \forall S & \text{supply constraint} \\
Y_S = \phi_S(\{y_S^q\}) & \forall S & \text{SFG relations} \\
x_S^i, y_S^q, Y_S \in \mathbb{Z}_+ & \forall S, \forall i, \forall q &
\end{cases}$$

The supply constraint must be binding at the optimum, so prices cancel from the buyers' and seller's objective. As a next step, I substitute for Y_S, and relax the integer programme to a linear programme 'LP'. The corresponding dual variables are listed to the right of the constraints.

LP

$$\max_{\{x_S^i, y_S^q\}} \left[\sum_{i,S} v_S^i x_S^i - \sum_{S,q} \mu_S^q y_S^q \right]$$

$$\text{s.t.} \left\{ \begin{array}{rcll} \sum_S x_S^i & \leq & \kappa^i \quad \forall i & b_i \quad \text{bid size constraint} \\ y_S^q & \leq & l_S^q \quad \forall q, \forall S & u_S^q \quad \text{step size constraint} \\ \sum_i x_S^i - \phi_S(\{y_S^q\}) & \leq & 0 \quad \forall S & z_S \quad \text{supply constraint} \\ x_S^i, y_S^q & \geq & 0 \quad \forall S, \forall i, \forall q & \end{array} \right.$$

Linear programming techniques allow me to write the corresponding dual problem 'DLP'. I denote by $\psi(\{z_S\})$ the "dual function" of $\phi(\{y_S^q\})$. Formally, let $\phi(\{y_S^q\}) = \Phi \mathbf{y}^\intercal$, where Φ is a $|2^N| \times |2^N|$-matrix, which can be determined by Algorithm 3, and $\mathbf{y} = \left(\sum_q y_S^q\right)_{S \in 2^N}$, i.e. a row vector each entry of which contains the total amount allocated on supply curve f_S. $\phi_S(\{y_S^q\}) = \Phi_S \mathbf{y}^\intercal$, i.e. the row corresponding to package S of Φ multiplied by $\mathbf{y}^\intercal$. Then $\psi(\{z_S\}) = \Phi^\intercal \mathbf{z}^\intercal$, where $\mathbf{z} = (z_S)_{S \in 2^N}$.

DLP

$$\min_{\{b^i, z_S, u_S^q\}} \left[\sum_i \kappa^i b^i + \sum_{q,S} l_S^q u_S^q \right]$$

$$\text{s.t.} \left\{ \begin{array}{rcll} b^i + z_S & \geq & v_S^i \quad \forall i, S & x_S^i \quad \text{surplus constraint} \\ u_S^q - \psi_S(\{z_S\}) & \geq & -\mu_S^q \quad \forall q, \forall S & y_S^q \quad \text{marginal cost constraint} \\ b^i, u_S^q, z_S & \geq & 0 \quad \forall S, \forall i, \forall q & \end{array} \right.$$

In the following, I define the final auction prices as part of the solution of DLP. I characterise the price structure and discuss some additional properties of the auction prices. Finally, I state my main theorem of this section.

The auction price of bundle S is given by the dual variable of the supply constraint on bundle S, z_S. In the interpretation of z_S as a shadow price, it is the value of the last unit the seller supplies on a given bundle. The auction prices z_S are uniform, anonymous, package-linear prices, i.e. a generalisation of the uniform pricing rule to package-goods. b^i is the surplus generated on bid i. The feasible set

of LP is a non-empty, convex polytope and therefore an optimal solution always exists; by strong duality, an optimal solution to DLP exists also.

Proposition 21 reveals an important property of the auction prices. The auction price is determined, not only for each package S of which a positive amount is allocated in the auction, but also for each package to which S has a cost relation.

Proposition 21. *For all $S \subseteq N$,*
(i) it holds that, for all $q \leq \bar{q}$,

$$z_S \leq \sum_{\gamma:\exists(S...\gamma)} \left(u_\gamma^q + \mu_\gamma^q\right)$$

(ii) and if $y_\gamma^{\widetilde{q_\gamma}} \neq 0 \ \forall\gamma : \exists(S...\gamma)$, it holds that, for all $q_\gamma \leq \widetilde{q_\gamma}$,

$$z_S = \sum_{\gamma:\exists(S...\gamma)} \left(u_\gamma^{q_\gamma} + \mu_\gamma^{q_\gamma}\right)$$

For single varieties, we have $z_j = u_j^q + \mu_j^q$ if $y_j^q \neq 0$. That is, the auction prices are given by marginal costs to the extent that they do not have to be adjusted to set non-allocated bids at least indifferent (the u_j^q provide the necessary flexibility); for packages, the z_S equal the sum of incremental marginal costs of each variety or package that package S has a cost relation to (plus adjustments through the u_j^q). There may exist a set of equilibrium prices. The complementary slackness conditions corresponding to primal and dual constraints generate a system of equations which the auction prices have to satisfy. This solution is not necessarily unique; and in addition, the u_S^q may leave some ambiguity. This creates more flexibility for the auctioneer because he can choose from a set of equilibrium prices, and she can specify additional rules to do so.[18] For example, she can define a rule to always choose the lowest equilibrium prices on certain packages,[19] or the highest, or a compromise between prices favouring bidders and prices favouring the seller. The following corollary provides an additional characterisation of the auction prices.

Corollary 1. Let q be the last step of supply function f_S that is allocated, i.e. $y_S^q \neq 0$ and $y_S^{q+1} = 0$. Let also $y_\gamma^{\widetilde{q_\gamma}} \neq 0 \ \forall\gamma : \exists(S...\gamma)$. Then, for all $q_\gamma \leq \widetilde{q_\gamma}$,

$$\text{(i)} \qquad u_S^q + \mu_S^q \leq \mu_S^{q+1}$$

$$\text{(ii)} \qquad \psi_S(\{z_S\}) = z_S - \sum_{\gamma:\exists(S...\gamma),\gamma\neq S} \left(u_\gamma^{q_\gamma} + \mu_\gamma^{q_\gamma}\right)$$

[18] Modern LP-solvers can return the set of all integer solutions.
[19] A set of lowest equilibrium prices may not always exists (see example 6 in the appendix).

The term $\psi_S(\{z_S\})$ describes the price gain from selling all objects in S together in package S, in addition to the price satisfying all other cost relations. If this price gain was strictly less than μ_S^q, the seller would prefer not to sell the package, and if this price gain was strictly greater than μ_S^{q+1}, the seller would prefer to sell an additional package S. Finally, the auction prices I characterised support indeed an equilibrium. Moreover, this equilibrium exists if and only if it can be characterised by an optimal solution of LP.

Theorem 1. A package-linear pricing Walrasian equilibrium exists if and only if any optimal solution to IP is also an optimal solution to LP, i.e. the optimum values of IP and LP coincide.

My theorem generalises the results by Bikhchandani and Mamer (1997) and Bikhchandani and Ostroy (2002) in that my seller's preferences are non-additive in varieties. My auction is a uniform price auction for multiple differentiated packages. Because the allocation rule is such that not only units of the same variety, but also packages composed of different varieties compete against one another for an efficient allocation, my auction can be seen as set of simultaneous uniform-price auctions for packages, with additional competition between packages. Thus, the results from Swinkels (2001) on asymptotic efficiency apply to my setting. Under asymptotic environmental similarity[20] and other standard assumptions, they show that any equilibrium must be asymptotically ex-ante efficient. If a large number of bidders *for each package* participates in my auction, with the assumptions of Swinkels (2001) being satisfied for each package separately, it appears my auction is also asymptotically ex-ante efficient. Related results have been shown by Cripps and Swinkels (2006) and Fan et al. (2003).

4.3.3.1 Complete supply function graphs

When cost savings from packaging two varieties together are independent of other varieties present, a special class of cost functions graphs can be formulated. This class of CFGs may be especially applicable to the procurement of factor inputs: if service A and service B are delivered by the same provider, the manufacturer incurs cost savings, and similarly for services A and C. However, if services B and C are delivered by the same provider, additional costs arise, e.g. because of a high degree of substitutability between B and C paired with a high contingency

[20] Asymptotic environmental similarity requires the probability of some bidder i winning the hth unit with bid b to converge uniformly to the probability of any bidder j winning the h'th unit with bid b, for all i, j, h, h', b.

risk. If all three services A, B, and C were delivered by the same provider, this could be modelled precisely: cost savings from A and B, A and C, and additional cost from B and C enter the cost function, plus an additional cost savings term to account for the interaction of A, B, and C together.

More generally, a cost function graph of this class is such that each package has a cost relation to all packages that are subsets of itself; and these cost relations are achieved with a minimal number of edges, i.e. there are directed paths from each package S to its subsets of size $|S|-1$ only. The following introduces some additional notation needed to generate results on these special graphs, including some closed form solutions.

Definition 20 (Levels)**.** Let $x, y \in 2^N$.

(i) $x \subset_r y := \{x \mid x \subseteq y, |y|-|x| = r\}$. Similarly, $y \supset_r x := \{y \mid y \supseteq x, |y|-|x| = r\}$. I say y is r levels above x, or x is r levels below y.

(ii) $x \subset_{\geq r} y := \{x \mid x \subseteq y, |y| - |x| \geq r\}$. Similarly, $y \supset_{\geq r} x := \{y \mid y \supseteq x, |y| - |x| \geq r\}$. I say y is at least r levels above x, or x is at least r levels below y.

(iii) $x \subset_{\leq r} y := \{x \mid x \subseteq y, |y| - |x| \leq r\}$. Similarly, $y \supset_{\leq r} x := \{y \mid y \supseteq x, |y| - |x| \leq r\}$. I say y is at most r levels above x, or x is at most r levels below y.

Note that $x \supset_0 y$ implies $x = y$.

Definition 21 (Complete CFG)**.** A CFG is complete if and only if for each S, for all $\gamma \subset_1 S$, we have $(S, \gamma) \in G$, and for all other $\gamma \subset_{\geq 2} S$, and for all $\gamma \not\subseteq S$, we have $(S, \gamma) \notin G$ (see Fig. C.3 in Appendix C.3.3).

Lemma 7 (Subadditivity in complete CFG)**.** *Suppose the CFG is complete. If the incremental marginal cost satisfies*

$$\Delta c(q, S) \geq 0 \ \forall S : |S| \ = 1 \quad \text{and} \quad \Delta c(q, S) \leq 0 \ \forall S : |S| \ > 1$$

then the marginal cost is subadditive.

The following lemma makes the crucial connection between the supply function graph and the overall allocation per variety.

Lemma 8 (ϕ-mapping for complete SFG)**.** *Let Y_S denote the overall amount of bundle S allocated. Given the supply function graph defined above, it holds that*

$$Y_S = \phi_S(\{y_S^q\}) = \sum_{r=0}^{n-|S|} \sum_{q, \gamma \supset_r S} (-1)^r y_\gamma^q$$

Duality then implies

Corollary 2 (ψ-mapping for complete SFG)**.**

$$\psi(\{z_S\}) = \sum_{r=0}^{|S|-1} \sum_{\gamma \subset_r S} (-1)^r z_\gamma$$

Following Corollary 1, the term $\sum_{r=0}^{|S|-1} \sum_{\gamma \subset_r S} (-1)^r z_\gamma$ describes the incremental price gain from selling all objects in S together in package S. In some cases the seller may wish to specify positive price increments. The subsequent corollary provides an additional characterisation result for this case, and is immediate from Proposition 21.

Corollary 3. If $\mu_S^q \geq 0$, and if $y_S^q \neq 0$, then $z_S \geq z_\gamma \; \forall \gamma \subseteq S$.

While a very general class of supply function graphs allows me to solve the partitioning problem, closed form solutions simplify the implementation of supply function graphs.

4.3.4 Complementarities and packaging costs between identical objects

If the market designer anticipates complementarities between identical objects, they may want to allow bidders to submit specific bids on packages containing those identical objects. To accommodate this, I simply introduce different notation relabelling identical objects. I demonstrate this "preprocessing" with an example. Suppose there are two distinct varieties A and B with supply $\Omega_A = \Omega_B = 2$. The two units of variety A may *either* be complementary for the buyer *or* a packaging cost may be associated with the multiset $\{AA\}$, i.e. the auctioneer wants to allow for the multiset $\{AA\}$ to be treated as a package with a separate price $p(AA) \neq 2p(A)$. An admissible XOR-bid is a vector $\boldsymbol{b} := (v_A^i, v_B^i, v_{AA}^i, v_{AB}^i)$, and, as usual, each bidder may submit a finite list of these XOR-bids. The relabelling is done by the auctioneer for the workings of the mechanism: bids are simply processed differently, but bidders and the seller are not presented with relabelled objects. I reformulate the setup such that *each unit of supply* of variety A obtains a different index, i.e. we have $N := \{A_1, A_2, B\}$, and a package is defined simply as $S \subseteq N$. To process the bid vector correctly, it is augmented as $\tilde{\boldsymbol{b}} = (v_{A_1}^i, v_{A_2}^i, v_B^i, v_{A_1A_2}^i, v_{A_1B}^i, v_{A_2B}^i, v_{A_1A_2B}^i)$, where $v_{A_1}^i = v_{A_2}^i = v_A^i$, $v_{A_1A_2}^i = v_{AA}^i$, $v_{A_1B}^i = v_{A_2B}^i = v_{AB}^i$, and $v_{A_1A_2B}^i = \max\{v_{AA}^i, v_{AB}^i\}$.

The seller submits supply functions f_A, f_B, f_{AB}, and f_{AA}, and a supply function graph defining the cost relations between those packages. Note that here only one admissible supply function graph exists (see Fig. C.4 in Appendix C.3.3). The supply functions and the supply function graph are then augmented as follows. The auctioneer defines supply functions f_{A_1}, f_{A_2}, f_{A_1B}, f_{A_2B}, $f_{A_1A_2}$, $f_{A_1A_2B}$ such that $\mu^1_{A_1} = \mu^1_A$, $\mu^1_{A_2} = \mu^2_A$, $\mu^1_{A_1B} = \mu^1_{AB}$, $\mu^1_{A_2B} = \mu^2_{AB}$, and $\mu^1_{A_1A_2} = \mu^1_{AA}$. All other steps heights μ^q_S are set to ∞ and f_B remains the same (step lengths are normalised to one). The supply function graph is augmented as shown in Fig. C.5 in Appendix C.3.3. Note that $f_{A_1A_2B}$ may be omitted, unless the seller wants to define a packaging cost associated with AAB. However, in that case the bidders should be offered to submit a separate bid price for v^i_{AAB} as well.

The linear programming arguments leading to Proposition 19, Corollary 1, and Theorem 1 can then be applied as before. Note that the auction prices must satisfy $z_A = \min\{z_{A_1}, z_{A_2}\}$, $z_{AA} = z_{A_1A_2}$, and $z_{AB} = \min\{z_{A_1B}, z_{A_2B}\}$. In addition, if A_1 and A_2 are both allocated it must be that $z_A = z_{A_1} = z_{A_2}$, and similarly for z_{AB}. This follows from $v^i_{A_1} = v^i_{A_2} = v^i_A$ and the surplus constraints of DLP.

4.4 Distinct complements and packaging costs

Equilibrium existence in the presence of complementarities *and* indivisibilities is not well understood in the literature, but important contributions have been made in recent years. For "gross substitutes and complements" preferences, existence of a linear pricing Walrasian equilibrium was shown in Sun and Yang (2006), and generalised by Teytelboym (2014). Both of these classes of preferences are generalised by the concept of demand types introduced in Baldwin and Klemperer (2019*b*). Candogan et al. (2016) study pricing equilibria for economies where buyers have graphical valuations with pairwise complementarities or substitutabilities, and Candogan et al. (2015) establish existence of a linear pricing Walrasian equilibrium for an environment where one unit per variety is sold and bidders have sign-consistent tree valuations. In all those studies, the seller cannot express preferences over the partitioning of supply.

Most notably, Sun and Yang (2014) (SY) establish existence of a non-linear pricing Walrasian equilibrium when buyers and the seller have superadditive values (not allowing for preferences over the partitioning of supply either).[21] I extend their ascending procedure to establish sufficient conditions for the existence of an

[21] SY also propose an incentive-compatible mechanism for their setting. Results in a similar spirit are established in Fujishige and Yang (2020) for unimodular demand types.

integer-valued equilibrium in my setting, restricting supply to one unit per variety. If the seller's costs are expressed through incremental marginal cost functions and CFGs, an equilibrium can also be found via linear programming.

The environment is restricted now such that $\Omega_j = 1$ for all j, i.e. one unit per variety is for sale. A set of packages is now characterised by $\mathbf{k} \in \{0,1\}^{|2^N|-1}$. Each bidder's aggregate value function is identical to his marginal value function $v^l(1,S), S \in 2^N$. I omit the first argument and define bidder l's value function as $v^l : 2^N \to \mathbb{Z}_+$. Every buyer's value function is superadditive, and his utility is quasi-linear. The seller's marginal cost function is identical to her incremental marginal cost for the first unit, i.e. $c^0(1,S,\mathbf{k}_{-S}) = \Delta c(1,S)$ for all $\mathbf{k}_{-S}$. Again, I simplify and define the seller's marginal cost from selling a package $S \in 2^N$ as $c^0 : 2^N \to \mathbb{Z}_+$. The seller's aggregate cost is defined as before and simplifies to $C^0(\mathbf{k}) = \sum_{S\in 2^N} c^0(S)k_S$. The seller's marginal cost function is subadditive and her utility is quasi-linear.

For comparability with SY, I use a different notation for sets of packages: instead of a vector $\mathbf{k}$, I view a set of packages as a partition δ of N into packages, i.e. $\delta = \{A_1, \ldots, A_k\}$ where $A_t \subseteq N, t = 1, \ldots, k$, $A_{t_1} \cap A_{t_2} = \emptyset$ for all $A_{t_1} \neq A_{t_2}$, and $\bigcup_{t=1}^k A_t \subseteq N$. As before, $\mathbb{K}$ denotes the universe of all partitions of N. Consequently, the seller's profit is given by $u^0(\delta, p) = \sum_{S\in\delta} (p(S) - c^0(S))$ when she sells the set of packages δ.

The definitions of demand and supply correspondences, and indirect utility and profits carry over from the previous section. Note that an allocation $\pi = (\pi(0), \pi(1), \ldots, \pi(L))$ can also be interpreted as a partition δ of N defined by $\delta = \{\pi(l) | \pi(l) \neq \emptyset \text{ and } l \in \mathcal{L}_0\}$, where $\pi(l)$ disappears if it is the empty set. The notion of package-linear pricing Walrasian equilibrium coincides with SY's notion of non-linear pricing Walrasian equilibrium.

4.4.1 Modelling the seller's preferences

The difference of this restricted environment to Sun and Yang (2014) are the seller's preferences. SY describe their seller as "revenue-maximising", while I model a profit-maximising seller. Examining how the models relate, I make two observations. First, note that in SY one needs to distinguish between the seller's utility and true reserve prices, whereas my reserve prices simply correspond to the seller's marginal cost function. Second, SY's seller preferences can be reformulated such that the seller acts as a profit-maximiser with a well-defined cost function.

In SY, the seller has a "reserve price" function $v^0 : 2^N \to \mathbb{Z}_+$. If she doesn't sell a package S, $v^0(S)$ is also the "utility" (or "revenue") she derives from keeping S. v^0 is assumed to be superadditive. Example 3 demonstrates the subtlety

in defining the seller's value functions and reserve price function when they are strictly super- or subadditive.

Example 3. I revisit the seller from Section 4.1.1. She possesses the bundle $\{AB\}$ which is worth 60 to her, and she values good A alone at 10, and good B alone at 20. Then, her reserve price for good A should be 40: at this price, she is indifferent between keeping the bundle and selling good A alone. Similarly, the reserve price for good B should be 50. In SY's basic ascending auction, by allowing the seller to choose her supply set *given the current prices*, these reserve prices are implicit. If (p_A, p_B, p_{AB}) are the auction prices, the seller chooses to sell good A individually only if $p_A + 20 \geq 60$ and $p_A + 20 \geq p_{AB}$, i.e., good A's true reserve price is $60 - 20 = 40$.

In my model, the seller's reserve price for a bundle corresponds to the marginal cost she incurs when supplying that bundle. If she is selling a set of packages, the cost of supplying the set is the sum of marginal costs of packages contained in the set. In contrast to SY, the cost of supplying a set of packages depends on the partition of supply into packages, where each package is allocated to a different bidder. I now describe the relationship between SY's seller and mine formally. Let S^c denote the complement for any set $S \in 2^N$, i.e. $S^c = N \setminus S$. First, I state the definition of the *dual* of a set function $f : 2^N \to \mathbb{R}$.[22]

Definition 22. For any bundle $S \in 2^N$, given $f : 2^N \to \mathbb{R}$ with $f(\emptyset) = 0$, define the transformation:

$$g(f, S) = f(N) - f(S^c)$$

$g(f, \cdot)$ is called the *dual* of f. Note that $g(f, N) = f(N)$.

Definition 23 (Set-cover submodularity)**.** Given a finite set N, and a function $f : 2^N \to \mathbb{R}$, f is *set-cover submodular* if $\forall\ S_1, S_2 \in 2^N$ with $S_1 \cup S_2 = N$

$$f(S_1) + f(S_2) \geq f(S_1 \cup S_2) + f(S_1 \cap S_2)$$

If the inequality sign in the above equation is reversed, f is *set-cover supermodular*, and if the equation holds with equality f is *set-cover modular*.

Note that set-cover submodularity is weaker than submodularity, because it is only required for every two subsets of N the union of which fully covers N. In particular, set-cover submodularity does not imply subadditivity. However, the following lemma exhibits a connection between superadditive and set-cover submodular functions.[23]

[22]This is standard in matroid theory, see e.g. Gul and Stacchetti (2000) or Fujishige (2005).

[23]Set-cover submodular and set-cover supermodular functions can both be subadditive or superadditive.

Lemma 9. *Given a superadditive (subadditive) function $v^0 : 2^N \to \mathbb{R}$, its dual $c^*(v^0, \cdot)$ is set-cover submodular (set-cover supermodular).*

The following proposition relates SY's and my model.

Proposition 22. *Let the seller's preferences be represented by a reserve price function $v^0 : 2^N \to \mathbb{Z}_+$, $v^0(\emptyset) = 0$. The seller's objective maximising revenue as defined in SY is equivalent to maximising quasi-linear profits with cost function $C^0 : \mathbb{K} \to \mathbb{Z}_+$, where*

(i) marginal costs $c^0 : 2^N \to \mathbb{Z}_+$ are such that c^0 is the dual of v^0

(ii) $C^0(\delta) = c^0\left(\bigcup_{S\in\delta} S\right)$

From (i) and Lemma 9 it follows that c^0 is set-cover submodular. A set-cover submodular cost function can have strictly subadditive and strictly superadditive elements, as shown in Example 7 in the appendix. Property (ii) in Proposition 22 states that the seller in SY only cares about which objects are contained in the set of objects she sells overall, in contrast to my seller whose preferences take into account how objects are partitioned among buyers. Thus, even with identical marginal cost functions, their and my ascending auction may generate different results.

4.4.2 The ascending auction and equilibrium existence

I use a modified version of the basic ascending procedure by SY to determine an indivisible, package-linear Walrasian equilibrium. The rules of my ascending procedure are identical to SY on the bidders' side, but differ for the seller. Bidders bid straightforwardly in this auction: in each round, every bidder bids as if the current round was the last round of the auction, i.e. he maximises his profits given the current auction prices. Let $p(t, S)$ denote the price of bundle $S \in 2^N$ at time t. Bidder l bids straightforwardly with respect to his valuation function v^l if at every time $t \in \mathbb{Z}_+$ for any price vector (pricing function) $p(t)$, he bids $A_l(t) \in D^l(p(t)) = \arg\max_{S\subseteq N}\left\{v^l(S) - p(t, S)\right\}$. I require $A_l(t) = \emptyset$ when $\emptyset \in D^l(p(t))$.

In the ascending procedure the seller states her supply and bidders state their demand at current auction prices. If a package is overdemanded, the price on this package is increased by one in the next round. The procedure stops as soon as no package is overdemanded. Formally, the procedure is described as follows:

Step 1: The seller states her initial reserve prices c^0, and the auctioneer fixes the initial price vector $p(0) : 2^N \to \mathbb{Z}$ such that $p(0, S) = c^0(S)$ for every $S \in 2^N$. Set $t = 0$ and continue with step 2.

Step 2: In each round of the auction process $t = 0, 1, \ldots$, the auctioneer announces the current price vector $p(t)$ and chooses a supply set $\delta \in S(p(t))$. Subsequently, every bidder l reports his demand $A_l(t)$ given the current price vector $p(t)$. Then, the auctioneer determines if any of the packages are overdemanded. Package S is overdemanded with respect to supply set δ and reported demand bundles $A_l, l \in \mathcal{L}$, if it is demanded by more than one bidder, or demanded by one bidder but not in the supply set δ. If no package is overdemanded, continue with step 3. If some package is overdemanded, increase the price of every package S that is overdemanded with respect to δ and $A_l, l \in \mathcal{L}$ by 1, i.e. $p(t+1, S) = p(t, S) + 1$. Every other price $p(t, S'), S' \neq S$ remains the same. Increase the counter $t = t + 1$ and continue with step 2.
Step 3: Every package $S \in A_l(t)$ is allocated to bidder l at price $p(t, S)$. If there is a package $B \in \delta(t)$ that is not demanded by any bidder at $p(t)$, B remains with the seller if $p(t, B) = c^0(B)$. If $p(t, B) > c^0(B)$, B is allocated for price $p(t, B)$ to a bidder who reported B in his demand set in a previous round was among the last to forfeit B. The auction terminates.

Theorem 2. If all bidders bid straightforwardly in this auction setting, the ascending auction terminates in a package-linear Walrasian equilibrium after a finite number of rounds.

Equilibrium existence follows for the model from section 4.2 when restricting supply to one unit per variety.

Corollary 4. When $\Omega_j = 1, j \in N$, there always exists a package-linear Walrasian equilibrium.

Example 8 in the appendix further illustrates that my ascending auction and SY's can lead to different auction outcomes, even with identical marginal cost values for each package. I also demonstrate that my ascending auction generalises SY's procedure, because their seller is similar to the buyers: she does not care about the partitioning of a set of varieties in her supply, as long the reserve price is met. Thus, their seller can "disguise" as a buyer.

4.4.3 The ascending buy-back auction

Suppose the seller from SY's model participates in my auction as a bidder. My seller's own supply is set to zero, but the SY-seller gives her supply to my seller, and then participates in the ascending procedure attempting to buy back her objects. If she wins a bundle, this bundle is effectively not sold, and she receives the

price of every bundle sold to a different bidder. I call this the "buy-back" auction to distinguish it from the standard SY-ascending auction. The seller attempting to buy back her objects is called the SY-seller. I denote objects relating to the SY-procedure with a superscript SY and show the following:

Lemma 10. *SY's auction and the buy-back auction terminate in the same allocation (up to ties).*

Proposition 23. *If the buy-back auction starts at prices* $p(-1, S) = v^0(S) - 1 \; \forall S \in 2^N$*, there exists a price path that is identical in the buy-back auction and the SY ascending auction, and the final auction prices at the end of this price path support the same allocation in both procedures.*

Hence, the buy-back auction strictly generalises SY by adding my seller with subadditive marginal costs over the partitioning of supply.

4.5 Conclusion

In many combinatorial auctions and markets, it is natural for the seller to have preferences over the partitioning of objects among buyers. I establish a new graph structure to express such preferences, while maintaining the tractability of the allocation problem through a linear programming formulation. To the best of my knowledge, mine is the first Walrasian-equilibrium auction setting in which the seller can express preferences over allocations.

I characterise the existence of an indivisible Walrasian equilibrium with package-linear pricing and provide necessary and sufficient conditions for the existence of such equilibrium. Moreover, I provide sufficient conditions for equilibrium existence relying only on the primitives of the model. I also uncover a relationship between valuations and costs defined on sets of indivisible goods. In order to characterise properties of these value and cost functions, I derive a new relationship between superadditive (subadditive) set functions and their duals.

While my results are theoretical, I aim to inspire practical auction design. My methods allow the seller to express vastly richer preferences than described in the previous literature and used in present-day auction design. Equilibrium prices have a modular structure: the price of a given package is tied to the marginal costs of varieties or subsets of varieties contained in the given package, according to the cost relations specified by the seller. This guarantees transparency in pricing between different packages that contain identical objects and addresses another important concern in combinatorial auctions with non-linear pricing: if not all bundles receive

bids, sensible bundle prices can be easily constructed. In practice, allowing sellers to express richer preferences will likely improve efficiency and a more accurate representation of marginal costs may be helpful to generate higher revenue.

Finally, the graph structure I develop in this chapter may be of independent interest in other allocation problems, where the distribution of goods, services, or matchings in the market matters. I hope that the ideas and tools developed in this chapter offer a new perspective on preferences over allocations and how they can be processed, with the goal of developing better solutions for a range of market design problems.

5

Divisible goods markets with budgets: unifying revenue and welfare

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5.1 Introduction

A central concern in markets for divisible goods is finding procedures for computing competitive equilibrium (Arrow and Debreu, 1954; Devanur et al., 2008; Eaves, 1976; Nisan et al., 2007; Orlin, 2010). While much of the literature has studied settings in which buyers have no financial limitations, some work has begun to explore the implications of imposing buyer budgets on the existence and

computation of equilibrium, mainly with quasilinear preferences (Chen et al., 2007; Conitzer et al., 2022, 2021; Dobzinski et al., 2012; Murray et al., 2020). Competitive equilibrium (CE) is then a standard, desirable market objective, as the fundamental theorems of welfare economics imply that a CE allocation also maximises social welfare. In many practical settings (e.g. the arctic auction (Klemperer, 2018)), however, the alternative objective of envy-free revenue maximisation, together with the option for buyers to express budget constraints, is often at least as attractive. In an envy-free allocation that maximises revenue, buyers receive bundles they demand at market prices, but the seller may prefer to allocate only a subset of her supply (Guruswami et al., 2005; Klemperer, 2008). Solutions for the two objectives of maximising revenue or welfare do not generally coincide; in fact, in many common economic settings the two objectives diverge significantly in the market outcome.[1]

This chapter is joint work with Paul Goldberg and Edwin Lock (Finster et al., 2022). It identifies an important market setting that unifies the two objectives of envy-free revenue maximisation and welfare maximisation. Our market contains multiple buyers and one seller. The seller supplies multiple goods in finite, divisible quantities. Buyers have quasi-linear utilities and budget constraints; that is, every buyer can set an upper limit on the amount they wish to spend. Each buyer has an additive valuation, i.e. a fixed per-unit value for each good.[2] While setups like this are important in practice (we elaborate on ad auctions below), they have received comparatively little attention in theoretical work. Our main contribution is to demonstrate that the two objectives of social welfare and envy-free revenue coincide in quasilinear budget-constrained markets: revenue is maximised at competitive equilibrium prices. To prove this, we introduce a novel geometric object, the *feasible region*, which is defined as the set of prices at which, for every good, either the market clears or there is excess supply. This region has a smallest element which, as we show in this chapter, is the revenue- and welfare-maximising price.

In applications, buyers with additive valuations are of particular interest. For example, Klemperer (2008) introduced the 'arctic auction', originally developed for the government of Iceland, in which buyers have budget-constrained preferences for multiple divisible goods. The market we consider can be interpreted as a special case of this auction. The government planned to use the arctic auction

[1]This is the case in divisible goods markets with concave preferences (see Section 5.1.2). This is also the case in standard indivisible goods markets, for instance, in the strong-substitutes product-mix auction, where competitive equilibrium can be computed efficiently Baldwin and Klemperer (2019*b*) but solving for revenue is APX-hard Guruswami et al. (2005).

[2]Note that budget constraints are especially natural for additive valuations on unbounded domains, as they are necessary to ensure finite quasi-linear demand. Infinite demand would make envy-free allocation impossible.

to exchange blocked accounts for other financial assets, e.g. cash or bonds of different quality, available with limited supply.[3] The auction was tailored to this use because buyers could submit a budget, as well as trade-offs between different assets through bid prices. The objective of Klemperer's auction was to maximise revenue subject to envy-freeness, which is attractive for the seller (government) as well as bidders. We show in Section 5.2.1 that each buyer in the arctic auction has preferences corresponding to the aggregation of multiple budget-constrained quasilinear buyers with additive valuations.

Our market is also a suitable model of many ad auctions as they occur in practice. An identical setting, but with an ex-ante different notion of equilibrium, has recently been addressed by Conitzer et al. (2022). When advertising companies compete for web space (goods) to display their ads, the decision of which publisher to choose and bid for is difficult. It may be intuitive, however, to choose an advertising budget and state demand in terms of 'limit market prices' for multiple, distinct goods, below which the seller, or the market platform, allocates those goods with the highest value for money. This is not only conceptually easier for advertisers, but also practical, feasible, and desirable from the perspective of social welfare *and* revenue, as demonstrated by our results for the budget-constrained quasilinear setting.

The market we study has recently been labelled a 'quasi-Fisher' market (Murray et al. (2020)), as it can be considered a generalisation of Fisher markets in which buyers have quasi-linear utilities. In standard Fisher markets, buyers spend their entire budget at any market prices, and so revenue is constant at all prices. In contrast, buyers with quasi-linear utility and budget constraints spend nothing when prices are unacceptably high. Hence, the notion of maximising revenue becomes a viable objective for the seller to pursue. Quasi-Fisher markets have been studied from a (robust) competitive equilibrium perspective (Chen et al., 2007; Murray et al., 2020) as well as from the perspective of Nash social welfare (Cole et al., 2017). While competitive equilibrium and Nash social welfare are desirable in terms of stability, designing an auction to maximise revenue while respecting envy-freeness may better reflect the primary interests of the seller without affecting the stability of the allocation.

Our contributions. In this chapter, we show that when buyers have budget-constrained utilities with additive valuations, the two objectives of maximising revenue and finding a competitive equilibrium coincide. The unique market-clearing

[3]Other related applications may include debt restructuring and the (re-)division of firms between shareholders Klemperer (2018).

prices are buyer-optimal among all revenue-maximising prices in the sense that they maximise the quantities allocated to each buyer, and thus maximise buyer utilities. Building on the work of Chen et al. (2007), our results imply that efficient inner-point methods can be used to find revenue-maximising prices and allocation. Our proof proceeds geometrically and may thus convey insights of independent interest. First we show that the set of 'feasible' prices (in the sense that aggregate demand does not exceed supply) contains an elementwise-minimal price vector. This is not immediately clear, as the set of feasible prices is not convex. We then complete the argument by showing that this elementwise-minimal price vector maximises revenue *and* clears the market. In order to demonstrate the latter, we adapt a price-scaling procedure from Adsul et al. (2012)'s simplex-like algorithm for linear Fisher markets to our market setting. Given non-minimal feasible prices, our procedure scales down the prices of a subset of goods while maintaining feasibility.[4] Just as Adsul et al. (2012) derive a simplex-like algorithm for solving linear algorithms by repeatedly calling their price-scaling procedure, we see that applying our procedure finitely many times leads to an algorithm for finding the elementwise-minimal prices that may be more efficient and easily implementable in practice than existing interior point methods.

Organisation. The remainder of this section discusses related literature (Section 5.1.1) and highlights the necessity of additive valuations for our result (Section 5.1.2). In Section 5.2, we define our model. Our main theorem on the coincidence of revenue and welfare is stated and proved in Section 5.3. The price-reduction procedure is described in Section 5.4. In Section 5.5, we state our theorem on finding market clearing prices, and Section 5.6 concludes.

5.1.1 Related literature

Computing equilibrium prices in multi-good markets has raised great interest among economists and computer scientists alike, most notably for the divisible goods market due to Léon Walras Walras (1874), for which Arrow and Debreu in their famous result Arrow and Debreu (1954) proved the existence of a general market equilibrium. Since then, much algorithmic machinery has been developed for the Arrow-Debreu market (e.g. Eaves (1976); Jain (2007)) and its important

[4]The procedure of Adsul et al. (2012) scales prices up instead of down. A second difference is that buyers in our setting can choose not to spend (a part of) their budget; recall that buyers in Fisher markets always spend their entire budget. This requires additional pre-processing for buyers that may be indifferent between spending all of the budget or only a fraction thereof at given prices.

special case, the Fisher market Brainard and Scarf (2005). Fisher markets are a simple, economic setting with well-known tractable algorithms for the computation of competitive market equilibria. The famous Eisenberg-Gale convex program (Eisenberg and Gale, 1959), originally introduced for linear utilities, has later been studied in the context of various types of preferences such as particular spending constraints (Birnbaum et al., 2011; Devanur and Vazirani, 2004; Garg et al., 2013) or additional transaction costs (Chakraborty et al., 2010).

Our model of budget-constrained buyers with quasilinear utilities and additive valuations can be understood as a 'quasi-Fisher' market Murray et al. (2020). In the linear Fisher setting, money has no intrinsic value: buyers do not gain any utility from any leftover budget. In contrast to the Arrow-Debreu market, prices are neither scalable in the linear nor in the quasilinear Fisher market setting. Moreover, contrasting with the linear Fisher market, in the budget-constrained buyer market with quasilinear utilities, demand is not invariant to scaling prices, in the sense that, for every buyer, there exists a limit price for each good beyond which the buyer does not demand the good.

While linear Fisher markets have received significant attention in the literature (e.g. Adsul et al. (2012); Devanur et al. (2008); Orlin (2010)), their counterpart with quasilinear utilities has appeared in various guises. The first results on quasilinear Fisher markets were developed by Chen et al. (2007). They show that competitive equilibria can be computed in polynomial time, using interior point algorithms for convex programmes. Cole et al. (2017) consider the problem of maximising Nash social welfare in, mainly in linear Fisher markets spending restricted utility models, but also with extensions to Leontief, CES utility, and quasilinear utility.

In contrast with this literature, this chapter considers not only competitive equilibrium and social welfare as solution concept and market objective, but also maximising revenue subject to envy-freeness. Our unifying result demonstrates the importance of the quasilinear setting, from a theoretical perspective as well as in applications. The practical relevance is highlighted also by a recent paper by Conitzer et al. (2022). Their setting is particularly inspired by online ad auctions, and the basic properties of the market, buyers, goods, and budget-constrained preferences with additive values are identical to ours. However, unlike in our setting, each divisible good is sold in an independent, single-unit first-price auction, where only the highest bidders can win a positive quantity. Conitzer et al. (2022) introduce the solution concept of *first price pacing equilibria* (FPPE), in which the bids submitted correspond to the buyers' values scaled (uniformly for each buyer) by a multiplicative

factor, the pacing multiplier.[5] Interestingly, this at first sight unrelated auction procedure can also be solved using the modified Eisenberg-Gale convex programme by Chen et al. (2007). Moreover, Conitzer et al. (2022) show that the unique FPPE corresponds to a competitive equilibrium in the sense of our setting, that is, in the overarching market for all goods with budget-constrained, quasilinear buyers. While the authors show that the FPPE is revenue-maximal among all budget-feasible pacing multipliers and corresponding allocations, our work implies that the FPPE is indeed revenue-maximising subject to envy-freeness in the entire market.

5.1.2 Introductory examples

The purpose of the first example is to demonstrate the difficulties one encounters with diminishing marginal values and motivates our interest in additive valuations. We also introduce some of the geometric properties of our framework in a two-good example with additive valuation, which are natural to work with for our objective.

Example 4. A seller (she) can provide a single good available with supply $s \in \mathbb{R}_+$ at zero marginal costs. There is one buyer (he) with a continuous valuation $v : \mathbb{R} \to \mathbb{R}$ for the good. The buyer has quasi-linear utility $u(p, x) = v(x) - px$, where p denotes the unit price of the good, and a budget $\beta \in \mathbb{R}_+ \cup \{\infty\}$. We aim to find an allocation $x \leq s$ and a price p that maximise social welfare (SW) or the seller's revenue (R). The buyer demands quantity x at price p if $x \in \arg\max_x v(x) - px$ such that $px \leq \beta$.

The revenue maximisation problem is constrained in the sense that the seller cannot enforce a specific allocation to the buyer. She can only set a price, anticipating the buyer's demand. Thus, the seller's revenue is maximised at $(x, p) \in \arg\max_{x,p} px$ such that $x \in \arg\max_x v(x) - px$ and $x \leq s$. Social welfare, on the other hand, is simply maximised at $(x, p) \in \arg\max_{x,p} v(x) - px$ such that $x \leq s$.

To derive optimal (for revenue or social welfare) allocations and prices, we first need to make some assumptions on the buyer's valuation. In the economics literature, diminishing marginal values are a typical assumption on preferences. In that case, the buyer always demands a quantity such that his marginal utility at this quantity is zero. Social welfare maximisation then implies $v'(x) = p$ and $x = s$; therefore, social-welfare maximising prices are always 'market-clearing'.[6] However, if

[5] An FPPE is defined as a set of pacing multipliers (one for each buyer) and allocations that satisfy the allocation and pricing rule of standard first-price auctions, as well as budget feasibility, supply feasibility, market clearing for demanded goods, and 'no unnecessary pacing', i.e. a buyer's multiplier equals one if she has unspent budget.

