ISSN 1471-0498

DEPARTMENT OF ECONOMICS

DISCUSSION PAPER SERIES

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Number 524
January 2011

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On the Smooth Ambiguity Model: A Reply*

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This version: January 14, 2011
First version: September 9, 2009

Abstract

We find that Epstein (2010)'s Ellsberg-style thought experiments pose, contrary to his claims, no paradox or difficulty for the smooth ambiguity model of decision making under uncertainty developed by Klibanoff, Marinacci and Mukerji (2005). Not only are the thought experiments naturally handled by the smooth ambiguity model, but our reanalysis shows that they highlight some of its strengths compared to models such as the maxmin expected utility model (Gilboa and Schmeidler, 1989). In particular, these examples pose no challenge to the model's foundations, interpretation of the model as affording a separation of ambiguity and ambiguity attitude or the potential for calibrating ambiguity attitude in the model.

JEL Classification Numbers: D800, D810.

Keywords: Ambiguity, Uncertainty, Knightian Uncertainty, Ambiguity Aversion, Uncertainty Aversion, Ellsberg Paradox, Smooth Ambiguity, Smooth model, Ambiguity hedging.

1 Introduction

Epstein (2010) describes two Ellsberg (1961)-style thought experiments and argues that they pose difficulties for the smooth ambiguity model of decision making under uncertainty developed by Klibanoff, Marinacci and Mukerji (2005) (henceforth KMM). We revisit these

*We thank Fabio Maccheroni, Bob Nau, Ben Polak and Peter Wakker for helpful discussions.

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thought experiments and find that they lend no support to the critical conclusions he draws from them. We find that the first thought experiment and all its suggested variations is handled quite naturally and completely by the smooth ambiguity model if one takes care to *formally* model the information the decision maker has available. Regarding the second experiment, we elaborate on the behavioral distinction that it provides between the smooth ambiguity model and models such as the maxmin expected utility (MEU) model (Gilboa and Schmeidler, 1989) and explain why it is the behavior predicted by the smooth ambiguity model that is intuitive. Our discussion of these examples highlights and reinforces the relative strengths of the smooth ambiguity model, including the degree of separation between ambiguity attitude and belief it affords and the range of ambiguity attitudes it accommodates.

To fix ideas and remind the reader, consider a separable metric state space Ω, a space of simple lotteries, X, over outcomes $\mathcal{C}$, where $\mathcal{C}$ is an interval of real numbers containing $[-1, 1]$, and the set Δ of all probability measures on Ω. The smooth ambiguity model represents preferences $\succsim$ over acts $f : \Omega \to X$ using the following functional:

$$V(f) = \int_{\Delta} \phi\left(\int_{\Omega} u\left(f\left(\omega\right)\right) d\pi\left(\omega\right)\right) d\mu(\pi), \tag{1.1}$$

where ϕ is continuous and strictly increasing on $u(\mathcal{C})$, $u : X \to \mathbb{R}$ is mixture linear and μ is a Borel probability measure on Δ, endowed with the vague topology. Let $\sigma(\Delta)$ be the associated Borel σ-algebra.[1] It also represents preferences $\succsim^2$ over second order acts $\mathfrak{f} : \Delta \to X$ using the functional

$$V^2(\mathfrak{f}) = \int_{\Delta} \phi\left(u\left(\mathfrak{f}\left(\pi\right)\right)\right) d\mu(\pi).$$

Notice that $\succsim$ and $\succsim^2$ agree when restricted to lotteries X (i.e., to constant acts and constant second order acts respectively).

As it will also be useful in what follows, recall that the α-MEU model represents preferences over acts according to

$$U(f) = \alpha \inf_{\pi \in C} \int_{\Omega} u\left(f\left(\omega\right)\right) d\pi\left(\omega\right) + (1-\alpha) \sup_{\pi \in C} \int_{\Omega} u\left(f\left(\omega\right)\right) d\pi\left(\omega\right), \tag{1.2}$$

where $\alpha \in [0, 1]$ is a weight and $C \subseteq \Delta$ is a set of probabilities. When $\alpha = 1$ we get back to the MEU model.

[1] See KMM for details. Here, as in Epstein (2010), we use an Anscombe-Aumann version of the original KMM model.

2 Thought Experiment 1: *Not* a Paradox

The experiment takes Ellsberg's (1961) 3-color urn (an urn with 3 balls divided among red (R), blue (B) and green (G)) and adds a construction urn,[2] containing 3 balls each of which has a label r, b or g. The individual is told that exactly one of the balls in the construction urn is labeled r. A draw from this construction urn will determine the composition of the Ellsberg 3-color urn. Specifically, if r is drawn from the construction urn the Ellsberg urn will contain one ball of each color (denoted $(1R, 1B, 1G)$), and, similarly, draws of b or g result in compositions $(1R, 2B, 0G)$ and $(1R, 0B, 2G)$ respectively in the Ellsberg urn. Apart from the usual bets on the color of a ball drawn from the Ellsberg urn, Epstein (2010) also considers bets on the composition of the Ellsberg urn (equivalently, bets on the type of ball drawn from the construction urn). He argues that the standard ambiguity averse choices over bets about the color drawn from the Ellsberg urn should imply ambiguity averse choices over bets about the color of the ball drawn from the construction urn. He claims that this behavior is incompatible with the smooth ambiguity model. All of his criticisms of the smooth ambiguity model stem from this alleged incompatibility. Below, we show that this behavior follows from the smooth ambiguity model quite naturally once one adopts, as Epstein (2010) fails to do, a state space adequate to formally and fully incorporate the information provided to the individual in the experiment. Thus the thought experiment is not a paradox for the smooth ambiguity model.

Epstein (2010) goes on to elaborate on this first thought experiment by describing two scenarios differing only in information about the composition of the construction urn. He argues that for the smooth ambiguity model to capture intuitive behavior in the two scenarios would require ambiguity attitudes (as identified through ϕ) to change across the scenarios. On this basis, he challenges the interpretation of ϕ as reflecting purely ambiguity attitude and μ as reflecting beliefs or information. This leads him to claim that efforts to calibrate an individual's ϕ in a context of interest (e.g., financial markets), by examining the behavior of that individual in another environment (e.g., real or hypothetical Ellsberg experiments), have no justification. We show, once one adopts a state space adequate in the sense mentioned above, that the intuitive behavior across his two informational scenarios is straightforwardly captured by changing *only* μ while leaving ϕ unchanged. Furthermore, the way in which μ changes is *exactly* what one would expect given the change in information. Hence, we argue, Epstein's first thought experiment and his variations thereof do not challenge in any way the interpretations of ϕ and μ in the smooth ambiguity model, and neither do they pose any

[2] Epstein (2010) calls this the "second-order urn". It will become obvious why we think this is not good terminology.

difficulties for calibration of ambiguity attitude or comparative statics in that model.

