

THE GEOMETRY OF ONE-RELATOR GROUPS SATISFYING A POLYNOMIAL ISOPERIMETRIC INEQUALITY

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ABSTRACT. For every pair of positive integers $p > q$ we construct a one-relator group $R_{p,q}$ whose Dehn function is $\simeq n^{2\alpha}$ where $\alpha = \log_2(2p/q)$. The group $R_{p,q}$ has no subgroup isomorphic to a Baumslag–Solitar group $BS(m, n)$ with $m \neq \pm n$, but is not automatic, not $CAT(0)$, and cannot act freely on a $CAT(0)$ cube complex. This answers a long-standing question on the automaticity of one-relator groups and gives counterexamples to a conjecture of Wise.

1. INTRODUCTION

A classical topic in combinatorial and geometric group theory is *one-relator groups*, that is, groups that can be defined by a presentation with only one relator. Magnus proved that a one-relator group has solvable word problem, but the algorithmic complexity of the word problem remains unknown. A geometric measure of this complexity is given by the Dehn function (see [Bri02] for a survey). The Dehn function of a one-relator group can grow very quickly: the group $\langle a, t \mid a^{(a^t)} = a^2 \rangle$ has Dehn function $\text{tower}_2(\log_2(n))$, which is not bounded by any finite tower of exponents, but its word problem is nonetheless solvable in polynomial time [MUW11]. This is conjecturally the largest Dehn function of a one-relator group; Bernasconi proved a weaker uniform upper bound, namely the Ackermann function [Ber94].

On the other hand, much less is known about the intricacies of the geometry of one-relator groups satisfying a polynomial isoperimetric inequality, that is, whose Dehn function is bounded by a polynomial. All previously known examples are hyperbolic or more generally automatic (see [ECH⁺92] for background on automatic groups), and thus have linear or quadratic Dehn function. For example, every one-relator group with torsion is hyperbolic, and Wise has proved they are virtually special [Wis11].

A standard obstruction to a group having desirable geometry is the presence of a subgroup isomorphic to the Baumslag–Solitar group $BS(m, n) = \langle a, t \mid t^{-1}a^mt = a^n \rangle$ for

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some $m \neq \pm n$: this group has a distorted cyclic subgroup, and its Dehn function is exponential. A distorted cyclic subgroup rules out being hyperbolic, acting properly cocompactly on a CAT(0) space, or acting freely on a CAT(0) cube complex (of possibly infinite dimension). A torsion-free one-relator group has geometric dimension 2 [Lyn50] so by a theorem of Gersten such a Baumslag–Solitar subgroup gives an exponential lower bound on Dehn function [Ger92a, Theorem C], further ruling out automaticity which would require a quadratic isoperimetric inequality.

It has been asked whether such Baumslag–Solitar subgroups are the only pathologies for one-relator groups:

Question 1. Is it true that one-relator groups with no subgroups isomorphic to $BS(m, n)$, for $m \neq \pm n$, are automatic?

Conjecture 2 (Wise, [Wis14, 1.9]). Every [torsion-free] one-relator group with no subgroup isomorphic to $BS(m, n)$, for $m \neq \pm n$, acts freely on a CAT(0) cube complex.

Question 1 was articulated when the theory of automatic groups was first developing [Ger92b, Problem 11 ff.] and was posed more recently by Myasnikov–Ushakov–Won [MUW11, 1.5]. If true, it would imply that all polynomial Dehn functions of one-relator groups are linear or quadratic.

The first-named author introduced in his thesis [Gar17] the one-relator groups

$$R(m, n, k, l) := \langle x, y, t \mid x^m = y^n, t^{-1}x^kt = x^ly \rangle \cong \langle x, t \mid x^m(x^{-l}t^{-1}x^kt)^{-n} \rangle$$

for $|m|, |n| \geq 2$, $k \neq 0$, $l \not\equiv 0 \pmod m$. He then proved that they have no distorted Baumslag–Solitar subgroups, and that they are CAT(0) precisely when $|k| > |l + \frac{m}{n}|$. In this paper we consider a subfamily of these groups: define

$$R_{p,q} := R(2, 2, 2q, 2p - 1) \cong \langle x, y, t \mid x^2 = y^2, t^{-1}x^{2q}t = x^{2p-1}y \rangle.$$

Theorem A. *Let $p > q$ be positive integers. The one-relator group $R_{p,q}$ has Dehn function $\simeq n^{2\alpha}$ where $\alpha = \log_2(2p/q)$. In particular, it has no subgroup isomorphic to a Baumslag–Solitar group $BS(m, n)$ with $m \neq \pm n$, but is not automatic and not CAT(0).*

This answers Question 1 negatively. The key observation is that $R_{p,q}$ is virtually a *tubular group*. A group is tubular if it splits as a finite graph of groups with \mathbb{Z}^2 vertex groups and \mathbb{Z} edge groups.