[6] The welfare theorems hold even in the general version of our model with multiple goods.

the seller were allowed to adjust the price subject to respecting the buyer's demand, she might be able to not sell the entire supply and extract more revenue. The following proposition shows that this is indeed the case for all strongly concave value functions if the buyer's budget is large.

Proposition 24. *Let v be differentiable and strongly concave with parameter m for some $m > 0$, and let $\beta = \infty$. Then there exists some supply $s \in \mathbb{R}$ so that revenue is not maximised at the market-clearing price.*

When the buyer has a finite budget, the situation is more intricate. Indeed, supply can only lie in the finite interval $X := [0, \max\{x \mid xv'(x) \leq \beta\}]$. We define $\tilde{x}$ implicitly by $\tilde{x}v'(\tilde{x}) = \beta$ and we call any bundle $x \in X$ *budget-feasible*.

Proposition 25. *Suppose the buyer has a strongly concave valuation v with parameter m and a finite budget β. If supply s is contained in X with $v'(s) < ms$, or if $m > \frac{v'(\tilde{x})}{\tilde{x}}$, then revenue is not maximised at the market-clearing price.*

This proposition tells us that for any strongly concave valuation, we can find a combination of budget and supply such that the maximisers of the social welfare and the revenue maximisation problem do not coincide. For example, we may require the valuation to be sufficiently concave relative to the budget. Therefore, we consider the class of constant marginal values, i.e. $v(x) = vx$.

Proposition 26. *Let the buyer's valuation be $v(x) = vx$ with budget β. Then the seller's revenue is maximised at market-clearing prices.*

The above proposition is straightforward. Social welfare is maximised at $p = \min\{v, \frac{\beta}{s}\}$ and $x = s$. Because the buyer spends his entire budget, the seller cannot extract more revenue, and the welfare maximising allocation and price are also revenue-optimal.

An immediate question to ask is whether this simple logic extends to more general environments and preferences. The answer we present in this chapter is affirmative: if any number of buyers have quasi-linear, budget-constrained utility and additive values for any number of goods, then revenue and social welfare are maximised at a unique set of elementwise-minimal prices. The result, however, is not immediate. In the following, we illustrate the difficulty in another simple example with two goods.

Example 5. Two goods A and B are for sale with a supply of $s_A = 3$ and $s_B = 2$. There are three buyers $1, 2$, and 3 with the following marginal values: $\boldsymbol{v}^1 = (v_A^1, v_B^1) = (2, 3)$, $\boldsymbol{v}^2 = (v_A^2, v_B^2) = (2, 2)$, and $\boldsymbol{v}^3 = (v_A^3, v_B^3) = (4, 2)$. Utilities

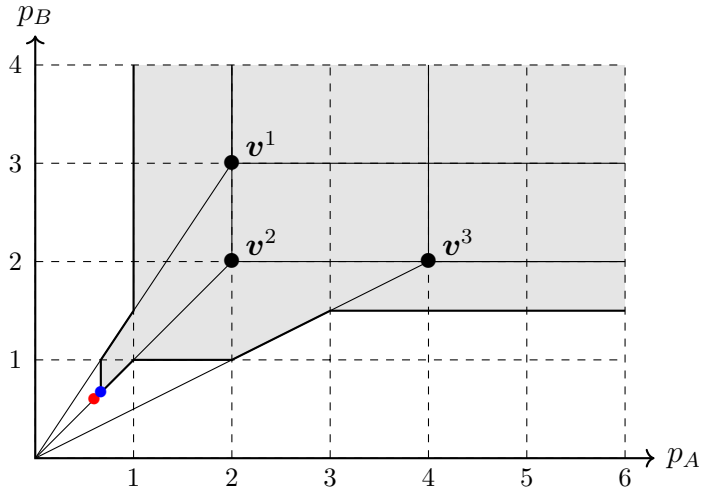


Figure 5.1: The feasible region in p_A-p_B-space

are quasi-linear, i.e. $u^i(x,p) = \sum_j (v^i_j - p_j)x_j$ for buyer i. Each buyer has a budget of $\beta^1 = \beta^2 = \beta^3 = 1$. We can allocate the 3 units of A and 2 units of B among the three buyers to maximise either revenue or social welfare, but need to respect individual demand. It is not hard to check that at given prices (p_A, p_B) each buyer i will demand a good $j \in \arg\max_{j=1,2} \frac{v^i_j}{p_j}$, if $v^i_j \geq p_j$. Any good k with $v^i_k < p_k$ will never be demanded by buyer i. This kind of individual demand can be easily represented in price space (more detail in Section 5.2.1). At some prices, aggregate demand is too large to be satisfied by supply. Prices at which aggregate demand does *not* exceed supply are called *supply-feasible*. The set of supply-feasible prices makes up the *feasible region*. The bids (black dots) and the feasible region (in grey) are illustrated in Fig. 5.1. Note that the feasible region also includes a short line segment between $p^* := (\frac{3}{5}, \frac{3}{5})$ (red dot) and $p' := (\frac{2}{3}, \frac{2}{3})$ (blue dot).

At prices $(\frac{3}{5}, \frac{3}{5})$, buyer 1 demands $\frac{5}{3}$ of B, buyer 3 demands $\frac{5}{3}$ of A, and buyer 2 demands $x^2_A \in [0, \frac{5}{3}]$ copies of A and $\frac{5}{3} - x^2_A$ of B. With supply $(s_A, s_B) = (3, 2)$, set $x^2_A = \frac{4}{3}$ to clear the market. It is easy to check that indeed any prices on $[p^*, p']$ induce a feasible allocation. All prices $[p^*, p']$ are revenue-maximising. However, only p^* clears the market and constitutes CE prices.

5.1.3 Preliminaries

For any two n-dimensional vectors $\boldsymbol{v} \in \mathbb{R}^n$ and $\boldsymbol{w} \in \mathbb{R}^n$, we write $\boldsymbol{v} \leq \boldsymbol{w}$ when the inequality holds element-wise. For any $j \in [n]$, $\boldsymbol{e}^j$ denotes the vector whose j-th entry is 1 and other entries are 0. For convenience, we also define $\boldsymbol{e}^0 = \boldsymbol{0}$. The dot product of $\boldsymbol{v}$ and $\boldsymbol{w}$ is denoted $\boldsymbol{v} \cdot \boldsymbol{w}$. For any function $f : A \times B \to \mathbb{R}^n$, we use the implicit summation $f(A', b) = \sum_{a \in A'} f(a, b)$, $f(a, B') = \sum_{b \in B'} f(a, b)$, and $f(A', B') = \sum_{a \in A', b \in B'} f(a, b)$, for any $A' \subseteq A$ and $B' \subseteq B$.

5.2 Markets, preferences, and objectives

We consider the following market. There are n goods $[n] := \{1, \dots, n\}$, typically denoted by j and k. We will also work with a notional *null good* 0, and let $[n]_0 = \{0, \dots, n\}$. A *bundle* of goods, typically denoted by $\boldsymbol{x}$ or $\boldsymbol{y}$ in this chapter, is a vector in $\mathbb{R}^n$ whose j-th entry denotes the *quantity* of good j. The seller has a supply bundle $\boldsymbol{s} \in \mathbb{R}^n$ that they wish to sell, partially or completely, by setting uniform market prices $\boldsymbol{p} \in \mathbb{R}^n_+$; a price vector $\boldsymbol{p} \in \mathbb{R}^n$ has a price entry for each of the n goods. When working with the notional null good 0, we implicitly define $p_0 = 0$.

We have m buyers $[m] = \{1, \dots, m\}$, typically denoted by i, with budgets β^i who compete for the supply of the n distinct goods. Throughout this chapter, we assume that every buyer $i \in [m]$ has an *additive valuation* $v^i : \mathbb{R}^n \to \mathbb{R}$ and budget β^i. That is, each buyer i possesses a vector $\boldsymbol{v}^i \in \mathbb{R}^n_+$ so that the valuation is given by $v^i(\boldsymbol{x}) = \boldsymbol{v}^i \cdot \boldsymbol{x}$ for all bundles $\boldsymbol{x} \in \mathbb{R}^n$. Moreover, without loss of generality we can assume that, for every good $j \in [n]$, there exists at least one buyer with a positive valuation v_j for the good. (Otherwise we can simply remove the good from the auction.) Utilities are quasi-linear, i.e. the buyer derives utility $u^i(\boldsymbol{x}, \boldsymbol{p}) = v^i(\boldsymbol{x}) - \boldsymbol{p} \cdot \boldsymbol{x} = (\boldsymbol{v}^i - \boldsymbol{p}) \cdot \boldsymbol{x}$ from receiving bundle $\boldsymbol{x} \in \mathbb{R}^n$ at prices $\boldsymbol{p} \in \mathbb{R}^n$. At any given prices $\boldsymbol{p}$, a buyer i demands the bundles $\boldsymbol{x}$ that maximise their utility $u^i(\cdot, \boldsymbol{p})$, subject to not exceeding their budget (expressed by the budget constraint $\boldsymbol{x} \cdot \boldsymbol{p} \leq \beta^i$). This leads to the *demand correspondence* $D^i(\boldsymbol{p}) = \max_{\boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{x} \cdot \boldsymbol{p} \leq \beta^i} (v^i(\boldsymbol{x}) - \boldsymbol{p} \cdot \boldsymbol{x}) = \max_{\boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{x} \cdot \boldsymbol{p} \leq \beta^i} (\boldsymbol{v}^i - \boldsymbol{p}) \cdot \boldsymbol{x}$.

As with prices, we define $v^i_0 = 1$ for the value of the null good. Moreover, for any buyer i and prices $\boldsymbol{p}$, we define the *demanded good set* $J^i(\boldsymbol{p}) = \arg\max_{j \in [n]_0} \frac{v^i_j}{p_j}$. This set of goods is also referred to in the literature (on Fisher markets) as maximising *bang-per-buck*. (Note that $J^i(\boldsymbol{p})$ contains the null good if $\max_{j \in [n]_0} \frac{v^i_j}{p_j} = 1$, as we have $v^i_0 = 1$ and $p_0 = 1$ by definition.) This allows us to provide an alternative characterisation of the demand correspondence that we rely on when proving our results below. In particular, Lemma 11 makes the initial observation that bundles only contain quantities of goods that maximise bang-per-buck; in other words, if $\boldsymbol{x}$ is a bundle demanded by buyer i at $\boldsymbol{p}$, then $x_j > 0$ implies $j \in J^i(\boldsymbol{p})$. Moreover, any demanded bundle is the convex combination of the 'extremal' bundles that arise when the entire budget is spent on a single demanded good in $J^i(\boldsymbol{p})$. In Section 5.2.1, we will see that the arctic product-mix auction introduces a bidding language which starts from this definition of demand to characterise a more general demand type.

Lemma 11. *Fix $\boldsymbol{p} \in \mathbb{R}^n_+$. Let $\boldsymbol{x} \in D(\boldsymbol{p})$ for some buyer with additive valuation $\boldsymbol{v}$, budget β and demanded good set $J(\boldsymbol{p})$. If $x_j > 0$, then $j \in J(\boldsymbol{p})$. Moreover $D(\boldsymbol{p}) = \operatorname{conv}\left\{\frac{\beta}{p_j}\boldsymbol{e}^j \mid j \in J(\boldsymbol{p})\right\}$.*

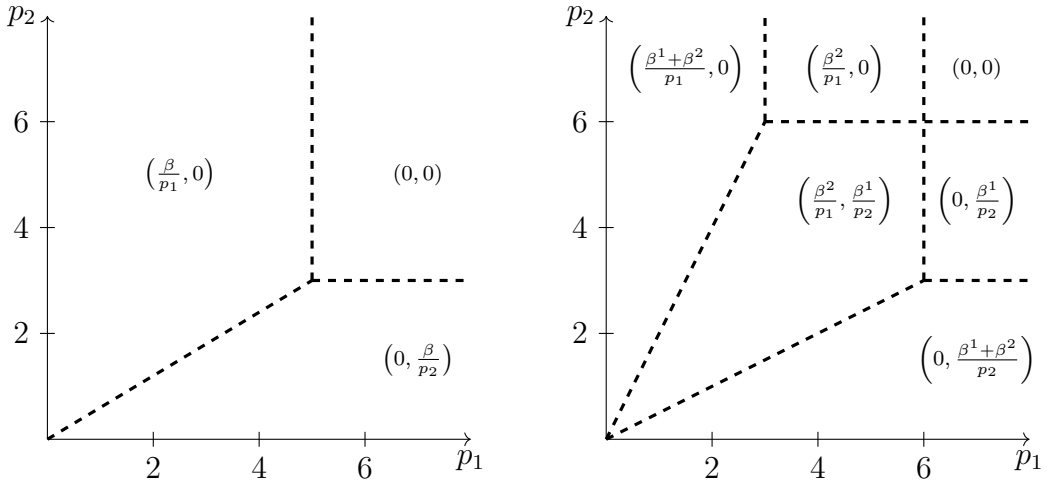


Figure 5.2: Illustrations of (aggregate) demand correspondences for buyers with quasi-linear utilities, additive valuations and budget constraints. Left: Demand for a single buyer with additive valuation $\boldsymbol{v} = (5, 3)$ and budget constraint β divides price-space into three regions. At prices that are element-wise larger than $\boldsymbol{v}$, the buyer demands nothing. At prices $\boldsymbol{p}$ that lie above the line connecting point $\boldsymbol{v}$ to the origin, the buyer demands a quantity β/p_1 of good 1. Conversely, at prices below the line, the buyer demands β/p_2 of good 2. Right: The aggregate demand correspondence of two buyers: the first has valuation $\boldsymbol{v}^1 = (3, 6)$ and budget β^1, and the second has valuation $\boldsymbol{v}^2 = (6, 3)$ and budget β^2.

Proof. Suppose, for contradiction, we have a non-null good j with $x_j > 0$ and $j \notin J(\boldsymbol{p})$. Then there exists another good $k \in J(\boldsymbol{p})$ with $\frac{v_j}{p_j} < \frac{v_k}{p_k}$. If k is the null good, then we define a new bundle $\boldsymbol{x}'$ from $\boldsymbol{x}$ by not spending on good j, so $\boldsymbol{x}' = \boldsymbol{x} - x_j \boldsymbol{e}^j$. It is immediate that $\boldsymbol{x}'$ is budget-feasible in the sense that it does not exceed the buyer's budget β at $\boldsymbol{p}$, as $\boldsymbol{x}' \cdot \boldsymbol{p} < \boldsymbol{x} \cdot \boldsymbol{p} \leq \beta$. Moreover, its utility is $(\boldsymbol{v} - \boldsymbol{p}) \cdot \boldsymbol{x}' = (\boldsymbol{v} - \boldsymbol{p}) \cdot \boldsymbol{x} + x_j(p_j - v_j) > (\boldsymbol{v} - \boldsymbol{p}) \cdot \boldsymbol{x}$. The last inequality follows from our assumption that $\frac{v_j}{p_j} < \frac{v_0}{p_0} = 1$, and it contradicts our assumption that $\boldsymbol{x}$ maximises utility. Now suppose k is not the null good. Then we let $\boldsymbol{x}'$ be the bundle obtained from $\boldsymbol{x}$ by spending nothing on good j and instead using the same amount of money to buy (additional) quantities of good k, so $\boldsymbol{x}' \coloneqq \boldsymbol{x} + x_j \left(\frac{p_j}{p_k} \boldsymbol{e}^k - \boldsymbol{e}^j\right) \geq 0$. Then $\boldsymbol{x}'$ does not exceed the buyer's budget, as we can verify $\boldsymbol{p} \cdot \boldsymbol{x}' = \boldsymbol{p} \cdot \boldsymbol{x} \leq \beta$. Moreover, its utility is $(\boldsymbol{v} - \boldsymbol{p}) \cdot \boldsymbol{x}' = (\boldsymbol{v} - \boldsymbol{p}) \cdot \boldsymbol{x} + x_j \left(\frac{p_j}{p_k} v_k - v_j\right) = (\boldsymbol{v} - \boldsymbol{p}) \cdot \boldsymbol{x} + x_j p_j \left(\frac{v_k}{p_k} - \frac{v_j}{p_j}\right) > (\boldsymbol{v} - \boldsymbol{p}) \cdot \boldsymbol{x}$.

Next we consider the second claim $D(\boldsymbol{p}) = \operatorname{conv}\left\{\frac{\beta}{p_j} \boldsymbol{e}^j \mid j \in J(\boldsymbol{p})\right\}$. Fix $\boldsymbol{x} \in D(\boldsymbol{p})$, and let β_j denote, for each good $j \in [n]_0$, the amount that the buyer spends on j to obtain x_j. If $0 \in J(\boldsymbol{p})$, then we let $\beta_0 = \beta - \beta_{[n]}$ be the remainder of the budget that the buyer doesn't spend. Hence we have $\beta_{J(\boldsymbol{p})} = \beta$ and $\beta_j = 0$ for $j \notin J(\boldsymbol{p})$. Moreover, $x_j = \frac{\beta_j}{p_j}$ for all $j \in [n]$, and thus $\boldsymbol{x} = \sum_{j \in J(\boldsymbol{p})} \frac{\beta_j}{\beta} \frac{\beta}{p_j} \boldsymbol{e}^j$, so $\boldsymbol{x}$ lies in

the convex hull. Now we show that all bundles in the convex hull lie in $D(\boldsymbol{p})$. Note that reallocating spending β_j on good j to another good $k \in J(\boldsymbol{p})$ leads to another bundle, the bundle $\boldsymbol{x}'$ as defined above, that does not exceed the buyer's budget and maximises utility. The latter follows as $\frac{v_k}{p_k} = \frac{v_j}{p_j}$ for $j, k \in J(\boldsymbol{p})$. Repeatedly reallocating spending until the bundle only contains positive quantities of a single good $j \in J(\boldsymbol{p})$ shows that the vertices of the convex hull are in $D(\boldsymbol{p})$. Finally, note that all bundles in the convex hull are convex combinations of the vertices, and taking convex combinations retains both budget-feasibility and maximal utility. □

Expenditure and allocation. When solving the auction, the seller sets market prices and determines an allocation to each buyer. Note that the allocation and prices directly control how much each buyer spends on each good. We capture this expenditure formally, as follows. An expenditure function $e : [m] \times [n]_0 \to \mathbb{R}_+$ expresses, for every buyer $i \in [m]$ and every good $j \in [n]_0$, the amount $e(i, j)$ that buyer i spends on good j. For each expenditure, we can define the corresponding *allocation* $\pi : [m] \times [n]_0 \to \mathbb{R}_+$ denoting the quantity $\pi(i, j) = \frac{e(i,j)}{p_j}$ of good j that buyer i receives. Moreover, for notational convenience we can define vectors $e(i)$ and $\pi(i)$ for each buyer i as $e(i)_j = e(i, j)$ and $\pi(i)_j = e(i, j)$ for all goods $j \in [n]$. The vector $e(i)$ captures how much buyer i spends on each good, while $\pi(i)$ denotes the bundle of goods that i receives. In particular, implicit summation allows us to express aggregate spending and demand as $e([m])$ and $\pi([m])$.

Given an expenditure e, we say that good $j \in [n]$ is *sated* if $e([m], j) = p_j s_j$, and *unsated* if $e([m], j) < p_j s_j$. If the seller wishes not to exceed supply and to allocate each buyer a bundle they demand at the chosen market prices, she needs to satisfy three criteria that are expressed in Definition 24.

Definition 24 (Feasibility and validity)**.** Let $e : [m] \times [n]_0 \to \mathbb{R}_+$ be an expenditure. Then e is:

(i) *Supply-feasible* at prices $\boldsymbol{p}$ if supply weakly exceeds the aggregate allocation for all goods j, so $e([m], j)/p_j \leq s_j$ for all $j \in [n]$.

(ii) *Budget-feasible* at prices $\boldsymbol{p}$ if every buyer i spends weakly less than its budget on non-null goods. The remaining (non-negative) amount is always spent on the null good, so $e(i, [n]_0) = \beta^i$ for all buyers $i \in [m]$.

(iii) *Demand-valid* at prices $\boldsymbol{p}$ if each buyer only spends on goods that maximise her bang-per-buck, so $e(i, j) > 0$ implies $j \in J^i(\boldsymbol{p})$.

An expenditure is *feasible* if (i) and (ii) hold, and it is *valid* if (iii) also holds. We extend the same terminology to the allocations π corresponding to feasible or valid expenditures defined by $\pi(i,j) = \frac{e(i,j)}{p_j}$. Note that for any valid allocation π at prices $\boldsymbol{p}$, the bundle of goods that buyer i is allocated under π lies in $D^i(\boldsymbol{p})$.

Definition 25. Prices $\boldsymbol{p}$ are *feasible* if there exists an expenditure e valid at $\boldsymbol{p}$.

In Section 5.3.1, we show that the set of feasible prices forms a lower semi-lattice, and in particular has an *elementwise-minimal price vector* that is dominated by all other feasible prices.

Market objectives. We consider two separate objectives for the seller. She may set prices with the objective of maximising her revenue, or she may set prices to maximise social welfare. Intuitively (and we prove this below), at the social optimum, the seller's entire supply is allocated. To maximise revenue, the seller may prefer to raise prices and retain some of her supply.

1. **Envy-free revenue maximisation.** Given a valid allocation π at prices $\boldsymbol{p}$, the revenue for the seller is given by $\boldsymbol{p} \cdot \pi([m]) = e([m])$. Hence, we define the indirect revenue function as $R(\boldsymbol{p}) := \max_\pi \boldsymbol{p} \cdot \pi([m])$ subject to π being valid. A solution to the *envy-free revenue maximisation problem* consists of a price vector $\boldsymbol{p}$ and allocation π that maximises $R(\boldsymbol{p})$.

2. **Welfare maximisation.** Recall that the buyers are price takers. Thus, social welfare is defined with respect to the submitted values. A solution to the *social welfare maximisation problem* is a price vector and allocation $(\boldsymbol{p}, \pi)$ that maximises social welfare $\sum_{i \in [m]} \boldsymbol{v}^i \cdot \pi(i)$ subject to π being a valid allocation.

3. **Competitive equilibrium.** Given a list of buyers $[m]$ and a supply vector $\boldsymbol{s}$, prices $\boldsymbol{p}$ and allocation π (at $\boldsymbol{p}$) form a competitive equilibrium if, and only if, π is valid at $\boldsymbol{p}$ and $\pi([m], j) = s_j$ for all goods j with $p_j > 0$.

It is well-known that the welfare theorems hold: competitive equilibrium maximises welfare, and any efficient allocation can be supported by prices to constitute a competitive equilibrium. We provide direct proofs of this in the appendix. For this reason, we focus on the two objectives of envy-free revenue maximisation and competitive equilibrium.

5.2.1 Arctic auctions

The market we consider can be interpreted as an important special case of the arctic product-mix auction market first proposed by Paul Klemperer Klemperer (2018) for the government of Iceland.[7] In this market, the seller has a given supply $\boldsymbol{s} \in \mathbb{R}^n_+$ of multiple divisible goods and wishes to find a valid allocation of some subset of this supply among a finite set of buyers with the goal of maximising revenue. The buyers express their demand preferences by submitting a collection of 'arctic bids', each of which consisting of an n-dimensional vector $\boldsymbol{b} \in \mathbb{R}^n_+$ and a monetary budget $\beta(\boldsymbol{b})$. At a market price *above the stated bid price*, an arctic bid *rejects the good*. At market prices below the stated bid prices, the bid spends its budget on goods that yield the highest 'bang-per-buck', i.e. the highest ratio of value to price $\frac{b_j}{p_j}$. When multiple goods maximise 'bang-per-buck', the bid can arbitrarily divide its expenditure between these goods. Moreover, if the maximal bang-per-buck is 1, then the bid may choose not to spend part of its budget (which we interpret as spending on the null good). Hence, Lemma 11 implies that each arctic bid $\boldsymbol{b}$, interpreted in isolation, induces a quasi-linear demand with additive valuation $v(\boldsymbol{x}) = \boldsymbol{b} \cdot \boldsymbol{x}$ and budget $\beta(\boldsymbol{b})$ as defined in Section 5.2. Moreover, if we denote the demand correspondence of bid $\boldsymbol{b}$ by $D_{\boldsymbol{b}}$, then the demand correspondence of a collection $[m]$ of bids is defined by the Minkowski sum of demands $D_{\mathcal{B}}(\boldsymbol{p}) = \left\{\sum_{\boldsymbol{b}\in[m]} \boldsymbol{x}^{\boldsymbol{b}} \mid \boldsymbol{x}^{\boldsymbol{b}} \in D_{\boldsymbol{b}}(\boldsymbol{p})\right\}$. Equivalently, $D_{\mathcal{B}}$ can be understood as the aggregate demand of $|\mathcal{B}|$ buyers with quasi-linear demand, additive valuations $\boldsymbol{b} \in \mathcal{B}$ and budgets $\beta(\boldsymbol{b})$. By submitting multiple arctic bids, buyers can express richer preferences. Nevertheless, it is straightforward to see from the definition of $D_{\mathcal{B}}$ that for the purposes of solving the auction (for welfare or for revenue), the seller can treat each bid independently, and we can assume without loss of generality that each buyer submits a single bid. Hence our main result on the coincidence of welfare and envy-free revenue holds also for the special case of the arctic auction we describe.

5.3 Main result

Theorem 3. If buyers have quasi-linear utility and additive valuations, revenue and welfare are maximised at the unique elementwise-minimal feasible price vector $\boldsymbol{p}^*$.

[7] The arctic product-mix auction is a variant of the original product-mix auction developed for the Bank of England by Paul Klemperer Klemperer (2008, 2010). See also Baldwin et al. (2019). Klemperer also chose the name 'arctic' to contrast with the fact that the original (but not the arctic) product-mix auction has close connections to tropical geometry. In the general arctic product-mix auction, the seller can additionally choose cost functions to fine-tune its preferences, while in our model, we assume the seller's costs to be zero. See Fichtl (2020) for some discussion of the general case.

To prove this in our general setting, we proceed as follows:

1. A key property of the feasible region is that it forms a lower semi-lattice: that is, there exists an elementwise-minimal price vector that is dominated by all other feasible prices. We show this by first proving that, for any two feasible prices, the element-wise minimum of these prices also belongs to the region.

2. We prove that revenue is maximised at elementwise-minimal feasible prices. Note that revenue may also be maximised at other prices.

3. It is known that market-clearing prices exist and are unique in our market (Chen et al., 2007). We also provide a direct proof of this in the appendix.

4. We provide a procedure that, given prices and a valid allocation with unsated goods, reduces prices and finds a new valid allocation. Our procedure is an adaptation of Adsul et al. (2012)'s price-scaling procedure designed for Fisher markets. Our procedure implies that markets must clear at elementwise-minimal feasible prices.

5.3.1 Elementwise-minimal prices

Recall that prices $\boldsymbol{p}$ are feasible if there exists a valid allocation π at $\boldsymbol{p}$. We show that set of feasible prices form a lower semi-lattice. In particular, there exists a special price vector $\boldsymbol{p}^*$ that is element-wise smaller than all other feasible prices (so that $\boldsymbol{p}^* \leq \boldsymbol{p}$ for all feasible $\boldsymbol{p}$). For any two price vectors $\boldsymbol{p}$ and $\boldsymbol{p}'$, we let $\boldsymbol{p} \wedge \boldsymbol{p}'$ denote their *element-wise minimum* defined as $(\boldsymbol{p} \wedge \boldsymbol{p}')_i = \min\{p_i, p'_i\}$. The following lemma is central to our proof.

Lemma 12. *If $\boldsymbol{p}$ and $\boldsymbol{p}'$ are feasible, then so is their element-wise minimum $\boldsymbol{p} \wedge \boldsymbol{p}'$.*

Fix feasible prices $\boldsymbol{p}$ and $\boldsymbol{p}'$ with element-wise minimum $\boldsymbol{r} = \boldsymbol{p} \wedge \boldsymbol{p}'$, and let π and π' respectively denote valid allocations at $\boldsymbol{p}$ and $\boldsymbol{p}'$. In order to prove Lemma 12, we construct an allocation τ at $\boldsymbol{r}$ and show that it is valid. We first define the set of goods A in which $\boldsymbol{p}$ is strictly dominated by $\boldsymbol{p}'$, and its complement B, so $A = \{j \in [n] \mid p_j < p'_j\}$ and $B = \{j \in [n] \mid p_j \geq p'_j\}$. Then our allocation τ at $\boldsymbol{r}$ is given by

$$\tau(i, \cdot) = \begin{cases} \pi'(i, \cdot) & \text{if buyer } i \text{ demands some good } j \in B \text{ at } \boldsymbol{r}, \\ \pi(i, \cdot) & \text{otherwise.} \end{cases} \tag{5.1}$$

In order to prove the validity of τ, we first state a technical lemma that establishes the connection between a buyer's demand at $\boldsymbol{p}$, $\boldsymbol{p}'$ and $\boldsymbol{r}$.

Lemma 13. *Suppose buyer i demands good $j \in A$ at $\boldsymbol{r}$. Then they also demands j at $\boldsymbol{p}$ and $J^i(\boldsymbol{p}) \subseteq J^i(\boldsymbol{r})$. Similarly, suppose buyer i demands $j \in B$ at $\boldsymbol{r}$. Then they also demand j at $\boldsymbol{p}'$, and $J^i(\boldsymbol{p}') \subseteq J^i(\boldsymbol{r})$. Moreover, we have $J^i(\boldsymbol{p}') \subseteq B$.*

Proof. Fix a buyer i who demands good $j \in A$ at $\boldsymbol{r}$. As $p_j = r_j$, this implies $\frac{v_j^i}{p_j} = \frac{v_j^i}{r_j} \geq \frac{v_j^i}{r_j} \geq \frac{v_j^i}{p_j}$ for all goods $j \in [n]_0$. The first inequality holds due to the definition of demand, and the second inequality follows from $r_j \leq p_j, \forall j \in [n]_0$. Hence, the buyer demands good j at $\boldsymbol{p}$. For the second claim that $J^i(\boldsymbol{p}) \subseteq J^i(\boldsymbol{r})$, fix a good $k \in J^i(\boldsymbol{p})$. Then we have $\frac{v_k^i}{r_k} \geq \frac{v_k^i}{p_k} \geq \frac{v_j^i}{p_j} = \frac{v_j^i}{r_j} \geq \frac{v_l^i}{r_l}$ for all goods $l \in [n]_0$. The first inequality holds due to $r_k \leq p_k$, and the second and third inequalities follow from the fact that i demands k at $\boldsymbol{p}$ and j at $\boldsymbol{r}$. Hence, if the buyer demands good k at $\boldsymbol{p}$, then they demand k at $\boldsymbol{r}$.

Now suppose that the buyer demands $j \in B$ at $\boldsymbol{r}$. The proof of the first claim is identical to the case $j \in A$. We prove the last claim that $J^i(\boldsymbol{p}') \subseteq B$. Suppose, for contradiction, that i demands a good $k \in A$ at $\boldsymbol{p}'$, and good $j \in B$ at $\boldsymbol{r}$. This implies $\frac{v_k^i}{p'_k} < \frac{v_k^i}{p_k} = \frac{v_k^i}{r_k} \leq \frac{v_j^i}{r_j} = \frac{v_j^i}{p'_j}$, in contradiction to the fact that k is demanded at $\boldsymbol{p}'$. □

We can now prove Lemma 12.

Proof of Lemma 12. Let $\boldsymbol{p}$ and $\boldsymbol{p}'$ be two feasible prices and $\boldsymbol{r} = \boldsymbol{p} \wedge \boldsymbol{p}'$ denote their element-wise minimum. As above, π and π' are valid allocations at $\boldsymbol{p}$ and $\boldsymbol{p}'$, and τ is defined as in (5.1). First we see that τ is indeed an allocation, as $\tau(i, j) \geq 0$ for all buyers i and goods j. It remains to prove that τ satisfies the three criteria of Definition 24 that define validity. We can partition buyers into two sets: the set $\mathcal{B} \subseteq [m]$ of buyers that demand a good in B at $\boldsymbol{r}$, and the set $\mathcal{A} = [m] \setminus \mathcal{B}$ of buyers that do not. Note that, by Lemma 13, the buyers in $\mathcal{B}$ demand only goods in B at $\boldsymbol{p}'$ and, by definition, the buyers in $\mathcal{A}$ demand only goods in A at $\boldsymbol{p}$. It follows that, under π, each buyer $i \in \mathcal{A}$ is only allocated quantities of goods in A (so $\pi(\mathcal{A}, j) > 0$ only if $j \in A$), and, similarly, every buyer $i \in \mathcal{B}$ only receives quantities of goods in B under π' (so $\pi(\mathcal{B}, j) > 0$ only if $j \in B$).

First we show that τ is supply-feasible at $\boldsymbol{r}$. Recall that τ is supply-feasible if $\tau([m], j) \leq s_j$ for all $j \in [n]$. Fix some good $j \in A$. Then by definition of τ, we have $\tau([m], j) = \pi(\mathcal{A}, j) + \pi'(\mathcal{B}, j)$. Recalling that $\pi'(\mathcal{B}, j) = 0$ and making use of the validity of π at $\boldsymbol{p}$, we get $\tau([m], j) \leq \pi([m], j) \leq s_j$. This immediately implies that τ satisfies part (i) of Definition 24 at $\boldsymbol{r}$ for all goods $j \in A$. Analogously, we can show that τ satisfies (i) at $\boldsymbol{r}$ for all goods $j \in B$ by recalling that $\pi(\mathcal{A}, j) = 0$ for any good $j \in B$. As $A \cup B = [n]$, we have shown that τ is supply-feasible.

Next we argue that τ is budget-feasible at $\boldsymbol{r}$. Indeed, note that any buyer in $\mathcal{A}$ only demands goods in A. Moreover, the prices of goods in A are the same at $\boldsymbol{p}$ and $\boldsymbol{r}$, by construction of $\boldsymbol{r}$. It follows that the buyer spends the same under τ at $\boldsymbol{r}$ as they do under π at $\boldsymbol{p}$. As π is budget-feasible at $\boldsymbol{p}$, we see that τ satisfies part (ii) of Definition 24 for all buyers in $\mathcal{A}$. Similarly, as the buyers in $\mathcal{B}$ only demand goods in B, we apply the same argument to see that τ satisfies part (ii) for all buyers in $\mathcal{B}$. As $[m] = \mathcal{A} \cup \mathcal{B}$, τ is budget-feasible.

Finally, we show the demand-validity of τ at $\boldsymbol{r}$. Lemma 13 and the validity of π at $\boldsymbol{p}$ together imply that, for any $j \in [n]$, we have $\tau(i,j) = \pi(i,j) > 0$ only if $j \in J^i(\boldsymbol{p}) \subseteq J^i(\boldsymbol{r})$, which implies that τ satisfies part (iii) of Definition 24 at $\boldsymbol{r}$ for all buyers in $\mathcal{A}$. Analogously, as $\tau(i,j) = \pi'(i,j)$ for all buyers $i \in \mathcal{B}$, we see that τ satisfies constraint (iii) at $\boldsymbol{r}$ for all buyers in $\mathcal{B}$. Hence, τ is demand-valid at $\boldsymbol{r}$, and thus valid. □

Corollary 5. There exists an elementwise-minimal price vector $\boldsymbol{p}^*$.

Proof. Suppose there exists no such price vector. This means that for all feasible $\boldsymbol{p}$, there exists some feasible $\boldsymbol{q}$ with $q_j < p_j$ for at least one good $j \in [n]$. Fix some feasible prices $\boldsymbol{p}$ with the property that $\boldsymbol{p}$ cannot be reduced any further in any direction without breaking feasibility. Such a point must exist, as the feasible region is closed and restricted to $\mathbb{R}^n_+$. By assumption, there exists a feasible price vector $\boldsymbol{q}$ with $q_j < p_j$ for some $j \in [n]$. Now consider $\boldsymbol{p}' = \boldsymbol{p} \wedge \boldsymbol{q}$. By Lemma 12, $\boldsymbol{p}'$ is feasible. But as $\boldsymbol{p}' \leq \boldsymbol{p}$ with $p'_j < p_j$, this contradicts our assumption that $\boldsymbol{p}$ cannot be reduced further. □

5.3.2 Maximising revenue

In Section 5.3.1, we established that the set of feasible prices contain a unique elementwise-minimal price vector $\boldsymbol{p}^*$. We now show that revenue is maximised at these prices. Note that we do not assume $\boldsymbol{p}^*$ to be the *only* prices at which revenue is maximised; indeed, there can be many revenue-maximising prices. However, $\boldsymbol{p}^*$ maximises the quantities of goods allocated.

Lemma 14. *For any two distinct feasible price vectors $\boldsymbol{p}$ and $\boldsymbol{p}'$ with $\boldsymbol{p} \leq \boldsymbol{p}'$ we have $R(\boldsymbol{p}) \geq R(\boldsymbol{p}')$. In other words, the maximum obtainable revenue at $\boldsymbol{p}$ is weakly greater than the revenue obtainable at $\boldsymbol{p}'$.*

Proof. Let π and π' be valid allocations that respectively maximise revenue at $\boldsymbol{p}$ and $\boldsymbol{p}'$. Our goal is to determine a valid allocation τ that achieves a weakly

greater revenue at $\boldsymbol{p}$ than π' does at $\boldsymbol{p}'$. As $R(\boldsymbol{p}) \geq \sum_{j\in[n]} p_j\tau([m],j)$ and $R(\boldsymbol{p}') = \sum_{j\in[n]} p'_j\pi'([m],j)$, this immediately implies the result.

If $\boldsymbol{p} = \boldsymbol{p}'$, there is nothing to prove. Hence we assume that $S := \{j \in [n] \mid p_j < p'_j\}$, the set of goods which are priced strictly lower at $\boldsymbol{p}$ than at $\boldsymbol{p}'$, is non-empty. Fix a buyer $i \in [m]$. In order to define the new allocation $\tau(i,\cdot)$ to i at $\boldsymbol{p}$, we distinguish between the two cases that $J^i(\boldsymbol{p})$ is, and is not, a subset of S.

Case 1: Suppose buyer i demands a subset of S at $\boldsymbol{p}$, so $J^i(\boldsymbol{p}) \subseteq S$. In this case, we set $\tau(i,\cdot) = \pi(i,\cdot)$. As π is valid, and τ and π are both allocations at the same prices, the buyer spends its entire budget under τ. Moreover, they are only allocated goods that they demand.

Case 2: Suppose $J^i(\boldsymbol{p}) \not\subseteq S$. We note that $J^i(\boldsymbol{p}') \cap S = \emptyset$ and buyer i still demands all goods in $J^i(\boldsymbol{p}')$, so $J^i(\boldsymbol{p}') \subseteq J^i(\boldsymbol{p})$. In this case, we set $\tau(i,\cdot) = \pi'(i,\cdot)$. As the buyer is only allocated goods not in S, and $p_j = p'_j$ for all goods $j \in [n]_0 \setminus S$, it follows that the buyer spends the same at both prices.

Note that, in both cases, the buyer is only allocated goods that they demand. To summarise, we define τ as

$$\tau(i,\cdot) = \begin{cases} \pi(i,\cdot) & \text{if buyer } i \text{ demands a subset of } S \text{ at } \boldsymbol{p}, \\ \pi'(i,\cdot) & \text{otherwise.} \end{cases}$$

We now prove that τ is valid. We have already argued above that all buyers satisfy the demand and budget conditions of Definition 24. It remains to show that aggregate demand does not exceed supply s_j for any goods $j \in [n]_0$ under τ. Note that a buyer is allocated a good j in S under τ if, and only if, they satisfy Case 1 above. Indeed, in this case we set $\tau(i,j) = \pi(i,j)$. Hence, for any $j \in S$, we have $\tau([m],j) \leq \pi([m],j) \leq s_j$. Similarly, for any $j \notin S$ the buyer will satisfy Case 2, and we get $\tau([m],j) \leq \pi'([m],j) \leq s_j$.

Finally, we see that τ achieves weakly greater revenue at $\boldsymbol{p}$ than π' does at $\boldsymbol{p}'$. To see this, note that each buyer satisfying Case 1 spends its entire budget on non-reject goods and thus contributes a weakly greater amount to revenue at $\boldsymbol{p}$ than at $\boldsymbol{p}'$, while a buyer satisfying Case 2 contributes the same amount to revenue. □

Corollary 6. Revenue is maximised at elementwise-minimal feasible prices $\boldsymbol{p}^*$.