2.1 Proper modeling of the first thought experiment

The only change we make to the description of the thought experiment is to have the construction urn contain six balls rather than three (and thus exactly two balls labeled r rather than one). We do this so as to treat both the basic thought experiment and Epstein's two scenario elaboration on it using the same set-up. In applying the smooth ambiguity model to analyze this experiment, Epstein considers the color of a ball drawn from the Ellsberg urn (R, B or G) to be the (first order) state of the world. Correspondingly, he considers the type of ball drawn from construction urn (r, b or g) to be the second order state, since it determines the composition of the Ellsberg urn and thus the probabilities of drawing R, B or G. Notice that by choosing the states in this way, the connection between the composition of the construction urn and the likelihoods of the color drawn from it is not formally incorporated into the state space. Thus, when we are given information about the composition of the construction urn (such as the information that exactly two of the six balls are labeled r), it cannot properly be incorporated in the sense that this information does not correspond to an event in the state space.

In state-space models (whether Savage (1954) or KMM or others) properly incorporating information requires that the information be modelled as ruling out some of the states. Often, in practice, this is done implicitly, with the "full state space" in the background and reduced form updating used in the calculations. This is perfectly fine as a shortcut as long as it leads to the same conclusions as an analysis using the full model. Epstein's analysis is an illustration of how this shortcut can lead one astray – with his chosen reduced form modeling, one obtains different results than when one uses the full model. With a full state space, the information in the thought experiment about the composition of the construction urn must correspond exactly to ruling out some states. Notice that with Epstein's choice of first and second order state spaces, this fails to hold. The fact that exactly two of the six balls in the construction urn are labelled r is consistent with all possible outcomes of draws from the construction urn and Ellsberg urn that are feasible given the description of how the two urns relate to each other.

We now present a full state space treatment for this thought experiment and show that applying the smooth ambiguity model delivers all of the behavior that Epstein suggests is intuitive.[3] The (first order) state space is $\{r, g, b\} \times \{R, G, B\}$ (and so includes the draws from

[3] Such a construction was not given as much prominence in an earlier version of this reply. We thank an anonymous referee and Bob Nau (Nau 2010) for emphasizing the importance of a more detailed treatment (and working out many of the details).

both the construction urn and the Ellsberg urn) while the second order state space is the set of all probability distributions over the first order state space.[4] In the thought experiment, the information the individual is given is: (1) how the distribution over draws from both urns is determined by the composition of the construction urn, and (2) that exactly two of the six balls in the construction urn are labeled r. This information rules out all but the five second order states $\pi_1, \ldots, \pi_5$ described in the five columns of the table below. The numbers in each column give the probabilities of the first order states. The numbers are derived by considering the color compositions consistent with (2) and then using the information in (1) to translate those into probabilities of the draws.

	Second order states				
	$\boldsymbol{\pi_1}$	$\boldsymbol{\pi_2}$	$\boldsymbol{\pi_3}$	$\boldsymbol{\pi_4}$	$\boldsymbol{\pi_5}$
	Composition of the construction urn				
Draws ↓	$(\boldsymbol{2r, 4b, 0g})$	$(\boldsymbol{2r, 3b, 1g})$	$(\boldsymbol{2r, 2b, 2g})$	$(\boldsymbol{2r, 1b, 3g})$	$(\boldsymbol{2r, 0b, 4g})$
$(\boldsymbol{r, R})$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$
$(\boldsymbol{r, B})$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$
$(\boldsymbol{r, G})$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$
$(\boldsymbol{b, R})$	$\frac{2}{9}$	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{1}{18}$	0
$(\boldsymbol{b, B})$	$\frac{4}{9}$	$\frac{1}{3}$	$\frac{2}{9}$	$\frac{1}{9}$	0
$(\boldsymbol{b, G})$	0	0	0	0	0
$(\boldsymbol{g, R})$	0	$\frac{1}{18}$	$\frac{1}{9}$	$\frac{1}{6}$	$\frac{2}{9}$
$(\boldsymbol{g, B})$	0	0	0	0	0
$(\boldsymbol{g, G})$	0	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{1}{3}$	$\frac{4}{9}$

To see that ambiguity aversion in the smooth ambiguity model implies the behavior posited by Epstein in this thought experiment, take ϕ strictly concave, let μ be any strictly positive probability distribution over $\pi_1, \ldots, \pi_5$ and normalize u so that $u(100) = 1$ and $u(0) = 0$. Consider bets with stakes 100 if win and 0 if lose. Suppose, as seems reasonable given symmetry of the situation, that betting on π_1 (i.e., betting that the construction urn has composition $(2r, 4b, 0g)$) is indifferent to betting on π_5, and, similarly, that betting on π_2

[4]It is worth recalling that a second order state space, by definition, is always isomorphic to the set of probability distributions over the first order states. This explains why, for example, considering the first order state space $\{R, G, B\}$ together with a putative second order state space $\{\text{composition of construction urn} \times \{r, g, b\}\}$ would not be a full state space. In terms of the probability of the first order states, this putative second order space collapses to Epstein's original second order space, $\{r, g, b\}$, and therefore suffers from the same failure to handle the information that two of the six balls in the construction urn are labeled r.

is indifferent to betting on π_4. These indifferences imply $\mu(\pi_1) = \mu(\pi_5)$ and $\mu(\pi_2) = \mu(\pi_4)$. Then, according to the smooth ambiguity model, betting on R is strictly preferred to betting on B while betting on $B \cup G$ is strictly preferred to betting on $R \cup G$ (i.e., $f_1 \succ f_2$ and $f_4 \succ f_3$ in Epstein's (2010, pp. 2088-89) notation) and, betting on r is strictly preferred to betting on b while betting on $b \cup g$ is strictly preferred to betting on $r \cup g$ (i.e., $F_1 \succ F_2$ and $F_4 \succ F_3$ in Epstein's notation). Furthermore, again as Epstein suggests is intuitive, the preferences are stronger in the case of bets on the color drawn from the construction urn compared to those on the Ellsberg urn since less is known about the composition of the construction urn.[5]

Next, consider Epstein's (2010, Section 2.4) extension of the first thought experiment (Scenario I) to consider a new scenario (Scenario II) in which additional information is provided about the composition of the construction urn. Recall, that Epstein used these scenarios to argue that this change in information required changing ϕ to get plausible behavior. We show that is false when the information is properly modeled. Take $u_I = u_{II} = u$ and normalize so that $u(100) = 1$ and $u(0) = 0$. Take $\phi_I = \phi_{II} = \phi$ strictly concave. Let μ_I be any strictly positive probability distribution over $\pi_1, \ldots, \pi_5$. In Scenario II, we are told that there is at least one b and at least one g ball in the construction urn. Let μ_{II} be the Bayesian update of μ_I reflecting this information (so π_1 and π_5 are given zero weight and the rest maintain the same relative weights as in Scenario I). Epstein asks for the following rankings to be satisfied: (1) a bet on b is indifferent to a bet on g in each scenario; (2) a bet on r has the same certainty equivalent in each scenario; (3) a bet on R is strictly preferred to a bet on B in each scenario; and (4) the certainty equivalent of a bet on B is higher in Scenario II than in Scenario I. One can calculate that given our assumptions above, all of these rankings follow. This shows that to accommodate the difference in behavior between the two scenarios, all that needs to change is μ, and furthermore, the required change is a natural reflection of exactly the information difference between the two situations. If anything, therefore, this example reinforces our interpretation that in the smooth ambiguity model there is a separation of beliefs and attitudes (towards ambiguity and towards risk), and that μ reflects information/belief. Epstein uses this example as the basis for his argument against this separation and against calibration of the model. Our discussion demonstrates that this, and similar examples, provide no such basis, and, if anything, reinforce the separate interpretations of ϕ and μ and the corresponding possibility for calibration of ϕ across informational environments.