Proof of Theorem A. It is demonstrated in Theorem B below that $R_{p,q}$ has an index two subgroup that is isomorphic to the Brady–Bridson snowflake (tubular) group $G_{p,q}$. These groups are discussed in full in Section 2, but the salient fact is that $G_{p,q}$ has Dehn function $\simeq n^{2\alpha}$ where $\alpha = \log_2(2p/q) > 1$; as the Dehn function is invariant up to finite index subgroups the first part of the statement holds. In contrast, automatic

and CAT(0) groups have at most quadratic Dehn function. Since $R_{p,q}$ is of geometric dimension 2 we conclude from Gersten's theorem that there are no such Baumslag–Solitar subgroups as their presence would force at least exponential Dehn function. \square

Remark 3. Jack Button has shown that for odd $q \geq 3$, the group $R_{1,q}$ is *not* residually finite, giving the first examples of one-relator groups that are CAT(0) (by [Gar17, Theorem G]) but not residually finite [But17]. The proof shows that $G_{1,q}$ is non-Hopfian, which implies that $G_{1,q}$ and thus $R_{1,q}$ are not equationally Noetherian, resolving [Bau99, Problem 1]; in fact one can extend Button's surjective endomorphism of $G_{1,q}$ with non-trivial kernel to show that $R_{1,q}$ itself is non-Hopfian, resolving [Bau99, Problem 7]. Button has informed us that he also has completely determined for which p and q the group $R_{p,q}$ is residually finite.

In [Wis14], Wise classified the tubular groups that act freely on CAT(0) cube complexes. In Section 3 we apply this classification to disprove Conjecture 2. The Dehn function is a quasi-isometry invariant, so there are infinitely many quasi-isometry types of counterexamples to Question 1 and Conjecture 2.

2. VIRTUALLY SNOWFLAKE GROUPS

In [BB00] Brady and Bridson studied the groups

$$G_{p,q} := \langle a, b, s, t \mid [a, b], s^{-1}a^q s = a^p b, t^{-1}a^q t = a^p b^{-1} \rangle$$

defined for positive integers p and q . Due to the suggestive nature of their van Kampen diagrams, these are called *snowflake groups*. The main theorem of their paper states that for $p \geq q$, the Dehn function of $G_{p,q}$ is $\simeq n^{2\alpha}$ where $\alpha = \log_2(2p/q)$. This gives the Dehn function of $R_{p,q}$, via the following:

Theorem B. *The snowflake group $G_{p,q}$ is an index 2 subgroup of the one-relator group $R_{p,q}$.*

Proof. First, we re-write the presentation of R to exploit the fact that $\langle x, y \mid x^2 = y^2 \rangle$ is the fundamental group of the Klein bottle: we map $x \mapsto a$ and $y \mapsto ab$ to get

$$R_{p,q} \cong \langle a, b, t \mid a^{-1}bab, t^{-1}a^{2q}t = a^{2p}b \rangle.$$

Let X be the graph of spaces for $R_{p,q}$ with a vertex space a Klein bottle and edge space a cylinder. We can assume that the attaching maps are geodesics in the Klein bottle as in Figure 1. Let $X' \rightarrow X$ be the index two regular cover corresponding to the map to $\mathbb{Z}/2$ defined by $a \mapsto 1$, $b \mapsto 0$ and $t \mapsto 0$, indicated in Figure 1; on the Klein bottle subspace this is just the oriented double cover.