Proof. Suppose $\boldsymbol{p} \neq \boldsymbol{p}^*$ is a revenue-maximising price vector. As $\boldsymbol{p}^*$ is elementwise-minimal, we have $\boldsymbol{p}^* \leq \boldsymbol{p}$. Then by Lemma 14, we can obtain weakly greater revenue at $\boldsymbol{p}^*$ than at $\boldsymbol{p}$, or $R(\boldsymbol{p}^*) \geq R(\boldsymbol{p})$. But as $\boldsymbol{p}$ maximises R, so does $\boldsymbol{p}^*$. □

5.4 Reducing prices

Suppose $\boldsymbol{p}$ is a feasible price vector at which no allocation exhausts supply $\boldsymbol{s}$ in all goods. In this case, we present a method to scale a subset of goods uniformly in price by a factor $0 < c < 1$ while retaining feasibility and increasing aggregate demand. As stated in Corollary 7, this implies that the elementwise-minimal prices $\boldsymbol{p}^*$ clear the market. We do this by modifying a price-scaling routine that forms part of a 'simplex-like' algorithm by Adsul et al. (2012) for Fisher markets to our market setting; this is not immediate, as buyers can choose not to spend their entire budget if prices are too high. In our approach, we scale prices down instead of up, and apply additional pre-processing for bids that may be indifferent between spending on the null good and 'real' goods.

5.4.1 The procedure for reducing prices

In the following, we describe our subroutine that, given prices $\boldsymbol{p}$ that are not market-clearing and an expenditure e, either modifies e in order to ensure that an unsated good becomes sated, or reduces prices of some goods. Moreover, we claim that after $O(n)$ repetitions of this subroutine, prices of some goods are reduced. Indeed, we start with at most n unsated goods. Each application of the subroutine that fails to reduce prices strictly reduces the number of unsated goods, as no sated goods can revert to being unsated. In particular, this implies that the elementwise-minimal prices $\boldsymbol{p}^*$ clear the market, as we could otherwise reduce $\boldsymbol{p}^*$ in price while maintaining feasibility.

Corollary 7. Competitive equilibrium is uniquely achieved at elementwise-minimal feasible prices $\boldsymbol{p}^*$.

Proof. It is well-known that market-clearing prices are unique in our market. (Theorem 6 in the appendix provides a direct proof of this.) Suppose that $\boldsymbol{p}^*$ is not market-clearing. Then by Theorem 4, and our discussion above, we can reduce prices of some goods while maintaining feasibility. But this contradicts our assumption that $\boldsymbol{p}^*$ is the elementwise-minimal feasible price vector. □

In order to describe our method of scaling prices, we first introduce the demand and expenditure graphs. Then we present a subroutine that – given prices $\boldsymbol{p}$, an expenditure e and an unsated good j under e at $\boldsymbol{p}$ – either returns scaled-down prices or an expenditure at $\boldsymbol{p}$ under which j is sated. After invoking this subroutine $O(n)$ times, prices will be scaled down or all goods are sated. We note that the subroutine defined here is also called in our algorithm (cf. Section 5.5) to find the unique market-clearing prices $\boldsymbol{p}^*$.

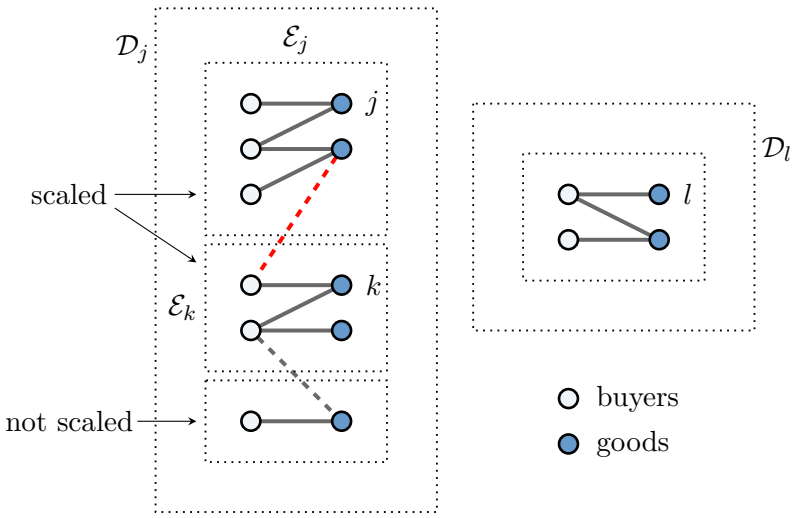


Figure 5.3: An illustration of the demand and expenditure graphs. Solid and dashed lines respectively indicate edges in $\mathcal{E}$ and $\mathcal{D} \setminus \mathcal{E}$. The component $\mathcal{D}_i$ contains three components of $\mathcal{E}$. If we root $\mathcal{D}_j$ at j, we note that there is an arc (dashed red) from a good in $\mathcal{E}_j$ to a buyer in $\mathcal{E}_k$, and an arc (dashed grey) from a buyer in $\mathcal{E}_k$ to the component below. Hence the tree constructed in Section 5.4.3, which contains all goods to be scaled down in price, consists of $\mathcal{E}_j$ and $\mathcal{E}_k$ but not the third component.

5.4.2 The demand and expenditure graphs

In order to describe demand and spending relationships between bids and goods, we introduce two bipartite graphs, the *demand graph* and the *expenditure graph*. Modulating expenditures in appropriately defined sub-trees of these graphs is the central component in our price reduction procedure.

Definition 26 (Demand graph and expenditure graph)**.** Let e be a valid expenditure at feasible prices $\boldsymbol{p}$. We denote by $\mathcal{D}$ the bipartite *demand graph* on vertex sets $[m]$ and $[n]_0$. There is an edge $(i, j) \in D$ if, and only if, buyer i demands j at $\boldsymbol{p}$. Moreover, we define the *expenditure graph* $\mathcal{E}$ of $\mathcal{D}$ induced by the edges (i, j) with positive expenditure $e(i, j) > 0$. Finally, $\mathcal{D}_j$ and $\mathcal{E}_j$ respectively denote the connected component of $\mathcal{D}$ and $\mathcal{E}$ that contains good j.

Note that $\mathcal{E}_j$ is a subgraph of $\mathcal{D}_j$ for all goods $j \in [n]_0$, and each $\mathcal{D}_j$ can contain multiple connected components of $\mathcal{E}$. Both graphs are illustrated in Fig. 5.3.

Breaking cycles. Our price reduction method assumes that the demand graph is acyclic. For this reason, we follow Orlin (2010) in perturbing buyer valuations by an infinitesimal amount in order to break cycles. Induce an ordering $[m] = \{i^1, i^2, \ldots\}$ on the buyers, and perturb each entry $v_j^{i^k}$ of buyer i^k by adding $\varepsilon^{kn} + \varepsilon^j$ for some infinitesimally small $\varepsilon > 0$. In our algorithms, this perturbation can

Algorithm 1 Tree-based price reduction

Input: Feasible prices $\boldsymbol{p}$, valid expenditure e and unsated good j.

Output: Prices $\boldsymbol{p}' \leq \boldsymbol{p}$ and expenditure e', where either $\boldsymbol{p}' = \boldsymbol{p}$ and j is sated, or $p'_k < p_k$ is for some goods $k \in [n]$.

1: Construct tree T by rooting $\mathcal{D}_j$ at j and removing subtrees rooted at goods k of arcs $(i,k) \in \mathcal{D}_j \setminus \mathcal{E}$.
2: **if** $0 \notin T$ **then** go to step 9. **end if**
3: Compute the path $j = g^1, b^1, \ldots, g^k, b^k, g^{k+1} = 0$ from j to null good 0 in T and the minimum m of $p_j s_j - e([m], j)$ and $\min_{i \in [k]} e(g^i, \boldsymbol{b}^i)$. Update e by $e(\boldsymbol{b}^i, g^i)$ += m and $e(\boldsymbol{b}^i, g^{i+1})$ −= m.
4: **if** good j is now sated under e **then**
5: **return** $\boldsymbol{p}$ and e.
6: **else**
7: Let i be the smallest such value $i \in [k]$ and remove from T the subtree rooted at g^{i+1}.
8: **end if**
9: Traverse T depth-first to recursively compute $\sum_{k \in T_v} \alpha_k p_k s_k$ for each vertex $v \in T$.
10: Compute $c^{(1)}, c^{(2)}$ and $c^{(3)}$ as specified in Section 5.4.3, and scaling factor $c^* = \max\left\{c^{(1)}, c^{(2)}, c^{(3)}\right\}$.
11: Traverse T depth-first to recursively compute $\Delta(i,k)$ for each arc in T. Compute e' as described in (5.2).
12: Let $p'_k = c^* p_k$ for all $k \in T$ and $p'_k = p_k$ otherwise.
13: **return** $\boldsymbol{p}'$ and e'.

be simulated without a running-time penalty by implementing lexicographic tie-breaking when constructing the demand and expenditure graphs. For details about the perturbation, we refer to (Orlin, 2010).

5.4.3 Reducing prices along a tree

We now describe the subroutine referred to above. The full procedure is stated in Algorithm 1. Correctness is proved in Theorem 4.

Identifying goods to reduce in price. Suppose j is an unsated good under expenditure e at $\boldsymbol{p}$. We describe a procedure which either identifies a set $J \subseteq [n]$ of goods to reduce in price, or finds a new expenditure e' at $\boldsymbol{p}$ under which good j is sated. We first construct a directed tree T by rooting the (undirected) tree $\mathcal{D}_j$ at good j and then remove all subtrees rooted at endpoint goods j of arcs $(i,j) \in \mathcal{D}_j \setminus \mathcal{E}$. This is illustrated in Fig. 5.3. For notational convenience, we also let T_v denote the subtree of T rooted at v for any vertex $v \in T$.

If T does not contain the null good 0, we let J consist of the goods of T and are done. Otherwise, we consider the directed path $(j = g^1, b^1, \ldots, g^k, b^k, g^{k+1} = 0)$ from good j to null good 0 in T and modify the expenditure e to redirect spending from 0 to j as follows. Let $m > 0$ be the minimum of $p_j s_j - e([m], j)$ and $\min_{i\in[k]} e(b^i, g^i)$. We let $e'(b^i, g^i) = e(b^i, g^i) + m$ and $e'(b^i, g^{i+1}) = e(b^j, g^i) - m$ for all $i \in [k]$. Note that the resulting expenditure is valid and redirecting spending in this way does not add edges to the expenditure graph, or cause the expenditure on any goods in $[n]$ to reduce. In particular, no sated good becomes unsated.

If $m = p_j s_j - e([m], j)$, then good j is sated under the new expenditure e', and we are done. Otherwise, if $m < p_j s_j - e([m], j)$, we obtain the tree corresponding to the new expenditure e' by identifying the smallest $i \in [k]$ such that $e'(b^i, g^{i+1}) = 0$ and removing the subtree rooted at g^{i+1} from T. As the tree no longer contains the null good 0, we let J denote the goods in T and are done.

Scaling prices. Now we suppose that we have identified the set of goods J of the tree T not containing 0, and good j is unsated. In order to fully specify our scaling method, it remains to determine the factor $0 < c < 1$ by which to uniformly scale the prices of goods in J so that the resulting prices remain feasible. Let $\boldsymbol{p}'$ denote the prices after scaling, so $p'_k = cp_k$ for $k \in J$ and $p'_k = p_k$ otherwise. We proceed by describing an expenditure e' at $\boldsymbol{p}'$ that is derived from e by adding, or removing, a difference term parameterised by c to, or from, each $e(i, k)$. The scaling factor c is then chosen to ensure that e' is valid. Recall that e' needs to satisfy the conditions in Definition 24 to be a valid expenditure at prices $\boldsymbol{p}'$. We recall, for instance, that the supply constraint imposes $e'([m], k) \leq p'_k s_k$ for all goods $k \in [n]$. In the following, we will impose a stronger supply constraint for all goods $k \in [n] \setminus \{j\}$. Suppose that the proportion of good k's supply aggregately demanded under e at $\boldsymbol{p}$ is given by $\alpha_k \in [0, 1]$, so that $e([m], k) = \alpha_k p_k s_k$ for good k. Then we require that this proportion is maintained under e' at the new prices $\boldsymbol{p}'$, so $e'([m], k) = \alpha_k p'_k s_k$ for all $k \in [n] \setminus \{j\}$. (Hence $e'([m], k) = c\alpha_k p_k s_k$ for $k \in J$ and $e'([m], k) = \alpha_k p_k s_k$ for $k \notin J$.) Note that this trivially implies the supply constraint, as $\alpha_k \leq 1$ and $c < 1$.

The stronger supply condition uniquely determines an expenditure e' for any given c, which can be expressed by adding, or subtracting, the differential terms $\Delta(i, k)$ to, or from, $e(i, k)$. For every arc in T between some buyer i and good k with endpoint $v \in \{i, k\}$, we define $\Delta(i, k) := (1 - c) \sum_{l\in T_v} \alpha_l p_l s_l$. For all other buyers i and k, we let $\Delta(i, k) = 0$. Finally, we define the new expenditure e' as

$$e'(i, k) = \begin{cases} e(i, k) - \Delta(i, k), & \text{if } i \text{ precedes } k \text{ in } T, \\ e(i, k) + \Delta(i, k), & \text{otherwise.} \end{cases} \tag{5.2}$$

It remains to find the smallest $0 < c < 1$ such that e' is valid. Conceptually, we choose c by reducing from 1 until one of three scenarios occurs. For each scenario $i\{1, 2, 3\}$, we determine the scaling factor $c^{(i)}$ at which this scenario occurs. Our scaling factor is then chosen by computing $c^* = \max\{c^{(1)}, c^{(2)}, c^{(3)}\}$. The scenarios are as follows.

1. Spending $e([m], j)$ on good j increases until it becomes sated. This strictly reduces the number of unsated goods in $\mathcal{D}$.

2. Some $e'(i, k)$ decreases to 0. This breaks up a component in $\mathcal{E}$.

3. Some bid not in T starts demanding a good in T (along with goods outside T). This connects two components in $\mathcal{D}$.

Scenario 1: We note that c must be chosen such that $e'([m], j) = e([m], j) + \Delta([m], j) \leq cp_j s_j$. Substituting $\Delta([m], j) = \sum_{k \in J \setminus \{j\}} (1 - c)\alpha_k p_k s_k$ and solving for c, we see that this holds for scaling factor

$$c^{(1)} = \frac{e([m], j) + \sum_{k \in J \setminus \{j\}} \alpha_k p_k s_k}{p_j s_j + \sum_{k \in J \setminus \{j\}} \alpha_k p_k s_k}. \tag{5.3}$$

As $0 \leq e([m], j) \leq p_j s_j$, factor $c^{(1)}$ is well-defined and $0 < c^{(1)} < 1$.

Scenario 2: Note that $\Delta(i, k) > 0$ for any scaling factor c that is strictly less than 1. Hence, $e'(i, k)$ is non-negative for all arcs from a good k to a buyer i, for any $c < 1$. Now fix an arc from buyer i to good k. In this case, we need to ensure that $\Delta(i, k) \leq e(i, k)$ to guarantee $e'(i, k) \geq 0$. Substituting $\Delta(i, k) = (1 - c) \sum_{l \in T_k} \alpha_l p_l s_l$, and solving for c, we get

$$c \geq 1 - \frac{e(i, j)}{\sum_{k \in T_j} \alpha_k p_k s_k}. \tag{5.4}$$

Hence $c^{(2)}$, as chosen below, guarantees $e'(i, j) \geq 0$ for every buyer i and good j, with equality for at least one pair (i, j).

$$c^{(2)} = \max_{(i,k) \in T} \left(1 - \frac{e(i, k)}{\sum_{l \in T_k} \alpha_l p_l s_l}\right). \tag{5.5}$$

Note that $0 < e(i, k) \leq \alpha_k p_k s_k \leq \sum_{l \in T_k} \alpha_l p_l s_l$ for all $(i, k) \in T$, so $c^{(2)}$ is well-defined and satisfies $0 < c^{(2)} < 1$.

Scenario 3: Fix a buyer i not in T. Then the buyer continues to demand goods outside T only if $\frac{1}{c} \max_{k \in T} \frac{v_k^i}{p_k} \leq \max_{l \notin T} \frac{v_l^i}{p_l}$. Hence, by solving for c and taking the maximum over all bids not in T, we get

$$c^{(3)} = \max_{i \notin T} \left[\max_{k \in T} \frac{v_k^i}{p_k} \min_{l \notin T} \frac{p_l}{v_l^i}\right]. \tag{5.6}$$

Note that $1 \leq \max_{k \in T} \frac{v_k^i}{p_k} < \max_{l \in [n]_0} \frac{v_l^i}{p_l}$ for all $i \notin T$, as these bids do not demand any goods in T at $\boldsymbol{p}$. Hence, $c^{(3)}$ is well-defined and $0 < c^{(3)} < 1$.

Theorem 4. Suppose $\boldsymbol{p}$ is not market-clearing, and e is a valid expenditure at $\boldsymbol{p}$ with an unsated good j. The price-reduction procedure above (Algorithm 1), when given $\boldsymbol{p}$, e and j, either returns a valid expenditure e' at $\boldsymbol{p}$ that sates good j, or it returns reduced prices $\boldsymbol{p}'$ and a valid expenditure e' at $\boldsymbol{p}'$, in polynomial time.

Proof. Suppose first that the subroutine terminates in line 5. Then the resulting expenditure e' that it returns is trivially valid, and we are done.

Now suppose that the subroutine does not terminate in line 5. Then we note that $\boldsymbol{p}'$ returned by the procedure is strictly smaller than $\boldsymbol{p}$ for all goods $k \in J$, as the scaling factor is strictly less than 1. Hence it suffices to show that the expenditure e' returned by Algorithm 1 after scaling prices of goods J by factor c^* is valid in the sense of Definition 24. By construction of c^*, $e'(i,k)$ is non-negative for all buyers $[m]$ and goods k. We also note that all arcs in T coincide with edges in the demand graph by construction, and if $e(i,k)$ changes, then there is an edge between i and k in the demand graph. Meanwhile, the choice of $c^{(3)}$ guarantees that all buyers $i \notin T$ demand, at $\boldsymbol{p}'$, a superset of the goods they demand at $\boldsymbol{p}$. It follows e is demand-valid.

Next, we verify that e is budget-feasible. Suppose first that i is a buyer not in T. Then, by construction of T, none of the prices of goods that i demands are scaled, and the spending of i is unchanged. Next suppose that buyer i is a leaf in T, and good k is their parent. Then $\Delta(i,k) = \sum_{l \in T_i} \alpha_l p_l s_l = 0$, as the tree T_i contains no goods. Hence, $e'(i,k) = e(i,k) = \beta^i$ and the condition for budget feasibility holds. Finally, suppose buyer i is not a leaf in T. Let g^1 be their parent and $g^2, \ldots, g^k$ be the children. We need to verify that $e'(i,[n]_0) = \beta^i$. As $e(i,[n]_0) = \beta^i$, it suffices to see that $-\Delta(i,g^1) = \sum_{j=2}^{k} \Delta(i,g^j)$. But this follows from the definition of Δ.

Next, we show that e is supply-feasible. For the originally unsated good j, this follows immediately from the choice of our scaling factor $c^* \geq c^{(1)}$ in (5.3). For all other goods, we verify the stronger supply condition introduced above. Suppose first that k is a good not in T. Then by construction of T, the amount that each bid spends on k remains unchanged. Next suppose good k is a leaf in T, and bid i is its parent. Then the stronger supply constraint requires that $e'(i,k) = ce(i,k) = c\alpha_k p_k s_k$, which agrees with our definition of $\Delta(i,k) = (1-c)\alpha_k p_k s_k$. Finally, suppose good k is not a leaf in T, and let i be its parent and $[m]'$ be its children. Then the stronger supply constraint is satisfied, as

$$e'([m],k) = e([m],k) - \Delta(i,k) + \Delta([m]',k) = \alpha_k p_k s_k - (1-c)\alpha_k p_k s_k = c\alpha_k p_k s_k.$$

Algorithm 2 Finding elementwise-minimal feasible prices $\boldsymbol{p}^*$

Input: List of buyers and their valuations in general position.
Output: Element-wise smallest feasible prices $\boldsymbol{p}'$.

1: Let $\boldsymbol{p}$ be the element-wise maximum over the buyers' valuations in $[m]$ and e be the expenditure at which all buyers spend their entire budget on the null good 0.
2: **while** there are unsated goods **do**
3: Identify an unsated good j under e at $\boldsymbol{p}$.
4: **while** good j is unsated **do**
5: Apply tree-based subroutine stated in Algorithm 1 with good j to update $(\boldsymbol{p}, e)$.
6: **end while**
7: **end while**
8: **return** prices $\boldsymbol{p}$ and expenditure e.

To see that the subroutine runs in polynomial time, note that the running time is dominated by traversing the tree with $n + m$ nodes depth-first, which is polynomial in n and m. □

5.5 Finding market-clearing prices

The simplex-like algorithm of Adsul et al. (2012) for linear Fisher markets proceeds by repeatedly applying a price-scaling subroutine until it returns the unique market-clearing prices. Similarly, we can repeatedly apply our subroutine from Section 5.4 in order to find the unique market-clearing prices in our market in finite time. Note that each call to Algorithm 1 makes progress in the sense that prices decrease or a good is sated. Moreover, the running time analysis we provide in Theorem 5 is analogous to that given in (Adsul et al., 2012), and shows that the algorithm terminates in at most exponential time.

Let $\boldsymbol{p} = \bigvee_{i\in[m]} \boldsymbol{v}^i$ be the element-wise maximum over all valuation vectors. Then it follows that all buyers demand the null good at $\boldsymbol{p}$, while possibly being indifferent between null and other goods. We define the expenditure e at $\boldsymbol{p}$ under which all buyers spend their entire budget on the null good, so for each buyer $i \in [m]$ we let $e(i, 0) = \beta^i$ and $e(i, j) = 0$ for all $j \in [n]$. This expenditure is trivially valid.

Of course, no market clears at this initial allocation. We pick any unsated good j and repeatedly call the subroutine of Algorithm 1 with good j, prices $\boldsymbol{p}$ and expenditure e to reduce prices until the market for good j clears (good j becomes sated). Subsequently, we select another unsated good, reduce prices by repeatedly calling Algorithm 1, and repeat this step until all goods are sated.

Theorem 5. Algorithm 2 finds the market-clearing prices $\boldsymbol{p}^*$ in exponential time.

Proof. We argue that the number of calls to Algorithm 1 is bounded. Note that the outer loop in Algorithm 2 iterates $O(n)$ times. Indeed, invoking Algorithm 1 does not cause any sated goods to become unsated. Every iteration of the outer loop reduces the number of unsated goods by one, and we start with at most n unsated goods.

We now show that the inner loop of Algorithm 2 applies Algorithm 1 an exponential number of times (in m and n) before good j is sated. Suppose, for contradiction, that the subroutine is called more often. In this case, we note that on each iteration, the scaling factor is chosen such that scenario 2, scenario 3 or both scenarios occur. Let $\mathcal{D}_j^{(k)}$ denote the connected component of $\mathcal{D}$ containing good j when scenario 2 occurs for the k-th time. In particular, we note that any two consecutive occurrences of scenario 2 can be separated by at most m iterations of scenario 3's. To see this, realise that every time only scenario 3 occurs, T grows through the addition of a buyer, and we have m buyers in total.

We now argue that all $\mathcal{D}_j^{(l)}$ are distinct. As there are exponentially many configurations of $\mathcal{D}_j$, this implies the result. Note that whenever the k-th scenario 2 occurs, we have $e'(i,k) = 0$ for some arc (i,k), and good $k \in T$ is not in T in the next iteration. Hence, the price of good k is not scaled down in the next iteration. Meanwhile, the price of good j is scaled down on every iteration of Algorithm 1. Now suppose, for any k, that $\mathcal{D}_i^{(k)}$ reoccurs at a later stage. This implies that p_j and p_k have been scaled down the same; indeed, this follows by considering the path from j to k in $\mathcal{D}_j^{(k)}$ and applying the conditions of demand. But we have just argued that p_j and p_k have not been scaled down the same, a contradiction.

Finally, we see that the algorithm finds market-clearing prices. To see this, note that it terminates only once all goods are sated. This concludes the proof. □

5.6 Conclusion

We analyse a market for multiple, divisible goods, in which a unique set of prices exists that induce a socially optimal and revenue-optimal allocation. This co-existence of revenue-optimality and efficiency makes our market compelling in theory as well as highly attractive for sellers, buyers, and market platforms in practice. We provide algorithmic results to derive the efficient and optimal price vector, and expect to improve on these results in future work.

6
Concluding remarks

This thesis presents new theoretical and empirical results on equilibria in multi-object auction markets. Chapter 2 and Chapter 3 study equilibria from a game-theoretic perspective and focus on strategic behaviour, with the underlying assumption that bidders strategise to maximise their own payoffs. In Chapter 2, a tractable model of a product-mix auction, a sequential auction, and a simultaneous auction is developed with a representative buyer with multi-object valuations, competing against several competitive fringes. The nature of this model enables a characterisation of Bayes-Nash equilibria, which is notoriously difficult in models with symmetric buyers with multi-variate strategies. Based on the equilibrium characterisation, I compute bidder surplus, revenue, welfare, and the representative, flexible bidder's payments. In the risk-neutral Bayes-Nash equilibrium, the three auction formats perform nearly identically (the simultaneous auction performs slightly worse).

The empirical data presented in Chapter 3 is in contrast with the theoretical findings: the product-mix format achieves higher bidder surplus and welfare, whereas the simultaneous format achieves the highest revenue and flexible bidder payments. The experimental study also comprises a second environment, in which groups of three symmetric, flexible bidders compete against each other. In this environment, the results are even more pronounced, with the product-mix auction generating almost double the bidder surplus and 1.5 times the bidder surplus of the sequential and the simultaneous auction. The product-mix auction also achieves 12% higher efficiency than both other formats, but the simultaneous auction generates the highest revenue (7% more than the product-mix auction). My empirical study is

first to compare these auction formats and to put them in context with a theoretical model of bidding behaviour. These results provide first insights, but also raise questions about the channels at work in the different auction formats. Future work may shed light on why the product-mix auction performs so well.

Chapter 4 and Chapter 5 focus on competitive equilibria and envy-free revenue-maximising allocations. The premise here is that, with a sufficiently large number of buyers in the market, they act as price-takers. Consequently, competitive equilibrium is a very good approximation of the market outcome. Because it leaves all market participants happy with what they bought or sold at the final auction prices, it also a good quality measure for a mechanism; if the allocation procedure results in a competitive equilibrium, some form of stability is guaranteed.

In Chapter 4, I generalise the existing notions of preferences over substitutes and complements. I develop a tool, the cost-function graph, to express preferences (of the seller) over allocations in the market in a way that allows the auctioneer to still compute a competitive equilibrium. This tool may be valuable in practical applications, as preferences over allocations are commonplace in procurement, spectrum auctions, conservation auctions, and many more. Future work may focus even more closely on computational aspects of the allocation problem. As the cost function graph may be restricted to represent only those preferences that seem 'relevant', algorithms may be able to make explicit use of this.

In Chapter 5, we provide a novel result on the equivalence of revenue and welfare, in a market containing any number of substitutes and budget-constrained bidders. Crucial for the equivalence to hold is that the bidders' valuations are additive, and we demonstrate how non-additivity may cause the equivalence to fail. The result is especially interesting in light of the many applications: the preferences we study naturally appear in many financial markets, in auctions for advertisement slots, or in the (re-)division of firms. The equivalence is surprising as in most known markets, the revenue-maximal allocation and competitive equilibrium do not coincide: from many settings, it is well-known that by raising prices, the seller can often extract more revenue at the cost of consumer surplus. However, as budgets are spent on different goods based on the corresponding auction prices, this strategy does not work for the seller: with increased prices, buyers demand less of each good, and the additivity in valuations guarantees sufficient elasticity for such a price increase not to be profitable. While existing inner-point methods can be used to find the competitive equilibrium, it is an open question if there exists a more immediate (perhaps combinatorial) polynomial-time algorithm. The geometric insights our work provides may be a stepping stone in this direction.

Appendices

A

Appendix for Chapter 2

A.1 Proofs for the 2-good case

Proof of Lemma 1. The proof hinges on the fact that there are (at least) two bidders bidding for good A only and two bidders bidding for good B only, as these two bidder engage in a Bertrand-type competition (the argument here is made for the case of $n = 2$, but is analogous for $n > 2$).[1] First, it is obvious that no equilibrium bid made by any bidder can be higher than their valuation under discriminatory pricing. It is also easy to see that in any equilibrium two bidders of group G_k have to bid the same amount. We call those bidders 1_k and 2_k. If their bid wins, the unit of good k is randomly allocated among them with equal probability. Suppose they make different bids in equilibrium, say 1_k bids higher than 2_k. Then, in the case where 2_k wins, bidder 1_k could improve his payoff by bidding at least as much as bidder 1_k. In the case where 2_k loses, 1_k's payoff would remain unchanged by this deviation, i.e. bidder 2_k could deviate to improve his expected payoff. Now suppose that the two bidders with an identical valuation $v_k \in \{a, b\}$ bid $s_k < v_k$. Then, again, 1_k for example could profitably deviate by bidding $s_k + \epsilon \leq v_k$. He would win the unit of good k *always*, not only half of the time, at an ϵ-higher price in the cases where s_k is a winning bid, which is a strict payoff improvement. In the cases where s_k isn't winning, his payoff would either remain zero (if $s_k + \epsilon$ remains a losing bid or is winning at $s_k + \epsilon = v_k$) or increase strictly (if $s_k + \epsilon < v_k$ becomes a winning bid). It follows that $s_k = v_k$ is the only possible candidate for a BNE. □

[1] Note that we are looking for pure strategy Bayes-Nash equilibria only.

A.1.1 Product-mix auctions

Proof of Proposition 1. The flexible bidder's preference for her ex-post allocation coincides with the efficient allocation: she wants to win good A if and only if

$$v_A > a \qquad \text{and} \tag{A.1}$$
$$v_A - a > v_B - b \tag{A.2}$$

and good B if and only if

$$v_B > b \qquad \text{and} \tag{A.3}$$
$$v_B - b > v_A - a. \tag{A.4}$$

□

We also establish bidder F's expected payoff function. We have

$$\begin{aligned}\Pi^F_{P2|v_A,v_B}(w_A,w_B) &= P\left(\text{F wins A}\right)\mathrm{E}[v_A - a \mid \text{F wins A}] \\ &\quad + P\left(\text{F wins B}\right)\mathrm{E}[v_B - b \mid \text{F wins B}] \\ &\quad + \int_0^{w_A}\int_{w_B-w_A+a}^{\bar{v}} v_A - a \,\mathrm{d}G(b)\,\mathrm{d}G(a) \\ &= \int_0^{w_A}\int_0^{w_B-w_A+a} v_B - b \,\mathrm{d}G(b)\,\mathrm{d}G(a) + \int_{w_A}^{\bar{v}}\int_0^{w_B} v_B - b \,\mathrm{d}G(b)\,\mathrm{d}G(a)\end{aligned}$$

Proof of Lemma 2. First, note that because a and b are *iid* distributed, we have $P_A(w_B,w_A) = P_B(w_A,w_B)$. Now suppose $w_A > w_B$. Then we can easily show that bidding $\widetilde{w_A} = w_B$ and $\widetilde{w_B} = w_A$ is a profitable deviation:

$$\begin{aligned}\Pi^F(\widetilde{w_A},\widetilde{w_B}) =& P_A(\widetilde{w_A},\widetilde{w_B})\left[v_A - \widetilde{w_A}\right] + P_B(\widetilde{w_A},\widetilde{w_B})\left[v_B - \widetilde{w_B}\right] \\ =& P_A(w_B,w_A)\left[v_A - w_B\right] + P_B(w_B,w_A)\left[v_B - w_A\right] \\ =& P_A(w_A,w_B)\left[v_B - w_A\right] + P_B(w_A,w_B)\left[v_A - w_B\right] \\ >& P_A(w_A,w_B)\left[v_A - w_A\right] + P_B(w_A,w_B)\left[v_B - w_B\right]\end{aligned}$$

The last inequality holds because $P_A(w_A,w_B) > P_B(w_A,w_B)$ for $w_A > w_B$. □

Proof of Proposition 2. Putting together equations (2.1), (2.2), and (2.3), we obtain

$$\begin{aligned}\Pi^F_{P1|v_A,v_B}(w_A,w_B) &= \left[\int_0^{w_A}\int_{w_B-w_A+a}^{\bar{v}} \mathrm{d}G(b)\,\mathrm{d}G(a)\right](v_A - w_A) \\ &\quad + \left[\int_0^{w_A}\int_0^{w_B-w_A+a} \mathrm{d}G(b)\,\mathrm{d}G(a) + \int_{w_A}^{\bar{v}}\int_0^{w_B} \mathrm{d}G(b)\,\mathrm{d}G(a)\right](v_B - w_B)\end{aligned}$$

The first-order conditions write

$$\begin{aligned}\frac{\partial \Pi^F_{P1|v_A,v_B}}{\partial w_A} &= (v_A - w_A)\left[g(w_A)\,(1 - G(w_B)) + \int_0^{w_A} g(w_B - w_A + a)\,\mathrm{d}G(a)\right] \\ &\quad - (v_B - w_B)\int_0^{w_A} g(w_B - w_A + a)\,\mathrm{d}G(a) - \int_0^{w_A}\int_{w_B - w_A + a}^{\bar{v}} \mathrm{d}G(b)\,\mathrm{d}G(a) = 0\end{aligned} \tag{A.5}$$

and

$$\begin{aligned}\frac{\partial \Pi^F_{P1|v_A,v_B}}{\partial w_B} &= (v_B - w_B)\left[g(w_B)\,(1 - G(w_A)) + \int_0^{w_A} g(w_B - w_A + a)\,\mathrm{d}G(a)\right] \\ &\quad - (v_A - w_A)\int_0^{w_A} g(w_B - w_A + a)\,\mathrm{d}G(a) \\ &\quad - \left[G(w_B)\,(1 - G(w_A)) + \int_0^{w_A}\int_0^{w_B - w_A + a} \mathrm{d}G(b)\,\mathrm{d}G(a)\right] = 0\end{aligned} \tag{A.6}$$

Simplifying yields

$$\begin{aligned}&(v_A - {w_A}^*)\,g({w_A}^*)\,(1 - G({w_B}^*)) - \int_0^{{w_A}^*}\int_{{w_B}^* - {w_A}^* + a}^{\bar{v}} \mathrm{d}G(b)\,\mathrm{d}G(a) \\ &+ (v_A - {w_A}^* - (v_B - {w_B}^*))\int_0^{{w_A}^*} g({w_B}^* - {w_A}^* + a)\,\mathrm{d}G(a) = 0\end{aligned} \tag{A.7}$$

$$\begin{aligned}&[(v_B - {w_B}^*)\,g({w_B}^*) - G({w_B}^*)]\,(1 - G({w_A}^*)) - \int_0^{{w_A}^*}\int_0^{{w_B}^* - {w_A}^* + a} \mathrm{d}G(b)\,\mathrm{d}G(a) \\ &+ (v_B - {w_B}^* - (v_A - {w_A}^*))\int_0^{{w_A}^*} g({w_B}^* - {w_A}^* + a)\,\mathrm{d}G(a) = 0\end{aligned} \tag{A.8}$$

Notice the asymmetry in the two first-order condition. This is because $w_A \leq w_B$. Setting $w_A = 0$, equation (A.8) reduces to the standard first-order condition we obtain in a single first-price auction in our setup, but the analogue is not possible with equation (A.7).

In order to prove that equations (A.5) and (A.6) indeed characterise a global optimum, we show that the maximum cannot be on the boundary of $[0, v_A] \times [0, v_B]$ and therefore must be a stationary point of $\Pi^F_{P1|v_A,v_B}$, characterised by the two equations above. First, we exploit symmetry properties of Figure 3.1 for the case where $w_A \geq w_B$ to write bidder F's payoff function:

$$\begin{aligned}\overline{\Pi}^F_{P1|v_A,v_B}(w_A, w_B) &= \left[\int_0^{w_B}\int_0^{w_A - w_B + b} \mathrm{d}G(a)\,\mathrm{d}G(b) + \int_{w_B}^{\bar{v}}\int_0^{w_A} \mathrm{d}G(a)\,\mathrm{d}G(b)\right](v_A - w_A) \\ &\quad + \left[\int_0^{w_B}\int_{w_A - w_B + b}^{\bar{v}} \mathrm{d}G(a)\,\mathrm{d}G(b)\right](v_B - w_B)\end{aligned}$$

Consider the line segment $\{w_A \in [0, v_A], w_B = 0\}$. Trivially, the local optimum is at $({w_A}^*, 0)$, with ${w_A}^* = v_A - \frac{G({w_A}^*)}{g({w_A}^*)}$, i.e. bidder F's optimal bid in a single first-price

auction. However, this cannot be a global optimum because of Lemma 2. For the line segment $\{w_A = 0, w_B \in [0, v_B]\}$ we obtain $\Pi^F_{P1|v_A,v_B}(0, {w_B}^*) = G({w_B}^*)(v_B - {w_B}^*)$. For the line segment $\{w_A = v_A, w_B \in [0, v_B]\}$ we can also derive a first-order condition characterising a local maximum, denoted by $\widetilde{{w_B}^*}$ (clearly $(v_A, 0)$ and (v_A, v_B) are not maximal on this line segment). Similarly, we find a local maximum $\widetilde{{w_A}^*}$ on the line segment $\{w_A \in [0, v_A], w_B = v_B\}$. We show

$$\Pi^F_{P1|v_A,v_B}(v_A - \epsilon, \widetilde{{w_B}^*}) > \Pi^F_{P1|v_A,v_B}(v_A, \widetilde{{w_B}^*}) \tag{A.9}$$

$$\Pi^F_{P1|v_A,v_B}(\widetilde{{w_A}^*}, v_B - \epsilon) > \Pi^F_{P1|v_A,v_B}(\widetilde{{w_A}^*}, v_B) \tag{A.10}$$

$$\Pi^F_{P1|v_A,v_B}(\epsilon, {w_B}^*) > \Pi^F_{P1|v_A,v_B}(0, {w_B}^*) \tag{A.11}$$

To show (A.9), we simply graph the areas describing the probabilities of winning good A and good B respectively and consider an small change in one of the bids. In Figure A.1, bidder F receives a positive payoff only in the lighter area. With an small deviation ϵ this area remains unchanged while at the same time a strictly positive payoff is gained in the dark area below the dashed line, and therefore (A.9) holds. Figure A.2 relates analogously to equation (A.10) (positive payoff is initially made only in the darker area).