Epstein (2010, Section 2.3) partially anticipates our resolution of the first thought exper-

[5]For the calculations behind the claims in this paragraph and the following one see Section 5.1 in the Appendix.

iment and claims that such a reformulation of the state space would render our assumption of expected utility over second order acts (KMM, Assumption 2) unfalsifiable. We disagree. In particular, once we have settled on a state space in any given problem, one can test, just as in a standard subjective environment, whether preferences over bets on the second order events (which, in our analysis of the thought experiment, correspond to compositions of the construction urn) are consistent with expected utility. In the next subsection, we elaborate on this point and show some distinctive behavioral implications of the smooth ambiguity model for behavior toward acts and second order acts.

2.2 Ambiguity of and ambiguity attitude toward second order events

Having shown that the first thought experiment is readily handled by the smooth ambiguity model, we turn to a more general question raised by the spirit of the example: Given that the smooth ambiguity model allows the individual to view some (first order) events as ambiguous (as evidenced by Ellsberg-type behavior), shouldn't such Ellsberg-type behavior toward (the intuitively more amorphous) second order events also be allowed? Not only is such behavior allowed, but, using a definition of ambiguous event that we proposed in KMM based on Ellsberg-type behavior, as in his famous two-urn thought experiment, we show that it occurs precisely when one would expect it to. Specifically, whenever, and only when, an event is ambiguous, the naturally associated second order events are also ambiguous. Moreover, ambiguity aversion for acts and second order acts is tied together: ϕ strictly concave implies strict ambiguity aversion in *both* domains.

To discuss ambiguity of second order events, we need to recall from KMM the notion of an *associated second order act* (adapted here to the Anscombe-Aumann setting):

Definition 2.1 *The second order act f^2 associated with an act f is defined as*

$$f^2(\pi) = l_f(\pi) \qquad \forall \pi \in \Delta$$

where $l_f(\pi) \in X$ is the reduced lottery generated by f together with π.

We now use this definition to define the notion of associated second order events:

Definition 2.2 *Given any $E \in \Sigma$, let I_E be the second order act associated with the act 1_E.*[6] *The collection of* associated second order events *is the sub σ-algebra $\sigma(I_E)$ of $\sigma(\Delta)$ generated by I_E.*

[6] 1_E is the indicator function for E. In this regard, note that throughout this section we adopt the normalization $u(0) = 0$ and $u(1) = 1$.

Observe that for any $\pi \in \Delta$, $I_E(\pi)$ is the lottery assigning probability $\pi(E)$ to the outcome 1 and the remaining probability to the outcome 0. Therefore, given E, the associated second order events are events like $\{\pi : \pi(E) \in B\}$ where B is a Borel subset of $[0, 1]$. We next write down the immediate adaptation to events in $\sigma(\Delta)$ of our (KMM, Definition 7) definition of unambiguous events in Ω.

Definition 2.3 *An event $A \in \sigma(\Delta)$ is* unambiguous *if, for each $p \in [0, 1]$ and each $x, y \in X$ such that $x \succ y$, either $[xAy \succ^2 px+(1-p)y$ and $py+(1-p)x \succ^2 yAx]$, $[xAy \prec^2 px+(1-p)y$ and $py + (1-p)x \prec^2 yAx]$ or $[xAy \sim^2 px + (1-p)y$ and $py + (1-p)x \sim^2 yAx]$. An event is* ambiguous *if it is not unambiguous.*

Notice that this definition declares an event to be ambiguous if it is impossible to calibrate the likelihood of the event against lotteries. The following results relate formally, within the smooth ambiguity model, the ambiguity/unambiguity of events in Ω with the ambiguity/unambiguity of their associated second order events.

Proposition 2.1 *Fix a smooth ambiguity model with ϕ that has some open interval of utility values over which it is strictly concave or strictly convex. An event $E \in \Sigma$ is unambiguous if and only if all the associated second order events are unambiguous.*

With this result, we have shown that ambiguity/unambiguity of events in Σ results in ambiguity/unambiguity of events in $\sigma(\Delta)$ in a natural way. Ambiguity of $E \in \Sigma$ implies that non-null and non-universal second order events concerning the probability of E are treated as ambiguous. This emphasizes the point that the smooth ambiguity model property of expected utility evaluation of second order acts does *not* mean that the decision maker treats these acts as based on unambiguous events. In particular, the following *intuitive hypothesis* holds in the smooth ambiguity model: If an event $E \in \Sigma$ is ambiguous (unambiguous), then naturally associated second order events are ambiguous (unambiguous).

Moving from ambiguity to ambiguity attitudes, given the definitions of ambiguous events (KMM, Definition 7 and Definition 2.3 of this paper), a natural test of (strict) ambiguity *averse* behavior is to see if bets on ambiguous events are treated as (strictly) *less* valuable than bets having their payoff determined by a known probability distribution (i.e., a lottery). We show (see Proposition 5.1 in the Appendix) that for the smooth ambiguity model with ϕ strictly concave, the individual is strictly ambiguity averse over both second order acts and (first order) acts. Thus, far from imposing ambiguity neutrality over second order acts, the smooth ambiguity model ties ambiguity aversion over the two domains together. In particular, this tells us that behavior reflecting, for example, strict ambiguity aversion over acts and ambiguity neutrality or seeking over second order acts is ruled out by the smooth ambiguity model.

2.3 Nonreduction of objective probabilities is not implied

As a final point related to the first thought experiment, Epstein (2010, Section 2.5), argues that nonreduction of compound lotteries is implicit in the smooth ambiguity model. To support this, he compares Scenario I above to a scenario (call it Scenario III) in which complete information about the composition of the construction urn is given to the individual. If this change in information were modeled (as Epstein suggests) by leaving μ unchanged but informally interpreting it as objective, then the individual would be facing an objective two-stage lottery and, Epstein argues, would be forced by the smooth ambiguity model to treat it just as he did when it was ambiguous and therefore differently than the corresponding reduced lottery. We find this analysis is flawed in the same way as Epstein's analysis of the comparison between Scenarios I and II above. Specifically, he carries out his analysis in a setting too sparse to incorporate the change in information (i.e., going from partial to full information about the composition of the construction urn). Given the state space we use above, such a change is easily seen to correspond to μ going from a non-degenerate to a degenerate distribution – there is no longer any uncertainty about the composition of the construction urn. In such a scenario, the smooth ambiguity model treats all events as unambiguous, reduces all uncertainty to risk, and becomes a standard expected utility preference. Thus, no nonreduction of objective probabilities is implied.[7]