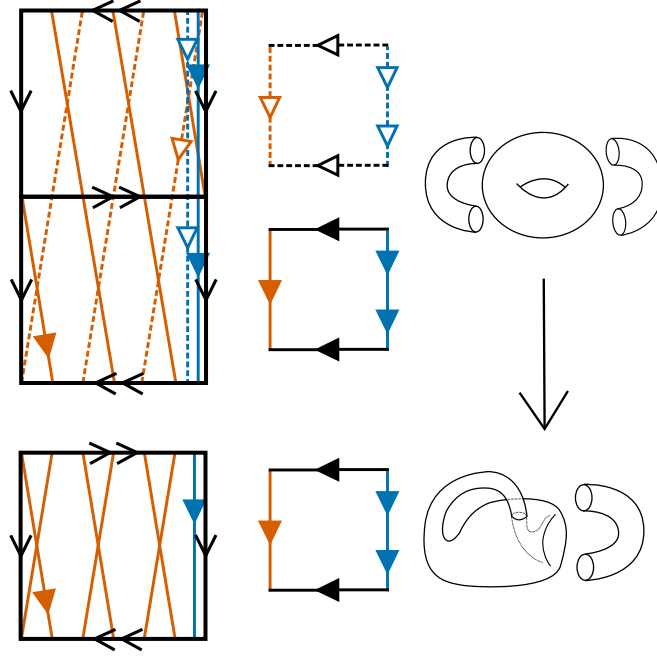


FIGURE 1. $G_{3,1}$ as an index 2 subgroup of $R_{3,1}$. The two blue arrows in each cylinder are attached along a and the orange arrows along geodesics representing (lifts of) a^6b .

The fundamental group of X' has the presentation

$$\langle x, y, s, t \mid [x, y], s^{-1}x^q s = x^p y, t^{-1}x^q t = x^p y^{-1} \rangle$$

where x and y are the generators of the fundamental group of the torus (corresponding to a^2 and b respectively). This group is none other than $G_{p,q}$. \square

Corollary C. *The set of exponents p such that n^p is the Dehn function of a one-relator group is dense in $[2, \infty)$.*

Remark 4. [MUW11, Problem 1.4] asks whether quadratic Dehn function implies that a one-relator group is automatic. The snowflake group $G_{1,1}$ is Gersten's non-CAT(0) free-by-cyclic group introduced in [Ger94]. It has been announced that this group is not automatic [BR06], which would settle this remaining problem as well.

3. NON-CUBULATED EXAMPLES

In [Wis14], Wise gave a necessary and sufficient condition for a tubular group to act freely on a CAT(0) cube complex. The condition is the existence of an *equitable set* which permits the construction of immersed walls in the graph of spaces associated to the group. A dual cube complex is then obtained from the corresponding wallspace.

Proposition 5. Let p and q be positive integers. The snowflake group $G_{p,q}$ acts freely on a CAT(0) cube complex if and only if $p \leq q$.

Proof. The existence of a free action of $G_{p,q}$ on a CAT(0) cube complex is equivalent to the existence of an equitable set: $S = \{(u_1, v_1), \dots, (u_k, v_k)\} \subseteq \mathbb{Z}^2 \setminus \{(0, 0)\}$ such that $[\mathbb{Z}^2 : \langle S \rangle] < \infty$ and

$$\sum_i \#[(q, 0), (u_i, v_i)] = \sum_i \#[(p, 1), (u_i, v_i)], \quad \sum_i \#[(q, 0), (u_i, v_i)] = \sum_i \#[(p, -1), (u_i, v_i)]$$

where $\#[(a, b), (c, d)]$ denote the “intersection number” $|ad - bc|$. Thus the problem reduces to solving

$$(\star) \quad \sum_i |qv_i| = \sum_i |pv_i - u_i| = \sum_i |pv_i + u_i|.$$

If $p \leq q$, then a solution is $\{(q, 1), (q, -1)\}$. If $p > q$, there is no solution:

$$\sum_i |pv_i - u_i| + |pv_i + u_i| \geq \sum_i |(pv_i - u_i) + (pv_i + u_i)| = 2 \sum_i |pv_i| \geq 2 \sum_i |qv_i|$$

and equality can only hold in this last inequality if all $v_i = 0$, in which case some $u_i \neq 0$ and (\star) clearly cannot hold. \square

Corollary D. *Let $p > q$ be positive integers. Then the one-relator group $R_{p,q}$ has no subgroup isomorphic to $BS(m, n)$ for $m \neq \pm n$ but does not act freely on a CAT(0) cube complex.*

Remark 6. One can also deduce that for $p > q$ the group $G_{p,q}$ does not act freely on a CAT(0) cube complex from the fact that $G_{p,q}$ has a cyclic subgroup with distortion n^α [BB00, Corollary 2.3] whereas cyclic subgroups are undistorted in groups admitting such actions by [Hag07, Theorem 1.5] (which was generalized to finitely generated virtually abelian subgroups in [Woo17]).

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