Finally, we show equation (A.11). In Figure A.3, the ϵ-deviation causes the probability of winning good B to decrease (light area shrinks by dark triangle), and the probability of winning good A to increase (dark triangle plus dark rectangle). The profits made on good A and good B are $v_A - \epsilon$ and $v_B - \widetilde{{w_B}^*}$, respectively. From the graph, it is easy to see that we can bound the changes in expected profits on good K, denoted by $\Delta\Pi^F_K$ for $K = A, B$, as follows:

$$\begin{aligned} \Delta\Pi^F_A &> (v_A - \epsilon)P(a < \epsilon)P(b > \widetilde{{w_B}^*}) \\ \Delta\Pi^F_B &< (v_B - \widetilde{{w_B}^*})P(a < \epsilon)P(b \in [\widetilde{{w_B}^*}, \widetilde{{w_B}^*} - \epsilon]) \end{aligned}$$

Thus, we have $\Delta\Pi^F_A - \Delta\Pi^F_B > 0$ for ϵ small. □

A.1.2 Sequential auctions

Proof of Proposition 3. (Good A sold first)
After having won the A-auction, bidder F's expected payoff function in the B-auction is

$$\begin{aligned} \Pi^{F|\text{won A}}_{SeqA2|v_A,v_B}(w_B) =& P\left(\text{F wins B}\right)\left(v_B - \text{E}[b \mid \text{F wins B}]\right) + P(\text{F loses B})v_A - a \\ =& G(w_B)v_B - \int_0^{w_B} b \, \mathrm{d}G(b) + (1 - G(w_B))v_A - a \end{aligned}$$

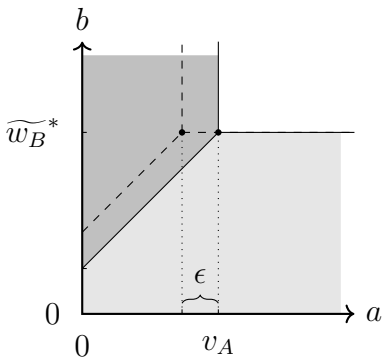


Figure A.1: ϵ-deviation from $(v_A, \widetilde{w_B}^*)$

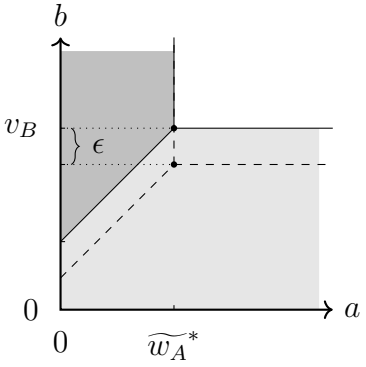


Figure A.2: ϵ-deviation from $(\widetilde{w_A}^*, v_B)$

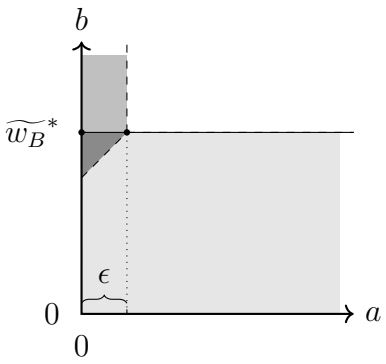


Figure A.3: ϵ-deviation from $(0, \widetilde{w_B}^*)$

The first-order condition yields ${w_B}^* = v_B - v_A$. The second-order condition $-g({w_B}^*) < 0$ is sufficient for a local maximum, and global optimality can be shown by an ϵ-deviation from $w_B = v_B$ and $w_B = 0$. Bidder F's conditional expected second stage payoff is $\Pi^{F|\text{won A}}_{SeqA2|v_A,v_B}({w_B}^*) = \int_0^{v_B - v_A} G(b)\,\mathrm{d}b + v_A - a$. After having lost the A-auction, bidder F bids her true valuation, i.e. $\overline{w_B}^* = v_B$ in the B-auction with a conditional expected second stage payoff of $\Pi^{F|\text{lost A}}_{SeqA2|v_A,v_B}(\overline{w_B}^*) = \int_0^{v_B} G(b)\,\mathrm{d}b$. Now we can analyse the first stage auction (A-auction). Bidder F's expected payoff is

$$\begin{aligned}
\Pi^{F}_{SeqA2|v_A,v_B}(w_A) =& P\left(\text{F wins A}\right) \mathrm{E}[\Pi^{F|\text{won A}}_{SeqA2|v_A,v_B}({w_B}^*) \mid \text{F wins A}] \\
&+ P\left(\text{F loses A}\right) \mathrm{E}[\Pi^{F|\text{lost A}}_{SeqA2|v_A,v_B}(\overline{w_B}^*) \mid \text{F loses A}] \\
=& \int_0^{w_A} \left(\int_0^{v_B - v_A} G(b)\,\mathrm{d}b + v_A - a \right) \mathrm{d}G(a) + \int_{w_A}^{\bar{v}} \left(\int_0^{v_B} G(b)\,\mathrm{d}b \right) \mathrm{d}G(a) \\
=& G(w_A) \left(v_A - w_A - \int_{v_B - v_A}^{v_B} G(b)\,\mathrm{d}b \right) + \int_0^{w_A} G(a)\,\mathrm{d}a + \int_0^{v_B} G(b)\,\mathrm{d}b
\end{aligned}$$

The first-order condition yields ${w_A}^* = v_A - \int_{v_B - v_A}^{v_B} G(b)\,\mathrm{d}b$. It is not hard to see that the RHS is always weakly greater than zero, and local second-order conditions and global optimality are satisfied. Bidder F bids her true first stage valuation for the good, that is the difference in the expected second stage payoffs conditional on winning or losing in the first auction, respectively, and not accounting for the sunk cost a. □

Proof of Proposition 4. (Good B is sold first)
After having won the B-auction, the flexible bidder cannot improve on her allocation anymore, so trivially ${w_A}^* = 0$. After having lost the B-auction, bidder F simply bids her true valuation, i.e. $\overline{w_A}^* = v_A$ in the A-auction with a conditional expected second stage payoff of $\Pi^{F|\text{lost B}}_{SeqB2|v_A,v_B}(\overline{w_A}^*) = \int_0^{v_A} G(a)\,\mathrm{d}a$. Now we can analyse the

first stage auction (B-auction). Bidder F's payoff is

$$\begin{aligned}\Pi^{F}_{SeqB2|v_A,v_B}(w_B) =& P\left(\text{F wins B}\right)\mathrm{E}[\Pi^{F|\text{won B}}_{SeqB2|v_A,v_B}(w_A{}^*) \mid \text{F wins B}]\\ &+P\left(\text{F loses B}\right)\mathrm{E}[\Pi^{F|\text{lost B}}_{SeqB2|v_A,v_B}(\overline{w_A}{}^*) \mid \text{F loses B}]\\ =&\int_0^{w_B}(v_B-b)\,\mathrm{d}G(b)+\int_{w_B}^{\bar{v}}\int_0^{v_A}G(a)\,\mathrm{d}a\,\mathrm{d}G(b)\\ =&G(w_B)\left(v_B-w_B-\int_0^{v_A}G(a)\,\mathrm{d}a\right)+\int_0^{w_B}G(b)\,\mathrm{d}b+\int_0^{v_A}G(a)\,\mathrm{d}a\end{aligned}$$

The first-order condition yields ${w_B}^* = v_B - \int_0^{v_A} G(a)\,\mathrm{d}a$. The RHS is always weakly greater than zero, and local second-order condition and global optimality are satisfied. Again, the flexible bidder bids her true first stage valuation for the good, that is the difference in the expected second stage payoffs conditional on winning or losing in the first auction, respectively, and not accounting for the sunk cost b. □

Proof of Proposition 5. (Good A is sold first)
After having won the A-auction, bidder F's expected payoff function in the B-Auction is

$$\begin{aligned}\Pi^{F|\text{won A}}_{SeqA1|v_A,v_B}(w_B) =&P\left(\text{F wins B}\right)(v_B-w_B)+P(\text{F loses B})v_A-w_A\\ =&G(w_B)\left(v_B-w_B\right)+(1-G(w_B))v_A-w_A\end{aligned}$$

The first-order condition yields ${w_B}^*+\frac{G({w_B}^*)}{g({w_B}^*)}=v_B-v_A$, the RHS of which is strictly positive. The function $H(x) := x+\frac{G(x)}{g(x)}$ is defined on $[0, v_B]$ and continuous, with $H(0)=0$ and $H(v_B)>v_B$. By the intermediate value theorem and Assumption 1 (weakly decreasing reverse hazard rate), a solution $0<{w_B}^*<v_B$ must exists, is unique, and is characterised by the first-order condition. Bidder F's conditional expected second stage payoff is $\Pi^{F|\text{won A}}_{SeqA1|v_A,v_B}({w_B}^*) = G({w_B}^*)(v_B-{w_B}^*-v_A)+v_A-w_A$. After having lost the A-auction, bidder F's expected payoff function in the B-Auction is

$$\begin{aligned}\Pi^{F|\text{lost A}}_{SeqA1|v_A,v_B}(w_B) =&P\left(\text{F wins B}\right)(v_B-w_B)\\ =&G(w_B)\left(v_B-w_B\right)\end{aligned}$$

Bidder F's bid then corresponds to the optimal bid in a single good first-price auction, i.e. $\overline{w_B}{}^* = v_B-\frac{G(\overline{w_B}{}^*)}{g(\overline{w_B}{}^*)}$. With the same argument as above, the unique solution must satisfy $0<\overline{w_B}{}^*<v_B$. Her conditional expected second stage

payoff of $\Pi^{F|\text{lost J}}_{SeqA1|v_A,v_B}(\overline{w_B}^*) = G(\overline{w_B}^*)(v_B - \overline{w_B}^*)$. In the first stage auction (A-auction), bidder F's payoff is then

$$\begin{aligned}\Pi^F_{SeqA1|v_A,v_B}(w_A) =& P\,(\text{F wins A})\,\mathrm{E}[\Pi^{F|\text{won A}}_{SeqA1|v_A,v_B}(w_B{}^*) \mid \text{F wins A}]\\ &+P\,(\text{F loses A})\,\mathrm{E}[\Pi^{F|\text{lost A}}_{SeqA1|v_A,v_B}(\overline{w_B}^*) \mid \text{F loses A}]\\ =&G(w_A)\left[G(w_B{}^*)(v_B - w_B{}^* - v_A) + v_A - w_A\right]\\ &+(1-G(w_A))G(\overline{w_B}^*)(v_B - \overline{w_B}^*)\end{aligned}$$

The first-order condition yields

$$w_A{}^* + \frac{G(w_A{}^*)}{g(w_A{}^*)} = G(w_B{}^*)(v_B - w_B{}^* - v_A) + v_A - G(\overline{w_B}^*)(v_B - \overline{w_B}^*) \quad \text{(A.12)}$$

$$\Leftrightarrow \quad w_A{}^* + \frac{G(w_A{}^*)}{g(w_A{}^*)} = \frac{G(w_B{}^*)^2}{g(w_B{}^*)} - \frac{G(\overline{w_B}^*)^2}{g(\overline{w_B}^*)} + v_A \quad \text{(A.13)}$$

$H(x) := \frac{G(x)^2}{g(x)}$ is an increasing function because of Assumption 1. From above, it is clear that $w_B{}^* < \overline{w_B}^*$, so the RHS of equation A.13 is weakly smaller than v_A. $\widetilde{G}(x) := x + \frac{G(x)}{g(x)}$ is defined on $[0, v_A]$, continuous, with $\widetilde{G}(0) = 0$ and $\widetilde{G}(v_A) > v_A$. So by the intermediate value theorem and Assumption 1, the unique solution $w_A{}^*$ must satisfy $w_A{}^* < v_A$. If the RHS of equation A.13 is negative, however, the FOC does not yield a solution. In that case, the optimum must lie on the boundary of $[0, v_A]$. In a standard first-price auction, both alternatives, bidding zero and your value yield zero profit. Here, however, bidding zero is strictly better than bidding your value, because the first-stage payment may be a sunk cost in case bidder F wins in the second stage. Formally, we show the following lemma.

Lemma 15. *$w_A{}^* = v_A$ is never optimal as bidder F's first stage auction bid.*

Proof.

$$\begin{aligned}&\Pi^F_{SeqA1|v_A,v_B}(v_A - \epsilon) =\\ &G(v_A - \epsilon)\left[G(w_B{}^*)(v_B - w_B{}^* - v_A) + \epsilon - G(\overline{w_B}^*)(v_B - \overline{w_B}^*)\right]\\ &+G(\overline{w_B}^*)(v_B - \overline{w_B}^*)\\ =&G(v_A - \epsilon)\left[\frac{G(w_B{}^*)^2}{g(w_B{}^*)} + \epsilon - \frac{G(\overline{w_B}^*)^2}{g(\overline{w_B}^*)}\right] + G(\overline{w_B}^*)(v_B - \overline{w_B}^*)\\ >&G(v_A)\left[\frac{G(w_B{}^*)^2}{g(w_B{}^*)} - \frac{G(\overline{w_B}^*)^2}{g(\overline{w_B}^*)}\right] + G(\overline{w_B}^*)(v_B - \overline{w_B}^*)\\ =&G(v_A)\left[G(w_B{}^*)(v_B - w_B{}^* - v_A) - G(\overline{w_B}^*)(v_B - \overline{w_B}^*)\right]\\ &+G(\overline{w_B}^*)(v_B - \overline{w_B}^*) = \Pi^F_{SeqA1|v_A,v_B}(v_A)\end{aligned}$$

for small ϵ. The inequality holds because $\overline{w_B}^* > w_B{}^*$ and the reverse hazard rate is weakly decreasing. □

Together with equation (A.12) it is not hard to see that the following holds: fix v_B and let $\widehat{v_A} := G(\overline{w_B}^*(\widehat{v_A}, v_B))(v_B - \overline{w_B}^*(\widehat{v_A}, v_B)) - G(w_B{}^*(\widehat{v_A}, v_B))(v_B - w_B{}^*(\widehat{v_A}, v_B) - v_A)$. Then for all $v_A \leq \widehat{v_A}$, $w_A^* = 0$ (followed by $\overline{w_B}^*$ in the second stage auction). □

Proof of Proposition 6. (Good B sold first)
After having won the B-auction, bidder F's expected payoff function in the A-Auction is

$$\begin{aligned}\Pi^{F|\text{won B}}_{SeqB1|v_A,v_B}(w_A) =& P\left(\text{F wins A}\right)(v_B - w_A) + P(\text{F loses A})v_B - w_B \\ =& G(w_A)\left(v_B - w_A\right) + (1 - G(w_A))v_B - w_B\end{aligned}$$

Clearly bidder F cannot improve upon her payoff by bidding again in the A-auction after having won the B-auction, so $w_A{}^* = 0$. After having lost the B-auction, bidder F's bid corresponds to the optimal bid in a single good first-price auction, i.e. $\overline{w_A}^* = v_A - \frac{G(\overline{w_A}^*)}{g(\overline{w_A}^*)}$. Again, the unique solution must satisfy $0 < \overline{w_A}^* < v_A$. Her conditional expected second stage payoff of $\Pi^{F|\text{lost B}}_{SeqB1|v_A,v_B}(\overline{w_A}^*) = G(\overline{w_A}^*)(v_A - \overline{w_A}^*)$. In the first stage auction (B-auction), bidder F's payoff is then

$$\begin{aligned}\Pi^{F}_{SeqB1|v_A,v_B}(w_B) =& P\left(\text{F wins B}\right)\mathrm{E}[\Pi^{F|\text{won B}}_{SeqB1|v_B,v_A}(w_A{}^*) \mid \text{F wins B}] \\ &+ P\left(\text{F loses B}\right)\mathrm{E}[\Pi^{F|\text{lost B}}_{SeqB1|v_B,v_A}(\overline{w_A}^*) \mid \text{F loses B}] \\ =& G(w_B)\left[v_B - w_B\right] + (1 - G(w_B))G(\overline{w_A}^*)(v_A - \overline{w_A}^*)\end{aligned}$$

The first-order condition yields

$$w_B{}^* + \frac{G(w_B{}^*)}{g(w_B{}^*)} = v_B - G(\overline{w_A}^*)(v_A - \overline{w_A}^*) \tag{A.14}$$

With the same reasoning as above, a unique solution $w_B{}^*$ exists and must satisfy $0 < w_B{}^* < v_B$. □

A.1.3 Simultaneous auctions

Proof of Proposition 7. Using simply the probabilities of winning and expected payments, we can define bidder F's expected payoff from bidding $w_A, w_B \geq 0$ in the A-auction and B-auction respectively as

$$\begin{aligned}\Pi^{F}_{Sim2|v_A,v_B}(w_A, w_B) =& P\left(\text{F wins A, not B}\right)\mathrm{E}[v_A - a \mid \text{F wins A}] \\ &+ P\left(\text{F wins B, not A}\right)\mathrm{E}[v_B - b \mid \text{F wins B}] \\ &+ P\left(\text{F wins A and B}\right)\mathrm{E}[v_B - a - b \mid \text{F wins A and B}] \\ =& G(w_A)\left(1 - G(w_B)\right)\left(v_A - \mathrm{E}[a \mid w_A]\right) \\ &+ G(w_B)\left(1 - G(w_A)\right)\left(v_B - \mathrm{E}[b \mid w_B]\right) \\ &+ G(w_A)G(w_B)\left(v_B - \mathrm{E}[a \mid w_A] - \mathrm{E}[b \mid w_B]\right)\end{aligned}$$

with $\mathrm{E}[x \mid w] = G(w)^{-1} \int_0^w x \, \mathrm{d}G(x)$ (the expected price conditional having won the good with a bid of $w > 0$). The payoff function can be simplified as shown in the main text and any stationary point of $\Pi^F_{Sim2|v_A,v_B}$ is characterised by

$$\boldsymbol{\nabla}\Pi^F_{Sim2|v_A,v_B}(w_A, w_B) = \begin{pmatrix} g(w_A)\,[v_A - w_A] - g(w_A)G(w_B)v_A \\ g(w_B)\,[v_B - w_B] - g(w_B)G(w_A)v_A \end{pmatrix} = 0, \tag{A.15}$$

which can be simplified to obtain

$$w_A{}^* = v_A(1 - G(w_B{}^*)) \tag{A.16}$$

$$w_B{}^* = v_B - v_A G(w_A{}^*) \tag{A.17}$$

We prove that the necessary conditions are indeed sufficient to characterise the global maximum of bidder F's maximisation problem. Instead of a second-order approach, we argue by eliminating other equilibrium candidates on the boundary of bidder F's choice set. Note that the objective function is continuous and the admissible range a compact set. Therefore, the maximum is obtained on $\mathcal{M} := [0, v_A] \times [0, v_B]$. This can either happen at the boundary or at stationary point of $\Pi^F_{Sim2|v_A,v_B}$. Below, we demonstrate that no point on the boundary of $\mathcal{M}$ can be optimal, and therefore, the unique stationary point must coincide with the global maximum. We transform bidder F's payoff function once more with integration by parts and obtain

$$\begin{aligned}\Pi^F_{Sim2|v_A,v_B}(w_A, w_B) =& G(w_A)\,[v_A - w_A] + \int_0^{w_A} G(a)\,\mathrm{d}a \\ &+ G(w_B)\,[v_B - w_B] + \int_0^{w_B} G(b)\,\mathrm{d}b - G(w_A)G(w_B)v_A\end{aligned}$$

First, consider the line segment $\{w_A \in [0, v_A], w_B = 0\}$. Trivially, all points in this set are payoff-dominated by the point $(v_A, 0)$. We obtain $\Pi^F_{Sim2|v_A,v_B}(v_A, 0) = \mathcal{F}(v_A)$. Similarly, for the line segment $\{w_A = 0, w_B \in [0, v_B]\}$ we obtain $\Pi^F_{Sim2|v_A,v_B}(0, v_B) = \mathcal{F}(v_B)$. Now consider the set $\{w_A = v_A, w_B \in [0, v_B]\}$. From the first-order condition on w_B and $\frac{\partial^2 \Pi^F_{Sim2|v_A,v_B}}{\partial {w_B}^2}$ we find and confirm a local maximum at $\widetilde{w_B}^* = v_B - v_A G(v_A)$. We have $\Pi^F_{Sim2|v_A,v_B}(v_A, \widetilde{w_B}^*) = \mathcal{F}(v_A) + \mathcal{F}(\widetilde{w_B}^*)$. Similarly, in the set $\{w_A \in [0, v_A], w_B = v_B\}$, we find a local maximum at $\widetilde{w_A}^* = v_A - v_A G(v_B)$ with $\Pi^F_{Sim2|v_A,v_B}(\widetilde{w_A}^*, v_B) = \mathcal{F}(\widetilde{w_A}^*) + \mathcal{F}(v_B)$. Finally, we have to consider the point (v_A, v_B) with $\Pi^F_{Sim2|v_A,v_B}(v_A, v_B) = \mathcal{F}(v_A) + \mathcal{F}(v_B) - v_A G(v_A)G(v_B)$. $(v_A, 0)$ and $(0, v_B)$ are obviously payoff-dominated.

We proceed to show

$$\Pi^F_{Sim2|v_A,v_B}(v_A - \epsilon, \widetilde{w_B}^*) > \Pi^F_{Sim2|v_A,v_B}(v_A, \widetilde{w_B}^*) \tag{A.18}$$

$$\Pi^F_{Sim2|v_A,v_B}(\widetilde{w_A}^*, v_B - \epsilon) > \Pi^F_{Sim2|v_A,v_B}(\widetilde{w_A}^*, v_B) \tag{A.19}$$

$$\Pi^F_{Sim2|v_A,v_B}(v_A - \epsilon, v_B) > \Pi^F_{Sim2|v_A,v_B}(v_A, v_B) \tag{A.20}$$

To show (A.18) to (A.20) we use linear and quadratic approximations of $G(v_k)$ and $g(v_k)$ at $v_k - \epsilon, k = A$ or B, respectively. We have

$$\begin{aligned}
\Pi^F_{Sim2|v_A,v_B}(v_A-\epsilon,\widetilde{w_B}^*) =& G(v_A-\epsilon)\epsilon + \int_0^{v_A-\epsilon} G(a)\,\mathrm{d}a + \int_0^{\widetilde{w_B}^*} G(b)\,\mathrm{d}b \\
&+ G(\widetilde{w_B}^*)v_A G(v_A) - G(\widetilde{w_B}^*)v_A G(v_A-\epsilon) \\
=& G(v_A-\epsilon)\epsilon + \int_0^{v_A-\epsilon} G(a)\,\mathrm{d}a + \int_0^{\widetilde{w_B}^*} G(b)\,\mathrm{d}b \\
&+ G(\widetilde{w_B}^*)v_A\left[G(v_A-\epsilon) + g(v_A-\epsilon)\epsilon + o(\epsilon^2) - G(v_A-\epsilon)\right] \\
>& \int_0^{\widetilde{w_B}^*} G(b)\,\mathrm{d}b + \int_0^{v_A-\epsilon} G(a)\,\mathrm{d}a + G(v_A-\epsilon)\epsilon + \frac{\epsilon^2}{2}g(v_A-\epsilon) \\
&+ o(\epsilon^3) \\
=& \Pi^F_{Sim2|v_A,v_B}(v_A,\widetilde{w_B}^*)
\end{aligned} \tag{A.21}$$

$$\begin{aligned}
\Pi^F_{Sim2|v_A,v_B}(v_A-\epsilon,v_B) =& G(v_A-\epsilon)\epsilon + \int_0^{v_A-\epsilon} G(a)\,\mathrm{d}a \\
&+ \int_0^{v_B} G(b)\,\mathrm{d}b - G(v_A-\epsilon)G(v_B)v_A \\
>& \int_0^{v_B} G(b)\,\mathrm{d}b + \int_0^{v_A-\epsilon} G(a)\,\mathrm{d}a + G(v_A-\epsilon)\epsilon + \frac{\epsilon^2}{2}g(v_A-\epsilon) \\
&+ o(\epsilon^3) \\
&- v_A G(v_B)\left[G(v_A-\epsilon) + g(v_A-\epsilon)\epsilon + o(\epsilon^2)\right] \\
=& \Pi^F_{Sim2|v_A,v_B}(v_A,v_B)
\end{aligned} \tag{A.22}$$

Showing (A.19) goes analogous to (A.21). Equations (A.21) and (A.22) hold for small deviations ϵ, hence, the global maximum cannot be obtained at the boundary of $\mathcal{M}$. □

Proof of Proposition 8. The flexible bidder's expected payoff from bidding $w_A, w_B \geq 0$ in the A-auction and B-auction is

$$\begin{aligned}
\Pi^F_{Sim1|v_A,v_A}(w_A,w_B) =& P\left(\text{F wins A, not B}\right)(v_A - w_A) \\
&+ P\left(\text{F wins B, not A}\right)(v_B - w_B) \\
&+ P\left(\text{F wins A and B}\right)(v_B - w_A - w_B) \\
=& G(w_A)\left(1 - G(w_B)\right)(v_A - w_A) \\
&+ G(w_B)\left(1 - G(w_A)\right)(v_B - w_B) \\
&+ G(w_A)G(w_B)\left(v_B - w_A - w_B\right)
\end{aligned}$$

The first-order conditions are

$$\boldsymbol{\nabla}\Pi^F_{Sim1|v_A,v_A}(w_A,w_B)=\begin{pmatrix} g(w_A)\left[v_A-w_A-G(w_B)v_A\right]-G(w_A) \\ g(w_B)\left[v_B-w_B-G(w_A)v_A\right]-G(w_B)\end{pmatrix}=0, \quad \text{(A.23)}$$

and can be simplified to

$$w_A{}^*=v_A(1-G(w_B{}^*))-\frac{G(w_A{}^*)}{g(w_A{}^*)} \quad \text{(A.24)}$$

$$w_B{}^*=v_B-v_AG(w_A{}^*)-\frac{G(w_B{}^*)}{g(w_B{}^*)} \quad \text{(A.25)}$$

Again, we prove that none of the points on the boundary of bidder F's choice set $\mathcal{M}:=[0,v_A]\times[0,v_B]$ is optimal, and hence the unique stationary point must coincide with the global maximum.

Consider the line segment $\{w_A\in[0,v_A],w_B=0\}$. The local optimum is at $(w_A{}^*,0)$, with $w_A{}^*=v_A-\frac{G(w_A{}^*)}{g(w_A{}^*)}$ (bidder F's optimal bid in a single first-price auction) and payoff $\Pi^F_{P1|v_A,v_B}(0,w_A{}^*)=\frac{G(w_A{}^*)^2}{g(w_A{}^*)}$. Similarly, for the line segment $\{w_A=0,w_B\in[0,v_B]\}$ we obtain $\Pi^F_{P1|v_A,v_B}(0,w_B{}^*)=\frac{G(w_B{}^*)^2}{g(w_B{}^*)}$. Hence the first local optimum is dominated by the latter. On the line segment $\{w_A=v_A,w_B\in[0,v_B]\}$ the first-order condition characterises a unique stationary point $\widetilde{w_B}{}^*$. Bidder F's payoff at this stationary point is $\Pi^F_{P1|v_A,v_B}(v_A,\widetilde{w_B}{}^*)=\frac{G(\widetilde{w_B}{}^*)^2}{g(\widetilde{w_B}{}^*)}$, which is dominated by $(0,w_B{}^*)$ because $\widetilde{w_B}{}^*<w_B{}^*$. Similarly, we find the stationary point $\widetilde{w_A}{}^*$ on the line segment $\{w_A\in[0,v_A],w_B=v_B\}$ also to be dominated by $(0,w_B{}^*)$. Therefore, we only show

$$\Pi^F(\epsilon,w_B{}^*) \quad > \quad \Pi^F(0,w_B{}^*) \quad \text{(A.26)}$$

We have

$$\begin{aligned}\Pi^F(\epsilon,w_B{}^*)=&G(w_B{}^*)(v_B-w_B{}^*)+G(\epsilon)(v_A(1-G(w_B{}^*))-\epsilon)\\ >&G(w_B{}^*)(v_B-w_B{}^*)=\Pi^F(0,w_B{}^*)\end{aligned} \quad \text{(A.27)}$$

(A.27) holds for small deviations ϵ. Overall, the global maximum cannot be obtained at the boundary of $\mathcal{M}$. □

Proof of Proposition 9. First we show the result for the second-price rule. Note that if G is convex or concave and weakly increasing, the flat parts of the function can only occur at the left or right end of the interval $[0,\overline{v}]$, and include the endpoints respectively. Thus, the G can be restricted to a strictly increasing function on a subset of $[0,\overline{v}]$. For simplicity we argue only for the case where G is

strictly increasing on $[0, \bar{v}]$. The inverse of G is defined on this interval and denoted G^{-1}. We substitute (A.16) into (A.17) and obtain

$$G^{-1}\left(\frac{v_B - {w_B}^*}{v_A}\right) = v_A(1 - G({w_B}^*))$$

Let $g({w_B}^*) := G^{-1}\left(\frac{v_B - {w_B}^*}{v_A}\right)$ and $h({w_B}^*) := v_A(1 - G({w_B}^*))$. g and h are both strictly decreasing functions and from standard composition rules it follows that whenever G is convex (concave), g and h are concave (convex). Two single valued convex (concave) functions may intersect at most twice on a given interval $[\underline{x}, \bar{x}]$. If they intersect twice, the ordering of the functions' values has to be the same at $\underline{x}$ and $\bar{x}$. g is defined on $G := [v_B - v_A, v_B]$ and h is defined on $H := [0, \bar{v}]$. Trivially, $G \subset H$. Because $g(v_B - v_A) = \bar{v} > v_A > h(v_B - v_A)$ and $g(v_B) = 0 = h(\bar{v}) < h(v_B)$, g and h intersect exactly once on $[v_B - v_A, v_B]$. Therefore, ${w_B}^*$ is uniquely defined, and so is ${w_A}^*$ by (A.16).

Under the first-price rule, the argument is similar. Remember that the reverse hazard rate is assumed to be weakly decreasing. As before we can restrict G to be a strictly increasing function on a subset of $[0, \bar{v}]$ and denote the inverse G^{-1} We simply define $\lambda(x) := x + \frac{G(x)}{g(x)}$, which is a strictly increasing function with inverse λ^{-1}. Substituting (A.24) into (A.25) we obtain

$$G^{-1}\left(\frac{v_B - \lambda({w_B}^*)}{v_A}\right) = \lambda^{-1}\left(v_A(1 - G({w_B}^*))\right) \tag{A.28}$$

Again, define the LHS of (A.28) as $g({w_B}^*)$ and the RHS as $h({w_B}^*)$. Further note that $\lambda^{-1}(v_B) < v_B < \bar{v}$ because λ^{-1} is below the 45-degree line. Then, by the same argument as before, g and h intersect exactly once on $[\lambda^{-1}(v_B - v_A), \lambda^{-1}(v_B)]$. The proof the second-price rule contains the full argument for $\lambda(x) = x$. □

A.2 Proofs for the general model

A.2.1 Standard PMA

Proof of Proposition 10. The flexible bidder's preference for her ex-post allocation coincides with the efficient allocation: she wants to win good j if and only if

$$v_j > x_j \qquad \text{and} \tag{A.29}$$

$$v_j - x_j = \max_{k \in \mathcal{J}} v_k - x_k \tag{A.30}$$

Suppose a realisation of $x_k, k = 1, .., M$ is such that bidder F wins good j. Then $w_j - x_j > w_k - x_k \ \forall k \in \mathcal{J}$ by the allocation rule. Substituting the optimal bids for

w_j and w_k, we obtain $v_j - x_j > v_k - x_k \; \forall k \in \mathcal{J}$, i.e. bidder F prefers to win good j. As bidder F wins for every realisation precisely what would be best for her to win, no profitable deviation is possible. Note that when $v_M - v_j \geq \bar{v}$, bidder F always wants to win good M because the maximal difference in payments is $\bar{v}$. In these cases, it may be optimal to bid $w_j{}^* = 0$ for some $j \leq \tilde{j}, j \neq M$, if $w_M{}^*$ is chosen not too large. □

A.2.2 First-price PMA

Proof of Lemma 4. We note again equation 2.4, which is key to this lemma, together with the fact that for any two goods $k, l \in \mathcal{J}$, x_k and x_l are *iid* distributed.

$$P_k(\mathbf{w}) = Prob(w_k > x_k \text{ and } w_k - x_k = \max_{j \in \mathcal{J}} w_j - x_j) \tag{A.31}$$

Because of the above equation, and because x_k and x_l are *iid*, we have

$$\begin{aligned} &P_k(w_1, ..., w_{l-1}, w_l, w_{l+1}, ..., w_{k-1}, w_k, w_{k+1}, ...w_M) \\ =&P_l(w_1, ..., w_{l-1}, w_k, w_{l+1}, ..., w_{k-1}, w_l, w_{k+1}, ...w_M) \end{aligned}$$

Now suppose $w_l > w_k$ for some $l < k$ (so $v_l \leq v_k$). Then we can easily show that bidding $\widetilde{w_l} = w_k$ and $\widetilde{w_k} = w_l$ is a profitable deviation. Let $\tilde{\mathbf{w}}$ denote the vector $(w_1, ..., w_M)$ where w_k is substituted by $\widetilde{w_k}$ and w_l is substituted by $\widetilde{w_l}$. Let $\mathbf{w}'$ denote the vector $(w_1, ..., w_M)$ where w_k is swapped with w_l.

$$\begin{aligned} \Pi^F(\tilde{\mathbf{w}}) &= \sum_{j \neq k,l} P_j(\tilde{\mathbf{w}})(v_j - w_j) + P_k(\tilde{\mathbf{w}})(v_k - \widetilde{w_k}) + P_l(\tilde{\mathbf{w}})(v_l - \widetilde{w_l}) \\ &= \sum_{j \neq k,l} P_j(\tilde{\mathbf{w}})(v_j - w_j) + P_k(\mathbf{w}')(v_k - w_l) + P_l(\mathbf{w}')(v_l - w_k) \\ &= \sum_{j \neq k,l} P_j(\tilde{\mathbf{w}})(v_j - w_j) + P_k(\mathbf{w})(v_l - w_k) + P_l(\mathbf{w})(v_k - w_l) \\ &> \sum_{j \neq k,l} P_j(\tilde{\mathbf{w}})(v_j - w_j) + P_k(\mathbf{w})(v_k - w_k) + P_l(\mathbf{w})(v_l - w_l) \end{aligned}$$

The last inequality holds because $P_l(\mathbf{w}) > P_k(\mathbf{w})$ for $w_l > w_k$ (this follows immediately from equation (A.31). The procedure to find profitable deviations can be applied repeatedly to restore the order $w_M \geq w_{M-1} \geq ... \geq w_1$. □

Proof of Proposition 11. To prove that bidder F's maximisation problem possesses indeed an interior solution, we simply show that the maximum cannot be obtained on the boundary of $[0, v_1] \times ... \times [0, v_M]$, and therefore must be a stationary point of $\Pi^F_{P1|\mathbf{v}}$, characterised by the system of first-order conditions $\boldsymbol{\nabla} \Pi^F_{P1|\mathbf{v}} = 0$.

Consider the border surfaces of $\mathcal{C}_M$. These surfaces are q-dimensional hyperplanes, with $q \in \{1, ..., M-1\}$. They can be described by vectors $(w_1, ..., w_M)$ where the j^{th} entry may be substituted by a constant $c \in \{0, v_j\}$, and overall $M - q$ elements of $\mathbf{w}$ are substituted when describing a q-dimensional hyperplane.

It is sufficient to consider $(M-1)$-dimensional hyperplanes because k-dimensional border surfaces are encompassed in $k+1$-dimensional border surfaces. If we do not put restrictions on the elements of $\mathbf{w}_{-j}$, which are not substituted by a constant, it is also sufficient to consider points on the hyperplane in which only the j^{th} entry is substituted by a constant $c \in \{0, v_j\}$.

First, let $c = v_j$. In this is case, it is obvious that an ϵ-deviation orthogonal to the hyperplane increases the flexible bidder's profits, i.e.

$$\Pi^F(w_1, ..., w_{j-1}, v_j - \epsilon, w_{j+1}, ..., w_M) > \Pi^F(w_1, ..., w_{j-1}, v_j, w_{j+1}, ..., w_M)$$

The flexible bidder makes zero profit on good j if she wins, so the deviation strictly increases her profits in those cases where she wins, while at the same time the probability of winning another good does not decrease. More formally, note that $\frac{dP_k(\mathbf{w})}{dw_j} < 0$ because $P_k(\mathbf{w}) = Prob(w_k > x_k \text{ and } w_k - x_k = \max_{j \in \mathcal{J}} w_j - x_j)$. Then, it is immediate that

$$\begin{aligned}
&\Pi^F_{P1|\mathbf{v}}(w_1, ..., w_{j-1}, v_j - \epsilon, w_{j+1}, ..., w_M) \\
&= \sum_{k \neq j} P_k(w_1, ..., w_{j-1}, v_j - \epsilon, w_{j+1}, ..., w_M)\,[v_k - w_k] \\
&+ P_j(w_1, ..., w_{j-1}, v_j - \epsilon, w_{j+1}, ..., w_M)\,[v_j - v_j + \epsilon] \\
&> \sum_{k \neq j} P_k(w_1, ..., w_{j-1}, v_j - \epsilon, w_{j+1}, ..., w_M)\,[v_k - w_k] \\
&\geq \Pi^F(w_1, ..., w_{j-1}, v_j, w_{j+1}, ..., w_M)
\end{aligned}$$

Second, we consider $(M-1)$-dimensional hyperplanes with $w_j = c = 0$. We need to establish that

$$\Pi^F(w_1, ..., w_{j-1}, \epsilon, w_{j+1}, ..., w_M) > \Pi^F(w_1, ..., w_{j-1}, 0, w_{j+1}, ..., w_M)$$

Let

$$\begin{aligned}
&\Delta P_k = [P_k(w_1, ..., w_{j-1}, \epsilon, w_{j+1}, ..., w_M) - P_k(w_1, ..., w_{j-1}, 0, w_{j+1}, ..., w_M) \text{ and} \\
&\Delta P_j = P_j(w_1, ..., w_{j-1}, \epsilon, w_{j+1}, ..., w_M)
\end{aligned}$$

Then

$$\begin{aligned}
\Delta P_k =& Prob(w_k - x_k > 0, w_k - x_k > w_{k'} - x_{k'}\ \forall k' \neq j, w_k - x_k > \epsilon - x_j) \\
& - Prob(w_k - x_k > 0, w_k - x_k > w_{k'} - x_{k'}\ \forall k' \neq j) \\
=& Prob(w_k - x_k > 0, w_k - x_k > w_{k'} - x_{k'}\ \forall k' \neq j, w_k - x_k > \epsilon - x_j, \epsilon - x_j > 0) \\
& + Prob(w_k - x_k > 0, w_k - x_k > w_{k'} - x_{k'}\ \forall k' \neq j, w_k - x_k > \epsilon - x_j, \epsilon - x_j < 0) \\
& - Prob(w_k - x_k > 0, w_k - x_k > w_{k'} - x_{k'}\ \forall k' \neq j) \\
>& Prob(w_k - x_k > \epsilon, w_k - x_k > w_{k'} - x_{k'}\ \forall k' \neq j, \epsilon - x_j > 0) \\
& + Prob(w_k - x_k > 0, w_k - x_k > w_{k'} - x_{k'}\ \forall k' \neq j, \epsilon - x_j < 0) \\
& - Prob(w_k - x_k > 0, w_k - x_k > w_{k'} - x_{k'}\ \forall k' \neq j) \\
=& Prob(w_k - x_k > \epsilon, w_k - x_k > w_{k'} - x_{k'}\ \forall k' \neq j, \epsilon - x_j > 0) \\
& - Prob(w_k - x_k > 0, w_k - x_k > w_{k'} - x_{k'}\ \forall k' \neq j, \epsilon - x_j > 0) \\
=& - Prob(x_j < \epsilon) Prob(x_k \in [w_k, w_k - \epsilon], w_k - x_k > w_{k'} - x_{k'}\ \forall k' \neq j)
\end{aligned}$$

and

$$\begin{aligned}
\Delta P_j =& P_j(w_1, ..., w_{j-1}, \epsilon, w_{j+1}, ..., w_M) \\
& > Prob(x_j < \epsilon) Prob(x_1 > w_1) ... Prob(x_{j-1} > w_{j-1}) Prob(x_{j+1} > w_{j+1}) ... Prob(x_M > w_M)
\end{aligned}$$

Using the above, we obtain

$$\begin{aligned}
& \Pi^F_{P1|\mathbf{v}}(w_1, ..., w_{j-1}, \epsilon, w_{j+1}, ..., w_M) - \Pi^F(w_1, ..., w_{j-1}, 0, w_{j+1}, ..., w_M) \\
=& \sum_{k \neq j} P_k(w_1, ..., w_{j-1}, \epsilon, w_{j+1}, ..., w_M) [v_k - w_k] + P_j(w_1, ..., w_{j-1}, \epsilon, w_{j+1}, ..., w_M) [v_j - \epsilon] \\
-& \sum_{k \neq j} P_k(w_1, ..., w_{j-1}, 0, w_{j+1}, ..., w_M) [v_k - w_k] \\
=& \sum_{k \neq j} [P_k(w_1, ..., w_{j-1}, \epsilon, w_{j+1}, ..., w_M) - P_k(w_1, ..., w_{j-1}, 0, w_{j+1}, ..., w_M)] [v_k - w_k] \\
+& P_j(w_1, ..., w_{j-1}, \epsilon, w_{j+1}, ..., w_M) [v_j - \epsilon] \\
=& \sum_{k \neq j} \Delta P_k [v_k - w_k] + \Delta P_j [v_j - \epsilon] \\
>& Prob(x_j < \epsilon) \Bigg[- \sum_{k \neq j} Prob(x_k \in [w_k, w_k - \epsilon], w_k - x_k > w_{k'} - x_{k'}\ \forall k' \neq j) [v_k - w_k] \\
+& Prob(x_1 > w_1) ... Prob(x_{j-1} > w_{j-1}) Prob(x_{j+1} > w_{j+1}) ... Prob(x_M > w_M) [v_j - \epsilon] \Bigg]
\end{aligned}$$

The last expression is greater than zero for ϵ small. □

A.2.3 Simultaneous second-price auction

Proof of Proposition 12. We need to show than on every border surface of the parallelepiped $\mathcal{C}_M$, Π^F is not maximal. We use the notational convention that $\prod_k^p x_k = 1$ if $k > p$. First, we transform the expected payoff function with integration by parts and obtain

$$\Pi^F(\mathbf{w}) = \sum_{k=1}^{M} \left[G(w_k)[v_k - w_k] + v_k G(w_k) \left(\prod_{l=k+1}^{M} (1 - G(w_l)) - 1 \right) + \int_0^{w_k} G(x)\,\mathrm{d}x \right]$$

Now consider the border surfaces of $\mathcal{C}_M$. These surfaces are q-dimensional hyperplanes, with $q \in \{1, ..., M-1\}$. They can be described by vectors $\mathbf{w} = (w_1, ..., w_M)$ where the jth entry may be substituted by a constant $c \in \{0, v_j\}$, and overall $M - q$ elements of $\mathbf{w}$ are substituted when describing a q-dimensional hyperplane. We consider $(M-1)$-dimensional hyperplanes and substitute *only* the jth entry by a constant $c \in \{0, v_j\}$. This is sufficient because we make no restriction on $\mathbf{w}_{-j}$.