3 Thought Experiment 2: Hedging across sources of ambiguity

Consider the second thought experiment proposed by Epstein (2010, Section 3). There are two urns, each containing 50 balls divided among red (R) and blue (B). An individual is told that the relative proportions of red and blue in each urn are determined independently. One ball is drawn from each urn. The individual considers bets on the colors of the drawn balls with outcomes $c^* > c$ and the 50-50 lottery $\left(c^*, \frac{1}{2}; c, \frac{1}{2}\right)$. Assume that lotteries are evaluated according to an expected utility function u, normalized so that $u(c^*) = 1$ and $u(c) = 0$. We can then write the acts that Epstein considers with utility payoffs as follows (where $R_1 B_2$

[7] As Epstein suggests, "Think of the corresponding exercise for a subjective expected utility agent in an abstract state space setting." The only formal sense in which one may "learn" that some distribution is "true" is through the process of updating beliefs over a "full" state space that includes all possible observations. This is the standard Bayesian model where the state space is the Cartesian product of parameters and signals. Learning in such a setting corresponds to updating by eliminating states including signals that did not occur. Thus, as more and more observations accumulate, the prior may become concentrated on the "true parameter". Exactly as we suggest here, the standard modeling of learning the truth corresponds to a prior becoming degenerate.

is the event that a red ball is drawn from the first urn while a blue ball is drawn from the second urn, etc.):

	R_1R_2	R_1B_2	B_1R_2	B_1B_2
f_1	1	1	0	0
f_2	1	0	1	0
$\frac{1}{2}f_1 + \frac{1}{2}f_2$	1	$\frac{1}{2}$	$\frac{1}{2}$	0
g_1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
g_2	$\frac{1}{2}$	0	1	$\frac{1}{2}$

Epstein argues that $\frac{1}{2}f_1 + \frac{1}{2}f_2 \sim f_1 \sim f_2$ and $g_1 \succ g_2$ are natural for a strictly ambiguity averse individual, and shows that these preferences are incompatible with any smooth ambiguity model with a concave ϕ. We agree with the intuition for $g_1 \succ g_2$, but disagree that $\frac{1}{2}f_1 + \frac{1}{2}f_2 \sim f_1 \sim f_2$ is natural for an ambiguity averse individual and think there is good reason to expect $\frac{1}{2}f_1 + \frac{1}{2}f_2 \succ f_1 \sim f_2$. The evaluation of f_1 depends on the ratio of red to blue in urn 1 but not on the composition of urn 2. Similarly, the evaluation of f_2 depends on only the ratio of red to blue in urn 2 and not on the composition of urn 1. In contrast, the evaluation of $\frac{1}{2}f_1 + \frac{1}{2}f_2$ depends on the color compositions of both urns, but has half the exposure to the uncertainty about the ratio in each urn compared to f_1 and f_2. Recall that the determination of the two urn compositions are viewed as independent. The act $\frac{1}{2}f_1 + \frac{1}{2}f_2$ thus diversifies the individual's exposure across the urns: it provides a hedging of the two independent ambiguities in the same sense as diversifying across bets on independent risks provides a hedging of the risks. To an individual who is averse to ambiguity (i.e., to subjective uncertainty about relative likelihoods), such diversification is naturally valuable. This value is reflected in the smooth ambiguity model with concave ϕ through the fact that mean-preserving spreads in the subjective distribution of expected utilities generated by an act are disliked.[8] However, preferences that ignore all except the minimum expected utility possibilities[9] will miss the diversification aspect of this situation, similar to an infinitely risk averse expected utility individual not valuing diversification across independent risks. This is extreme behavior. Models such as MEU and α-MEU force this extreme devaluation of diversification across these independent but not necessarily identical ambiguities. The smooth ambiguity model delivers more moderate and, to us, reasonable behavior, as it implies that

[8] Epstein remarks (p. 2096) that our intuition does not rely on ambiguity and claims it would equally apply to cases where there was an objective distribution over expected utilities (i.e., an "objective μ"). His reasoning ignores the fact that the individual's dislike of variation in expected utility is *only* when the variation comes from an ambiguous source – this is why it is ambiguity aversion. Just as we discussed in Section 2.3, what happens when μ becomes "objective" is that, properly modeled, learning eliminates the ambiguity (μ becomes degenerate) and thus the variation in expected utility coming from an ambiguous source disappears.

[9] Or a fixed weighted average of the minimum and maximum expected utility possibilities.

such diversification is valued by ambiguity averse individuals, while this value may vary in size as ambiguity aversion varies.[10]

The next result formally verifies this difference in behavior between the models. Let $\Omega = \{R_1, B_1\} \times \{R_2, B_2\}$ and, given a set $C \subseteq \Delta$ of probabilities, denote the set of probabilities of drawing red from urn i by $\Gamma_i = \{p(R_i) : p \in C\}$.[11] Consider the following properties on C:

(1) $\Gamma_1 = \Gamma_2$;

(2) Γ_i nonsingleton; and

(3) if $x \in \Gamma_1$ and $y \in \Gamma_2$, there is $p \in C$ such that $p(R_1) = x$ and $p(R_2) = y$.

Property (1) reflects symmetry across the urns as it says that the same set of compositions are considered for each urn. Without it, there is no reason to expect $f_1 \sim f_2$. Property (2) says there is ambiguity about the color composition of the urns. Without it, all of the acts in the example are unambiguous. Property (3) seems a necessary condition for independence of the urn compositions as it says that any color composition of urn 1 could be combined with any composition of urn 2.

We can now state the following result, which is proved in the Appendix. The result references the condition

$$\mu(p \in C : p(R_1) \in B) = \mu(p \in C : p(R_2) \in B) \quad \text{for all Borel sets } B \subseteq [0,1], \tag{3.1}$$

which is meant to reflect the perceived symmetry across urns, and without which, again, there is no reason to presume $f_1 \sim f_2$.[12]

Proposition 3.1 *Suppose $C \subseteq \Delta$ is nonempty, closed, and satisfies properties (1)-(3). Then,*

(i) Any smooth ambiguity preference with ϕ strictly concave and μ with support C and

[10] By no means do we claim that the smooth ambiguity model (and its close relatives Nau, 2006, Ergin and Gul, 2009, Seo, 2009 and Neilson, 2010) is the only model capturing these intuitive choices. Many other models in the ambiguity aversion literature – e.g., invariant biseparable preferences (Ghirardato, Maccheroni and Marinacci, 2004, and Amarante, 2009), variational preferences (Maccheroni, Marinacci and Rustichini, 2006), and vector expected utility preferences (Siniscalchi, 2009) – have cases compatible with the choices that we claim are intuitive.

[11] Note the use of $p(R_i)$ in place of the more formal $p(R_i \times \{R_{j \neq i}, B_{j \neq i}\})$.