First, let $w_j = c = 0$. To show that the flexible bidder can do better submitting a strictly positive bid, we first show the following lemma.

Lemma 16. *Let $\mathbf{w} = (w_{-j}, w_j = 0)$ be the flexible bidder's bid. Then she can improve her expected payoff by bidding $\mathbf{w}' = (w_{-j,-(j-1)}, w'_{j-1} = 0, w'_j = w_{j-1})$.*

Proof. We have

$$\begin{aligned}
\Pi^F(\mathbf{w}) = &\sum_{k=1}^{j-2} \left[v_k G(w_k) \prod_{l=k+1}^{M} (1 - G(w_l)) - \int_0 w_k x g(x)\,\mathrm{d}x \right] \\
&+ v_{j-1} G(w_{j-1}) \prod_{l=j}^{M} (1 - G(w_l)) - \int_0^{w_{j-1}} x g(x)\,\mathrm{d}x \\
&+ \sum_{k=j+1}^{M} \left[v_k G(w_k) \prod_{l=k+1}^{M} (1 - G(w_l)) - \int_0 w_k x g(x)\,\mathrm{d}x \right] \text{ and} \\
\Pi^F(\mathbf{w}') = &\sum_{k=1}^{j-2} \left[v_k G(w_k) \prod_{l=k+1}^{M} (1 - G(w_l)) - \int_0 w_k x g(x)\,\mathrm{d}x \right] \\
&+ v_j G(w_{j-1}) \prod_{l=j+1}^{M} (1 - G(w_l)) - \int_0^{w_{j-1}} x g(x)\,\mathrm{d}x \\
&+ \sum_{k=j+1}^{M} \left[v_k G(w_k) \prod_{l=k+1}^{M} (1 - G(w_l)) - \int_0 w_k x g(x)\,\mathrm{d}x \right]
\end{aligned}$$

Because of the substitution from $\mathbf{w}$ to $\mathbf{w}'$ the first lines of $\Pi^F(\mathbf{w})$ and $\Pi^F(\mathbf{w}')$ are identical, and trivially the third lines are identical too. Because $w_j = 0$ and $v_j \geq v_{j-1}$, the second line of $\Pi^F(\mathbf{w}')$ is weakly greater than the second line of $\Pi^F(\mathbf{w})$. □

From the lemma above it follows that every bid $(w_1, ..., w_{j-1}, 0, w_{j+1}, ..., w_M)$ can be replaced by a bid $(0, w_1, w_2, ..., w_{j-1}, w_{j+1}, ..., w_M)$. Therefore, we only need to show that $\Pi^F(\widetilde{w_1}, w_2, ..., w_M) > \Pi^F(0, w_2, ..., w_M)$. We simply choose $\widetilde{w_1} = v_1 \prod_{l=2}^{M}(1 - G(w_l))$. Then we have

$$\Pi^F(0, w_2, ..., w_M) = \sum_{k=2}^{M} \left[G(w_k)[v_k - w_k] + v_k G(w_k) \prod_{l=k+1}^{M} (1 - G(w_l)) + \int_0^{w_k} G(x)\,\mathrm{d}x \right]$$

$$\text{and } \Pi^F(\widetilde{w_1}, w_2, ..., w_M) = \int_0^{\widetilde{w_1}} G(x)\,\mathrm{d}x + \sum_{k=2}^{M} \left[G(w_k)[v_k - w_k] + v_k G(w_k) \prod_{l=k+1}^{M} (1 - G(w_l)) + \int_0^{w_k} G(x)\,\mathrm{d}x \right]$$

Now consider hyperplanes where $c = v_j$. Then, we have

$$\Pi^F(w_1, ..., w_{j-1}, v_j, w_{j+1}, ..., w_M) =$$
$$\sum_{k=1}^{j-1} \left[G(w_k)[v_k - w_k] + v_k G(w_k) \left((1 - G(v_j)) \prod_{\substack{l=k+1,\\ l\neq j}}^{M} (1 - G(w_l)) - 1 \right) + \int_0^{w_k} G(x)\,\mathrm{d}x \right]$$
$$+ \sum_{k=j+1}^{M} \left[G(w_k)[v_k - w_k] + v_k G(w_k) \left(\prod_{l=k+1}^{M} (1 - G(w_l)) - 1 \right) + \int_0^{w_k} G(x)\,\mathrm{d}x \right]$$
$$+ v_j G(v_j) \left(\prod_{l=j+1}^{M} (1 - G(w_l)) - 1 \right) + \int_0^{v_j} G(x)\,\mathrm{d}x$$

For an ϵ-deviation orthogonal to the hyperplane we obtain

$$\Pi^F(w_1, ..., w_{j-1}, v_j - \epsilon, w_{j+1}, ..., w_M) =$$
$$\sum_{k=1}^{j-1} \left[G(w_k)[v_k - w_k] + v_k G(w_k) \left((1 - G(v_j - \epsilon)) \prod_{\substack{l=k+1,\\ l\neq j}}^{M} (1 - G(w_l)) - 1 \right) + \int_0^{w_k} G(x)\,\mathrm{d}x \right]$$
$$+ \sum_{k=j+1}^{M} \left[G(w_k)[v_k - w_k] + v_k G(w_k) \left(\prod_{l=k+1}^{M} (1 - G(w_l)) - 1 \right) + \int_0^{w_k} G(x)\,\mathrm{d}x \right]$$
$$+ G(v_j - \epsilon)\epsilon + v_j G(v_j - \epsilon) \left(\prod_{l=j+1}^{M} (1 - G(w_l)) - 1 \right) + \int_0^{v_j - \epsilon} G(x)\,\mathrm{d}x$$

Now define

$$\psi := \sum_{k=1}^{j-1} \left[v_k G(w_k) \prod_{\substack{l=k+1,\\ l\neq j}}^{M} (1 - G(w_l)) \right] - v_j \left(\prod_{l=j+1}^{M} (1 - G(w_l)) - 1 \right).$$

ψ is strictly positive if at least one $w_k, k \in \mathcal{J} \setminus \{j\}$ is strictly positive, and we demonstrated above below that indeed all $w_j, j \in \mathcal{J}$ are strictly positive. Denote the difference in payoffs as $\Delta := \Pi^F(w_1, ..., w_{j-1}, v_j - \epsilon, w_{j+1}, ..., w_M) - \Pi^F(w_1, ..., w_{j-1}, v_j, w_{j+1}, ..., w_M)$. Then we can simplify Δ to obtain

$$\Delta = \psi \left(G(v_j) - G(v_j - \epsilon)\right) + G(v_j - \epsilon)\epsilon + \int_0^{v_j - \epsilon} G(x)\,\mathrm{d}x - \int_0^{v_j} G(x)\,\mathrm{d}x$$

A simple linear approximation of $G(v_j)$ and a quadratic approximation of $\int_0^{v_j} G(x)\,\mathrm{d}x$ (after Taylor) around $v_j - \epsilon$ yields

$$\begin{aligned}\Delta = \psi \Big(G(v_j - \epsilon) + g(v_j - \epsilon)\epsilon + o(\epsilon^2) - G(v_j - \epsilon)\Big) + G(v_j - \epsilon)\epsilon + \int_0^{v_j - \epsilon} G(x)\,\mathrm{d}x \\ - \int_0^{v_j - \epsilon} G(x)\,\mathrm{d}x - G(v_j - \epsilon)\epsilon - g(v_j - \epsilon)\frac{\epsilon^2}{2} - o(\epsilon^3),\end{aligned}$$

which is strictly positive as $\epsilon \to 0$ as long as $\psi > 0$, i.e. bidding truthfully in any of the M simultaneous auction is not optimal if the flexible bidder makes at least one other strictly positive bid for a another good. The continuous function Π^F must attain its maximum on the compact set $\mathcal{C}_M$ by the extreme value theorem. Let $\overline{\mathcal{C}_\mathcal{M}}$ denote the open set defined by $\overline{\mathcal{C}_\mathcal{M}} := (0, v_1) \times ... \times (0, v_M)$. The global maximum must also be a local maximum in an ϵ-neighbourhood on $\overline{\mathcal{C}_\mathcal{M}}$. Since Π^F is also differentiable everywhere on $\overline{\mathcal{C}_\mathcal{M}}$, by Fermat's theorem, the global maximum must be a stationary point of Π^F, i.e. it is characterised by the first-order conditions. □

Calculations for Example 1. Let $M = 2$ and $v_1 = v_2 = v < 1$. We assume a probability distribution with support $[0, 1]$ and a uniform spike around $x \in [0, 1]$, where $x < v$. Let $h := \frac{1-\epsilon+2\epsilon^2}{2\epsilon}$ and $\epsilon < x$. Formally, the probability density function is

$$g(t) = \begin{cases} \epsilon & \text{if} \quad t < x - \epsilon \\ h & \text{if} \quad t \in [x - \epsilon, x + \epsilon) \\ \epsilon & \text{if} \quad t \geq x + \epsilon \end{cases}$$

and the probability distribution function is

$$G(t) = \begin{cases} \epsilon t & \text{if} \quad t < x - \epsilon \\ (\epsilon - h)(x - \epsilon) + ht & \text{if} \quad t \in [x - \epsilon, x + \epsilon) \\ 1 - \epsilon + \epsilon t & \text{if} \quad t \geq x + \epsilon \end{cases}$$

To find the solution one has to distinguish six different cases. $\mathbf{H}\Pi^F$ denotes the hessian of Π^F.

Case 1: $w_A, w_B < x - \epsilon$

Solving equations (A.16) and (A.17) on this domain gives us ${w_A^1}^* = {w_B^1}^* = \frac{v}{1+v\epsilon}$,

which is a global maximum (on this domain) because $\mathbf{H}\Pi^{F,1}(w_1, w_2)$ is negative definite. We obtain $\Pi^{F,1}({w_A^1}^*, {w_B^1}^*) = \frac{v^2\epsilon}{1+v\epsilon}$.

Case 2: $w_A, w_B \in [x-\epsilon, x+\epsilon)$
Solving equations (A.16) and (A.17) on this domain gives us ${w_A^2}^* = {w_B^2}^* = \frac{v(1+h(x-\epsilon)-x\epsilon+\epsilon^2}{1+hv}$, but $\mathbf{H}\Pi^{F,2}(w_1, w_2)$ is only negative definite if $h < v^{-1}$, i.e. if the spike is not too prominent.

Case 3: $w_A, w_B \geq x+\epsilon$
Solving equations (A.16) and (A.17) on this domain gives us ${w_A^3}^* = {w_B^3}^* = \frac{v\epsilon}{1+v\epsilon}$, which, again, is a global maximum because $\mathbf{H}\Pi^{F,3}(w_1, w_2)$ is negative definite. We obtain $\Pi^{F,3}({w_A^3}^*, {w_B^3}^*) = \epsilon(2\epsilon(2h-2\epsilon+1)-3) - \frac{1}{v^2\epsilon+v} + v + \frac{1}{v} + 2x(\epsilon-1)$.

Case 4: $w_A < x-\epsilon, w_B \in [x-\epsilon, x+\epsilon)$
Solving equations (A.16) and (A.17) on this domain gives us ${w_A^4}^* = \frac{v\left(h(v-x+\epsilon)+x\epsilon-\epsilon^2-1\right)}{hv^2\epsilon-1}$ and ${w_B^4}^* = \frac{v\left(v\epsilon\left(h(x-\epsilon)-x\epsilon+\epsilon^2+1\right)-1\right)}{hv^2\epsilon-1}$, but, again, $\mathbf{H}\Pi^{F,4}(w_1, w_2)$ is only negative definite if $h < (v^2\epsilon)^{-1}$, i.e. if the spike is not too prominent.

Case 5: $w_A < x-\epsilon, w_B \geq x+\epsilon$
Solving equations (A.16) and (A.17) on this domain gives us ${w_A^5}^* = \frac{(v-1)v\epsilon}{v^2\epsilon^2-1}$ and ${w_B^5}^* = \frac{v(v\epsilon^2-1)}{v^2\epsilon^2-1}$, which is a global maximum because $\mathbf{H}\Pi^{F,5}(w_1, w_2)$ is negative definite. We obtain
$\Pi^{F,5}({w_A^5}^*, {w_B^5}^*) = \frac{v^2\epsilon\left(\epsilon^2(2\epsilon(2h-2\epsilon+1)-3)+2x(\epsilon-1)\epsilon-1\right)-2\epsilon(\epsilon(2h-2\epsilon+1)+x)+2v^3\epsilon^2+2v(\epsilon-1)+2(x+\epsilon)}{2v^2\epsilon^2-2}$.

Case 6: $w_A \in [x-\epsilon, x+\epsilon), w_B \geq x+\epsilon$
Solving equations (A.16) and (A.17) on this domain gives us ${w_A^6}^* = \frac{v\epsilon\left(v\left(h(x-\epsilon)-x\epsilon+\epsilon^2+1\right)-1\right)}{hv^2\epsilon-1}$ and ${w_B^6}^* = \frac{v\left(h(v\epsilon-x+\epsilon)+x\epsilon-\epsilon^2-1\right)}{hv^2\epsilon-1}$, and $\mathbf{H}\Pi^{F,6}(w_1, w_2)$ is only negative definite if $h < (v^2\epsilon)^{-1}$, i.e. if the spike is not too prominent. We obtain $\Pi^{F,6}({w_A}^*, {w_B}^*) =$
$\frac{2hv^3\epsilon+2v\left(-\epsilon^2(h+x)+hx\epsilon+\epsilon^3+\epsilon-1\right)-\epsilon^2(5h+2x+2)+\epsilon(x(2h+x-2)+2)+x(2-hx)+5\epsilon^3}{2hv^2\epsilon-2}$
$\frac{-v^2\epsilon\left(-4h^2\epsilon^2+h\left(-x^2\epsilon+2x\left(\epsilon^2-\epsilon+2\right)+\epsilon\left(3\epsilon^2-2\epsilon+1\right)\right)+\left(-x\epsilon+\epsilon^2+1\right)^2\right)}{2hv^2\epsilon-2}$

Now let $\epsilon = 0.1$ (implying $h = 4.6$), $v = 0.7$, and $x = 0.5$. The six cases give the following candidates for the optimal bid:

$({w_A^1}^*, {w_B^1}^*) \approx (0.65, 0.65)$, which is not admissible on $[0, x-\epsilon]$. Case 2 does not fulfil the sufficient condition for a global maximum; the stationary point is $({w_A^2}^*, {w_B^2}^*) \approx (0.464, 0.464)$ and yields an expected payoff of 0.120. The boundary

point $(x-\epsilon, x+\epsilon)$ yields a higher payoff of 0.197. $({w_A^3}^*, {w_B^3}^*) \approx (0.065, 0.065)$ fulfils the second-order conditions, but is not admissible on $[x+\epsilon, 1]$. $({w_A^4}^*, {w_B^4}^*) \approx (-0.380, 0.727)$, which also fulfils the second-order conditions, but is not admissible on $[0, x-\epsilon) \times [x-\epsilon, x+\epsilon)$. We obtain $({w_A^5}^*, {w_B^5}^*) \approx (0.021, 0.699)$, which is a global optimum and yields a payoff of 0.205. $({w_A^6}^*, {w_B^6}^*) \approx (-0.087, 2.239)$ also fulfils the second-order conditions, but is again not admissible on $[x-\epsilon, x+\epsilon) \times [x+\epsilon, 1]$.

Clearly, the global maximum is $({w_A^5}^*, {w_B^5}^*) \approx (0.021, 0.699)$. For the general case, one can easily verify that ${w_A^5}^* \to 0$, ${w_B^5}^* \to v$, and $\Pi^{F,5}({w_A^5}^*, {w_B^5}^*) \to v - x$ as $\epsilon \to 0$.

However, depending on how much probability mass concentrates around the spike, we may also obtain a symmetric equilibrium. For example, let $\epsilon = 0.4$ (implying $h = 1.15$), $v = 0.7$, and $x = 0.5$. The six cases give the following candidates for the optimal bid:

$({w_A^1}^*, {w_B^1}^*) \approx (0.547, 0.547)$, which is not admissible on $[0, x-\epsilon]$. Case 2 fulfils the sufficient condition for a global maximum; the stationary point is $({w_A^2}^*, {w_B^2}^*) \approx (0.417, 0.417)$ and yields an expected payoff of 0.259. $({w_A^3}^*, {w_B^3}^*) \approx (0.219, 0.219)$ fulfils the second-order conditions, but is not admissible on $[x+\epsilon, 1]$. $({w_A^4}^*, {w_B^4}^*) \approx (0.244, 0.632)$, which also fulfils the second-order conditions, but is not admissible on $[0, x-\epsilon) \times [x-\epsilon, x+\epsilon)$. We obtain $({w_A^5}^*, {w_B^5}^*) \approx (0.091, 0.674)$, which is a global optimum and yields a payoff of 0.220. $({w_A^6}^*, {w_B^6}^*) \approx (0.089, 0.680)$ also fulfils the second-order conditions, but is again not admissible on $[x-\epsilon, x+\epsilon) \times [x+\epsilon, 1]$.

The global maximum here is $({w_A^2}^*, {w_B^2}^*) \approx (0.417, 0.417)$.

A.2.4 Simultaneous first-price auction

Proof of Proposition 13. It is obvious that under the first-price rule bidding one's true value on any good can never be optimal, because expected profit on that good would always be less than zero. So we only show that any bid $(w_1, ..., w_{j-1}, 0, w_{j+1}, ..., w_M)$ cannot be optimal. With probability $P := \prod_{l=1, l\neq j}(1 - G(w_l))$ bidder F wins nothing in the auctions she participates in. Now simply choose $\widetilde{w_j} = \alpha P v_j$, with $0 < \alpha < 1$. Then we have

$$
\begin{aligned}
&\Pi^F((w_1, ..., w_{j-1}, \widetilde{w_j}, w_{j+1}, ..., w_M) \\
\geq &\Pi^F((w_1, ..., w_{j-1}, 0, w_{j+1}, ..., w_M) + P(v_j - \widetilde{w_j}) - (1-P)\widetilde{w_j} \\
= &\Pi^F((w_1, ..., w_{j-1}, 0, w_{j+1}, ..., w_M) + P(v_j - \alpha P v_j) - (1-P)\alpha P v_j \\
= &\Pi^F((w_1, ..., w_{j-1}, 0, w_{j+1}, ..., w_M) + P v_j(1 - \alpha P - \alpha(1-P)) \\
> &\Pi^F((w_1, ..., w_{j-1}, 0, w_{j+1}, ..., w_M)
\end{aligned}
$$

□

A.2.5 Sequential second-price auction

Proof of Proposition 14. In the last auction for good M, she knows the past history of outcomes of auctions $1, ..., M-1$. Denote by v_{M-1}^{max} the value of the highest-value object bidder F obtained in the history H_{M-1}. Then, her payoff in the last auction for good M is

$$\Pi^F_{M,v_{M-1}^{max}}(w_M) = G(w_M)v_M - \int_0^{w_M} x \,\mathrm{d}G(x) + (1 - G(w_M))v_{M-1}^{max}$$

She bids ${w_{M,v_{M-1}^{max}}}^* = v_M - v_{M-1}^{max}$ and obtains $\Pi^{F*}_{M,v_{M-1}^{max}} = v_{M-1}^{max} + \int_0^{v_M - v_{M-1}^{max}} G(x)\,\mathrm{d}x$. In the $M-1$-auction, the flexible bidder takes possible future gains into account. She maximises

$$\Pi^F_{M-1,v_{M-2}^{max}}(w_{M-1}) = G(w_{M-1})\Pi^{F*}_{M,v_{M-1}} - \int_0^{w_{M-1}} x \,\mathrm{d}G(x) + (1 - G(w_{M-1}))\Pi^{F*}_{M,v_{M-2}^{max}}$$

In the auction for good $j < M$, given history H_{j-1}, and thus v_{j-1}^{max}, she maximises

$$\Pi^F_{j,v_{j-1}^{max}}(w_j) = G(w_j)\Pi^{F*}_{j+1,v_j} - \int_0^{w_j} x \,\mathrm{d}G(x) + (1 - G(w_j))\Pi^{F*}_{j+1,v_{j-1}^{max}}$$

and her optimal bid is given by

$${w_{j,v_{j-1}^{max}}}^* = \max\left\{\Pi^{F*}_{j+1,v_j} - \Pi^{F*}_{j+1,v_{j-1}^{max}}, 0\right\} \tag{A.32}$$

We then have $\Pi^{F*}_{j,v_{j-1}^{max}} = \Pi^{F*}_{j+1,v_{j-1}^{max}} + \int_0^{\max\{\Pi^{F*}_{j+1,v_j} - \Pi^{F*}_{j+1,v_{j-1}^{max}}, 0\}} G(x)\,\mathrm{d}x$. Clearly, bidding truthfully cannot be optimal because of the opportunity to improve one's payoff in the auction for good $j+1$. So the maximum must be characterised by the first-order condition given above. □

Equilibrium bidding for sales order $M, ..., 1$. In the last auction of good 1, the flexible bidder trivially bids her valuation ${w_1}^* = v_1$ and obtains $\Pi_1^{F*} = \int_0^{v_1} G(x)\,\mathrm{d}x$. In the auction of good $j > 1$, the flexible bidder bids w_j to maximise her expected stage-payoff

$$\Pi^F_j(w_j) = G(w_j)v_j - \int_0^{w_j} x \,\mathrm{d}G(x) + (1 - G(w_j))\,\Pi^{F*}_{j+1}$$

The first-order condition yields ${w_j}^* = v_j - \Pi^F_{j+1}$. This is the optimal bid if the flexible bidder did not win the auction for good $j-1$, and she bids $w_j^* = 0$ in all subsequent auctions for good j if she won good $j-1$. The second-order condition guarantees a local maximum at ${w_j}^*$ and it can indeed be verified that this is also a global maximum (neither $w_j = 0$ nor $w_j = v$ are optimal as can be shown by a simple ϵ-deviation). Simplifying the expected equilibrium profit, we obtain $\Pi_j^{F*} := \Pi^F_j({w_j}^*) = \Pi^{F*}_{j+1} + \int_0^{{w_j}^*} G(x)\,\mathrm{d}x$.

A.2.6 Sequential first-price auction

Proof of Proposition 15. In the last auction for good M, she knows the past history of outcomes of auctions $1, ..., M-1$. Denote by v_{M-1}^{max} the value of the highest-value object bidder F obtained in the history H_{M-1}. Then, her payoff in the last auction for good M is

$$\Pi^F_{M,v_{M-1}^{max}}(w_M) = G(w_M)(v_M - w_M) + (1 - G(w_M))v_{M-1}^{max}$$

The first-order condition characterises ${w_{M,v_{M-1}^{max}}}^* = v_M - v_{M-1}^{max} - \frac{G(w_M)}{g(w_M)}$ and obtains $\Pi^{F*}_{M,v_{M-1}^{max}} = v_{M-1}^{max} + \frac{G({w_{M,v_{M-1}^{max}}}^*)^2}{g({w_{M,v_{M-1}^{max}}}^*)}$. In the $M-1$-auction, the flexible bidder takes possible future gains into account. She maximises

$$\Pi^F_{M-1,v_{M-2}^{max}}(w_{M-1}) = G(w_{M-1})\left(\Pi^{F*}_{M,v_{M-1}} - w_{M-1}\right) + (1 - G(w_{M-1}))\Pi^{F*}_{M,v_{M-2}^{max}}$$

In the auction for good $j < M$, given history H_{j-1}, and thus v_{j-1}^{max}, she maximises

$$\Pi^F_{j,v_{j-1}^{max}}(w_j) = G(w_j)\left(\Pi^{F*}_{j+1,v_j} - w_j\right) + (1 - G(w_j))\Pi^{F*}_{j+1,v_{j-1}^{max}}$$

and her optimal bid is given by

$${w_{j,v_{j-1}^{max}}}^* = \max\left\{\Pi^{F*}_{j+1,v_j} - \Pi^{F*}_{j+1,v_{j-1}^{max}} - \frac{G(w_j)}{g(w_j)}, 0\right\} \tag{A.33}$$

and we have $\Pi^{F*}_{j,v_{j-1}^{max}} = \Pi^{F*}_{j+1,v_{j-1}^{max}} + \frac{G({w_{j,v_{j-1}^{max}}}^*)^2}{g({w_{j,v_{j-1}^{max}}}^*)}$. Of course, truthful bidding truthfully cannot be optimal under the first-price payment rule. So the maximum must be characterised by the first-order condition. □

A.2.7 Identical values

Assuming symmetric equilibria, we obtain the following equilibrium characterisations:

$$\begin{aligned}
w^*_{P2} &= v \\
w^*_{P1} &= v - \frac{\left(1 - (1 - G(w^*_{P1}))^M\right)}{M\left(1 - G(w^*_{P1})\right)^{M-1} g(w^*_{P1})} \\
w^*_{Sim2} &= v\left(1 - G(w^*_{Sim2})\right)^{M-1} \\
w^*_{Sim1} &= v\left(1 - G(w^*_{Sim1})\right)^{M-1} - \frac{G(w^*_{Sim1})}{g(w^*_{Sim1})} \\
{w_{j,Seq2}}^* &= v - \Pi^F_{Seq2,j+1} \\
{w_{j,Seq1}}^* &= v - \frac{G({w_{j,Seq1}}^*)}{g({w_{j,Seq1}}^*)} - \Pi^F_{Seq1,j+1}
\end{aligned}$$

where $\Pi^F_{Seq2,M+1} = \Pi^F_{Seq1,M+1} = 0$. Note that if the flexible bidder's value $v = 1$ and the inflexible bidders' values are uniformly distributed on $[0, 1]$, i.e. $G(x) = x$ $v = 1$, we find closed form expression for the equilibrium bid in the first-price PMA; we have $w^*_{P1} = 1 - \exp\left(\frac{\log\left(\frac{1}{1+M}\right)}{M}\right)$.

Product-mix auctions. In the standard PMA, the flexible bidder's payoff can be written as

$$\Pi^F_{P2}(\mathbf{w}) = \left(1 - (1 - G(w))^M\right) v - \int_0^w xM\,(1 - G(x))^{M-1}\, g(x)\,\mathrm{d}x$$

The first-order conditions yield $w^* = v$, which is of course the dominant strategy.

In the first-price PMA, assuming a symmetric equilibrium $w_j = w\ \forall j \in \mathcal{J}$, we have

$$\Pi^F_{P1}(\mathbf{w}) = \left(1 - (1 - G(w))^M\right)(v - w)$$

The first-order conditions yield $(v-w^*)M\,(1 - G(w^*))^{M-1}\, g(w^*)-(1-(1 - G(w^*))^M)$ $= 0$.

To state the auctioneer's revenue, let $x_{(r)}$ denote the $r-$statistic of the M random variables x, which are all iid distributed with probability distribution function G.[2]

Lemma 17. *The auctioneer's revenue in the first-price PMA with identical values in a symmetric equilibrium is given by*

$$R_{P1}(\mathbf{w}) = \int_0^w (w - x)M(1 - G(x))^{M-1}g(x)\,\mathrm{d}x + M\int_0^{\bar{v}} x\,\mathrm{d}G(x)$$

Proof. The probability density function of $x_{(r)}$ is given by $\frac{M!}{(r-1)!(M-r)!}g(x)G(x)^{r-1}(1-G(x))^{M-r}$. Let $b_{(r)}$ denote the $r-$statistic of the $M+1$ bids the auctioneer receives. Then her expected revenue, given a bid $\mathbf{w}$ from the flexible bidder, is

$$\begin{aligned}
R_{P1}(\mathbf{w}) &= \sum_{r=1}^{M} \mathbb{E}\left[b_{(r)}\right] \\
&= \int_0^w wM(1 - G(x))^{M-1}g(x)\,\mathrm{d}x + \\
&\quad \int_w^{\bar{v}} xM(1 - G(x))^{M-1}g(x)\,\mathrm{d}x + \sum_{r=2}^{M} \mathbb{E}\left[x_{(r)}\right] \\
&= \int_0^w wM(1 - G(x))^{M-1}g(x)\,\mathrm{d}x + \\
&\quad \int_w^{\bar{v}} xM(1 - G(x))^{M-1}g(x)\,\mathrm{d}x + M\mathbb{E}\left[x\right] - \mathbb{E}\left[x_{(1)}\right] \\
&= \int_0^w (w - x)M(1 - G(x))^{M-1}g(x)\,\mathrm{d}x + M\mathbb{E}\left[x\right]
\end{aligned}$$

□

[2]We denote by $x_{(1)}$ the minimum of the M RVs, and by $x_{(M)}$ the maximum of the M RVs.

Simultaneous auctions. Assuming a symmetric equilibrium $w_j = w \ \forall j \in \mathcal{J}$, the flexible bidder's payoff in the simultaneous second-price auction is

$$\Pi^F_{Sim2}(\mathbf{w}) = \left(1 - (1 - G(w))^M\right) v - M \int_0^w x g(x) \, \mathrm{d}x$$

The first-order conditions yield $w^* = v \left(1 - G(w^*)\right)^{M-1}$.

In the simultaneous first-price auction, again assuming a symmetric equilibrium $w_j = w \ \forall j \in \mathcal{J}$, we have

$$\Pi^F_{Sim1}(\mathbf{w}) = \left(1 - (1 - G(w))^M\right) v - MG(w)w$$

The first-order conditions yield $w^* = v \left(1 - G(w^*)\right)^{M-1} - \frac{G(w^*)}{g(w^*)}$.

The auctioneer's expected revenue is straightforward to compute. Given a bid $\mathbf{w}$ from the flexible bidder, we have

$$\begin{aligned} R_{Sim1}(\mathbf{w}) =& M \mathbb{E}\left[\max\{w, x\}\right] \\ =& M \left[\int_0^w w g(x) \, \mathrm{d}x + \int_w^{\overline{v}} x g(x) \, \mathrm{d}x\right] \\ =& \int_0^w (w - x) M g(x) \, \mathrm{d}x + M\mathbb{E}[x] \end{aligned}$$

Sequential auctions. The sequential auctions with identical values can be seen as a special case of the sales order $M, ..., 1$. The first-order condition characterises ${w_j}^* = v - \Pi^F_{j+1}$, where $\Pi^F_{M+1} = 0$. This is the optimal bid if the flexible bidder did not win the auction for good $j - 1$, and she bids $w^*_f = 0$ in all subsequent auctions if she won good $j - 1$.

The sequential first-price auction is solved analogously. The first-order condition yields ${w_j}^* = v - \frac{G({w_j}^*)}{g({w_j}^*)} - \Pi^F_{j+1}$, where $\Pi^F_{M+1} = 0$. This is the optimal bid if the flexible bidder did not win the auction for good $j - 1$, and she bids $w^*_f = 0$ in all subsequent auctions if she won good $j - 1$.

The auctioneer's revenue is also computed recursively. We have

$$R_j(w_j) = G(w_j) \left[w_j + (M - j)\mathbb{E}[x]\right] + \int_{w_j}^{\overline{v}} x g(x) + (1 - G(w_j)) R_{j+1}(w_{j+1})$$

A.3 Closed form solutions for 2-good model with uniform priors

The closed form solutions for the various auction types (whenever available) are listed below. We compare them using the functions *Reduce* and *Simplify* in

Mathematica and find the ordering posited in section 2.4.

$$\Pi^F_{P2|v_A,v_B} = \frac{1}{6}({v_A}^3 + 3{v_A}^2(1-v_B) + 3{v_B}^2)$$
$$\Pi^F_{Sim2|v_A,v_B} = \frac{{v_A}^2 - 2{v_A}^2 v_B + {v_B}^2}{2 - 2{v_A}^2}$$
$$\Pi^F_{SeqA2|v_A,v_B} = \frac{1}{8}\left({v_A}^4 - 4{v_A}^3(v_B-1) + 4{v_A}^2(v_B-1)^2 + 4{v_B}^2\right)$$
$$\Pi^F_{SeqB2|v_A,v_B} = \frac{1}{8}\left({v_A}^4 - 4{v_A}^2(v_B-1) + 4{v_B}^2\right)$$
$$\Pi^F_{Sim1|v_A,v_B} = \frac{{v_A}^2 - {v_A}^2 v_B + {v_B}^2}{4 - {v_A}^2}$$
$$\Pi^F_{SeqA1|v_A,v_B} = \frac{1}{64}\left({v_A}^2(4+v_A)^2 - 4{v_A}^2(4+v_A)v_B + 4(4+{v_A}^2){v_B}^2\right)$$
$$\Pi^F_{SeqB1|v_A,v_B} = \frac{1}{64}\left({v_A}^4 - 8{v_A}^2(v_B-2) + 16{v_B}^2\right)$$

$$\begin{aligned} R_{P1|v_A,v_B}(w_A,w_B) = &\int_0^{w_A}\int_{w_B-w_A+a}^1 (w_A+b)\,\mathrm{d}G(b)\,\mathrm{d}G(a) \\ &+ \int_0^{w_A}\int_0^{w_B-w_A+a} (w_B+a)\,\mathrm{d}G(b)\,\mathrm{d}G(a) \\ &+ \int_{w_A}^1\int_0^{w_B} (w_B+a)\,\mathrm{d}G(b)\,\mathrm{d}G(a) \\ &+ \int_{w_A}^1\int_{w_B}^1 (a+b)\,\mathrm{d}G(b)\,\mathrm{d}G(a) \end{aligned}$$

$$\begin{aligned} R_{Sim1|v_A,v_B} =& G(w_A)\,(1-G(w_B))\,\mathrm{E}[w_A+b \mid \text{F wins A}] \\ &+ G(w_B)\,(1-G(w_A))\,\mathrm{E}[w_B+a \mid \text{F wins B}] \\ &+ G(w_A)G(w_B)\,\mathrm{E}[w_A+w_B \mid \text{F wins A and B}] \\ &+ (1-G(w_A))\,(1-G(w_B))\,\mathrm{E}[a+b \mid \text{F wins nothing}] \end{aligned}$$

$$\begin{aligned} R_{SeqJ1|v_A,v_B} =& G({w_J}^*)\Big(G({w_K}^*)\,\mathrm{E}[{w_J}^*+{w_K}^* \mid \text{F wins J and K}] \\ &+ (1-G({w_K}^*))\,\mathrm{E}[{w_J}^*+k \mid \text{F wins J, not K}]\Big) \\ &+ (1-G({w_J}^*))\Big(G(\overline{{w_K}}^*)\,\mathrm{E}[j+\overline{{w_K}}^* \mid \text{F wins K, not J}] \\ &+ (1-G(\overline{{w_K}}^*))\,\mathrm{E}[j+k \mid \text{F wins nothing}]\Big) \end{aligned}$$

$$R_{P2|v_A,v_B} = R_{Sim2|v_A,v_B} = R_{SeqA2|v_A,v_B} = R_{SeqB2|v_A,v_B} = 1$$
$$R_{Sim1|v_A,v_B} = \frac{3{v_A}^4 + 4(8 + {v_B}^2) + {v_A}^2({v_B}^2 - 8v_B - 12)}{2({v_A}^2 - 4)^2}$$
$$R_{SeqA1|v_A,v_B} = \frac{1}{128}(3{v_A}^4 - 4{v_A}^3(-4 + 3v_B) + 16(8 + {v_B}^2) + 4{v_A}^2(4 - 8v_B + 3{v_B}^2))$$
$$R_{SeqB1|v_A,v_B} = \frac{1}{128}(3{v_A}^4 - 16{v_A}^2(-1 + v_B) + 16(8 + {v_B}^2))$$

$$\mathcal{W}_{P2|v_A,v_B} = \frac{1}{6}({v_A}^3 + 3{v_A}^2(1 - v_B) + 3(2 + {v_B}^2))$$
$$\mathcal{W}_{Sim2|v_A,v_B} = \frac{2 + {v_B}^2 - {v_A}^2(1 + 2v_B)}{2 - 2{v_A}^2}$$
$$\mathcal{W}_{SeqA2|v_A,v_B} = \frac{1}{8}\left({v_A}^4 - 4{v_A}^3(v_B - 1) + 4{v_A}^2(v_B - 1)^2 + 4{v_B}^2 + 8\right)$$
$$\mathcal{W}_{SeqB2|v_A,v_B} = \frac{1}{8}\left({v_A}^4 - 4{v_A}^2(v_B - 1) + 4{v_B}^2 + 8\right)$$
$$\mathcal{W}_{Sim1|v_A,v_B} = \frac{{v_A}^4(1 + 2v_B) - {v_A}^2({v_B}^2 + 16v_B + 4) + 4(8 + 3{v_B}^2)}{2({v_A}^2 - 4)^2}$$
$$\mathcal{W}_{SeqA1|v_A,v_B} = \frac{1}{128}(5{v_A}^4 - 4{v_A}^3(-8 + 5v_B) + 16(8 + 3{v_B}^2) + 4{v_A}^2(12 - 16v_B + 5{v_B}^2))$$
$$\mathcal{W}_{SeqB1|v_A,v_B} = \frac{1}{128}(128 + 5{v_A}^4 + 48{v_B}^2 - 16{v_A}^2(-3 + 2v_B))$$

A.4 Tables

Table A.1: Equilibrium bids and interim payoffs (flexible bidder), revenue, and welfare; the multi-column for the sequential auction displays the second-stage bid conditional on winning or losing the first stage.