[12] The sets $\{p \in C : p(R_i \times \{R_{j \neq i}, B_{j \neq i}\}) \in B\}$ belong, for all Borel sets $B \subseteq [0,1]$, to the Borel σ-algebra of Δ (see, e.g., Theorem 15.13 of Aliprantis and Border, 2006).

such that condition (3.1) holds,[13] *will have*

$$\frac{1}{2}f_1 + \frac{1}{2}f_2 \succ f_1 \sim f_2 \quad \textit{and} \quad g_1 \succ g_2.$$

(ii) Any α-MEU preference will have

$$\frac{1}{2}f_1 + \frac{1}{2}f_2 \sim f_1 \sim f_2,$$

while $g_1 \succ g_2$ if and only if $\alpha > 1/2$.

When, *unlike* in this thought experiment, the two urns are known to have *identical* color compositions (and thus, as discussed by Walley (1991) and Epstein and Schneider (2003), are independently and identically distributed (iid) and not simply independently and indistinguishably distributed (IID)) the events R_1B_2 and B_1R_2 have unambiguously equal likelihoods. Thus $\frac{1}{2}f_1 + \frac{1}{2}f_2$ would not be expected to provide a hedge against ambiguity in this case as it diversifies only across these two events when compared to f_1 and f_2. Proposition 3.1 does not apply to this iid case, since the restriction to identical color compositions violates the conjunction of properties (2) and (3) above. It may be shown that the smooth ambiguity model indeed delivers $\frac{1}{2}f_1 + \frac{1}{2}f_2 \sim f_1 \sim f_2$ in this case.

To summarize our respective arguments regarding this interesting thought experiment and its implications for the smooth ambiguity model: Epstein argues that $\frac{1}{2}f_1 + \frac{1}{2}f_2 \sim f_1 \sim f_2$ and $g_1 \succ g_2$ are natural for a strictly ambiguity averse individual, leading to a seeming inconsistency in the modelling of ambiguity attitude in the smooth ambiguity model through ϕ. We argue that under strict ambiguity aversion, $\frac{1}{2}f_1 + \frac{1}{2}f_2 \succ f_1 \sim f_2$ is the more natural behavior. In this case, there is no conflict at all with $g_1 \succ g_2$, since both strict preferences are generated by a strictly concave ϕ in the smooth ambiguity model. Hence, we conclude, contrary to Epstein (2010), that the intuitive ambiguity averse choices in thought experiment 2 are indeed captured by the smooth ambiguity model, whereas they are not captured by the MEU (or α-MEU) model. Beyond the specific issue of compatibility with the smooth ambiguity model, this discussion and thought experiment highlights a point we feel is fundamental in thinking about ambiguity aversion – hedging across independent sources of ambiguity makes a lot of sense.

[13] Here the support of μ is defined as $\operatorname{supp}\mu = \bigcap \{D \text{ closed} : \mu(D) = 1\}$.

4 Conclusion

There are, at least, two significant ways in which the smooth ambiguity model adds to older frameworks, such as MEU, in order to facilitate the analysis of the effect of ambiguity in economic contexts. The first is the ability to do meaningful comparative statics in ambiguity and ambiguity aversion while allowing great flexibility in the ambiguity of (first order) events and in ambiguity attitude and also a quite tractable functional form. This ability stems in part from the degree of separation of beliefs and taste attributes in the representation; a separation that is, as was demonstrated in our analysis, not challenged by either of Epstein's (2010) thought experiments. Second, the model allows us to explore implications of ambiguity aversion that do not have their source in preference kinks. Kinks are not implied by ambiguity averse or Ellsbergian behavior (and, indeed, may be present without such behavior), yet they are what drive behavior in many applications of models like MEU or Choquet expected utility (Schmeidler, 1989) to economics and finance. Such kinks may indeed be important, but are a conceptually separate phenomenon from ambiguity attitude *per se,* and it is valuable to have models that separate the two.

Our analysis of Epstein's first thought experiment shows that his results are due to the failure to use a state space allowing the incorporation of the key information defining the experiment. When one analyzes the thought experiment and the suggested variations using the a full model, the "paradox", the counter intuitive results obtained in Epstein's analyses, all go away. The criticisms Epstein draws from his results (about foundations, interpretation, separation and calibration) similarly disappear.

In analyzing the second thought experiment, we clarify the differences in behavior across models that the experiment illustrates and tie these differences to the intuitive idea that an ambiguity averse individual would want to hedge across separate sources of ambiguity unless their ambiguity attitude were extreme or the sources were guaranteed to have identical realizations of the ambiguity. The smooth ambiguity model delivers this behavior while MEU and α-MEU models cannot.

All models have strengths and weaknesses, and the smooth ambiguity model is no exception. However, this reply has shown that the thought experiments at the heart of Epstein (2010) justify none of the criticisms he offers of the model.

5 Appendix

5.1 Calculations supporting Section 2.1

To see that $f_1 \succ f_2$, $f_4 \succ f_3$, $F_1 \succ F_2$ and $F_4 \succ F_3$ observe that $f_1 \succ f_2$ iff $\phi\left(\frac{1}{3}\right) > \mu(\pi_1)\phi\left(\frac{5}{9}\right) + \mu(\pi_2)\phi\left(\frac{4}{9}\right) + \mu(\pi_3)\phi\left(\frac{1}{3}\right) + \mu(\pi_4)\phi\left(\frac{2}{9}\right) + \mu(\pi_5)\phi\left(\frac{1}{9}\right)$; $f_4 \succ f_3$ iff $\phi\left(\frac{2}{3}\right) > \mu(\pi_1)\phi\left(\frac{4}{9}\right) + \mu(\pi_2)\phi\left(\frac{5}{9}\right) + \mu(\pi_3)\phi\left(\frac{2}{3}\right) + \mu(\pi_4)\phi\left(\frac{7}{9}\right) + \mu(\pi_5)\phi\left(\frac{8}{9}\right)$; $F_1 \succ F_2$ iff $\phi\left(\frac{1}{3}\right) > \mu(\pi_1)\phi\left(\frac{2}{3}\right) + \mu(\pi_2)\phi\left(\frac{1}{2}\right) + \mu(\pi_3)\phi\left(\frac{1}{3}\right) + \mu(\pi_4)\phi\left(\frac{1}{6}\right) + \mu(\pi_5)\phi(0)$; and $F_4 \succ F_3$ iff $\phi\left(\frac{2}{3}\right) > \mu(\pi_1)\phi\left(\frac{1}{3}\right) + \mu(\pi_2)\phi\left(\frac{1}{2}\right) + \mu(\pi_3)\phi\left(\frac{2}{3}\right) + \mu(\pi_4)\phi\left(\frac{5}{6}\right) + \mu(\pi_5)\phi(1)$. Since $\mu(\pi_1) = \mu(\pi_5)$ and $\mu(\pi_2) = \mu(\pi_4)$, each of the four inequalities hold because the subjective distribution of expected utilities on the right-hand side is a mean-preserving spread of the (degenerate) distribution of expected utilities on the left-hand side and ϕ is strictly concave.

That the differences in evaluations are larger for the bets on the draws from the construction urn follows from strict concavity and the fact that the subjective distributions of expected utilities from F_2 and F_3 are mean-preserving spreads of those from f_2 and f_3 respectively given that $\mu(\pi_1) = \mu(\pi_5)$ and $\mu(\pi_2) = \mu(\pi_4)$.

The four behaviors Epstein suggests as desirable in the two scenarios may be verified as follows: The symmetry of μ_I is inherited by μ_{II} through Bayes' rule and together they assure (1); $u_I = u_{II} = u$ ensures (2); strict concavity of ϕ plus symmetry of μ_I and μ_{II} (which ensures that the induced distribution of expected utilities from betting on B is a mean-preserving spread of the distribution of expected utilities from betting on R in each scenario) implies (3); and (4) follows from the fact that the induced distribution of expected utilities from betting on B in Scenario I is a mean-preserving spread of that in Scenario II together with strict concavity of ϕ.

5.2 Proof of Proposition 2.1

The proof makes use of the following two lemmas.

Lemma 5.1 *Let (S, Σ, P) be any probability space. A Σ-measurable function $\xi : S \to \mathbb{R}$ is constant P-a.e. if and only if $P(A) \in \{0, 1\}$ for all $A \in \sigma(\xi)$.*

Proof Suppose $\xi : S \to \mathbb{R}$ is constant P-a.e., i.e., there is $\bar{t} \in \mathbb{R}$ such that $P(\xi = \bar{t}) = 1$. Set $E_t = (\xi \leq t)$ for $t \in \mathbb{R}$. The σ-algebra $\sigma(\xi)$ is generated by the chain $\{E_t\}$ of all lower contour sets. Since $P(\xi = \bar{t}) = 1$, we have $P(E_t) \in \{0, 1\}$ for $t \in \mathbb{R}$. Moreover, the collection $\Lambda = \{A \in \Sigma : P(A) \in \{0, 1\}\}$ is a λ-class. By the Dynkin Lemma, $\sigma(\xi) \subseteq \Lambda$.

As to the converse, suppose $P(A) \in \{0, 1\}$ for all $A \in \sigma(\xi)$. Define $F : \mathbb{R} \to \mathbb{R}$ by $F(t) = P(E_t)$. The cumulative density function F is increasing and right continuous. Consider the interval $I = \{t \in \mathbb{R} : F(t) = 1\}$. Set $\alpha = \inf I$. The right continuity of F implies $\alpha \in I$. Then, $P(\xi = \alpha) = 1$. For, $P(\xi \leq \alpha) = 1$ and $P(\xi < \alpha) = P\left(\bigcup_n (\xi \leq \alpha - 1/n)\right) = \lim_n P(\xi \leq \alpha - 1/n) = 0$. ■

Lemma 5.2 *Fix a smooth ambiguity model with ϕ strictly concave or strictly convex over some open interval of utility values. An event $A \in \sigma(\Delta)$ is ambiguous if and only if it is such that $0 < \mu(A) < 1$.*

Proof Let $A \in \sigma(\Delta)$ be such that $0 < \mu(A) < 1$. Without loss of generality, assume $\mu(A) \geq 1/2$ (if it is not, simply swap the roles of A and A^c). Let J be an open interval of utility values over which ϕ is strictly concave or strictly convex. For $p \in [0,1]$ and $x, y \in X$ such that $x \succ y$ and $u(x), u(y) \in J$, xAy is evaluated as $\mu(A)\phi(u(x)) + (1 - \mu(A))\phi(u(y))$, while $px + (1-p)y$ is evaluated as $\phi(pu(x) + (1-p)u(y))$. By continuity of ϕ and the fact that $0 < \mu(A) < 1$, there exists a $\hat{p} \in (0,1)$ such that

$$\mu(A)\phi(u(x)) + (1 - \mu(A))\phi(u(y)) = \phi(\hat{p}u(x) + (1 - \hat{p})u(y)).$$

If ϕ is strictly concave on J, this equality implies $\mu(A) > \hat{p}$. Similarly, strict convexity on J implies $\mu(A) < \hat{p}$.

Similarly, there exists a $\hat{q} \in (0,1)$ such that

$$\mu(A)\phi(u(y)) + (1 - \mu(A))\phi(u(x)) = \phi(\hat{q}u(y) + (1 - \hat{q})u(x)).$$

If ϕ is strictly concave on J, this equality implies $1 - \mu(A) > 1 - \hat{q}$, and so $\mu(A) < \hat{q}$. Strict convexity on J similarly implies $\mu(A) > \hat{q}$. Therefore, either $\hat{q} > \hat{p}$ and $yAx \sim^2 \hat{q}y + (1 - \hat{q})x \prec^2 \hat{p}y + (1-\hat{p})x$ (under strict concavity) or $\hat{q} < \hat{p}$ and $yAx \sim^2 \hat{q}y + (1-\hat{q})x \succ^2 \hat{p}y + (1-\hat{p})x$ (under strict convexity). This shows that A is ambiguous since $xAy \sim^2 \hat{p}x + (1 - \hat{p})y$ and $yAx \nsim^2 \hat{p}y + (1 - \hat{p})x$.

For the other direction, it is enough to observe that $\mu(A) \in \{0, 1\}$ implies that A is unambiguous. ■

Proof of Proposition 2.1 Observe that, denoting any lottery between the outcomes 0 and 1 by the probability assigned to 1, we can view I_E as a real-valued function given by $I_E(\pi) = \pi(E)$ for all $\pi \in \Delta$. Since ϕ is strictly concave or strictly convex on some open interval of utility values, by Theorem 3 of KMM an event $E \in \Sigma$ is unambiguous if and only if I_E is constant μ-a.e. By Lemma 5.1, this happens if and only if $\mu(A) \in \{0, 1\}$ for all $A \in \sigma(I_E)$. By Lemma 5.2, this is equivalent to requiring that all $A \in \sigma(I_E)$ are unambiguous. We conclude that $E \in \Sigma$ is unambiguous if and only if all $A \in \sigma(I_E)$ are unambiguous, as desired. ■

5.3 Linking ambiguity aversion over acts and second order acts

Observe from the definition of unambiguous event and continuity that for an event E to be ambiguous, there must be some $p \in [0,1]$ and $x, y \in X$ with $x \succ y$ such that the lottery $px + (1-p)y$ is either strictly better than both xEy and yEx, strictly worse than both, or indifferent to one and strictly ranked relative to the other. Note that in any model where preference for the lottery $px + (1-p)y$ is increasing and continuous in p, we can ignore the cases involving indifference, as when they exist one of the strict cases occurs as well. As xEy and yEx involve ambiguity but the lottery does not, it is natural to call strictly ambiguity averse the case where the lottery is strictly better than both, and strictly ambiguity seeking the case where the lottery is strictly worse than both. In this vein, let us call a preference $\hat{\succsim}$ *strictly ambiguity averse* if, given any ambiguous event E and $x, y \in X$ such that $x \hat{\succ} y$, there exists a $p \in (0,1)$ such that

$$xEy \hat{\prec} px + (1-p)y \text{ and } yEx \hat{\prec} py + (1-p)x \tag{5.1}$$

and for no $p \in [0,1]$ is it true that

$$xEy \hat{\succ} px + (1-p)y \text{ and } yEx \hat{\succ} py + (1-p)x. \tag{5.2}$$

The result below implies what we claimed regarding ambiguity aversion in Section 2.2 – ϕ strictly concave implies preferences over acts and over second order acts are strictly ambiguity averse. An analogous proposition holds for strict ambiguity seeking when ϕ is convex rather than concave.