	(P2)	(Sim2)	(SeqA2)	(SeqB2)	(P1)	(Sim1)	(SeqA1)	(SeqB1)
				$v_A = 0, v_B = 0.5$				
${w_A}^*/\overline{w_A}^*$	0.000	0.000	0.000	0.000 0.000	0.000	0.000	0.000	0.000 0.000
${w_B}^*/\overline{w_B}^*$	0.500	0.500	0.500 0.500	0.500	0.250	0.250	0.250 0.250	0.250
Π_M^F	0.125	0.125	0.125	0.125	0.063	0.063	0.063	0.063
R_M	1.000	1.000	1.000	1.000	1.031	1.031	1.031	1.031
W_M	1.125	1.125	1.125	1.125	1.094	1.094	1.094	1.094
				$v_A = 0.5, v_B = 0.5$				
${w_A}^*/\overline{w_A}^*$	0.500	0.333	0.375	0.000 0.500	0.232	0.200	0.219	0.000 0.250
${w_B}^*/\overline{w_B}^*$	0.500	0.333	0.000 0.500	0.375	0.232	0.200	0.000 0.250	0.219
Π_M^F	0.208	0.167	0.195	0.195	0.110	0.100	0.110	0.110
R_M	1.000	1.000	1.000	1.000	1.050	1.040	1.048	1.048
W_M	1.208	1.167	1.195	1.195	1.160	1.140	1.159	1.159
				$v_A = 0.5, v_B = 1$				
${w_A}^*/\overline{w_A}^*$	0.500	0.000	0.125	0.000 0.500	0.193	0.133	0.156	0.000 0.250
${w_B}^*/\overline{w_B}^*$	1.000	1.000	0.500 1.000	0.875	0.480	0.467	0.250 0.500	0.469
Π_M^F	0.521	0.500	0.508	0.508	0.276	0.267	0.274	0.282
R_M	1.000	1.000	1.000	1.000	1.126	1.118	1.123	1.126
W_M	1.521	1.500	1.508	1.508	1.402	1.384	1.397	1.409
				$v_A = 0.9, v_B = 1$				
${w_A}^*/\overline{w_A}^*$	0.900	0.000	0.405	0.000 0.900	0.375	0.282	0.326	0.000 0.450
${w_B}^*/\overline{w_B}^*$	1.000	1.000	0.100 1.000	0.595	0.437	0.373	0.050 0.500	0.399
Π_M^F	0.622	0.500	0.582	0.582	0.354	0.313	0.356	0.362
R_M	1.000	1.000	1.000	1.000	1.144	1.109	1.138	1.140
W_M	1.622	1.500	1.582	1.582	1.498	1.423	1.494	1.502

Table A.2: Relative differences for first-price PMA vs. first-price simultaneous auction

pma/sim		Efficiency	v_A^{max}	v_B^{max}	Revenue	v_A^{max}	v_B^{max}	Bidder surplus	v_A^{max}	v_B^{max}
max diff	$\widetilde{g}_4$	0.092	1	1	0.049	0.94	0.94	0.138	0.56	0.56
	$\widetilde{g}_3$	0.092	1	1	0.051	1	1	0.142	0.74	0.74
	$\widetilde{g}_2$	0.087	1	1	0.049	1	1	0.150	1	1
	uniform	0.065	1	1	0.038	1	1	0.155	1	1
	g_2	0.061	1	1	0.035	1	1	0.260	1	1
	g_3	0.055	1	1	0.031	1	1	0.351	1	1
	g_4	0.049	1	1	0.027	1	1	0.433	1	1
avg diff	$\widetilde{g}_4$	0.023			0.010			0.043		
	$\widetilde{g}_3$	0.023			0.011			0.047		
	$\widetilde{g}_2$	0.020			0.010			0.052		
	uniform	0.012			0.006			0.052		
	g_2	0.007			0.004			0.068		
	g_3	0.004			0.002			0.072		
	g_4	0.002			0.001			0.072		

Table A.3: Relative differences for first-price PMA vs. first-price sequential auction, good A first

pma/seqA		Efficiency	v_A^{max}	v_B^{max}	Revenue	v_A^{max}	v_B^{max}	Bidder surplus	v_A^{max}	v_B^{max}
max diff	$\widetilde{g}_4$	0.010	0.5	0.72	0.022	0.92	1	-0.023	1	1
	$\widetilde{g}_3$	0.009	0.56	0.8	0.016	0.88	1	-0.023	1	1
	$\widetilde{g}_2$	0.008	0.62	0.9	0.011	0.86	1	-0.021	1	1
	uniform	0.005	0.64	0.94	0.005	0.84	1	-0.015	1	1
	g_2	0.003	0.7	0.98	0.003	1	1	-0.014	1	1
	g_3	0.002	0.76	0.98	0.003	1	1	-0.014	1	1
	g_4	0.001	0.96	0.96	0.002	1	1	-0.014	1	1
avg diff	$\widetilde{g}_4$	0.004			0.007			0.001		
	$\widetilde{g}_3$	0.004			0.005			0.002		
	$\widetilde{g}_2$	0.003			0.004			0.002		
	uniform	0.002			0.002			0.003		
	g_2	0.001			0.001			0.002		
	g_3	0.000			0.000			0.001		
	g_4	0.000			0.000			0.000		

Table A.4: Relative differences for first-price PMA vs. first-price sequential auction, good B first

pma/seqB		Efficiency	v_A^{max}	v_B^{max}	Revenue	v_A^{max}	v_B^{max}	Bidder surplus	v_A^{max}	v_B^{max}
max diff	$\widetilde{g}_4$	-0.019	0.74	1	0.019	1	1	-0.042	0.74	1
	$\widetilde{g}_3$	-0.017	0.72	1	0.014	1	1	-0.040	0.74	1
	$\widetilde{g}_2$	-0.012	0.7	1	0.009	1	1	-0.036	0.72	1
	uniform	-0.005	0.64	1	0.004	1	1	-0.025	0.7	1
	g_2	0.001	0.86	0.86	0.003	1	1	-0.017	0.78	1
	g_3	0.001	0.92	0.92	0.003	0.98	1	-0.014	1	1
	g_4	0.001	0.94	1	0.002	0.98	1	-0.014	1	1
avg diff	$\widetilde{g}_4$	-0.008			0.003			-0.024		
	$\widetilde{g}_3$	-0.007			0.001			-0.023		
	$\widetilde{g}_2$	-0.004			0.001			-0.019		
	uniform	-0.001			0.000			-0.013		
	g_2	0.000			0.000			-0.007		
	g_3	0.000			0.000			-0.005		
	g_4	0.000			0.000			-0.003		

Table A.5: Efficiency

W	(P1)	(Sim1)	(SeqA1)	(SeqB1)
$\widetilde{g}_4$	0.825	0.806	0.821	0.832
$\widetilde{g}_3$	0.883	0.864	0.880	0.889
$\widetilde{g}_2$	0.988	0.969	0.985	0.993
g_1	1.219	1.205	1.217	1.221
g_2	1.449	1.439	1.447	1.449
g_3	1.571	1.565	1.571	1.571
g_4	1.649	1.645	1.648	1.648

Table A.6: Revenue

R	(P1)	(Sim1)	(SeqA1)	(SeqB1)
$\widetilde{g}_4$	0.496	0.491	0.492	0.494
$\widetilde{g}_3$	0.594	0.588	0.591	0.593
$\widetilde{g}_2$	0.755	0.747	0.752	0.754
g_1	1.069	1.062	1.067	1.069
g_2	1.378	1.373	1.377	1.378
g_3	1.530	1.527	1.529	1.530
g_4	1.621	1.619	1.621	1.621

Table A.7: Bidder surplus

Π^F	(P1)	(Sim1)	(SeqA1)	(SeqB1)
$\widetilde{g}_4$	0.329	0.316	0.329	0.337
$\widetilde{g}_3$	0.289	0.276	0.288	0.296
$\widetilde{g}_2$	0.234	0.222	0.233	0.238
g_1	0.150	0.143	0.150	0.152
g_2	0.070	0.066	0.070	0.071
g_3	0.041	0.038	0.041	0.041
g_4	0.027	0.025	0.027	0.027

B

Appendix for Chapter 3

B.1 Proofs

Proof of Proposition 16.

The flexible bidder's payoff function simplifies to

$$\mathcal{U}^{pma}(w_A, w_B) = \\ -\frac{1}{2}{w_A}^3 + {w_A}^2\left(\frac{1}{2}v_A - \frac{1}{2}v_B + \frac{3}{2}w_B - 1\right) + v_B w_B - {w_B}^2 + v_A w_A(1 - w_B)$$

First-order conditions are

$$-\frac{3}{2}{w_A}^2 + w_A\left(-2 + 3w_B + v_A - v_B\right) + v_A(1 - w_B) = 0 \tag{B.1}$$

$$\frac{3}{2}{w_A}^2 + v_B - 2w_B - v_A w_A = 0 \tag{B.2}$$

Substituting (B.2) into (B.1) yields

$$9{w_A}^3 - \left(9v_A + 6\right){w_A}^2 + \left(2{v_A}^2 + 4v_A + 2v_B - 8\right)w_A - 2v_A v_B + 4v_A = 0 \tag{B.3}$$

If an interior solution exists, it must be a stationary point of $\mathcal{U}^{pma}(w_A, w_B)$, i.e. the optimal w_A must be a root of equation (B.3). Let $p(w_A) := \alpha{w_A}^3 + \beta{w_A}^2 + \gamma w_A + \delta$ denote the polynomial on the left hand side of (B.3), where

$$\begin{aligned}\alpha :=& 9\\ \beta :=& -\left(9v_A + 6\right)\\ \gamma :=& 2{v_A}^2 + 4v_A + 2v_B - 8\\ \delta :=& -2v_A v_B + 4v_A\end{aligned}$$

Using *Mathematica*, one can easily verify that the discriminant of $p(w_A)$

$$\Delta_p = 18\alpha\beta\gamma\delta - 4\beta^3\delta + \beta^2\gamma^2 - 4\alpha\gamma^3 - 27\alpha^2\delta^2$$

is strictly positive for $v_A, v_B < 2$, hence $p(w_A)$ possesses three distinct real roots. Existence of at least one real root follows from the argument given below for the existence of an interior solution. Furthermore, we know that $p(w_A) \to -\infty$ as $w_A \to -\infty$ and $p(w_A) \to \infty$ as $w_A \to \infty$. It is also easy to check that

$$\begin{aligned} p(0) &= \delta && > 0 && \text{if } v_B < 2 \text{ and} \\ p(1) &= -5 - v_A + 2{v_A}^2 + 2v_B - 2v_A v_B && < -5 + 2v_B < 0 && \text{if } v_B < 2 \end{aligned}$$

Thus, by the intermediate value theorem, $p(w_A)$ has exactly one real root on $(0, 1)$. Moreover, one can easily check that $p(v_A) < 0$ for $v_A < 2$, hence the solution is such that $w_A < v_A$. From equation (B.2) one can derive w_B as a function of w_A. Therefore, if w_A is uniquely defined, and so is w_B. Global optimality is immediate from Proposition 2. □

Comparative statics for the product-mix auction. The implicit function theorem applied to the first-order conditions yields

$$\begin{aligned} \frac{\partial w_A}{\partial v_A} &= \frac{\frac{1}{2}w_A(3w_A - v_A) + w_B - w_A - 1}{\frac{1}{2}(3w_A - v_A)^2 + 3(w_B - w_A) + v_A - v_B - 2} \\ \frac{\partial w_A}{\partial v_B} &= \frac{\frac{1}{2}(v_A - w_A)}{\frac{1}{2}(3w_A - v_A)^2 + 3(w_B - w_A) + v_A - v_B - 2} \\ \frac{\partial w_B}{\partial v_A} &= \frac{\partial w_A}{\partial v_A}\frac{3w_A - v_A}{2} - \frac{w_A}{2} \\ \frac{\partial w_B}{\partial v_B} &= \frac{\partial w_A}{\partial v_B}\frac{3w_A - v_A}{2} + \frac{1}{2} \end{aligned}$$

First, we note that $w_A \leq \frac{v_A}{2}$. This follows because $p(\frac{v_A}{2})$ is strictly negative. From Eq. (3.4), it follows then that $w_B \leq \frac{v_B}{2}$. Using these upper bounds on the optimal bids w_A and w_B, one can establish that the following holds: $\frac{\partial w_A}{\partial v_A} > 0$, $\frac{\partial w_A}{\partial v_B} < 0$, $\frac{\partial w_B}{\partial v_A} < 0$, and $\frac{\partial w_B}{\partial v_B} > 0$.

Proof of Proposition 17. Assuming $v_B \leq 1$, we can restrict the analysis to $0 \leq w_B < 1$.

Case 1: Bidder won the auction for object A. The payoff function is given by

$$\begin{aligned} &Prob\,(\text{bidder wins } B)\,(v_B - w_B - w_A) + Prob(\text{bidder loses } B)(v_A - w_A) \\ =&w_B(v_B - w_B - w_A) + (1 - w_B)(v_A - w_A) \end{aligned}$$

The first-order condition yields ${w_B}^* = \frac{v_B - v_A}{2}$ and this is the global maximum if $v_B < 2 + v_A$. The bidder's conditional expected second stage payoff is $\frac{(v_B - v_A)^2}{4} + v_A - w_A$.

Case 2: Bidder lost the auction for object A. She bids her half of her valuation, i.e. $\frac{v_B}{2}$ with a conditional expected second stage payoff of $\frac{{v_B}^2}{4}$. This is a valid solution for $v_B < 2$.

We can now analyse the first stage auction. The bidder's payoff function is

$$\mathcal{U}^{seq}(w_A, w_B) = w_A \left(\frac{(v_B - v_A)^2}{4} + v_A - w_A \right) + (1 - w_A) \frac{{v_B}^2}{4}$$

The first-order condition yields ${w_A}^* = \frac{1}{4}\left(\frac{{v_A}^2}{2} + v_A(2 - v_B)\right)$, which is a global and interior solution. □

Proof of Proposition 18.

The bidder's payoff function is given by

$$\mathcal{U}^{sim}(w_A, w_B) = w_A(1 - w_B)v_A - {w_A}^2 + w_B(v_B - w_B)$$

First-order conditions yield ${w_A}^* = v_A \frac{2 - v_B}{4 - {v_A}^2}$ and ${w_B}^* = \frac{2v_B - {v_A}^2}{4 - {v_A}^2}$. Second-order conditions hold if $v_A < 2$. For an interior solution, we need $v_B < 2$ as an additional necessary and sufficient condition. □

Comparative statics for the simultaneous auction. We have $\frac{\partial w_A}{\partial v_A} = \frac{(2 - v_B)(4 + {v_A}^2)}{(4 - {v_A}^2)} > 0$, $\frac{\partial w_A}{\partial v_B} = \frac{-v_A(4 - {v_A}^2)}{(4 - {v_A}^2)} < 0$, $\frac{\partial w_B}{\partial v_A} = \frac{4v_A(v_B - 2)}{(4 - {v_A}^2)} < 0$, $\frac{\partial w_B}{\partial v_B} = 2 > 0$, for all $v_A, v_B \in (0, 1)$.

B.2 Tables

B.2.1 Expected outcomes by covariates for computerised treatments

B.2.2 Equivalence tests for computerised treatments

Table B.1: Expected mean outcomes by gender

		Female	Male	Other	Prefer not to say
#subjects		123	85	1	2
$\mathcal{U}$	pma	9.71	10.15	-	-
	seq	7.62	9.47	-	17.38
	sim	5.31	7.96	13.04	9.76
$\mathcal{R}$	pma	114.55	115.19	-	-
	seq	114.55	112.56	-	113.40
	sim	117.35	115.19	114.39	116.60
$\mathcal{W}$	pma	124.27	125.34	-	-
	seq	122.18	122.03	-	130.78
	sim	122.66	123.15	127.43	126.37
$\mathcal{P}$	pma	27.32	28.51	-	-
	seq	29.11	25.13	-	26.80
	sim	34.70	30.39	28.78	33.20

Table B.2: Expected mean outcomes by age

		18-24	25-34	35-44	45-54	55-64
#subjects		40	53	42	44	32
$\mathcal{U}$	pma	10.66	9.64	9.38	10.58	8.61
	seq	10.95	7.06	7.58	8.58	8.61
	sim	7.89	6.97	7.40	4.50	8.00
$\mathcal{R}$	pma	113.04	115.56	114.74	114.19	116.68
	seq	112.78	113.52	115.13	112.59	115.17
	sim	113.23	117.40	114.89	117.69	116.09
$\mathcal{W}$	pma	123.71	125.20	124.12	124.77	125.29
	seq	123.72	120.58	122.72	121.16	123.78
	sim	121.12	124.37	122.29	122.18	124.09
$\mathcal{P}$	pma	24.52	29.26	27.77	26.47	31.22
	seq	25.55	27.04	30.26	25.18	30.34
	sim	26.46	34.80	29.79	35.38	32.18

Table B.3: Expected mean outcomes by ethnicity

		Arabic	Black	Central Asian	East Asian	Latin American	South Asian	White	Other	Prefer not to say
#subjects		1	4	2	9	1	18	158	11	7
$\mathcal{U}$	pma	-	9.55	-	9.83	-	10.64	9.71	9.36	11.43
	seq	-	-	-	5.86	-	9.94	8.67	-	6.40
	sim	14.13	5.61	2.54	3.07	2.82	0.16	7.44	7.32	10.90
$\mathcal{R}$	pma	-	118.03	-	112.27	-	113.38	114.76	117.62	113.33
	seq	-	-	-	118.04	-	113.79	113.58	-	111.64
	sim	111.98	116.44	113.39	118.07	121.03	123.10	115.72	116.89	112.47
$\mathcal{W}$	pma	-	127.58	-	122.10	-	124.01	124.48	126.98	124.76
	seq	-	-	-	123.90	-	123.73	122.26	-	118.04
	sim	126.11	122.05	115.94	121.15	123.85	123.26	123.16	124.21	123.37
$\mathcal{P}$	pma	-	33.42	-	23.33	-	24.96	27.76	32.70	25.26
	seq	-	-	-	36.08	-	27.57	27.16	-	23.28
	sim	23.96	32.88	26.79	36.14	42.07	46.20	31.44	33.78	24.94

Table B.4: Expected mean outcomes by socio-economic status

		Upper class	Upper middle class	Lower middle class	Working class	Lower class	Prefer not to say
#subjects		0	59	78	46	5	23
$\mathcal{U}$	pma	-	9.82	9.36	10.30	14.18	10.57
	seq	-	10.57	7.67	8.08	8.72	7.44
	sim	-	8.64	6.98	5.63	9.09	1.32
$\mathcal{R}$	pma	-	114.54	115.17	113.92	108.38	115.27
	seq	-	113.80	113.75	114.63	108.97	113.34
	sim	-	113.79	116.78	117.95	105.61	121.64
$\mathcal{W}$	pma	-	124.36	124.53	124.22	122.57	125.84
	seq	-	124.37	121.41	122.72	117.69	120.78
	sim	-	122.43	123.76	123.58	114.70	122.96
$\mathcal{P}$	pma	-	27.49	28.40	26.12	16.06	28.61
	seq	-	27.60	27.49	29.27	17.93	26.68
	sim	-	27.59	33.56	35.90	11.22	43.28

Table B.5: Expected mean outcomes by income

		£0	£1 to £9,999	£10,000 to £24,999	£25,000 to £49,999	£50,000 to £99,999	£100,000 to £149,999	£150,000 and more	Prefer not to say
#subjects		7	44	46	62	22	3	0	27
$\mathcal{U}$	pma	10.45	10.50	9.49	9.26	10.28	14.89	-	10.16
	seq	9.48	9.20	7.14	8.82	8.47	16.03	-	8.27
	sim	-5.18	6.82	8.02	6.24	9.12	2.64	-	5.39
$\mathcal{R}$	pma	114.22	113.26	115.08	115.15	116.45	113.41	-	113.51
	seq	110.53	113.00	114.55	114.38	114.38	110.01	-	113.23
	sim	124.50	114.75	115.73	117.86	112.91	116.50	-	118.37
$\mathcal{W}$	pma	124.67	123.76	124.57	124.41	126.73	128.31	-	123.67
	seq	120.01	122.20	121.69	123.20	122.85	126.04	-	121.50
	sim	119.33	121.57	123.75	124.10	122.03	119.14	-	123.76
$\mathcal{P}$	pma	26.09	24.90	28.53	28.37	30.76	25.51	-	25.37
	seq	21.06	26.01	29.09	28.76	28.75	20.02	-	26.46
	sim	49.00	29.50	31.45	35.73	25.82	33.00	-	36.73

Table B.6: Expected mean outcomes by education

		Early childhood education / no education	Primary education	Lower secondary education	Upper secondary education	Post-secondary non-tertiary education	Short-cycle tertiary education	Bachelor or equivalent	Master or equivalent	Doctoral or equivalent
#subjects		0	0	2	21	12	6	92	68	10
$\mathcal{U}$	pma	-	-	-	7.61	8.99	7.73	10.74	9.37	7.71
	seq	-	-	-	9.59	6.71	9.65	9.16	8.45	6.32
	sim	-	-	5.02	5.72	5.17	9.67	6.43	7.18	11.48
$\mathcal{R}$	pma	-	-	-	117.25	115.44	116.96	113.36	115.79	116.17
	seq	-	-	-	111.02	113.53	113.81	114.34	114.08	112.43
	sim	-	-	116.69	117.84	113.69	115.81	116.10	116.24	114.16
$\mathcal{W}$	pma	-	-	-	124.87	124.43	124.69	124.10	125.15	123.88
	seq	-	-	-	120.61	120.24	123.46	123.50	122.53	118.75
	sim	-	-	121.72	123.56	118.86	125.48	122.53	123.42	125.64
$\mathcal{P}$	pma	-	-	-	31.20	28.88	32.03	25.25	29.60	30.23
	seq	-	-	-	22.04	27.06	27.62	28.68	28.16	24.86
	sim	-	-	33.39	35.67	27.38	31.63	32.20	32.48	28.31

Table B.7: Expected mean outcomes by numeracy score, 4 being the best score

		1	2	3	4
#subjects		46	75	28	62
$\mathcal{U}$	pma	9.01	9.06	9.88	11.60
	seq	6.38	9.34	9.95	8.58
	sim	5.40	6.45	7.20	8.20
$\mathcal{R}$	pma	115.72	114.88	115.39	113.43
	seq	113.54	112.70	114.20	114.49
	sim	117.90	116.09	114.15	115.59
$\mathcal{W}$	pma	124.73	123.94	125.27	125.02
	seq	119.92	122.04	124.15	123.07
	sim	123.29	122.54	121.35	123.79
$\mathcal{P}$	pma	29.62	27.94	28.54	25.33
	seq	27.08	25.40	28.40	28.97
	sim	35.79	32.17	28.30	31.19

Table B.8: Expected mean outcomes by number of comprehension errors

		0	1-2	3-5	6-10	≥ 11
#subjects		84	68	34	19	6
$\mathcal{U}$	pma	11.07	9.10	8.11	10.15	9.38
	seq	10.59	7.96	4.44	10.01	9.00
	sim	9.41	8.69	2.43	1.94	0.84
$\mathcal{R}$	pma	113.06	116.08	114.46	115.56	119.62
	seq	113.39	114.18	115.14	112.82	105.50
	sim	114.28	114.51	120.02	118.62	124.42
$\mathcal{W}$	pma	124.13	125.18	122.57	125.71	129.00
	seq	123.98	122.14	119.59	122.84	114.50
	sim	123.69	123.20	122.45	120.56	125.26
$\mathcal{P}$	pma	24.65	30.08	27.30	28.98	36.28
	seq	26.78	28.37	30.29	25.65	10.99
	sim	28.56	29.03	40.05	37.23	48.84

Table B.9: Expected mean outcomes by self-reported risk preferences, 10 being the most risk-averse

		0-1	2-3	4-5	6-7	8-9	10
#subjects		17	53	86	38	17	0
$\mathcal{U}$	pma	8.03	10.28	9.87	10.21	8.63	-
	seq	6.51	9.32	9.11	7.48	6.36	-
	sim	5.56	6.76	9.30	3.91	4.81	-
$\mathcal{R}$	pma	116.64	114.67	114.36	113.81	118.05	-
	seq	110.29	114.16	114.02	112.89	113.24	-
	sim	115.59	115.90	115.05	118.36	119.59	-
$\mathcal{W}$	pma	124.67	124.95	124.23	124.02	126.69	-
	seq	116.80	123.49	123.13	120.36	119.60	-
	sim	121.15	122.66	124.36	122.27	124.40	-
$\mathcal{P}$	pma	31.15	27.54	26.95	26.11	33.36	-
	seq	20.57	28.33	28.05	25.78	26.48	-
	sim	31.18	31.80	30.11	36.73	39.18	-

Table B.10: Expected mean outcomes by risk aversion measure with test after Drichoutis et al. (2015), 10 being the most risk-averse

		0-1	2-3	4-5	6-7	8-9	10
#subjects		41	21	65	52	21	11
$\mathcal{U}$	pma	7.59	10.79	10.93	9.86	9.73	9.34
	seq	4.41	6.80	8.05	10.64	14.04	7.54
	sim	5.24	3.18	7.05	9.86	4.35	7.53
$\mathcal{R}$	pma	117.84	117.43	113.66	113.07	114.55	113.82
	seq	115.75	114.56	113.46	113.19	110.97	111.46
	sim	117.55	118.86	116.27	113.22	118.44	116.27
$\mathcal{W}$	pma	125.44	128.22	124.59	122.93	124.28	123.16
	seq	120.15	121.36	121.51	123.83	125.01	119.00
	sim	122.79	122.04	123.32	123.08	122.80	123.80
$\mathcal{P}$	pma	33.29	32.30	25.78	24.59	27.50	25.88
	seq	31.49	29.13	26.93	26.39	21.94	22.92
	sim	35.10	37.72	32.54	26.44	36.88	32.55

Table B.11: Expected mean outcomes by risk aversion (probability weights) measure after test by Holt and Laury (2002), 10 being the most risk-averse

		0-1	2-3	4-5	6-7	8-9	10
#subjects		20	12	76	66	27	10
$\mathcal{U}$	pma	7.14	8.05	9.70	11.38	10.86	8.26
	seq	4.63	9.66	6.94	10.52	10.30	10.10
	sim	5.29	1.28	7.47	7.98	5.27	4.91
$\mathcal{R}$	pma	119.35	118.16	115.18	112.67	112.11	115.03
	seq	112.19	105.58	115.36	113.21	113.20	112.92
	sim	115.35	119.03	116.59	114.19	120.30	118.06
$\mathcal{W}$	pma	126.49	126.21	124.89	124.05	122.98	123.29
	seq	116.82	115.24	122.30	123.74	123.51	123.02
	sim	120.64	120.31	124.06	122.17	125.57	122.96
$\mathcal{P}$	pma	36.29	33.75	28.34	23.95	22.95	28.45
	seq	24.38	11.16	30.72	26.43	26.40	25.83
	sim	30.70	38.07	33.17	28.38	40.60	36.11

Table B.12: Expected mean outcomes by ambiguity aversion measure after Gneezy et al. (2015), 20 being the most ambiguity-averse

		0-2	3-5	6-8	9-11	12-14	15-17	18-20
#subjects		26	53	25	23	16	38	30
$\mathcal{U}$	pma	8.11	10.18	9.08	10.89	9.60	10.25	10.02
	seq	5.13	9.09	5.53	11.24	9.88	10.53	8.26
	sim	5.46	7.58	4.22	7.34	9.17	6.94	6.88
$\mathcal{R}$	pma	118.64	112.56	116.26	115.67	117.25	114.04	113.87
	seq	113.40	114.45	116.62	112.92	112.50	112.54	112.44
	sim	118.56	116.14	118.88	114.06	114.69	115.43	114.70
$\mathcal{W}$	pma	126.75	122.74	125.34	126.56	126.86	124.29	123.89
	seq	118.53	123.54	122.14	124.16	122.37	123.07	120.70
	sim	124.02	123.72	123.10	121.40	123.85	122.37	121.58
$\mathcal{P}$	pma	34.91	23.73	30.45	29.41	32.03	26.40	25.99
	seq	26.79	28.91	33.23	25.84	24.99	25.08	24.88
	sim	37.13	32.27	37.76	28.12	29.37	30.86	29.40

Table B.13: Expected mean outcomes by self-reported, perceived competition in auction

		0-1	2-3	4-5	6-7	8-9	10
#subjects		10	20	31	68	63	19
$\mathcal{U}$	pma	6.06	10.03	9.99	9.94	9.51	12.96
	seq	9.62	7.75	7.93	8.34	9.18	9.34
	sim	10.38	5.61	9.20	7.85	4.93	5.16
$\mathcal{R}$	pma	117.73	115.47	114.88	114.49	114.81	111.34
	seq	112.05	112.78	114.07	113.68	113.98	113.64
	sim	113.08	119.09	112.74	116.44	117.64	117.22
$\mathcal{W}$	pma	123.79	125.50	124.87	124.43	124.32	124.30
	seq	121.67	120.54	122.00	122.02	123.17	122.98
	sim	123.46	124.70	121.94	124.29	122.57	122.38
$\mathcal{P}$	pma	33.36	28.99	28.09	27.17	27.73	21.54
	seq	24.10	25.57	28.14	27.36	27.97	27.28
	sim	26.16	38.18	25.48	32.88	35.29	34.44

Table B.14: Expected mean outcomes by self-reported, perceived difficulty of bidding in auction

		0-1	2-3	4-5	6-7	8-9	10
#subjects		7	28	30	68	66	12
$\mathcal{U}$	pma	11.18	10.59	10.92	10.19	8.81	5.38
	seq	7.38	9.31	8.70	8.30	9.17	3.08
	sim	10.44	8.13	10.59	6.71	5.11	4.81
$\mathcal{R}$	pma	113.29	112.22	115.23	113.89	116.06	122.41
	seq	115.12	110.33	111.87	114.99	113.69	115.65
	sim	111.94	116.40	112.55	116.58	117.88	114.93
$\mathcal{W}$	pma	124.47	122.81	126.15	124.08	124.87	127.79
	seq	122.50	119.64	120.57	123.29	122.86	118.73
	sim	122.37	124.53	123.14	123.29	122.99	119.74
$\mathcal{P}$	pma	25.30	23.00	28.80	26.12	30.10	40.83
	seq	30.24	20.65	23.73	29.98	27.37	31.31
	sim	23.87	32.80	25.11	33.17	35.76	29.86

Table B.15: Expected mean outcomes by experience in auction experiments (number of auction experiments participated in)

		0	1	2	3	4	5 or more
#subjects		144	37	14	8	6	2
$\mathcal{U}$	pma	9.81	9.77	10.39	9.13	11.09	-
	seq	9.07	7.01	7.87	-	3.05	11.86
	sim	7.03	7.43	7.96	2.96	4.75	-
$\mathcal{R}$	pma	115.22	113.95	112.61	115.17	107.38	-
	seq	113.53	117.53	110.87	-	106.82	111.44
	sim	116.26	115.32	115.67	117.14	119.54	-
$\mathcal{W}$	pma	125.04	123.72	123.00	124.30	118.47	-
	seq	122.60	124.54	118.74	-	109.87	123.30
	sim	123.29	122.75	123.63	120.10	124.30	-
$\mathcal{P}$	pma	28.54	26.09	24.23	28.57	14.22	-
	seq	27.05	35.07	21.73	-	13.64	22.88
	sim	32.51	30.64	31.34	34.29	39.09	-

Table B.16: Expected mean outcomes by experience in real life auctions (number of real life auctions participated in)

		0	1	2	3	4	5 or more
#subjects		111	35	19	5	6	35
$\mathcal{U}$	pma	9.68	9.78	11.86	11.33	8.00	10.15
	seq	8.50	8.83	9.83	13.06	3.05	8.28
	sim	8.03	6.27	4.50	-0.73	3.98	8.24
$\mathcal{R}$	pma	114.73	114.58	112.90	110.05	120.63	114.67
	seq	113.97	114.76	109.23	111.27	106.82	114.82
	sim	115.59	117.31	118.97	118.72	119.51	113.14
$\mathcal{W}$	pma	124.42	124.36	124.76	121.38	128.63	124.82
	seq	122.46	123.59	119.06	124.33	109.87	123.10
	sim	123.62	123.58	123.48	117.99	123.49	121.38
$\mathcal{P}$	pma	27.66	27.29	24.63	19.13	37.96	27.58
	seq	27.93	29.51	18.47	22.55	13.64	29.64
	sim	31.19	34.62	37.95	37.44	39.03	26.27

Table B.17: Equivalence tests of expected outcomes in PMA vs. SEQ, computerised

TOST Results PMA/c vs. SEQ/c								
							Equivalence Bounds	
		estimate	SE	t	df	p-value	CI lower	CI upper
payoff							*-1.746*	*1.746*
	t-test	1.24	0.33	3.70	2285.85	2.22E-04	0.687	1.788
	TOST Lower		0.33	8.92	2285.85	4.77E-19		
	TOST Upper		0.33	-1.52	2285.85	6.44E-02		
revenue							*-0.746*	*0.746*
	t-test	1.08	0.45	2.40	2649.20	1.65E-02	0.338	1.817
	TOST Lower		0.45	4.06	2649.20	2.53E-05		
	TOST Upper		0.45	0.74	2649.20	7.69E-01		
welfare							*-2.487*	*2.487*
	t-test	2.32	0.55	4.19	2754.83	2.86E-05	1.406	3.224
	TOST Lower		0.55	8.69	2754.83	2.95E-18		
	TOST Upper		0.55	-0.31	2754.83	3.78E-01		
payment							*-1.458*	*1.458*
	t-test	0.35	0.86	0.40	2512.79	6.88E-01	-1.069	1.760
	TOST Lower		0.86	2.10	2512.79	1.80E-02		
	TOST Upper		0.86	-1.29	2512.79	9.80E-02		

Note: Welch Two Sample t-test, $\alpha = 0.05$*, 90% confidence interval*

Table B.18: Equivalence tests of expected outcomes in PMA vs. SIM, computerised

TOST Results PMA/c vs. SIM/c								
							Equivalence Bounds	
		estimate	SE	t	df	p-value	CI lower	CI upper
payoff							*-1.680*	*1.680*
	t-test	3.03	0.34	8.96	2532.32	6.37E-19	2.475	3.589
	TOST Lower		0.34	13.92	2532.32	8.43E-43		
	TOST Upper		0.34	3.99	2532.32	1.00E+00		
revenue							*-0.714*	*0.714*
	t-test	-1.49	0.46	-3.22	2837.31	1.32E-03	-2.254	-0.728
	TOST Lower		0.46	-1.68	2837.31	9.53E-01		
	TOST Upper		0.46	-4.76	2837.31	1.04E-06		
welfare							*-2.385*	*2.385*
	t-test	1.54	0.52	2.95	2931.26	3.18E-03	0.682	2.400
	TOST Lower		0.52	7.52	2931.26	3.56E-14		
	TOST Upper		0.52	-1.62	2931.26	5.29E-02		
payment							*-1.391*	*1.391*
	t-test	-4.79	0.89	-5.39	2701.82	7.86E-08	-6.255	-3.327
	TOST Lower		0.89	-3.82	2701.82	1.00E+00		
	TOST Upper		0.89	-6.95	2701.82	2.30E-12		

Note: Welch Two Sample t-test, $\alpha = 0.05$*, 90% confidence interval*

Table B.19: Equivalence tests of expected outcomes in SEQ vs. SIM, computerised

TOST Results SEQ/c vs. SIM/c								
							Equivalence Bounds	
		estimate	SE	t	df	p-value	CI lower	CI upper
payoff							*-1.680*	*1.680*
	t-test	1.79	0.39	4.54	2691.26	5.75E-06	1.144	2.443
	TOST Lower		0.39	8.80	2691.26	1.19E-18		
	TOST Upper		0.39	0.29	2691.26	6.13E-01		
revenue							*-0.701*	*0.701*
	t-test	-2.57	0.49	-5.24	2697.59	1.74E-07	-3.375	-1.762
	TOST Lower		0.49	-3.81	2697.59	1.00E+00		
	TOST Upper		0.49	-6.67	2697.59	1.55E-11		
welfare							*-2.371*	*2.371*
	t-test	-0.77	0.52	-1.49	2606.46	1.37E-01	-1.632	0.083
	TOST Lower		0.52	3.06	2606.46	1.11E-03		
	TOST Upper		0.52	-6.04	2606.46	9.08E-10		
payment							*-1.403*	*1.403*
	t-test	-5.14	0.98	-5.24	2697.59	1.74E-07	-6.750	-3.523
	TOST Lower		0.98	-3.81	2697.59	1.00E+00		
	TOST Upper		0.98	-6.67	2697.59	1.55E-11		

Note: Welch Two Sample t-test, $\alpha = 0.05$*, 90% confidence interval*

B.2.3 Random effects model

Table B.20: Models with subject-specific random effects, testing for difference in expected outcomes, computerised environment

	Dependent variable:			
	Exp. payoff	Exp. revenue	Exp. welfare	Exp. payments
	(1)	(2)	(3)	(4)
PMA/c vs. SEQ/c	1.238*	1.078	2.315***	0.346
	(0.736)	(0.933)	(0.728)	(1.771)
SEQ/c	8.597***	113.651***	122.248***	27.301***
	(0.544)	(0.690)	(0.539)	(1.310)
Random effects:				
sd(Subject)	3.97	4.924	2.822	9.357
sd(Residual)	7.589	10.691	14.358	20.243
# subjects	139	139	139	139
PMA/c vs. SIM/c	3.032***	−1.491	1.541**	−4.791***
	(0.790)	(0.962)	(0.626)	(1.844)
SIM/c	6.804***	116.219***	123.023***	32.438***
	(0.566)	(0.689)	(0.449)	(1.321)
Random effects:				
sd(Subject)	4.463	5.262	2.134	10.094
sd(Residual)	7.965	11.415	14.092	21.833
# subjects	148	148	148	148
SEQ/c vs. SIM/c	1.794*	−2.568**	−0.775	−5.137**
	(0.978)	(1.017)	(0.701)	(2.034)
SIM/c	6.804***	116.219***	123.023***	32.438***
	(0.668)	(0.695)	(0.479)	(1.389)
Random effects:				
sd(Subject)	5.315	5.283	2.801	10.567
sd(Residual)	8.834	11.688	13.174	23.377
# subjects	135	135	135	135
Note:				*p<0.1; **p<0.05; ***p<0.01

B.2.4 Linear model with covariates

Table B.21: Linear model for expected outcomes. Reference category for 'Auction experiments participated in' and 'Real auctions participated in': 0, for all other categorical variables: 'Prefer not to say'

		Dependent variable:			
		Payoff	Revenue	Welfare	Payments
	PMA/c vs. SIM/c	3.589***	−1.330***	2.259***	−4.418***
	SEQ/c vs. SIM/c	1.751***	−2.689***	−0.938***	−5.293***
	valueA	−0.048***	0.135***	0.087***	0.239***
	valueB	0.202***	0.291***	0.493***	0.577***
Gender	Female	−5.547***	4.344***	−1.203	7.954***
	Male	−4.094***	2.679*	−1.414	4.687*
	Other	−1.200	−0.332	−1.532	−1.475
	Age	−0.054***	0.071***	0.017**	0.139***
Ethnicity	Arabic	4.117**	−2.867	1.251	−5.299
	Black	0.994	1.366	2.360***	2.132
	Central Asian	−1.840	0.740	−1.100	1.496
	East Asian	−3.241***	5.273***	2.032***	10.245***
	Latin American	−2.860	4.538**	1.678	9.189**
	South Asian	−2.610***	4.232***	1.622***	8.424***
	White	−0.908	1.803**	0.894*	3.518**
	Other	−0.093	1.533	1.439**	2.819
	Lower class	3.685***	−6.665***	−2.980***	−12.942***
	Working class	2.296***	−1.823***	0.473	−3.376***
	Lower middle class	1.898***	−1.697***	0.201	−3.325***
	Upper middle class	3.078***	−2.484***	0.593*	−4.770***
Income	£0	−1.967**	0.577	−1.389**	0.726
	£1 to £9,999	−0.140	1.272**	1.132***	2.395**
	£10,000 to £24,999	0.276	1.120**	1.396***	2.209**
	£25,000 to £49,999	−1.463***	3.208***	1.745***	6.189***
	£50,000 to £99,999	0.498	0.194	0.692*	0.155
	£100,000 to £149,999	−0.074	1.428	1.354*	2.693
Education	Upper secondary education	3.071**	0.503	3.574***	0.813
	Post-secondary non-tertiary education	3.938***	0.387	4.325***	0.703
	Short-cycle tertiary education	3.397**	2.129	5.527***	4.226
	Bachelor or equivalent	3.593***	−0.152	3.441***	−0.084
	Master or equivalent	3.136**	0.905	4.041***	1.818
	Doctoral or equivalent	2.785*	0.337	3.122***	0.677
	Risk av. self report (0-10)	−0.318***	0.297***	−0.021	0.580***
	Risk aversion (0-10)	0.297***	−0.313***	−0.017	−0.627***
	Risk av./Prob. weight (0-10)	0.167***	−0.228***	−0.060	−0.394***
	Ambiguity av. (0-20)	0.056**	−0.015	0.041***	−0.039
	Perceived competition (0-10)	0.120*	−0.111*	0.008	−0.169
	Perceived difficulty (0-10)	−0.420***	0.645***	0.225***	1.188***
Auction experiments participated in	1	−0.884**	1.342***	0.458*	2.768***
	2	0.970*	−2.484***	−1.514***	−4.589***
	3	−4.631***	3.176***	−1.455**	6.273***
	4	−1.232	−2.027**	−3.259***	−3.472*
	5 or more	2.650**	−1.547	1.103	−3.153
Real auctions participated in	1	−0.294	0.907**	0.613**	1.803**
	2	−0.151	−0.406	−0.557	−0.714
	3	1.274	−1.931**	−0.657	−3.532*
	4	−1.652	−1.053	−2.705***	−2.686
	5 or more	−0.077	0.029	−0.047	0.090
	Numeracy score (1-4)	0.068	0.258*	0.326***	0.516*
	#comprehension errors	−0.207***	0.215***	0.009	0.428***
	SIM/c	−1.173	80.768***	79.595***	−36.215***

Note: *p<0.1; **p<0.05; ***p<0.01

B.2.5 Additional tables

Table B.22: Average probabilities of winning both objects with optimal bidding

		Mean	SD	Median	Q1	Q3
P(win AB)	pma	0	0	0	0	0
	seq	0.015	0.010	0.013	0.006	0.023
	sim	0.036	0.028	0.031	0.013	0.056

Table B.23: Average probabilities of winning both objects with worst possible overbidding

		Mean	SD	Median	Q1	Q3
P(win AB)	pma	0	0	0	0	0
	seq	0.257	0.223	0.194	0.072	0.395
	sim	0.257	0.223	0.194	0.072	0.395

Table B.24: Average loss with worst possible overbidding

		Mean	SD	Median	Q1	Q3
$\mathcal{U}$	pma	0	0	0	0	0
	seq	-13.80	18.25	-5.70	-20.03	-1.00
	sim	-13.80	18.25	-5.70	-20.03	-1.00

C

Appendix for Chapter 4

C.1 Proofs

Proof of Lemma 5. This proof proceeds by analogy with Sun and Yang (2014) and Bikhchandani and Mamer (1997) who show the result for an environment with a single unit per variety for sale, and without explicit seller's costs. Let $\mathcal{A}$ denote the universe of all feasible allocations, i.e.

$$\mathcal{A} := \left\{ \pi \in \mathbb{Z}_+^{|2^N| \times |\mathcal{L}_0|} : \bigdoublecup_{l \in \mathcal{L}} \pi(l) = \mathbf{k} \text{ for some } \mathbf{k} \in \mathbb{K}, \text{ and } \pi(0) = \mathcal{N} - \mathbf{k}^* \right\}$$

(p^*, π^*) is a package-linear pricing Walrasian equilibrium, so for any bidder $l \in \mathcal{L}$ and any allocation $\pi' \in \mathcal{A}$, we have

$$V^l(\pi^*(l)) - \sum_{S \in \pi^*(l)} p^*(S) \geq V^l(\pi'(l)) - \sum_{S \in \pi'(l)} p^*(S)$$

Let $\bigdoublecup_{l \in \mathcal{L}} \pi^*(l) = \mathbf{k}$ and $\bigdoublecup_{l \in \mathcal{L}} \pi'(l) = \mathbf{k}'$. I sum over $l \in \mathcal{L}$ and add and subtract the seller's cost

$$\begin{aligned} &\sum_{l \in \mathcal{L}} V^l(\pi^*(l)) - C^0(\mathbf{k}) - \left(\sum_{l \in \mathcal{L}} V^l(\pi'(l)) - C^0(\mathbf{k}') \right) \\ \geq &\sum_{l \in \mathcal{L}} \sum_{S \in \pi^*(l)} p^*(S) - C^0(\mathbf{k}) - \left(\sum_{l \in \mathcal{L}} \sum_{S \in \pi'(l)} p^*(S) - C^0(\mathbf{k}') \right) \end{aligned} \tag{C.1}$$

Because $\pi^* \in S(p^*)$, we have, for all $\pi' \in \mathcal{A}$,

$$\sum_{l \in \mathcal{L}} \sum_{S \in \pi^*(l)} p^*(S) - C^0(\mathbf{k}) \geq \sum_{l \in \mathcal{L}} \sum_{S \in \pi'(l)} p^*(S) - C^0(\mathbf{k}')$$

From equation (C.1), it follows that, for all $\pi' \in \mathcal{A}$,

$$\sum_{l\in\mathcal{L}} V^l(\pi^*(l)) - C^0(\mathbf{k}) - \left(\sum_{l\in\mathcal{L}} V^l(\pi'(l)) - C^0(\mathbf{k}')\right) \geq 0$$

and so π^* is efficient.

Now let π' be an efficient allocation. Then $V(\mathcal{N}) = \sum_{l\in\mathcal{L}} V^l(\pi'(l)) - C^0(\mathbf{k}')$. It also holds that $V(\mathcal{N}) = \sum_{l\in\mathcal{L}} V^l(\pi^*(l)) - C^0(\mathbf{k})$ because π^* is efficient as part of the equilibrium. The equilibrium is also bidder-optimal and seller-optimal, given prices. Thus, we obtain the following two inequalities:

$$\begin{aligned}
&\mathcal{V}^l(p^*) \geq V^l(\pi'(l)) - \sum_{S\in\pi'(l)} p^*(S), \qquad \text{for all } l \in \mathcal{L} \text{ and} \\
&\sum_{l\in\mathcal{L}} \sum_{S\in\pi^*(l)} p^*(S) - C^0(\mathbf{k}) = \Pi(p^*) \geq \sum_{l\in\mathcal{L}} \sum_{S\in\pi'(l)} p^*(S) - C^0(\mathbf{k}')
\end{aligned}$$

Suppose that one of these two inequalities were strict, then we would obtain

$$\begin{aligned}
V(\mathcal{N}) &= \sum_{l\in\mathcal{L}} V^l(\pi^*(l)) - C^0(\mathbf{k}) \\
&= \sum_{l\in\mathcal{L}} \left[V^l(\pi^*(l)) - \sum_{S\in\pi^*(l)} p^*(S) + \sum_{S\in\pi^*(l)} p^*(S)\right] - C^0(\mathbf{k}) \\
&= \sum_{l\in\mathcal{L}} \mathcal{V}^l(p^*) + \Pi(p^*) \\
&> V^l(\pi'(l)) - \sum_{S\in\pi'(l)} p^*(S) + \sum_{l\in\mathcal{L}} \sum_{S\in\pi'(l)} p^*(S) - C^0(\mathbf{k}') \\
&= \sum_{l\in\mathcal{L}} V^l(\pi'(l)) - C^0(\mathbf{k}') \\
&= V(\mathcal{N}),
\end{aligned}$$

This is a contradiction, and consequently it holds that

$$\begin{aligned}
\mathcal{V}^l(p^*) = V^l(\pi'(l)) - \sum_{S\in\pi'(l)} p^*(S), \qquad &\text{for all } l \in \mathcal{L} \text{ and ,} \\
\Pi(p^*) = \sum_{l\in\mathcal{L}} \sum_{S\in\pi'(l)} p^*(S) - C^0(\mathbf{k}'), \quad &\text{i.e., } \pi' \in S(p^*).
\end{aligned}$$

It follows that (p^*, π') is also a package-linear pricing Walrasian equilibrium. □

Proof of Proposition 19. To prove Proposition 19 I show two things:

(a) There exists a mapping from V^l to a list $\{v_S^i, S \in 2^N, i \leq \overline{M}\}$.

(b) Suppose we have an allocation to bidder l given by $x_S^i, S \in 2^N, i \in \mathcal{I}^l$ and mapped into the multiset $\mathbf{k}^l$. Then the bidder's value V^l from this allocation is identical to the value entering the auctioneer's objective function.

(a) is straightforward given the structure on V^l, which is defined by bidder l's marginal values $v^l(q,S), S \in 2^N, q \leq \overline{M}$. The bidder can simply make a set of bids $\{(v^l(q,S),1)_{S\in 2^N}, q = 1,...,\overline{M}\}$. (b) follows from the bidder's aggregation of marginal values. The value entering the welfare maximising auctioneer's objective is simply given by $\sum_{S,i\in\mathcal{I}^l} v^i_S x^i_S$. An allocation to a bidder is easily aggregated to the corresponding multiset such that $k^l_S = \sum_{i\in\mathcal{I}^l} x^i_S$. For a given allocation $x^i_S, S \in 2^N, i \in \mathcal{I}^l$, the bidder's overall value is defined as

$$V^l(\{x^i_S\}) = \max_{\{x_{q,S}\}} \sum_{S\in 2^N}\sum_{q=1}^{\overline{M}} v^l(q,S)x_{q,S} \quad \text{s.t.} \sum_{q=1}^{\overline{M}} x_{q,S} \leq \sum_{i\in\mathcal{I}^l} x^i_S, \quad \sum_{S\in 2^N} x_{q,S} \leq 1 \ \forall q, x_{q,S} \in \{0,1\}$$

Note that every allocation $x^i_S, S \in 2^N, i \in \mathcal{I}^l$ respects $\sum_S x^i_S \leq \kappa^i$ because of Definition 18, and we can normalise $\kappa^i = 1$. Thus, $x_{q,S} = x^i_S$ is feasible in bidder l' value aggregation for some order of bids in $\mathcal{I}^l$. It follows immediately that $V^l(\{x^i_S\}) \geq \sum_{S,i\in\mathcal{I}^l} v^i_S x^i_S$. Suppose that $V^l(\{x^i_S\}) > \sum_{S,i\in\mathcal{I}^l} v^i_S x^i_S$. Then, the order in the marginal value aggregation of at least two packages must have switched compared to the auctioneer's assignment $\mathcal{I}^l$. But all marginal values were available to the auctioneer to choose from, i.e. the bidder's marginal value aggregation was a feasible allocation in the social welfare maximisation problem. Thus $\{x^i_S\}$ was not optimal. □

Proof of Proposition 20. Because the bidder's marginal values are superadditive between varieties and disjoint packages, the bidder could never gain from repackaging a given package into two (or more) disjoint subsets of that package. Consequently, repackaging can only occur if the bidder is allocated two disjoint packages and merges them.

Because the seller's marginal cost is subadditive between varieties and disjoint packages, she would never have a strict incentive to sell two disjoint packages separately if she could sell them bundled in one larger package. For contradiction, suppose that the bidder received two disjoint packages S_1 and S_2 on two different bids. Let $v^l(S_1,q)$ and $v^l(S_2,q')$ denote the corresponding marginal values submitted, with $q \geq q'$. Then, by assumption of MU-concave superadditive preferences, it must be that

$$v^l(S_1 \cup S_2, q) \geq v^l(S_1,q) + v^l(S_2,q') \tag{C.2}$$

By subadditivity of the seller's marginal cost, we have

$$c^0(q, S_1 \cup S_2, \mathbf{k}_{-(S_1 \cup S_2, S_1, S_2)}) \leq c^0(q, S_1, \mathbf{k}_{-(S_1 \cup S_2, S_1, S_2)}) + c^0(q, S_2, \mathbf{k}_{-(S_1 \cup S_2, S_1, S_2)}) \tag{C.3}$$

Adding equation C.2 and C.3 yields the auctioneer's (partial) social welfare objective

$$v^l(S_1 \cup S_2, q) - c^0(q, S_1 \cup S_2, \mathbf{k}_{-(S_1 \cup S_2, S_1, S_2)}) \geq$$
$$v^l(S_1, q) + v^l(S_2, q') - c^0(q, S_1, \mathbf{k}_{-(S_1 \cup S_2, S_1, S_2)}) - c^0(q, S_2, \mathbf{k}_{-(S_1 \cup S_2, S_1, S_2)}),$$

hence a contradiction to allocating S_1 and S_2 on two different bids. In case of indifference, the bidder must also be indifferent between repackaging S_1 and S_2 to $S_1 \cup S_2$, and leaving them as separate bundles. □

Proof of Lemma 6. First, I show that given a supply function graph G, any supply function allocation $(y_S^q)_{S \in 2^N, q \leq \bar{q}}$ maps into a unique vector $(k_S)_{S \in 2^N}$. The proof is constructive; I provide an algorithm to determine k_S.

Algorithm 3 Construct multiset from allocation on SFG

Input: supply function graph $G = (V, E)$, allocation $(y_S^q)_{S \in G, q \leq \bar{q}}$.
Output: $(k_S)_{S \in 2^N}$.

```
Initialise list of successfully visited nodes 𝒱 := ∅.
while V ≠ 𝒱 do
    Select a node S ∈ V \ 𝒱
    if ∃A ∈ V \ 𝒱 : ∃(A...S) then
        skip S
    else
        k_S = Σ_{q=1}^{q̄} y_S^q − Σ_{A∈𝒱:∃(A...S)} k_A
        mark S as successfully visited: 𝒱 = 𝒱 ∪ S
    end if
end while
```

To proof that algorithm 0 constructs a unique image from any permissible input, I demonstrate that (a) algorithm 0 stops, (b) each node S is successfully visited at some point, (c) the time at which S is successfully visited is irrelevant. That the mapping is linear is obvious from the definition of k_S in the algorithm.

(a) and (b) follow because Assumption 4(i) implies that there are no cycles in G. Thus, as long as $V \neq \mathscr{V}$, there exists some $S \in V \setminus \mathscr{V}$ for which the if-condition is false. Consequently, all nodes are added to $\mathscr{V}$ at some point. It is without loss of generality to assume that the algorithm does not get stuck in a trivial loop, i.e. it does not select a sequence of nodes that allow no successful visit and are

skipped and revisited indefinitely. (c) follows because once the if-condition is false for a given node S, the set of nodes $A \in \mathscr{V} : \exists(A...S)$ remains unaltered. Once S *could* be successfully visited, is does not matter when it is actually selected for the successful visit, i.e. other nodes may be selected first.

The reverse mapping is straightforward. Given a supply function graph G, a multiset $\mathbf{k} = (k_S)_{S \in 2^N}$ maps in the following supply function allocation: for all $S \in 2^N$, $y_S^{q_S} = 1 \; \forall q_S \leq \sum_{A:\exists(A...S)} k_A$ and $y_S^{q_S} = 0 \; \forall q_S > \sum_{A:\exists(A...S)} k_A$. □

Proof of Proposition 21. To prove this proposition I use a slightly different formulation of the auctioneer's partitioning problem. The supply function graph required that whenever a step on supply function f_S is allocated, a step on every supply function f_γ for all $\gamma : \exists(S...\gamma)$ is allocated as well. Therefore, I reformulate objective and constraints in the following way: Summing over the steps q, I introduce the identity of the package due to which a step on a supply curve is allocated. Formally, I call these steps $q'(S)$; they are the steps which are allocated on a given supply curve due to the allocation of package S. I suppress S in the problem; but it is important that the steps q' are different for each S.

Note that this problem formulation is not practical for solving the auction because the q' would have to be linked to the corresponding package S, and additional rules would have to be specified for which packages are allocated on which steps. However, this alternative problem formulation is useful to find out more about the price structure.

With the new definition of q', it holds that $\sum_{q'} y_S^{q'} = Y_S$. I rewrite LP as follows:

LP

$$\max_{\{x_S^i, y_S^{q'}\}} \left[\sum_{i,S} v_S^i x_S^i - \sum_{q'} y_S^{q'} \left(\sum_{\gamma:\exists(S...\gamma)} \mu_\gamma^{q'} \right) \right]$$

$$\text{s.t.} \quad \begin{cases} \sum_S x_S^i \leq \kappa^i & \forall i & b_i & \text{constraint on bid size} \\ y_S^{q'} \leq l_\gamma^{q'} & \forall q', \forall S, \forall \gamma : \exists(S...\gamma) & u_\gamma^{q'} & \text{constraint on step size} \\ \sum_i x_S^i - \sum_{q'} y_S^{q'} \leq 0 & \forall S & z_S & \text{supply constraint} \end{cases}$$

and all variables are non-negative. Then, the corresponding dual problem is

DLP

$$\min_{\{b^i, z_S, u_S^q\}} \left[\sum_i \kappa^i b^i + \sum_{q', S, \gamma:\exists(S...\gamma)} l_\gamma^{q'} u_\gamma^{q'} \right]$$

$$\text{s.t.} \quad \begin{cases} b^i + z_S \geq v_S^i & \forall i, S & x_S^i & \text{surplus constraint} \\ \sum_{\gamma:\exists(S...\gamma)} u_\gamma^{q'} - z_S \geq -\sum_{\gamma:\exists(S...\gamma)} \mu_\gamma^{q'} & \forall q', \forall S & y_S^{q'} & \text{marginal cost constraint} \end{cases}$$

and all variables b^i, $u_S^{q'}$ and z_S are non-negative. From the marginal cost constraint it follows that $z_S \leq \sum_{\gamma:\exists(S\ldots\gamma)} \left(u_\gamma^{q'} + \mu_\gamma^{q'}\right)$ for all S and for all q'. Complementary slackness implies that, if $y_S^{q'} > 0$, $z_S = \sum_{\gamma:\exists(S\ldots\gamma)} \left(u_\gamma^{q'} + \mu_\gamma^{q'}\right)$. As described above, $q' = q'(S)$ refers to the specific steps on which package S is allocated. However, in the original problem formulation, it does not matter which package is allocated on which step, because only the entirety of allocated steps is relevant for the seller's cost. Because $q'(S)$ could indeed be any of the steps on f_S, on which a positive amount is allocated, the equality holds for all $q_\gamma \leq \widetilde{q_\gamma}$, when $y_\gamma^{\widetilde{q_\gamma}} \neq 0$. The inequality holds for any $q \leq \bar{q}$ because of the marginal cost constraint. □

Proof of Corollary 1. From CC 7 and CC 8 follows

$$\mu_S^q \leq \psi(\{z_S\}) \leq \mu_S^{q+1}$$

Together with CSC 4, (i) follows. (ii) is immediate from Proposition 21 and CSC 4. □

Proof of Theorem 1. First, I note the complementary slackness conditions (CSC) from LP and DLP. The CSC are standard in linear programming; they are derived and used by analogy with Baldwin and Klemperer (2019*a*), and the subsequent argument follows Bikhchandani and Mamer (1997).

CSC 1 : If $x_S^i \neq 0$, then $b^i = v_S^i - z_S$, for all $i \in \mathcal{I}, S \subseteq N$.

CSC 2 : If $\sum_S x_S^i < \kappa^i$, then $b^i = 0$, for all $i \in \mathcal{I}$.

CSC 3 : If $\sum_i x_S^i - \phi(\{y_S^q\}) < 0$, then $z_S = 0$, for all $S \subseteq N$.

CSC 4 : If $y_S^q \neq 0$, then $u_S^q + \mu_S^q = \psi(\{z_S\})$, for all $q \leq \bar{q}, S \subseteq N$.

CSC 5 : If $y_S^q < l_S^q$, then $u_S^q = 0$, for all $q \leq \bar{q}, S \subseteq N$.

I establish the following lemma to help characterise a solution of LP.

Lemma 18. *If $\{x_S^i, y_S^q\}$ and $\{b^i, u_S^q, z_S\}$, respectively, are solutions to LP and DLP, $\{x_S^i, y_S^q\}$ and $\{b^i, u_S^q, z_S\}$ satisfy the following "characteristic conditions":*

CC 1 : $\sum_S x_S^i \leq \kappa^i$, for all $i \in \mathcal{I}$.

CC 2 : $y_S^q \leq l_S^q$, for all $q \leq \bar{q}, S \subseteq N$.

CC 3 : $\sum_{i,\gamma:\exists(\gamma\ldots S)} x_\gamma^i \leq \sum_q y_S^q$, for all $S \subseteq N$.

CC 4 : If $z_S > 0$, *then* $\sum_{i,\gamma:\exists(\gamma \ldots S)} x^i_\gamma = \sum_q y^q_S$, *for all* $S \subseteq N$.

CC 5 : If $v^i_S - z_S < \max_{S' \neq S}\{v^i_{S'} - z_{S'}, 0\}$, *then* $x^i_S = 0$, *for all* $i \in \mathcal{I}, S \subseteq N$.

CC 6 : If $\max_S\{v^i_S - z_S\} > 0$, *then* $\sum_S x^i_S = \kappa^i$, *for all* $i \in \mathcal{I}, S \subseteq N$.

CC 7 : If $\mu^q_S < \psi(\{z_S\})$, *then* $y^q_S = l^q_S$, *for all* $S \subseteq N$.

CC 8 : If $\mu^q_S > \psi(\{z_S\})$, *then* $y^q_S = 0$, *for all* $S \subseteq N$.

I also note again Corollary 1(ii): If $y^q_S > 0$, then for all $q' \leq q$, it holds that

$$\psi(\{z_S\}) = z_S - \sum_{\gamma:\exists(S \ldots \gamma),\gamma \neq S} \left(u^{q'}_\gamma + \mu^{q'}_\gamma\right) \tag{C.4}$$

The subsequent lemma makes the well known connection between a primal-dual solution and Walrasian equilibrium.

Lemma 19. *Prices* $\{z_S\}$ *given by the solution of DLP support the allocation* $\{x^i_S, y^q_S\}$ *given by the solution of LP as a package-linear pricing Walrasian equilibrium.*

Proof. Assume the allocation $\{x^i_S, y^q_S\}$ and prices $\{z_S\}$ are solutions of LP and DLP as defined above. Then, by Lemma 18, conditions CC 1 - CC 8 hold. I show that CC 1 - CC 8, together with the CSC and the constraints of LP and DLP, imply that the $\{z_S\}$ support $\{x^i_S, y^q_S\}$ as a package-linear pricing Walrasian equilibrium. I prove (a) that there is no surplus improvement possible for any bid i, (b) that for a bidder who received a multiset of packages no surplus improvement is possible from receiving a different partition of the multiset of packages, (c) that, given a partition of the allocated supply and given the final auction prices, the seller cannot improve her profit by allocating more or less of a given package, and (d) that, given a partition of the allocated supply and given the final auction prices, the seller cannot improve her profit by choosing a different partition of supply.

(a): If there is no positive surplus on any of the packages of a bid i, CC 5 ensures that this bid is allocated nothing. CC 6 implies that, if a strictly positive surplus can be made on any of the packages, the bid is allocated the maximum quantity bid for, and from CC 5 follows that the maximum quantity is allocated only on packages that maximise the bid's surplus. CC 1 ensures that the maximum quantity bid for is always respected.

(b): A bidder does not desire to repackage goods that he received in a certain partition (Proposition 20), or is prohibited from doing so, i.e. if he received the multiset of objects $\mathbf{k}$, he does not rearrange single varieties to obtain a different multiset $\mathbf{k}'$ with $\mathbf{k}^* = \mathbf{k}'^*$. However, I still have to show that the bidder would not be

strictly better off with a different partitioning of supply at the given auction prices. Let $\mathbf{k}$ denote the multiset of objects received by bidder l through the set of winning bids $\mathcal{W}^l$ (this set contains only those bids of l with a strictly positive allocation), i.e. $\mathbf{k} = (\sum_{i\in\mathcal{W}^l} x^i_S)_{S\in 2^N}$. First, recall Proposition 19: given an allocation $\mathbf{k}$, the bidder's value derived from $\mathbf{k}$ equals its contribution to social welfare as it appears in the auctioneer's objective, i.e. $V^l(\mathbf{k}) = \sum_{i\in\mathcal{W}^l,S} x^i_S v^i_S$. Then, the bidder's utility is given by $u^l(\mathbf{k}, z) = \sum_{i\in\mathcal{W}^l,S}(v^i_S - z_S)x^i_S$. Suppose there is a different partitioning $\mathbf{k}' \neq \mathbf{k}$ where $\mathbf{k}^* = \mathbf{k}'^*$, with a corresponding set of winning bids $\mathcal{W}'^l$ and allocation $\{x'^i_S\}$, which gives bidder l strictly higher utility, i.e. $u(\mathbf{k}', z) = \sum_{i\in\mathcal{W}'^l,S}(v^i_S - z_S)x'^i_S > u(\mathbf{k}, z)$. There exist at least two bids $i, j \in \mathcal{W}'$, for which $x'^i_S \neq x^i_S$ and $x'^j_S \neq x^j_S$ for some S. Let $\mathcal{D} := \{i \in \mathcal{W}^l \cap \mathcal{W}'^l : \exists S \text{ s.t. } x'^i_S \neq x^i_S\}$, i.e. $\mathcal{D}$ contains the winning bids in $\mathcal{W}$ which are also in $\mathcal{W}'$, but with some allocations shifted to different packages. By the surplus constraint of DLP we have $v^i_S - z_S \leq b^i$ for all $i \in \mathcal{D}, S \in 2^N$; so no additional surplus can be generated through shifting allocations on those bids. Let $\mathcal{C} := \mathcal{W}' \setminus \mathcal{W}$, i.e. the bids in $\mathcal{W}'$ that were not winning in the auction. By the contraposition of CC 6 it must hold that $z_S \geq v^i_S$ for all $i \in \mathcal{C}, S \in 2^N$. It follows that $u^l(\mathbf{k}', z) = \sum_{i\in\mathcal{W}',S} (v^i_S - z_S)\, x'^i_S \leq \sum_{i\in\mathcal{W}^l,S} b^i = u^l(\mathbf{k}, z)$, i.e. bidder l maximises his surplus with the multiset received in the partition prescribed by the auction.

(c): If a step on supply curve f_S is allocated, i.e. $y^q_S > 0$, then CC 8 implies $\psi(\{z_S\}) \geq \mu^q_S$. Together with equation (C.4) we have $z_S \geq \sum_{\gamma:\exists(S\ldots\gamma),\gamma\neq S} \left(u^q_\gamma + \mu^q_\gamma\right) + \mu^q_S$, i.e. the seller always sells package S at a weakly positive surplus. Furthermore, if $z_S > 0$, it follows by CC 4 that $\sum_{i,\gamma:\exists(\gamma\ldots S)} x^i_\gamma = \sum_q y^q_S \;\forall S \subseteq N$, i.e. the amount of all packages with cost relation to S sold equals $\sum_q y^q_S$. Suppose $z_S > \mu^q_S + \sum_{\gamma:\exists(S\ldots\gamma),\gamma\neq S} \left(u^q_\gamma + \mu^q_\gamma\right) \Leftrightarrow z_S - \sum_{\gamma:\exists(S\ldots\gamma),\gamma\neq S} \left(u^q_\gamma + \mu^q_\gamma\right) > \mu^q_S$, i.e. a strictly positive surplus is made on package S. Then, Proposition 21(i) implies that

$$\sum_{\gamma:\exists(S\ldots\gamma)} \left(u^q_\gamma + \mu^q_\gamma\right) - \sum_{\gamma:\exists(S\ldots\gamma),\gamma\neq S} \left(u^q_\gamma + \mu^q_\gamma\right) \geq z_S - \sum_{\gamma:\exists(S\ldots\gamma),\gamma\neq S} \left(u^q_\gamma + \mu^q_\gamma\right) > \mu^q_S$$

and thus $u^q_S + \mu^q_S > \mu^q_S$. With CSC 5, $u^q_S > 0$ implies that the entire supply function step is allocated. Conversely, let $z_S < \mu^q_S + \sum_{\gamma:\exists(S\ldots\gamma),\gamma\neq S} \left(u^q_\gamma + \mu^q_\gamma\right)$, i.e. a strict loss would be made on a package. For contradiction, suppose $y^q_S > 0$. Then, by equation (C.4), $\psi(\{z_S\}) < \mu^q_S$, and therefore, by CC 8, $y^q_S = 0$, a contradiction. Hence, the supply function step cannot be not allocated at all. Finally, by CC 2, a supply step is never allocated more than its maximum step size.

(d) I claim that, given a partition of supply that is a solution to LP, the seller cannot improve her profit by choosing a different partition of supply. To see this,

recall Lemma 6, which states that the mapping from the allocation on the supply function graph to a multiset of packages, i.e. $(y_S^q)_{S\in 2^N, q\leq \bar{q}} \to (Y_S)_{S\in 2^N}$, is one-to-one. In (c), I have shown that the seller cannot improve upon the allocation on the supply function graph, given the dual prices. Then, it is immediate that the resulting partition of allocated supply is seller-optimal also. □

Using the lemma above, a standard argument establishes the theorem. O_P denotes the value of an optimal solution to P, O_{LP} the value of an optimal solution to LP, and O_{DLP} the value of an optimal solution to DLP. First, let $O_P = O_{LP}$. Then, there exists $\{x_S^i, y_S^q\}$ as an optimal solution to IP and LP, and $\{x_S^i, y_S^q\}$ is efficient. By Lemma 19, the dual variables $\{z_S\}$ of DLP support this allocation as a package-linear pricing Walrasian equilibrium.

Now suppose there exists an equilibrium, i.e. prices $\{z_S\}$ that support $\{x_S^i, y_S^q\}$ as an equilibrium allocation. By Lemma 5, the allocation is efficient. Let $b_i := v_S^i - z_S$ for all $i \in \mathcal{I} : x_S^i > 0$, and let $u_S^q := \psi_S(\{z_S\}) - \mu_S^q$ for all $q \leq \bar{q}_S : y_S^q > 0, S \in 2^N$. Note that the u_S^q are defined recursively by equation C.4: for $j \in N$, we have $\psi(\{z_j\}) = z_j$, so $u_j^q := z_j - \mu_j^q$ for all $q \leq \bar{q}_j, j \in N$. Given the u_j^q, one can go on to define u_S^q for all S, for which it holds that for all $\{\gamma : \exists(S...\gamma), \gamma \neq S\}$, $\gamma \in N$, and so on. The u_S^q are the seller's surplus on each individual supply step, and the b^i are the buyers' surplus on each bid. Because the $\{z_S\}$ are Walrasian equilibrium prices, it must be that $u_S^q, b^i \geq 0$. Efficiency implies that

$$\sum_{i,S} v_S^i x_S^i - \sum_{S,q} \mu_S^q y_S^q \;\geq\; \sum_{i,S} v_S^i (x_S^i)' - \sum_{S,q} \mu_S^q (y_S^q)' \quad \forall \quad (x_S^i)', (y_S^q)'$$

i.e. u_S^q and b^i are feasible in DLP *and* $\{x_S^i, y_S^q\}$ are optimal in LP. By strong duality it holds that $O_{LP} = O_{DLP}$. Thus, we have

$$\begin{aligned}
O_{LP} &= O_{DLP} \\
&\overset{(1)}{\leq} \sum_i \kappa^i b^i + \sum_{q,S} l_S^q u_S^q \\
&\overset{(2)}{=} \sum_{S,i:x_S^i=\kappa^i} \kappa^i (v_S^i - z_S) + \sum_{q,S:y_S^q=l_S^q} l_S^q (\psi_S(\{z_S\}) - \mu_S^q) \\
&\overset{(3)}{=} \sum_{S,i:x_S^i=\kappa^i} \kappa^i (v_S^i - z_S) + \sum_{q,S:y_S^q=l_S^q} l_S^q (z_S - \sum_{\gamma:\exists(S...\gamma),\gamma\neq S} \left(u_\gamma^{q'} + \mu_\gamma^{q'}\right) - \mu_S^q) \\
&\overset{(4)}{=} \sum_{S,i} x_S^i (v_S^i - z_S) + \sum_S Y_S z_S - \sum_{q,S} y_S^q \mu_S^q \\
&\quad + \underbrace{\sum_S \left(\sum_{\gamma:\exists(\gamma...S),\gamma\neq S} Y_\gamma \right) z_S - \sum_S \left(\sum_{\gamma:\exists(\gamma...S)} Y_\gamma \right) \left(\sum_{\gamma:\exists(S...\gamma),\gamma\neq S} \left(u_\gamma^{q'} + \mu_\gamma^{q'}\right) \right)}_{:=F}
\end{aligned}$$

$$\overset{(5)}{=} \sum_{i,S} v_S^i x_S^i - \sum_{S,q} \mu_S^q y_S^q$$
$$\overset{(6)}{=} O_{IP}$$

(1) follows from DLP's objective function. (2) follows by definition of b_i and u_S^q above. (3) follows by equation (C.4). (4) follows because the amount on a given supply curve allocated equals the sum of the quantities of all packages sold that have a cost relation to the given supply curve, i.e.

$$\sum_{q,S:y_S^q=l_S^q} l_S^q z_S = \sum_S Y_S z_S + \sum_S \left(\sum_{\gamma:\exists(\gamma...S),\gamma\neq S} Y_\gamma \right) z_S$$

(5) follows because $\sum_i x_S^i = Y_S$ for all S for which $z_S > 0$, and because, using Proposition 21 (substituting for z_S) and simplifying sums, we have

$$F = \sum_S Y_S \left(\sum_{\gamma:\exists(S...\gamma)} \left(u_\gamma^{q'} + \mu_\gamma^{q'} \right) \right) - \sum_S \left(\sum_{\gamma:\exists(\gamma...S)} Y_\gamma \right) \left(\mu_S^{q'} + u_S^{q'} \right) = 0$$

The previous equation is equal to zero simply because of the distributive law in arithmetic. Finally, (6) follows from the definition of the auctioneer's indivisible allocation problem P.

I have shown that $O_{LP} \leq O_{IP}$. It also holds that $O_{LP} \geq O_{IP}$ because any solution of IP is feasible in LP, and the claim follows.

□

Proof of Lemma 18. CC 1 and CC 2 follow from the constraint on the bid size, and the constraint on the step size in LP. CC 3 is obtained by summing up the bundle S supply constraints from LP. In particular, I sum over all $\gamma : \exists(\gamma...S)$. By definition of the supply functions, $\sum_{\gamma:\exists(\gamma...S)} Y_\gamma = \sum_q y_S^q$ for any package $S \subseteq N$. CC 4 follows from CSC 3: the contrapositive of CSC 3 states that, if $z_S > 0$, $\sum_i x_S^i - \phi(\{y_S^q\}) = 0$ (> 0 is ruled out by the supply constraint of LP). If $z_S > 0$, then by Proposition 21, Requirement 1, and because $u_S^q \geq 0 \ \forall S, q$, it holds that $z_\gamma > 0$ for all $\gamma : \exists(\gamma...S)$ *on which some* y_γ^q is allocated. Therefore, I can take the sum of tight supply constraints across $\gamma : \exists(\gamma...S)$ (if $x_\gamma^i > 0$ for some i, then we must also have $y_\gamma^q > 0$ for some q); we obtain CC 4. CC 5 is derived from CSC 1: Assume $x_S^i > 0$ and $x_{S'}^i > 0$ for some $S \neq S'$. Then, by CSC 1, $v_S^i - z_S = v_{S'}^i - z_{S'} = b^i$. So if $v_S^i - z_S < v_{S'}^i - z_{S'}$ for some S', then $x_S^i = 0$. Also note that it always holds that $b^i \geq 0$ and $b^i \geq v_S^i - z_S$, by the surplus constraint of DLP. So if $v_S^i - z_S < 0$, then it cannot be that $v_S^i - z_S = b^i$, and thus $x_S^i = 0$. Together, we obtain CC 5.

$b^i \geq v^i_S - z_S$ also implies that, if $\max_S\{v^i_S - z_S\} > 0$, then $b^i > 0$. The contrapositive of CSC 2 then implies $\sum_S x^i_S = \kappa^i$, and thus CC 6. The marginal cost constraint of DLP is $u^q_S \geq \psi(\{z_S\}) - \mu^q_S$. Thus, if $\mu^q_S < \psi(\{z_S\})$, then $u^q_S > 0$, and so CC 7 follows by the contrapositive of CSC 5. Finally, if $\mu^q_S > \psi(\{z_S\})$, then we cannot have $u^q_S = \psi(\{z_S\}) - \mu^q_S$, because u^q_S is positive. Hence, the contrapositive of CSC 4 implies $y^q_S = 0$, and therefore CC 8. □

Proof of Lemma 7. First, note that for complete CFG, the definition of marginal cost simplifies to

$$c^0(k, S, \mathbf{k}_{-(B_1,\ldots,B_t)}) = \sum_{\gamma \subseteq S} \Delta c(q_\gamma, \gamma), \qquad \text{where } q_\gamma := \sum_{A \supset \gamma, A \in 2^N \setminus (B_1,\ldots,B_t)} k_A + k$$

Let $B := (S_1 \cup S_2, S_1, S_2)$ and let $q_\gamma := \sum_{A \supset \gamma, A \in 2^N \setminus B} k_A + k$. Then we have, for any disjoint sets $S_1, S_2 \in 2^N$,

$$\begin{aligned}
&c^0(k, S_1, \mathbf{k}_{-B}) + c^0(k, S_2, \mathbf{k}_{-B}) \\
=&\sum_{\gamma \subseteq S_1} \Delta c\,(q_\gamma, \gamma) + \sum_{\gamma \subseteq S_2} \Delta c\,(q_\gamma, \gamma) \\
=&\sum_{\gamma \subseteq S_1 \cup S_2} \Delta c\,(q_\gamma, \gamma) - \sum_{\gamma \subseteq S_1 \cup S_2, \gamma \not\subseteq S_1, S_2} \Delta c\,(q_\gamma, \gamma) \\
\geq&\sum_{\gamma \subseteq S_1 \cup S_2} \Delta c\,(q_\gamma, \gamma) \\
=&c^0(k, S_1 \cup S_2, \mathbf{k}_{-S})
\end{aligned}$$

□

Proof of Lemma 8. Let $W \subseteq S \subseteq \delta \subseteq N$, and let $y(S, \delta, t, W)$ denote the amount allocated on supply function f_S that is due to the allocation of a bundle $\delta \supset_t W$, t levels above W. Let $dist(x, y) := ||x| - |y||$ for any $x, y \subseteq N$. I first establish a series of facts.

Fact (1). Given any reference supply set $W \subseteq S$, I can write the amount allocated on supply function f_S as

$$\sum_q y^q_S = \sum_{t=dist(S,W)}^{n-|W|} \sum_{\delta \supset_t W} y(S, \delta, t, W).$$

Lemma 20. *Given are sets $x \subseteq z \subseteq N$, and a number q with $|x| \leq q \leq |z|$. Let $R := \{y \subseteq N | x \subseteq y \subseteq z, |y| = q\}$. Then, $|R| = \binom{|z|-|x|}{q-|x|}$.*

Proof. This is a standard combinatorics problem. First, note that $q - |x|$ objects can be added to x such that y contains q objects. These objects also need to be different from those contained in x, and they need to be contained in z. Hence, there are $|z| - |x|$ different objects, of which $q - |x|$ many can be added to x. This is possible in $\binom{|z|-|x|}{q-|x|}$ different ways. □

Fact (2). Given S, δ, W, let $r := dist(S, W)$ and $t := dist(\delta, W)$. By Lemma 20 above, there exist $\binom{t}{r}$ supply functions f_S relative to f_W, on which the amount $y(S, \delta, t, W)$ is allocated. Of course, t (or δ or W) is redundant; it is only written for clarity of exposition.

Fact (3). $y(S, \delta, t, W)$ does not depend on S. If a step on supply function f_δ is allocated, then a step on each supply function $\gamma \subseteq \delta$ is allocated. Thus, the allocation of bundles on supply functions in the graph "between" W and δ *due to the allocation of bundle* δ has to be the same amount.

Fact (4). For any $t \geq 1$, we have, by the binomial theorem, $\sum_{r=0}^{t} \binom{t}{r}(-1)^r = 0$.

We now have

$$
\begin{aligned}
& \sum_{r=0}^{n-|S|} \sum_{q} \sum_{\gamma \supset_r S} (-1)^r y_\gamma^q \\
= \; & \sum_{r=0}^{n-|S|} \sum_{\gamma \supset_r S} (-1)^r \sum_{q} y_\gamma^q \\
\overset{(1)}{=} \; & \sum_{r=0}^{n-|S|} \sum_{\gamma \supset_r S} (-1)^r \sum_{t=dist(\gamma,S)}^{n-|S|} \sum_{\delta \supset_t S} y(\gamma, \delta, t, S) \\
= \; & \sum_{r=0}^{n-|S|} \sum_{\gamma \supset_r S} (-1)^r \sum_{t=r}^{n-|S|} \sum_{\delta \supset_t S} y(\gamma, \delta, t, S) \\
\overset{(2),(3)}{=} \; & \sum_{r=0}^{n-|S|} (-1)^r \sum_{t=r}^{n-|S|} \binom{t}{r} \sum_{\delta \supset_t S} y(\cdot, \delta, t, S) \\
\overset{(4)}{=} \; & \sum_{t=0}^{n-|S|} \sum_{\delta \supset_t S} y(\cdot, \delta, t, S) \sum_{r=0}^{t} (-1)^r \binom{t}{r} \\
= \; & \sum_{\delta \supset_0 S} y(\cdot, \delta, 0, S) \sum_{r=0}^{0} (-1)^r \binom{0}{r} \\
= \; & y(\cdot, S, 0, S) \\
= \; & Y_S
\end{aligned}
$$

□

Proof of Lemma 9. To simplify notation, I write $c^*(v^0, S)$ as $c^*(S)$ for any $S \in 2^N$. Let $S_1^c, S_2^c \in 2^N$ and $S_1^c \cap S_2^c = \emptyset$. Note that $c^*(N) = v^0(N)$. Also note

that $S_1^c \cap S_2^c = \emptyset \Leftrightarrow S_1 \cup S_2 = N$. Because v^0 is superadditive we have

$$\begin{aligned} & v^0(S_1^c \cup S_2^c) \geq v^0(S_1^c) + v^0(S_2^c) \\ \Leftrightarrow \quad & v^0(N) - c^*((S_1^c \cup S_2^c)^c) \geq 2v^0(N) - c^*(S_1) - c^*(S_2) \\ \Leftrightarrow \quad & c^*(S_1) + c^*(S_2) \geq c^*(S_1 \cup S_2) + c^*(S_1 \cap S_2) \end{aligned}$$

The proof for subadditive v^0 is analogous to the above. □

Proof of Proposition 22. In SY, the seller's supply correspondence is defined as

$$S^{SY}(p) = \arg\max_{\delta \in \mathbb{K}} \left\{ \sum_{A \in \delta} p(A) \right\}$$

In my ascending auction the seller's supply correspondence is defined as

$$S(p) = \arg\max_{\delta \in \mathbb{K}} \left\{ \sum_{A \in \delta} \Big(p(A) - c^0(A) \Big) \right\}$$

By definition of SY's ascending auction, $p(B) = v^0(B)$ for any bundle B that is assigned to the seller during the procedure; and from the proof of SY's Theorem 2, we know that $p(\pi(0)) = v^0(\pi(0))$. In partition δ, where bundle $B \in \delta$ goes to the seller, we have $p(B) = v^0(B)$. Hence, we have

$$\begin{aligned} S^{SY}(p) &= \arg\max_{\delta \in \mathbb{K}} \left\{ \sum_{A \in \delta \setminus B} p(A) + v^0(B) \right\} \\ &= \arg\max_{\delta \in \mathbb{K}} \left\{ \sum_{A \in \delta \setminus B} p(A) + v^0(B) - v^0(N) \right\} \\ &= \arg\max_{\delta \in \mathbb{K}} \left\{ \sum_{A \in \delta \setminus B} p(A) - c^*(N \setminus B) \right\} \\ &= \arg\max_{\delta \in \mathbb{K}} \left\{ \sum_{A \in \delta \setminus B} p(A) - c^* \left(\bigcup_{A \in \delta \setminus B} A \right) \right\} \end{aligned}$$

Part (i) and (ii) of the proposition immediately follow, as c^* is by definition the dual of v^0, and $c^* \left(\bigcup_{A \in \delta \setminus B} A \right)$ may be interpreted as the seller's cost function. □

Proof of Theorem 2. The proof proceeds exactly as the proof of Theorem 2 in Sun and Yang (2014). The auction procedure terminates at some time t^*, because demand will cease entirely as soon as the price of a package exceeds the package's maximum value among all bidders. The price of the empty package is always zero. Let $p^* = p(t^*)$ and let $A_l^* = A_l(t^*)$. Furthermore, let $\delta^* = \delta(t^*) \in S(p^*)$ denote

the supply set in $S(p^*)$ that is chosen at time t^* by the seller. First, I establish an allocation π^* such that (p^*, π^*) constitutes a package-linear Walrasian equilibrium. Because at p^* no package is overdemanded, for any bidder $l \in \mathcal{L}$ with $A_l^* \neq \emptyset$, his demand A_l^* must be in δ^*. If $\bigcup_{l\in\mathcal{L}} A_l^* = N$ holds, define the allocation as follows: $\pi^*(l) = A_l^*$ for all $l \in \mathcal{L}$ and $\pi^*(0) = \emptyset$. Then (p^*, π^*) is a package-linear Walrasian equilibrium.