Proposition 5.1 *Fix a smooth ambiguity model with ϕ concave. The following are equivalent:*

(i) $\succsim$ is strictly ambiguity averse;

(ii) $\succsim^2$ is strictly ambiguity averse;

(iii) ϕ is strictly concave.

Proof We prove separately the equivalence of (i) and (iii) and of (ii) and (iii).

(iii) implies (i): Suppose ϕ is strictly concave. We want to show that $\succsim$ is strictly ambiguity averse. Let $x, y \in X$, with $y \prec x$, and $E \in \Sigma$ be an ambiguous event. By setting $p = \int \pi(E)\, d\mu(\pi)$ it is easy to see that

$$xEy \prec px + (1-p)y \quad \text{and} \quad yEx \prec py + (1-p)x$$

It remains to show that there is no $p \in [0,1]$ such that

$$xEy \succ px + (1-p)y \quad \text{and} \quad yEx \succ py + (1-p)x.$$

Suppose *per contra* there is such a p. Since $y \prec x$, by the continuity of ϕ there is $1 \geq p' > p$ such that

$$\int \phi\left(\pi(E)\, u(x) + (1 - \pi(E)\, u(y))\right) d\mu(\pi) = \phi(p'u(x) + (1-p')u(y)).$$

Since ϕ is strictly concave, this equality implies $\int \pi(E)\, d\mu(\pi) > p' > p$.

Similarly, there is $0 < p'' < p$ such that

$$\int \phi\left(\pi(E)\, u(y) + (1 - \pi(E)\, u(x))\right) d\mu(\pi) = \phi(p''u(y) + (1-p'')u(x)).$$

Since ϕ is strictly concave, this equality implies $\int \pi(E)\, d\mu(\pi) < p'' < p$, a contradiction.

(i) implies (iii): Suppose $\succsim$ is strictly ambiguity averse. Suppose *per contra* that ϕ is not strictly concave. Then there exist $u(x) > u(y)$ such that, for all $\alpha \in [0,1]$,

$$\phi\left(\alpha u(x) + (1-\alpha)\, u(y)\right) = \alpha \phi(u(x)) + (1-\alpha)\, \phi(u(y))\,. \tag{5.3}$$

Let $E \in \Sigma$ be ambiguous. For each $\pi \in \operatorname{supp} \mu$ it holds that

$$\begin{aligned} \phi\left(\pi(E)\, u(x) + (1-\pi(E))\, u(y)\right) &= \pi(E)\, \phi(u(x)) + (1-\pi(E))\, \phi(u(y))\,, \text{ and} \\ \phi\left((1-\pi(E))\, u(x) + \pi(E)\, u(y)\right) &= (1-\pi(E))\, \phi(u(x)) + \pi(E)\, \phi(u(y))\,, \end{aligned}$$

and so, by setting $p = \int \pi(E)\, d\mu(\pi)$,

$$\begin{aligned} \int \phi\left(\pi(E)\, u(x) + (1-\pi(E))\, u(y)\right) d\mu(\pi) &= p\phi(u(x)) + (1-p)\, \phi(u(y)) \\ &= \phi(pu(x) + (1-p)\, u(y)) \end{aligned}$$

and

$$\begin{aligned}\int \phi\left(\left(1-\pi\left(E\right)\right)u\left(x\right)+\pi\left(E\right)u\left(y\right)\right)d\mu\left(\pi\right) &= \left(1-p\right)\phi\left(u\left(x\right)\right)+p\phi\left(u\left(y\right)\right)\\ &= \phi\left(\left(1-p\right)u\left(x\right)+pu\left(y\right)\right).\end{aligned}$$

Hence, both $xEy \sim px+(1-p)y$ and $yEx \sim py+(1-p)x$. Since $\succsim$ is strictly ambiguity averse, there is a $q \in (0,1)$ such that

$$px+(1-p)y \sim xEy \prec qx+(1-q)y \quad \text{and} \quad py+(1-p)x \sim yEx \prec qy+(1-q)x.$$

In turn, this implies $pu\left(x\right)+\left(1-p\right)u\left(y\right) < qu\left(x\right)+\left(1-q\right)u\left(y\right)$ and $pu\left(y\right)+\left(1-p\right)u\left(x\right) < qu\left(y\right)+\left(1-q\right)u\left(x\right)$, that is,

$$\begin{aligned}u\left(y\right)+u\left(x\right) &= p\left(u\left(x\right)+u\left(y\right)\right)+\left(1-p\right)\left(u\left(y\right)+u\left(x\right)\right)\\ &< q\left(u\left(x\right)+u\left(y\right)\right)+\left(1-q\right)\left(u\left(y\right)+u\left(x\right)\right)=u\left(x\right)+u\left(y\right),\end{aligned}$$

a contradiction. We conclude that ϕ is strictly concave.

(iii) implies (ii) Suppose ϕ is strictly concave. We want to show that $\succsim^2$ is strictly ambiguity averse. Let $x, y \in X$, with $y \prec^2 x$, and $A \in \sigma(\Delta)$ be an ambiguous event, i.e., $0 < \mu(A) < 1$. By setting $p = \mu\left(A\right)$ it is easy to see that

$$xAy \prec^2 px+(1-p)y \quad \text{and} \quad yAx \prec^2 py+(1-p)x.$$

It remains to show that there is no $p \in [0,1]$ such that

$$xAy \succ^2 px+(1-p)y \quad \text{and} \quad yAx \succ^2 py+(1-p)x.$$

Suppose *per contra* there is such a p. Since $y \prec^2 x$, by the continuity of ϕ there is $1 \geq p' > p$ such that

$$\mu(A)\phi(u(x))+(1-\mu(A))\phi(u(y)) = \phi(p'u(x)+(1-p')u(y)).$$

Since ϕ is strictly concave, this equality implies $\mu(A) > p' > p$.

Similarly, there is $0 < p'' < p$ such that

$$\mu(A)\phi(u(y))+(1-\mu(A))\phi(u(x)) = \phi(p''u(y)+(1-p'')u(x)).$$

Since ϕ is strictly concave, this equality implies $\mu(A) < p'' < p$, a contradiction.

(ii) implies (iii) Suppose $\succsim^2$ is strictly ambiguity averse. Suppose *per contra* that ϕ is not strictly concave. Then there exist $u(x) > u(y)$ such that (5.3) holds for all $\alpha \in [0,1]$. Let $A \in \sigma(\Delta)$ be an ambiguous event. It holds that

$$\begin{aligned} \phi(\mu(A)u(x) + (1-\mu(A))u(y)) &= \mu(A)\phi(u(x)) + (1-\mu(A))\phi(u(y)), \text{ and} \\ \phi((1-\mu(A))u(x) + \mu(A)u(y)) &= (1-\mu(A))\phi(u(x)) + \mu(A)\phi(u(y)) \end{aligned}$$

and so, by setting $p = \mu(A)$,

$$px + (1-p)y \sim^2 xAy \quad \text{and} \quad py + (1-p)x \sim^2 yAx.$$

Since $\succsim^2$ is strictly ambiguity averse, there is a $q \in (0,1)$ such that

$$qx + (1-q)y \succ^2 xAy \sim^2 px + (1-p)y \quad \text{and} \quad qy + (1-q)x \succ^2 yAx \sim^2 py + (1-p)x.$$

In turn, this implies $pu(x) + (1-p)u(y) < qu(x) + (1-q)u(y)$ and $pu(y) + (1-p)u(x) < qu(y) + (1-q)u(x)$, which, as seen before, leads to a contradiction. We conclude that ϕ is strictly concave. ■

5.4 Proof of Proposition 3.1

Abbreviate $p(R_1 \times \{R_2, B_2\})$ by $p(R_1)$ and so on. Observe that properties (2) and (3) imply that there exist $p \in C$ such that $p(R_1) \neq p(R_2)$.

(i) Suppose $\operatorname{supp}\mu = C$ and $\mu(p \in C : p(R_1) \in B) = \mu(p \in C : p(R_2) \in B)$ for all Borel sets B in $[0,1]$. Since ϕ is strictly increasing, by (1) we have $\{(\phi \circ p)(R_1) : p \in C\} = \{(\phi \circ p)(R_2) : p \in C\}$, and so $\int_\Delta (\phi \circ p)(R_1)\, d\mu(p) = \int_\Delta (\phi \circ p)(R_2)\, d\mu(p)$ because of the assumption on μ. Hence, $f_1 \sim f_2$. On the other hand,

$$\phi\left(\frac{1}{2}p(R_1) + \frac{1}{2}p(R_2)\right) \geq \frac{1}{2}(\phi \circ p)(R_1) + \frac{1}{2}(\phi \circ p)(R_2), \qquad \forall p \in \operatorname{supp}\mu,$$

with strict inequality if $p(R_1) \neq p(R_2)$.

Claim There is a Borel set $A \subseteq \operatorname{supp}\mu$, with $\mu(A) > 0$, such that $p(R_1) \neq p(R_2)$ for all $p \in A$.

Proof of the Claim As shown at the start of the proof, there is $\overline{p} \in \operatorname{supp}\mu$ such that $\overline{p}(R_1) \neq \overline{p}(R_2)$. Suppose first that $\overline{p}$ is an isolated point in $\operatorname{supp}\mu$. Then, $\mu(\overline{p}) > 0$ and the claim trivially holds. Suppose that $\overline{p}$ is not an isolated point in $\operatorname{supp}\mu$. Then, $B_\varepsilon(\overline{p}) \cap \operatorname{supp}\mu \neq \emptyset$ for every neighborhood $B_\varepsilon(\overline{p})$ of $\overline{p}$. Since $\overline{p}(R_1) \neq \overline{p}(R_2)$, by taking

ε small enough there is $B_{\varepsilon}(\overline{p})$ such that $p(R_1) \neq p(R_2)$ for all $p \in B_{\varepsilon}(\overline{p})$. By setting $A = B_{\varepsilon}(\overline{p}) \cap \operatorname{supp}\mu$, this proves the claim since $\mu(A) > 0$ because $B_{\varepsilon}(\overline{p}) \cap \operatorname{supp}\mu \neq \emptyset$. For, if $\mu(A) = 0$, then $\mu(B_{\varepsilon}(\overline{p})) = \mu(A) + \mu(B_{\varepsilon}(\overline{p}) \cap (\operatorname{supp}\mu)^c) = 0$, and so $\operatorname{supp}\mu \subseteq B_{\varepsilon}(\overline{p})^c$, a contradiction (see Aliprantis and Border, 2006, p. 442). $\triangle$

The Claim implies

$$\int \phi\left(\frac{1}{2}p(R_1) + \frac{1}{2}p(R_2)\right) d\mu(p) > \frac{1}{2}\int (\phi \circ p)(R_1)\, d\mu(p) + \frac{1}{2}\int (\phi \circ p)(R_2)\, d\mu(p),$$

that is, $\frac{1}{2}f_1 + \frac{1}{2}f_2 \succ f_1 \sim f_2$.

Act g_1 is evaluated as $\phi(1/2)$. Act g_2 is evaluated as $\int \phi(1/2 + (p(B_1R_2) - p(R_1B_2))/2)\, d\mu(p)$. Define $\gamma : \Delta \to \mathbb{R}$ by $\gamma(p) = 1/2 + (p(B_1R_2) - p(R_1B_2))/2$. Since $p(B_1R_2) - p(R_1B_2) = p(R_2) - P(R_1)$, the Claim implies $\gamma(p) \neq 1/2$ for all $p \in A$. Therefore, by the Jensen inequality and the assumption on μ, we have

$$\int (\phi \circ \gamma)(p)\, d\mu(p) < \phi\left(\int \gamma(p)\, d\mu(p)\right) = \phi\left(\int \left(\frac{1}{2} + \frac{1}{2}(p(R_2) - p(R_1))\right) d\mu(p)\right) = \phi\left(\frac{1}{2}\right),$$

that is, $g_1 \succ g_2$.

(ii) By properties (1) and (3), $\max_{p\in C} p(R_1) = \max_{p\in C} p(R_2)$ and $\min_{p\in C} p(R_1) = \min_{p\in C} p(R_2)$, as well as

$$\begin{aligned}
\max_{p\in C}\left(\frac{1}{2}p(R_1) + \frac{1}{2}p(R_2)\right) &= \frac{1}{2}\max_{p\in C} p(R_1) + \frac{1}{2}\max_{p\in C} p(R_2) \\
\min_{p\in C}\left(\frac{1}{2}p(R_1) + \frac{1}{2}p(R_2)\right) &= \frac{1}{2}\min_{p\in C} p(R_1) + \frac{1}{2}\min_{p\in C} p(R_2)
\end{aligned}$$

Hence, $\frac{1}{2}f_1 + \frac{1}{2}f_2 \sim f_1 \sim f_2$. From $\min_{p\in C}(p(R_2) - p(R_1)) = -\max_{p\in C}(p(R_2) - p(R_1))$,

$$\begin{aligned}
&\alpha \min_{p\in C}\left(\frac{1}{2} + \frac{1}{2}(p(R_2) - p(R_1))\right) + (1-\alpha)\max_{p\in C}\left(\frac{1}{2} + \frac{1}{2}(p(R_2) - p(R_1))\right) \\
= \; &\frac{1}{2} + \frac{1-2\alpha}{2}\max_{p\in C}(p(R_2) - p(R_1)),
\end{aligned}$$

and so $g_1 \succ g_2$ if and only if $1/2 > 1/2 + (1/2 - \alpha)\max_{p\in C}(p(R_2) - p(R_1))$. By properties (2) and (3), $\max_{p\in C}(p(R_2) - p(R_1)) > 0$, so that $g_1 \succ g_2$ if and only if $\alpha > 1/2$. ■

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