If $\bigcup_{l\in\mathcal{L}} A_l^* \subset N$, there is at least one package B in the chosen supply set δ^* which is not demanded by any bidder at time t^* (in SY such package is called a *squeezed-out* package). Now we have to distinguish multiple cases:

Case 1: $p^*(B) = c^0(B)$

The final auction price of bundle B is still fixed at the initial reserve price. This means, B was never overdemanded. If it was demanded in some earlier round by some bidder, this bidder demands now a different, more profitable package. Let

$$\delta_0^* = \{B \in \delta^* \mid p^*(B) = c^0(B) \text{ and } B \neq A_l^* \text{ for all } l \in \mathcal{L}\}$$

be the set of all squeezed-out packages. Let $\pi^*(0) = \bigcup_{B\in\delta_0^*} B$ and allocate $\pi^*(0)$ to the seller (at zero cost). Let $\mathbb{K}_0^*$ denote the universe of all partitions of the elements contained in δ_0^*. Because $\delta^* \in S(p^*)$, we have

$$\sum_{B\in\gamma} \left[p^*(B) - c^0(B)\right] \leq \sum_{B\in\delta_0^*} \left[p^*(B) - c^0(B)\right] = 0$$

for all $\gamma \in \mathbb{K}_0^*$. Trivially, the partition of objects retained by the seller is irrelevant, and the seller is indifferent between selling $\pi^*(0)$ or not.

Case 2: $p^*(B) > c^0(B)$

Because $p^*(B) > c^0(B)$, package B was demanded by some bidder in an earlier round. I denote by t the last round in which B was demanded by some bidder l. Just as in SY, the auction rule determines that package B may be allocated to bidder l, at the current price $p^*(B)$. Hence, we need to demonstrate that it is still profit-maximising for bidder l to receive package B at the current price. By the auction rule and Assumption 3, we must have

$$\mathcal{V}^l(p(t)) = u^l(B, p(t)) = v^l(B) - p(t, B) \geq 1 \tag{C.5}$$

and $p^*(B) = p(t, B)$ or $p^*(B) = p(t, B) + 1$. Given equation (C.5) we have for bidder l, who is allocated the squeezed-out package B,

$$u^l(B, p^*) = v^l(B) - p^*(B) \geq 0 \tag{C.6}$$

By analogy with SY, I distinguish the following two sub-cases.

Case 2A: If $A_l^* = \emptyset$, simply assign bidder l the squeezed-out bundle, i.e. $\pi^*(l) = B$. $A_l^* \in D^l(p^*)$ and $A_l^* = \emptyset$ imply that $\mathcal{V}^l(p^*) = 0$. By definition of $\mathcal{V}^l$ we have $\mathcal{V}^l(p^*) \geq u^l(B, p^*)$. Together with equation (C.6) this implies $u^l(B, p^*) = 0$, and hence $\pi^*(l) \in D^l(p^*)$.

Case 2B: If $A_l^* \neq \emptyset$, we assign bidder l what he demanded at time t^* *and* the squeezed-out bundle, i.e. $\pi^*(l) = A_l^* \cup B$. Because the seller chose a supply set $\delta^* \ni \{A_l^*, B\}$, we have

$$p^*(A_l^*) - c^0(A_l^*) + p^*(B) - c^0(B) \geq p^*(\pi^*(l)) - c^0(\pi^*(l)). \tag{C.7}$$

Superadditivity of bidder l's utility implies that

$$v^l(\pi^*(l)) \geq v^l(A_l^*) + v^l(B). \tag{C.8}$$

Subadditivity of the seller's cost implies that

$$c^0(\pi^*(l)) \leq c^0(A_l^*) + c^0(B). \tag{C.9}$$

And because $A_l^* \in D^l(p^*)$, we have

$$v^l(A_l^*) - p^*(A_l^*) \geq v^l(\pi^*(l)) - p^*(\pi^*(l)) \text{ and} \tag{C.10}$$

$$v^l(A_l^*) - p^*(A_l^*) \geq 0. \tag{C.11}$$

From (C.7) and (C.9) follows

$$p^*(A_l^*) + p^*(B) \geq p^*(\pi^*(l)) \tag{C.12}$$

Then, using equation (C.12), (C.8), and (C.6) (in this order), we obtain

$$\begin{aligned} v^l(\pi^*(l)) - p^*(\pi^*(l)) &\geq v^l(\pi^*(l)) - [p^*(A_l^*) + p^*(B)] \\ &\geq \left[v^l(A_l^*) - p^*(A_l^*)\right] + \left[v^l(B) - p^*(B)\right] \\ &\geq v^l(A_l^*) - p^*(A_l^*), \end{aligned}$$

Using equation (C.10), it follows that

$$\begin{aligned} v^l(\pi^*(l)) - p^*(\pi^*(l)) &= v^l(\pi^*(l)) - [p^*(A_l^*) + p^*(B)] \\ &= v^l(A_l^*) - p^*(A_l^*), \end{aligned}$$

and therefore

$$p^*(\pi^*(l)) = p^*(A_l^*) + p^*(B). \tag{C.13}$$

Bidder l is therefore happy to receive bundle B in addition to his demanded bundle A_k^*, and pay the price that is set for the bundle $\pi^*(l)$.
Exactly as in SY, this process can be repeated for every squeezed-out bundle B with $p^*(B) > c^0(B)$. Every bidder l who is not allocated any squeezed-out bundle receives his demanded package, i.e. $\pi^*(l) = A_l^*$. δ^* is a partition of N chosen by the seller, and thus $(\pi^*(0), \ldots, \pi^*(L))$ is an allocation of N. The seller's profit is, from equation (C.13),

$$\sum_{l\in\mathcal{L}} \Big[p^*(\pi^*(l)) - c^0(\pi^*(l))\Big] = \sum_{A\in\delta^*} \Big[p^*(A) - c^0(A)\Big] = \Pi(p^*)$$

and consequently, the allocation π^* is in the seller's supply correspondence. It follows that (p^*, π^*) is a package-linear pricing Walrasian equilibrium. □

Proof of Lemma 10. In SY, an allocation π is efficient if it holds for every allocation π' that

$$\sum_{l\in\mathcal{L}_0} \Big[v^l(\pi(l))\Big] \geq \sum_{l\in\mathcal{L}_0} \Big[v^l(\pi'(l))\Big] \tag{C.14}$$

In my model, an allocation π is efficient if for every allocation π' it holds that

$$\sum_{l\in\mathcal{L}} \Big[v^l(\pi(l)) - c^0(\pi(l))\Big] \geq \sum_{l\in\mathcal{L}} \Big[v^l(\pi'(l)) - c^0(\pi'(l))\Big] \tag{C.15}$$

The SY auction terminates in an allocation π that is efficient in the sense of their definition (theorem 2 in SY), i.e. if equation (C.14) holds for all allocations π'. In my model, the ascending auction procedure also terminates in an efficient allocation π in the sense of my definition, i.e. if equation (C.15) holds for all allocations π'. The efficient allocation in the buy-back auction procedure is equivalent to the efficient allocation in SY: setting $c^0(S) := 0\ \forall S \in 2^N$ and adding the seller as a participant in the auction, i.e. running the auction with the set of buyers $\mathcal{L}' = \mathcal{L} + \{0\} = \mathcal{L}_0$, (C.14) and (C.15) are equivalent. The auction rules apart from the seller's choosing of the supply set are identical in SY and my auction, so the claim follows. □

Proof of Proposition 23. Note that every conventional bidder bids identically in both procedures, up to ties. I simply split up the SY-seller into 2^n dummy bidders, denoted by $l_S, S \in 2^N$. Define bidder l_S's utility function as follows:

$$v^{l_S}(B) := \begin{cases} v^0(S) & \text{if } B \supseteq S \\ 0 & \text{otherwise} \end{cases}$$

Each bidder l_S has the highest bid on bundle S among all dummy bidders, because v^0 is superadditive. Let bidder l_S demand bundle S whenever he weakly prefers to demand bundle S to another bundle, but let him demand the empty set when she weakly prefers to do so.

Let the ascending auction start at $t = -1$ with starting prices $p(-1, S) = v^0(S) - 1 \ \forall S \in 2^N$. Let two instances of each bidder l_S participate. Bidders $l_S, S \in 2^N$ each demand bundle S. The seller (my seller) offers some supply set. Regardless of the non-dummy bidders' demand, each bundle $S \in 2^N$ is overdemanded in $t = -1$, so prices in $t = 0$ are increased by one. The dummy bidders all demand the empty set for all $t = 0, 1, \ldots$, so if at some round $t \geq 0$ the auction ends with squeezed-out bundles, they can be allocated to the dummy bidders if they were the last to demand them. It is without loss of generality to stipulate that such squeezed-out bundles *are* allocated to dummy bidders (if they were the last to demand them), and not to regular bidders who might have demanded them at $t = -1$ as well. Then, in all rounds $t = 0, 1, \ldots$, the supply correspondence and the demand correspondences are chosen to maximise identical profit and utility functions in both auction procedures. Thus, the supply correspondence and demand correspondences are identical in every round of both auction procedures, and it immediately follows that an identical price path resulting in the same allocation exists. □

C.2 The seller's demand type

If all buyers have strong substitutes valuations and supply is fixed, it is well known that Walrasian equilibrium exists. If the seller is partitioning a supply of different objects into packages, however, her preferences are not strong-substitutes-between-packages under non-linear (or package-linear) pricing (see also Footnote 8). Consequently, a Walrasian non-linear pricing equilibrium need not exist. Example 2 in Sun and Yang (2014) (also Beviá et al. (1999)) illustrates this: There are three objects A, B, C for sale, one unit of each, and there are three bidders 1,2, and 3. The bidders' values are given in Table C.1, and the seller's reserve prices are zero for all possible packages. It can be verified that Walrasian equilibrium does not exists under linear pricing Beviá et al. (1999) or non-linear pricing Sun and Yang (2014). All bidders demand only one package, hence their valuation is trivially strong-substitutes-between-packages type. However, the seller's valuation is *not* strong-substitutes-between-packages. The reason is that she may be induced by an infinitesimal price change to switch from, e.g., selling the partition $\{AB\}$ to selling

the partition $\{A, B\}$. Baldwin and Klemperer (2019*b*) show that a valuation is strong-substitutes if and only if it is concave and corresponds to a strong substitutes demand type, which is defined by strong substitute vectors with at most one +1 entry, at most one -1 entry, and no other nonzero entries. A demand type contains all the vectors describing the directions in which demand could change due to an infinitesimal generic price change. The seller's "demand" type characterisation would contain the vectors $\pm(1, 1, 0, -1, 0, 0, 0)$. The entries of the vector correspond to the change in supply of packages $(A, B, C, AB, AC, BC, ABC)$ induced by an arbitrary, small price change, i.e. there exists an arbitrary, small price change that could make the seller prefer selling the partition $\{AB\}$ to selling the partition $\{A, B\}$. Hence, the seller's demand type and valuation are not strong-substitutes-between packages.

	$\emptyset$	A	B	C	AB	AC	BC	ABC
Bidder 1	0	10	8	2	13	11	9	14
Bidder 2	0	8	5	10	13	14	13	15
Bidder 3	0	1	1	8	2	9	9	10

Table C.1: Bidders' values over objects

C.3 Examples and additional figures

C.3.1 Example for Section 4.3

Example 6. Consider the sale of two units of good A and two units of good B, which may be sold separately or in packages. Only $\{AB\}$ is considered a package; the multiset $\{A, A, B\}$, could only be bundled as $\{\{AB\}, \{A\}\}$, but not as $\{\{AA\}, \{B\}\}$.

Bidder values. Suppose there is one bidder, "bidder 1", with the MU-concave superadditive value function given by Table C.2. Then, for $\mathbf{k} = (k_A, k_B, k_{AB})$, we have, e.g., $V^1(1, 0, 0) = 3$, $V^1(1, 0, 1) = 9 + 3$, $V^1(0, 1, 1) = 5 + 9$, $V^1(1, 1, 0) = 5 + 1$, and $V^1(1, 1, 1) = 5 + 9 + 0$. Note that the bidder cannot repackage objects A and B if he receives them separately, so if he receives three distinct packages one of them is worthless to him. Consider now a different MU-concave superadditive value function given in Table C.3. In Table C.2 we had $V^1(0, 1, 1) = v^1(1, B) + v^1(2, AB) = 5 + 9$; in this example, we have $V^1(0, 1, 1) = v^1(1, AB) + v^1(2, B) = 9 + 2$. Of course a convention could be introduced according to which $V^1(0, 1, 1) = v^1(1, B) + v^1(2, AB) = 5 + 5$ in this example also. However, it would be difficult to argue why to choose such convention over another. The marginal value aggregation by

Table C.2: Bidder 1's marginal values

q	$v^1(q,A)$	$v^1(q,B)$	$v^1(q,AB)$
1	3	5	9
2	1	2	9
>2	0	0	0

Table C.3: Bidder 1's marginal values

q	$v^1(q,A)$	$v^1(q,B)$	$v^1(q,AB)$
1	3	5	9
2	1	2	5
>2	0	0	0

Definition 5 seems least restrictive to us. One may reasonably require, however, that bidders are capable of repackaging the objects they receive, in accordance with Definition 6. Suppose this is the case; then, if the bidder was given A and B in separate packages, we have $V^1(1,1,0) = v^1(1,B) + v^1(2,A) = 5+1$ and $\widetilde{V}^1(1,1,0) = v^1(1,AB) = 9$.

Seller's cost. Suppose there is a seller with two units of A and two units of B available for sale. Her preference is given first as her marginal cost function in Table C.4. Note that $k_{-S} = (k_B, k_{AB})$ if $S = A$, $k_{-S} = (k_A, k_{AB})$

Table C.4: Seller's marginal cost

k_{-S}	$c^0(1,A,k_{-S})$	$c^0(2,A,k_{-S})$	$c^0(1,B,k_{-S})$	$c^0(2,B,k_{-S})$	$c^0(1,AB,k_{-S})$	$c^0(2,AB,k_{-S})$
(0,0)	1	2	1	2	1	4
(1,0)	1	2	1	2	2	∞
(0,1)	2	∞	2	∞	2	∞
(1,1)	2	∞	2	∞	3	∞
(0,2)	∞	∞	∞	∞	∞	∞
(2,0)	1	2	1	2	∞	∞
(2,2)	∞	∞	∞	∞	∞	∞
(1,2)	∞	∞	∞	∞	∞	∞
(2,1)	2	∞	2	∞	∞	∞

if $S = B$, and $k_{-S} = (k_A, k_B)$ if $S = AB$. The overall cost function is not straightforwardly defined; take for example $C^0(1,1,1)$. We could plausibly have $C^0(1,1,1) = c^0(1,A,(0,0)) + c^0(1,B,(1,0)) + c^0(1,AB,(1,1)) = 1+1+3$, or $C^0(1,1,1) = c^0(1,AB,(0,0)) + c^0(1,B,(0,1)) + c^0(1,A,(1,1)) = 1+2+2$, or $C^0(1,1,1) = c^0(1,A,(1,1)) + c^0(1,B,(1,1)) + c^0(1,AB,(1,1)) = 2+2+3$, etc., depending on how one defines marginal costs of different goods to interact with one another.

The representation of the seller's preferences in terms of incremental costs and cost relations has two advantages: first, it is much more compact, and second, it allows for a simple and *unambiguous* aggregation of incremental costs to overall costs. The incremental cost representation of the same preference is given in Table 5 and the graph in Figure 6. Then, by Definition 13 above, the overall cost is given by, e.g., $C^0(2,1,0) = \Delta c(1,A) + \Delta c(2,A) + \Delta c(1,B)$, and

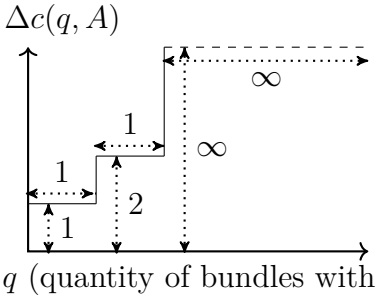


Figure C.2: Marginal cost function/supply function f_A submitted by the seller

$C^0(1,1,1) = \Delta c(1,A) + \Delta c(1,B) + \Delta c(2,A) + \Delta c(2,B) + \Delta c(1,AB).$

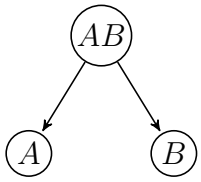


Figure C.1: Cost function graph

q	$\Delta c(q,A)$	$\Delta c(q,B)$	$\Delta c(q,AB)$
1	1	1	-1
2	2	2	0
> 2	∞	∞	∞

Table C.5: Incremental marginal cost

Value representation. Bidder 1's preferences (Table C.2) can be expressed simply by submitting his two sets of marginal values in two XOR-bids, $(v_A, v_B, v_{AB}, \kappa) = (3,5,9,1)$ and $(1,2,9,1)$. These bids can be represented geometrically; e.g. $(v_A, v_B, v_{AB}, \kappa) = (3,5,9,1)$ is shown in Fig. C.6 in Appendix C.3.3.
Cost representation. The preference given in Table 6 and Example 6 can be directly submitted to the auctioneer and serves as input in the auction algorithm. Note again the compactness of the bidding language compared to expressing marginal costs directly (Table C.4). $\Delta c(q,S)$ is now simply renamed into μ_S^q. The incremental marginal cost function corresponding to package A from Table 6 is given in Fig. C.2.
Solving the auctions. Bidder 1 submits the bids $i = 1, 2$, and there are two additional bidders who submit bids 3 and 4, respectively. The seller submits the supply schedule discussed above. Supply schedule and bids are listed as auction input in Table C.6. The efficient allocation is underlined in the bid list. The seller has a cost saving if she sells the first units of A and B as a package, hence bid 4 obtains $\{AB\}$. For the second units, the seller is indifferent between selling separately or as a package, so bid 1 and 3 win. Prices have to satisfy the set of equations corresponding to the constraints of DLP. One can verify that the set of equilibrium prices is given by

$(z_A, z_B, z_{AB}) \in \{(4,5,9),(5,4,9),(5,5,9),(5,5,10)\}$, which all support the unique equilibrium allocation. Note that a lowest equilibrium price vector does not exist, but a highest equilibrium price vector does. Proposition 1 (ii) tells us, e.g., that $z_{AB} = z_A + z_B + \mu^1_{AB} + u^1_{AB}$. Corollary 1 (i) tells us that $\mu^1_{AB} + u^1_{AB} \leq \mu^2_{AB} = 0$. Thus, equilibrium prices must satisfy either $z_A + z_B = z_{AB}$ or $z_A + z_B = z_{AB} - 1$. The surplus constraints pin down the set of equilibrium prices precisely.

Supply functions f_A, f_B, f_{AB}

q	μ^q_A	μ^q_B	μ^q_{AB}
1	1	1	-1
2	2	2	0
> 2	∞	∞	∞

List of XOR-bids submitted

i	v^i_A	v^i_B	v^i_{AB}	κ^i
1	3	$\underline{5}$	9	1
2	1	2	9	1
3	$\underline{5}$	3	8	1
4	6	2	$\underline{11}$	1

Table C.6: Auction input

C.3.2 Examples for Section 4.4

Example 7. In Table C.7, c^0 is a subadditive. Its dual $v^*(c^0, \cdot)$ is set-cover supermodular by Lemma 9, but of course it is also subadditive. The strictly subadditive elements A, B, and AB are untouched by the set-cover requirement.

In Table C.8, v^0 is superadditive. Its dual is set-cover submodular, and elements A, BC, and ABC are strictly subadditive. However, it also exhibits strictly superadditive elements B,C, and BC. Strictly superadditive and strictly subadditive elements of a cost function correspond to negative and positive synergies between objects that are sold together. Selling B costs the seller 2, and selling C costs the seller 0. Selling $\{BC\}$ or $\{B, C\}$ (each object to a different bidder) costs the seller 3. But adding A to the set of objects sold, i.e. selling any partition of $\{A, B, C\}$, takes away the cost increment of 1 the seller incurred from selling B and C: Any partition of $\{A, B, C\}$ now costs merely 4. Hence, in this example both negative and positive synergies are present.

Table C.7: Set-cover supermodular and subadditive function

S	A	B	C	AB	AC	BC	ABC
$c^0(S)$	3	2	0	4	2	1	4
$v^*(c^0, S)$	3	2	0	4	2	1	4

Table C.8: Superadditive v^0 and set-cover submodular c^*

S	A	B	C	AB	AC	BC	ABC
$v^0(S)$	1	2	0	4	2	2	4
$c^*(v^0, S)$	2	2	0	4	2	3	4

Example 8. Table C.9 states the bidders' and the seller's values for goods A, B, and the package AB. Bidders are numbered from B1 to B6, the seller is denoted by S. The dual of these values yields $c^0(A) = 4$, $c^0(B) = 6$, and $c^0(AB) = 8$. Table C.10 illustrates the basic ascending auction from SY (the seller chooses a revenue-maximising supply set wrt. v^0, or a profit-maximising supply set wrt. $C^0(\delta) = c^0\left(\bigcup_{S\in\delta} S\right)$, in each round). Table C.11 illustrates my procedure, i.e. the ascending auction where the seller chooses a profit-maximising supply set wrt. $C^0(\delta) = \sum_{S\in\delta} c^0(S)$ in each round. Both procedures terminate in the respective efficient allocation defined in the corresponding environment: in SY's model, the two individual objects A and B are allocated, e.g. to B1 and B3; in my model, the bundle AB is allocated, e.g. to B5.

Table C.9: Values and costs

	A	B	AB
B1	5	0	5
B2	5	0	5
B3	0	7	7
B4	0	7	7
B5	0	0	11
B6	0	0	11
S (values)	2	4	8
S (costs)	4	6	8

Table C.10: Revenue-maximising ascending auction

current price	supply set	demand B1	B2	B3	B4	B5	B6	squeezed-out
$p(0) = (2, 4, 8)$	$\{AB\}$	A	A	B	B	AB	AB	
$p(1) = (3, 5, 9)$	$\{AB\}$	A	A	B	B	AB	AB	
$p(2) = (4, 6, 10)$	$\{AB\}$	A	A	B	B	AB	AB	
$p(3) = (5, 7, 11)$	$\{A, B\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	A,B

Table C.11: Profit-maximising ascending auction, cost = sum of individual package costs

current price	supply set	demand B1	B2	B3	B4	B5	B6	squeezed-out
$p(0) = (4, 6, 8)$	$\{AB\}$	A	A	B	B	AB	AB	
$p(1) = (5, 7, 9)$	$\{A, B\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	AB	AB	
$p(2) = (5, 7, 10)$	$\{AB\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	AB	AB	
$p(3) = (5, 7, 11)$	$\{AB\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	AB

C.3.3 Additional figures

Different examples for cost/supply function graphs, referred to in the main text, are shown in Figs. C.3 to C.5.

Fig. C.6 illustrates the package XOR-bid $(v_A, v_B, v_{AB}, \kappa) = (3, 5, 9, 1)$ geometrically. The black dot at $(3, 5, 9)$ corresponds to the bid. Beyond the light blue faces,

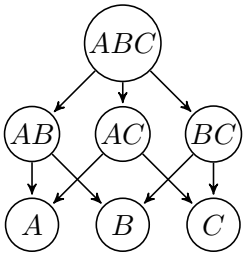


Figure C.3: Complete CFG with three different varieties A, B, C

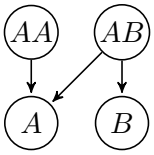


Figure C.4: SFG with package AA

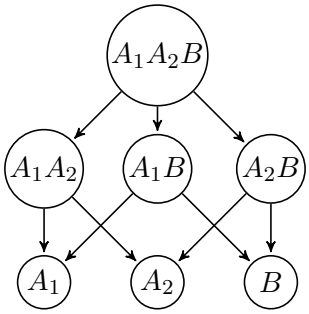


Figure C.5: Augmented SFG with three different varieties A_1, A_2, B

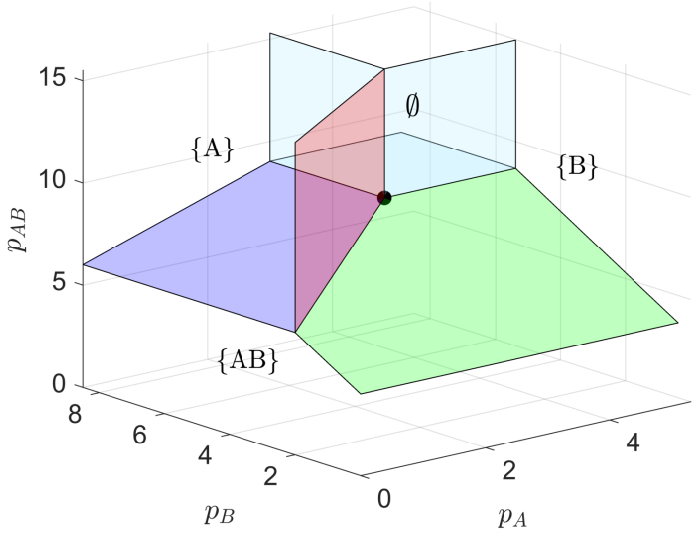


Figure C.6: Combination XOR-bid for three packages

the bid demands nothing because prices are too high. Consider now the area that is not beyond the light blue faces; it is divided in three areas. For prices above the blue face and left of the red face bundle $\{A\}$ is demanded, for prices above the green face and right of the red face bundle $\{B\}$ is demanded, and for priced below the blue and the green face bundle $\{AB\}$ is demanded. At prices that lie on the red face, the bid is indifferent between receiving $\{A\}$ or $\{B\}$, and similarly for prices on the green or blue face. At prices that lie on the intersection of all three faces, the bid is indifferent between all three bundles. At prices on any of the light blue faces, the bid is indifferent between receiving nothing and the bundle from the neighbouring demand region. A detailed description of the geometric representation of such preferences is given in Baldwin and Klemperer (2019*b*).

D
Appendix for Chapter 5

D.1 On concave functions

The following example can be used to show revenue and welfare do not generally coincide when we move away from additive valuations, even when we have only one good. The valuation function is chosen to be fairly 'benign'. Note that the budget constraints never apply, so this example also applies to general quasi-linear utilities (without budget constraints).

Example 9 (Concave valuation)**.** Consider an auction with a single good available in $s = 3$ units that has a single buyer. The buyer has valuation $v : \mathbb{R} \to \mathbb{R}$ given by $v(x) = \frac{4}{\log 2}(1 - 2^{-x})$ and budget 2. Then revenue is not maximised at market-clearing prices.

Indeed, note that the utility of the buyer for quantity q at price p is $u(x, p) = v(x) - px$, so the buyer's demand $D(p)$ at p is found by solving $v'(x) = p$, which yields $x = -\log_2(\frac{p}{4})$. At $p = 0.5$, we have demand $x = 3$, so p clears the market. Revenue at p is $px = 1.5$. At price $q = 1$, we have demand $y = 2$, so q does not clear the market, but revenue is $qy = 2$, which is greater. Revenue is maximised at $p = \frac{4}{e}$ with a demanded quantity of $\frac{1}{\log(2)}$ and a revenue of $\frac{4}{e \log(2)}$.

Proof of Proposition 24. Recall that at any price p, the buyer demands the bundle x that maximises $v(x) - p$. Given valuation $v(x)$, revenue is maximised at $(x, p) \in \arg\max_{x,p} px$ such that $v'(x) = p$ and $x \leq s$. Thus, maximal revenue given v and s is $v'(x)x$ for some $x \leq s$. Social welfare is maximised at $(x, p) \in \arg\max_{x,p} v(x) - px$ such that $x \leq s$, i.e. revenue at the social optimum is $v'(s)s$.

We show that there exists x and s with $x < s$ and $v'(x)x > v'(s)s$. Recall that a differentiable function $f : \mathbb{R} \to \mathbb{R}$ is strongly concave if it satisfies $|f'(x) - f'(y)| \geq m\|x - y\|$ for all distinct $x, y \in \mathbb{R}$ (v does not need to be twice differentiable).

First, note that there exists $\overline{x} < \infty$ such that $v'(x) < mx$ for all $x \geq \overline{x}$, due to strict concavity of v. Fix some supply $s \geq \overline{x}$ and let $\varepsilon = \frac{1}{2}(ms - v'(s))$. As v is strongly concave, we also have $v'(s - \varepsilon) \geq v'(s) + m\varepsilon$ for any ε. Hence,

$$\begin{aligned} v'(s-\epsilon)(s-\epsilon) &\geq (v'(s) + m\epsilon)(s-\epsilon) \\ &= v'(s)s + \epsilon(ms - v'(s) - m\epsilon) \\ &> v'(s)s. \end{aligned}$$

□

Proof of Proposition 25. First, suppose that there exists supply $s \in X$ with $v'(s) < ms$. Then the result follows analogous to the proof of Proposition 24.

We now prove the second part of the statement. The budget constraint is given by $px \leq \beta$. At the demanded quantity, $v'(x) = p$ holds. Thus, for all feasible x, it must hold that $v'(x) \leq \beta/x$, i.e. $x \leq \tilde{x}$. Now we demonstrate that $v'(\tilde{x} - \epsilon) \leq m(\tilde{x} - \epsilon)$ for some small ϵ. Then the result follows from Proposition 25. For some $\delta > 0$, we have

$$\begin{aligned} m(\tilde{x} - \epsilon) &\geq \left(\frac{v'(\tilde{x})}{\tilde{x}} + \delta\right)(\tilde{x} - \epsilon) \\ &= \frac{(v'(\tilde{x}) + \delta\tilde{x})(\tilde{x} - \epsilon)}{\tilde{x}} \\ &\geq v'(\tilde{x} - \epsilon) \end{aligned}$$

The last inequality holds for $\epsilon \to 0$ and some $\delta > 0$. □

D.2 Omitted proofs

It is known (Chen et al., 2007) that the market we consider has a unique market-clearing price vector (but allocations at these prices may not be unique). Here we provide a direct proof of this.

Theorem 6. There is a unique price vector $\boldsymbol{p}^*$ at which the market is cleared.

Proof. Suppose we have a valid allocation π that exhausts supply $\boldsymbol{s}$. We want to find the corresponding prices $\boldsymbol{p}$ that support π. First, we formulate the problem of finding $\boldsymbol{p} \geq 0$ as a system of linear inequalities by writing out the budget constraints as equalities and the demand constraints as inequalities. The latter is achieved by adding, for every buyer i who demands a positive amount of good j, the inequalities

$\frac{v_j^i}{p_j} \geq \frac{v_k^i}{p_k}, \forall k \in [n]_0$. Our goal is to prove that there is exactly one solution to this system of inequalities. In Section 5.4, we show that the elementwise-minimal prices $\boldsymbol{p}^*$ clear the market, so the polytope defined by this system has at least one solution.[1]

For contradiction, suppose there is a second feasible price vector $\boldsymbol{p} \geq \boldsymbol{p}^*$ that also clears the market. Then by convexity of the polytope, it follows that $\boldsymbol{p}' = \boldsymbol{p}^* + \varepsilon(\boldsymbol{p} - \boldsymbol{p}^*)$ is also market-clearing for any $0 \leq \varepsilon \leq 1$. From now on, we assume that $\boldsymbol{p}'$ is obtained by letting ε be infinitesimally small. Let $S := \{j \in [n] \mid p_j^* < p_j\}$ denote the prices that change when we move from $\boldsymbol{p}^*$ to $\boldsymbol{p}'$. As $\boldsymbol{p} \neq \boldsymbol{p}^*$, S is non-empty. We first make a technical observation about the demand of each bid at $\boldsymbol{p}^*$ and $\boldsymbol{p}'$.

Observation 1. For any buyer $i \in [m]$, we note: (i) If $J^i(\boldsymbol{p}^*) \not\subseteq S$, then $J^i(\boldsymbol{p}') = J^i(\boldsymbol{p}^*) \setminus S$. (ii) If $J^i(\boldsymbol{p}^*) \subseteq S$, then $J^i(\boldsymbol{p}') \subseteq J^i(\boldsymbol{p}^*)$. This holds because ε was chosen to be infinitesimally small.

Now suppose e' is a valid expenditure at $\boldsymbol{p}'$ that clears the market. Let $\mathcal{S} := \{i \in [m] \mid J^i(\boldsymbol{p}^*) \subseteq S\}$ denote the set of buyers satisfying case (ii) of Observation 1. We take a look at the revenue contributions from each buyer towards goods in S. At $\boldsymbol{p}'$, Observation 1 tells us that only buyers $i \in \mathcal{S}$ spend on goods in S, and they each spend their entire budget on S. Hence, the budget and supply feasibility of e' imply $\sum_{i \in \mathcal{S}} \beta^i = e'(\mathcal{S}, S) = e'([m], S) = \sum_{j \in S} p'_j s_j$. At $\boldsymbol{p}^*$, the buyers of $\mathcal{S}$ also spend their entire budget on S, but it may be the case that other buyers also spend part of their budget on S. Hence, $\sum_{i \in \mathcal{S}} \beta^i = e(\mathcal{S}, S) \leq e([m], S) = \sum_{j \in S} p_j^* s_j$. It follows that $\sum_{j \in S} p'_j s_j \leq \sum_{j \in S} p_j^* s_j$, in contradiction to the fact that $p'_j > p_j^*$ for all $j \in S$. □

D.2.1 Welfare theorems

We show that the first and second welfare theorem hold in our market. First we define efficiency formally.

Definition 27 (Efficiency)**.** A feasible allocation π is *efficient* if $\sum_{i \in [m], j \in [n]} v_j \pi(i,j) \geq \sum_{i \in [m], j \in [n]} v_j \pi'(i,j)$ for all feasible allocations $\pi' \neq \pi$.

Proposition 27. *Suppose π is a budget-feasible allocation at prices $\boldsymbol{p}$ and fix buyer i. Then $\sum_{j \in [n]_0} (v_j^i - p_j)\pi(i,j) \geq \sum_{j \in [n]_0} (v_j^i - p_j)\pi'(i,j)$ for all budget-feasible allocations π' if, and only if, $\pi(i) \in D^i(\boldsymbol{p})$.*

[1] Indeed, the algorithm in Section 5.5 finds these market-clearing prices.

Proof. This follows immediately from budget-constrained quasi-linearity. □

Theorem 7 (First welfare theorem)**.** If $(\boldsymbol{p}, \pi)$ is a competitive equilibrium, π is an efficient allocation.

Proof. Let $(\boldsymbol{p}, \pi)$ be a competitive equilibrium with corresponding expenditure e. Let π' be a feasible allocation at $\boldsymbol{p}$ with expenditure e'. Proposition 27 implies

$$\sum_{i\in[m],j\in[n]} (v_j^i - p_j)\pi(i,j) \geq \sum_{i\in[m],j\in[n]} (v_j^i - p_j)\pi'(i,j).$$

Rearranging and using the fact that $e([m],[n]) = \boldsymbol{p}\cdot\boldsymbol{s} \geq e'([m],[n])$, we have

$$\sum_{i\in[m],j\in[n]} v_j^i\pi(i,j) - \sum_{i\in[m],j\in[n]} v_j^i\pi'(i,j) \geq e([m],[n]) - e'([m],[n]) \geq 0.$$

□

Theorem 8 (Second welfare theorem)**.** Let $(\boldsymbol{p}, \pi)$ be a competitive equilibrium and π' be another efficient and feasible allocation. Then $(\boldsymbol{p}, \pi')$ is also a competitive equilibrium.

Proof. By definition of efficiency, we first note that

$$\sum_{i\in[m],j\in[n]} v_j^i\pi(i,j) = \sum_{i\in[m],j\in[n]} v_j^i\pi'(i,j). \tag{D.1}$$

Secondly, as π is a competitive equilibrium and π' is feasible, we have $\pi([m]) = \boldsymbol{s}$ and $\pi'([m]) \leq \boldsymbol{s}$. Multiplying both terms with $\boldsymbol{p}$, we get

$$\boldsymbol{p}\cdot\pi([m]) \geq \boldsymbol{p}\cdot\pi'([m]). \tag{D.2}$$

We now show that equality must hold for (D.2) by demonstrating that strict inequality contradicts (D.1). This proves our claim. Indeed, using (D.2) with strict inequality and Proposition 27, we obtain

$$\begin{aligned}
\sum_{i\in[m],j\in[n]} v_j^i\pi(i,j) &= \sum_{i\in[m],j\in[n]} (v_j^i - p_j)\pi(i,j) + \boldsymbol{p}\cdot\pi([m]) && \text{(D.3)}\\
&> \sum_{i\in[m],j\in[n]} (v_j^i - p_j)\pi'(i,j) + \boldsymbol{p}\cdot\pi'([m]) && \text{(D.4)}\\
&= \sum_{i\in[m],j\in[n]} v_j^i\pi'(i,j). && \text{(D.5)}
\end{aligned}$$

□

D.3 Further examples

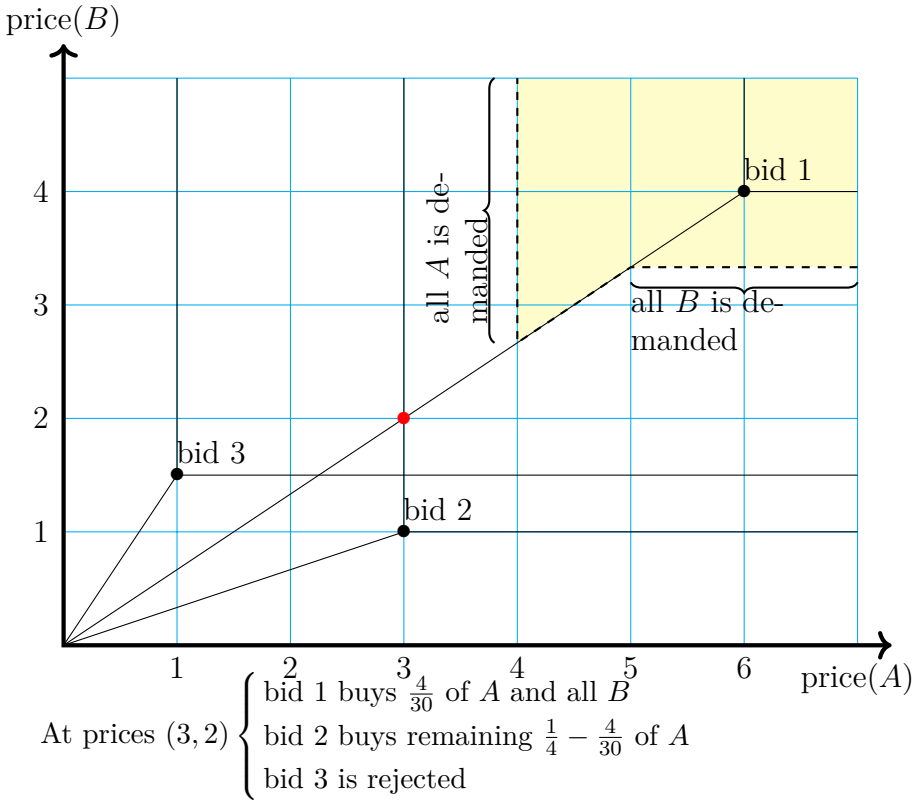


Figure D.1: Example with 3 bids each of weight (or, budget) 1. With a supply of $\frac{1}{4}$ units of A and $\frac{3}{10}$ of B, the yellow region is "feasible prices" where at most this supply is demanded. It includes a 1-dimensional line between $(4, \frac{8}{3})$ and $(3, 2)$. It has the key property that for any pair of feasible price vectors, the coordinate-wise minimum of them also belongs to the region. The red dot at $(3, 2)$ is the unique coordinate-wise minimal price where this supply is demanded. Bid 2 is partially fulfilled: we prioritise bid 1 to maximise welfare. Notice that if we increase the weights of bids 2 or 3, that has no effect on the final price vector.

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